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Zeros-interlacing-poles systems and applications to model reduction by moment matching

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To my brother, Riccardo.

ABSTRACT

This thesis is concerned with the analysis of the “zero-interlacing-poles”(ZIP) systems, a particular, yet fascinating, class of linear dynamical systems. The present work is divided in four parts. The first part contains the definitions and the basic properties of ZIP systems. In the second part, the model order reduction of ZIP systems by moment matching is studied. Therein, under suitable assumptions, the inheritance of the ZIP property in the reduced order model is proved. The third part is devoted to a nonlinear enhancement of the notion of ZIP system. In the fourth part, examples and applications arising in the engineering domain are considered.

Acknowledgements

Many people have contributed one way or another to the birth of this manuscript.

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Chapter 1

Introduction

This thesis focuses on the class of linear dynamical systems satisfying the so-called “zeros-interlacing-poles” (ZIP) property. From a theoretical point of view, the peculiar class of ZIP systems is composed, by definition, by systems whose transfer function has interlaced real zeros and poles. From a practical point of view, ZIP systems arise in the field of electrical networks, precisely when considering parallel interconnections of RC circuits. However, the existence of ZIP systems in different frameworks cannot be excluded.

In this work, the novel notion of left ZIP system is introduced and an originally developed state-space realization of such systems is proposed. The mentioned representation allows one to easily parameterize zeros and poles of an arbitrary left ZIP system. A characterization of the zeros of a ZIP systems is discussed, linking the notion of ZIP and left ZIP system. Subsequently, the model reduction of ZIP systems by moment matching is investigated. Among a number of properties holding for ZIP systems, the inheritance of the ZIP property in the reduced order model undoubtedly stands out. The inheritance issue has been previously discussed in [15]. Therein, it has been shown that several SVD-based model reduction methods preserve the ZIP property. Consistently with this previous study, reduced order models of order n of ZIP systems matching n finite moments and n Markov parameters preserve the ZIP property. Next, the problem of determining a ZIP reduced order model with prescribed poles is investigated. Necessary and sufficient conditions for the feasibility of the pole placement are provided and a polynomial interpretation of the sufficient conditions is explored. Also, an attempt to define a nonlinear enhancement of the notion of ZIP system is presented. Finally, in order to validate the proposed results, some examples of ZIP systems arising in the engineering domain have been selected.

Notation

Throughout this work we use standard notation. $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{p\times m}$ respectively denote the set of real numbers, of n-dimensional vectors with real components, and of $p \times m$-dimensional matrices with real entries, respectively. $\mathbb{R}_+$ ($\mathbb{R}_-$) denotes the set of non-negative (non-positive) real numbers. $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{C}_0$ denotes the set of complex numbers with zero real part, $\mathbb{C}_-$ denotes the set of complex numbers with negative real part. $\sigma(M)$ denotes the spectrum of the matrix $M \in \mathbb{R}^{n\times n}$, while $\emptyset$ denotes the empty set. $\mathbf{e}_i$ denotes the ith vector of the standard basis of $\mathbb{R}^n$, the vector with a 1 in the ith coordinate and 0's elsewhere. $L_f^k h$ denotes the kth Lie derivative of the smooth function h along the smooth vector field f, as defined in [11, Chapter 1].

Chapter 2

Preliminaries

In this chapter we review some basic results concerning *linear dynamical systems*. In a broad outline, a *dynamical system* is a mathematical model that describes the variability of a *state* over time. The behavior of the modeled process is condensed in a function representing the time dependence of a point, the state, in an appropriate geometrical space. A dynamical system is normally intended to absorb *inputs*, process them in some way and produce *outputs*. When the function characterizing the dynamical system qualifies as *linear*, the equations governing the evolution of the process are analytically and numerically well manageable. In addition, properly choosing the inputs, one may easily predict or even control the state and the output of the system. The reader is referred to [8, 10] for a clear and exhaustive introduction to linear dynamical systems.

2.1 Linear dynamical systems

2.1.1 State-space representation of linear systems

Given the linear spaces $\mathbb{X} = \mathbb{R}^n$, $\mathbb{U} = \mathbb{R}^m$, $\mathbb{Y} = \mathbb{R}^p$, respectively called the *state space*, the *input space* and the *output space*, the *state equations* describing a linear continuous-time system are a set of first-order linear differential equations

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad t \in \mathbb{R}, \tag{2.1}$$

where $x(t) \in \mathbb{X}$ is the value of the *state (function)* of the system $x\colon \mathbb{R} \to \mathbb{X}$ at time t, while $u(t) \in \mathbb{U}$ is the value of the *input (function)* $u\colon \mathbb{R} \to \mathbb{U}$ at time t. The *output equations* are a set of linear algebraic equations

$$y(t) = Cx(t) + Du(t), \qquad t \in \mathbb{R}, \tag{2.2}$$

where $y(t) \in \mathbb{Y}$ is the value of the *output (function)* $y\colon \mathbb{R} \to \mathbb{Y}$ at time t. In what follows, the term *linear system* will be used to denote a linear, finite-dimensional, time-invariant, continuous time system: linear because $\mathbb{X}$, $\mathbb{U}$, $\mathbb{Y}$ are linear spaces; finite-dimensional because n, m, p are all finite positive integers; time-invariant because A, B, C, D do not depend on time, hence their matrix representations are constant $n \times n$, $n \times m$, $p \times n$ and $p \times m$ matrices; finally continuous-time since $t \in \mathbb{R}$.

Definition 1 (State-space description)**.** *The* state-space description *of a linear system Σ, is a quadruple of matrices*

$$\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \qquad A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}. \tag{2.3}$$

The dimension *of the system is defined as the dimension of the associated state space*

$$\dim \Sigma = n. \tag{2.4}$$

We will also use the notation $\Sigma = (A, B, C, D)$ or more shortly $\Sigma = (A, B, C)$ to denote a system Σ where $D = 0$ or D is irrelevant for the argument pursued. Similarly, $\Sigma = (A, B)$ and $\Sigma = (A, C)$ will denote a system $\Sigma = (A, B, C)$ where respectively C and B are irrelevant for the argument pursued.

2.1.2 Impulse response and transfer function

In what follows we will give the solution of (2.1). Then, we will derive the notion of *impulse response* and consequently define the concept of *transfer function* of a linear system. To this end, we briefly recall the definition of *matrix exponential.* Given $M \in \mathbb{R}^{n\times n}$ and $t \in \mathbb{R}$, the matrix exponential of M is defined as

$$e^{Mt} := I_n + \frac{t}{1!}M + \frac{t^2}{2!}M^2 + \dots. \tag{2.5}$$

Let us denote by $x(t; x_0, u)$ the solution of the state equations (2.1), namely the state of the system at time t attained from the initial state x_0 a time t_0, under the influence of the input u. Then, the solution of the state equations (2.1) has the following expression

$$x(t; x_0, u) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau, \qquad t \geq t_0. \tag{2.6}$$

Substituting (2.6) in (2.2), it follows that the output is given by

$$\begin{aligned} y(t) &= Cx(t; x_0, u) + Du(t) \\ &= Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)d\tau, \ t \geq t_0. \end{aligned} \tag{2.7}$$

If we set $t_0 = -\infty$, $x_0 = 0$ and $u = \delta$, the δ-distribution, equation (2.7) yields the so-called *impulse response*

$$h(t) = \begin{cases} Ce^{At}B + \delta(t)D, & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{2.8}$$

Finally, taking the Laplace transform of the impulse response, returns

$$\boxed{W(s) = C(sI_n - A)^{-1}B + D} \tag{2.9}$$

which is called the *transfer function* of Σ. The transfer function of a linear time-invariant system is a powerful tool, it incorporates the relation between the input and output. Expanding the transfer function in Laurent series[1] in the neighborhood of infinity, we get

$$\begin{aligned} W(s) &= D + CBs^{-1} + CABs^{-2} + CA^2Bs^{-3} + \dots && (2.10) \\ &= h_0 + h_1 s^{-1} + h_2 s^{-2} + h_3 s^{-3} + \dots. && (2.11) \end{aligned}$$

The matrices $h_0 = D \in \mathbb{R}^{p\times m}$ and $h_k = CA^{k-1}B \in \mathbb{R}^{p\times m}$ are often referred to as *Markov parameters* of the system Σ.

[1] The Laurent series of a complex function $f : \mathbb{C} \to \mathbb{C}$ at a point $c \in \mathbb{C}$ is given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-c)^n, \quad z \in \mathbb{C},$$

where the $a_n \in \mathbb{C}$ are constants, defined by

$$a_n := \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z-c)^{n+1}} \mathrm{d}z.$$

If there exists a holomorphic function g and a positive integer n, such that

$$f(z) = \frac{g(z)}{(z-a)^n} \quad (f(z) = (z-a)^n g(z)), \quad g(a) \neq 0,$$

hold, then $a \in \mathbb{C}$ is called a *pole* (*zero*) of f. The smallest such n is called the *order* of the pole (zero). A pole of order 1 is called a *simple* pole.

2.1.3 Reachability, observability and realization

In this subsection we introduce the fundamental concepts of *reachability* and *observability*. The former describes the ability of an external input to move the internal state of a system from any initial state to any other final state in a finite time interval. The latter gives a measure for how well internal states of a system can be inferred by knowledge of its external outputs. The observability and reachability of a system are dual concepts, and both contribute to the solution of the *realization problem*. For a survey on reachability, observability and realization see [1, 8].

Definition 2 (Reachability). *Consider the system $\Sigma = (A, B)$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. A state $x^\star \in \mathbb{X}$ is* reachable *from the zero state if there exist an input $u^\star$ and a time $T^\star < \infty$ such that*

$$x^\star = x(T^\star; 0, u^\star). \tag{2.12}$$

The reachable subspace $\mathbb{X}^R \subseteq \mathbb{X}$ *of Σ is the set containing all reachable states of Σ. The system Σ is said* reachable *if $\mathbb{X}^R = \mathbb{X}$. Finally, the matrix*

$$\mathcal{R} := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{2.13}$$

is said the reachability matrix *of Σ.*

Hereafter we list some equivalent conditions to reachability.

Proposition 1 (Reachability conditions). *The following statements are equivalent.*

1. *The system $\Sigma = (A, B)$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, is reachable.*

2. *The reachability matrix of Σ has full rank:* $\operatorname{rank}(\mathcal{R}) = n$.

3. *No left eigenvector v of A is in the left kernel of B:*

$$v^\top A = \lambda v^\top \Rightarrow v^\top B \neq 0. \tag{2.14}$$

4. $\operatorname{rank}\Big(\left[\begin{array}{c|c} sI_n - A & -B \end{array}\right]\Big) = n$ *for all $s \in \mathbb{C}$.*

Definition 3 (Observability). *Consider the system $\Sigma = (A, C)$, $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$. A state $x^\star \in \mathbb{X}$ is* undistinguishable *from the zero state if*

$$0 = y(t) = Cx(t; x^\star, 0), \qquad \text{for all } t \geq 0, \tag{2.15}$$

namely, if the output produced by the initial state $x^\star$ is undistinguishable from the output produced by a zero initial state for all $t \geq 0$. The unobservable subspace $\mathbb{X}^{no} \subseteq \mathbb{X}$ *of Σ is the set of all undistinguishable states (from the zero state) of Σ. The system Σ is said* observable *if $\mathbb{X}^{no} = \{0\}$. Finally, the matrix*

$$\mathcal{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{2.16}$$

is said to be the observability matrix *of Σ.*

Hereafter we list some equivalent conditions to observability.

Proposition 2 (Observability conditions)**.** *The following statements are equivalent.*

1. *The system $\Sigma = (A, C)$, $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, is observable.*
2. *The observability matrix of Σ has full rank:* $\operatorname{rank}(\mathcal{O}) = n$.
3. *No right eigenvector v of A is in the right kernel of C:*

$$Av = \lambda v \Rightarrow Cv \neq 0. \tag{2.17}$$

4. $\operatorname{rank}\Big(\begin{bmatrix} sI_n - A \\ \hline C \end{bmatrix}\Big) = n$ *for all $s \in \mathbb{C}$.*

Given the sequence of $p \times m$ matrices h_k, $k > 0$, the *realization problem* consists of finding a positive integer n and constant matrices (A, B, C) such that

$$h_k = CA^{k-1}B, \qquad A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, k = 1, 2, \ldots. \tag{2.18}$$

The system $\Sigma = (A, B, C)$ is then called a *realization* of the sequence h_k, and the latter is called a *realizable* sequence. The system $\Sigma = (A, B, C)$ is a *minimal* realization if among all realizations of the sequence, its dimension n is the smallest possible. Without loss of generality we assumed for all sequences $h_0 = 0$.[2]

[2]It readily follows without computations that if $h_0 \neq 0$, then $D = h_0$.

The key tool for solving the realization problem is the *Hankel matrix*

$$\mathcal{H} = \begin{bmatrix} h_1 & h_2 & \dots & h_k & \dots \\ h_2 & h_3 & \dots & h_{k+1} & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_k & h_{k+1} & \dots & h_{2k-1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.19}$$

composed by the Markov parameters. It can be proven [1] that the sequence h_k, $k > 0$, is realizable if and only if the rank of $\mathcal{H}$ is finite. More specifically, if the sequence h_k, $k > 0$, is realizable by means of the system $\Sigma = (A, B, C)$, $\mathcal{H}$ can be factored as the product of the observability matrix $\mathcal{O}$ and of the reachability matrix $\mathcal{R}$, namely $\mathcal{H} = \mathcal{O}\mathcal{R}$. Hence, the rank of $\mathcal{H}$ is finite. The converse, that if the rank of $\mathcal{H}$ is finite then the sequence h_k, $k > 0$, is realizable can be proved using the *Silverman realization algorithm*, and is considerably more difficult.

2.1.4 Interconnections

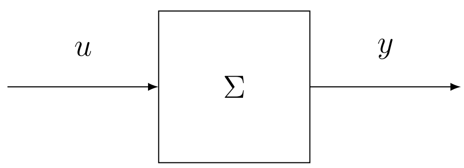

Figure 2.1: The two-port block represents a system with input u and output y, where the directions of the arrows specify which is which.

It is convenient to represent systems by block diagrams as in Figure 2.1. These diagrams generally serve as compact representations for complex equations. Although not explicitly represented in the diagram, one must keep in mind the existence of the state, which affects the output through the initial condition.

Interconnections of block diagrams are useful to highlight special structures in state-space equations. To clarify, consider the systems

$$\Sigma_1 : \ \dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t), \quad y_1(t) = C_1 x_1(t) + D_1 u_1(t), \tag{2.20}$$
$$\Sigma_2 : \ \dot{x}_2(t) = A_2 x_2(t) + B_2 u_2(t), \quad y_2(t) = C_2 x_2(t) + D_2 u_2(t), \tag{2.21}$$

where $x_1(t) \in \mathbb{R}^{n_1}$, $u_1(t) \in \mathbb{R}^{m_1}$, $y_1(t) \in \mathbb{R}^{p_1}$, $x_2(t) \in \mathbb{R}^{n_2}$, $u_2(t) \in \mathbb{R}^{m_2}$, $y_2(t) \in \mathbb{R}^{p_2}$, $t \in \mathbb{R}$. Denote by $H_1(s)$ and $H_2(s)$ the transfer functions of these two systems respectively. So to speak, the general procedure to obtain the state-space representation for the interconnected system consists of "stacking" the states of the individual subsystems in a "tall" vector x and computing x using the state and output equations of the individual blocks. The output equation is also obtained from the output equations of the subsystems.

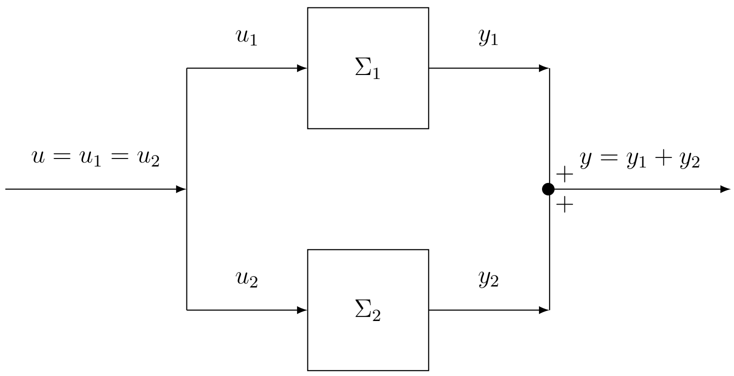

Figure 2.2: Parallel interconnection of Σ_1 and Σ_2.

Parallel interconnection The *parallel interconnection* Σ_p of Σ_1 and Σ_2 is shown in Figure 2.2. The interconnection is described by equations

$$u = u_1 = u_2, \tag{2.22}$$

$$y = y_1 + y_2. \tag{2.23}$$

Assuming $\mathbb{X} = \mathbb{X}_1 \oplus \mathbb{X}_2$ as state space for Σ_p, the equations governing the evolution of system Σ_p are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{2.24}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_1 + D_2 \end{bmatrix} u \tag{2.25}$$

The transfer function of the parallel interconnection is $H_p(s) = H_1(s) + H_2(s)$. We will also use the notation $\Sigma_p = \Sigma_1 + \Sigma_2$ to denote the parallel interconnection of systems Σ_1 and Σ_2.

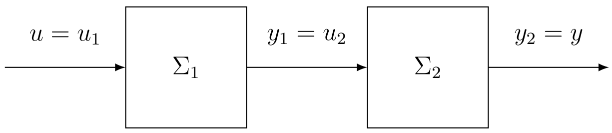

Figure 2.3: Series interconnection of Σ_1 and Σ_2.

Series interconnection The *series interconnection* Σ_s of Σ_1 and Σ_2 is shown in Figure 2.3. The interconnection is described by equations

$$u = u_1, \tag{2.26}$$
$$y_1 = u_2, \tag{2.27}$$
$$y = y_2. \tag{2.28}$$

Assuming $\mathbb{X} = \mathbb{X}_1 \oplus \mathbb{X}_2$ as state space for Σ_s and eliminating the connection variable $y_1 = u_2$, the equations governing the evolution of system Σ_s are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2D_1 \end{bmatrix} u \tag{2.29}$$
$$y = \begin{bmatrix} D_2C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_2D_1 \end{bmatrix} u \tag{2.30}$$

The transfer function of the series interconnection is $H_s(s) = H_1(s)H_2(s)$. We will also use the notation $\Sigma_s = \Sigma_1 \to \Sigma_2$ to denote the series interconnection of systems Σ_1 and Σ_2.

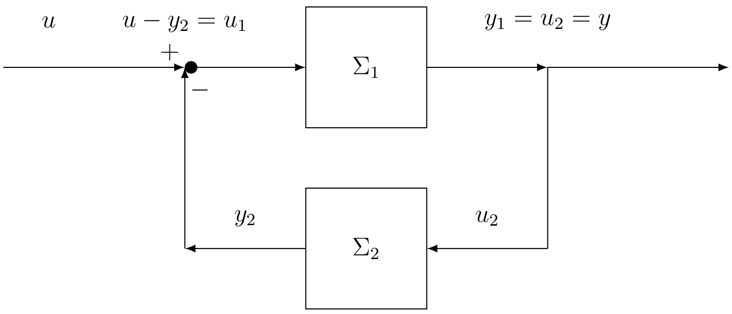

Figure 2.4: Feedback interconnection of Σ_1 and Σ_2.

Feedback interconnection The *(negative) feedback interconnection* Σ_f of Σ_1 and Σ_2 is shown in Figure 2.4. The interconnection is described by equations

$$u_1 = u - y_2, \tag{2.31}$$
$$y = y_1 = u_2. \tag{2.32}$$

For our purposes it will be sufficient to consider Σ_1 and Σ_2 as single-input single-output (SISO), namely $m_1 = m_2 = p_1 = p_2 = 1$. Assuming $\mathbb{X} = \mathbb{X}_1 \oplus \mathbb{X}_2$ as state space for Σ_f and eliminating the connection variable $u_1 = u - y_2$, the equations governing the evolution of system Σ_f are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left(\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} + \frac{1}{1 - D_1 D_2} \begin{bmatrix} 0 & B_1 \\ B_2 & 0 \end{bmatrix} \begin{bmatrix} C_1 & D_1 C_2 \\ D_2 C_1 & C_2 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{1 - D_1 D_2} \begin{bmatrix} B_1 \\ D_1 G_2 \end{bmatrix} u \tag{2.33}$$

$$y = \frac{1}{1 - D_1 D_2} \begin{bmatrix} C_1 & D_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{D_1}{1 - D_1 D_2} \end{bmatrix} u \tag{2.34}$$

When $D_1 D_2 = 1$ the feedback interconnections is ill-posed, and the interconnection is not feasible. The transfer function of the feedback interconnection is $H_f(s) = \frac{H_1(s)}{1 + H_1(s) H_2(s)}$. We will also use the notation $\Sigma_f = \Sigma_1 \lozenge \Sigma_2$ to denote the feedback interconnection of systems Σ_1 and Σ_2.

2.2 The Sylvester equation

Given the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{k \times k}$, $C \in \mathbb{R}^{n \times k}$, the matrix equation

$$\boxed{AX + XB = C} \tag{2.35}$$

in the unknown $X \in \mathbb{R}^{n \times k}$ is the *Sylvester equation*, and it is one of the most important matrix equations in theory and applications.
Let us recall some of its well-known properties that may be found in standard references on matrix analysis as [1]. The Sylvester equation has a unique solution for each C if and only if A and B have no eigenvalues in common. Roth proved in [14] that the Sylvester equation has some solution (possibly nonunique) if and only if

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \text{ and } \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{ are similar.} \tag{2.36}$$

The concepts of reachability and observability are intimately related with properties of the Sylvester equation, see for instance [6].

Chapter 3

ZIP systems

In this chapter, we study the family of linear systems satisfying the so-called "zeros-interlacing-poles" (ZIP) property. These systems may arise in the synthesis of RC networks[12, 15] or engineering-related issues, such as the modeling of axial bearings [9]. Hereafter, we formally define the concept of ZIP system, extend this notion to left ZIP systems and characterize some properties of their state-space realizations.

3.1 ZIP systems

Definition 4 (ZIP system)**.** *Consider an n-dimensional single-input-single-output linear continuous time system $\Sigma = (A, B, C)$ and let its trasfer function be*

$$W(s) = K\frac{\prod_{j=i}^{n-1}(s+z_j)}{\prod_{i=i}^{n}(s+a_i)}. \tag{3.1}$$

The system Σ, and correspondingly the transfer function $W(s)$, are ZIP *if $0 < a_i < z_i < a_{i+1}$ holds for $i = 1, \dots, n-1$.*

Without loss of generality K is assumed to be positive in the sequel. By referring to the previous definition, consider the system $\Sigma = (A, B, C)$ and define

$$\mathcal{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{3.2}$$

$$\mathcal{R} := \begin{bmatrix} B \ AB \ \cdots \ A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{3.3}$$

$$\mathcal{H} := \begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \\ CAB & CA^2B & \cdots & CA^nB \\ \vdots & \ddots & \ddots & \vdots \\ CA^{n-1}B & CA^nB & \cdots & CA^{2n-2}B \end{bmatrix} = \mathcal{O}\mathcal{R} \in \mathbb{R}^{n\times n}, \tag{3.4}$$

$$\sigma\mathcal{H} := \begin{bmatrix} CAB & CA^2B & \cdots & CA^nB \\ CA^2B & CA^3B & \cdots & CA^{n+1}B \\ \vdots & \ddots & \ddots & \vdots \\ CA^nB & CA^{n+1}B & \cdots & CA^{2n-1}B \end{bmatrix} = \mathcal{O}A\mathcal{R} \in \mathbb{R}^{n\times n}. \tag{3.5}$$

Definition 5 (Simple compartment)**.** *A* simple compartment *is a one-dimensional system described by the transfer function*

$$C(s) = \frac{b}{s+a}, \tag{3.6}$$

with $a > 0$ and $b > 0$.

Proposition 3 (Properties of ZIP systems)**.** *The following statements are equivalent.*

1. *W(s) is a ZIP transfer function.*
2. *W(s) describes a system composed by the parallel interconnection of n simple compartments with distinct (negative) poles.*
3. *W(s) can be written as*

$$W(s) = \sum_{i=1}^{n} \frac{b_i^2}{s+a_i}, \tag{3.7}$$

   with $a_i > 0$, $b_i^2 > 0$, $a_i \neq a_j$ $\forall i \neq j$.

4. *W(s) admits an asymptotically stable minimal diagonal realization*

$$\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -a_1 & & & b_1 \\ & \ddots & & \vdots \\ & & -a_n & b_n \\ \hline b_1 & \cdots & b_n & 0 \end{array}\right], \tag{3.8}$$

   where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$ with $a_i > 0, b_i > 0, a_i \neq a_j \forall i \neq j$.

5. *$\mathcal{H} > 0$ and $\sigma\mathcal{H} < 0$.*

Proof. The proof of Proposition 3 can be deduced combining [3, p.332] and [15, pp.1-2]. □

3.2 Left ZIP systems

Parallel to what was done for ZIP systems, we introduce the notions of *simple left/right compartment* and *left/right ZIP system.* Then, we study the properties of left ZIP systems and discuss their relation with ZIP systems.

Definition 6 (Left/Right simple compartment)**.** *A* left simple compartment (LSC) *is a one-dimensional system described by the transfer function*

$$C(s) = \frac{s+z}{s+a}, \tag{3.9}$$

with $a > 0$ *and* $z > a$*. The system is said a* right simple compartment (RSC) *if* $a > 0$ *and* $z < a$*.*

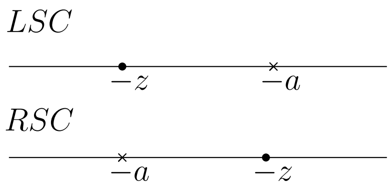

Figure 3.1: The zero $-z$ of a LSC is *smaller* than the pole $-a$, conversely the pole $-a$ of a RSC is *smaller* than the zero $-z$.

Remark 3.1 (Realization of a left simple compartment)**.** The transfer function of an arbitrary left simple compartment can be written as

$$C(s) = \frac{s+z}{s+a} = 1 + \frac{z-a}{s+a}. \tag{3.10}$$

Defining $b := \sqrt{z-a}$, a minimal diagonal realization of $C(s)$ is given by

$$\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} -a & \sqrt{z-a} \\ \hline \sqrt{z-a} & 1 \end{array}\right] = \left[\begin{array}{c|c} -a & b \\ \hline b & 1 \end{array}\right] \tag{3.11}$$

Finally, every zero-pole pair $(-z, -a)$ of a transfer function satisfying $-z < -a$, as shown in Figure 3.1, can be regarded as a simple compartment and hence represented by (3.11) with $b > 0$. ▲

Definition 7 (Left/Right ZIP system)**.** *Consider a single-input-single-output linear continuous time system* $\Sigma = (A, B, C, D)$ *and let its transfer function be*

$$W(s) = K \frac{\prod_{j=i}^{n}(s + z_j)}{\prod_{i=i}^{n}(s + a_i)} \tag{3.12}$$

Notice that the The system Σ *is* left ZIP (LZ) *if*

$$-z_1 < -a_1 < -z_2 < -a_2 < \cdots < -z_n < -a_n < 0. \tag{3.13}$$

The system Σ *is* right ZIP (RZ) *if*

$$-a_1 < -z_1 < -a_2 < -z_2 < \cdots < -a_n < -z_n < 0. \tag{3.14}$$

By definition, a left/right simple compartment is a one-dimensional left/right ZIP system. Notice that the transfer function of a left/right ZIP system (as well as its inverse) is *biproper*, namely the degree of its numerator equals the degree of its denominator.

Henceforward, we will refer to *left* simple compartments and *left* ZIP systems when not specified. We now investigate the properties of left ZIP systems and their connection with simple compartments. We will assume hereafter that $K = 1$.[1]

3.2.1 Left ZIP systems representations

We begin with a simple example incorporating the main idea that we will expand throughout this section.

Example 8 (Series interconnection of two left simple compartments)**.** Consider the series interconnection of two arbitrary left simple compartments

$$C_1(s) = \frac{s+z_1}{s+a_1}, \quad a_1 > 0, z_1 > a_1, \tag{3.15}$$

$$C_2(s) = \frac{s+z_2}{s+a_2}, \quad a_2 > 0, z_2 > a_2, \tag{3.16}$$

with different poles and whose corresponding minimal realizations are respectively

$$\Sigma_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] = \left[\begin{array}{c|c} -a_1 & b_1 \\ \hline b_1 & 1 \end{array}\right], \tag{3.17}$$

$$\Sigma_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{c|c} -a_2 & b_2 \\ \hline b_2 & 1 \end{array}\right], \tag{3.18}$$

where without loss of generality $a_1 > a_2$ and $b_i^2 = z_i - a_i$ for $i = 1, 2$.

The transfer function of the series interconnection $\Sigma_s := \Sigma_1 \to \Sigma_2$ is

$$C_s(s) = C_1(s)C_2(s) = \Big(\frac{s+z_1}{s+a_1}\Big)\Big(\frac{s+z_2}{s+a_2}\Big). \tag{3.19}$$

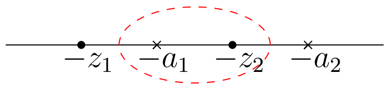

Figure 3.2: Assuming $a_1 > a_2$, $\Sigma_s = \Sigma_1 \to \Sigma_2$ is left ZIP if and only if $-a_1 < -z_2$.

The corresponding state-space description is given by

$$\begin{aligned}
\Sigma_s = \left[\begin{array}{c|c} A_s & B_s \\ \hline C_s & D_s \end{array}\right] &= \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2 B_1^\top & A_2 & B_2 \\ \hline B_1^\top & B_2^\top & 1 \end{array}\right] \\
&= \left[\begin{array}{cc|c} -a_1 & 0 & b_1 \\ b_2 b_1 & -a_2 & b_2 \\ \hline b_1 & b_2 & 1 \end{array}\right], \qquad (3.20)
\end{aligned}$$

having exploited equations (2.29) and (2.30). As shown in Figure 3.2, Σ_s is left ZIP if and only if

$$\begin{gathered}
a_1 > a_2 > 0; \\
b_i \neq 0, \quad i = 1, 2; \\
a_1 > z_2 = a_2 + b_2^2.
\end{gathered} \qquad (3.21)$$

or, equivalently,

$$\begin{gathered}
a_1 > a_2 > 0; \\
b_i \neq 0, \quad i = 1, 2; \\
b_2 < \sqrt{a_1 - a_2}
\end{gathered} \qquad (3.22)$$

Note that the ordering of a_i's, and correspondingly the ordering of b_i's, is relevant to obtain the state-space description given by equation (3.20). Finally, the *zeros* of Σ_s are straightforwardly given by $z_i = a_i + b_i^2$ for $i = 1, 2$. ∎

The reasoning of the previous example will be exploited now to derive a new state-space representation for n-dimensional left ZIP systems.

Consider the series interconnection $\Sigma_s := \Sigma_1 \to \Sigma_2 \to \cdots \to \Sigma_n$ of n simple compartments $\Sigma_i = (-a_i, b_i, b_i, 1)$, indexed so that the corresponding poles verify

$$a_1 > a_2 > \cdots > a_n > 0. \qquad (3.23)$$

[1]If $K \neq 0$, it is sufficient to consider $\bar{W}(s) = \frac{W(s)}{K}$. Assuming $K = 1$ corresponds to a scaling in the measurement unit of u.

The transfer function of Σ_s is

$$C_s(s) = \prod_{i=1}^{n} C_i(s) = \prod_{i=1}^{n} \left(\frac{s + z_i}{s + a_i} \right) \tag{3.24}$$

and necessary and sufficient conditions for system Σ_s to left ZIP are[2]

$$\begin{gathered} a_1 > a_2 > \cdots > a_n > 0; \\ b_i \neq 0, \quad i = 1, 2, \ldots, n; \\ a_i > z_{i+1} = a_{i+1} + b_{i+1}^2, \quad i = 1, 2, \ldots, n-1. \end{gathered} \tag{3.25}$$

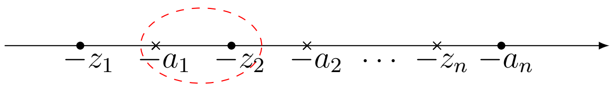

Figure 3.3: The system $\Sigma_s = \Sigma_1 \to \Sigma_2 \to \cdots \to \Sigma_n$ is left ZIP if and only if $-a_i < -z_{i+1}$ for all $i = 1, \ldots, n-1$.

In light of equations (2.29) and (2.30), given two linear systems of the form $\Sigma_i = (A_i, B_i, B_i^\top, D_i)$, for instance $i = 1, 2$, their series interconnection is

$$\Sigma_1 \to \Sigma_2 = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2 B_1^\top & A_2 & B_2 D_1 \\ \hline D_2 B_1^\top & B_2^\top & D_1 D_2 \end{array} \right]. \tag{3.26}$$

Now, think of Σ_s as the repeated interconnection of n consecutive systems

$$\Sigma_s := ((\ldots((\Sigma_1 \to \Sigma_2) \to \Sigma_3) \to \ldots) \to \Sigma_n). \tag{3.27}$$

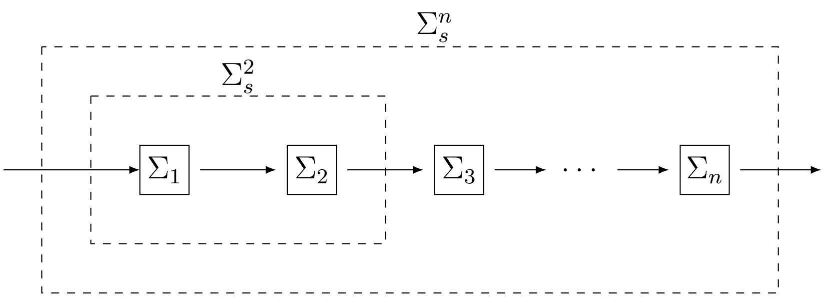

Figure 3.4: Iterative construction of Σ_s^n.

[2]These conditions translate on the coefficients b_i the left ZIP property; see Figure 3.3.

Denoting the series of the first k systems by $\Sigma_s^k := \Sigma_1 \to \Sigma_2 \to \cdots \to \Sigma_k$ and repeatedly exploiting (3.26), we have

$$\Sigma_s^1 = \left[\begin{array}{c|c} A_s^1 & B_s^1 \\ \hline C_s^1 & D_s^1 \end{array}\right] = \left[\begin{array}{c|c} -a_1 & b_1 \\ \hline b_1 & 1 \end{array}\right]$$

$$\Sigma_s^2 = \left[\begin{array}{c|c} A_s^2 & B_s^2 \\ \hline C_s^2 & D_s^2 \end{array}\right] = \left[\begin{array}{cc|c} A_s^1 & 0 & B_s^1 \\ B_2(B_s^1)^\top & A_2 & B_2 D_s^1 \\ \hline D_2(B_s^1)^\top & B_2^\top & D_s^1 D_2 \end{array}\right] = \left[\begin{array}{cc|c} -a_1 & 0 & b_1 \\ b_1 b_2 & -a_2 & b_2 \\ \hline b_1 & b_2 & 1 \end{array}\right]$$

$$\Sigma_s^3 = \left[\begin{array}{c|c} A_s^3 & B_s^3 \\ \hline C_s^3 & D_s^3 \end{array}\right] = \left[\begin{array}{cc|c} A_s^2 & 0 & B_s^2 \\ B_3(B_s^2)^\top & A_3 & B_3 D_s^2 \\ \hline D_3(B_s^2)^\top & B_3^\top & D_s^2 D_3 \end{array}\right] = \left[\begin{array}{ccc|c} -a_1 & 0 & 0 & b_1 \\ b_1 b_2 & -a_2 & 0 & b_2 \\ b_1 b_3 & b_2 b_3 & -a_3 & b_3 \\ \hline b_1 & b_2 & b_3 & 1 \end{array}\right]$$

$$\vdots$$

$$\Sigma_s^k = \left[\begin{array}{c|c} A_s^k & B_s^k \\ \hline C_s^k & D_s^k \end{array}\right] = \left[\begin{array}{cc|c} A_s^{k-1} & 0 & B_s^{k-1} \\ B_k(B_s^{k-1})^\top & A_k & B_k D_s^{k-1} \\ \hline D_k(B_s^{k-1})^\top & B_k^\top & D_s^{k-1} D_k \end{array}\right]$$

$$= \left[\begin{array}{ccccc|c} -a_1 & 0 & \cdots & 0 & 0 & b_1 \\ b_1 b_2 & -a_2 & 0 & \cdots & 0 & b_2 \\ b_1 b_3 & b_2 b_3 & -a_3 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & b_{k-1} \\ b_1 b_k & b_2 b_k & \cdots & b_{k-1} b_k & -a_k & b_k \\ \hline b_1 & b_2 & \cdots & b_{k-1} & b_k & 1 \end{array}\right]$$

The sequence terminates at the $n-$th step, yielding

$$\Sigma_s = \Sigma_s^n = \left[\begin{array}{c|c} A_s^n & B_s^n \\ \hline C_s^n & D_s^n \end{array}\right] \tag{3.28}$$

$$= \left[\begin{array}{ccccc|c} -a_1 & 0 & \cdots & 0 & 0 & b_1 \\ b_1 b_2 & -a_2 & 0 & \cdots & 0 & b_2 \\ b_1 b_3 & b_2 b_3 & -a_3 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & b_{n-1} \\ b_1 b_n & b_2 b_n & \cdots & b_{n-1} b_n & -a_n & b_n \\ \hline b_1 & b_2 & \cdots & b_{n-1} & b_n & 1 \end{array}\right]. \tag{3.29}$$

In this form, the zeros of the system can be straightforwardly calculated using the relation $z_i = a_i + b_i^2$. The minimality of the system is guaranteed as the entries b_i are nonzero. The ordering of a_i's, and correspondingly of b_i's, is important, since it implies the cascade of inequalities $b_{i+1} < \sqrt{a_i - a_{i+1}}$,

$i = 1, 2, \ldots, n$. Finally, provided that K is unitary, there is a one to one correspondence between a system represented by the previous state space realization and a left ZIP transfer function.

A fortiori we can state the following

Proposition 4 (Left ZIP state-space canonical form)**.** *An n-dimensional single-input single-output linear continuous time system Σ is left ZIP if and only if there exist $a_1, a_2, \ldots, a_n \in \mathbb{R}_+$ and $b_1, b_2, \ldots, b_n \in \mathbb{R}_+$ such that*

$$\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccccc|c} -a_1 & 0 & \cdots & 0 & 0 & b_1 \\ b_1 b_2 & -a_2 & 0 & \cdots & 0 & b_2 \\ b_1 b_3 & b_2 b_3 & -a_3 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & b_{n-1} \\ b_1 b_n & b_2 b_n & \cdots & b_{n-1} b_n & -a_n & b_n \\ \hline b_1 & b_2 & \cdots & b_{n-1} & b_n & 1 \end{array}\right], \tag{3.30}$$

where

$$\begin{array}{ll} a_1 > a_2 > \cdots > a_n > 0; & \\ b_i \neq 0, & i = 1, 2, \ldots, n; \\ b_{i+1} < \sqrt{a_i - a_{i+1}}, & i = 1, 2, \ldots, n-1. \end{array} \tag{3.31}$$

3.3 ZIP systems and left ZIP systems

In this section, we focus on the relation between ZIP systems and left ZIP systems in order to get different state-space descriptions of ZIP systems and characterize their zeros.

3.3.1 Diagonal state-space description of a left ZIP systems

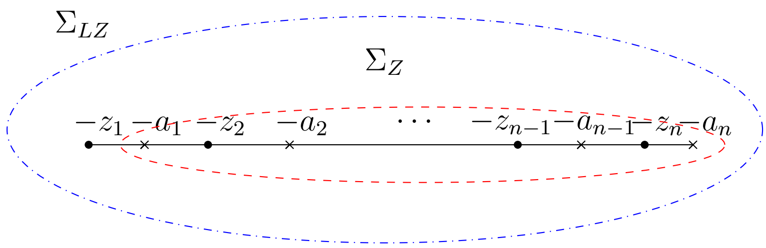

Figure 3.5: Common zeros and poles of Σ_Z (dashed) and Σ_{LZ} (dash-dotted).

Assume Σ_{LZ} is a left ZIP system, whose transfer function is

$$W_{LBZ}(s) = \frac{\prod_{j=i}^{n}(s+z_j)}{\prod_{i=i}^{n}(s+a_i)}. \tag{3.32}$$

Without loss of generality we assumed $W_{LZ}(\infty) = 1$. If not, we can consider $\tilde{W}_{LZ}(s) := \frac{W_{LZ}(s)}{\sum_{i=1}^{n} \bar{b}_i^2}$ instead. As displayed by Figure 3.5, $W_Z(s) := \frac{W_{LZ}(s)}{(s+z_1)}$ is a ZIP transfer function. By Proposition 3, there exist $\bar{b}_1, b_2, \dots b_n \in \mathbb{R}$ such that

$$W_Z(s) = \sum_{i=1}^{n} \frac{b_i^2}{s+a_i}. \tag{3.33}$$

Combining what stated until now

$$\begin{aligned}
W_Z(s) := \frac{W_{LZ}(s)}{(s+z_1)} &\Leftrightarrow W_{LZ}(s) = (s+z_1)W_Z(s) \\
&\Leftrightarrow W_{LZ}(s) = (s+z_1)\Big(\sum_{i=1}^{n} \frac{b_i^2}{s+a_i}\Big) \\
&\Leftrightarrow W_{LZ}(s) = \sum_{i=1}^{n} b_i^2\Big(\frac{s+z_1}{s+a_i}\Big).
\end{aligned} \tag{3.34}$$

Hence, a *left* ZIP system can be decomposed as the convex combination[3] of n simple *left*[4] compartments with n distinct poles, a (unique) common zero z_1. This result allows us to calculate the residues[5] of $W_{LZ}(s)$ at its (simple) poles

$$\begin{aligned}
\mathrm{Res}(W_{LZ}(s), -a_k) &= \lim_{s\to -a_k} (s+a_k)W_{LZ}(s) \\
&= \lim_{s\to -a_k} (s+a_k)\sum_{i=1}^{n} b_i^2\Big(\frac{s+z_1}{s+a_i}\Big) \\
&= b_k^2(z_1 - a_k) \\
&=: r_k > 0, \qquad k = 1, 2, \dots, n.
\end{aligned} \tag{3.35}$$

Expanding the transfer function $W_{LZ}(s)$ in partial fractions, the left ZIP

[3]Indeed, $1 = W_{LZ}(\infty) = \sum_{i=1}^{n} b_i^2$ and $b_i^2 > 0$ for all $i = 1, \dots, n$.

[4]Indeed, $-z_1 < -a_i$ for all $i = 1, 2, \dots, n$.

[5]The residue $Res(f, c)$ of $f : \mathbb{C} \to \mathbb{C}$ at $c \in \mathbb{C}$ is the coefficient a_{-1} of $(z-c)^{-1}$ in the Laurent series expansion of f around c.

transfer function $W_{LZ}(s)$ can be decomposed as

$$\begin{aligned} W_{LZ}(s) &= W_{LZ}(\infty) + \sum_{k=1}^{n} \frac{r_k}{s+a_k} \\ &= 1 + \sum_{k=1}^{n} \frac{r_k}{s+a_k} \\ &= 1 + \frac{r_1}{s+a_1} + \frac{r_2}{s+a_2} + \cdots + \frac{r_n}{s+a_n} \end{aligned} \tag{3.36}$$

Since $r_k > 0$ for all k, by equation (3.36), a minimal realization of $W_{LZ}(s)$ is

$$\Sigma_{LZ} := \left[\begin{array}{c|c} A & R \\ \hline R^\top & 1 \end{array}\right] = \left[\begin{array}{ccc|c} -a_1 & & & \sqrt{r_1} \\ & \ddots & & \vdots \\ & & -a_n & \sqrt{r_n} \\ \hline \sqrt{r_1} & \cdots & \sqrt{r_n} & 1 \end{array}\right]. \tag{3.37}$$

This argument allows us to draw several conclusions. Given an arbitrary left ZIP system Σ_{LZ}, the (strictly proper) system (drawn from Σ_{LZ}), $(A, R, R^\top)$, is a ZIP system. Conversely, adding 1 (or more generally a constant $K > 0$) to a ZIP transfer function yields a left ZIP transfer function. Finally, if the zeros of the numerator and the zeros of the denominator of $W_{LZ}(s)$ "interlace", meaning

$$-z_1 < -a_1 < -z_2 < -a_2 < \cdots < -z_n < -a_n < 0, \tag{3.38}$$

then the zeros of the numerator and the denominator of the transfer function obtained by subtracting the constant $W_{LZ}(\infty)$ (representing the so-called *DC gain*) still "interlace", having

$$-a_1 < -\tilde{z}_2 < -a_2 < \cdots < -\tilde{z}_n < -a_n < 0. \tag{3.39}$$

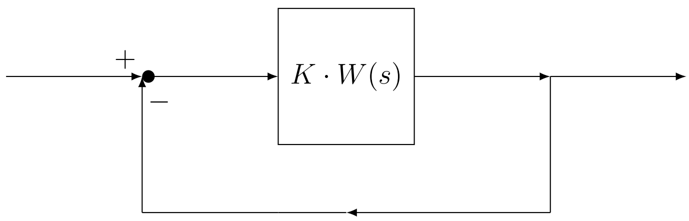

Figure 3.6: A closed-loop transfer function.

Remark 3.2 (Root locus of a ZIP transfer function)**.** Figure 3.6 depicts the (negative) feedback interconnection of $KW(s)$, with $K \in \mathbb{R}$, and a unitary gain. In this case, $\frac{KW(s)}{1+KW(s)}$ is said to be a *closed-loop transfer function*. Assuming without loss of generality that $n(s)$ and $d(s)$ are coprime polynomials such that

$$W(s) = \frac{n(s)}{d(s)}, \tag{3.40}$$

the *closed-loop poles* are the poles of the closed-loop transfer function

$$\frac{KW(s)}{1+KW(s)} = \frac{Kn(s)}{d(s)+Kn(s)} \tag{3.41}$$

namely, the roots of

$$d(s) + Kn(s) = 0 \tag{3.42}$$

and the *root locus* is the set of values of $s \in \mathbb{C}$ for which the equation (3.42) is satisfied as $K \in \mathbb{R}$ varies. The *root locus analysis* provides a graphical method for examining how the poles of the closed-loop transfer function move as the value of the gain K varies. The root locus is said *positive* (*negative*) if $K > 0$ ($K < 0$). Using a few basic rules, the root locus can be sketched in the complex plane; for a more detailed survey on the root locus analysis the reader is referred to [4, 7]. Given a ZIP transfer function $W(s)$, its root locus can be easily determined. We will describe henceforward the positive locus of an arbitrary ZIP transfer function $W(s)$, being its negative locus the complementary part of the real axis. This fact is true since zeros and poles of a ZIP transfer function are real and interlaced, but untrue in general. Assuming $K > 0$, recall the locus starting points are at the (open-loop) poles, the locus ending points are at the (open-loop) zeros and $\deg(n(s)) - \deg(d(s)) = n - (n-1) = 1$ branch end at infinity. As all roots must be real, all the points of the locus are *real*. To conclude, a point belongs to the positive locus of a ZIP transfer function if and only if it is a point on the real axis such that it is to the left of an odd number of poles and zeros. Conversely, a point belongs to the negative locus of a ZIP transfer function if and only if it is a point on the real axis such that it is to the right of an odd number of poles and zeros.

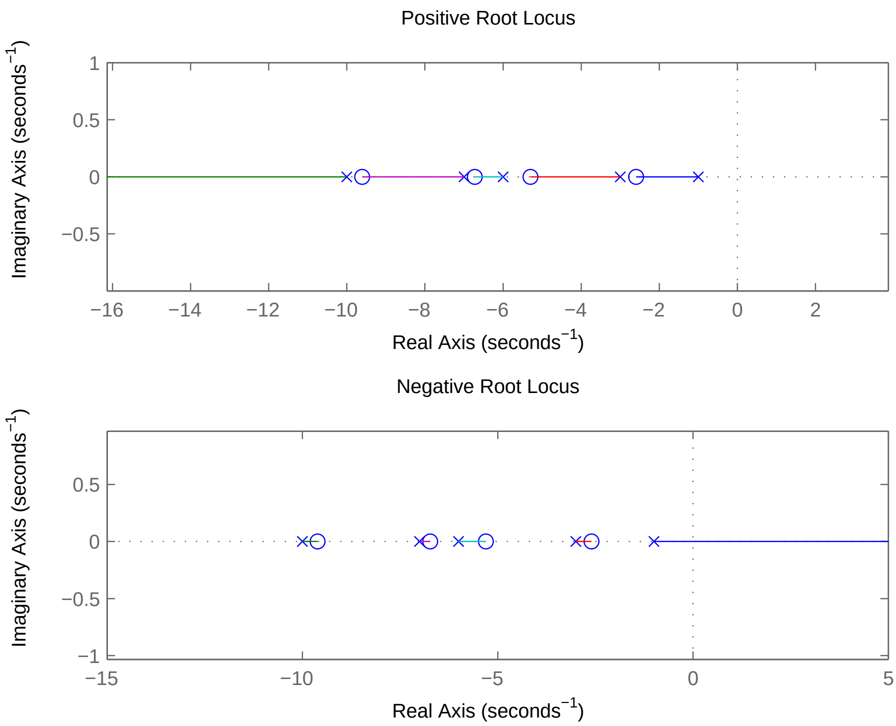

Figure 3.7: Positive root locus (top) and negative root locus (bottom) of a 5−dimensional ZIP system.

▲

3.3.2 Triangular state-space realization of ZIP systems

Assume we are given a minimal ZIP system

$$\hat{\Sigma} = \left[\begin{array}{ccc|c} -a_1 & & & \hat{b}_1 \\ & \ddots & & \vdots \\ & & -a_n & \hat{b}_n \\ \hline \hat{b}_1 & \cdots & \hat{b}_n & 0 \end{array}\right] \tag{3.43}$$

whose transfer function is

$$\hat{W}(s) = \sum_{i=1}^{n} \frac{\hat{b}_i^2}{s + a_i}. \tag{3.44}$$

Hereafter we investigate the link between the representation (3.43) and the state-space representation (3.30). We show that there exists a similar version of (3.30) for ZIP systems, though not so powerful.

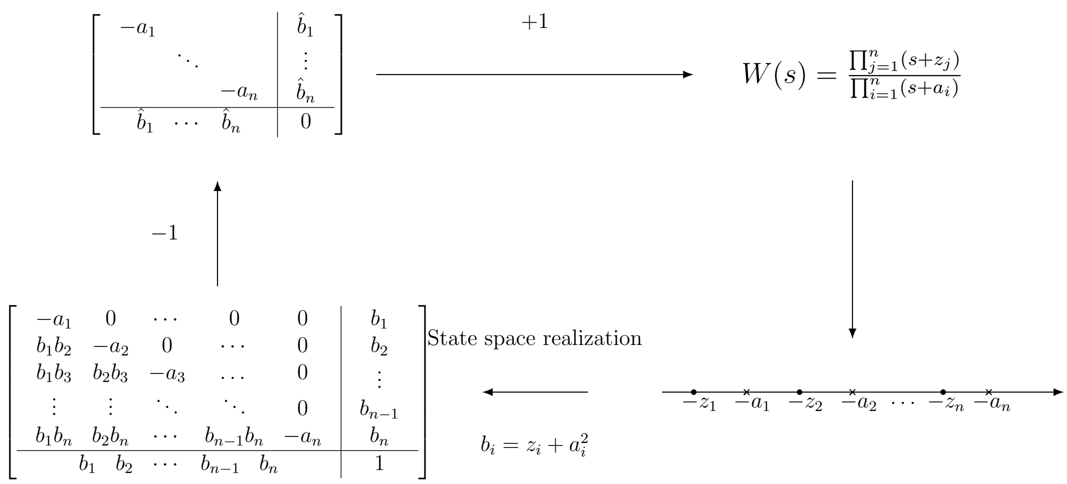

Figure 3.8: Triangular state space realization of ZIP systems.

As already stated, adding 1 (or more generally a constant $K > 0$) to the ZIP transfer function $\hat{W}(s)$ yields a left ZIP transfer function $W(s) = \hat{W}(s) + 1$. Being the poles of the left ZIP transfer function $W(s)$ the known a_i's, computing its zeros with any efficient algorithm, we get

$$W(s) = \frac{\prod_{j=1}^n (s + z_j)}{\prod_{i=1}^n (s + a_i)}. \tag{3.45}$$

Defining $b_i := \sqrt{z_i - a_i}$, by (3.30), $W(s)$ admits the following the state-space realization

$$\left[\begin{array}{ccccc|c} -a_1 & 0 & \cdots & \cdots & 0 & b_1 \\ b_1 b_2 & -a_2 & 0 & \cdots & 0 & b_2 \\ b_1 b_3 & b_2 b_3 & -a_3 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & b_{n-1} \\ b_1 b_n & b_2 b_n & \cdots & b_{n-1} b_n & -a_n & b_n \\ \hline b_1 & b_2 & \cdots & b_{n-1} & b_n & 1 \end{array}\right]. \tag{3.46}$$

Now, subtracting the unitary constant (or the constant K) that we initially added, yields the following state-space realization for $\hat{W}(s)$

$$\left[\begin{array}{ccccc|c} -a_1 & 0 & \cdots & \cdots & 0 & b_1 \\ b_1 b_2 & -a_2 & 0 & \cdots & 0 & b_2 \\ b_1 b_3 & b_2 b_3 & -a_3 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & b_{n-1} \\ b_1 b_n & b_2 b_n & \cdots & b_{n-1} b_n & -a_n & b_n \\ \hline b_1 & b_2 & \cdots & b_{n-1} & b_n & 0 \end{array}\right]. \tag{3.47}$$

Since we added a unitary constant, changed the coordinates in the modified system Σ and subtracted the unitary constant again, we did not change the input-output behavior of the system $\hat{\Sigma}$, as displayed in Figure 3.9. Thus, the state-space realizations (3.43) and (3.47) represent the same system.[6] Importantly, the state-space realization (3.47) does not allow one to calculate the zeros of the system with the formula $z_i = b_i + a_i^2$.

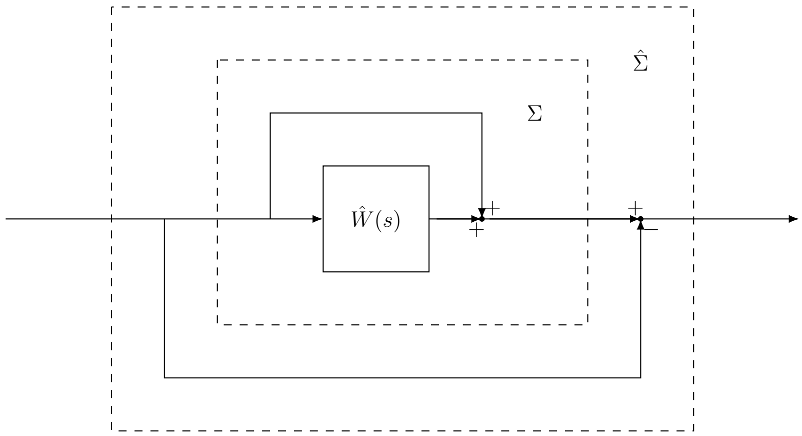

Figure 3.9: The input-output behavior does not change by adding and subtracting the same quantity to a transfer function.

3.3.3 Characterization of the zeros of a ZIP transfer function

Given an arbitrary ZIP transfer function

$$\bar{W}(s) = \sum_{i=1}^{n} \frac{\bar{b}_i^2}{s + a_i}, \tag{3.48}$$

[6] Consequently there exists a change of coordinates $T \in \mathbb{R}^{n \times n}$ that leads system (3.43) to the form (3.47). The change of coordinates T is discussed in Appendix B.1.

we will describe a procedure to characterize the zeros of $\bar{W}(s)$ by means of the state-space representation (3.30).

$$
\bar{W}(s) = \sum_{i=1}^{n} \frac{\bar{b}_i^2}{s+a_i}, \xrightarrow{\quad \cdot(s+z_1) \quad} W(s) = \sum_{i=1}^{n} \frac{b_i^2}{s+a_i},
$$

$$
W(s) \xrightarrow{\text{State space realization}}
\left[\begin{array}{ccccc|c}
-a_1 & 0 & \cdots & \cdots & 0 & b_1 \\
b_1 b_2 & -a_2 & 0 & \cdots & 0 & b_2 \\
b_1 b_3 & b_2 b_3 & -a_3 & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 & b_{n-1} \\
b_1 b_n & b_2 b_n & \cdots & b_{n-1} b_n & -a_n & b_n \\
\hline
b_1 & b_2 & \cdots & b_{n-1} & b_n & 0
\end{array}\right]
\xrightarrow{\cdot(s+z_1)^{-1}} \bar{W}(s)
$$

Figure 3.10: Characterization of the zeros of a ZIP system.

Choosing $z_1 > a_1$, the multiplication of the ZIP transfer function $\bar{W}(s)$ by $(s+z_1)$ yields a left ZIP transfer function $W(s) := (s+z_1)\bar{W}(s)$. By means of any efficient realization algorithm $W(s)$ admits a state space representation of the form (3.30). This representation allows one to calculate the zeros as $z_i = -(a_i + b_i^2)$. Hence, the transfer function $W(s)$ may be represented as

$$
W(s) = \frac{\prod_{j=1}^{n}(s+z_j)}{\prod_{i=1}^{n}(s+a_i)}. \tag{3.49}
$$

Finally, dividing $W(s)$ by $(s+z_1)$ yields a factorized expression of the initial transfer function

$$
\bar{W}(s) = \frac{W(s)}{s+z_1} = \frac{\prod_{j=2}^{n}(s+z_j)}{\prod_{i=1}^{n}(s+a_i)} \tag{3.50}
$$

and consequently the zeros of $\bar{W}(s)$, z_j for $j = 2, \ldots, n$.

3.4 Further properties of left/right ZIP transfer functions

Hereafter we outline some additional properties of left/right ZIP transfer functions.

Proposition 5 (Properties of left/right ZIP transfer functions)**.** *The following properties hold.*

(i) *$W(s)$ is LZ/RZ $\Rightarrow \frac{1}{W(s)}$ is RZ/LZ.*

(ii) *$W(s)$ is LZ/RZ $\Rightarrow W(s+\alpha)$ is LZ/RZ for all $\alpha \in \mathbb{R}_+$.*

(iii) *$W(s)$ is LZ/RZ $\Rightarrow \alpha W(s)$ is LZ/RZ for all $\alpha \in \mathbb{R}_+$.*

(iv) *If $W_1(s)$ and $W_2(s)$ are LZ/RZ, and the rightmost pole/zero of $W_1(s)$ is strictly less than the leftmost zero/pole of $W_2(s)$, then $W_1(s)W_2(s)$ is LZ/RZ.*

(v) *If $W_1(s) = \sum_{i=1}^{n} \frac{b_i^2}{s+a_i}$ and $W_2(s) = \sum_{i=n+1}^{2n} \frac{b_i^2}{s+a_i}$ are two ZIP transfer functions with distinct poles and non zero b_i for all $i = 1, \dots, 2n$, then $W_1(s) + W_2(s)$ is ZIP;*

(vi) *If $W_1(s) = W_1(\infty) + \sum_{i=1}^{n} \frac{b_i^2}{s+a_i}$ and $W_2(s) = W_2(\infty) + \sum_{i=n+1}^{2n} \frac{b_i^2}{s+a_i}$ are LZ/RZ transfer functions with distinct poles and non zero b_i for all $i = 1, \dots, 2n$, then $W_1(s) + W_2(s)$ is LZ/RZ;*

(vii) *$\frac{1}{W_1(s)} + W_2(s)$ is LZ/RZ$\Leftrightarrow W_1(s)\diamondsuit W_2(s) = \frac{W_1(s)}{1+W_1(s)W_2(s)}$ is RZ/LZ;*

(viii) *If $W_1(s)$ is RZ/LZ and $W_2(s)$ is LZ/RZ and $W_1(s)$ has no zero equal to a pole of $W_2(s) \Rightarrow W_1(s)\diamondsuit W_2(s) = \frac{W_1(s)}{1+W_1(s)W_2(s)}$ is RZ/LZ.*

Proof. (vii). $W_1(s)\diamondsuit W_2(s) = \frac{W_1(s)}{1+W_1(s)W_2(s)} = \frac{1}{\frac{1}{W_1(s)}+W_2(s)} = (W_1(s)+W_2(s))^{-1}$ is RZ/LZ if and only if $\frac{1}{W_1(s)} + W_2(s)$ is LZ/RZ.
(viii). If $W_1(s)$ is RZ/LZ and $W_2(s)$ is LZ/RZ and $W_1(s)$ has no zero equal to a pole of $W_2(s)$, by (i), $\frac{1}{W_1(s)} + W_2(s)$ is LZ/RZ. By (viii), $W_1(s)\diamondsuit W_2(s) = \frac{W_1(s)}{1+W_1(s)W_2(s)}$ is RZ/LZ. $\square$

Chapter 4

Model reduction by moment matching for ZIP systems

Model reduction, also called dimensional model reduction or model order reduction (MOR), is a technique that aims at distilling a simpler substitute model for a (possibly large scale) complex system, while preserving the input-output behavior.

Model reduction techniques are of paramount importance for many practical purposes and have been used extensively in the fields of control theory [1], electrical circuits simulation [1, 2], microelectromechanical systems [1, 5], weather forecasting [1, 16], etc. For a detailed overview the reader is referred to [1].

Two classes of methods are currently in use, namely

- Singular Values Decomposition (SVD) based methods and,
- moment matching based methods.

The former techniques center the issue of system approximation around the singular values of the associated Hankel operator. A very popular SVD-based method is *model reduction by balancing*. Roughly speaking, model reduction by balancing consists in finding a state space representation where the states which are difficult to reach are simultaneously difficult to observe. Then, the reduced model is obtained simply by truncating the states which have this property. Unfortunately the SVD methods are computationally rather demanding.

In systems theory language, the latter techniques generalize the well-known partial realization problem. Even if moment matching based methods do not automatically preserve stability and have no global error bounds, these methods can be iteratively implemented yielding numerically efficient algorithms. In addition, model reduction by moment matching admits a nonlinear enhancement [2].

In what follows, we study the problem of model reduction by moment matching for linear systems. Then, we apply this technique to the class of ZIP systems to show that, under suitable assumptions, the reduced order model inherits the ZIP property. Finally, we consider the possibility of placing prescribed poles in the reduced order model, with the additional constraint of maintaining the ZIP property.

4.1 Model reduction by moment matching for linear systems

Consider an N-dimensional single-input single-output continuous time linear system described by

$$\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right], \tag{4.1}$$

where $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times 1}$, $C \in \mathbb{R}^{1\times N}$ and whose transfer function is

$$W(s) = C(sI_N - A)^{-1}B. \tag{4.2}$$

Definition 9 (0-moment, k-moment)**.** *The* 0*-moment of system (4.1) at* $s^\star \in \mathbb{C}$ *is the complex number* $\eta_0(s^\star) = C(s^\star I - A)^{-1}B$*. The* k*-moment of system (4.1) at* $s^\star \in \mathbb{C}$ *is the complex number*

$$\eta_k(s^\star) = \frac{(-1)^k}{k!}\left[\frac{d^k}{ds^k}(C(sI-A)^{-1}B)\right]_{s=s^\star} = C(s^\star I - A)^{-(k+1)}B \tag{4.3}$$

Since the moments are associated with the transfer function of the system (4.1), in what follows we assume that the system (4.1) is a minimal realization of its transfer function.

Moments can be also characterized, for almost all $s^\star \in \mathbb{C}$, by means of suitable Sylvester equations [2]. Assuming $s^\star \notin \sigma(A)$, the moments $\eta_0(s^\star), \ldots, \eta_k(s^\star)$ are in one-to-one relation with the matrix $C\Pi$, where $\Pi \in \mathbb{R}^{N\times n}$ is the unique solution of the Sylvester equation

$$A\Pi + BL = \Pi S \tag{4.4}$$

with $S \in \mathbb{R}^{k+1\times k+1}$ any non-derogatory[1] matrix such that $\det(sI - S) = (s - s^\star)^{k+1}$ and $L \in R^{1\times k+1}$ such that (S, L) is observable.

[1]A matrix is non-derogatory if its characteristic and minimal polynomials coincide.

In addition, the moments $\eta_0(s_1), \dots, \eta_0(s_k)$ are in one-to-one relation with the matrix $C\Pi$ if we choose any non-derogatory matrix $S \in \mathbb{R}^{k\times k}$ such that

$$\det(sI - S) = \prod_{i=1}^{n}(s - s_i). \tag{4.5}$$

We now give the precise definition of (reduced order) model.

Definition 10 (Model, reduced order model)**.** *The system*

$$\left[\begin{array}{c|c} F & G \\ \hline H & 0 \end{array}\right], \tag{4.6}$$

where $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times 1}$, $H \in \mathbb{R}^{1\times n}$, is a model *of system (4.1) at $\sigma(S)$, with $S \in \mathbb{R}^{n\times n}$ and $\sigma(S) \cap \sigma(A) = \emptyset$ if*

$$\sigma(S) \cap \sigma(F) = \emptyset \tag{4.7}$$

and

$$\boxed{C\Pi = HP} \tag{4.8}$$

where L is such that (S, L) is observable, and Π and P are respectively the unique solutions of the equations

$$A\Pi + BL = \Pi S \tag{4.9}$$

$$FP + GL = PS \tag{4.10}$$

Furthermore system (4.6) is a reduced order model *of system (4.1) if $n < N$.*

As mentioned in [2] system (4.6) solves the model reduction problem with moment matching at $\sigma(S)$ for system (4.1).

For the sake of ease, we assume henceforward that S and A do not have common eigenvalues and S and F do not have common eigenvalues. Now, selecting $P = I$ yields a family of reduced order models for system (4.1) at S, hence achieving moment matching, described by matrices of the form

$$\boxed{P = I} \Longrightarrow \begin{cases} F = S - \Delta L \\ G = \Delta \\ H = C\Pi \end{cases}$$

namely

$$\Sigma_\Delta = \left[\begin{array}{c|c} S - \Delta L & \Delta \\ \hline C\Pi & 0 \end{array}\right], \tag{4.11}$$

with Δ any matrix such that $\sigma(S) \cap \sigma(S - \Delta L) = \emptyset$.

Finally, the family of reduced order models achieving moment matching is parameterized directly by the matrix Δ, which has to satisfy a generic constraint. As a consequence, the relations between the matrix Δ and the properties of the reduced order model are straightforward and easy to characterize.

4.2 Inheritance of the ZIP property by the reduced order model

Without loss of generality an N-dimensional single-input single-output linear continuous time ZIP system $\Sigma = (A, B, C)$ has the form

$$\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -a_1 & & & b_1 \\ & \ddots & & \vdots \\ & & -a_N & b_N \\ \hline b_1 & \cdots & b_N & 0 \end{array}\right], \tag{4.12}$$

where $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times 1}$, $C \in \mathbb{R}^{1\times N}$ with $a_i > 0, b_i > 0, a_i \neq a_j \ \forall i \neq j$.

Define

$$\mathcal{O}_{n-1} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{n\times N}, \tag{4.13}$$

$$\mathcal{R}_{n-1} := \begin{bmatrix} B \ AB \ \cdots \ A^{n-1}B \end{bmatrix} \in \mathbb{R}^{N\times n}, \tag{4.14}$$

$$\mathcal{H}_{n-1} := \begin{bmatrix} CB & CAB & \cdots & CA^{n-1}B \\ CAB & CA^2B & \cdots & CA^nB \\ \vdots & \ddots & \ddots & \vdots \\ CA^{n-1}B & CA^nB & \cdots & CA^{2n-2}B \end{bmatrix} = \mathcal{O}_{n-1}\mathcal{R}_{n-1} \in \mathbb{R}^{n\times n}, \tag{4.15}$$

and, since $A = A^\top = \operatorname{diag}\{-a_1, -a_2, \cdots, -a_N\}$ and $B = C^\top$, we have

$$\mathcal{O}_{n-1} = \mathcal{R}_{n-1}^\top \tag{4.16}$$
$$\mathcal{H}_{n-1} = \mathcal{O}_{n-1}\mathcal{O}_{n-1}^\top \tag{4.17}$$

Consider the linear system (4.12). Let $s_1, s_2, \dots, s_n \in \mathbb{C}$ be given *interpolation points* such that the presence of $s_i \in \mathbb{C} \setminus \mathbb{R}$ implies the presence of its complex conjugate $\bar{s}_i \in \mathbb{C} - \mathbb{R}$. Assuming $0 < 2n < N$, consider any non-derogatory matrix $S \in \mathbb{R}^{n\times n}$ such that

$$\det(sI - S) = \prod_{i=1}^{n}(s - s_i) \tag{4.18}$$

and $L \in \mathbb{R}^{1\times n}$ such that the pair (S, L) is observable.

Define the reduced order system $\Sigma_G = (F, G, H) := (S - GL, G, C\Pi)$ where $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times 1}$, $H \in \mathbb{R}^{1\times n}$ and $\Pi \in \mathbb{R}^{N\times n}$ is the (unique) solution of the Sylvester equation

$$A\Pi + BL = \Pi S \tag{4.19}$$

Remark 4.1 (Matching the first n Markov parameters)**.** For the linear system (4.1) the $k-$moments at infinity are defined as $\eta_k(\infty) = CA^{k-1}B$, i.e. the first k moments at infinity coincide with the first k Markov parameters [1]. To match the first n moments at infinity of Σ and Σ_G, G must satisfy

$$\begin{bmatrix} C\Pi \\ C\Pi(S-GL) \\ \vdots \\ C\Pi(S-GL)^{n-1} \end{bmatrix} G = \begin{bmatrix} CB \\ CAB \\ \vdots \\ CA^{n-1}B \end{bmatrix} \Leftrightarrow \begin{bmatrix} C\Pi \\ CA\Pi \\ \vdots \\ CA^{n-1}\Pi \end{bmatrix} G = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} B \tag{4.20}$$

$$\Leftrightarrow \mathcal{O}_{n-1}\Pi G = \mathcal{O}_{n-1} B \tag{4.21}$$

hence

$$\boxed{G = (\mathcal{O}_{n-1}\Pi)^{-1}\mathcal{O}_{n-1}B} \tag{4.22}$$

As a consequence, the reduced order model can be cast as

$$\begin{aligned} F &= S - GL \\ &\overset{(4.22)}{=} S - (\mathcal{O}_{n-1}\Pi)^{-1}\mathcal{O}_{n-1}BL \\ &\overset{(4.19)}{=} (\mathcal{O}_{n-1}\Pi)^{-1}\mathcal{O}_{n-1}A\Pi \end{aligned} \tag{4.23}$$

$$G = (\mathcal{O}_{n-1}\Pi)^{-1}\mathcal{O}_{n-1}B \tag{4.24}$$

$$H = C\Pi = \mathbf{e}_1^\top \mathcal{O}_{n-1}\Pi \tag{4.25}$$

▲

Remark 4.2 (Observability matrix with moment matching at infinity)**.** Matching the first n Markov parameters drastically simplifies the calculation of the observability matrix of Σ_G. Indeed, consider equation (4.20). We have

$$\begin{aligned} C\Pi(S-GL) &= C\Pi S - C\Pi GL \\ &\overset{(4.19)}{=} C(A\Pi + BL - \Pi GL) \\ &= CA\Pi + \underbrace{(CB - C\Pi G)}_{=0 \Leftrightarrow CB = C\Pi G} L. \end{aligned}$$

Assuming that the first Markov parameter is matched, i.e. $CB = C\Pi G$, we have

$$\begin{aligned} C\Pi(S-GL)^2 &= \underbrace{C\Pi(S-GL)}_{CA\Pi}(S-GL) \\ &= CA(\Pi S - \Pi GL) \\ &\overset{(4.19)}{=} CA^2\Pi + \underbrace{(CAB - CA\Pi G)}_{=0 \Leftrightarrow CAB = CA\Pi G} L. \end{aligned}$$

Inductively, the structure

$$\begin{bmatrix} C\Pi \\ C\Pi(S-GL) \\ \vdots \\ C\Pi(S-GL)^{n-1} \end{bmatrix} = \begin{bmatrix} C\Pi \\ CA\Pi \\ \vdots \\ CA^{n-1}\Pi \end{bmatrix} = \mathcal{O}_{n-1}\Pi$$

is achieved if and only if the first n Markov parameters are matched. ▲

Letting n be the order of the reduced model, we will prove that the ZIP property is inherited by the reduced order model when matching the first n Markov parameters.

Proposition 6 (Inheritance of the ZIP property)**.** *Let (A, B, C) be an asymptotically stable minimal realization of an arbitrary ZIP system Σ of order $N > 1$. Given $0 < n < \frac{N}{2}$ interpolation points $s_1, s_2, \ldots, s_n \in \mathbb{C}$ consider any non-derogatory matrix $S \in \mathbb{R}^{n\times n}$ such that*

$$det(sI - S) = \prod_{i=1}^{n}(s - s_i)$$

and $L \in \mathbb{R}^{1\times n}$ such that the pair (S, L) is observable.
Define the reduced order system $\Sigma_G = (F, G, H) := (S - GL, G, C\Pi)$ where

$F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times 1}$, $H \in \mathbb{R}^{1\times n}$ and $\Pi \in \mathbb{R}^{N\times n}$ is the (unique) solution of the Sylvester equation

$$A\Pi + BL = \Pi S. \tag{4.26}$$

Finally, choose G such that the first n Markov parameters are matched. Then Σ_G is an asymptotically stable (minimal) ZIP system.

Proof. Proof provided in Appendix A.1. □

4.3 Pole placement in the reduced order model

Consider a single-input single-output linear continuous time ZIP system

$$\Sigma = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -a_1 & & & b_1 \\ & \ddots & & \vdots \\ & & -a_N & b_N \\ \hline b_1 & \cdots & b_N & 0 \end{array}\right], \tag{4.27}$$

where $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times 1}$, $C \in \mathbb{R}^{1\times N}$ with $a_i > 0, b_i > 0, a_i \neq a_j \ \forall i \neq j$.

Given $0 < n < \frac{N}{2}$, define the reduced order system as $\Sigma_G = (F, G, H)$, where $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times 1}$, $H \in \mathbb{R}^{1\times n}$. Recall that choosing G so that the first n Markov parameters (of Σ_G) are matched (with the ones of Σ) implies that Σ_G is a ZIP system.

We wish to investigate whether, properly choosing the triple (F, G, H), the spectrum of F

$$\sigma(F) = \{f_1, \dots, f_n\}, \quad f_i > 0, \ f_i \neq f_j, \ i \neq j, \tag{4.28}$$

with $\sigma(F) \cap \sigma(A) = \emptyset$, can be freely assigned, while preserving the ZIP property and matching the first n Markov parameters.[2]

4.3.1 The bidimensional case

Assume we are matching the first $n = 2$ Markov parameters of Σ and Σ_G, namely

$$\begin{bmatrix}\alpha_1 \\ \alpha_2\end{bmatrix} := \begin{bmatrix} CB \\ CAB \end{bmatrix} = \begin{bmatrix} HG \\ HFG \end{bmatrix}. \tag{4.29}$$

[2]By Proposition 6, if the first n Markov parameters matched, Σ_G inherits the ZIP property.

Since we are matching the first 2 Markov parameters, Σ_G is guaranteed to be ZIP. Hence, without loss of generality, we may assume

$$H^\top = G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \tag{4.30}$$

and $F = \mathrm{diag}(f_1, f_2)$. Substituting (4.30) in equation (4.29) yields

$$\begin{aligned} \begin{cases} g_1^2 & +g_2^2 & = \alpha_1 \\ f_1 g_1^2 & +f_2 g_2^2 & = \alpha_2 \end{cases} & \Leftrightarrow \begin{cases} \frac{g_1^2}{\alpha_1} & +\frac{g_2^2}{\alpha_1} & = 1 \\ \frac{g_1^2}{\alpha_2/f_1} & +\frac{g_2^2}{\alpha_2/f_2} & = 1 \end{cases} \\ & \Leftrightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ f_1 & f_2 \end{bmatrix}}_{:=V \in \mathbb{R}^{2\times 2}} \begin{bmatrix} g_1^2 \\ g_2^2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}. \end{aligned} \tag{4.31}$$

Since we are assuming that $f_i \neq f_j$ for all $i \neq j$, the Vandermonde matrix V is non-singular. Inverting V yields

$$\begin{bmatrix} g_1^2 \\ g_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ f_1 & f_2 \end{bmatrix}^{-1} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \frac{1}{f_2 - f_1} \begin{bmatrix} \alpha_1 f_2 - \alpha_2 \\ -\alpha_1 f_1 + \alpha_2 \end{bmatrix}. \tag{4.32}$$

The left hand entries of equation (4.32) are positive. Hence, assuming without loss of generality $f_2 - f_1 > 0$[3], the pole placement is feasible if and only if

$$f_1 < \frac{\alpha_2}{\alpha_1} < f_2. \tag{4.33}$$

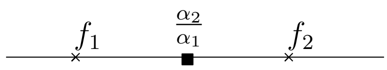

Figure 4.1: Pole placement conditions for $n = 2$: $f_1 < \frac{\alpha_2}{\alpha_1} < f_2$.

Conversely, equations (4.31) yield a degenerate situation if and only if one of the minors (of order $n = 2$) of the following matrix is zero

$$C := \left[\begin{array}{cc|c} 1 & 1 & \alpha_1 \\ f_1 & f_2 & \alpha_2 \end{array}\right]. \tag{4.34}$$

Geometrically speaking, equations (4.31) represent the intersection of a circle and an ellipse. In this case, the length of a semi-axe of the ellipse equals the radius of the other circle, see Figure 4.3.1. Correspondingly, there exists g_i

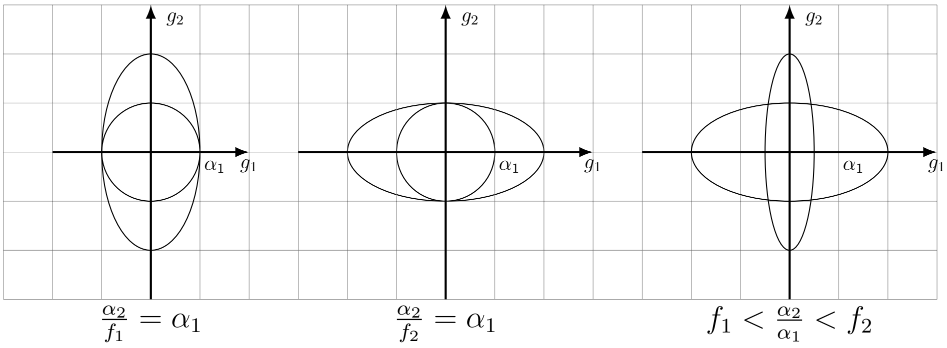

Figure 4.2: Geometric interpretation of pole placement for $n = 2$: degenerate situations (left, center) and feasible situation (right).

which is zero (hence Σ_G is not minimal). This is nothing but a consequence of Cramer's rule.[4] Finally, the (unique) zero is given by

$$z_1 = \frac{g_1^2}{g_1^2 + g_2^2} f_2 + \frac{g_2^2}{g_1^2 + g_2^2} f_1 \tag{4.35}$$

which is a convex combination of the two (negative) poles. As expected, the system Σ_G is ZIP.

4.3.2 The n-dimensional case

Assume we are matching the first n Markov parameters of Σ and Σ_G, namely

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} := \begin{bmatrix} CB \\ CAB \\ \vdots \\ CA^{n-1}B \end{bmatrix} = \begin{bmatrix} HG \\ HFG \\ \vdots \\ HF^{n-1}G \end{bmatrix}. \tag{4.36}$$

[3] If $f_2 - f_1 < 0$, it is sufficient to change the basis by resorting to a suitable permutation matrix.

[4] Consider the linear system

$$Ax = b, \quad x, b \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n}, \det(A) \neq 0,$$

in the unknown $x = [x_1, \ldots, x_n]^\top$. Cramer's rule states that the system has a unique solution, whose individual values for the unknowns are given by:

$$x_i = \frac{\det(A_i)}{\det(A)}, \qquad i = 1, \ldots, n,$$

where A_i is the matrix formed by replacing the $i-$th column of A by the column vector b.

Since we are matching the first n Markov parameters, Σ_G is guaranteed to be a ZIP system. Hence, without loss of generality, we may assume

$$H^\top = G = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix} \tag{4.37}$$

and $F = \mathrm{diag}(f_1, f_2, \ldots, f_n)$. Substituting (4.37) in equation (4.36) yields

$$\begin{cases} g_1^2 & +g_2^2 & +\ldots & +g_n^2 & = \alpha_1 \\ f_1 g_1^2 & +f_2 g_2^2 & +\ldots & +f_n g_n^2 & = \alpha_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_1^{n-1} g_1^2 & +f_2^{n-1} g_2^2 & +\ldots & +f_n^{n-1} g_n^2 & = \alpha_n \end{cases} \tag{4.38}$$

which is equivalent to

$$\underbrace{\begin{bmatrix} 1 & 1 & \ldots & 1 \\ f_1 & f_2 & \ldots & f_n \\ \vdots & \vdots & \ldots & \vdots \\ f_1^{n-1} & f_2^{n-1} & \ldots & f_n^{n-1} \end{bmatrix}}_{:=V \in \mathbb{R}^{n\times n}} \begin{bmatrix} g_1^2 \\ g_2^2 \\ \vdots \\ g_n^2 \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}}_{:=\alpha \in \mathbb{R}^{n\times 1}} \tag{4.39}$$

Since we are assuming that $f_i \neq f_j$ for all $i \neq j$, the Vandermonde matrix V is non-singular and the solution can be easily found.[5] However, the pole placement is feasible if and only if the solution of $Vx = \alpha$ is a (strictly) positive vector.
Parallel to the case $n = 2$, the following question arises:

- given $\sigma(F) = \{f_1, \ldots, f_n\}$, $f_i \neq f_j$, $i \neq j$,, under which conditions on $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ do there exist strictly positive solutions $\bar{x}$ to equation $Vx = \alpha$?

[5] Denoting by $B = [b_{ij}]$ the inverse matrix of V, then b_{ij} has the following expression

$$b_{jk} = (-1)^{k-1} \left(\frac{\displaystyle\sum_{\substack{1 \leq m_1 < \ldots < m_{n-k} \leq n \\ m_1, \ldots, m_{n-k} \neq j}} f_{m_1} \cdots f_{m_{n-k}}}{\displaystyle\prod_{\substack{1 \leq m \leq n \\ m \neq j}} (f_m - f_j)} \right). \tag{4.40}$$

Several factorisations of V, and consequently of V^{-1}, have been proposed in literature, see for instance [13]; however, handling these equations might be tricky.

In the next subsection, we analyze some necessary and sufficient conditions for the existence of positive solutions to equation $Vx = \alpha$. When such $\bar{x}$ exists, the pole placement is feasible and the zeros and the poles of Σ_G are guaranteed to interlace.

Remark 4.3 (On the minimality of the reduced order model)**.** Parallel to what we found for $n = 2$, by Cramer's rule, $g_j = 0$ if and only if the determinant of the matrix C_j is zero, where C_j is obtained by eliminating the j-th column of

$$C := \left[\begin{array}{cccc|c} 1 & 1 & \dots & 1 & \alpha_1 \\ f_1 & f_2 & \dots & f_n & \alpha_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_1^{n-1} & f_2^{n-1} & \dots & f_n^{n-1} & \alpha_n \end{array}\right]. \tag{4.41}$$

Geometrically speaking, equations (4.38) represent the intersection of n ellipsoids. In this case, the length of a semi-axe of an ellipsoid equals the the length of a semi-axe of another ellipsoid. Correspondingly, there exists g_i which is zero (hence Σ_G is not minimal). ▲

4.3.3 Pole placement and the Farkas' lemma

Provided that $f_i < 0$ and $\text{sign}(\alpha_j) = \text{sign}(CA^{j-1}B) = (-1)^{j-1}$, it is not restrictive to assume that all the terms in the equation $Vx = \alpha$ are positive. Indeed, it is sufficient to premultiply the equation $Vx = \alpha$ by a sign matrix $P = (1, -1, 1, -1, \dots, (-1)^{n-1}) \in \mathbb{R}^{n\times n}$. Furthermore, we stress that this assumption is equivalent to assume that $f_i > 0$ and $a_j > 0$ for all i, j.

Theorem 1 (Farkas' lemma)**.** *Let $A \in \mathbb{R}^{m\times n}$ and let $b \in \mathbb{R}^m$ be nonzero. Then exactly one of the following holds:*

1. *There is a positive solution $\bar{x} \in \mathbb{R}^n$ to the system $Ax = b$.*

2. *There is a vector $w \in \mathbb{R}^m$ for which $A^\top w \geq 0$ and $w^\top b < 0$.*

Focusing on the existence of positive solutions for the equation $Vx = \alpha$, letting $A = V$ and $b = \alpha$, we can state that the following

Corollary 1 (Pole placement in the reduced order model)**.** *Exactly one of the following holds:*

1. *There is a positive solution $\bar{x} \in \mathbb{R}^n_+$ to the system $Vx = \alpha$.*

2. *There is a vector $w \in \mathbb{R}^n$ for which $V^\top w \geq 0$ and $w^\top \alpha < 0$.*

Notice that Corollary 1 states a necessary and sufficient condition for the feasibility of the pole placement in the $n-$dimensional reduced order model matching the first n Markov parameters. More precisely, the pole placement is feasible if and only if there is no vector $w \in \mathbb{R}^n$ for which $V^\top w \geq 0$ and $w^\top \alpha < 0$.

On the non-feasibility of the pole placement

We now analyze the previously found alternative condition to the non-feasibility of the pole placement. Namely, the pole placement is not feasible if and only if there exists a vector $w \in \mathbb{R}^n$ for which $V^\top w \geq 0$ and $w^\top \alpha < 0$.

Proposition 7 (Non-feasibility of the pole placement)**.** *The following statements are equivalent.*

1. *There is a vector $w \in \mathbb{R}^n$ for which $V^\top w \geq 0$ and $w^\top \alpha < 0$.*
2. *There exists a polynomial*

$$w(s) = \sum_{k=0}^{n-1} w_k s^k, \quad w_i \in \mathbb{R}, \tag{4.42}$$

such that

- $w(f_1) \geq 0,\ w(f_2) \geq 0,\ \dots,\ w(f_{n-1}) \geq 0,\ w(f_n) \geq 0,$
- $\sum_{j=1}^{n} b_j^2 w(a_j) < 0,$

where the b_j's are the entries of B in the realization (4.27).

Proof. Given $w \in \mathbb{R}^n$ such that $V^\top w \geq 0$ and $w^\top \alpha < 0$ hold, define the polynomial

$$\begin{aligned} w(s) &:= w_0 + w_1 s + \cdots + w_{n-1} s^{n-1} && (4.43) \\ &= \begin{bmatrix} w_0 & w_1 & \dots & w_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{bmatrix} && (4.44) \end{aligned}$$

where $w_i \in \mathbb{R}$ for all $i = 1, \dots, n$. As a preliminary remark, note that

$$V^\top w = \begin{bmatrix} w(f_1) \\ w(f_2) \\ \vdots \\ w(f_{n-1}) \end{bmatrix}. \tag{4.45}$$

It follows that

$$V^\top w \geq 0 \Leftrightarrow w(f_1) \geq 0,\ w(f_2) \geq 0,\ \ldots,\ w(f_{n-1}) \geq 0. \tag{4.46}$$

Additionally, notice that

$$w^\top \alpha = \begin{bmatrix} w_0 & w_1 & \ldots & w_{n-1} \end{bmatrix} \begin{bmatrix} CB \\ CAB \\ \vdots \\ CA^{n-1}B \end{bmatrix} \tag{4.47}$$

$$= C(\sum_{k=0}^{n-1} w_k A^k)B = Cw(A)B \tag{4.48}$$

$$= \sum_{j=1}^{n} b_j^2 w(a_j). \tag{4.49}$$

The claim is an obvious consequence of equations (4.46) and (4.49). □

Noting that

$$\sum_{j=1}^{n} b_j^2 w(a_j) < 0 \Leftrightarrow \frac{\sum_{j=1}^{n} b_j^2 w(a_j)}{\sum_{j=1}^{n} b_j^2} < 0, \tag{4.50}$$

Proposition 7 can interpreted as follows: the pole placement is non-feasible if and only if we are able to find a polynomial in one complex variable with real coefficients $w(s)$ such that

- $w(s)$ is non-negative when evaluated at $s = f_i$ for $i = 1, \ldots, n$;
- the convex combination of the points of $w(a_j)$, with coefficients $\frac{b_j^2}{\sum_{j=1}^{n} b_j^2}$, is negative.

Example 11 (A non-feasible pole placement)**.** Figure 4.3 displays a non-feasible pole placement. Indeed, since $N = 4$ and $n = 3$, the polynomial mentioned in Proposition 7 has the form

$$w(s) = w_2 s^2 + w_1 s + w_0, \quad w_0, w_1, w_2 \in \mathbb{R}. \tag{4.51}$$

Due to the particular structure depicted in Figure 4.3, it is easy to realize that we are always able to find a parabola such that $w(f_i) \geq 0$ for $i = 1, 2, 3$ and $w(a_j) < 0$ for all $j = 1, \ldots, 4$. Indeed, all the poles a_j lie in the open interval (f_2, f_3). Consequently, it is always possible to find two points $s_1, s_2 \in$

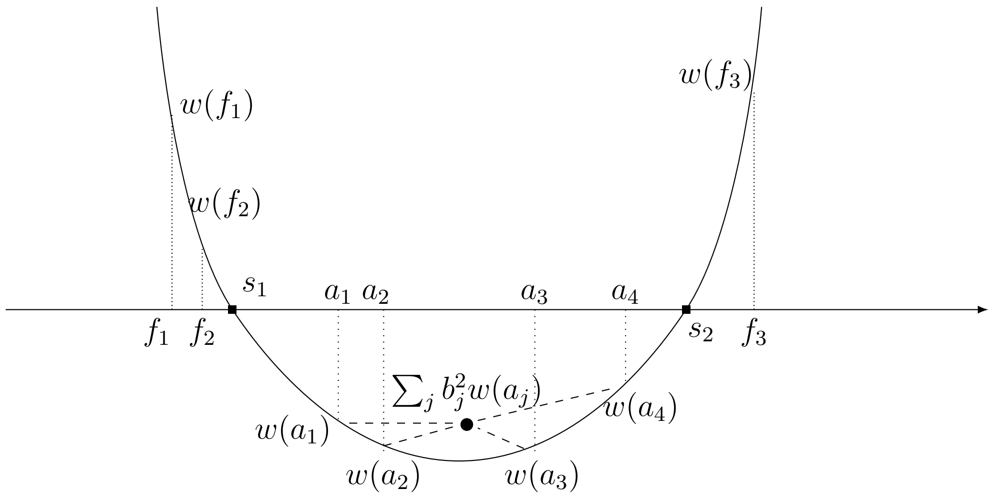

Figure 4.3: A non-feasible pole placement: $w(f_i) \geq 0$ and $\frac{\sum_{j=1}^n b_j^2 w(a_j)}{\sum_{j=1}^n b_j^2} < 0$.

(f_2, f_3) such that the inclusions $(a_1, a_2) \subsetneq (s_1, s_2) \subsetneq (f_2, f_3)$ hold. Choosing w_0, w_1, w_2 such that $w_2 > 0$ and $w(s_1) = w(s_2) = 0$ yields a poynomial $w(s)$ satisfying $w(f_i) \geq 0$ and $w(a_j) < 0$. Thus, for each choice of the entries b_i, $w(s)$ satisfies the conditions

- $w(f_1) \geq 0,\ w(f_2) \geq 0,\ w(f_3) \geq 0,$
- $\sum_{j=1}^4 b_j^2 w(a_j) < 0,$

then, by Proposition 7, the pole placement is non-feasible. ■

Example 12 (The bidimensional case)**.** Assuming $f_1 < f_2$, in subsection 4.3.1 we stated that for $n = 2$ the pole placement is feasible if and only if $f_1 < \frac{\alpha_2}{\alpha_1} < f_2$. In this example, we will give a graphical interpretation of Proposition 7 also checking the correspondence with the non-feasibility of the pole placement.

Being $n = 2$, the polynomial mentioned in Proposition 7 has the form

$$w(s) = w_1 s + w_0, \quad w_0, w_1 \in \mathbb{R}. \tag{4.52}$$

It is not restrictive to assume $w_0 \in \mathbb{R}_+$. As depicted in Figure 4.4, condition $f_1 < \frac{\alpha_2}{\alpha_1} < f_2$ imply $w(f_1) \leq w(\frac{\alpha_2}{\alpha_1}) \leq w(f_2)$. This means that conditions $w(f_1) \geq 0$, $w(f_2) \geq 0$ and $w(\frac{\alpha_2}{\alpha_1}) = \sum_{j=1}^2 b_j^2 w(a_j) < 0$ are never met simultaneously. Conversely, if $\frac{\alpha_2}{\alpha_1} < f_1 < f_2$ ($f_1 < f_2 < \frac{\alpha_2}{\alpha_1}$), it is easy to verify that there always exists a line meeting the conditions $w(f_1) \geq 0$, $w(f_2) \geq 0$ and $w(\frac{\alpha_2}{\alpha_1}) = \sum_{j=1}^2 b_j^2 w(a_j) < 0$. Notably, given the position of $\frac{\alpha_2}{\alpha_1}$ the feasibility of the pole placement is decidable.

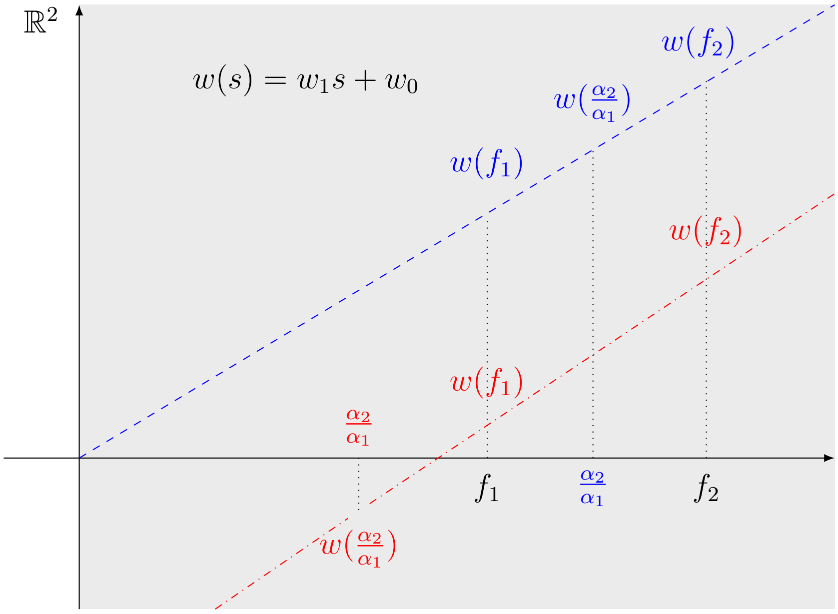

Figure 4.4: A feasible pole placement (dashed) and a non-feasible pole placement (dash-dotted) for $n = 2$.

$\blacksquare$

Remark 4.4 (Sign conditions and pole placement)**.** The polynomial $w(s) = \sum_{k=0}^{n-1} w_k s^k$, mentioned in Proposition 7 has real coefficients. Thus, $w(s)$ can be written as

$$w(s) = w_{n-1}\Big(\prod_{i=1}^{\nu}(s-\mu_i)\Big)\Big(\prod_{j=1}^{\eta}(s^2+\omega_j^2)\Big), \quad \mu_i, \omega_j \in \mathbb{R}, \quad \nu + 2\eta = n-1. \tag{4.53}$$

Without loss of generality, assume $w_{n-1} > 0$.[6] What determines the sign of $w(s)$ when evaluated along the real axis is the product

$$\prod_{i=1}^{\nu}(s-\mu_i) = (s-\mu_1)(s-\mu_2)\cdots(s-\mu_n). \tag{4.54}$$

In particular, along the real axis

$$\operatorname{sign}(s-\mu_i) = \begin{cases} 1, & s > \mu_i, \\ -1, & s < \mu_i. \end{cases} \tag{4.55}$$

This means that

- $w(s) < 0$ if and only if there is an odd number of real zeros $\mu_i > s$;
- conversely, $w(s) > 0$ if and only if there is an even number (possibly zero) of real zeros $\mu_i > s$.

Conversely $w(s) > 0$ if and only if there is an even number (possibly zero) of $\mu_i > s$ (at the right of s). With a similar reasoning, we infer that $w(a_j) < 0$ for all $j = 1, \ldots, n$ if and only if there is an odd number of zeros of $w(s)$ at the right of a_j for all j. Finally, note that $w(a_j) < 0$ for all $j = 1, \ldots, n$ implies $\sum_{j=1}^{n} b_j^2 w(a_j) < 0$.
Thus, if we find a polynomial $w(s)$ with real coefficients and $w_{n-1} > 0$ such that

- there is an even number of zeros of $w(s)$ (possibly zero) at the right of f_i for all i,
- there is an odd number of zeros of $w(s)$ at the right of a_j for all j,

then, the pole placement is not feasible. Dually, if we find a polynomial $w(s)$ with real coefficients and $w_{n-1} < 0$ such that

- there is an odd number of zeros of $w(s)$ (eventually zero) at the right of f_i for all i,

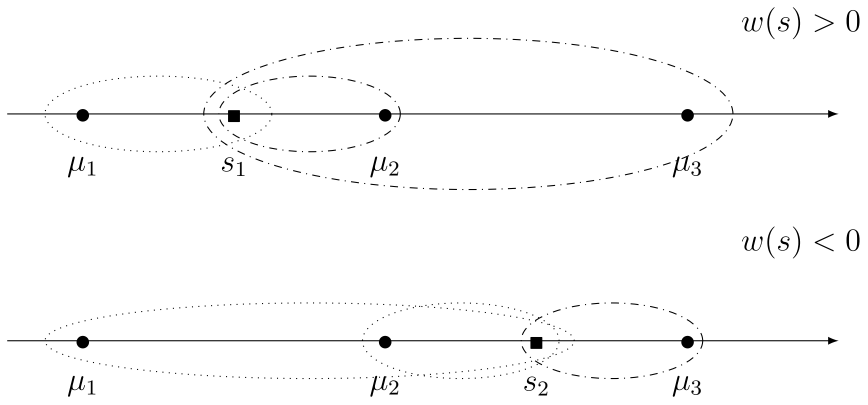

Figure 4.5: Pictorial representation of the positive (dotted) and negative (dash-dotted) terms determining the sign of $w(s)$, when $w_{n-1} > 0$.

- there is an even number of zeros of $w(s)$ at the right of a_j for all j,

then, the pole placement is not feasible.

▲

The conclusions of Remark 8 are summarized in the following

Proposition 8 (Sign conditions and pole placement)**.** *If there exists a polynomial*

$$w(s) = \sum_{k=0}^{n-1} w_k s^k, \quad w_i \in \mathbb{R}, \tag{4.56}$$

such that

1. *there is an even (respectively odd) number of zeros of $w(s)$ (possibly zero) at the right of f_i for all i,*

2. *there is an odd (respectively even) number of zeros of $w(s)$ at the right of a_j for all j,*

then there is a vector $w \in \mathbb{R}^n$ for which $V^\top w \geq 0$ and $w^\top \alpha < 0$. Consequently, the pole placement is not feasible.

[6]If $w_{n-1} < 0$, the direction of the inequalities has to be switched. We will take into account this situation later.

Chapter 5

Nonlinear ZIP systems

Herein, we propose a nonlinear enhancement of the notion of ZIP system.

5.1 Nonlinear ZIP system definition

Consider a nonlinear affine system described by equations of the form

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{5.1}$$

with $x(t) \in \mathbb{R}^N$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ and $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ smooth mappings. Following the arguments presented in [2, p. 2329] and [11], it is possible to define the k-th Markov parameter, for $k \geq 0$, of the nonlinear system as

$$\eta_k(\infty) = y_I^{(k)}(0) = y_{F,g(0)}^{(k)}(0) = L_f^k h \circ g(0), \qquad k \geq 0. \tag{5.2}$$

This definition allows us to derive a reduced order model which matches the first k moments at infinity of system (5.1) for any positive integer k.

In the linear case the reduced order model $\Sigma_G = (F, G, H)$ built from a ZIP system matching the first n moments at infinity is still ZIP. We investigate whether, suitably defining the notion of "nonlinear ZIP system", the inheritance property is preserved. To combine the notion of moment at infinity of a nonlinear system with the fact that a linear ZIP system satisfies $\mathcal{H} > 0$ and $\sigma\mathcal{H} < 0$, we give the following definition.

Definition 13 (Nonlinear ZIP system)**.** *The system (5.1) is said to be a nonlinear ZIP system of order n (at least) if there exists a neighborhood of* *the origin U such that for all $x \in U$ the following hold*

$$\mathcal{H}^{nl}_{n-1} := \begin{bmatrix} L_f^0 h \circ g(x) & L_f^1 h \circ g(x) & \cdots & L_f^{n-1} h \circ g(x) \\ L_f^1 h \circ g(x) & L_f^2 h \circ g(x) & \cdots & L_f^n h \circ g(x) \\ \vdots & \ddots & \ddots & \vdots \\ L_f^{n-1} h \circ g(x) & L_f^n h \circ g(x) & \cdots & L_f^{2n-2} h \circ g(x) \end{bmatrix} > 0, \tag{5.3}$$

$$\sigma\mathcal{H}^{nl}_{n-1} := \begin{bmatrix} L_f^1 h \circ g(x) & L_f^2 h \circ g(x) & \cdots & L_f^n h \circ g(x) \\ L_f^2 h \circ g(x) & L_f^3 h \circ g(x) & \cdots & L_f^{n+1} h \circ g(x) \\ \vdots & \ddots & \ddots & \vdots \\ L_f^n h \circ g(x) & L_f^{n+1} h \circ g(x) & \cdots & L_f^{2n-1} h \circ g(x) \end{bmatrix} < 0. \tag{5.4}$$

Remark 5.1. If system (5.1) is a linear system $\Sigma = (A, B, C)$ of order $N = n$, i.e. $f(x) = Ax$, $g(x) = B$ and $h(x) = Cx$, then $\mathcal{H}^{nl}_{n-1} = \mathcal{H}_{n-1}$ and $\sigma\mathcal{H}^{nl}_{n-1} = \sigma\mathcal{H}_{n-1}$. Indeed $y_I^{(k)}(0) = \frac{d^k}{dt^k}(Ce^{At}B)\Big|_{t=0} = CA^kB$ for any positive integer k. Hence, the linear system Σ is nonlinear ZIP of order n if and only if $\mathcal{H}_{n-1} > 0$ and $\sigma\mathcal{H}_{n-1} < 0$, thus, by Proposition 3, if and only if Σ is ZIP. ▲

Remark 5.2. If system (5.1) is nonlinear ZIP of order n, then it is nonlinear ZIP of order k, for any $1 \leq k \leq n$. To show this, note that eliminating from $\mathcal{H}^{nl}_{n-1}$ and $\sigma\mathcal{H}^{nl}_{n-1}$ the last row and column we get $\mathcal{H}_{n-2} > 0$ and $\sigma\mathcal{H}_{n-2} < 0$, then (5.1) is nonlinear ZIP of order $n-1$. By induction, system (5.1) is nonlinear ZIP of order k, for any $1 \leq k \leq n$. Conversely, if system (5.1) is not nonlinear ZIP of order k, for some $\bar{k} \geq 1$, then it not nonlinear ZIP for any $k \geq \bar{k}$. To prove this, assume that $\mathcal{H}^{nl}_{\bar{k}}$ is not positive definite. Thus, note that $\mathcal{H}^{nl}_{\bar{k}}$ is a principal submatrix of $\mathcal{H}^{nl}_{k}$, for any $k \geq \bar{k}$. For any $k \geq \bar{k}$, since of the principal minors of $\mathcal{H}^{nl}_{k}$ is not positive, namely $\det \mathcal{H}^{nl}_{\bar{k}}$, the system cannot be nonlinear ZIP of order k. The proof is similar if $\sigma\mathcal{H}^{nl}_{\bar{k}}$ is not negative definite. ▲

Remark 5.3. A necessary condition for system (5.1) to be nonlinear ZIP of order n, for some $n \geq 1$, is to have $y_I^{(0)}(0) = y_I(0) > 0$. ▲

Proposition 9 (Inheritance of the ZIP property by the reduced order model)**.** *Given a nonlinear ZIP system of order n described by equations of the form (5.1), consider the linear system Σ_G described by*

$$\Sigma_G = \left[\begin{array}{c|c} F & G \\ \hline H & 0 \end{array}\right], \tag{5.5}$$

where $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times 1}$, $H \in \mathbb{R}^{1\times n}$, and assume $HF^iG = y_I^{(i)}(0)$, for $i = 0, 1, \cdots, 2n-1$. The linear system thus constructed is

1. *a model of the nonlinear system achieving moment matching at infinity;*
2. *an asymptotically stable (minimal strictly proper) ZIP system.*

Proof. The first part is a direct consequence of the above discussion combined with [2, pp. 2229-2230].
To prove the second part, it is sufficient to note that $\mathcal{H}_{n-1}^{nl} = \mathcal{H}_{n-1}$ and $\sigma\mathcal{H}_{n-1}^{nl} = \sigma\mathcal{H}_{n-1}$, since $HF^iG = y_I^{(i)}(0)$, for $i = 0, 1, \cdots, 2n-1$. The nonlinear ZIP definition implies $\mathcal{H}_{n-1} = \mathcal{H}_{n-1}^{nl} > 0$ and $\sigma\mathcal{H}_{n-1} = \sigma\mathcal{H}_{n-1}^{nl} < 0$. By Proposition 3, Σ_G is a(n asymptotically stable minimal) ZIP system. □

Remark 5.4. By Proposition 6, the (linear) ZIP system (5.5) can be further reduced to a smaller (linear) ZIP system, still preserving matching at infinity. ▲

In future works, it might be interesting to investigate whether letting k grow there is a bound from which system (5.1) is not nonlinear ZIP anymore, i.e. there exists a positive integer k such that either $\mathcal{H}_k \ngtr 0$ or $\sigma\mathcal{H}_k \nless 0$.

Chapter 6

Examples and applications

In this chapter, we describe a few examples and applications arising in network analysis of electrical circuits, such as the parallel interconnection of series RC circuits, or in engineering-related problems, such as the modeling of non-laminated axial magnetic bearings [9] that can be modeled as ZIP systems. The behavior of the simulated ZIP systems and the model reduction algorithms described in the previous sections have been respectively simulated and implemented in MATLAB 2012 and were run on an Intel i5 CPU running at 1.7 GHz.

Example 14 (Parallel interconnection of series RC circuits)**.** Consider the series resistor−capacitor (RC) circuit in Figure (6.1). Straightforwardly, the

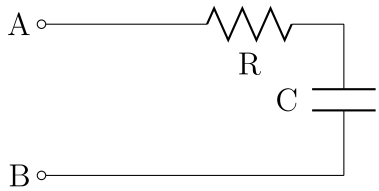

Figure 6.1: Series RC circuit.

transfer function from the input voltage between A and B, v_{in}, to the voltage across the capacitor C, v_C, is

$$W(s) = \frac{V_C(s)}{V_{in}(s)} = \frac{1}{1 + RCs} = \frac{\frac{1}{RC}}{s + \frac{1}{RC}}. \tag{6.1}$$

Defining $a := \frac{1}{RC}$, it is easy to see that $W(s)$ is a simple compartment.

$$W(s) = \frac{a}{s + a}, \qquad a > 0. \tag{6.2}$$

The parallel interconnection of N series RC circuit, with different $a_i := \frac{1}{R_i C_i}$, is a ZIP system. Indeed, denoting by $W_i(s)$ the transfer function of the $i-$th series RC circuit, the overall transfer function, $W(s)$, is given by

$$W(s) = \sum_{i=1}^{N} W_i(s) = \sum_{i=1}^{N} \frac{a_i}{s + a_i}, \qquad 0 < a_1 < a_2 < \cdots < a_N. \tag{6.3}$$

By Proposition 3, $W(s)$ is a ZIP transfer function. In what follows, a description of how we simulated systems of this form is given. Then, pictorial evidence of the good level of approximation achieved with model reduction by moment matching is provided.

In these experiments, the $N-$dimensional ZIP system to be reduced Σ, corresponding to (6.3), has been randomly generated as follows. The N coefficients a_i were selected and ordered through the MATLAB functions `randperm` and `sort`. The continuous time system has been created, by means of the MATLAB functions `zpk` and `ss`, summing the N simple compartments $W_i(s)$. Lastly, two $n-$dimensional reduced order models have been created, respectively denoted by Σ_G and Σ_{PP}. In the former functions, the n interpolation points $s_i \in \mathbb{R}$ are i.i.d. random variables drawn from a uniform distribution over the interval $[-10^5, 0]$. In the latter, we tested the pole placement assigning n distinct poles $f_i \in \mathbb{R}$, selected as i.i.d. random variables drawn from a uniform distribution over the interval $[-10^5, 0]$.

Denoting by $W(s)$, $W_G(s)$ and $W_{PP}(s)$ the transfer functions of Σ, Σ_G and Σ_{PP} respectively, Figure 6.2 displays the Bode plots of $W(s)$, $W_G(s)$ and $W_{PP}(s)$. It is easy to see that the Bode plots of $W(s)$ and $W_G(s)$ almost coincide, while $W(s)$ and $W_{PP}(s)$ show a slight difference. In all likelihood, forcing the reduced order model to have prescribed poles might not be the optimal choice in terms of distance between the Bode plots.

■

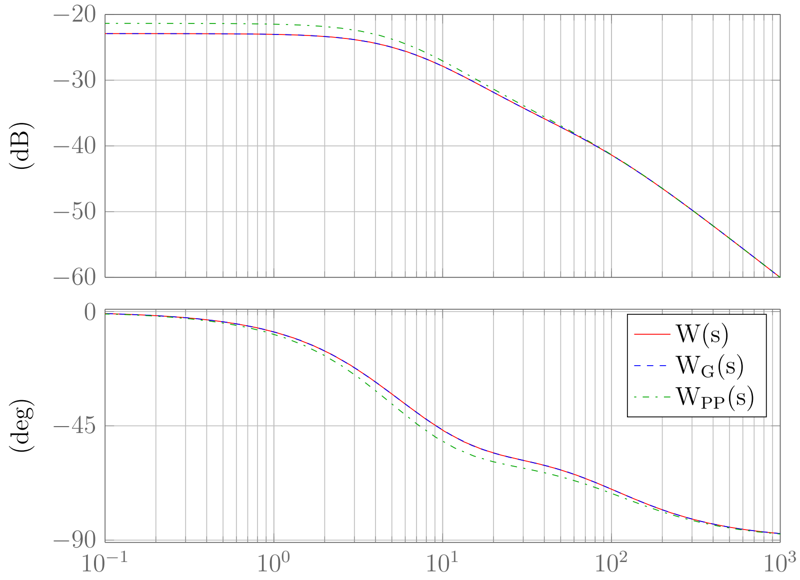

Figure 6.2: Amplitude and phase Bode plots of $W(s)$ (solid), $W_G(s)$ (dashed) and $W_{PP}(s)$ (dash-dotted) with $N = 11$ and $n = 4$.

Example 15 (Model reduction)**.** In these experiments, we reduced a ZIP system $\Sigma = (A, B, C)$ of dimension N. The system Σ has been randomly generated as follows. The $2N - 1$ (different and) interlacing zeros and the poles of Σ, respectively denoted by `z` and `p`, have been selected and ordered through the MATLAB functions `randperm` and `sort`. By means of the MATLAB functions `zpk` and `ss`, the continuous time system with zeros `z`, poles `p` has been created, where for simplicity the gain `k` has been chosen unitary. Lastly, to create the $n-$dimensional reduced order model Σ_G, the n interpolation points $s_i \in \mathbb{R}$ are i.i.d. random variables drawn from a uniform distribution over the interval $[10^5, 0]$.

Denoting by $W(s)$ and $W_G(s)$ the transfer functions of Σ and Σ_G respectively, Figure 6.3 displays the Bode plots of $W(s)$ and $W_G(s)$ for different values of N and n. One may notice that the Bode plots of $W(s)$ and $W_G(s)$ almost overlap.

■

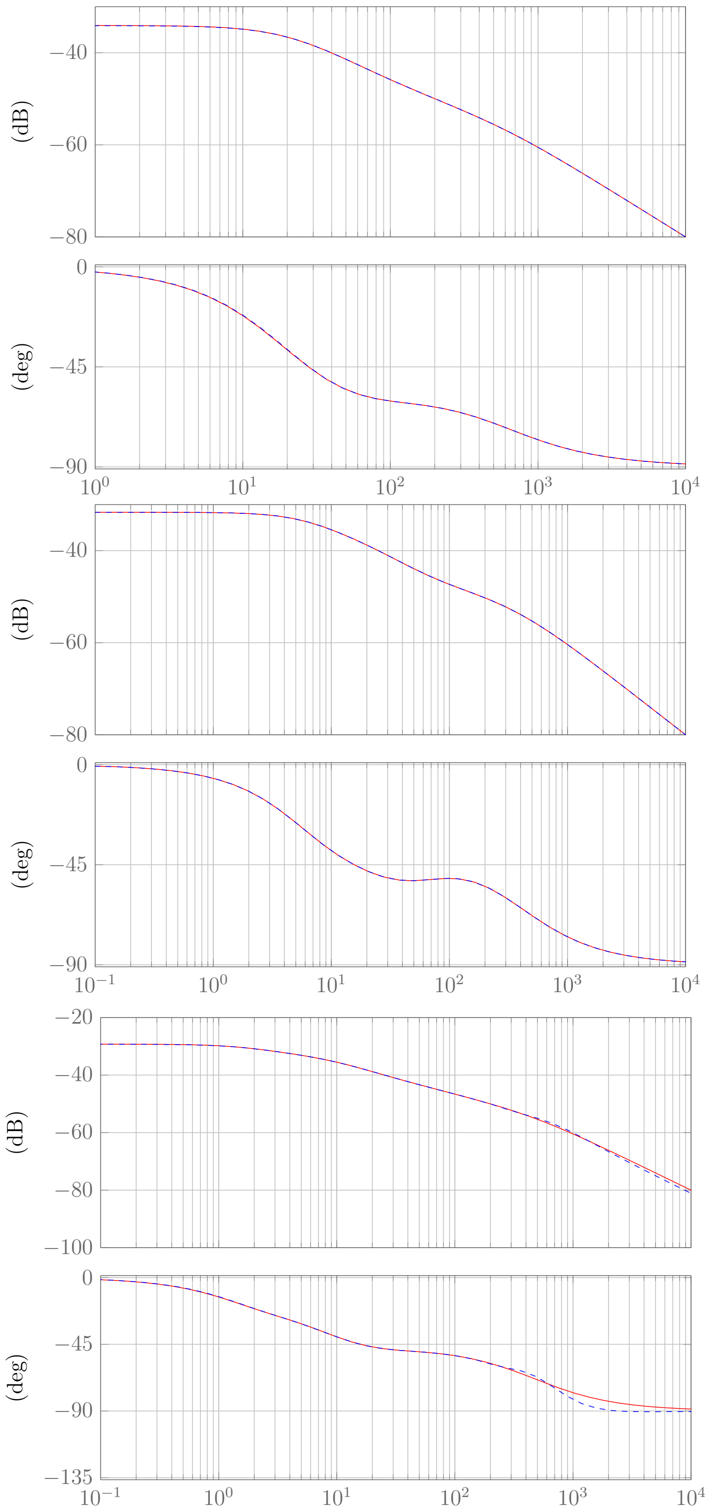

Figure 6.3: Amplitude and phase Bode plots of $W(s)$ (solid) and $W_G(s)$ (dashed) with $N = 7$, $n = 3$ (top), $N = 15$, $n = 5$ (middle), $N = 105$, $n = 10$ (bottom).

Appendix A

Proofs

A.1 Proof of Proposition 6

To prove the claim, we wish to derive a transformation $T \in \mathbb{R}^{n\times n}$, with $\det(T) \neq 0$, such that the following equations hold

$$T^{-1}FT = \mathrm{diag}(-\bar{f}_1, -\bar{f}_2, \cdots, -\bar{f}_n) =: \bar{F} \tag{A.1}$$
$$\bar{G} := T^{-1}G = (HT)^\top =: \bar{H}^\top \tag{A.2}$$

where $\bar{f}_i > 0, \bar{g} = \bar{h}_i > 0, \bar{f}_i \neq \bar{f}_j \ \forall i \neq j$.

The proof of the claim will proceed as follows:

(i) we will construct a change of coordinates T simultaneously verifying (A.1) and (A.2);

(ii) we will verify that the diagonal entries of $\bar{F}$, $\bar{f}_i$, are *real negative* numbers;

(iii) we will prove that all the diagonal entries of $\bar{F}$ are distinct: $\bar{f}_i \neq \bar{f}_j \ \forall i \neq j$;

(iv) we will complete the proof showing that the entries of $\bar{H}$, $\bar{h}_i$, can be assumed positive for all $i = 1, \ldots, n$.

(i). To begin with, consider equation (A.2). The following equivalences hold

$$\begin{aligned}(HT)^\top = T^{-1}G &\Leftrightarrow (C\Pi T)^\top = T^{-1}(\mathcal{O}_{n-1}\Pi)^{-1}\mathcal{O}_{n-1}B = (\mathcal{O}_{n-1}\Pi T)^{-1}\mathcal{O}_{n-1}B \\ &\Leftrightarrow (\mathcal{O}_{n-1}\Pi T)(T^\top\Pi^\top\mathcal{O}_{n-1}^\top)\mathbf{e}_1 = \mathcal{O}_{n-1}B \\ &\Leftrightarrow (\mathcal{O}_{n-1}\Pi)TT^\top(\Pi^\top\mathcal{O}_{n-1}^\top)\mathbf{e}_1 = \mathcal{O}_{n-1}\mathcal{O}_{n-1}^\top\mathbf{e}_1\end{aligned}$$

Choosing

$$\boxed{T = (\mathcal{O}_{n-1}\Pi)^{-1}\mathcal{O}_{n-1}Q} \tag{A.3}$$

equation (A.2) holds for any matrix $Q \in \mathbb{R}^{N\times n}$ such that $QQ^\top = I_n$.

Assuming T has the form (A.3), we now investigate the existence of Q such that equations (A.1) and (A.3) are verified.

Define

$$\boxed{X := \Pi(\mathcal{O}_{n-1}\Pi)^{-1}\mathcal{O}_{n-1}} \in \mathbb{R}^{N\times N}. \tag{A.4}$$

Since $X^2 = X$, X is a projection matrix onto $\mathrm{Im}(\Pi)$ along $\ker(\mathcal{O}_{n-1})$. Note that $\mathrm{rank}(X) = n$. In addition, the following properties hold

$$X\Pi = \Pi, \tag{A.5}$$

$$\mathcal{O}_{n-1}X = \mathcal{O}_{n-1}. \tag{A.6}$$

Consider now equation (A.1). The following identities hold

$$\begin{aligned}
\bar{F} &\overset{(A.1)}{=} T^{-1}FT \\
&\overset{(A.3)}{=} [(\mathcal{O}_{n-1}\Pi)^{-1}\mathcal{O}_{n-1}Q]^{-1}F[(\mathcal{O}_{n-1}\Pi)^{-1}\mathcal{O}_{n-1}Q] \\
&\overset{(4.23)}{=} (\mathcal{O}_{n-1}Q)^{-1}\mathcal{O}_{n-1}A\Pi(\mathcal{O}_{n-1}\Pi)^{-1}\mathcal{O}_{n-1}(\mathcal{O}_{n-1}Q) \\
&\overset{(A.4)}{=} (\mathcal{O}_{n-1}Q)^{-1}\mathcal{O}_{n-1}AXQ
\end{aligned} \tag{A.7}$$

for any matrix $Q \in \mathbb{R}^{N\times n}$ such that $QQ^\top = I_n$.

We now wish to show that there exist $Q \in \mathbb{R}^{N\times n}$, respecting $Q^\top Q = I_n$, such that the following equation holds

$$\bar{F} \overset{?}{=} (\mathcal{O}_{n-1}Q)^{-1}\mathcal{O}_{n-1}AXQ. \tag{A.8}$$

Equation (A.8) can be cast as

$$\begin{aligned}
\bar{F} = (\mathcal{O}_{n-1}Q)^{-1}\mathcal{O}_{n-1}AXQ &\Leftrightarrow \mathcal{O}_{n-1}Q\bar{F} = \mathcal{O}_{n-1}AXQ \\
&\overset{(A.6)}{\Leftrightarrow} \mathcal{O}_{n-1}XQ\bar{F} = \mathcal{O}_{n-1}AXQ
\end{aligned} \tag{A.9}$$

A sufficient condition for equation (A.9) to hold is

$$XQ\bar{F} = AXQ \tag{A.10}$$

Premultiplying equation (A.10) by $X^\top$ yields

$$X^\top XQ\bar{F} = X^\top AXQ \tag{A.11}$$

Since $X^\top X$ is a symmetric matrix and $\operatorname{rank}(X^\top X) = n$, there exists a orthogonal matrix $U \in \mathbb{R}^{N\times N}$, such that $U^\top U = UU^\top = I_N$, and a diagonal matrix

$$D := \left[\begin{array}{c|c} D_1 & \\ \hline & 0_{N-n} \end{array}\right] = \left[\begin{array}{ccc|c} d_1 & & & \\ & \ddots & & \\ & & d_n & \\ \hline & & & 0_{N-n} \end{array}\right] \in \mathbb{R}^{N\times N} \tag{A.12}$$

such that

$$U^\top X^\top X U = \left[\begin{array}{c|c} D_1 & 0 \\ \hline 0 & 0 \end{array}\right] \tag{A.13}$$

Premultiplying equation (A.11) by $U^{-1} = U^\top$ yields

$$\begin{aligned} X^\top X Q \bar{F} = X^\top A X Q &\Leftrightarrow U^{-1}(X^\top X)(UU^{-1})Q\bar{F} = U^\top X^\top AX(UU^{-1})Q \\ &\Leftrightarrow \underbrace{[U^\top(X^\top X)U]}_{=D}\underbrace{[U^\top Q]}_{:=\bar{Q}}\bar{F} = \underbrace{[U^\top X^\top AXU]}_{\bar{A}=\bar{A}^\top := \left[\begin{array}{c|c} \bar{A}_{11} & \bar{A}_{12} \\ \hline \bar{A}_{12}^\top & \bar{A}_{22} \end{array}\right]} \underbrace{[U^\top Q]}_{:=\bar{Q}} \\ &\Leftrightarrow D\bar{Q}\bar{F} = \bar{A}\bar{Q} \\ &\Leftrightarrow \left[\begin{array}{c|c} D_1 & \\ \hline & \end{array}\right] \left[\begin{array}{c} \bar{Q}_1 \\ \hline \bar{Q}_2 \end{array}\right] \bar{F} = \left[\begin{array}{c|c} \bar{A}_{11} & \bar{A}_{12} \\ \hline \bar{A}_{12}^\top & \bar{A}_{22} \end{array}\right] \left[\begin{array}{c} \bar{Q}_1 \\ \hline \bar{Q}_2 \end{array}\right] \\ &\Leftrightarrow \left[\begin{array}{c} D_1\bar{Q}_1\bar{F} \\ \hline 0 \end{array}\right] = \left[\begin{array}{c} \bar{A}_{11}\bar{Q}_1 \\ \hline \bar{A}_{12}^\top\bar{Q}_1 \end{array}\right] \end{aligned} \tag{A.14}$$

It is always possible to choose $\bar{Q} \in \mathbb{R}^{N\times n}$ such that $\bar{Q}_1$ is a (square) orthogonal matrix, namely $\bar{Q}_1\bar{Q}_1^\top = \bar{Q}_1^\top\bar{Q}_1 = I_n$. Invertibility of $\bar{Q}_1$ together with equation (A.14) imply $\bar{A}_{12}^\top = 0_{N-n\times n}$. Thus, to verify equation (A.14), $\bar{A}$ must be a block-diagonal matrix. It can be shown that this assumption is not restrictive[1].

To prove that equation (A.14) holds for some $\bar{Q}_1$, choose $\bar{Q}_1 = D_1^{-\frac{1}{2}}\tilde{Q}_1$. Then

$$\begin{aligned} D_1\bar{Q}_1\bar{F} = \bar{A}_{11}\bar{Q}_1 &\Leftrightarrow D_1 D_1^{-\frac{1}{2}}\tilde{Q}_1\bar{F} = \bar{A}_{11}D_1^{-\frac{1}{2}}\tilde{Q}_1 \\ &\Leftrightarrow D_1^{\frac{1}{2}}\tilde{Q}_1\bar{F} = \bar{A}_{11}D_1^{-\frac{1}{2}}\tilde{Q}_1 \\ &\Leftrightarrow \bar{F} = \tilde{Q}_1^\top \underbrace{D_1^{-\frac{\top}{2}}\bar{A}_{11}D_1^{-\frac{1}{2}}}_{:=\tilde{A}_{11}}\tilde{Q}_1 \\ &\Leftrightarrow \bar{F} = \tilde{Q}_1^\top\tilde{A}_{11}\tilde{Q}_1 \end{aligned} \tag{A.15}$$

[1]Proof provided in Appendix A.2.

Observe that $\tilde{A}_{11}$ is a symmetric matrix, thus all its eigenvalues are *real*, so there exists an orthogonal matrix $\tilde{Q}_1 \in \mathbb{R}^{n\times n}$ that diagonalizes it. Properly choosing $\tilde{Q}_1$, we get

$$\begin{aligned} T &= (\mathcal{O}_{n-1}\Pi)^{-1}\mathcal{O}_{n-1}Q & \text{(A.16)}\\ &= (\mathcal{O}_{n-1}\Pi)^{-1}\mathcal{O}_{n-1}(U\left[\frac{\bar{Q}_1}{\bar{Q}_2}\right]) & \text{(A.17)}\\ &= (\mathcal{O}_{n-1}\Pi)^{-1}\mathcal{O}_{n-1}(U\left[\frac{D_1^{-\frac{1}{2}}\tilde{Q}_1}{\bar{Q}_2}\right]), & \text{(A.18)} \end{aligned}$$

the change of coordinates we were looking for. The degree of freedom on $\bar{Q}_2$ is exploited to satisfy condition $Q^\top Q = I_n$. Namely, we have to choose $\bar{Q}_2 = -\tilde{Q}_1^\top D_1^{-1}\tilde{Q}_1$.

(ii). To show that the diagonal entries of $\bar{F}$ are negative, observe that for every nonzero $x \in \mathbb{R}^n$

$$\begin{aligned} x^\top \bar{F} x &\overset{(A.15)}{=} x^\top \tilde{Q}_1^\top \tilde{A}_{11}\tilde{Q}_1 x & \text{(A.19)}\\ &= \underbrace{x^\top \tilde{Q}_1^\top D_1^{-\frac{\top}{2}}}_{=y^\top} \bar{A}_{11} \underbrace{D_1^{-\frac{1}{2}}\tilde{Q}_1 x}_{:=y} & \text{(A.20)}\\ &= y^\top \bar{A}_{11} y < 0. & \text{(A.21)} \end{aligned}$$

Indeed, for every nonzero $y \in \mathbb{R}^n$

$$y^\top \bar{A}_{11} y = (y^\top \left[\begin{array}{c|c} I_n & 0 \end{array}\right] U^\top X^\top)\underbrace{A}_{<0}(XU\left[\frac{I_n}{0}\right]y) \le 0. \qquad \text{(A.22)}$$

On the other hand, $\text{rank}(\bar{A}_{11}) = n$.[2] Thus, inequality (A.22) must be strict and the eigenvalues of $\bar{F}$ are all *negative* and *real*.

(iii). To show that the diagonal entries of $\bar{F}$ are distinct, note that $\sigma(S) \cap \sigma(S - GL) = \emptyset$.[3] Thus the pair $(S - GL, G) = (F, G)$ is reachable[4] and the algebraically equivalent pair $(\bar{F}, \bar{G})$ is reachable as well. Since $\bar{F}$ is diagonal, the single-input single-output system $(\bar{F}, \bar{G})$ is reachable if and only if $\bar{g}_i = \bar{h}_i \neq 0$ for all $i = 1, \ldots, n$ and the eigenvalues $\bar{f}_i$ are distinct.

[2]Proof provided in Appendix A.2.
[3]Proof provided in Appendix A.3.
[4]See [p.6][3].

(iv). To show that all entries of $\bar{H} = \bar{G}^\top$ can be assumed positive, consider the transformation

$$R = \text{diag}(\text{sign}(\bar{h}_1), \cdots, \text{sign}(\bar{h}_n)). \tag{A.23}$$

Note that $R^{-1} = R$, that $R\bar{F}R^{-1} = \bar{F}$ and that

$$\bar{H}R^{-1} = (R\bar{G})^\top = \left[|\bar{h}_1| \cdots |\bar{h}_n|\right], \tag{A.24}$$

hence the claim.

A.2 Proof of $U^\top X^\top AXU = diag(\bar{A}_{11}, \bar{A}_{22})$

Recall $U \in \mathbb{R}^{N\times N}$, $UU^\top = U^\top U = I_N$ and

$$U^\top X^\top XU = \left[\begin{array}{c|c} D_1 & 0 \\ \hline 0 & 0 \end{array}\right]. \tag{A.25}$$

In addition, X is a projection matrix, hence there exists a non-singular matrix V, such that

$$V^{-1}XV = \left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & 0 \end{array}\right] \Leftrightarrow X = V\left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & 0 \end{array}\right]V^{-1}. \tag{A.26}$$

We wish to show that $U^\top X^\top AXU$ is block diagonal, i.e. that

$$U^\top X^\top AXU = \left[\begin{array}{c|c} \bar{A}_{11} & 0 \\ \hline 0 & \bar{A}_{22} \end{array}\right]. \tag{A.27}$$

Preliminarily observe that

$$
\begin{aligned}
\left[\begin{array}{c|c} D_1 & \\ \hline & 0 \end{array}\right] &\overset{(A.13)}{=} U^\top X^\top X U \\
&= U^\top V^{-\top} \left[\begin{array}{c|c} I_n & \\ \hline & 0 \end{array}\right] \underbrace{V^\top V}_{:=\bar{V}} \left[\begin{array}{c|c} I_n & \\ \hline & 0 \end{array}\right] V^{-1} U \\
&= U^\top V^{-\top} \left[\begin{array}{c|c} I_n & \\ \hline & 0 \end{array}\right] \left[\begin{array}{c|c} \bar{V}_{11} & \bar{V}_{12} \\ \hline \bar{V}_{12}^\top & \bar{V}_{22} \end{array}\right] \left[\begin{array}{c|c} I_n & \\ \hline & 0 \end{array}\right] V^{-1} U \\
&= \underbrace{U^\top V^{-\top}}_{:=\Omega^\top} \left[\begin{array}{c|c} \bar{V}_{11} & 0 \\ \hline 0 & 0 \end{array}\right] \underbrace{V^{-1} U}_{:=\Omega} \\
&= \left[\begin{array}{c|c} \Omega_{11}^\top & \Omega_{21}^\top \\ \hline \Omega_{12}^\top & \Omega_{22}^\top \end{array}\right] \left[\begin{array}{c|c} \bar{V}_{11} & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} \Omega_{11} & \Omega_{12} \\ \hline \Omega_{21} & \Omega_{22} \end{array}\right] \\
&= \left[\begin{array}{c|c} \Omega_{11}^\top \bar{V}_{11} \Omega_{11} & \Omega_{11}^\top \bar{V}_{11} \Omega_{12} \\ \hline (\Omega_{11}^\top \bar{V}_{11} \Omega_{12})^\top & \Omega_{12}^\top \bar{V}_{11} \Omega_{12} \end{array}\right] \\
&\Rightarrow \Omega_{12} = 0 \\
&\Rightarrow \Omega = V^{-1} U = \left[\begin{array}{c|c} \Omega_{11} & 0 \\ \hline \Omega_{21} & \Omega_{22} \end{array}\right].
\end{aligned} \tag{A.28}
$$

Now consider the following equalities

$$
\begin{aligned}
U^\top X^\top A X U &= \underbrace{U^\top V^{-\top}}_{=\Omega^\top} \left[\begin{array}{c|c} I_n & \\ \hline & 0 \end{array}\right] V^\top A V \left[\begin{array}{c|c} I_n & \\ \hline & 0 \end{array}\right] \underbrace{V^{-1} U}_{=\Omega} \\
&= \left[\begin{array}{c|c} * & * \\ \hline & * \end{array}\right] \left[\begin{array}{c|c} * & \\ \hline & \end{array}\right] \left[\begin{array}{c|c} * & * \\ \hline * & * \end{array}\right] \left[\begin{array}{c|c} * & \\ \hline & \end{array}\right] \left[\begin{array}{c|c} * & \\ \hline * & * \end{array}\right] \\
&= \left[\begin{array}{c|c} * & \\ \hline & \end{array}\right] \left[\begin{array}{c|c} * & * \\ \hline * & * \end{array}\right] \left[\begin{array}{c|c} * & \\ \hline & \end{array}\right] \\
&= \left[\begin{array}{c|c} * & \\ \hline & \end{array}\right] \\
&:= \left[\begin{array}{c|c} \bar{A}_{11} & \\ \hline & 0 \end{array}\right]
\end{aligned} \tag{A.29}
$$

hence the claim. Note that $\bar{A}_{22} = 0$. Since $U^\top X^\top A X U = (U^\top X^\top A X U)^\top$ and $\operatorname{rank}(U^\top X^\top A X U) = n$, equation (A.29) implies $\bar{A}_{11} = \bar{A}_{11}^\top$ and $\operatorname{rank}(\bar{A}_{11}) = n$.

A.3 Proof of $\sigma(S) \cap \sigma(S - GL) = \emptyset$

By contradiction assume that there exists $\lambda \in \sigma(S) \cap \sigma(S-GL) \neq \emptyset$. Then, there exist two nonzero vectors $v, w \in \mathbb{R}^n$ such that

$$\begin{cases} Sv = \lambda v \\ w^\top (S - GL) = \lambda w^\top \end{cases} \Rightarrow \begin{cases} w^\top S v = \lambda w^\top v \\ w^\top (S - GL) v = \lambda w^\top v \end{cases} \tag{A.30}$$

$$\Rightarrow (\underbrace{w^\top G}_{\in \mathbb{R}})(\underbrace{Lv}_{\in \mathbb{R}}) = 0 \tag{A.31}$$

$$\Rightarrow \begin{cases} w^\top G = 0 \\ Lv = 0 \end{cases} \tag{A.32}$$

In either case equations in (A.32) lead to a contradiction. Indeed, if it was $Lv = 0$, then

$$\begin{bmatrix} L \\ LS \\ \vdots \\ LS^{n-1} \end{bmatrix} v = \begin{bmatrix} Lv \\ L(Sv) \\ \vdots \\ L(S^{n-1}v) \end{bmatrix} = \begin{bmatrix} Lv \\ L(\lambda v) \\ \vdots \\ L(\lambda^{n-1}v) \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda \\ \vdots \\ \lambda^{n-1} \end{bmatrix} (Lv) = 0.$$

By the observability of (S, L), v must be zero, a contradiction. By using a dual version, being $(S - GL, G)$ reachable, w must be zero, again a contradiction.

Appendix B

Further properties

B.1 The coordinate transformation T

Given the vector $B = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}^\top \in \mathbb{R}^n$ we derive a coordinate transformation $T = T(A, B)$, such that

$$\underbrace{\begin{bmatrix} -a_1 & 0 & \cdots & 0 & 0 \\ b_1 b_2 & -a_2 & 0 & \cdots & 0 \\ b_1 b_3 & b_2 b_3 & -a_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b_1 b_n & b_2 b_n & \cdots & b_{n-1} b_n & -a_n \end{bmatrix}}_{=:A_s \in \mathbb{R}^{n\times n}} T = T^{-1} \underbrace{\begin{bmatrix} -a_1 & 0 & \cdots & 0 & 0 \\ 0 & -a_2 & 0 & \cdots & 0 \\ 0 & 0 & -a_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & -a_n \end{bmatrix}}_{=:A \in \mathbb{R}^{n\times n}} T. \tag{B.1}$$

Obviously, the coordinate transformation T must be triangular. It is not restrictive to assume that T has unitary diagonal entries, namely

$$T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ t_{21} & 1 & 0 & \cdots & 0 \\ t_{31} & t_{32} & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ t_{n1} & t_{n2} & \cdots & t_{nn-1} & 1 \end{bmatrix}. \tag{B.2}$$

Premultiplying equation (B.1) by T yields

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ t_{21} & 1 & 0 & \cdots & 0 \\ t_{31} & t_{32} & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ t_{n1} & t_{n2} & \cdots & t_{nn-1} & 1 \end{bmatrix} \begin{bmatrix} -a_1 & 0 & \cdots & 0 & 0 \\ b_1b_2 & -a_2 & 0 & \cdots & 0 \\ b_1b_3 & b_2b_3 & -a_3 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b_1b_n & b_2b_n & \cdots & b_{n-1}b_n & -a_n \end{bmatrix}$$
$$= \begin{bmatrix} -a_1 & 0 & 0 & \cdots & 0 \\ -a_2t_{21} & -a_2 & 0 & \cdots & 0 \\ -a_3t_{31} & -a_3t_{32} & -a_3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -a_nt_{n1} & -a_nt_{n2} & \cdots & -a_nt_{nn-1} & -a_n \end{bmatrix}.$$

Provided that $t_{ii} = 1$ for all $i = 1, \ldots, n$, the entries of the m-th row of T satisfy

$$-t_{ml}a_l + b_l\Big(\sum_{k=l+1}^{m} t_{mk}b_k\Big) = -a_m t_{ml}, \qquad \forall m > l. \tag{B.3}$$

Equivalently

$$t_{ml} = \frac{b_l}{a_l - a_m}\Big(\sum_{k=l+1}^{m} t_{mk}b_k\Big), \qquad \forall m > l. \tag{B.4}$$

For $m = 1, 2, \ldots, n$, equation (B.4) yields the entries of the $m-$th row of T, namely

$$\begin{aligned} t_{m,m} &= 1, \\ t_{m,m-1} &= \frac{b_{m-1}b_m}{a_{m-1} - a_m}, \\ t_{m,m-2} &= \frac{b_{m-2}b_m}{a_{m-2} - a_m}\Big(1 + \frac{b_{m-1}^2}{a_{m-1} - a_m}\Big), \\ t_{m,m-3} &= \frac{b_{m-3}b_m}{a_{m-3} - a_m}\Big(1 + \frac{b_{m-1}^2}{a_{m-1} - a_m} + \frac{b_{m-2}^2}{a_{m-2} - a_m} + \frac{b_{m-1}^2 b_{m-2}^2}{(a_{m-1} - a_m)(a_{m-2} - a_m)}\Big), \\ &\vdots \end{aligned}$$

Hence, $T = T(A, B)$ has the form

$$T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{b_2 b_1}{a_1 - a_2} & 1 & 0 & \cdots & 0 \\ \frac{b_3 b_1}{a_1 - a_2}(1 + \frac{b_2^2}{a_2 - a_3}) & \frac{b_3 b_2}{a_3 - a_2} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ & \cdots & \cdots & \frac{b_n b_{n-1}}{a_{n-1} - a_n} & 1 \end{bmatrix}. \tag{B.5}$$

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