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ON LOVÁSZ-SAKS-SCHRIJVER IDEALS

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Abstract

This work is divided into two parts. In the first part, we present some known results about Lovász-Saks-Schrijver ideals associated to graphs. The properties of being prime, radical or a complete intersection play a central role in our investigation. We also show how LSS ideals are related to both orthogonal representations of graphs and to hyperplane sections of determinantal varieties. In the second part, we give a partial affirmative answer to a question of A. Conca and V. Welker ([CW19]) regarding the radicality of LSS ideals associated to complete graphs. Finally, we compute a Gröbner basis of the minimal prime ideals.

Contents

Preface

In the current trends of commutative algebra, the role of combinatorics is prominent, and, in particular, graphs are one of the key topics.

It has been proved to be fruitful to attach algebraic objects to graphs in order to retrieve meaningful information about the underlying graphs: broadly speaking, graph-theoretic problems are encoded into these objects and can therefore be investigated by means of techniques in commutative algebra. The easiest example is given by edge ideals, which are monomial ideals whose generators are defined from the edges of a graph.
A generalization of edge ideals is given by *Lovász-Saks-Schrijver ideals* (abbreviated to LSS), which were introduced by L. Lovász, M. Saks and A. Schrijver in [LSS89] in connection with orthogonal representations of graphs. To be more precise, given a simple undirected graph G of n vertices and an integer $d > 0$, the LSS ideal of G is the ideal $\mathrm{L}_{\mathbb{K}}(G, d)$ in the polynomial ring $\mathbb{K}[x_{1,1}, \dots, x_{n,d}]$ generated by the polynomials

$$g_{i,j} := \sum_{k=1}^{d} x_{i,k} x_{j,k}$$

for $\{i, j\}$ belonging to the edges of G.

In this work, we firstly present some of the known algebraic properties of LSS ideals, and then focus on a question that A. Conca and V. Welker posed in [CW19].

Conjecture 0.0.1 ([CW19]). *The LSS ideal* $\mathrm{L}_{\mathbb{K}}(K_n, d)$ *associated to the complete graph* K_n *is radical for every* n *and every* d*, at least when* $\operatorname{char} \mathbb{K} = 0$.

We will concentrate on the smallest open case, which is $d = 3$, and give a contribution to this problem by computing an irredundant primary decomposition of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$, up to the embedded components. We will deduce that, if $\operatorname{char} \mathbb{K} \neq 2$, then $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ is *generically* radical (Theorem 3.1.6). This means that its scheme is reduced almost everywhere.

In Chapter 1, we recall some of the tools in commutative algebra that will be employed in this work.

Chapter 2 is a survey about LSS ideals. We will firstly give some motivations to investigate them. These include the connection between LSS ideals and orthogonal representations of graphs, or hyperplane sections of determinantal varieties. After that, we discuss the main known results about some algebraic properties of LSS ideals, namely being prime, radical or complete intersection. We observe that these properties are related to combinatorial properties of the underlying graph.

In Chapter 3, we focus on the study of the radicality of LSS ideals. After a brief summary on the state of the art, we present our contribution to Conjecture 0.0.1. The main result of this work is Theorem 3.1.6:

Theorem 3.1.6. *Over a field* $\mathbb{K}$ *with* $\operatorname{char}\mathbb{K} \neq 2$*, the primary components corresponding to minimal primes of* $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ *for* $n > 3$ *are of the following two types:*

- *For any* 3*-subset* $\Gamma \subseteq [n]$*, the ideal*
$$A_\Gamma := (g_{i,j} \mid i < j, i, j \in \Gamma) + (x_{i,k} \mid i \notin \Gamma, k \in [3]);$$
- *For any* $k \in [n]$*, the ideal*
$$B_k := (g_{i,j} \mid i \leq j, (i,j) \neq (k,k)) + I_2(X_k)$$
where X_k *denotes the matrix obtained from* X *by deleting the* k*-th row.*

Moreover, the above ideals are the minimal primes of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$*.*

In order to prove this result, we will proceed in the following way:

1. The ideals A_Γ and B_k appearing in the statement are actually prime;
2. The ideals A_Γ and B_k are primary components and minimal primes of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$;
3. There are no other minimal primes of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$.

We conclude the chapter by computing a Gröbner basis of B_k.

Notation

Rings will always be assumed to be commutative and unitary. The symbol $\mathbb{K}$ will always denote a field. Given an ideal I of a polynomial ring R in n variables over $\mathbb{K}$, we denote by $V(I)$ the algebraic set of I in the affine space $\mathbb{K}^n$, which is

$$V(I) = \{x \in \mathbb{K}^n \mid f(x) = 0 \text{ for every } f \in I\}.$$

Vectors are assumed to be row vectors and we denote by $\mathbf{v}_i \cdot \mathbf{v}_j$ the componentwise product of the vectors $\mathbf{v}_i$ and $\mathbf{v}_j$, that is

$$\mathbf{v}_i \cdot \mathbf{v}_j = \mathbf{v}_i \mathbf{v}_j^\top.$$

Observe that, in general, this is not an inner product, since $\mathbf{v} \cdot \mathbf{v} = 0$ does not imply $\mathbf{v} = 0$, for instance if $\mathbb{K} = \mathbb{C}$. Nevertheless, if $\mathbf{v}_i \mathbf{v}_j^\top = 0$, with a little abuse of notation we will say that $\mathbf{v}_i$ and $\mathbf{v}_j$ are orthogonal.

Chapter 1

Preliminaries

In this chapter, we recall some of the notions in commutative algebra that will be required to understand this work.

1.1 Primary decomposition

Let R be a ring. In this section, we will recall the definition of primary decomposition of an ideal of R, and how it is related to the radicality of I. These results will be used to prove the main result of this work, Theorem 3.1.6. The omitted proofs in this section can be easily found in the literature, see for example [AM69] or [Eis13].

Definition 1.1.1. A proper ideal I of R is called **primary** if for every $a, b, \in R$ with $ab \in I$ we have either $a \in I$ or $b \in \sqrt{I}$.

Every prime ideal is trivially primary. The converse is not true, as seen in the next

Example 1.1.2. *Let $R = \mathbb{K}[x]$ for some field $\mathbb{K}$. Then the ideal (x^2) is primary: if $ab \in (x^2)$, then $ab = fx^2$ for some $f \in \mathbb{K}[x]$. This means that either x^2 divides a, in which case $a \in (x^2)$, or x divides b, in which case $b \in (x) = \sqrt{(x^2)}$.*

Actually, a generalization of the above proof shows that, in every principal ideal domain, primary ideals are exactly the powers of prime ideals.

Proposition 1.1.3. *The radical ideal of a primary ideal is prime.*

Proof. Let I be a primary ideal, and let $P = \sqrt{I}$. Let $a, b \in R$ such that $ab \in P$. Then $a^n b^n \in I$ for some $n \in \mathbb{N}$. Since I is primary, this means that either $a^n \in I$ or $b^n \in \sqrt{I}$, hence $a \in P$ or $b \in P$. □

In order to specify the underlying prime ideal P of a primary ideal I, we will say that I is P-**primary**.

We are going to study when an ideal can be written as an intersection of primary ideals. This is a matter of interest because of its geometric interpretation, as we will see later on.

Definition 1.1.4. Let I be an ideal of R. A **primary decomposition** is an expression of I as a finite intersection of primary ideals:

$$I = \bigcap_{k=1}^{n} I_k.$$

If an ideal has a primary decomposition, we will call it **decomposable**.

Proposition 1.1.5. *In a Noetherian ring, every proper ideal is decomposable.*

From now on, we will suppose R to be Noetherian.

In general, a primary decomposition is not unique. We are interested in primary decompositions with nice minimal properties.

Definition 1.1.6. A primary decomposition $I = \bigcap_{k=1}^{n} I_k$ is called **irredundant** if:

- the $\sqrt{I_k}$ are all distinct,
- for every k, we have $\bigcap_{h \neq k} I_h \nsubseteq I_k$.

Any primary decomposition can be reduced to an irredundant one: since the intersection of P-primary ideals is again P-primary, we may firstly consider the intersection of primary ideals with the same radical and then leave out superfluous terms.

Example 1.1.7. *An irredundant primary decomposition is not unique: let $I = (x,y) \cdot (x) = (x^2, xy) \in \mathbb{K}[x,y]$. Then it can be checked that*

$$I = (x) \cap (x^2, xy, y^2) \quad \text{and} \quad I = (x) \cap (x^2, y)$$

are two distinct irredundant primary decompositions of I.

However, we are going to see that the number of components and the underlying primes in an irredundant primary decomposition of an ideal are uniquely determined.

Proposition 1.1.8. *Let $I = \bigcap_{k=1}^{n} I_k$ be an irredundant primary decomposition of I, and let $P_k = \sqrt{I_k}$ for every k. Then the P_i are precisely the prime ideals of the form*

$$\sqrt{I : a}$$

for some $a \in R$, and, in particular, they do not depend on the chosen decomposition.

Proof. See [AM69]. □

The primes described in the above proposition are called **associated prime ideals** of I. Among these, the minimal elements with respect to inclusion are exactly the minimal prime ideals of I, i.e. the prime ideals P containing I such that there is no prime Q with $I \subseteq Q \subset P$. The non minimal associated primes are called **embedded prime ideals**.

The primary components associated to minimal primes are uniquely determined as well.

Proposition 1.1.9. *Given a minimal prime P of I, the primary component of I associated to P is the kernel of the localization map $R \to (R/I)_P$.*

Proof. See [Eis13]. □

We can characterize the primary components which are also minimal primes in a way that will be more convenient for us. Let I be an ideal of R and let $x \in R \setminus I$. Observe that

$$I : x \subseteq I : x^2 \subseteq I : x^3 \subseteq \dots$$

Since R is Noetherian, there is an $n \in \mathbb{N}$ such that $I : x^n = I : x^m$ for all $m \geq n$. We write $I : x^\infty := I : x^n$. We need the following

Lemma 1.1.10. *For any prime ideal P, we have*

1. $P : x = P$ *if* $x \notin P$*;*
2. $P : x = R$ *if* $x \in P$.

Proposition 1.1.11. *Given a prime ideal P, if $P = I : x = I : x^\infty$ for some $x \in R$, then P is a both minimal prime of I and the P-primary component of I.*

Proof. Suppose that there is a prime ideal Q with $I \subseteq Q \subset P$. Then

$$P = I : x \subseteq Q : x \subset P : x = I : x^2 = P,$$

contradiction. Hence P is a minimal prime of P.

We claim that $I = J \cap P$, where $J = I + (x)$. The inclusion $I \subseteq J \cap P$ is trivial. Conversely, let $z \in J \cap P$. Since $z \in P = I : x$, we have that $zx \in I$. Since $z \in J$, we can write

$$z = hx + ki \tag{1.1}$$

where $i \in I$, $h, k \in R$. It follows that $zx = hx^2 + kix \in I$, hence $hx^2 \in I$, which is equivalent to $h \in I : x^2 = P = I : x$. This means that $hx \in I$ and thus, by (1.1), that $z \in I$.

We have proved that $I = J \cap P$ where J and P are strictly larger than I. Moreover, from $P : x = P$ and Lemma 1.1.10 we also deduce that $x \notin P$, hence J is not contained in P. This means that $J_P = R_P$ and $I_P = J_P \cap P_P = P_P$. It follows that P is the P-primary component of I. □

We conclude that the irredundant primary decomposition of an ideal is unique up to the primary components corresponding to embedded primes.

Remark 1.1.12. There is a geometric interpretation of the primary decomposition of an ideal. For an ideal J, let $\mathscr{V}(J) = \{P \in \mathrm{Spec}(R) \mid P \supseteq J\}$ be a subset of the spectrum $\mathrm{Spec}(R)$ endowed with the Zariski topology. The irreducible closed subsets of Spec(R) are the subsets $\mathscr{V}(P)$ with P prime. It follows that the irreducible *components* of Spec(R) are precisely the subsets $\mathscr{V}(P)$ with P minimal prime of R. Hence, for an ideal I, the minimal primes of I are in one-to-one correspondence with the irreducible components of Spec(R/I).

If R is a polynomial ring over an algebraically closed field, we can rewrite the one-to-one correspondence as

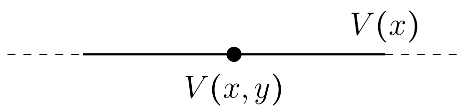
$$\{\text{minimal primes of } I\} \leftrightarrow \{\text{irreducible components of } V(I)\}.$$

Indeed, if $I = \bigcap_{k=1}^{n} I_k$ is an irredundant primary decomposition of I, then

$$V(I) = \bigcup_{k=1}^{n} V(I_k). \tag{1.2}$$

Hence, if I_k is a primary component associated to an embedded prime P, this means that the algebraic set $V(I_k)$ will be contained in another algebraic set $V(I_h)$ where I_h is a primary component associated to a prime containing P. For this reason, $V(I_h)$ will not be "visible" in (1.2).

Example 1.1.13. *Let $I = (x,y)\cdot(x) = (x^2, xy) \in \mathbb{K}[x,y]$ as in Example 1.1.7. Observe that $V(I) = V(x)$, i.e. the algebraic set of I is just the line $x = 0$. However, any primary decomposition of I contains an embedded component associated to the prime (x,y), whose algebraic set is the origin $\{(0,0)\}$ and hence already contained in $V(x)$. Therefore the embedded component is not "visible" in $V(I)$.*



From Remark 1.1.12 we get that, intuitively, if an ideal I is radical, then writing its primary decomposition will just correspond to writing the algebraic set $V(I)$ as the intersection of its irreducible components. We are going to prove this fact in algebraic terms.

Proposition 1.1.14. *Let I be a radical ideal and let $I = \bigcap_{k=1}^n I_k$ be an irredundant primary decomposition of I. Then, for every k, I_k is a minimal prime of I.*

Proof. Let $P_k = \sqrt{I_k}$ for every i. Since I is radical, we have

$$I = \sqrt{I} = \sqrt{\bigcap_{k=1}^n I_k} = \bigcap_{k=1}^n \sqrt{I_k} = \bigcap_{k=1}^n P_k,$$

hence $\bigcap_{k=1}^n P_k$ is another primary decomposition of I. Suppose there are r, s such that $P_r \subseteq P_s$. Then P_s is superfluous and $I = \bigcap_{k \neq s} P_k$, contradicting the irredundance of $\bigcap_{k=1}^n I_k$. □

Not only have we just proved that a radical ideal has no embedded primes, but also that its primary components are all prime.

1.2 Gradings and multigradings

Definition 1.2.1. Let R be a ring. An $\mathbb{N}$-**grading**, or simply **grading**, on R is a direct sum decomposition

$$R = \bigoplus_{i \in \mathbb{N}} R_i$$

where

(1) R_i is an additive subgroup of R for every $i \in \mathbb{N}$,

(2) $R_i R_j \subseteq R_{i+j}$ for every $i, j \in \mathbb{N}$.

We say that R is **graded**. An element $u \in R_i$ is called **homogeneous** of degree i, and we write $\deg(u) = i$.

Notice that the condition (2) is equivalent to

(2') $\deg(uv) = \deg(u) + \deg(v)$ for every two homogeneous elements $u, v \in R$.

The easiest example of grading is the standard polynomial grading.

Example 1.2.2. *Le* $\mathbb{K}$ *field,* $R = \mathbb{K}[x_1, \dots, x_n]$*. Then* $R = \bigoplus_{i \in \mathbb{N}} R_i$ *where* R_i *denotes the* $\mathbb{K}$*-vector space generated by monomials of the form* $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ *with* $\sum_{k=1}^{n} \alpha_k = i$*. We call this grading the* ***standard grading*** *on* R*.*

By definition of direct sum, every $f \in R$ can be written uniquely as a finite sum

$$f = \sum_i f_i \text{ with } f_i \in R_i,$$

and f_i is called the **homogeneous component** of f of degree i.

Definition 1.2.3. An ideal I of a graded ring R is called **graded** or **homogeneous** if one of the following equivalent conditions holds:

- I has a system of homogeneous generators;
- $I = \bigoplus_{i \in \mathbb{N}} I_i$, where $I_i = R_i \cap I$.

Definition 1.2.4. A module M over a graded ring R is called a **graded module** if it can be written as a direct sum decomposition

$$\bigoplus_{i \in \mathbb{N}} M_i$$

such that $R_i M_j \subseteq M_{i+j}$ for every $i, j \in \mathbb{N}$.

Let I be a graded ideal of a graded ring R. The quotient ring R/I is graded as well:

$$R/I = \bigoplus_{i \in \mathbb{N}} R_i/I_i.$$

Definition 1.2.5. An algebra A over a ring R is called a **graded algebra** if it is a graded R-module that is also a graded ring.

Definition 1.2.6. Let R be a graded ring with R_0 Artinian, and let $M = \bigoplus_{i \in \mathbb{N}} M_i$ be a graded R-module such that M_i is finitely generated over R_0 for every $i \in \mathbb{N}$. The **Hilbert function** of M is the function

$$\begin{aligned} H(M, -) : \mathbb{N} &\to \mathbb{N} \\ i &\mapsto H(M, i) := \ell(M_i) \end{aligned}$$

where $\ell(M_i)$ denotes the length of M_i over R_0.

Notice that, if R is a graded algebra over a field $\mathbb{K}$, then the M_i's are $\mathbb{K}$-vector spaces and $H(M,i) = \dim_{\mathbb{K}}(M_i)$. This happens, for instance, for polynomial rings $R = \mathbb{K}[x_1,\dots,x_n]$, which is the case we are interested in.

Definition 1.2.7. The **Hilbert series** of M is the power series

$$H_M(t) = \sum_{i\in\mathbb{N}} H(M,i)t^i.$$

The Hilbert series yields important information about the underlying module and is a central notion in commutative algebra. See for example Proposition 1.3.15 for an application.

Definition 1.2.8. Let R be a graded ring and M,N be graded R-modules. We say that a module homomorphism $\varphi : M \to N$ has **degree** i if $\deg(\varphi(m)) = i + \deg(m)$ for every homogeneous $m \in M \setminus \ker(\varphi)$. A **graded homomorphism** is a module homomorphism of degree i for some $i \in \mathbb{N}$.

We can generalize the theory seen until now and grade a ring R over an arbitrary abelian group A. This gives rise to a theory of multigradings. For the scopes of this work, we will only need the case $A = \mathbb{Z}^n$.

Definition 1.2.9. Let R be a ring. A $\mathbb{Z}^n$**-grading** on R is a direct sum decomposition

$$R = \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} R_{\mathbf{a}}$$

where the $R_{\mathbf{a}}$'s are additive subgroups of R, and $R_{\mathbf{a}}R_{\mathbf{b}} \subseteq R_{\mathbf{a}+\mathbf{b}}$ for every $\mathbf{a},\mathbf{b} \in \mathbb{Z}^n$. We say that R is **multigraded**, or, specifically, $\mathbb{Z}^n$**-graded**. The elements of $R_{\mathbf{a}}$ are called **homogeneous elements** of multidegree $\mathbf{a}$.

The notions of multigraded module, multigraded algebra, multigraded Hilbert series and multigraded homomorphism are defined analogously. See [MS04] for the details.

In the following, we are going to see that the property of being a complete intersection behaves nicely in graded rings.

Definition 1.2.10. A sequence $x_1, \dots, x_n$ of elements of a ring R is called R-**regular** if the ideal $(x_1, \dots, x_n)$ is proper and, for each i the element x_i is a nonzerodivisor in the quotient $R/(x_1, \dots, x_{i-1})$.

Notice that the order of the terms in $x_1, \dots, x_n$ is significant: for instance, the sequence x, $y(1-x)$, $z(1-x)$ is regular in $\mathbb{C}[x, y, z]$, while $y(1-x)$, $z(1-x)$, x is not. However, under certain conditions, the order does not matter. For example, we will work in the following setting.

Definition 1.2.11. Let R be a $\mathbb{Z}^n$-graded $\mathbb{K}$-algebra. We say that R is **positively graded** if $R_0 = \mathbb{K}$ and $R_{\mathbf{a}} = 0$ for every $\mathbf{a} \in \mathbb{Z}^n \setminus \mathbb{N}^n$.

Proposition 1.2.12. *Given a regular sequence $x_1, \dots, x_n$ in a positively graded algebra R, if the x_i are homogeneous of positive degree, then any permutation of $x_1, \dots, x_n$ is again a regular sequence.*

Definition-Proposition 1.2.13. An ideal I in a positively graded algebra R is a **complete intersection** if one of the following equivalent conditions hold:

- there is a regular sequence that generates I;
- every minimal set of generators of I is a regular sequence.

To see why the two above conditions are equivalent, see for example [BH98].

1.3 Monomial orders and Gröbner bases

Throughout this section, let $\mathbb{K}$ be a field, and let be $S := \mathbb{K}[x_1, \dots, x_n]$ the polynomial ring of n variables over $\mathbb{K}$. Moreover, let $\mathrm{Mon}(S)$ be the set of monomials of S.

We will denote by $\mathbb{N}^n$ the set of vectors $\mathbf{a} = (a_1, \dots, a_n)$ of $\mathbb{Z}^n$ such that $a_i \geq 0$ for every i. Furthermore, we will write a monomial $u = x_1^{a_1} \dots x_n^{a_n}$ as $\mathbf{x}^{\mathbf{a}}$. Notice that

$$\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}} = \mathbf{x}^{\mathbf{a}+\mathbf{b}}.$$

Finally, given a polynomial f, we will call **support** of f the set $\mathrm{supp}(f)$ of monomials appearing in f with a nonzero coefficient.

Definition 1.3.1. A **monomial order** on S is a total order $<$ on $\mathrm{Mon}(S)$ such that:

- $1 < u$ for every $u \in \mathrm{Mon}(S)$, $u \neq 1$;
- if $u, v \in \mathrm{Mon}(S)$ are such that $u < v$, then $uw < vw$ for every $w \in \mathrm{Mon}(S)$.

Example 1.3.2. *We will now give some examples of standard monomial orders which will be useful later on.*

The lexicographic order. *Let* $\mathbf{a} = (a_1, \dots, a_n)$ *and* $\mathbf{b} = (b_1, \dots, b_n)$ *be two distinct vectors in* $\mathbb{N}^n$. *We define the order* $<_{\mathrm{lex}}$ *on* S *by setting* $\mathbf{x^a} <_{\mathrm{lex}} \mathbf{x^b}$ *if and only if* $a_i < b_i$ *for the smallest index* i *such that* $a_i \neq b_i$. *This is called the lexicographic order induced by* $x_1 > \dots > x_n$.

The graded lexicographic order. *Let* $\mathbf{a} = (a_1, \dots, a_n)$ *and* $\mathbf{b} = (b_1, \dots, b_n)$ *be two distinct vectors in* $\mathbb{N}^n$. *We define the order* $<_{\mathrm{glex}}$ *on* S *by setting* $\mathbf{x^a} <_{\mathrm{glex}} \mathbf{x^b}$ *if either*

- $\sum_{i=0}^n a_i < \sum_{i=0}^n b_i <$ *or*
- $\sum_{i=0}^n a_i = \sum_{i=0}^n b_i <$ *and* $\mathbf{x^a} <_{\mathrm{lex}} \mathbf{x^b}$.

This is called the graded lexicographic order induced by $x_1 > \dots > x_n$.

The graded reverse lexicographic order. *Let* $\mathbf{a} = (a_1, \dots, a_n)$ *and* $\mathbf{b} = (b_1, \dots, b_n)$ *be two distinct vectors in* $\mathbb{N}^n$. *We define the order* $<_{\mathrm{grevlex}}$ *on* S *by setting* $\mathbf{x^a} <_{\mathrm{grevlex}} \mathbf{x^b}$ *if either*

- $\sum_{i=0}^n a_i < \sum_{i=0}^n b_i <$ *or*
- $\sum_{i=0}^n a_i = \sum_{i=0}^n b_i <$ *and* $a_i > b_i$ *for the biggest index* i *such that* $a_i \neq b_i$.

This is called the graded reverse lexicographic order induced by $x_1 > \cdots > x_n$.

For the rest of this section, let us fix a monomial order $<$ on S. Let $f \in S$ be a nonzero polynomial.

Definition 1.3.3. The **initial monomial** or **leading term** of f with respect to $<$ is the biggest monomial $u \in \operatorname{supp}(f)$ with respect to $<$. We will denote it by $\operatorname{in}_<(f)$. The **leading coefficient** of f is the coefficient of $\operatorname{in}_<(f)$ in f.

Definition 1.3.4. Let I be a nonzero ideal of S. The **initial ideal** of I with respect to $<$ is the ideal

$$\operatorname{in}_<(I) := (\operatorname{in}_<(f) \mid f \in I, f \neq 0).$$

However, since S is a Noetherian ring, $\operatorname{in}_<(I)$ is generated by finitely many monomials $\operatorname{in}_<(f_1), \dots, \operatorname{in}_<(f_m)$ with $f_1, \dots, f_m \in I$.

Definition 1.3.5. Let $\mathcal{G} = \{f_1, \dots, f_m\} \subseteq I$. If $\operatorname{in}_<(I)$ is generated by the monomials $\operatorname{in}_<(f_1), \dots, \operatorname{in}_<(f_m)$, $\mathcal{G}$ is called a **Gröbner basis** of I.

Observe that, if $\mathcal{G} = \{f_1, \dots, f_m\}$ is a Gröbner basis of I and $\mathcal{G}'$ is a finite set with $\mathcal{G} \subseteq \mathcal{G}' \subseteq I$, then $\mathcal{G}'$ is a Gröbner basis of I as well. Moreover, if $\{g_1, \dots, g_m\}$ is a set of polynomials of I with $\operatorname{in}_<(f_i) = \operatorname{in}_<(g_i)$, then $\{g_1, \dots, g_m\}$ is again a Gröbner basis of I. We are interested in Gröbner bases with certain minimal properties.

Definition 1.3.6. A Gröbner basis $\mathcal{G} = \{f_1, \dots, f_m\}$ of I is called **reduced** if:

- The leading coefficient of f_i is 1 for every i;
- None of the monomials in $\operatorname{supp}(f_j)$ is divisible by $\operatorname{in}_<(f_i)$, for every $i \neq j$.

Theorem 1.3.7. *Any nonzero ideal of* S *has a unique reduced Gröbner basis.*

We are now going to describe a criterion for determining whether a given set of polynomials is a Gröbner basis. We firstly need some additional notions.

Definition 1.3.8. Let $f, g \in S$ be nonzero polynomials, let c_f and c_g be the leading coefficients of f and g respectively. We define the **S-polynomial** of f and g as

$$S(f,g) := \frac{\operatorname{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{c_f \,\mathrm{in}_<(f)} f - \frac{\operatorname{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{c_g \,\mathrm{in}_<(g)} g.$$

Theorem 1.3.9 (The division algorithm)**.** *Let $f_1, \dots, f_m \in S$ be nonzero polynomials. For every polynomial $f \neq 0$ of S, there are polynomials $g_1, \dots, g_m$ and r in S such that:*

- *$f = g_1 f_1 + \cdots + g_m f_m + r$;*
- *if $r \neq 0$ and $u \in \operatorname{supp}(r)$, then none of $\mathrm{in}_<(f_1), \dots, \mathrm{in}_<(f_m)$ divides u;*
- *for every i such that $g_i \neq 0$, we have $\mathrm{in}_<(f) \geq \mathrm{in}_<(g_i f_i)$.*

The polynomial r in the statement is called **remainder** of f with respect to $f_1, \dots, f_m$. We say that f **reduces** to r with respect to $f_1, \dots, f_m$.

Remark 1.3.10. If $f_1, \dots, f_m$ is a Gröbner basis of $I = (f_1, \dots, f_m)$, then the remainder is unique. Moreover, in this case, f belongs to I if and only if the unique remainder is 0.

Theorem 1.3.11 (Buchberger's criterion)**.** *Let $\mathcal{G} = \{f_1, \dots, f_m\}$ a system of generators of I. Then $\mathcal{G}$ is a Gröbner basis of I if and only if $S(f_i, f_j)$ reduces to 0 with respect to $\mathcal{G}$ for every $i \neq j$.*

Actually, there is no need to check whether all S-polynomials reduce to 0, thanks to the next

Proposition 1.3.12. *With the above notation, if $\mathrm{in}_<(f_i)$ and $\mathrm{in}_<(f_j)$ are coprime for some $i \neq j$, then $S(f_i, f_j)$ reduces to 0 with respect to $\{f_i, f_j\}$ and hence to $\mathcal{G}$.*

Proof. For the sake of simplicity, let us rename $f := f_i$, $g := f_j$ and omit the subscript $<$. We want to prove that $\{f, g\}$ is a Gröbner basis of the ideal (f, g). Write

$$f = a\,\mathrm{in}(f) + f', \quad g = b\,\mathrm{in}(g) + g'$$

where a is the leading coefficient of f and b is the leading coefficient of g. Then, since $\mathrm{in}(f)$ and $\mathrm{in}(g)$ are coprime, we have

$$S(f,g) = \frac{\mathrm{lcm}(\mathrm{in}(f),\mathrm{in}(g))}{a\,\mathrm{in}(f)}f - \frac{\mathrm{lcm}(\mathrm{in}(f),\mathrm{in}(g))}{b\,\mathrm{in}(g)}g = \frac{\mathrm{in}(g)}{a}f - \frac{\mathrm{in}(f)}{b}g =$$

$$= \frac{g-g'}{ab}f - \frac{f-f'}{ab}g = \frac{\mathrm{in}(f)}{b}g = \frac{-g'}{ab}f + \frac{f'}{ab}g$$

which is a reduction equation. Hence $S(f, g)$ reduces to zero.

□

Corollary 1.3.13. *If $f_1, \dots, f_n$ are nonzero polynomials with pairwise coprime leading terms, then $f_1, \dots, f_n$ is a Gröbner basis of the ideal $(f_1, \dots, f_n)$.*

Many of the features of $\mathrm{in}_<(I)$ are transferred to I, so, in a lot of cases, Gröbner bases reduce the study of an ideal to the study of its initial ideal. This is a monomial ideal and hence much easier to work with. For instance, a monomial ideal is radical if and only if it is squarefree; it is prime if and only if it is generated by distinct variables, and so on. For the proofs of the following results, see for example [Eis13].

Proposition 1.3.14. *If $\mathrm{in}_<(I)$ is a radical (resp. prime, complete intersection) ideal, then I is radical (resp. prime, complete intersection).*

Proposition 1.3.15 ([Eis13]). *The Hilbert series of $S/\mathrm{in}_<(I)$ is the same as the Hilbert series of S/I.*

1.4 Determinantal ideals and varieties

In this section, let X be the $n \times d$ generic matrix of variables $(x_{i,j})$, i.e.

$$X = \begin{pmatrix} x_{1,1} & \dots & x_{1,d} \\ \vdots & & \vdots \\ x_{n,1} & \dots & x_{n,d} \end{pmatrix}.$$

Let $S := \mathbb{K}[X]$ be the polynomial ring in nd variables over $\mathbb{K}$.

Definition 1.4.1. Let $r \leq \min\{n, d\}$ a positive integer. The r-**determinantal ideal** is the ideal $I_r(X)$ generated by the r-minors of X.

Observe that the affine space $V(S) = \mathbb{A}^{nd}_{\mathbb{K}}$ can be identified with the space $\mathrm{M}_{\mathbb{K}}(n, d)$ of $n \times d$ matrices with coefficients in $\mathbb{K}$. This justifies the next

Definition 1.4.2. The r-**determinantal variety** is the affine variety defined by $I_r(X)$, which is

$$V(I_r(X)) = \{M \in \mathrm{M}_{\mathbb{K}}(n, d) \mid \mathrm{rank}(M) < r\}.$$

We are now going to state some classical facts about determinantal ideals.

Proposition 1.4.3. *For any r, $I_{r+1}(X) \subseteq I_r(X)$.*

Proof. It follows from the Laplace expansion of minors. □

Proposition 1.4.4. *The height of the ideal $I_r(X)$ is $(n + 1 - r)(d + 1 - r)$.*

Let $<$ be a **diagonal monomial order**, i.e. a monomial order on $\mathbb{K}[X]$ for which the leading term of a minor of X is always the product of the elements on the diagonal of the minor. For instance, $<_{\mathrm{lex}}$ is a diagonal monomial order. The following is a classical result.

Proposition 1.4.5. *The r-minors of X are a Gröbner basis for $I_r(X)$ with respect to any diagonal monomial order.*

Proof. See for example [EH11, Theorem 6.33]. □

Let now $\mathrm{char}\,\mathbb{K} = 0$ and consider two matrices of variables Y and Z of size $n \times r$ and $r \times d$ respectively. Let $G = \mathrm{GL}_r(\mathbb{K})$ be the general linear group of dimension r and let and $A \in G$. Let ϕ_A be the following $\mathbb{K}$-algebra automorphism of $\mathbb{K}[Y, Z]$:

$$\begin{aligned} \phi_A : \mathbb{K}[Y, Z] &\to \mathbb{K}[Y, Z] \\ Y &\mapsto YA \\ Z &\mapsto A^{-1}Z \end{aligned}$$

and let ω be the following action of the group G on $\mathbb{K}[Y, Z]$:

$$\begin{aligned} \omega : G \times \mathbb{K}[Y, Z] &\to \mathbb{K}[Y, Z] \\ (A, f) &\mapsto \phi_A(f). \end{aligned}$$

Theorem 1.4.6 (First main theorem of invariant theory). *Let $\mathbb{K}[Y, Z]^\omega$ be the ring of invariants of the action ω. Then $\mathbb{K}[Y, Z]^\omega = \mathbb{K}[YZ]$.*

Then, we have a surjective $\mathbb{K}$-algebra homomorphism

$$\begin{aligned} \varphi : \mathbb{K}[X] &\twoheadrightarrow \mathbb{K}[Y, Z]^\omega = \mathbb{K}[YZ] \\ X &\mapsto YZ. \end{aligned}$$

Theorem 1.4.7 (Second main theorem of invariant theory). *The kernel of φ is $I_{r+1}(X)$, hence $\mathbb{K}[X]/I_{r+1}(X) \simeq \mathbb{K}[YZ]$.*

In our context, the most important consequence is the following

Corollary 1.4.8. *The ideal $I_r(X)$ is prime for every r.*

Now, let us study the case $r = 2$, which is the space of matrices with rank either 0 or 1. There are various interpretations of the ideal $I_2(X)$, as the ideal defining the Segre embedding or a diagonal algebra. We firstly recall an elementary result in linear algebra.

Proposition 1.4.9. *Let M be an $n \times d$ matrix. Then $\operatorname{rank}(M) \leq 1$ if and only if $M = \mathbf{a}^\intercal \mathbf{b}$ for some vectors $\mathbf{a} \in \mathbb{K}^n$ and $\mathbf{b} \in \mathbb{K}^d$.*

Proof. The case $\operatorname{rank}(M) = 0$ is trivial. Assume $\operatorname{rank}(M) = 1$. Then every row of M is a multiple of the same vector $\mathbf{b} = (b_1, \dots, b_d) \in \mathbb{K}^d$ and we can write

$$M = \begin{pmatrix} a_1\mathbf{b} \\ \vdots \\ a_n\mathbf{b} \end{pmatrix} = \mathbf{a}^\top \mathbf{b}$$

for some $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{K}^n$. Conversely, let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{K}^n$ and $\mathbf{b} = (b_1, \dots, b_d) \in \mathbb{K}^d$ be nonzero vectors and notice that the matrix

$$M = \mathbf{a}^\top \mathbf{b} = \begin{pmatrix} a_1 b_1 & \dots & a_1 b_d \\ \vdots & \vdots & \vdots \\ a_n b_1 & \dots & a_n b_d \end{pmatrix} = \begin{pmatrix} a_1\mathbf{b} \\ \vdots \\ a_n\mathbf{b} \end{pmatrix}$$

has rank 1. □

Therefore, we can express the determinantal variety $I_2(X)$ as the set

$$V(I_2(X)) = \{\mathbf{a}^\top \mathbf{b} \mid \mathbf{a} = (a_1, \dots, a_n) \in \mathbb{K}^n, \mathbf{b} = (b_1, \dots, b_d) \in \mathbb{K}^d\}.$$

Observe that this result holds for fields of arbitrary characteristics.

The ring $\mathbb{K}[\mathbf{a}^\top \mathbf{b}] = \mathbb{K}[a_i b_j \mid i \in [n], j \in [d]]$ defined by $I_2(X)$ is also called **Segre ring** because it is the homogeneous coordinate ring of the embedding

$$\sigma : \mathbb{P}^{n-1} \times \mathbb{P}^{d-1} \to \mathbb{P}^{nd-1}$$

given, in homogeneous coordinates, by

$$((a_1 : \dots : a_n), (b_1 : \dots : b_d)) \mapsto (a_1 b_1 : a_0 b_1 : \dots : a_i b_j : \dots : a_n b_d)$$

and called **Segre embedding** (see for example [Eis13]).

We will now give a different interpretation of the ring $\mathbb{K}[\mathbf{a}^\top \mathbf{b}]$. Let $\mathcal{C}$ be the category of $\mathbb{N}^2$-graded $\mathbb{K}$-vector spaces: an object in $\mathcal{C}$ is of the form

$$V = \bigoplus_{(i,j) \in \mathbb{N}^2} V_{i,j}$$

where the $V_{i,j}$'s are $\mathbb{K}$-vector spaces, and morphisms of $\mathcal{C}$ are $\mathbb{N}^2$-graded homomorphisms of vector spaces. Notice that $\mathbb{N}^2$-graded $\mathbb{K}$-algebras and their homogeneous ideals are objects of $\mathcal{C}$.

Definition-Proposition 1.4.10. There is an exact functor

$$\begin{aligned} \Delta : \mathcal{C} &\to \mathcal{C} \\ V &\to \bigoplus_{i \in \mathbb{N}} V_{i,i}. \end{aligned}$$

We call Δ the **diagonal functor**.

We endow the ring $\mathbb{K}[\mathbf{a}^\intercal \mathbf{b}]$ with the $\mathbb{N}^2$-grading induced by $\deg(y_{i,j}) = (1,0)$ and $\deg(z_{j,k}) = (0,1)$ for every $i \in [n], j \in [r], k \in [d]$. The following is an easy, well-known fact.

Proposition 1.4.11. $\Delta(\mathbb{K}[a_i, b_i \mid i \in [n], j \in [d]]) = \mathbb{K}[\mathbf{a}^\intercal \mathbf{b}]$.

Chapter 2

Lovász-Saks-Schrijver ideals

In this chapter, we will introduce LSS ideals and present some of the motivations that have brought to their investigation. Furthermore, we will discuss some results on LSS ideals, with a focus on the properties of being prime or a complete intersection. We are going to see that these algebraic properties are linked to combinatorial properties of the underlying graph.

2.1 Generalities on graphs

In this section, we set the notation used in this work with regard to graphs. Any graph $G = (V, E)$ will be a simple and undirected graph on a finite set of vertices V. Hence E will be a subset of $\binom{V}{2}$, where, for a set X, $\binom{X}{k} := \{Y \subseteq 2^X \mid |Y| = k\}$ denotes the k-subsets of X. Moreover, we will usually assume V to be the set $[n] := \{1, \ldots, n\}$.

Definition 2.1.1. A **bipartite graph** is a graph whose vertices can be divided into two disjoint sets $[a]$ and $[b']$ (where $[b'] = \{1', \ldots, b'\}$) such that there is no edge $\{i, j\}$ with i, j both belonging to $[a]$ or $[b']$.

These are some special graphs which we will deal with:

- K_n, the **complete graph** on n vertices, i.e. the graph such that $\{i, j\} \in E$ for every distinct $i, j \in [n]$;
- $K_{a,b}$, the **complete bipartite graph** of vertices $[a] \cup [b']$ and edges $\{\{i, j'\} \mid i \in [a], j' \in [b']\}$, i.e. every vertex of $[a]$ is connected to every vertex of $[b']$;

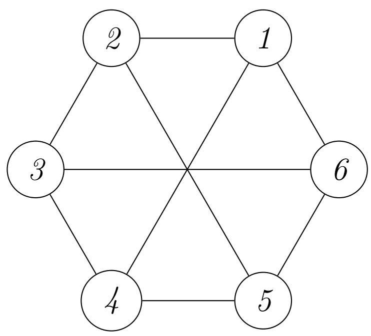
- B_n, the subgraph of $K_{n,n}$ obtained by removing the edges of the form $\{i, i'\}$;
- C_n, the **cycle** on n vertices, whose edges are $\{1,2\}$, $\{2,3\}$, ..., $\{n-1,n\}$, $\{n,1\}$.

Moreover, $\overline{G}$ will denote the **complement graph** of G, i.e. the graph on the same set of vertices as G such that an edge $\{i,j\}$ belongs to $\overline{E}$ if and only if it does not belong to E.

A **subgraph** of G is a graph $H = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$. If W is a subset of vertices of V, we call **subgraph induced by** W the graph $G_W = (W, E_W)$ where $E_W = \{e \in E \mid e \subseteq W\}$.

Definition 2.1.2. Given an integer $k \leq n-1$, the graph $G = ([n], E)$ is called k**-connected** if, for every $(k-1)$-subset W of V, the graph $G_{V \setminus W}$ is connected.

Example 2.1.3. *The graph G in the picture below is 3-connected, because deleting two vertices and the corresponding incident edges yields a connected graph. No proper subgraph of G is 3-connected.*



The graph G.

Definition 2.1.4. A subset M of edges of $G = ([n], E)$ is called **matching** if its elements are pairwise non-adjacent, i.e. they have no vertices in common. The graph G is called a matching if E is a matching.

Observe that G is $(n-2)$-connected if and only if its complement graph consists only of isolated vertices and disjoint edges, i.e. $\overline{G}$ is a matching. This is a special case of the following, more general fact which arises easily from the definitions.

Proposition 2.1.5. *For a graph $G = ([n], E)$ and an integer $1 \leq d \leq n$, G is $(n-d)$-connected if and only if $\overline{G}$ does not contain $K_{a,b}$ for any a, b such that $a + b = d + 1$.*

2.2 LSS ideals

From now on, let $n, d \geq 1$ be integers, $G = ([n], E)$ a graph, $\mathbb{K}$ a field and $S_n(d) := \mathbb{K}[X]$ where X is the $n \times d$ generic matrix of variables $(x_{i,j})$, namely

$$X = \begin{pmatrix} x_{1,1} & \dots & x_{1,d} \\ \vdots & & \vdots \\ x_{n,1} & \dots & x_{n,d} \end{pmatrix}.$$

Definition 2.2.1. Let $\{i, j\}$ be an edge of G. We define the polynomial $g_{i,j}$ as the componentwise product of the rows i, j of X, that is

$$g_{i,j} := x_{i,1}x_{j,1} + \dots + x_{i,d}x_{j,d}.$$

The **Lovász-Saks-Schrijver ideal** or **LSS ideal** of G is the ideal

$$\mathrm{L}_{\mathbb{K}}(G, d) := (g_{i,j} \mid \{i, j\} \in E) \subseteq S_n(d).$$

Notice that, since

$$XX^\top = \begin{pmatrix} g_{1,1} & \dots & g_{1,n} \\ \vdots & & \vdots \\ g_{n,1} & \dots & g_{n,n} \end{pmatrix},$$

$\mathrm{L}_{\mathbb{K}}(G, d)$ can also be seen as the ideal whose generators are the entries of $XX^\top$ in positions (i, j) with $\{i, j\} \in E$.

Remark 2.2.2. Consider the $\mathbb{Z}^n$-grading on $S_n(d)$ with $\deg(x_{i,k}) = \mathbf{e}_i$ for every $(i,k) \in [n]\times[d]$, where $\mathbf{e}_i$ is the i-th unit vector in $\mathbb{K}^n$. The polynomial $g_{i,j}$ has multidegree $\mathbf{e}_i + \mathbf{e}_j$. It follows that the ideal $\mathrm{L}_{\mathbb{K}}(G,d)$ is positively $\mathbb{Z}^n$-graded. This will be useful in Section 3.2.

There are various reasons to study LSS ideals of graphs, coming from different branches of mathematics. The first one is that they generalize other algebraic objects coming from graph theory, namely edge ideals, binomial edge ideals and permanental edge ideals. These are well-known objects that have been investigated widely in combinatorial commutative algebra (see for example [Her+10]).

Definition 2.2.3. Let $R = \mathbb{K}[x_1, \dots, x_n]$. The **edge ideal** of the graph G is the ideal $(x_i x_j \mid \{i,j\} \in E)$ of R.

Definition 2.2.4. Let $S = \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_n]$. The **binomial edge ideal** of the graph G is the ideal of S generated by all polynomials $x_i y_j - x_j y_i$, with $\{i,j\} \in E$ and $i < j$. The **permanental edge ideal** of the graph G is the ideal of S generated by all polynomials $x_i y_j + x_j y_i$, with $\{i,j\} \in E$ and $i < j$.

Trivially, $\mathrm{L}_{\mathbb{K}}(G,d)$ is precisely the edge ideal of G for $d = 1$, and the permanental edge ideal for $d = 2$. Moreover, for $d = 2$, we have the following result.

Proposition 2.2.5 ([Her+15, Remark 1.5])**.** *If $G = (V,E)$ is a bipartite graph and $\mathbb{K}$ contains a square root of -1, then $\mathrm{L}_{\mathbb{K}}(G,2)$ can be identified with the binomial edge ideal of G.*

Proof. Let $n = |V|$ and let $V = V_1 \cup V_2$ be the bipartition of G, with $|V_1| = a$, $|V_2| = b$. It is easy to see that the following ring homomorphism

$$\begin{aligned} \mathbb{K}[x_{1,1}, x_{1,2}, \dots, x_{n,1}, x_{n,2}] &\to \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_n] \\ x_{i,1} &\mapsto x_i \\ x_{i,2} &\mapsto \sqrt{-1}\, y_i \end{aligned}$$

is in fact an isomorphism. The image of $\mathrm{L}_{\mathbb{K}}(G,2)$ is the binomial edge ideal of G attached to the matrix

$$\begin{pmatrix} x_1 & \dots & x_a & \sqrt{-1}y_{a+1} & \dots & \sqrt{-1}y_n \\ \sqrt{-1}y_1 & \dots & \sqrt{-1}y_a & x_{a+1} & \dots & x_n \end{pmatrix}.$$

□

There are at least two more reasons why LSS ideals of graphs are interesting: they are related both to orthogonal representations of graphs and to hyperplane sections of determinantal varieties. We will clarify this in the next two sections.

2.3 Orthogonal representations of a graph

In this section, we are going to present the basics about orthogonal representations of graphs, which were introduced by L. Lovász in [Lov79]. We will then establish their connection with LSS ideals over real numbers. Let G be a graph, $d > 0$ an integer. In this section, we assume $\mathbb{K} = \mathbb{R}$.

Definition 2.3.1. An **orthogonal representation** of G in $\mathbb{R}^d$ is an assignment

$$V \to \mathbb{R}^d$$
$$i \mapsto \mathbf{v}_i$$

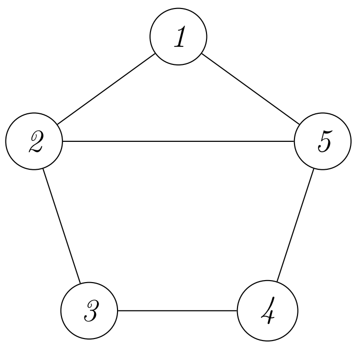
such that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ if $\{i,j\} \in E(\overline{G})$.

Notice that we do not ask $\mathbf{v}_i \cdot \mathbf{v}_j \neq 0$ for $\{i,j\} \in E(G)$. If this happens, the orthogonal representation is called **faithful**.

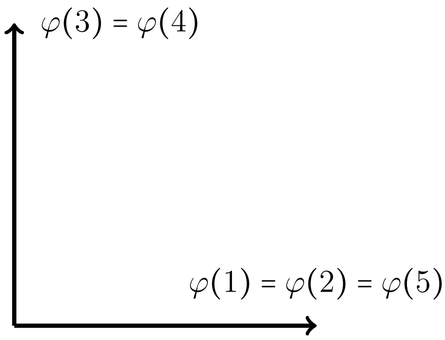
Example 2.3.2. *Let G be a graph with n vertices. Then G has a trivial orthogonal representation in $\mathbb{R}^n$ with $\mathbf{v}_i = \mathbf{e}_i$ for every $i \in V$. Clearly, this is a faithful orthogonal representation if and only if G has no edges.*

Example 2.3.3. *Let $d = |E|$. Then G has an orthogonal representation in $\mathbb{R}^d$ where each vertex is assigned to the indicator vector of the edges incident to it. This is always a faithful orthogonal representation.*

Example 2.3.4. *In the picture below, we show an orthogonal representation φ of the graph G in $\mathbb{R}^2$: the vertices $1, 2, 5$ are mapped to $\mathbf{e}_1$, while the vertices $3, 4$ are mapped to $\mathbf{e}_2$.*



The graph G.



An orthogonal representation of G.

The ideal $\mathrm{L}_{\mathbb{R}}(G, d)$ defines the variety of orthogonal representations of $\overline{G}$ in $\mathbb{R}^d$: let

$$\begin{aligned} V &\to \mathbb{R}^d \\ i &\mapsto \mathbf{v}_i = (v_{i,1}, \dots, v_{i,d}) \end{aligned}$$

be a vector assignment. This can be seen as the point in $\mathbb{R}^{nd}$ of coordinates $(v_{i,k})_{i \in [n], k \in [d]}$. With this point of view in mind, we can consider the algebraic variety $\mathcal{V} \subseteq \mathbb{R}^{nd}$ of orthogonal representations of $\overline{G}$ in $\mathbb{R}^d$, defined by all equations

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0 \text{ for } \{i, j\} \in E.$$

By the definition of $g_{i,j}$, it follows that $\mathcal{V}$ is the variety defined by $\mathrm{L}_{\mathbb{R}}(G, d)$.

Orthogonal representations are a ubiquitous notion in graph theory and they find applications in many, diverse branches of mathematics, such as information theory or quantum physics (see [Lov19]). Therefore, it is not surprising that the study of Lovász-Saks-Schrijver ideals was firstly introduced in [LSS89] (hence the name) in relation with the variety $\mathcal{V}$ of orthogonal representations. The variety $\mathcal{V}$ contains many "degenerate" orthogonal representations, for instance those mapping some of the vertices to the zero vector, and it is for this reason that the authors introduced the following condition of "non-degeneracy".

Definition 2.3.5. An orthogonal representation $i \mapsto \mathbf{v}_i$ of G in $\mathbb{R}^d$ is said to be in **general position** if every set of d vectors $\mathbf{v}_i$ is linearly independent.

In the same paper, the authors proved the following

Proposition 2.3.6 ([LSS89, Theorem 1.1])**.** *A graph G has a general-position orthogonal representation in $\mathbb{R}^d$ if and only if it is $(n-d)$ connected.*

This property is connected with the primality of $\mathrm{L}_{\mathbb{K}}(\overline{G}, d)$, as we will see in Proposition 2.5.1.

2.4 Hyperplane sections of determinantal varieties

Another reason why we are interested in investigating LSS ideals is that, in characteristic 0, their primality is related to that of hyperplane sections of determinantal ideals. In this section we will hence assume $\operatorname{char} \mathbb{K} = 0$.

Let X^{sym} be the symmetric $n \times n$ matrix of variables $x_{i,j}$, that is,

$$X^{\mathrm{sym}} = \begin{pmatrix} x_{1,1} & \dots & x_{n,1} \\ \vdots & \dots & \vdots \\ x_{n,1} & \dots & x_{n,n} \end{pmatrix}.$$

We have seen in 1.4.7 that

$$\mathbb{K}[X]/I_{r+1}(X) \simeq \mathbb{K}[YZ].$$

There is a correspondent result for X^{sym} (see for example [CW19]).

Theorem 2.4.1. *Let Y be a $n \times r$ matrix of variables. Then*

$$\mathbb{K}[X^{sym}]/I_{r+1}(X) \simeq \mathbb{K}[YY^{\top}]. \tag{2.1}$$

The following proposition is the symmetric counterpart of Proposition 1.4.4.

Proposition 2.4.2. *The height of the ideal $I_r(X^{\text{sym}})$ is $\binom{n-r+2}{2}$.*

We want to know whether the ideals defining hyperplane sections (i.e. intersections with hyperplanes) of determinantal varieties are radical or prime. The answer to this question concerns LSS ideals.

Let $G = ([n], E)$ be a graph. For every $\{i,j\} \in E$, we replace the entries (i,j) and (j,i) of X^{sym} with 0, and denote by X_G^{sym} the matrix obtained in this way. We aim to determine the aforementioned properties of $I_r(X_G^{\text{sym}})$ in terms of properties of the graph G. In [CW19], A. Conca and V. Welker proved the following

Theorem 2.4.3 ([CW19, Proposition 7.5])**.** *Over a field $\mathbb{K}$ of characteristic 0, if $\mathrm{L}_{\mathbb{K}}(G,d)$ is radical (resp. prime), then $I_{d+1}(X_G^{\text{sym}})$ is radical (resp. prime).*

Proof. Let Y as in Theorem 2.4.1 with $r = d$, and let J_G be the ideal of $\mathbb{K}[YY^{\top}]$ generated by the entries $(YY^{\top})_{i,j}$ for $\{i,j\} \in E$. Observe that $\mathrm{L}_{\mathbb{K}}(G,d)$ can be seen as an ideal of $\mathbb{K}[Y]$. It is a fact in invariant theory that $\mathbb{K}[YY^{\top}]$ is a direct summand of $\mathbb{K}[Y]$. It follows that

$$J_G = \mathrm{L}_{\mathbb{K}}(G,d) \cap \mathbb{K}[YY^{\top}].$$

Hence J_G is the pullback of $\mathrm{L}_{\mathbb{K}}(G,d)$ in the map $\mathbb{K}[YY^{\top}] \hookrightarrow \mathbb{K}[Y]$. Since $\mathrm{L}_{\mathbb{K}}(G,d)$ is radical (resp. prime) by assumption, J_G is radical (resp. prime) as well. Furthermore, we know from Theorem 2.4.1 that $\mathbb{K}[X^{sym}]/I_{d+1}(X^{sym}) \simeq \mathbb{K}[YY^{\top}]$.

From the isomorphisms

$$\frac{\mathbb{K}[YY^{\top}]}{J_G} \simeq \frac{\mathbb{K}[X^{sym}]/I_{d+1}(X^{sym})}{J_G} \simeq \frac{\mathbb{K}[X^{sym}]}{I_{d+1}(X^{sym}) + (x_{i,j} \mid \{i,j\} \in E)} \simeq$$

$$\simeq \frac{\mathbb{K}[X^{sym}]/(x_{i,j} \mid \{i,j\} \in E)}{I_{d+1}(X^{sym}) + (x_{i,j} \mid \{i,j\} \in E)/(x_{i,j} \mid \{i,j\} \in E)} \simeq \frac{\mathbb{K}[X_G^{sym}]}{I_{d+1}(X_G^{sym})}$$

we conclude that $I_{d+1}(X_G^{sym})$ is radical (resp. prime) as well.

□

For bipartite graphs, we can work with matrices of smaller size. Let G be a subgraph of $K_{a,b}$, and let X be a generic $a \times b$ matrix of variables. Let X_G be the matrix obtained by X by replacing the entries (i,j) with 0 for every $\{i,j\} \in E$. The proof of the next theorem follows the same argument as Theorem 2.4.3.

Theorem 2.4.4 ([CW19, Proposition 7.4]). *Let G be a subgraph of $K_{a,b}$. Over a field $\mathbb{K}$ of characteristic 0, if $\mathrm{L}_{\mathbb{K}}(G,d)$ is radical (resp. prime), then $I_{d+1}(X_G)$ is radical (resp. prime).*

2.5 Primality and complete intersection

In this section, we present some known results about LSS ideals, concerning in particular the properties of being prime or complete intersection. We will see that the algebraic properties of $\mathrm{L}_{\mathbb{K}}(G,d)$ are intimately connected with the graph-theoretic features of G.

A question that arises naturally is whether the ideals $\mathrm{L}_{\mathbb{K}}(G,d)$ for which a certain property holds, for example primality, can be characterized in terms of the graph G.

As anticipated in Section 2.3, a first result shows that the $(n-d)$-connectivity of $\overline{G}$ is a necessary condition for the primality of $\mathrm{L}_{\mathbb{K}}(G,d)$.

Theorem 2.5.1 ([CW19, Proposition 4.4]). *If the ideal $\mathrm{L}_{\mathbb{K}}(G,d)$ is prime, then G does not contain $K_{a,b}$ for any a,b such that $a+b=d+1$, or, equivalently, $\overline{G}$ is $(n-d)$-connected.*

Proof. We are going to show that, if G contains $K_{a,b}$ for any a,b such that $a+b=d+1$, then $\mathrm{L}_{\mathbb{K}}(G,d)$ is not prime. Suppose by contradiction that $\mathrm{L}_{\mathbb{K}}(G,d)$ is prime. We observe that $a+b\leq n$. It is convenient to assume that $b\leq a$ and the bipartition of $K_{a,b}$ is $V_1\cup V_2$, where $V_1=\{1,\dots,a\}$ and $V_2=\{a+1,\dots,a+b\}$. The ring $R:=S_n(d)/\mathrm{L}_{\mathbb{K}}(G,d)$, is a domain since R is prime. Let $Y=(y_{i,k})\in \mathrm{M}_R(a,d)$, $Z=(z_{k,i})\in \mathrm{M}_R(d,b)$ two matrices of variables with $z_{l,i}=y_{a+i,l}$, so that

$$YZ=\begin{pmatrix} y_{1,1} & \cdots & y_{1,d}\\ \vdots & \cdots & \vdots\\ y_{a,1} & \cdots & y_{a,d}\end{pmatrix}\begin{pmatrix} y_{a+1,1} & \cdots & y_{a+b,1}\\ \vdots & \cdots & \vdots\\ y_{a+1,d} & \cdots & y_{a+b,d}\end{pmatrix}=$$

$$=\begin{pmatrix} y_{1,1}y_{a+1,1}+\cdots+y_{1,d}y_{a+1,d} & \cdots & y_{1,1}y_{a+b,1}+\cdots+y_{1,d}y_{a+b,d}\\ \vdots & \cdots & \vdots\\ y_{a,1}y_{a+1,1}+\cdots+y_{a,d}y_{a+1,d} & \cdots & y_{a,1}y_{a+b,1}+\cdots+y_{a,d}y_{a+b,d}\end{pmatrix}.$$

Since the edge $\{i,a+j\}$ belongs to $K_{a,b}$ for every $i\in[a], j\in[b]$ and therefore to G, it follows that $YZ=0$ in the quotient ring R. This means that one between Y and Z does not have full rank, therefore either $I_a(Y)=0$ or $I_b(Z)=0$ as ideals of R. It follows that $I_a(Y)\subseteq \mathrm{L}_{\mathbb{K}}(G,d)$ or $I_b(Z)\subseteq \mathrm{L}_{\mathbb{K}}(G,d)$. But minors of Y or Z do not vanish modulo $\mathrm{L}_{\mathbb{K}}(G,d)$ because they do not belong to the ideal generated by the terms $x_{i,k}x_{j,k}$, which is an ideal containing $\mathrm{L}_{\mathbb{K}}(G,d)$. □

Hence we have a graph-theoretic necessary condition for $\mathrm{L}_{\mathbb{K}}(G,d)$ to be prime. However, the converse of Theorem 2.5.1 is not true for $d>3$.

Example 2.5.2 ([CW19]). *Let $G=B_d$ with $d\geq 4$ and $\operatorname{char}\mathbb{K}=0$. We are going to show that $\mathrm{L}_{\mathbb{K}}(G,d)$ is not prime, despite not containing $K_{a,b}$ for $a+b=d+1$. Suppose by contradiction that $\mathrm{L}_{\mathbb{K}}(G,d)$ is prime. Then, by Proposition 2.4.4 $I_{d+1}(Y_G)$ would be prime as well, where Y is a generic matrix of arbitrary size. We are going to show that this is not true.*

Let Y have size $(d+2)\times(d+2)$. Then

$$Y_G = \begin{pmatrix} x_{1,1} & 0 & \dots & 0 & x_{1,d+1} & x_{1,d+2} \\ 0 & x_{2,2} & \dots & 0 & x_{1,d+1} & x_{1,d+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & x_{d,d} & x_{d,d+1} & x_{d,d+2} \\ x_{d+1,1} & x_{d+1,2} & \dots & \dots & x_{d+1,d+1} & x_{d+1,d+2} \\ x_{d+2,1} & x_{d+2,2} & \dots & \dots & x_{d+2,d+1} & x_{d+2,d+2} \end{pmatrix}.$$

We want to show that $I := I_{d+1}(Y_G)$ has height 4 as an ideal of $\mathbb{K}[Y_G]$. Since I is strictly contained in $(x_{1,1}, x_{2,2}, x_{3,3}, x_{4,4})$, it would follow that I is not prime. Let $P \supseteq I$ be a prime ideal. If P contains $(x_{1,1}, x_{2,2}, x_{3,3}, x_{4,4})$ then the height of P is at least 4. If P does not contain $(x_{1,1}, x_{2,2}, x_{3,3}, x_{4,4})$, say $x_{1,1} \notin P$, then it can be shown that $P\mathbb{K}[Y_G]_{x_{1,1}}$ contains $I_d(Y_{d-1})$, up to a change of variables. It follows that $4 \le \operatorname{ht}(P\mathbb{K}[Y_G]_{x_{1,1}}) = \operatorname{ht}(P)$ and hence that $\operatorname{ht}(I) = 4$.

Nevertheless, for $d \le 3$, Theorem 2.5.1 can be reversed, as the two following results show. Their proof is rather technical and falls outside the scopes of this work.

Theorem 2.5.3 ([Her+15, Corollary 1.4]). *The ideal $\mathrm{L}_{\mathbb{K}}(G,2)$ is prime if and only if G is a matching.*

Theorem 2.5.4 ([CW19, Theorem 1.4]). *The ideal $\mathrm{L}_{\mathbb{K}}(G,3)$ is prime if and only if G does not contain $K_{1,3}$ nor $K_{2,2}$.*

To be more precise, in [Her+15] J. Herzog et al. proved Theorem 2.5.3 in the case $\operatorname{char}\mathbb{K} \ne 1,2 \bmod 4$, but in [CW19] Conca and Welker generalized this theorem to any field.

These results characterize the graphs with prime LSS ideals for $d = 2,3$ (for $d = 1$, it is trivial to see that no LSS ideal is prime), and show how the primality, an algebraic property, can be related to combinatorial properties of the graph. The next interesting case is $d = 4$, for which the characterization remains open. Conca and Welker formulated the following conjecture in [CW19], motivated by Example 2.5.2.

Conjecture 2.5.5. *The ideal* $\mathrm{L}_{\mathbb{K}}(G,4)$ *is prime if and only if* G *does not contain* $K_{a,b}$ *for any* a,b *such that* $a+b=d+1$*, nor* B_4*.*

There is a result analogous to Theorem 2.5.1 for the property of being complete intersection.

Proposition 2.5.6 ([CW19, Proposition 4.4])**.** *The ideal* $\mathrm{L}_{\mathbb{K}}(G,2)$ *is a complete intersection if and only if* G *does not contain* $K_{1,3}$ *nor* C_{2m} *for some* $m \geq 2$*.*

For the property of being complete intersection, the smallest case for which the characterization remains open is $d=3$. To the author's knowledge, no conjecture has been formulated regarding this matter.

2.6 Asymptotic behaviour of LSS ideals

In this section, we are going to see that, for LSS ideals, the properties of being prime or a complete intersection are closely related and they stabilize for large d. We will make use of a graph-theoretic invariant called positive matching decomposition.

Definition 2.6.1. Let $\mathcal{P}$ be an algebraic property of an ideal. We denote

$$\operatorname{asym}_{\mathbb{K}}(\mathcal{P},G)=\inf\{d>0 \mid \mathrm{L}_{\mathbb{K}}(G,d') \text{ has the property } \mathcal{P} \text{ for every } d'\geq d\}.$$

We are interested in studying $\mathcal{P}$ = radical, prime, complete intersection. The first questions that arise naturally are: is $\operatorname{asym}_{\mathbb{K}}(\mathcal{P},G)$ finite for $\mathcal{P}$ = radical, prime, complete intersection? Are there interesting bounds for it?

In [CW19], Conca and Welker proved the following

Theorem 2.6.2 ([CW19, Theorem 1.1])**.**

- *If* $L_{\mathbb{K}}(G,d)$ *is prime, then* $L_{\mathbb{K}}(G,d)$ *is complete intersection.*
- *If* $L_{\mathbb{K}}(G,d)$ *is complete intersection, then* $L_{\mathbb{K}}(G,d+1)$ *is prime.*

In particular, if $L_{\mathbb{K}}(G,d)$ *is a prime ideal (resp. complete intersection) then* $L_{\mathbb{K}}(G,d+1)$ *is a prime ideal (resp. complete intersection).*

In the same paper, the authors introduced a graph-theoretic invariant, called $\mathrm{pmd}(G)$ or positive matching decomposition number of G, and showed that this invariant gives an upper bound for the values $\mathrm{asym}_{\mathbb{K}}(\mathrm{prime}, G)$ and $\mathrm{asym}_{\mathbb{K}}(\text{complete intersection}, G)$.

Definition 2.6.3. Let $G = (V,E)$ be a graph. A matching M of G is called **positive matching** if there exists a weight function $w : V \to \mathbb{R}$ with $w(i) + w(j) > 0$ if $\{i,j\} \in M$ and $w(i) + w(j) < 0$ if $\{i,j\} \in E \setminus M$.

Definition 2.6.4. A **positive matching decomposition** of G is a partition of E into pairwise disjoint subsets $\{E_1,\dots,E_k\}$ such that E_i is a positive matching of $(V, E \setminus \bigcup_{j=1}^{i-1} E_j)$ for $i = 1,\dots k$. The smallest k for which such a decomposition exists is denoted by $\mathrm{pmd}(G)$.

Theorem 2.6.5 ([CW19, Theorem 1.3])**.** *For* $d \geq \mathrm{pmd}(G)$, *the ideal* $L_{\mathbb{K}}(G,d)$ *is a radical complete intersection, and hence* $L_{\mathbb{K}}(G,d+1)$ *is prime.*

Therefore, the values $\mathrm{asym}(\text{complete intersection}, G)$, $\mathrm{asym}(\mathrm{prime}, G)$ and $\mathrm{asym}(\mathrm{radical}, G)$ are finite for every graph G and, combining Theorems 2.6.5 and 2.6.2, we get the following

Corollary 2.6.6. *Let* G *be any graph,* $c(G) = \mathrm{asym}(\textit{complete intersection}, G)$ *and* $p(G) = \mathrm{asym}(\textit{prime}, G)$. *Then* $c(G) \leq p(G) \leq c(G) + 1$.

Example 2.6.7. *Let* G *be an even cycle. Then* $c(G) = p(G)$*: by Theorems 2.5.3 and 2.5.4 we know that* $L_{\mathbb{K}}(G,2)$ *is not prime and* $L_{\mathbb{K}}(G,3)$ *is prime, hence* $p(G) = 3$. *By 2.5.6,* $L_{\mathbb{K}}(G,2)$ *is not a complete intersection, therefore* $p(G) = c(G)$. *The same argument shows that, if* G *is an odd cycle, then* $p(G) = c(G) + 1$.

Other bounds for $\operatorname{asym}(\mathcal{P}, G)$ have been given in [CW19]. For some classes of graphs, the authors found their exact values:

Theorem 2.6.8 ([CW19, Theorem 8.6]). *For the complete bipartite graph $K_{a,b}$ we have:*

- $\operatorname{asym}(\mathit{radical}, G) = 1$;
- $\operatorname{asym}(\mathit{prime}, G) = m + n$;
- $\operatorname{asym}(\mathit{complete\ intersection}, G) = m + n - 1$.

Chapter 3

Radicality of LSS ideals

After having discussed the basics and principal results about LSS ideals, we want to focus on the study of the radicality of $\mathrm{L}_{\mathbb{K}}(G,d)$. In general, this has proved to be a hard task and very little is known about it.

3.1 State of the art and main conjecture

For fixed d, we have the following results:

1. $\mathrm{L}_{\mathbb{K}}(G,1)$ is trivially radical for every graph G, since it is a monomial squarefree ideal.

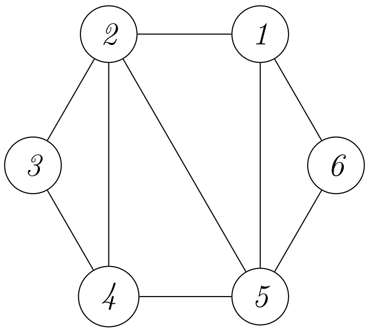
2. For $d = 2$, J. Herzog et al. gave a complete picture in [Her+15] proving the following

   Theorem 3.1.1 ([Her+15, Theorems 1.1, 1.2]). *Let G be a graph.*

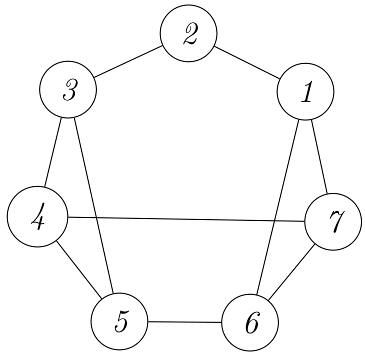
   - *If* $\operatorname{char}\mathbb{K} \neq 2$, *then* $\mathrm{L}_{\mathbb{K}}(G,2)$ *is always radical;*
   - *If* $\operatorname{char}\mathbb{K} = 2$, *then* $\mathrm{L}_{\mathbb{K}}(G,2)$ *is radical if and only if G is bipartite.*

3. For $d \geq 3$, we do not even have a conjecture for the characterization of graphs with radical LSS ideals. Unlike primality and complete intersection, the radicality of LSS ideals for large d does not seem to behave nicely: if G is a graph such that $\mathrm{L}_{\mathbb{K}}(G,d)$ is radical, $\mathrm{L}_{\mathbb{K}}(G,d+1)$ need not be radical. The following examples were presented by Conca and Welker in [CW19].

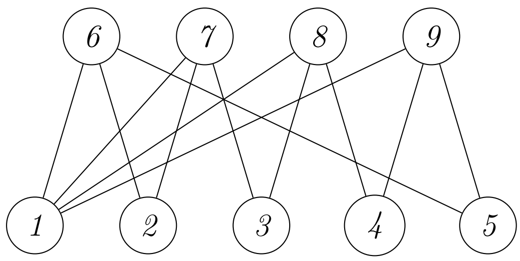
   Example 3.1.2 ([CW19]).



The graph G_1.



The graph G_2.



The graph G_3.

The graphs in the picture above have nonradical LSS ideal for $d = 3$ *and* $\operatorname{char}\mathbb{K} = 0$. *The graph G_1 is the smallest known example in terms of number of vertices. The graph G_2 has complete intersection* $\mathrm{L}_{\mathbb{K}}(G,3)$, *while the graph G_3 is bipartite. In all of these examples, the authors have proved the nonradicality of* $\mathrm{L}_{\mathbb{K}}(G,3)$ *by finding an element* $g \in S_n(d)$ *such that* $\mathrm{L}_{\mathbb{K}}(G,3) : g^2 \neq \mathrm{L}_{\mathbb{K}}(G,3) : g$.

This means, in particular, that $\operatorname{asym}(\text{radical}, G)$ might be harder to find. A starting point to investigate the radicality of LSS ideals for $d > 3$ could be working with specific classes of graphs. For complete graphs, we have a concrete conjecture, stated by Conca and Welker in [CW19].

Conjecture 3.1.3. *The LSS ideal of K_n is radical for every n and d, at least when* $\operatorname{char}\mathbb{K} = 0$*; or, equivalently,* $\operatorname{asym}_{\mathbb{K}}(\textit{radical}, K_n) = 1$.

In other words, the conjecture says that the ideal of $S_n(d)$ generated by the off-diagonal entries of the matrix $XX^{\intercal}$ is always radical. For $d = 1$, the conjecture is trivially true and for $d = 2$ it follows from 3.1.1. In this chapter, we will analyze the smallest open case, which is $d = 3$.

One of the first approaches that comes to mind when attacking Conjecture 3.1.3 is trying to find a monomial order $<$ on $S_n(3)$ such that $\operatorname{in}_<(\mathrm{L}_{\mathbb{K}}(K_n, 3))$ is squarefree, hence radical. From Proposition 1.3.15, it would follow that $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ is radical. However, for arbitrary n, there is little hope of finding a nice monomial squarefree ideal with the same Hilbert series as $\mathrm{L}_{\mathbb{K}}(K_n, 3)$: with the computer software Macaulay2 ([GS]) we have obtained the following result.

Proposition 3.1.4. *If* $\operatorname{char}\mathbb{K} = 0$*, there is no quadratic monomial squarefree ideal with the same Hilbert series as* $\mathrm{L}_{\mathbb{K}}(K_5, 3)$.

A monomial order for which $\operatorname{in}_<(\mathrm{L}_{\mathbb{K}}(K_n, 3))$ is squarefree could still exist, but the generators of $\operatorname{in}_<(\mathrm{L}_{\mathbb{K}}(K_n, 3))$ would have degree at least 3. It is unclear and hard to understand whether such an initial ideal exists.

A different approach consists in computing an irredundant primary decomposition of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$, showing that every primary component of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ is prime and applying Proposition 1.1.14. We formulate the following

Conjecture 3.1.5. *Over a field $\mathbb{K}$ with* $\operatorname{char}\mathbb{K} \neq 2$*, the primary components of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ for $n > 3$ are of the following two types:*

- *For any 3-subset $\Gamma \subseteq [n]$, the ideal*

$$A_\Gamma := (g_{i,j} \mid i < j, i, j \in \Gamma) + (x_{i,k} \mid i \notin \Gamma, k \in [3]);$$

- *For any $k \in [n]$, the ideal*

$$B_k := (g_{i,j} \mid i \leq j, (i,j) \neq (k,k)) + I_2(X_k)$$

where X_k denotes the matrix obtained from X by deleting the k-th row.

Moreover, the above ideals are minimal primes of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$, hence $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ is radical.

In this work, we will give a proof of the following, slightly weaker

Theorem 3.1.6. *Over a field $\mathbb{K}$ with $\operatorname{char} \mathbb{K} \neq 2$, the primary components corresponding to minimal primes of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ for $n > 3$ are of the following two types:*

- *For any 3-subset $\Gamma \subseteq [n]$, the ideal*

$$A_\Gamma := (g_{i,j} \mid i < j, i, j \in \Gamma) + (x_{i,k} \mid i \notin \Gamma, k \in [3]);$$

- *For any $k \in [n]$, the ideal*

$$B_k := (g_{i,j} \mid i \leq j, (i,j) \neq (k,k)) + I_2(X_k)$$

where X_k denotes the matrix obtained from X by deleting the k-th row.

Moreover, the above ideals are the minimal primes of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$.

This means that the only way for $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ not to be radical would be to have primary components corresponding to embedded primes. The reason why we will not be able to "see" the embedded components is that we will solve the problem geometrically, i.e. by proving that $V(\mathrm{L}_{\mathbb{K}}(K_n, 3))$ has no other irreducible components apart from those corresponding to the conjectured minimal prime ideals (see Remark 1.1.12). Remember that the zero locus of an ideal is reduced and therefore gives no information about the embedded points.

However, thanks to Theorem 3.1.6, we will be able to say that the scheme $X := \operatorname{Spec}(S_n(3)/\mathrm{L}_{\mathbb{K}}(K_n, 3))$ is *generically reduced*, i.e. $\mathcal{O}_{X,x}$ is reduced for every $x \in U$, where U is some Zariski dense open subset of X. For this reason, we will say that $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ is *generically radical.*

The next three sections will be devoted to proving Theorem 3.1.6.

3.2 Primality of A_Γ and B_k

Fix $n \in \mathbb{N}$, $n > 3$. The first step will be proving that the ideals in the set $\{A_\Gamma \mid \Gamma \subseteq [n], |\Gamma| = 3\} \cup \{B_k \mid k \in [n]\}$ are prime.

Proposition 3.2.1. *The ideal A_Γ is prime for every 3-subset $\Gamma \subseteq [n]$.*

Proof. The quotient

$$\frac{S_n(d)}{A_\Gamma} \simeq \frac{S_n(d)/(x_{i,k} \mid i \notin \Gamma, k \in [3])}{(g_{i,j} \mid i < j, i,j \in \Gamma)/(x_{i,k} \mid i \notin \Gamma, k \in [3])} \simeq \frac{S_3(3)}{(g_{i,j} \mid i < j, i,j \in [3])} \simeq \frac{S_3(3)}{\mathrm{L}_{\mathbb{K}}(K_3, 3)}$$

is a domain since $\mathrm{L}_{\mathbb{K}}(K_3, 3) \subseteq S_3(3)$ is prime by Theorem 2.5.4. □

Now we prove the same for the ideals of the form B_k. We firstly need a lemma.

Lemma 3.2.2. *The ideal $Q = (b_1^2 + b_2^2 + b_3^2, b_1x_1 + b_2x_2 + b_3x_3)$ in the six-variable polynomial ring $T := \mathbb{K}[x_1, x_2, x_3, b_1, b_2, b_3]$ is prime.*

Proof. Consider the canonical localization map

$$T \to T_{b_1}$$

for which the image of Q is Q_{b_1}. We have

$$T_{b_1}/Q_{b_1} \simeq \mathbb{K}[b_1, \tfrac{1}{b_1}, b_2, b_3, x_1, x_2, x_3]/Q_{b_1} \simeq \mathbb{K}[b_1, \tfrac{1}{b_1}, b_2, b_3, x_2, x_3]/(b_1^2 + b_2^2 + b_3^2)$$

which is a domain since $b_1^2 + b_2^2 + b_3^2$ is irreducible and hence prime (because the ring $\mathbb{K}[b_1, \frac{1}{b_1}, b_2, b_3, x_2, x_3]$ is a UFD). Therefore Q_{b_1} is prime in T_{b_1}, and its pullback is prime as well. We only need to check that the pullback of Q_{b_1} is actually Q.

By [AM69, Proposition 3.11] this is equivalent to showing that b_1 is not a zerodivisor in T/Q, which is in turn equivalent to proving that

$$(b_1^2 + b_2^2 + b_3^2, b_1x_1 + b_2x_2 + b_3x_3, b_1)$$

is a T-regular sequence. By Definition-Proposition 1.2.13, we only have to show that $Q + (b_1)$ is a complete intersection. We choose a system of generators which is more convenient for the purpose, namely $Q + (b_1) = (b_2^2 + b_3^2, b_2x_2 + b_3x_3, b_1)$. Let $<$ be the lexicographic order induced by $x_3 > x_2 > x_1 > b_1 > b_2 > b_3$. Then the leading terms of the generators of $Q + (b_1)$, namely

$$\begin{aligned}
&\mathrm{in}_<(b_2^2 + b_3^2) = b_2^2,\\
&\mathrm{in}_<(b_2x_2 + b_3x_3) = b_3x_3,\\
&\mathrm{in}_<(b_1) = b_1,
\end{aligned}$$

are coprime. By Proposition 1.3.13, $\{b_2^2+b_3^2, b_2x_2+b_3x_3, b_1\}$ is a Gröbner basis for $Q+(b_1)$ and its initial ideal is $\mathrm{in}_<(Q+(b_1)) = (b_2^2, b_3x_3, b_1)$. Hence $Q+(b_1)$ is a complete intersection by Proposition 1.3.14. □

Proposition 3.2.3. *If* $\mathrm{char}\,\mathbb{K} \neq 2$*, the ideal* B_k *is prime.*

Proof. We may assume $k = 1$, possibly after rearranging the rows of X. Consider the canonical projection

$$\mathbb{K}[X] \twoheadrightarrow \frac{\mathbb{K}[X]}{I_2(X_1)}$$

and notice that $\frac{\mathbb{K}[X]}{I_2(X_1)} \simeq \frac{\mathbb{K}[X_1]}{I_2(X_1)}[x_{1,1}, x_{1,2}, x_{1,3}]$ since the generators of $I_2(X_1)$ do not involve the variables $x_{1,1}, x_{1,2}, x_{1,3}$. Let $R := \mathbb{K}[a_ib_j \mid i \in [n-1], j \in [3]]$ and $S := \mathbb{K}[a_i, b_j \mid i \in [n-1], j \in [3]]$. Clearly, R is a subring of S. By Theorem 1.4.7, we know

$$\frac{\mathbb{K}[X_1]}{I_2(X_1)}[x_{1,1}, x_{1,2}, x_{1,3}] \simeq R[x_{1,1}, x_{1,2}, x_{1,3}].$$

We hence have the following diagram:

$$\mathbb{K}[X] \twoheadrightarrow \frac{\mathbb{K}[X]}{I_2(X_1)} \xrightarrow{\sim} \frac{\mathbb{K}[X_1]}{I_2(X_1)}[x_{1,1}, x_{1,2}, x_{1,3}] \xrightarrow{\sim} R[x_{1,1}, x_{1,2}, x_{1,3}] \hookrightarrow S[x_{1,1}, x_{1,2}, x_{1,3}]. \tag{3.1}$$

Consider the ideal

$$P := (b_1^2 + b_2^2 + b_3^2, b_1 x_{1,1} + b_2 x_{1,2} + b_3 x_{1,3})$$

in the ring $S[x_{1,1}, x_{1,2}, x_{1,3}]$. This is a prime ideal: let $Q := (b_1^2 + b_2^2 + b_3^2, b_1 x_{1,1} + b_2 x_{1,2} + b_3 x_{1,3})$ an ideal of the ring $T := \mathbb{K}[b_1, b_2, b_3, x_{1,1}, x_{1,2}, x_{1,3}]$. It is clear that $QS[x_{1,1}, x_{1,2}, x_{1,3}] = P$. The isomorphism of quotient rings

$$\frac{S[x_{1,1}, x_{1,2}, x_{1,3}]}{P} \simeq \frac{T}{Q}[a_i \mid i \in [n-1]]$$

implies that Q is prime if and only if P is prime. But Q is prime by Lemma 3.2.2. Now we are going to show that B_k is the pullback of P in (3.1). In order to do so, we retrieve the diagonal functor seen in Definition-Proposition 1.4.10. The ring S is a multigraded $\mathbb{K}$-algebra with the $\mathbb{N}^2$-multigrading generated by

- $\deg(a_i) = (1, 0)$ for every $i \in [n-1]$;
- $\deg(b_j) = (0, 1)$ for every $j \in [3]$;
- $\deg(x_{1,k}) = (0, 0)$ for every $k \in [3]$.

In our context, the statement of Proposition 1.4.11 says that $\Delta(S) = R$. Moreover, from the definition of Δ, it is not difficult to check that for every ideal I of R the equality $\Delta(I) = S \cap I$ holds.

Let J be the ideal generated by the image of B_k in $R[x_{1,1}, x_{1,2}, x_{1,3}]$. It is easy to see that

$$J = (a_i b_1 x_{1,1} + a_i b_2 x_{1,2} + a_i b_3 x_{1,3}, a_i a_j b_1^2 + a_i a_j b_2^2 + a_i a_j b_3^2 \mid i, j \in [n-1]).$$

The claim is therefore equivalent to the equality $\Delta(P) = J$. It is straightforward to show that $J = P_{1,1} + P_{2,2}$, where $P_{i,j}$ is the ideal generated by the homogeneous component of P of bidegree (i,j). Therefore, we only have to check whether $\Delta(P) \subseteq J$, that is

$$\sum_{n\in\mathbb{N}} P_{n,n} \subseteq P_{1,1} + P_{2,2} = J.$$

The reverse inclusion is trivial.

Fix $n \in \mathbb{N}$ and let $f \in P_{n,n}$. We want to show that $f \in P_{2,2} + P_{1,1}$. Since $f \in P$, f can be written as

$$p(b_1^2 + b_2^2 + b_3^2) + q(b_1 x_{1,1} + b_2 x_{1,2} + b_3 x_{1,3}) \tag{3.2}$$

for some homogeneous polynomials $p, q \in R$. Looking at the multidegrees, we deduce $\deg(p) = (n, n-2)$ and $\deg(q) = (n, n-1)$. Hence, for every monomial $u \in \operatorname{supp}(p)$, we can write $u = u' a_i a_j$ for some $i, j \in [n-1]$, $u' \in \operatorname{Mon}(R_{n-2,n-2})$, and similarly, for every monomial $v \in \operatorname{supp}(q)$, we can write $v = v' a_i$ for some $i, j \in [n-1]$, $u' \in \operatorname{Mon}(R_{n-1,n-1})$. The conclusion follows by substituting into 3.2. Hence B_k is prime, being the pullback of a prime ideal. □

In characteristic 2, the above statement does not hold.

Proposition 3.2.4. *If* $\operatorname{char} \mathbb{K} = 2$*, then* B_k *is not prime.*

Proof. The ideal B_k contains elements of the form

$$g_{i,i} = x_{i,1}^2 + x_{i,2}^2 + x_{i,3}^2 = (x_{i,1} + x_{i,2} + x_{i,3})^2$$

for $i \neq k$, but it does not contain $x_{i,1} + x_{i,2} + x_{i,3}$. Hence B_k is not prime. □

From now on, we will assume $\operatorname{char} \mathbb{K} \neq 2$. Before proving that A_Γ and B_k are in fact minimal primes of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$, let us give a geometric description of the algebraic sets that they define, by parameterizing the points with matrices. We start with A_Γ. The algebraic set $V(A_\Gamma)$ is just

$$V(A_\Gamma) = V(g_{i,j} \mid i < j, i, j \in \Gamma) \cap V(x_{i,k} \mid i \notin \Gamma, k \in [3]),$$

which means that the rows with indices in Γ are pairwise orthogonal, and the rows with indices not in Γ are zero. To sum up,

$$V(A_\Gamma) = \left\{ \begin{pmatrix} \mathbf{m}_1 \\ \vdots \\ \mathbf{m}_n \end{pmatrix} \;\middle|\; \begin{array}{ll} \mathbf{m}_i \in \mathbb{K}^3 & \text{for } i \in [n] \\ \mathbf{m}_i = 0 & \text{for } i \notin \Gamma \\ \mathbf{m}_i \cdot \mathbf{m}_j = 0 & \text{for } i \neq j \end{array} \right\}.$$

Now we describe the ideal B_k geometrically. We have seen in Section 1.4 that

$$V(I_2(X)) = \{\mathbf{a}^\intercal \mathbf{b} \mid \mathbf{a} = (a_1, \dots, a_n) \in \mathbb{K}^n, \mathbf{b} = (b_1, b_2, b_3) \in \mathbb{K}^3\}.$$

Recall that

$$B_k = (g_{i,j} \mid i \leq j, (i,j) \neq (k,k)) + I_2(X_k),$$

hence

$$V(B_k) = V(g_{i,j} \mid i \leq j, (i,j) \neq (k,k)) \cap V(I_2(X_k)).$$

First of all, let us describe $V(I_2(X_k))$. Without loss of generality, we will assume $k = 1$, thus a point $M \in V(I_2(X_1))$ will be an $n \times 3$ matrix where there is no restriction on the first row, and the remaining rows form a submatrix of rank at most 1:

$$V(I_2(X_1)) = \left\{ \begin{pmatrix} \mathbf{c} \\ \mathbf{a}^\intercal \mathbf{b} \end{pmatrix} \in \mathrm{M}_{\mathbb{K}}(n,3) \;\middle|\; \begin{array}{l} \mathbf{a} \in \mathbb{K}^{n-1} \\ \mathbf{b} \in \mathbb{K}^3 \\ \mathbf{c} \in \mathbb{K}^3 \end{array} \right\}.$$

Now we add the condition $M \in V(g_{i,j} \mid i \leq j, (i,j) \neq (1,1))$: this means that the rows of M are pairwise orthogonal, excluding the pair $(1,1)$, but including the other pairs (i,i) with $i = 2, \dots, n$. Indeed, for every $i \in [n], i \neq 1$, the condition $g_{1,i} = 0$ means

$$a_i \mathbf{b} \cdot \mathbf{c} = 0$$

while, for $i, j \neq 1$, the condition $g_{i,j} = 0$ translates as

$$a_i \mathbf{b} \cdot a_j \mathbf{b} = 0.$$

If $a_i \neq 0$ for some $i \neq 1$, we must have

$$\begin{cases} \mathbf{b} \cdot \mathbf{b} = 0 \\ \mathbf{b} \cdot \mathbf{c} = 0 \end{cases} \tag{3.3}$$

which means

$$\begin{cases} b_1^2 + b_2^2 + b_3^3 = 0 \\ b_1 c_1 + b_2 c_2 + b_3 c_3 = 0. \end{cases}$$

If, instead, $a_i = 0$ for every $i \neq 1$, then $\mathbf{a} = 0$ and

$$M = \begin{pmatrix} & \mathbf{c} & \\ 0 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & 0 \end{pmatrix}.$$

In this case, we can choose $\mathbf{b} = 0$ and the equations in 3.3 still hold true.

To sum up, we have

$$V(B_k) = \left\{ \begin{pmatrix} \mathbf{m}_1 \\ \vdots \\ \mathbf{m}_n \end{pmatrix} \middle| \, \exists\, \mathbf{b} \in \mathbb{K}^3 \text{ such that } \begin{array}{l} \mathbf{b} \cdot \mathbf{b} = 0 \\ \mathbf{m}_i = a_i \mathbf{b} \text{ for } i \neq k \\ \mathbf{m}_k \cdot \mathbf{b} = 0 \end{array} \right\}.$$

3.3 Minimal primes of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$

In this section, we show that the prime ideals A_Γ and B_k are in fact minimal primes and primary components of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$, using Proposition 1.1.11.

Lemma 3.3.1. *For every 3-subset Γ of $[n]$, the ideal A_Γ is a primary component and a minimal prime of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$.*

Proof. Let X_Γ be the 3×3 matrix obtained from X by selecting the 3 rows whose indices belong to Γ. Let w_Γ be the determinant of X_Γ. We are going to show that

$$A_\Gamma = (\mathrm{L}_{\mathbb{K}}(K_n, 3) : w_\Gamma) = (\mathrm{L}_{\mathbb{K}}(K_n, 3) : w_\Gamma^\infty).$$

The conclusion will follow from Proposition 1.1.11.

Without loss of generality, we may assume $\Gamma = \{1,2,3\}$, up to renaming the variables. therefore

$$X_\Gamma = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix}$$

and

$$w_\Gamma = x_{1,1}x_{2,2}x_{3,3}+x_{1,2}x_{2,3}x_{3,1}+x_{1,3}x_{2,1}x_{3,2}-x_{1,3}x_{2,2}x_{3,1}-x_{1,2}x_{2,1}x_{3,3}-x_{1,1}x_{2,3}x_{3,2}.$$

Firstly we show that $A_\Gamma \subseteq (\mathrm{L}_{\mathbb{K}}(K_n,3) : w_\Gamma)$, which is equivalent to $w_\Gamma A_\Gamma \subseteq \mathrm{L}_{\mathbb{K}}(K_n,3)$, i.e. for every generator g of A_Γ, gw_Γ belongs to $\mathrm{L}_{\mathbb{K}}(K_n,3)$. If $g = g_{i,j}$ for some $i \neq j$, then g belongs to $\mathrm{L}_{\mathbb{K}}(K_n,3)$ and there is nothing to prove. So let $g = x_{i,k}$ for some $i > 3, k \in [3]$. We want to write gw_Γ as a combination of the generators $g_{i,j}$ of $\mathrm{L}_{\mathbb{K}}(K_n,3)$. Let $X_{\Gamma,k}$ be the 3×2 matrix obtained from X_Γ by deleting the k-th column. We claim that

$$w_\Gamma x_{i,k} = m_{i_2,i_3}g_{i_1,i} - m_{i_1,i_3}g_{i_2,i} + m_{i_1,i_2}g_{i_3,i}$$

where $m_{i,j}$ denotes the unique 2-minor of $X_{\Gamma,k}$ corresponding to the rows i,j.

In the following, we give a proof for $k = 1$. For $k = 2,3.$ the claim follows from analogous computations.

$$
\begin{aligned}
&m_{i_2,i_3}g_{i_1,i} - m_{i_1,i_3}g_{i_2,i} + m_{i_1,i_2}g_{i_3,i} = \\
&(x_{2,2}x_{3,3} - x_{3,2}x_{2,3})g_{1,i} + (x_{3,2}x_{1,3} - x_{1,2}x_{3,3})g_{2,i} + (x_{1,2}x_{2,3} - x_{2,2}x_{1,3})g_{3,i} = \\
&= (x_{2,2}x_{3,3} - x_{3,2}x_{2,3})(x_{1,1}x_{i,1} + x_{1,2}x_{i,2} + x_{1,3}x_{i,3}) \\
&+ (x_{1,2}x_{3,3} - x_{3,2}x_{1,3})(x_{2,1}x_{i,1} + x_{2,2}x_{i,2} + x_{2,3}x_{i,3}) \\
&+ (x_{1,2}x_{2,3} - x_{2,2}x_{1,3})(x_{3,1}x_{i,1} + x_{3,2}x_{i,2} + x_{3,3}x_{i,3}) = \\
&= x_{2,2}x_{3,3}x_{1,1}x_{i,1} + x_{2,2}x_{3,3}x_{1,2}x_{i,2} + x_{2,2}x_{3,3}x_{1,3}x_{i,3} \\
&- x_{3,2}x_{2,3}x_{1,1}x_{i,1} - x_{3,2}x_{2,3}x_{1,2}x_{i,2} - x_{3,2}x_{2,3}x_{1,3}x_{i,3} \\
&+ x_{3,2}x_{1,3}x_{2,1}x_{i,1} + x_{3,2}x_{1,3}x_{2,2}x_{i,2} + x_{3,2}x_{1,3}x_{2,3}x_{i,3} \\
&- x_{1,2}x_{3,3}x_{2,1}x_{i,1} - x_{1,2}x_{3,3}x_{2,2}x_{i,2} - x_{1,2}x_{3,3}x_{2,3}x_{i,3} \\
&+ x_{1,2}x_{2,3}x_{3,1}x_{i,1} + x_{1,2}x_{2,3}x_{3,2}x_{i,2} + x_{1,2}x_{2,3}x_{3,3}x_{i,3} \\
&- x_{2,2}x_{1,3}x_{3,1}x_{i,1} - x_{2,2}x_{1,3}x_{3,2}x_{i,2} - x_{2,2}x_{1,3}x_{3,3}x_{i,3} = \\
&x_{2,2}x_{3,3}x_{1,1}x_{i,1} - x_{3,2}x_{2,3}x_{1,1}x_{i,1} + x_{3,2}x_{1,3}x_{2,1}x_{i,1} \\
&- x_{1,2}x_{3,3}x_{2,1}x_{i,1} + x_{1,2}x_{2,3}x_{3,1}x_{i,1} - x_{2,2}x_{1,3}x_{3,1}x_{i,1} = w_\Gamma x_{i,1}.
\end{aligned}
$$

This shows that $A_\Gamma \subseteq (\mathrm{L}_{\mathbb{K}}(K_n, 3) : w_\Gamma) \subseteq (\mathrm{L}_{\mathbb{K}}(K_n, 3) : w_\Gamma^\infty)$. In order to show that $(\mathrm{L}_{\mathbb{K}}(K_n, 3) : w_\Gamma^\infty) \subseteq A_\Gamma$, let $g \in \mathbb{K}[X]$ such that $g w_\Gamma^m \in \mathrm{L}_{\mathbb{K}}(K_n, 3) \subseteq A_\Gamma$ for some $m \in \mathbb{N}$. If we prove that $w_\Gamma \notin A_\Gamma$, from the primality of A_Γ it will follow that $g \in A_\Gamma$, hence the conclusion.

Suppose by contradiction that $w_\Gamma \in A_\Gamma$. We want to write it as a combination of some generators of A_Γ. Since the only variables appearing in w_Γ are of the form $x_{i,k}$ with $i \in \{1, 2, 3\}$, w_Γ must be a combination of some $g_{i,j}$, with $i < j$ and $i, j \in \Gamma$. This is impossible since no monomial in $\mathrm{supp}(w_\Gamma)$ has two factors $x_{i,k}$, $x_{j,k}$ for some $k \in [3]$. □

Lemma 3.3.2. *For every $k \in [n]$, the ideal B_k is a primary component and a minimal prime of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$.*

Proof. The proof is analogous to that of Lemma 3.3.1. Again, we may assume $k = 1$. We ought to find a $u \in S$ such that

$$B_1 = (\mathrm{L}_{\mathbb{K}}(K_n, 3) : u) = (\mathrm{L}_{\mathbb{K}}(K_n, 3) : u^\infty).$$

We call such u a *witness*. Let $\mathcal{A}$ be the set of 2-minors of X involving the first row, i.e the elements of $\mathcal{A}$ are polynomials of the form

$$x_{1,k_1}x_{i,k_2} - x_{1,k_2}x_{i,k_1}$$

with $k_1, k_2 \in [3]$ distinct, and $i \in [n], i \neq 1$. Then we claim that

$$u := \prod_{m \in \mathcal{A}} m$$

is a witness.

Again, we want to prove that $B_1 \subseteq (\mathrm{L}_{\mathbb{K}}(K_n, 3) : u)$ by showing that, for every generator g of B_1, the product gu belongs to $\mathrm{L}_{\mathbb{K}}(K_n, 3)$. It will be sufficient to show $gv \in \mathrm{L}_{\mathbb{K}}(K_n, 3)$ for some $v \in \mathbb{K}[X]$ dividing u. This time there are two nontrivial cases.

- g is a minor of X_1, i.e.

  $$g = x_{i,h_1}x_{j,h_2} - x_{i,h_2}x_{j,h_1}$$

  with $1 \neq i < j$, $h_1 \neq h_2 \in [3]$. Let $k \in [n], k \neq 1, i, j$ (remember that $n > 3$) and let $h_3 \in [3]$ such that h_1, h_2, h_3 is a permutation of $1, 2, 3$. We choose

  $$u = x_{1,h_2}x_{k,h_3} - x_{1,h_3}x_{k,h_2}$$

  and show that gu can be written as a combination of generators of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$:

  $$\begin{aligned}
  &x_{j,h_1}x_{1,h_3}g_{k,i} - x_{i,h_1}x_{1,h_3}g_{k,j} - x_{j,h_1}x_{k,h_3}g_{1,i} + x_{i,h_1}x_{k,h_3}g_{1,j} = \\
  &= x_{j,h_1}x_{1,h_3}x_{k,h_1}x_{i,h_1} + x_{j,h_1}x_{1,h_3}x_{k,h_2}x_{i,h_2} + x_{j,h_1}x_{1,h_3}x_{k,h_3}x_{i,h_3} \\
  &- x_{i,h_1}x_{1,h_3}x_{k,h_1}x_{j,h_1} - x_{i,h_1}x_{1,h_3}x_{k,h_2}x_{j,h_2} - x_{i,h_1}x_{1,h_3}x_{k,h_3}x_{j,h_3} \\
  &- x_{j,h_1}x_{k,h_3}x_{1,h_1}x_{i,h_1} - x_{j,h_1}x_{k,h_3}x_{1,h_2}x_{i,h_2} - x_{j,h_1}x_{k,h_3}x_{1,h_3}x_{i,h_3} \\
  &+ x_{i,h_1}x_{k,h_3}x_{1,h_1}x_{j,h_1} + x_{i,h_1}x_{k,h_3}x_{1,h_2}x_{j,h_2} + x_{i,h_1}x_{k,h_3}x_{1,h_3}x_{j,h_3} = \\
  &= x_{j,h_1}x_{1,h_3}x_{k,h_2}x_{i,h_2} - x_{i,h_1}x_{1,h_3}x_{k,h_2}x_{j,h_2} \\
  &- x_{j,h_1}x_{k,h_3}x_{1,h_2}x_{i,h_2} + x_{i,h_1}x_{k,h_3}x_{1,h_2}x_{j,h_2} = \\
  &(x_{1,h_2}x_{k,h_3} - x_{1,h_3}x_{k,h_2})(x_{i,h_1}x_{j,h_2} - x_{i,h_2}x_{j,h_1}) = vg.
  \end{aligned}$$

- Let $g = g_{i,i}$ for some $i \in [n], i \neq 1$. If $i \neq 2, 3$, then we take $v = (x_{1,1}x_{2,2} - x_{1,2}x_{2,1})(x_{1,1}x_{3,3} - x_{1,3})$. We show that $gv \in \mathrm{L}_{\mathbb{K}}(K_n, 3)$ by writing it as a combination of the generators of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$. We claim that

$$\begin{aligned}
&x_{i,3}(x_{1,1}(x_{2,1}x_{i,2} - x_{2,2}x_{i,1}) + x_{1,3}(x_{2,3}x_{i,2} - x_{2,2}x_{i,3}))g_{0,2}\\
&+ x_{3,3}(x_{1,1}(x_{2,2}x_{i,1} - x_{2,1}x_{i,1}) + x_{1,3}(x_{2,2}x_{i,3} - x_{2,3}x_{i,2}))g_{1,i}\\
&+ (x_{1,1}x_{i,1}x_{i,2}(x_{1,2} + x_{1,3}) + x_{1,3}(x_{i,2}^2x_{1,2} - x_{1,1}^2x_{i,2} + x_{i,3}^2x_{1,2}))g_{2,3}\\
&+ (x_{1,1}(x_{1,1}x_{3,3}x_{i,2} - x_{1,2}x_{3,3}x_{i,1} - x_{1,3}x_{3,1}x_{i,2})\\
&- x_{1,2}x_{1,3}(x_{3,2}x_{i,2} - x_{3,3}x_{i,3}))g_{2,i}+\\
&(x_{1,1}x_{i,3} - x_{1,3}x_{i,1})(x_{1,1}x_{2,2} - x_{1,2}x_{2,1})g_{3,i} = vg.
\end{aligned}$$

  Again, renaming the variables (i.e. rearranging the rows of X), we may assume $i = 3$. If $gv \in \mathrm{L}_{\mathbb{K}}(K_n, 3)$, then $gv \in \mathrm{L}_{\mathbb{K}}(K_{n+1}, 3) \cap S_n(3)$, hence it is sufficient to prove the claim for the base case $n = 4$. This is an explicit computation and it can be verified with Macaulay2 ([GS]). The same goes for the cases $i \neq 2, 3$.

The rest of the proof is analogous to that of Lemma 3.3.1. □

We have shown that the ideals of the form A_Γ and B_k are minimal primes and primary components of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$. In terms of their algebraic sets, this means that

$$\mathcal{V} \supseteq \left(\bigcup_{\Gamma \subseteq [n]} V(A_\Gamma)\right) \cup \left(\bigcup_{k \in [n]} V(B_k)\right) \tag{3.4}$$

where $\mathcal{V} := V(\mathrm{L}_{\mathbb{K}}(K_n, 3))$. Observe that none of the components in (3.4) is redundant because A_Γ and B_k are minimal primes, and hence pairwise incomparable. We are going to give another proof of this.

Proposition 3.3.3. *Any two distinct elements in the set*

$$\{A_\Gamma \mid \Gamma \subseteq [n], |\Gamma| = 3\} \cup \{B_k \mid k \in [n]\}$$

are pairwise incomparable with respect to inclusion.

Proof. Let I, J two distinct ideals in $\{A_\Gamma \mid \Gamma \subseteq [n], |\Gamma| = 3\} \cup \{B_k \mid k \in [n]\}$.

- If I and J are of the same type, then I, J are isomorphic, hence they have the same dimension. Since they are prime and distinct, it follows that $I \nsubseteq J$ and $J \nsubseteq I$.

- Let $I = A_\Gamma$ and $J = B_k$ for some Γ and k. From the description of $V(I)$ and $V(J)$ in Section 3.2, we know that $V(I)$ contains matrices of rank 3, for example a matrix whose nonzero rows are $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, and $V(J)$ does not. This shows that $V(I) \nsubseteq V(J)$, and, since I, J are prime, that $J \nsubseteq I$. Conversely, I contains variables and J does not, which means $I \nsubseteq J$.

□

In order to prove Theorem 3.1.6, we are left to show that there are no other minimal primes.

3.4 Primary decomposition of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$

Finally, in this section, we show that the minimal primes of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ are *precisely* those of the form A_Γ and B_k. After this, we will have determined the irredundant primary decomposition of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ up to embedded components.

3.4.1 Case $\mathbb{K}$ algebraically closed

We will use the one-to-one correspondence between minimal primes of an ideal and irreducible components of its algebraic set described in Remark 1.1.12. Up until now, we have shown that A_Γ and B_k are minimal primes of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$, or, equivalently,

$$\mathcal{V} \supseteq \left(\bigcup_{\Gamma \subseteq [n]} V(A_\Gamma) \right) \cup \left(\bigcup_{k \in [n]} V(B_k) \right).$$

In order to show that there are no other minimal primes, it is sufficient to prove the reverse inclusion. Firstly, assume $\mathbb{K} = \overline{\mathbb{K}}$.

Proposition 3.4.1. *We have*

$$\mathcal{V} = \left(\bigcup_{\Gamma \subseteq [n]} V(A_\Gamma) \right) \cup \left(\bigcup_{k \in [n]} V(B_k) \right) \tag{3.5}$$

where $\mathcal{V} := V(\mathrm{L}_{\mathbb{K}}(K_n, 3)) \subseteq \mathbb{A}^{nd}$.

Proof. One inclusion has already been proved. Conversely, we want to show that

$$\mathcal{V} \subseteq \left(\bigcup_{\Gamma \subseteq [n]} V(A_\Gamma) \right) \cup \left(\bigcup_{k \in [n]} V(B_k) \right).$$

Let $M \in \mathcal{V} \subseteq \mathbb{A}^{nd}$ be a nonzero $n \times d$ matrix. If $\mathrm{rank}(M) = 1$, then by Proposition 1.4.9 $M = \mathbf{a}^\intercal \mathbf{b}$ where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{K}^n$ and $\mathbf{b} \in \mathbb{K}^3$.

- If there is at most one i such that $a_i \neq 0$, then M has at most one nonzero row. It follows that $M \in A_\Gamma$ for any $\Gamma \ni i$.

- If at least two distinct rows of M are nonzero, say $a_i\mathbf{b}$ and $a_j\mathbf{b}$, then from $M \in \mathcal{V}$ we get $a_i\mathbf{b} \cdot a_j\mathbf{b} = 0$ and hence $\mathbf{b} \cdot \mathbf{b} = 0$. Then every row of M is self-orthogonal and $M \in B_k$ for every $k \in [n]$.

We are left with the case $\mathrm{rank}(M) \geq 2$. Let us denote by $\mathbf{m}_1, \ldots, \mathbf{m}_n$ the rows of M. Remember that, by the definition of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$, every two distinct rows $\mathbf{m}_i$ and $\mathbf{m}_j$ of M are orthogonal. Since $\mathrm{rank}(M) \geq 2$, two of its rows are linearly independent, say $\mathbf{m}_i, \mathbf{m}_j$.

If there is at most one $k \neq i, j$ such that the row $\mathbf{m}_k$ is nonzero, then in this case $M \in V(A_\Gamma)$ for $\Gamma = \{i, j, k\}$. Suppose that at least two rows $\mathbf{m}_k, \mathbf{m}_h$ with $h, k \notin \{i, j\}$ are nonzero. The rows $\mathbf{m}_k, \mathbf{m}_h$ are orthogonal to both $\mathbf{m}_i$ and $\mathbf{m}_j$, hence

$$\mathbf{m}_k, \mathbf{m}_h \in \ker \begin{pmatrix} \mathbf{m}_i \\ \mathbf{m}_j \end{pmatrix}$$

where $\ker \begin{pmatrix} \mathbf{m}_i \\ \mathbf{m}_j \end{pmatrix}$ has dimension 1 (since $\mathbf{m}_i$ and $\mathbf{m}_j$ are linearly independent), i.e. it is generated by a single nonzero vector $\mathbf{b} \in \mathbb{K}^3$. Then we have $\mathbf{m}_k, \mathbf{m}_h \in$

span$(\mathbf{b})$ and we can write

$$\mathbf{m}_k = \alpha\mathbf{b}, \mathbf{m}_h = \beta\mathbf{b}, \text{for some } \alpha, \beta \in \mathbb{K}.$$

Since $\mathbf{m}_h$ and $\mathbf{m}_k$ are orthogonal as well, we get $\alpha\mathbf{b} \cdot \beta\mathbf{b} = 0$, which implies $\mathbf{b} \cdot \mathbf{b} = 0$, therefore $\mathbf{b}$ belongs to $\ker\left(\mathbf{b}\right)$. This is a vector space of dimension 2, generated by $\mathbf{b}$ and another vector of $\mathbb{K}^3$ that we will call $\mathbf{c}$.

The rows $\mathbf{m}_i$ and $\mathbf{m}_j$ are orthogonal to $\mathbf{m}_k$, and hence to $\mathbf{b}$. Therefore we have $\mathbf{m}_i, \mathbf{m}_j \in \ker\left(\mathbf{b}\right) = \mathrm{span}(\mathbf{b}, \mathbf{c})$. We can write

$$\begin{cases} \mathbf{m}_i = \delta_1\mathbf{b} + \gamma_1\mathbf{c} \\ \mathbf{m}_j = \delta_2\mathbf{b} + \gamma_2\mathbf{c} \end{cases} \tag{3.6}$$

for some $\delta_1, \delta_2, \gamma_1, \gamma_2 \in \mathbb{K}$.

Finally, substituting (3.6) in the condition $\mathbf{m}_i \cdot \mathbf{m}_j = 0$, we get

$$\gamma_1\mathbf{c} \cdot \gamma_2\mathbf{c} = 0.$$

This leads to two distinct cases.

- If γ_1 or $\gamma_2 = 0$, then one between $\mathbf{m}_i$ or $\mathbf{m}_j$, say $\mathbf{m}_i$, is a multiple of $\mathbf{b}$. Since the rows with index different from i and j are orthogonal to $\mathbf{m}_i, \mathbf{m}_j$, they belong to span$(\mathbf{b})$. Hence all rows but $\mathbf{m}_j$ are multiples of $\mathbf{b}$, and $\mathrm{rank}(M_j) = 1$, where M_j is the matrix obtained from M by deleting the row j. It follows that $M \in V(B_j)$.

- If $\mathbf{c} \cdot \mathbf{c} = 0$, then the system of equations

$$\begin{cases} \mathbf{x} \cdot \mathbf{x} = 0 \\ \mathbf{y} \cdot \mathbf{y} = 0 \\ \mathbf{x} \cdot \mathbf{y} = 0 \end{cases} \tag{3.7}$$

  has $\mathbf{x} = \mathbf{b}$, $\mathbf{y} = \mathbf{c}$ as a solution. But this means that any pair of linear combination of $\mathbf{b}$ and $\mathbf{c}$ satisfies (3.7) as well. However, the locus

$$\{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{K}^3 \mid x_1^2 + x_2^2 + x_3^2 = 0\}$$

defined by the first equation in (3.7) is an irreducible quadric and it cannot contain $\mathrm{span}(\mathbf{b}, \mathbf{c})$ which is a plane, contradiction.

This shows that, in any case, $M \in \left(\bigcup_{\Gamma \subseteq [n]} V(A_\Gamma)\right) \cup \left(\bigcup_{k \in [n]} V(B_k)\right)$. □

This means that (3.5) is the decomposition of $\mathcal{V}$ into irreducible components, or, equivalently, that the minimal primes of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ are precisely those of the two types A_Γ and B_k. This concludes the proof of Theorem 3.1.6 in the case of algebraically closed fields.

3.4.2 Case $\mathbb{K}$ arbitrary

Now let $\mathbb{K}$ be any field. We want to prove that $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ has no other minimal primes other than those in $\{A_\Gamma \mid \Gamma \subseteq [n], |\Gamma| = 3\} \cup \{B_k \mid k \in [n]\}$. We have just shown that the claim holds true for $\mathrm{L}_{\overline{\mathbb{K}}}(K_n, 3)$. We firstly recall some results about tensor products.

Lemma 3.4.2. *For a ring R and a field $\mathbb{K} \subseteq R$, the homomorphism*

$$i : R \to R \otimes_{\mathbb{K}} \overline{\mathbb{K}}$$

is faithfully flat.

Lemma 3.4.3. *A faithfully flat ring extension $R \to S$ satisfies the lying over property, i.e. the map $i^\# : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective.*

Proposition 3.4.4. *The minimal prime ideals of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ are*

$$\{A_\Gamma \mid \Gamma \subseteq [n], |\Gamma| = 3\} \cup \{B_k \mid k \in [n]\}.$$

Proof. Let $S := \mathbb{K}[X]$, $\overline{S} := \overline{\mathbb{K}}[X] = S \otimes_K \overline{\mathbb{K}}$. We have $S \subseteq \overline{S}$. Observe that

$$\mathrm{L}_{\overline{\mathbb{K}}}(K_n, 3) = \mathrm{L}_{\mathbb{K}}(K_n, 3)\overline{S} = \mathrm{L}_{\mathbb{K}}(K_n, 3) \otimes_{\mathbb{K}} \overline{\mathbb{K}}.$$

Let $R := S/\mathrm{L}_{\mathbb{K}}(K_n, 3)$. The statement is equivalent to finding the minimal primes of R. Let $\overline{R} := R \otimes_{\mathbb{K}} \overline{\mathbb{K}} = \overline{S}/\mathrm{L}_{\mathbb{K}}(K_n, 3)\overline{S} = \overline{S}/\mathrm{L}_{\overline{\mathbb{K}}}(K_n, 3)$. We already know that the minimal primes of $\overline{R}$ are of the desired form. By Lemmas

3.4.2 and 3.4.3, the map $i^{\#} : \mathrm{Spec}(\overline{R}) \to \mathrm{Spec}(R)$ is surjective. We want to prove that every minimal prime of R is the contraction of a minimal prime of $\overline{R}$. Let then P be a minimal prime of R, there is a $Q \in \mathrm{Spec}(\overline{R})$ such that $P = Q \cap R$. Now suppose that Q' is a minimal prime of $\overline{R}$ such that $Q' \subseteq Q$. Then $Q' \cap R \subseteq Q \cap R = P$, and since Q' is prime, it follows that $Q' \cap R = P$ because $Q' \cap R$ is prime and P is minimal. This means that the minimal prime ideals of R are those of the form $A_\Gamma \overline{R} \cap R = A_\Gamma$, and $B_k \overline{R} \cap R = B_k$. □

This concludes the proof of Theorem 3.1.6 for arbitrary fields (of characteristics different from 2). We have proved that, for $\mathrm{char}\, \mathbb{K} \neq 2$, $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ is generically radical, in the sense specified in Section 3.1: the primary decomposition of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ consists only of minimal primes and, possibly, embedded components.

3.5 Gröbner basis of B_k

We conclude this chapter by computing a Gröbner basis for the ideal B_k introduced in the previous section. A Gröbner basis could be useful in the future for computing some invariants of B_k, and consequently of $\mathrm{L}_{\mathbb{K}}(K_n, 3)$, such as its dimension, Betti numbers and so on. In this section, we suppose $\mathrm{char}\, \mathbb{K} = 0$.

We may suppose $k = 1$, possibly substituting the k-th row of X with the first one.

Theorem 3.5.1. *Let $\mathcal{G}_n$ denote the set whose elements are the following polynomials of $S_n(3)$:*

- *the minimal generators of B_1, i.e. $g_{i,j}$ for $i, j \in [n]$, $(i, j) \neq (1, 1)$ and the 2-minors of the matrix X_1 obtained from X by deleting the first row;*
- *the cubics*

$$x_{1,1}(x_{i,2}x_{j,2} + x_{i,3}x_{j,3}) - x_{j,1}(x_{i,2}x_{0,2} + x_{i,3}x_{0,3})$$

for $1 < i \leq j \leq n$.

Then $\mathcal{G}_n$ *is a Gröbner basis for* B_1 *with respect to the order* $<_{\text{lex}}$ *induced by* $x_{1,1} > x_{1,2} > x_{1,2} > \cdots > x_{n,1} > x_{n,2} > x_{n,2}$.

Proof. By the Buchberger's criterion (Theorem 1.3.11), the conclusion will follow once we have shown that all S-polynomials of $\mathcal{G}_n$ reduces to 0. We do not have to check this for every pair of polynomials: firstly, the 2-minors of X_1 are a Gröbner basis of $I_2(X_1)$ by Proposition 1.4.5, so we already now that the S-polynomial of a pair of 2-minors reduces to 0. Secondly, by Proposition 1.3.12, if the leading terms of a pair of polynomials do not have a common factor, then their S-polynomial reduces to 0. Hence we only have to consider the pairs of polynomials whose leading terms have a common factor. In short, the pairs to check are:

- $g_{i,j}$ and $g_{j,k}$ for $i \leq j \leq k, i \neq k$;
- $g_{i,j}$ and $x_{i,1}x_{k,h} - x_{i,h}x_{k,1}$ for $i < k$;
- $c_{i,j}$ and $x_{i,2}x_{k,h} - x_{i,h}x_{k,2}$ for $i < k$;
- $g_{1,i}$ and $c_{j,k}$;
- $c_{i,j}$ and $c_{h,k}$ for $i \leq j \leq h \leq k, i \neq k$.

We will proceed by induction. We are going to write a proof for the first pair by showing a method that works for every pair of the list.

First of all, for the base case $n = 4$, we check with Macaulay2 ([GS]) that $\mathcal{G}_4$ is a Gröbner basis for B_1 in $S_4(3)$ over $\mathbb{Q}$. The code is the following:

```
--- setup:
K=QQ
n=4
d=3
R=K[x_(0,0) .. x_(n-1,d-1), MonomialOrder => Lex]
X=transpose(genericMatrix(R,x_(0,0),d,n));
```

```
g=X*(transpose X);
L={};

--- adding g_(i,j) to the generators of B1:
for i from 0 to n-1 do (for j from i+1 to n-1 do
L=append(L, g_(i,j)));
for i from 1 to n-1 do L=append(L, g_(i,i));

--- adding the minors to the generators of B1:
X1=submatrix'(X,{0},{});
mins=minors(2,X1);

--- groebner basis of B1:
B1=mins + ideal L
print gens gb B1
```

Notice that the fact that $\mathcal{G}_4$ is a Gröbner basis of B_1 does not depend on the field: the leading coefficients of the polynomials in $\mathcal{G}_4$ are all equal to 1, and this means that the S-polynomials will be the same over any field $\mathbb{K} \supseteq \mathbb{Q}$.

So let $i \leq j \leq k$, $i \leq k$ and denote by S the S-polynomial $S(g_{i,j}, g_{j,k})$. We may assume, possibly by renaming the variables, that $k \leq 4$, so that $S \in S_4(3)$. By the induction hypothesis, S reduces to 0 in the ring $S_4(3)$. Hence, we will have a reduction equation of the form

$$S = h_1 f_1 + \cdots + h_m f_m \tag{3.8}$$

where $f_s \in \mathcal{G}_4$, $0 \neq h_s \in S_4(3)$ and $\text{in}_{<_{\text{lex}}}(S) \geq \text{in}_{<_{\text{lex}}}(h_s f_s)$ for every $s \in [m]$. Clearly, the identity 3.8 still holds in any ring containing $S_4(3)$, and hence in $S_n(3)$; moreover, $\mathcal{G}_n \supseteq \mathcal{G}_4$. We are left to check that $\text{in}_{<_{\text{lex}}}(S) \geq \text{in}_{<_{\text{lex}}}(h_s f_s)$ in $S_n(3)$. But this is clearly true since the order $<_{\text{lex}}$ on $S_4(3)$ is the restriction of the order $<_{\text{lex}}$ on $S_n(3)$. Hence $S(g_{i,j}, g_{j,k})$ reduces to 0 in $S_n(3)$. The proof for the other pairs is analogous. □

As an immediate consequence, we can describe the initial ideal $\text{in}_{<_{\text{lex}}}(B_1)$.

Corollary 3.5.2. *The initial ideal of B_1 with respect to $<_{\text{lex}}$ is generated by the following monomials:*

- $x_{i,1}x_{j,1}$ *for* $i, j \in [n]$, $(i, j) \neq (1, 1)$;
- $x_{i,h_1}x_{j,h_2}$ *for* $1 < i < j \leq n$, $1 \leq h_1 < h_2 \leq 3$;
- $x_{1,1}x_{i,2}x_{j,2}$ *for* $1 < i \leq j \leq n$.

Conclusion and future developments

In Chapter 3, we have partially solved Conjecture 3.1.3, which was posed by A. Conca and V. Welker in [CW19], for the case $d = 3$. The conjecture asked whether LSS ideals associated to complete graphs were always radical. What we have found in Theorem 3.1.6 is that, in $\mathrm{L}_{\mathbb{K}}(K_n, 3)$, all the primary components associated to minimal primes are the minimal primes themselves. This means that the only way for $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ not to be radical would be to have embedded components and, unfortunately, these cannot be detected geometrically, as explained in Section 3.1.

However, what we have proved is equivalent to saying that the scheme $X = \mathrm{Spec}(S_n(3)/\mathrm{L}_{\mathbb{K}}(K_n, 3))$ has the property that $\mathcal{O}_{X,x}$ is reduced for every $x \in U$, where U is some Zariski dense open subset of X. For this reason, we say that X is *generically reduced*, or that $\mathrm{L}_{\mathbb{K}}(K_n, 3)$ is *generically radical*. Moreover, the original conjecture was stated for fields of characteristic 0 but our argument holds for any characteristic different from 2.

Hence, the first problem that arises naturally is studying what happens in characteristic 2. Another potential future direction could be trying to prove the general case of the conjecture, for arbitrary d. Most of the methods used in this work could be generalized to any d. However, there seem to be some obstructions: the proof of Proposition 3.2.1 suggests that the radicality of $\mathrm{L}_{\mathbb{K}}(K_n, d)$ is related to the primality of $\mathrm{L}_{\mathbb{K}}(K_d, d)$ and Example 8.5 in [CW19] shows that $\mathrm{L}_{\mathbb{K}}(K_d, d)$ is not necessarily prime. Alternatively, one could stick to the case $d = 3$ and study the radicality of arbitrary graphs: to this day, the characterization of graphs G such that $\mathrm{L}_{\mathbb{K}}(G, 3)$ is radical remains open. A different direction in the investigation of LSS ideals could be studying their homological invariants. Section 3.5 might represent a starting point.

Bibliography

[AM69] Michael Francis Atiyah and I. G. MacDonald. *Introduction to commutative algebra.* Addison-Wesley-Longman, 1969, pp. I–IX, 1–128.

[BH98] W. Bruns and H.J. Herzog. *Cohen-Macaulay Rings*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1998.

[CW19] A. Conca and V. Welker. "Lovász–Saks–Schrijver ideals and coordinate sections of determinantal varieties". In: *Algebra & Number Theory* 13.2 (2019), pp. 455–484.

[Eis13] D. Eisenbud. *Commutative Algebra: with a View Toward Algebraic Geometry*. Graduate Texts in Mathematics. Springer New York, 2013.

[EH11] Viviana Ene and Jürgen Herzog. *Gröbner Bases in Commutative Algebra*. Vol. 130. American Mathematical Society, Jan. 2011.

[GS] Daniel R. Grayson and Michael E. Stillman. *Macaulay2, a software system for research in algebraic geometry*. Available at `http://www.math.uiuc.edu/Macaulay2/`.

[Her+15] J. Herzog et al. "On the ideal of orthogonal representations of a graph in R2". In: *Advances in Applied Mathematics* 71 (2015), pp. 146–173.

[Her+10] Jürgen Herzog et al. "Binomial edge ideals and conditional independence statements". In: *Advances in Applied Mathematics* 45.3 (2010), pp. 317–333.

[Lov19] L. Lovász. *Graphs and Geometry*. Colloquium Publications. American Mathematical Society, 2019.

[LSS89] L. Lovász, M. Saks, and A. Schrijver. “Orthogonal representations and connectivity of graphs”. In: *Linear Algebra and its Applications* 114-115 (1989). Special Issue Dedicated to Alan J. Hoffman, pp. 439–454.

[Lov79] László Miklós Lovász. “On the Shannon capacity of a graph”. In: *IEEE Trans. Inf. Theory* 25 (1979), pp. 1–7.

[MS04] E. Miller and B. Sturmfels. *Combinatorial Commutative Algebra*. Graduate Texts in Mathematics. Springer New York, 2004.

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