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Un approccio topos-teoretico alla logica quantistica

A topos-theoretic approach to quantum logic

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Abstract

In the thesis is discussed an approach to quantum logic, advocated especially by von Neumann as the logic underlying the foundations of quantum physics, in the formalism of category theory, in particular in terms of elementary topos, with main reference the papers of Doring and Isham.

Keywords: quantum mechanics, category theory, topos theory, C*-algebras, quantum logic

Abstract (italiano)

Nella tesi si discute di un possibile approccio alla logica quantistica, delineata da von Neumann come la struttura logica sottesa alla fondazione della meccanica quantistica. L'approccio proposto, che ha come riferimento principale gli articoli di Doring e Isham, utilizza il formalismo della teoria delle categorie e dei topoi elementari per riprodurre la struttura di calcolo proposizionale su contesti di misura quantististici.

Contents

Introduction

A physical reality is the realization of a rational discourse over an irrational matter. In this sense a physical theory may be seen as a logical structure expressing a perspective over an empirical object under the philosophical assumption of a defined realism. The foundations of a physical theory then enclose and clarify the manner in which the empirical, the philosophical and the formal structures agree each other.

Given this interpretation, a foundation for the Quantum Mechanics is extremely problematic by virtue of the fact that the reality emerging from its formalism and its empirical structure (discussed in chapter III) contrasts with the assumptions made by Einstein, Podolsky and Rosen [1] in the attempt of giving a realist model for physical theories. While this model depicts the reality as deterministic and independent from the context of measurement, Quantum Mechanics reproduces an indeterministic and deeply contextual informational theory (see uncertainty principle [11, sec.13] and [17, ch.12]).

The underlying matter of this debate consists in the affirmation of a possible existence of two environments arising in the quantum formalism, which generate two different informational theories. The first one, called in the present work *global context*, describes the formal object in which is contained the entire possible information of a physical system, a complete information for the system, whose algebraic definition arises in the Hilbertian formalism as related to the space of bounded operators $B(H)$ and to the algebra of projectors $P(H)$, as explained in III.2 and IV.4. The character of the ”global” informational theory, defined in the work of von Neumann and Birkhoff [7, 34], is that of a formal algebraic language structure built as over $P(H)$ seen as a lattice, where this construction shows critical issues with respect of the aforementioned realist conception and this because it reproduces a non-contextual but indeterministic informational theory. The second one, called *local context*, is the result of the attempt of building a mathematical object compatible with the operatorial Hilbertian environment and the hypothesis of the EPR realism. This object (described III.3) takes the form of the category $V(H)$ of commutative W*-sub-algebras in $B(H)$ (defined in II.5). Operators in these sub-algebras have the property of commuting with each other, de facto reproducing an environment in which the uncertainty principle has the same meaning as in classical theories. For this reason the local context, with an essentially different behavior from global one, may be seen as a semi-classical construction approaching the deterministic hypothesis but proposing a deeply contextual informational theory. The adoption of categorical representation, from the work of Isham and Döring [12, 18], allows to identify algebraic logic structures in this ”local” framework. Building the ”local” informational theory requires then a more general way of representing physical inferences than the classical one (see III.1) and a generalised algebraic environment in which it is possible to reproduce and study formal languages and their relations, identified with the *elementary topos* environment (treated in I). Results of this is an entire new approach to quantum mechanics and its problems, a categorical approach that brings in the physical debate additional mathematical tools as global elements, which in building relationships between local and global contexts are capable of demonstrating easily theorems over global and local differences as Kochen-Specker one (see III.4 and IV.6), and that furnishes an environment where complex mathematical structures expressed by the ordinary quantum formalism have a simpler representation.

Chapter I

Topos and logics: categorical preliminaries

A central role in this discussion belongs to Topos conception as a general way to reproduce an algebraic context sufficiently structured to manage the building of both logical structures and physical models.

I.1 Algebraic contexts as categories. Topos conception needs an a-priori general definition of *algebraic context* that condenses algebraic properties and a logical axiomatization of mathematical objects into a structuralist point of view, an *arrow-theoretic* vision of mathematical objects[1]. This definition has historically occurred to be the following, as in [3, 22].

Definition I.1.1 (Category) A **category** C is an algebraic structure with:

1. a collection of *objects*, denoted C_0 ;
2. a collection of *arrows* between objects, denoted C_1 ;
3. an associative *operation of composition* over C_1, denoted $\circ_C$, such that given any arrow $f \in Hom_C(X,Y)$ and $g \in Hom_C(Y,Z)$ [2], the composition of these arrows is the arrow $g \circ_C f \in Hom_C(X,Z)$;
4. for all $X \in C_0$ there exists an *identity arrow* $id_X \in Hom_C(X,X)$, such that for $f \in Hom_C(Z,X)$ and any $g \in hom_C(X,W)$ it follows: $id_X \circ_C f = f$ and $g \circ_C id_X = g$.

Example. Important examples of categories are the following:

- the category Set whose objects are sets and arrows are functions between sets, axiomatized by Zermelo-Fraenkel theory;
- given an *enriched set* (A,S) , where A is a set and S is an algebraic structure on it, then there exists a category C whose objects are these enriched sets and arrows, called S*-homomorphisms*, are functions between sets which preserve S structures, i.e. which map S-structure in S-structure[3];
- the category Top whose objects are topological spaces and arrows are continuous functions between topological spaces;
- the category Pos whose objects are partially ordered sets[4] (or posets) and arrows preserve poset structure. It is notable the fact that each poset $(P,\leq)$ is itself a category whose objects are the elements of P and for each couple $x,y \in P$ there exists a unique arrow from x to y if $x \leq y$.

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[1] In this discussion the term *structure* refers to a defined collection or composition of arrows or formalized relationships built *on* an object, see [9, pp. 313-318]

[2] In a category C given two objects $X,Y \in C_0$ then the set $Hom_C(X,Y) \subseteq C_1$ is the set of all arrows from x to y in C.

[3] Important examples of these categories are *Grp*, the category of groups, and *Ring*, the category of rings

[4] A *poset* is a set P equipped with a reflexive, transitive and anti-symmetric binary relation

- the category *Cat* whose objects are small categories[5] and arrows are called *covariant functors* between categories. A covariant functor $F \in Hom_{Cat}(C,D)$ has the following definition:
  1. F maps each object $X \in C_0$ into an object $FX \in D_0$;
  2. F maps each arrow $f \in Hom_C(X,Y)$ into an arrow $Ff \in Hom_D(FX,FY)$, preserving identities arrows and compositions (i.e. preserving the category structure).
- given a small category $C \in Cat_0$ and any category D, the category D^C or $[C,D]$ is the category whose objects are covariant functors $F : C \to D$ and whose arrows are called *natural transformations* between functors. A natural transformation $\zeta \in Hom_{[C,D]}(F,G)$ has the following definition:
  1. to every $X \in C_0$ the transformation ζ associates an arrow $\zeta_X \in Hom_D(FX,GX)$;
  2. for every $f \in Hom_C(X,Y)$ it follows that: $\zeta_Y \circ_D Ff = Gf \circ_D \zeta_X$.

An important property that needs to be mentioned in the categorical framework is called **duality** (see [3]). Any property of a given category C corresponds to a dual property for the *opposite category* C^{op}, which is the category with same objects as C but with all arrows reversed. Therefore given a structure S in C, the act of reversing its arrows generates a *co-structures* S^{op} that acquires in C^{op} the same role and characterization of S in C. This allows the definition of *contravariant functors* as funtors $F : C \to D$ that map arrows of C but reversing them, i.e. they are covariant functors but in $[C^{op},D]$.

A category as formalization of an algebraic context could be characterized via its structure in *Cat*, i.e. via its arrow-relationship with other categories. But the essential nature of a category is caught in the existence of special objects in the category. This special objects called **limits** are objects that *universally* embody structural properties of the category. In this context *universality* coincides with the concept of *arrow-theoretic uniqueness*, where this formally means that: given a object $A \in C_0$ with a structure s of type S[6], A is said to be universal with respect to S, or to have a *universal mapping property* (UMP) with respect to S, if for any objects $Z \in C_0$ equipped with a structure s' of type S, there is a *unique* arrow $u \in Hom_C(Z,A)$ such that the u preserves the S-structure[7].

Example. Important and clarifying limits that could exist in a category are the following:

- *initial object*, denoted by $\mathbb{O}$, which embodies the UMP of "being source of arrow", formally: $\forall Z \in C_0 : \ \exists! u \in Hom_C(\mathbb{O}, Z)$;
- *final object*, denoted by $\mathbb{I}$, which embodies the UMP of "being target of arrow", formally: $\forall Z \in C_0 : \ \exists! u \in Hom_C(Z, \mathbb{I})$ [8];
- *product* of two objects $A, B \in C_0$ is an object denoted $A \times B$ and equipped with a structure of two *projection* arrow $\pi_A \in Hom_C(A \times B, A)$ and $\pi_B \in Hom_C(A \times B, B)$, where the UMP is formalized as :

  $$\forall Z \in C_0 \ with \ \phi_A \in Hom_C(Z,A), \phi_B \in Hom_C(Z,B) :$$

  $$\exists! u \in Hom_C(Z, A \times B) \ \ such\ that \ \ \phi_A = \pi_A \circ_C u \ and \ \phi_B = \pi_B \circ_C u$$

  where this construction could be generalized to any finite n-tuple of objects in C.

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[5] A *small category* is a category where C_0 and C_1 are sets.

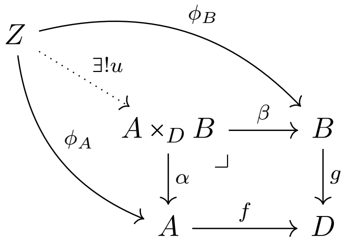
[6] In this context a *structural type* S is the characterization logical or algebraic of a structure over A. This concept allows building the same structure over different objects.

[7] For a more formal discussion see [3, ch.5].

[8] Final objects are co-structures of initial objects and they both might be seen as all limits and co-limits formal structure, as universality is a non trivial construction of final objects in ad-hoc built categories, see [3, sec.5.4].

I.2 A definition for elementary topos. Finding a category characterization for which it is possible a representation of logical structures is therefore a search for right objects with specific limit's structure that communicate a sort of *regularity* for the context. The attempt is specifically to insulate and reproduce weakly the *Set* category nature, recognised to be an ideal category to work in. For this are fundamental the following definitions.

Definition I.2.1 (Pullback) Given a category C and a diagram made by three objects $A, B, D \in C_0$ with two arrows $f \in Hom_C(A, D)$ and $g \in Hom_C(B, D)$, a *pullback* of this diagram is an object denoted $A \times_D B$ equipped with two arrows $\alpha \in Hom_C(A \times_D B, A)$ and $\beta \in Hom_C(A \times_D B, B)$, with $f \circ_C \alpha = g \circ_C \beta$, such that the following UMP holds: for every $Z \in C_0$ equipped with two arrows $\phi_A \in Hom_C(Z, A)$ and $\phi_B \in Hom_C(Z, B)$, with $f \circ_C \phi_A = g \circ_C \phi_B$, there exists a unique arrow $u \in Hom_C(Z, A \times_D B)$ such that the following diagram commutes:



Proposition I.2.1 Given a category C, then in C there exist a terminal object and all pullbacks if and only if all finite limits exist, i.e. all limits involving finitely many objects and arrows[9].

The definition of pullback and specifically the reported proposition are related to a fact: the categorical environment and its main matter are structural extensions of the category *Set*, thus the concept itself of limit come directly from the properties and the existence of peculiar objects in *Set*, most of them represented by finite limits. The most powerful character of regularity of an algebraic context is therefore the possibility of reproducing in itself objects with this nature and the existence of pullbacks communicate and insulates this regular behavior of *Set*.

Definition I.2.2 (Exponential object) Given a category C with binary products for all pair of objects, given two objects $X, Y \in C_0$, an **exponential object** of Y in X is an object $X^Y \in C_0$ equipped with an arrow $e \in Hom_C(X^Y \times Y, X)$, called *evaluation*, with the following UMP: $\forall Z \in C_0, \forall g \in Hom_C(Z \times Y, X)$:

$$\exists! u \in Hom_C(Z, X^Y) \;\; \textit{such that} \;\; e \circ_C (u \times id_Y) = g$$

where the arrow-product is an arrow that acts component-wise on a product.

Such definition encloses the categorical generalization of *functional spaces* or arrow-spaces. In fact in *Set* the exponentiation of objects acts as $\_^Y = Hom_{Set}(Y, \_)$ and $e(f, y) = f(y)$. Thus for categories with exponential objects spaces of arrows between objects are themself objects.

Definition I.2.3 (Sub-object) Given a category C and and object $X \in C_0$ and given two monic arrows[10] $f \in Hom_C(A, X)$ and $g \in Hom_C(B, X)$ they are said to be:

$$f \leq g \;\; if \;\; \exists h \in Hom_C(A, B) \;\; monic \; such \; that \;\; f = g \circ_C h$$

where f and g are said to be *isomorphic* if h is an isomorphism[11]. Then the class of all monic arrows

[9]For the formal proof see [3, prop.5.23]

[10]A *monic arrow* is an arrow $\iota \in Hom_C(A, X)$ with the following property: $\forall Z \in C_0$ *and* $\forall f, g \in Hom_C(Z, A)$: $\iota \circ_C f = \iota \circ_C g \implies f = g$. This concept extends injectivity idea from *Set* to every category.

[11]An *isomorphism* is an arrow $f \in Hom_C(A, X)$ for which exists a unique inverse arrow $f^{-1} \in Hom_C(X, A)$ such that $f \circ_C f^{-1} = id_A$ and $f^{-1} \circ_C f = id_X$.

to X quotiented[12] via isomorphism is denoted with $Sub_C(X)$, where an isomorphic class of monic arrows is called a **sub-object** of X in C.

Such definition encloses the categorical generalization of *subsets* especially for categories with object without global elements (see I.5). It is important to underline that in *Set* the concept of subobject and *inclusion* arrow are categorically the same, i.e. the subset is the inclusion and not the set included.

Definition I.2.4 (Sub-object classifier) Given a category C with all pullbacks, a terminal object and an object $\Omega \in C_0$ called *truth object*, then a **sub-object classifier** is a monic arrow $\iota \in Hom_C(\mathbb{I}, \Omega)$ such that for every $X, Y \in C_0$ and every monic arrow $s \in Hom_C(X, Y)$ there exists a unique $u \in Hom_C(Y, \Omega)$ that makes the following diagram a pullback:

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{I} \\ \downarrow{\scriptstyle s} & \lrcorner & \downarrow{\scriptstyle \iota} \\ Y & \xrightarrow{\exists ! u} & \Omega \end{array}$$

Finally such definition encloses the categorical generalization of *characteristic function*[13]. Such generalisation is fundamental for the construction of *valuation process* or logically for propositional calculus formalization, from where the truth object is named. A direct consequence of the existence of a sub-object classifier is the following (see [22, prop. 1 at page 33]).

Proposition I.2.2 Given a category C with all finite limits and small Hom-sets, then C have a sub-object classifier Ω if and only if for each $X \in E_0$ there exists an isomorphism natural in X :

$$Sub_C(X) \cong Hom_C(X, \Omega)$$

Summarizing regularity for categories as a weak *Set* characterization, the following *regular context* arises.

Definition I.2.5 (Elementary topos) An **elementary topos** E is a category that :

1. is *finitely complete*, i.e. it has all finite limits;
2. is *Cartesian closed*, i.e it has exponential objects for each couple of objects;
3. has a sub-object classifier.

For the complexity of the matter, this definition is not unique and could be formulated in various ways, making different properties more clear (see [22, ch.IV]).

For the proposition I.2.2, in the topos-environment there exists a canonical process of *internalization* for the spaces $Sub_C(X)$. In fact in a generic category C one can conceptually defines *externally* the concept of space of sub-objects for an object X, but in general such an external account of the powerset has not an internal representation in C. However, when C is a topos, one can consider the *power object* of an object X defined as the exponential object Ω^X, which is an adequate internalization of $Sub_E(X)$, since the definition of subobject classifier determines a bijective correspondance between subobjects of X and arrows from X to Ω, and $Hom_E(X, \Omega)$ can be internalized as Ω^X by means of the exponential construction.

Notice also that the correspondence between $Sub_E(X)$ and $Hom(X, \Omega)$ is well-behaved with respect to arrows of , that is, from the sub-object classifier definition, given any other object Y and any $f \in Hom_E(X, Y)$, the following diagram with displayed pullbacks

[12] This construction, which is foundationally problematic, can be carry out by using the so-called *Scott's trick*, see [31].

[13] In *Set* the truth object is $\Omega = \{0, 1\}$ and the final object is $\mathbb{I} = \{*\}$, i.e. a singleton.

$$\begin{array}{ccccc} & \longrightarrow & & \longrightarrow & \mathbb{I} \\ {\scriptstyle x}\big\downarrow & \lrcorner & {\scriptstyle y}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \iota} \\ X & \xrightarrow{f} & Y & \longrightarrow & \Omega \end{array}$$

shows that the pullback of subobjects (which represents the counterimage operation) corresponds with pre-composition once read in terms of arrows towards Ω.

For further details on the so-called *power-object* construction see [22].

I.3 A concrete topos construction. Not all categories are topoi. This forces to seek a general way to associate a topos category to any category. For the purpose of the present discussion this process could be restricted to only small categories[14], where it holds:

Definition I.3.1 (Presheaf on a category) Given a small category C, a **presheaf** P on C is a controvariant Set-valued functor, i.e. $P \in Hom_{Cat}(C^{op}, Set)$.
Presheaves on a category C with their natural transformations form the functor category $[C^{op}, Set]$.

Proposition I.3.1 (Topos of presheaves on a category) Given a small category C, the presheaves category $[C^{op}, Set]$ has a natural elementary topos structure.[15]

The construction comes clear as a representation of a category into a *Set* environment, which inherits *Set* nature of topos.

Given the regular context construction and the way of associating such structure to categories, it needs to be clarified how logical structures belongs to topoi and what category might be associated to a quantum environment model.

I.4 Algebraic structure for logics. For the purpose of this discussion it becomes fundamental to understand how *propositional calculus* and *typed formal languages* might be represented into algebraic structures and what is the place of this structures into categorical environment.

Exploring the role and the sense of a propositional calculus amounts to explain the nature itself of the present discussion. A propositional calculus is a logical theory, a set of rules, over a collection of linguistic constructions called *propositions*, which are said to have a defined *value of truth.* The definition of the above-mentioned value represents the raison d'être of logical theories, where a propositional calculus is then a process of evaluation over linguistic structures, or more precisely over a *space of propositions.*
The focus of this discussion converges then on the categorical characterization and physical identification of such spaces on which it can be build a propositional calculus. A space of propositions is an algebraic environment closed with respect to the "classical" logic operators representing the connectives $\wedge$,$\vee$, $\neg$ and $\Rightarrow$, where it's important to notice that any characterization of a logical theory consists of a relational theory between these connectives, a reproduction of all connectives using only a few of them.
In *intuitionistic logics* the connectives of interest are only $\wedge$,$\vee$ and $\Rightarrow$, therefore the following algebraic structures' definition hold.

Definition I.4.1 (Lattice) A **lattice** l is a finitely complete and co-complete poset, where binary products and binary co-products represent respectively connectives $\wedge$ and $\vee$.
A lattice l is said to be:

[14] The majority of categorical structures treated in this discussion are poset categories, which are important examples of small categories.
[15] For a detailed discussion see [22].

1. *bounded* if it has initial and final objects such that for every $x \in l_0$ it follows $0 \leq_l x \leq_l 1$;
2. *distributive* if connectives $\wedge$ and $\vee$ are distributive.

A lattice then reproduces an algebraic context that interprets perfectly $\wedge$ and $\vee$ as operations over objects, but the implication definition requires a stronger characterization.

Definition I.4.2 (Heyting algebra) An **Heyting algebra** is a Cartesian closed, bounded and distributive[16] lattice l, where the exponential object of $x, y \in l_0$ is denoted by $x \Rightarrow y$. In particular:

$$\forall z \in l_0 : \quad z \leq_l (x \Rightarrow y) \quad if\ and\ only\ if \quad z \wedge x \leq_l y$$

In any Heyting algebra, as an intuitionistic environment, the negation connective is defined via implication as for any proposition $\psi \in l_0$:

$$\neg\psi \equiv (\psi \implies \bot)$$

In such environment the negation is not necessarily involutive. Such a property refers in fact to the inclusion of the axiom of *reductio ad absurdum* [17] that belongs only to classical propositional calculi. Therefore there exists a peculiar class of Heyting algebras, called *Boolean algebras*, characterized by the internal definition of a $\neg$ operation as a unary involutive operation such that:

$$\forall x \in l_0 : \quad x \wedge \neg x = 0 \quad and \quad x \vee \neg x = 1 \quad .$$

From their definition Heyting algebras seem to be deeply related to topoi, as Heyting algebras have a "weaker structure" than the one of topoi. Relations between this two algebraic environments are shown clearly in the following statement (see [22, pp. 198-204]).

Proposition I.4.1 (Internal logic for a topos) Given a topos E and an object $X \in E_0$, the poset $Sub_E(X)$ has an Heyting algebra structure l, where the posetal structure $\leq_l$ coincides with the ordering structure defined in I.2.3.

Example. As an important example of this, $Sub_{Set}(X)$ has a canonical Boolean algebra structure given by the operations $\cap$, $\cup$ and the complement that respectively reproduce the operations of $\wedge$,$\vee$ and $\neg$.

From these results, a topos-theoretical approach is an exceptional way to recognise spaces of propositions over objects in a specific algebraic context. In the opposite point of view, if a collection of entities (for example physical ones) are recognised to belong to a topos structure, then propositional theories could be built on such entities. Starting from this point, it is clarifying to understand what a topos structure stands for in a logical conception.

A topos reproduces an *interpretation* for a *typed formal language*, where conceptually this interpretation consists of the following correspondences:

- objects in the topos are *types*, where a type could be defined as a symbolic or linguistic space, where this means that a type encloses the idea of a closed informational structure, a "kind" or a "way" in which the information could be expressed;

[16] The distributive property is not necessary because it could be demonstrated by the other assumptions. In fact for every $x, y, z \in l_0$ it follows that:

- $x \wedge (y \vee z) \geq_l (x \wedge y) \vee (x \wedge z)$ from the connectives definitions;
- $y \vee z \leq_l x \Rightarrow (x \wedge y) \vee (x \wedge z)$ from implication definition;

and then the equality is obtained from the properties of implication (and similar for $\vee$ distributivity).

[17] Also called *negation introduction* this axioms states that given a list of propositions Γ and a proposition ψ then: $\frac{\Gamma, \neg\psi\ \vdash \bot}{\Gamma \vdash \psi} ra$.

- arrows in the topos are *terms*, where a term could be conceptually associated to a "context" for logical structures. As category theory is an arrow-theoretical approach to mathematical matter, in a topos-theoretical interpretation of logical entities what really matter is the terms' structure, where the idea of "inference" or "proposition" lies upon the idea of a relation between closed informational structures, as a projection or an act of belonging of part of a structure to the others (see IV.1 for a physical interpretation of this).

The latter proposition combined with the existence of a sub-object classifier specifies that propositions are somehow special among terms. A proposition in fact encloses specifically the idea of a "belonging", of an inclusion, and this from the identification with sub-objects. Each types is then associated with its own propositional space, where this recovers the idea of a logical closure for a single type and where the act of mapping each type to its propositional structure is the basic idea for the edification of the *hyperdoctrines of the first order*, i.e. the covariant functor $Sub : E^{op} \to Hey$ from a topos E to the category Hey of Heyting algebras (a sub-category of Pos) is the basic structure of a categorical conception of formal languages (see [23, ch. 16-18]).
As a final glimpse on topos as a propositional environment, the existence of a truth value object Ω and a sub-object classifier in a topos E communicates the idea that a topos is a context in which propositional evaluations, as arrows to the truth value object, and spaces of propositions, with the property $Sub_E(X) \cong Hom_E(X, \Omega)$ for every object X, are both internal. This means that a topos could be seen as an evaluation context for spaces of propositions built over logical closed environments as types are.

A topos as a typed formal language interpretation is also capable of reproducing *predicative* structures over terms making deeper the relation between topos environment and logic one, but this kind of further structuration is out of the interest for this discussion[18].

I.5 Global elements. The last categorical concept useful for the present discussion is that of *global element*.

Definition I.5.1 (Global element) Given a category C with a terminal object $\mathbb{I}$, then a **global element** of an object $X \in C_0$ is an arrow $\gamma(X) \in Hom_C(\mathbb{I}, X)$. We denote with $\Gamma(X)$ the collection of global elements of X.

To understand this definition it is useful to look at the situation in Set, where the terminal object is the singleton $\{*\}$ and a global element of a set $X \in Set_0$ represents conceptually the action of isolation of an element $x \in X$. Therefore a global element is the categorical generalisation of element's isolation as deprivation of all membership relations, or simply the concept of element itself.

In the category of interest for this discussion, i.e. $[C^{op}, Set]$, limits and co-limits are built object-wise[19] and then $\mathbb{I}_{[C^{op},Set]} : C^{op} \to \{*\}$ is the presheaf that associates each object in C to the singleton $\{*\}$. A global element $\gamma(P)$ of a presheaf P on C is a natural transformation $\mathbb{I}_{[C^{op},Set]} \to P$ in Fun category that for each $X \in C_0$:

1. the object $\gamma_P(X) \in P(X)$;
2. for each $f \in Hom_C(Y, X)$ it follows $P(f)(\gamma_P(X)) = \gamma_P(Y)$.

Physically speaking globality or non-contextuality could be seen as the isolation of an element of the theory from all contexts to which it belongs as explained in IV.6.

[18] For a formal and complete construction of topoi as types formal langauges see [19,22,23].

[19] A limit or co-limit, with index category J, in the category of presheaves on a category C follows for each $X \in C_0$ the equivalence: $(\lim_{j\in J_0} F_j)(X) \cong \lim_{j\in J_0}(F_j(X))$, see [3,22].

Chapter II

C*-algebras and von Neumann algebras

The attempt of categorically reproduce propositional structures over a physical quantum system requires an algebraic model capable of embodying a "closed" informational structure compatible with the operatorial environment introduced by quantum mechanics formalism. The algebraic model of interest comes to be the *C*-algebra* one for the reasons explained in the succeeding chapters, where a reliable role in this discussion belongs to the relation between this algebraic context with the complex and real environments, as a physical theory over quantities might be seen as a reproduction of a physical informational structure into $\mathbb{R}$, specifically due to the canonical ordering structure of this mathematical entity.

II.1 The definition of C*-algebras. A way of realizing a connection between algebraic structures consists in recognizing common parts of these structures that belong to the same algebraic context, i.e. the same category. Given then an object with interesting properties, decomposing and insulating the part of its structure related to these properties is then a way to reproduce abstractly the properties themselves. Finally recognizing or reconstructing this structure in other objects allows to make a representation of these properties and to build relations between the given object and the others. An example is given in the first chapter, where *Set* from its construction is in a natural way a reproduction of an algebraic logical environment, insulating then the "logical kind" structure from *Set* it is possible to obtain a topos structure, which embodies the logical structure itself.

In physical environments (especially for physical informational theories) two fundamental algebraic objects are $\mathbb{R}$, for its ordering and topological structures, and $\mathbb{C}$, especially for its algebraically closed field structure. Since $\mathbb{R} = \{z \in \mathbb{C} \mid a = \bar{a}\}$, then insulating the algebraic structure of $\mathbb{C}$ becomes indispensable for such theories.

Definition II.1.1 (*-ring and *-algebra) A ***-ring** is an algebraic structure $R = (\underline{R}, +_R, \cdot_R, 0_R, 1_R, *)$ where $(\underline{R}, +_R, \cdot_R, 0_R, 1_R)$ is a unital ring and $* : \underline{R} \to \underline{R}$ is a function with the property of being an antiautomorphism and an involution (i.e. an anti-involution), that is for every $x, y \in \underline{R}$:

1. $*(x +_R y) = *(x) +_R *(y)$;
2. $*(x \cdot_R y) = *(y) \cdot_R *(x)$ (antiautomorphism);
3. $*(1_R) = 1_R$ and $*(*(x)) = x$ (involution).

A ***-algebra** A over a unital commutative *-ring R is an algebraic structure $A = (\underline{A}, +_A, 0_A, \cdot_A, \times_A, 1_A)$, where:

1. the structure $(\underline{A}, +_A, 0_A)$ is an Abelian group;
2. the map $\cdot_A : \underline{R} \times \underline{A} \to \underline{A}$, called *scalar multiplication*, satisfies the conditions that for every $r, s \in \underline{R}$ and $x, y \in \underline{A}$:

- $r \cdot_A (x +_A y) = (r \cdot_A x) +_A (r \cdot_A y)$;
- $(r +_R s) \cdot_A x = (r \cdot_A x) +_A (s \cdot_A x)$;
- $(r \cdot_R s) \cdot_A x = r \cdot_A (s \cdot_A x)$;
- $1_R \cdot_A x = x$;

The Abelian gruop equipped with this operation forms an *R-module*[1];

3. the structure $(\times_A, 1_A)$ is a monoid, where the map $\times_A : \underline{A} \times \underline{A} \to \underline{A}$, called *multiplication*, satisfies the condition that for every $r \in \underline{R}$ and $x, y \in \underline{A}$:

$$r \cdot_A (x \times_A y) = (r \cdot_A x) \times_A y = x \times_A (r \cdot_A y) \quad ;$$

The R-module equipped with this monoidal structure forms a *unital associative algebra* over R;

4. the map $*_A : \underline{A} \to \underline{A}$ is an anti-involution compatible with the algebra structure, i.e. for every $r \in \underline{R}$ and $x \in \underline{A}$ it holds: $*_A(r \cdot_A x) = *_R(r) \cdot_A *_A(x)$.

In this discussion it is assumed that *-rings and *-algebras are *unital*, where multiplicative unit requirement is a strong hypothesis over these constructions that brings sufficient regularity to get propositions as Gel'fand duality (cfr. [26]).

Example. Important examples of *-rings and *-algebras are the following:

- the *complex *-ring* over $\mathbb{C}$ where:

  1. the ring structure is given by the canonical definition of $\mathbb{C}$ as a field, with complex sum and multiplication;
  2. the anti-involution structure is given by the conjugate operation $*$ over $\mathbb{C}$;

  As every ring is an algebra over itself, this algebraic structure is trivially also a *-algebra over $\mathbb{C}$, called *complex *-algebra* and denoted $(\mathbb{C}, *)$, where the monoid structure is given by the complex multiplication and the anti-involution is the complex conjugate;

- **-algebras of square matrices*, where given $n \in \mathbb{N}$ and the space $M_{n\times n}(\mathbb{C})$ of complex square matrices of dimension n :

  1. the associative algebra structure is given by the vector space structure of $M_{n\times n}(\mathbb{C})$ over $\mathbb{C}$ equipped with the canonical multiplication of matrices;
  2. the anti-involution structure is given by the hermitian conjugate transpose operation over $M_{n\times n}(\mathbb{C})$.

This structuration as a pure algebraic one lacks of basic metric and topological characterizations of the objects involved, fundamental for a $\mathbb{C}$-valued informational theory. For this purpose it comes into question the nature of the environment into which it is to be build the structure that is researched, i.e. the operatorial formalism over Hilbert spaces (see III.2). For reconciling the structure with this environment, it could be added a Banach space[2] structure, which gives a metric characterization for operator spaces[3].

Definition II.1.2 (C*-algebra) A **C*-algebra** A is a *-algebra over the *-ring of complex numbers equipped with a Banach space structure with a norm $\|\cdot\|_A : M \to \mathbb{R}$ compatible with the *-algebra structure, namely:

$$\forall x \in \underline{A} : \quad \| *_A (x) \times_A x \|_A = \|x\|_A^2 \quad .$$

[1] If R is a field, an R-module is called *R-vector space.*

[2] A *Banach space* is a normed vector space $(X, \|\cdot\|)$ that is complete with respect of the metric induced by the norm, i.e. the distance $d(x,y) = \|x - y\|$ with $x, y \in X$.

[3] See [5].

The C*-algebra model is then capable of reproducing the algebraic structure of $\mathbb{C}$ into an operatorial environment.
Example. The most reliable example of C*-algebras are the following:

- given the complex *-algebra $(\mathbb{C}, *)$, then the *complex C*-algebra*, denoted simply $\mathbb{C}$, is obtained adding the Banach space structure induced by the norm $|\cdot|_{\mathbb{C}}$, i.e. the complex absolute value;
- given an Hilbert space H[4], the space of all bounded linear operators on H, called $B(H)$, forms a C*-algebra where:
  1. the associative algebra structure is given by the complex vector space structure (the same as $M_{n\times n}(\mathbb{C})$) and the operation of associative composition of operators;
  2. the anti-involution structure is given by the ad-joint operation, denoted by †;
  3. the Banach space structure is given by the operator norm, i.e. for any $x \in B(H)$ the norm $\|x\|_{B(H)} = sup_{\mathbb{R}}\{\|x(\psi)\|_H \in \mathbb{R}^+ \mid \|\psi\|_H = 1 \; for \; \psi \in H\}$; [5]
- given a compact Hausdorff topological[6] space X, the space of all continuous complex-valued functions on X, called $\mathbf{C}(X)$, forms a C*-algebra (see [8, ch.II]) where:
  1. the associative algebra structure is given by the complex pointwise addition, multiplication and scalar multiplication[7];
  2. the anti-involution structure is given by complex pointwise conjugation;
  3. the Banach space structure is given by the uniform norm, i.e. for any $f \in \mathbf{C}(X)$ the norm: $\|f\|_\infty = sup_{\mathbb{R}}\{|f(x)|_{\mathbb{C}} \mid x \in X\}$;

  This example shows a possible connection between C*-algebra and compact topological spaces, a connection that is the main matter of Gel'fand theory, treated later in this section.

II.2 Categories of C*-algebras. Before building any kind of representation theory over C*-algebras, such structures need to be framed into their algebraic context and this makes arise to the *C*-algebras category.*

Definition II.2.1 (C*-algebras category) The **C*-algebras category**, denoted C^*Alg , is the category whose objects are C*-algebras and whose arrows are called *C*-homomorphisms.* Such arrows have to preserve all structures that form a C*-algebra and so a C*-homomorphism $\phi \in Hom_{C^*Alg}(X,Y)$ needs to be:

1. *linear* to preserve the vector space structure, or for every $a, b \in X$ and $\alpha, \beta \in \mathbb{C}$ it follows that: $\phi(\alpha \cdot_X a +_X \beta \cdot_X b) = \alpha \cdot_Y \phi(a) +_Y \beta \cdot_Y \phi(b)$;
2. *multiplicative* to preserve the algebra structure, or for every $a, b \in X$ it follows that: $\phi(a \times_X b) = \phi(a) \times_Y \phi(b)$;
3. a **-homomorphism* to preserve anti-involution structure, or for every $a, b \in X$ it follows that: $\phi(*_X(a)) = *_Y(\phi(a))$;
4. an *isometry* as a condition to preserve the Banach space structure, or for every $a, b \in X$ it follows $\|a -_X b\|_X = \|\phi(a) -_Y \phi(b)\|_Y$.

[4] A *complex Hilbert space* is a complex vector space that is complete with respect of the metric induced by an *inner product* $(\cdot,\cdot)_H : H \times H \to \mathbb{C}$, which is linear, positive and conjugate symmetric. The inner product induces a norm over the space, where for any $x \in H$ the norm is $\|x\|_H = \sqrt{(x,x)_H}$. Any Hilbert space is then a Banach space.

[5] An important property of B(H) algebras is the fact that they are not commutative for $dim(H) \geq 2$ (cfr. [21, 24])

[6] An Hausdorff topological space is a topological space (X, τ) where points are separated by neighbourhoods, i.e. given any two points $x, y \in X$ there exist two open sets $U, V \in \tau$ that are respectively neighbourhood of x and y and that are disjoint, see [35, ch.5]. A compact topological space is a topological space (X, τ) where every open cover of X has a finite sub-cover, where an open cover for X is a family of open sets $\{U_i\}_{i\in J} \subseteq \tau$ such that $\bigcup_{i\in I} U_i = X$, see [35, ch.6].

[7] Given an operation $\phi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ it could be extended to a *pointwise operation* $\Phi : C(X) \times C(X) \to C(X)$ such that for any $f, g \in C(X)$: $\Phi(f,g)(\_) = \phi(f(\_), g(\_))$.

By the definition of C*-algebras and their category, it comes clearly that topological properties of C*-algebras depend strictly on the environment to which these algebras belong and not by the definition of C*-algebra itself. For this reason in this discussion it will be always specified the topological structure assumed. Thinking categorically, as the metric of a Banach space induces a metric topology[8], then each C*-algebra is naturally a *topological vector space* (or TVS, see [16, sec.1.5]) and a topological space. This requires a better understanding of functors of the form $C^*Alg \to Top$, given lately in this section.

As shown in the following proposition, C*-algebras are also naturally associated with partially ordered sets (see [25, cor.2.2]).

Proposition II.2.1 (C*-algebras as categories) Given the category C^*Alg and Pos the category of posets, then there exists a functor:

$$\eta : C^*Alg \to Pos$$

which sends:

1. any C*-algebra A to the poset $\eta(A)$, whose objects are all commutative sub-algebras of A, i.e. the sub-algebras with a commutative multiplication $\times$, and whose arrows are inclusions of sub-algebras.
2. any arrow $f \in Hom_{C^*Alg}(A, A')$ to $\eta(f) \in Hom_{Pos}(\eta(A), \eta(A'))$ that maps every commutative subalgebra B of A to its image $f(B)$, which is a commutative subalgebra of A'.

II.3 Spectra and Gel'fand duality. Established the model and the its algebraic context, it is now important to understand the way a C*-algebra might interact with $\mathbb{C}$ and its structures. From the definition of C*-algebra if follows that any object A of $(C^*Alg)_0$:

1. is a complex vector space, i.e. $A \in (Vect(\mathbb{C}))_0$ [9], and then the set $A^* = Hom_{Vect(\mathbb{C})}(A, \mathbb{C})$ is called *algebraic dual space* of A and its elements are called *linear functional* over A. This space is naturally a complex vector space;
2. might have a topological space structure, i.e. $\mathbb{A} = (A, \tau) \in (Top)_0$, and then the set $\mathbb{A}' = A^* \cap Hom_{Top}(\mathbb{A}, \mathbb{C})$ [10] is called *continuous dual space* of $\mathbb{A}$. In this construction the topology choice defines the relation between the algebraic and the continuous dual spaces; if $\mathbb{A}' = A^*$ then all linear functionals are also countinuous. As continuity is preserved if the topology becomes weaker[11], then requiring the condition $\mathbb{A}' = A^*$ is equivalent to chose the weakest topology that makes all linear functionals continuous, called *weak topology* on A.
3. has a set called *Gel'fand spectrum*, denoted $\sigma(A)$, whose elements, called the *characters* of A, are the non-zero C*-homomorphisms $\phi \in Hom_{C^*Alg}(A, \mathbb{C})$. It follows that $Hom_{C^*Alg}(A, \mathbb{C}) \subseteq \mathbb{A}'$ for the weak topology, as C*-homomorphisms are isometries that preserves the metric topologies, which are stronger than the weak topology.

To understand the relation between a C*-algebra and its own Gel'fand spectrum, the space A^* must be topologically characterized. Given the double dual space $A^{**} = Hom_{Vect(\mathbb{C})}(A^*, \mathbb{C})$, then there exists an injective homomorphism $\Gamma \in Hom_{Vect(\mathbb{C})}(A, A^{**})$ that maps every $x \in A$ into a linear functional $\Gamma_x \in A^{**}$ such that $\Gamma_x(\phi) = \phi(x)$ for every $\phi \in A^*$. The weakest topology on A^* that makes Γ_x continuous for every $x \in A$ is called *weak-*topology* (for more details see [16, sec.2.8 and 2.9]).

[8] Given a metric space (X, d) , the *metric topology* on X induced by d has as open sets the balls $O(y, c) = \{x \in X \mid d(y, x) \leq c\}$, with $y \in X$ and $c \in \mathbb{R}$.

[9] $Vect(\mathbb{C})$ is the category whose objects are all complex vector spaces and whose arrows are linear functions between these spaces.

[10] This expression has a meaning in *Set* and this because $Vect(\mathbb{C})$ and Top are both *locally small* categories, i.e. each collection of arrows between two objects forms a set.

[11] Given two topologies $\tau_1, \tau_2 \in Set_0$, if it follows that $\tau_1 \subseteq \tau_2$ then τ_1 is said to be *weaker* than τ_2. From this if a function f is continuous for the topology τ, then f is continuous on its subsets and so for any topology weaker than τ .

In this topological framework the Gel'fand spectrum $\sigma(A)$ forms a locally compact Hausdorff space and then the space $\mathbf{C}(\sigma(A))$ equipped with pointwise operations and the uniform norm forms a C*-algebra. As the inherited C*-algebra structure on $\mathbf{C}(\sigma(A))$ is commutative, it is possible to think about a duality between commutative C*-algebras and compact Hausdorff topological space, recovering the following results (see [2, 8, 24, 26]).

Proposition II.3.1 (Gel'fand isomorphism theorem) Given a commutative C*-algebra A there exists an isomorpshism in the category C^*alg from A to $\mathbf{C}(\sigma(A))$.
Proof. Let $\omega \in \sigma(A)$ and claim that for every $x \in A$ it follows $\omega(*(x)) = \omega(x)^*$. This is equivalent to claim that when $x = *(x)$ it follows $\omega(x) \in \mathbb{R}$. Chosen then $x = *(x)$ and $t \in \mathbb{R}$ it can be defined $u_t = e^{itx} = \sum_{n=0}^{\infty} \frac{(itx)^n}{n!}$. From the power series definition, it follows that $u_t^* = e^{-itx}$ and then $\|u_t\|^2 = \|u_t^* u_t\| = 1$. Since ω has unitary norm one has that $e^{tRe(i\omega(x))} = |e^{\omega(itx)}| = |\omega(u_t)| \leq 1$ for every $t \in \mathbb{R}$ and then $Re(i\omega(x)) = 0$ with ω real. Let now define the map $\gamma(x)(\omega) = \omega(x)$, where $\gamma(*(x)) = *(\gamma(x))$ for every $\omega \in \sigma(A)$. One has to claim that $\|\gamma(x)\| = \|x\|$ and by the Gel'fand spectral radius formula[12] the left side is equal to the spectral radius of x, i.e. $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$. If $x = *(x)$ it follows $\|x\|^2 = \| *(x)x\| = \|x^2\|$ and then by induction $\|x\|^{2n} = \|x^{2n}\|$ for $n \geq 1$ and then $\|\gamma(x)\|^2 = \| *(\gamma(x))\gamma(x)\| = \|\gamma(*(x)x)\| = \| *(x)x\| = \|x\|^2$. Thus γ preserves Banach space and anti-involution structures of A onto a unital sub-algebra of $\mathbf{C}(\sigma(A))$, but since $\gamma(A)$ separates points and for the Stone-Weierstrass theorem[13] , γ is the searched isomorphism.

This theorem brings to the following equivalence of categories:
Proposition II.3.2 (Gel'fand duality) Given C^*Alg_c the category of commutative C*-algebra and Top_{cpt} the subcategory of compact topological space, then there is an equivalence of categories:

$$C^*Alg_c^{op} \longleftrightarrow Top_{cpt}$$

where:

1. $\alpha : Top_{cpt} \to C^*Alg_c^{op}$ sends any compact topological space X to $\alpha(X) = \mathbf{C}(X) \subseteq Hom_{Top}(X, \mathbb{C})$ equipped with a the pointwise C*-algebra structure;
2. $\sigma : C^*Alg_c^{op} \to Top_{cpt}$ (full and faithfull) sends a C*-algebra A to $\sigma(A)$ Gel'fand spectrum of A.

As Top represents a weak idea of algebraic geometric context, then the meaning of Gel'fand duality lies upon its possible generalisations to non commutative C*-algebras as foundation of a non commutative geometry (see [24, 26]).
As an important results of these two propositions, for a commutative C*-algebra A there exists a canonical way to associate the Gel'fand spectrum $\sigma(A)$ with the usual *Banach spectrum* of an element $a \in A$ defined as:

$$sp(a) = \{\lambda \in \mathbb{C} \mid (a - \lambda\mathbb{I}) \text{ is not invertible}\}$$

In fact the isomorphism γ defined in the proof, usually called *Gel'fand map*, allows to write:
$sp(a) = \gamma(a)(\sigma(A))$ where a simple proof of this could be find in [10, ch.VII ex.3.2].

II.4 States and concrete representations. Clarified the role of the complex C*-algebra, it is important to recover the connection between the C*-algebras' environment and the operatorial one, represented by the $B(H)$ space and this is achieved in the theory of states of a C*-algebra.
Given a C*-algebra A equipped with the weak-topology, then the space $\mathbb{A}'$ is a (complete) normed space with the *dual norm* defined for each $f \in \mathbb{A}'$ by $\|f\|_D = sup_{\mathbb{R}}\{|f(x)|_{\mathbb{C}} \mid \|x\|_A = 1 \ for \ x \in A\}$ and in this context one has that:

[12] For more details see Gel'fand-Mazur theorem in [10, ch.VII, prop.8.1] and Gel'fand formula theorem in [10, ch.VII, prop.3.8 and 8.9].
[13] Which states that for X compact Hausdorff space and B a sub-set of $\mathbf{C}(X)$ separating points,i.e. there exists a function $f \in B$ such that if $x \neq y$ then $f(x) \neq f(y)$, then the (commutative) unital *-algebra generated by B is dense in $\mathbf{C}(X)$, see [35, sec.44].

Definition II.4.1 (State of a C*-algebra) An element $\rho \in \mathbb{A}'$ is called a *state* of A if :

1. has unital norm (or it's normalised), i.e. $\|\rho\|_D = 1$;
2. is positive, i.e. ρ maps every $a \in A$ with $a = *(a)$ into $\mathbb{R}^+$ [14].

An operatorial or concrete representation of a C*-algebra, as presented in [10, p.248], is instead:
Definition II.4.2 (Representation of C*-algebra) Given a C*-algebra A, a *representation* of A is the couple (π, H) where H is an Hilbert space and $\pi : A \to B(H)$ is a C*-homomorphism, with $\pi(1) = \mathbb{I}_H$. Such a representation is said to be *cyclic* if there exists a vector $\psi \in H$, called cyclic vector, such that $\pi(A)\psi = \{\pi(a)\psi \mid a \in A\}$ is dense in H.

The theory exposed by Gel'fand, Neumark and Segal, in [14, 32], ensures the existence of such cyclic representations for every C*algebra bonding them with a nice property of states.

Proposition II.4.1 (GNS representation) Given a C*-algebra A and a state $\rho \in \mathbb{A}'$, then there exists a (class of equivalent) cyclic representation (π, H) such that:

$$(\forall a \in A) : \quad \rho(a) = (\pi(a)\psi, \psi)_H$$

where $\psi \in H$ is the cyclic vector of the representation.
Proof. The outline of the proof consists essentially in the construction of the Hilbert space H for which the map π has the property expressed by $\rho(a)$. To do this lets define for $x, y \in A$ the sesquilinear form $(x, y)_A = \rho(*(y)x)$. Then the set $L = \{x \in A \mid \rho(*(x)x) = 0\}$ forms a closed vector subspace on A and a left ideal in A. Let now consider the quotient space A/L equipped with the inner product $(x+L, y+L) = \rho(*(y)x)$ for every $x, y \in A$, then the Cauchy-completion of A/L with this inner product is the wanted Hilbert space, where the action $\pi : A \to A/L$ is defined for all $a \in A$ as $\pi(a)(b+L) = ab+L$ and a cyclic vector is represented by the class $\psi = [1_A] \in A/L$ [15].

This association has a resonant physical meaning, as the inner product of an Hilbert space (as a state space) formalises the concept of *average value* for a quantity in a precise physical state, as discussed in the following chapter. Therefore the GNS representation communicates clearly a correspondence between states and concrete operatorial representations for a C*-algebra.
In this sense special representations could reproduce special states as in the following case:
Definition II.4.3 (Pure states) Given a C*-algebra A, a *pure state* on A is a state $\rho \in \mathbb{A}'$ with one of the following equivalent conditions:

1. is an extreme point of the set of states of A [16];
2. corresponds to a (class of equivalent) non degenerate irreducible cyclic representation (π, H), i.e. a representation where the only closed and invariant under $\pi(A)$ sub-spaces are trivial (i.e $\{0\}$ and H)[17].

Pure states are central in this discussion as they produce a physical meaning for Gel'fand spectrum as one has:
Proposition II.4.2 Given a commutative C*-algebra A, then the space of pure states of A, denoted $PS(A)$, coincides with the Gel'fand spectrum of A.

[14] Such a property is a conceptual extension of positive elements definition for complex ordered vector space, where in the case of C*-algebras is a sufficient assumption for the preservation of self-adjointness of elements as $\mathbb{R} = \{a \in \mathbb{C} \mid a = *(a)\}$.

[15] For a complete proof see [10, pp.251-252]

[16] Given a complex vector space V and three points $x, y, z \in V$ with $y \neq z$, then it is to be said that x lies between y and z if there exists a real number $t \in]0, 1[$ such that $x = ty + (1-t)z$. If $U \subseteq_{SET} V$ and $x \in U$, then x is an *extreme point* of U if it does not lie between any two distinct points of U. The existence of such points is ensured by the condition of Krein-Milman theorem for compact convex spaces [16, pp.187-188] and indeed the set of states of A is a compact convex set, as for every two states $\rho_1, \rho_2 : A \to \mathbb{C}$ and any real number $t \in [0, 1]$ the linear functional $p\rho_1 + (1-p)\rho_2$ is trivially a state.

[17] The equivalence of these properties of pure states is left in [32]

Proof. The idea under this proposition comes by the definition of an element of the spectrum $\sigma(A)$ as a linear continuous positive normalized functional that is also multiplicative on A. This means that if pure states are multiplicative than they belongs to the spectrum of A. To show this a self-adjoint $a \in A$ with $a \leq 1$, then $a^{\frac{1}{2}} \geq 0$. Let consider a pure state ρ with $\rho(a) \neq 0$ and $\rho(1-a) \neq 0$. Let define $\gamma_a(x) = \frac{\rho(ax)}{\rho(a)}$ for every self-adjoint $x \in A$, then it follows by commutativity of A that: $\gamma_a(x) = \frac{\rho(a^{1/2}xa^{1/2})}{\rho(a)} \geq 0$ and $\gamma_a(1) = 1$, then indeed γ_a is a state. If one puts $t = \rho(a)$, it follows that $\rho(x) = t\gamma_a(x) + (1-t)\gamma_{1-a}(x)$, but since ρ is pure then ρ is maximal with respect to the parameter t and this implies that $\rho(x) = \gamma_a(x)$ and equivalently $\rho(ax) = \rho(a)\rho(x)$ or ρ is multiplicative.

In the commutative environment all the treated representations take then the shape of the following chain of conceptual relationships:

$$A \xleftrightarrow{G.duality} \mathbf{C}(\sigma(A)) \overset{\mathbf{C}(\_)}{\leftarrow\cdots\rightarrow} \sigma(A) \xleftrightarrow{GNS} PS(A) \overset{G.map}{\leftarrow\cdots\rightarrow} sp(a)$$

II.5 Von Neumann algebras. A quantum physical model for observables requires a more refined algebraic structure, compatible with measure spaces conception. Such structures are therefore described formally by *von Neumann algebras*.

Definition II.5.1 (W*-algebra) Given an Hilbert space H, a **von Neumann algebra** or **W*-algebra** is a C*-algebra in $B(H)$ closed in the weak operator topology over H[18] and with a unit element equal to the identity operator of H.

For commutative W*-algebras, Gel'fand duality sets a controvariant equivalence between the category of W*-algebras and the category of spaces $L^\infty(X)$ of essentially bounded measurable functions, where X is a σ-finite measure space and that under point-wise almost everywhere multiplication has a Banach space structure (see [20]).
Another fundamental characterization of W*-algebras is contained in the *bicommutant theorem*, which establishes a connection between W*-algebras and the operation of commutation in $B(H)$.

Proposition II.5.1 (Bicommutant theorem) Given a *self-adjoint* C*-subalgebra A in $B(H)$, i.e. a concrete C*-algebra A where for every $a \in A$ one has $a = *(a)$, then it holds:

1. the closures of A in the weak and strong operator topology over H are equal to the bicommutant $c(c(A))$ of A, where $c(A) = \{F \in B(H) \mid \forall a \in A : [F, a] = 0\}$ and $[\cdot,\cdot]$ is the commutator on $B(H)$;
2. A is a W*-algebra in $B(H)$ [19].

As shown in III, the structure of $B(H)$ entangles the essence of observables' structures over a physical quantum system, which is a W*-algebra structure as for $B(H)$ the commutant $c(B(H)) = \{\emptyset\}$ and then $c(c(B(H))) = B(H)$. From II.2.1 it follows that $B(H)$ has the following categorical structure fundamental for the present discussion.

Definition II.5.2 (Category *V(H)*) Given an Hilbert space H, the category of **von-Neumann algebras** $V(H)$ is the category whose:

1. objects are *commutative* W*-algebras on H;
2. arrows are the inclusions of sub-algebras;

where the categorical structure is inherited by the poset $B(H)$[20].

[18] The *weak operator topology* over H is the weakest topology on $B(H)$ that makes $(Ax, y)_H$ continuous for every operator $A \in B(H)$ and every $x, y \in H$.
[19] For a proof see [20, sec.3.2].
[20] Thus $V(H)$ is a small category.

Chapter III

A quantum valuation theory

The matter on which the development of a physical logic theory is meaningful is certainly a *valuation theory*, definable as the formal structure arising from an empirical theory of measurements or as a formal physical information theory.

III.1 A definition of valuation. Defining principles for a general understanding of any valuation theory is the conceptual matter which any physical formalism is based on and then strictly related to the foundational debates of physical representation of reality.
Defining a general valuation theory requires then a conceptual asset capable of explaining what a valuation on a physical system is and how it acts on the physical information. For this purpose the base of any valuation theory is a formal definition of the physical information that acquires the following conceptual definition:

Definition III.1.1 (State and space of states) Given a physical system Λ in a physical theory T, the *state* of Λ is the object in the formalism of T containing the possible physical information on Λ, i.e. is the formalized object representing Λ in T. As the physical information on a system Λ in T could change, the collection of possible states of Λ defines a *state space* $S(\Lambda)$ with a specific mathematical structure defined in the formalism of T.

Acquiring this wide definition[1] allows to reproduce a very general conception of a valuations as the formal counterpart of empirical *processes of measurement*[2]. The aim of a measure is to reproduce a physical information in a mathematical environment, generally in a numerical one, in which there could be build ordering and algebraic structures, or conceptually in which a comparison or a relation between two different pieces of information is possible.
But defining a valuation requires a further conceptual step concerning a possible qualitative differentiation among the physical information over a system. This characterization of the informational structure takes the shape of a variety of different *physical quantities* and then a valuation is a process of mathematical representation of a collection of these quantities into a value structure. Then one has the following definition[3]:

Definition III.1.2 (Quantity value and valuation) Given a system Λ with $S(\Lambda)$ as its state space in the theory T, given a physical quantity F in T, the *quantity value* of F over Λ is an arrow:

$$\tilde{F} : S(\Lambda) \to R(\Lambda)$$

[1] That has a bland representation in the literature, see for example [11, pp. 15-16] and [30].

[2] A process of measurement reproduces usually a more complex formal structure then a valuation in the sense explained in this work and examples of this are the usual quantum measure theory and statistical mechanics theories [11, 17, 34]. However valuations catch the idea of an elementary or simple formal process of measurement and for this reason they have been chosen as a representative structure for an elementary formal measure theory.

[3] Inspired by the work of Isham and Doering [12].

where $R(\Lambda)$ is called *value space*[4].
Given a collection of physical quantities $M = \{F_i\}_i$ in T, called *context of measurement*, a *valuation* over M is an arrow:

$$U : S(\Lambda) \times M \to R(\Lambda)$$

such that for each $\psi \in S(\Lambda)$ and $F_j \in M$ the arrow acts as $U(\psi, F_j) = \tilde{F}_j(\psi)$.

The nature of these arrows and the definition of the value space depends strongly on the formalism of the theory T, especially on the representation that T provides of the physical reality and on its principles. However it's possible to think about a canonical required structure for making a valuation somehow physical and usually the following properties are required:

1. Classically the value space is recognized as $\mathbb{R}$ (or in classical statistical mechanics as $Sub_{Set}(\mathbb{R})$) for its exceptional regularity in terms of ordering and algebraic and topological structures and for its historical importance[5]. Acquiring then $R(\Lambda) = \mathbb{R}$, an important issue comes from the possibility of gauge reshaping and then it's generally adopted the following axiom for valuation, called *functional composition principle*[6]:
   Definition III.1.3 (FUNC axiom) Given a quantity $F \in M$ and any Borel function $h \in Hom_{Set}(\mathbb{R}, \mathbb{R})$ [7], then for a valuation over M and for any $\psi \in S(\Lambda)$ it follows that there exists a quantity $h(F) \in M$ with:
   $$U(\psi, h(F)) = h(U(\psi, F))$$
   or equivalently any valuation is compatible with respect of the composition with map preserving measurable-topological structures[8];

2. For a matter of consistency of the valuation, as well explained in the given definition, is required a characterization for the context of measurement M. In fact the quantities in M have to be *commensurable* or in other words it's possible the construction of an empirical process of measurement of all the quantities in M but *without interfering with the state* $\psi \in S(\Lambda)$ *of* Λ. Commensurability between quantities could be defined then as:
   Definition III.1.4 (Commensurability) Given two quantities $F_1, F_2 \in M$ and a valuation U on M (with the FUNC axiom), then F_1 and F_2 are said to be *commensurable* if for every $\psi \in S(\Lambda)$ there exist $U(\psi, F_1)$ and $U(\psi, F_2)$ and they're unique.

   An example of commensurable quantities is expressed by the quantities generated assuming the FUNC axiom, as a valuation for a quantity $F_2 = h(F_1)$ in the state ψ exists and it's unique if a valuation of F_1 in the state ψ exists and it's unique. Imposing commensurability among M allows to fix a state $\psi \in S(\Lambda)$ and then to consider every valuation as an arrow $U_\psi : M \to R(\Lambda)$.

These two properties directly define and influence the construction of "physical" valuations and more important of contexts of measurement, where if the FUNC axiom or the commensurability do not hold for a contexts of measurement, then it is to be said that a "physical" valuation does not exists for that context.

III.2 Quantum states and observables. Principles of quantum mechanics, exposed in [11, 17, 34], reproduce a formalism in which it's difficult to visualize quantity values as in III.1.2, then this forces to focalize the attention on the definition of contexts of measurement and valuations, which next

[4] This general definition for value space hides the possibility for a state $\psi \in S(\Lambda)$ to be associated a single numerical value (as for $R(\Lambda) = \mathbb{R}$) or to an interval of numerical values (as for $R(\Lambda) = Sub_{Set}(\mathbb{R})$) or to an abstract object in a category.

[5] But in view of the next chapter it's useful to think loosely about the definition of the value space.

[6] See [12, 13].

[7] A Borel fuction over a topological space $X \in Top_0$ is a function that preserves the Borel measurable structure over X, i.e. the measurable space structure (X, σ) where σ is the smallest sigma-algebra that contains the topology of X. Such functions then preserve measurable constructions compatible with topologies.

[8] For a generalization of this axiom one has to consider a map $h : R(\Lambda) \to R(\Lambda)$ preserving measure structures with $h(\tilde{F}) = h \circ \tilde{F}$.

sections will show to be respectively commutative W*-subalgebras of $B(H)$ (for an abstract Hilbert space H) and the pure states of these subalgebras, i.e. the elements of the Gel'fand spectrum of these subalgebras.

Quantum mechanics theories consider the informational structure of a system Λ being connected to an abstract[9] complex and separable Hilbert space H of finite or infinite dimension and to its algebra $B(H)$. Assuming from now on only finite dimensional H for a simpler treatment of the object involved, given a system Λ then the states of this system could be seen as elements of H with:

1. *pure states*, defined conceptually as representations of a maximal information structure for Λ, where formally the space of pure states of the system is the projective space $\{\psi \in H \mid \|\psi\|_H = \sqrt{(\psi,\psi)_H} = 1\}$ and chosen an orthogonal base $\{\phi_i\}_i$ for H it follows that any pure state is of the form $\psi = \sum_i c_i\phi_i$ with $c_i \in \mathbb{C}$ and $\sum_i |c_i|^2_{\mathbb{C}} = 1$;
2. *mixed states*, defined conceptually as a statistical ensemble of pure states, where formally these states are convex combination of pure states, i.e. any mixed state Ψ could be expressed as $\Psi = \sum_k p_k\psi_k$ where ψ_k are pure states and $p_k \in [0,1]$ with $\sum_k p_k = 1$. As the inner product of H acts over any mixed state Ψ as: $(\Psi,\Psi)_H = \sum_k p_k(\Psi,\psi_k)_H = \sum_k \sum_j p_k p_j(\psi_j,\psi_k)_H$ as every p_k is real, then all the constructions involving only the inner product and linear structures are completely determined by inner products of pure states, so this allows the use from now on of *only* pure states for the purpose of this work.

It's possible to represent states as maps over the algebra $B(H)$. Given in fact any pure state $\psi \in H$ one can consider the following construction of *average value* in ψ as:

$$\rho_\psi : B(H) \to \mathbb{R} \ \ \textit{such that} \ \ \rho_\psi(a) = \frac{(a(\psi),\psi)_H}{(\psi,\psi)_H}$$

where for the properties of the scalar product of H and according with the GNS construction this map is a normalized positive linear functional over $B(H)$, i.e. a state on $B(H)$ view as a C*-algebra.

On the other hand the formalism of quantum mechanics proposes a representation of physical quantities via *observables*[10] defined as operators. Given a physical quantity F and given the Hilbert space H describing the informational structure of a system Λ, then the representation of F is an operator $\hat{F}$ on H such that:

1. has a domain $D(\hat{F}) = H$, where the physical meaning of this is related to the fact that one wants $\hat{F}$ to act over any element ψ of H [11];
2. $\hat{F}$ is linear on $D(\hat{F})$, where this is conform to the superposition principle of quantum mechanics;
3. $\hat{F}$ is self-adjoint, i.e. $\hat{F}^\dagger = \hat{F}$, where from the symmetry of self-adjoint operators one obtains that every average value of $\hat{F}$ is real and from the spectral theorem one has that there exists an orthogonal base for H of *eigenvectors* $\{\phi_\lambda\}_\lambda$, with $\lambda \in \mathbb{R}$ and $\hat{F}(\phi_\lambda) = \lambda\phi_\lambda$, such that for every Borel function $h \in Hom_{Set}(\mathbb{R},\mathbb{R})$ there exists an observable $h(\hat{F})$ that acts on ϕ_λ as $h(\hat{F})(\phi_\lambda) = h(\lambda)\phi_\lambda$, reproducing an environment for the FUNC axiom.

Considering for the economy of the present discussion only finite dimensional Hilbert space H and the bounded observables in $B(H)$, then:

Definition III.2.1 (Quantum context of observables, a first definition) A *quantum context of observables* M is defined to be a C*-subalgebra of self-adjoint operators in $B(H)$, and by the bicommutant theorem M is a W*-subalgebra of $B(H)$.

[9]In this context "abstract" has quite the same meaning of "up to isomorphisms" as for any Hilbert space H changing the orthogonal base realizes an automorphism on H, but physically it realizes a change in terms of representation of the information.

[10]Where the term "observable" indicates the formal object denoted $\hat{F}$ representing a quantity F in the formalism of the theory T.

[11]In the case of an infinite dimensional H this condition is chosen to be weaker as $D(\hat{F})$ is assumed to be a dense affine sub-variety of H.

In such subalgebras the Banach spectrum of every observable $\hat{F} \in M$, defined in II.3, coincides with the usual spectrum defined in quantum theory as the set of *eigenvalues* of $\hat{F}$ expressed in the spectral theorem or equivalently $sp(\hat{F}) = \{\lambda \in \mathbb{C} \mid (\hat{F} - \lambda\mathbb{I})^{-1} \notin B(H)\}$.
On these sub-algebras quantum states $\psi \in H$ acts as C*-algebra states thanks to the given average value functional definition and then the state space of this sub-algebras appears to catch the idea of a possible space of valuations for quantum contexts of observables.

In fact on a quantum context M the FUNC axiom need a stronger requirement of compatibility with the operations that define the W*-algebra structure over M, and then a quantum version of this axiom imposes to any valuation $U_\psi : M \to \mathbb{R}$ with an appropriate $\psi \in H$, that for any $\alpha, \beta \in \mathbb{R}$ and for any two observables $\hat{F}_1, \hat{F}_2 \in M$:

$$U_\psi(\alpha\hat{F}_1 +_{B(H)} \beta\hat{F}_2) = \alpha U_\psi(\hat{F}_1) +_{\mathbb{R}} \beta U_\psi(\hat{F}_2) \quad (\mathbb{R}-linearity)$$

and:

$$U_\psi(\hat{F}_1 \times_{B(H)} \hat{F}_2) = U_\psi(\hat{F}_1) \times_{\mathbb{R}} U_\psi(\hat{F}_2) \quad (multiplicativity)$$

essentially reproducing the properties of linear and multiplicative states on M viewed as a W*-algebra. Then one has:
Definition III.2.2 (Quantum valuations) Given a quantum context of observables M and imposing the quantum version of the FUNC axiom, the objects reproducing valuations over M are all the pure states of M viewed as an W*-algebra, i.e. all positive normalized linear functional over M that are also multiplicative.

The condition then for the choice of an appropriate $\psi \in H$ is that the associated average value ρ_ψ, chosen as a valuation, needs to be multiplicative or equivalently ψ is an eigenvector for every element of the algebra M. The condition of existence of such a valuation is the condition of the existence of such a state ψ and this is possible only in the case of a commutative M, as shown in the next section. Moreover, if the context M has the property of being commutative, for II.4, one has a clear representation of the space valuations in terms of the Gel'fand spectrum, and also a clear relationship between valuations and elements of the Banach spectrum for elements in M. These are reasons for adopting only *commutative* W*-subalgebras of $B(H)$, but commensurability will give a much stronger point for this assumption.

III.3 Quantum commensurability and local contexts. The conditions describing the a contexts of measurement require a quantum version of commensurability.
Such a structure is expressed in the statement of the *uncertainty principle*, where two observables $\hat{F}_1$ and $\hat{F}_2$ are said to be commensurable if and only if their commutator is zero or formally:

$$[\hat{F}_1, \hat{F}_2] = (\hat{F}_1 \times_{B(H)} \hat{F}_2) -_{B(H)} (\hat{F}_2 \times_{B(H)} \hat{F}_1) = 0$$

This kind of commensurability is equivalent to the one given in III.1.4, and it is compatible with the FUNC axiom from an application of the spectral theorem.

Proposition III.3.1 Given two observables $\hat{F}_1$ and $\hat{F}_2$ such that there exist a (Borel) function $h \in Hom_{Set}(\mathbb{R}, \mathbb{R})$ with $\hat{F}_2 = h(\hat{F}_1)$, then they commutes.
Proof. This statement descends trivially from the conjunction of the uncertainty principle and the spectral theorem for which two observables $\hat{F}_1$ and $\hat{F}_2$ commutes if and only if there exists an orthogonal base $\{\phi_{ij}\}_{i,j}$ of common eigenvectors of the two observables, i.e. it follows that $\hat{F}_1\phi_{ij} = \lambda_i\phi_{ij}$ and $\hat{F}_2\phi_{ij} = \mu_j\phi_{ij}$ with λ_i and μ_j eigenvalues respectively of $\hat{F}_1$ and $\hat{F}_2$ (see [17]). Assuming then $\mu_j = h(\lambda_i)$ one has the statement.

Assuming that a contexts of measurement is a context of all commensurable quantities, then:
Definition III.3.1 (Quantum context of observables, a final definition) A *quantum context of observables* M respecting the FUNC axiom and the assumption of commensurability is defined to be

a W*-subalgebra of $B(H)$, where given any two observables $\hat{F}_1, \hat{F}_2 \in M$ then they commute, or equivalently $\hat{F}_1 \times_{B(H)} \hat{F}_2 = \hat{F}_2 \times_{B(H)} \hat{F}_1$ with the multiplication $\times_M$ commutative.

A valuation U_ψ over such a context M should then preserve the commutative structure generated by the $\mathbb{C}$-linear derivation $[\cdot,\cdot]_H$ and with the following property, given $[\hat{F}_1, \hat{F}_2] = 0$ then:

$$[\hat{F}_1, \hat{F}_1 \times_M \hat{F}_2] = [\hat{F}_1, \hat{F}_2 \times_M \hat{F}_1] = [\hat{F}_2, \hat{F}_1 \times_M \hat{F}_2] = [\hat{F}_2, \hat{F}_2 \times_M \hat{F}_1] = 0 \quad .$$

explaining another time the adoption of the properties of linearity and multiplicativity for quantum valuations. As far as the spectral theorem holds in terms of the proof of III.3.1, given a context M there exists an orthogonal base of common eigenvectors $\{\phi_\lambda\}_\lambda$ for all the elements in M and then a valuation U_ψ exists as the average value ρ_ψ if $\psi \in H$ is an element of this base.

This drives to the adoption of the category $V(H)$ of commutative W*-subalgebras of $B(H)$ as a category in which is possible to have a well defined notion of quantum valuation, where each context of measurement $M \in V(H)_0$ encloses the idea of a "closed" informational environment that contains the whole possible information of a system that could be measured *without changing the state of the system*. As this amount of information is not necessarily *complete*, in the sense that does not necessarily contains all the constructible physical quantities, then a context $M \in V(H)_0$ will be called *local context* to be distinguished from the *global context* associated to the complete information, i.e. $B(H)$ itself.
In this sense the informational locality of contexts in $V(H)$ is a way of preserving a weak idea of determinism and then of reproducing a "local" realism in the quantum environment. From this idea the comparison between $V(H)$ and $B(H)$ in their informational and logical structures appears a fundamental step for the comprehension of the realist debate.

III.4 Issues on the global context. The hardship faced in the attempt of building a unique notion of realism in the quantum environment arises plastically from the impossibility to build a valuation on the global environment, that is a more complex proposition equivalent to the assumption of the existence of non commutative collection of observables.
This perspective assumes the role of a no-go theorem exposed by Kochen and Specker in [21], which the papers of Isham and Doering [12].

Proposition III.4.1 (Kochen-Specker theorem) Given an Hilbert space H with dimension $n > 2$, then it does not exists any algebra-homomorphism $\rho : B(H) \to \mathbb{C}$, or equivalently it does not exists any valuation on $B(H)$.
Proof. A formal proof of this statement given in [21] is far out of the aim of the present work, but the underlying conceptual suggestion for the statement is the property of $B(H)$ of not being commutative for $dim(H) \geq 2$. After that, given an orthogonal base $\{\phi_i\}_i$ one can consider the collection of projectors $\{P_i\}_i$ associated to this base. For the linearity of a valuation ρ on $B(H)$ and the properties of the base projectors, then it follows that $1 = \rho(\mathbb{I}) = \rho(\sum_i P_i) = \sum_i \rho(P_i)$, since $\rho(P_i) \in \{0, 1\}$ then there exists only one $\rho(P_k) = 1$ and all the other are null. The collection $\{P_i\}_i$ is a collection of mutually commutative observables that generates a local context. One then can think about a change in the base corresponding to the generation of another context of projectors $\{Q_j\}_j$ and under the same valuation this new projectors should reproduces new zeros and ones. As soon the same projectors appear in different basis, if a well defined valuation exists then its maps the same projector to the same value in any possible base with respect of the sum rule mentioned. The proof of the theorem simply shows that this is an impossible situation unless one has only one projector (the case of $dim(H) = 1$) or there exists a unique orthogonal complement for any projectors (the case of $dim(H) = 2$).

The Kochen-Specker theorem tells that the behaviour of the global context is completely different from the local one with respect of a valuation theory, where this situation produces two divergent approaches in terms of informational structure and simply two distinct logic theories on the same physical system. The aim of the next chapter is to show these two different logical structure in the language of topos theory to understand the connections between them.

Chapter IV

Quantum logics

A physical logic theory is a theoretical environment of interaction between a physical model and a formal language, where the deductive structure of the formal language is represented or depicted in the formalism of a physical theory, where this process of imaging and embedding a language into a formal construction is in reality the foundation and the definition of the physical theory itself.

IV.1 A language for physical systems. Given a physical system Λ and a generic theory T, the aim to identify and structure the empirical information into a deductive environment requires to understand how an empirical statement could be depicted in a formal way. Assuming the above definition of physical quantities III.1, an *empirical inference* deriving from a process of measurement over the system Λ assumes the shape of a statement like:

"The value of the quantity F is in Δ"

where Δ is a "sub-object" of the value space $R(\Lambda)$. An empirical inference over Λ is then an affirmation of the act of belonging of a quantity value to a specific value space and in general representing this act formally with the symbol ϵ, then the adopted notation for any empirical inference is:

"$\tilde{F}\epsilon\Delta$"

As the reason for the existence of an empirical inference is to give a description of the informational structure of the system Λ, contained in the formal object of the state space $S(\Lambda)$, but using the value space $R(\Lambda)$ as a reference space for measurement, then one expects any empirical inferences to be associated to some structure proper of $S(\Lambda)$. This forces an important definition around which the topos approach to physical logic gravitates, i.e. the *interpretation map.* A well defined physical logic theory requires in fact the existence of a map Π sending any empirical inference "$\tilde{F}\epsilon\Delta$" to a sub-object[1] of $S(\Lambda)$ as:

$$\Pi[\tilde{F}\epsilon\Delta] = \{\phi \in S(\Lambda) \mid \tilde{F}(\phi) \in \Delta\} \equiv \tilde{F}^{-1}(\Delta)$$

where "ϵ" is meaningful in *Set*-environment and $\tilde{F}^{-1}(\Delta)$ is a sub-object of $S(\Lambda)$. This map works as a "reification" of inferences into a "concrete" informational structure expressed by the state of space and its sub-objects, with $Sub(S(\Lambda))$ or its internalization acquires the role of *space of inferences.*

Assuming finally empirical inferences as atomic propositions for any physical language, then it appears clearly the nature of such a language as a typed formal language $L(\Lambda)$ closed under the construction of powerset $\mathcal{P}$ and finite products $\times$, with the adequate constructions for the respective terms, and generated by four *ground types* (see [12, ch.4] and [19]):

1. A physical state type $S(\Lambda)$, whose terms ϕ represents states;
2. A physical value type $R(\Lambda)$, whose terms δ represent values;

[1] Already thinking about a categorical implementation of this structures.

3. A truth-values type Ω, where an example of a term of this type is "$\tilde{F}(\phi) \in \Delta$" which depends on a variable ϕ of type $S(\Lambda)$ and a variable Δ of type $\mathcal{P}(R(\Lambda))$;

4. The singleton type I.

The next step consists in providing a concrete interpretation of this language for physical formalism into an algebraic structure. An adequate structure turns out to be that of a topos.

IV.2 A topos for physical theories. Topoi work well for the purposes of this work, essentially because they can be seen as ghosts of formal typed languages and moreover as inference languages were designed having in mind some of the properties of a topos.

In fact, given the physical language of inferences $L(\Lambda)$, one can interpret it in a topos E as follows:

1. types T are interpreted as objects $\|T\|$ of the topos by sending $R(\Lambda)$ and $S(\Lambda)$ to objects $\|R(\Lambda)\|$ and $\|S(\Lambda)\|$ of the topos, I to a final object and Ω to a sub-object classifier, and then using products to intepret the product constructor and interpreting the powerset of a type T as the exponential $\|\Omega\|^{\|T\|}$;

2. a term of type T depending on $x_1 \in T_1, ..., x_n \in T_n$ is interpreted as an arrow from the product of the interpretations of T_1,...,T_n to T, using the obvious interpretations for term constructors.

In such a topos E a quantity value $\tilde{F}$ is interpreted as an arrow from $\|S(\Lambda)\|$ to $\|R(\Lambda)\|$ and then the collection of arrows between these two objects, i.e $Hom_E(\|S(\Lambda)\|, \|R(\Lambda)\|)$, reproduces all possible quantity values in E (which can be also internalized by means of an exponential which is available in E).
The interpretation of the space of inferences, that is the type $\mathcal{P}(S(\Lambda))$ has a canonical internal structure of Heyting algebra, while the sub-objects of the interpretation of $S(\Lambda)$ form a Heyting algebra. The internal logic of the topos E then provides a suitable and general logic environment for physical interpretations[2].

As a physical theory T on a system Λ proposes essentially a collection of different physical quantities to translate the physical information, and a mathematical formalism to represent these quantities; as the topos E inteprets the mathematical environment of the formalism of T and it hosts an object $\|R(\Lambda)\|^{\|S(\Lambda)\|}$ representing the variety of physical quantities, then every topos E provides essentially a possible mathematical "reification" of the theory T on Λ.
In this profound sense, the aim of a topos approach to physical logic is then essentially to see the structure of $L(\Lambda)$ as part of that of a logically and mathematically rich (while general) structure: a topos.[3]
Example. Given any classical theory on the system Λ then it follows that:

1. Any quantity value of any physical quantity F is of the form $\tilde{F}: \mathbb{R}^m \to \mathbb{R}$ with $m \in \mathbb{N}$, where the state space is $S(\Lambda) = \mathbb{R}^m$, usually called *state of micro-states*, and the value space is $R(\Lambda) = \mathbb{R}$;

2. The physical logic topos in which our language is interpreted is Set whose internal first-order logic is exactly the standard first-order classical logic.

From this example, it could be built a first conceptual bridge to quantum environment. Given the space $S(\Lambda) = \mathbb{R}^m$, for every $K \subseteq (S(\Lambda))$ there exists an idempotent function of the form $P_K : S(\Lambda) \to K$, called *projector* over K and for an inference "$\tilde{F}\epsilon\Delta$" for which $K = \tilde{F}^{-1}(\Delta)$ it follows that:

$$\Pi_{cl}[\tilde{F}\epsilon\Delta] = \tilde{F}^{-1}(\Delta) \equiv K = P_K(S(\Lambda))$$

[2]Notice that one could extend the language $L(\Lambda)$ on such a way that one can organize it as a topos. However this needs some more intricate rules other than the proposed ones, including some rules of introduction for types in the form of comprehension, see proposed structures in [19, 22].

[3]From this point for the economy of notation there will be use the same notation for indicates a type of $L(\Lambda)$ and its interpretation in a topos E.

where this is a first sight of the role of projectors in physical logic theory, driving a projectors formalism for quantum logic theories (see IV.4).

IV.3 The spectral presheaf and a quantum contextual logic. In quantum environment the objects that take the role of $S(\Lambda)$ and $R(\Lambda)$ are not immediately clear due to the operatorial definition of observables. In fact in the usual quantum state space H it is not possible to reproduce the formalism related to the conception of contexts of measurement as an "informational closed structure", therefore this space is not appropriate for the foundation of a quantum logic theory. Also in the definition of quantum valuations there is not a unique way of defining the state space and the value space as valuations seem to act as a substitute of usual quantum states.
In fact from chapter II and III, given a quantum context of observables $M \in V(H)_0$, it is known that a state $\phi \in H$, with the property of being an eigenvector for the element of M, is "perceived" by M as a valuation $\rho_\phi : M \to \mathbb{R}$, i.e. as an element of the spectrum $\sigma(M)$. Thus the sense of such a usual state ϕ is that of an association between contexts and valuations, or pictorially in a abuse of notation the state ϕ acts as an arrow $\phi : M \mapsto \rho_\phi \in \sigma(M)$. Thus given an observable $\hat{F}$ in the context M and given an inference of the kind "$\tilde{F} \epsilon \Delta$", then this inference is associated to a collection of states denoted informally $\tilde{F}^{-1}(\Delta) \subseteq H$ for which the inference is true, where in the C*-algebraic representation one has $\tilde{F}^{-1}(\Delta) : M \mapsto Sub(\sigma(M))$.
Even if this considerations are a naive description of the situation, they clarify the sense of the following definition, as an idea of a "representation" of the usual state space H in a categorical formalism.

Definition IV.3.1 (Spectral Presheaf) Given the category $V(H)$ of all contexts in $B(H)$, the **spectral presheaf** on $V(H)$ is the presheaf $\Sigma : V(H)^{OP} \to Set$ such that:

1. For each $M \in V(H)_0$ the object $\Sigma(M) \in Set_0$ is the set of all multiplicative linear functionals $f : M \to \mathbb{R}$;
2. A sub-algebra inclusion $i \in Hom_{V(H)}(M, M')$ is mapped into $\Sigma(i) \in Hom_{Set}(\Sigma(M'), \Sigma(M))$ which acts as the restriction $f_{|M'} \mapsto f_{|M}$.

Categorically speaking the spectral presheaf is the functor $\sigma : C^*Alg_c^{op} \to Top_{cpt}$ restricted to $V(H)$ and composed with a functor that forgets the topological structure (see [3,12,18]) and clearly it might be chosen to represent the state space object $S(\Lambda)$.

This definition suggests that the suitable topos E in which to interpret a local quantum logic theory is the topos of presheaves $[V(H)^{op}, Set]$ and then all the other types of the physical language $L(\Lambda)$ are represented by presheaves on $V(H)$.
Such an interpretation is then determined when one assigns a presheaf to $R(\Lambda)$. The first possible candidate for this would be the constant presheaf with value $\mathbb{R}$, which is exactly the object of real numbers in the presheaf topos (see [22]). However, this is in contrast with the definition of physical valuation over a context because a valuation does not produces the same value for all the observables in a context, i.e. it's not constant over a context. In [12], the authors propose a different construction: instead of the constant presheaf $\mathbb{R}$, $R(\Lambda)$ is taken to be the followimg presheaf:
Definition IV.3.2 Given the category $V(H)$, the $\mathbb{R}-$*valued presheaf* on $V(H)$, denoted $\mathbb{R}^{\leftrightarrow}$, is the presheaf that acts as:

1. for every $M \in V(H)_0$ it follows

$$\mathbb{R}^{\leftrightarrow}(M) = \{(\mu, \nu) \mid \mu \in OP(P_M, \mathbb{R}), \nu \in OR(P_M, \mathbb{R}) \ with \ \mu \leq \nu\}$$

where: P_M is the poset of all sub-algebras of M and $\mathbb{R}$ is taken as poset with its canonical order; $OP(P_M, \mathbb{R})$ is the collection of all $\mu : P_M \to \mathbb{R}$ monotonically increasing functions and $OR(P_M, \mathbb{R})$ is the collection of all $\nu : P_M \to \mathbb{R}$ monotonically decreasing functions[4];

[4]Monotonic functions are the arrows in Pos, the category of posets.

2. for every $\iota \in Hom_{V(H)}(M', M)$ it follows $\mathbb{R}^{\leftrightarrow}(\iota) : \mathbb{R}^{\leftrightarrow}(M) \to \mathbb{R}^{\leftrightarrow}(M')$ is the restriction on M' of the monotonic arrows on M.

The reasons why $\mathbb{R}^{\leftrightarrow}$ is a good candidate for the value space $R(\Lambda)$ is explained better with the structures introduced in IV.5 and IV.6, but anticipating these notions, a way of reproducing an observable $\hat{F}$ in every context in $M \in V(H)$ is obtained by approximating it with observables in M using posetal structures over M (i.e. via the structure called $\delta_M(\hat{F})$). This approximation can be built refining observables to $\hat{F}$ from top to bottom (*outer* approximation defined by $\delta^O_M(\hat{F})$) or from bottom to top (*inner* approximation defined by $\delta^I_M(\hat{F})$). Then the definition of the quantity value of $\hat{F}$ is affected by these two approximations construction, and for any element in state space object Σ and any context $M \in V(H)$, the value of $\hat{F}$ is made of two monotonic real functions: ν decreasing (for the outer approximation) and μ increasing (for the inner approximation), with a loose form of convergence expressed by the condition $\mu \leq \nu$.[5]

The definition of the spectral presheaf Σ as the base for a *unique* logical quantum structure faces two main problems:

1. it is a "deeply contextual" construction of physical inferences, therefore one needs a global interpretation for inferences;

2. it does not reproduce inferences as operators on H, therefore one needs an operatorial construction of inferences.

These problems are solved introducing the *projectors formalism.*

IV.4 The logical nature of projectors. Before searching for their role in the logical structure of quantum theories, it's fundamental to precisely define what a projector is in the Hilbertian formalism.

Definition IV.4.1 (Projector) Given an Hilbert space H, a *projector* P on H is a bounded self-adjoint operator with the property of being idempotent, i.e. $P \in B(H)$ with $P^2 = P$.
The collection of projectors over H is denoted $P(H)$.[6]

The collection $P(H)$ has an interesting structure derived from the correspondence between a projector P and its range $Ran(P)$, i.e. the image of its entire domain. The range of a projector is a closed subspace in H and for any closed subspace K there exists a unique projector that has as its range K (see [7, 29]). From this the collection $P(H)$ is a poset where the order structure is inherited by the inclusion of closed sub-spaces in H :

$$P \leq_{P(H)} Q \iff Ran(P) \subseteq Ran(Q)$$

This poset is a bounded lattice if one considers the following operations:

1. $P \wedge Q$ is the projector with range $Ran(P) \cap Ran(Q)$;

2. $P \vee Q$ is the projector with range $Ran(P) + Ran(Q)$;

where these operations could be iterated on families of projectors. This lattice is not distributive for $dim(H) > 1$ and it's an atomic and modular lattice[7]. As every closed subspace $K \subseteq H$ may be complemented topologically by $K^{\perp}$, with $H = K \oplus K^{\perp}$, then the lattice on $P(H)$ is ortho-complemented,

[5] It follows that $\sigma(M) \subseteq \mathbb{R}$ and then Gel'fand spectra has a poset structure inherited by $\mathbb{R}$, making the presheaf $\sigma(M)^{\leftrightarrow}$ a sub-object of $\mathbb{R}^{\leftrightarrow}$, where this harmonizes the usual quantum value construction with the assumption of $R(\Lambda) = \mathbb{R}^{\leftrightarrow}$. For more details over this construction and for the other types construction see [12].

[6] Analogous notations and definitions will be used for the collection of projectors in any C*-algebra.

[7] A lattice l is *atomic* if for every $x \in l$ with $x >_l 0$ there exists an atom $y \in l$ such that $x \geq_l y$, where an atom is a non-zero element $y \in l$ such that there *not* exists any non-zero element $z \in l$ with $y > z$. A lattice l is *modular* if given $x, y, z \in l$ then $x \leq_l y$ implies $x \vee (z \wedge y) = (x \vee z) \wedge y$.

i.e. $P(H)$ is closed for the complemention $\perp$ that has the property of being involutive and order reversing and then of acting as a negation operation over $P(H)$.

All of these properties makes $P(H)$ a suitable but exotic environment in which one can think at a logical structure, that differs from Heyting algebras remarkably for the property of being non-distributive. As discussed by von Neumann and Birkhoff in [7, sec.10], this non distributivity is proper of a pure-quantum environment, where the possibility of the existence for non-commutating observables implies that different arrangements of observables could produce contrasting sets of measurements (see for example the mental experiment in [6]) and then a distributivity law on the inferences that reproduce these measurements is not possible. From this, $P(H)$ seems to reproduce a well defined operatorial logical structure over the global context $B(H)$.
In addition to this and supported by the use of projectors in the spectral theorem [11, 17], an important clue over the choice of $P(H)$ as a suitable logical environment for the global context comes from Gleason's theorem [15, 27] that conceptually states that any generalized valuation[8] on the global context may be reproduces via an additive family of projector-like operators.

These suggestions brings to the adoption of $P(H)$ as a possible model for a global and non-contextual logical theory, where an inference "$\tilde{F}\epsilon\Delta$" over a system Λ is formally described by a projector (or a set of projectors) denoted $\hat{P}[\tilde{F}\epsilon\Delta] \in P(H)$.

IV.5 Contextual projectors. As local contexts of measurement are "closed" under valuations, or are "closed" informational environments, then the choice of a global defined lattice $P(H)$ for representing inferences contrasts with this closed definition.

In fact an inference associated to $\hat{P}[\tilde{F}\epsilon\Delta] \in P(H)$ has a representation in a local context $M \in V(H)_0$ if the projector belongs to $P(M)$, which is the sub-lattice of $P(H)$ containing only the projectors in M. If it does *not* do so it is required an "approximation" of $\hat{P}[\tilde{F}\epsilon\Delta]$ in M.
Logically such approximation is possible introducing an operation of inferences *coarse-graining* (see [12, sec.5.1.2] and [13, sec.8.1]) that produces a representation of any projector-inference in any local context of measurement as follows:

Definition IV.5.1 (Contextual projector) Given a context $M \in V(H)_0$ and a projector $\hat{P} \in P(H)$, the *contextual projector* of $\hat{P}$ in M is:

$$\delta_M(\hat{P}) = \bigwedge\{\hat{q} \in P(M) \mid \hat{q} \geq_{P(H)} \hat{P}\}$$

where the $\wedge$ operation is inherited on $P(M)$ by the one defined on $P(H)$.

Categorically speaking the family $\{\delta_M(\hat{P})\}_{M \in V(H)_0}$ could be seen as a family of approximations of $\hat{P}$ with respect to different objects M of $V(H)$. However this approximation, called *outer*, is not unique, in fact one could consider an *inner* approximation made by:

$$\delta^I_M(\hat{P}) = \bigvee\{\hat{q} \in P(M) \mid \hat{q} \leq_{P(H)} \hat{P}\}$$

This idea leads to the definition of a presheaf on $V(H)$ called *outer presheaf* (or respectively *inner presheaf*).

Definition IV.5.2 (Outer presheaf) Given the category $V(H)$, the *outer presheaf* on $V(H)$ is the presheaf $O \in [V(H)^{OP}, Set]_0$ such that:

1. For each $M \in V(H)_0$ the object $O(M) \in Set_0$ is the set $P(M)$ of all projectors in M;
2. A sub-algebra inclusion $i \in Hom_{V(H)}(M, M')$ is mapped into a set function $O_i \in Hom_{Set}(O(M'), O(M))$ which acts on $\hat{P} \in O(M')$ as $O_i(\hat{P}) = \delta_M(\hat{P})$.

[8] One may in fact think to a state for the C*-algebra $B(H)$ that forms locally a valuation on commutative W*-subalgebra, but that is not globally a valuation, see quasi-state in [24, 28].

With this definition, as $P(H)$ is assumed to describe the logical structure in the global environment, then the outer presheaf O seems to be a first approach of a representation of $P(H)$ in the local structure defined by the spectral presheaf Σ.

IV.6 Linking the global environment: desainization map. The power of the topos-theoretical approach to quantum logic theories may be shown as said in the simplicity in managing relations between local and global contexts. However it is to be noted that this approach has as its own algebraic environment not the category $V(H)$ but the category of preshaves $[V(H)^{op}, Set]$, where this means that the approach does not study an individual local context but the whole category which it belongs to. This property emphasizes the nature of the topos-theoretical approach as a "bridge" between locality and globality.

For this purpose recalling the definition I.5.1 of global element especially in the case of presheaves, a global element of a presheaf in $[V(H)^{op}, Set]$ is an algebraic entity independent from $V(H)$, i.e. independent from the single context of measurement, but only dependent on the structure over $V(H)$. Therefore the existence of global elements in $[V(H)^{op}, Set]$ is related to the existence of a property for the global context or for a non-contextual environment.
Example. A clear example of this is the categorical formulation of Kochen-Specker's theorem, that states:

Given H with $dim\ H > 2$, then the spectral presheaf Σ on $V(H)$ has no global elements.

as demonstrated in [12, sec.5.1.2] [9].

Then for the purpose of the construction of a quantum logical theory from both local and global environment, it needs to be built a global element $\gamma_O^{\hat{P}}$ of the outer presheaf on $V(H)$ for every projector P. This can be obtained using the definition of contextual projector as follows (see [12, sec.5.2]):

$$\gamma_O^{\hat{P}}(\_) = \delta\_(\hat{P})$$

Thus we use $\delta\_(\hat{P})$ which is a non-contextual representation of $\hat{P}$. As any non-contextual inference is represented by an element in $P(H)$, then the inclusion $\delta : P(H) \to \Gamma(O)$, called *daseinization map*[10], is the non-contextual reproduction of $P(H)$ and its structures into the local presheaf environment, or equivalently a topos-theoretical representation of von Neumann and Birkhoff quantum logic structure. This is more evident when we compose δ with a map from $\Gamma(O)$ to $Sub_{cl}(\Sigma)$ obtaining a new map $\overline{\delta}$ as it is done in [12], where $Sub_{cl}(\Sigma)$ is the collection of sub-objects S of Σ whose components S_V are clopen subsets of Σ_V.

Once this is done, from the point of view of the physical language $L(\Lambda)$, given an inference "$\tilde{F}\epsilon\Delta$" one has *non-contextual quantum logic theory* on Λ as:

$$\Pi_{ncQ}(\tilde{F}\epsilon\Delta) = \overline{\delta}(\hat{P}[\tilde{F}\epsilon\Delta])$$

where the lattice structure over $P(H)$ furnishes the algebraic representation of the propositional calculus.[11]

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[9] Referring to this paper, for the Σ presheaf a global element is a function γ that maps every operator $\hat{F}$ to an element $\gamma(\hat{F}) \in sp(\hat{F})$ such that if $\hat{F}_2 = f(\hat{F})$ then $f(\gamma(\hat{F})) = \gamma(\hat{F}_2)$, but this function coincides with an evaluation over B(H) under the assumption of FUNC axiom.

[10] "Dasein" is a term in Heideggerian lexicon translatable in english as "existence-in" or "being-in". Used in this context, "daseinization" indicates a process of "reification" or "transposition to reality".

[11] For more details over the desainization map see [12, sec.5.3].

Conclusion

The topos-theoretic approach proposed briefly in these pages has two main purposes that justify its conception and its adoption.

The first purpose concerns a technical resolution for the specific problem of foundation for quantum mechanics. In fact the proposed attempt is clearly the conception of a formal system, that takes the form of the topos $[V(H)^{op}, Set]$, where two concurrent perspectives on the physical reality are contemplated not resolving the primordial philosophical questions over the physical reality, but absolving the need for a formalism in which the debate could be cultivated.
In fact in this topos, the construction of the spectral presheaf Σ and of the Heyting algebra on $Sub_{[V(H)^{op},Set]}(\Sigma)$ is the proposition of a pure local informational theory, emptying the quantum logical theory from all statements of global nature; but the existence of the desainization map δ is a reconstruction of the formal language proposed by von Neumann and Birkhoff, esplicitly of a global nature.
The assumption then of one realist construction instead of another is formally the choice of one of this two formal languages instead of another, both with a formal representation in $[V(H)^{op}, Set]$.

The second purpose has a different but more airy nature. The approach is a first attempt of *categorification* of the physical reality. The introduction of such a powerful mathematical tool, as category theory is, brings in the world of physics not only new formal possibilities, but a renovate perspective on reality, introducing a structuralist vision for physical theories. This is the case in which an instrument loudly communicates a matter of content that only that specific perspective reveals.
This perspective has been whispered in the conclusive words of [12], where as the matter of the realist debate appeared to be strongly related to a relativistic foundation for the quantum mechanics (from the work of Bell [6]), Isham and Döring dream about a "topos of the universe".

Finally, it is my personal opinion that a structuralist approach to physics in the shape of a categorical formalisation would renovate the intimate relations between mathematical, philosophical and scientific perspectives on reality.

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