UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Master Degree in Physics

Final Dissertation

Higher form symmetries and orbifolds in quantum gravity

Thesis supervisor	Candidate
Prof. Roberto Volpato	Matteo Pretto

Academic Year 2021/2022

Contents

Introduction **1**

1 Higher form global symmetries **5**
- 1.1 Generalities . . . 5
  - 1.1.1 Ordinary symmetries . . . 5
  - 1.1.2 Higher form global symmetries . . . 6
  - 1.1.3 Higher form gauge symmetries . . . 8
- 1.2 A paradigmatic example: U(1) gauge theory in 4 D . . . 8
  - 1.2.1 A clarification on $U(1)$ group . . . 8
  - 1.2.2 Electric and magnetic 1-form global symmetries . . . 9
  - 1.2.3 Gauging electric and magnetic 1-form symmetries . . . 10
  - 1.2.4 Electric and magnetic 1-form symmetries with charged matter . . . 11
  - 1.2.5 $\mathbb{Z}_N$ gauge symmetry . . . 12
  - 1.2.6 Gauging a $\mathbb{Z}_N$ subgroup of the U(1) 1-form global symmetries . . . 13

2 2-group global symmetries with continuous groups **15**
- 2.1 The abelian $U(1)^{(0)} \times U(1)^{(1)}$ 2-group global symmetry . . . 15
- 2.2 A brief introduction to 't Hooft anomalies in conventional symmetries . . . 16
- 2.3 2-group symmetries from mixed 't Hooft anomalies . . . 17
- 2.4 2-group 't Hooft anomalies . . . 19
- 2.5 On gauging 2-group global symmetries . . . 20
- 2.6 A useful example: $U(1)^{(0)}_{A^{(1)}} \times U(1)^{(0)}_{A^{(2)}} \times_{\hat{k}_{12}} U(1)^{(1)}_{B}$ . . . 22

3 Brief introduction to singular (co)homology and group cohomology **25**
- 3.1 Manifold triangulation and simplicial complexes . . . 25
- 3.2 Simplicial homology and singular homology . . . 26
- 3.3 Singular cohomology . . . 28
- 3.4 Group cohomology . . . 29
- 3.5 Group cohomology for finite abelian groups . . . 32

4 2-group global symmetries with discrete groups **37**
- 4.1 Symmetry defects . . . 37
- 4.2 2-group symmetry in the formalism of defects . . . 38
- 4.3 Gauge transformations and 2-group symmetry . . . 41
- 4.4 Example: $G = \mathbb{Z}_N$ . . . 42
- 4.5 Example: $G = \mathbb{Z}_N \times \mathbb{Z}_N$ . . . 44

5 Elements of string theory **47**
- 5.1 The action principle . . . 47
- 5.2 The light-cone quantization and the spectrum . . . 49

5.2.1 Open string with Neumann boundary conditions . . . 50
5.2.2 Closed string . . . 52
5.3 The σ-model in complex coordinates . . . 53
5.4 What about fermions? . . . 54
5.5 Strings in curved spacetime . . . 54
5.6 Compactification of extra dimensions . . . 57
5.6.1 Toroidal compactification . . . 58
5.6.2 Orbifolds . . . 60

6 't Hooft anomalies cancellation in toroidal compactification of 1 dimension 63
6.1 Spacetime action after toroidal compactification . . . 63
6.2 World-sheet action after toroidal compactification . . . 65
6.3 't Hooft anomaly for a periodic free scalar in 2D . . . 66
6.4 't Hooft anomaly cancellation in $U(1)_A^{(0)} \times U(1)_B^{(0)}$ . . . 69
6.5 More on 't Hooft anomalies and their cancellation . . . 71
6.6 't Hooft anomalies in coordinate inversion . . . 74
6.7 't Hooft anomaly cancellation in $\mathbb{Z}_2^C \ltimes (U(1)_A^{(0)} \times U(1)_B^{(0)})$ . . . 76

7 't Hooft anomalies cancellation in toroidal compactification with $\mathbb{Z}_3$ symmetry 81
7.1 Compactification on a torus with $\mathbb{Z}_3$ symmetry . . . 81
7.2 Gauge field transformations . . . 85
7.3 't Hooft anomalies . . . 87
7.4 't Hooft anomaly cancellation . . . 91

A Elements of group theory 95

Bibliography 99

Introduction

Quantum field theories sit at the heart of the modern physics and symmetries play a central role in their definition. In the last few years, there has been remarkable progress in understanding the concept of global symmetry which has been generalized in many directions [1] [2] [3] [4] [5]. Ordinary global symmetries, which essentially consist in the action of a group on the space of local point-like operators, have been extended to higher form symmetries that can act on higher dimensional objects like lines, surfaces or higher dimensional submanifolds of the spacetime. Furthermore, the action of a single group has been generalized to non-trivial combinations of various q-form symmetries of different degree q called n-group global symmetries, where n is the degree of the background field of the symmetry of higher order [6] [7] [8]. These developments have had consequences in various fields of physics and their applications range from condensed matter physics to high energy physics up to quantum gravity and string theory.

In the study of global symmetries, a fundamental question is to ask when such symmetries can be gauged. We know there can be impediments in the gauging of an ordinary global symmetry and these obstacles are called 't Hooft anomalies. For instance, if we consider a theory with a 0-form global symmetry coupled to its background fields, it can happen that the partition function is no longer invariant under a gauge tranformation of the backgrounds. In this case, we cannot make the gauge fields dynamical. The same can happen in presence of higher form global symmetries and the discussion of their 't Hooft anomalies is a fruitful research topic [9] [10] [11] [12] [13].

In quantum gravity there is a deep conjecture, based principally on black holes physics, that forbids the presence of global symmetries in a consistent UV complete theory [14]. This means that any global symmetries arising in low energy effective models of gravity must be either gauged or broken at some higher energy scale. This conjecture has been also extended to all higher form global symmetries [15] [16]. Therefore, the possible presence of 't Hooft anomalies represents a relevant problem in a quantum gravity theory. Focusing essentially on string theory, we know that the dynamics of the string can be described by a conformal field theory defined on a 2-dimensional world-sheet and such a world-sheet theory does admit the presence of global symmetries. However, when we consider the string theory from the spacetime point of view, such global symmetries must be local symmetries of the spacetime action; thus, there cannot be obstructions to their gauging.

Moreover, 't Hooft anomalies can be dangerous also when we consider the orbifold of a theory. Roughly speaking, an orbifold is a way to obtain a new QFT from a 'parent' one by gauging a discrete global symmetry. The idea of orbifold has been generalized to include higher form symmetries. In string theory, starting from a world-sheet CFT with a global discrete symmetry, we can obtain a new CFT by taking this orbifold. From the spacetime point of view, this is a way to obtain a new string theory background from an old one; therefore, it is also a procedure to obtain a new string model from an old one. Obviously, the orbifold procedure fails if the symmetry subgroup that we want to 'quotient' by has a 't Hooft anomaly [17] [18].

The goal of this thesis is to study the presence of such anomalies in some simple quantum field theory models inspired by string theory. In particular, we focus on models obtained from toroidal compactification of string theory. Indeed, a consistent string theory requires a number of spacetime dimensions that is greater than the four that we usually experiment; therefore, we have to suppose that some directions must be compact and invisible at our length scale. In toroidal compactification, the compactified directions can be described by (free) scalars with symmetry group the product of different $U(1)$ subgroups. For instance, if we consider a single scalar compactified on a circle, the 0-form symmetry group is the direct product of two $U(1)$, the first due to the invariance under translations on the circle and the other whose charge is the winding number of the string around the circle. These $U(1)$ symmetries are global from the world-sheet viewpoint, but must be gauged in spacetime. As we will see, apparently these two subgroups on the world-sheet have a mixed t'Hooft anomaly that prohibits the gauging of both. However, the 0-form group is not all the symmetry content of such a model. We will verify that the theory is endowed of a non-trivial 2-group global symmetry and the transformation of the background field for the 1-form symmetry exactly cancels the mixed anomaly coming from the ordinary 0-form. These gauge transformations have been known for a long time and are called Nicolai-Townsend transformation but they were not initially interpreted as the presence of a 2-group structure which combines a 1-form symmetry with the ordinary one. A similar mechanism is well known also in superstring theory under the name of Green-Schwarz mechanism [19] [7].

In the case of continuos symmetries the non-trivial higher group structures of the string compactification models, that make the theory consistent, have been known for a long time. More subtle issues arise if we consider discrete symmetries. This case is also relevant because gauging discrete symmetries corresponds to making an orbifold. The description of such symmetries is not obvious, since there are no associated Noether currents and the usual spacetime effective action description is not useful because all the associated gauge fields have no propagating degrees of freedom. Furthermore, the 2-group structure for these discrete gauge fields is still unexplored and, in principle, it is not obvious that a discrete global symmetry on the world-sheet corresponds to a discrete gauge symmetry of the spacetime action. For instance, since the world-sheet description is valid only at the perturbative level, one can think that the global world-sheet symmetry may be broken in a non-perturbative theory. The idea that, also for discrete symmetries, the 't Hooft anomalies of the world-sheet theory are canceled by the presence of a non-trivial global 2-group has been introduced in [18]. If for the continuous symmetries the 2-group transformations have been derived by imposing the invariance of the effective action, in the case of discrete symmetries we cannot proceed in the same way. The opportune formalism to study discrete symmetries is their description in terms of a network of topological defects that act on charged operators and describe the fiber bundle associated with the background gauge fields for such symmetries. These operators can be defined even without currents. Besides, with the tools of algebraic topology, we can characterizes the anomalies for discrete symmetries with classes of opportune cohomology groups and we can describe the 2-group symmetries in a precise way.

In our work, after the development of the necessary background of discrete symmetries, we focus on two simple examples of string compactification endowed with such symmetries. First, following [18] we consider the invariance under coordinate inversion of a periodic single scalar ($\mathbb{Z}_2$ symmetry). If we consider only the 0-form symmetries in the world-sheet theory, we discover the presence of a mixed 't Hooft anomaly between $\mathbb{Z}_2$ and the two $U(1)$ subgroups. On the other hand, the presence of a non-trivial 2-group structure allows us to cancel this anomaly. Second, we consider the CFT of two scalars corresponding to the compactification of two directions on a torus with a particular $\mathbb{Z}_3$ rotational symmetry. In this case there is a non-trivial mixed anomaly in the world-sheet theory between the 0-form $U(1)$ groups and $\mathbb{Z}_3$. However, in a similar way we can verify that the presence of an higher form symmetry cancels the anomalous phase.

The thesis is divided into two main parts. The first one is an introduction on higher form symmetries and 2-groups for both continuous and discrete groups in a generic quantum field theory framework. In the second part, after presenting some elements of string theory, we discuss extensively the 't Hooft anomalies cancellation in the two string compactification models presented above.

In *Chapter 1* we introduce the concept of higher form global symmetries using from the beginning the language of topological operators. Starting from the description of ordinary global symmetries, we present the main features of higher form symmetries and discuss the electric and magnetic 1-form symmetries in 4 dimensions Maxwell theory as paradigmatic example. These global 1-form symmetries present a mixed 't Hooft anomaly that prevents their gauging. Such 1-form symmetries can be broken by the insertion of electric or magnetic charged matter. Of particular interest is the discussion of the $\mathbb{Z}_N$ gauge symmetry that arise when we consider the Higgsing by a charge N Higgs field.

Chapter 2 is devoted to the introduction of 2-group global symmetries with continuos groups, which can be presented via the lagrangian formulation by describing the coupling of the theory with background gauge fields. We mainly focus on the so called 'Abelian' 2-group, where a 0-form $U(1)^{(0)}$ symmetry is combined with a 1-form $U(1)^{(1)}$ group. We discuss its origin from a parent theory with 2 $U(1)^{(0)}$ symmetry group and a mixed 't Hooft anomaly. We present also the gauging of the global 2-group and its obstructions since 2-groups may in turn have 't Hooft anomalies. Finally, a preparatory example is considered since we discuss the 2-group $U(1)^{(0)}_{A^{(1)}} \times U(1)^{(0)}_{A^{(2)}} \times_{\hat{k}_{12}} U(1)^{(1)}_{B}$.

Chapter 3 is a sort of mathematical chapter that introduces rudimental concepts in algebraic topology. In order to discuss discrete symmetries, it is useful to use singular cohomology to describe the flat gauge backgrounds over manifold triangulations. Furthermore, anomalies and 2-group symmetries are labeled by classes in group cohomology; we principally focus on the study of the third cohomology group $H^3(\mathbb{Z}_N^k, U(1))$ which labels the Postnikov classes of the 2-group discrete symmetries.

In *Chapter 4* we exploit the results of the *Chapter 3* to present in the more general way discrete global symmetries via topological defects and 2-group discrete symmetries with the related gauge transformations. The finite groups $\mathbb{Z}_N$ and $\mathbb{Z}_N \times \mathbb{Z}_N$ are the 0-form part of the two basic examples that we study at the end of the chapter.

Chapter 5 is a basic introduction to string theory that we need in order to introduce the physical content of the models that we will study. We present the world-sheet action for a relativistic bosonic string on a flat background and its massless spectrum, that is the field content of the theory. We consider also the world-sheet action and the spacetime effective action for strings on a curved background. Then, we focus on string compactification: toroidal compactification of one dimension is the simplest case that we discuss in detail, but we present also a brief sketch on orbifolds.

In *Chapter 6* we discuss the 't Hooft anomalies in the string model for a single direction compactified on a circle. Initially, we present the standard result about the anomaly between the two $U(1)$ groups for a single periodic scalar and then we focus on the mixed anomaly under coordinate inversion. Both these gauging obstructions are canceled by the presence of a non-trivial 2-group.

Finally, in *Chapter 7* we discuss the anomalies cancellation in a theory with two direction compactified in a torus with $\mathbb{Z}_3$ rotational symmetry. This example constitutes the completely original part of our work.

In *Appendix A* we provide the basic definitions in group theory that are been used in our thesis.

Chapter 1

Higher form global symmetries

Global symmetries in quantum field theory usually act on operators that are supported on points in space-time. It has been recently understood that there exist symmetries that only act on operators supported on higher dimensional submanifolds of space-time, such as lines, surfaces... These are called higher form symmetries. We want to introduce the concept of higher form global symmetries using from the beginning the language of topological operators. We start by describing ordinary global symmetries; then, we present the main features of higher form symmetries and we discuss the electric and magnetic 1-form symmetries in 4 dimensions Maxwell theory as paradigmatic example. We essentially follow [2] as main reference.

1.1 Generalities

1.1.1 Ordinary symmetries

In this section we present ordinary global symmetries in the language of differential forms in order to introduce the most appropriate formalism to describe generalized symmetries.

We start by considering a generic QFT supported in a D-dimensional manifold. In general, symmetry transformations form a group G that can be Abelian or non Abelian. If the group is continuous, for every continuos generator there is a conserved Nother current, which we express as a (D-1)-form j_{D-1}. The conservation of the current is equivalent to the closure of the form associated: $\mathrm{d}j_{D-1} = 0$. For those who are more familiar with the tensor notation, we observe that the Nother current j_μ is usually associated with a 1-form j_1 connected by Hodge duality with the one we have considered so far ($j_1 = *j_{D-1}$).

The charged operators are local operators, supported on manifolds of dimension 0, i.e., points in space-time and we denote them as $V(\mathcal{P})$. The conserved charge is the integral

$$Q(M^{(D-1)}) = \oint_{M^{(D-1)}} j_{D-1} \tag{1.1}$$

where $M^{(D-1)}$ is a (D-1)-dimensional manifold, typically space (at fixed time t). In general we consider it as a closed (D-1)-dimensional space.

In order to describe symmetry transformations, it is useful to consider a topological operator associated with the manifold $M^{(D-1)}$ and labeled by $g \in G$, a group element of the global symmetry. We denote this charge operator as $U_g(M^{(D-1)})$ and if G is continuos, we can define it by exponentiating $Q(M^{(D-1)})$. In general, both for discrete and for continuos symmetries, we can define $U_g(M^{(D-1)})$ by cutting spacetime along $M^{(D-1)}$ and assigning a symmetry transformation to charged objects that cross $M^{(D-1)}$. These transformations must satisfy the group multiplication law

$$U_g(M^{(D-1)}) \times U_{g'}(M^{(D-1)}) = U_{g''}(M^{(D-1)}) \tag{1.2}$$

where $g, g', g'' \in G$, with $g'' = gg'$ in the group.

$U_g(M^{(D-1)})$ is said to be topological because a slight modification of $M^{(D-1)}$ without crossing charged operators $V(\mathcal{P})$ does not affect the correlator in which the symmetry operator is inserted. Specifically, considering a sphere $\mathbb{S}^{D-1}$ surrounding $\mathcal{P}$ we have

$$U_g(\mathbb{S}^{D-1})V_i(\mathcal{P}) = R_i^j(g)V_j(\mathcal{P}) \tag{1.3}$$

where $R_i^j(g)$ is an appropriate representation of $g \in G$ determined by the charge of V. This is the Ward identity associated with this symmetry. For example, if the group is $U(1)$ the group element is a phase and $R_i^j(g) = g^{q(V)}$ where $q(V)$ is the charge of V.

1.1.2 Higher form global symmetries

A generalized global symmetry, also called q-form global symmetry, is a global symmetry that acts on charged operator $V(\mathcal{C}^q)$ supported on q-dimensional manifolds $\mathcal{C}^q$. If the group is continuos, the symmetry parameter is a closed q-form λ_q. The pairing $\int_{\mathcal{C}^q} \lambda_q$ determines the operator tranformations and the theory can be coupled to a background $(q+1)$-form connection.

Like for the ordinary symmetries, if the symmetry group G is continuos, the q-form global symmetry is associated to a closed $(D-q-1)$-form current j_{D-q-1} (or a conserved current of rank $(q+1)$ in tensorial notation). We can define the standard charge operators integrating the $(D-1)$-form $\lambda \wedge j$ on space. However, it is more useful to consider charge operators $U_g(M^{(D-q-1)})$ defined in a co-dimension $(q+1)$ manifold $M^{(D-q-1)}$, obtained by integrating on it the current j_{D-q-1}.

In general, both for continuos and both for discrete symmetry groups, a higher form global symmetry is defined by the existence of topological operators $U_g(M^{(D-q-1)})$ associated with co-dimension (q+1) manifolds $M^{(D-q-1)}$ subject to the group law

$$U_g(M^{(D-q-1)}) \times U_{g'}(M^{(D-q-1)}) = U_{g''}(M^{(D-q-1)}) \tag{1.4}$$

where $g, g', g'' \in G$, with $g'' = gg'$ in the group, as before.

This relation can be interpreted as the multiplication rule of two operators acting at a given time t along the manifold $M^{(D-q-1)}$. In particular, if we insert the two operators at arbitrarily small different times t and $t+\epsilon$, we can study the ordering of operators using the standard time-ordering. For $q = 0$ (ordinary symmetries) $M^{(D-q-1)}$ is of co-dimension 1 and the operators $U_g(M^{(D-q-1)})$ at the different times do not necessarily commute. Therefore, the symmetry group G can be non Abelian. In contrast, for $q > 0$ the manifold $M^{(D-q-1)}$ at time $t+\epsilon$ can be continuosly deformed to the one at time t. Hence, the two operators in the multiplication law must commute and G must be Abelian.

Since the charged objects are q-branes rather than particles, the ordinary charges obtained by integrating the current on the entire space is infinite; only the charge per unit q-volume (along the brane) is finite.

The Ward identity for the generalized symmetries is given by the following expression

$$U_g(\mathbb{S}^{D-q-1})V(\mathcal{C}^q) = g(V)V(\mathcal{C}^q) \tag{1.5}$$

where $U_g(\mathbb{S}^{D-q-1})$ is supported on a small $(D-q-1)$-dimensional sphere which links once with $V(\mathcal{C}^q)$, supported on a q-dimensional manifold, and $g(V)$ is the representation of g that is simply a phase determined by the charge of V.

We can also consider the special case of the equal-time commutation relation given by

$$U_g(M^{(D-q-1)})V(\mathcal{C}^q) = g(V)^{\mathcal{I}(\mathcal{C}^q, M^{(D-q-1)})}V(\mathcal{C}^q)U_g(M^{(D-q-1)}) \tag{1.6}$$

where $\mathcal{I}(\mathcal{C}^q, M^{(D-q-1)})$ is the intersection number. We can easily see that the symmetry transformation is implemented whenever a charged object crosses $M^{(D-q-1)}$.

In order to remove such generalized global symmetries we can gauge these symmetries or break them explicitly (for more details one can see [16]). The gauging of a q-form global symmetry with current j_{D-q-1} can be achieved by coupling the current to a dynamical $(q+1)$-form gauge field B_{q+1}:

$$S \supset \int \Big(-\frac{1}{2g^2}H_{q+2} \wedge *H_{q+2} + B_{q+1} \wedge j_{D-q-1}\Big) \tag{1.7}$$

with $H_{q+2} = \mathrm{d}B_{q+1}$ locally.
The gauge transformation for B_{q+1} is a q-form gauge symmetry (see below) given by:

$$B_{q+1} \to B_{q+1} + \mathrm{d}\lambda_q, \tag{1.8}$$

with λ_q a q-form. Since j_{D-q-1} is closed, the action is invariant under these gauge transformation up to boundary terms. The equation of motion for B_{q+1} tells us that the current j_{D-q-1} is also exact:

$$j_{D-q-1} = (-1)^{q+1}\frac{1}{g^2}\mathrm{d} * H_{q+2}. \tag{1.9}$$

This equation generalizes the Gauss law and implies that every integral of j_{D-q-1} on a closed manifold vanish.

In terms of charged operators, gauging a q-form symmetry implies that these operators are no longer genuine operators once the symmetry has been gauged. Instead the charged operators will represent boundaries of higher-dimensional gauge-invariant operators. For instance, considering the gauging of a local operator with charge N under a $U(1)$ 0-form global symmetry, we get that the local operator is no longer gauge invariant, but it becomes the endpoint of a Wilson line of charge N.

A global symmetry can also be broken by adding terms to the Lagrangian that violate the Noether current conservation. Starting from the action defined above, we can have an example observing that, in the absence of the coupling $B_{q+1} \wedge j_{D-q-1}$ the current $j_{D-q-2} := *H_{q+2}$ is a conserved current for a $(q+1)$-form symmetry. The coupling $B_{q+1} \wedge j_{D-q-1}$ explicitly breaks this $(q+1)$-form global symmetry. We will find another example studying the $U(1)$ gauge theory.

In terms of charged operators, the explicit breaking of a q-form symmetry is a modification of the theory that allows the charged operators to live on q-dimensional manifold with non trivial boundary. For instance, in the pure $U(1)$ gauge theory the Wilson lines are supported on closed manifolds without boundary; in contrast, in the presence of charged matter Wilson lines can end on a local excitation of the charged particles.

1.1.3 Higher form gauge symmetries

Without going into detail, we want to briefly introduce some terminology and some idea about higher form gauge symmetries. Ordinary gauge field are represented by a 1-form A_1 that is precisely the 1-form of connection on the principal bundle of the symmetry group. For a $U(1)$ group, the gauge transformation are parametrized by a scalar function $\lambda \in U(1)$:

$$A_1 \to A_1 + \mathrm{d}\lambda. \tag{1.10}$$

In general, a principal bundle can be described by choosing a covering $\{U_i\}$ of the manifold in which the theory lives and by specifying the transition functions f_{ij} for the local trivialization of the principal bundle in the overlapping of U_i and U_j, being careful about the cocycle condition

$$f_{ij} f_{jk} = f_{ik} \tag{1.11}$$

on triple overlapping $U_i \cap U_j \cap U_k$.

Higher form gauge symmetries are generalization of the above formalism. For the simple case of $U(1)$, a q-form gauge symmetry is naturally associated to a $(q+1)$-form A_{q+1} subjected to the gauge transformation

$$A_{q+1} \to A_{q+1} + \mathrm{d}\lambda_q \tag{1.12}$$

where λ_q is a q-form. More precisely, λ_q is a q-form gauge field with the opportune transition functions associated with its own gauge symmetry. For instance, a $U(1)$ q-form gauge symmetry has the q-form transition functions λ_q in turn subject to transformation $\lambda_q \to \lambda_q + \mathrm{d}\lambda_{q-1}$. The process stops when reaches the 0-form λ, interpreted as $\mathbb{S}^1$-valued function.

1.2 A paradigmatic example: U(1) gauge theory in 4 D

1.2.1 A clarification on $U(1)$ group

Before starting the discussion about the 1-form symmetry of a $U(1)$ gauge theory, we will focus on the specificity of the group $U(1)$. $U(1)$ is the group of 1-dimensional unitary transformations and it is a compact group since it is isomorphic to the circle $\mathbb{S}^1$. Its irreducible representations are all 1-dimensional and act by multiplication by a phase $e^{i\alpha}$.

The fact that $U(1)$ is compact means that the gauge parameter λ takes values in $\mathbb{S}^1$; therefore, it is subjected to the identification $\lambda \sim \lambda + 2\pi$. Thus, if we consider the integral over 1-cycle of $\mathrm{d}\lambda$ it is non trivially 0, but it is an integer:

$$\frac{1}{2\pi} \oint_\gamma \mathrm{d}\lambda \in \mathbb{Z}. \tag{1.13}$$

Roughly speaking, this can be understood thinking that each time λ travels through a closed path, it wraps around the circle $\mathbb{S}^1$ an integer number of times. A more elegant explanation is given by considering a non trivial $U(1)$ bundle which is the topological setting of the monopoles. Let us consider a principal bundle with fibre $U(1)$ and base space $\mathbb{S}^2$. We consider the open charts on the sphere

$$\begin{aligned} U_N &= \{(\theta,\phi) | 0 \le \theta \le \pi/2 + \epsilon, 0 \le \phi < 2\pi\} \\ U_S &= \{(\theta,\phi) | \pi/2 - \epsilon \le \theta \le \pi, 0 \le \phi < 2\pi\} \end{aligned} \tag{1.14}$$

where θ, ϕ are the usual polar angles and the intersection $U_N \cap U_S$ is essentially the equator. The transition function t_{NS} is a $U(1)$ valued function of the form $e^{in\phi}$ where n must be integer since we want that t_{NS} is uniquely defined on the equator. Therefore, t_{NS} is a map from $\mathbb{S}^1$ to $U(1) \simeq \mathbb{S}^1$ and the integer n labels the different maps that are classified by the homotopy group $\pi_1(U(1)) \simeq \mathbb{Z}$. If $n = 0$, the transition function is the identity in the structure group and the bundle are trivial, i.e. $\mathbb{S}^2 \times \mathbb{S}^1$. If $n \neq 0$, the $U(1)$ bundle is twisted and the integer characterizes how two local section are pasted togheter at the equator.

The compactness of $U(1)$ is at the basis of the quantization of the $U(1)$ charges. Taking, for example, the magnetic charge in the Wu-Yang potential, we want that in the intersection $U_N \cap U_S$ the potential differs by pure gauge since the physics does not depend on the choice of the gauge. Thus,

$$A_N - A_S = -e^{-i\lambda(\phi)} \mathrm{d} e^{i\lambda(\phi)} = \mathrm{d}\lambda(\phi) \tag{1.15}$$

with λ the map from $\mathbb{S}^1$ to $\mathbb{R}$ with the identification $\lambda \sim \lambda + 2\pi n$, $n \in \mathbb{Z}$. If $A_N = Q_m(1-\cos\theta)\mathrm{d}\phi$ and $A_S = -Q_m(1+\cos\theta)\mathrm{d}\phi$, we obtain $A_N - A_S = 2Q_m \mathrm{d}\phi$ and

$$\int_0^{2\pi} 2Q_m \mathrm{d}\phi = 4\pi Q_m = \int \mathrm{d}\lambda(\phi) \in 2\pi\mathbb{Z} \tag{1.16}$$

so we have the quantization of the magnetic charge $2Q_m \in \mathbb{Z}$. As usual, the quantization of the electric charge can be shown considering the holonomy of a electrically charged particle moving around a monopole.

Finally, we have to notice that also the background field strength $F_2 = \mathrm{d}A_1$ must have integer fluxes through closed 2-cycle Σ

$$\frac{1}{2\pi} \oint_\Sigma F_2 \in \mathbb{Z}. \tag{1.17}$$

This can be easily checked in the particular topology of the monopole considered above. In fact,

$$\int_{\mathbb{S}^2} F_2 = \int_{U_N} \mathrm{d}A_N + \int_{U_S} \mathrm{d}A_S = \int_{\mathbb{S}^1} A_N - \int_{\mathbb{S}^1} A_S = \int_{\mathbb{S}^1} \mathrm{d}\lambda \in 2\pi\mathbb{Z}. \tag{1.18}$$

1.2.2 Electric and magnetic 1-form global symmetries

Let us consider the Maxwell $U(1)$ theory without matter in $D = 4$. The action reads:

$$S = -\frac{1}{2g^2} \int F_2 \wedge *F_2 \tag{1.19}$$

where F_2 is the curvature associated of the $U(1)$ bundle with 1-form connection A_1

$$F_2 = \mathrm{d}A_1 \tag{1.20}$$

and g the gauge coupling constant. The gauge transformation is $A_1 \to A_1 + \mathrm{d}\lambda$ with $\lambda \in U(1)$, i.e. a $\mathbb{S}^1$-valued function. The equation of motion and the Bianchi identity are respectively

$$\mathrm{d} * F_2 = 0 \qquad \text{and} \qquad \mathrm{d}F_2 = 0. \tag{1.21}$$

In this theory we have two $U(1)$ 1-form symmetries. The first one is the electric $U(1)$ 1-form global symmetry associated with the 2-form current $j_2^e = \frac{1}{g^2} * F_2$ closed by the equation of motion. The symmetry operator is

$$U^E_{g=e^{i\alpha}}(M^{(2)}) = e^{i\frac{\alpha}{g^2}\int_{M^{(2)}} *F_2} \tag{1.22}$$

where the integral is the eletric flux through $M^{(2)}$ and, clearly, $\alpha \in [0, 2\pi)$ is a phase. The charged objects are Wilson loops supported on closed 1-dim manifold γ and labeled by an electric charge $p \in \mathbb{Z}$:

$$W_p(\gamma) = e^{ip \oint_\gamma A_1}. \tag{1.23}$$

The action of the 1-form symmetry operator U^E on the Wilson loops is in accordance with the Ward identity. If the Wilson loop is supported on a curve γ surrounded by an $\mathbb{S}^2$ in which $U^E(\mathbb{S}^2)$ is defined, we have $U^E_{e^{i\alpha}}(\mathbb{S}^2) W_p(\gamma) = e^{i\alpha p} W_p(\gamma)$.

The second 1-form global symmetry is called the $U(1)$ 1-form magnetic symmetry. The closed current by the Bianchi identity is $j_2^m = \frac{1}{2\pi} F_2$ and the symmetry is generated by

$$U^M_{g=e^{i\eta}}(M^{(2)}) = e^{i\frac{\eta}{2\pi}\int_{M^{(2)}} F_2}. \tag{1.24}$$

The integral $\int_{M^{(2)}} F_2$ is the magnetic flux through $M^{(2)}$ and the charged operator are 't Hooft loops V_m supported on closed 1-dim manifold and labeled by a magnetic charge $m \in \mathbb{Z}$. The Ward identity holds in the same way.

We note that in D dimensions the global electric symmetry is still a 1-form symmetry with current j^e_{D-2}, while the magnetic $U(1)$ symmetry is a $(D-3)$-form symmetry with the usual 2-form current j^m_2.

If we include in our theory also symmetry operators defined on open 2-dimensional manifolds, then we need to consider the boundary of such manifolds. In the case of the magnetic operator U^M supported on a open manifold Σ_2, the boundaries of such manifold are improperly quantized Wilson loops. We can easily see this by applying the Stoke's theorem without taking into account topological subtleties:

$$U^M_{g=e^{i\eta}}(\Sigma_2) = e^{i\frac{\eta}{2\pi}\int_{\Sigma_2} F_2} = e^{i\frac{\eta}{2\pi}\int_{\Sigma_2} \mathrm{d}A_1} = e^{i\frac{\eta}{2\pi}\oint_{\gamma=\partial\Sigma_2} A_1}. \tag{1.25}$$

The loop is said to be improperly quantized since it does not satisfy the Dirac quantization condition. Similarly, an open manifold supporting an electric 1-form charge operator U^E is bounded by an improperly quantized 't Hooft loop. Notice that these improperly quantized operators depend also on Σ_2 and not only on γ; therefore, they are not 'genuine' line operators.

1.2.3 Gauging electric and magnetic 1-form symmetries

In order to remove the two 1-form $U(1)$ global symmetries we try to gauge them with some background gauge field. Following the standard procedure exposed above we couple the j^e_2 and j^m_2 respectively to the gauge field B_2 and C_2 subjected to the gauge tranformation $B_2 \to B_2 + \mathrm{d}b_1$ and $C_2 \to C_2 + \mathrm{d}c_1$. The resulting action, neglecting the kinetic term for the 2-form gauge fields, reads:

$$S = -\frac{1}{2g^2}\int (\mathrm{d}A_1 + B_2) \wedge *(\mathrm{d}A_1 + B_2) + \frac{i}{2\pi}\int C_2 \wedge \mathrm{d}A_1. \tag{1.26}$$

To see the explicit gauging of the current, we note that from the first part of 1.26 we obtain the kinetic term for A_1 and the gauging for the electric global 1-form:

$$\begin{aligned} &-\frac{1}{2g^2}\int (\mathrm{d}A_1 \wedge *\mathrm{d}A_1 + \mathrm{d}A_1 \wedge *B_2 + B_2 \wedge *\mathrm{d}A_1 + B_2 \wedge *B_2) = \\ &= -\frac{1}{2g^2}\int (\mathrm{d}A_1 \wedge *\mathrm{d}A_1 + 2B_2 \wedge *\mathrm{d}A_1 + B_2 \wedge *B_2) \end{aligned} \tag{1.27}$$

where we used the property of the Hodge duality into the exterior product.

In order to make gauge invariant the first term of 1.26 we require that under the gauge transformation for B_2 also A_1 transforms as $A_1 \to A_1 - b_1$. This is precisely the transformation of A_1 induced by the 1-form global symmetry. In contrast, the second term is not invariant under this transformation of A_1. We can modify the second term in such a way to making invariant under the gauge transformation of B_2 and the related transformation of A_1. If we consider $\int C_2 \wedge (\mathrm{d}A_1 + B_2)$, we reach our goal, but then it is no longer invariant under the gauge symmetry of C_2. Therefore, one of the transformation of B_2 or C_2 is always violated. This fact is a signal of the presence of a mixed 't Hooft anomaly for the two 1-form $U(1)$ global symmetries (see [13]). A 't Hooft anomaly, as we will see, is an obstruction in the transformation of a global symmetry into a local one.

To conclude, either one of the two 1-form $U(1)$ symmetries can be separately gauged through coupling the current to a background gauge field, but an anomaly prevent us from gauging both of them.

1.2.4 Electric and magnetic 1-form symmetries with charged matter

The only way to remove both the two $U(1)$ 1-form symmetries is to break explicitly both the symmetries adding charged particles. We will study only the breaking of the electric one.

Consider the $U(1)$ action with the addition of electrically charged massive particles represented by a complex 0-form ϕ.

$$S = \int \Big(-\frac{1}{2g^2} F_2 \wedge *F_2 - (\mathrm{d}\phi^\dagger + iA_1) \wedge *(\mathrm{d}\phi - iA_1) - m^2 \phi^\dagger \wedge *\phi \Big). \tag{1.28}$$

The equation of motion for the gauge field is

$$\frac{1}{g^2} \mathrm{d} * F_2 = *(i\phi^\dagger(\mathrm{d}\phi - iA_1) - i\phi(\mathrm{d}\phi^\dagger - iA_1)) \neq 0. \tag{1.29}$$

It is evident that the 2-form current j_2^e of the electric global symmetry is no longer conserved and then the correspondent 1-form global symmetry is broken.

Of greater interest may be the case of adding a field ϕ of electric charge N. We will study this case first from a topological point of view (see for example [15]) and then from the explicit Higgsing of the $U(1)$ action.

From a topological point of view, the addition of matter charge N fields allows the Wilson line of charge N to be supported on a manifold with boundary. The boundaries of the Wilson line γ are the pointlike support of the matter fields that create and destroy the charge "transported" on γ. Now, we have to familiarize with a general properties of topological operators.

Following [20] [15], we define as endable an operator $V(\mathcal{C}^{(q)})$ supported on a manifold $\mathcal{C}^{(q)}$ with boundary. If a topological operator $U_g(\mathbb{S}^{D-q-1})$ surrounds an endable operator $V(\mathcal{C}^{(q)})$, it may be either shrunk to a point getting a factor $g(V)$ according to Ward identity, or unlink from $V(\mathcal{C}^{(q)})$ and then shrunk to a point, yelding a factor $g(1)$. Therefore, this implies that $g(V) = g(1)$, i.e. any endable operator must link trivially with any topological operator.

This is the case of endable Wilson line of charge N. The electric 1-form symmetry operators $U_{g=e^{i\alpha}}(M^{D-q-1})$ that survive are the ones which have $\alpha = \frac{i\pi k}{N}$ with $k \in \mathbb{Z}$, since they remain topological. The others, instead, are no longer topological. We see that the $U(1)$ electric 1-form global symmetry is broken to $\mathbb{Z}_N$ in the presence of charge N matter fields.

If we consider the Higgsing of the $U(1)$ action with charge N Higgs field the consequence are more dramatic since we also break the $U(1)$ gauge symmtery to a $\mathbb{Z}_N$ gauge symmetry. We will study this case in detail in the next section.

1.2.5 $\mathbb{Z}_N$ gauge symmetry

The standard description of a $\mathbb{Z}_N$ gauge theory is in terms of charts and $\mathbb{Z}_N$ transition functions between them for the local trivialization. In this formulation there is no continuos degrees of freedom and the action vanishes. As we will see, one can describe the gauge bundle in term of the insertion of topological defects that represent the transition functions of the bundle (see 4.1).

However, we can also describe such a theory in terms of a continuos $U(1)$ gauge theory higgsed by a Higgs field ϕ of charge N using the Lagrangian formalism (we essentially follow [14], [1]). We can consider the complex scalar field in its polar form $\phi = \rho e^{i\varphi}$ with $\varphi \in \mathbb{S}^1$. We can consider only the angular variable taking into account of the vacuum expectation value in the costant $t \in \mathbb{R}$ in the following lagrangian:

$$\mathcal{L} = t^2(\mathrm{d}\varphi - NA_1) \wedge *(\mathrm{d}\varphi - NA_1) - \frac{1}{2g^2} F_2 \wedge *F_2 \tag{1.30}$$

where the phase is subjected to the identification $\varphi \sim \varphi + 2\pi$. Since ϕ carries charge N under the $U(1)$ gauge symmetry the gauge transformation for φ is $\varphi \to \varphi + N\lambda$ with $\lambda \in \mathbb{S}^1$, while for A_1 it is the usual $A_1 \to A_1 + \mathrm{d}\lambda$. From this lagrangian we can easily see the effect of the higgsing in the breaking down of the $U(1)$ gauge symmetry into $\mathbb{Z}_N$.

In the low energy limit $t^2 \to \infty$ we have $A_1 = \frac{1}{N}\mathrm{d}\varphi$ and then the connection is flat and there are no local degrees of freedom. However, the holonomy around any non contractible loop is $\frac{1}{2\pi}\oint A_1 \in \frac{1}{N}\mathbb{Z}$.

In order to understand more deeply, we can dualize φ. We introduce a 3-form H_3 with quantized period, i.e. whose integral over a 3-cycle is integer, and we obtain the lagrangian

$$\mathcal{L} = \frac{1}{(4\pi)^2 t^2} H_3 \wedge *H_3 + \frac{i}{2\pi} H_3 \wedge (\mathrm{d}\varphi - NA_1) - \frac{1}{2g^2} F_2 \wedge *F_2 \tag{1.31}$$

that is equivalent to the starting one integrating out H_3 by its equation of motion: $*H_3 = 4\pi i t^2(\mathrm{d}\varphi - NA_1)$. In the low energy limit we can see H_3 as a Lagrange multiplier imposing $A_1 = \frac{1}{N}\mathrm{d}\varphi$.

The equation of motion $\mathrm{d}H_3 = 0$ guarantees that we can write locally $H_3 = \mathrm{d}B_2$. Therefore, integrating by parts up to boundary terms [1] , we get

$$\mathcal{L} = \frac{1}{(4\pi)^2 t^2} H_3 \wedge *H_3 + \frac{iN}{2\pi} B_2 \wedge \mathrm{d}A_1 - \frac{1}{2g^2} F_2 \wedge *F_2. \tag{1.32}$$

This lagrangian is known as BF theory (given by the coupling $\frac{iN}{2\pi} B_2 \wedge F_2$), a topological quantum field theory representing a $\mathbb{Z}_N$ gauge theory. The gauge transformation are the 0-form gauge $A_1 \to A_1 + \mathrm{d}\lambda$ and the 1-form gauge $B_2 \to B_2 + \mathrm{d}\lambda_1$.

To making explicit the $\mathbb{Z}_N$ 1-form gauge symmetry of B_2 we can dualize A_1

$$\mathcal{L} = \frac{1}{(4\pi)^2 t^2} H_3 \wedge *H_3 - \frac{1}{2g^2} F_2 \wedge *F_2 + \frac{i}{2\pi} F_2 \wedge (\mathrm{d}\hat{A}_1 - NB_2) \tag{1.33}$$

[1] We use the fact that $\mathrm{d}B_2 \wedge \mathrm{d}\varphi = \mathrm{d}(B_2 \wedge \mathrm{d}\varphi)$ and $\mathrm{d}B_2 \wedge A_1 = \mathrm{d}(B_2 \wedge A_1) + B_2 \wedge \mathrm{d}A_1$.

with gauge transformation $\hat{A}_1 \to \hat{A}_1 + \mathrm{d}\lambda - N\lambda_1$ and $B_2 \to B_2 + \mathrm{d}\lambda_1$. Integrating out F_2 using the equation of motion $*F_2 = -\frac{ig^2}{2\pi}(\mathrm{d}\hat{A}_1 - NB_2)$ we get

$$\mathcal{L} = \frac{1}{(4\pi)^2 t^2} H_3 \wedge *H_3 + \frac{g^2}{8\pi^2}(\mathrm{d}\hat{A}_1 - NB_2) \wedge *(\mathrm{d}\hat{A}_1 - NB_2) \tag{1.34}$$

where the gauge transformation are $\hat{A}_1 \to \hat{A}_1 + \mathrm{d}\lambda - N\lambda_1$ and $B_2 \to B_2 + \mathrm{d}\lambda_1$. We can now easily interpret the vector field $\hat{A}_1$ as a matter field charged under the 1-form gauge symmetry given by B_2. This 1 form gauge symmetry is broken from $U(1)$ to $\mathbb{Z}_N$ by the higgsing with $\hat{A}_1$ and it is a sort of "emergent" symmetry always associated with the $\mathbb{Z}_N$ ordinary gauge symmetry.

We have to note that both the fields of the BF-theory are $U(1)$ gauge fields, but the distinct observables are labeled by $\mathbb{Z}_N$. The local gauge invariant operator are trivial by the equation of motion, indeed we can consider $\mathrm{d}\varphi - NA_1 \sim *H_3$ and $\mathrm{d}\hat{A}_1 - NB_2 \sim *F_2$. However, there are two electric (Wilson) operator: a Wilson line and a "Wilson surface":

$$W_{A_1}(\gamma, n_{A_1}) = e^{in_{A_1}\oint_\gamma A_1} \qquad W_{B_2}(M^{(2)}, n_{B_2}) = e^{in_{B_2}\oint_{M^{(2)}} B_2} \tag{1.35}$$

The Wilson line describe a particle of charge n_{A_1} and worldline γ, while the "Wilson surface" the insertion of a vortex string with worldsheet $M^{(2)}$. Considering the equation of motion $B_2 = g^2 \frac{2\pi i}{N} * F_2$ we have

$$\langle W_{A_1}(\gamma, n_{A_1}) W_{B_2}(M^{(2)}, n_{B_2})\rangle \sim \exp \frac{2\pi i n_{A_1} n_{B_2} \mathcal{I}(\gamma, M^{(2)})}{N} \tag{1.36}$$

with $\mathcal{I}(\gamma, M^{(2)})$ is the linking number of γ and $M^{(2)}$. Therefore only $n_{A_1} \mod N$ and $n_{B_2} \mod N$ label distinct operators. A $\mathbb{Z}_N$ gauge transformation shifts the charges $n_{A_1,B_2} \to n_{A_1,B_2} + N$. This is the non trivial content of the $\mathbb{Z}_N$ gauge theory.

1.2.6 Gauging a $\mathbb{Z}_N$ subgroup of the U(1) 1-form global symmetries

Finally, we can show how to gauge a $\mathbb{Z}_N$ subgroup of the two U(1) 1-form global symmetries. Putting together what has been studied in the gauging of a global symmetry and in the topological BF theory, we can gauge the electric $\mathbb{Z}_N$ 1-form global symmetry by gauging the global symmetry in the standard way and adding the BF-coupling in order to make the 2-form background field B_2 a $\mathbb{Z}_N$ gauge 2-form. The action becomes:

$$S = -\frac{1}{2g^2}\int \left((\mathrm{d}A_1 + B_2) \wedge *(\mathrm{d}A_1 + B_2) + \frac{iN}{2\pi} B_2 \wedge F_2^e\right) \tag{1.37}$$

where $F_2^e = \mathrm{d}A_1^e$ is an ordinary $U(1)$ gauge field. As we have shown in the previous section, the equation of motion of F_2^e makes B_2 a $\mathbb{Z}_N$ 2-form gauge field associated to the emergent 1-form gauge symmetry in the BF-theory.

For the magnetic 1-form symmetry we have

$$S = -\frac{1}{2g^2}\int \left(F_2 \wedge *F_2 + \frac{i}{2\pi} C_2 \wedge F_2 + \frac{iN}{2\pi} C_2 \wedge F_2^m\right) \tag{1.38}$$

where $F_2^m = \mathrm{d}A_1^m$ is an ordinary $U(1)$ gauge field whose equation of motion makes C_2 a $\mathbb{Z}_N$ 2-form gauge field.

Chapter 2

2-group global symmetries with continuous groups

Higher form symmetries have many applications in various fields of physics, from condensed matter to high energy physics and they often arise togheter with ordinary 0-form symmetries and other higher form symmetries of different order, combined in a non-trivial way. When this happens we are in the presence of a n-group symmetry. The integer n is the order of the form which constitutes the background of the higher form symmetry with the highest order. We will mainly focus in the case of 2-groups, where 1-form symmetry and 0-form symmetry transformations are linked togheter.

As an introduction, we will focus on the most simple case of 2 group global symmetry, the abelian $U(1)^{(0)} \times U(1)^{(1)}$, presenting the main properties and features. Subsequently, we analyze a more complex case preparatory to our study. In this chapter we will also introduce the concept of 't Hooft anomaly. In our discussion we essentially follow [7].

2.1 The abelian $U(1)^{(0)} \times U(1)^{(1)}$ 2-group global symmetry

Roughly speaking, a 2-group global symmetry consists in the invariance of the theory under a global transformation that involves both 0-form and 1-form global symmetries. In order to make the 2-group transformation manifest, we consider the theory with the coupling of the conserved current with the correspondent background gauge field (not dynamical). Conventionally, the gauge trasformations for the background gauge fields do not mix the fields with each other. A 2-group global symmetry is a tranformation that allows such mixing.

The simplest example is the abelian 2-group global symmetry that involves the mixing of a background 1-form gauge field A_1 for a $U(1)^{(0)}_A$ ordinary global symmetry with a background 2-form gauge field B_2 for a $U(1)^{(1)}_B$ 1-form global symmetry. We denote this 2-group with the following notation

$$U(1)^{(0)}_A \times_{\hat{k}_A} U(1)^{(1)}_B \tag{2.1}$$

where $\hat{k}_A \in \mathbb{Z}$ for reasons that we explain in the following and the superscript (-) on the group name represents the type of higher form symmetry (0 for ordinary symmetries, 1 for 1-form. . .).

The gauge transformations for the background fields are

$$A_1 \to A_1 + \mathrm{d}\lambda_0 \quad \text{with} \quad \lambda_0 \in U(1) \tag{2.2}$$

$$B_2 \to B_2 + \mathrm{d}\lambda_1 + \frac{\hat{k}_A}{2\pi}\lambda_0 F_2^A \quad \text{with} \quad F_2^A = \mathrm{d}A_1 \tag{2.3}$$

and λ_1 the usual 1-form for 1-form gauge transformation. In such a way, the gauge shift for B_2 involves a transformation under $U(1)_A^{(0)}$ proportional to the field strength F_2^A in addition to the usual 1-form gauge transformation.

The conventional 3-form field strength $\mathrm{d}B_2$ is not invariant under the 2-group transformation. However, we can define a different field strength H_3 fully gauge invariant that satisfies a modified Bianchi identity:

$$H_3 = \mathrm{d}B_2 - \frac{\hat{k}_A}{2\pi}A_1 \wedge F_2^A \quad , \qquad \mathrm{d}H_3 = -\frac{\hat{k}_A}{2\pi}F_2^A \wedge F_2^A. \tag{2.4}$$

When $\hat{k}_A = 0$ the 2-group transformation is trivial and decomposes into a conventional product symmetries $U(1)_A^{(0)} \times U(1)_B^{(1)}$. A more general introduction to the 2-group symmetries will be presented using the language of defects and group cohomology in *Section* 4.24.3.

The fact that $\hat{k}_A \in \mathbb{Z}$ is essentially due to the compactness of $U(1)$. As opposed to its universal covering group $\mathbb{R}$, $U(1)^{(0)}$ is compact. This implies that all charges are quantized since the gauge parameter is subject to the identification $\lambda_0 \sim \lambda_0 + 2\pi$, so that $\frac{1}{2\pi}\int_{\Sigma_1} \mathrm{d}\lambda_0 \in \mathbb{Z}$, where Σ_1 is a 1-cycle. Similarly for $U(1)^{(1)}$, $\frac{1}{2\pi}\int_{\Sigma_2} \mathrm{d}\lambda_1 \in \mathbb{Z}$, where Σ_2 is a 2-cycle. The gauge parameters are ambiguous, e. g. $\lambda_0 \sim \lambda_0 + 2\pi$, but since they parametrize elements of a compact group, these ambiguities must disapper, even at the level of the gauge shifts. For the transformation of A_1 there is no problem because it depends only on $\mathrm{d}\lambda_0$; in contrast, the shift for B_2 implies that the ambiguity for λ_0 induces an ambiguity in $B_2 \sim B_2 + \hat{k}_A F_2^A$. Fortunately, this is not a problem since the period of F_2^A is quantized (because the group is $U(1)_A^{(0)}$) with the same quantization condition as the B_2 gauge parameter λ_1, i.e. $\frac{1}{2\pi}\int_{\Sigma_2} F_2^A \in \mathbb{Z}$. It is possible to adsorb the ambiguity $B_2 \sim B_2 + \hat{k}_A F_2^A$ if and only if the 2-group structure constant $\hat{k}_A$ is an integer, simply performing an appropriate choice of the gauge parameter λ_1. For this reason, the structure constant is forced to be an integer.

2.2 A brief introduction to 't Hooft anomalies in conventional symmetries

In a generic QFT with global symmetries the 't Hooft anomalies are essentially impediments to gauging this theory. These anomalies can be shown coupling the theory with a background field $\mathcal{B}$ for the global symmetry and performing a gauge transformation for this field, i.e. $\mathcal{B} \to \mathcal{B} + \delta\mathcal{B}$: if the effective action $W[\mathcal{B}] = -\log Z[\mathcal{B}]$ is not gauge invariant and the variation cannot be canceled by adjusting local counterterms, the global symmetry has a 't Hooft anomaly.

$$W[\mathcal{B} + \delta\mathcal{B}] = W[\mathcal{B}] + \mathcal{A}[\mathcal{B}] \tag{2.5}$$

where the anomaly $\mathcal{A}[\mathcal{B}]$ is a c-number and vanishes when the background fields $\mathcal{B}$ are turned off. This is the opposite of the ABJ anomalies, which does not vanish in the absence of background fields.

In D-spacetime dimensions the 't Hooft anomalies for continuous global symmetries are encoded in the D+2-form anomaly polynomial $\mathcal{I}_{D+2}[\mathcal{B}]$ constructed out of various characteristic classes. The anomalous shift is determined from the anomaly polynomial via the descent equation:

$$\mathcal{A}[\mathcal{B}] = 2\pi i \int_{M^D} \mathcal{I}_D[\mathcal{B}, \delta\mathcal{B}] \quad , \qquad \mathrm{d}\mathcal{I}_D[\mathcal{B}, \delta\mathcal{B}] = \delta\mathcal{I}_{D+1}[\mathcal{B}] \quad , \qquad \mathrm{d}\mathcal{I}_{D+1}[\mathcal{B}] = \mathcal{I}_{D+2}[\mathcal{B}] \tag{2.6}$$

where $M^{(D)}$ is the spacetime manifold and $\mathcal{I}_D[\mathcal{B}, \delta\mathcal{B}]$ and $\mathcal{I}_{D+1}[\mathcal{B}]$ are local expression in the background field. In many cases, 't Hooft anomalies in local QFT admit a description in terms of anomaly inflow: we can consider the D-dim spacetime manifold of our theory as the boundary of a (D+1)-dim manifold (bulk) with euclidean action $S_{D+1}[\mathcal{B}] = \int_{M^{(D+1)}} \mathcal{I}_{D+1}[\mathcal{B}]$. This action is gauge invariant modulo $2\pi i\mathbb{Z}$ when $M^{(D+1)}$ is a compact manifold without boundary; however, if $\partial M^{(D+1)} = M^{(D)}$, the action S_{D+1} induces the anomaly $\mathcal{A}[\mathcal{B}]$ on the boundary by anomaly inflow.

An anomaly polynomial $\mathcal{I}^{red}_{D+2}[\mathcal{B}]$ is called reducible if it factorizes into a product of closed gauge invariant polynomials $\mathcal{J}_p[\mathcal{B}]$ and $\mathcal{K}_{D+2-p}[\mathcal{B}]$ of lower degree:

$$\mathcal{I}^{red}_{D+2}[\mathcal{B}] = \mathcal{J}_p[\mathcal{B}] \wedge \mathcal{K}_{D+2-p}[\mathcal{B}]. \tag{2.7}$$

If the anomaly polynomial is reducible there is an ambiguity in the first step of the descent procedure $\mathrm{d}\mathcal{I}_{D+1}[\mathcal{B}] = \mathcal{I}_{D+2}[\mathcal{B}]$ because we can remove the differential from either factors. One can show that $\mathcal{I}^{red}_{D+1}[\mathcal{B}]$ depends on a free real parameter s:

$$\mathcal{I}^{red}_{D+1}[\mathcal{B}] = \mathcal{J}_{p-1}[\mathcal{B}] \wedge \mathcal{K}_{D+2-p}[\mathcal{B}] + s\mathrm{d}(\mathcal{J}_{p-1}[\mathcal{B}] \wedge \mathcal{K}_{D+1-p}[\mathcal{B}]) \tag{2.8}$$

where $\mathcal{J}_p[\mathcal{B}] = \mathrm{d}\mathcal{J}_{p-1}[\mathcal{B}]$ and $\mathcal{K}_{D+2-p}[\mathcal{B}] = \mathrm{d}\mathcal{K}_{D+1-p}[\mathcal{B}]$. The s parameter multiplies an exact term in the (D+1)-dim anomaly inflow action $S_{D+1}[\mathcal{B}]$ and so it corresponds to a local counterterm in D-dimension. We can modify this counterterm changing the form of the anomaly, but if the anomaly is genuine, we cannot remove it.

As an example, the anomaly polynomials in 4 dimensions is a 6-form and we can sketch their basic ingredients. Since they must be gauge invariant, they naturally involved field strengths assembling into characteristic classes. If the gauge group is an ordinary $U(1)^{(0)}_A$, it contributes to $\mathcal{I}_6$ via the first Chern classes $c_1(F_2)^A = \frac{1}{2\pi} F^A_2$, where $F^A_2 = \mathrm{d}A_1$ is the associated field strength. If there are more than one 0-form $U(1)$ gauge group, each of them contributes via its first Chern class and we consider their possible combinations. For instance, if the group is $U(1)^{(0)}_I \times U(1)^{(0)}_J \times U(1)^{(0)}_K$, the contributions to the anomaly polynomial are

$$\sum_{IJK} k_{IJK} c_1(F^I_2) \wedge c_2(F^J_2) \wedge c_1(F^K_2) \tag{2.9}$$

where the indices I, J, K may coincide and all the terms are always reducible.

A $SU(N)^{(0)}$ 0-form symmetry contributes to $\mathcal{I}_6$ via the Chern classes $c_k(F^A_2) = \frac{1}{(2\pi)^k} \operatorname{tr}\Big((F^A_2)^k\Big)$ with $k \geq 2$, that are $2k$-forms constructed from the $SU(N)^{(0)}$ field strength. Finally, a q-form symmetry $U(1)^{(q)}_B$ contributes to the anomaly polynomial through the field strength $\mathrm{d}B_{q+1}$ which is invariant under q-form background gauge transformation $B_{q+1} \to B_{q+1} + \mathrm{d}\zeta_q$.

2.3 2-group symmetries from mixed 't Hooft anomalies

In this section we exploit the link between 't Hooft anomalies and 2-group global symmetries. In particular we present the arising of the abelian 2-group $U(1)^{(0)}_A \times_{\hat{k}_A} U(1)^{(1)}_B$ by gauging the global $U(1)^{(0)}_C$ in a theory with ordinary global symmetry under $U(1)^{(0)}_A \times U(1)^{(0)}_C$ with mixed 't Hooft anomaly.

Considering the theory in 4-dimensions, the most general 6-form anomaly polynomial $\mathcal{I}_6$ can be constructed using the field strength of the background fields:

$$\mathcal{I}_6 = \frac{1}{(2\pi)^3}\left(\frac{k_{A^3}}{3!}F_2^A\wedge F_2^A\wedge F_2^A + \frac{k_{A^2C}}{2!}F_2^A\wedge F_2^A\wedge F_2^C + \frac{k_{AC^2}}{2!}F_2^A\wedge F_2^C\wedge F_2^C + \frac{k_{C^3}}{3!}F_2^C\wedge F_2^C\wedge F_2^C\right) \tag{2.10}$$

where the anomaly coefficients are real parameters that can be extracted from the three point functions of the correspondent conserved currents (e.g. $\langle j_3^A j_3^A j_3^C\rangle$). These coefficients are quantized since they correspond to sum of integer charges, but their quantization is also a general property of 't Hooft anomalies.

All the terms contained in $\mathcal{I}_6$ are reducible and the terms containing both F_2^A and F_2^C lead to the ambiguity 2.8 in the descent procedure. For instance, considering the term proportional to $F_2^A\wedge F_2^A\wedge F_2^C$ in $\mathcal{I}_6$ and applying the descent procedure, we obtain

$$\mathcal{I}_5 \supset \frac{k_{A^2C}}{(2\pi)^3 2!}A_1\wedge F_2^A\wedge F_2^C + s\mathrm{d}(A_1\wedge F_2^A\wedge C_1) \tag{2.11}$$

with $s\in\mathbb{R}$. The ambiguity is an exact 5-form and it corresponds to a local counterterms in four dimensions:

$$S_{C.T.} = 2\pi i s\int_{M^{(4)}} A_1\wedge F_2^A\wedge C_1 \tag{2.12}$$

where $A_1\wedge F_2^A$ is the Chern-Simons 3-form. Using the formalism of the previous section, we find that $\mathcal{J}_{p-1} = \mathcal{J}_3 \sim A_1\wedge F_2^A\wedge C_1$ and $\mathcal{K}_{D+2-p} = \mathcal{K}_1 \sim C_1$. Similarly, for the term proportional to $F_2^A\wedge F_2^C\wedge F_2^C$ we have an ambiguity parametrized by $t\in\mathbb{R}$. Therefore, the descent 5-form $\mathcal{I}_5$ that we obtain from the anomaly polinomial $\mathcal{I}_6$ is

$$\mathcal{I}_5 = \frac{1}{(2\pi)^3}\left(\frac{k_{A^3}}{3!}A_1\wedge F_2^A\wedge F_2^A + \frac{k_{A^2C}}{2!}A_1\wedge F_2^A\wedge F_2^C + \frac{k_{AC^2}}{2!}A_1\wedge F_2^C\wedge F_2^C + \frac{k_{C^3}}{3!}C_1\wedge F_2^C\wedge F_2^C\right). \tag{2.13}$$

Now, the corresponding anomalies can be easily computed following 2.6. We denote with λ_0^A and λ_0^C the gauge parameter for the background field A_1 and C_1 and the anomalies are:

$$\mathcal{A}_A = \frac{i}{4\pi^2}\int_{M^{(4)}}\lambda_0^A\left(\frac{k_{A^3}}{3!}F_2^A\wedge F_2^A + \left(\frac{k_{A^2C}}{2!} - s\right)F_2^A\wedge F_2^C + \left(\frac{k_{AC^2}}{2!} - t\right)F_2^C\wedge F_2^C\right) \tag{2.14}$$

$$\mathcal{A}_C = \frac{i}{4\pi^2}\int_{M^{(4)}}\lambda_0^C\left(\frac{k_{C^3}}{3!}F_2^C\wedge F_2^C + sF_2^A\wedge F_2^A + tF_2^A\wedge F_2^C\right). \tag{2.15}$$

From the computation of the anomalies we can obtain the non conservation equation for the currents:

$$\mathrm{d}j_3^A = -\frac{i}{4\pi^2}\left(\frac{k_{A^3}}{3!}F_2^A\wedge F_2^A + \left(\frac{k_{A^2C}}{2!} - s\right)F_2^A\wedge F_2^C + \left(\frac{k_{AC^2}}{2!} - t\right)F_2^C\wedge F_2^C\right) \tag{2.16}$$

$$\mathrm{d}j_3^C = -\frac{i}{4\pi^2}\left(\frac{k_{C^3}}{3!}F_2^C\wedge F_2^C + sF_2^A\wedge F_2^A + tF_2^A\wedge F_2^C\right). \tag{2.17}$$

Since we want to gauge $U(1)_C^{(0)}$ we have to require that $\mathcal{A}_C$ is trivial. In order to obtain this result we impose $k_{C^3} = 0$ and we adjust the counterterms such that $s = t = 0$. Therefore, the remaining anomaly for A is:

$$\mathcal{A}_A = \frac{i}{4\pi^2}\int_{M^{(4)}}\lambda_0^A\left(\frac{k_{A^3}}{3!}F_2^A\wedge F_2^A + \frac{k_{A^2C}}{2!}F_2^A\wedge F_2^C + \frac{k_{AC^2}}{2!}F_2^C\wedge F_2^C\right). \tag{2.18}$$

Gauging $U(1)^{(0)}_C$ means to promote the background field to a dynamical field: $C_1 \to c_1$ and $F_2^C \to f_2^c$. In this way the anomalous shifts proportional to k_{A^2C} and k_{AC^2} become 'operator valued' shifts and no longer c-numbers. We have to reintrepret this terms, since they are no longer conventional 't Hooft anomalies. The term proportional to k_{A^3} remains a c-number, but its meaning is related to 't Hooft anomaly for 2-group symmetry (see below).

After gauging, the mixed term proportional to k_{AC^2} gives rise to an ABJ anomlay for the $U(1)^{(0)}_A$ current:

$$\mathrm{d}j_3^A \supset -\frac{ik_{AC^2}}{8\pi^2} f_2^C \wedge f_2^C. \tag{2.19}$$

The violation of current conservation remains even if the background field vanishes and also at separated points inside correlators, since $f_2^C \wedge f_2^C$ is a non trivial operator. We will focus in theory without symmetries explicitly broken by an ABJ anomaly, so we require $k_{AC^2} = 0$.

The anomaly due to k_{A^2C} leads to a different violation of current conservation:

$$\mathrm{d}j_3^A \supset -\frac{ik_{A^2C}}{8\pi^2} F_2^A \wedge f_2^C. \tag{2.20}$$

If the background field A_1 is turned off, the violation disappears. Therefore, this anomaly is deeply different from an ABJ anomaly. However, it is also distinct from 't Hooft anomalies, since it is an 'operator valued' shift and we can use an appropriate background field to remove this shift from the effective action. As we have presented in the previous section (main reference [2]), gauging $U(1)^{(0)}_C$ leads to a 1-form global symmetry called the magnetic $U(1)^{(1)}_B$ symmetry with 2-form current $j_2^B = \frac{i}{2\pi} f_2^C$. It is conserved because the Bianchi identity $\mathrm{d}f_2^C = 0$ and the magnetic 1-form charges are integer since the symmetry group is $U(1)$.

The appropriate background gauge field that can be coupled to the magnetic 2-form current is the 2-form B_2 subject to the usual 1-form gauge transformation. The standard coupling is $\int B_2 \wedge j_2^B = \int B_2 \wedge f_2^C$ and the gauge transformation is $B_2 \to B_2 + \mathrm{d}\lambda_1$ with $\frac{1}{2\pi}\int_{\Sigma_2} \mathrm{d}\lambda_1 \in \mathbb{Z}$. The invariance under large gauge transformations (with non trivial flux for λ_1) is ensured by the fact that the magnetic 1-form charges are quantized.

Eventually, it is possible to cancel the 'operator valued' shift $\frac{ik_{A^2C}}{8\pi^2}\int \lambda_0 F_2^A \wedge f_2^C$ exploiting the gauge trasformation for the source of the magnetic 1-form symmetry, by imposing:

$$B_2 \to B_2 + \frac{\hat{k}_A}{2\pi}\lambda_0 F_2^A \quad , \qquad \hat{k}_A = -\frac{1}{2}k_{A^2C} \tag{2.21}$$

This is precisely the 2-group transformation for $U(1)^{(0)}_A \times_{\hat{k}_A} U(1)^{(1)}_B$ and the 2 group structure constant is proportional to the mixed anomaly coefficient k_{A^2C}.

2.4 2-group 't Hooft anomalies

Like ordinary global symmetries, 2-group symmetries can have 't Hooft anomalies. Following [7] we note that there is no candidate terms in the anomaly polynomials $\mathcal{I}_6$ that mixes a $U(1)^{(1)}_B$ global symmetry with ordinary 0-form symmetries. Therefore, the 6-form anomaly polynomial does not involve B_2 and takes the form:

$$\mathcal{I}_6 = \frac{1}{(2\pi)^3}\frac{k_{A^3}}{3!} F_2^A \wedge F_2^A \wedge F_2^A. \tag{2.22}$$

It gives rise to an anomaly shift

$$\mathcal{A}_A = \frac{ik_{A^3}}{24\pi^2}\int \lambda_0 F_2^A \wedge F_2^A. \tag{2.23}$$

So far there is no difference with the case of $U(1)^{(0)}$ symmetry, but now we have to consider the counterterms that can adsorbe the anomaly in the new framework of 2-group transformation. We will analyze the behavior of the so called Green-Schwarz (GS) counterterm under the abelian 2-group shift.

$$S_{GS} = \frac{in}{2\pi}\int B_2 \wedge F_2^A \quad , \qquad S_{GS} \to S_{GS} + \frac{in\hat{k}_A}{4\pi^2}\int \lambda_0 F_2^A \wedge F_2^A \tag{2.24}$$

where n is a parameter that we will discuss below. Since the gauge transformation of the GS counterterm takes the same form of the anomaly term proportional to k_{A^3}, we observe that adding this term produces a shift in the anomaly coefficient that becomes no longer scheme independent: $k_{A^3} \to k_{A^3} + 6n\hat{k}_A$. In order to understand if this counterterm is enough to cancel the anomaly we have to specify the value of n.

If we require the invariance under large $U(1)_B^{(1)}$ gauge transformation, i. e. with non trivial $\int d\lambda_1$, we have to impose $n \in \mathbb{Z}$. If n is quantized, the 't Hooft anomaly coefficient k_{A^3} is scheme independent only $\mod 6\hat{k}_A$. In other words, only the fractional part $\frac{k_{A^3}}{6\hat{k}_A}$ (mod 1) is the genuine anomaly that cannot be cancelled, while the integer part can be set to any value (either zero) by an appropriate GS counterterm.

If we give up on invariance under large $U(1)_B^{(1)}$ gauge transformation, we no longer have any constraints on n. Since $n \in \mathbb{R}$, we can fix $n = -\frac{k_{A^3}}{6\hat{k}_A}$ to set $k_{A^3} = 0$ and to cancel apparently the anomaly. However, under a large gauge $U(1)_B^{(0)}$ gauge transformation the partition function picks up an anomalous phase $\frac{k_{A^3}}{6\hat{k}_A}$ (mod 1) in the same way.

To summarize, the abelian 2-group 't Hooft anomaly is characterized by the fractional part $\frac{k_{A^3}}{6\hat{k}_A}$ and not by an integer as the usual 't Hooft anomalies. Furthermore, the 2-group 't Hooft anomaly arises from the matching between $U(1)_A^{(0)}$ and large $U(1)_B^{(0)}$ because we can preserve one of them, but not both.

2.5 On gauging 2-group global symmetries

Gauging a global symmetry consists in promoting the background gauge field to dynamical field and doing the path intergration over their orbits. If we consider a 2-group global symmetry, one may wonder if we can gauge the entire 2-group and either the groups that compose it. Considering the abelian 2-group as usual, we can easily observe that we cannot gauge $U(1)_A^{(0)}$ without gauging $U(1)_B^{(1)}$. This arises from the simple fact that the 2-group shift mixes B_2 with A_1 and because $U(1)_A^{(0)}$ is not a good subgroup of the entire 2-group, while $U(1)_B^{(1)}$ is. Another proof comes from considering the 2-group theory arising by the gauging of the $U(1)_C^{(0)}$ symmetry in a $U(1)_A^{(0)} \times U(1)_C^{(0)}$ theory with mixed 't Hooft anomaly. Gauging $U(1)_A^{(0)}$ in the 2-group symmetry means to gauge both $U(1)_A^{(0)}$ and $U(1)_C^{(0)}$ in the parent theory; however, this is impossible because a non-trivial 2-group is related to a mixed 't Hooft anomaly $k_{A^2C} \neq 0$. Therefore, the allowed possibilities are:

$$U(1)_A^{(0)} \times_{\hat{k}_A} U(1)_B^{(1)} \to U(1)_a^{(0)} \times_{\hat{k}_A} U(1)_b^{(1)} \quad , \qquad A_1 \to a_1 \quad , \quad B_2 \to b_2 \tag{2.25}$$

or

$$U(1)^{(0)}_A \times_{\hat{k}_A} U(1)^{(1)}_B \to U(1)^{(0)}_A \times_{\hat{k}_A} U(1)^{(1)}_b \quad , \qquad B_2 \to b_2 \tag{2.26}$$

where we indicate with the upper case the background gauge fields and with the lower case the dynamical gauge fields.

Before focusing into gauging of global 2-group, we study the simple case of gauging a single $U(1)^{(0)}_A$. Gauging $U(1)^{(0)}_A$ leads to a new theory with a 1-form symmetry $U(1)^{(1)}_B$ with current $j^B_2 = \frac{i}{2\pi} f^A_2$ where f^A_2 is the Maxwell field strength. Subsequently, we can also gauge the $U(1)^{(1)}_B$ 1-form symmetry of the $U(1)^{(0)}_a$ gauge theory and we go back to the original theory with $U(1)^{(0)}_A$ global symmetry with current $j^A_3 = \frac{i}{2\pi}\mathrm{d}b_2$. In order to appreciate that gauging $U(1)^{(0)}_A$ and $U(1)^{(1)}_B$ are inverse operations, we consider the partition functions in the presence of the background fields. Starting from the partition function $Z[A_1]$ for the $U(1)^{(0)}_A$ theory, we can construct $\tilde{Z}[B_2]$ by coupling B_2 with the current and performing the path integration over a_1:

$$\tilde{Z}[B_2] = \int \mathcal{D}a_1 Z[a_1] \exp\left(\frac{i}{2\pi}\int B_2 \wedge \mathrm{d}a_1\right). \tag{2.27}$$

This expression can be thought as functional Fourier transform and can be easily inverted:

$$Z[A_1] = \int \mathcal{D}b_2 \tilde{Z}[b_2] \exp\left(-\frac{i}{2\pi}\int b_2 \wedge \mathrm{d}A_1\right). \tag{2.28}$$

where in principle we do not include any extra term for b_2 (e.g. kinetic term) to ensure the invertibility. We can check that by inserting one expression into the other; exploiting the properties of the delta-function, we obtain the identity as we expect for inverse operation.

As a first case we consider the gauging of the subgroup $U(1)^{(1)}_B$, but we begin by remembering its inverse operation: the gauging of $U(1)^{(0)}_C$ in a theory with $U(1)^{(0)}_A \times U(1)^{(0)}_C$ symmetry and mixed 't Hooft anomaly k_{A^2C}. As we have already seen in the previous section, gauging $U(1)^{(0)}_C$ in this theory leads to a theory with 2-group symmetry $U(1)^{(0)}_A \times_{\hat{k}_A} U(1)^{(1)}_B$ with the identification $\hat{k}_A = -\frac{1}{2}k_{A^2C}$. The partition function for the theory with 2-group symmetry can be obtained from the partition function $Z[A_1, C_1]$ of the $U(1)^{(0)}_A \times U(1)^{(0)}_C$ theory:

$$Z[A_1, B_2] = \int \mathcal{D}c_1 Z[A_1, c_1] \exp\left(\frac{i}{2\pi}\int B_2 \wedge \mathrm{d}c_1\right). \tag{2.29}$$

Therefore, if we gauge $U(1)^{(1)}_B$ in the abelian 2-group theory, we return to the previous theory $U(1)^{(0)}_A \times U(1)^{(0)}_C$. Since we have supposed that we have only mixed 't Hooft anomaly k_{A^2C} in the $U(1)^{(0)}_A \times U(1)^{(0)}_C$ theory, the 2-group is 't Hooft anomaly free and we can write the partition function for the gauging of $U(1)^{(1)}_B$ as in the simple case considered before.

$$Z[A_1, C_1] = \int \mathcal{D}b_2 Z[A_1, b_2] \exp\left(-\frac{i}{2\pi}\int b_2 \wedge \mathrm{d}C_1\right). \tag{2.30}$$

The invariance under $U(1)^{(0)}_C$ background gauge transformation is trivial. However, if we consider the transformation under $U(1)^{(0)}_A$ accompanied by a change of variable in the path integral $b_2 \to b_2 + \frac{\hat{k}_A}{2\pi}\lambda_0 F^A_2$ and we use the invariance of $Z[A_1, b_2]$ under this shift, we find the mixed k_{A^2C} 't Hooft anomaly arising from the non invariance of the exponential factor:

$$Z[A_1, C_1] \to Z[A_1, C_1] - \frac{i\hat{k}_A}{4\pi^2}\int \lambda_0 F^A_2 \wedge F^C_2 = Z[A_1, C_1] + \frac{ik_{A^2C}}{8\pi^2}\int \lambda_0 F^A_2 \wedge F^C_2. \tag{2.31}$$

In this way we correctly reproduce the mixed 't Hooft anomaly of the $U(1)^{(0)}_A \times U(1)^{(0)}_C$ theory and we show that any 2-group $U(1)^{(0)}_A \times_{\hat{k}_A} U(1)^{(1)}_B$ theory arises from a theory with $U(1)^{(0)}_A \times U(1)^{(0)}_C$ symmetry and mixed 't Hooft anomaly. This theory can be simply found by gauging $U(1)^{(1)}_B$.

As a second choice, we can try to gauge the entire 2-group if it is anomaly free. In order to avoid the obstruction of gauging we have to impose that the 2-group theory comes from a $U(1)^{(0)}_A \times U(1)^{(0)}_C$ with 't Hooft anomaly coefficient $k_{A^3} = 0 \pmod{6\hat{k}_A}$. Gauging $U(1)^{(0)}_A$ we obtain a theory with $U(1)^{(0)}_a$ gauge symmetry and a new global $U(1)^{(1)}_X$ 1-form symmetry with conserved current $j^X_2 = \frac{i}{2\pi}\mathrm{d}a_1$ and background 2-form field X_2. Gauging $U(1)^{(1)}_B \to U(1)^{(1)}_b$ we obtain a 0-form symmetry $U(1)^{(0)}_C$ with background field C_1 as in the previous case. However, the global $U(1)^{(0)}_C$ symmetry suffers from an ABJ anomaly because we cannot couple the $U(1)^{(0)}_C$ background field with the current $j^C_3 \sim \mathrm{d}b_2$, since it is not gauge invariant under $U(1)^{(0)}_a$ which now is a dynamical symmetry.

We must use the gauge invariant current

$$\tilde{j}^C_3 = \frac{i}{2\pi}h_3 \quad \text{where} \quad h_3 = \mathrm{d}b_2 - \frac{\hat{k}_A}{2\pi}a_1 \wedge f^a_2, \tag{2.32}$$

but the modified Bianchi identity leads to a violation of the current conservation:

$$\mathrm{d}\tilde{j}^C_3 = -\frac{i\hat{k}_A}{4\pi^2}f^a_2 \wedge f^a_2. \tag{2.33}$$

This non conservation equation constitutes an ABJ anomaly for the $U(1)^{(0)}_C$ symmetry and can be possibly treated by introducing the θ term in the Yang-Mills theory for a_1 and by promoting it into a background field with an appropriate shift. Omitting these details, the partition function for the 2-group gauged theory is:

$$Z[X_2, C_1] = \int \mathcal{D}a_1\mathcal{D}b_2 Z[a_1, b_2] \exp\left(\frac{i}{2\pi}\int X_2 \wedge \mathrm{d}a_1\right) \exp\left(\frac{i}{2\pi}\int \left(\mathrm{d}b_2 - \frac{\hat{k}_A}{2\pi}a_1 \wedge f^a_2\right) \wedge C_1\right). \tag{2.34}$$

Performing the integration over b_2 we obtain the $U(1)^{(0)}_A \times U(1)^{(0)}_C$ theory with the mixed 't Hooft anomaly encoded in the gauge invariant current $\tilde{j}^C_3$. In fact, before gauging $U(1)^{(0)}_A$, we can observe that $\tilde{j}^C_3$ is invariant under $U(1)^{(0)}_A$ background gauge transformations, but not conserved in a $U(1)^{(0)}_A$ background gauge field. This is an alternative presentation of the k_{A^2C} 't Hooft anomaly. Finally, gauging also $U(1)^{(0)}_A$ we obtain a theory with $U(1)^{(1)}_X \times U(1)^{(0)}_C$ symmetry and with an ABJ anomaly for $U(1)^{(0)}_C$. Obviously, we can consider the inverse: if we gauge $U(1)^{(1)}_X$ we return to the parent theory $U(1)^{(0)}_A \times U(1)^{(0)}_C$ with mixed 't Hooft anomaly and gauging also $U(1)^{(0)}_C$ we return to the original 2-group theory.

2.6 A useful example: $U(1)^{(0)}_{A^{(1)}} \times U(1)^{(0)}_{A^{(2)}} \times_{\hat{k}_{12}} U(1)^{(1)}_B$

The abelian $U(1)^{(0)}_A \times_{\hat{k}_A} U(1)^{(1)}_B$ 2-group symmetry studied so far is the most simple example of 2-group. Now we consider a theory with a slightly more complicated symmetry group and we analyze it generalyzing the above results.

The 2-group symmetry is

$$U(1)^{(0)}_{A^{(1)}} \times U(1)^{(0)}_{A^{(2)}} \times_{\hat{k}_{12}} U(1)^{(1)}_B \tag{2.35}$$

and the background gauge tranformations are

$$A^{(i)}_{(1)} \to A^{(i)}_1 + \mathrm{d}\lambda^{(i)}_0 \quad \text{with} \quad \lambda^{(i)}_0 \in U(1) \quad , \quad i = 1, 2 \tag{2.36}$$

$$B_2 \to B_2 + \mathrm{d}\lambda_1 + \frac{\hat{k}_{12}}{2\pi}\big(\lambda^{(1)}_0 F^{A^{(2)}}_2 + \lambda^{(2)}_0 F^{A^{(1)}}_2\big) \quad \text{with} \quad F^{A^{(i)}}_2 = \mathrm{d}A^{(i)}_1 \tag{2.37}$$

where λ_1 is the usual 1-form for 1-form gauge transformation.

This is a particular case of an obvious generalization of the abelian 2-group studied before. If we consider the 2-group

$$\Pi_I U(1)^{(0)}_I \times_{\hat{k}_{IJ}} U(1)^{(1)}_B \tag{2.38}$$

where $I = 1, 2, \ldots n$, with background tranformations

$$A^{(I)}_1 \to A^{(I)}_1 + \mathrm{d}\lambda^{(I)}_0 \quad \text{and} \quad B_2 \to B_2 + \mathrm{d}\lambda_1 + \frac{1}{2\pi}\sum_{IJ} \hat{k}_{IJ}\lambda^{(I)}_0 F^{(J)}_2 \tag{2.39}$$

where $\hat{k}_{IJ} \in \mathbb{Z}$ is an element of a symmetric $n \times n$ matrix, our theory is a particular case with $n = 2$ and symmetric matrix:

$$\begin{pmatrix} 0 & \hat{k}_{12} \\ \hat{k}_{12} & 0 \end{pmatrix}. \tag{2.40}$$

The gauge invariant field strength for the 2-form background field B_2 is

$$H_3 = \mathrm{d}B_2 - \frac{\hat{k}_{12}}{2\pi}\big(A^{(1)}_1 \wedge F^{A^{(2)}}_2 + A^{(2)}_1 \wedge F^{A^{(1)}}_2\big) \tag{2.41}$$

and it implies the 'modified' Bianchi identity:

$$\mathrm{d}H_3 = -\frac{\hat{k}_{12}}{2\pi}\big(F^{A^{(1)}}_2 \wedge F^{A^{(2)}}_2 + F^{A^{(2)}}_2 \wedge F^{A^{(1)}}_2\big) = -\frac{\hat{k}_{12}}{\pi} F^{A^{(1)}}_2 \wedge F^{A^{(2)}}_2. \tag{2.42}$$

Each continuos 2-group symmetry arises from gauging a theory with ordinary symmetries and mixed 't Hooft anomalies. In our case, the parent theory has a $U(1)^{(0)}_{A^{(1)}} \times U(1)^{(0)}_{A^{(2)}} \times U(1)^{(0)}_C$ symmetry group and mixed 't Hooft anomaly with non trivial anomaly coefficient $k_{A^{(1)}A^{(2)}C}$. Without writing all the long anomaly polynomials and following the previous discussion, we can argue that, in order to gauge the $U(1)^{(0)}_C$ symmetry, we need $k_{C^3} = 0$ and in order to avoid ABJ anomalies for $A^{(i)}_1$ we need $k_{A^{(i)}C^2} = 0$. Therefore, gauging C_1 we obtain the anomaly contribution

$$\mathcal{A}_{A^{(i)}}(F^C_2 \to f^c_2) = \frac{i}{4\pi}\sum_{j=1}^{2} \frac{k_{A^{(i)}A^{(j)}C}}{2!}\int \lambda^{(i)}_0 F^{A^{(j)}}_2 \wedge f^c_2 \tag{2.43}$$

and, to cancel the 'operator valued' shift, we impose the gauge trasformation of the background gauge field for the magnetic 1-form symmetry arising from gauging $U(1)^{(0)}_C$: $B_2 \to B_2 + \frac{1}{2\pi}\sum_{i,j} \hat{k}_{ij}\lambda^{(i)}_0 F^{A^{(j)}}_2$. The 2-group structure constant can be identified $\hat{k}_{ij} = -\frac{1}{2}k_{A^{(i)}A^{(j)}C}$. In order to correctly reproduce our 2-group theory we require that the anomaly coefficient $k_{A^{(1)2}C} = k_{A^{(2)2}C} = 0$. Therefore, we have $\hat{k}_{12} = -\frac{1}{2}k_{A^{(1)}A^{(2)}C} = \hat{k}_{21}$.

To study the 't Hooft anomalies for the 2-group symmetry we start from the parent theory and we write the terms of the anomaly 6-form polynomial that can contribute to the anomalies.

$$\begin{aligned} \mathcal{I}_6 \supset &\frac{1}{(2\pi)^3}\bigg(\frac{k_{A^{(1)3}}}{3!}F^{A^{(1)}}_2 \wedge F^{A^{(1)}}_2 \wedge F^{A^{(1)}}_2 + \frac{k_{A^{(1)2}A^{(2)}}}{2!}F^{A^{(1)}}_2 \wedge F^{A^{(1)}}_2 \wedge F^{A^{(2)}}_2 + \\ &+ \frac{k_{A^{(1)}A^{(2)2}}}{2!}F^{A^{(1)}}_2 \wedge F^{A^{(2)}}_2 \wedge F^{A^{(2)}}_2 + \frac{k_{A^{(2)3}}}{3!}F^{A^{(2)}}_2 \wedge F^{A^{(2)}}_2 \wedge F^{A^{(2)}}_2\bigg). \end{aligned} \tag{2.44}$$

Calculating via descent equation the anomaly shifts and fixing the free parameter in order to equally divide the contributions, we obtain

$$\mathcal{A}_{A^{(1)}} = \frac{i}{4\pi^2}\int \lambda_0^{A^{(1)}}\left(\frac{k_{A^{(1)3}}}{3!}F_2^{A^{(1)}}\wedge F_2^{A^{(1)}}+\frac{k_{A^{(1)2}A^{(2)}}}{4}F_2^{A^{(1)}}\wedge F_2^{A^{(2)}}+\frac{k_{A^{(1)}A^{(2)2}}}{4}F_2^{A^{(2)}}\wedge F_2^{A^{(2)}}\right) \tag{2.45}$$

$$\mathcal{A}_{A^{(2)}} = \frac{i}{4\pi^2}\int \lambda_0^{A^{(2)}}\left(\frac{k_{A^{(2)3}}}{3!}F_2^{A^{(2)}}\wedge F_2^{A^{(2)}}+\frac{k_{A^{(1)2}A^{(2)}}}{4}F_2^{A^{(1)}}\wedge F_2^{A^{(1)}}+\frac{k_{A^{(1)}A^{(2)2}}}{4}F_2^{A^{(1)}}\wedge F_2^{A^{(2)}}\right). \tag{2.46}$$

Adding the GS counterterms $S_{GS}^{(i)} = \frac{in}{2\pi}\int B_2\wedge F_2^{A^{(i)}}$, $i=1,2$, and performing a 2-group symmetry transformation we obtain

$$\begin{aligned} S_{GS}^{(1)}+S_{GS}^{(2)} \to S_{GS}^{(1)}+S_{GS}^{(2)}+\frac{in\hat{k}_{12}}{4\pi^2}\int\Big(&\lambda_0^{A^{(1)}}\big(F_2^{A^{(1)}}\wedge F_2^{A^{(2)}}+F_2^{A^{(2)}}\wedge F_2^{A^{(2)}}\big)\\ &+\lambda_0^{A^{(2)}}\big(F_2^{A^{(1)}}\wedge F_2^{A^{(1)}}+F_2^{A^{(1)}}\wedge F_2^{A^{(2)}}\big)\Big). \end{aligned} \tag{2.47}$$

Therefore, we cannot adjust the value of the $k_{A^{(1)3}}$ and $k_{A^{(2)3}}$ anomaly coefficients adding the counterterms since there is no terms of the same form as a result of the gauge transformation of $S_{GS}^{(i)}$. However, we can modify the others: if we add both the GS counterterms, we induce a shift in the anomaly coefficients:

$$k_{A^{(1)2}A^{(2)}} \to k_{A^{(1)2}A^{(2)}}+4n\hat{k}_{12} \quad \text{and} \quad k_{A^{(1)}A^{(2)2}} \to k_{A^{(1)}A^{(2)2}}+4n\hat{k}_{12}. \tag{2.48}$$

Consequently, we can conclude that only the fractional parts of $\frac{k_{A^{(1)2}A^{(2)}}}{4\hat{k}_{12}}$ (mod 1) and $\frac{k_{A^{(1)}A^{(2)2}}}{4\hat{k}_{12}}$ (mod 1) constitute genuine 't Hooft anomaly for the 2-group symmetry, toghether with the $k_{A^{(1)3}}$ and $k_{A^{(2)3}}$ terms. If we set all this terms to 0, we can gauge the 2-group symmetry.

The gauging of the $U(1)_B^{(1)}$ leads to the parent theory $U(1)_{A^{(1)}}^{(0)}\times U(1)_{A^{(2)}}^{(0)}\times U(1)_C^{(0)}$ with mixed 't Hooft anomaly and in the path integral formalism the partition function is

$$Z[A_1^{(1)},A_1^{(2)},C_1] = \int \mathcal{D}b_2 Z[A_1^{(1)},A_1^{(2)},b_2]\exp\left(-\frac{i}{2\pi}\int b_2\wedge \mathrm{d}C_1\right). \tag{2.49}$$

If the 2-group is 't Hooft anomaly free, we can gauge the entire symmetry group:

$$\begin{aligned} Z[X_2^{(1)},X_2^{(2)},C_1] = \int &\mathcal{D}a_1^{(1)}\mathcal{D}a_1^{(2)}\mathcal{D}b_2 Z[a_1^{(1)},a_1^{(2)},b_2]\exp\left(\frac{i}{2\pi}\int X_2^{(1)}\wedge \mathrm{d}a_1^{(1)}\right)\\ &\exp\left(\frac{i}{2\pi}\int X_2^{(2)}\wedge \mathrm{d}a_1^{(2)}\right)\exp\left(\frac{i}{2\pi}\int\left(\mathrm{d}b_2-\frac{\hat{k}_{12}}{2\pi}\big(a_1^{(1)}\wedge F_2^{a^{(2)}}+a_1^{(2)}\wedge F_2^{a^{(1)}}\big)\wedge C_1\right)\right). \end{aligned} \tag{2.50}$$

The use of the gauge invariant field strength h_3 gives rise to an ABJ anomaly for the global $U(1)_C^{(0)}$ symmetry.

Chapter 3

Brief introduction to singular (co)homology and group cohomology

The study of 't Hooft anomalies in quantum field theory and 2-group symmetry requires familiarity with the concept of cohomology and group cohomology; however, the study of discrete symmetries makes these concepts indispensable. Thus, we want to briefly but rigorously introduce these topics in order to provide a self-consistent introduction and present some important results that will be useful later. Before introducing the algebraic definitions of homology and cohomology, we start by presenting manifold triangulation which is another way to describe a topological space. Using the simplicial complexes introduced for the triangulation, we can define the simplest homology groups that constitute simplicial homology. Then we discuss singular homology that is the basis of the first important tool: singular cohomology. Finally, we will focus on group cohomology and in particular on group cohomology for finite group. The main reference are [21], [22], [23] and [24].

3.1 Manifold triangulation and simplicial complexes

We start by defining what is the meaning of manifold triangulation, which is an alternative description of the open charts that we use to cover the manifold. We initially define what are simplices and simplicial complexes. A n-simplex σ_n is the smallest convex set in $\mathbb{R}^m$ containing $n+1$ points $p_0, p_1, \dots, p_n$ that do not lie in any $(n-1)$-dimensional hyperplane, or, in other words, such that the vectors $p_1 - p_0, \dots, p_n - p_0$ are linearly independent. Such a set of points is said to be geometric independent. We denote it with angular brackets:

$$\sigma_n = \langle p_0 p_1 \dots p_n \rangle. \tag{3.1}$$

For instance, the 0-simplex $\langle p_0 \rangle$ is a point, the 1-simplex $\langle p_0 p_1 \rangle$ is a line, the 2-simplex $\langle p_0 p_1 p_2 \rangle$ is a triangle with its interior included, a 3-simplex $\langle p_0 p_1 p_2 p_3 \rangle$ is a solid tetrahedron, and so on.

If we take $q+1$ points $p_{i_0}, \dots, p_{i_q}$ of a n-simplex $\sigma_n = \langle p_0 \dots p_n \rangle$, with $0 \leq q \leq n$, we can define the q-simplex $\sigma_q = \langle p_{i_0} \dots p_{i_q} \rangle$, which is called a q-face of σ_n. One can verify that the number of $q-$faces in a n-simplex are $\begin{pmatrix} n+1 \\ q+1 \end{pmatrix}$.

A simplicial complex K is a set of finite number of simplices in $\mathbb{R}^m$ with the two following properties: first, an arbitary face σ' of a simplex $\sigma \in K$ must belong to K ($\sigma' \in K$); second, if σ and σ' are two simplices in K, the intersection $\sigma \cap \sigma'$ is either empty or a common face of σ and σ'. For instance, taking the triangle $\langle p_0, p_1, p_2 \rangle$, then the simplicial complex K is $K = \{\langle p_0 \rangle, \langle p_1 \rangle, \langle p_2 \rangle, \langle p_0 p_1 \rangle, \langle p_1 p_2 \rangle, \langle p_0 p_2 \rangle, \langle p_0 p_1 p_2 \rangle\}$. If each simplex of a simplicial complex K is regarded as a subset of $\mathbb{R}^m$, the union of all the simplices are a subset of $\mathbb{R}^m$ called polyhedron and denoted as $|K|$. The dimension of the polyhedron is the same of the simplicial complex which is defined to be the largest dimension of simplices in K.

A triangulation of a topological space $M^{(D)}$ (for instance a manifold) is the pair (K, t), where K is a simplicial complex and $t : |K| \to M^{(D)}$ is a homeomorphism. If such a pair exists, the space $M^{(D)}$ is called triangulable. Notice that, given a topological space, its triangulation is far from unique. For instance, in Figure 3.1 we report the simplest triangulation of a cylinder $\mathbb{S}^1 \times [0, 1]$.

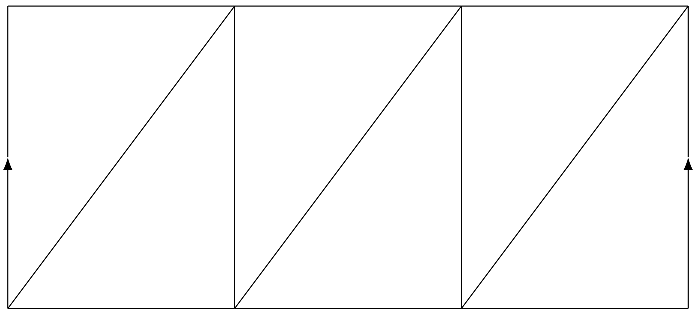

Figure 3.1: Triangulation of a cylinder $\mathbb{S}^1 \times [0, 1]$. Notice that a simpler choice does not exist since other possibilities do not respect the condition that intersection of simplices in the simplicial complex must be empty or a simplex.

Finally, we define the orientation of a n-simplex. Taking the sequence of $n+1$ geometrically independent points $\{p_0, p_1, \ldots, p_n\}$ we say that the sequence

$$\{p_{i_0}, p_{i_1}, \ldots, p_{i_n}\} = \{p_{P(1)}, p_{P(2)}, \ldots, p_{P(n)}\} \tag{3.2}$$

is equivalent to the original one, if the permutation P is an even permutation, i.e. it is a permutation originated from an even number of neighbors exchanges. This is an equivalence relation and we can define an oriented n-simplex as the equivalence class of this relation. Obviously, there are two equivalence classes and we denote with $\sigma_n = \langle p_0 p_1 \ldots p_n \rangle$ the equivalence class that contains $\{p_0, p_1, \ldots, p_n\}$ and with $-\sigma_n$ the other which contains the simplices obtained by an odd permutation of the prevoius sequence. For instance, we have $\langle p_0 p_1 \rangle = -\langle p_1 p_0 \rangle$.

3.2 Simplicial homology and singular homology

The simplest and more intuitive homology groups are the ones of the simplicial homology defined via the simplicial complexes. We present the definition of these groups since the structure of other types of homology and cohomology that we will use are similar.

Let K a k-dimensional simplicial complex composed by oriented simplices σ_i. The n-chain group $C_n(K)$ of a simplicial complex K is a free abelian group generated by the oriented n-simplices of K. If $n >$ dim$K$ we set $C_n(K) = 0$. An element $c \in C_n(K)$ is called an n-chain and it can be written as

$$c = \sum_{i=1}^{N} c_i \sigma_{n,i} \tag{3.3}$$

where $c_i \in \mathbb{Z}$ are the coefficients of c, $\sigma_{n,i}$ are different n-simplices in K and N is their number. The addition of two chains, the neutral element and the inverse one are defined in the obvious way by acting on the coefficients $c_i \in \mathbb{Z}$.
$C_n(K)$ is a free abelian group of rank N: $C_n(K) \simeq \bigoplus_{i=1}^{N} \mathbb{Z}$.

We define the boundary of a n-simplex as $(n-1)$-chain obtained by the action of the boundary operator ∂_n:

$$\partial_n \sigma_n := \sum_{i=0}^{n} (-1)^i \langle p_0 p_1 \ldots \hat{p}_i \ldots p_n \rangle \tag{3.4}$$

where $\hat{p}_i$ is omitted. A 0-simplex has no boundary, thus we set $\partial_0 \langle p_0 \rangle = 0$. For instance, $\partial_2 \langle p_0 p_1 p_2 \rangle = \langle p_1 p_2 \rangle - \langle p_0 p_2 \rangle + \langle p_0 p_1 \rangle$. We can notice that with this definition we preserve the orientation of the boundary, since we walk around the triangle via p_0, p_1, p_2 and return to p_0. The boundary operator acts linearly on the chains and it defines an homomorphism

$$\partial_n : C_n(K) \to C_{n-1}(K) \qquad \text{with} \quad \partial_n c = \sum_i c_i \partial_n \sigma_{n,i}. \tag{3.5}$$

Therefore, we are in presence of a sequence of homomorphisms of abelian groups called chain complex:

$$\cdots \to C_{n+1}(K) \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0 \tag{3.6}$$

If $c \in C_n(K)$ satisfies $\partial_n c = 0$, it is called a n-cycle and belongs to the n-cycle group $Z_n(K) \subset C_n(K)$ with the property that $Z_n(K) = \ker \partial_n$. Let $c \in C_n(K)$, if there exists an element $\tilde{c} \in C_{n+1}(K)$ such that $c = \partial_{n+1}\tilde{c}$, then c is called a n-boundary. The set of n-boundaries $B_n(K) \subset C_n(K)$ is called the n-boundary group and $B_n(K) = \operatorname{im} \partial_{n+1}$. The crucial property of the boundary operator is that

$$\partial_n \circ \partial_{n+1} : C_{n+1}(K) \to C_{n-1}(K) \quad \text{is a zero map, i.e.} \quad \partial_n(\partial_{n+1} c) = 0. \tag{3.7}$$

Therefore, we have the inclusion $B_n(K) \subset Z_n(K)$, since an element of $B_n(K)$ is the boundary of a $(n+1)$-chain. This fact is according to the geometrical interpretation that a boundary has in turn no boundaries.

Finally, we can define the n-homology group $H_n(K)$ as the quotient group

$$H_n(C) = \frac{Z_n(K)}{B_n(K)} = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}. \tag{3.8}$$

This is the set of the equivalence classes of n-cycles; two n-cycles are said to be homologous if they are in the same equivalence class, i.e. if they differ only by a n-coboundary. We denote the equivalence class of a n-cycle z as $[z]$. One can prove that homology groups are topological invariant since they do not depend on the triangulation of the topological space.

The simplicial homology of simplicial complex is the most intuitive homology that one can define on a topological space. However, there exists another formulation based on the Δ-complexes, which has the advantage of simpler computations since fewer simplices are required. For example, to consider a triangulation of the torus $\mathbb{T}^2$ one need at least 14 triangles, 21 edges and 7 vertices, while a Δ-complex describing the torus is composed by 2 triangles, 3 edges and 1 vertices. Without entering into details, a Δ-complex is a quotient space of a collection of disjoint simplices obtained by identifying certain of their faces via the canonical linear homeomorphisms that preserve the ordering of vertices.

As we have anticipated, the most important homology theory in algebraic topology is the singular homology. We sketch its main features since it is the base of the singular cohomology that we use to describe discrete gauge symmetries. A singular n-simplex in a topological space X is a continuous map $\sigma : \Delta^n \to X$, where Δ^n is the standard n-simplex

$$\Delta^n = \left\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \middle| \sum_i t_i = 1 \text{ and } t_i \geq 0 \ \forall i\right\} \tag{3.9}$$

whose vertices are the unit vectors along the coordinate axes. The word 'singular' means that σ is not necessarily a nice embedding but can have 'singularities' where its image does not look at all like a simplex. The only requirement for σ is continuity. As for simplicial homology, we define the singular n-chain group $C_n(X)$ as the free abelian group with basis the set of singular n-simplices in X. A singular n-chain is a finite formal sum $\sum_i c_i\sigma_i$ with coefficients $c_i \in \mathbb{Z}$. The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined by the action on the the basis elements

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{\langle t_0 \dots \hat{t}_i \dots t_n\rangle} \tag{3.10}$$

where $|$ means the restriction of σ on the standard $(n-1)$-simplex where $\hat{t}_i$ is omitted. This boundary map has the usual properties that we expect in homology: $\partial_n\partial_{n+1} = 0$, the singular n-cycles are the n-chains belonging to the kernel of ∂_n, i.e. $c \in C_n(X)$ such that $\partial_n c = 0$, and the singular n-boundaries are the n-chains in the $\operatorname{im}\partial_{n+1}$, i.e the n-chains in $C_n(X)$ that can be expressed as $c = \partial_{n+1}c'$ for some $c' \in C_{n+1}(X)$. These subgroups are denoted respectively $Z_n(X)$ and $B_n(X)$ and the singular n-homology group is

$$H_n(X) = \frac{Z_n(X)}{B_n(X)} = \frac{\ker\partial_n}{\operatorname{im}\partial_{n+1}}. \tag{3.11}$$

From the definitions, it is evident that homeomorphic spaces have isomorphic singular homology groups and that singular homology are defined also for non triangulable spaces. However, one can prove the equivalence of the singular and the simplicial homology with standard techniques in algebraic topology.

Finally, there is an easy generalization of the homologies considered so far. Instead of using chains of the form $\sum_i c_i\sigma_i$ with $c_i \in \mathbb{Z}$, we can take the coefficients c_i in a fixed abelian group G. Such n-chains form an abelian group $C_n(X, G)$. All the properties and definitions based on the boundary operator remain the same; thus, we can define the n-homology group with coefficient in G, $H_n(X, G)$, by quotienting the n-cycle group $Z_n(X, G)$ by the n-boundary group $B_n(X, G)$ as usual. We denote simply with $H_n(X)$ the n-homology group with $\mathbb{Z}$ coefficients, otherwise we indicate explicitly the group; for instance we can have $H_n(X, \mathbb{R})$ or $H_n(X, \mathbb{Z}_2)$ and so on.

3.3 Singular cohomology

Cohomology is an alternative algebraic variant of homology and corresponds to the dualization of homology. The structure and the definitions are very similar, except that the induced homomorphisms go in the opposite direction after dualization. A particular type of cohomology, the De Rham cohomology, constitutes also a link between this algebraic structure and differential geometry.

Let us start by defining the singular cohomology. Let X a topological space and $\mathcal{A}$ an abelian group, we define the group $C^n(X, \mathcal{A})$ of singular n-cochains with coefficients in $\mathcal{A}$ to be the dual group $\mathrm{Hom}(C_n(X), \mathcal{A})$ of the singular chain group $C_n(X)$. A n-cochain $f \in C^n(X, \mathcal{A})$ assigns to each singular n-simplex $\sigma : \Delta^n \to X$ a value $f(\sigma) \in \mathcal{A}$. Since the singular n-simplices form a basis for $C_n(X)$, these values can be chosen arbitrarily and a n-cochain is exactly equivalent to a function from singular n-simplices to $\mathcal{A}$. We denote these cochains with the subscripts that correspond to the simplices whose they are functions; i.e. $f_{i_0 i_1 \dots i_n}$ is a function of the singular simplex σ_i with standard simplex $\langle i_0 i_1 \dots i_n \rangle$.

The coboundary map $\mathrm{d}_n : C^n(X, \mathcal{A}) \to C^{n+1}(X, \mathcal{A})$ is the dual ∂^*:

$$\langle \mathrm{d}_n f, c \rangle = (-1)^n \langle f, \partial_{n+1} c \rangle \tag{3.12}$$

where $f \in C^n(X, \mathcal{A})$, $c \in C_{n+1}(X)$ and the pairing $\langle f, c \rangle$ represents the linear $\mathcal{A}$-valued action of $f \in C^n(X, \mathcal{A})$ on $c \in C_n(X)$. In other words, the coboundary $\mathrm{d}_n f$ for $f \in C^n(X, \mathcal{A})$ is the composition $C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{f} \mathcal{A}$. This leads to the following formula for the coboundary $\mathrm{d}_n f$ on a n-cochain $f \in C^n(X, \mathcal{A})$:

$$(\mathrm{d}_n f)_{i_0 i_1 \dots i_{n+1}} = \sum_{j=0}^{n+1} (-1)^j f_{i_0 \dots \hat{i}_j \dots i_{n+1}} \tag{3.13}$$

where $\hat{i}_j$ is omitted. As for homology, we obtain the sequence of homomorphisms between the cochain groups called cochain complex:

$$\dots \leftarrow C^{n+1}(X, \mathcal{A}) \xleftarrow{\mathrm{d}_n} C^n(X, \mathcal{A}) \xleftarrow{\mathrm{d}_{n-1}} C^{n-1}(X, \mathcal{A}) \leftarrow \dots \leftarrow C^0(X, \mathcal{A}) \leftarrow 0 \tag{3.14}$$

Since $\partial_n \partial_{n+1} = 0$, it is automatic that $\mathrm{d}_{n+1} \mathrm{d}_n = 0$. Therefore, we can define the n-cohomology group $H^n(X, \mathcal{A})$ with coefficients in $\mathcal{A}$ to be the quotient

$$H^n(X, \mathcal{A}) = \frac{Z^n(X, \mathcal{A})}{B^n(X, \mathcal{A})} = \frac{\ker \mathrm{d}_n}{\mathrm{im}\ \mathrm{d}_{n-1}} \tag{3.15}$$

where elements of $Z^n(X, \mathcal{A}) = \ker \mathrm{d}_n$ are n-cocycles or closed n-cochains, that is n-cochains f that vanish on boundary or in other words that $\mathrm{d}_n f = 0$, and elements in $B^n(X, \mathcal{A}) = \mathrm{im}\ \mathrm{d}_{n-1}$ are n-coboundaries or exact n-cochains. $f \in B^n(X, \mathcal{A})$ is an exact n-cochain if there exists $h \in C^{n-1}(X, \mathcal{A})$ such that $f = \mathrm{d}_{n-1} h$. In the following we will omit the subscripts of d.

If the topological space X is a smooth manifold, De Rham's theorem says that the singular cohomology with real coefficients is isomorphic to the De Rham cohomology of X constructed with the differential forms. A n-cochain is a n-form, the boundary operator is the exterior differential d that induces the De Rham complex. An element of $\ker \mathrm{d}_n$ is a closed n-form and belongs to the cocycle group $Z^n(X, \mathbb{R})$, while an element of $\mathrm{im}\ \mathrm{d}_{n-1}$ is an exact n-form and belongs to $B^n(X, \mathbb{R})$. The n-De Rham cohomology group is the quotient

$$H^n(X, \mathbb{R}) = \frac{Z^n(X, \mathbb{R})}{B^n(X, \mathbb{R})} = \frac{\ker \mathrm{d}_n}{\mathrm{im}\ \mathrm{d}_{n-1}}. \tag{3.16}$$

3.4 Group cohomology

We want to introduce a particular type of cohomology groups which are useful in the study of gauge groups and their backgrounds. Brief introductions are present in [12], [11], [25] and [8]. In order to define the group cohomology, we have to introduce the Eilenberg-Mac Lane space which serves as a universal classifying space. Let G be a group and q a non negative integer.

The Eilenberg-Mac Lane space $K(G, q+1)$ is a connected topological space with the $(q+1)$-th homotopy group isomorphic to G, i.e. $\pi_{q+1} \simeq G$, and all other homotopy groups trivial. It is defined up to homotopy equivalence and when $q > 0$ G must be abelian.

Let us initially focus on the case $q = 0$ to present the concept of classifying space. A classifying space BG is the base space of a particular principal G-bundle EG called universal bundle. A universal bundle is a principal G-bundle with the property that every principal G-bundle E over a base space M is a pull-back bundle by a continuos map $\gamma : M \to BG$. In other words, any principal G-bundle E admits a bundle map $\Gamma : E \to EG$ into the universal bundle with the property that any two such morphism are smoothly homotopic, i.e. $\Gamma \sim \Gamma'$ $\forall \Gamma, \Gamma' : E, E' \to EG$ with $\sim$ the homotopy equivalence relation. A representation of these maps is in the following diagram, where we denotes with γ the induces map of Γ.

$$\begin{array}{ccc} E & \xrightarrow{\Gamma} & EG \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi_E} \\ M & \xrightarrow{\gamma} & BG \end{array} \tag{3.17}$$

The topology of the bundle E is completely determined by the homotopy class of the classifying map γ, that is the different classes of maps $[M, BG]$ are in bijective correspondence with distinct G-bundles on M.

One can show that BG is uniquely determined up to homotopy by requiring EG to be contractible and the action of G to be free. We remember that an action of a group G on M, i.e. a map $\rho : G \times M \to M$ with $\rho(e, p) = p$ $\forall p \in M$ and $\rho(g_1, \rho(g_2, p)) = \rho(g_1 g_2, p)$ $\forall g_1, g_2 \in G$, $p \in M$, is free if any non trivial $g \in G$ has no fixed points in M, that is if $\rho(g, p) = p \Rightarrow g \equiv e$. In general the classifying space for compact group are infinite dimensional space; for instance $BU(1) = \mathbb{CP}^\infty$ and $B\mathbb{Z}_2 = \mathbb{RP}^\infty$, where $\mathbb{P}$ stands for projective plane.

We can now define the group cohomology of a group, which is distinct from its cohomology as topological space. It consists of the cohomology groups $H^n(BG, \mathcal{A})$, where BG is the Eilenberg Mac Lane space $K(G, 1)$ and $\mathcal{A}$ is an abelian group. When $\mathcal{A}$ is finite one can take the singular cohomology of BG; in contrast, when $\mathcal{A}$ is continuos the topology of $\mathcal{A}$ should be considered, and we have to use the sheaf cohomology. For our purposes, we will derive an isomorphism useful when $\mathcal{A} = U(1)$.

The elements in $H^n(BG, \mathcal{A})$ are cohomology classes called universal characteristic classes for a background field, since under the pullback γ^* they give rise to cohomology classes in $H^n(M, \mathcal{A})$ that depends only on the topology of the bundle E. Remember that characteristic classes are subset of the cohomology of the base space that measure the non triviality or twisting of a bundle. In general elements in $H^n(K(G, q+1), \mathcal{A})$ are classifying classes for the q-form gauge field of a symmetry G.

With particular regard to physics, the universal classifying space $K(G, q+1)$ serves as classifying space for the background fields for the q-form finite symmetry group G. We have already seen, that when $q = 0$ the background is a G-bundle and distinct G bundles are in corrispondence with the homotopy class of maps $[M, K(G, 1) = BG]$. When $q \geq 0$, G is abelian and the background q-form fields are element of $H^{q+1}(M, G)$ cohomology group, whose distinct elements are in bijective correspondence with the homotopy classes of maps $[M, K(G, q+1) = B^q G]$.

If we regard a background field on M as a homotopy class of maps, $g \in [M, K(G, q+1)]$, and we take an $\mathcal{A}$-valued cohomology class $\alpha \in H^n(K(G, q+1), \mathcal{A})$, we can pullback $\mathcal{A}$ via g to M and find a cohomology class $g^*(\alpha) \in H^n(M, \mathcal{A})$. We can denote by $\alpha(g)$ this pullback and regard α as an operation:

$$\alpha : [-, K(G, q+1)] \to H^n(-, \mathcal{A}). \tag{3.18}$$

If we use the simplicial notation for the transition functions of the background field (4.1), we usually denote by A_{ij} the elements $g \in G$, but the expression $\alpha(A) = A^*\alpha$ has the same meaning as above.

In the following we will focus only on group cohomology; however the computation and characterization of the different cohomology groups $H^n(BG, \mathcal{A})$ are cumbersome. We will present some episodic results and in the next section we will focus on finite abelian groups.

If the group G is compact all odd cohomology groups with real coefficients vanish

$$H^n(BG, \mathbb{R}) = 0 \quad \text{with} \quad n\,\text{odd}; \tag{3.19}$$

therefore, the odd cohomology consists completely of torsion. Remember that an element h of a group H is a torsion element if it has finite order, that is if there exists $m \in \mathbb{N}$ such that $h^m = e$ with e the identity group. A group is a torsion group if all its elements are torsion elements. If the cohomology groups are even there exists an important isomorphism

$$\bigoplus_{n\ \text{even}} H^n(BG, \mathbb{R}) = I(G) \quad \text{with} \quad n\,\text{even} \tag{3.20}$$

where $I(G)$ is the ring of polynomials on the Lie algebra of G invariant under the adjoint action of G. Using the Chern-Weil homomorphism we map a polynomial $P \in I(G)$ to the class $[P(F)]$, where F is the curvature of an arbitrary connection in the universal bundle. The class $[P(F)]$ is independent of the choice of the connection. For instance, the group cohomology of the unitary group $U(N)$ contains no torsion and is given by the polynomial ring in the Chern classes $c_k(F_2)$:

$$\bigoplus_n H^n(BU(N), \mathbb{Z}) = P[c_1, \ldots, c_N]. \tag{3.21}$$

We are mainly interested in $U(1)$ whose cohomology is $\bigoplus_n H^n(BU(1), \mathbb{Z}) = P[c_1]$ with $c_1(F_2) = \frac{1}{2\pi}F_2$.

Finally, we will present an important result for finite groups, based on the fact that all real group cohomology are trivial: $H^n(BG, \mathbb{R}) = 0$ when G is finite. Before that we have to introduce some other concepts.

A sequence of homomorphisms between generic groups H_i

$$\cdots \to H_{i+1} \xrightarrow{h_{i+1}} H_i \xrightarrow{h_i} H_{i-1} \to \cdots \tag{3.22}$$

is said to be exact if the kernel of each map is the image of the previous one, $\ker h_i = \operatorname{im} h_{i+1}$. Since $\operatorname{im} h_{i+1} \subset \ker h_i$ is equivalent to $h_i h_{i+1} = 0$, the sequence is a chain complex; on the other hand, since $\ker h_i \subset \operatorname{im} h_{i+1}$ the homology group of this chain complex are trivial. Using exact sequences we can express some basic algebraic concepts; for example:

1. $0 \to A \xrightarrow{\alpha} B$ is an exact sequence if $\ker \alpha = 0$ (since the first map can be thought as an inclusion), that is α is injective;
2. $A \xrightarrow{\alpha} B \to 0$ is an exact sequence if $\operatorname{im} \alpha = B$ since B is the kernel of the null map which must be the image of α, thus, α is surjective;
3. $0 \to A \xrightarrow{\alpha} B \to 0$ is an exact sequence if α is an isomorphism by (1) and (2);
4. $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is an exact sequence if α is injective, β is surjective and $\ker \beta = \operatorname{im} \alpha$. Therefore, β induces an isomorphism $C \simeq B/\operatorname{im} \alpha$ which can be written $C \simeq B/A$ if we think of α as an inclusion of A as a subgroup of B. An exact sequence of this type is called short exact sequence and the map β is a projection on the quotient.

A short exact sequence induces a exact sequence in cohomology if we consider cohomology with values in those groups; in fact, for each group we can take the set $\mathrm{Hom}(C_n(X), -) = C^n(X, -)$ where $C_n(X)$ is the n-chain group. This gives rise to a long exact sequence in cohomology since $C_n(X)$ is free. Let

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \simeq B/A \to 0 \tag{3.23}$$

a short exact sequence that induces a long exact sequence in cohomology:

$$\cdots \to H^n(X, A) \xrightarrow{\alpha} H^n(X, B) \xrightarrow{\beta} H^n(X, B/A) \xrightarrow{\text{Bock}} H^{n+1}(X, A) \to \cdots \tag{3.24}$$

whose map

$$\text{Bock} : H^n(X, B/A) \to H^{n+1}(X, A) \tag{3.25}$$

is called Bockstein homomorphism. This map can be always constructed and depends only on the choice of a class in $H^n(X, B/A)$. Let us take a representative $c \in Z^n(X, B/A)$ of a cohomology class $[c] \in H^n(X, B/A)$ and lift to an element $b \in C^n(X, B)$, namely $\beta(b) = c$. b is not necessarily closed, but $\beta(\mathrm{d}b) = 0$ because it is equal to $\mathrm{d}\beta(b) = \mathrm{d}c = 0$. Since $\mathrm{d}b \in \ker\beta$, it is in the image of α, thus $\mathrm{d}b = \alpha(a)$ for some $a \in C^{n+1}(X, A)$. Since α is injective, $\mathrm{d}a = 0$ and a defines a class $[a] \in H^{n+1}(X, A)$. Therefore, the Bockstein homomorphism is defined as $[c] \xrightarrow{Bock} [a]$ where $a \in C^{n+1}(X, A)$ such that $\alpha(a) = \mathrm{d}b$ for $b \in C^n(X, B)$ such that $\beta(b) = c$. Since the kernel of Bockstein homomorphism is the image of β, the Bockstein represent the obstruction to lifting a class in $H^n(X, B/A)$ to a class in $H^n(X, B)$.

Let us go back to group cohomology. Using the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \simeq U(1) \to 0 \tag{3.26}$$

that induces in group cohomology the following exact sequence

$$\cdots H^n(BG, \mathbb{Z}) \to H^n(BG, \mathbb{R}) \to H^n(BG, \mathbb{R}/\mathbb{Z}) \to H^{n+1}(BG, \mathbb{Z}) \to \cdots \tag{3.27}$$

and the fact that $H^n(BG, \mathbb{R}) = 0 \ \forall n$ when G is finite, we obtain the exact sequence

$$\cdots H^n(BG, \mathbb{Z}) \to 0 \to H^n(BG, \mathbb{R}/\mathbb{Z}) \to H^{n+1}(BG, \mathbb{Z}) \to 0 \to \cdots \tag{3.28}$$

Therefore, we can conclude that for finite G the following isomorphism holds

$$H^n(BG, \mathbb{R}/\mathbb{Z}) \simeq H^{n+1}(BG, \mathbb{Z}). \tag{3.29}$$

Notice that in the long sequence 3.27 the kernel of the homomorphism $H^n(BG, \mathbb{Z}) \to H^n(BG, \mathbb{R})$ is the torsion elements of $H^n(BG, \mathbb{Z})$. Since for finite group the kernel of this map is the entire $H^n(BG, \mathbb{Z})$, we can conclude via the above homomorphism that $H^n(BG, \mathbb{R}/\mathbb{Z})$ is constitute only of torsion.

3.5 Group cohomology for finite abelian groups

For finite groups the cohomology group $H^n(BG, \mathcal{A})$ has an algebraic description similar to those considered above. In the definition we consider the general case where $\mathcal{A}$ is a G-module, that is an abelian group on which G acts compatibly with its abelian group structure. We denote this homomorphism as $\rho : G \to \mathrm{Aut}\mathcal{A}$.

We define an n-cochain $f \in C^n(BG, \mathcal{A})$ to be a function

$$f : \underbrace{G \times \cdots \times G}_{n} \to \mathcal{A} \tag{3.30}$$

The differential is defined to be

$$\begin{aligned}(\mathrm{d}_\rho f)(g_1,\dots,g_{n+1}) = \rho_{g_1} f(g_2,\dots,g_{n+1}) + \sum_{j=1}^{n}(-1)^j f(g_1,\dots,g_j g_{j+1},\dots,g_{n+1}) + \\ + (-1)^{n+1} f(g_1,\dots,g_n).\end{aligned} \tag{3.31}$$

For instance, we can have

$$\begin{aligned}(\mathrm{d}_\rho f)(g_1,g_2) &= \rho(g_1)f(g_2) - f(g_1g_2) + f(g_1)\\ (\mathrm{d}_\rho f)(g_1,g_2,g_3) &= \rho(g_1)f(g_2,g_3) - f(g_1g_2,g_3) + f(g_1,g_2g_3) - f(g_1,g_2)\\ (\mathrm{d}_\rho f)(g_1,g_2,g_3,g_4) &= \rho(g_1)f(g_2,g_3,g_4) - f(g_1g_2,g_3,g_4) + f(g_1,g_2g_3,g_4)-\\ &\quad - f(g_1,g_2,g_3g_4) + f(g_1,g_2,g_3).\end{aligned} \tag{3.32}$$

The differential is nilpotent, that is $\mathrm{d}^2 = 0$. Therefore, we can define the n-cocycle group $Z^n(BG,\mathcal{A})$ as the kernel of the differential and the n-coboundary group $B^n(BG,\mathcal{A})$ as the image of the previous differential in the cochain complex. The cohomology group $H^n(BG,\mathcal{A})$ is defined as usual by taking the quotient, $Z^n(BG,\mathcal{A})/B^n(BG,\mathcal{A})$. These cochains are assumed to be normalized, i.e. $f(g_1,\dots,g_n) = 0$ if $g_i = e$ for some $1 \le i \le n$. The equivalence between algebraic cocycles and simplicial cocycles of BG is proved using Milnor's construction of BG.

For the simple case of finite abelian groups we can explicitly provide an expression for the cocycles of some cohomology groups. Since any finite abelian group can be expressed as the product of cyclic abelian groups, we can consider the cohomology groups of $\mathbb{Z}_N$, the prototypical cyclic groups. In particular we will focus on the third cohomology group of the direct product of k cyclic groups. We will use these results for the calculation of some anomalies in *Chapter 6* and *7*.

Initially, we observe that in the case of finite groups the cohomology of the universal classifying space BG is the same of the group itself, since the following isomorphism holds:

$$H^n(BG,\mathbb{Z}) \simeq H^n(G,\mathbb{Z}). \tag{3.33}$$

Furthermore, we have proved that also the following isomorphism is valid

$$H^n(G,\mathbb{Z}) \simeq H^{n-1}(G,U(1)) \qquad \forall n > 1 \tag{3.34}$$

therefore, we can derive the results for the group $H^n(G,U(1))$, which is more useful for our goals, from the features of $H^{n+1}(G,\mathbb{Z})$.

Using the Kunneth formula, a result in algebraic topology, one can prove abstractly the content of the relevant cohomology groups $H^n(\mathbb{Z}_N^k,U(1))$. We only report the results and provide an euristical motivation by considering explicitly the 3-cocycles for the third cohomology group (see [26] for details).

$$\begin{aligned}&H^1(\mathbb{Z}_N^k,U(1)) \simeq \mathbb{Z}_N^k\\ &H^2(\mathbb{Z}_N^k,U(1)) \simeq \mathbb{Z}_N^{\frac{1}{2}k(k-1)}\\ &H^3(\mathbb{Z}_N^k,U(1)) \simeq \mathbb{Z}_N^{k+\frac{1}{2}k(k-1)+\frac{1}{3!}k(k-1)(k-2)}\end{aligned} \tag{3.35}$$

In order to write the 3-cocycles, we have to fix some notational conventions. We denote the group elements with the capital letters $A,B,C \in \mathbb{Z}_N^k$, and with the lowercase letters the elements of the different subgroups; e.g. for A we have

$$A := (a^{(1)},a^{(2)},\dots,a^{(k)}) \quad \text{with} \quad a^{(i)} \in \mathbb{Z}_N \quad \forall i = 1,\dots,k. \tag{3.36}$$

We adopt the additive notation for $\mathbb{Z}_N$ and we denote its elements with the first $N-1$ natural numbers: $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$. When we write $a^{(i)} + b^{(j)}$ for generic elements $a^{(i)}, b^{(i)} \in \{0, 1, \ldots, N-1\}$ of A and B, we mean the usual addition with values in $\mathbb{Z}$. The group operation is defined as following

$$A + B = ([a^{(1)} + b^{(1)}], \ldots, [a^{(k)} + b^{(k)}]) \tag{3.37}$$

where $[\cdot] : \mathbb{Z} \to \{0, 1, \ldots, N-1\}$ denotes the reduction $\mod N$.

From the abstract derivation of the isomorphism for $H^3(\mathbb{Z}_N^k, U(1))$, one can argue that there are three different types of 3-cocycles that belong to the third cohomology group. They differ in the number and type of subgroups $\mathbb{Z}_N^{(i)}$ involved in their definition.

$$\omega_I^{(i)}(A, B, C) = \exp\left(\frac{2\pi i p_I^{(i)}}{N^2} a^{(i)}(b^{(i)} + c^{(i)} - [b^{(i)} + c^{(i)}])\right) \qquad 1 \leq i \leq k \tag{3.38}$$

$$\omega_{II}^{(ij)}(A, B, C) = \exp\left(\frac{2\pi i p_{II}^{(ij)}}{N^2} a^{(i)}(b^{(j)} + c^{(j)} - [b^{(j)} + c^{(j)}])\right) \qquad 1 \leq i < j \leq k \tag{3.39}$$

$$\omega_{III}^{(ijl)}(A, B, C) = \exp\left(\frac{2\pi i p_{III}^{(ijl)}}{N} a^{(i)} b^{(j)} c^{(l)}\right) \qquad 1 \leq i < j < l \leq k \tag{3.40}$$

where $p_I^{(i)}, p_{II}^{(ij)}, p_{III}^{(ijl)}$ are integer parameters that label the different 3-cocycles. These cocycles are periodic of period N, in accordance with the isomorphisms above; therefore, we can consider $p_I^{(i)}, p_{II}^{(ij)}, p_{III}^{(ijl)} \in \mathbb{Z}_N$. The periodicity for the cocyle of type III is obvious, while for the type I and II it follows from the fact that the factor $(b^{(i)} + c^{(i)} - [b^{(i)} + c^{(i)}])$ only vanishes or is equal to N.

The 3-cocycles of type I describe the contribution arising from a single $\mathbb{Z}_N$ subgroup and the $N-1$ non-vanishing terms are labelled by p_I^i with $1 \leq i \leq k$; thus, we can understand the first term $\mathbb{Z}_N^k$ in the isomorphism above. The type II cocycles establish pairwise couplings between two different $\mathbb{Z}_N$ subgroups and ω_{II}^{ij} is equivalent to ω_{II}^{ji} since they differ by a 3-coboundary. In the counting of different terms we have to consider k terms for the indices i and $k-1$ for j, which must be different from i, and we have to divide by 2 since they do not depend on the order; therefore, we obtain $\frac{1}{2}k(k-1)$ inequivalent terms. For the same reason, in the cocycles of type III we can permute the indices i, j, l that must be different; hence, we end up with $\frac{1}{3!}k(k-1)(k-2)$ different 3-cocycles.

Finally, we can consider the cocycles for general abelian groups which are the direct product of cyclic groups of different order. If the group is the product of k different cyclic groups, from the algebraic topological analysis, we can argue that there are again k inequivalent cocycles of type I, $\frac{1}{2}k(k-1)$ of type II and $\frac{1}{3!}k(k-1)(k-2)$ of type III. Therefore, one can easily generalize the previous expressions:

$$\begin{aligned}
\omega_I^{(i)}(A, B, C) &= \exp\left(\frac{2\pi i p_I^{(i)}}{N^{(i)2}} a^{(i)}(b^{(i)} + c^{(i)} - [b^{(i)} + c^{(i)}])\right) \qquad 1 \leq i \leq k \\
\omega_{II}^{(ij)}(A, B, C) &= \exp\left(\frac{2\pi i p_{II}^{(ij)}}{N^{(i)}N^{(j)}} a^{(i)}(b^{(j)} + c^{(j)} - [b^{(j)} + c^{(j)}])\right) \qquad 1 \leq i < j \leq k \\
\omega_{III}^{(ijl)}(A, B, C) &= \exp\left(\frac{2\pi i p_{III}^{(ijl)}}{\gcd(N^{(i)}, N^{(j)}, N^{(j)})} a^{(i)} b^{(j)} c^{(l)}\right) \qquad 1 \leq i < j < l \leq k
\end{aligned} \tag{3.41}$$

where $N^{(i)}$ denotes the order of the i-th cyclic factor of the direct product that constitutes the group G and gcd means greatest common divisor. We can make explicit our discussion considering as specific example the group $G = \mathbb{Z}_N \times \mathbb{Z}_M \times \mathbb{Z}_K$. The first three cohomology groups are characterized by

$$\begin{aligned}
&H^1(\mathbb{Z}_N \times \mathbb{Z}_M \times \mathbb{Z}_K, U(1)) \simeq \mathbb{Z}_N \oplus \mathbb{Z}_M \oplus \mathbb{Z}_K \\
&H^2(\mathbb{Z}_N \times \mathbb{Z}_M \times \mathbb{Z}_K, U(1)) \simeq \mathbb{Z}_{\gcd(N,M)} \oplus \mathbb{Z}_{\gcd(N,K)} \oplus \mathbb{Z}_{\gcd(M,k)} \\
&H^3(\mathbb{Z}_N \times \mathbb{Z}_M \times \mathbb{Z}_K, U(1)) \simeq \mathbb{Z}_N \oplus \mathbb{Z}_M \oplus \mathbb{Z}_K \oplus \mathbb{Z}_{\gcd(N,M)} \oplus \mathbb{Z}_{\gcd(N,K)} \oplus \mathbb{Z}_{\gcd(M,k)} \oplus \mathbb{Z}_{\gcd(N,M,K)}
\end{aligned} \tag{3.42}$$

The cocycles of type I are labeled by $p_I^{(i)} \in \mathbb{Z}_N \oplus \mathbb{Z}_M \oplus \mathbb{Z}_K$ and the periodicity is the order $N^{(i)}$ of the correspondent cyclic group. The type II are labeled by $\mathbb{Z}_{\gcd(N,M)}, \mathbb{Z}_{\gcd(N,K)}$ and $\mathbb{Z}_{\gcd(M,k)}$ and with period $\gcd(N^{(i)}, N^{(j)})$. This can be shown by considering the equality

$$\frac{\gcd(N^{(i)}, N^{(j)})}{N^{(i)} N^{(j)}} = \frac{x}{N^{(i)}} + \frac{y}{N^{(j)}}, \quad x, y \in \mathbb{Z} \tag{3.43}$$

and by observing that $\omega_{II}^{(ij)}$ becomes a trivial 3-cocycle or a 3-coboundary when we set $p_{II}^{(ij)} = \gcd(N^{(i)}, N^{(j)})$. Finally, the type III cocycles have period the greatest common divisor of N, M and K and are labeled by $p_{III}^{NMK} \in \mathbb{Z}_{\gcd(N,M,K)}$.

We can wonder what is the meaning of this three fundamental cohomology groups. The 1-cochains belonging to $H^1(G, U(1))$ are maps from G to $U(1)$ and with the cocycle condition turn them into group homomorphisms. In fact, let $c \in H^1(G, U(1))$ a map $c : G \to U(1)$, since it satisfies the cocycle condition $\mathrm{d}c = 0$, i.e. in multiplicative notation

$$c(g_1)c(g_2) = c(g_1, g_2) \quad g_1, g_2 \in G \tag{3.44}$$

the group law of G is preserved in $U(1)$ and c is a group homomorphism. Therefore, $H^1(G, U(1))$ is the group composed by all the inequivalent irreducible 1-dimensional unitary representations of G. When G is an abelian finite group, $H^1(G, U(1)) \simeq \mathrm{Hom}(G, U(1)) = \hat{G}$ where $\hat{G}$ is the Pontryagin dual group.

The second cohomology group $H^2(G, U(1))$ labels inequivalent projective representations of the group G. A representation R of a group G in a vectorial space V is a map

$$\begin{aligned}
R : G &\to \mathrm{Aut}V \\
g &\mapsto R(g)
\end{aligned} \tag{3.45}$$

where $\mathrm{Aut}V$ is the space of linear operators acting on V. R satisfies the properties: $R(e) = \mathbb{I}_X$ with e the identity in G and $R(g_1) \cdot R(g_2) = R(g_1 g_2)\ \forall g_1, g_2 \in G$. A projective representation is a group representation defined up to a phase. We can denote the projective operators $\mathcal{R}$ defining

$$\mathcal{R}(g) = \{e^{i\phi(g)} R(g); e^{i\phi(g)} \in U(1), R(g) \text{ a representation of } G\} \tag{3.46}$$

with composition rule $\mathcal{R}(g_1) \cdot \mathcal{R}(g_2) = \mathcal{R}(g_1 g_2)$. If we take a representative R of the projective representation, we can rewrite the composition rule

$$R(g_1) \cdot R(g_2) = \epsilon(g_1, g_2) R(g_1 g_2) \tag{3.47}$$

where $\epsilon(g_1, g_2)$ is a phase depending on both the group elements g_1, g_2. We have also to require that the representation preserves the associative property of the group; since $\forall g_1, g_2, g_3 \in G$ $g_1(g_2 g_3) = (g_1 g_2) g_3$ we have to require $R(g_1(g_2 g_3)) = R((g_1 g_2) g_3)$, but this leads to a condition for the phase ϵ. In fact, in order to have the equality of these relations

$$\begin{aligned}
R(g_1(g_2 g_3)) =& \epsilon(g_1, g_2 g_3) R(g_1) R(g_2 g_3) = \epsilon(g_1, g_2 g_3) \epsilon(g_2, g_3) R(g_1) R(g_2) R(g_3) \\
R((g_1 g_2) g_3) =& \epsilon(g_1 g_2, g_3) R(g_1 g_2) R(g_3) = \epsilon(g_1 g_2, g_3) \epsilon(g_1, g_2) R(g_1) R(g_2) R(g_3)
\end{aligned} \tag{3.48}$$

we obtain the condition

$$\epsilon(g_1, g_2)\epsilon(g_1 g_2, g_3) = \epsilon(g_1, g_2 g_3)\epsilon(g_2, g_3) \tag{3.49}$$

that is precisely the cocycle condition for the 2-cochain ϵ. Therefore, since we have to satisfy the associative property of the group, we have to require that $\epsilon \in Z^2(G, U(1))$. On the other hand, such a cocycle can be shifted by a coboundary of the form

$$(\mathrm{d}\zeta)(g_1, g_2) = \frac{\zeta(g_1)\zeta(g_2)}{\zeta(g_1 g_2)} \tag{3.50}$$

which corresponds to a rephasing of the group action on the projective representation. Thus, we have found that the inequivalent projective phases are defined up to coboundaries and belong to $H^2(G, U(1))$: the different phases in projective representation are classified by the second cohomology group $H^2(G, U(1))$. Notice that the usual representations (without phases) corresponds to the trivial element of this cohomology group.

Finally, the group $H^3(G, U(1))$ is useful in different contest, as we will see. It labels the 't Hooft anomaly in a 2-dimensional quantum field theory and the Postnikov classes in a 2-group symmetry. More generally, since $H^3(G, U(1)) \simeq H^4(BG, \mathbb{Z})$, the $3D$-Chern-Simons actions for a compact gauge group G are in one-to-one correspondence with the cohomology group elements [25].

Chapter 4

2-group global symmetries with discrete groups

If we want to discuss discrete symmetries, we cannot use the Noether's formalism since there is no conserved current. On the other hand, the language of topological operators introduced in the previous chapter is the basis of the discrete symmetry studies since, even though we cannot write an explicit form of these operators in terms of the Noether's currents, they successfully implement these symmetries. To describe these topological operators we couple the theory with a background gauge field, that is the connection of the principal bundle whose structure group is the symmetry group. The simplicial calculation is a useful tool to represent this bundle, even if it is necessary to introduce quite a lot of formalism. However, using topological defects in this formalism, the characterization of 2-group symmetries may be be deeper and more meaningful. We follow [8] as main reference.

4.1 Symmetry defects

A 0-form global symmetry is implemented by unitary topological operators $U_g(M^{(D-1)})$, $g \in G$ with G the symmetry group, supported along co-dimension 1 manifold $M^{(D-1)}$. When a charged object passes through one of these operators, it transforms according to the symmetry. We can always couple the theory with a background gauge field for the global symmetry. The principal bundle associated to the gauge field can be described covering the manifold $M^{(D)}$ with open contractible patches V_i. The order of them is arbitrary and their intersections are denoted with ordered indices by $V_{ij} = V_i \cap V_j$, $V_{ijk} = V_i \cap V_j \cap V_k$ and so on. We choose the covering in such a way that all possible multiple intersections are either empty or contractible. The transition function $A_{ij} \in G$ must satisfy the condition

$$A_{ij}A_{jk} = A_{ik} \quad \text{on triple intersections} \quad V_{ijk} \tag{4.1}$$

for ordered $\langle ijk \rangle$, i.e. are 1-cocycle in the simplicial formalism.

The simplicial formulation of the principal bundle starts from the triangulation of the manifold $M^{(D)}$ with the following identifications: vertices are 0-simplices $\langle i \rangle$, lines or edges are 1-simplices $\langle ij \rangle$, faces are 2-simplices $\langle ijk \rangle$ and so on up to D-simplex. The vertex $\langle i \rangle$ corresponds to the open chart V_i, the edge $\langle ij \rangle$ to the intersection V_{ij} and so on for the multiple intersections. Notice that each p-simplex ($p > 0$) gives an orientation following the order of the vertices from that with lower label to the one with higher index. A q-form on the manifold is represented as a q-cochain $f \in C^q(M^{(D)}, G)$ which is a function on q-simplices taking values in the group G. We denote this q-cochain as $f_{i_0 i_1 \ldots i_q}$ and we assume that $\{i_0 i_1 \ldots i_q\}$ are ordered. As we have already presented, we can define singular cohomology groups starting from these cochains (see section 3.3). Therefore, A_{ij} is a 1-cocycle since it satisfies $\mathrm{d}A = 0$ (in 4.1 we use multiplicative notation).

In order to describe a topological operator $U_g(M^{(D-1)})$ we couple the theory with a G-bundle with transition functions equal to g if the double intersections cross the support $M^{(D-1)}$ of the defect. More precisely, we construct the bundle in such a way that when an edge $\langle ij \rangle$ cuts the manifold $M^{(D-1)}$, we assign $A_{ij} = g$ if the edge crosses the hypersurface with positive orientation, or $A_{ij} = g^{-1}$ if with negative orientation. Otherwise, we write $A_{ij} = \mathbb{I}$. The cocycle condition is satisfied since for every 2-simplices there is either no edge or two edges with opposite orientation that cross the defect.

The action of the symmetry operator (in correlators) can be implemented as gauge transformation. Suppose we have a charged object $\mathcal{O}$ located at one of the vertices of the triangulation and surrounded by a support of a topological operator $U_g(M^{(D-1)})$. We can remove $U_g(M^{(D-1)})$ by performing a gauge tranformation

$$A_{ij} \to A^f_{ij} := f_i A_{ij} f_j^{-1} \tag{4.2}$$

with $f_i = g^{-1}$ if the vertex i is inside $M^{(D-1)}$ or $f_i = \mathbb{I}$ if it is outside. Therefore, the operator $\mathcal{O}$ is mapped to its transformation under g (see eq. 1.3).

Topological operators for a 1-form symmetry with (necessarily) abelian group $\mathcal{A}$ are unitary operators $U_a(M^{(D-2)})$ supported on codimension-2 manifolds. The background gauge bundle is realized by assigning an element of $\mathcal{A}$, $B_{ijk} \in \mathcal{A}$, on triple intersection V_{ijk} such that they satisfy the cocycle condition:

$$B_{jkl} - B_{ikl} + B_{ijl} - B_{jkl} = (\mathrm{d}B)_{ijkl} = 0 \tag{4.3}$$

for ordered $\langle ijkl \rangle$. Notice that we indicate with "+" the group operation since $\mathcal{A}$ is abelian. Therefore, $B \in Z^2(M^{(D)}, \mathcal{A})$. As for 0-form symmetries, the operator $U_a(M^{(D-2)})$ is described by assigning the value $a \in \mathcal{A}$ to the transition function of triple intersections that cross $M^{(D-2)}$. We write $B_{ijk} = a$ if the defect crosses the triangle $\langle ijk \rangle$ with positive orientation, $B_{ijk} = -a$ if with negative orientation and $B_{ijk} = \mathbb{I}$ otherwise.

The charged objects of 1-form symmetries are line operators. If a 1-form symmetry operator $U_a(M^{(D-2)})$ winds around a line operator $\mathcal{O}(M^{(1)})$ of charge α, we can remove such charge operator from the correlators by performing a gauge transformation:

$$B_{ijk} \to B_{ijk} + \gamma_{jk} - \gamma_{ik} + \gamma_{ij} = B_{ijk} + (\mathrm{d}\gamma)_{ijk} \tag{4.4}$$

on V_{ijk} for ordered vertices. γ is defined by assigning $\gamma_{ij} = a$ if the edge $\langle ij \rangle$ crosses with positive orientaton a generic surface Σ whose boundary is $M^{(D-2)}$, $\gamma_{ij} = -a$ if with negative orientation. Since Σ also cuts an edge along $\mathcal{O}(M^{(1)})$, in the symmetry transformation $\mathcal{O}(M^{(1)})$ acquires a phase $e^{2\pi i \alpha(a)}$, where $\alpha \in \hat{\mathcal{A}} = \mathrm{Hom}(\mathcal{A}, \mathbb{R}/\mathbb{Z})$, the Pontryagin dual of $\mathcal{A}$ (see 1.5).

4.2 2-group symmetry in the formalism of defects

If we regard symmetries as topological defects acting on charged objects, the group law is encoded in the idea of junctions of defects. When two topological operators U_g, $U_{g'}$ meet, they can fuse into one defect $U_{gg'}$ according to the group law. In correlators such junctions usually have the associative property, i.e. $U_{gh}U_k = U_g U_{hk} = U_{ghk}$. However, there exist junctions without this property and this is a possible signal of the presence of both 0-form and 1-form defects combined in a non-trivial way (2-group symmetry).

When we have two symmetries, a 0-form with group G and a 1-form with group $\mathcal{A}$, we can mix their actions. Firstly, the group G can act on $\mathcal{A}$ by a group automorphism (it preserves the group structure):

$$\rho : G \to \mathrm{Aut}A \tag{4.5}$$

where $\mathrm{Aut}A$ is the group of automorphisms of $\mathcal{A}$. In particular, when an operator $U_a(M^{(D-2)})$ with $a \in \mathcal{A}$ crosses the support of $U_g(M^{(D-1)})$, $g \in G$, it transforms in a new operator $U_{\rho_g a}(M^{(D-2)})$ and ρ acts as a permutation of the 1-dimensional representations of $\mathcal{A}$. On the other hand, when a line operator $\mathcal{O}(M^{(1)})$ of charge α crosses $U_g(M^{(D-1)})$, it becomes a line operator with charge $\rho_g \alpha := \alpha \rho_g^{-1}$ in order to assure that $\alpha(a)$ is invariant. This is precisely the phase that a line operator acquires when it crosses a 1-form defect.

If the mixing is due only to this homomorphism, the cocycle condition becomes twisted since it involves ρ:

$$\rho(A_{ij})B_{jkl} - B_{ikl} + B_{ijl} - B_{ijk} := (\mathrm{d}_A B)_{ijkl} = 0 \quad , \quad \text{on}\, V_{ijkl} \tag{4.6}$$

d_A is the twisted differential that is defined as follow:

$$(\mathrm{d}_A f)_{i_0 i_1 \dots i_{q+1}} = \rho(A_{i_0 i_1}) f_{i_1 \dots i_{q+1}} + \sum_{j=1}^{n+1} (-1)^j f_{i_0 \dots \hat{i}_j \dots i_{q+1}} \tag{4.7}$$

for ordered vertices $\langle i_j \rangle$. This differential is nilpotent and leads to the definition of twisted cocycles, twisted coboundaries and twisted cohomology classes, as usual.

The second element of a 2-group is the Postnikov class $[\beta]$ which is a group cohomology class

$$[\beta] \in H^3_\rho(BG, \mathcal{A}) \tag{4.8}$$

or, more concretely, it is a function

$$\beta : G \times G \times G \to \mathcal{A}. \tag{4.9}$$

We can understand the meaning of β considering the junction of three topological defects into one. Three defects U_g, U_h, U_k with $g, h, k \in G$ can merge into U_{ghk} in two different ways that differ from each other by a codimension-2 symmetry operator $\beta(g, h, k) \in \mathcal{A}$.

In D=2 we can view graphically the defects configuration (Figure 4.1): on the left we have the fusion $(U_g U_h)U_k = U_{ghk}$ and if we move topologically the defect U_h to the right (F-move) we have $U_g(U_h U_k) = U_{ghk} + \beta(g, h, k)$ as depicted on the right. This is necessary to mantain the correlators invariant. In other words, when U_h passes through the junction a codimension-2 symmetry operator $\beta(g, h, k)$ is created. In $D > 2$ considering the bordism between the two configuration in figure 4.1, the codimension-3 locus where the four operators meet acts as a source for the symmetry operator $\beta(g, h, k) \in \mathcal{A}$. The presence of such a 1-form defect starting from this triple junction is the signal of the presence of a global 2-group symmetry.

The function β is normalized such that $\beta(g, h, k) = 0$ if g or h or k are the group's identity and satisfies the twisted cocyle condition

$$(\mathrm{d}_\rho \beta)(g, h, k, l) = \rho_g \beta(h, k, l) - \beta(gh, k, l) + \beta(g, hk, l) - \beta(g, h, kl) + \beta(g, h, k) = 0$$

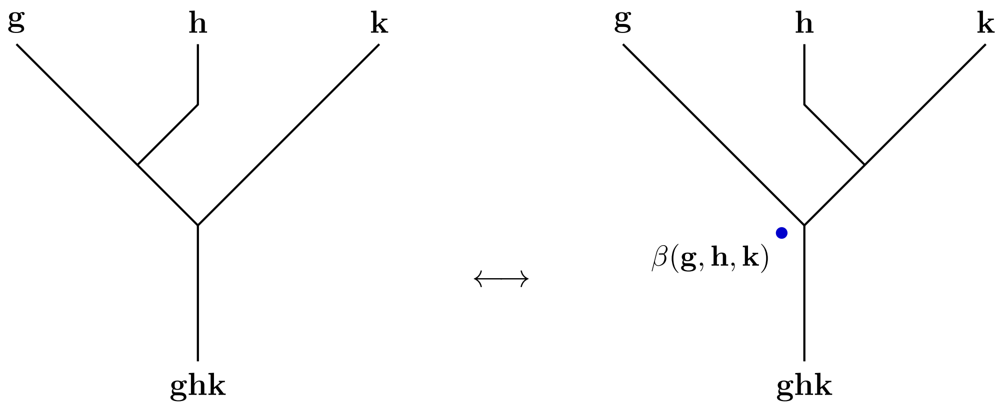

Figure 4.1: The lines are codimension-1 topological defects representig the symmetry operators of G. The blue dot is the 1-form symmetry operator $\beta(g,h,k) \in \mathcal{A}$. On the left the original configuration of the junction of the three defects g, h, k, on the right the same defect fusion but with h on the other side of the junction. They are equivalent only if we insert the 1-form symmetry β. The figure is inspired from [8]

for $g, h, k, l \in G$. This condition is a consequence of the pentagon identity. The physical theory depends only on the equivalence class $[\beta]$ where we identify

$$\beta \sim \beta + \mathrm{d}_\rho \mu \quad \text{with} \quad \mu : G \times G \to \mathcal{A}. \tag{4.10}$$

Therefore, the physical invariant of a 2-group symmetry is the equivalence class $[\beta] \in H^3_\rho(BG, \mathcal{A})$.

If the Postnikov class is non trivial, i.e. we have a 2-group symmetry, the background field for $\mathcal{A}$ is no longer closed and β fixes its coboundary:

$$(\mathrm{d}_A B)_{ijkl} = \beta(A_{ij}, A_{jk}, A_{kl}). \tag{4.11}$$

Since $A \in Z^1(M^{(D)}, G)$ is a cocycle, we can view it as a homotopy class of maps $A : M^{(D)} \to BG$ and we can write $\mathrm{d}_A B = A^*\beta$, where the pull-back A^* corresponds to substituting $g_j \to A_{i_{j-1}, i_j}$ for each simplex (see 3.4). In this way, we can see that the G-bundle acts as a source for B. The twisted differential of B depends on the representative β of the cohomology class $[\beta]$; if we change the representative $\beta \to \beta + \mathrm{d}_\rho \mu$, with μ a 1-cochain, we have to redefine simultaneously the 2-cochain $B \to B + A^*\mu$, so that the modified cocycle condition is still satisfied. This redefinition corresponds to a modification of the theory by adding local counterterms. Notice that we use the property $\mathrm{d}_A A^* = A^* \mathrm{d}_\rho$ valids as long as the cocycle condition for A_{ij} is satisfied.

In the end, a 2-group global symmetry is described by the quadruplet

$$\mathcal{G} = (G, \mathcal{A}, \rho, [\beta]) \tag{4.12}$$

The 2-group bundle consists of two type of transition functions: the usual transition function $A_{ij} \in G$ for the G-bundle with the cocycle condition $A_{ij}A_{jk} = A_{ik}$ for each triangle $\langle ijk \rangle$ with ordered vertices and the background field for $\mathcal{A}$ that is an $\mathcal{A}$-valued 2-cochain satysfying the modified twisted cocycle condition

$$\rho(A_{ij})B_{jkl} - B_{ikl} + B_{ijl} - B_{ijk} = \beta(A_{ij}, A_{jk}, A_{kl}) \tag{4.13}$$

for each 3-simplices $\langle ijkl \rangle$ with ordered vertices.

The gauge transformations in a 2-group bundle are of two types. The first is the usual one for A_{ij}, involving also the field 2-cochain B. It depends on a 0-cochain $f \in G$, i.e. for each vertex $\langle i \rangle$ we associate $f_i \in G$. The 1-cocycle A transforms as

$$A_{ij} \to A^f_{ij} := f_i A_{ij} f_j^{-1} \tag{4.14}$$

The transformation for the 2-cochain B is:

$$B \to B^f := \rho(f)B + \zeta(A, f) \tag{4.15}$$

where $\rho(f)B = \rho_{f_i} B_{ijk}$ for each 2-simplices $\langle ijk \rangle$ with ordered vertices and $\zeta(A, f)$ is a 2-cochain that satisfies

$$\mathrm{d}_{A^f}\zeta(A, f) = A^{f*}\beta - \rho(f)A^*\beta \tag{4.16}$$

and vanishes when $f = \mathbb{I}$. This equation always has solution because the cohomology class for β does not change under a gauge transformation of A. In this way, using the identity $\mathrm{d}_{A^f}\rho(f) = \rho(f)\mathrm{d}_A$, we can see that B^f still satisfies the modified cocycle condition.

The second type of gauge tranformation is the one that involves the 1-form gauge symmetry: the 1-cocycle A does not transform, while B shifts as

$$B \to B^\gamma := B + \mathrm{d}_A\gamma \tag{4.17}$$

where $\gamma \in \mathcal{A}$ is a 1-cochain. Since the definition of $\zeta(A, f)$ may leads to ambiguity, we can reabsorb it by an appropriate choice of γ.

4.3 Gauge transformations and 2-group symmetry

In this section we want to interpret the so-called F-move as a particular gauge transformation. In the formalism of defects, we will explicitly see the emergence of a 1-form symmetry defect in the triple 0-form junction.

Let us consider the two defect configurations in figure 4.2.

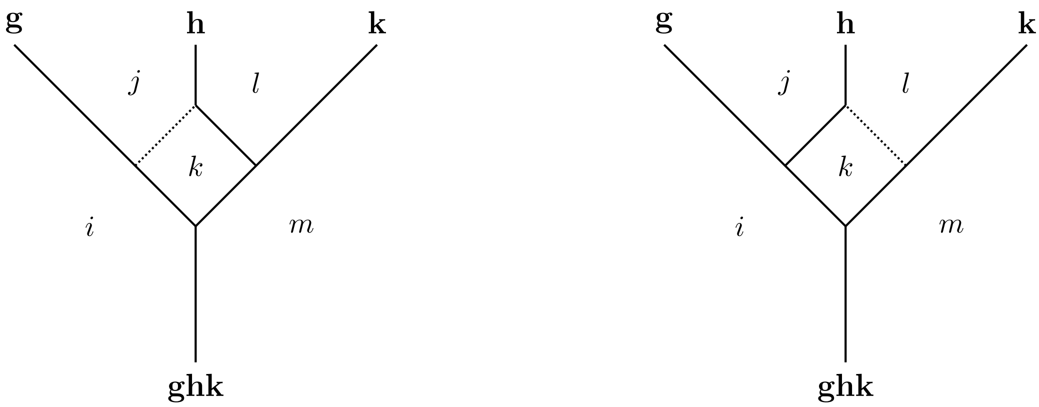

Figure 4.2: Junctions of defects in two dimensions: $g + h + k \to ghk$. On the left, h joins first k and then g, in contrast on the right h joins first g and then k. i, j, k, l, m represent the open patches of the gauge bundle and the dotted lines represent the defect having as transition function the group identity. The two pictures are related by a gauge transformation in the open patch k given by $f_k = h^{-1}$.

They represent six open patches in two dimensions with the correspondent transition functions: on the left $A_{ij} = A_{ik} = g$, $A_{jl} = A_{kl} = h$, $A_{lm} = k$, $A_{jk} = \mathbb{I}$, $A_{km} = hk$ and $A_{im} = ghk$; on the right $A_{ij} = g$, $A_{jl} = A_{jk} = h$, $A_{lm} = A_{km} = k$, $A_{kl} = \mathbb{I}$, $A_{ik} = hk$ and $A_{im} = ghk$. They are related by a gauge transfomation in the open patch k given by $f_k = h^{-1}$:

$$\begin{aligned} &A_{ik} \to A_{ik} f_k^{-1} : g \to gh, \qquad && A_{jk} \to A_{jk} f_k^{-1} : \mathbb{I} \to h, \\ &A_{kl} \to f_k A_{kl} : h \to \mathbb{I}, && A_{km} \to f_k A_{km} : hk \to k. \end{aligned} \tag{4.18}$$

The gauge transformation for B is given by $B \to B + \zeta(f, A)$ and we have to find the 2-cochain $\zeta(f, A)$ for every triple intersections $\langle ijk\rangle$, $\langle ikm\rangle$, $\langle jkl\rangle$ and $\langle klm\rangle$. To make sense of the equation to which ζ is subject ($\mathrm{d}_A\zeta(A, f) = A^{f*}\beta - \rho(f)A^*\beta$), we must consider the theory in one more dimension setting to $\mathbb{I}$ the 2-cochains out of the plane. We also set $\rho = \mathbb{I}$ and we use the multiplicative notation.

$$
\begin{aligned}
\langle ijkl\rangle : \quad & \frac{\zeta_{jkl}\zeta_{ijl}}{\zeta_{ikl}\zeta_{ijk}} = \frac{\beta(A^f_{ij}, A^f_{jk}, A^f_{kl})}{\beta(A_{ij}, A_{jk}, A_{kl})} \Rightarrow \frac{\zeta_{jkl}}{\zeta_{ijk}} = \frac{\beta(g, h, \mathbb{I})}{\beta(g, \mathbb{I}, h)} \\
\langle ijkm\rangle : \quad & \frac{\zeta_{jkm}\zeta_{ijm}}{\zeta_{ikm}\zeta_{ijk}} = \frac{\beta(A^f_{ij}, A^f_{jk}, A^f_{km})}{\beta(A_{ij}, A_{jk}, A_{km})} \Rightarrow \frac{1}{\zeta_{ikm}\zeta_{ijk}} = \frac{\beta(g, h, k)}{\beta(g, \mathbb{I}, hk)} \\
\langle iklm\rangle : \quad & \frac{\zeta_{klm}\zeta_{ikm}}{\zeta_{ilm}\zeta_{ikl}} = \frac{\beta(A^f_{ik}, A^f_{kl}, A^f_{lm})}{\beta(A_{ik}, A_{kl}, A_{lm})} \Rightarrow \zeta_{klm}\zeta_{ikm} = \frac{\beta(gh, \mathbb{I}, k)}{\beta(g, h, k)} \\
\langle jklm\rangle : \quad & \frac{\zeta_{klm}\zeta_{jkm}}{\zeta_{jlm}\zeta_{jkl}} = \frac{\beta(A^f_{jk}, A^f_{kl}, A^f_{lm})}{\beta(A_{jk}, A_{kl}, A_{lm})} \Rightarrow \frac{\zeta_{klm}}{\zeta_{jlm}} = \frac{\beta(h, \mathbb{I}, k)}{\beta(\mathbb{I}, h, k)}
\end{aligned}
\tag{4.19}
$$

We observe that we have obtained four equations for four unknowns and we can therefore have hope of determining the solutions. Exploiting the fact that β is normalized such that $\beta = \mathbb{I}$ if at least one of the entrances is trivial we obtain

$$
\frac{\zeta_{jkl}}{\zeta_{ijk}} = \mathbb{I}, \quad \frac{1}{\zeta_{ikm}\zeta_{ijk}} = \beta(g, h, k), \quad \zeta_{klm}\zeta_{ikm} = \frac{1}{\beta(g, h, k)}, \quad \frac{\zeta_{klm}}{\zeta_{jlm}} = \mathbb{I}. \tag{4.20}
$$

Therefore, we have

$$
\zeta_{jkl} = \zeta_{ijk} = \zeta_{klm} \quad \text{and} \quad \zeta_{ikm} = \frac{1}{\zeta_{ijk}\beta(g, h, k)} \tag{4.21}
$$

and if $\beta(g, h, k)$ is non trivial, also B changes under the gauge transformation.

We have the freedom of choosing $\zeta_{jkl} = \zeta_{ijk} = \zeta_{klm} = \mathbb{I}$ so that the only non trivial gauge transformation is

$$
B_{ikm} \to B_{ikm}\beta^{-1}(g, h, k) \tag{4.22}
$$

in multiplicative notation.

If we interpret this two configurations as the initial and final point of the F-move, we can justify the difference between the two junctions. The arising of the inverse of the Postnikov class in the gauge transfomation is compensated by the insertion of a 1-form symmetry defects $\beta(g, h, k)$ in the triple junction in order to satisfy the modified cocycle condition for B. This fact is shown explicitly in [8] considering the triangulation that implements the modified cocycle condition.

Finally, we want to consider two simple examples that will be useful in the following. We will discuss the 2-group structure with 0-form symmetry group $\mathbb{Z}_N$ and $\mathbb{Z}_N \times \mathbb{Z}_N$ by applying the formalism developed so far.

4.4 Example: $G = \mathbb{Z}_N$

The 2-group structure is given by

$$
\mathcal{G} = (\mathbb{Z}_N^{(0)}, U(1)^{(1)}, \mathbb{I}, [\beta]) \tag{4.23}
$$

where we assume $\rho = \mathbb{I}$ and $\beta \in H^3(\mathbb{Z}_N, U(1)) \cong \mathbb{Z}_N$.

Now we rewrite the cocycle condition using the adictive notation for G and the multiplicative notation for $U(1)$. The cocycle condition for $A \in \mathbb{Z}_N$ is $A_{ij} + A_{jk} = A_{ik}$ for each triangle $\langle ijk \rangle$. The modified cocycle condition for $B \in U(1)$ is

$$\frac{B_{jkl} B_{ijl}}{B_{ikl} B_{ijk}} = \beta(A_{ij}, A_{jk}, A_{kl}) \tag{4.24}$$

for each 3-simplices $\langle ijkl \rangle$ with ordered vertices. The first type of gauge transformation takes the form of

$$A_{ij} \to A^f_{ij} = f_i + A_{ij} - f_j \qquad B_{ijk} \to B_{ijk} + \zeta_{ijk}(A, f) \tag{4.25}$$

where $f \in \mathbb{Z}_2$ for each 0-simplices and ζ is a $U(1)$ valued 2-cochain that satisfies the relation

$$\frac{\zeta_{jkl}\zeta_{ijl}}{\zeta_{ikl}\zeta_{ijk}} = \frac{\beta(A^f_{ij}, A^f_{jk}, A^f_{kl})}{\beta(A_{ij}, A_{jk}, A_{kl})} \tag{4.26}$$

The second type of gauge transformation is

$$B_{ijk} \to B^{\gamma}_{ijk} = \frac{B_{ijk}\gamma_{jk}\gamma_{ij}}{\gamma_{ik}} \tag{4.27}$$

where $\gamma \in U(1)$ is a 1-cochain.

From 3.38, a representative of the Postnikov class in $H^3(\mathbb{Z}_N, U(1))$ is given by

$$\beta(A_{ij}, A_{jk}, A_{kl}) = \exp\left(\frac{2\pi i \hat{k}}{N^2} A_{ij}(A_{jk} + A_{kl} - [A_{jk} + A_{kl}])\right) \tag{4.28}$$

where $\hat{k} \in \mathbb{Z}_N$ is the 2-group structure and labels the different classes. On the right hand side of the formula $A_{ij}, A_{jk}, A_{kl} \in \mathbb{Z}_N$ are identified with elements in $\{0, 1, \dots, N-1\} \in \mathbb{Z}$, the operations are to be understood as the usual operations in $\mathbb{R}$ and not modulo N, while the square brackets $[\cdot] : \mathbb{Z} \to \{0, 1, \dots, N-1\}$ indicate the reduction $\mod N$.

Before moving on to the general case, we will begin by analyzing the simple case of $G = \mathbb{Z}_2$ in order to make the steps and calculations explicit. $\mathbb{Z}_2$ is an abelian finite cyclic group. In the adictive notation $\mathbb{Z}_2 = \{0, 1\}$, the group operation is denoted by "+", the neutral element is 0, the inverse of 1 is $1^{-1} = 1$ such that $1 + 1 = 0$. We calculate explicitly the value of β in order to find the group configuration (A_{ij}, A_{jk}, A_{kl}) that gives a non trivial β (Table 4.1). Notice that we choose $\hat{k} = 1$ otherwise β is always the identity and the 2-group symmetry vanishes.

A_{ij}	A_{jk}	A_{kl}	$\beta(A_{ij}, A_{jk}, A_{kl})$	A_{ij}	A_{jk}	A_{kl}	$\beta(A_{ij}, A_{jk}, A_{kl})$
0	0	0	1	1	0	0	1
0	0	1	1	1	0	1	1
0	1	0	1	1	1	0	1
0	1	1	1	1	1	1	$e^{i\pi} = -1$

Table 4.1: Explicit calculation of $\beta(A_{ij}, A_{jk}, A_{kl})$ with $A \in \mathbb{Z}_2$ and $\hat{k} = 1$.

In order to have a non trivial β we have to satisfy two different conditions: A_{ij} must be different from the neutral element 0 and the sum $A_{jk} + A_{kl}$ must be equal 2 in such a way the difference with the same sum $\mod 2$ does not vanish.

The necessary condition in order to have a non trivial ζ is that $\beta(A^f_{ij}, A^f_{jk}, A^f_{kl}) \neq \beta(A_{ij}, A_{jk}, A_{kl})$. We can easily check there are many gauge transfomations that satisfy this condition, even if β is trivial. In our case β can take only two values, $\beta = \pm 1$, and a non trivial ζ must satisfy $\frac{\zeta_{jkl}\zeta_{ijl}}{\zeta_{ikl}\zeta_{ijk}} = -1$; Therefore, there is at least one ζ different from 1. The ambiguity in the choice of ζ is due to the fact that ζ is defined up to coboundaries, $\zeta \sim \zeta + \mathrm{d}_A\gamma$ where γ is a 1-cochain, and it can be absorbed by the second type of gauge transformation for B. After a gauge transformation the modified cocycle condition is still satisfied: e.g. if we choose $\zeta_{ikl} = \zeta_{ijl} = \zeta_{ijk} = 1$ and $\zeta_{jkl} = -1$ we obtain

$$\frac{B_{jkl}B_{ijl}}{B_{ikl}B_{ijk}} = \beta(A_{ij}, A_{jk}, A_{kl}) \to \frac{-B_{jkl}B_{ijl}}{B_{ikl}B_{ijk}} = \beta(A^f_{ij}, A^f_{jk}, A^f_{kl}) = -\beta(A_{ij}, A_{jk}, A_{kl}) \tag{4.29}$$

If we consider the particular gauge transformation presented above (Figure 4.2), we can verify the necessity of the insertion of the 1-form symmetry defect β to mantain the equivalence of the two configurations. β is non trivial if $g = h = k = 1$. The transition functions for the picture on the left are $A_{ij} = A_{ik} = A_{im} = A_{jl} = A_{kl} = A_{lm} = 1$ and $A_{jk} = A_{km} = 0$. The gauge transformation in the open set k is given by $f_k = 1$ and the 1-cocycles transformed are $A_{ik} \to A_{ik} + 1 = 0$, $A_{jk} \to A_{jk} + 1 = 1$, $A_{kl} \to 1 + A_{kl} = 0$ and $A_{km} = 1 + A_{km} = 1$. The others remain untouched. From the above discussion, for the gauge transformation for B we can choose $\zeta_{ijk} = \zeta_{jkl} = \zeta_{klm} = 1$ and therefore $\zeta_{ikm} = \beta(1,1,1)^{-1}$. The 2-cochains $B_{ijk}, B_{jkl}, B_{klm}$ remain invariant, while $B_{ikm} \to B_{ikm}\beta(1,1,1)^{-1} = -B_{ikm}$. Apparently the gauge transformation given by f seems to change by the phase $\beta(1,1,1)^{-1}$ the value of the correlators; however, we note that this gauge transfomation corresponds to a F-move, so we have to insert the 1-form symmetry defect $\beta(1,1,1)$ in the triple junction. Therefore, this 1-form symmetry defect compensate exactly the gauge transformation for B and the 2-group symmetry is encoded in this compensation.

For the general case $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$ we can generalize the results obtained above. We note that if $A_{jk} + A_{kl} \geq N$, then $A_{jk} + A_{kl} = [A_{jk} + A_{kl}] + N$ and we obtain

$$\beta(A_{ij}, A_{jk}, A_{kl}) = \exp\left(\frac{2\pi i\hat{k}}{N} A_{ij}\right) \quad \text{with} \quad \hat{k} \in \mathbb{Z}_N \tag{4.30}$$

that correspond to the roots of unity. Therefore, the Postnikov class is non trivial if

- $\hat{k} \neq 0$;
- $A_{ij} \neq 0$;
- $A_{jk} + A_{kl} \geq N$;
- $\frac{\hat{k}A_{ij}}{N} \notin \mathbb{Z}$.

The last condition ensures that the particular combination between $\hat{k}$ and A_{ij} does not give rise to the trivial phase. The discussion of the gauge transformation for $\mathbb{Z}_2$ can be directly applied also for the general case for every non trivial β.

4.5 Example: $G = \mathbb{Z}_N \times \mathbb{Z}_N$

In this case the 2-group structure is given by

$$\mathcal{G} = \left((\mathbb{Z}^{(1)}_N \times \mathbb{Z}^{(2)}_N)^{(0)}, U(1)^{(1)}, \mathbb{I}, [\beta]\right) \tag{4.31}$$

where we assume $\rho = \mathbb{I}$ and $\beta \in H^3(\mathbb{Z}_N \times \mathbb{Z}_N, U(1)) \cong \mathbb{Z}_N \times \mathbb{Z}_N \times \mathbb{Z}_N$. We indicate the transition functions $A_{ij} \in G$ as $A_{ij} = (A^{(1)}_{ij}, A^{(2)}_{ij})$. The representatives of the Postnikov class are of two types (see 3.5): the first involves only one of the subgroups $\mathbb{Z}^{(1)}_N$ or $\mathbb{Z}^{(2)}_N$, while the second mixes them. Therefore, the functions $\beta : G \times G \times G \to U(1)$ take the form

$$\begin{aligned}
\beta^{1}_{I}(A_{ij}, A_{jk}, A_{kl}) &= \exp\left(\frac{2\pi i\hat{k}_1}{N^2} A^{(1)}_{ij}(A^{(1)}_{jk} + A^{(1)}_{kl} - [A^{(1)}_{jk} + A^{(1)}_{kl}])\right) \\
\beta^{(2)}_{I}(A_{ij}, A_{jk}, A_{kl}) &= \exp\left(\frac{2\pi i\hat{k}_2}{N^2} A^{(2)}_{ij}(A^{(2)}_{jk} + A^{(2)}_{kl} - [A^{(2)}_{jk} + A^{(2)}_{kl}])\right) \\
\beta^{(1,2)}_{II}(A_{ij}, A_{jk}, A_{kl}) &= \exp\left(\frac{2\pi i\hat{k}_{12}}{N^2} A^{(1)}_{ij}(A^{(2)}_{jk} + A^{(2)}_{kl} - [A^{(2)}_{jk} + A^{(2)}_{kl}])\right) \\
\beta^{(2,1)}_{II}(A_{ij}, A_{jk}, A_{kl}) &= \exp\left(\frac{2\pi i\hat{k}_{21}}{N^2} A^{(2)}_{ij}(A^{(1)}_{jk} + A^{(1)}_{kl} - [A^{(1)}_{jk} + A^{(1)}_{kl}])\right)
\end{aligned} \tag{4.32}$$

where $\hat{k}_1, \hat{k}_2, \hat{k}_{12}, \hat{k}_{21} \in \mathbb{Z}_N$ label the different classes and $\hat{k}_{12} = \hat{k}_{21}$. $\beta^{(1,2)}_{II}$ and $\beta^{(2,1)}_{II}$ are equivalent since they differ by a 3-coboundary.

We will focus only on the second type of Postnikov classes, so we set $\hat{k}_1 = \hat{k}_2 = 0$. As in the above section, we present the full calculation for the simple case $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, then we generalize the results. In the simple case of $\mathbb{Z}_2 \times \mathbb{Z}_2$ we can compute explicitly the values of β_{II}. We choose $\hat{k}_{12} = 1$ in order to have non trivial results; we also note that if $A_{ij} = (0, 0)$, β_{II} is always 1. The calculations are presented in Table 4.2.

A non trivial $\beta^{(1,2)}_{II}$ requires the usual two conditions at the same time: $A^{(1)}_{ij}$ must be different from the neutral element 0 and the sum $A^{(2)}_{jk} + A^{(2)}_{kl}$ must be equal 2 in such a way the difference with the same sum mod 2 does not vanish. For $\beta^{(2,1)}_{II}$ it sufficient to swap the indices in the conditions above. We have to note that except for the case with all the transition functions A_{ij}, A_{jk}, A_{kl} equal to $(1, 1)$, the Postnikov classes are trivial even if a single representative $\beta^{(1,2)}_{II}$ or $\beta^{(2,1)}_{II}$ is different from the identity. Indeed, since they are equivalent, we can always find a 2-coboundary that connects them:

$$\beta^{(1,2)}_{II}(A_{ij}, A_{jk}, A_{kl}) \sim \beta^{(2,1)}_{II}(A_{ij}, A_{jk}, A_{kl})(\mathrm{d}_\rho\mu)(A_{ij}, A_{jk}, A_{kl}) \tag{4.33}$$

where $\mu : G \times G \to U(1)$ is a 1-cochain. Notice that we also have to redefine $B_{ijk} \to B_{ijk}\mu(A_{ij}, A_{jk})$ in order to satisfy the cocycle condition for B.

For example if $A_{ij} = (1, 0)$, $A_{jk} = (0, 1)$ and $A_{kl} = (0, 1)$ we can choose

$$\mu(X, Y) = \exp\frac{\pi i}{2}\left([X^{(1)} + X^{(2)}] - [Y^{(1)} + Y^{(2)}]\right) \tag{4.34}$$

with $X, Y \in G$. Indeed,

$$(\mathrm{d}_\rho\mu)(A_{ij}, A_{jk}, A_{kl}) = \frac{\mu(A_{jk}, A_{kl})\mu(A_{ij}, A_{jk}A_{kl})}{\mu(A_{ij}A_{jk}, A_{kl})\mu(A_{ij}, A_{jk})} \tag{4.35}$$

and $(\mathrm{d}_\rho\mu)((1, 0), (0, 1), (0, 1)) = -1$.

Secondly, we can consider the gauge tranformation of the section 4.3 in the only non-trivial configuration. The transition functions for the picture on the left (Figure 4.2) are $A_{ij} = A_{ik} = A_{im} = A_{jl} = A_{kl} = A_{lm} = (1, 1)$ and $A_{jk} = A_{km} = (0, 0)$. The gauge transformation in the open k is given by $f_k = (1, 1)$ and the transformed 1-cocycles are $A_{ik} \to A_{ik} + (1, 1) = (0, 0)$, $A_{jk} \to$

A_{ij}	A_{jk}	A_{kl}	$\beta_{II}^{(1,2)}$	$\beta_{II}^{(2,1)}$
$(1,0)$	$(0,0)$	$(0,0)$	1	1
$(1,0)$	$(0,0)$	$(0,1)$	1	1
$(1,0)$	$(0,0)$	$(1,0)$	1	1
$(1,0)$	$(0,0)$	$(1,1)$	1	1
$(1,0)$	$(1,0)$	$(0,0)$	1	1
$(1,0)$	$(1,0)$	$(0,1)$	1	1
$(1,0)$	$(1,0)$	$(1,0)$	1	1
$(1,0)$	$(1,0)$	$(1,1)$	1	1
$(1,0)$	$(0,1)$	$(0,0)$	1	1
$(1,0)$	$(0,1)$	$(0,1)$	-1	1
$(1,0)$	$(0,1)$	$(1,0)$	1	1
$(1,0)$	$(0,1)$	$(1,1)$	-1	1
$(1,0)$	$(1,1)$	$(0,0)$	1	1
$(1,0)$	$(1,1)$	$(0,1)$	-1	1
$(1,0)$	$(1,1)$	$(1,0)$	1	1
$(1,0)$	$(1,1)$	$(1,1)$	-1	1
$(0,1)$	$(0,0)$	$(0,0)$	1	1
$(0,1)$	$(0,0)$	$(0,1)$	1	1
$(0,1)$	$(0,0)$	$(1,0)$	1	1
$(0,1)$	$(0,0)$	$(1,1)$	1	1
$(0,1)$	$(1,0)$	$(0,0)$	1	1
$(0,1)$	$(1,0)$	$(0,1)$	1	1
$(0,1)$	$(1,0)$	$(1,0)$	1	-1
$(0,1)$	$(1,0)$	$(1,1)$	1	-1
$(0,1)$	$(0,1)$	$(0,0)$	1	1
$(0,1)$	$(0,1)$	$(0,1)$	1	1
$(0,1)$	$(0,1)$	$(1,0)$	1	1
$(0,1)$	$(0,1)$	$(1,1)$	1	1
$(0,1)$	$(1,1)$	$(0,0)$	1	1
$(0,1)$	$(1,1)$	$(0,1)$	1	1
$(0,1)$	$(1,1)$	$(1,0)$	1	-1
$(0,1)$	$(1,1)$	$(1,1)$	1	-1
$(1,1)$	$(0,0)$	$(0,0)$	1	1
$(1,1)$	$(0,0)$	$(0,1)$	1	1
$(1,1)$	$(0,0)$	$(1,0)$	1	1
$(1,1)$	$(0,0)$	$(1,1)$	1	1
$(1,1)$	$(1,0)$	$(0,0)$	1	1
$(1,1)$	$(1,0)$	$(0,1)$	1	1
$(1,1)$	$(1,0)$	$(1,0)$	1	-1
$(1,1)$	$(1,0)$	$(1,1)$	1	-1
$(1,1)$	$(0,1)$	$(0,0)$	1	1
$(1,1)$	$(0,1)$	$(0,1)$	-1	1
$(1,1)$	$(0,1)$	$(1,0)$	1	1
$(1,1)$	$(0,1)$	$(1,1)$	-1	1
$(1,1)$	$(1,1)$	$(0,0)$	1	1
$(1,1)$	$(1,1)$	$(0,1)$	-1	1
$(1,1)$	$(1,1)$	$(1,0)$	1	-1
$(1,1)$	$(1,1)$	$(1,1)$	-1	-1

Table 4.2: Explicit calculation of $\beta_{II}^{(1,2)}(A_{ij}, A_{jk}, A_{kl})$ and $\beta_{II}^{(2,1)}(A_{ij}, A_{jk}, A_{kl})$ with $A \in (Z_2^{(1)} \times \mathbb{Z}_2^{(2)})$ and $\hat{k}_{12} = 1$.

$A_{jk} + (1,1) = (1,1)$, $A_{kl} \to (1,1) + A_{kl} = (0,0)$ and $A_{km} = (1,1) + A_{km} = (1,1)$.
For B we can choose $\zeta_{ijk} = \zeta_{jkl} = \zeta_{klm} = 1$ and therefore $\zeta_{ikm} = \beta_{II}((1,1),(1,1),(1,1))^{-1}$. The 2-cochains $B_{ijk}, B_{jkl}, B_{klm}$ remain invariant, while $B_{ikm} \to B_{ikm}\beta_{II}((1,1),(1,1),(1,1))^{-1} = -B_{ikm}$ and we see the necessity of the insertion of $\beta_{II}((1,1),(1,1),(1,1))$ to preserve the invariance of the correlators in the F-move.

If we consider the general case $G = (\mathbb{Z}_N \times \mathbb{Z}_N)^{(0)}$ we can extend the discussion for $\mathbb{Z}_N$. In particular if $A_{jk}^{(a)} + A_{kl}^{(a)} \geq N$ at the same time for $a = 1, 2$ we obtain that $[\beta_{II}]$ corresponds to the roots of unity. Thus, to have a non trivial Postnikov class we have the following conditions:

- $\hat{k}_{12} = \hat{k}_{21} \neq 0$;
- $A_{ij}^{(1)}$ and $A_{ij}^{(2)} \neq 0$;
- $A_{jk}^{(1)} + A_{kl}^{(1)} \geq N$ and $A_{jk}^{(2)} + A_{kl}^{(2)} \geq N$;
- $\frac{\hat{k}_{12}A_{ij}^{(1)}}{N}$ and $\frac{\hat{k}_{12}A_{ij}^{(2)}}{N} \notin \mathbb{Z}$.

Chapter 5

Elements of string theory

This chapter is not an introduction to string theory. It contains only a brief presentation of some string theory topics which constitute the inspiring motive of this thesis work and which are the prerequisite for understanding the following discussions. In fact, despite its capital importance in string theory, we will never use the formalism of conformal quantum field theory and we will try to make the presentation clear and coherent even without this important tool.

We will start by presenting the relativistic action for the string and we will give the fundamental results of the light-cone quantization which allows to become familiar with some fundamental features such as the critical dimension and the string's spectrum. After a very brief mention of the superstring action, we will discuss the action for a string moving in a curved background and introduce some ideas about compacting extra dimensions.

We essentially follow the two Polchinski's books [27], [28], the Tong's lectures notes [29] and the recent book written by K. Becker, M. Becker and J.H. Schwarz [30] .

5.1 The action principle

We start by introducing the action describing the relativistic dynamics of a one dimensional object, a string, moving in the spacetime. Firstly we present the action for the bosonic string, secondly we briefly mention the action for the superstring (type IIA or IIB). We initially consider a D dimensional spacetime with Minkowskian metric $\eta_{\mu\nu} = \mathrm{diag}(-, +, +, \cdots, +)$ and then we take into account also curved spacetime.

In order to consider a one dimensional object, it is useful to review the action for a point particle (0 dimensional object). We can parametrize the position in the spacetime of a particle by using D functions $X^\mu(\tau)$, with τ an arbitary real parameter. The simplest Poincarè invariant action would be proportional to the proper time along the world-line, the trajectory of the particle in the space time:

$$S_{pp} = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} \tag{5.1}$$

where $\cdot$ denote τ-derivative and the normalization costant m is the particle's mass. The invariance under reparametrization is a gauge invariance of the theory, since it is a redundacy in the description of the dynamic.

We can also consider another useful form (quadratic in the derivative and without square root) of the action by introducing an independent world-line metric $\gamma_{\tau\tau}(\tau)$ working with the tetrad

$e(\tau) = \sqrt{-\gamma_{\tau\tau}(\tau)}$:

$$S'_{pp} = \frac{1}{2}\int d\tau \big(e^{-1}\dot{X}^{\mu}\dot{X}_{\mu} - em^2\big) \tag{5.2}$$

By varying the tetrad, we can eliminate e and find the earlier S_{pp}.

A string moving in the spacetime sweeps out a two-dimensional world-sheet and we have to use two parameters to describe it. Let us denote by $X^{\mu}(\tau,\sigma)$ the embedding in spacetime of the world-sheet. Generalizing the point-particle action, the simplest invariant action is proportional to the area of the world-sheet and is called the Nambu-Goto action. Let $h_{ab} = \partial_a X^{\mu}\partial_b X^{\nu}\eta_{\mu\nu}$ with $a, b = \tau$ or σ, the induced metric on the world-sheet, the Nambu-Goto action is

$$S_{NG} = -T\int_M d\tau d\sigma \sqrt{-\det h_{ab}} \tag{5.3}$$

where M is the world-sheet and $T = \frac{1}{2\pi\alpha'}$ is the tension of the string. α' is called the Regge slope for historical reasons and it has units of length-squared. The action is invariant under Poincarè tranformations and under reparametrizations.

Similarly to the point particle, we can simplfy the action by introducing an independent world-sheet metric $\gamma_{ab}(\tau,\sigma)$ with Lorentzian signature $(-,+)$. Therefore we obtain the most important action for the bosonic string, the Polyakov action:

$$S_P = -T\int_M d\tau d\sigma \sqrt{-\gamma}\gamma^{ab}\partial_a X^{\mu}\partial_b X^{\nu}\eta_{\mu\nu} \tag{5.4}$$

where $\gamma = \det \gamma_{ab}$. The Polyakov action is equivalent to the Nambu-Goto using the equation of motion obtained by varying the metric to eliminate it. The symmetries of S_P are:

- D-dimensional Poincarè invariance:

$$\begin{aligned} X'^{\mu}(\tau,\sigma) &= \Lambda^{\mu}_{\nu}X^{\nu}(\tau,\sigma) + a^{\mu} \\ \gamma'_{ab}(\tau,\sigma) &= \gamma_{ab}(\tau,\sigma) \end{aligned} \tag{5.5}$$

  with Λ^{μ}_{ν} a Lorentz transformation and a^{μ} a translation.

- reparametrization or diffeomorphism invariance:

$$\begin{aligned} X'^{\mu}(\tau',\sigma') &= X^{\mu}(\tau,\sigma) \\ \frac{\partial\sigma'^c}{\partial\sigma^a}\frac{\partial\sigma'^d}{\partial\sigma^b}\gamma'_{cd}(\tau',\sigma') &= \gamma_{ab}(\tau,\sigma) \end{aligned} \tag{5.6}$$

  where $\sigma'^a(\tau,\sigma)$ are the new coordinates.

- 2-dim Weyl invariance:

$$\begin{aligned} X'^{\mu}(\tau,\sigma) &= X^{\mu}(\tau,\sigma) \\ \gamma'_{ab}(\tau,\sigma) &= e^{2\omega(\tau,\sigma)}\gamma_{ab}(\tau,\sigma) \end{aligned} \tag{5.7}$$

  for arbitary $\omega(\tau,\sigma)$.

There is no analog of the Weyl invariance in the Nambu-Goto action. This invariance under local rescaling of the world-sheet metric can be explained by the relation between the Polykov and the Nambu-Goto action. If we vary the metric in the Polyakov action and impose $\delta_{\gamma} = 0$, we obtain $\gamma_{ab} = 2\frac{h_{ab}}{\gamma^{cd}h_{cd}}$, where h_{ab} is the induced metric of the Nambu-Goto action. We see by comparison that the worldsheet metric is only proportional to the induced metric, because the presence of $\frac{1}{\gamma^{cd}h_{cd}}$, called the conformal factor. The conformal factor does not appear in the equation of motion for X^μ, therefore the two description are equivalent but with the adjoint of this extra redundancy.

If there is no topological obstruction, the reparametrization invariance and the Weyl invariance allow us to gauge fix γ_{ab} completely setting $\gamma_{ab} = \eta_{ab}$. We also notice that, from the point of view of the world-sheet, the coordinates tranformation law $X'^\mu(\tau', \sigma') = X^\mu(\tau, \sigma)$ shows that X^μ is a scalar and μ is an internal index since the Poincarè invariance acts at fixed τ, σ.

From the Polyakov action we can obtain the equations of motion by varying the action. The variation with respect the metric defines the energy-momentum tensor

$$T^{ab}(\tau, \sigma) = -4\pi\sqrt{-\gamma}\frac{\delta}{\delta\gamma_{ab}}S_P = -\frac{1}{\alpha'}\left(\partial^a X^\mu \partial^b X^\nu \eta_{\mu\nu} - \frac{1}{2}\gamma^{ab}\partial^c X^\mu \partial^d X^\nu \gamma_{cd}\eta_{\mu\nu}\right) \tag{5.8}$$

and the equation of motion for γ_{ab} is $T^{ab} = 0$. Varying X^μ gives the Euler-Lagrange equation:

$$\partial_a(\sqrt{-\gamma}\gamma^{ab}\partial_b X^\mu) = \sqrt{-\gamma}\nabla^2 X^\mu = 0. \tag{5.9}$$

However, to have a complete variational discussion, we have to consider the boundary conditions. Let's take the coordinate region to be $-\infty < \tau < \infty$ and $0 \leq \sigma \leq l$, thinking of τ as time variable and σ as space. So,

$$\delta_X S_P = T\int_{-\infty}^{+\infty} d\tau \int_0^l d\sigma \sqrt{-\gamma}\ \nabla^2 X_\mu \delta X^\mu - T\int_{-\infty}^{+\infty} d\tau \sqrt{-\gamma}\ \partial_\sigma X_\mu \delta X^\mu|_{\sigma=0}^{\sigma=\pi} \tag{5.10}$$

The boundary term can vanish in some different ways and we can have three different type of strings:

- *closed string*: if we impose the conditions $X^\mu(\tau, 0) = X^\mu(\tau, l)$, $\partial^\sigma X^\mu(\tau, 0) = \partial^\sigma X^\mu(\tau, l)$ and $\gamma_{ab}(\tau, 0) = \gamma_{ab}(\tau, l)$, the fields are periodic and so there is no boundary and the string forms a closed loop.
- *open string with Neumann boundary conditions*: if the component of the momentum normal to the boundary vanishes, i.e. $n^a\partial_a X_\mu = 0$ on ∂M where n^a is normal to the boundary ∂M. We don't have restriction on δX^μ, therefore the end-points of string can move freely. These conditions respect the Poincarè invariance.
- *open string with Dirichlet boundary conditions*: if we impose that the position of the two string end-points are fixed, we have that $\delta X^\mu = 0$. It is sufficient to impose that $X^\mu|_{\sigma=0} = X_0^\mu$ and $X^\mu|_{\sigma=l} = X_l^\mu$ where X_0^μ and X_l^μ are constants and $\mu = 1, \ldots, D-p-1$. Neumann boundary conditions are imposed for the other $p+1$ functions. This breaks the Poincarè invariance; however, following the modern interpretation, we interpret X_0^μ and X_l^μ as belonging to a *Dp*-branes, special *p*-branes on which the fundamental strings can end.

5.2 The light-cone quantization and the spectrum

Our aim is to introduce the spectrum of the open and the closed strings. We use the light-cone gauge in order to eliminate the reparametrization and Weyl redundancy; even if this gauge fixing is not manifestly covariant, it allows us to define an Hilbert space with states with non negative norm and to describe some important general features of the bosonic string (e.g. the critical dimension). We do not discuss this method in detail since we are interested in the main results.

We define the light-cone coordinate in space-time:

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^1), \qquad x^i \quad \text{with } i = 2, \ldots, D-1 \tag{5.11}$$

We use the lower case for spacetime coordinates and the upper case for the embedding functions. The convenient choice is to set $x^+ = \tau$, so that x^+ will play the role of time and the correspondent momenta p^- the role of energy. The longitudinal variables x^- and p^+ are spatial coordinate and momentum, as for the transverse x^i, p^i.

5.2.1 Open string with Neumann boundary conditions

We will start by considering the open string with Neumann boundary conditions. The gauge fixing that simplfies the equations of motion is

$$X^+ = \tau, \quad \partial_\sigma \gamma_{\sigma\sigma} = 0, \quad \det \gamma_{ab} = -1. \tag{5.12}$$

We have three conditions to fix the two world-sheet parameter and the Weyl scaling. We impose these gauge conditions and we split $X^-(\tau,\sigma)$ into two pieces: the mean value of X^- at fixed τ $x^-(\tau) = \frac{1}{l}\int_0^l d\sigma X^-(\tau,\sigma)$ and $Y^-(\tau,\sigma) = X^-(\tau,\sigma) - x^-(\tau)$ that acts as a Lagrange multiplier. Therefore, the Polyakov action becomes:

$$S_P = T\int_{-\infty}^{+\infty} d\tau \Big(-l\gamma_{\sigma\sigma}\partial_\tau x^- + \frac{1}{2}\int_0^l d\sigma \big(\gamma_{\sigma\sigma}\partial_\tau X^i \partial_\tau X^i - \gamma_{\sigma\sigma}^{-1}\partial_\sigma X^i \partial_\sigma X^i\big)\Big). \tag{5.13}$$

The degree of freedom is represented by $x^-(\tau), \gamma_{\sigma\sigma}(\tau)$ and $X^i(\tau,\sigma)$ with $i = 2, \ldots, D-1$. If L_P is the correpondent Polyakov lagrangian, the conjugate momenta are

$$\begin{aligned} p_- = -p^+ &= \frac{\partial L_P}{\partial(\partial_\tau x^-)} = -Tl\gamma_{\sigma\sigma} \\ \Pi^i &= \frac{\delta L_P}{\delta(\partial_\tau X^i)} = T\gamma_{\sigma\sigma}\partial_\tau X^i = \frac{p^+}{l}\partial_\tau X^i. \end{aligned} \tag{5.14}$$

The hamiltonian H_P can be derived by Legendre transform

$$H_P = p_-\partial_\tau x^- + \int_0^l d\sigma \Pi^i \partial_\tau X^i - L_P = \frac{lT}{2p^+}\int_0^l d\sigma\Big(2\pi\alpha'\Pi^i\Pi^i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i \partial_\sigma X^i\Big) \tag{5.15}$$

and we can easily derive the equation of motion:

$$\partial_\tau x^- = \frac{H_P}{p^+}, \quad \partial_\tau p^+ = 0, \quad \partial_\tau X^i = \frac{c}{T}\Pi^i, \quad \partial_\tau \Pi^i = cT\partial_\sigma^2 X^i \tag{5.16}$$

from which we have the wave equation

$$\partial_\tau^2 X^i = c^2 \partial_\sigma^2 X^i \tag{5.17}$$

with velocity $c = \frac{l}{2\pi\alpha' p^+}$.

Since the transverse coordinates satisfy a free wave equation, it is useful to expand them in normal modes. Defining the center of mass variables $x^i = \frac{1}{l}\int_0^l d\sigma X^i(\tau,\sigma)$ and $p^i(\tau) = \int_0^l d\sigma \Pi^i(\tau,\sigma)$, we can write the general solution with Neumann boundary conditions:

$$X^i(\tau,\sigma) = x^i + \frac{p^i}{p^+}\tau + i\sqrt{2\alpha'}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n}\alpha_n^i \exp\Big(-\frac{\pi i n c\tau}{l}\Big)\cos\Big(\frac{\pi n\sigma}{l}\Big) \tag{5.18}$$

where $\alpha^i_{-n} = (\alpha^i_n)^\dagger$ to assure the reality of X^i.

We impose the equal time canonical commutation relations to quantize the string modes. The only non vanishing are:

$$[x^-, p^+] = i\eta^{-+} = -i, \quad [X^i(\sigma), \Pi^j(\sigma)] = i\delta^{ij}\delta(\sigma - \sigma') \tag{5.19}$$

and in terms of normal mode components

$$[x^i, p^j] = i\delta^{ij}, \quad [\alpha^i_m, \alpha^j_n] = m\delta^{ij}\delta_{m,-n}. \tag{5.20}$$

If we fix m and i the modes satisfy a harmonic oscillator algebra with non standard normalization: $[a, a^\dagger] = 1$ with $\alpha^i_m \sim \sqrt{m}a$ and $\alpha^i_{-m} = \sqrt{m}a^\dagger$. Therefore, we can define the state $|0; k\rangle$, where $k = (k^+, k^i)$, that is the state annihilated by the lowering operators and the eigenstate of the center-of-mass momenta k:

$$p^+|0; k\rangle = k^+|0; k\rangle, \quad p^i|0; k\rangle = k^i|0; k\rangle \quad \text{and} \quad \alpha^i_m|0; k\rangle = 0 \quad \text{with} \quad m > 0. \tag{5.21}$$

To build a general state we can act on $|0; k\rangle$ with the raising operator and each independent state can be labeled by the center of mass momenta (k^+, k^i), which are the degrees of freedom of a point particle, and the occupation numbers N_{in} with $i = 2, \ldots, D-1$ and $n = 1, \ldots, \infty$ which represent the internal degrees of freedom of the oscillation modes. From the spacetime point of view, this different states represent different particles or spin states. Notice that the state $|0; 0\rangle$ is the state of an open string with 0 momentum and not the vacuum or zero-string state.

In order to study the spectrum of the string, we can insert the mode expansion in the Hamiltonian 5.15 setting the normal ordering of the operators (lowering operator on the right and raising operator on the left) and leaving A as unknown constant from the commutators:

$$H_P = \frac{p^i p^i}{2p^+} + \frac{1}{2p^+\alpha'}\left(\sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + A\right). \tag{5.22}$$

Without the pretense of a systematic treatment, we can easily guess, in analogy to the armonic oscillator, that $A = \frac{D-2}{2}\sum_{n=1}^{\infty} n$, since we are considering only the $D-2$ transverse directions[1]. In order to regularize the result, we taking into account the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ with $\text{Re}(s) > 0$ which admits an unique analytic continuation except for the pole in $s = 1$. Therefore, we find $\zeta(-1) = -\frac{1}{12}$ and

$$A = \frac{2-D}{24}. \tag{5.23}$$

Considering the relativistic invariant mass $m^2 = 2p^+p^- - \sum_{i=2}^{D-1} p^i p^i$ and recalling that $p^- = H_P$ we have

$$m^2 = 2p^+ H_P - p^i p^i = \frac{1}{\alpha'}\sum_{i=2}^{D-1}\sum_{\substack{n=-\infty \\ n\neq 0}} \alpha^i_{-n}\alpha^i_n = \frac{1}{\alpha'}\left(N + \frac{2-D}{24}\right) \tag{5.24}$$

[1]A sketch of calculation is given by:

$$\frac{1}{2}\sum_{i=2}^{D-1}\sum_{\substack{n=-\infty \\ n\neq 0}} \alpha^i_{-n}\alpha^i_n = \frac{1}{2}\sum_{i=2}^{D-1}\left(\sum_{n>0}\alpha^i_{-n}\alpha^i_n + \sum_{n<0}\alpha^i_{-n}\alpha^i_n\right) = \frac{1}{2}\sum_{i=2}^{D-1}\left(2\sum_{n>0}\alpha^i_{-n}\alpha^i_n + \sum_{n>0}[\alpha^i_{-n}, \alpha^i_n]\right) =$$

$$= \sum_{i=2}^{D-1}\sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n + \frac{D-2}{2}\sum_{n=1}^{\infty} n$$

with the level $N = \sum_{i=2}^{D-1} \sum_{n=1}^{\infty} n N_{in}$: the mass is determined by the level of excitation of the string.

Starting from the lightest state, $|0,k\rangle$, we have $m^2 = \frac{2-D}{24\alpha'}$ and the mass-squared is negative if $D > 2$. This state is a tachyon and it is problematic for the bosonic string, because it implies that the string vacuum is actually unstable, since it means that we are expanding around a maximum of the potential. We still do not know if the bosonic string has another vacuum that is stable, but we know that there exist string that is tachyon-free, the superstring; so we can overlook this issue.

Before continuing let us recall the results of the Wigner's theorem: in a D-dimensions Minkowski spacetime, the Poincarè group is represented by massive particles, classified by representations of $SO(D-1)$, and by massless particles organized according to the representations of the little group $SO(D-2)$. Therefore, if we consider the first excited state $n = 1$ we have the state $|N,k\rangle = \alpha_{-1}^{i}|0,k\rangle$ with mass-squared $m^2 = \frac{26-D}{24\alpha'}$. Since in D dimensions there is no way to organise $D-2$ states in a massive representation, we can argue that these states must be massless. The correpondent particle is a massless vector boson. In such a way, we obtain the critical values $D = 26$ and $A = -1$ and we discover the critical spacetime dimension for having a Lorentz invariant spectrum (the classical theory is always Lorentz invariant, but there is an anomaly, except for $D = 26$). The higher excited states of the open string are massive.

5.2.2 Closed string

To quantize the closed string we proceed similarly, but in addition to the gauge fixing conditions 5.12 it is necessary an additional constraint because there is an additional coordinate freedom:

$$\sigma' = \sigma + s(\tau) \mod l \tag{5.25}$$

since the point $\sigma = 0$ can be chosen anywhere along the string. If we impose in addition $\gamma_{\tau\sigma}(\tau,0) = 0$, we fix all the gauge freedom, except for τ-independent translations $\sigma' = \sigma + s \mod l$.

Proceeding in parallel to the open string, we obtain the general periodic solution to the equation of motion:

$$X^i(\tau,\sigma) = x^i + \frac{p^i}{p^+}\tau + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left\{ \frac{\alpha_n^i}{n} \exp\left(-\frac{2\pi i n(\sigma + c\tau)}{l} \right) + \frac{\tilde{\alpha}_n^i}{n} \exp\left(\frac{2\pi i n(\sigma - c\tau)}{l} \right) \right\} \tag{5.26}$$

In the closed string there are two different and independent sets of oscillators represented by α_n^i and $\tilde{\alpha}_n^i$ and corresponding to the left-moving and right-moving waves along the string. We can also impose the canonical commutation relations to the independent degrees of freedom:

$$[x^-, p^+] = -1, \quad [x^i, p^j] = i\delta^{ij}, \quad [\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m,-n}, \quad [\tilde{\alpha}_m^i, \tilde{\alpha}_n^j] = m\delta^{ij}\delta_{m,-n}. \tag{5.27}$$

The state annihilated by both the operators α_m^i and $\tilde{\alpha}_m^i$ with $m > 0$ is denoted by $|0,0,k\rangle$ and we have to distinguish the level number for the right and left mode: N, $\tilde{N}$. Therefore, the mass formula is now given by

$$m^2 = \frac{2}{\alpha'}(N + \tilde{N} + A + \tilde{A}) \quad \text{with} \quad A = \tilde{A} = \frac{2-D}{24}. \tag{5.28}$$

Eventually, the residual gauge freedom for τ-independent σ-translations allows us to fix $N = \tilde{N}$.

As for the open string, the lighest state $|0,0,k\rangle$ is a tachyon for $D > 2$.

The first excited states are $\alpha^i_{-1}\tilde{\alpha}^i_{-1}|0,0,k\rangle$, them mass-squared is $m^2 = \frac{26-D}{6\alpha'}$ and they does not form a complete representation of $SO(D-1)$. Therefore, these states are massless, $D = 26$ and $A = \tilde{A} = -1$. They are a representation of $SO(24)$. This representation can be decomposed into a traceless symmetric tensor $G_{\mu\nu}$, an anti-symmetric tensor $B_{\mu\nu}$ and a scalar Φ. $G_{\mu\nu}$ represents a particles with spin 2, the graviton, $B_{\mu\nu}$ is called the Kalb-Ramond field and Φ is called the dilaton. By applying the creation operators, it is possible to generate the other excited massive states of the theory.

From the study of the interactions, it is known that only theories with closed string or closed plus open strings are consistent. Consequently, the principal field content of the bosonic string are $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ for the closed string and $G_{\mu\nu}$, $B_{\mu\nu}$, Φ and A_μ for the closed plus open string, where A_μ is a massless vector boson of spin 1.

5.3 The σ-model in complex coordinates

The Polyakov action is the action that describes the relativistic dynamics of the bosonic string. It is useful to rewrite it in the Euclidean spacetime using complex coordinates. In such a way, from the point of view of the world-sheet, the σ-model describes the dynamics of D free massless scalar fields in two dimensions.

We can substitute the world-sheet metric with the Euclidean metric, $\gamma_{ab} \to \delta_{ab}$, in the Polyakov action 5.4 because we have enough gauge freedom to set the world-sheet metric to the Minkowski one. The relation between Minkowski (τ, σ) and Euclidean (σ^1, σ^2) spacetime is given by a standard analytic continuation; we can easily change from Euclidean to Minkowskian replacing $\sigma^2 \to i\tau$. Hence, the action takes the form:

$$S_\sigma = \frac{T}{2}\int d^2\sigma\Big(\partial_1 X^\mu \partial_1 X_\mu + \partial_2 X^\mu \partial_2 X_\mu\Big). \tag{5.29}$$

Furthermore, it is useful to adopt complex coordinates:

$$z = \sigma^1 + i\sigma^2\ , \quad \bar{z} = \sigma^1 - i\sigma^2 \tag{5.30}$$

where $\bar{z}$ means the complex conjugate of z. We can define also the derivatives

$$\partial_z = \partial = \frac{1}{2}(\partial_1 - i\partial_2)\ , \quad \partial_{\bar{z}} = \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2) \tag{5.31}$$

which have the obvious properties: $\partial z = 1, \partial\bar{z} = 0, \bar{\partial}z = 0$ and $\bar{\partial}\bar{z} = 1$. Then the metric becomes

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}, \quad g_{zz} = g_{\bar{z}\bar{z}} = 0, \quad g^{z\bar{z}} = g^{\bar{z}z} = 2, \quad g^{zz} = g^{\bar{z}\bar{z}} = 0 \tag{5.32}$$

and we have to pay attention when we lower or raise indices. From the Jacobian, we have the measure

$$d^2 z = 2d\sigma^1 d\sigma^2. \tag{5.33}$$

Therefore, the action becomes

$$S = T\int d^2 z \partial X^\mu \bar{\partial} X_\mu \tag{5.34}$$

The classical equations of motion are

$$\partial\bar{\partial}X^\mu(z,\bar{z}) = 0 \tag{5.35}$$

From the scriptures $\partial(\bar{\partial}X^\mu) = \bar{\partial}(\partial X^\mu) = 0$, it follows that ∂X^μ is holomorphic and $\bar{\partial}X^\mu$ is anti-holomorphic. In the Minkowski spacetime an holomorphic field is function only of $\tau - \sigma$ and an anti-holomorphic field is function only on $\tau + \sigma$. Thus, an holomorphic field is left-moving while an anti-holomorphic field is right-moving.

5.4 What about fermions?

The bosonic string does not contain fermions in its spectrum. In order to introduce fermions in string theory, it is necessary to introduce fermionic spacetime fields in addition to the scalars X^μ. In this section we only want to introduce the action for the superstring theory, without any detail, calculation or development.

In the bosonic string the mass-shell condition is the Klein-Gordon equation for the scalar field in the momentum space:

$$p_\mu p^\mu + m^2 = 0 \tag{5.36}$$

In order to introduce spacetime fermions, we should satisfy the Dirac equation

$$ip_\mu \Gamma^\mu + m = 0 \tag{5.37}$$

where Γ is the gamma matrices satisfying the Clifford algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$. Furthermore, we can guess that the gamma matrices are the center-of-mass mode of the anticommuting world-sheet field ψ^μ. Then, we can consider the following action:

$$S = \frac{1}{4\pi} \int d^2z \left(\frac{2}{\alpha'} \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu \right) \tag{5.38}$$

where ψ^μ and $\tilde{\psi}^\mu$ are D anticommuting world-sheet fields, superparteners of X^μ, respectively holomorphic and anti-holomorphic.

One can show that from this action the resulting string theory has spacetime fermions and no tachyon. The critical spacetime dimension is $D = 10$, so the world-sheet fields for the superstring are 10 free scalars and 10 free fermions. Regarding the continuation of our work, we are not interested in this fermionic fields and we mainly focus on the content of the bosonic string.

5.5 Strings in curved spacetime

Since string theory aims to be a theory that can also describe gravity and gravity is described as a geometric deformation of the flat spacetime, it is interesting to consider the theory that describes strings moving in a curved spacetime.

Considering the simple case of a point particle in a curved background, we observe that we have to substitute the spacetime flat metric with a general metric $G_{\mu\nu}$ in the point-particle action. Hence, variating X^μ, we obtain the geodesic equation that represents the motion of a particle in a gravitational field. We can make the same replacement in the Polyakov action. In this section we consider the Polyakov action in Euclidean spacetime and we indicate with g_{ab}, instead of γ_{ab}, the Euclidean world-sheet metric with signature $(+,+)$; we do not take $g_{ab} = \delta_{ab}$ because we do not fix the Weyl gauge redundancy. The correspondent action is:

$$S_\sigma = \frac{T}{2} \int d^2\sigma \sqrt{g} g^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu. \tag{5.39}$$

This replacement can be further justified in the conformal field theory framework. Without entering into details, we can consider the spacetime metric close to the flat one: $G_{\mu\nu}(X) = \eta_{\mu\nu} + \chi_{\mu\nu}(X)$ with $\chi_{\mu\nu}$ small. Exponentiating S_σ in the path integral

$$\exp(-S_\sigma) = \exp(S_P)\left(1 - \frac{T}{2}\int d^2\sigma\sqrt{g}g^{ab}\chi_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu\right) \tag{5.40}$$

one can show that the term proportional to χ is the vertex operator for the graviton state of the string. In other words, putting a curved background in the action it is not an insertion of the gravity by hand, but it is precisely the background for the graviton states already contained in the string.

This may suggest a natural generalization obtained by including backgrounds of the other massless string states:

$$S_\sigma = \frac{T}{2}\int d^2\sigma\sqrt{g}\Big\{\big(g^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}(X)\big)\partial_a X^\mu \partial_b X^\nu + \alpha' R\Phi(X)\Big\} \tag{5.41}$$

where $B_{\mu\nu}$ is the antisymmetric Kalb-Ramond field, ϵ^{ab} is the Levi-Civita pseudotensor normalized $\sqrt{g}\epsilon^{12} = 1$, Φ is the dilaton and R is the Ricci world-sheet scalar.

The Lorentz invariance of the theory is ensured if $G_{\mu\nu}$ and $B_{\mu\nu}$ transform as tensors and Φ as a scalar. From the path integral point of view this corresponds to a field redefinition $X'^\mu(X)$; in contrast, from the spacetime point of view it is a coordinate transformation. The action is also invariant under a generalization of the $U(1)$ gauge transformation, which we will study in details in the following:

$$B_{\mu\nu}(X) \to B_{\mu\nu}(X) + \partial_\mu\zeta_\nu(X) - \partial_\nu\zeta_\mu(X). \tag{5.42}$$

Under this tranformation the Lagrangian is inviariant up to a total derivative. Therefore, we can consider a three-index field strength

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \tag{5.43}$$

which is invariant under this gauge tranformation.

This theory, dependent only on gauge invariant objects, describes an embedding given by the field coordinates X^μ from the world-sheet to the target space. Since the kinetic term is field dependent, i.e. $G_{\mu\nu}(X)$ depends on X^μ, the field space is a curved manifold and the theory in the world-sheet is an interacting quantum field theory. This theory is called nonlinear σ-model. If we expand the kinetic term around the classical solution $X^\mu(\sigma) = x_0^\mu$ and denote by $X^\mu(\sigma) = x_0^\mu + Y^\mu(\sigma)$ we obtain the form

$$G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu = \left(G_{\mu\nu}(x_0) + \partial_\rho G_{\mu\nu}(x_0)Y^\rho + \frac{1}{2}\partial_\rho\partial_\omega G_{\mu\nu}(x_0)Y^\rho Y^\omega + \dots\right)\partial_a Y^\mu \partial_b Y^\nu \tag{5.44}$$

in which we can see the interaction terms, starting from the cubic and so on. One can see that the nonlinear σ-model is also renormalizable: the dimension of fields Y^μ is zero and all interactions have dimension two; nevertheless the couplings are infinite in number.

If the characteristic radius of curvature of a target space is r_c, the effective dimensionless coupling in the theory is $\sqrt{\alpha'}r_c^{-1}$ because derivatives of the metric are of order r_c^{-1}. Perturbation theory in 2-dim quantum field theory can be a useful tool, if the coupling is small, i.e. if the characteristic length scale of the string is much smaller than the radius of curvature. Notice that, since the characteristic length scale is longer than the string, we can ignore the internal structure of the string and develop a low energy effective field theory whose coupling are determined by the string theory. We will discuss the action for such a theory at the end of this section. The assumption $\sqrt{\alpha'}r_c^{-1} \ll 1$ is also implicit in the restriction only to massless backgrounds.

From the point of view of the 2-dimensional world-sheet, changing the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ seems to give a new different theory. However, from the point of view of the full string theory, these field are merely different backgrounds, a sort of different states in the same theory. This is an important feature of string theory: there is no free parameters and the coupling constants depend on the states and are determined in principle by the dynamics.

The action 5.41 is classically Weyl invariant under global transformation (independent on σ); the terms containing $G_{\mu\nu}$ and $B_{\mu\nu}$ are also classically local Weyl invariant, but the dilaton term is not. Furthermore, it is known that the Weyl symmetry is anomalous. In order to correctly quantize the theory, not all configurations of the fields are admitted: we have to consider the Weyl transformation of the dilaton and the quantum effects for all the fields to obtain some constraints for the background fields.

We can express the Weyl invariance requiring that the energy-momentum tensor is traceless and the presence of an anomaly is phrased in $T^a_a \neq 0$. On the other hand, regarding the string theory in a curved background as an interacting theory in 2-dimensions, we can interpret the Weyl anomaly in term of the renormalization group flow. If we try to regulate the occurrent divergences in the interacting process as usual, the renormalization scheme often introduces some energy scale under which Weyl invariance is broken. Weyl invariance is the independence of the theory on the energy scale, hence, intuitively, we have to require that the corrispondent renormalization group β-functions must vanish. In fact, the β-functions govern the dependence of the physics on world-sheet scale. This fact can be formalized expressing the trace of the energy momentum tensor in terms of the β-functions, starting from the linear approximation and then adding loops contributions. We report only the results, without any derivation.

$$T^a_a = -\frac{1}{2\alpha'}\beta^G_{\mu\nu} g^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{i}{2\alpha'}\beta^B_{\mu\nu}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu - \frac{1}{2}\beta^\Phi R. \tag{5.45}$$

From the one-loop β-functions computations, we have

$$\begin{aligned}
\beta^G_{\mu\nu} =& \alpha' \boldsymbol{R}_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\lambda\omega}H_\nu{}^{\lambda\omega} + O(\alpha'^2)\\
\beta^B_{\mu\nu} =& -\frac{\alpha'}{2}\nabla^\omega H_{\omega\mu\nu} + \alpha'\nabla^\omega\Phi H_{\omega\mu\nu} + O(\alpha'^2)\\
\beta^\Phi =& -\frac{D-26}{6} - \frac{\alpha'}{2}\nabla^\omega\nabla_\omega\Phi + \alpha'\nabla_\omega\Phi\nabla^\omega\Phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + O(\alpha'^2)
\end{aligned} \tag{5.46}$$

where $\boldsymbol{R}_{\mu\nu}$ is the spacetime Ricci tensor, distinguished from the world-sheet Ricci tensor R_{ab}. The conditions that the world-sheet theory be Weyl invariant is thus

$$\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0. \tag{5.47}$$

These are a sort of equations of motion. $\beta^G_{\mu\nu} = 0$ resembles Einstein's equation in presence of sources from the antisymmetric tensor field and the dilaton. $\beta^B_{\mu\nu} = 0$ is the generalization of the Maxwell's equation for the Kalb-Ramond field determinig the divergence of the field strength.

We can look at the equation of motion 5.47 in another different perspective; they can be derived from a spacetime effective action. This action takes the form:

$$\boldsymbol{S} = \frac{1}{2k_0^2}\int d^Dx\sqrt{-G}e^{-2\Phi}\left\{-\frac{2(D-26)}{3\alpha'} + \boldsymbol{R} - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\partial_\mu\Phi\partial^\mu\Phi + O(\alpha')\right\} \tag{5.48}$$

where $\boldsymbol{R}$ is the spacetime Ricci scalar. The normalization constant k_0 is not determined by the field equation and has no physical significance because it can be changed by a redefinition of Φ. One can take the variation

$$\delta \boldsymbol{S} = -\frac{1}{2k_0^2\alpha'}\int d^D x\sqrt{-G}e^{-2\Phi}\left\{\beta^{G\mu\nu}\delta G_{\mu\nu} + \beta^{B\mu\nu}\delta B_{\mu\nu} + \left(2\delta\Phi - \frac{1}{2}G^{\mu\nu}\delta G_{\mu\nu}\right)\left(\beta^{G\omega}{}_{\omega} - 4\beta^{\Phi}\right)\right. \tag{5.49}$$

and verify that one can derive the equations 5.47.

Finally, we can consider another form of the effective spacetime action which contains exactly the Hilbert-Einstein action. If we redefine the metric by a Weyl transformation

$$\tilde{G}_{\mu\nu}(x) = \exp(2\omega(x))G_{\mu\nu}(x) \tag{5.50}$$

the Ricci scalar becomes

$$\tilde{\boldsymbol{R}} = \exp(2\omega)\Big\{\boldsymbol{R} - 2(D-1)\nabla^2\omega - (D-2)(D-1)\partial_\mu\omega\partial^\mu\omega\Big\}. \tag{5.51}$$

Setting

$$\omega = \frac{2\Phi_0 - \Phi}{D-2} \tag{5.52}$$

with Φ_0 the constant part of the dilaton and

$$\tilde{\Phi} = \Phi - \Phi_0 \tag{5.53}$$

which has vanishing expectation value, we obtain the action:

$$\begin{aligned}\boldsymbol{S} = \frac{1}{2k^2}\int d^D x\sqrt{-\tilde{G}}\Bigg\{ & -\frac{2(D-26)}{3\alpha'}e^{4\tilde{\Phi}/(D-2)} + \tilde{\boldsymbol{R}} - \frac{1}{12}e^{-8\tilde{\Phi}/(D-2)}H_{\mu\nu\rho}\tilde{H}^{\mu\nu\rho} - \\ & - \frac{4}{D-2}4\partial_\mu\tilde{\Phi}\tilde{\partial}^\mu\tilde{\Phi} + O(\alpha')\Bigg\}.\end{aligned} \tag{5.54}$$

Tildes has been inserted to remind that one has to raise the indices by the new metric $\tilde{G}^{\mu\nu}$. We find the Hilbert action $\sqrt{-\tilde{G}}\tilde{\boldsymbol{R}}/2k^2$, therefore the coupling $k = k_0 e^{\Phi_0}$ is the gravitational coupling that in four dimensions can be expressed in terms of the Newton constant. $G_{\mu\nu}$ is called the sigma-metric or string metric, instead $\tilde{G}_{\mu\nu}$ is called the Einstein metric.

5.6 Compactification of extra dimensions

One important bad feature of the string theory is the need of extra dimensions to have a gauge invariant quantizable theory, 26 dimensions for the bosonic string and 10 for the superstring. The presence of general relativity in the low energy effective spacetime action provides a sort of natural way to account for the extra dimensions. Since the geometry of the spacetime is dynamical, there may be special solutions in which some dimensions are large and flat as usual and others are small, curved and fold in on themselves. As an example, the metric would be

$$G_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & G_{mn}(x^p) \end{pmatrix} \tag{5.55}$$

where $M, N = 0, \ldots, 25$, $\mu\nu = 0, \ldots, 3$ and $m, n, p = 4, \ldots, 25$. The coordinates are divided into four flat 'spacetime' coordinates and 22 'internal' coordinates, assumed to be compact.

More generally, the D-dim spacetime is of the form $M^d \times K$, where M^d is a d-dimensional Minkowsky space and K is some $(D-d)$-dimensional compact Riemannian space. The physics at length scales much longer than the size of K is the same as in d-dimensional Minkowski and the $(D-d)$ dimensions are said to be compactified. These compact dimensions are so small that they have never been observed.

We will present the simplest compactification of string theory consisting in a periodical identification for the compact dimensions. Then, we will sketch a more complicated procedure called orbifold.

5.6.1 Toroidal compactification

The toroidal compactification has not been introduced into the realm of string theory. The first appearence was in the 1914 in the attempt of unifying the electromagnetic and gravitational fields as components of a single higher dimensional field.

As a starting point let us consider a 5-dim theory with x^4 periodic: $x^4 \sim x^4 + 2\pi R$ and the others x^μ with $\mu = 0, \ldots, 3$ non compact. The 5-dim metric separates into three components :$G_{\mu\nu}$, $G_{\mu 4}$ and G_{44} which, from the 4-dim point of view, are a metric, a vector and a scalar.

Generalizing to $D = d+1$ dimensions we take the d coordinate to be periodic and we leave the others unchanged.

$$x^d \sim x^d + 2\pi R \tag{5.56}$$

The metric tensor can be splitted in compact and non compacts directions and it is parametrized by

$$ds^2 = G^D_{MN} dx^M dx^N = G_{\mu\nu} dx^\mu dx^\nu + G_{dd}(dx^d + A_\mu dx^\mu)^2 \tag{5.57}$$

where $M, N = 0, \ldots, d$ and $\mu\nu = 0, \ldots, d-1$. We establish that $G_{\mu\nu}$, G_{dd} and A_μ depend only on the noncompact coordinates x^μ. Notice also that $G^D_{\mu\nu} \neq G_{\mu\nu}$ and that the metric 5.57 is the most general metric invariant under translations of x^d. The other symmetries are the change of coordinates for the compact $x'^\mu(x)$ and a gauge transformation

$$\begin{cases} x^\mu & \to x^d + \lambda(x) \\ A_\mu & \to A_\mu - \partial_\mu \lambda(x) \end{cases} \tag{5.58}$$

This is the first $U(1)$ symmetry that appears. This is called the Kaluza-Klein mechanism.

We can see the effect of this periodical identification by considering a massless scalar ϕ in D dimensions. We assume for simplicity that $G_{dd} = 1$. Since the d dimension is periodic, the momentum in this direction is quantized $p_d = \frac{n}{R}$. Expanding the x^d dependence in a complete set

$$\phi(x^M) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) e^{\frac{inx^d}{R}} \tag{5.59}$$

the D-dimensional wave equation $\partial_M \partial^M \phi = 0$ becomes

$$\partial_\mu \partial^\mu \phi_n(x^\mu) = \frac{n^2}{R^2} \phi_n(x^\mu) \tag{5.60}$$

and the modes ϕ_n of the D-dim field ϕ becomes an infinite set of d-dim fields labeled by n. The mass-shell condition

$$p^\mu p_\mu = -\frac{n^2}{R} \tag{5.61}$$

does not vanish if $p_d \neq 0$. Therefore, at energy smaller than R^{-1} the physics is the d-dimensional and only the x^d-independent fields remain. At energy above R^{-1} the Kaluza-Klein states are visible. These state are charged under the gauge transformation 5.58 with charge p_d and they are massive.

One can also have massless charged fields by considering the spacetime effective action in a curved background 5.48. Setting $G_{dd} = e^{2\sigma}$, the Ricci scalar for the metric 5.57 takes the form:

$$\boldsymbol{R} = \boldsymbol{R}_d - 2e^{-\sigma}\nabla^2 e^{\sigma} - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu} \tag{5.62}$$

with $\boldsymbol{R}$ is the Ricci scalar from G^D_{MN} and $\boldsymbol{R}_d$ from $G_{\mu\nu}$. Considering only the terms of the action 5.48 containing the graviton and the dilaton and integrating over the d-direction, one obtains:

$$\begin{aligned}\boldsymbol{S}_{G\Phi} =& \frac{1}{2k_0^2}\int d^D x\sqrt{-G^D}e^{-2\Phi}(\boldsymbol{R} + 4\partial_\mu\Phi\partial^\mu\Phi)\\ =& \frac{\pi R}{k_0^2}\int d^d x\sqrt{-G_d}e^{-2\Phi+\sigma}\left(\boldsymbol{R}_d - 4\partial_\mu\sigma\partial^\mu\sigma + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}\right)\\ =& \frac{\pi R}{k_0^2}\int d^d x\sqrt{-G_d}e^{-2\Phi_d}\left(\boldsymbol{R}_d - \partial_\mu\sigma\partial^\mu\sigma + 4\partial_\mu\Phi_d\partial^\mu\Phi_d - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}\right).\end{aligned} \tag{5.63}$$

In the integration we have assumed that all the fields do not depend on x^d. G_d denotes the determinant of the d-dimensional metric $G_{\mu\nu}$ and Φ_d denotes the d-dimensional dilaton $\Phi_d = \Phi - \frac{\sigma}{2}$. There is no potential for Φ and σ, thus these fields are massless.

One can define the covariant derivative

$$\partial_\mu + ip_d A_\mu = \partial_\mu + in\tilde{A}_\mu \tag{5.64}$$

with $\tilde{A}_\mu = \frac{A_\mu}{R}$. We can also define the d-dimensional gauge and gravitational couplings. The coefficient of $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$ is conventionally $-\frac{1}{4g_d^2}$ and the coefficient of the Hilbert action is $\frac{1}{2k^2}$. Therefore, the couplings are

$$g_d^2 = \frac{k_0^2 e^{2\Phi_d}}{\pi R^3 e^{2\sigma}}, \qquad k^2 = \frac{k_0^2}{2\pi R e^{-2\Phi_d}} \tag{5.65}$$

and they are related via the invariant radius $\rho = Re^{\sigma}$:

$$g_d^2 = \frac{2k_d^2}{\rho^2}. \tag{5.66}$$

We can also consider the antisymmetric tensor which gives rise to another $U(1)$ gauge symmetry. We split B_{MN} into $B_{\mu\nu}$ and $B_{d\mu} =: B_\mu$ in the usual way. The gauge parameters ζ_M of the transformation $B_{MN} \to \partial_M\zeta_N - \partial_N\zeta_M$ separates into ζ_μ with the transformation

$$B_{\mu\nu} \to \partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu \tag{5.67}$$

and ζ_d with the ordinary $U(1)$ gauge transformation

$$B_\mu \to B_\mu + \partial_\mu\zeta_d. \tag{5.68}$$

The field strength for the $U(1)$ gauge field B_μ is $H_{d\mu\nu} = \partial_\mu B_{d\nu} - \partial_\nu B_{d\mu}$. We also define a sort of Chern-Simons term

$$\tilde{H}_{\mu\nu\lambda} = (\partial_\mu B_{\nu\lambda} - A_\mu H_{d\nu\lambda}) + \text{cyclic permutations} \tag{5.69}$$

to taking into account the effect of the metric G^{MN} in raising the indices of H_{MNL}. $\tilde{H}_{\mu\nu\lambda}$ is gauge invariant because the variation $A_\mu \to A_\mu - \partial_\mu \lambda$ can be compensated by the transformation $B_{\nu\lambda} \to B_{\nu\lambda} - \lambda H_{d\nu\lambda}$ that is precisely of the type 5.67. This type of double transformation will be discussed in detail in the following chapter.

Eventually, the action for the antisymmetric tensor becomes

$$\begin{aligned}\boldsymbol{S}_B =& \frac{1}{24k_0^2}\int d^Dx\sqrt{-G^D}e^{-2\Phi}H_{MNL}H^{MNL}\\ =& \frac{\pi R}{12k_0^2}\int d^dx\sqrt{-G_d}e^{-2\Phi_d}\left(\tilde{H}_{\mu\nu\lambda}\tilde{H}^{\mu\nu\lambda}+3e^{2\sigma}H_{d\mu\nu}H_d^{\ \mu\nu}\right).\end{aligned} \tag{5.70}$$

There is no way to couple the potential B_{MN} minimally to others fields.

So far we have covered toroidal compactification in a generic quantum field theory. Now, we want to specialize to the case of string theory considering the conformal field theory with a single periodic scalar field. We do not discuss this topic in detail, but we only focus into two main effects that toroidal compactification produces. We consider the periodical identification for the scalar

$$X^d \sim X^d + 2\pi R \tag{5.71}$$

and we set $G_{dd} = 1$; the world-sheet action and the equation of motion remain the same of the non compact theory. However, there are two effects due to the periodicity related to the two $U(1)$ symmetries described above.

First, string states must be single-valued under the periodical identification 5.71. If $e^{2\pi i R p_d}$ is the unitary operator that translates strings once around the periodic dimension, it must leave states invariant. Therefore, the center-of-mass momentum in the compact direction is quantized

$$k_d = \frac{n}{R}, \quad n \in \mathbb{Z} \tag{5.72}$$

like the discussion before. The first $U(1)$ symmetry, the one related to A_μ, is the symmetry under translation around the compact circe $\mathbb{S}^1$ and the conserved charge is the center-of-mass momentum in the compact direction.

The second effect is specific for string theory. One can think of a closed string that wraps itself numerous times around the compact direction. In this case we say that the closed string winds around this direction and the winding number $w \in \mathbb{Z}$ indicates the number of windings. Matematically we write

$$X^d(\sigma + 2\pi) = X^d(\sigma) + 2\pi R w. \tag{5.73}$$

From the point of view of the world-sheet, strings of nonzero winding number are topological solitons, i.e. topologically nontrivial field configurations like vortices. A consistent string theory must include strings with nonzero winding number. w is the conserved charge of the second $U(1)$ symmetry considered before.

5.6.2 Orbifolds

We briefly sketch another compactification procedure that is fundamental in string theory since it is more than a simple way to compactify some dimensions, but it allows to produce new different string theories.

Instead of a periodic identification, we can identify points under the reflection

$$X^d \sim -X^d \tag{5.74}$$

where d is the last coordinate in a $D = d + 1$-dim spacetime. Therefore, the half-space region $X^d \geq 0$ is sufficient to describe the theory and the hyperplane $X^d = 0$ is the boundary of this region. The points of the hyperplane are fixed points under reflection, i.e. they do not change under this transformation. More generally, we could identify points under a simultaneous reflection of k coordinates

$$X^m \sim -X^m, \quad D-k \leq m \leq d \tag{5.75}$$

and the space of fixed points is given by $X^{D-k} = \cdots = X^d = 0$. For $k \geq 2$ this space is a singularity, e.g. a conic singularity for $k = 2$.

The combination of the reflection 5.74 after the periodical identification 5.71 generates a compact space. The action of the toroidal identification reduces the non compact d direction to the compact $\mathbb{S}^1$, then the identification under reflection of the circle produces a line segment with $0 \leq X^d \leq \pi R$ as a fundamental region. The fixed points under both the transformations are the end points of the segment. More generally, we can consider a generic periodical identification:

$$X^d \sim X^d + 2\pi m R, \quad \text{and} \quad X^d \sim 2\pi R m - X^d. \tag{5.76}$$

Similarly we can identify the k-torus under 5.75. We obtain 2^k fixed points $X^m = 0, \pi R$ for each compact directions. Orbifolds is the name of the singular space obtained in this way. We indicate $\mathbb{R}^k/\mathbb{Z}_2$ if the starting space is not compact and $\mathbb{T}^k/\mathbb{Z}_2$ if the identification is in a compact space, the k-torus.

One can show that string theory on such singular space makes sense and it is not so different from toroidal compactification, even if more symmetry is broken. As in the previous case, 5.74 has two effects. The first is the invariance under reflection of the string fields, i.e. they must be equal at identified points. Therefore, the center-of-mass string momentum becomes discrete. Second, there is a new sector in the closed string spectrum, in which

$$X^d(\sigma^1 + 2\pi) \sim -X^d(\sigma^1). \tag{5.77}$$

This identification is possible because these are the same points after the reflection in spacetime. The strings in this sector are called twisted states. This has a parallel in the toroidal compactification in which strings are closed only up to periodic identification labeled by the winding number.

Furthermore, we want to only mention that the orbifold procedure allows us to construct new string theory from one other theory. If a string theory is described by a CFT and H is a discrete symmetry group of this theory, we can form a new CFT in two steps. First, we add the twisted sectors, in which F, a generic closed string world-sheet fields, is periodic only up to some transfomation $h \in H$

$$F(\sigma^1 + 2\pi) \sim h \cdot F(\sigma^1). \tag{5.78}$$

Second, we restrict the spectrum to H-invariant states by a projection. This procedure can be also thought of as gauging the discrete group H. Intuitively, the twisted sectors can be considered as non trivial gauge configurations on the world-sheet and the projection corresponds to consider only physical operators that are gauge invariant.

Chapter 6

’t Hooft anomalies cancellation in toroidal compactification of 1 dimension

In this chapter we want to focus on the symmetries of a string theory with one direction compactified on a circle. As we will see, after the periodical identification of one direction, mixed ’t Hooft anomalies potentially arise between different subgroups of the global symmetry group of the world-sheet theory. In principle, these anomalies are not an issue for the world-sheet theory; however, they become dangerous when we consider the theory from the spacetime point of view because global symmetry on the world-sheet are coupled to dynamical gauge field in spacetime. Nevertheless, the presence of higher form symmetries combined in a non-trivial 2-group conspires in such a way the anomalous phase is exactly canceled. We will verify this explicitly for the global symmetry arising from the toroidal compactification.

6.1 Spacetime action after toroidal compactification

The first step of our discussion is to recall the spacetime effective action of the string and to make explicit the gauge symmetries due to the dimensional reduction by periodical identification of one coordinate. We refer to 5.6.1 for a complete discussion; here the attention will be aimed at explaining the gauge transformations to which the different fields are subjected. The starting point is the spacetime effective action describing the low-energy behavior of a string moving in a curved background. We only consider bosonic string or the NS-NS background of type II superstring.

$$\boldsymbol{S} = \frac{1}{2k_0^2}\int d^D x\sqrt{-G}e^{-2\Phi}\left\{-\frac{2(D-26)}{3\alpha'} + \boldsymbol{R} - \frac{1}{12}H_{MNR}H^{MNR} + 4\partial_\mu\Phi\partial^\mu\Phi + O(\alpha')\right\} \tag{6.1}$$

where $H_{MNR} = \partial_M B_{NR} + \partial_N B_{RM} + \partial_R B_{MN}$. H_{MNR} is gauge invariant under

$$B_{MN} \to B_{MN} + \partial_M\zeta_N - \partial_N\zeta_M; \tag{6.2}$$

therefore, the action is invariant under this gauge symmetry.

After the periodical identification of the d-direction in a $D = d + 1$ dimensional spacetime, $x^d \sim x^d + 2\pi R$, the metric splits in three parts: the d dimensional metric $G_{\mu\nu}$, the vector

$A_\mu = G_{d\mu}$ and a scalar G_{dd} (see 5.57). The indices run in the usual way: $M, N = 0, \dots, d$ and $\mu\nu = 0, \dots, d-1$. The vector A_μ transforms under a $U(1)$ gauge symmetry as

$$A_\mu \to A_\mu + \partial_\mu \lambda. \tag{6.3}$$

The origin of this $U(1)$ symmetry is the requirement of the invariance under general transformation of the compact coordinate whose infinitesimal form is $x^d \to x^d + \lambda(x)$.

Also the Kalb-Ramond field B_{MN} splits in two parts, $B_{\mu\nu}$ and $B_{d\mu} =: B_\mu$, but the identification is a bit trickier. To justify the definition of $\tilde{H}_{\mu\nu\rho}$ in 5.69, one should first identify the field strength using tangent-space-indices, i.e. orthonormal Vielbein basis, and then go back to the original coordinate basis. Therefore, without entering in detail, we obtain

$$H_{d\mu\nu} =: H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{6.4}$$

and

$$\tilde{H}_{\mu\nu\lambda} = (\partial_\mu B_{\nu\lambda} - A_\mu H_{\mu\nu}) + \text{cyclic permutations.} \tag{6.5}$$

The gauge parameters ζ_M of the transfomation 6.2 separates into ζ_μ and ζ_d. The transformation for B_μ is

$$B_\mu \to B_\mu + \partial_\mu \zeta_d \tag{6.6}$$

and that for $B_{\mu\nu}$ is

$$B_{\mu\nu} \to \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu - \lambda H_{\mu\nu}. \tag{6.7}$$

In the last variation, the terms dependent on ζ_μ is due to the original gauge symmetry 6.2, instead the term dependent on λ is required to assure the invariance of $\tilde{H}_{\mu\nu\lambda}$. Since the identification in tangent-space-indices concerns the field strength, we have some freedom in the definition of $B_{\mu\nu}$. Instead of assuminig $B_{\mu\nu} = B^D_{\mu\nu}$, where $B^D_{\mu\nu}$ are the components of the D-dim tensor, we can define

$$B_{\mu\nu} = B^D_{\mu\nu} + \frac{1}{2}(A_\mu B_\nu - A_\nu B_\mu). \tag{6.8}$$

The new field strength, that we indicate $H_{\mu\nu\rho}$ again, becomes in a certain sense more symmetric:

$$H_{\mu\nu\rho} = (\partial_\mu B_{\nu\lambda} - \frac{1}{2} A_\mu H_{\mu\nu} - \frac{1}{2} B_\mu F_{\nu\rho}) + \text{cyclic permutations} \tag{6.9}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The invariance under the gauge transformations 6.6 and 6.3 is guaranteed if we set the transformation for $B_{\mu\nu}$:

$$B_{\mu\nu} \to \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu + \frac{1}{2}(B_\mu \partial_\nu \lambda - B_\nu \partial_\mu \lambda + A_\mu \partial_\nu \zeta_d - A_\nu \partial_\mu \zeta_d) \tag{6.10}$$

The set of transformation 6.3, 6.6 and 6.10 are called the Nicolai-Townsend transformations. Finally, following the discussion in 5.6.1, after the integration over x^d, we obtain the spacetime effective action in $d = D - 1$ dimensions:

$$\begin{aligned} \boldsymbol{S} = \frac{\pi R}{k_0^2} \int d^d x \sqrt{-G_d} e^{-2\Phi_d} \Big(& \boldsymbol{R}_d - \partial_\mu \sigma \partial^\mu \sigma + 4 \partial_\mu \Phi_d \partial^\mu \Phi_d \\ & - \frac{1}{4} e^{2\sigma} F^A_{\mu\nu} F^{A\mu\nu} - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{4} e^{-2\sigma} F^B_{\mu\nu} F^{B\mu\nu} \Big) \end{aligned} \tag{6.11}$$

6.2 World-sheet action after toroidal compactification

Let us consider the non-linear sigma model for a string moving in a curved spacetime. The world-sheet action is

$$S_\sigma = \frac{T}{2}\int d^2\sigma\sqrt{g}\Big\{\big(g^{ab}G_{MN}(X) + i\epsilon^{ab}B_{MN}(X)\big)\partial_a X^M\partial_b X^N + \alpha' R\Phi(X)\Big\}. \tag{6.12}$$

We want to consider the world-sheet action for a string moving in a D-dimensional curved spacetime with one dimension constrained in a circle, i.e. subjected to the identification $X^d \sim X^d + 2\pi R$. We follow [31] as main reference. Since we want to focus on the terms containing the periodic scalar X^d, we can neglect the dilaton term and, at low energy, we assume that the spacetime fields are independent of X^d. The metric and the Kalb-Ramond field split as we have already seen (5.57 and 6.8); therefore, we can substitute the decomposed fields into the action above and we obtain the following action:

$$S_\sigma \supset \frac{T}{2}\int d^2\sigma\sqrt{g}\Big\{g^{ab}h_{ab} - G_{dd}F^2 + i\epsilon^{ab}(B_{ab} + A_a B_b - 2F_a B_b)\Big\} \tag{6.13}$$

where $F_a = \partial_a X^d + A_a$ and

$$h_{ab} = G_{\mu\nu}\partial_a X^\mu\partial_b X^\nu, \quad B_{ab} = B_{\mu\nu}\partial_a X^\mu\partial_b X^\nu, \quad A_a = A_\mu\partial_a X^\mu \quad \text{and} \quad B_a = B_\mu\partial_a X^\mu \tag{6.14}$$

are the pull-backs of the $(D-1)$-dimensional fields to the world-sheet.

The following step is to write the world-sheet action for the periodic scalar in terms of differential forms, that represent a more useful formalism for our purposes. A generic differential form can be written as $F = F_i dx^i$ and the integral $\int d^2x F_i G_j\epsilon^{ij}$ can be written with the wedge product: $\int d^2x F_i G_j \epsilon^{ij} = \int F_i G_j dx^i \wedge dx^j = \int F \wedge B$. We set $G_{dd} = 1$ for simplicity and we neglect the terms containing h_{ab} that do not depend on X^d. We denote with B_2 the 2-form $B_{ab}d\sigma^a \wedge d\sigma^b$ and with A_1, B_2 and X^d the other two 1-form and the scalar 0-form. Therefore, the world-sheet action for the compactified scalar is

$$\begin{aligned} S \supset &-\frac{T}{2}\int\Big\{(\mathrm{d}X^d + A_1)\wedge *(\mathrm{d}X^d + A_1) + i[B_2 + A_1\wedge B_1 - 2(\mathrm{d}X^d + A_1)\wedge B_1]\Big\} = \\ = &-\frac{T}{2}\int\Big\{(\mathrm{d}X^d + A_1)\wedge *(\mathrm{d}X^d + A_1) + i[B_2 - 2\mathrm{d}X^d\wedge B_1 - A_1\wedge B_1]\Big\}. \end{aligned} \tag{6.15}$$

The square-root of the determinant of the metric is encoded in the Hodge dual operator $*$ since the Levi-Civita pseudotensor is normalized $\sqrt{g}\epsilon^{12} = 1$. The gauge transformations written in terms of forms become

$$\begin{aligned} A_1 &\to A_1 + \mathrm{d}\lambda \\ X^d &\to X^d - \lambda \end{aligned} \tag{6.16}$$

$$B_1 \to B_1 + \mathrm{d}\zeta_d \tag{6.17}$$

and

$$B_2 \to B_2 + \mathrm{d}\zeta_1 + B_1\wedge\mathrm{d}\lambda + A_1\wedge\mathrm{d}\zeta_d + \mathrm{d}\zeta_d\wedge\mathrm{d}\lambda \tag{6.18}$$

with ζ_1 the 1-form correspondent to the gauge parameter ζ_μ. The term $\mathrm{d}\zeta_d\wedge\mathrm{d}\lambda$ is necessary since the group symmetry is the compact $U(1)$ and the integral over 1-cycle of $\mathrm{d}\lambda$ or $\mathrm{d}\zeta_d$ does not vanish but are integer. In the following we will abandon the subscrit d in ζ_d and the superscript d in X^d.

6.3 't Hooft anomaly for a periodic free scalar in 2D

In this section we will discuss a general fact concerning a periodic scalar field in a 2-dimensional field theory [13]. Let us consider a free scalar X in a 2-dimensional space. The two 1-form current $\mathrm{d}X$ and $*\mathrm{d}X$ are the Noether currents that correspond to two 0-form global symmetries. The currents are conserved since the scalar is free.

If the target space of X is $\mathbb{S}^1$, i.e. we have the identification $X \sim X + 2\pi R$ for some $R > 0$, the symmetry group is $U(1) \times U(1)$: one $U(1)$ with current $*\mathrm{d}X$ is associated with the momentum in the circle and the other with current $\mathrm{d}X$ with the winding around $\mathbb{S}^1$. The crucial fact is that there is a mixed 't Hooft anomaly between these two $U(1)$ symmetries. If we couple the currents with the background gauge fields for the two $U(1)$ symmetries, the action is not invariant under the $U(1)\times U(1)$ gauge transformations. This means that the gauge fields cannot be made dynamical. Let us analyze it in detail [13].

The action for the free scalar and the background fields coupled to the currents is

$$-S = -\frac{2\pi}{2g^2}\int (\mathrm{d}X + A_1) \wedge *(\mathrm{d}X + A_1) + 2\pi i \int B_1 \wedge \mathrm{d}X \tag{6.19}$$

with g a free positive parameter. The gauge transformations for the background fields are given by:

$$A_1 \to A_1 + \mathrm{d}\lambda, \quad B_1 \to B_1 + \mathrm{d}\zeta \tag{6.20}$$

while the scalar transforms as

$$X \to X - \lambda. \tag{6.21}$$

The action is invariant under the B_1 gauge transformation; however, the second term is not invariant under the A_1 gauge transformation

$$2\pi i \int B_1 \wedge \mathrm{d}X \to 2\pi i \int (B_1 \wedge \mathrm{d}X - B_1 \wedge \mathrm{d}\lambda) \tag{6.22}$$

and this term cannot be canceled by a gauge transformation for B_1.

We can try to modify the second term so as to make it invariant under the A_1 gauge transfomation

$$2\pi i \int B_1 \wedge (\mathrm{d}X + A_1), \tag{6.23}$$

but, unfortunately, it is no longer invariant under the transformation for B_1:

$$2\pi i \int B_1 \wedge (\mathrm{d}X + A_1) \to 2\pi i \int (B_1 + \mathrm{d}\zeta) \wedge (\mathrm{d}X + A_1) = 2\pi i \int \Big\{ B_1 \wedge (\mathrm{d}X + A_1) + \mathrm{d}\zeta \wedge A_1 \Big\} \tag{6.24}$$

and there is no way to adsorb $\mathrm{d}\zeta \wedge A_1$ in a total derivative or to cancel with other terms.

Therefore, one of the $U(1)$ symmetry is always violated: when we couple the action to background fields A_1, B_1 the partition function is no longer invariant under gauge transformations. This is precisely the mixed 't Hooft anomaly for the periodic free scalar in two dimensions. A similar argument hold for the mixed 't Hooft anomaly for the electric and magnetic $U(1)$ 1-form symmetries (1.2.3).

This argument is valid only on a classical level and we can wonder if there are modifications that arise at quantum level. In order to discuss this problem, we can consider the partition function in presence of a non-trivial background for the periodic scalar, that is described by an Euclidean action defined on a 2-dimensional world-sheet torus $\mathbb{T}^2$.

Firstly, we set the background fields to 0 and we consider the theory in the absence of background. Let θ and ϕ the two coordinates in the world-sheet with period 1 subjected to the identification $\theta \sim \theta + 1$ and $\phi \sim \phi + 1$. The scalar is a map $X : \mathbb{T}^2 \to \mathbb{S}^1_R$, where $\mathbb{S}^1_R$ is a circle of radius R, i.e. it is subjected to the identification $X \sim X + 2\pi R$. Since the world-sheet is a torus, X must transform as

$$\begin{aligned} X &\to X + 2\pi n_1 R \quad \text{when} \quad \theta \to \theta + 1 \\ X &\to X + 2\pi n_2 R \quad \text{when} \quad \phi \to \phi + 1 \end{aligned} \tag{6.25}$$

where $n_1, n_2 \in \mathbb{Z}$ count the number of times the worldsheet $\mathbb{T}^2$ is wrapped around the target space $\mathbb{S}^1_R$. Therefore, there are infinite solitonic sectors labeled by the integers n_1, n_2 and we have to sum them in the partition function:

$$Z[A = B = 0] = \sum_{n_1,n_2} \int DX e^{-S[X;n_1,n_2]}. \tag{6.26}$$

We can decompose the boson field in two components: $X = X_{cl} + X_q$, where X_{cl} is the solution of the equation of motion with periodicity depending on n_1, n_2 and X_q is the quantum fluctuation with standard periodicity $X_q \to X_q$ when $\theta \sim \theta + 1$ or $\phi \sim \phi + 1$. The expansion of the action

$$S[X] = \frac{1}{2} \int d\theta d\phi X \partial_a \partial^a X \tag{6.27}$$

around the classical solution is

$$S[X_{cl}+X_q] = S[X_{cl}] + \int d\sigma' \frac{\delta S}{\delta X(\sigma')}\bigg|_{X_{cl}} X_q(\sigma') + \frac{1}{2}\int d\sigma' d\sigma'' X_q(\sigma') \frac{\delta S}{\delta X(\sigma')\delta X(\sigma'')}\bigg|_{X_{cl}} X_q(\sigma'') + \dots \tag{6.28}$$

where we have denoted with σ' or σ'' the set of the two world-sheet coordinates (θ, ϕ). The term $\frac{\delta S}{\delta X(\sigma')}\big|_{X_{cl}}$ vanishes since the solution of the classical equation of motion is the solution that minimize the action. The second term takes the form

$$\frac{\delta S}{\delta X(\sigma)\delta X(\sigma')} = \int d\sigma \{\delta^{(2)}(\sigma - \sigma')\partial_a \partial^a \delta^{(2)}(\sigma - \sigma'')\}. \tag{6.29}$$

Since the action is quadratic in the fields X, the partition function factorizes:

$$Z[A = B = 0] = \sum_{n_1,n_2} e^{-S[X_{cl};n_1,n_2]} Z_q \quad \text{with} \quad Z_q = \int DX_q \exp\Big(-\int d\theta d\phi X_q \partial_a \partial^a X_q\Big) \tag{6.30}$$

and Z_q depends only on the quantum fluctuations which are periodic in the standard way.

Now we consider the same partition function in presence of non trivial background fields. For simplicity, we take only backgrounds that are flat when restricted on the world-sheet, i.e. $\mathrm{d}A = \mathrm{d}B = 0$. We study only flat background since we are mainly interested in the study of discrete groups and the discussion is simpler than the one with non-trivial curvature. A useful choice is

$$\begin{aligned} A_1 &= \alpha_1 \mathrm{d}\theta + \alpha_2 \mathrm{d}\phi \\ B_2 &= \beta_1 \mathrm{d}\theta + \beta_2 \mathrm{d}\phi \end{aligned} \tag{6.31}$$

where $\alpha_i, \beta_i \in \mathbb{R}$ are real constants. Flat backgrounds are characterized by the holonomy of the gauge fields along the two non-trivial cycle of the world-sheet, one in each of the periodic directions of $\mathbb{T}^2$. We denote with γ_1 the cycle along θ coordinate and γ_2 the one along the ϕ direction. The holonomies of the gauge fields are:

$$\oint_{\gamma_i} A_1 = \alpha_i \quad \text{and} \quad \oint_{\gamma_i} B_1 = \beta_i. \tag{6.32}$$

Since the gauge group is $U(1)$ we can shift the holonomies by integer; in fact, considering the phase

$$g = e^{2\pi i(m_1\theta + m_2\phi)} \in U(1) \tag{6.33}$$

well defined on $\mathbb{T}^2$ if $m_1, m_2 \in \mathbb{Z}$ and making the gauge transformation

$$A_1, B_1 \to A_1, B_1 + \frac{i}{2\pi} g^{-1}\mathrm{d}g \tag{6.34}$$

we can shift α_i and β_i by the integer m_i. Therefore, the different flat gauge backgrounds are parametrized by $\alpha_i \mod \mathbb{Z}$ and $\beta_i \mod \mathbb{Z}$.

In order to study how the partition function changes in presence of such non-trivial backgrounds, we can do a 'singular' gauge transformation to eliminate the field A_1. Considering

$$g(\theta, \phi) = e^{2\pi i(\alpha_1\theta + \alpha_2\phi)} \tag{6.35}$$

we obtain

$$A_1 + \frac{i}{2\pi} g^{-1}\mathrm{d}g = \alpha_1 \mathrm{d}\theta + \alpha_2 \mathrm{d}\phi - \alpha_1 \mathrm{d}\theta - \alpha_2 \mathrm{d}\phi = 0. \tag{6.36}$$

This transformation is well defined only for $\theta, \phi \neq 0$ and it modifies the periodicity of the field:

$$\begin{aligned} X &\to X + 2\pi(n_1 + \alpha_1)R \quad \text{when} \quad \theta \to \theta + 1 \\ X &\to X + 2\pi(n_2 + \alpha_2)R \quad \text{when} \quad \phi \to \phi + 1. \end{aligned} \tag{6.37}$$

In the same way as before, we can split $X = X_{cl} + X_q$, where X_{cl} has the above periodicity depending on α_1 and α_2, while X_q has the standard periodicity.

Considering also the field B_1, we observe that in the partition function it induces a phase depending on the winding indices n_i and the background field itself. Denoting the component of $\mathrm{d}X$ in the usual way $\mathrm{d}X = X_1\mathrm{d}\theta + X_2\mathrm{d}\phi$ where $X_1 = \frac{\partial X}{\partial \theta}$ and $X_2 = \frac{\partial X}{\partial \phi}$, we can rewrite the second term of the action

$$\begin{aligned} 2\pi i \int B \wedge dX =& 2\pi i \int (\beta_1 \mathrm{d}\theta + \beta_2 \mathrm{d}\phi) \wedge (X_1 \mathrm{d}\theta + X_2 \mathrm{d}\phi) \\ =& 2\pi i \int (\beta_1 X_2 \mathrm{d}\theta \wedge \mathrm{d}\phi + \beta_2 X_1 \mathrm{d}\phi \wedge \mathrm{d}\theta) \\ =& 2\pi i \int (\beta_1 X_2 - \beta_2 X_1) \mathrm{d}\theta \wedge \mathrm{d}\phi \\ =& 2\pi i \int d\theta d\phi \ \epsilon_{ij} \beta_i X_j \\ =& 2\pi i \epsilon_{ij} \beta_i (n_j + \alpha_j) \end{aligned} \tag{6.38}$$

where in the last equality we exploit the integration

$$\int_0^1 d\theta \frac{\partial X}{\partial \theta} = X(1, \phi) - X(0, \phi) = 2\pi(n_1 + \alpha_1)R. \tag{6.39}$$

Therefore, the presence of B_1 induces the phase $e^{2\pi i\epsilon_{ij}\beta_i(n_j+\alpha_j)}$ in the partition function and we can observe that only the classical part of Z is modified by the introduction of the background.

$$Z[A,B] = \sum_{n_1,n_2} e^{-S[X_{cl};n_1,n_2,\alpha_1,\alpha_2,\beta_1,\beta_2]} Z_q. \tag{6.40}$$

The crucial fact is that only the classical partition function changes, while Z_q remains untouched after the introduction of non trivial gauge backgrounds. Therefore, any anomalous phase that can affect the partition function under gauge transformations depends only on the classical part of the action. Thus, the previous totally classical discussion on the presence of a mixed 't Hooft anomaly is complete and we do not have to consider quantum effects in this treatment.

6.4 't Hooft anomaly cancellation in $U(1)^{(0)}_A \times U(1)^{(0)}_B$

So far we have presented the action for the toroidal compactification of the bosonic string and we have discussed the presence of a mixed 't Hooft anomaly for the periodic scalar in 2D. Now we want to reinterpret the results in string compactification at light of the concepts of 't Hooft anomaly and of 2-group symmetry.

When we consider a string moving in a curved $D-$dimensional spacetime with one direction periodically identified, the momentum in the compact direction is quantized and conserved since it is the charge of a $U(1)$ symmetry. It is a 0-form symmetry and, from the point of view of the world-sheet, it is just a global symmetries with the non-dynamical background 1-form A_1. There is also another $U(1)$ symmetry whose charge is the winding number and its background 1-form is B_1. Therefore, the symmetry group is exactly $U(1)^{(0)}_A \times U(1)^{(0)}_B$ and, if we couple the background 1-forms with the scalar Noether currents $\mathrm{d}X$ and $*\mathrm{d}X$, we are in the same situation of the previous section: there is a mixed 't Hooft anomaly and we cannot gauge the entire group $U(1)^{(0)}_A \times U(1)^{(0)}_B$.

From the point of view of the world-sheet the 't Hooft anomaly is not so problematic since we can consider the 1-form fields as fixed non-dynamical backgrounds and the symmetries can remain only global. However, from the point of view of the spacetime, we know that there is a deep conjecture stating that in quantum gravity there cannot be any global symmetries (e.g. see [14]); therefore, they must be broken or gauged. In the spacetime the 1-forms A_1 and B_1 must be dynamical gauge fields. Unfortunately, the mixed 't Hooft anomaly on the world-sheet would make the coupling with dynamical fields inconsistent and without the presence of other terms we will find ourselves facing an impasse.

Furthermore, often it is useful to consider the orbifold of a worldsheet theory to obtain a new theory from the original one. In general the orbifold consists in the gauging of a global symmetry of the parent theory; however, if this symmetry has 't Hooft anomaly, the orbifold procedure fails.

The solution to this apparent paradox consists in the presence in the world-sheet action 6.15 of the 2-form B_2 that is subject to a gauge transformation with the same parameter of A_1 and B_1. The 2-form B_2 also transforms under a 1-form global symmetry with parameter ζ_1, but the relevant fact is the non-trivial mixing of the gauge parameters λ and ζ with the 1-forms A_1 and B_1. This non-trivial mixing

$$B_2 \to B_2 + \mathrm{d}\zeta_1 + B_1 \wedge \mathrm{d}\lambda + A_1 \wedge \mathrm{d}\zeta \tag{6.41}$$

has exactly the form of the 2-group symmetry

$$U(1)^{(0)}_A \times U(1)^{(0)}_B \times_{\hat{k}_{AB}} U(1)^{(1)}_{B_2} \tag{6.42}$$

studied before in 2.6, with $\hat{k}_{AB} = 1$. The 2-group transformation is the same with the only difference that here the exterior derivative acts on the gauge parameters, but by integrating by parts we can move d to the gauge fields and obtain exactly 2.37 up to a total derivative that can be absorbed in the parameter ζ_1.

Even if, in principle, one can set $B_2 = 0$, it cannot rest null after a non-trivial transformation for A_1 or B_1. The presence of the 2-group structure guarantees the possibility of the gauging of these global symmetries, because the variation of B_2 cancels out the mixed 't Hooft anomaly for $U(1)^{(0)}_A \times U(1)^{(0)}_B$.

One can see this anomaly cancellation directly in the action by applying 6.16, 6.17 and 6.18:

$$\begin{aligned}
&-\frac{T}{2}\int\Big\{(\mathrm{d}X + A)\wedge *(\mathrm{d}X + A) + i[B_2 - 2\mathrm{d}X\wedge B - A\wedge B]\Big\} \to \\
&\to -\frac{T}{2}\int\Big\{(\mathrm{d}X - \mathrm{d}\lambda + A + \mathrm{d}\lambda)\wedge *(\mathrm{d}X - \mathrm{d}\lambda + A + \mathrm{d}\lambda) + i[B_2 + \mathrm{d}\zeta_1 + B\wedge \mathrm{d}\lambda + A\wedge \mathrm{d}\zeta \\
&\quad + \mathrm{d}\zeta\wedge\mathrm{d}\lambda - 2(\mathrm{d}X - \mathrm{d}\lambda)\wedge(B + \mathrm{d}\zeta) - (A + \mathrm{d}\lambda)\wedge(B + \mathrm{d}\zeta)]\Big\} = \\
&= -\frac{T}{2}\int\Big\{(\mathrm{d}X + A)\wedge *(\mathrm{d}X + A) + i[B_2 + B\wedge\mathrm{d}\lambda + \cancel{A\wedge\mathrm{d}\zeta} + \mathrm{d}\zeta\wedge\mathrm{d}\lambda - 2\mathrm{d}X\wedge B - 2\mathrm{d}X\wedge\mathrm{d}\zeta \\
&\quad + 2\mathrm{d}\lambda\wedge B + 2\mathrm{d}\lambda\wedge\mathrm{d}\zeta - A\wedge B - \mathrm{d}\lambda\wedge B - \cancel{A\wedge\mathrm{d}\zeta} - \mathrm{d}\lambda\wedge\mathrm{d}\zeta]\Big\} = \\
&= -\frac{T}{2}\int\Big\{(\mathrm{d}X + A)\wedge *(\mathrm{d}X + A) + i[B_2 - \cancel{\mathrm{d}\lambda\wedge B} - \cancel{\mathrm{d}\lambda\wedge\mathrm{d}\zeta} - 2\mathrm{d}X\wedge B + \cancel{2\mathrm{d}\lambda\wedge B} + \cancel{2\mathrm{d}\lambda\wedge\mathrm{d}\zeta} \\
&\quad - A\wedge B - \cancel{\mathrm{d}\lambda\wedge B} - \cancel{\mathrm{d}\lambda\wedge\mathrm{d}\zeta}]\Big\} = \\
&= -\frac{T}{2}\int\Big\{(\mathrm{d}X + A)\wedge *(\mathrm{d}X + A) + i[B_2 - 2\mathrm{d}X\wedge B - A\wedge B]\Big\}.
\end{aligned} \tag{6.43}$$

The term $-2\mathrm{d}X\wedge\mathrm{d}\zeta$ vanishes up to a total derivative, in fact $2\mathrm{d}\zeta\wedge\mathrm{d}X = 2\mathrm{d}(\zeta\mathrm{d}X) - \cancel{2\zeta\wedge\mathrm{d}^2X}$.

This mechanism of anomaly cancellation by a 2-group symmetry occurs in more general situations than the simple case of a periodic free scalar with $U(1)^{(0)}_A \times U(1)^{(0)}_B$ symmetry. Let us generically consider a D-dimensional spacetime having the structure of $M^{(D-d)} \times K$, where $M^{(D-d)}$ is a $(D-d)$-dimensional Minkowsky space and K is some d-dimensional compact Riemannian space. Therefore, the world-sheet string theory is the product of a trivial theory for the $(D-d)$ scalars in the Minkowskian spacetime times a potentially cumbersome quantum field theory describing the compact directions. The theory for the compact direction may have a symmetry group G in analogy with the $U(1)^{(0)}_A \times U(1)^{(0)}_B$ group in the case of the toroidal compactification of one dimension.

Under general assumptions, the global symmetry G for the world-sheet action, should be a gauge symmetry in the $(D-d)$ dimensional Minkowski spacetime $M^{(D-d)}$, in agreement with the conjecture that forbids global symmetry in a quantum gravity theory. However, the group G may have a 't Hooft anomaly on the world-sheet and its gauging may be problematic. The solution is once again in presence of the B-field after its dimensional reduction in the $(D-d)$-spacetime. This field must transform under a non-trivial transformation that allows us to cancel the anomaly of G. In other words, the symmetry group is not a simple 0-form symmetry, but there are also a 1-form symmetry combined in a structure of 2-group.

If this fact can be easily checked in the case where G is a continuos group, the mechanism is more subtle in the case of discrete symmetry. For continuos symmetry the correspondent background fields have degrees of freedom which propagate and are usually described in the spacetime effective action. In fact, the Nicolai-Townsend 2-group transformation for the B_2 field was known for a long time and can be derived from the form of the spacetime action. In superstring an analogous mechanism is known as Green-Schwarz mechanism. In contrast, the background gauge fields for discrete symmetries are not included into the effective action and the 2-group structure is unknown and must be derived in a new way.

6.5 More on 't Hooft anomalies and their cancellation

We have already seen that in a generic D-dimensional QFT on a manifold $M^{(D)}$ with global symmetries G the 't Hooft anomalies can be shown coupling the theory with a background field $\mathcal{B}$ for the global symmetry and performing a gauge transformation for this field, i.e. $\mathcal{B} \to \mathcal{B} + \delta\mathcal{B}$: if the partition function $Z[\mathcal{B}]$ is not gauge invariant even after the attempt of making it gauge invariant by adjusting local counterterms, the global symmetry has a 't Hooft anomaly.

$$Z[\mathcal{B} + \delta\mathcal{B}] = e^{i\int_{M^{(D)}} \alpha(\mathcal{B},\delta\mathcal{B})} Z[\mathcal{B}] \tag{6.44}$$

where $\alpha(\mathcal{B}, \delta\mathcal{B})$ is the anomalous phase depending only on the background field and its transfomation. If we add local D-dimensional counterterms we can modify the anomalous phase, but we cannot cancel it if it is a genuine anomaly.

There is also another powerful way to describe 't Hooft anomalies using a classical field theory in one dimension higher. This description is a mechanism to remove the obstruction of gauging called the anomaly inflow mechanism. The inflow mechanism is based on the extension of the spacetime manifold $M^{(D)}$ of the theory to one higher dimensional manifold $N^{(D+1)}$ with $\partial N^{(D+1)} = M^{(D)}$. We assume that the theory is gauge invariant when N is a closed manifold. In general, if the partition function for the theory in $D + 1$ dimensions is

$$e^{i\int_{N^{(D+1)}} F(\mathcal{B})}, \tag{6.45}$$

where F denotes some functions of the background field, it is gauge invariant by construction up to boundary terms. Therefore, if we take the gauge transformation of the theory in N we can cancel the anomalous phase of the original theory.

$$e^{i\int_{N^{(D+1)}} F(\mathcal{B}+\delta\mathcal{B})} = e^{-i\int_{M^{(D)}} \alpha(\mathcal{B},\delta\mathcal{B})} e^{i\int_{N^{(D+1)}} F(\mathcal{B})}. \tag{6.46}$$

The original theory coupled to a $D + 1$ dimensional gauge invariant bulk theory is invariant under the background gauge transformation.

Notice that, even if the theory is gauge invariant in N, the anomaly is encoded into the different way of taking the extension N. Let us discuss this in the previuos example of a periodic free scalar with $U(1)^{(0)}_A \times U(1)^{(0)}_B$ symmetry.

If we extend the second term of the action 6.19

$$2\pi i \int_{M^{(D)}} B_1 \wedge (\mathrm{d}X + A_1) \tag{6.47}$$

to the extended manifold N, we obtain the gauge invariant term

$$2\pi i \int_{N^{(D+1)}} \mathrm{d}B_1 \wedge (\mathrm{d}X + A_1). \tag{6.48}$$

Even if it is gauge invariant, it depends on the choiche of N. In fact, let us take another N' and compute the difference between the two extensions:

$$2\pi i\int_N \mathrm{d}B_1\wedge(\mathrm{d}X+A_1)-2\pi i\int_{N'}\mathrm{d}B_1\wedge(\mathrm{d}X+A_1)=2\pi i\int_{N_{closed}}\mathrm{d}B_1\wedge(\mathrm{d}X+A_1) \quad (6.49)$$

where $N_{closed}=N\cup\overline{N'}$ is a closed manifold obtained by gluing N and the orientation reversal $\overline{N'}$ of N' along the common boundary M. The Stokes's theorem assures that $\int_{N_{closed}} dB_1\wedge dX=0$; therefore, the anomaly does not depends on the dynamical field X but only on

$$\int_{N_{closed}}\mathrm{d}B_1\wedge A_1=-\int_{N_{closed}}B_1\wedge\mathrm{d}A_1. \quad (6.50)$$

We can also see the anomaly cancellation for the scalar by taking a gauge trasformation in the bulk:

$$\int_{N^{(D+1)}}B_1\wedge\mathrm{d}A_1\to\int_{N^{(D+1)}}(B_1+\mathrm{d}\zeta)\wedge\mathrm{d}A_1=\int_{N^{(D+1)}}B_1\wedge\mathrm{d}A_1-\int_{M^{(D)}}\mathrm{d}\zeta\wedge A_1 \quad (6.51)$$

where the last term is precisely the anomalous one 6.24.

We want also to consider the case where the symmetry group G for the world-sheet theory of the compactified scalar is discrete. Gauging a discrete symmetry does not lead to an action with kinetical term for the gauge fields and so the discussion of the 't Hooft anomaly is more subtle than the continuous case. The anomaly for a discrete symmetry can be described using the formalism of defects (see. [12], 4.1).

Focusing on a 2-dimensional quantum field theory, a discrete symmetry can be described using a network of topological line defects. This network describes the fiber bundle associated to the background field of the global discrete symmetry. To each line we associate an element $g\in G$ that represents the transition functions for the bundle. The anomaly arises when we consider the fusion of three lines: three defects g,h and k can be joined to form the line ghk in two different ways that differ for a phase $\alpha(g,h,k)\in U(1)$. The two configurations are depicted in figure 6.1: on the left h merges first with g and then gh merges to k, in contrast, on the right h glues first with k and suddenly with g. α can in principle be view as a function

$$\alpha: G\times G\times G\to U(1) \quad (6.52)$$

and it is precisely a $U(1)$-valued 3-chain belonging to $C^3(G,U(1))$.

Furthermore, the phase α must satisfy the pentangon identity when we consider the merger of four defects. In particular, we can switch between two different configurations of the fusion of four lines in two different way that leads to the same results:

$$g(h(kl))\to g((hk)l)\to(g(hk))l\to((gh)k)l \quad (6.53)$$

or

$$g(h(kl))\to(gh)(kl)\to((gh)k)l \quad (6.54)$$

Since the two final configuration are identical, we require for consitency that

$$\alpha(h,k,l)+\alpha(g,hk,l)+\alpha(g,h,k)=\alpha(gh,k,l)+\alpha(g,h,kl) \quad (6.55)$$

where we use the additive notation for the $U(1)$ group. This condition is exactly the cocycle condition with the twisted differential (3.32 with $\rho=\mathbb{I}$):

$$(d\alpha)(g,h,k,l)=\alpha(h,k,l)-\alpha(gh,k,l)+\alpha(g,hk,l)-\alpha(g,h,kl)+\alpha(g,h,k)=0. \quad (6.56)$$

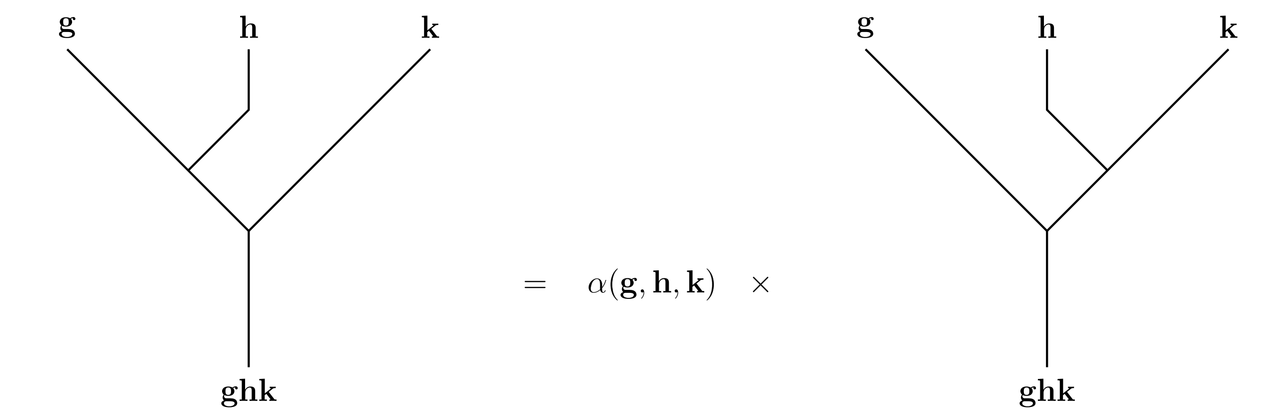

Figure 6.1: The lines are line defects representing the symmetry operators of G. On the left the original configuration of the junction of the three defects g, h, k, on the right the same defect fusion but with h on the other side of the junction. They are equivalent up to a phase $\alpha(\mathbf{g},\mathbf{h},\mathbf{k}) \in H^3(G, U(1))$.

Therefore, α is a $U(1)$-valued 3-cocycle: $\alpha \in Z^3(G, U(1))$. Furthermore, we can try to modify α by adding a 3-coboundary $d\mu \in B^3(G, U(1))$ where $\mu : G \times G \to U(1)$. In the continuos case, this corresponds to the adjoint of local counterterms. If the attempt to delete the anomaly is unsuccessful, we are in presence of a genuine 't Hooft anomaly. As a consequence, the possible 't Hooft anomalies of a theory are described by a class in the third-group cohomology:

$$[\alpha] \in H^3(G, U(1)). \tag{6.57}$$

This is in accordance to the general definition valid also for continuos symmetry since the anomaly is always a phase which depends both on the background gauge field and the gauge transformation. We can see that also for continuos symmetry 't Hooft anomalies in 2 dimensions are labeled by the same cohomology calsses. Initially, we note that $H^3(G, U(1)) \simeq H^4(G, \mathbb{Z})$ for finite groups. Furthermore, we assume that we can describe the anomaly of a D dimensional theory as boundary term in the gauge variation of the Chern-Simons action for G in $D + 1$ dimensions [10]. This is precisely the fact that we can assume that 't Hooft anomaly can be canceled by anomaly inflow from one dimension higher. We know from the work of Dijkgraaf and Witten [25] that Chern-Simons theories in $D + 1$ dimensions are classified by elements of $H^{D+2}(BG, \mathbb{Z})$ and this means that the 't Hooft anomalies for compact connected semi-simple Lie groups take values in $H^{D+2}(BG, \mathbb{Z})$. Therefore, in a 2-dimensional theory the 't Hooft anomalies are labeled by classes in $H^4(BG, \mathbb{Z})$, like the discrete case.

The fact that the possible 't Hooft anomalies for discrete symmetries are labeled by classes in the third cohomology group is crucial to their cancellation. In fact, the Postnikov class $[\beta]$ that characterizes a 2-group symmetry is precisely a class in the same group cohomology. We can notice that this is possible since we are considering anomalies in two dimensions. In a generic D-dimensional space, the 't Hooft anomaly is given by an element $\omega \in H^{D+1}(G, U(1))$, so there is not a direct correspondence with the Postnikov classes of the 2-groups. In this case, one can hope that a similar mechanism exists by considering more complicate n-groups.

The cancellation of the anomaly is possible if we require the presence of a non-trivial 2-group symmetry that combines the 0-form symmetry with a higher form one. In string theory, the $U(1)^{(1)}$ 1-form symmetry of the field B_2 is the higher form symmetry that constitutes a 2-group structure with G. Therefore, the Postnikov class $[\beta]$ of this 2-group can exactly cancel the 't Hooft anomaly $[\alpha]$. In other words, the non-trivial transfomation for B_2 shows that the spacetime gauge group is not simply the product of a 0-form symmetry G with a $U(1)$ 1-form symmetry, but these symmetry are mixed in a 2-group structure. Following 4.2, the 2-group that cancels the anomaly can be denoted by

$$\mathcal{G} = (G, U(1), \mathbb{I}, [\beta]) \tag{6.58}$$

with $[\beta] = [\alpha] \in H^3(G, U(1))$.

We can view the anomaly cancellation by assuming that the gauge invariant field strength for B_2 is

$$H_3 = dB_2 + A^*\alpha \tag{6.59}$$

where A^* is the pull-back of a class in $H^3(BG, U(1))$ to one in $H^3(M, U(1))$; indeed, as presented in 3.4, a G-bundle with flat connection on a space M can always be defined by a map $A : M \to BG$. If the world-sheet is a closed manifold M and the 3-dimensional manifold N is the extension of M such that $\partial N = M$, the B_2 term in the action gives us

$$\int_M B_2 = \int_N dB_2 = \int_N H_3 - \int_N A^*\alpha. \tag{6.60}$$

$\int_N H_3$ is gauge invariant and the gauge variation of the last term exactly cancels the 't Hooft anomaly α by inflow mechanism.

As a first example of the presence of this anomaly cancellation for discrete symmetries, we can consider again the free periodic boson which in addition to the continuos $U(1)^{(0)}_A \times U(1)^{(0)}_B$ is also symmetric under the inversion of the periodic coordinate.

6.6 't Hooft anomalies in coordinate inversion

A scalar field compactified on a circle $\mathbb{S}^1$ has a larger symmetry group than the one considered so far. We can always invert the coordinate on the circle, i.e. $X \to -X$, acting with a $\mathbb{Z}_2$ group symmetry. The coordinate inversion acts by charge conjugation on all the $U(1)$ factors. The 0-form global symmetry group of the world-sheet theory is the semidirect product:

$$G = \mathbb{Z}_2^C \ltimes (U(1)^{(0)}_A \times U(1)^{(0)}_B). \tag{6.61}$$

The normal subgroup $U(1)^{(0)}_A \times U(1)^{(0)}_B$ is exactly the same considered before: the first $U(1)^{(0)}_A$ factor represents the invariance under translation along the circle with the internal momentum as conserved charge and the second $U(1)^{(0)}_B$ is the translations in the dual circle with winding number as charge. As before, A_1 and B_1 are the 1-form gauge fields and there is a mixed 't Hooft anomaly that forbids the gauging of both these fields. $\mathbb{Z}_2^C$ is the subgroup of the coordinate inversion and we denote by $\mathcal{C}$ its generator that it is not uniquely defined. In fact, we can take $\mathcal{C}$ as an element in G with non-trivial image under the projection $G \to \mathbb{Z}_2^C$ and we identify $\mathbb{Z}_2^C$ with the subgroup of G generated by $\mathcal{C}$. Since, as we will see, all such elements are conjugate to each other in G, the freedom in the choice of $\mathcal{C}$ is perfectly licit. $\mathcal{C}$ acts non-trivially on the gauge fields and on the field strengths:

$$\begin{aligned} \mathcal{C}A_1 = -A_1 \quad &, \quad \mathcal{C}F_2 = \mathcal{C}\mathrm{d}A_1 = -F_2 \\ \mathcal{C}B_1 = -B_1 \quad &, \quad \mathcal{C}H_2 = \mathcal{C}\mathrm{d}B_1 = -H_2; \end{aligned} \tag{6.62}$$

therefore, we can easily see that G is non-abelian.

We want to study the presence of mixed 't Hooft anomaly between the $\mathbb{Z}_2^C$ subgroup and the normal subgroup $U(1)^{(0)}_A \times U(1)^{(0)}_B$. In order to do this we can consider the abelian subgroup

$$\tilde{G} = \mathbb{Z}_2^C \times \mathbb{Z}_2^A \times \mathbb{Z}_2^B \tag{6.63}$$

where the first is the charge conjugation and the others $\mathbb{Z}_2$ are finite subgroups of order 2 of the two $U(1)$ groups. We can visualize these two subgroups as the choice of one point and its opposite in the circle $\mathbb{S}^1 \cong U(1)$ or, in other words, the elements of these subgroups are the phases e^0 and $e^{i\pi}$. These subgroups commute with the charge conjugation.

The restriction to finite groups allows us to use the tools developed in the previous chapter (section 3.5), since we have an explicit expression for the cocycle in $H^3(\tilde{G}, U(1))$. We denote by α the anomalous phase, which is just a representative of a class in the third cohomology group, $H^3(\tilde{G}, U(1))$. We know (see eq. 3.35) that

$$H^3(\mathbb{Z}_2^C \times \mathbb{Z}_2^A \times \mathbb{Z}_2^B, U(1)) \simeq \mathbb{Z}_2^7 \tag{6.64}$$

and we can choose the generators of $\mathbb{Z}_2^7$ to be divided into three different types, corresponding to the generators in eqs. 3.38 3.39 3.40. The classes of type I are non-trivial when restricted to a $\mathbb{Z}_2^A$, $\mathbb{Z}_2^B$ or $\mathbb{Z}_2^C$ subgroups. On the other hand, the cocycles of type II are trivial if restricted to a single $\mathbb{Z}_2^A$, $\mathbb{Z}_2^B$ or $\mathbb{Z}_2^C$ subgroups but non trivial when restricted to a product of two subgroups: $\mathbb{Z}_2^A \times \mathbb{Z}_2^B$, $\mathbb{Z}_2^A \times \mathbb{Z}_2^C$ or $\mathbb{Z}_2^B \times \mathbb{Z}_2^C$. Finally, the class of type III becomes trivial if restricted to any of these subgroups. A generic class in $H^3(\mathbb{Z}_2^C \times \mathbb{Z}_2^A \times \mathbb{Z}_2^B, U(1))$ can be written by summing the different contributions coming from these seven components which we denote by

$$\alpha_I^A\,, \quad \alpha_I^B\,, \quad \alpha_I^C\,, \quad \alpha_{II}^{AB}\,, \quad \alpha_{II}^{AC}\,, \quad \alpha_{II}^{BC}\,, \quad \alpha_{III}^{ABC} \tag{6.65}$$

where the subscript represents the type and the superscript the groups where the generator has non-trivial restriction. Therefore, we have seven possible sources of anomaly for $\tilde{G}$ and we have to study, in our particular QFT, which contributions are trivial and which are significant.

From the study of the $U(1)_A^{(0)} \times U(1)_B^{(0)}$ possible anomalies we know that the single $U(1)$ is not anomalous, while there is a mixed 't Hooft anomaly between the two $U(1)$ subgroups. Since $\mathbb{Z}_2^A \times \mathbb{Z}_2^B$ is a subgroup of order two of $U(1)_A^{(0)} \times U(1)_B^{(0)}$, we can easily conclude that α_I^A and α_I^B are trivial, while we have a non-vanishing contribution to the $\tilde{G}$ anomaly from α_{II}^{AB}.

A crucial observation is that the anomaly restricted to any $\mathbb{Z}_2$ subgroup of G generated by the multiplication of $\mathcal{C}$ by any element of $\mathbb{Z}_2^A \times \mathbb{Z}_2^B$, is the same of the subgroup generated exclusively by $\mathcal{C}$. In other words, denoting with g an element of $\mathbb{Z}_2^A$ and with h an element of $\mathbb{Z}_2^B$, we have that

$$\alpha(\mathcal{C}, \mathcal{C}, \mathcal{C}) = \alpha(\mathcal{C}gh, \mathcal{C}gh, \mathcal{C}gh). \tag{6.66}$$

This is due to the fact that in the original group G all such $\mathbb{Z}_2$ subgroups are conjugated to each other; therefore they have the same anomalies. We can show this by considering the most general case, i.e. $\mathcal{C}$ is conjugated to $\mathcal{C}gh$. Let us set $g = e^{i\mu Q_A}$ and $h = e^{i\nu Q_B}$ with $0 \leqslant \mu, \nu \leqslant 2\pi$ and Q_A, Q_B respectively the charge of the $U(1)_A^{(0)}, U(1)_B^{(0)}$ subgroup. Since $\mathcal{C}$ acts by charge conjugation on the $U(1)$ charges, i.e. it changes their sign, the following equality holds:

$$\mathcal{C}e^{i\mu Q_A + i\nu Q_B} = e^{-i\frac{\mu}{2}Q_A - i\frac{\nu}{2}Q_B}\,\mathcal{C}\,e^{i\frac{\mu}{2}Q_A i\frac{\nu}{2}Q_B}. \tag{6.67}$$

This is precisely the definition of the conjugation between the two elements $\mathcal{C}e^{i\mu Q_A + i\nu Q_B}$ and $\mathcal{C}$, i.e. there exists $k \in G$ such that $\mathcal{C}e^{i\mu Q_A + i\nu Q_B} = k^{-1}\mathcal{C}k$.

Returning to the anomalies, we can consider the simple case $\alpha(\mathcal{C}, \mathcal{C}, \mathcal{C}) = \alpha(\mathcal{C}g, \mathcal{C}g, \mathcal{C}g)$ with $g \in \mathbb{Z}_2^A$. The possible contributions are only of the first type for the left-hand-side and of the type I and type II for the right-hand-side; we have

$$\alpha_I^C = \alpha_I^C + \alpha_I^A + \alpha_{II}^{AC} \tag{6.68}$$

where we used the additive notation. Since $\alpha_I^A = 0$ we can argue that also α_{II}^{AC} must be trivial; therefore, there is no mixed 't Hooft anomaly between $\mathbb{Z}_2^A$ and $\mathbb{Z}_2^C$. With a similar argument we can prove that also α_{II}^{BC} must vanish and there is no mixed 't Hooft anomaly between $\mathbb{Z}_2^B$ and $\mathbb{Z}_2^C$.

With the more general equality $\alpha(\mathcal{C},\mathcal{C},\mathcal{C}) = \alpha(\mathcal{C}gh, \mathcal{C}gh, \mathcal{C}gh)$ we can deduce that

$$\alpha_I^C = \alpha_I^C + \alpha_{II}^{AB} + \alpha_{III}^{ABC} \tag{6.69}$$

where we have already omitted the vanishing terms. Since α_{II}^{AB} is non-trivial, we find that also α_{III}^{ABC} must be non-trivial. We have proved that there is a mixed anomaly between $\mathbb{Z}_2^C$ and $\mathbb{Z}_2^A \times \mathbb{Z}_2^B$. We can figure the presence of the anomaly through the formalism of defects. Let $g \in \mathbb{Z}_2^A$, $h \in \mathbb{Z}_2^B$ and $\mathcal{C}$ be three defects and let us consider their triple junction. We can observe the presence of the anomaly in the arising of a non trivial phase α, which in practice is a minus sign if it is non-trivial. In fact, the explicit expression for the 3-cocycle that represents the mixed anomaly is:

$$\alpha_{III}^{ABC}(g,h,\mathcal{C}) = \exp\left(\frac{2\pi i p_{III}^{ABC}}{2} ghc\right) \tag{6.70}$$

where $c = 1$ is the non-trivial elements in $\mathbb{Z}_2$ corresponding to the generator. If $g = h = 1$, then $\alpha_{III}^{ABC}(1,1,1) = -1$.

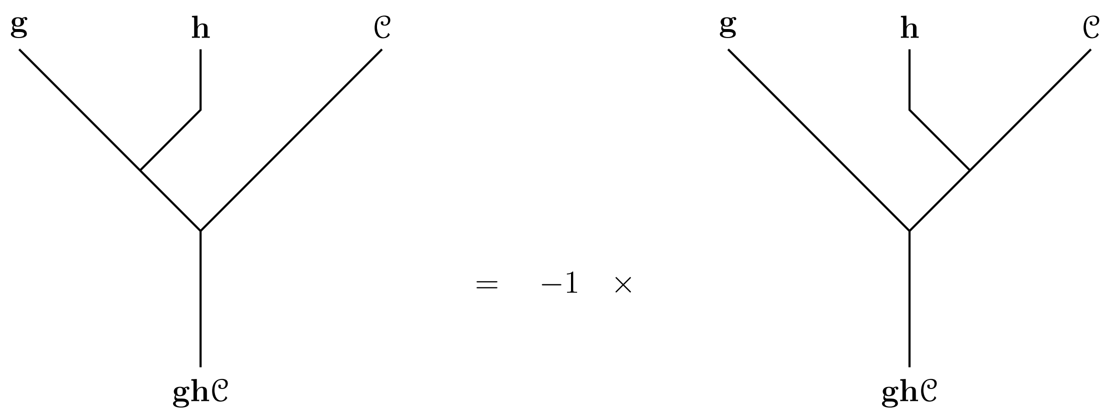

Figure 6.2: Moving the defect h through the triple junction causes the onset of an anomlous phase $\alpha_{III}^{ABC}(g,h,\mathcal{C}) = -1$ if $g = h = 1$. The phase is precisely a minus sign.

Notice that, in order to identify completely the cocycle for the anomaly of $\tilde{G}$, we should consider also the anomaly for the $\mathbb{Z}_2^C$ subgroup α_I^C. Using conformal field theory techniques one can prove that α_I^C is trivial. We will not need this fact in the following.

Finally, we can observe that a mixed 't Hooft anomaly must be present also between $\mathbb{Z}_2^C$ and $U(1)_A^{(0)} \times U(1)_B^{(0)}$. Since this anomaly is visible even restricting to the abelian subgroup $\mathbb{Z}_2^C \times \mathbb{Z}_2^A \times \mathbb{Z}_2^B$, we can conclude that it is present also in the original group G.

6.7 't Hooft anomaly cancellation in $\mathbb{Z}_2^C \ltimes (U(1)_A^{(0)} \times U(1)_B^{(0)})$

The 't Hooft anomaly in the world-sheet theory costitutes a problem when we take into account the theory in spacetime: the global symmetry of the world-sheet theory must become gauge symmetry in the spacetime theory. As we have already see for the case of $U(1)_A^{(0)} \times U(1)_B^{(0)}$, the solution of this problem is given by the presence of the B_2 field and by the non-trivial 2-group symmetry in which it is involved. For the mixed $\mathbb{Z}_2^C$ anomaly the cancellation is more subtle, since we are in presence of discrete groups.

In order to study the effect of the charge conjugation on B_2, we firstly focus on the restriction to the subgroup $\tilde{G} = \mathbb{Z}_2^C \times \mathbb{Z}_2^A \times \mathbb{Z}_2^B$. This subgroup is abelian and it is the centralizer $Z_G(\mathcal{C})$ of $\mathbb{Z}_2^C$ in G, that is all its elements commute with $\mathcal{C}$: $\mathcal{C}g = g\mathcal{C}$ $\forall g \in \tilde{G}$. The restriction of the gauge fields to $Z_G(\mathcal{C})$ is equivalent to the restriction of the path integral to gauge bundles that commute with $\mathcal{C}$. Let us prove this claim.

To have $\mathcal{C}$-invariant field strengths we have to impose

$$F_2 = -F_2\,, \quad H_2 = -H_2 \qquad \text{with} \quad F_2 = \mathrm{d}A_1\,, \quad H_2 = \mathrm{d}B_1 \tag{6.71}$$

and this implies that the $U(1)_A^{(0)} \times U(1)_B^{(0)}$ connections must be flat, as we expected for discrete group:

$$F_2 = 0\,, \quad H_2 = 0. \tag{6.72}$$

Since $\mathcal{C}$ inverts the sign of the gauge 1-form, we have to impose also:

$$A_1 \sim -A_1\,, \quad B_1 \sim -B_1 \tag{6.73}$$

where $\sim$ denotes equality up to $U(1)_A^{(0)} \times U(1)_B^{(0)}$ gauge tranformations and this implies

$$2A_1 \sim 0\,, \quad 2B_1 \sim 0 \tag{6.74}$$

i.e. $2A_1$ and $2B_1$ must be pure gauge. Since we are considering flat $U(1)$ connections, the condition on the pure gauge is

$$\frac{1}{2\pi}\oint_\gamma 2A \in \mathbb{Z}\,, \quad \frac{1}{2\pi}\oint_\gamma 2B_1 \in \mathbb{Z} \tag{6.75}$$

with γ is a 1-cycle. The consequence of these requirements is not that the gauge fields are trivial, since they can have half integer holonomies; therefore, since $e^{i\oint_\gamma A_1}$ and $e^{i\oint_\gamma B_1}$ must be signs, we can conclude that the gauge fields takes values on the subgroup $\mathbb{Z}_2^A \times \mathbb{Z}_2^B$.

Since we want to study the effects of $\mathcal{C}$ on the 2-group gauge fields, we have to consider also the transformation for the field B_2. In this case we refer to transformation 6.7 that is slightly different from the one considered before 6.10. The two transfomations are related by a field redefinition: $B_2 \to B_2 - \alpha B_1 \wedge A_1$, $\alpha \in \mathbb{R}$, that is always possibile if we insert $\alpha A_1 \wedge B_1$ in the world-sheet action. In the redefinition before, we have setted $\alpha = 1$. Therefore, in this case the gauge transformations are:

$$A_1 \to A_1 + \mathrm{d}\lambda\,, \quad B_1 \to B_1 + \mathrm{d}\zeta\,, \quad B_2 \to B_2 + \mathrm{d}\zeta_1 + B_1 \wedge \mathrm{d}\lambda \tag{6.76}$$

where λ and ζ are $U(1)$ gauge 0-forms, i.e. they have values in $\mathbb{R}/\mathbb{Z}$, and ζ_1 is a 1-form defined modulo $U(1)$ gauge transformations. Notice that no transformation for the field B_2 can cancel the dependence on the B_1 field and gauge parameters λ and ζ.

The generators of the charge conjugation $\mathcal{C}$ acts on these fields as

$$A_1 \to -A_1\,, \quad B_1 \to -B_1\,, \quad B_2 \to B_2. \tag{6.77}$$

The field B_2 remains untouched by the coordinate inversion on $\mathbb{S}^1$ because it corresponds to the non compact components of the Kalb-Ramond field, i.e. the tensor $B_{\mu\nu}$ whose indices are not along the circle.

In order to restrict to the $\mathbb{Z}_2$ subgroup of the two $U(1)$ we have to impose the conditions 6.73 on the gauge fields. In particular, grouping the fields into a triplet, we set

$$(A_1, B_1, B_2) \sim (-A_1, -B_1, B_2) \tag{6.78}$$

where $\sim$ denotes equality up to 6.76 gauge tranformations. At this point we can exploit the fact that $2A_1$ and $2B_1$ must be pure gauge. Using a simple trick we can write

$$(A_1, B_1, B_2) \sim (-A_1, -B_1, B_2) = (A_1 - 2A_1, B_1 - 2B_1, B_2) = (A_1 + \mathrm{d}\lambda, B_1 + \mathrm{d}\zeta, B_2) \quad (6.79)$$

where we $-2A_1 = \mathrm{d}\lambda$ and $-2B_1 = \mathrm{d}\zeta$ for some gauge $U(1)$ 0-forms λ and ζ. Then, we can make a gauge transformation 6.76 in order to restore A_1 and B_1 in the triplet:

$$(A_1+\mathrm{d}\lambda, B_1+\mathrm{d}\zeta, B_2) \sim (A_1+\mathrm{d}\lambda-\mathrm{d}\lambda, B_1+\mathrm{d}\zeta-\mathrm{d}\zeta, B_2-B_1\wedge\mathrm{d}\lambda) = (A_1, B_1, B_2-2A_1\wedge B_1) \quad (6.80)$$

where we set $\zeta_1 = 0$.

We can wonder if a gauge transformation with parameter ζ_1 can readsorbe the extra term:

$$-2A_1 \wedge B_1 = -\frac{1}{2}\mathrm{d}\lambda \wedge \mathrm{d}\zeta. \quad (6.81)$$

In order to cancel this term we have to choose ζ_1 with the correct quantization condition and such that $\mathrm{d}\zeta_1 = \frac{1}{2}\mathrm{d}\lambda \wedge \mathrm{d}\zeta$. However, this is not possible since the integral over a 2-cocycle γ_2 $\oint_{\gamma_2} \frac{1}{2}\mathrm{d}\lambda \wedge \mathrm{d}\zeta$ has values in $\frac{1}{2}\mathbb{Z}$ instead of $\mathbb{Z}$. Also the field redefinition $B_2 \to B_2 - \alpha B_1 \wedge A_1$ does not work since $B_1 \wedge A_1$ is inviariant under the gauge transformations with parameters $-2A_1 = \mathrm{d}\lambda$ and $-2B_1 = \mathrm{d}\zeta$.

Let us explain what we have found out so far. Apparently, the conditions 6.73, which allow us to restrict to the subgroup $\mathbb{Z}_2^A \times \mathbb{Z}_2^B$, seem to be sufficient in order that the background fields are $\mathcal{C}$-invariant; in reality we have to impose 6.78 to taking into account also the field B_2. The condition 6.78 is stronger than the condition 6.73. In fact, if we impose only 6.73 we obtain that the right-hand side of 6.78 is gauge equivalent to the last term of 6.80. However, this is the same as the left-hand side of 6.78 only if, in addition to the condition that $2A_1$ and $2B_1$ are pure gauge, one also requires that the integral of $2A_1 \wedge B_1$ over any closed 2-manifold is even.

In general, eq.6.80 tells us that if A_1 and B_1 are restricted to $\mathbb{Z}_2^A \times \mathbb{Z}_2^B$, we can describe the action of $\mathcal{C}$ on the triplet of fields as

$$(A_1, B_1, B_2) \to (A_1, B_1, B_2 - 2A_1 \wedge B_1) \quad (6.82)$$

because it is equivalent to 6.77.

Since there is no way to cancel the extra term $-2A_1 \wedge B_1$ by a $\mathrm{d}\zeta_1$ term, we conclude that, when we restrict to the subgroup $\mathbb{Z}_2^C \times \mathbb{Z}_2^A \times \mathbb{Z}_2^B$, the transformation $\mathcal{C}$ acts on B_2 by a shift

$$B_2 \to B_2 - 2A_1 \wedge B_1 \quad (6.83)$$

which can be non-zero in the presence of a non-trivial background for A_1 and B_1.

This transformation of B_2 under $\mathcal{C}$ gives precisely the term to cancel the mixed 't Hooft anomaly considered before. Taking into account again the defects configuration in figure 6.2, we can consider the defects as $(d-1)-$dimensional manifolds in spacetime. Each pairs of defects cross in a $(d-2)$-dimensional intersection and the triple junction point represents a $(d-3)$-dimensional triple intersection. We know that when B_2 crosses $\mathcal{C}$ it jumps by $2A_1 \wedge B_1$ which represents an half integer flux localized at the intersection of a g- and h-defect. In fact, if we consider a small sphere $\mathbb{S}^2$ surrounding a triple intersection between a $\mathcal{C}$, a $\mathbb{Z}_2^A$ and a $\mathbb{Z}_2^B$ defect, we have that

$$e^{i2\pi \int_{\mathbb{S}^2} B_2} = -1 \quad (6.84)$$

at least in the limit where $\mathbb{S}^2$ is small. In fact, the defect $\mathcal{C}$ divides $\mathbb{S}^2$ into two hemispheres and the difference of B_2 from the two sides of $\mathcal{C}$ is $2A_1 \wedge B_1$. If the value of B_2 is about constant in the small sphere, the flux $\int_{\mathbb{S}^2} B_2$ reduces to the integral of $2A_1 \wedge B_1$ over one of the two hemispheres and we know from the previous discussion that this integral has values in $\frac{1}{2}\mathbb{Z}$. Therefore, the transformation of B_2 under coordinate inversion provides exactly the minus sign that cancels the mixed 't Hooft anomaly that are present when we consider the F-moving between these three defects. The mixed anomaly, that is problematic from the point of view of coupling the global symmetry of the world-sheet to dynamical fields in spacetime, does not constitute a problem since it is canceled by the non-trivial 2-group symmetry that characterized any string compactification. However, if we consider the orbifold of the world-sheet theory, the presence of a mixed anomaly is still relevant and suggests that the ordinary orbifold is not consistent, but we have to make an operation that involves the 1-form symmetry of B_2 in a non trivial way [18].

Chapter 7

’t Hooft anomalies cancellation in toroidal compactification with $\mathbb{Z}_3$ symmetry

A more complicated case is the compactification of two dimensions on a torus with a particular discrete rotational symmetry. More generally in (super)string theory one can consider the compactification of six dimensions in $\mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$ with the same geometry (Z-model) or one can consider the orbifold by this discrete symmetry [32]. Before presenting this particular torus, we briefly introduce the ideas of compactification of several dimensions.

We consider the periodic identification of k dimensions in the bosonic string:

$$X^m \sim X^m + 2\pi R\,, \quad 26-k \leq m \leq 25. \tag{7.1}$$

If $d = 26-k$ is the number of noncompact dimensions, the spacetime is $M^d \times \mathbb{T}^k$. Let us consider the spacetime effective action 5.48 introduced before. Following the same procedure as in 5.6.1, we can obtain the low dimensional action by integrating over the compact dimensions. Notice that with more than one compact dimension, the antisymmetric tensor B_{MN} also has scalar components B_{mn} and the total number of scalars from the Kalb-Ramond field and the metric is k^2. There are also k Kaluza-Klein gauge bosons A^m_μ with field strength $F^m_{\mu\nu}$ and k antisymmetric tensor gauge bosons B^m_μ with field strength $H^m_{\mu\nu}$. The low energy action is:

$$\begin{aligned}\boldsymbol{S} = \frac{(2\pi R)^k}{2k_0^2}\int d^dx\sqrt{-G_d}e^{-2\Phi_d}\Big\{&\boldsymbol{R}_d + 4\partial_\mu\Phi_d\partial^\mu\Phi_d - \frac{1}{4}G^{mn}G^{pq}(\partial_\mu G_{mp}\partial^\mu G_{nq} + \partial_\mu B_{mp}\partial^\mu B_{nq})\\ &- \frac{1}{4}G_{mn}F^m_{\mu\nu}F^{n\mu\nu} - \frac{1}{4}G^{mn}H_{m\mu\nu}H^{\mu\nu}_n - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\Big\}\end{aligned} \tag{7.2}$$

where $\Phi_d = \Phi - \frac{1}{4}\ln\det G_{mn}$ and the latin indices run in the compact directions $26-k \leq m \leq 25$, while the greek indices in the $26-k$ non compact.

7.1 Compactification on a torus with $\mathbb{Z}_3$ symmetry

We want to consider the compactification of two dimensions on a torus $\mathbb{T}^2$ with a particular geometry. The two compact scalars on which we focus on are

$$X_1 \sim X_1 + 2\pi R_1\,, \quad X_2 \sim X_2 + 2\pi R_2. \tag{7.3}$$

The target space is $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ with a special geometry that allows a discrete rotational symmetry. In order to describe it we consider the torus as a complex manifold. Let us define

$$Z = X_1 + iX_2 \qquad Z \in \mathbb{C} \simeq \mathbb{R}^2 \tag{7.4}$$

and assume the euclidean metric in $\mathbb{C}$. We set to zero the value of the single scalar that we get from the compactification of the Kalb-Ramond field B_{MN} and we consider the identifications

$$Z \sim Z + 2\pi R \qquad Z \sim Z + 2\pi R e^{i\frac{\pi}{3}}. \tag{7.5}$$

With this identification the torus can be identified with a fundamental region in the complex plane, a parallelogram with the opposite sides identified (see Figure 7.1). The sides are equally long, thus we have $R_1 = R_2 = R$.

There are two types of symmetries that one can consider in such a structure. If we rotate the complex manifold by $\frac{2}{3}\pi$ we obtain an equivalent torus; the same happens if we consider the reflection with respect the origin, i.e. the coordinates inversion. If we combine these two discrete symmetries we obtain a group generated by a rotation by $\frac{\pi}{3}$. Let us study in detail these symmetries and explain why the transformed torus is equivalent to the original one.

Let us consider a lattice in the complex plane $\mathbb{C}$

$$\Lambda(\omega_1, \omega_2) = \{\omega_1 n_1 + \omega_2 n_2 \,|\, n_1, n_2 \in \mathbb{Z}\} \subset \mathbb{C} \tag{7.6}$$

where ω_1 and ω_2 are non vanishing complex numbers such that $\omega_2/\omega_1 \notin \mathbb{R}$. The manifold $\mathbb{C}/\Lambda(\omega_1, \omega_2)$ is obtained by identifying the points $z_1, z_2 \in \mathbb{C}$ such that $z_1 - z_2 = \omega_1 n_1 + \omega_2 n_2$ for some $n_1, n_2 \in \mathbb{Z}$ and the metric on $\mathbb{T}^2$ is then induced by the Euclidean metric on $\mathbb{C}$. We know that a lattice $\Lambda(\omega_1', \omega_2')$ define the same $\mathbb{T}^2$ if and only if there exists a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{Z}) = \frac{SL(2,\mathbb{Z})}{\mathbb{Z}_2} \tag{7.7}$$

such that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \tag{7.8}$$

This happens because (ω_1, ω_2) and (ω_1', ω_2') are different generators of the same lattice $\Lambda(\omega_1, \omega_2) = \Lambda(\omega_1', \omega_2')$, thus also the torus is the same.

The torus that we are considering is described by the lattice $\Lambda(1, e^{i\frac{\pi}{3}})$, where we have setted $2\pi R = 1$ for simplicity. It corresponds to the identifications of $z = a + ib$

$$\begin{aligned} z \sim z + n_1 \quad &\Rightarrow \quad a \sim a + n_1\,, \quad b \sim b \\ z \sim z + n_2 e^{i\frac{\pi}{3}} \quad &\Rightarrow \quad a \sim a + \frac{n_2}{2}\,, \quad b \sim b + \frac{\sqrt{3}}{2} n_2. \end{aligned} \tag{7.9}$$

A rotation of $\frac{2}{3}\pi$ corresponds to a transfomation

$$z \to z e^{i\frac{2}{3}\pi} \quad \Rightarrow \quad a \to -\frac{a}{2} - \frac{\sqrt{3}}{2} b\,, \quad b \to \frac{\sqrt{3}}{2} a - \frac{b}{2}. \tag{7.10}$$

The lattice of the torus after this rotation is described by $\omega_1' = e^{i\frac{2}{3}\pi}$ and $\omega_2' = -1$. The tori defined by these two different lattices are equivalent, in fact we can find a $SL(2,\mathbb{Z})$ matrix that relates the two lattices. We search for a matrix such that

$$\begin{pmatrix} -\frac{1}{2} + \frac{\sqrt{3}}{2} i \\ -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{2} + \frac{\sqrt{3}}{2} i \end{pmatrix} = \begin{pmatrix} a + \frac{b}{2} + \frac{\sqrt{3}}{2} ib \\ c + \frac{d}{2} + \frac{\sqrt{3}}{2} id \end{pmatrix} \tag{7.11}$$

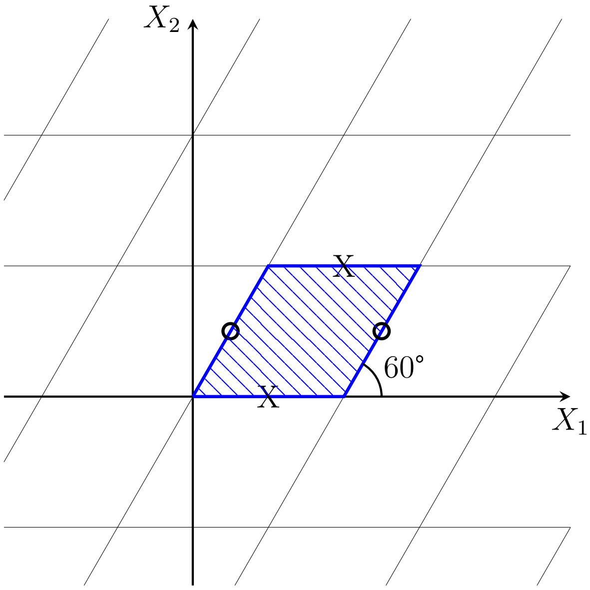

Figure 7.1: The lattice $\Lambda(1, e^{i\frac{\pi}{3}})$ on the complex plane. $\mathbb{C}/\Lambda(1, e^{i\frac{\pi}{3}})$ is homeomorphic to the torus $\mathbb{T}^2$ which can be described by the blue rhombus with the opposite sides identified.

and we obtain

$$b = 1 \Rightarrow a + \frac{1}{2} = -\frac{1}{2} \rightarrow a = -1\,, \quad d = 0\,, \quad c = -1. \tag{7.12}$$

We denote with g^{-1} such a matrix for reasons that will become clear afterwards,

$$g^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \tag{7.13}$$

and we can easily verify that $\det g^{-1} = 1$ and $(g^{-1})^3 = \mathbb{I}$. Therefore, if we repeat three times this rotations we return to the original configuration, thus this is a $\mathbb{Z}_3$ symmetry. Notice also that

$$(g^{-1})^2 = g = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \tag{7.14}$$

and $\{\mathbb{I}, g^{-1}, g\}$ are the elements of a $SL(2,\mathbb{Z})$ representation of $\mathbb{Z}_3$. Also the integer n_1, n_2 transform with the matrix g^{-1}

$$\begin{cases} n_1' = -n_1 + n_2 \\ n_2' = -n_1 \end{cases} \tag{7.15}$$

and this result can be depicted in the Figure 7.2. This transformation will be particularly important in the following since n_i represents the winding number of the string along the i-circle and via this transformation we will deduce the transformation for the background fields.

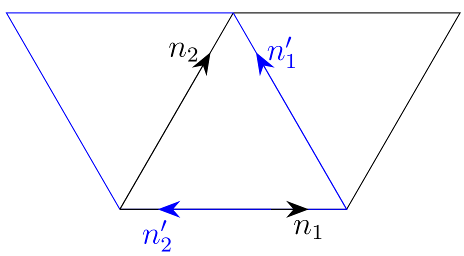

Figure 7.2: In black the original fundamental region describing the torus with the winding numbers n_1, n_2 in the two directions. In blue the rotated lattice with the winding numbers n_1', n_2'.

We can also find the fixed points of the transformation $z \to z e^{i\frac{2}{3}\pi}$ by imposing $z \sim z e^{i\frac{2}{3}\pi}$ which corresponds to

$$a + n_1 + \frac{n_2}{2} + i\Big(b + \frac{\sqrt{3}}{2} n_2\Big) = -\frac{a}{2} - \frac{\sqrt{3}}{2} b + i\Big(\frac{\sqrt{3}}{2} a - \frac{b}{2}\Big). \tag{7.16}$$

We have to impose that

$$\begin{cases} a + n_1 + \frac{n_2}{2} = -\frac{a}{2} - \frac{\sqrt{3}}{2} b \\ b + \frac{\sqrt{3}}{2} n_2 = \frac{\sqrt{3}}{2} a - \frac{b}{2} \end{cases} \quad \Rightarrow \quad \begin{cases} a = -\frac{n_1}{2} \\ b = -\frac{n_1}{2\sqrt{3}} - \frac{n_2}{\sqrt{3}} \end{cases} \tag{7.17}$$

and we obtain three fixed points

$$\begin{aligned} n_1 = 0 \quad n_2 = 0 \quad &\Rightarrow \quad \tilde{z}_1 = 0 \\ n_1 = -1 \quad n_2 = 0 \quad &\Rightarrow \quad \tilde{z}_2 = \frac{1}{2} + \frac{1}{2\sqrt{3}} i = \frac{1}{\sqrt{3}} e^{i\frac{\pi}{6}} \\ n_1 = -2 \quad n_2 = 0 \quad &\Rightarrow \quad \tilde{z}_3 = 1 + \frac{1}{\sqrt{3}} i = \frac{2}{\sqrt{3}} e^{i\frac{\pi}{6}}. \end{aligned} \tag{7.18}$$

The torus that we are considering is also symmetric under coordinates inversion, since if we map $z \to -z$ we obtain a metric that is equivalent to the previous one. Indeed, we have

$$\begin{pmatrix} -1 \\ -e^{i\frac{\pi}{3}} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ e^{i\frac{\pi}{3}} \end{pmatrix}. \tag{7.19}$$

We denote

$$g_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{7.20}$$

and we note that it is of order two, thus tori's coordinates inversion corresponds to a $\mathbb{Z}_2$ symmetry. The fixed points can be found by solving

$$a + n_1 + \frac{n_2}{2} + i\Big(b + \frac{\sqrt{3}}{2} n_2\Big) = -a - ib \tag{7.21}$$

and they are

$$\tilde{z}'_1 = 0\,, \quad \tilde{z}'_2 = \frac{1}{2}\,, \quad \tilde{z}'_3 = \frac{1}{2} e^{i\frac{\pi}{3}}\,, \quad \tilde{z}'_4 = \frac{\sqrt{3}}{2} e^{i\frac{\pi}{6}}. \tag{7.22}$$

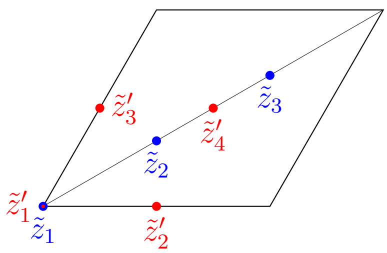

Figure 7.3: In blue the fixed points of the transformation $z \to z e^{i\frac{2}{3}\pi}$, in red the ones under $z \to -z$.

Finally, we can consider the combination of the $\mathbb{Z}_3$ symmetry with the $\mathbb{Z}_2$ to obtain a $\mathbb{Z}_6$ symmetry that represents the rotation in the complex plane by $\frac{\pi}{3}$. In Figure 7.4 we depict the different lattices that are related by the matrix g_6 that we obtain by considering

$$\begin{pmatrix} \frac{1}{2} + \frac{\sqrt{3}}{2} i \\ -\frac{1}{2} + \frac{\sqrt{3}}{2} i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{2} + \frac{\sqrt{3}}{2} i \end{pmatrix}. \tag{7.23}$$

All these lattice transformations are generated by the matrix g_6 for the $\mathbb{Z}_6$ symmetry:

$$\begin{aligned} g_6 &= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} & (g_6)^2 &= \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = g^{-1} & (g_6)^3 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = g_2 \\ (g_6)^4 &= \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = g & (g_6)^5 &= \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = g^{-1} g_2 & (g_6)^6 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}. \end{aligned} \tag{7.24}$$

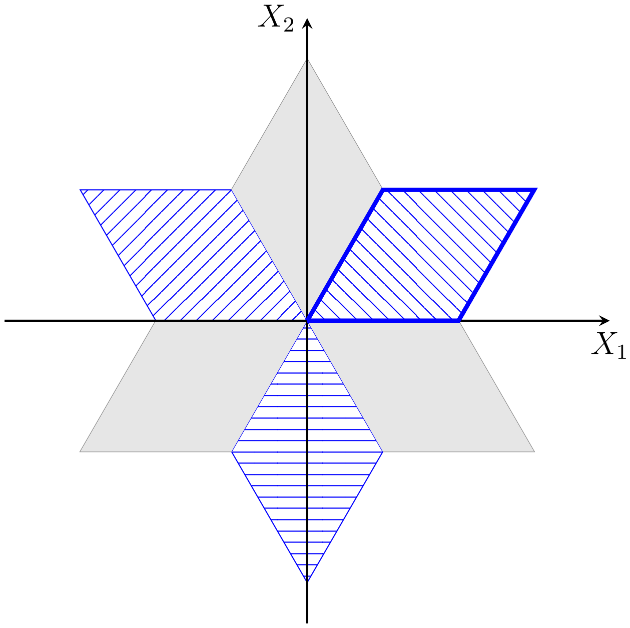

Figure 7.4: The different $\mathbb{Z}_6$ rotation of the fundamental region of $\mathbb{T}^2$ in the complex plane. In blue with a thick line the original torus and with thin lines the torus after $\mathbb{Z}_3$ rotation. In gray the results of $\mathbb{Z}_2$ coordinates inversion.

7.2 Gauge field transformations

Let us consider the world-sheet theory describing the two scalars on $\mathbb{T}^2$. Each of the two scalars compactified on $\mathbb{T}^2$ has the same global symmetries considered for a single free scalar on a circle. For each boson there is a $U(1)^{(0)}_{A^{(i)}}$ symmetry due to the invariance under translations on the i-circle with charge the center of mass momentum and a $U(1)^{(0)}_{B^{(i)}}$ symmetry due to transaltions in the dual torus whose charge is the winding number n_i. Due to the particular geometry of $\mathbb{T}^2$ there are also the discrete $\mathbb{Z}_3^R$ rotational symmetry and the $\mathbb{Z}_2^C$ coordinates inversion symmetry. Therefore, the 0-form global symmetry group is the semi-direct product:

$$G = \left(\mathbb{Z}_3^R \times \mathbb{Z}_2^C\right) \ltimes \left(U(1)^{(0)}_{A^{(1)}} \times U(1)^{(0)}_{A^{(2)}} \times U(1)^{(0)}_{B^{(2)}} \times U(1)^{(0)}_{B^{(2)}}\right). \tag{7.25}$$

The gauge tranformations for the $U(1)$ background gauge fields are the usual ones:

$$A_1^{(i)} \to A_1^{(i)} + \mathrm{d}\lambda^{(i)} \qquad B_1^{(i)} \to B_1^{(i)} + \mathrm{d}\zeta^{(i)} \qquad \text{with} \quad \lambda^{(i)}, \zeta^{(i)} \in U(1) \tag{7.26}$$

and they are independent since each of them is related to one of the two free periodic scalars. We denote with $F_2^{(i)} = \mathrm{d}A_1^{(i)}$ and with $H_2^{(i)} = \mathrm{d}B_1^{(i)}$ the field strengths for the gauge fields $A_1^{(i)}$ and $B_1^{(i)}$, respectively.

The generator $\mathcal{C}$ of $\mathbb{Z}_2^C$ acts as charge conjugation on the $U(1)$ gauge fields: $\mathcal{C}A_1^{(i)} = -A_1^{(i)}$ and $\mathcal{C}B_1^{(i)} = -B_1^{(i)}$. We refer to the previous chapter for a discussion about this symmetry.

The action of the $\mathbb{Z}_3^R$ can be derived by studying the action on the B_1 charges. In fact, if we consider the transfomation of the charges, we know that it must be compensated by the transformation of the gauge fields, as one can easily see thinking to the standard covariant derivative of gauge theories. Since the charges of the $B_1^{(i)}$ $U(1)$ fields are the winding numbers n_i, we can derive from their transformation law the action of the $\mathbb{Z}_3^R$ on $B_1^{(i)}$. From the previous discussion we know that the transformation is described by the matrix g^{-1} (7.13), thus

$$\begin{pmatrix} B_1^{(1)'} \\ B_1^{(2)'} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} B_1^{(1)} \\ B_1^{(2)} \end{pmatrix} = \begin{pmatrix} -B_1^{(1)} + B_1^{(2)} \\ -B_1^{(1)} \end{pmatrix}. \tag{7.27}$$

Since the background gauge fields A_1^i are the duals of the $B_1^{(i)}$, they transform with the inverse matrix g:

$$\begin{pmatrix} A_1^{(1)'} \\ A_1^{(2)'} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} A_1^{(1)} \\ A_1^{(2)} \end{pmatrix} = \begin{pmatrix} -A_1^{(2)} \\ A_1^{(1)} - A_1^{(2)} \end{pmatrix}. \tag{7.28}$$

We can denote these $\mathbb{Z}_3$ transformations using the generator $\mathcal{R}$:

$$\begin{aligned} \mathcal{R}A_1^{(1)} &= -A_1^{(2)} & \qquad \mathcal{R}B_1^{(1)} &= -B_1^{(1)} + B_1^{(2)} \\ \mathcal{R}A_1^{(2)} &= A_1^{(1)} - A_1^{(2)} & \qquad \mathcal{R}B_1^{(2)} &= -B_1^{(1)}. \end{aligned} \tag{7.29}$$

We can verify that the multiple applications of the generator $\mathcal{R}$ are cyclic and after three actions we return to the original configuration. In other words, we can write that $\mathcal{R}^3 = \mathbb{I}$.

$$\begin{aligned} \mathcal{R}A_1^{(1)} &= -A_1^{(2)} \\ \mathcal{R}^2 A_1^{(1)} &= \mathcal{R}(-A_1^{(2)}) = -A_1^{(1)} + A_1^{(2)} \\ \mathcal{R}^3 A_1^{(1)} &= \mathcal{R}(-A_1^{(1)} + A_1^{(2)}) = A_1^{(2)} + A_1^{(1)} - A_1^{(2)} = A_1^{(1)} \\ \mathcal{R}A_1^{(2)} &= A_1^{(1)} - A_1^{(2)} \\ \mathcal{R}^2 A_1^{(2)} &= \mathcal{R}(A_1^{(1)} - A_1^{(2)}) = -A_1^{(2)} - A_1^{(1)} + A_1^{(2)} = -A_1^{(1)} \\ \mathcal{R}^3 A_1^{(2)} &= \mathcal{R}(-A_1^{(1)}) = A_1^{(2)} \\ \\ \mathcal{R}B_1^{(1)} &= -B_1^{(1)} + B_1^{(2)} \\ \mathcal{R}^2 B_1^{(1)} &= \mathcal{R}(-B_1^{(1)} + B_1^{(2)}) = B_1^{(1)} - B_1^{(2)} - B_1^{(1)} = -B_1^{(2)} \\ \mathcal{R}^3 B_1^{(1)} &= \mathcal{R}(-B_1^{(2)}) = B_1^{(1)} \\ \mathcal{R}B_1^{(2)} &= -B_1^{(1)} \\ \mathcal{R}^2 B_1^{(2)} &= \mathcal{R}(-B_1^{(1)}) = B_1^{(1)} - B_1^{(2)} \\ \mathcal{R}^3 B_1^{(2)} &= \mathcal{R}(B_1^{(1)} - B_1^{(2)}) = -B_1^{(1)} + B_1^{(2)} + B_1^{(1)} = B_1^{(2)} \end{aligned} \tag{7.30}$$

The world-sheet action for these two periodic scalars contains also the Kalb-Ramond field B_2 that transforms under a $U(1)^{(1)}$ global gauge transformation $B_2 \to B_2 + \mathrm{d}\zeta_1$. As usual, there is also a 2-group non-trivial structure that affects the transfomation of B_2:

$$B_2 \to B_2 + B_1^{(1)} \wedge \mathrm{d}\lambda^{(1)} + B_1^{(2)} \wedge \mathrm{d}\lambda^{(2)}. \tag{7.31}$$

Notice that other forms of this transformation are possible because we can always redefine $B_2 \to B_2 - \alpha^{(1)} B_1^{(1)} \wedge A_1^{(1)} - \alpha^{(2)} B_1^{(2)} \wedge A_1^{(2)}$ with $\alpha^{(1)}, \alpha^{(2)} \in \mathbb{R}$ by inserting $\alpha^{(1)} B_1^{(1)} \wedge A_1^{(1)} + \alpha^{(2)} B_1^{(2)} \wedge A_1^{(2)}$ in the action. For our purposes, it will be more useful the more symmetric form:

$$B_2 \to B_2 + A_1^{(1)} \wedge \mathrm{d}\zeta^{(1)} + A_1^{(2)} \wedge \mathrm{d}\zeta^{(2)} + B_1^{(1)} \wedge \mathrm{d}\lambda^{(1)} + B_1^{(2)} \wedge \mathrm{d}\lambda^{(2)} + \mathrm{d}\zeta^{(1)} \wedge \mathrm{d}\lambda^{(1)} + \mathrm{d}\zeta^{(2)} \wedge \mathrm{d}\lambda^{(2)}. \quad (7.32)$$

7.3 't Hooft anomalies

The 0-form global symmetry with group G for the world-sheet theory of two periodic scalars is affected by different 't Hooft anomalies that apparently forbid the gauging. Notice that this happens if we neglect the coupling with B_2, that is if we consider only the 0-form symmetries of the theory. Some of them are the same discussed in the previous chapter and we only recall them for completeness. For each scalar there is a 't Hooft anomaly between the two $U(1)$ subgroups: we cannot gauge both the $U(1)^{(0)}_{A^{(i)}}$ and $U(1)^{(0)}_{B^{(i)}}$ since in the gauge transfomations for the background fields the action varies by an extra term $-B_1^{(i)} \wedge \mathrm{d}\lambda^{(i)}$ that makes non invariant the partition function (see 6.3). However, once the coupling with the B_2 field is considered, the 2 group transformation for the B_2 field exactly cancels this anomaly thanks to the presence of the $B_1^{(i)} \wedge \mathrm{d}\lambda^{(i)}$ terms. There is also a mixed 't Hooft anomaly between the subgroup $\mathbb{Z}_2^C$ of coordinates inversion and the $U(1)$ subgroups. Its study and its cancellation is quite similar to the one discussed in the previous chapter for a single scalar and can be easily generalized by extending the analysis for two $U(1)^{(0)}_{A^{(i)}} \times U(1)^{(0)}_{B^{(i)}}$ subgroups.

More interesting is the case that involves the $\mathbb{Z}_3^R$ subgroups since under this symmetry there is a non trivial mixing between the fields $A_1^{(1)}, A_1^{(2)}$ and $B_1^{(1)}, B_1^{(2)}$. This is different from the other anomalies that can be split on each $U(1)^{(0)}_{A^{(i)}} \times U(1)^{(0)}_{B^{(i)}}$ subgroups. For simplicity, we neglect the $\mathbb{Z}_2^C$ by restricting to

$$\tilde{G} = \mathbb{Z}_3^R \ltimes \left(U(1)^{(0)}_{A^{(1)}} \times U(1)^{(0)}_{A^{(2)}} \times U(1)^{(0)}_{B^{(2)}} \times U(1)^{(0)}_{B^{(2)}} \right), \quad (7.33)$$

the subgroup of G with trivial $\mathbb{Z}_2^C$ elements, since we are mainly interesting in the rotational symmetry. We adopt the previous strategy to study the anomalies; therefore, we search for the subgroup of $\tilde{G}$ that commutes with $\mathbb{Z}_3^R$, i.e. we search for the centralizer subgroup

$$Z_{\tilde{G}}(\mathcal{R}) = \{ \tilde{g} \in \tilde{G} \mid \mathcal{R}\tilde{g} = \tilde{g}\mathcal{R} \}. \quad (7.34)$$

From the path integral point of view, this means that we restrict to the background configurations that are invariant under $\mathbb{Z}_3^R$.

Let us initially focus on the fields $A_1^{(i)}$. In order to make the background $\mathbb{Z}_3^R$-gauge invariant, we have to impose

$$A_1^{(i)} \sim \mathcal{R} A_1^{(i)} \qquad \Rightarrow \qquad (1 - \mathcal{R}) A_1^{(i)} \sim 0 \quad (7.35)$$

where $\sim$ means up to $U(1)^{(0)}_{A^{(i)}}$ gauge transformations, and, in matrix notation, corresponds to

$$\left(\mathbb{I} - g\right) \begin{pmatrix} A_1^{(1)} \\ A_1^{(2)} \end{pmatrix} \sim 0 \sim \begin{pmatrix} \mathrm{d}\lambda^{(1)} \\ \mathrm{d}\lambda^{(2)} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} A_1^{(1)} \\ A_1^{(2)} \end{pmatrix} = \left(\mathbb{I} - g\right)^{-1} \begin{pmatrix} \mathrm{d}\lambda^{(1)} \\ \mathrm{d}\lambda^{(2)} \end{pmatrix}. \quad (7.36)$$

Thus, the background gauge fields must be pure gauge and must satisfy the following quantization conditions when integrated over a 1-cycle γ:

$$\frac{1}{2\pi} \oint_\gamma (1 - \mathcal{R}) A_1^{(i)} \in \mathbb{Z} \quad \Rightarrow \quad \frac{1}{2\pi} \oint_\gamma \left(A_1^{(1)} + A_1^{(2)} \right) \in \mathbb{Z} \quad \text{and} \quad \frac{1}{2\pi} \oint_\gamma \left(2A_1^{(2)} - A_1^{(1)} \right) \in \mathbb{Z}. \quad (7.37)$$

We can also find the holonomies for the fields $A_1^{(i)}$; indeed, since

$$\left(\mathbb{I} - g\right)^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \tag{7.38}$$

we obtain that $A_1^{(i)}$ is equal to $\frac{1}{3}\mathrm{d}\lambda$ for some combination of the gauge parameters $\lambda^{(i)} \in U(1)$ and the quantization condition for the gauge parameters implies

$$\frac{1}{2\pi}\oint_\gamma 3A_1^{(i)} = \frac{1}{2\pi}\oint_\gamma \mathrm{d}\lambda \in \mathbb{Z} \quad \Rightarrow \quad e^{i\oint_\gamma A_1^{(i)}} = e^{\frac{2\pi k}{3}} \quad \text{with} \quad k \in \mathbb{Z}, \tag{7.39}$$

i.e. the holonomies for $A_1^{(i)}$ take values in $\mathbb{Z}_3$.

The same argument holds for $B_1^{(i)}$: if we impose up to gauge transformations that $B_1^{(i)} \sim \mathcal{R}B_1^{(i)}$, we obtain that the background fields must be proportional to pure gauge:

$$\left(\mathbb{I} - g^{-1}\right)\begin{pmatrix} B_1^{(1)} \\ B_1^{(2)} \end{pmatrix} \sim 0 \sim \begin{pmatrix} \mathrm{d}\zeta^{(1)} \\ \mathrm{d}\zeta^{(2)} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} B_1^{(1)} \\ B_1^{(2)} \end{pmatrix} = \left(\mathbb{I} - g^{-1}\right)^{-1}\begin{pmatrix} \mathrm{d}\zeta^{(1)} \\ \mathrm{d}\zeta^{(2)} \end{pmatrix} \tag{7.40}$$

with the quantization conditions

$$\frac{1}{2\pi}\oint_\gamma \left(B_1^{(1)} + B_1^{(2)}\right) \in \mathbb{Z} \qquad \text{and} \qquad \frac{1}{2\pi}\oint_\gamma \left(2B_1^{(1)} - B_1^{(2)}\right) \in \mathbb{Z}. \tag{7.41}$$

The holonomies can be found by considering the matrix

$$\left(\mathbb{I} - g^{-1}\right)^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \tag{7.42}$$

from which we obtain that $B_1^{(i)}$ is equal to $\frac{1}{3}\mathrm{d}\zeta$ for some $U(1)$-valued function and

$$\frac{1}{2\pi}\oint_\gamma 3B_1^{(i)} = \frac{1}{2\pi}\oint_\gamma \mathrm{d}\zeta \in \mathbb{Z} \quad \Rightarrow \quad e^{i\oint_\gamma B_1^{(i)}} = e^{\frac{2\pi k}{3}} \quad \text{with} \quad k \in \mathbb{Z}. \tag{7.43}$$

Therefore, if we consider all the background fields that are invariant under $\mathbb{Z}_3^R$, from the study of their holonomies we discover that they must take values in the $\mathbb{Z}_3^{A^{(1)}} \times \mathbb{Z}_3^{A^{(2)}} \times \mathbb{Z}_3^{B^{(1)}} \times \mathbb{Z}_3^{B^{(2)}}$ subgroup of $U(1)^{(0)}_{A^{(1)}} \times U(1)^{(0)}_{A^{(2)}} \times U(1)^{(0)}_{B^{(2)}} \times U(1)^{(0)}_{B^{(2)}}$; thus the centralizer is

$$Z_{\tilde{G}}(\mathcal{R}) = \mathbb{Z}_3^R \times \mathbb{Z}_3^{A^{(1)}} \times \mathbb{Z}_3^{A^{(2)}} \times \mathbb{Z}_3^{B^{(1)}} \times \mathbb{Z}_3^{B^{(2)}}. \tag{7.44}$$

Since the $\mathbb{Z}_3^R$ inviariant gauge fields are proportional to pure gauge, it is straightforward that the background curvatures are null: $F_2^{(i)} = H_2^{(i)} = 0$.

In order to discuss the anomalies, a crucial preliminary observation is that the subgroup $\mathbb{Z}_3^R$ is conjugated with other $\mathbb{Z}_3$ subgroups derived from different $U(1)$. If $h \in \mathbb{Z}_3^{A^{(1)}} \times \mathbb{Z}_3^{A^{(2)}} \times \mathbb{Z}_3^{B^{(1)}} \times \mathbb{Z}_3^{B^{(2)}}$, we have that $\mathcal{R}$ and $\mathcal{R}h$ are in the same conjugacy class, that is there exists $\tilde{g} \in \tilde{G}$ such that

$$\mathcal{R} = \tilde{g}\mathcal{R}h\tilde{g}^{-1}. \tag{7.45}$$

In fact, there is an ambiguity in the vary choice of $\mathcal{R}$ in the group $\tilde{G}$ since we can choose any representative of such conjugacy class. In order to prove 7.45, we have to consider the general form of the group elements and we have to introduce some notation.

Let us denote $\tilde{g} \in U(1)^{(0)}_{A^{(1)}} \times U(1)^{(0)}_{A^{(2)}} \times U(1)^{(0)}_{B^{(2)}} \times U(1)^{(0)}_{B^{(2)}} \subset \tilde{G}$ by $e^{2\pi i \alpha_i T^i}$ where $\{T^i\}, i = 1, \dots 4$ are the generators of the different $U(1)^{(0)}_i$ subgroups of $\tilde{G}$ ordered as in the original group 7.25 and $\{\alpha_i\}$ are real numbers. We can denote $h \in Z_{\tilde{G}}(\mathcal{R}) \cap U(1)^{(0)}_{A^{(1)}} \times U(1)^{(0)}_{A^{(2)}} \times U(1)^{(0)}_{B^{(2)}} \times U(1)^{(0)}_{B^{(2)}}$ in the same way since $Z_{\tilde{G}}(\mathcal{R}) \subset \tilde{G}$, so $h = e^{2\pi i \beta_i T^i}$ with $\beta_i \in \mathbb{Z}/3$. We want to prove that for any h we can find $\tilde{g} \in \tilde{G}$ such that the conjugacy relation 7.45 holds.

Let us rewrite 7.45:

$$\tilde{g}^{-1}\mathcal{R}\tilde{g} = \mathcal{R}h \quad \longleftrightarrow \quad \mathcal{R}^{-1}\tilde{g}^{-1}\mathcal{R}\tilde{g} = h. \tag{7.46}$$

In order to describe the action of $\mathcal{R}$ on $\tilde{g}$ we have to use the matrix notation. Let us define the 4×4 matrix

$$\bar{g} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}, \tag{7.47}$$

thus, we get the action:

$$\mathcal{R}\tilde{g} = \mathcal{R}e^{2\pi i \alpha_i T^i} = e^{2\pi i T^i \bar{g}_{ij}\alpha_j}\mathcal{R}. \tag{7.48}$$

Now we can simplify the previous relation:

$$e^{2\pi i \beta_i T^i} = \mathcal{R}^{-1}e^{-2\pi i \alpha_i T^i}\mathcal{R}e^{2\pi i \alpha_i T^i} = e^{-2\pi i T^i(\bar{g}^{-1})_{ij}\alpha_j}\mathcal{R}^{-1}\mathcal{R}e^{2\pi i \alpha_i T^i} = e^{2\pi i T^i(\mathbb{I}_4 - \bar{g}^{-1})_{ij}\alpha_j} \tag{7.49}$$

and, given the parameters $\{\beta_i\}$ that characterize $h \in Z_{\tilde{G}}(\mathcal{R})$, we can find the paramenters $\{\alpha_i\}$ if $(\mathbb{I}_4 - \bar{g}^{-1})$ is invertible:

$$\alpha_i = (\mathbb{I}_4 - \bar{g}^{-1})^{-1}_{ij}\beta_j. \tag{7.50}$$

As we have already seen, the matrix $(\mathbb{I}_2 - g)$ and $(\mathbb{I}_2 - g^{-1})$, that constitute the blocks of the 4×4 matrix $\bar{g}$, are invertible, therefore the conjugacy 7.45 is proved. Notice that the different conjugacy classes are given by the presence of $\mathcal{R}$, $\mathcal{R}^2$ or the $\mathbb{Z}_3^R$ identity in $Z_{\tilde{G}}(\mathcal{R})$ elements.

Being reduced to studying finite groups we can exploit the results presented in 3.5 and discuss esplicitely the anomalies. Denoting the anomaly by α, we know that it is a representative of a class in the third cohomology group

$$H^3(\mathbb{Z}_3^R \times \mathbb{Z}_3^{A^{(1)}} \times \mathbb{Z}_3^{A^{(2)}} \times \mathbb{Z}_3^{B^{(1)}} \times \mathbb{Z}_3^{B^{(2)}}, U(1)) \simeq \mathbb{Z}_3^{25}. \tag{7.51}$$

We can list the different contributions to α from the different types of 3-cocycles: 5 cocycles of type I,

$$\alpha_I^R,\ \alpha_I^{A^{(1)}},\ \alpha_I^{A^{(2)}},\ \alpha_I^{B^{(1)}},\ \alpha_I^{B^{(2)}} \tag{7.52}$$

10 cocycles of type II,

$$\alpha_{II}^{RA^{(1)}}, \alpha_{II}^{RA^{(2)}}, \alpha_{II}^{RB^{(1)}}, \alpha_{II}^{RB^{(2)}}, \alpha_{II}^{A^{(1)}A^{(2)}}, \alpha_{II}^{B^{(1)}B^{(2)}}, \alpha_{II}^{A^{(1)}B^{(1)}}, \alpha_{II}^{A^{(2)}B^{(2)}}, \alpha_{II}^{A^{(1)}B^{(2)}}, \alpha_{II}^{A^{(2)}B^{(1)}} \tag{7.53}$$

and 10 cocycle of type III,

$$\begin{gathered} \alpha_{III}^{RA^{(1)}A^{(2)}}, \alpha_{III}^{RA^{(1)}B^{(1)}}, \alpha_{III}^{RA^{(1)}B^{(2)}}, \alpha_{III}^{RA^{(2)}B^{(1)}}, \alpha_{III}^{RA^{(2)}B^{(2)}}, \alpha_{III}^{RB^{(1)}B^{(2)}}, \\ \alpha_{III}^{A^{(1)}A^{(2)}B^{(1)}}, \alpha_{III}^{A^{(1)}A^{(2)}B^{(2)}}, \alpha_{III}^{A^{(1)}B^{(1)}B^{(2)}}, \alpha_{III}^{A^{(2)}B^{(1)}B^{(2)}}. \end{gathered} \tag{7.54}$$

We have denoted with the superscript the different $\mathbb{Z}_3^R$ subgroups that contribute and with the subscript the type of cocycle (see section 6.6).

Since we know that for each boson there is a mixed 't Hooft anomaly between the $U(1)^{(0)}_{A^{(i)}}$ and $U(1)^{(0)}_{B^{(i)}}$ gauge groups, we argue that $\alpha_{II}^{A^{(1)}B^{(1)}}$ and $\alpha_{II}^{A^{(2)}B^{(2)}}$ are non-trivial. In contrast, the other contributions that involve the $U(1)$ fields, $\alpha_{II}^{A^{(1)}A^{(2)}}, \alpha_{II}^{B^{(1)}B^{(2)}}, \alpha_{II}^{A^{(1)}B^{(2)}}$ and $\alpha_{II}^{A^{(2)}B^{(1)}}$, are trivial since the two scalars are free and the subgroups $U(1)^{(0)}_{A^{(1)}} \times U(1)^{(0)}_{B^{(1)}}$ and $U(1)^{(0)}_{A^{(2)}} \times U(1)^{(0)}_{B^{(2)}}$ are independent. Also the contribution of type I for the various $U(1)$ subgroups are trivial.

Since $\mathbb{Z}_3^R$ is conjugated to other $\mathbb{Z}_3$ subgroups, we can deduce that the restriction of the anomaly to $\mathbb{Z}_3^R$ and the restriction to the group generated by $\mathcal{R}a^{(1)}a^{(2)}b^{(1)}b^{(2)}$ are the same. Here we have denoted with the lower case the generic elements of the $\mathbb{Z}_3$ subgroups of the $U(1)$ gauge fields (e.g. $a^{(1)} \in \mathbb{Z}_3^{A^{(1)}}$ and so on). Thus, if we consider the equality $\alpha(\mathcal{R},\mathcal{R},\mathcal{R}) = \alpha(\mathcal{R}a^{(1)},\mathcal{R}a^{(1)},\mathcal{R}a^{(1)})$ we obtain

$$\alpha_I^R = \alpha_I^R + \alpha_I^{A^{(1)}} + \alpha_{II}^{RA^{(1)}} \tag{7.55}$$

and, since $\alpha_I^{A^{(1)}}$ is trivial because the single $U(1)^{(0)}$ is not anomalous, we can conclude that also $\alpha_{II}^{RA^{(1)}}$ is trivial. The same holds for the other cocycles of type II that involve $\mathcal{R}$: $\alpha_{II}^{RA^{(2)}}$, $\alpha_{II}^{RB^{(1)}}$ and $\alpha_{II}^{RB^{(2)}}$ vanish.

For the cocycle of type III we focus on those containing $\mathcal{R}$. Considering for instance

$$\alpha(\mathcal{R},\mathcal{R},\mathcal{R}) = \alpha(\mathcal{R}a^{(1)}a^{(2)},\mathcal{R}a^{(1)}a^{(2)},\mathcal{R}a^{(1)}a^{(2)}), \tag{7.56}$$

we obtain

$$\alpha_I^R = \alpha_I^R + \alpha_I^{A^{(1)}} + \alpha_I^{A^{(2)}} + \alpha_{II}^{RA^{(1)}} + \alpha_{II}^{RA^{(2)}} + \alpha_{II}^{A^{(1)}A^{(2)}} + \alpha_{III}^{RA^{(1)}A^{(2)}}. \tag{7.57}$$

Notice that there are no anomalies for the single $U(1)^{(0)}_{A^{(i)}}$ and all the cocycles of type II are trivial, so we can conclude that $\alpha_{III}^{RA^{(1)}A^{(2)}}$ must be trivial. The same happens for the 3-cocycles $\alpha_{III}^{RA^{(1)}B^{(2)}}$, $\alpha_{III}^{RA^{(2)}B^{(1)}}$ and $\alpha_{III}^{RB^{(1)}B^{(2)}}$. Differently, if we consider the combination of the two gauge fields of the same scalar we have, for instance,

$$\alpha(\mathcal{R},\mathcal{R},\mathcal{R}) = \alpha(\mathcal{R}a^{(1)}b^{(1)},\mathcal{R}a^{(1)}b^{(1)},\mathcal{R}a^{(1)}b^{(1)}), \tag{7.58}$$

from which we obtain

$$\alpha_I^R = \alpha_I^R + \alpha_I^{A^{(1)}} + \alpha_I^{B^{(1)}} + \alpha_{II}^{RA^{(1)}} + \alpha_{II}^{RB^{(1)}} + \alpha_{II}^{A^{(1)}B^{(1)}} + \alpha_{III}^{RA^{(1)}B^{(1)}}. \tag{7.59}$$

Since $\alpha_{II}^{A^{(1)}B^{(1)}}$ are non trivial, we can argue that also $\alpha_{III}^{RA^{(1)}B^{(1)}}$ must be non-trivial. With the same argument, we observe that $\alpha_{III}^{RA^{(2)}B^{(2)}}$ is not null.

Therefore, from the study of the discrete group $Z_{\tilde{G}}(\mathcal{R})$ we can deduce that there must be a mixed 't Hooft anomaly between $\mathbb{Z}_3^R$ and $U(1)^{(0)}_{A^{(1)}} \times U(1)^{(0)}_{A^{(2)}} \times U(1)^{(0)}_{B^{(2)}} \times U(1)^{(0)}_{B^{(2)}}$, since this anomaly is visible even when restricting to the abelian subgroup $Z_{\tilde{G}}(\mathcal{R})$. Finally, we can observe that also the cocycles $\alpha_{III}^{A^{(1)}A^{(2)}B^{(1)}}$, $\alpha_{III}^{A^{(1)}A^{(2)}B^{(2)}}$, $\alpha_{III}^{A^{(1)}B^{(1)}B^{(2)}}$ and $\alpha_{III}^{A^{(2)}B^{(1)}B^{(2)}}$ are trivial because the partition function factorizes and there cannot be a mixed anomaly between groups acting on different factors.

The explicit form of the cocycles $\alpha_{III}^{RA^{(1)}B^{(1)}}$ and $\alpha_{III}^{RA^{(2)}B^{(2)}}$ are

$$\alpha_{III}^{RA^{(1)}B^{(1)}} = \exp\Big(\frac{2\pi i}{3} p_{III}^{RA^{(1)}B^{(1)}} r a^{(1)} b^{(1)}\Big) \qquad \alpha_{III}^{RA^{(2)}B^{(2)}} = \exp\Big(\frac{2\pi i}{3} p_{III}^{RA^{(2)}B^{(2)}} r a^{(2)} b^{(2)}\Big) \tag{7.60}$$

where $r \in \mathbb{Z}_3^R$ and $p_{III}^{RA^{(1)}B^{(1)}}, p_{III}^{RA^{(2)}B^{(2)}} \in \mathbb{Z}_3$ label the different cocycles. We have non-trivial cocycle for $p_{III}^{RA^{(1)}B^{(1)}}, p_{III}^{RA^{(2)}B^{(2)}} = 1, 2$ and we can compute the non-trivial values of the anomaly by focusing for instance in the first case $\alpha_{III}^{RA^{(1)}B^{(1)}}$ (see Table 7.1).

We have a non trivial anomalous phase every time that we have non trivial entries in the cocycle, i.e. $r, a^{(1)}, b^{(1)} \neq 0$. The possible values of the anomaly cocycles are the non trivial cubic root of units in the complex plane that correspond precisely to the non trivial elements of the $\mathbb{Z}_3$ cyclic group. The exact values of $p_{III}^{RA^{(1)}B^{(1)}}, p_{III}^{RA^{(2)}B^{(2)}}$ can be fixed by considering the anomalies $\alpha_{II}^{A^{(1)}B^{(1)}}$ and $\alpha_{II}^{A^{(2)}B^{(2)}}$.

$p_{III}^{RA^{(1)}B^{(1)}} = 1$			
r	$a^{(1)}$	$b^{(1)}$	$\alpha_{III}^{RA^{(1)}B^{(1)}}$
1	1	1	$e^{\frac{2\pi i}{3}}$
1	1	2	$e^{\frac{4\pi i}{3}}$
1	2	1	$e^{\frac{4\pi i}{3}}$
1	2	2	$e^{\frac{2\pi i}{3}}$
2	1	1	$e^{\frac{4\pi i}{3}}$
2	1	2	$e^{\frac{2\pi i}{3}}$
2	2	1	$e^{\frac{2\pi i}{3}}$
2	2	2	$e^{\frac{4\pi i}{3}}$

$p_{III}^{RA^{(1)}B^{(1)}} = 2$			
r	$a^{(1)}$	$b^{(1)}$	$\alpha_{III}^{RA^{(1)}B^{(1)}}$
1	1	1	$e^{\frac{4\pi i}{3}}$
1	1	2	$e^{\frac{2\pi i}{3}}$
1	2	1	$e^{\frac{2\pi i}{3}}$
1	2	2	$e^{\frac{4\pi i}{3}}$
2	1	1	$e^{\frac{2\pi i}{3}}$
2	1	2	$e^{\frac{4\pi i}{3}}$
2	2	1	$e^{\frac{4\pi i}{3}}$
2	2	2	$e^{\frac{2\pi i}{3}}$

Table 7.1: Explicit calculation of the non trivial anomaly cocycle $\alpha_{III}^{RA^{(1)}B^{(1)}}$ for the mixed $\mathbb{Z}_3^R$ 't Hooft anomaly.

7.4 't Hooft anomaly cancellation

As we have already discussed, the presence of a 't Hooft anomaly in the world-sheet theory does not costitute an immediate problem for the theory, but is dangerous when we consider the spacetime action or if we want to consider the orbifold of the theory. However, as for the previous cases, the presence of a non trivial 2-group structure between the 0-form gauge fields and the Kalb-Ramond 1-form provides the way to get rid of this mixed anomaly. Let us explain how.

We have to search for the transformation of the B_2 field under the rotation of $\frac{2}{3}\pi$ in the complex plane where the torus are defined. If we want to restrict to the $\mathbb{Z}_3^R$ invariant configurations we have to impose on the quintet of gauge fields that

$$\big(A_1^{(1)}, A_1^{(2)}, B_1^{(1)}, B_1^{(2)}, B_2\big) \sim \big(\mathcal{R}A_1^{(1)}, \mathcal{R}A_1^{(2)}, \mathcal{R}B_1^{(1)}, \mathcal{R}B_1^{(2)}, B_2\big). \tag{7.61}$$

The rotation of the circles in the compact directions does not affect the Kalb-Ramond field B_2 since its tensor has no indices in the compact spacetime directions. Using the same trick as in the previous chapter, we can rewrite the second term

$$\begin{aligned}&\big(A_1^{(1)} - (1-\mathcal{R})A_1^{(1)}, A_1^{(2)} - (1-\mathcal{R})A_1^{(2)}, B_1^{(1)} - (1-\mathcal{R})B_1^{(1)}, B_1^{(2)} - (1-\mathcal{R})B_1^{(2)}, B_2\big) = \\ &\big(A_1^{(1)} + \mathrm{d}\lambda^{(1)}, A_1^{(2)} + \mathrm{d}\lambda^{(2)}, B_1^{(1)} + \mathrm{d}\zeta^{(1)}, B_1^{(2)} + \mathrm{d}\zeta^{(2)}, B_2\big)\end{aligned} \tag{7.62}$$

and we require that $-(1-\mathcal{R})A_1^{(i)}$ and $-(1-\mathcal{R})B_1^{(1)}$ are pure gauge, as in 7.36 and 7.40. Therefore, we can make a gauge transformation with parameter $-\mathrm{d}\lambda^{(i)}, -\mathrm{d}\zeta^{(i)}$ in order to cancel these pure gauge terms, but the transformation involves also B_2 (7.32):

$$\begin{aligned}\big(&A_1^{(1)} + \mathrm{d}\lambda^{(1)} - \mathrm{d}\lambda^{(1)}, A_1^{(2)} + \mathrm{d}\lambda^{(2)} - \mathrm{d}\lambda^{(2)}, B_1^{(1)} + \mathrm{d}\zeta^{(1)} - \mathrm{d}\zeta^{(1)}, B_1^{(2)} + \mathrm{d}\zeta^{(2)} - \mathrm{d}\zeta^{(2)}, \\ &B_2 - A_1^{(1)} \wedge \mathrm{d}\zeta^{(1)} - A_1^{(2)} \wedge \mathrm{d}\zeta^{(2)} - B_1^{(1)} \wedge \mathrm{d}\lambda^{(1)} - B_1^{(2)} \wedge \mathrm{d}\lambda^{(2)}\big).\end{aligned} \tag{7.63}$$

Developing the calculations for the field B_2 we obtain:

$$\begin{aligned}
&B_2 + A_1^{(1)} \wedge (1-\mathcal{R})B_1^{(1)} + A_1^{(2)} \wedge (1-\mathcal{R})B_1^{(2)} + B_1^{(1)} \wedge (1-\mathcal{R})A_1^{(1)} + B_1^{(2)} \wedge (1-\mathcal{R})A_1^{(2)} = \\
&= B_2 + A_1^{(1)} \wedge (2B_1^{(1)} - B_1^{(2)}) + A_1^{(2)} \wedge (B_1^{(1)} + B_1^{(2)}) + B_1^{(1)} \wedge (A_1^{(1)} + A_1^{(2)}) + B_1^{(2)} \wedge (2A_1^{(2)} - A_1^{(1)}) \\
&= B_2 + 2A_1^{(1)} \wedge B_1^{(1)} - A_1^{(1)} \wedge B_1^{(2)} + A_1^{(2)} \wedge B_1^{(1)} + A_1^{(2)} \wedge B_1^{(2)} + \\
&\qquad + B_1^{(1)} \wedge A_1^{(1)} + B_1^{(1)} \wedge A_1^{(2)} + 2B_1^{(2)} \wedge A_1^{(2)} - B_1^{(2)} \wedge A_1^{(1)} = \\
&= B_2 + A_1^{(1)} \wedge B_1^{(1)} + B_1^{(2)} \wedge A_1^{(2)}.
\end{aligned} \tag{7.64}$$

Finally, we obtain that 7.63 equals

$$\big(A_1^{(1)}, A_1^{(2)}, B_1^{(1)}, B_1^{(2)}, B_2 + A_1^{(1)} \wedge B_1^{(1)} + B_1^{(2)} \wedge A_1^{(2)}\big) \tag{7.65}$$

that in principle is different from the original quintet 7.61. In order to make sense of these equality and transfomation we have to impose that under the $\mathbb{Z}_3^R$ rotation the field B_2 transforms as

$$B_2 \to B_2 + A_1^{(1)} \wedge B_1^{(1)} + B_1^{(2)} \wedge A_1^{(2)}. \tag{7.66}$$

This means that even if $A_1^{(i)}$ and $B_1^{(i)}$ are restricted as in 7.36 and 7.40, the quintet $(A_1^{(1)}, A_1^{(2)}, B_1^{(1)}, B_1^{(2)}, B_2)$ is not invariant under $\mathbb{Z}_3^R$, but transforms as

$$\big(A_1^{(1)}, A_1^{(2)}, B_1^{(1)}, B_1^{(2)}, B_2\big) \to \big(A_1^{(1)}, A_1^{(2)}, B_1^{(1)}, B_1^{(2)}, B_2 + A_1^{(1)} \wedge B_1^{(1)} + B_1^{(2)} \wedge A_1^{(2)}\big). \tag{7.67}$$

Let us express this transformation in terms of the gauge parameters in order to study its quantization. Notice that we have neglected the last two term in 7.32 since they only contribute by integer to the flux of B_2 so they do not affect the phase. From 7.36 and 7.40, we get

$$\begin{aligned}
A_1^{(1)} &= \frac{2}{3}\mathrm{d}\lambda^{(1)} - \frac{1}{3}\mathrm{d}\lambda^{(2)} \qquad & A_1^{(2)} &= \frac{1}{3}\mathrm{d}\lambda^{(1)} + \frac{1}{3}\mathrm{d}\lambda^{(2)} \\
B_1^{(1)} &= \frac{1}{3}\mathrm{d}\zeta^{(1)} + \frac{1}{3}\mathrm{d}\zeta^{(2)} \qquad & B_1^{(2)} &= -\frac{1}{3}\mathrm{d}\zeta^{(1)} + \frac{2}{3}\mathrm{d}\zeta^{(2)}
\end{aligned} \tag{7.68}$$

and we can substitute the fields in 7.66 with them.

$$\begin{aligned}
&A_1^{(1)} \wedge B_1^{(1)} + B_1^{(2)} \wedge A_1^{(2)} = \\
&= \frac{1}{9}(2\mathrm{d}\lambda^{(1)} - \mathrm{d}\lambda^{(2)}) \wedge (\mathrm{d}\zeta^{(1)} + \mathrm{d}\zeta^{(2)}) + \frac{1}{9}(-\mathrm{d}\zeta^{(1)} + 2\mathrm{d}\zeta^{(2)}) \wedge (\mathrm{d}\lambda^{(1)} + \mathrm{d}\lambda^{(2)}) \\
&= \frac{1}{9}\big(2\mathrm{d}\lambda^{(1)} \wedge \mathrm{d}\zeta^{(1)} + 2\mathrm{d}\lambda^{(1)} \wedge \mathrm{d}\zeta^{(2)} - \mathrm{d}\lambda^{(2)} \wedge \mathrm{d}\zeta^{(1)} - \mathrm{d}\lambda^{(2)} \wedge \mathrm{d}\zeta^{(2)} - \\
&\quad - \mathrm{d}\zeta^{(1)} \wedge \mathrm{d}\lambda^{(1)} - \mathrm{d}\zeta^{(1)} \wedge \mathrm{d}\lambda^{(2)} + 2\mathrm{d}\zeta^{(2)} \wedge \mathrm{d}\lambda^{(1)} + 2\mathrm{d}\zeta^{(2)} \wedge \mathrm{d}\lambda^{(2)}\big) = \\
&= \frac{1}{3}\big(\mathrm{d}\lambda^{(1)} \wedge \mathrm{d}\zeta^{(1)} + \mathrm{d}\zeta^{(2)} \wedge \mathrm{d}\lambda^{(2)}\big).
\end{aligned} \tag{7.69}$$

Therefore, the non-trivial transfomation of B_2 under $\mathbb{Z}_3^R$ cannot be reabsorbed by a gauge transformation of B_2 with parameter ζ_1 because the integral of 7.69 over a 2-dimensional closed manifold is not integer. So, the holonomy of B_2 changes.

The non trivial transformation of B_2 is the solution for the cancellation of the 't Hooft anomaly between $\mathbb{Z}_3^R$ and the different $U(1)$ subgroups. Let us consider a $(d-1)-$dimensional topological defect in spacetime $M^{(d)}$ representing the generator $\mathcal{R}$ of the $\mathbb{Z}_3^R$-rotations. If the fields $A_1^{(i)}$ and $B_1^{(i)}$ are restricted as in 7.36 and 7.40, the field B_2 differs by $A_1^{(1)} \wedge B_1^{(1)} + B_1^{(2)} \wedge A_1^{(2)}$ from the two sides of the defect, that is by $\frac{1}{3}\big(\mathrm{d}\lambda^{(1)} \wedge \mathrm{d}\zeta^{(1)} + \mathrm{d}\zeta^{(2)} \wedge \mathrm{d}\lambda^{(2)}\big)$ if we use the gauge parameters. If we described the $Z_{\tilde{G}}(\mathcal{R})$ background by a network of $(d-1)-$dimensional defects in spacetime labeled by elements in $Z_{\tilde{G}}(\mathcal{R})$, then $A_1^{(i)} \wedge B_1^{(i)}$ (or more precisely $\frac{1}{3}\mathrm{d}\lambda^{(i)} \wedge \mathrm{d}\zeta^{(i)}$) is the Poincarè dual of the intersection between the defect labeled by $a^{(i)}$ and the one labeled by $b^{(i)}$. Taking into account, for instance, the $(d-3)$-dimensional triple junction of three defects $\mathcal{R}$, $a^{(i)}$ and $b^{(i)}$ we can consider the integral of B_2 over a small sphere $\mathbb{S}^2$ which encircles the triple intersection. Therefore, we know that the defect $\mathcal{R}$ divides the sphere into two parts and, in the limit where the radius of $\mathbb{S}^2$ is infinitesimal so that B_2 is about constant, the flux $\oint_{\mathbb{S}^2} B_2$ is equivalent to the integral of $A_1^{(1)} \wedge B_1^{(1)} + B_1^{(2)} \wedge A_1^{(2)}$ over one of the two hemispheres. Since $\frac{1}{3}\mathrm{d}\lambda^{(i)} \wedge \mathrm{d}\zeta^{(i)}$ is the Poincarè dual of the intersection between the defects $a^{(i)}$ and $b^{(i)}$ and since this intersection necessarily crosses the hemisphere, we obtain that the integral of $A_1^{(1)} \wedge B_1^{(1)} + B_1^{(2)} \wedge A_1^{(2)}$ over such hemisphere has values in $\mathbb{Z}/3$. Therefore, in the triple intersection of the three defects $\mathcal{R}$, $a^{(i)}$ and $b^{(i)}$, we have

$$e^{2\pi i \oint_{\mathbb{S}^2} B_2} = e^{i\frac{2k\pi}{3}} \qquad k \in \mathbb{Z}. \tag{7.70}$$

In such a way, we obtain exactly the phase that cancels out the anomalous phase that we want to eliminate. Once again the presence of a non-trivial 2 group structure solves the problem of the presence of mixed 't Hooft anomalies between subgroups of the 0-form global symmetry group of compactified string theory models.

Appendix A

Elements of group theory

We want essentially to recall without systematicity some definitions regarding the theory of groups that are used in the previous chapter. In particular, we will recall the notion of product and quotient between groups and that of finite, cyclic and finitely generated groups.

Definition 1. A group $(G, \circ)$ is a set G with a composition law, called multiplication

$$\begin{aligned} \circ: \quad & G \times G \to G \\ & (g_1, g_2) \mapsto g_1 \circ g_2 \end{aligned}$$

satisfying the following properties:

- associativity: $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3 \; \forall \, g_1, g_2, g_3 \in G$;
- existence of neutral element (or identity) e: $g \circ e = e \circ g = g \; \forall \, g \in G$;
- existence of the inverse: $\forall g \in G, \exists g^{-1} \,|\, g^{-1} \circ g = g \circ g^{-1} = e$.

Definition 2. A group $(G, \circ)$ is said to be Abelian (commutative) if the composition law is commutative: $g_1 \circ g_2 = g_2 \circ g_1 \, \forall \, g_1, g_2 \in G$.

Definition 3. A subset of a group $H \subset G$ is a subgroup $H < G$ if it is closed under the composition law of G.

Definition 4. The center of a group is the subgroup $Z_G < G$ that contains all the elements that commute with all the others:

$$Z_G = \{a \in G \,|\, a \circ g = g \circ a \; \forall g \in G\}$$

By definition, the center of a group is Abelian and it is straightforward to see that if G is Abelian than the Z_G is the whole group.

Definition 5. The centralizer of a subset $H \subset G$ in the group G is defined as the set $Z_G(H)$ of elements of G such that each member $g \in Z_G(H)$ commutes with each element of H:

$$Z_G(H) = \{g \in G \,|\, gh = hg \; \forall h \in H\}.$$

If a group is finite, i.e. it has a finite number of elements ($|G| < \infty$, with $|G|$ the cardinality of G), when we multiply an element by itself enough times, we recover the group identity. In fact, multiplying any element of a finite group by itself more than $|G|$ times must lead to a recurrence of the product, because there are at most $|G|$ distinct elements.

Definition 6. The order of a finite group G is its cardinality, i.e. the number of its elements $|G|$. The order of an element $g \in G$ is the smallest integer n such that $g^n = e$.

Ler us consider the notion of finite generated abelian groups.

Definition 7. Let $g_1, \dots g_r \in G$, the elements of G of the form $n_1 g_1 + \dots + n_r g_r$ with $n_i \in \mathbb{Z}$ form a subgroup of G. This subgroup is said to be generated by $g_1, \dots g_r$ and these elements are called generators.
A group G is said to be finitely generated if it is generated by a finite number of generators.

If $n_1 g_1 + \dots + n_r g_r = 0$ is satisfied only when $n_1 = \dots n_r = 0$, $g_1, \dots g_r$ are said to be linearly independent.

Definition 8. A group G is called a free Abelian group of rank r if it is finitely generated by r linearly independent generators.

Definition 9. G is a cyclic group if it is generated by one element g: $G = \{0, \pm g, \pm 2g, \dots\}$. If $ng \neq 0$ for any $n \in \mathbb{Z} - \{0\}$, G is a infinite cyclic group, while if $ng = 0$ for some $n \in \mathbb{Z} - \{0\}$ it is a finite cyclic group.

Any infinite cyclic group is isomorphic to $\mathbb{Z}$ while any finite cyclic group is isomorphic to $\mathbb{Z}_N$. Furthermore, any finitely generated Abelian group G (not necessarily free) with m generators is isomorphic to the direct sum of cyclic groups:

$$G \simeq \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \oplus \mathbb{Z}_{N_1} \oplus \dots \oplus \mathbb{Z}_{N_k} \tag{A.1}$$

In order to define the quotients of groups we recall the definition of equivalence relations, since the quotient is essentially an identification of elements via equivalence relation.

Definition 10. A binary relation $\sim$ on a set X is said to be an equivalence relation if and only if it satisfies the following properties:

- $a \sim a$
- $a \sim b \Rightarrow b \sim a$
- $a \sim b$ and $b \sim c \Rightarrow a \sim c$

An equivalence class for an element $a \in X$ is the set of all elements in X equivalent to a:

$$[a] := \{x \in X \mid x \sim a\}$$

.

Definition 11. Let $H < G$ and $r \in G$. The equivalence classes

$$Hr = \{hr \mid h \in H\} \quad Hr = \{rh \mid h \in H\}$$

are said respectively right cosets and left cosets.

Note that in general gH and Hg are not in H unless also $g \in H$ and they are not even subgroups of G. One can prove that two cosets of a subgroups have the same element or they have not any element in common; therefore, we can write G as the disjoint union of cosets of its subgroup H and we can define

Definition 12. The right quotient G/H of a group G by its subgroup H is the set of all the right cosets of H, i.e. the set of all the equivalence classes of elements of G obtained by right multiplications with H: $g_1 \sim g_2$ if $\exists\, h \in H \mid g_1 = g_2 h$.
The left quotient $H \setminus G$ is the set of all the left cosets of H, i.e. the set of all the equivalence classes of elements of G defined by the relation: $g_1 \sim g_2$ if $\exists\, h \in H \mid g_1 = h g_2$.

In general $G/H \neq H \setminus G$ and they do not form a group with respect to the group multiplication in G. This may happen only if H is a normal subgroup, i.e. it is self-conjugated:

Definition 13. A subgroup $N < G$ is said to be normal or invariant ($N \lhd G$) if all the right and left cosets coincide:

$$Hg = gH \;\; \forall\, g \in G$$

or equivalently

$$\forall\, g \in G, h_1 \in H \;\; \exists h_2 \in H \mid g h_1 = h_2 g$$

Definition 14. Two elements $g_1, g_2 \in G$ are said to be conjugate if $\exists g \in G \mid g_2 = g g_1 g^{-1}$

The conjugation is an equivalence relation.

Definition 15. A subgroup $H < G$ is said to be self-conjugate if $\forall g \in G \; gHg^{-1} = H$, i.e.

$$\forall\, g \in G, h_1 \in H \;\; \exists h_2 \in H \mid g h_1 g^{-1} = h_2$$

Obviously, a subgroup $N < G$ is normal if and only if it is self-conjugate. Now, we can define the group quotient since, one can prove that if $N \lhd G$ is a normal subgroup, then G/N is a group.

Definition 16. If $N < G$, the group G/H is the group whose elements are the the equivalence classes given by the cosets and the composition law is

$$g_1 H \circ g_2 H = g_1 g_2 H$$

with $g_1 g_2 \in G$ in according to the multiplication in G.

With respect to conjugation, we can also observe that if a group G is abelian, all its subgroups are normal. Furthermore, we can notice that the center Z_G of a group G is normal in the group itself ($Z_G \lhd G$).

Definition 17. A group G is said to be simple if it has no non-trivial normal subgroups. A group G is said to be semi-simple if it has no normal abelian subgroups.

Groups can be combined to form a more general group via the product of different groups.

Definition 18. Given two groups G_1 and G_2, the group $G_1 \times G_2$ is the group whose elements are the couple

$$(g_1 \in G_1, g_2 \in G_2) \in G_1 \times G_2$$

with the composition law

$$(g_1, g_2) \circ (g_1', g_2') = (g_1 g_1', g_2 g_2')$$

and

$$e_{G_1 \times G_2} = (e_1, e_2) \qquad (g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1}).$$

For finite groups, the order of the product group is the product of the order of the groups: $|G_1 \times G_2| = |G_1||G_2|$.

Definition 19. Let N and H two groups and ϕ a map from H to the automorphisms of N

$$\begin{aligned} \phi : & H \to \text{Aut } N \\ & h \mapsto \phi_h \end{aligned}$$

the semi-direct product of N and H determined by ϕ is denoted by $N \rtimes_\Phi H$ and defined to be the group whose elements are $(n \in N, h \in H) \in N \rtimes_\Phi H$ and the composition law is

$$(n_1, h_1) \circ (n_2, h_2) = (n_1 \phi_{h_1}(n_2), h_1 h_2)\,, \quad e_{N \rtimes_\Phi H} = (e_N, e_H)\,, \quad (n, h)^{-1} = (\phi_{h^{-1}}(n^{-1}), h^{-1})$$

Notice that, if we denote with $G = N \rtimes_\Phi H$, we have $N \lhd G$ and $H < G$.

Bibliography

[1] A. Kapustin, N. Seiberg, Coupling a QFT to a TQFT and duality, Journal of High Energy Physics 2014 (4) (apr 2014). doi:10.1007/jhep04(2014)001.
URL https://doi.org/10.1007%2Fjhep04%282014%29001

[2] D. Gaiotto, A. Kapustin, N. Seiberg, B. Willett, Generalized global symmetries, Journal of High Energy Physics 2015 (2) (feb 2015). doi:10.1007/jhep02(2015)172.
URL https://doi.org/10.1007%2Fjhep02%282015%29172

[3] E. Sharpe, Notes on generalized global symmetries in QFT, Fortschritte der Physik 63 (11-12) (2015) 659–682. doi:10.1002/prop.201500048.
URL https://doi.org/10.1002%2Fprop.201500048

[4] C. Cordova, T. T. Dumitrescu, K. Intriligator, S.-H. Shao, Snowmass White Paper: Generalized Symmetries in Quantum Field Theory and Beyond (5 2022). arXiv:2205.09545.

[5] D. S. Freed, G. W. Moore, C. Teleman, Topological symmetry in quantum field theory (2022). doi:10.48550/ARXIV.2209.07471.
URL https://arxiv.org/abs/2209.07471

[6] A. Kapustin, R. Thorngren, Higher symmetry and gapped phases of gauge theories (2013). doi:10.48550/ARXIV.1309.4721.
URL https://arxiv.org/abs/1309.4721

[7] C. Córdova, T. T. Dumitrescu, K. Intriligator, Exploring 2-group global symmetries, Journal of High Energy Physics 2019 (2) (feb 2019). doi:10.1007/jhep02(2019)184.
URL https://doi.org/10.1007%2Fjhep02%282019%29184

[8] F. Benini, C. Córdova, P.-S. Hsin, On 2-group global symmetries and their anomalies, Journal of High Energy Physics 2019 (3) (mar 2019). doi:10.1007/jhep03(2019)118.
URL https://doi.org/10.1007%2Fjhep03%282019%29118

[9] A. Kapustin, R. Thorngren, Anomalous discrete symmetries in three dimensions and group cohomology, Physical Review Letters 112 (23) (jun 2014). doi:10.1103/physrevlett.112.231602.
URL https://doi.org/10.1103%2Fphysrevlett.112.231602

[10] A. Kapustin, R. Thorngren, Anomalies of discrete symmetries in various dimensions and group cohomology (2014). doi:10.48550/ARXIV.1404.3230.
URL https://arxiv.org/abs/1404.3230

[11] L. Bhardwaj, Y. Tachikawa, On finite symmetries and their gauging in two dimensions, Journal of High Energy Physics 2018 (3) (mar 2018). doi:10.1007/jhep03(2018)189.
URL https://doi.org/10.1007%2Fjhep03%282018%29189

[12] Y. Tachikawa, On gauging finite subgroups, SciPost Physics 8 (1) (jan 2020). doi:10.21468/scipostphys.8.1.015.
URL https://doi.org/10.21468%2Fscipostphys.8.1.015

[13] C.-T. Hsieh, Y. Tachikawa, K. Yonekura, Anomaly inflow and p-form gauge theories (2020). doi:10.48550/ARXIV.2003.11550.
URL https://arxiv.org/abs/2003.11550

[14] T. Banks, N. Seiberg, Symmetries and strings in field theory and gravity, Physical Review D 83 (8) (apr 2011). doi:10.1103/physrevd.83.084019.
URL https://doi.org/10.1103%2Fphysrevd.83.084019

[15] B. Heidenreich, J. McNamara, M. Montero, M. Reece, T. Rudelius, I. Valenzuela, Non-invertible global symmetries and completeness of the spectrum, Journal of High Energy Physics 2021 (9) (sep 2021). doi:10.1007/jhep09(2021)203.
URL https://doi.org/10.1007%2Fjhep09%282021%29203

[16] B. Heidenreich, J. McNamara, M. Montero, M. Reece, T. Rudelius, I. Valenzuela, Chern-weil global symmetries and how quantum gravity avoids them, Journal of High Energy Physics 2021 (11) (nov 2021). doi:10.1007/jhep11(2021)053.
URL https://doi.org/10.1007%2Fjhep11%282021%29053

[17] J. FRÖHLICH, J. FUCHS, I. RUNKEL, C. SCHWEIGERT, DEFECT LINES, DUALITIES AND GENERALISED ORBIFOLDS (mar 2010). doi:10.1142/9789814304634_0056.
URL https://doi.org/10.1142%2F9789814304634_0056

[18] S. Giaccari, R. Volpato, A fresh view on string orbifolds (2022). doi:10.48550/ARXIV.2210.10034.
URL https://arxiv.org/abs/2210.10034

[19] P. Candelas, G. T. Horowitz, A. Strominger, E. Witten, Vacuum configurations for superstrings, Nuclear Physics B 258 (1985) 46–74. doi:https://doi.org/10.1016/0550-3213(85)90602-9.
URL https://www.sciencedirect.com/science/article/pii/0550321385906029

[20] T. Rudelius, S.-H. Shao, Topological operators and completeness of spectrum in discrete gauge theories, Journal of High Energy Physics 2020 (12) (dec 2020). doi:10.1007/jhep12(2020)172.
URL https://doi.org/10.1007%2Fjhep12%282020%29172

[21] M. Nakahara, Geometry topology and physics / Mikio Nakahara, 2nd Edition, Graduate student series in physics, IOP, Bristol Philadelphia, c2003.

[22] A. Hatcher, Algebraic topology / Allen Hatcher, Cambridge University Press, Cambridge [etc, c2001.

[23] J. F. Davis, P. Kirk, Lecture notes in algebraic topology / James F. Davis, Paul Kirk, Graduate studies in mathematics, American Mathematical Society, Providence, c2001.

[24] K. S. Brown, Cohomology of groups / Kenneth S. Brown, Graduate texts in mathematics, Springer-Verlag, New York Heidelberg Berlin, c1982.

[25] R. Dijkgraaf, E. Witten, Topological gauge theories and group cohomology, Communications in Mathematical Physics 1990 (129) (apr 1990). doi:10.1007/BF02096988.
URL https://doi.org/10.1007/BF02096988

[26] M. d. W. Propitius, Topological interactions in broken gauge theories (1995). doi:10.48550/ARXIV.HEP-TH/9511195.
URL https://arxiv.org/abs/hep-th/9511195

[27] J. Polchinski, String theory 1: An introduction to the bosonic string / Joseph Polchinski, Cambridge University Press, Cambridge, 2005.

[28] J. Polchinski, String theory 2: Superstring theory and beyond / Joseph Polchinski, Cambridge University Press, Cambridge, 2005.

[29] D. Tong, Lectures on string theory, Lectures notes (2009).
URL https://www.damtp.cam.ac.uk/user/tong/string.html

[30] K. Becker, String theory and M-theory : a modern introduction / Katrin Becker ; Melanie Becker ; John H. Schwarz, Cambridge University Press, Cambridge, 2007.

[31] T. Ortin, Gravity and strings / Tomas Ortin, Cambridge monographs on mathematical physics, Cambridge University press, Cambridge, c2004.

[32] L. Dixon, J. Harvey, C. Vafa, E. Witten, Strings on orbifolds, Nuclear Physics B 261 (1985) 678–686. doi:https://doi.org/10.1016/0550-3213(85)90593-0.
URL https://www.sciencedirect.com/science/article/pii/0550321385905930