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Conservatism and Scalability for Distributed Controller Synthesis of Interconnected Systems

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Abstract

This Master project concerns the problem of distributed controller synthesis for a special class of interconnected systems known as "decomposable systems" in a large-scale setting. Due to computational limitations, scalability constitutes one of the major concerns of the controller synthesis of such large-scale systems. One possibility to deal with this limitation is to decompose the synthesis conditions by enforcing some structure on the Lyapunov and multiplier matrices. However, this approach in general introduces conservatism. The aim of this project is to analyze the conservatism introduced by imposing specific structures on the matrices involved and then to reduce the conservatism imposing less structure, however still achieving some degree of decomposition. The strongest limitation in terms of performance, in the state of the art of the controller synthesis for this kind of systems, comes from the structure imposed on the Lyapunov matrix that is needed both for the decomposition and for the achievement of the desired distributed controller structure. During this project a new method, based on the Full Block S-Procedure and on the extended formulation of the Bounded Real Lemma, has been designed. It will be shown that, with the proposed method, it is possible to impose less restrictive structures on the Lyapunov matrix by still keeping the same degree of decomposition, and thus reduce the conservatism.

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Chapter 1

Introduction

There is a wide variety of applications, that have been made possible by modern technologies, that are characterised by their large scale, their distributed nature and the sparse structure of their physical interconnections [30]. This is the case for systems such as new smart grids, the internet, automated highways [30], satellite formation flying [17, 31], car platoons [24], unmanned aerial vehicles [1, 23] and large telescopes [28], [36]. For this reason, in recent times, the Systems and Control community has cultivated an increasing interest in the modelling and control of these systems in which many agents interacting through a network are involved. Distributed systems constitute one class of interconnected systems. When systems are physically interconnected we refer to them as "spatially distributed systems" and this is the case of automated highway systems [35], airplane formation flight, satellite constellations [17, 31]. We call, instead, "virtually interconnected systems" those systems in which all the agents are not spatially interconnected, but they share information in order to reach a common goal as it happens, for instance, when dealing with mobile robots [33]. Groups of identical subsystems, that are connected either physically or virtually, constitute a "homogeneous system", see [3].

When dealing with interconnected systems, three possible control strategies can be adopted, namely centralized, distributed and decentralized control. In a centralized control model, one component plays the role of controller and it is responsible of managing and coordinating the tasks of all the agents in the network. In a decentralised approach, instead, each component of the network adopts an independent controller calibrated exclusively on measured local signals. In a distributed control strategy, finally, each agent of the network has its autonomous controller and all the local controllers are allowed to communicate among themselves.

Very often the control of large-scale networked systems results to be a challenging problem since a wide number of inputs and outputs can be involved. Whenever the number of interconnected systems grows the controller synthesis becomes more complex and, in order to render it feasible, it is fundamental to reach an adequate level of decomposition and scalability of the synthesis equations. When the subsystems in the network are subject to communication constraints [6], in fact, both the synthesis and the implementation of a centralised controller can be not feasible in practice and either a decentralised or a distributed control architecture may be needed [29]. However, in applications in which a strong interaction among the subsystems is needed, a completely decentralised architecture of the controller may not guarantee good performance [12].

This project focuses on linear time invariant homogeneous systems in a large scale setting. The peculiarity of these systems is that they can be expressed in a state space form through system matrices with identical diagonal blocks and non zero off-diagonal terms. Moreover we will focus our attention on a specific class of systems known in literature as "decomposable systems" for which a special, simplified, control synthesis procedure can be adopted. For a decomposable system, in fact, as explained in [29], the off-diagonal terms assume a block structure whose disposition mirrors the interconnection topology of the network and thus they can be formalised

through a Kronecker product in which a "pattern matrix" is involved. It must be said that the idea of decomposing a system in order to simplify the controller synthesis has been widely exploited in the past and has found many applications in different contexts such as symmetrically interconnected systems [22, 40] and in SVD (Singular Value Decomposition) controllers [4]. From this formalisation, in fact, many interesting properties can be deduced and a controller with a distributed architecture can be easily designed. In the context of interconnected systems, exploiting the graph theory, each subsystem can be thought as a node of the graph, the interconnections as the edges and the pattern matrix as the adjacency matrix. Many properties related to decomposable systems will be presented and widely exploited in this work, especially when dealing with the LMIs that incorporate the constraints to be fulfilled to meet the stability and $\mathcal{H}_\infty$ performance requirements for the systems. It will be shown, then, that a MIMO (Multi Input Multi Output) system with n inputs and n outputs can be treated as n SISO (Single Input Single Output) systems.
Goal of this project is to bring some improvements with respect to the state of the art of the research related to the distributed controller synthesis for interconnected decomposable systems, reducing the conservatism introduced by the decomposition of the conditions to be solved. The constraints to be satisfied are derived from the Bounded Real Lemma and the extended Lyapunov stability inequality exposed in [8] and will be reformulated according to the Full Block S-Procedure presented by C. W. Schrerer in [39].
This thesis is organised as follows. Chapter 2 contains the preliminary notions and the mathematical tools exploited in this project, while in Chapter 3 an alternative stability condition and the extended version of the Bounded Real Lemma are introduced. In Chapter 4 the setting in which our problem is formulated is introduced, while Chapter 5 is the core of this work. It explains the $\mathcal{H}_\infty$ controller synthesis method derivation as a reformulation of the Full Block S-Procedure [39], the extended Lyapunov inequality [8] and the Bounded Real Lemma. In Chapter 6 some numerical results are shown, while Chapter 7 concludes this work.

Chapter 2

Preliminaries

In this chapter, the mathematical tools exploited in this Master Degree project are introduced.

2.1 Vector Norms and Matrix Norms

In this section the definitions of vector norms and matrix norms are recalled since they play a leading role in the synthesis and design of $\mathcal{H}_\infty$ controllers that are the core of this project. These concepts will be widely exploited whenever we will consider the $\mathcal{H}_\infty$ norm of transfer functions or transfer matrices.
Let $\mathcal{X}$ be a vector space. A function $\|\cdot\|$: $\mathcal{X} \to \mathbb{R}$ is said to be a *norm* on $\mathcal{X}$ if it satisfies the following properties:

(i) $\|x\| \geq 0$ (nonnegativity);

(ii) $\|x\| = 0$ if and only if $x = 0$ (positive definiteness);

(iii) $\|\alpha x\| = |\alpha|\ \|x\|$ for any scalar α (homogeneity);

(iv) $\|x+y\| \geq \|x\| + \|y\|$ for any $x,\ y \in \mathcal{X}$ (triangle inequality).

Let $x \in \mathbb{C}^n$, then the vector p-norm of x is defined as

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}, \text{ for } 1 \leq p < \infty. \tag{2.1}$$

The norms that are most commonly used in control as performance criteria are the following ones.

Vector 1-norm

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|, \tag{2.2}$$

vector 2-norm

$$\|x\|_2 := \sqrt{\sum_{i=1}^{n} |x_i|^2}, \tag{2.3}$$

vector ∞-norm

$$\|x\|_\infty := \max_{1 \leq i \leq n} |x_i|. \tag{2.4}$$

Let $Y = [y_{ij}] \in \mathbb{C}^{m \times n}$, then the matrix norm induced by the vector p-norm is called *induced* $p-$*norm* and is defined as

$$\|Y\|_p := \sup_{x \neq 0} \frac{\|Yx\|_p}{\|x\|_p}. \tag{2.5}$$

For matrix norms, in addition to the properties (i)-(iv), it holds that:
(v) $\|XY\| \geq \|X\|\|Y\|$ for any $X,\ Y \in \mathbb{C}^{m\times n}$ (Cauchy-Schwarz inequality).

From a system theory perspective, the induced norm can be interpreted as an input/output amplification gain.

2.1.1 Signal Norms

This mathematical concept will be exploited when we will investigate the $\mathcal{L}_2$ gain of a system (see Section 2.2) and when we will give a physical interpretation of the $\mathcal{L}_\infty$ norm of a system (Section 2.5).
Consider a multivariate signal $x(t)$ that maps $t \in]-\infty, \infty[$ to $\mathbb{C}^n$. Signals norms fulfil the norms properties (i)-(iv) and are defined as follows
signal p-norm

$$\|x(t)\|_p := \left(\int_{-\infty}^{\infty}\sum_{i=1}^{n}|x_i(\tau)|^p d\tau\right)^{\frac{1}{p}}. \tag{2.6}$$

In the following, some signals norms of particular interest are enlisted.
Signal 1-norm

$$\|x(t)\|_1 := \int_{-\infty}^{\infty}\sum_{i=1}^{n}|x_i(\tau)|d\tau = \int_{-\infty}^{\infty}\|x(\tau)\|_1 d\tau, \tag{2.7}$$

signal 2-norm

$$\|x(t)\|_2 := \sqrt{\int_{-\infty}^{\infty}\sum_{i=1}^{n}|x_i(\tau)|^2 d\tau} = \sqrt{\int_{-\infty}^{\infty}\|x(\tau)\|_2 d\tau}, \tag{2.8}$$

signal ∞-norm

$$\|x(t)\|_\infty := \max_\tau \left(\max_i|x_i(\tau)|\right) = \max_\tau(\|x(\tau)\|_\infty). \tag{2.9}$$

2.2 $\mathcal{L}_2$ Gain

Before introducing the *Bounded Real Lemma* in Section 2.5, we first need to recall the concept of $\mathcal{L}_2$ gain of a system. To do so we first recall that the $\mathcal{L}_2$ gain of a signal $z(t) : (-\infty, +\infty) \to \mathbb{C}$ is defined as

$$\|z\|_2 := \left(\int_{-\infty}^{\infty}\|z(t)\|^2 dt\right)^{\frac{1}{2}}. \tag{2.10}$$

Consider now a system of transfer function $T(s)$ with input w and output z as shown in Figure 2.1.

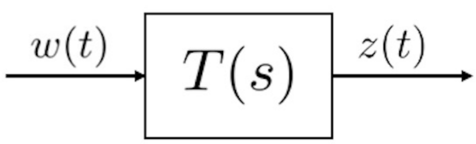

Figure 2.1: Transfer function from w to z

Its $\mathcal{L}_2$ gain is defined as

$$\|T(s)\|_2 := \max_{w\neq 0}\frac{\|z\|_2}{\|w\|_2}. \tag{2.11}$$

If we look at the signal w as a disturbance and at the signal z as a performance output, bounding the $\mathcal{L}_2$ gain of the system $T(s)$ amounts to bounding the effects of the disturbance on the performance output. Let us introduce thus a Lyapunov method to bound $\mathcal{L}_2$ gain [25].

Theorem 2.2.1 *Let us consider a generic non-linear system*

$$\dot{x} = f(x, w), \quad x(0) = 0 \tag{2.12}$$

$$z = g(x, w) \tag{2.13}$$

with $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{m_w}$ and $z(t) \in \mathbb{R}^{r_z}$ and $f : \mathbb{R}^{n \times m_w} \to \mathbb{R}^n$ a class $\mathcal{C}^1$ function. Suppose there exists $\gamma \geq 0$ and a differentiable Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ with $\dot{V} : \mathbb{R}^{n \times m_w} \to \mathbb{R}$ defined as

$$\dot{V}(x, w) := \nabla V(x)^T f(x, w) \tag{2.14}$$

such that

$$(i) V(x) \geq 0 \text{ for all } x,\ V(0) = 0; \tag{2.15}$$

$$(ii) \dot{V}(x, w) \leq \gamma^2 w^T w - z^T z \text{ for all } z,\ w. \tag{2.16}$$

Then, the $\mathcal{L}_2$ gain of the system is not greater than γ.
Proof:

$$V(x(\infty)) - V(x(0)) = \int_0^\infty \dot{V}(x(t), w(t))dt \leq \int_0^\infty (\gamma^2 w(t)^T w(t) - z(t)^T z(t))dt \tag{2.17}$$

and since $V(x(0)) = V(0) = 0$ and $V(x(\infty)) \geq 0$ it holds that

$$0 \leq V(x(\infty)) - V(x(0)) \leq \int_0^\infty (\gamma^2 w(t)^T w(t) - z(t)^T z(t))dt \tag{2.18}$$

from which

$$\int_0^\infty z(t)^T z(t)dt \leq \gamma^2 \int_0^\infty w(t)^T w(t)dt \tag{2.19}$$

that is

$$\|z\|_2^2 \leq \gamma^2 \|w\|_2^2 \tag{2.20}$$

What has just been stated holds for all z, w and therefore also for the w such that the ratio $\frac{\|z\|_2}{\|w\|_2}$ is maximized. This guarantees that $\|T(s)\|_2 < \gamma$ and this concludes the proof. □

2.3 $\mathcal{H}_\infty$ Space

As stated in [44], robustness to disturbances and uncertainties can be a challenging problem to be addressed in feedback control and hence developing robust control methods is an attractive objective for control engineers. One way to formulate performance requirements for a certain system is in terms of the norm of specific signals of interest: it is in this scenario that $\mathcal{H}_\infty$ robust control techniques have to be located and this is the reason why a first introduction to the Hardy $\mathcal{H}_\infty$ space is presented.

$\mathcal{L}_\infty(j\mathbb{R})$ Space

If $\mathcal{X}$ is a vector space and $\|\cdot\|$ is a norm defined on it, then $\mathcal{X}$ is a *normed vector space*.
A sequence $\{x\}_n$ is a *Cauchy sequence* if $\forall\epsilon$ there exists an integer number N such that for every pair of positive integer numbers $n, m > N$, it holds $|x_n - x_m| < \epsilon$.
A vector space $\mathcal{X}$ is called *complete*, if every Cauchy sequence $\{x\}_n$ in $\mathcal{X}$ converges to a point belonging to $\mathcal{X}$ for $n \to \infty$.
A complete normed vector space is called *Banach space*.
A $\mathcal{L}_\infty(j\mathbb{R})$ space, or briefly $\mathcal{L}_\infty$, is a Banach space of matrix-valued (or scalar-valued) functions $F(s) : \mathbb{C} \to \mathbb{C}^{n\times m}$ that are bounded on $j\mathbb{R}$, with norm

$$\|F\|_\infty := \sup_{\omega\in\mathbb{R}} \bar{\sigma}[F(j\omega)], \tag{2.21}$$

where the symbol $\bar{\sigma}[F(j\omega)]$ indicates the maximum singular value of $F(j\omega)$.
The rational subspace of $\mathcal{L}_\infty$, denoted by $\mathcal{RL}_\infty(j\mathbb{R})$, consists of all proper and real rational transfer matrices with no poles on the imaginary axis.

$\mathcal{H}_\infty$ Space

$\mathcal{H}_\infty$ is a closed[1] subspace of $\mathcal{L}_\infty$ consisting of all functions $F(s) : \mathbb{C} \to \mathbb{C}^{n\times m}$ that are analytic and bounded in the open right-half plane. The $\mathcal{H}_\infty$ norm is defined as

$$\|F\|_\infty := \sup_{Re(s)>0} \bar{\sigma}[F(s)] = \sup_{\omega\in\mathbb{R}} \bar{\sigma}[F(j\omega)], \tag{2.22}$$

where the second equality follows from a generalisation, for matrix functions, of the following theorem.

Maximum modulus theorem [9] *Let $S \subset \mathbb{C}$ be an open set, and let $f(s) : S \to \mathbb{C}$ be a complex function on S analytic in its domain.*[2] *If $f(s)$ is continuous on a closed-bounded set S and analytic in the interior of S, then $|f(s)|$ can not reach its maximum in the interior of S unless $f(s)$ is a constant function.*

The maximum modulus theorem implies that $|f(s)|$ can only achieve its maximum on the boundary of its domain (∂S) and hence

$$\max_{s\in S}|f(s)| = \max_{s\in\partial S}|f(s)|. \tag{2.23}$$

The real rational subspace of $\mathcal{H}_\infty$ is denoted by $\mathcal{RH}_\infty$ and consists of all proper and real rational stable[3] transfer matrices.

2.4 Hamiltonian Matrices

Hamiltonian matrices constitute a category of matrices that exhibit some interesting properties. The *Bounded Real Lemma* in Section 2.5 will refer to them.

[1] A vector space $\mathcal{X}$ is said to be *closed* if it contains all its accumulation points.

[2] A function $f(s) : S \to \mathbb{C}$ is said to be *analytic at a point* z_0 if it is differentiable at z_0 and at each point in some neighbourhood of z_0, and thus it admits a power series representation around the point z_0. A function $f(s) : S \to \mathbb{C}$ is said to be *analytic in S* if it is analytic at each point of S. A matrix-valued function is said to be analytic in S if every element of the matrix is analytic in S.

[3] For a continuous time system, a transfer matrix is said to be *stable* if all its poles lie in the open left half-plane

Consider the square matrix $J \in \mathbb{R}^{2n\times 2n}$ defined by

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}, \tag{2.24}$$

where the symbol $0_n \in \mathbb{R}^{n\times n}$ indicates the zero matrix of dimension n by n, while the symbol $I_n \in \mathbb{R}^{n\times n}$ indicates the identity matrix of the same dimensions.

Definition 2.4.1 (from [42]) A matrix $A \in \mathbb{R}^{2n\times 2n}$ is said to be *Hamiltonian* if JA is symmetric and therefore

$$JA = (JA)^T \implies A^T J + JA = 0 \tag{2.25}$$

where the implication follows directly from the fact that $J^T = -J$.
The set of $2n \times 2n$ Hamiltonian matrices is denoted by

$$\mathcal{H}^n = \{A \in \mathbb{R}^{2n\times 2n} | A^T J + JA = 0\}$$

Properties [9]

In the following a list of some interesting properties of Hamiltonian matrices is provided.
Given a matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.26}$$

where $A, B, C, D \in \mathbb{M}^{n\times n}$, the following properties hold:

(i) M is Hamiltonian if B and C are symmetric matrices and $A + D^T = 0$;

(ii) the transpose of a Hamiltonian matrix is still a Hamiltonian matrix;

(iii) the trace of a Hamiltonian matrix is zero;

(iv) the eigenvalues of a Hamiltonian matrix are symmetric with respect to the imaginary axis.

2.5 $\mathcal{L}_\infty$ Norm of a System and the Bounded Real Lemma

2.5.1 $\mathcal{L}_\infty$ Norm of a System

The $\mathcal{L}_\infty$ norm of a transfer function $T(s)$ [9], is the maximum norm of the matrix $T(j\omega)$ over all possible ω and thus, it is well defined if and only if $T(s)$ does not have poles on the imaginary axis. For a SISO system it corresponds to the maximum value of the Bode amplitude plot of the transfer function. For MIMO systems, instead, it corresponds to the maximum of the maximum singular value over all possible ω. In other words, for a system with transfer function $T(s) \in \mathcal{L}_\infty$ it is defined as

$$\|T(s)\|_\infty = \sup_\omega \|T(j\omega)\|$$

In case $T(s) \in \mathcal{H}_\infty$, this norm can be expressed as

$$\|T(s)\| = \sup_{Re(s)>0} \|T(s)\|.$$

Unfortunately there is not a direct way to calculate the $\mathcal{L}_\infty$ norm of a system, but there are some tests that allow one to establish if the $\mathcal{L}_\infty$ norm is smaller than a certain positive value γ, namely if $\|T(s)\|_\infty < \gamma$. This is equivalent to verify if

$$I - \frac{1}{\gamma^2} T(-j\omega)' T(j\omega) > 0, \quad \forall\omega \tag{2.27}$$

and the smallest γ for which this inequality is satisfied represents the $\mathcal{L}_\infty$ norm (in the following, simply “the infinity norm") of $T(s)$.

2.5.2 Physical Interpretation of the $\mathcal{L}_\infty$ norm of a System

Let us consider a system with transfer function $T(s)$, as in Figure 2.1, that takes as input the signal $w(t)$, for example an exogenous input, and gives as output the signal $z(t)$. In this context, in which w is the disturbance input and z the performance output, the infinity norm of a system can be interpreted either as the worst case gain in frequency domain

$$\|T(s)\|_\infty = \max_{\omega} \max_{w(\omega)\neq 0} \frac{\|z(\omega)\|_2}{\|w(\omega)\|_2},$$

or as the worst case induced 2-norm in time domain

$$\|T(s)\|_\infty = \max_{w(t)\neq 0} \frac{\|z(t)\|_2}{\|w(t)\|_2}.$$

The worst case input is a sinusoid with frequency ω^* in the direction to which it corresponds as output the highest peak of the maximum singular value of $T(j\omega^*)$ [11]. More generally, we can write

$$\|z\|_2 \leq \|T\|_\infty \|w\|_2. \tag{2.28}$$

So, the $\mathcal{H}_\infty$ norm of $T(s)$ provides a bound on the energy throughput of the system. The objective is to minimise the infinity norm of this system in order to keep as small as possible the energy of the output signal with respect to the energy of the exogenous input.
Tests to compute the $\mathcal{H}_\infty$ norm of a system follow directly from the *Bounded Real Lemma* to which the following section is dedicated.

2.5.3 The Bounded Real Lemma

The Bounded Real Lemma [10] is the starting point for the derivation of a controller synthesis method in the context of $\mathcal{H}_\infty$ control. It will be widely exploited in Chapter 5.
Let us consider the closed-loop system

$$\dot{x} = A_{cl}x + B_w w \tag{2.29}$$
$$z = C_{cl}x \tag{2.30}$$

with strictly proper transfer function $T(s) = C_{cl}(sI - A_{cl})^{-1}B_w$*, then the following are equivalent*

(i) $\|T\|_\infty < \gamma$;
(ii) The matrix $\Phi(s) := \gamma^2 I - T^T(-s)T(s)$ *satisfies* $\Phi(j\omega) > 0, \quad \forall\omega \in \mathbb{R}$;
(iii) The Hamiltonian Matrix

$$M_\gamma := \begin{pmatrix} A_{cl} & \frac{1}{\gamma^2}B_w B_w^T \\ -C_{cl}^T C_{cl} & -A_{cl} \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$$

has no purely imaginary eigenvalues;
(iv) There exists a matrix $X \succ 0$ *such that*

$$\begin{pmatrix} A_{cl}^T X + X A_{cl} + C_{cl}^T C_{cl} & X B_w \\ B_w^T X & -\gamma^2 I \end{pmatrix} \prec 0 \tag{2.31}$$

In Appendix A some comments about this theorem and its time domain interpretation are given.

2.6 Linear Fractional Transformation

A Linear Fractional Transformation (LFT) is a useful way to standardise and make different block diagrams comparable for robust control analysis and design. It will be exploited in Section 4.3 for the calculation of the $\mathcal{H}_\infty$ system norm.
A mapping $F : \mathbb{C} \to \mathbb{C}$ of the form:

$$F(s) = \frac{a + bs}{c + ds} \tag{2.32}$$

with a, b, c and $d \in \mathbb{C}$ is called an LFT. If $c \neq 0$ then $F(s) = \alpha + \beta s(1 - \gamma s)^{-1}$ for some $\alpha, \beta, \gamma \in \mathbb{C}$.
Definition 2.6.1 Let M be a complex matrix block-partitioned as

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in \mathbb{C}^{(p_1+p_2)\times(q_1+q_2)} \tag{2.33}$$

and let $\Delta_l \in \mathbb{C}^{p_1 \times q_1}$ and $\Delta_u \in \mathbb{C}^{p_2 \times q_2}$. A *lower LFT* is defined as

$$F_l(M, \cdot) : \mathbb{C}^{p_2 \times q_2} \to \mathbb{C}^{p_1 \times q_1} \text{ with } F_l(M, \Delta_l) = M_{11} + M_{12}\Delta_l(I - M_{22}\Delta_l)^{-1}M_{21} \tag{2.34}$$

provided that $(I - M_{22}\Delta_l)^{-1}$ exists.
An *upper LFT* is defined as

$$F_u(M, \cdot) : \mathbb{C}^{p_1 \times q_1} \to \mathbb{C}^{p_2 \times q_2} \text{ with } F_u(M, \Delta_u) = M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12} \tag{2.35}$$

provided that $(I - M_{11}\Delta_u)^{-1}$ exists.
The diagrams in figures 2.2-2.3, taken from [26], clarify the origin of the terminologies "lower and upper LFT".

Lower LFT

$$T_{zw1} := F_l(M, \Delta_l)$$

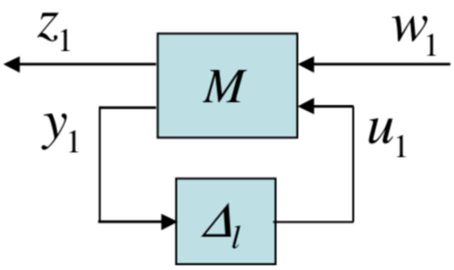

Figure 2.2: Lower LFT

$$\begin{bmatrix} z_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ u_1 \end{bmatrix}, u_1 = \Delta_l y_1 \tag{2.36}$$

$$u_1 = \Delta_l y_1 \tag{2.37}$$
$$y_1 = M_{21}w_1 + M_{22}u_1 \tag{2.38}$$

from which

$$y_1 = M_{21}w_1 + M_{22}\Delta_l y_1, \tag{2.39}$$

that brings

$$y_1 = (I - M_{22}\Delta_l)^{-1} M_{21} w_1 \tag{2.40}$$

$$u_1 = \Delta_l (I - M_{22}\Delta_l)^{-1} M_{21} w_1 \tag{2.41}$$

and then, from (2.36),

$$z_1 = M_{11} w_1 + M_{12} u_1 = \underbrace{[M_{11} + M_{12}\Delta_l (I - M_{22}\Delta_l)^{-1} M_{21}]}_{F_l(M,\Delta_l)} w_1. \tag{2.42}$$

Upper LFT

$$T_{zw2} := F_u(M, \Delta_u)$$

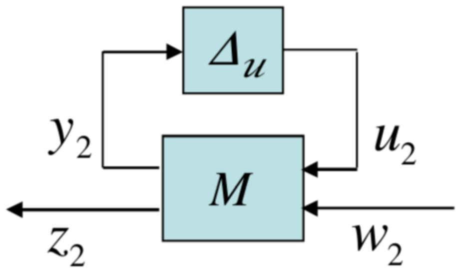

Figure 2.3: Upper LFT

2.7 Schur Complement

The Schur complement is a very powerful mathematical tool that allows to convert nonlinear inequalities into Linear Matrix Inequalities (LMIs). This is a reformulation of what is discussed in detail in Section 2 of [19]. This mathematical tool will come in handy in Section 5.7 for the derivation of the extended version of the Full Block S-Procedure.
Suppose to deal with the system

$$Ma = b, \tag{2.43}$$

with $M \in \mathbb{R}^{n \times n}$,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.44}$$

a block-partitioned matrix, where $A \in \mathbb{R}^{p \times p}$, $D \in \mathbb{R}^{q \times q}$ and $n = p + q$. Moreover $a = (x, y)^T \in \mathbb{R}^n$, $b = (c, d)^T \in \mathbb{R}^n$. Assuming that the submatrix D is non-singular, the matrix $(A - BD^{-1}C)$ is called the *Schur complement of* D *in* M, while, if A is invertible, the Schur complement of A in M is defined as $(D - CA^{-1}B)$. The following explains why these matrices deserve attention. System (2.43) can be equivalently written as

$$Ax + By = c \tag{2.45}$$

$$Cx + Dy = d \tag{2.46}$$

and, under the assumption that D is invertible, it follows from (2.46) that

$$y = D^{-1}(d - Cx) \tag{2.47}$$

and substituting in (2.45) we get

$$Ax + B(D^{-1}(d - Cx)) = c, \tag{2.48}$$

that can be rewritten as

$$(A - BD^{-1}C)x = c - BD^{-1}d. \tag{2.49}$$

If $(A - BD^{-1}C)^{-1}$ exists then we get the solutions

$$x = (A - BD^{-1}C)^{-1}(c - BD^{-1}d) \tag{2.50}$$

$$y = D^{-1}(d - C(A - BD^{-1}C)^{-1}(c - BD^{-1}d)) \tag{2.51}$$

Analogously, if A is invertible, starting from (2.45), we can write $x = A^{-1}(c - By)$, and plugging it in (2.46) and repeating the same steps as above, it turns out that the Schur complement of A in M is $(D - CA^{-1}B)$.
Equation(2.43) yields to

$$\begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} c \\ d \end{pmatrix}. \tag{2.52}$$

While, rewriting the equations (2.50) and (2.51) as

$$x = (A - BD^{-1}C)^{-1}c - (A - BD^{-1}C)^{-1}BD^{-1}d \tag{2.53}$$

$$y = D^{-1}C(A - BD^{-1}C)^{-1}c + (D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1})d \tag{2.54}$$

comparing (2.53), (2.54) with (2.52) it turns out that

$$M^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{pmatrix} \tag{2.55}$$

can be factorised as follows

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & 0 \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} \end{pmatrix} \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix} \tag{2.56}$$

and then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix} \begin{pmatrix} (A - BD^{-1}C)^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix} \tag{2.57}$$

and from the property of the inverse of a product of matrices it turns out that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}. \tag{2.58}$$

2.7.1 Schur Complement as Check for Positive Definiteness of Symmetric Matrices

Let us suppose now that M is symmetric, which means $A = A^T$, $D = D^T$ and $C = B^T$ so that

$$M = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}B^T & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}^T \tag{2.59}$$

This leads to the following proposition.

Proposition 3.2.1 *For any symmetric matrix M of the form*

$$M = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

if D is invertible, then the following properties hold:

(i) $M \succ 0$ *iff* $D \succ 0$ *and* $A - BD^{-1}B^T \succ 0$,
(ii) If $D \succ 0$, *then* $M \succeq 0$ *iff* $A - BC^{-1}B^T \succeq 0$.

For the proof, see Appendix B.

2.8 Kronecker Product

The Kronecker product is a mathematical operator that is widely used when dealing with distributed and multiagent control. It has some interesting properties that will be exploited in Chapter 4, when dealing with the properties of decomposable systems, and in Section 5.8 where a decomposition approach for the Full Block S-Procedure will be introduced.
Given two matrices $A \in \mathbb{M}^{m\times n}$ and $B \in \mathbb{M}^{p\times q}$, the Kronecker product of

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \text{ and } B = \begin{pmatrix} = b_{11} & \dots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{p1} & \dots & b_{pq} \end{pmatrix},$$

denoted by the symbol $A \otimes B$, returns a matrix $C \in \mathbb{M}^{mp\times nq}$ of the type

$$C = (A \otimes B) := \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix}$$

2.8.1 Properties

The Kronecker product has a lot of interesting properties, some of them are listed in this section and can be easily verified by the reader. For more specific details about the Kronecker product the reader can refer to [21].

Bilinearity and Associativity

Given the matrices $A \in \mathbb{M}^{m\times n}$, $B \in \mathbb{M}^{p\times q}$ and $C \in \mathbb{M}^{r\times s}$ it holds that

$$A \otimes (B + C) = (A \otimes B) + (A \otimes C) \tag{2.60}$$
$$(A + B) \otimes C = (A \otimes C) + (B \otimes C) \tag{2.61}$$

Mixed Product

Given the matrices $A \in \mathbb{M}^{m\times n}$, $B \in \mathbb{M}^{p\times q}$, $C \in \mathbb{M}^{n\times r}$, $D \in \mathbb{M}^{q\times s}$, it holds that

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \tag{2.62}$$

Inverse of a Kronecker Product

Given two invertible matrices A, B then $(A \otimes B)^{-1}$ exists and it is given by

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \tag{2.63}$$

This property holds for the Moore-Penrose pseudoinverse as well.

2.9 Dissipativity of a System

Before introducing an extended version of the Bounded Real Lemma in Section 3.2, it is worth to define the concept of *dissipativity* of a system.

Definition 2.9.1 The system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{2.64}$$

is said to be *dissipative* with respect to a supply rate $s(z, w) : \mathbb{R}^{r_z \times m_w} \to \mathbb{R}$ if there exists a continuously differentiable storage function $V(x) : \mathbb{R}^n \to R$ such that $V(0) = 0$, $V(x) > 0 \; \forall x \neq 0$ and

$$\dot{V}(x) + s(z, w) \leq 0. \tag{2.65}$$

Definition 2.9.2 System (2.64) is said to be *strictly dissipative* with respect to the supply rate $s(z, w)$ if there exists a continuously differentiable storage function $V(x)$ such that $V(0) = 0$, $V(x) > 0 \; \forall x \neq 0$ and

$$\dot{V}(x) + s(z, w) < 0. \tag{2.66}$$

Chapter 3

An Alternative Stability Condition and the Extended Bounded Real Lemma

In this chapter an extended and less conservative stability condition with respect to the classic Lyapunov stability inequality for continuous time systems is shown. Afterwards an extended version of the Bounded Real Lemma, that incorporates the extended stability condition, is presented. This chapter is a reformulation of the results presented in the research works [8, 13, 14] that, together with the Full-Block S-Procedure in [39], constitute the inspiration of this master project.

3.1 Extended Stability Condition for Continuous-time Systems

Consider a generic continuous time system expressed in state space form:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times l}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times l}$.

Definition 3.1.1 System (3.1) is *asymptotically stable* if all the eigenvalues of the A matrix lie in the (open) left half of the complex plane.

If the system is asymptotically stable, there exists a positive definite matrix P such that the Lyapunov matrix inequality

$$PA + A^T P \prec 0 \tag{3.2}$$

holds.

Theorem 3.1.1 [8]: *If there exist a positive definite matrix* $P \in \mathbb{M}^{n\times n}$[1] *and a matrix* $F \in \mathbb{M}^{n\times n}$ *such that (i)-(ii) hold, the following conditions are equivalent*:

(i) $PA + A^T P \prec 0$;

(ii) $\begin{pmatrix} FA + A^T F & P - F + A^T F^T \\ P - F^T + FA & -F - F^T \end{pmatrix} \prec 0.$

Proof : Referring to the system (3.1), let us define $z := \dot{x}$.

The equation

$$\dot{x} = Ax + Bu$$

can be rewritten as:

$$\underbrace{\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}}_{:=E} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & -I \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix} u \tag{3.3}$$

[1]Note that the matrices P such that (i) and (ii) are satisfied are not necessarily the same.

and defining $\xi := \begin{pmatrix} x \\ z \end{pmatrix}$ the quadratic Lyapunov function can be expressed as:

$$V(\xi) = x^T P x = \xi^T \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \xi. \tag{3.4}$$

Let us call E the descriptor matrix of the system (3.3), namely

$$E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.5}$$

It holds that

$$\begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P & F \\ 0 & F \end{pmatrix} E = E^T \begin{pmatrix} P & 0 \\ F^T & F^T \end{pmatrix}, \tag{3.6}$$

and thus $\dot{V}$ can be expressed as:

$$\dot{V}(\xi) = \xi^T \begin{pmatrix} P & F \\ 0 & F \end{pmatrix} E\dot{\xi} + \dot{\xi}^T E^T \begin{pmatrix} P & 0 \\ F^T & F^T \end{pmatrix} \xi, \tag{3.7}$$

from which the new stability condition turns out to be

$$\begin{pmatrix} P & F \\ 0 & F \end{pmatrix} \begin{pmatrix} 0 & I \\ A & -I \end{pmatrix} + \begin{pmatrix} 0 & A^T \\ I & -I \end{pmatrix} \begin{pmatrix} P & 0 \\ F^T & F^T \end{pmatrix} \prec 0 \tag{3.8}$$

that brings to

$$\begin{pmatrix} FA + A^T F & P - F + A^T F^T \\ P - F^T + FA & -F - F^T \end{pmatrix} \prec 0 \tag{3.9}$$

and this completes the proof. □
The advantage of this extended formulation is that it eliminates the product between the matrix A and the Lyapunov matrix P, introducing extra degrees of freedom by means of the matrix F, that is not restricted to be symmetric. The decoupling between the Lyapunov matrix and the system matrices will prove to be very beneficial to obtain a less conservative controller synthesis procedure.

3.2 Extended LMI for $\mathcal{H}_\infty$ Controller Synthesis

Consider the continuous time closed-loop time-invariant linear system

$$\dot{x} = A_{cl}x + B_w w \tag{3.10}$$

$$z = C_{cl}x + D_w w \tag{3.11}$$

in which w represents the exogenous input and z the performance output. The control input u in (3.1), has been replaced by the expression $u = Kx$ and $A_{cl} = A + B_u K$, $C_{cl} = C + B_u K$. In the following $T(s)$ denotes the transfer function from the input w to the output z.
Lemma 3.2.1 ($\mathcal{H}_\infty$ norm)[13] Let $\|T\|_\infty$ denote the $\mathcal{H}_\infty$ norm of T. A_{cl} is stable and the $\mathcal{H}_\infty$ norm of T is smaller than γ if and only if there exists a symmetric matrix $P \succ 0$ such that

$$\begin{pmatrix} A_{cl}^T P + P A_{cl} & P B_w & C_{cl}^T \\ B_w^T P & -\gamma I & D_w \\ C_{cl} & D_w & -\gamma I \end{pmatrix} \prec 0. \tag{3.12}$$

Notice that this is an equivalent formulation of the Bounded Real Lemma presented in Section 2.5.3. In fact, by applying the Schur complement, and imposing $D_w = 0$, from (3.12) we obtain:

$$(i) - \gamma I \prec 0 \tag{3.13}$$

$$(ii) \begin{pmatrix} A_{cl}^T P + PA_{cl} + \frac{1}{\gamma} C_{cl}^T C_{cl} & PB_w \\ B_w^T P & -\gamma I \end{pmatrix} \prec 0 \tag{3.14}$$

That is equivalent to (2.31) after the substitution $D = 0$ up to a γ factor on the last diagonal element and a $\sqrt{\gamma}$ factor on the channel $z \to w$.
Adapting the inequality (3.1) to system (3.3) it turns out that the expression (3.8) implies that

$$\dot{V}(\xi, w) = \begin{pmatrix} \xi \\ w \end{pmatrix}^T \begin{pmatrix} FA_{cl} + A_{cl}^T F^T & P - F + A_{cl}^T F^T & FB_w \\ P - F^T + FA_{cl} & -F - F^T & FB_w \\ B_w^T F^T & B_w^T F^T & 0 \end{pmatrix} \begin{pmatrix} \xi \\ w \end{pmatrix} \prec 0. \tag{3.15}$$

In fact, after defining a new vector $\varrho := \begin{pmatrix} x \\ \dot{x} \\ w \end{pmatrix} = \begin{pmatrix} \xi \\ w \end{pmatrix}$, we can rewrite the system (3.3) as

$$\underbrace{\begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{:=E'} \begin{pmatrix} \dot{x} \\ \ddot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & I & 0 \\ A_{cl} & -I & B_w \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \\ w \end{pmatrix} \tag{3.16}$$

and the Lyapunov function as

$$V(\varrho) = x^T P x = \varrho^T \begin{pmatrix} P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \varrho. \tag{3.17}$$

So now it holds that

$$\begin{pmatrix} P & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} P & F & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{pmatrix} E' = E'^T \begin{pmatrix} P & 0 & 0 \\ F^T & F^T & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.18}$$

from which

$$\dot{V}(\varrho) = \varrho^T \begin{pmatrix} P & F & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{pmatrix} E' \dot{\varrho} + \dot{\varrho}^T E'^T \begin{pmatrix} P & 0 & 0 \\ F^T & F^T & 0 \\ 0 & 0 & 0 \end{pmatrix} \varrho \tag{3.19}$$

and the new stability condition turns out to be

$$\begin{pmatrix} P & F & 0 \\ 0 & F & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & I & 0 \\ A_{cl} & -I & B_w \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & A_{cl}^T & 0 \\ I & -I & 0 \\ 0 & B_w^T & 0 \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ F^T & F^T & 0 \\ 0 & 0 & 0 \end{pmatrix} \prec 0, \tag{3.20}$$

that leads to

$$\begin{pmatrix} FA_{cl} + A_{cl}^T F^T & P - F + A_{cl}^T F^T & FB_w \\ P - F^T + FA_{cl} & -F - F^T & FB_w \\ B_w^T F^T & B_w^T F^T & 0 \end{pmatrix} \prec 0. \tag{3.21}$$

Theorem 3.2.1 (Extended $\mathcal{H}_\infty$ Norm) [13] *System* (3.1) *is asymptotically stable and* $\|T\|_\infty$ *is smaller than* γ *if and only if there exist a matrix* $P \succ 0$ *and a matrix* F *such that*

$$\begin{pmatrix} FA_{cl} + A_{cl}^T F^T & P - F + A_{cl}^T F^T & FB_w & C_{cl}^T \\ P - F^T + FA_{cl} & -F - F^T & FB_w & 0 \\ B_w^T F^T & B_w^T F^T & -\gamma I & D_w^T \\ C_{cl} & 0 & D_w & -\gamma I \end{pmatrix} \prec 0. \tag{3.22}$$

Proof: (Necessity) If the norm of $\|T\|_\infty$ is less than γ then the system with transfer function T is strictly dissipative with respect to the supply function

$$s(z,w) = \begin{pmatrix} z \\ w \end{pmatrix}^T \begin{pmatrix} \frac{1}{\gamma} I & 0 \\ 0 & -\gamma I \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \frac{1}{\gamma} z^T z - \gamma w^T w \tag{3.23}$$

that is the same as in (2.16) except for a multiplication by a factor $\sqrt{\gamma}$ along the performance channel $z \to w$, that does not invalidate what has been stated.
Recalling that

$$z = \begin{pmatrix} C_{cl} & 0 \end{pmatrix} \xi + Dw \tag{3.24}$$

it turns out that the supply function in (3.23) can be expressed as a function of the variables ξ and w as

$$s(\xi,w) = \begin{pmatrix} \xi \\ w \end{pmatrix}^T \begin{pmatrix} C_{cl} & 0 & D_w \\ 0 & 0 & I \end{pmatrix}^T \begin{pmatrix} \frac{1}{\gamma} & 0 \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} C_{cl} & 0 & D_w \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \xi \\ w \end{pmatrix} \tag{3.25}$$

that is equal to

$$s(\xi,w) = \begin{pmatrix} \xi \\ w \end{pmatrix}^T \begin{pmatrix} \frac{C^T C}{\gamma} & 0 & C_{cl}^T D_w \\ 0 & 0 & 0 \\ \frac{D_w^T C_{cl}}{\gamma} & 0 & D_w^T D_w - \gamma \end{pmatrix} \begin{pmatrix} \xi \\ w \end{pmatrix} \tag{3.26}$$

and from the *strict dissipativity* of the system it directly follows that

$$\dot{V}(\xi) + s(\xi,w) < 0, \tag{3.27}$$

that is, from (2.65) and (3.26),

$$\begin{pmatrix} FA_{cl} + A_{cl}^T F^T + \frac{C_{cl}^T C_{cl}}{\gamma} & P - F + A_{cl}^T F^T & FB_w + C_{cl}^T D_w \\ P - F^T + FA_{cl} & -F - F^T & FB_w \\ B_w^T F^T + \frac{D_w^T C_{cl}}{\gamma} & B_w^T F^T & D_w^T D_w - \gamma \end{pmatrix} \prec 0. \tag{3.28}$$

This is equivalent to (3.22) after applying the Schur complement with respect to $-\gamma I$, namely

$$\begin{pmatrix} FA_{cl} + A_{cl}^T F^T & P - F + A_{cl}^T F^T & FB_w \\ P - F^T + FA_{cl} & -F - F^T & FB_w \\ B_w^T F^T & B_w^T F^T & -\gamma I \end{pmatrix} + \begin{pmatrix} C_{cl}^T \\ 0 \\ D_w^T \end{pmatrix} \frac{1}{\gamma} I \begin{pmatrix} C_{cl} & 0 & D_w \end{pmatrix}. \tag{3.29}$$

(Sufficiency) Suppose there exist $P \succ 0$ and F satisfying the matrix inequality (3.22). Then the asymptotic stability of the system follows directly from the fact that one of the diagonal blocks of the matrix in (3.22) is the extended stability condition (3.9). For what concerns the performance requirements, instead, by reversing the previous steps, we have

$$\underbrace{\begin{pmatrix} FA_{cl} + A_{cl}^T F^T & P - F + A_{cl}^T F^T & FB_w \\ P - F^T + FA_{cl} & -F - F^T & FB_w \\ B_w^T F^T & B_w^T F^T & 0 \end{pmatrix}}_{\dot{V}(\xi)} + \underbrace{\begin{pmatrix} \frac{C_{cl}^T C_{cl}}{\gamma} & 0 & C_{cl}^T D_w \\ 0 & 0 & 0 \\ \frac{D_w^T C_{cl}}{\gamma} & 0 & D_w^T D_w - \gamma \end{pmatrix}}_{s(\xi,w)} \prec 0, \tag{3.30}$$

or equivalently, exploiting the expression (3.23),

$$\dot{V}(\xi) + \begin{pmatrix} z \\ w \end{pmatrix}^T \begin{pmatrix} \frac{1}{\gamma}I & 0 \\ 0 & -\gamma I \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} < 0. \tag{3.31}$$

Integrating from $t = 0$ to $t = T \geq 0$, and assuming $x(0) = 0$[2], we get

$$x(T)^T P x(T) + \int_0^T \begin{pmatrix} z \\ w \end{pmatrix}^T \begin{pmatrix} \frac{1}{\gamma}I & 0 \\ 0 & -\gamma I \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} dt < 0. \tag{3.32}$$

And since $x(T)^T P x(T) \geq 0$, this implies that

$$\int_0^T \gamma w^T w - \frac{1}{\gamma} z^T z \, dt > 0 \tag{3.33}$$

and thus it is a valid supply function for the performance requirements according to what expressed by the Bounded Real Lemma. This concludes the proof. □

[2]By definition of Lyapunov function, $V(0) = 0$.

Chapter 4

Decomposable Systems

This thesis focuses on the problem of designing a distributed controller for a specific class of systems called "decomposable systems". They are characterised by the fact that they are obtained from the interconnection of identical subsystems and are such that their state space matrices satisfy a certain structural property. This allows us to derive a procedure to design a distributed controller whose structure mirrors the interconnections among the subsystems and to decompose the constraints to be satisfied for stability and performance purpose, improving the scalability of the controller that, when dealing with large-scale systems, results to be a major concern [29].

Definition 4.1 Assume that $P \in \mathbb{R}^{N\times N}$ is diagonalizable. We define $\wp^{P,p,q}$ as the set of all matrices $M \in \mathbb{R}^{Np\times Nq}$ for which there exist two matrices $M^d, M^i \in \mathbb{R}^{p\times q}$ such that [29]

$$M = I_N \otimes M^d + P \otimes M^i. \tag{4.1}$$

An interesting property of this set is given by the following lemma.

Lemma 4.1 Let $P \in \mathbb{R}^{N\times N}$ be a diagonalizable matrix and let $Z \in \mathbb{C}^{N\times N}$ be a non singular matrix such that $\Lambda = Z^{-1}PZ$ is diagonal. If $M \in \wp^{P,p,q}$ then

$$\mathbf{M} = (Z \otimes I_p)^{-1} M (Z \otimes I_q) \tag{4.2}$$

is block diagonal.
Proof: From (4.1) it follows that

$$\mathbf{M} = (Z \otimes I_p)^{-1}(I_N \otimes M^d + P \otimes M^i)(Z \otimes I_q) \tag{4.3}$$

and applying the properties of the Kronecker product, listed in Section 2.8, we can write

$$\begin{aligned}
\mathbf{M} &= (Z \otimes I_p)^{-1}(I_N \otimes M^d + P \otimes M^i)(Z \otimes I_q) = \\
&= (Z^{-1} \otimes I_p)(I_N \otimes M^d + P \otimes M^i)(Z \otimes I_q) = \\
&= [(Z^{-1} \otimes I_p)(I_N \otimes M^d) + (Z^{-1} \otimes I_p)(P \otimes M^i)](Z \otimes I_q) \\
&= (Z^{-1} \otimes I_p)(I_N \otimes M^d)(Z \otimes I_q) + (Z^{-1} \otimes I_p)(P \otimes M^i)(Z \otimes I_q) = \\
&= (Z^{-1} I_N \otimes I_p M^d)(Z \otimes I_q) + (Z^{-1} P \otimes I_p M^i)(Z \otimes I_q) = \\
&= (Z^{-1} I_N Z \otimes I_p M^d I_q) + (Z^{-1} P Z \otimes I_p M^i I_q) = I_N \otimes M^d + \Lambda \otimes M^i
\end{aligned} \tag{4.4}$$

and since I_N and Λ are diagonal, then $\mathbf{M}$ is block diagonal.□
It is worth noticing that the matrix $\mathbf{M}$ has a special structure since in addition to being block diagonal, each of its diagonal blocks $\mathbf{M}_i$, admits the following parametrization

$$\mathbf{M}_i = M^d + \lambda_i M^i \tag{4.5}$$

where λ_i is the ith eigenvalue of P.

4.1 Interconnected Systems

Let us consider N identical subsystems, each of them of order l. Suppose they are interconnected in accordance with a pattern matrix P. We call P a "pattern matrix" since it mirrors the interconnections among the subsystems, as the adjacency matrix does for the nodes of a non-oriented graph. Exploiting the analogy between interconnected systems and graphs, in fact, we can think of each subsystem as a node of the graph, the interconnections as the edges and the pattern matrix P as the adjacency matrix.

Definition 4.1.1 Let us consider the Nl-th order dynamical system described by the following equations:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_w w(t) + B_u u(t) \\ z(t) = C_z x(t) + D_{zw} w(t) + D_{zu} u(t) \\ y(t) = C_y x(t) + D_{yw} w(t) \end{cases} \tag{4.6}$$

where $x \in \mathbb{R}^{Nl}$ is the state, $w \in \mathbb{R}^{Nm_w}$ is the disturbance, $u \in \mathbb{R}^{Nm_u}$ is the control input, $y \in \mathbb{R}^{Nr_y}$ is the measured output and $z \in \mathbb{R}^{Nr_z}$ the performance output. The system is said to be a *decomposable system* if there exists a state space representation as in (4.6) such that $A, B_w, B_u, C_y, C_z, D_{zw}, D_{zu}, D_{yw}$ have the same structure as in (4.1) with P diagonalisable.
If the pattern matrix P is symmetric we call the system a *symmetric decomposable system*. In this case the matrix Z that diagonalizes P is real and orthonormal. The diagonal blocks of the matrices, marked with the superscript "d", represent the internal dynamics of each subsystem, while the off-diagonal, marked with the superscript "i", model the interactions among the subsystems.
An interesting property of this kind of systems is stated by the following theorem.
Theorem 4.1.1 [29]: *The decomposable system* (4.6)*, of order Nl, is equivalent to N independent subsystems of order l, each of which with m_u inputs, m_w disturbances, r_z performance outputs and r_y measurement outputs.*
Proof From the equation (4.2) it follows directly that every matrix M can be rewritten as

$$M = (Z \otimes I_p)\mathbf{M}(Z \otimes I_q)^{-1}$$

where $\mathbf{M}$ is block diagonal, and I_p, I_q of adequate dimensions. Then after the following manipulations:

$$\begin{cases} (Z \otimes I_l)^{-1}\dot{x}(t) = (Z \otimes I_l)^{-1} A \overbrace{(Z \otimes I_l)(Z \otimes I_l)^{-1}}^{=I_{Nl}} x(t) + (Z \otimes I_l)^{-1} B_w (Z \otimes I_{m_w})(Z \otimes I_{m_w})^{-1} w(t) + \\ +(Z \otimes I_l)^{-1} B_u (Z \otimes I_{m_u})(Z \otimes I_{m_u})^{-1} u(t) \\ \\ (Z \otimes I_{r_z})^{-1} z(t) = (Z \otimes I_{r_z})^{-1} C_z (Z \otimes I_l)(Z \otimes I_l)^{-1} x(t) + \\ +(Z \otimes I_{r_z})^{-1} D_{zw} (Z \otimes I_{m_w})(Z \otimes I_{m_w})^{-1} w(t) + (Z \otimes I_{m_w})^{-1} D_{zu} (Z \otimes I_{m_u})(Z \otimes I_{m_u})^{-1} u(t) \\ \\ (Z \otimes I_{r_y})^{-1} y(t) = (Z \otimes I_{r_y})^{-1} C_y (Z \otimes I_l)(Z \otimes I_l)^{-1} x(t) + \\ +(Z \otimes I_{r_y})^{-1} D_{yw} (Z \otimes I_{m_w})(Z \otimes I_{m_w})^{-1} w(t) \end{cases} \tag{4.7}$$

and exploiting the relation (4.2) we come to

$$\begin{cases} (Z \otimes I_l)^{-1}\dot{x}(t) = \mathbf{A}(Z \otimes I_l)^{-1} x(t) + \mathbf{B}_w (Z \otimes I_{m_w})^{-1} w(t) + \mathbf{B}_u (Z \otimes I_{m_u})^{-1} u(t) \\ (Z \otimes I_{r_z})^{-1} z(t) = \mathbf{C}_z (Z \otimes I_l)^{-1} x(t) + \mathbf{D}_{zw} (Z \otimes I_{m_w})^{-1} w(t) + \mathbf{D}_{zu} (S \otimes I_{m_u})^{-1} u(t) \\ (Z \otimes I_{r_y})^{-1} y(t) = \mathbf{C}_y (Z \otimes I_l)^{-1} x(t) + \mathbf{D}_{yw} (Z \otimes I_{m_w})^{-1} w(t) \end{cases} \tag{4.8}$$

with $\mathbf{A}, \mathbf{B}_w, \mathbf{B}_u, \mathbf{C}_z$ block diagonal.
Finally, after the following change of variables

$$\begin{aligned}\hat{x} &= (Z \otimes I_l)^{-1} x, \\ \hat{w} &= (Z \otimes I_{m_w})^{-1} w, \\ \hat{y} &= (Z \otimes I_{r_y})^{-1} y, \\ \hat{u} &= (Z \otimes I_{m_u})^{-1} u, \\ \hat{z} &= (Z \otimes I_{r_z})^{-1} z,\end{aligned} \tag{4.9}$$

the system becomes

$$\begin{cases} \dot{\hat{x}}(t) = \mathbf{A}\hat{x}(t) + \mathbf{B}_w \hat{w}(t) + \mathbf{B}_u \hat{u}(t) \\ \hat{z}(t) = \mathbf{C}_z \hat{x}(t) + \mathbf{D}_{zw} \hat{w}(t) + \mathbf{D}_{zu} \hat{u}(t) \\ \hat{y}(t) = \mathbf{C}_y \hat{x}(t) + \mathbf{D}_{yw} \hat{w}(t) \end{cases} \tag{4.10}$$

and this is equivalent to the following set of N equations, one for each subsystem:

$$\begin{cases} \dot{\hat{x}}_i(t) = \mathbf{A}_i \hat{x}_i(t) + \mathbf{B}_{w,i} \hat{w}_i(t) + \mathbf{B}_{u,i} \hat{u}_i(t) \\ \hat{z}_i(t) = \mathbf{C}_{z,i} \hat{x}_i(t) + \mathbf{D}_{zw,i} \hat{w}_i(t) + \mathbf{D}_{zu,i} \hat{u}_i(t) \quad i = 1, \dots, N \\ \hat{y}_i(t) = \mathbf{C}_{y,i} \hat{x}_i(t) + \mathbf{D}_{yw,i} \hat{w}_i(t) \end{cases} \tag{4.11}$$

where $\hat{x}_i$ is the i-th element of size $l \times 1$ of $\hat{x}$, $\hat{w}_i$ is the i-th element of size $m_w \times 1$ of $\hat{w}$, $\hat{u}_i$ is the i-th element of size $m_u \times 1$ of $\hat{u}$, $\hat{z}_i$ is the i-th element of size $r_z \times 1$ of $\hat{z}$ and $\hat{y}_i$ is the i-th element of size $r_y \times 1$ of $\hat{y}$. □
Subsystems in (4.11) are called "modal subsystems". It is worth noticing that these subsystems are not the same as the physical subsystems that compose the global plant (i.e. the diagonal part of A). In fact, according to the result obtained in (4.4), it holds that

$$\begin{cases} \dot{\hat{x}}_i(t) = (A^d + \lambda_i A^i)\hat{x}_i(t) + (B^d_{w,i} + \lambda_i B^i_{w,i})\hat{w}_i(t) + (B^d_{u,i} + \lambda_i B^i_{u,i})\hat{u}_i(t) \\ zi(t) = (C^d_{z,i} + \lambda_i C^i_{z,i})\hat{x}_i(t) + (D^d_{zw,i} + \lambda_i D^i_{zw,i})\hat{w}_i(t) + (D^d_{zu,i} + \lambda_i D^d_{zu,i})\hat{u}_i(t) \quad i = 1, \dots, N \\ y_i(t) = (C^d_{y,i} + \lambda_i C^i_{y,i})\hat{x}_i(t) + (D^d_{yw,i} + \lambda_i D^i_{yw,i})\hat{w}_i(t) \end{cases} \tag{4.12}$$

with λ_i the i_{th} eigenvalue of P. This property of decomposition, is what gives this kind of systems its name. This is their main characteristic and it is what allows us to face many control design problems decomposing them into N smaller problems independently.

4.2 Static State Feedback Controller for Decomposable Systems

Goal of this project is to design a stabilizing static state feedback distributed controller whose structure mirrors the interconnections among the subsystems, i.e. a controller taking the form

$$u(t) = Kx(t), \tag{4.13}$$

with

$$K = I_N \otimes K^d + P \otimes K^i, \tag{4.14}$$

To this end, techniques from robust control will be exploited. More specifically, we will exploit the $Full\ Block\ S-Procedure$ that provides a convex formulation for the state feedback synthesis problem under some structural assumptions. In order to bring the system model in a form such that the Full Block S-Procedure is applicable, we introduce a new channel $p \to q$ in the model that in a decomposed form becomes

$$\begin{cases} \dot{x}_i(t) = A^d x_i(t) + B_1 p_i(t) + B_2 w_i(t) + B u_i(t) \\ q_i(t) = C_1 x_i(t) + D_1 p_i(t) + D_{12} w_i(t) + E_1 u_i(t) \quad\quad i = 1, \dots, N \\ z_i(t) = C_2 x_i(t) + D_{21} p_i(t) + D_2 w_i(t) + E_2 u_i(t) \end{cases} \tag{4.15}$$

where $x_i(t) \in \mathbb{R}^l$ is the i_{th} state variable, $w_i(t) \in \mathbb{R}^{m_w}$ the i_{th} exogenous input, $u_i(t) \in \mathbb{R}^{m_u}$ the i_{th} control input, $z_i(t) \in \mathbb{R}^{r_z}$ the i_{th} performance output and $q_i(t) \in \mathbb{R}^{n_p}$, $p_i(t) \in \mathbb{R}^{n_p}$ signals such that, for the original system, it holds that

$$p(t) = P \otimes I_{n_p} q(t) \tag{4.16}$$

and for the decomposed system it holds that

$$p_i(t) = \lambda_i q_i(t). \tag{4.17}$$

Starting from (4.16), repeating the steps of decomposition,

$$(Z \otimes I_{n_p})^{-1} p(t) = (Z \otimes I_{n_p})^{-1} (P \otimes I_{n_p})(Z \otimes I_{n_p})(Z \otimes I_{n_p})^{-1} q(t), \tag{4.18}$$

$$\hat{p}(t) = \Lambda \otimes I_{n_p} \hat{q}(t), \tag{4.19}$$

it follows that

$$p_i(t) = \lambda_i q_i(t) \tag{4.20}$$

The state space matrices for the new model are

$$\left[\begin{array}{c|c|c|c} A^d & B_1 & B_2 & B \\ \hline C_1 & D_1 & D_{12} & E_1 \\ \hline C_2 & D_{21} & D_2 & E_2 \end{array}\right] = \left[\begin{array}{c|c|c|c|c} A^d & A^i & B_u^d & B_w^d & B_u^d \;\; 0 \\ \hline I & 0 & 0 & 0 & 0 \;\; 0 \\ 0 & 0 & 0 & 0 & 0 \;\; I \\ \hline C_p^d & C_p^i & D_{pu}^d & 0 & D_{pu}^d \;\; 0 \end{array}\right] \tag{4.21}$$

and we will exploit the fact that the system

$$\begin{cases} \dot{x}(t) = (I_N \otimes A^d + P \otimes A^i) x(t) + (I_N \otimes B_u) u(t) + (I_N \otimes B_w) w(t) \\ z(t) = (I_N \otimes C_z^d + P \otimes C_z^i) x(t) + (I_N \otimes D_{zu}) u(t) \end{cases} \tag{4.22}$$

in closed loop with

$$u(t) = (I_N \otimes K^d + P \otimes K^i) x(t) \tag{4.23}$$

is equivalent to

$$\begin{cases} \dot{x}(t) = (I_N \otimes A^d + P \otimes A^i) x(t) + (I_N \otimes [B_u \quad 0] + P \otimes [0 \quad B_u]) \tilde{u}(t) + (I_N \otimes B_w) w(t) \\ z(t) = (I_N \otimes C_z^d + P \otimes C_z^i) x(t) + (I_N \otimes [D_{zu} \quad 0] + P \otimes [0 \quad D_{zu}]) \tilde{u}(t) \end{cases} \tag{4.24}$$

in closed loop with

$$\tilde{u}(t) = I_N \otimes \begin{bmatrix} K^d \\ K^i \end{bmatrix} x(t) \tag{4.25}$$

and both of them yield

$$\begin{cases} \dot{x}(t) = (I_N \otimes (A^d + B_u K^d) + P \otimes (A^i + B_u K^i)) x(t) + (I_N \otimes B_w) w(t) \\ z(t) = (I_N \otimes (C_z^d + D_{zu} K^d) + P \otimes (C_z^i + D_{zu} K^i)) x(t). \end{cases} \tag{4.26}$$

For this reason we can synthesize a decentralized controller (4.25) for the system (4.27) and from it compute the distributed controller (4.23) for the original system.
It is important to notice that this equivalence holds under the hypothesis that both B_u^i and D_u^i are equal to zero. In fact if this hypothesis is violated the original system in closed loop takes the form

$$\begin{cases} \dot{x}(t) = (I_N \otimes A^d + P \otimes A^i)x(t) + (I_N \otimes B_u^d + P \otimes B_u^i)(I_N \otimes K^d + P \otimes K^i)x(t) + (I_N \otimes B_w)w(t) \\ z(t) = (I_N \otimes C_z^d + P \otimes C_z^i)x(t) + (I_N \otimes D_{zu}P \otimes D_{zu}^i)(I_N \otimes K^d + P \otimes K^i)x(t) \end{cases}$$

that is

$$\begin{cases} \dot{x}(t) = ((I_N \otimes (A^d + B_u^d K^d) + P \otimes (A^i + B_u^i K^d + B_u^d K^i) + P^2 \otimes B_u^i K^i)x(t) + (I_N \otimes B_w)w(t) \\ z(t) = ((I_N \otimes (C_z^d + D_{zu}^d K^d) + P \otimes (C_z^i + D_{zu}^i K^d + D_{zu}^d K^i) + P^2 \otimes D_{zu}^i K^i)x(t) \end{cases}.$$

While, for the transformed system, it holds

$$\begin{cases} \dot{x}(t) = (I_N \otimes A^d + P \otimes A^i)x(t) + (I_N \otimes [B_u \quad 0] + P \otimes [0 \quad B_u])I_N \otimes \begin{bmatrix} K^d \\ K^i \end{bmatrix} x(t) + (I_N \otimes B_w)w(t) \\ z(t) = (I_N \otimes C_z^d + P \otimes C_z^i)x(t) + (I_N \otimes [D_{zu} \quad 0] + P \otimes [0 \quad D_{zu}])I_N \otimes \begin{bmatrix} K^d \\ K^i \end{bmatrix} x(t)) \end{cases} \tag{4.27}$$

and the presence of the terms depending on P^2, when B_u^i and D_u^i are different from zero, in the first equations, would prevent the two formulations to be equivalent.
Another important assumption is the constraint that the matrix P is symmetric. In fact, if we denote by $\hat{T}_{wz}$ the transfer function from w to z of the transformed system it holds:

$$\frac{\underline{\sigma}(Z)}{\bar{\sigma}(Z)}\|\hat{T}_{wz}\|_\infty \leq \|T_{wz}\|_\infty \leq \frac{\bar{\sigma}(Z)}{\underline{\sigma}(Z)}\|\hat{T}_{wz}\|_\infty \tag{4.28}$$

where $Z \in \mathbb{R}^{n\times n}$ is the matrix that diagonalizes P, and $\bar{\sigma}$ and $\underline{\sigma}$ indicate the maximum and minimum eigenvalues of Z, respectively. Since P is symmetric, Z is orthonormal, and this guarantees that $\underline{\sigma}(Z) = \bar{\sigma}(Z) = 1$ and thus that $\|\hat{T}_{wz}\|_\infty = \|T_{wz}\|_\infty$. This means that dealing with the transformed system does not lead to a loss of information about the norm of the real system. Another advantage of dealing with symmetric pattern matrices lies in the fact that they have real eigenvalues and this makes the decomposed system and the decomposition approach, that will be treated later, more manageable.

4.3 $\mathcal{H}_\infty$ Norm of Decomposable Systems

The $\mathcal{H}_\infty$ norm of the system is calculated by applying a linear fractional transformation to the decomposed system. More explicitly, let us define for the system in its decomposed form the state space matrices as follows

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{B} \\ \mathcal{C}_1 & \mathcal{D}_1 & \mathcal{D}_{12} & \mathcal{E}_1 \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_2 & \mathcal{E}_2 \end{bmatrix} := \left[\begin{array}{c|c|c|c} I_N \otimes A^d & I_N \otimes B_1 & I_N \otimes B_2 & I_N \otimes B \\ \hline I_N \otimes C_1 & I_N \otimes D_1 & I_N \otimes D_{12} & I_N \otimes E_1 \\ \hline I_N \otimes C_2 & I_N \otimes D_{21} & I_N \otimes D_2 & I_N \otimes E_2 \end{array}\right] \tag{4.29}$$

that, in closed-loop, become

$$\begin{bmatrix} \dot{x} \\ q \\ z \end{bmatrix} = \begin{bmatrix} \mathcal{A} + \mathcal{B}K & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 + \mathcal{E}_1 K & \mathcal{D}_1 & \mathcal{D}_{12} \\ \mathcal{C}_2 + \mathcal{E}_2 K & \mathcal{D}_{21} & \mathcal{D}_2 \end{bmatrix} \begin{bmatrix} x \\ p \\ w \end{bmatrix} \tag{4.30}$$

and after a reordering of the variables

$$\left[\begin{array}{c}\dot{x}\\ z\\ \hline q\end{array}\right] = \left[\begin{array}{cc|c}\mathcal{A}+\mathcal{B}K & \mathcal{B}_2 & \mathcal{B}_1\\ \mathcal{C}_2+\mathcal{E}_2K & \mathcal{D}_2 & \mathcal{D}_{21}\\ \hline \mathcal{C}_1+\mathcal{E}_1K & \mathcal{D}_{12} & \mathcal{D}_1\end{array}\right]\left[\begin{array}{c}x\\ w\\ \hline p\end{array}\right] = \left[\begin{array}{c|c}G_{11} & G_{12}\\ \hline G_{21} & G_{22}\end{array}\right]\left[\begin{array}{c}x\\ w\\ \hline p\end{array}\right]. \tag{4.31}$$

By exploiting the relation

$$p = \underbrace{P \otimes I_{n_p}}_{\Delta} q \tag{4.32}$$

and referring to the lower LFT as shown in Figure 4.1,

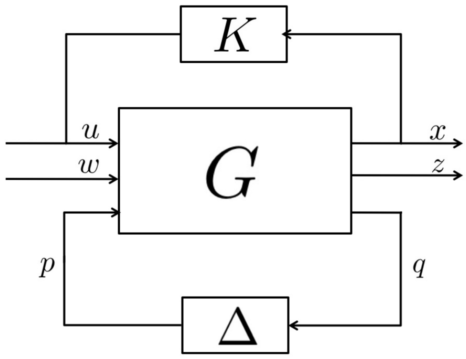

Figure 4.1: Lower LFT for decomposable systems.

we have

$$\begin{bmatrix}\dot{x}\\ z\end{bmatrix} = \mathcal{F}_l(G,\Delta)\begin{bmatrix}x\\ w\end{bmatrix} \text{ with } p = \Delta q. \tag{4.33}$$

From (4.31) it follows

$$\begin{cases} p = \Delta q \\ q = G_{21}\begin{bmatrix}x\\ w\end{bmatrix} + G_{22}\Delta q \end{cases} \tag{4.34}$$

from which

$$p = \Delta(I - G_{22}\Delta)^{-1}G_{21}\begin{bmatrix}x\\ w\end{bmatrix} \tag{4.35}$$

and thus

$$\begin{bmatrix}\dot{x}\\ z\end{bmatrix} = G_{11}\begin{bmatrix}x\\ w\end{bmatrix} + G_{12}p = \underbrace{(G_{11} + G_{12}\Delta(I - G_{22}\Delta^{-1}G_{21}))}_{\mathcal{F}_l(G,\Delta)}\begin{bmatrix}x\\ w\end{bmatrix} \tag{4.36}$$

From $\mathcal{F}_l(G,\Delta)$ in (4.36), we can extract the closed-loop matrices from which we calculate the $\mathcal{H}_\infty$ norm of the system through the Matlab function
`norm(ss(A_cl,B_cl,C_cl,D_cl), Inf)` with A_cl,B_cl,C_cl,D_cl, extracted from the relation (4.33) block-partitioning $\mathcal{F}_l(G,\Delta)$ according to the dimensions of the vectors x, w.

Chapter 5

Full Block S-Procedure

In this chapter a sub-optimal approach for the design of distributed controllers for decomposable systems is presented. This technique is based on a robust control synthesis procedure called "Full Block S-Procedure" [39].

5.1 Introduction

The Full Block S-Procedure is a powerful mathematical tool that applies to more general problems with respect to that one addressed in this work. The main objective for which it was conceived is the construction of controllers that take into account online measurements of time-varying parameters that influence the system dynamics. Linear Parameter Varying (LPV) systems are systems described by the equations

$$\begin{cases} \dot{x}(t) = \hat{A}(\delta(t))x(t) + \hat{B}_1(\delta(t))w(t) + \hat{B}_2(\delta(t))u(t) \\ z(t) = \hat{C}_1(\delta(t))x(t) + \hat{D}_1(\delta(t))w(t) + \hat{D}_{12}(\delta(t))u(t) \\ y(t) = \hat{C}_2(\delta(t))x(t) + \hat{D}_{21}(\delta(t))w(t) \end{cases} \tag{5.1}$$

where $x(t) \in \mathbb{R}^n, w(t) \in \mathbb{R}^{m_w}, u(t) \in \mathbb{R}^{m_u}, y(t) \in \mathbb{R}^{r_y}, z(t) \in \mathbb{R}^{r_z}$ are the state, the disturbance, the control input, the measured output and the performance output, respectively, while $\delta(t) \in \mathbb{R}^k$ is the time-varying parameter. They are the object of considerable attention since they are well-suited to model and synthesize problems related to robust control and non linear control. In LPV control it is assumed that the time-varying parameter $\delta(t)$ belongs to an a priori given set $\boldsymbol{\delta}$, whereas the actual parameter curve is not known a priori but can be measured on-line. The objective is to design an LPV controller of the type

$$\begin{cases} \dot{x}_c(t) = \hat{A}_c(\delta(t))x(t) + \hat{B}_c(\delta(t))y(t) \\ u(t) = \hat{C}_c(\delta(t))x_c(t) + \hat{D}_c(\delta(t))y(t), \end{cases} \tag{5.2}$$

that takes on-line decisions based on the measured input $y(t)$ and the time-varying parameter $\delta(t)$ according to which it generates the control input $u(t)$.
The closed loop system then, takes the form

$$\begin{cases} \dot{\xi}(t) = \hat{\mathscr{A}}(\delta(t))\xi(t) + \hat{\mathscr{B}}(\delta(t))w(t) \\ u(t) = \hat{\mathscr{C}}(\delta(t))\xi(t) + \hat{\mathscr{D}}(\delta(t))w(t) \end{cases} . \tag{5.3}$$

The design goal is to stabilise the controlled system (5.3) and to satisfy the performance requirements on the channel $w \to z$ generalized by the expressions (5.4)-(5.6). From a general point of view, in fact, the closed-loop system is required to satisfy

$$\int_0^T \begin{pmatrix} z \\ w \end{pmatrix}^T \begin{pmatrix} Q_p & S_p \\ S_p^T & R_p \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} \le 0 \tag{5.4}$$

for all $T \ge 0$ and all $\delta(t) \in \boldsymbol{\delta}$ if $\xi(0) = 0$.
If the objective is to bound the $\mathcal{H}_\infty$ gain of the channel $w \to z$, a possible choice in (5.4) is

$$\begin{pmatrix} Q_p & S_p \\ S_p^T & R_p \end{pmatrix} = \begin{pmatrix} -\gamma^2 I & 0 \\ 0 & I \end{pmatrix}. \tag{5.5}$$

Performance requirements together with stability can be guaranteed by the existence of a matrix $\mathscr{X} > 0$ such that the following (infinite) family of LMIs is feasible

$$\left(\begin{array}{cc} I & 0 \\ \hat{\mathscr{A}}(\delta(t)) & \hat{\mathscr{B}}(\delta(t)) \\ \hline 0 & I \\ \hat{\mathscr{C}}(\delta(t)) & \hat{\mathscr{D}}(\delta(t)) \end{array}\right)^T \left(\begin{array}{cc|cc} 0 & \mathscr{X} & 0 & 0 \\ \mathscr{X} & 0 & 0 & 0 \\ \hline 0 & 0 & Q_p & S_p \\ 0 & 0 & S_p^T & R_p \end{array}\right) \left(\begin{array}{cc} I & 0 \\ \hat{\mathscr{A}}(\delta(t)) & \hat{\mathscr{B}}(\delta(t)) \\ \hline 0 & I \\ \hat{\mathscr{C}}(\delta(t)) & \hat{\mathscr{D}}(\delta(t)) \end{array}\right) < 0 \tag{5.6}$$

for all $\delta \in \boldsymbol{\delta}$.
It is immediate to see how these LMIs can be interpreted as a generalisation of the Bounded Real Lemma. In fact, developing the calulations in (5.6) for the time invariant case, we obtain

$$\begin{pmatrix} A^T X + XA + C^T R_p C & XB + C^T S_p^T + C^T R_p D \\ B^T X + S_p C + D^T R_p C & Q_p + D^T S_p^T + S + D^T R_p D \end{pmatrix} < 0. \tag{5.7}$$

Plugging the (5.5) in (5.6) in order to guarantee the satisfying of the performance requirements and $D = 0$ in accordance with (2.30) it immediately follows that (5.7) is nothing but (2.31).

5.2 Full Block S-Procedure

The Full Block S-Procedure represents a general result on families of quadratic forms which allows to rewrite (5.6) into one LMI in which $\mathscr{X}$ and the so called multipliers are the decision variables.

Let $\mathscr{B}$ be a closed subspace of $\mathbb{R}^n$, let $U \in \mathbb{R}^{p \times n}$ and $V \in \mathbb{R}^{q \times n}$, let $Q \in \mathbb{R}^{p \times p}$ be symmetric, and let $\mathscr{S}(\Delta) \subset \mathbb{R}^q$ be the subspace that depends continuously on the parameter Δ which varies in the compact path-connected set $\boldsymbol{\Delta}$. Moreover define

$$\mathscr{B}(\Delta) := \{x \in \mathscr{B} | Vx \in \mathscr{S}(\Delta)\}. \tag{5.8}$$

Theorem 5.2.1 (from [39])
(a) The performance specification

$$U^T Q U < 0 \text{ on } \mathscr{B}(\Delta) \text{ for all } \Delta \in \boldsymbol{\Delta} \tag{5.9}$$

holds if and only if there exists a symmetric multiplier P with

$$P > 0 \text{ on } \mathscr{S}(\Delta) \text{ for all } \Delta \in \boldsymbol{\Delta} \tag{5.10}$$

which satisfies

$$U^T Q U + V^T P V < 0 \text{ on } \mathscr{B}; \tag{5.11}$$

(b) suppose there exists a subspace $\mathscr{B}_0 \subset \mathscr{B}$ on which U^TQU is positive semi-definite, and whose dimension is large enough to satisfy

$$\dim(V\mathscr{B}_0) + \dim(\mathscr{S}(\Delta)) \geq q. \tag{5.12}$$

Then (5.10) *and* (5.11) *imply that $V\mathscr{B}_0$ and $\mathscr{S}(\Delta)$ are in direct sum (symbol $\oplus$), namely*

$$V\mathscr{B}_0 \oplus \mathscr{S}(\Delta) = \mathbb{R}^q \text{ for all } \Delta \text{ in } \boldsymbol{\Delta}. \tag{5.13}$$

Proof. (a) The proof of "if" in (a) is trivial. In fact, if we choose any $x \in \mathscr{B}(\Delta)$ with $x \neq 0$, since $x \in \mathscr{B}$, it follows directly from (5.11) that

$$x^TU^TQUx < -x^TV^TPVx \tag{5.14}$$

Since $x \in \mathscr{B}(\Delta)$ it holds that $Vx \in \mathscr{S}(\Delta)$ and therefore from (5.10) we deduce that

$$x^TV^TPVx \geq 0 \tag{5.15}$$

in which the equality holds when $Vx = 0$. Condition (5.14) yields $x^TU^TQUx < 0$, that is (5.9), and this concludes the proof.
The proof of "only if" in (a) consists of six steps.

Step 1

Let us denote by $S(\Delta)$ the orthogonal projector onto $\mathscr{S}(\Delta)$. Since $\mathscr{S}(\Delta)$ depends continuously on Δ, also $S(\Delta)$ does. Moreover

$$\mathscr{S}(\Delta) = ker(I - S(\Delta)). \tag{5.16}$$

To see that this relation is always satisfied it is worth noticing that this is equivalent to saying that

$$\mathscr{S}(\Delta) = \{x \in \mathbb{R}^q | (I - S(\Delta))x = 0\} = \{x \in \mathbb{R}^q | Ix = S(\Delta)x\} \tag{5.17}$$

that means that $\mathscr{S}(\Delta)$ is the vectorial space generated by all the vectors $x \in \mathbb{R}^q$ whose orthogonal projection is the vector x itself, that is a redundant way to say that $x \in \mathscr{S}(\Delta)$.
From (5.16) we obtain an alternative description for $\mathscr{B}(\Delta)$:

$$\mathscr{B}(\Delta) = \mathscr{B} \cap ker([I - S(\Delta)]V). \tag{5.18}$$

In fact, similarly to what has been done before, this is equivalent to

$$\mathscr{B}(\Delta) = \{x \in \mathscr{B} | Vx = S(\Delta)Vx\}, \tag{5.19}$$

i.e. $x \in \mathscr{B}$ and $Vx \in \mathscr{S}(\Delta)$.

Step 2

We claim that there exists $\delta_0 > 0$ such that

$$U^TQU + \delta_0 I < 0 \text{ on } \mathscr{B}(\Delta) \text{ for all } \Delta \in \boldsymbol{\Delta}. \tag{5.20}$$

If not true, there exists for any $j = 1, 2, \dots$ a $\Delta_j \in \boldsymbol{\Delta}$ and a vector $x_j \in \mathscr{B}(\Delta_j)$, $x_j \neq 0$ such that

$$x_j^T[U^TQU + j^{-1}I]x_j \geq 0. \tag{5.21}$$

By the compactness of $\mathbf{\Delta}$, there exists a subsequence j_v such that $\Delta_{j_v} \to \Delta_0 \in \mathbf{\Delta}$ and $x_{j_v} \to x_0$ with $x_0 \neq 0$ for $v \to \infty$. Moreover, since $x_j \in \mathscr{B}(\Delta_j)$, we infer that

$$[I - S(\Delta_{jv})]Vx_{j_v} = 0 \tag{5.22}$$

which implies that, since $\mathscr{S}(\Delta)$ depends continuously on Δ, for $v \to \infty$

$$[I - S(\Delta_0)]Vx_0 = 0 \tag{5.23}$$

Since $\mathscr{B}$ is closed, we also infer that $x_0 \in \mathscr{B}$ which, together with (5.23), yields $x_0 \in \mathscr{B}(\Delta_0)$. Moreover the inequality (5.21) for $v \to \infty$ becomes $x_0^T U^T Q U x_0 \geq 0$. Thus we have found a non zero vector $x_0 \in \mathscr{B}(\Delta_0)$ such that hypothesis (5.9) is contradicted.

Step 3

Let B_1 be a basis matrix of $ker(V) \cap \mathscr{B}$ and B_2 an extension to a basis B of $\mathscr{B}$, so that $B = (B_1 B_2)$. This means that $VB_1 = 0$ and, since B_1 is a basis for $ker(V) \cap \mathscr{B}$ and B_2 is a set of generators for $\mathscr{B}$, that VB_2 has full column rank. By definition of $\mathscr{B}(\Delta)$ and of vector space in general, it also holds that

$$ker(V) \cap \mathscr{B} \subset \mathscr{B}(\Delta) \tag{5.24}$$

that, together with (5.20), yields

$$B_1^T[U^T Q U + \delta_0 I]B_1 < 0. \tag{5.25}$$

Therefore there exists a symmetric matrix M such that

$$\xi_2^T M \xi_2 = \max_{\xi_1} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}^T B^T[U^T Q U + \delta_0 I]B \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{5.26}$$

Let us define

$$\hat{P} := -[(VB_2)^\dagger]^T M (VB_2)^\dagger, ^1 \tag{5.27}$$

from which

$$M = -(VB_2)^T \hat{P} (VB_2). \tag{5.28}$$

Due to $(VB_2)^\dagger (VB_2) = I$ we deduce that, for all $\xi = (\xi_1^T, \xi_2^T)^T$,

$$\xi^T B^T V^T \hat{P} V B \xi = \xi^T (B_1 B_2)^T V^T \hat{P} V (B_1 B_2) \xi = \xi_2^T \underbrace{(VB_2)^T \hat{P} (VB_2)}_{-M} \xi_2 = -\xi_2^T M \xi_2, \tag{5.29}$$

where the fact that $VB_1 = 0$ has been exploited. Finally, combining (5.29) with (5.26), we come to the conclusion that

$$\underbrace{\xi^T B^T V^T \hat{P} V B \xi}_{-\xi_2^T M \xi_2} \leq -\xi^T B^T [U^T Q U + \delta_0 I] B \xi \tag{5.30}$$

where the equality holds if ξ_1 is the argmax in (5.26). The (5.30) can be equivalently expressed as

$$\xi^T B^T [U^T Q U + V^T \hat{P} V] B \xi \leq -\underbrace{\delta_0}_{>0} \xi^T \underbrace{B^T B}_{\succ 0} \xi, \tag{5.31}$$

which implies

$$B^T [U^T Q U + V \hat{P} V] B < 0 \tag{5.32}$$

[1]The symbol †indicates the Moore-Penrose inverse.

Step 4

In this step we prove that

$$\hat{P} > 0 \text{ on } \mathscr{S}(\Delta) \cap V\mathscr{B} \text{ for all } \Delta \in \mathbf{\Delta}. \tag{5.33}$$

Indeed, for any vector $z \neq 0$ such that $z \in \mathscr{S}(\Delta) \cap V\mathscr{B}$ there exists $\xi_2 \neq 0$ for which $z = VB_2\xi_2$. Let us take the corresponding maximising vector ξ_1 in (5.26) and define $\xi := (\xi_1^T \xi_2^T)^T$. Since $VB_1 = 0$ we have that $VB\xi = VB_2\xi_2 = z$ from which, due to (5.29),

$$z^T\hat{P}z = \xi^T B^T V^T \hat{P} V B\xi = -\xi_2^T M\xi_2 = -\xi^T B^T[U^T QU + \delta_0 I]B\xi. \tag{5.34}$$

where the last equality holds since ξ_1 is the argmax in (5.26). Moreover, since $VB\xi = z \in \mathscr{S}(\Delta)\backslash\{0\}$, we deduce, by definition of $\mathscr{B}(\Delta)$, that $B\xi \in \mathscr{B}(\Delta)\backslash\{0\}$ which implies by (5.20) that

$$\xi^T B^T[U^T QU + V^T \hat{P} V]B\xi < 0. \tag{5.35}$$

Thus from (5.34) we obtain $z^T\hat{P}z > 0$ that confirms the thesis (5.33).

Step 5

Let L be a full row rank matrix such that $ker(L) = V\mathscr{B}$. We claim that there exist some $r_0 > 0$ such that

$$P = \hat{P} + r_0 L^T L > 0 \text{ on } \mathscr{S}(\Delta) \text{ for all } \Delta \in \mathbf{\Delta}. \tag{5.36}$$

The proof follows by contradiction. In fact, if the claim was false, we could find for each $j = 1, 2, \ldots$ a $\Delta_j \in \mathbf{\Delta}$ some $z_j \in \mathscr{S}(\Delta)$, $z_j \neq 0$ such that

$$z_j^T[\hat{P} + jL^T L]z_j \leq 0 \tag{5.37}$$

Take a subsequence j_v such that Δ_{j_v} and z_{j_v}, converge to $\Delta_0 \in \mathbf{\Delta}$ and $z_0 \neq 0$ respectively. Since $z_{j_v} \in \mathscr{S}(\Delta_{j_v})$

$$[I - S(\Delta_{j_v})]z_{j_v} = 0 \tag{5.38}$$

and, since $\mathscr{S}(\Delta)$ depends continuously on Δ, for $v \to \infty$ we obtain

$$[I - S(\Delta_0)]z_0 = 0 \tag{5.39}$$

and hence $z_0 \in \mathscr{S}(\Delta_0)$. Notice that (5.37) allows to write

$$z_{j_v}^T[j_v^{-1}\hat{P} + L^T L]z_{j_v} \leq 0 \tag{5.40}$$

from which we conclude, for $v \to \infty$

$$z_0^T \underbrace{L^T L}_{\succ 0} z_0 \leq 0 \tag{5.41}$$

which implies $Lz_0 = 0$ and hence z_0 belongs to ker(L) or alternatively $z_0 \in V\mathscr{B}$. Since we also have

$$z_{j_v}^T\hat{P}z_{j_v} \leq -j_v z_{j_v}^T L^T L z_{j_v} \leq 0 \tag{5.42}$$

we infer $z_0^T\hat{P}z_0 \leq 0$. Therefore we have $z_0 \neq 0, z_0 \in \mathscr{S}(\Delta_0) \cup V\mathscr{B}$ for which $z_0^T\hat{P}z_0$ and this contradicts (5.33).

Step 6

In (5.36) we have constructed a multiplier in accordance with (5.10). Due to

$$(VB)^T PVB = (VB)^T[\hat{P} + r_0 L^T L]VB = (VB)^T \hat{P} VB \tag{5.43}$$

we can rewrite (5.32) as

$$B^T[U^T QU + VPV]B < 0 \tag{5.44}$$

which is an equivalent formulation of (5.11), and this concludes the proof. □

Proof. (b) Let $z \neq 0$ be a vector in $V\mathscr{B}_0 \cap \mathscr{S}(\Delta)$, and let x be a non-zero vector, $x \in \mathscr{B}_0$, such that $z = Vx$. Since $x \in \mathscr{B}$ we infer from (5.11) that

$$x^T U^T QUx < -x^T V^T PVx = -z^T Pz < 0 \tag{5.45}$$

where the last inequality holds due to (5.10). Thus we end up with

$$x^T U^T QUx < 0 \text{ on } \mathscr{B}_0 \tag{5.46}$$

That contradicts the hypothesis that $x^T U^T QUx \geq 0$ on the same set. Therefore $V\mathscr{B}_0 \cap \mathscr{S}(\Delta) = \{0\}$ and so $V\mathscr{B}_0$ and $\mathscr{S}(\Delta)$ are in direct sum, that means

$$\dim(V\mathscr{B}_0 + \mathscr{S}(\Delta)) = \dim(V\mathscr{B}_0) + \dim(\mathscr{S}(\Delta)) = q, \tag{5.47}$$

and thus they generate $\mathbb{R}^q$. □

The previous result allows to replace the family of (infinite) constraints (5.6), with the following LMIs:

$$X \succ 0 \tag{5.48}$$

$$[2] \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \succ 0 \text{ on } \mathscr{S}(\Delta) \text{ for all } \Delta \in \boldsymbol{\Delta} \tag{5.49}$$

$$J^T HJ \prec 0. \tag{5.50}$$

With

$$[3] H = \left[\begin{array}{cc|cc|cc} 0 & X & 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & Q & S & 0 & 0 \\ 0 & 0 & S^T & R & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right] \tag{5.51}$$

and

$$J = \left[\begin{array}{ccc} 0 & 0 & 0 \\ A & B_1 & B_2 \\ \hline 0 & I & 0 \\ C_1 & D_1 & D_{12} \\ \hline 0 & 0 & I \\ C_2 & D_{21} & D_2 \end{array}\right] \tag{5.52}$$

with $A, B_1, B_2, C_1, D_1, D_{12}, C_2, D_{21}, D_2$ expressed as in (5.1) and $\begin{pmatrix} w(t) \\ z(t) \end{pmatrix} \in \mathscr{S}(\Delta(t)) \subset \mathbb{R}^{m_w + r_z}$. The main advantage of (5.48)-(5.50) over (5.6) is that the Full Block S-Procedure provides a standard linear matrix inequality in the unknowns Q, R, S, X and that the uncertainties come into play only on the multipliers (5.49). This separation brought an important contribution to the theory of robust control.

[2] This is the P matrix in the Full Block S-Procedure theorem.

[3] This is the $U^T QU + V^T PV$ matrix in the Full Block S-Procedure theorem.

5.3 Dualisation

When the state matrices belong to a closed loop system with state feedback control, the state matrix A in (5.7) will be replaced by $A+BK$. This means that, when developing the calculation, we will end up with terms of the following kind

$$X(A+BK), \quad (A+BK)^T X \tag{5.53}$$

in the unknowns X and K. This is undesirable because it makes the problem non-convex in the unknowns X, K and thus not solvable via convex optimisation tools. For this reason the dual system needs to be considered. Moreover a trick will be applied. It exploits the fact that the Lyapunov stability inequality that appears in (5.7)

$$X(A+BK) + (A+BK)^T X \prec 0, \tag{5.54}$$

can be factorised as

$$X(AX^{-1} + BKX^{-1} + X^{-1}A^T + X^{-1}K^T B^T)X \prec 0 \tag{5.55}$$

and upon defining $Y = X^{-1}$ and introducing the auxiliary variable $M = KY$, we can equivalently rewrite the Lyapunov inequality as

$$AY + BM + YA^T + M^T B^T \prec 0 \tag{5.56}$$

that is affine in the variables Y, M and thus convex. After computing the optimisation variables Y, M, we can obtain the controller K through the relation $K = MY^{-1}$.

5.3.1 Dual Bounded Real Lemma

Lemma 5.3.1.1 *Let us consider the dual closed-loop system*

$$\begin{cases} \dot{x} = A_{cl}^T x + C_{cl}^T w \\ z = B_w^T x \end{cases} \tag{5.57}$$

with strictly proper transfer function $T(s) = B_w^T(sI - A_{cl}^T)^{-1}C_{cl}^T$, then the following are equivalent:

(i) $\|T\|_\infty < \gamma$;

(ii) Given $\Phi(s) := \gamma^2 I - T^T(-s)T(s)$ *then* $\Phi(j\omega) > 0 \quad \forall \omega \in \mathbb{R}$;

(iii) The Hamiltonian Matrix

$$M_\gamma := \begin{pmatrix} A^T & \frac{1}{\gamma^2}H^T H \\ -GG^T & -A^T \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$$

has no purely imaginary eigenvalues;

(iv) There exists a matrix $Y \succ 0$ *such that*

$$\begin{pmatrix} YA_{cl}^T + A_{cl}Y + B_w B_w^T & YC_{cl}^T \\ C_{cl}Y & -\gamma^2 I \end{pmatrix} \prec 0. \tag{5.58}$$

In order to formalise the dual Full Block S-Procedure, we first need to introduce the following lemma.

Dualization Lemma (from [38]) *Let P be a non-singular symmetric matrix in $\mathbb{R}^{n\times n}$ and let $\mathscr{U}$ and $\mathscr{V}$ be two complementary* [4] *subspaces of $\mathbb{R}^n$. Then*

$$x^T Px < 0 \;\forall\; x \in \mathscr{U}\backslash\{0\} \text{ and } x^T Px \geq 0 \;\forall\; x \in \mathscr{V} \tag{5.59}$$

is equivalent to

$$x^T P^{-1}x > 0 \;\forall\; x \in \mathscr{U}^{\perp}\backslash\{0\} \text{ and } x^T P^{-1}x \leq 0 \;\forall\; x \in \mathscr{V}^{\perp} \tag{5.60}$$

Proof Since $\mathscr{U} \oplus \mathscr{V} = \mathbb{R}^n$ is equivalent to $\mathscr{U}^{\perp} \oplus \mathscr{V}^{\perp} = \mathbb{R}^n$, we only need to prove that (5.59) implies (5.60). The converse follows by symmetry.
(5.59) $\Rightarrow$ (5.60)
Let us assume that $\mathscr{U}$ and $\mathscr{V}$ have dimensions k and l, respectively. From (5.59) it follows that P has at least k negative eigenvalues and at least l non-negative eigenvalues. Since $\mathscr{U}$ and $\mathscr{V}$ are in direct sum, $k+l=n$ and P is non-singular, we infer that P has exactly k negative eigenvalues and l positive eigenvalues. Let us prove now that P^{-1} is positive definite on $\mathscr{U}^{\perp}$, by proceeding by contradiction. Assume that there exists a vector $y \in \mathscr{U}^{\perp}\backslash\{0\}$ such that $y^T P^{-1} y \leq 0$. Upon defining the vector $z := P^{-1}y$, we deduce from (5.59) that z does not belong to $\mathscr{U}$. In fact, if z belonged to $\mathscr{U}$, it would have followed that $z^T Pz < 0$ and $z \perp y = Pz$. Since z is in $\mathscr{U}$ and y in $\mathscr{U}^{\perp}$ that would have implied

$$\langle z, y\rangle = \langle z, Pz\rangle = 0 = z^T Pz \tag{5.61}$$

thus contradicting the fact that P is non singular. Therefore the vector subspace $\mathscr{U}_z = span(z)+\mathscr{U}$ has dimension $k+1$. Moreover, for any $x \in \mathscr{U}$ we have

$$(z+x)^T P(z+x) = (P^{-1}y+x)^T P(P^{-1}y+x) = y^T P^{-1}y + \underbrace{y^T x}_{\perp} + \underbrace{x^T y}_{\perp} + x^T Px = \underbrace{y^T P^{-1} y}_{\leq 0} + \underbrace{x^T Px}_{\leq 0}{}^{5} \tag{5.62}$$

that implies

$$(z+x)^T P(z+x) \leq 0. \tag{5.63}$$

This implies that P has at least $k+1$ non-positive eigenvalues, which contradicts what has already been established, namely that P has exactly k negative eigenvalues and that P has not zero eigenvalues.
Let us prove now that P^{-1} is negative definite on $\mathscr{V}^{\perp}$. Note that

$$x^T (P + I\epsilon)x < 0 \;\forall\; x \in \mathscr{U}\backslash\{0\} \text{ and } x^T (P + \epsilon I)x > 0 \;\forall\; x \in \mathscr{V}\backslash\{0\} \tag{5.64}$$

for $\epsilon > 0$ small. Moreover due to what has already been proved and by symmetry, this implies

$$x^T (P + I\epsilon)^{-1}x > 0 \;\forall\; x \in \mathscr{U}^{\perp}\backslash\{0\} \text{ and } x^T (P + \epsilon I)^{-1}x < 0 \;\forall\; x \in \mathscr{V}^{\perp}\backslash\{0\} \tag{5.65}$$

for all small $\epsilon > 0$. Then we exploit the fact that

$$(P + \epsilon I)^{-1} \to P^{-1} \text{ for } \epsilon \to 0 \tag{5.66}$$

since P is non singular. Therefore, after taking the limit, we get from (5.65)

$$x^T P^{-1}x \geq 0 \;\forall\; x \in \mathscr{U}^{\perp}\backslash\{0\} \text{ and } x^T P^{-1}x \leq 0 \quad \forall x \in \mathscr{V}^{\perp}\backslash\{0\} \tag{5.67}$$

where we already know that the first inequality is strict. □

The dual Full Block S-Procedure is structured as follows.

[4] The therm *complementary* means that the subspaces are in direct sum.
[5] The equality holds if $x = 0$.

5.3.2 Dual Full Block S-Procedure

Conditions (5.48)-(5.50) are equivalent to

$$Y \succ 0 \tag{5.68}$$

$$\begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix}^{6} \prec 0 \text{ on } \mathscr{S}^{\perp}(\Delta) \text{ for all } \Delta \in \mathbf{\Delta} \tag{5.69}$$

$$\tilde{J}^T \tilde{H} \tilde{J} \succ 0 \tag{5.70}$$

with

$$\tilde{H} = \left[\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} & 0 & 0 \\ 0 & 0 & \tilde{S}^T & \tilde{R} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\frac{1}{\gamma^2} I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right] \tag{5.71}$$

and

$$\tilde{J} = \left[\begin{array}{ccc} -YA^T & -YC_1^T & -YC_2^T \\ I & 0 & 0 \\ \hline -B_1^T & -D_1^T & -D_{21}^T \\ 0 & I & 0 \\ \hline -B_2^T & -D_{21}^T & -D_2^T \\ 0 & 0 & I \end{array}\right] \tag{5.72}$$

where in (5.68) $Y = X^{-1}$, while (5.69) is a direct consequence of the dualization lemma applied to (5.49). As regards (5.70), it can be derived as it has been done for (5.50) starting from the dual Bounded Real Lemma and exploiting the relations (5.55)-(5.56).

5.4 Dual Full Block S-Procedure for Infinity Norm Minimisation of LTI Decomposable Systems

As already mentioned in Chapter 4, in order to apply the Full Block S-Procedure, we need to equivalently rewrite the system (4.6) by introducing an interconnection channel $q \to p$ so that the new mathematical model for the interconnected system in a decomposed form becomes

$$\begin{cases} \dot{x}_i(t) = A^d x_i(t) + B_1 p_i(t) + B_2 w_i(t) + B u_i(t) \\ q_i(t) = C_1 x_i(t) + D_1 p_i(t) + D_{12} w_i(t) + E_1 u_i(t) \quad\quad i = 1, \ldots, N \\ z_i(t) = C_2 x_i(t) + D_{21} p_i(t) + D_2 w_i(t) + E_2 u_i(t) \end{cases} \tag{5.73}$$

where $x_i(t)$ is the i_{th} state variable, $w_i(t)$ the i_{th} disturbance, $u_i(t)$ the i_{th} control input, $z_i(t)$ thei_{th} performance output, and $q_i(t)$, $p_i(t)$ are such that

$$q_i(t) = \lambda_i p_i(t). \tag{5.74}$$

P is the pattern matrix and it is assumed to be symmetric.
In the following we recall the nomenclature for the state space matrices of the decomposed system:

$$\left[\begin{array}{c|c|c|c} A^d & B_1 & B_2 & B \\ \hline C_1 & D_1 & D_{12} & E_1 \\ \hline C_2 & D_{21} & D_2 & E_2 \end{array}\right] := \left[\begin{array}{c|c|c|c|c|c} A^d & A^i & B_u^d & B_w^d & B_u^d & 0 \\ \hline I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ \hline C_p^d & C_p^i & D_{pu}^d & 0 & D_{pu}^d & 0 \end{array}\right] \tag{5.75}$$

[6] This is the inverse of the P matrix in the Full Block S Procedure theorem.

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{B} \\ \mathcal{C}_1 & \mathcal{D}_1 & \mathcal{D}_{12} & \mathcal{E}_1 \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_2 & \mathcal{E}_2 \end{bmatrix} := \left[\begin{array}{c|c|c|c} I_N \otimes A^d & I_N \otimes B_1 & I_N \otimes B_2 & I_N \otimes B \\ \hline I_N \otimes C_1 & I_N \otimes D_1 & I_N \otimes D_{12} & I_N \otimes E_1 \\ \hline I_N \otimes C_2 & I_N \otimes D_{21} & I_N \otimes D_2 & I_N \otimes E_2 \end{array}\right] \tag{5.76}$$

The objective of the distributed control problem, that we investigate in this work, is to minimise the $\mathcal{H}_\infty$ norm of the transfer function T_{wz} between the exogenous input $w(t)$ and the performance output $z(t)$, thus guaranteeing robust stability. As claimed in [28], the feasibility of the LMIs (5.77)-(5.79) is a sufficient condition for the existence of a controller that solves the afore mentioned problem. In the following the matrix constraints for the synthesis of the desired controller are listed.

$$Y \succ 0 \tag{5.77}$$

$$\begin{bmatrix} I \\ P \otimes I \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} I \\ P \otimes I \end{bmatrix} \prec 0 \tag{5.78}$$

$$\tilde{J}^T \tilde{H} \tilde{J} \succ 0 \tag{5.79}$$

with

$$\tilde{H} = \left[\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} & 0 & 0 \\ 0 & 0 & \tilde{S}^T & \tilde{R} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\frac{1}{\gamma^2} I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right], \tag{5.80}$$

$$\tilde{Q} = I_N \otimes \tilde{Q}_i, \quad \tilde{R} = I_N \otimes \tilde{R}_i, \quad \tilde{S} = I_N \otimes \tilde{S}_i \quad i = 1, \ldots, N \tag{5.81}$$

and

$$\tilde{J} = \left[\begin{array}{ccc} -(\mathcal{A}Y + \mathcal{B}M)^T & -(\mathcal{C}_1 Y + \mathcal{E}_1 M)^T & -(\mathcal{C}_2 Y + \mathcal{E}_2 M)^T \\ I & 0 & 0 \\ \hline -\mathcal{B}_1^T & -\mathcal{D}_1^T & -\mathcal{D}_{21}^T \\ 0 & I & 0 \\ \hline -\mathcal{B}_2^T & -\mathcal{D}_{12}^T & -\mathcal{D}_2^T \\ 0 & 0 & I \end{array}\right] \tag{5.82}$$

Anyway it is worth noticing that the use of this procedure, as soon as some structure is imposed on the decision variables, brings some conservatism and therefore the solvability of the LMIs (5.77)-(5.79) is just a sufficient condition for the fulfilling of the performance requirements.
Exploiting the Full Block S-Procedure it is possible to obtain a static state feedback controller of the form

$$u(t) = (I_N \otimes K^d + P \otimes K^i) x(t) \tag{5.83}$$

by extracting the controller parameters from the relation

$$\begin{bmatrix} K^d \\ K^i \end{bmatrix} = M_i Y_i^{-1} \tag{5.84}$$

in which both the optimization variables M and Y are imposed to be blockdiagonal. A controller as the one in (5.83) is called "distributed controller". In this case, the control action on each subsystem depends on both the subsystem itself and all those ones with which it communicates and with which the controller can interact. Under the additional constraint $K^i = 0$, that can be enforced by imposing that the lower part of each diagonal block in M is equal to zero, the controller is instead called "decentralized controller". In this case the controller interacts only with the single agent and therefore the control decisions are taken only on the basis of the local outputs.

5.5 Decentralised Controller Synthesis

In this section we face the problem of synthesising a decentralised controller for decomposable systems through the Full Block S-Procedure. The objective of this controller is still to guarantee the stability of the system, meanwhile minimising the $\mathcal{H}_\infty$ norm of T_{wz}.

Theorem 5.5.1 (from [28]) *Let us consider the continuous time decomposable system described by the equations*

$$\begin{cases} \dot{x}(t) = (I_N \otimes A^d + P \otimes A^i)x(t) + I_N \otimes B_u^d u(t) + I_N \otimes B_w^d w(t) \\ z(t) = (I_N \otimes C^d + P \otimes C^i)x(t) + I_N \otimes D^d u(t) \end{cases} \tag{5.85}$$

here $x(t)$ is the state variable, $w(t)$ the disturbance, $u(t)$ the control input, $z(t)$ the performance output. There exists a sub-optimal controller of the form

$$u(t) = (I_N \otimes K)x(t) \tag{5.86}$$

with $\|T_{wz}\| < \gamma$ if there exist some matrices $Y = Y^T, M, \tilde{R} = \tilde{R}^T, \tilde{S}, \tilde{Q} = \tilde{Q}^T$ such that the LMIs (5.87)-(5.90) *are feasible. The controller gain K is obtained as $K = MY^{-1}$.*

$$Y \succ 0, \tag{5.87}$$

$$\begin{bmatrix} I \\ P \otimes I \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} I \\ P \otimes I \end{bmatrix} \prec 0 \tag{5.88}$$

$$\tilde{G}^T \tilde{H} \tilde{G} \succ 0 \tag{5.89}$$

with

$$\tilde{H} = \left[\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} & 0 & 0 \\ 0 & 0 & \tilde{S}^T & \tilde{R} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\frac{1}{\gamma^2}I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right], \tag{5.90}$$

$$\tilde{Q} = I_N \otimes \tilde{Q}_i, \quad \tilde{R} = I_N \otimes \tilde{R}_i, \quad \tilde{S} = I_N \otimes \tilde{S}_i \quad i = 1, \ldots, N \tag{5.91}$$

and

$$\tilde{G} = \left[\begin{array}{ccc} -(A^d Y + B_u^d M)^T & -(A^i Y)^T & -(C^d Y + D^d M)^T \\ I & 0 & 0 \\ \hline -I & 0 & -C^{iT} \\ 0 & I & 0 \\ \hline -B_w^{dT} & 0 & 0 \\ 0 & 0 & I \end{array}\right] \tag{5.92}$$

5.6 Distributed Controller Synthesis

To make the synthesis of a distributed controller for decomposable systems

$$u(t) = (I_N \otimes K^d + P \otimes K^i)x(t),$$

compatible with the Full Block S-Procedure, we will exploit the fact, already showed at the end of Chapter 4, that, the following system

$$\begin{cases} \dot{x}(t) = (I_N \otimes A^d + P \otimes A^i)x(t) + (I_N \otimes B_u)u(t) + (I_N \otimes B_w)w(t) \\ z(t) = (I_N \otimes C_z^d + P \otimes C_z^i)x(t) + (I_N \otimes D_{zu})u(t) \end{cases}, \tag{5.93}$$

in closed loop with

$$u(t) = (I_N \otimes K^d + P \otimes K^i)x(t) \tag{5.94}$$

is equivalent to

$$\begin{cases} \dot{x}(t) = (I_N \otimes A^d + P \otimes A^i)x(t) + (I_N \otimes [B_u \quad 0] + P \otimes [0 \quad B_u])\tilde{u}(t) + (I_N \otimes B_w)w(t) \\ z(t) = (I_N \otimes C_z^d + P \otimes C_z^i)x(t) + (I_N \otimes [D_{zu} \quad 0] + P \otimes [0 \quad D_{zu}])\tilde{u}(t) \end{cases}, \tag{5.95}$$

in closed loop with

$$\tilde{u}(t) = I_N \otimes \begin{bmatrix} K^d \\ K^i \end{bmatrix} x(t) \tag{5.96}$$

and both of them yield

$$\begin{cases} \dot{x}(t) = (I_N \otimes (A^d + B_u K^d) + P \otimes (A^i + B_u K^i))x(t) + (I_N \otimes B_w)w(t) \\ z(t) = (I_N \otimes (C_z^d + D_{zu} K^d) + P \otimes (C_z^i + D_{zu} K^i))x(t) \end{cases}. \tag{5.97}$$

The previous reasoning ensures that we can design a controller of the form (5.96) for the decomposed system (5.95) and from that one go back to a distributed controller as in (5.94) for the system of interest (5.93). This is summarised by the following theorem.

Theorem 5.6.1 (from [28]) *Let us consider the continuous time symmetric decomposable system*

$$\begin{cases} \dot{x}(t) = (I_N \otimes A^d + P \otimes A^i)x(t) + I_N \otimes B_u u(t) + I_N \otimes B_w w(t) \\ z(t) = (I_N \otimes A^d + P \otimes A^i)x(t) + I_N \otimes D_{zu} u(t) \end{cases} \tag{5.98}$$

where $x(t)$ is the state variable, $w(t)$ the disturbance, $u(t)$ the control input, $z(t)$ the performance output. There exists a distributed controller with the following structure

$$u(t) = (I_N \otimes K^d + P \otimes K^i)x(t) \tag{5.99}$$

such that $\|T_{wz}\|_\infty < \gamma$ if there exist some matrices $Y = Y^T, M, \tilde{R} = \tilde{R}^T, \tilde{S}, \tilde{Q} = \tilde{Q}^T$ such that the LMIs (5.100)-(5.102) *are feasible.*

$$Y \succ 0, \tag{5.100}$$

$$\begin{bmatrix} I \\ P \otimes I \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} I \\ P \otimes I \end{bmatrix} \prec 0 \tag{5.101}$$

$$\tilde{L}^T \tilde{H} \tilde{L} \succ 0 \tag{5.102}$$

with

$$\tilde{H} = \left[\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tilde{Q} & \tilde{S} & 0 & 0 \\ 0 & 0 & \tilde{S}^T & \tilde{R} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\frac{1}{\gamma^2} I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right], \tag{5.103}$$

$$\tilde{Q} = I_N \otimes \tilde{Q}_i, \quad \tilde{R} = I_N \otimes \tilde{R}_i, \quad \tilde{S} = I_N \otimes \tilde{S}_i \quad i = 1, \ldots, N \tag{5.104}$$

and

$$\tilde{L} = \left[\begin{array}{ccc} -(\mathcal{A}Y + \mathcal{B}M)^T & -(\mathcal{C}_1 Y + \mathcal{E}_1 M^T) & -(\mathcal{C}_2 Y + \mathcal{E}_2 M)^T \\ I & 0 & 0 \\ \hline -\mathcal{B}_1^T & -\mathcal{D}_1^T & -\mathcal{D}_{21}^T \\ 0 & I & 0 \\ \hline -\mathcal{B}_2^T & -\mathcal{D}_{12}^T & -\mathcal{D}_2^T \\ 0 & 0 & I \end{array}\right] \tag{5.105}$$

in which

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}_1 & \mathcal{B}_2 & \mathcal{B} \\ \mathcal{C}_1 & \mathcal{D}_1 & \mathcal{D}_{12} & \mathcal{E}_1 \\ \mathcal{C}_2 & \mathcal{D}_{21} & \mathcal{D}_2 & \mathcal{E}_2 \end{bmatrix} = \left[\begin{array}{c|c|c|c} I_N \otimes A^d & I_N \otimes B_1 & I_N \otimes B_2 & I_N \otimes B \\ \hline I_N \otimes C_1 & I_N \otimes D_1 & I_N \otimes D_{12} & I_N \otimes E_1 \\ \hline I_N \otimes C_2 & I_N \otimes D_{21} & I_N \otimes D_2 & I_N \otimes E_2 \end{array}\right] \tag{5.106}$$

The controller parameters K^d, K^i are extracted from the relation

$$\begin{bmatrix} K^d \\ K^i \end{bmatrix} = M_i Y_i^{-1} \tag{5.107}$$

For the sake of clarity we recall that the LMI (5.100) impose the Lyapunov matrix to be positive definite, the LMIs (5.101), (5.102) come directly from the Full Block S-Procedure and the Dualization Lemma and take the role of the LMIs (5.49),(5.50) respectively in the primal version of the Full Block S-Procedure.

Proof Consider the decomposed system described as in (5.73), where λ_i is the i_{th} eigenvalue of P:

$$\begin{cases} \dot{x}_i(t) = (A^d + \lambda_i A^i)x_i(t) + ([B_u \quad 0] + \lambda_i [0 \quad B_u])\tilde{u}_i(t) + B_w w_i(t) \\ z_i(t) = (C_z^d + \lambda_i C_z^i)x_i(t) + ([D_{zu} \quad 0] + \lambda_i [0 \quad D_{zu}])\tilde{u}_i(t) \end{cases} \tag{5.108}$$

with $\tilde{u}_i = [\tilde{u}_i^{dT} \quad \tilde{u}_i^{iT}]^T$. Let us define the signal q_i as follows

$$q_i(t) = [x_i^T \quad \tilde{u}_i^{iT}]^T \tag{5.109}$$

and the signal p_i as

$$p_i(t) = \lambda_i q_i(t) \tag{5.110}$$

then the system (5.108) is equivalently written as

$$\begin{cases} \dot{x}_i(t) = A^d x_i(t) + [A^i \quad B_u]p_i(t) + [B_u \quad 0]\tilde{u}_i(t) + B_w w_i(t) \\ q_i(t) = \begin{bmatrix} I \\ 0 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \tilde{u}_i(t) \\ z_i(t) = C_z^d x_i(t) + [C_z^i \quad D_{zu}]p_i(t) + [D_{zu} \quad 0]\tilde{u}_i(t), \end{cases} \qquad p_i = \lambda_i q_i \tag{5.111}$$

or, in a more compact form:

$$\begin{bmatrix} \dot{x}_i \\ q_i \\ z_i \end{bmatrix} = \left[\begin{array}{c|cc|c|cc} A^d & A^i & B_u^d & B_w^d & B_u^d & 0 \\ \hline I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ \hline C_p^d & C_p^i & D_{pu}^d & 0 & D_{pu}^d & 0 \end{array}\right] \begin{bmatrix} x_i \\ p_i \\ w_i \\ \tilde{u}_i \end{bmatrix} \tag{5.112}$$

in accordance with (5.75) and this concludes the proof. □

It is worth recalling that, as already mentioned in Section 4.2, considering the decomposed system, since P is symmetrical, does not lead to a loss of information about the norm of the real system.

5.7 Extended Full Block S-Procedure

In order to derive the extended version of the Full Block S-Procedure we follow the steps explained in [39] and summarized in Sections 5.1-5.2, relying on the extended version of the dual Bounded

Real Lemma presented in Section 3.2. As demonstrated in [14], the LMI in the dual Bounded Real Lemma (3.22) can be equivalently rewritten as[7]

$$\begin{pmatrix} -A_{cl}F - F^T A_{cl}^T & -Y + F - F^T A_{cl}^T & -B_w & -F^T C_{cl}^T \\ -Y + F^T - A_{cl}F & F + F^T & -B_w & 0 \\ -B_w^T & -B_w^T & \gamma^2 I & -D_w^T \\ -C_{cl}F & 0 & -D_w & I \end{pmatrix} \succ 0 \tag{5.113}$$

By applying a similarity transformation that permutes the last two columns and rows

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -A_{cl}F - F^T A_{cl}^T & -Y + F - F^T A_{cl}^T & -B_w & -F^T C_{cl}^T \\ -Y + F^T - A_{cl}F & F + F^T & B_w & 0 \\ -B_w^T & -B_w^T & \gamma^2 I & -D_w^T \\ -C_{cl}F & 0 & -D_w & I \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \tag{5.114}$$

$$= \begin{pmatrix} -A_{cl}F - F^T A_{cl}^T & -Y + F - F^T A_{cl}^T & -F^T C_{cl}^T & -B_w \\ -Y + F^T - A_{cl}F & F + F^T & 0 & -B_w \\ -C_{cl}F & 0 & I & -D_w \\ -B_w^T & -B_w^T & -D_w^T & \gamma^2 I \end{pmatrix}$$

and taking the Schur complement with respect to the last diagonal entry

$$\begin{pmatrix} -A_{cl}F - F^T A_{cl}^T & -Y + F - F^T A_{cl}^T & -F^T C_{cl}^T \\ -Y + F^T - A_{cl}F & F + F^T & 0 \\ -C_{cl}F & 0 & I \end{pmatrix} - \begin{pmatrix} -B_w \\ -B_w \\ -D_w \end{pmatrix} \frac{1}{\gamma^2} \begin{pmatrix} -B_w^T & -B_w^T & -D_w^T \end{pmatrix}$$

we obtain

$$\begin{pmatrix} -F^T A_{cl}^T - A_{cl}F - \frac{B_w B_w^T}{\gamma^2} & -F^T A_{cl}^T + F - Y - \frac{B_w B_w^T}{\gamma^2} & -F^T C_{cl} - \frac{B_w D_w^T}{\gamma^2} \\ -AF + F^T - Y - \frac{B_w B_w^T}{\gamma^2} & F + F^T - \frac{B_w B_w^T}{\gamma^2} & -\frac{B_w D_w^T}{\gamma^2} \\ -C_{cl}F - \frac{D_w B_w^T}{\gamma^2} & -\frac{D_w B_w^T}{\gamma^2} & -\frac{D_w D_w^T}{\gamma^2} + I \end{pmatrix}. \tag{5.115}$$

Referring to the linear time invariant case, the matrix (5.115) rewritten in the same form as (5.6), becomes

$$\star \left(\begin{array}{cccc|cc} 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & F - Y & 0 & 0 \\ 0 & I & F^T - Y & F + F^T & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\frac{I}{\gamma^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right) \left(\begin{array}{ccc} -F^T A^T & -F^T A^T & -F^T C^T \\ 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ \hline -B^T & -B^T & -D^T \\ 0 & 0 & I \end{array}\right), \tag{5.116}$$

from which, introducing the multipliers, as done in [39], the new constraints to be fulfilled become

$$Y \succ 0 \tag{5.117}$$

$$\begin{bmatrix} I \\ -P \otimes I \end{bmatrix}^T \begin{bmatrix} \tilde{Q} & \tilde{S} \\ \tilde{S}^T & \tilde{R} \end{bmatrix} \begin{bmatrix} I \\ -P \otimes I \end{bmatrix} \prec 0 \tag{5.118}$$

$$\tilde{N}^T \tilde{L} \tilde{N} \succ 0 \tag{5.119}$$

[7]Here the Lyapunov matrix is called Y and not P, as was done in Section 3, in order not to create ambiguity with the pattern matrix.

with

$$\tilde{L} = \left[\begin{array}{cccc|cc|cc} 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & F-Y & 0 & 0 & 0 & 0 \\ 0 & I & F^T - Y & F + F^T & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \tilde{Q} & \tilde{S} & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{S}^T & \tilde{R} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\gamma^2} I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{array}\right] \tag{5.120}$$

and

$$\tilde{N} = \left[\begin{array}{cccc} -(\mathcal{A}F + \mathcal{B}M)^T & -(\mathcal{A}F + BM)^T & -(\mathcal{C}_1 F + \mathcal{E}_1 M)^T & -(\mathcal{C}_2 F + \mathcal{E}_2 M)^T \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline -\mathcal{B}_1^T & -\mathcal{B}_1^T & -\mathcal{D}_1^T & -\mathcal{D}_{21}^T \\ 0 & 0 & I & 0 \\ \hline -\mathcal{B}_2^T & -\mathcal{B}_2^T & -\mathcal{D}_{12}^T & -\mathcal{D}_2^T \\ 0 & 0 & 0 & I \end{array}\right] \tag{5.121}$$

We can now introduce the following statement.
Let us consider the continuous time symmetric decomposable system

$$\begin{cases} \dot{x}(t) = (I_N \otimes A^d + P \otimes A^i)x(t) + I_N \otimes B_u u(t) + I_N \otimes B_w w(t) \\ z(t) = (I_N \otimes A^d + P \otimes A^i)x(t) + I_N \otimes D_{zu} u(t) \end{cases} \tag{5.122}$$

where $x(t)$ is the state variable, $w(t)$ the disturbance, $u(t)$ the control input, $z(t)$ the performance output. There exists a distributed controller with the following structure

$$u(t) = (I_N \otimes K^d + P \otimes K^i)x(t) \tag{5.123}$$

such that $\|T_{wz}\|_\infty < \gamma$ if there exist some matrices $Q = Q^T, R = R^T, S, M, F, Y = Y^T$ such that the LMIs (5.117)-(5.119) *are feasible.*

Comments

It is worth noticing that as soon as some structure is imposed on the optimization variables F, Y, Q, R, S, M, the satisfaction of the constraints (5.117)-(5.119) is just a sufficient condition for the existence of a controller described as in (5.123) that allows to satisfy the stability and performances requirements. The imposition of the block diagonal structure on the optimization variables F, M is necessary in order to be able to go back to the controller parameters K^d, K^i that in the Extended Full Block S-Procedure are extracted from the relation

$$\begin{bmatrix} K^d \\ K^i \end{bmatrix} = M_i F_i^{-1} \tag{5.124}$$

where M_i and F_i are the i_{th} diagonal blocks of the matrices M and F, respectively. At the same time, restricting the optimization variables to assume some particular structures, even if this can bring to some loss in terms of controller performance, is advantageous to reduce the computational burden of the controller synthesis that is a major concern when dealing with large scale systems. This new formulation has some benefits with respect to the standard formulation of the Full Block S-Procedure. In this case, in fact, thanks to the introduction of a slack variable F,

the Lyapunov matrix appears decoupled from any other optimization variable and this allows us to impose a less restrictive structure on it. In the standard version of the Full Block S-Procedure, in fact, in order to be able to extract the parameters K^d, K^i of the distributed controller it was necessary to restrict the Lyapunov matrix to be block diagonal and, as it will be shown in Section 6.1, this is a significant source of conservatism in the controller synthesis. In this formulation, instead, to extract the controller parameters, the matrix M and the slack variables F are involved and the variable F is not restricted to be symmetric. In this way, by keeping the same level of scalability, the degree of conservatism is reduced by adding more degrees of freedom.
Anyway the extended method showed in this section, in the context of decomposable and homogeneous systems, can be interpreted as a more general result applicable to LPV systems in a formulation analogous to the one in [39]. Moreover, when dealing with interconnected system, this approach is still valid even when the pattern matrix P depends continuously on a parameter Δ that varies in a compact set.

5.8 Decomposition

The decomposition approach presented in this section is a method that allows to impose less restrictive structures on the optimization variables by still reaching the same level of scalability as in the case when all the optimization variables are restricted to assume a block diagonal structure.
Let us call Z a matrix such that

$$\Lambda = Z^{-1}PZ = \text{diag}\{\lambda_1, \ldots, \lambda_N\} \tag{5.125}$$

and let us suppose that the eigenvalues of the pattern matrix P are ordered as follows

$$\lambda_1 \leq \cdots \leq \lambda_N. \tag{5.126}$$

Multiplier Condition

Let us apply a similarity transformation by means of the matrix $Z \otimes I_{np}$ to the multiplier condition. And let us assume that the multipliers Q, R, S get the structure $M = I_N \otimes M^d + P \otimes M^i$. The multiplier condition in the Full Block S-Procedure takes the form:

$$Z^{-1} \otimes I_{n_p} \begin{bmatrix} I_N \otimes I_{n_p} \\ -P \otimes I_{n_p} \end{bmatrix}^T \begin{bmatrix} I_N \otimes \tilde{Q}^d + P \otimes \tilde{Q}^i & I_N \otimes \tilde{S}^d + P \otimes \tilde{S}^i \\ I_N \otimes \tilde{S}^{dT} + P \otimes \tilde{S}^{iT} & I_N \otimes \tilde{R}^d + P \otimes \tilde{R}^i \end{bmatrix} \begin{bmatrix} I_N \otimes I_{n_p} \\ -P \otimes I_{n_p} \end{bmatrix} Z \otimes I_{n_p} \prec 0 \tag{5.127}$$

and exploiting the properties of bilinearity and associativity of the Kronecker product,

$$\begin{bmatrix} Z^{-1} \otimes I_{n_p} \\ -Z^{-1}P \otimes I_{n_p} \end{bmatrix}^T \begin{bmatrix} I_N \otimes \tilde{Q}^d + P \otimes \tilde{Q}^i & I_N \otimes \tilde{S}^d + P \otimes \tilde{S}^i \\ I_N \otimes \tilde{S}^{dT} + P \otimes \tilde{S}^{iT} & I_N \otimes \tilde{R}^d + P \otimes \tilde{R}^i \end{bmatrix} \begin{bmatrix} Z \otimes I_{n_p} \\ -PZ \otimes I_{n_p} \end{bmatrix} =$$

$$= I_N \otimes Q^d + \Lambda \otimes Q^i - \Lambda \otimes S^{dT} - \Lambda^2 S^{iT} - \Lambda \otimes S^d - \Lambda^2 \otimes S^i + \Lambda^2 \otimes R^d + \Lambda^3 \otimes R^i,$$

we end up with a block diagonal matrix of the form

$$\text{diag}\{Q^d + \lambda_i(Q^i - S^{dT} - S^d) + \lambda_i^2(-S^{iT} - S^i + R^d) + \lambda_i^3 R^i\}_{i=1,\ldots,N} \prec 0 \tag{5.128}$$

where in the writing above, "diag" indicates a block-diagonal matrix in which all the diagonal blocks assume the same structure, as the one written in brackets, depending on λ_i. The constraint in the expression (5.128) is not convex in the variable λ_i due to the presence of the cubic term

$\lambda_i^3 R^i$ and the quadratic term $\lambda_i^2(-S^{iT} - S^i + R^d)$ whose definiteness is a priori not defined. At the price of introducing some conservatism, it is possible to make this constraint convex by imposing

$$R^i = 0 \tag{5.129}$$

$$(-S^{Ti} - S^i + R^d) \succ 0. \tag{5.130}$$

This brings the advantage that, the fulfilment of this new constraint

$$\text{diag}\{Q^d + \lambda_i(Q^i - S^{dT} - S^d) + \lambda_i^2(-S^{iT} - S^i + R^d)\}_{i=1,\dots,N} \prec 0 \tag{5.131}$$

is equivalent to the fullfillment of the same constraint only for the vertices of the convex set $\{\lambda_1, \dots, \lambda_N\}$ on which the constraint is defined. This means that we can take into account only these two LMIs:

$$\text{diag}\{Q^d + \lambda_1(Q^i - S^{dT} - S^d) + \lambda_1^2(-S^{iT} - S^i + R^d)\} \prec 0 \tag{5.132}$$

$$\text{diag}\{Q^d + \lambda_N(Q^i - S^{dT} - S^d) + \lambda_N^2(-S^{iT} - S^i + R^d)\} \prec 0 \tag{5.133}$$

It has been found empirically that, adding the constraints (5.129), (5.130) to the optimization problem, does not induce much conservatism. For example, for a bound γ of the order 10^{-1}, the loss in terms of performance brought by adding these constraints is of the order of 10^{-3} even when dealing with strongly coupled networked systems (i.e. P matrix quite dense).

Nominal Condition

The decomposition approach can be applied to the nominal condition as well and, also in this case, exploiting the diagonalizability of the pattern matrix, we can obtain the same level of decomposition and scalability as if all the optimization matrices were block diagonal. Let us consider a matrix V of the form

$$V = \left[\begin{array}{c|c|c} Z \otimes I_{V_1} & & \\ \hline & Z \otimes I_{V_2} & \\ \hline & & Z \otimes I_{V_3} \end{array}\right], \tag{5.134}$$

and a matrix U of the form

$$U = \left[\begin{array}{c|c|c} Z \otimes I_{U_1} & & \\ \hline & Z \otimes I_{U_2} & \\ \hline & & Z \otimes I_{U_3} \end{array}\right], \tag{5.135}$$

where the dimensions of the identity matrices I_{V_i} and I_{U_i}, $i = 1, 2, 3$, are such that the diagonal blocks match the dimensions of the blocks in the matrices $\tilde{L}$ and $\tilde{H}$.

Applying a similarity transformation by means of the matrix V and multiplying on the left and on the right the matrix $\tilde{H}$ by the identity written in the form UU^{-1} we can equivalently rewrite the nominal condition as follows

$$\underbrace{V^{-1}\tilde{L}^T U}_{\hat{L}^T}\underbrace{U^{-1}\tilde{H}U}_{\hat{H}}\underbrace{U^{-1}\tilde{L}V}_{\hat{L}} \succ 0 \tag{5.136}$$

and recalling that

$$\tilde{L} = \left[\begin{array}{ccc} -I_N \otimes (A^d Y_i + BM_i)^T & -I_N \otimes (C_1 Y_i + E_1 M_i)^T & -I_N \otimes (C_2 Y_i + E_2 M_i)^T \\ I & 0 & 0 \\ \hline -I_N \otimes B_1^T & -I_N \otimes D_1^T & -I_N \otimes D_{21}^T \\ 0 & I & 0 \\ \hline -I_N \otimes B_2^T & -I_N \otimes D_{12}^T & -I_N \otimes D_2^T \\ 0 & 0 & I \end{array}\right], \quad (5.137)$$

$$\tilde{H} = \left[\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_N \otimes Q^d + P \otimes Q^i & I_N \otimes S^d + P \otimes S^i & 0 & 0 \\ 0 & 0 & I_N \otimes S^{dT} + P \otimes S^{iT} & I_N \otimes R^d + P \otimes R^i & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\frac{1}{\gamma^2} I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right], \quad (5.138)$$

it turns out that

$$\hat{H} = \left[\begin{array}{c|c|c} Z^{-1} \otimes I_{U_1} & & \\ \hline & Z^{-1} \otimes I_{U_2} & \\ \hline & & Z^{-1} \otimes I_{U_3} \end{array}\right] \left[\begin{array}{cc|cc|cc} 0 & I & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_N \otimes Q^d + P \otimes Q^i & I_N \otimes S^d + P \otimes S^i & 0 & 0 \\ 0 & 0 & I_N \otimes S^{dT} + P \otimes S^{iT} & I_N \otimes R^d + P \otimes R^i & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -\frac{1}{\gamma^2} I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right] \left[\begin{array}{c|c|c} Z \otimes I_{U_1} & & \\ \hline & Z \otimes I_{U_2} & \\ \hline & & Z \otimes I_{U_3} \end{array}\right],$$

$$\hat{H} = \left[\begin{array}{cc|cc|cc} 0 & I_N \otimes I & 0 & 0 & 0 & 0 \\ I_N \otimes I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_N \otimes Q^d + \Lambda \otimes Q^i & I_N \otimes S^d + \Lambda \otimes S^i & 0 & 0 \\ 0 & 0 & I_N \otimes S^{dT} + \Lambda \otimes S^{iT} & I_N \otimes R^d + \Lambda \otimes R^i & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_N \otimes -\frac{1}{\gamma^2} I & 0 \\ 0 & 0 & 0 & 0 & 0 & I_N \otimes I \end{array}\right],$$

$$\hat{L} = \left[\begin{array}{ccc} -I_N \otimes (A^d Y_i + BM_i)^T & -I_N \otimes (C_1 Y_i + E_1 M_i)^T & -I_N \otimes (C_2 Y_i + E_2 M_i)^T \\ I & 0 & 0 \\ \hline -I_N \otimes B_1^T & -I_N \otimes D_1^T & -I_N \otimes D_{21}^T \\ 0 & I & 0 \\ \hline -I_N \otimes B_2^T & -I_N \otimes D_{12}^T & -I_N \otimes D_2^T \\ 0 & 0 & I \end{array}\right],$$

and in this way the product $\hat{L}^T \hat{H} \hat{L}$ takes the following structure

$$J = \hat{L}^T \hat{H} \hat{L} = \left[\begin{array}{c|c|c} I_N \otimes J_{11}^d + \Lambda \otimes J_{11}^i & I_N \otimes J_{12}^d + \Lambda \otimes J_{12}^i & I_N \otimes J_{13}^d + \Lambda \otimes J_{13}^i \\ \hline I_N \otimes J_{21}^d + \Lambda \otimes J_{21}^i & I_N \otimes J_{22}^d + \Lambda \otimes J_{22}^i & I_N \otimes J_{32}^d + \Lambda \otimes J_{32}^i \\ \hline I_N \otimes J_{31}^d + \Lambda \otimes J_{31}^i & I_N \otimes J_{32}^d + \Lambda \otimes J_{32}^i & I_N \otimes J_{33}^d + \Lambda \otimes J_{33}^i \end{array}\right] =$$

$$= \left[\begin{array}{c|c|c} \mathrm{diag}\{J_{11}^d + \lambda_i J_{11}^i\}_{i=1,\ldots,N} & \mathrm{diag}\{J_{12}^d + \lambda_i J_{12}^i\}_{i=1,\ldots,N} & \mathrm{diag}\{J_{13}^d + \lambda_i J_{13}^i\}_{i=1,\ldots,N} \\ \hline \mathrm{diag}\{J_{21}^d + \lambda_i J_{21}^i\}_{i=1,\ldots,N} & \mathrm{diag}\{J_{22}^d + \lambda_i J_{22}^i\}_{i=1,\ldots,N} & \mathrm{diag}\{J_{23}^d + \lambda_i J_{23}^i\}_{i=1,\ldots,N} \\ \hline \mathrm{diag}\{J_{31}^d + \lambda_i J_{31}^i\}_{i=1,\ldots,N} & \mathrm{diag}\{J_{32}^d + \lambda_i J_{32}^i\}_{i=1,\ldots,N} & \mathrm{diag}\{J_{33}^d + \lambda_i J_{33}^i\}_{i=1,\ldots,N} \end{array}\right], \quad (5.139)$$

with

$$J_{11}^d = -(A^dY_i + BM_i) - (A^dY_i + BM_i)^T + B_1Q^dB_1^T - \frac{1}{\gamma^2}B_2B_2^T, \quad J_{11}^i = B_1Q^iB_1^T,$$

$$J_{21}^d = -(C_1Y_i + E_1M_i) + D_1Q^dB_1^T - S^dB_1^T - \frac{1}{\gamma^2}D_{12}B_2^T, \quad J_{21}^i = D_1Q^iB_1^T - S^{iT}B_1^T,$$

$$J_{31}^d = -(C_2Y_i + E_2M_i) + D_{21}Q^dB_1^T - \frac{1}{\gamma^2}D_2B_2^T, \quad J_{31}^i = D_{21}Q^iB_1^T,$$

$$J_{12}^d = -(C_1Y_i + E_1M_i)^T + B_1Q^dD_1^T - B_1S^d - \frac{1}{\gamma^2}B_2D_{12}^T, \quad J_{12}^i = B_1Q^iD_1^T - B_1S^i,$$

$$J_{22}^d = D_1Q^dD_1^T - D_1S^d - S^{dT}D_1^T + R^d - \frac{1}{\gamma^2}D_{12}D_{12}^T, \quad J_{22}^i = D_1Q^iD_1^T - D_1S^{iT} - S^{iT}D_1^T + R^i,$$

$$J_{23}^d = +D_{21}Q^dD_1^T - D_{21}S^d - \frac{1}{\gamma^2}D_2D_{12}^T, \quad J_{23}^i = D_{12}Q^iD_1^T - D_{21}S^i,$$

$$J_{13}^d = -(C_2Y_i + E_2M_i)^T + B_1Q^dD_{21}^T - \frac{1}{\gamma^2}B_2D_2^T, \quad J_{13}^i = B_1Q^iD_{21}^T,$$

$$J_{23}^d = +D_1Q^dD_{21}^T - S^dD_{21}^T - \frac{1}{\gamma^2}D_{12}D_2^T, \quad J_{23}^i = D_1Q^iD_{21}^T - S^iD_{21}^T,$$

$$J_{33}^d = D_{21}Q^dD_{21}^T - \frac{1}{\gamma^2}D_2D_2^T + I, \quad J_{12}^i = D_{21}Q^iD_{21}^T.$$

The constraints in (5.139) turn out to be affine in the variable λ_i and thus convex. For this reason in order to be sure that the nominal condition is fulfilled for each λ_i it is necessary and sufficient to check if it is satisfied just for λ_1 and λ_N.
The decomposition approach, here introduced for the standard version of the Full Block S-Procedure, is still applicable when considering its extended version.
Moreover, it is possible to extend the decomposition approach to matrices that have an even less restrictive structure. For this purpose we first recall the following definition and theorem.

Definition 5.8.1 *A set of matrices $\mathcal{P}$ is said to be simultaneously diagonalizable if there exists an invertible matrix Z such that, for every $P \in \mathcal{P}$, $Z^{-1}PZ$ is diagonal.*

Theorem 5.8.1 [20] *Two diagonalizable matrices A, B are simultaneously diagonalizable if and only if $AB = BA$.*

In light of what it has just been mentioned, it is evident that, whenever the optimization variables take the following structures

$$M = I \otimes M^d, \quad M = I \otimes M^d + P \otimes M^i, \quad M = I \otimes M^d + P_1 \otimes M_1^i + P_2 \otimes M_2^i,$$
$$M = I \otimes M^d + P_1 \otimes M_1^i + \dots + P_z \otimes M_z^i$$

with P_1, P_2, $\dots$, P_z simultaneously diagonalizable matrices, the decomposition approach is still applicable and, generally, including non-null simultaneously diagonalizable matrices in the structure of the variables can bring improvements in the performance since more degrees of freedom are included.

Chapter 6

Numerical Results

In this chapter some numerical results are shown. The Matlab parser Yalmip that relies on the external solver SDPT3 has been exploited.

6.1 Numerical Analysis for the Sources of Conservatism in the Standard Full Block S-Procedure

The following results refer to a homogeneous system composed of $N = 6$ interconnected subsystems of third order. The numerical example we are going to refer to is taken from [2]. In the following the state space realization has been reported

$$A^d = \begin{bmatrix} 0.1 & -0.2 & 0.7 \\ -0.9 & -0.6 & -0.4 \\ -0.9 & 0.6 & -0.5 \end{bmatrix}, \quad A^i = \begin{bmatrix} 0.1 & 0.1 & -0.1 \\ -0.3 & -0.1 & -0.1 \\ -0.2 & -0.1 & 0 \end{bmatrix},$$

$$B_z^d = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \quad B_z^i = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_u^d = \begin{bmatrix} 0.7 \\ 0 \\ 0.5 \end{bmatrix}, \quad B_u^i = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C_z^d = \begin{bmatrix} 0.9 & 0.2 & 0.3 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_z^i = \begin{bmatrix} 0.5 & 0.3 & 0.5 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D_{zw}^d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D_{zw}^i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D_{zu}^d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{zu}^i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

All the subsystems are interconnected according to the following interconnection matrix

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{6.1}$$

to which the interconnection graph in Figure 6.1 corresponds.

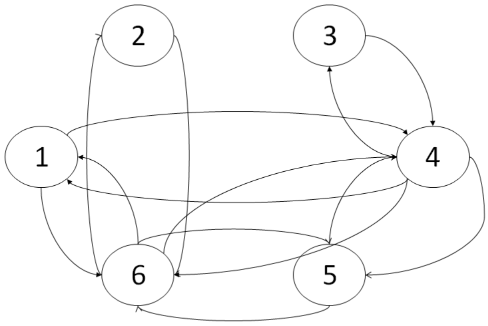

Figure 6.1: Interconnection graph among the subsystems.

In the following tables the symbol $M \otimes_P$ stands for $M = I_N \otimes M^d + P \otimes M^i$.

		Y full	
	Multipliers full	Multipliers $\otimes_P$	Multipliers blockdiag
γ	0.1888	0.2175	0.2779
$\mathcal{H}_\infty$	0.1888	0.2107	0.2624

Table 6.1: Distributed Controller from Standard Dual FBSP without structure on the Lyapunov matrix

		Y $\otimes_P$	
	Multipliers full	Multipliers $\otimes_P$	Multipliers blockdiag
γ	0.2126	0.2175	0.2779
$\mathcal{H}_\infty$	0.2062	0.2107	0.2624

Table 6.2: Distributed Controller from Standard Dual FBSP with $Y = I_N \otimes Y^d + P \otimes Y^i$

		Y blockdiag	
	Multipliers full	Multipliers $\otimes_P$	Multipliers blockdiag
γ	0.2779	0.2779	0.2779
$\mathcal{H}_\infty$	0.2624	0.2624	0.2624

Table 6.3: Decentralized Controller from Standard Dual FBSP with block diagonal Lyapunov Matrix

The results above show that whenever either the Lyapunov matrix Y or the multipliers Q, R, S are restricted to assume a block diagonal structure, the value of the bound γ assumes the highest value among all possible combinations of structures imposed in the standard version of the Full Block S-Procedure. Moreover it is worth noticing that, in order to be able to extract the parameters K^d, K^i of the distributed controller from this method, it is necessary, to the best of our knowledge, to impose a block diagonal structure on the Lyapunov matrix. This justifies the need to adopt a strategy that allows to bypass what can be considered as a limit of the standard Full Block S-Procedure when applied to homogeneous interconnected decomposable systems. This is what has led us to reinvent this method based on the extended version of the Bounded Real Lemma. In this way we could exploit the advantage of decoupling the Lyapunov matrix from all the variables involved, by the introduction of a new slack variable that is not restricted to be symmetric.

6.2 Numerical Comparison between Standard and Extended Full Block S-Procedure

The following results refer to a homogeneous system composed of $N = 6$ interconnected subsystems of third order. The state space matrix elements have been randomly generated from a uniform aleatory variable defined on an an interval of amplitude 0.2 centered around the example taken from [2]. We will consider a strictly proper system ($D^d_{zw} = D^i_{zw} = 0$) and we recall that B^i_u and D^i_{zu} need to be imposed equal to zero in order to render the Full Block S-Procedure applicable. Without loss of generality, the matrices B^i_z and C^i_z have been imposed to be zero since this choice led to more explicative results. The realisation we are going to refer to is:

$$A^d = \begin{bmatrix} 0.1903 & -0.1159 & 0.6105 \\ -0.8524 & -0.6462 & -0.4154 \\ -0.8904 & 0.5115 & -0.5165 \end{bmatrix}, \quad A^i = \begin{bmatrix} 0.1966 & 0.0603 & -0.1402 \\ -0.2667 & -0.1078 & -0.1396 \\ -0.1667 & -0.0356 & 0 \end{bmatrix},$$

$$B^d_z = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}, \quad B^i_z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad B^d_u = \begin{bmatrix} 0.2976 \\ -0.8983 \\ 0.3289 \end{bmatrix}, \quad B^i_u = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C^d_z = \begin{bmatrix} 0.8008 & 0.2844 & 0.3115 \\ -0.0787 & 0.0924 & -0.0991 \end{bmatrix}, \quad C^i_z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$D^d_{zw} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D^i_{zw} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D^d_{zu} = \begin{bmatrix} 0.0550 \\ 0.9365 \end{bmatrix}, \quad D^i_{zu} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

All the subsystems are interconnected according to the adjacency matrix

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{6.2}$$

corresponding to the interconnection graph in Figure 6.2.

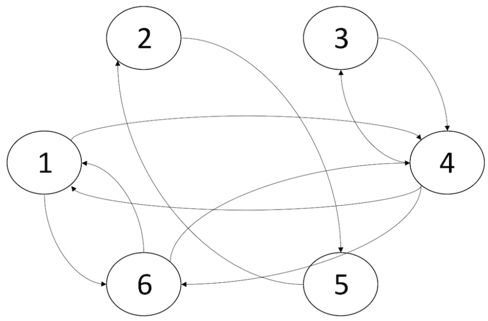

Figure 6.2: Interconnection graph among the subsystems.

In this example, two simultaneously diagonalizable matrices are involved, namely $P_1(= P)$ and

P_2 and they assume the following structures

$$P_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$P_2 = [p_{ij}]_{i,j=1,\ldots N}$ has been found by solving the optimization problem

$$\max_{\substack{s.t. \\ P_2 = P_2^T \\ P_1 P_2 = P_2 P_1 \\ p_{ii} = 0 \\ p_{ij}(1-p_{ij})=0}} \sum_{i<j} p_{ij}. \tag{6.3}$$

Tables 6.4, 6.5, 6.6, 6.7 refer to the standard version of the Full Block S-Procedure while the results in Tables 6.8, 6.9, 6.10, 6.11 refer to the extended Full Block S-Procedure.
In the following the symbol $M \otimes_{P_1,P_2}$ stands for $M = I_N \otimes M^d + P_1 \otimes M_1^i + P_2 \otimes M_2^i$.

		Y full	
	Multipliers full	Multipliers $\otimes_P$	Multipliers blockdiag
γ	1.5816	2.2280	2.7440
$\mathcal{H}_\infty$	1.5816	2.0986	2.5787

Table 6.4: Distributed Controller from Standard Dual FBSP without structure on the Lyapunov matrix

		Y $\otimes_P$	
	Multipliers full	Multipliers $\otimes_P$	Multipliers blockdiag
γ	2.1802	2.2280	2.7440
$\mathcal{H}_\infty$	2.0723	2.0986	2.5787

Table 6.5: Distributed Controller from Standard Dual FBSP with $Y = I_N \otimes Y^d + P \otimes Y^i$

		Y $\otimes_{P_1,P_2}$	
	Multipliers full	Multipliers $\otimes_{P_1,P_2}$	Multipliers blockdiag
γ	2.1802	2.2228	2.7440
$\mathcal{H}_\infty$	2.0723	2.0960	2.5787

Table 6.6: Distributed Controller from Standard Dual FBSP with $Y = I_N \otimes Y^d + P_1 \otimes Y_1^i + P_2 \otimes Y_2^i$

		Y blockdiag	
	Multipliers full	Multipliers $\otimes_P$	Multipliers blockdiag
γ	2.7440	2.7440	2.7440
$\mathcal{H}_\infty$	2.5787	2.5787	2.5787

Table 6.7: Decentralized Controller from Standard Dual FBSP with block diagonal Lyapunov Matrix

	F full								
	Multipliers full			Multipliers $\otimes_P$			Multipliers blockdiag		
	Y full	Y $\otimes_P$	Y blockdiag	Y full	Y $\otimes_P$	Y blockdiag	Y full	Y $\otimes_P$	Y blockdiag
γ	1.5816	1.7774	1.8270	2.2379	2.2379	2.2482	2.8053	2.8053	2.8053
$\mathcal{H}_\infty$	1.5816	1.6787	1.6994	2.0940	2.0940	2.1196	2.5823	2.5823	2.5823

Table 6.8: Distributed Controller from Extended Dual FBSP without structure on the slack variable F

	F $\otimes_P$								
	Multipliers full			Multipliers $\otimes_P$			Multipliers blockdiag		
	Y full	Y $\otimes_P$	Y blockdiag	Y full	Y $\otimes_P$	Y blockdiag	Y full	Y $\otimes_P$	Y blockdiag
γ	1.9828	2.0683	2.1012	2.2428	2.2428	2.2549	2.8053	2.8053	2.8053
$\mathcal{H}_\infty$	1.8704	1.9309	1.9900	2.1062	2.1062	2.1129	2.5823	2.5823	2.5823

Table 6.9: Distributed Controller from Extended Dual FBSP with slack variable $F = I_N \otimes F^d + P \otimes F^i$

	F $\otimes_{P_1,P_2}$								
	Multipliers full			Multipliers $\otimes_{P_1,P_2}$			Multipliers blockdiag		
	Y full	Y $\otimes_{P_1,P_2}$	Y blockdiag	Y full	Y $\otimes_{P_1,P_2}$	Y blockdiag	Y full	Y $\otimes_{P_1,P_2}$	Y blockdiag
γ	1.9417	2.0654	2.1013	2.2428	2.2428	2.2549	2.8053	2.8053	2.8053
$\mathcal{H}_\infty$	1.8302	1.9234	1.9899	2.1062	2.1062	2.1129	2.5823	2.5823	2.5823

Table 6.10: Distributed Controller from Extended Dual FBSP with slack variable $F = I_N \otimes F^d + P_1 \otimes F_1^i + P_2 \otimes F_2^i$

	F blockdiag								
	Multipliers full			Multipliers $\otimes_P$			Multipliers blockdiag		
	Y full	Y $\otimes_P$	Y blockdiag	Y full	Y $\otimes_P$	Y blockdiag	Y full	Y $\otimes_P$	Y blockdiag
γ	2.2135	2.2135	2.7739	2.3550	2.3550	2.7757	2.8053	2.8053	2.8053
$\mathcal{H}_\infty$	2.1695	2.1694	2.5812	2.3186	2.3186	2.1196	2.5823	2.5823	2.5823

Table 6.11: Decentralized Controller from Extended Dual FBSP with block diagonal slack variable

Comments

The numerical results show that the more restrictive the structure imposed on the optimisation variables the lower the performance. In fact, it is evident that the optimal value is reached from both the procedures when all the optimisation variables are not restricted to assume any particular structure while the performance gets much worse when imposing the block diagonal structure on all the optimisation variables. Moreover Tables 6.2, 6.4 and Tables 6.6, 6.7 show that involving the Kronecker product with simultaneously diagonalisable matrices in the structure of the optimisation variables can bring improvements. However the improvement brought in this case are not so consistent since the matrix P_2 is very sparse and thus not so many degrees of freedom are introduced. More consistent improvement could have been reached for other P_1, P_2 in which P_2 is a less sparse matrix with respect to the one involved in this numerical example.
Another fact that deserves attention is that, in the standard Full Block S-Procedure, whenever one of the matrices is set to assume a block diagonal structure, the performance bound γ assumes the highest value among all the γ obtained in the same procedure while this does not happen in the extended Full Block S-Procedure and this is a considerable advantage. This is a considerable improvement with respect to the classical approach since the block diagonal structure is always needed at least on one of the optimization variables (Y in the standard case, F in the extended case) in order to be able to extract the controller parameters.

The structure involving the simultaneously diagonalisable matrices on the slack variable F and on the Lyapunov matrix in the extended and standard Full Block S-Procedures respectively is the less restrictive structure that allows to decompose the controller synthesis inequalities. In that case the distributed controller parameters would assume the form

$$K^d = M^d(F^d + \lambda_i F^i)^{-1}, \quad K^i = M^i(F^d + \lambda_i F^i)^{-1}. \tag{6.4}$$

In this case better performances with respect to the case in which the variable F (or Y for the standard Full Block S-Procedure) are constrained to be block diagonal can be achieved. However, since in this case the controller parameters depend on λ_i, it generally happens that $K_i^d \neq K_j^d$ $\forall i \neq j$ and $K_i^i \neq K_j^i$ $\forall i \neq j$. A still open question is if in this case it would be still possible to design a distributed controller for the untransformed system and, if yes, it would be interesting to understand how to arrange the different K^i corresponding to the non-zero elements of the pattern matrix and the different K^d on the diagonal entries. The fact that different controller parameters K_i^d, K_i^i, for different subsystems, allow to reach a lower bound for the $\mathcal{H}_\infty$ norm with respect to the case in which the controller parameters are all equal, even when dealing with homogeneous systems, is not surprising. In the physical plant, in fact, each subsystem can be connected to a different number of other subsystems and therefore dealing with different controllers in the plant would allow to reach better performances.

Chapter 7

Conclusions and Future Works

During this master project two approaches for the synthesis of distributed controller for homogeneous decomposable systems have been developed. An extended version of the already existing Full Block S-Procedure has been presented and a decomposition approach in order to improve the computational efficiency has been proposed.
The Extended Full Block S-Procedure has been derived from the extended version of the Bounded Real Lemma and it has the advantage of reducing the conservatism with respect to the standard version by introducing a slack variable that renders the Lyapunov matrix decoupled from any other optimization variable. In this way we overcame the limit of the standard Full Block S-Procedure in which the Lyapunov matrix was constrained to assume a block diagonal form in order to let us extract the distributed controller parameters. In this way the extended Full Block S-Procedure bypasses one of the main sources of conservatism in the state of the art of distributed controller synthesis for this class of interconnected systems.
The decomposition approach can be applied to both the versions of the Full Block S-Procedure. With this method we exploit the diagonalizability of the pattern matrix so that, by applying a similarity transformation, we can impose less restrictive structures on the optimization variables by still reaching the same level of scalability with respect to the case in which all the optimization variables were block diagonal. Both the proposed methods show the advantage of reducing the conservatism with equal level of scalability with respect to the state of the art. Anyway it is worth remembering that the Full Block S-Procedure is a powerful robust control method that, in general, can be applied to linear parameter varying systems depending on uncertain parameters in a compact domain which means that, when it is applied to the restrictive case of deterministic linear time invariant systems, some more conservatism is induced, as said in [28].
A possible extension to this project could be to consider the more general case of $\alpha - \beta$ *heterogeneous systems* in which α different types of systems and β different kinds of interconnections are involved. In a more general context, in fact, different system variables could be interconnected according to different pattern matrices and β different interconnection channels would be introduced. Moreover it could happen that the controller interacts with the system according to a different pattern matrix with respect to the one that mirrors the physical interconnections among the subsystems. In a large scale setting, it is more likely, for the controller patter matrix, to involve less communication links. Anyway, whenever the β (or a subset of them) interconnection matrices are simultaneously diagonalizable, the decomposition approach is still a valid approach.

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Appendix A

Considerations About the Bounded Real Lemma and its Time Domain Interpretation

A.0.1 The Bounded Real Lemma

Let us consider the closed-loop system

$$\dot{x} = A_{cl}x + B_w w \tag{A.1}$$

$$z = C_{cl}x \tag{A.2}$$

with strictly proper transfer function $T(s) = C_{cl}(sI - A_{cl})^{-1}B_w$, *then the following are equivalent*

(*i*) $\|T\|_\infty < \gamma$;

(*ii*) *Given* $\Phi(s) := \gamma^2 I - T^T(-s)T(s)$ *then* $\Phi(j\omega) > 0 \quad \forall \omega \in \mathbb{R}$;

(*iii*) *The Hamiltonian Matrix*

$$M_\gamma := \begin{pmatrix} A_{cl} & \frac{1}{\gamma^2}B_w B_w^T \\ -C_{cl}^T C_{cl} & -A_{cl} \end{pmatrix} \in \mathbb{R}^{2n\times 2n}$$

has no purely imaginary eigenvalues;

(*iv*) *There exists a matrix* $X \succ 0$ *such that*

$$\begin{pmatrix} A_{cl}^T X + X A_{cl} + C_{cl}^T C_{cl} & X B_w \\ B_w^T X & -\gamma^2 I \end{pmatrix} \prec 0 \tag{A.3}$$

Considerations

If $\gamma > 0$ then M_γ has a clear division of eigenvalues, in the sense that they respect a certain symmetry on the complex plan. To make it more evident, we apply a similarity transformation that, as known, does not change the eigenvalues of the matrix. Let us consider the matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{A.4}$$

Notice that the J matrix is such that it is equal to the negative inverse, in fact

$$J^2 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix} \tag{A.5}$$

and, since $J^2 = -I$, multiplying both the terms in (A.5) by J^{-1}, it is evident that $J = -J^{-1}$. Therefore it holds that

$$J^{-1} M_\gamma J = -J M_\gamma J = -M_\gamma^T \tag{A.6}$$

where the last equality holds because a multiplication, both on the left and on the right, by J corresponds to permute rows and columns of the matrix. Therefore

$$det(sI - M_\gamma) = det(sI + M_\gamma^T) = det(sI + M_\gamma), \tag{A.7}$$

where the last relation is due to the fact that, since M_γ is a square matrix, its eigenvalues coincide with those of its transpose. Since M_γ is hamiltonian, its eigenvalues are symmetric with respect to the imaginary axes and thus we have a clear separation among stable and unstable eigenvalues showed in the following.
Let us consider the following similarity transformation

$$\begin{pmatrix} V_s & V_u \end{pmatrix}^{-1} M_\gamma \begin{pmatrix} V_s & V_u \end{pmatrix} = \begin{pmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{pmatrix} \tag{A.8}$$

where V_s indicates the eigenspaces associated with the stable eigenvalues, V_u the eigenspace associated with the unstable eigenvalues, Λ_s a block diagonal matrix, where all the diagonal blocks are upper triangular matrices with stable eigenvalues on the diagonal and Λ_u has the same structure as Λ_s but with unstable eigenvalues on the diagonal.
Equivalently, multiplying on the left both the expressions in (A.8) by $\begin{pmatrix} V_s & V_u \end{pmatrix}$ it results that

$$M_\gamma \begin{pmatrix} V_s & V_u \end{pmatrix} = \begin{pmatrix} V_s & V_u \end{pmatrix} \begin{pmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{pmatrix}. \tag{A.9}$$

Partitioning the eigenvectors of M_γ as follows

$$\begin{pmatrix} V_s & V_u \end{pmatrix} = \begin{pmatrix} V_{s1} & V_{u1} \\ V_{s2} & V_{u2} \end{pmatrix} \in \mathbb{R}^{2n \times 2n} \tag{A.10}$$

in which each block has dimension $2n$, we assume also V_{s1} to be non singular and define

$$P := V_{s2} V_{s1}^{-1} \tag{A.11}$$

From (A.9) it follows

$$M_\gamma V_s = V_s \Lambda_s \tag{A.12}$$

and multiplying on the right by V_{s1}^{-1} we get

$$M_\gamma \begin{pmatrix} V_{s1} \\ V_{s2} \end{pmatrix} V_{s1}^{-1} = \begin{pmatrix} V_{s1} \\ V_{s2} \end{pmatrix} V_{s1}^{-1} \underbrace{V_{s1} \Lambda_s V_{s1}^{-1}}_{:=A_c}. \tag{A.13}$$

(A.13) together with (A.11) yield to

$$M_\gamma \begin{pmatrix} I \\ P \end{pmatrix} = \begin{pmatrix} I \\ P \end{pmatrix} A_c \tag{A.14}$$

and thus $\begin{pmatrix} I \\ P \end{pmatrix}$ is another basis for the eigenspace associated with the stable eigenvalues. Expressing the above relation in an extended form

$$\begin{pmatrix} A_{cl} & \frac{1}{\gamma^2} B_w B_w^T \\ -C_{cl}^T C_{cl} & -A_{cl}^T \end{pmatrix} \begin{pmatrix} I \\ P \end{pmatrix} = \begin{pmatrix} I \\ P \end{pmatrix} A_c \tag{A.15}$$

we can derive the following two equations

$$A_{cl} + \frac{1}{\gamma^2} B_w B_w^T P = A_c \tag{A.16}$$

$$-C_{cl}^T C_{cl} - A_{cl}^T P = P A_c \tag{A.17}$$

and plugging the first equation into the second one we get the following Algebraic Riccati Equation (ARE)

$$A_{cl}^T P + P A_{cl} + \frac{1}{\gamma^2} P B_w B_w^T P + C_{cl}^T C_{cl} = 0 \tag{A.18}$$

It is worth noticing the similarity between the ARE (A.18) and the Lyapunov equation

$$A_{cl}^T P + P A_{cl} + \underbrace{C_{cl}^T C_{cl}}_{:=Q \succ 0} = 0 \tag{A.19}$$

to which the extra term $\frac{1}{\gamma^2} P B_w B_w^T P$, quadratic in P, has been added and which is known to be solved by the observability Gramian

$$P = \int_0^\infty e^{A_{cl}^T \tau} C_{cl}^T C_{cl} e^{A_{cl} \tau} d\tau \tag{A.20}$$

The ARE (A.18) admits just one solution that is a *stabilizing* solution, namely a solution P such that there exists a control matrix depending on P, $K_d(P)$, that makes $A_{cl} + B_w K_d$ Hurwitz and it results to be

$$K_d = \frac{1}{\gamma^2} B_w^T P. \tag{A.21}$$

Recalling equation (A.16), we have that

$$A_{cl} + \frac{1}{\gamma^2} B_w B_w^T P = A_{cl} + B_w K_d = A_c \tag{A.22}$$

where A_c is known to be stable since it has as eigenvalues all the stable eigenvalues of M_γ. In order to ensure the stability of the sytstem from the ARE (A.18) we come to the Quadratic Matrix Inequality (QMI) in P

$$A_{cl}^T P + P A_{cl} + \frac{1}{\gamma^2} P B_w B_w^T P + C_{cl}^T C_{cl} \prec 0 \tag{A.23}$$

that is equivalent to the LMI

$$\begin{pmatrix} A_{cl}^T P + P A_{cl} + C_{cl}^T C_{cl} & P B_w \\ B_w^T P & -\gamma^2 I \end{pmatrix} \prec 0 \tag{A.24}$$

in fact, the Schur complement of $-\gamma^2$ in (A.24) brings to the constraints

$$(i) - \gamma^2 I \prec 0 \tag{A.25}$$

$$(ii) A_{cl}^T P + P A_{cl} + C_{cl}^T C_{cl} - P B_w (-\gamma^2 I)^{-1} B_w^T P \prec 0 \tag{A.26}$$

where the first constraint is trivial, while the second one is equivalent to the QMI (A.23). What afore mentioned ensures that the P that solves the (A.23) is a stabilizing solution.
As regards the second statement of the Bounded Real Lemma, instead, we will exploit the equivalence

$$A_{cl}^T P + P A_{cl} \pm j\omega P = -(-j\omega I - A_{cl})^T P - P(j\omega - A_{cl}) \tag{A.27}$$

and the LMI

$$0 \succ \begin{pmatrix} B_w^T(-j\omega I - A_{cl})^{-T} & I \end{pmatrix} \underbrace{\begin{pmatrix} A_{cl}^T P + PA_{cl} + C_{cl}^T C_{cl} & PB_w \\ B_w^T P & -\gamma^2 I \end{pmatrix}}_{\prec 0} \begin{pmatrix} (j\omega - A_{cl})^{-1} B_w \\ I \end{pmatrix} \quad \text{(A.28)}$$

that is equal to

$$B_w^T(-j\omega I - A_{cl})^{-T} C_{cl}^T \underbrace{C_{cl}(j\omega - A_{cl})^{-1} B_w}_{T(j\omega)} - \gamma^2 I = T^*(j\omega)T(j\omega) - \gamma^2 I. \quad \text{(A.29)}$$

Condition (A.28) holds since the matrix in the center is the same as the one that appears in the statement of the Bounded Real Lemma, while the matrices that multiply it are one the transpose conjugate of the other. The equivalence between the (A.28) and (A.29), instead, comes from the relation (A.27). In fact, developing the calculations in (A.28), we obtain

$$\begin{pmatrix} B_w^T(-j\omega I - A_{cl})^{-T} & I \end{pmatrix} \underbrace{\begin{pmatrix} A_{cl}^T P + PA_{cl} + C_{cl}^T C_{cl} & PB_w \\ B_w^T P & -\gamma^2 I \end{pmatrix}}_{\prec 0} \begin{pmatrix} (j\omega - A_{cl})^{-1} B_w \\ I \end{pmatrix} =$$

$$= B_w^T(-j\omega I - A_{cl})^{-T} A_{cl}^T P(j\omega I - A_{cl})^{-1} B_w + B_w^T(-j\omega I - A_{cl})^{-T} PA_{cl}(j\omega I - A_{cl})^{-1} B_w +$$
$$+ G^T(-j\omega I - A_{cl})^{-T} C_{cl}^T C_{cl}(j\omega I - A_{cl})^{-1} B_w + B_w^T P(j\omega - A_{cl})^{-1} B_w + B_w^T(-j\omega I - A_{cl})^{-T} PB_w - \gamma^2 I$$

in which

$$B_w^T(-j\omega I - A_{cl})^{-T} A_{cl}^T P(j\omega I - A_{cl})^{-1} G + G^T(-j\omega I - A_{cl})^{-T} PA_{cl}(j\omega I - A_{cl})^{-1} G +$$
$$+ G^T P(j\omega I - A_{cl})^{-1} G - G^T(-j\omega I - A_{cl})^{-T} PG = 0 \quad \text{(A.30)}$$

To see that, we rewrite this expression in a more compact way as

$$B_w^T(-j\omega I - A_{cl})^{-T} \underbrace{\left[A_{cl}^T P + PA_{cl} + (-j\omega I - A_{cl})^T P + P(j\omega I - A_{cl})\right]}_{=0} (j\omega I - A_{cl})^{-1} B_w \quad \text{(A.31)}$$

and exploiting the equivalence (A.27) it is immediate to see that the expression in between square brackets is equal to zero. Therefore from (A.28) and (A.29) it follows that

$$\begin{pmatrix} A_{cl}^T P + PA_{cl} + C_{cl}^T C_{cl} & PB_w \\ B_w^T P & -\gamma^2 I \end{pmatrix} \prec 0 \quad \text{(A.32)}$$

if and only if $\forall \omega \in \mathbb{R}\ \Phi(j\omega) = \gamma^2 I - T^*(j\omega)T(j\omega) \succ 0$

The Time Domain Interpretation

Let us consider the system (A.1)-(A.2) and the differentiable quadratic Lyapunov function $V(x) = x^T P x$, where P is the stabilising solution for the ARE (A.18). In order to bound the $\mathcal{L}_2$ gain of the system, according to what expressed in (2.16), we impose that

$$\dot{V}(x, w) \leq \gamma^2 w^T w - z^T z \quad \text{(A.33)}$$

that is

$$\dot{V}(x, w) = \dot{x}^T P x + x^T P \dot{x} = (A_{cl}x + B_w w)^T P x + x^T P(A_{cl}x + B_w w) \leq \gamma^2 w^T w - x^T C_{cl}^T C_{cl} x \quad \text{(A.34)}$$

that can be rewritten as

$$x^T A_{cl}^T P x + w^T B_w^T P x + x^T P A_{cl} x + x^T P B_w w - \gamma^2 w^T w + x^T C_{cl}^T C_{cl} x \leq 0 \tag{A.35}$$

or equivalently

$$\begin{pmatrix} x^T & w^T \end{pmatrix} \begin{pmatrix} A_{cl}^T P + P A_{cl} + C_{cl}^T C_{cl} & P B_w \\ B_w^T P & -\gamma^2 I \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \preceq 0 \tag{A.36}$$

and the inequality is strict as soon as x, w are both different from zero. This relation provides an upper bound on the $\mathcal{L}_2$ gain of a system.

Appendix B

Schur Complement as Check for Positive Definiteness of Symmentrc Matrices

Proposition 3.2.1 *For any symmetric matrix, M, of the form*

$$M = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \tag{B.1}$$

if D is invertible then the following properties hold:

(i) $M \succ 0$ *iff* $D \succ 0$ *and* $A - BD^{-1}B^T \succ 0$.
(ii) If $D \succ 0$, *then* $M \succeq 0$ *iff* $A - BC^{-1}B^T \succeq 0$.

Proof (i) We recall that (B.1) can be equivalently rewritten as

$$M = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}B^T & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}^T \tag{B.2}$$

From equation (B.2) it directly follows that proving the positive definiteness of M is equivalent to prove the positive definiteness of

$$G := \begin{pmatrix} A - BD^{-1}B^T & 0 \\ 0 & D \end{pmatrix}$$

In fact, $x^T M x > 0 \; \forall \; x \neq 0$ if and only if $y^T G y > 0 \; \forall \; y \neq 0$ where

$$y = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}^T x$$

And since G is block diagonal, it holds that G is positive definite if and only if each diagonal block is such, which concludes the proof.
(ii) Similarly to what has been proven above, proving that $M \succeq 0$ is equivalent to prove that $G \succeq 0$ and since $D \succ 0$ by hypothesis, it has to hold that $A - BD^{-1}B^T \succeq 0$. □

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