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Investigating parity violation in Cosmology

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Chapter 1

State of the art and introduction

At early times, the universe is strongly believed to have undergone a phase of accelerated expansion called inflation. This phase can be achieved if a scalar field, called inflaton, dominates the energy density budget of the early universe. This epoch was originally proposed in the early 1980s to overcome a number of shortcomings of standard big bang cosmology, such as the flatness, horizon, and monopole problem [1, 2] More importantly, it provides an elegant mechanism for the generation of the first density perturbations in the early universe, seeding the large-scale structure (LSS) in the distribution of galaxies and the underlying dark matter, and for the cosmic microwave background (CMB) temperature anisotropies that we observe today. Those are in fact generated from quantum fluctuations of the inflaton field stretched over cosmological scales by the exponentially fast expansion of the universe [3, 4, 5].

Although a plethora of inflationary models has been proposed so far, slow-roll single-field inflation, the simplest inflationary scenario, is in extremely good agreement CMB observations. This model, characterized by a single scalar field slowly rolling along a smooth potential, predicts the generation of almost scale invariant primordial fluctuations, with spectral index n_s given by the combination of slow-roll parameters. Moreover, the fluctuations are predicted to be nearly Gaussian distributed, with small deviation from Gaussianity of the order of slow-roll parameters [6, 7]. These predictions have been widely tested from CMB observations [8, 9, 10], finding that the primordial scalar power spectrum indeed slightly deviates from a scale invariant regime, $n_s = 0.9603 \pm 0.0073$, and does not manifest a significant deviation from Gaussianity. Inflation predicts the generation of a background of gravitational waves (GW), that can be interpreted as ripples of space-time around the background metric. Even if they have not been detected yet, Their detection and constraint shall be decisive for the determination of the energy scale of inflation, which are for sure higher than the MeV scale, and provide striking proof of single-field inflation if the consistency relation is fulfilled.

Although single-filed slow-roll inflation is favored by CMB data, the power spectrum is tightly constrained only at very large scales. This means that, at a certain range, the potential has to be slightly tilted to realize slow-roll conditions. Beyond such rage, we are free to consider theory which violates SR conditions, resulting in an amplification of fluctuations at small scales [11]. The simple case of violation of SR condition is ultra slow-roll (USR) [12, 13, 14, 15], which can be generalized to the constant-roll phase [16]. This phase occurs when the inflaton potential exhibits an extremely flat region, either a plateau or a point of inflection, and they are justified in theories of strings and supergravity [17]. These "exotic" phases of inflation have caught the attention of the scientific community since are able to generate a copious amount of primordial black holes (PBH) [17, 18]. Such a model is of great interest since PBHs play a significant role not only in cosmology, but also from the astrophysical and fundamental physics perspective. In fact, they are considered good dark matter candidates, can potentially source the gravitational signal detected by LIGO (as discussed in [19]), and possibly seed super massive black holes in galactic nuclei [20, 21, 22].

A part from the mechanism of inflation, other puzzling questions remain unanswered. Among all is the existence of the universe as we know it, made only by baryons and not antibaryons. To explain this, a mechanism of *baryogenesis* needs to be active, such that a slightly overabundance of baryon w.r.t. antibaryons was produced in the early universe. Sakharov [23] introduced three conditions leading baryogenesis, if satisfied. These conditions are related to two of the three fundamental discrete symmetries governing quantum fields: charge conjugation (C), that acts reversing the charge of the particle, and parity (P) which corresponds to the inversion of all spatial coordinates. In order to have baryogenesis, the early universe needed to simultaneously violate CP. Unfortunately, standard model (SM) interactions are not capable to generate the amount of baryon asymmetry needed to explain the universe: CP violating phase in the CKM matrix is too small. Although there are on-going efforts to understand whether a CP violating phase can be present in the neutrino sector, it appears clear the urgency to investigate new physics interactions and possible way to test them. Whatever interaction violates CP may violate pure parity as well.

It is still an open question whether parity symmetry is broken on large cosmological scales. If observationally confirmed, this would constitute a compelling evidence for new physics beyond the standard model, pointing towards new interactions either in the gravitational or in the electromagnetic sector. The most favored candidates are Chern-Simons type interactions among a (pseudo-)scalar filed and either the photon or the graviton [24]. In particular, the study of parity violating interactions in the gravitational sector has flourished over the past few years mainly for two reasons. First, gravitational Chern-Simons term arises naturally in the effective filed theory (EFT) As originally demonstrated in [24], the CMB is sensitive to violation of parity symmetry through its EB and TB power spectra, and recent hints of parity violation on the EB angular spectral power of CMB seems to be present from Planck data [25, 26]. Parity violating interactions may also manifest themselves in N-point correlation functions of perturbations, with the lowest order (in case of correlators of scalar perturbations) sensitive to parity being the four-point correlator. Hints of parity violation have been found in the four-point statistics of galaxy clustering from the BOSS and DESI surveys, with reported detection significance ranging from 2–7 σ [27, 28]. Finally, it is possible that parity violation may be encoded in the in the large-scale correlation of galaxy spins, as investigated in [29].

Turning on the hints of violation of the parity symmetry from the four point correlation function (4PCF) of galaxies found in BOSS, it led to a growing interest in models of the early universe responsible for the generation of such a signal [30, 31, 32]. However, studies of primordial scalar trispectrum generated by single-field slow-roll inflation and CS revealed that this model cannot be the source of any (possible) parity violating signature in the LSS: the primordial trispectrum is too small to leave any observable imprints [30, 32].

Since USR dynamics is able to amplify curvature perturbations, in this thesis we shall study the observational imprints of a single inflaton driving the initial expansion of the universe whose potential admits a brief phase of USR and which couples non trivially to the graviton through a Chern-Simons interaction. Specifically, we will focus on the computation of the parity-odd (PO) contribution to the primordial scalar graviton-mediated trispectrum: we shall see that the dynamics of SR-USR-SR inflation allows to circumvent no-go theorems for cosmological parity odd trispectrum [33] generating an enhanced parity violating signal at small scales.

The thesis is organized as follows:

- **Introduction to slow-roll inflation**: A single scalar field slowly rolling down its potential generates a phase of exponential expansion of the early universe if its potential energy dominates the energy content of the universe. In this chapter we review the simplest though striking model of single-field inflation, with focus on its background dynamics.
- **Perturbation theory**: We provide an introduction to linear perturbation theory in the contest of inflationary universe. We shall introduce the ADM formalism, which is most suitable for studying perturbation in a general relativistic setting.
- **Ultra slow-roll Inflation**: We provide a study of the background dynamics of models of inflation admitting a brief phase of USR sandwiched between two SR epochs. Moreover, we investigate the how scalar and tensor perturbations are affected by such dynamics.
- **Modifying general relativity**: We address the origin of gravitational CS correction and study its effects on tensor perturbations.
- **Cosmological correlators**: The problem of computing cosmological correlators differs from

scattering amplitudes in quantum field theory. After introducing the in-in formalism for the computation of cosmological correlators we will use it for computing the graviton mediated trispectrum in single-field slow-roll inflation and gravity described by the theory of general relativity.

- **Parity violation in slow-roll inflation**: We compute the PO and PE trispectrum in slow-roll inflation and CS correction, approximating the spacetime as pure de Sitter, results that are already known in the literature. We confirm the validity of no-go theorems, finding a vanishing PO trispectrum. We proceed providing an estimate of the PO trispectrum in a quasi-de Sitter universe.
- **Signatures of parity violation in ultra slow-roll inflation**: We shall evaluate the primordial parity odd graviton-mediated scalar trispectrum in the SR-USR-SR inflation. We will study the large and small-scale behavior checking their agreement with no-go theorems for parity odd trispectrum and visualize the scale and angular dependence by studying the equilater configuration. We will complement this analysis by studying the parity even primordial scalr trispectrum

Chapter 2

Introduction to slow-roll inflation

Einstein's general relativity (GR) is the most beautiful physical theory ever proposed. It describes one of the most pervasive features of the world, gravity, as a manifestation of the curvature of spacetime, rather than as force in Newtonian sense. It plays a crucial role in cosmology, which can be viewed as the application of GR to the entire cosmos. In fact, despite being the weakest among all interactions, gravity has literally shaped the universe: the matter content displays itself in filaments and knots, leaving large empty regions called "voids". For scales exceeding 100 Mpc, the pattern of the cosmic web appears to be homogeneous, and, upon applying the *Copernican principle*, we can assume that the universe is homogeneous and isotropic *everywhere*. This is the foundation for every cosmological theory in modern physics and we approximate the universe as a spatially homogeneous and isotropic three-dimensional space which expands as a function of time: distant galaxies are receding from us with a velocity that increases with the distance [34].

In GR this translates into the statement that the universe can be foliated into space-like slices, also called hypersurfaces, such that each three-dimensional slice is maximally symmetric. The more general metric describing such a spacetime can be written as

$$ds^2 = -dt^2 + a(t)\left[\frac{dr^2}{1-kr^2} + r^2\big(d\theta^2 + \sin^2\theta d\phi^2\big)\right] \tag{2.1}$$

where $a(t)$ is a function of time known as the scale factor, and k can assume values $k = \pm 1, 0$. This is the famous Robertson-Walker (RW) metric. Geometrically, k describes the curvature of spatial sections, slices of constant cosmic time t. $k = +1$ correspond to positively curved spatial sections (closed universe); $k = 0$ to an universe which is flat, and $k = -1$ corresponds to negatively curved spatial sections (open universe). Since matter and gravity are tied through the Einstein's equations (EE), the amount of matter in the universe influences its shape, not only locally, but globally[1].

Current CMB observations combined with BAO data [10]

$$\Omega_k = 0.0007 \pm 0.0019 \qquad \text{at } 68\% \text{ CL} \tag{2.5}$$

[1]This can be seen once the dynamical equations for the scale factor are derived. These equations are called Friedman equations and can be derived directly from the EE. They are

$$\left(\frac{\dot a}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{2.2}$$

which is derived from the $(0,0)$ EE, and

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{2.3}$$

arising from combining $(0,0)$ and (i,i) EE. Above, G is the Newton constant, k the curvature parameter introduced in (2.1) ρ and p the density and pressure of the cosmic fluid, comprised of matter, radiation and cosmological constant. If the dependence of ρ on a is known, the first (2.2) is enough to solve for $a(t)$. In terms of the density parameter $\Omega = 8\pi G\rho/3H^2$ the first Friedman equation can be written in the form

$$\Omega - 1 = \frac{k}{H^2a^2}, \tag{2.4}$$

and it is clear that, if $\rho = \rho_{crit} = 3H^2/8\pi G$, $k = 0$. Whether the universe is open ($k = -1$) or closed ($k = +1$) depends on if $\rho < \rho_{crit}$ or $\rho > \rho_{crit}$ respectively. Setting $\Omega_k = k/a^2H^2$, Ω_k closed to zero is proper to spatially flat spacetimes.

point towards a universe which is spatially flat, and the line element takes the easy form

$$ds^2 = -dt^2 + a^2(t)\big(dx^2 + dy^2 + dz^2\big) = a^2(\eta)\big(- d\eta^2 + dx^2 + dy^2 + dz^2\big). \tag{2.6}$$

In the second equality we have introduce $d\eta = dt/a$, infinitesimal time interval of the conformal time, which allows to identify the spatially flat expanding universe as the conformally transformed Minkowski spacetime.

The picture appears to be clear: going backward in time the universe grows smaller and denser. Its components, matter, radiation and cosmological constant Λ, evolve very differently with respect to time (and thus on the scale factor a). While matter density is diluted by the universe expansion $\Omega_m \propto a^{-3}$, radiation suffers also from cosmological redshift, which can be interpret as an "energy dilution" effect, and it scales with the scale factor as $\Omega_r \propto a^{-4}$. While at early times the latter is dominating, as the universe expands, it decreases faster than the former, which result to dominate afterwards. On the contrary, cosmological constant Λ remains constant as the universe expands: in the present time the cosmos is dominated by this component. These are the pillars of the hot big bang (HBB) model, which had been *the* cosmological model for many years. Its most outstanding prediction? The *big bang nucleosynthesis*, which refers to the production of light isotopes[2], by nuclear reactions ignited during the first seconds due to the extremely high temperature. The measured abundance of these elements cannot be explained only by the nuclear reactions in the core of stars [35].

However, other observations showed shortcomings of BBN. In particular, in order to explain the universe that we observe, a fine tuning of cosmological parameters would be needed. The largest CMB scales share the same temperature, even if they were not in causal contact when photon decoupled from the thermal bath, roughly 380 000 years after the big bang. Moreover, the small value of the Ω parameter, which is the density parameter of all components of the universe, suggested by observations at present time, implies that, at very early times, its value was even smaller by sixty orders of magnitude. These shortcomings of the hot big bang model are known as the *horizon* and *flatness problems.* Without relying on fine tunings for the value of the temperature T and the density parameter Ω, cosmologists searched for dynamical mechanism capable of generating a universe whose geometry is extremely closed to spatial flatness even starting from general values for the density. This physical process, which acts as an attractor mechanism, is called inflation.

2.1 Inflation

To solve shortcomings of the HBB model, the mechanism of *inflation* was introduced by Guth [1] for the first time in 1981. It consists of an era of accelerated expansion ($\ddot{a} > 0$) in the very early universe driven by some component other than matter or radiation that redshifts away slowly as the universe expands. In this way, the flatness and horizon problems can be solved simultaneously. The simplest case to realize it is when it is driven by a constant vacuum energy, which acts as a cosmological constant leading to an exponential expansion. Then, during the vacuum era, $\Omega \propto$ const. grows rapidly w.r.t. $\Omega_k \propto a^{-2}$, so the universe becomes flatter with time. If this process lasts long enough, then the density parameter will be extremely closed to unity. Regarding the horizon problem, it can be traced to the fact that comoving length scales[3] are constant during the expansion of the universe, while the comoving Hubble radius[4] $r_{\rm H} = 1/\dot{a}$. If a period of accelerating expansion occurs, $dr_{\rm H}/dt = -\ddot{a}/\dot{a}^2$ decreases with time. It means that at very early times the horizon could have been as large as the horzon today, or even bigger. Also, a minimum duration of this period is required, such that the

[2] Such as D, ^{4}He, ^{4}He, ^{7}Li

[3] A comoving length-scale λ_{com} is defined as $\lambda_{com} = \lambda_{phys}/a$, where a is the scale factor. Comoving lengthscales have the property that remain constant over time.

[4] The Hubble radius is a crucial quantity during inflation as it represent the characteristic length scale beyond which causal processes cannot operate[36]. It is defined as the radius of the sphere centered on the observerthe comoving distance within which all the spatial points has been in causal contact with the observer during an Hubble time $t_H = 1/H$, which is the characteristic time of expansion. It is written as follows:

$$r_H = \frac{1}{aH} = \frac{1}{\dot{a}}. \tag{2.7}$$

horizon had been in much larger than its size today, such that the biggest scale observed in the CMB (the quadrupole scales) had been in causal contact and could thermalize.

How is it possible to realize such an epoch? Consider a single real scalar field $\phi(\mathbf{x}, t)$, with associated energy-momentum tensor is $T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \mathcal{L}_\phi$, $\mathcal{L}_\phi = -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - V(\phi)$, and suppose that the field is located in the minimum of its potential

$$\phi(\mathbf{x}, t) = \langle \phi \rangle = \langle 0 | \phi(\mathbf{x}, t) | 0 \rangle = \text{const.} \tag{2.8}$$

This implies that stress-energy tensor associated to such a field is simply given by

$$T_{\mu\nu} = g_{\mu\nu} \mathcal{L}_\phi = -V(\langle \phi \rangle) g_{\mu\nu}. \tag{2.9}$$

Turn to the EE: in the presence of a cosmological constant Λ they are modified as

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} \tag{2.10}$$

since $g_{\mu\nu}\Lambda$ is perfectly consistent with the idea of general covariance. Meanwhile, for a given particle species

$$\langle 0 | T_{\mu\nu} | 0 \rangle = -\langle 0 | \rho | 0 \rangle g_{\mu\nu} \tag{2.11}$$

with ρ being its energy density. This equation is valid because of Lorentz symmetry and it is clear that Λ can be interpreted as the energy density associated to the vacuum state $\Lambda = 8\pi G \langle \rho \rangle$. Therefore, if the energy density of the universe is dominated by the potential of the scalar field ϕ and it is stuck in its vacuum expectation value, the universe appear dominated by cosmological constant leading to a phase of accelerated expansion. A natural candidate for the generation of a period of inflation is a real scalar field. Notice few things:

1. We considered scalar fields only because their vacuum do not break isotropy. If the expectation value of a vector field A_μ were non-vanishing $\langle A_\mu \rangle \neq 0$, it would mean having a non-vanishing preferred direction. Models of inflation driven by a vector field have been proposed, but they have to account for empirical constraints.
2. Some exotic models of inflation take into account that the scalar particle leading to an epoch of inflation is not a elementary, but a spinor condensate [37]. These models are not excluded by data in the present days.

However, the scenario presented above displays a crucial problem: if inflation were driven by the vacuum energy density of a scalar field, it would never end. This means that the universe would not be able to recover the radiation domination and the subsequent matter domination phases that have been widely tested and validated. This problem can be solved by requiring that the scalar field, that form now on we will call inflaton, is not stuck in its minimum but evolves slowly with time, extremely slowly though. This condition can be achieved if the inflaton is traversing an extremely flat region of its potential and slowly rolls down it. After that, since we want inflation to end, it will reach the minimum of its potential, decaying into standard model particles, reconstituting the thermal bath that has been washed out by the accelerated expansion of the universe. This is *slow-roll inflation.* Before diving into this scenario, it is useful to study how a scalar field behaves in an expanding universe.

We assume that the total action of the universe is given by

$$\begin{aligned} S^{tot} &= S_{\mathrm{EH}} + S_\phi + S_{matter} \\ &= -\frac{M_{\mathrm{Pl}}}{2} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \mathcal{L}_\phi[\phi, g_{\mu\nu}] + \sum_{matter} \int d^4x \sqrt{-g} \mathcal{L}_{matter} \end{aligned} \tag{2.12}$$

where S_{EH} is the Einstein-Hilbert action, capturing the dynamics gravity, S_{matter} is the action of all standard model particles (matter, gauge bosons, Higgs field) and any other d.o.f. that may be present (for example dark matter candidates), and S_ϕ is the action for the inflaton field ϕ. As we saw before, inflation takes place when the energy budget of the universe is dominated by the potential of the inflaton. This means that, at very early times, we can neglect the contribution coming from $\mathcal{L}_{matter}$. The EH action leads to the Friedman equations (2.2) and (2.3). Assuming that the inflaton field ϕ

is minimally coupled to gravity and is canonically normalized[5], its action is the Klein-Gordon action generalized to curved spacetimes

$$S_\phi = -\int d^4x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right). \tag{2.13}$$

From the minimal action principle, e.o.m. are obtained requiring that the action is minimized. Therefore

$$\frac{\delta S_\phi}{\delta\phi} = 0 \qquad \Longrightarrow \qquad \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right] = \Box\phi = \frac{\partial V}{\partial\phi} \tag{2.14}$$

and, if we assume that the metric induces the line element (2.6),

$$\sqrt{-g} = a^3, \qquad\qquad \Box\phi = -\ddot{\phi} - 3\frac{a^2\dot{a}}{a^3}\dot{\phi} + \frac{1}{a^2}\nabla^2\phi = \frac{\partial V}{\partial\phi}. \tag{2.15}$$

Recalling that $H = \dot{a}/a$ entails the expansion rate of the universe, the inflaton field ϕ satisfies

$$\ddot{\phi}(\mathbf{x},t) + 3H\dot{\phi}(\mathbf{x},t) - \frac{\nabla^2\phi(\mathbf{x},t)}{a^2} = -\frac{\partial V}{\partial\phi} \tag{2.16}$$

where $\nabla^2\phi = \partial^i\partial_i\phi$, $i = 1,2,3$ spatial index. Since the universe is homogeneous and isotropic we take the inflaton to depend only on time, $\phi(\mathbf{x},t) = \phi(t)$, and obtain [36]

$$\ddot{\phi}(t) + 3H\dot{\phi}(t) = -\frac{\partial V}{\partial\phi}. \tag{2.17}$$

Eq.(2.17) is the Klein-Gordon equation for the (homogeneous and isotropic) inflaton field in an expanding spatially flat spacetime. The term $3H\dot{\phi}$ acts as a friction term, and, if spacetime was flat, it would be vanishing, reducing the equation to the its Minkowskian counterpart.

2.2 Slow-roll

As aforementioned, inflation takes place if, in the early universe, the potential od a real scalar d.o.f. ϕ dominates, slowly rolls down its potential. In this way, the inflaton mimics a cosmological constant for a given amount of time, since its equation of state reads

$$p_\phi \approx -\rho_\phi \quad \text{if} \quad \dot{\phi}^2 \ll V(\phi) \tag{2.18}$$

where the pressure of the inflaton is given by $p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$ and its energy density by $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$. This condition can be realized if the potential is *sufficiently* flat for a certain region of its domain. Suppose that the inflaton reaches a very flat region after having traversed a steep part of its potential: as it starts rolling over the flat region its energy density is dominated by the kinetic term $\rho_\phi \approx \frac{1}{2}\dot{\phi}^2$, and the equation of motion reads $p_\phi = \rho_\phi$, $\omega_\phi = p_\phi/\rho_\phi = 1$. Since the density depends on the scale factor as $\rho_\phi \propto a^{-3(1+\omega_\phi)} = a^{-6}$[6], it decreases extremely fast, eventually allowing for the potential to dominate. Moreover we require that the second derivative of ϕ is small enough to allow this state of affairs to be maintained for a sufficient period. Therefore, we claim that inflation can

[5]This means that the kinetic term is of the form

$$K = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$

[6]The dependence of the density of a give particle species and the scale factor can be obtained from the continuity equation $\nabla_\mu T^\mu_0$ in a spatially flat expanding spacetime, once the equation of state is made explicit.

be realized for a sufficient amount of time, or number of e-folds[7], if the following two conditions are satisfied [36]:

$$\dot{\phi}^2 \ll V(\phi) \tag{2.20}$$
$$|\ddot{\phi}| \ll 3H\dot{\phi} \tag{2.21}$$

these conditions are indeed called *slow-roll conditions*.

The properties that the field and its potential need to satisfy are entailed in model independent parameters, called *slow-roll* parameters, that are defined as:[38]

$$\epsilon_{n+1} \equiv \frac{d\ln\epsilon_n}{dN} = \frac{1}{H}\frac{\dot{\epsilon}_n}{\epsilon_n}, \tag{2.22}$$

and

$$\epsilon_1 \equiv -\frac{\dot{H}}{H^2} \tag{2.23}$$

with ϵ_1 it quantifies how much the Hubble rate H changes with time during inflation [36]. We state that slow-roll inflation can take place only if all slow roll parameters ϵ_n are much smaller than unity.

Starting from the first slow-roll parameter ϵ_1, we can show requiring it to be small correspond to fulfill the condition (2.20). By using the fact that the inflaton dominates the energy budget of the early universe during inflation, the first Friedman equation (2.2) yields

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{2.24}$$

and we can extract the expression for the derivative of the Hubble parameter H

$$\dot{H} = \frac{4\pi G}{3H}\left(\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi}\right) = -4\pi G\dot{\phi}^2 \tag{2.25}$$

upon substituting $\ddot{\phi}$ using the KG equation. Using the first slow-roll approximation to the Hubble parameter in the denominator of (2.24), we find

$$\epsilon_1 \simeq \frac{3}{2}\frac{\dot{\phi}^2}{V(\phi)}. \tag{2.26}$$

Therefore, the first slow-roll condition (2.20) corresponds to impose the first slow-roll parameter to be much smaller than one, $\epsilon_1 \ll 1$. Moreover, we can write this parameter in terms of the inflaton potential and its derivative. In order to express it only as a function of the potential, we substitute $\dot{\phi}^2$ by using the equation of motion of the field in the slow-roll regime and the Friedman equation again, obtaining

$$\epsilon_1 = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2, \tag{2.27}$$

which makes manifest the fact that the first slow-roll condition is achieved if the potential is flat.

We turn to the successive parameter,

$$\epsilon_2 = \frac{1}{H}\frac{\dot{\epsilon_1}}{\epsilon_1} = 2\epsilon_1 + \frac{\ddot{H}}{H\dot{H}} = 2\epsilon_1 + 2\frac{\ddot{\phi}}{H\dot{\phi}} \tag{2.28}$$

[7] In early universe cosmology, the time intervals are evaluated in number of e-folds N, which are defined as

$$N = \int_{t_1}^{t_2} H dt = \ln\left(\frac{a_2}{a_1}\right) \tag{2.19}$$

where $t_2 > t_1$, a_1 and a_1 are the scale factor evaluated at time t_1 and t_2, and, to obtain the second equality the definition of the Hubble parameter H is used. Therefore, the number of e-folds provide a clear information on how the universe expanded during the time interval $\delta t = t_2 - t_1$.

and we see that, if both the first and the second (2.21) are satisfied, $|\epsilon_2| \ll 1$. Therefore, the smallness of the parameter ϵ_1 and ϵ_2 is necessary to develop slow-roll inflation. In particular, ϵ_1 is related to the existence of the inflationary phase itself: as soon as $\epsilon_1 \sim \mathcal{O}(1)$, inflation ends, whereas the condition $\epsilon_2 \ll 1$ is needed in order for inflation to last enough to solve horizon and flatness problems [36].

Eventually, we can extend these conditions to a stronger one, demanding for *all* slow-roll parameters to be much smaller than unity during inflation. Notice that these conditions implies that the slow-roll parameters are approximately constant: their derivative is higher-order. In fact, since slow-roll parameters are required to be small in order for inflation to take place and provide the solution to fine-tuning problems, $\epsilon_1, \epsilon_2, \ldots, \epsilon_m = \mathcal{O}(\epsilon)$, $\epsilon \ll 1$. From the definition of the nth slow-roll parameter, $1 < n \leq m$ we get an expression of its time variation during the characteristic time of expansion $1/H$:

$$\frac{\dot{\epsilon}_n}{H} = \epsilon_{n-1}\epsilon_n = O(\epsilon^2). \tag{2.29}$$

Therefore, at first order in ϵ, which is equivalent to the first order in slow-roll parameters, they are constant.

Since the slow-roll parameters are proportional to the derivatives of the potential, which are very small, the de Sitter approximation is often employed. This means that we can approximate H as constant and the relation between the scale factor, the conformal time and the Hubble parameter reads

$$\eta = -\frac{1}{aH}. \tag{2.30}$$

If this approximation is relaxed, the approximation is modified as such

$$\eta = -\frac{1}{aH(1-\epsilon_1)}. \tag{2.31}$$

2.3 Generation of primordial fluctuations

In the sections above, since the universe is homogeneous and isotropic, we accounted only for a time dependence in the inflaton field. However, this is not exactly the case. More precisely, what we previously identified as $\phi(t)$, is only the background of the field, which can be written as:

$$\phi(\mathbf{x}, t) = \phi(t) + \delta\phi(\mathbf{x}, t) \tag{2.32}$$

where $\delta\phi$ are the quantum fluctuation of the inflaton which will give rise to cosmological perturbations seeding the CMB temperature anisotropies and the LSS. The capability of generating coismological perturbations is indeed the most striking prediction of inflation The splitting (2.32) is a perturbative treatment, fully justified from CMB observations. Since $\delta T/T \approx 10^{-5}$ and it is demonstrate that they can be seeded by $\delta\phi$, it means that the perturbations of the inflaton are much smaller than ϕ of several order of magnitude.

To study the dynamics of perturbations, we linearize the equation of motion (2.16). Isolating the zeroth order we get (2.17), which indeed describes the evolution of the background. The first order terms determine the e.o.m. for perturbations, which is

$$\delta\ddot{\phi}(\mathbf{x}, t) + 3H\delta\dot{\phi}(\mathbf{x}, t) - \frac{\nabla^2\delta\phi(\mathbf{x}, t)}{a^2} = -\frac{\partial^2 V}{\partial\phi^2}\delta\phi(\mathbf{x}, t). \tag{2.33}$$

The e.o.m. for perturbations, once the Laplacian is neglected[8], is of the same form of the time derivative of (2.17)

$$\dddot{\phi}(t) + 3H\ddot{\phi}(t) = -\frac{\partial^2 V}{\partial\phi^2}\dot{\phi}(t) \tag{2.35}$$

[8]The Laplacian can be neglected once the large scale limit is taken. In fact, partial derivatives correspond to wavenumbers associated to a comoving length scale. In fact, if we require that the physical length scale is much larger than the Hubble horizon $r_H = 1/H$, then

$$a\lambda_{com} \equiv \lambda_{phys} \gg \frac{1}{H} \implies \frac{a}{k} \gg \frac{1}{H} \implies k \ll aH. \tag{2.34}$$

and therefore we can ask whether $\delta\phi$ and $\dot{\phi}$ are independent variables. To do this, we take the Wronkian finding that

$$w(\delta\phi, \dot{\phi}) = \delta\dot{\phi}\dot{\phi} - \delta\phi\ddot{\phi} \simeq e^{-3Ht} \to 0, \tag{2.36}$$

which means that, after a short period of time after the beginning of inflation, the perturbation $\delta\phi(\mathbf{x}, t)$ is indeed proportional to the derivative of the background field $\phi(t)$. This means that the inflaton field, background and quantum fluctuation, can be written as

$$\phi(\mathbf{x}, t) = \phi(t) - \dot{\phi}(t)\delta t(\mathbf{x}) \tag{2.37}$$

$\delta t(\mathbf{x})$ being a space dependent time shift. What the expression above tells us is that, if we consider patches of the universe separated by a distance larger than the Hubble radius, the scalar field assumes the values dictated by its background trajectory at slightly different times.

The presence of fluctuations has profound implications in cosmology: the strength and correlations of those fluctuations can be predicted in various model of inflation and entail information of the physics of the early universe, the particle content and its symmetries, providing the best tool to unveiling its first instants.

Chapter 3

Perturbation theory

The greatest prediction of inflation is the generation of primordial perturbations capable of seeding the large scale distribution of matter and the CMB temperature fluctuations. As mentioned in the previous chapter, those perturbations originate form quantum fluctuations of the fields in the early universe. In a single-field model of inflation, in which the potential energy of the inflaton dominates the energy density of the universe, the quantum fluctuations of the inflaton field are those giving rise to cosmic structures.

To study primordial perturbation in cosmology we cannot separate the inflaton fluctuations from metric perturbations: indeed, matter content of the universe and the geometry of spacetime are strictly connected through the Einstein's equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$. In the following, we provide a formalism enabling us to study cosmological perturbation in a general relativistic setting, and we will apply it to compute the power spectrum of primordial fluctuations. This chapter is written mainly following [39, 40, 36, 7]

3.1 Gauge issue and gauge transformations

A subtlety of the study of cosmological perturbations is that the split into background and perturbations is not unique and depends on the *gauge choice*. To see this consider a tensor T describing either geometry or the matter field (it can be $g_{\mu\nu}$, $T_{\mu\nu}$ of some matter field, or the inflaton $\phi(\mathbf{x}, t)$), then the cosmological perturbations are defines as

$$\Delta T = T - T_0 \tag{3.1}$$

where T is the value of the tensor in the perturbed spacetime, the physical one, and T_0 is the value of the tensor T in the unperturbed spacetime, described by the FRW metric. The problem is manifest already at this stage: the perturbation ΔT is defined from the comparison of two tensors T and T_0, which live in two different spacetimes. In order this comparison to be meaningful, T and T_0 should be considered at the same point. However, they are defined in different spacetimes, thus, they can be compared only after providing a prescription to associate to each point of the physical spacetime, each point of the unperturbed one. A *gauge choice* is defined as a one-to-one correspondence (a map) between the points in the background $\mathcal{M}_0$ and those in the physical spacetime $\mathcal{M}$ [40]. A change of this map corresponds to a *gauge transformation* and the freedom we have in choosing it gives rise to an arbitrariness in the value of the perturbation ΔT, unless T is gauge invariant.

In perturbation theory, a basic assumption is the existence of a parametric family of solutions of the Einstein's field equations, to which the unperturbed spacetime belongs. On each $\mathcal{M}_\lambda$ we define the physical and geometrical quantities T_λ and we use the parameter λ to Taylor expand them. The map between $\mathcal{M}_0$ and $\mathcal{M}_\lambda$ is a one-parameter function in λ: we can provide two of these maps ψ_λ and ϕ_λ. Let us assign coordinates x^μ to the background spacetime $\mathcal{M}_0$. A one-to-one correspondence, for example ψ_λ, carries the points of $\mathcal{M}_0$ over $\mathcal{M}_\lambda$ and defines a gauge choice, thus we can call the map *gauge*. A point P in the unperturbed spacetime is mapped into a point Q in the physical spacetime, $Q = \psi_\lambda(P)$. If change the gauge, say we take the gauge associated to ϕ_λ, then the function will map the point P in $\mathcal{M}_0$ into another point S in $\mathcal{M}_\lambda$, $S = \phi_\lambda(P)$. Therefore, if we choose the gauge ψ_λ, the point Q is the representative of the point P in the physical perturbed space. The same is for the point S if we fix the gauge ϕ_λ. Point S and Q are related as [40]:

$$S = \phi_\lambda(\psi_\lambda^{-1}(Q)) = \Phi_\lambda(Q). \tag{3.2}$$

Relation 3.2 defines the *gauge transformation* since we relate the representative of gauge ϕ_λ to the one of gauge ψ_λ.

Such a transformation is also called an *active coordinate transformation.* To better understand it and relate it to the *passive coordinate transformation*, which is the coordinate transformation we are used to, consider point P on the background $\mathcal{M}_0$, and Q and S on the manifold $\mathcal{M}_\lambda$ describing the physical spacetime. Let us call $x^\mu(Q)$ and $y^\mu(S)$ the coordinates of Q and S in the perturbed spacetime, such that $y^\mu(S) = x^\mu(Q)$. We have that the coordinates of S, $y^\mu(S) = \Phi^\mu(x^\alpha(Q))$ are one parameter functions of those of those of Q. This transformation, which, in a given coordinate system, moves each point to another, is an active coordinate transformation, which deffers from the passive coordinate transformation, which is just a relabeling of points.

Consider a tensor field T_λ on each $\mathcal{M}_\lambda$, by means of the gauges ϕ_λ and ψ_λ we can define two different representations of T_λ on $\mathcal{M}_0$ that can be denoted, for simplicity, $T(\lambda)$ and $\tilde{T}(\lambda)$. $T(\lambda)$ and $\tilde{T}(\lambda)$ are related by Φ_λ, giving rise to a relation between the two tensors of the type [40]

$$\tilde{T}(\lambda) = T + \lambda \mathcal{L}_\xi T + \dots \tag{3.3}$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field $\xi = \mathrm{d}x^\mu/\mathrm{d}\lambda$. This is indeed the definition of the action of a gauge transformation on a tensor quantity.

3.2 Perturbations of the inflaton and of the metric

In this section, we introduce the inflaton and metric perturbations in full generality, mainly following [40, 36, 41]. Generically, perturbations of the inflaton field can be written as

$$\delta\phi(\mathbf{x}, t) = \sum_{n=1}^{\infty} \frac{\delta\phi^{(n)}(\mathbf{x}, t)}{n!} \tag{3.4}$$

where the n-th term in this series corresponds to the contribution to n-th order in perturbation. If we consider small fluctuations around the background, and this seems to be the case if we consider that those fluctuations generate temperature anisotropies of CMB which $\Delta T/T \simeq 10^{-5}$, we can keep only the first term in the series. Thus, $\delta\phi^{(1)}(\mathbf{x}, t) = \delta\phi(\mathbf{x}, t)$

$$\phi(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t). \tag{3.5}$$

The same expansion is valid also for the metric tensor $g_{\mu\nu}$

$$\delta g_{\mu\nu}(\mathbf{x}, t) = \sum_{n=1}^{\infty} \frac{\delta g_{\mu\nu}^{(n)}(\mathbf{x}, t)}{n!} \tag{3.6}$$

and we can keep only linear order terms: $\delta g_{\mu\nu}^{(1)}(\mathbf{x}, t) = \delta g_{\mu\nu}(\mathbf{x}, t)$. We can formally add these perturbation to the FRW metric, whose components become

$$g_{00} = -a^2(\eta)\left[1 + 2\psi^{(1)}(\mathbf{x}, \eta)\right] \tag{3.7}$$

$$g_{0,i} = g_{i,0} = a^2(\eta) w_i^{(1)} \tag{3.8}$$

$$g_{ij} = a^2(\eta)\left[(1 - 2\phi^{(1)}(\mathbf{x}, \eta))\delta_{ij} + \chi_{ij}^{(1)}\right] \tag{3.9}$$

where ψ, ϕ, w_i and χ_{ij} are the perturbations of the metric. Note that ϕ here should not be confused with the inflaton field. In cosmology these perturbations are usually split into their scalar, vector and tensor components, according to the SVT decomposition. This is very useful since, at linear order, the perturbations evolve independently and can be studied separately. According to the SVT splitting,

$$w_i = \partial_i w^{\parallel} + w_i^{\perp} \tag{3.10}$$

$$\chi_{ij} = D_{ij}\chi^{\parallel} + \partial_j \chi_i^{\perp} + \partial_i \chi_j^{\perp} + \chi_{ij}^{T} \tag{3.11}$$

where $w^{\parallel}$ and $\chi^{\parallel}$ are scalars, $w_i^{\perp}$ and $\chi_i^{\perp}$ vectors, and χ_{ij}^{T} is a tensor. D_{ij} is a traceless derivative operator $D_{ij} = \partial_i\partial_j - \delta_{ij}\nabla^2/3$. Above the definition of scalar, vector and tensor perturbations. *Scalar* perturbations can always be constructed from a scalar, or its derivatives, and a background quantity. *Vector* perturbations are divergence free, thus $\partial_i w_i^{\perp} = \partial_i \chi_i^{\perp} = 0$. *Tensor* perturbations are contributions to $\delta g_{ij} = a^2 h_{ij}$ that are both transverse $\partial_i \chi_{ij}^{T}$ (i.e. divergence free) and traceless ${\chi^T}^i_{\ i} = 0$.

3.3 Power spectrum of fluctuations

To characterize the properties of perturbations, it is fundamental to employ statistical tools. One example is the power spectrum, which is an extremely important tool in case we want to compare theoretical predictions for the early universe with observations. In the following, we are going to formally define this quantity.

Consider a given random field $\delta(\mathbf{x}, t)$. The *autocorrelation function*, or two-point correlation function, is defined as the joint ensemble average of the field at two different locations $\mathbf{x}_1 = \mathbf{x}$ and $\mathbf{x}_2 = \mathbf{x} + \mathbf{r}$,

$$\xi(\mathbf{x}, \mathbf{x} + \mathbf{r}) = \langle \delta(\mathbf{x})\delta(\mathbf{x} + \mathbf{r})\rangle. \tag{3.12}$$

We underline that the autocorrelation function depends only on $r = |\mathbf{r}|$ ($\xi(\mathbf{x}, \mathbf{x} + \mathbf{r}) = \xi(r)$), because of statistical homogeneity and isotropy[1].

Usually, the field is expressed in terms of its spatial Fourier transform defined by[2]:

$$\delta_{\mathbf{k}} \equiv \int \delta(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{r} \tag{3.13}$$

with inverse

$$\delta(\mathbf{r}) \equiv \int \delta_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d^3\mathbf{k}}{(2\pi)^3} \tag{3.14}$$

and $\delta_{\mathbf{k}}$ denoting a complex random variable. Imposing the reality of $\delta_{\mathbf{k}}$, the variable is related to its complex conjugate as follows:

$$\delta_{\mathbf{k}} = \delta^{\star}_{-\mathbf{k}}. \tag{3.15}$$

The field is therefore determined by the statistical properties of the random variable $\delta(\mathbf{k})$. We can then compute the two-point correlator in Fourier space, which is given by:

$$\begin{aligned}
\langle \delta(\mathbf{k})\delta(\mathbf{k}')\rangle &= \int d^3\mathbf{x} \int d^3\mathbf{r} \langle \delta(\mathbf{x})\delta(\mathbf{x} + \mathbf{r})\rangle \exp\left[-i(\mathbf{k} + \mathbf{k}')\cdot\mathbf{x} - i\mathbf{k}'\cdot\mathbf{r}\right] && (3.16)\\
&= \int d^3\mathbf{x} \int d^3\mathbf{r}\, \xi(r) \exp\left[-i(\mathbf{k} + \mathbf{k}')\cdot\mathbf{x} - i\mathbf{k}'\cdot\mathbf{r}\right] && (3.17)\\
&= (2\pi)^3 \delta_D(\mathbf{k} + \mathbf{k}') \int d^3\mathbf{r}\, \xi(r) \exp\left[i\mathbf{k}\cdot\mathbf{r}\right] && (3.18)\\
&\equiv (2\pi)^3 \delta_D(\mathbf{k} + \mathbf{k}') P(k) && (3.19)
\end{aligned}$$

[1]The *cosmological principle* states that all expectation value, and so all correlation functions, are invariant under global translations $\mathbf{x} \to \mathbf{x} + \mathbf{r}$ on a fixed time slice; this is referred as *statistical homogeneity.* The fact that the Universe is isotropic implies that the autocorrelation function cannot depend on a quantity which have a privileged direction. Thus $\xi(\mathbf{x}, \mathbf{x} + \mathbf{r}) = \xi(r)$.

[2]For the Fourier transform to exist the integral $\int |\delta(\mathbf{r})| d^3\mathbf{r}$ needs to be finite. It means that we must set δ to be zero outside some large volume.

where $P(k)$ is by definition the density *power spectrum* which is a well-defined quantity for all homogeneous random fields. Thus, the relation between the two-point correlation function and the power spectrum reads

$$\xi(r) = \int P(k)e^{i\mathbf{k}\cdot\mathbf{r}}\frac{d^3\mathbf{k}}{(2\pi)^3} \tag{3.20}$$

with inverse

$$P(k) = \int \xi(r)e^{-i\mathbf{k}\cdot\mathbf{r}}d^3\mathbf{r}. \tag{3.21}$$

Note that in the literature there are two conventions for which the definition of the power spectrum differ by a factor of $(2\pi)^3$, due to different definitions of the Fourier transform (3.13) and (3.14). Defining $\delta(\mathbf{r}) \equiv \int \delta_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}d^3\mathbf{k}$, we find $\langle\delta_\mathbf{k}\delta_{\mathbf{k}'}\rangle \equiv \delta_D(\mathbf{k}+\mathbf{k}')P(k)$.

The *variance* of the density field is given by

$$\sigma^2 \equiv \langle\delta(\mathbf{r})^2\rangle = \xi(0) = \int P(k)\frac{d^3\mathbf{k}}{(2\pi)^3} \tag{3.22}$$

and it is convenient to define the quantity $\Delta^2(k)$

$$\sigma^2 = \int \Delta^2(k)d\ln k \tag{3.23}$$

which is the contribution to the variance per $\ln k$. With the conventions on the Fourier transform defined in (3.13) and (3.14) and using $d^3\mathbf{k} = 4\pi k^3 d\ln k$ we can make more explicit the relationship between $\Delta^2(k)$ and the power spectrum

$$\Delta^2(k) = \frac{k^3P(k)}{2\pi^2}. \tag{3.24}$$

Instead of $P(k)$, is more advantageous to use $\Delta^2(\mathbf{k})$ since its numerical values do not depend on the specific conventions on the Fourier transform. An extremely important quantity is the spectral index of the power spectrum

$$n-1 = \frac{d\ln\Delta(k)}{d\ln k} \tag{3.25}$$

and entails information on the scale dependence of the power spectrum. If $n-1<1$, the power decreasing at increasing values of k, and $\Delta(k)$ is said to be *red tilted.* In contrast, if $n-1>1$, for increasing values of k the power also increases, and the dimensioless spectral power is said to be *blue tilted.* For $n\equiv 1$, the power spectrum is perfectly scale-invariant. Moreover, in case the spectral index does not depend on the scale k, the adimensional power spectrum can be written in the following way

$$\Delta(k) = \Delta(k_0)\left(\frac{k}{k_0}\right)^{n-1} \tag{3.26}$$

where k_0 is a certain pivot scale.

In the early universe the perturbations are quantum. This implies that the power spectrum is defined as the Fourier transform of the vacuum expectation value $\langle\delta_\mathbf{k}\delta^*_{\mathbf{k'}}\rangle = \langle 0|\delta_\mathbf{k}\delta^*_{\mathbf{k'}}|0\rangle$, rather than an ensemble average. The perturbation δ is then canonically quantized

$$\hat{\delta}_\mathbf{k}(\eta) = \delta_k(\eta)\hat{a}_\mathbf{k} + \delta^*_k(\eta)\hat{a}^\dagger_{-\mathbf{k}} \tag{3.27}$$

where the $\hat{a}^\dagger_\mathbf{k}$ and $\hat{a}_\mathbf{k}$ are the creation and annihilation operators satisfying $[\hat{a}_\mathbf{k},\hat{a}^\dagger_{\mathbf{k'}}] = (2\pi)^3\delta(\mathbf{k}-\mathbf{k'})$ under the normalization condition

$$\delta'^*_k\delta_k - \delta'_k{\delta^*}_k = i\hbar. \tag{3.28}$$

The hat, which indicates that they are indeed operators, shall be omitted from now on. If we evaluate the vacuum expectation value $\langle 0|\delta_\mathbf{k}\delta^*_{\mathbf{k'}}|0\rangle$, expanding the perturbation operator in terms of creation and annihilation operators and knowing that $[a_\mathbf{k},a_{\mathbf{k'}}] = 0 = [a^\dagger_\mathbf{k},a^\dagger_{\mathbf{k'}}]$, it is easy to show that it can be written as:

$$\langle 0|\delta_\mathbf{k}\delta^*_{\mathbf{k'}}|0\rangle = (2\pi)^3|\delta_\mathbf{k}|^2\delta^{(3)}(\mathbf{k}-\mathbf{k'}) \tag{3.29}$$

and so [36],

$$P_\delta(k) = |\delta_k|^2 \tag{3.30}$$

3.4 The ADM formalism

The dynamical content of general relativity is fully expressed by Einstein's field equation, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, which provides a fully covariant formulation of the theory of gravitation. In this form, a precise isolation of the equations for the dynamical variables from those that are constraints is unclear. For this reason, it is useful to analyze the dynamics as the evolution of a three-dimensional hypersurface where the fields are defined. This reformulation of Einstein's general relativity was carried out by Richard Arnowitt, Stanley Deser and Charles W. Misner and takes the name of ADM formalism [42].

The goal is to find a parametrization of the spacetime metric adapted to a given choice of foliation of the spacetime by spacelike hypersurfaces. We start by assuming that the hypersurfaces of the foliation of spacetime are "constant-time" hypersurfaces, i.e. given some time function $t(x^\alpha)$, the hypersurfaces are defined by

$$\Sigma_{t_0} = \{x^\alpha : t(x^\alpha) = t_0\} \tag{3.31}$$

with timelike and future-oriented normal vector n^α, $n_\alpha \sim \partial_\alpha t$[3]. We can now introduce a new coordinate system (t, y^a) via coordinate transformation $x^\alpha = x^\alpha(t, y^\alpha)$ by the defined as follows. For any fixed value of $t = t_0$, we notice that

$$x^\alpha_{t_0}(y^a) := x^\alpha(t_0, y^a) \tag{3.32}$$

gives us the embedding of an hypersurface Σ with coordinates y^a as the hypersurface Σ_{t_0} in spacetime. Furthermore we observe that the curves

$$x^\alpha_{y_0}(t) := x^\alpha(t, y^a_0) \tag{3.33}$$

connect points on different hypersurfaces with the same values of the spatial coordinates $y^a = y^a_0$, providing us with a notion of time evolution from one hypersurfaces to the other.

For some choice of foliation and time evolution,

$$E^\alpha_a = \left(\frac{\partial x^\alpha}{\partial y^a}\right)_t \tag{3.34}$$

are tangent vectors to Σ_t, and

$$(\partial_t)^\alpha = \left(\frac{\partial x^\alpha}{\partial t}\right)_y \tag{3.35}$$

gives us the components of the time-evolution vector field that generates the integral curves (3.33). In general these curves are not orthogonal to the hypersurface, so the vector field ∂_t can be decomposed into a normal and tangential part (to the hypersurface) in this way:

$$(\partial_t)^\alpha = N n^\alpha + E^\alpha_a N^a \tag{3.36}$$

with the function N being the *lapse function* and the vector field N^a the *shift vector*. They parametrize the freedom in the choice of the time-evolution vector. From equations (3.34), (3.35) and (3.36), we have:

$$\begin{aligned} dx^\alpha &= (\partial_t)^\alpha dt + E^\alpha_a dy^a \\ &= (N n^\alpha + E^\alpha_a N^a) dt + E^\alpha_a dy^a \\ &= N n^\alpha dt + E^\alpha_a (N^a dt + dy^a). \end{aligned} \tag{3.37}$$

By inserting this value spacetime coordinates into the line element, we find

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -N^2 dt^2 + h_{ab}(dy^a + N^a dt)(dy^b + N^b dt) \tag{3.38}$$

[3]By building the normal vector field in this way, we ensure its orthogonality to the hypersurface Σ_t. To check that we just need to notice that $\forall x = x^\mu \partial_\mu$ that is tangent to Σ_t, $x^\mu \partial_\mu t(x)\Big|_{\Sigma_t} = 0$

where we have used $g_{\alpha\beta}n^\alpha n^\beta = -1$ (the vector field normal to the hypersurface is time-like), $g_{\alpha\beta}E^\alpha_a n^\beta = E^\alpha_a n_\beta = 0$ and $h_{ab} = g_{\alpha\beta}E^\alpha_a E^\beta_b$ which is the metric induced on the hypersurface. The metric then takes the following form

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^i N_i & N_i \\ N_j & h_{ij} \end{pmatrix} , \qquad g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^i/N^2 \\ N^j/N^2 & h^{ij} - N^i N^j/N^2 \end{pmatrix} . \tag{3.39}$$

With the metric written in this form, the Einstein-Hilbert action

$$S_{\text{EH}} = \frac{M_{\text{Pl}}}{2} \int d^4x \sqrt{-g} R \tag{3.40}$$

, once $M_{\text{Pl}} = 1$, can be written as

$$S = \frac{1}{2} \int dt d^3x \sqrt{h} N \big[{}^{(3)}R + 2X - 2V \big] + \frac{1}{2} \int dt d^3x \sqrt{h} N^{-1} \big[E_{ij}E^{ij} - E^2 \big], \tag{3.41}$$

with

$$E_{ij} = \frac{1}{2}\big(\dot{h}_{ij} - D_i N_j - D_j N_i\big) = N K_{ij}, \qquad E = h^{ij} E_{ij}. \tag{3.42}$$

In the definitions above, h_{ij} is the metric on the tridimensional space-like hypersurfaces Σ_t, and K_{ij} is the extrinsic curvature, measuring the "bending" of the hypersurfaces on the spacetime. Moreover, ${}^{(3)}R$ and D_i are the Ricci scalar and the covariant derivative calculated by means of the metric h_{ij}, and $X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$. Note that N and N_i do not have any time derivative acting on them. They are indeed Lagrange multipliers and can be written in terms of the dynamical variables. Within single-field models of inflation we expect to have three d.o.f. propagating in the early universe: one is the inflaton field, which is a real scalar field, and two carried by the graviton, a massless gauge field. The two d.o.f. of the graviton are indeed the two polarization state of gravitational waves. Eventually, we want to study how scalar and tensor perturbations evolve in early times, and, as explained in Sec.3.1, this brings the problem of gauge redundancies. We can proceed as follows:

1. Remove extra d.o.f. associated to gauge redundancies we fix the gauge. This will allow us to have h_{ij} written as function of physical scalar and tensor perturbations.
2. Compute the ADM variables K_{ij}, ${}^{(3)}R$, N and N_i using h_{ij}, keeping only terms of the desired order in perturbation theory.
3. Extract the e.o.m. of the non dynamical variables N and N_i in terms of scalar and tensor perturbation.
4. Substituting N and N_i back into (3.41), we will obtain the inflationary action for scalar and tensor perturbations at the desired order in perturbations.

3.5 Perturbation theory and gauge fixing

Before continuing, it is necessary to remove the d.o.f which are gauge redundancies. This can be done by fixing the gauge, which corresponds to a suitable choice of two scalar perturbations and one vector (divergence-free) perturbation. These conditions are eventually equivalent to fix the foliation of the spacetime by choosing the space-like hypersurfaces. We introduce two gauges which will be utilized in the calculations presented further: the comoving gauge and the spatially flat gauge.

In the **spatially flat gauge**, the scalar perturbations of the tridimensional metric h_{ij} are set to be zero, $\phi^{(1)} = \chi^{\|} = 0$. To fix the gauge, we still have a vector perturbation to fix: we set the vector perturbation of h_{ij} to zero, $\chi_i^{\perp} = 0$. In this way, we are left only with the perturbation of the inflaton field $\delta\phi(\mathrm{x}, t)$, and the ones contained in the lapse function and shift vector. In this gauge, the metric induced on the space-like hypersurfaces can be written as

$$\phi(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t) \quad h_{ij} = a^2(\delta_{ij} + \gamma_{ij} + \dots), \quad \gamma^i_i = 0, \quad \partial_i \gamma_{ii} = 0 \tag{3.43}$$

where $\gamma_{ij} = \chi^T_{ij}$. This gauge is equivalent to choose space-like hypersurfaces which are unperturbed: the Ricci scalar computed by means of the metric h_{ij} identically vanishes.

The second gauge that we introduce is the **comoving gauge**, which corresponds to set $\delta\phi$, $\chi^{\parallel}$ and $\chi_i^{\perp}$ to zero. The tridimensional metric is of the form

$$h_{ij} = a^2\big[(1-2\phi)\delta_{ij} + \gamma_{ij} + \dots\big], \quad \gamma^i_i = 0, \quad \partial_i\gamma_{ii} = 0 \tag{3.44}$$

It is useful to introduce the gauge invariant scalar quantity ζ (3.45)[4], also known as *curvature perturbation on comoving hypersurfaces* for reason that will be clear later on.

$$\zeta = -\mathcal{H}\frac{\delta\phi}{\phi_0'} - \hat{\phi} \tag{3.45}$$

In the definition of ζ where $\mathcal{H} = a'/a$ and $\hat{\phi} = \phi + \nabla^2\chi^{\parallel}/6$. In the comoving gauge ζ can be written as $\zeta = -\phi$; it is then clear that the induced metric on the space-like hypersurfaces is given by [7]

$$h_{ij} = a^2\big[(1+2\zeta)\delta_{ij} + \gamma_{ij} + \dots\big], \quad \gamma^i_i = 0, \quad \partial_i\gamma_{ii} = 0. \tag{3.46}$$

In the following we will choose to work with the comoving gauge: the physical d.o.f. are ζ and γ. We do so since both ζ and γ are gauge invariant (γ is gauge invariant at least at first order in perturbation) and ζ remains approximately constant on super-horizon scales.

3.6 Constraint equations

As emphasized in the previous sections, the ADM formalism is particularly useful since it allows for a clear distinction between physical d.o.f., ζ and γ, and non dynamical ones N and N^i. In order to find the action of the physical d.o.f., we need to solve the equations of motion for N and N^i, which can be found by variating (3.41), and substitute the result back in (3.41).
Taking into account that, in the comoving gauge

$$X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = \frac{N^{-2}}{2}\dot{\phi}_0^2 \tag{3.47}$$

and that the expression of E_{ij} is given by

$$\begin{aligned} E_{ij} &= \frac{1}{2}\big(\dot{h}_{ij} - D_iN_j - D_jN_i\big) \\ &= \frac{1}{2}\big(\dot{h}_{ij} - h_i^l h_j^m \nabla_l N_m - h_j^l h_i^m \nabla_l N_m\big) \\ &= \frac{1}{2}\big(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i\big), \end{aligned} \tag{3.48}$$

by variating (3.41) with respect to N we get the *Hamiltonian constraint*

$${}^{(3)}R - 2V - N^{-2}\big(E_{ij}E^{ij} - E^2\big) - N^{-2}\dot{\phi}_0^2 = 0. \tag{3.49}$$

By taking the variation of the action with respect to N^i we get the *momentum constraint*

$$\nabla_i\big[N^{-1}\big(E^i_j - \delta^i_j E\big)\big] = 0. \tag{3.50}$$

Notice that both (3.49) and (3.50) do not contain any time derivative of the field N and N^i, so they are not physical degrees of freedom that propagate in time. Let us now expand the lapse function and the shift vector perturbatively

$$N = N^{(0)} + N^{(1)} + N^{(2)} + \dots\ , \qquad N_i = N_i^{(0)} + N_i^{(1)} + N_i^{(2)} + \dots \tag{3.51}$$

and, furthermore, rewrite the shift function according to the SVT splitting

$$N_i^{(n)} = \partial_i\psi^{(n)} + \bar{N}_i^{(n)}\ , \qquad \nabla_i\bar{N}^{i\,(n)} = 0. \tag{3.52}$$

[4]This definition is valid only in single-field inflation.

We now proceed solving the two constraint equations perturbatively.

At zeroth order (unperturbed universe), the solution is given by $N^{(0)} = 1$ and $N_i^{(0)} = 0$. The momentum constraint vanishes trivially due to the isotropy of the background and the energy constraint reduces to Friedman equations.

Before solving the (3.49) and (3.50) at first order, it is useful to make some considerations to simplify calculations. Being N and $N^{(i)}$ non physical, they can be written as scalar and vector function of the physical dof ζ and γ_{ij}. At first order, the only scalar and vector quantities that can be constructed out of γ_{ij} are $\partial_i\partial_j\gamma^{ij}$ and $\partial_j\gamma^{ij}$ that are zero because of transversality of gravitational waves. As a result, $N^{(1)}$ and $N^{i^{(1)}}$ will not depend on γ and so we can set it to zero. Moreover, the only first order vector we can build out of ζ is $\partial^i f$ where f is a function of ζ and its derivatives. From (3.52) we have that $\partial^i\partial_i f = 0$, which is a Laplace equation solved by $f = 0$, once the boundary conditions are properly chosen, and then $\bar{N}^i = 0$. Substituting the values $N = 1 + N^{(1)}$ and $N^i = \partial^i\psi^{(1)}$ in the constraint equations and taking just first order quantities we get

$$\partial^2\psi^{(1)} = -a^{-2}\frac{\partial^2\zeta}{H} + \frac{\dot{\phi}_0^2}{2H^2}\dot{\zeta} \ , \qquad \nabla_i\big(HN^{(1)} - \dot{\zeta}\big) = 0 \tag{3.53}$$

which are solved by

$$\psi^{(1)} = -a^{-2}\frac{\zeta}{H} + \frac{\dot{\phi}_0^2}{2H^2}\partial^{-2}\dot{\zeta} \ , \qquad N^{(1)} = \frac{\dot{\zeta}}{H} \ , \qquad \bar{N}^{(1)} = 0. \tag{3.54}$$

Proceeding in the same fashion, it is possible to find the solutions of the constraint at n-th order in perturbation. As noted and explicitly shown in [7], if we are interested in computing the action up to third order in perturbations, we just need the first order solution of the constraints. In fact, second order terms in N will multiply the hamiltonian constraint $\partial L/\partial N$ evaluated at the zeroth order which eventually vanishes since it obeys the equations of motions. The same argument holds for N^i.

3.7 Scalar power spectrum

We are interested to analyze the power spectrum of primordial perturbations generated at the end of inflation. To do so, we need to know how perturbations evolve with time in the early universe. For this reason, we focus now on the derivation of the quadratic action in ζ, which, according the least action principle, will lead to the equation of motion for that quantity. In the following, we shall present the derivation of the quadratic action in ζ, the associated equation of motion and its solution, and the primordial scalar power spectrum.

3.7.1 Second order scalar action

It is our aim to compute the second order action which can be found by replacing the first order expression for the constraints (3.54), and expanding the that to the second order in ζ.

Let us focus on the first term in (3.41). We start with the evaluation of the extrinsic curvature ${}^{(3)}R$ first. The extrinsic curvature is defined as

$$\begin{aligned} {}^{(3)}R_{ij} &= {}^{(3)}R^l_{ilj} \\ &= \partial_l^{(3)}\Gamma^l_{ij} - \partial_j^{(3)}\Gamma^l_{il} + {}^{(3)}\Gamma^l_{lk}{}^{(3)}\Gamma^k_{ij} - {}^{(3)}\Gamma^l_{jk}{}^{(3)}\Gamma^k_{il} \\ &= -\partial_i\partial_j\zeta - \delta_{ij}\partial^2\zeta + \partial_i\zeta\partial_j\zeta - \delta_{ij}(\partial\zeta)^2 \end{aligned} \tag{3.55}$$

were we used:

$$
\begin{aligned}
{}^{(3)}\Gamma^l_{ij} &= \frac{1}{2}h^{lk}\big(\partial_i h_{jk} + \partial_j h_{ki} - \partial_k h_{ij}\big) \\
&= \delta^l_j\partial_i\zeta + \delta^l_i\partial_j\zeta - \delta_{ij}\partial^l\zeta, \\
\partial_l {}^{(3)}\Gamma^l_{ij} &= 2\partial_i\partial_j\zeta - \delta_{ij}\partial^2\zeta, \\
\partial_j {}^{(3)}\Gamma^l_{il} &= 3\partial_j\partial_i\zeta, \\
{}^{(3)}\Gamma^l_{lk}{}^{(3)}\Gamma^k_{ij} &= 6\partial_i\zeta\partial_j\zeta - 3\delta_{ij}(\partial\zeta)^2, \\
{}^{(3)}\Gamma^l_{jk}{}^{(3)}\Gamma^k_{il} &= 5\partial_i\zeta\partial_j\zeta - \delta_{ij}(\partial\zeta)^2.
\end{aligned}
\tag{3.56}
$$

We can the find the expression for the extrinsic curvature by contracting the two indices with the induced metric:

$$
\begin{aligned}
{}^{(3)}R &= h^{ij\,(3)}R_{ij} \\
&= a^{-2}e^{-2\zeta}\delta^{ij}\big[-\partial_i\partial_j\zeta - \delta_{ij}\partial^2\zeta + \partial_i\zeta\partial_j\zeta - \delta_{ij}(\partial\zeta)^2\big] \\
&= a^{-2}e^{-2\zeta}\big[-4\partial^2\zeta - 2(\partial\zeta)^2\big]
\end{aligned}
\tag{3.57}
$$

It is now worth to make a small digression regarding the physical meaning of the variable ζ. We notice that, by expanding (3.57) to linear order in ζ, we get

$$
{}^{(3)}R = -\frac{4}{a^2}\nabla^2\zeta. \tag{3.58}
$$

Going to Fourier space to make the meaning of the latter equation more manifest, $\partial \to ik$ and the scalar curvature can be written as

$$
{}^{(3)}R = \frac{4}{a^2}k^2\zeta \tag{3.59}
$$

and it is clear that, to linear order, ζ determines the curvature of the spatial slices the spacetime has been foliated in: for this reason the variable ζ *curvature perturbation on uniform energy density hypersurfaces*.

Then we have that

$$
\frac{1}{2}\int dtd^3x\sqrt{h}N\big[{}^{(3)}R + 2X - 2V\big] = \frac{1}{2}\int dtd^3xa^3e^{3\zeta}N\big[-4a^{-2}e^{-2\zeta}\partial^2\zeta - 2a^{-2}e^{-2\zeta}(\partial\zeta)^2 - 2V + N^{-1}\dot{\phi}_0^2\big] \tag{3.60}
$$

In order to get the expression of the second term in (3.41) as function of the ADM variables we need to evaluate the term $E_{ij}E^{ij} - E^2$ up to first order in ζ. Starting from (3.48)and considering that $\bar{N}_i = 0$ we get

$$
\begin{aligned}
E_{ij} &= a^2(H+\dot{\zeta})e^{2\zeta}\delta_{ij} - \partial_{(i}N_{j)} + (2N_{(i}\partial_{j)}\zeta - N_k\partial_k\zeta\delta_{ij}) \\
&= a^2\big(H+\dot{\zeta}\big)e^{2\zeta}\delta_{ij} - \partial_i\partial_j\psi + \partial_i\zeta\partial_j\psi + \partial_j\zeta\partial_i\psi - \partial_k\psi\partial^k\zeta\delta_{ij}
\end{aligned}
\tag{3.61}
$$

and

$$
\begin{aligned}
E^{ij} &= a^{-4}e^{-4\zeta}\delta^{il}\delta^{jm}\big[a^2e^{2\zeta}\big(H+\dot{\zeta}\big)\delta_{lm} - \partial_l\partial_m\psi + \partial_l\zeta\partial_m\psi + \partial_m\zeta\partial_l\psi - \partial_k\psi\partial^k\zeta\delta_{lm}\big] \\
&= a^{-4}e^{-4\zeta}\big[a^2e^{2\zeta}(H+\dot{\zeta})\delta^{ij} - \partial^i\partial^j\psi + \partial^i\zeta\partial^j\psi + \partial^j\zeta\partial^i\psi - \partial_k\psi\partial^k\zeta\delta^{ij}\big]
\end{aligned}
\tag{3.62}
$$

therefore

$$
E_{ij}E^{ij} = 3\big(H+\dot{\zeta}\big)^2 - 2a^{-2}e^{-2\zeta}\big(H+\dot{\zeta}\big)\partial^2\psi - 2a^{-2}e^{-2\zeta}\big(H+\dot{\zeta}\big)\partial_i\zeta\partial^i\psi + a^{-4}e^{-4\zeta}\partial_i\partial_j\psi\partial^i\partial^j\psi. \tag{3.63}
$$

We can now take the contraction of (3.112) with the spatial metric to obtain E

$$
\begin{aligned}
E = h^{ij}E_{ij} &= a^{-2}e^{-2\zeta}\delta^{ij}\big[a^2\big(H+\dot{\zeta}\big)e^{2\zeta}\delta_{ij} - \partial_i\partial_j\psi + \partial_i\zeta\partial_j\psi + \partial_j\zeta\partial_i\psi - \partial_k\psi\partial^k\zeta\delta_{ij}\big] \\
&= 3\big(H+\dot{\zeta}\big) - a^{-2}e^{-2\zeta}\partial^2\psi - a^{-2}e^{-2\zeta}\partial_i\zeta\partial^i\psi
\end{aligned}
\tag{3.64}
$$

and compute its square

$$E^2 = 9\big(H+\dot\zeta\big)^2 - 6a^{-2}e^{-2\zeta}\big(H+\dot\zeta\big)\partial^2\psi + a^{-4}e^{-4\zeta}\big(\partial^2\psi\big)^2 - a^{-2}e^{-2\zeta}\big(H+\dot\zeta\big)\partial_i\zeta\partial^i\psi. \tag{3.65}$$

Putting the pieces together we find

$$\begin{aligned} E_{ij}E^{ij} - E^2 = & -6\big(H+\dot\zeta\big)^2 + 4a^{-2}e^{-2\zeta}\big(H+\dot\zeta\big)\partial^2\psi + 4a^{-2}e^{-2\zeta}\big(H+\dot\zeta\big)\partial_i\zeta\partial^j\psi \\ & + a^{-4}e^{-4\zeta}\partial_i\partial_j\psi\partial^i\partial^j\psi - a^{-4}e^{-4\zeta}\big(\partial^2\psi\big)^2 \end{aligned} \tag{3.66}$$

and the second term in the inflationary action becomes

$$\begin{aligned} \frac{1}{2}\int dtd^3x\sqrt{h}N^{-1}\big[E_{ij}E^{ij}-E^2\big] = \frac{1}{2}\int dtd^3xa^3e^{3\zeta}N^{-1}\big[& -6\big(H+\dot\zeta\big)^2 + 4a^{-2}e^{-2\zeta}\big(H+\dot\zeta\big)\partial^2\psi \\ & + 4a^{-2}e^{-2\zeta}\big(H+\dot\zeta\big)\partial_i\zeta\partial^j\psi + a^{-4}e^{-4\zeta}\partial_i\partial_j\psi\partial^i\partial^j\psi - a^{-4}e^{-4\zeta}\big(\partial^2\psi\big)^2\big]. \end{aligned} \tag{3.67}$$

Let us now make some considerations in order to simplify the integral (3.67). First of all, notice that

$$\begin{aligned} & \int dtd^3xa^3e^{3\zeta}N^{-1}\big[a^{-4}e^{-4\zeta}\partial_i\partial_j\psi\partial^i\partial^j\psi - a^{-4}e^{-4\zeta}\big(\partial^2\psi\big)^2\big] \\ & = \int dtd^3xa^{-1}\left(1+\frac{\dot\zeta}{H}\right)^{-1}\big[(1-\zeta)\partial_i\partial_j\psi\partial^i\partial^j\psi - (1-\zeta)\big(\partial^2\psi\big)^2\big] \\ & \approx \int dtd^3xa^{-1}\big[\partial_i\partial_j\psi\partial^i\partial^j\psi - \big(\partial^2\psi\big)^2\big] = 0 \end{aligned} \tag{3.68}$$

because $\psi \sim \mathcal{O}(\zeta)$ and $\int dtd^3xa^{-1}\partial_i\partial_j\psi\partial^i\partial^j\psi = \int dta^{-1}\int d^3x\partial_i\partial_j\psi\partial^i\partial^j\psi = -\int dta^{-1}\int d^3x\partial_i\psi\partial_j\partial^i\partial^j\psi$ $= \int dtd^3xa^{-1}\big(\partial\psi\big)^2$ where we have neglected total derivative terms. Second, considering only terms up to second order and integrating by parts, we have

$$\begin{aligned} & \int \mathrm{d}t\mathrm{d}^3xa^3e^{3\zeta}N^{-1}\Big[4a^{-2}e^{-2\zeta}\big(H+\dot\zeta\big)\partial^2\psi + 4a^{-2}e^{-2\zeta}\big(H+\dot\zeta\big)\partial_i\zeta\partial^j\psi\Big] \\ & = 4\int \mathrm{d}t\mathrm{d}^3xN^{-1}\Big[ae^{\zeta}\big(H+\dot\zeta\big)\partial^2\psi + ae^{\zeta}\big(H+\dot\zeta\big)\partial_i\zeta\partial^j\psi\Big] \\ & \approx 4\int \mathrm{d}t\mathrm{d}^3xa\Big[e^{\zeta}\big(H+\dot\zeta\big)^{-1}\big(H+\dot\zeta\big)\partial^2\psi + e^{\zeta}H\big(H+\dot\zeta\big)^{-1}\partial_i\zeta\partial^i\psi\Big] \\ & \approx 4\int \mathrm{d}t\mathrm{d}^3xaH\Big[\partial^2\psi + \zeta\partial^2\psi + \partial_i\zeta\partial^i\psi\Big] = 4\int \mathrm{d}t\mathrm{d}^3x\partial_i\big(aH\partial^i\psi\big) \end{aligned} \tag{3.69}$$

which is a total derivative linear in ψ and can be neglected. So the second term in the action can be rewritten as

$$\frac{1}{2}\int dtd^3x\sqrt{h}N^{-1}\big[E_{ij}E^{ij}-E^2\big] = \frac{1}{2}\int \mathrm{d}t\mathrm{d}^3xa^3e^{3\zeta}N^{-1}\big[-6(H+\dot\zeta)^2\big]. \tag{3.70}$$

The total action is then given by

$$S_\zeta = \frac{1}{2}\int \mathrm{d}t\mathrm{d}^3xae^{\zeta}N\big[-4\partial^2\zeta - 2(\partial\zeta) - 2a^2e^{2\zeta}V\big] + \frac{1}{2}\int \mathrm{d}t\mathrm{d}^3xa^3e^{3\zeta}N^{-1}\big[-6(H+\dot\zeta)^2 + \dot\phi_0\big]. \tag{3.71}$$

Focusing on the first integral we write the potential V in terms of H^2 and $\dot\phi_0$ using the first Friedmann equation (2.2), then we can expand everything up to second order in ζ[5] obtaining

$$\begin{aligned}
I_1 &= \frac{1}{2}\int \mathrm{dtd}^3 x a e^{\zeta} N\big[-4\partial^2\zeta - 2(\partial\zeta) - 2a^2 e^{2\zeta} V\big] \\
&= \frac{1}{2}\int \mathrm{dtd}^3 x a e^{\zeta} N\Big[-4\partial^2\zeta - 2(\partial\zeta)^2 - 2a^2 e^{2\zeta}\Big(3H - \frac{\dot\phi_0^2}{2}\Big)\Big] \\
&\approx \frac{1}{2}\int \mathrm{dtd}^3 x a N\Big(1+\zeta+\frac{\zeta^2}{2}\Big)\Big[-4\partial^2\zeta - 2\big(\partial\zeta\big)^2 - 2a^2\Big(3H - \frac{\dot\phi_0^2}{2}\Big)\Big(1+\zeta+\frac{\zeta^2}{2}\Big)(1+2\zeta+2\zeta^2)\Big] \\
&\approx \frac{1}{2}\int \mathrm{dtd}^3 x a\Big[-4\partial^2\zeta - 4H^{-1}\dot\zeta\partial^2\zeta - 4\zeta\partial^2\zeta - 2\big(\partial\zeta\big)^2 + a^2\big(-6H^2+\dot\phi_0^2\big)\Big(1+3\zeta+\frac{9}{2}\zeta^2+\frac{\dot\zeta}{H}+3\frac{\zeta\dot\zeta}{H}\Big)\Big].
\end{aligned} \tag{3.72}$$

By considering only $\mathcal{O}(\zeta^2)$ terms we get

$$I_1^{(2)} = \frac{1}{2}\int \mathrm{dtd}^3 x a\Big[-2\big(\partial\zeta\big)^2 - 4H^{-1}\dot\zeta\partial^2\zeta - 4\zeta\partial^2\zeta + \frac{9}{2}a^2\big(-6H^2+\dot\phi_0^2\big)\zeta^2 + 3a^2H^{-1}\big(-6H^2+\dot\phi_0^2\big)\zeta\dot\zeta\Big] \tag{3.73}$$

By doing the same with the second integral we find

$$\begin{aligned}
I_2 &= \frac{1}{2}\int \mathrm{dtd}^3 x a^3 e^{3\zeta} N^{-1}\big[-6(H+\dot\zeta)^2 + \dot\phi_0^2\big] \\
&\approx \frac{1}{2}\int \mathrm{dtd}^3 x a^3\Big(1+3\zeta+\frac{3}{2}\zeta^2\Big)\Big(1+\frac{\dot\zeta}{H}\Big)^{-1}\Big[-6\big(H^2+2H\dot\zeta+\dot\zeta^2\big)+\dot\phi_0^2\Big] \\
&\approx \frac{1}{2}\int \mathrm{dtd}^3 x a^3\Big(1-\frac{\dot\zeta}{H}+\frac{\dot\zeta^2}{H^2}\Big)\Big[-6\big(H^2+\dot\zeta^2+2H\dot\zeta\big)+\dot\phi_0^2 - 6\big(3H^2\zeta+6H\zeta\dot\zeta\big)+3\zeta\dot\phi_0^2 - 27H^2\zeta^2+\frac{9}{2}\zeta^2\dot\phi_0^2\Big] \\
&\approx -6\big(H^2+\dot\zeta^2+2H\dot\zeta\big)+\dot\phi_0^2 - 6\big(3H^2\zeta+6H\dot\zeta\zeta\big)+3\zeta\dot\phi_0^2 - 27H^2\zeta^2+\frac{9}{2}\zeta^2\dot\phi_0^2 \\
&+6\big(H\dot\zeta+2\dot\zeta^2\big) - H^{-1}\dot\zeta\dot\phi_0^2 + 6\big(3H\zeta\dot\zeta\big) - 3H^{-1}\zeta\dot\zeta\dot\phi_0^2 - 6\dot\zeta^2 + H^{-2}\dot\zeta^2\dot\phi_0^2
\end{aligned} \tag{3.74}$$

and

$$\begin{aligned}
I_2^{(2)} &= -6\dot\zeta^2 - 36H\dot\zeta\zeta - 27H^2\zeta^2 + \frac{9}{2}\zeta^2\dot\phi_0^2 + 12\dot\zeta^2 + 18H\zeta\dot\zeta - 3H^{-1}\zeta\dot\zeta\dot\phi_0^2 - 6\dot\zeta^2 + H^{-2}\dot\zeta^2\dot\phi_0^2 \\
&= H^{-2}\dot\zeta\dot\phi_0^2 - 3\Big(6H+\frac{\dot\phi_0^2}{H}\Big)\zeta\dot\zeta - 9\Big(3H^2-\frac{\dot\phi_0^2}{2}\Big)\zeta^2.
\end{aligned} \tag{3.75}$$

We can then write the total action as follows

$$\begin{aligned}
S_\zeta^{(2)} &= I_1^{(2)} + I_2^{(2)} \\
&= \frac{1}{2}\int \mathrm{dtd}^2 x\Big\{a\Big[-2\big(\partial\zeta\big)^2 - 4H^{-1}\dot\zeta\partial^2\zeta - 4\zeta\partial^2\zeta + \frac{9}{2}a^2\big(-6H^2+\dot\phi_0^2\big)\zeta^2 + 3a^2H^{-1}\big(-6H^2+\dot\phi_0^2\big)\zeta\dot\zeta\Big] \\
&+ a^3\Big[H^{-2}\dot\zeta^2\dot\phi_0^2 - 9\Big(3H^2-\frac{1}{2}\dot\phi_0^2\Big)\zeta^2 - 3\Big(6H+\frac{\dot\phi_0^2}{H}\Big)\zeta\dot\zeta\Big]\Big\} \\
&= \frac{1}{2}\int \mathrm{dtd}^3 x\big[-2a(\partial\zeta)^2 - 4aH^{-1}\dot\zeta\partial^2\zeta - 4a\zeta\partial^2\zeta + a^3H^{-2}\dot\zeta^2\dot\phi_0^2 + 9a^3(-6H^2+\dot\phi_0^2)\zeta^2 - 36a^3H\zeta\dot\zeta\big] \\
&= \frac{1}{2}\int \mathrm{dtd}^3 x\big[2a(\partial\zeta)^2 - 2a(1+\epsilon)\big(\partial\zeta\big)^2 + a^3H^{-2}\dot\zeta^2\dot\phi_0^2 + 9a^3(-6H^2+\dot\phi_0^2)\zeta^2 + 18a^3(3H^2+\dot H)\zeta^2\big] \\
&= \frac{1}{2}\int \mathrm{dtd}^3 x\big[-2a\epsilon\big(\partial\zeta\partial\big)^2 + a^3H^{-2}\dot\zeta^2\dot\phi_0^2 + 9a^3\big(-6H^2+\dot\phi_0^2\big)\zeta^2 + 9a^3\big(6H^2-\dot\phi_0^2\big)\zeta^2\big] \\
&= \frac{1}{2}\int \mathrm{dtd}^3 x\big[-2a\epsilon\big(\partial\zeta\big)^2 + 2a^3\epsilon\dot\zeta^2\big]
\end{aligned} \tag{3.76}$$

[5]A part from the lapse function that we only need to expand up to $\mathcal{O}(\zeta)$.

where, in the second passage, we made use of

$$\begin{aligned}\int \mathrm{d}t\mathrm{d}^3x36a^3H\zeta\dot{\zeta} &= \int \mathrm{d}t\mathrm{d}^3x36a^3H\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\zeta^2}{2}\right)\\ &= \int \mathrm{d}t\mathrm{d}^3x\frac{\mathrm{d}}{\mathrm{d}t}\left(36a^3H\frac{\dot{\zeta}}{2}\right) - \int \mathrm{d}t\mathrm{d}^3x36\frac{\mathrm{d}}{\mathrm{d}t}\left(a^3H\right)\frac{\zeta^2}{2}\\ &= -\int \mathrm{d}t\mathrm{d}^3x18\left(3H^2+\dot{H}\right)\zeta^2,\end{aligned} \tag{3.77}$$

$$\begin{aligned}\int \mathrm{d}t\mathrm{d}^3x4aH^{-1}\dot{\zeta}\partial^2\zeta &= -\int \mathrm{d}t\mathrm{d}^3x4aH^{-1}\partial_i\dot{\zeta}\partial^i\zeta\\ &= -\int \mathrm{d}t\mathrm{d}^3x4aH^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\left(\partial\zeta\right)^2}{2}\right]\\ &= \int \mathrm{d}t\mathrm{d}^3x\frac{\mathrm{d}}{\mathrm{d}t}\left(2aH^{-1}\right)\left(\partial\zeta\right)^2\\ &= \int \mathrm{d}t\mathrm{d}^3x2\left(\frac{\dot{a}}{H} - a\frac{\dot{H}}{H^2}\right)\left(\partial\zeta\right)^2\\ &= \int \mathrm{d}t\mathrm{d}^3x2a\left(1+\epsilon\right)\left(\partial\zeta\right)^2,\end{aligned} \tag{3.78}$$

and in the third and the fourth that $\dot{\phi}_0^2 = -2\dot{H}$[6].
Therefore, the second order action for scalar perturbations reads:

$$S_\zeta^{(2)} = \int \mathrm{d}t\mathrm{d}^3x\left[a^3\epsilon\dot{\zeta}^2 - a\epsilon\left(\partial\zeta\right)^2\right]. \tag{3.80}$$

3.7.2 Equations of motion for scalar perturbations

The action quadratic in ζ is the starting point for studying the evolution of scalar perturbations in the early universe. To make the action canonically normalized, we introduce the Mukhanov-Sasaki variable $v = zM_{\mathrm{Pl}}\zeta$, where $z = a\sqrt{2\epsilon_1}$. Once we write (3.80 in terms of u we find

$$S_u^{(2)} = \frac{1}{2}\int \mathrm{d}\eta\mathrm{d}^3x\left[(u')^2 - (\partial_i u)^2 + \frac{z''}{z}u^2\right], \tag{3.81}$$

where the prime denotes derivatives w.r.t. conformal time η. Note that the quantity z''/z behaves as an effective time-dependent mass squared, since it can be expressed as [43]:

$$\frac{z''}{z} = (aH)^2\left(2 - \epsilon_1 + \frac{3\epsilon_2}{2} + \frac{\epsilon_2^2}{4} - \frac{\epsilon_1\epsilon_2}{2} + \frac{\epsilon_2\epsilon_3}{2}\right). \tag{3.82}$$

Since the Mukhanov-Sasaki variable u is related to the curvature perturbation ζ, which is quantum in the very early universe as it has a quantum origin, we canonically quantize it:

$$\hat{u}(\mathbf{x},\eta) = M_{\mathrm{Pl}}z(\eta)\hat{\zeta}(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[u_k(\eta)\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + u_k^*(\eta)\hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \tag{3.83}$$

where the $\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{a}_{\mathbf{k}}$ are the creation and annihilation operators satisfying $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k'}}^\dagger] = (2\pi)^3\delta(\mathbf{k} - \mathbf{k'})$ under the normalization condition

$$u'^*_k u_k - u'_k u^*{}_k = i\hbar. \tag{3.84}$$

[6]This relation is obtained by taking the time derivative of the background equation

$$H^2 = \frac{1}{3}\left(V(\phi) + \frac{1}{2}\dot{\phi}_0^2\right). \tag{3.79}$$

Note that we have passed to armonic space, which is a standard procedure in dealing with perturbations since each k-mode u_k evolves independently to the others at linear order.
The mode function $u_k(\eta)$ satisfies the Mukhanov-Sasaki equation [44]:

$$u_k'' + \left(k^2 - \frac{z''}{z}\right)u_k = 0 \tag{3.85}$$

which is derived by applying the variational principle to (3.81). In slow-roll inflation we can neglect the terms of the type $\epsilon_1\epsilon_2$ and $\epsilon_1\epsilon_3$, since all slow-roll parameters are small quantities in slow-roll inflation [38].Thus, the time-dependent mass term can be written as

$$\frac{z''}{z} = (aH)^2\left(2 - \epsilon_1 + \frac{3}{2}\epsilon_2 + O(\epsilon^2)\right), \tag{3.86}$$

and, since up to first order in slow-roll parameter

$$aH = -\frac{1}{\eta(1-\epsilon_1)}, \tag{3.87}$$

the effective mass can be expressed in terms of the conformal time as

$$\frac{z''}{z} = \frac{1}{\eta^2}\left(2 + 3\epsilon_1 + \frac{3}{2}\epsilon_2\right) = \frac{1}{\eta^2}\left(\nu_s^2 - \frac{1}{4}\right) \tag{3.88}$$

with $\nu_s = 3/2 + \delta_s = 3/2 + \epsilon_1 + \epsilon_2/2$.
The Mukhanov-Sasaki variable can be rewritten as

$$u_k'' + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\eta^2}\right)u_k = 0 \tag{3.89}$$

which is a Bessel equation. Before solving it exactly, it is useful to understand how perturbations behave in the super-horizon and sub-horizon regimes.

Sub-horizon regime

Consider $k \gg aH$, which corresponds to the sub-horizon regime. In this case k^2 dominates over $z''/z \propto (aH)^2$, and the equation of motion (3.85) can be simplified as

$$u_k'' + k^2 u_k = 0. \tag{3.90}$$

This is the equation of motion of a classical harmonic oscillation with frequency $\omega = k$, with solution

$$u_k^{k\gg aH}(\eta) = c_1 e^{-ik\eta} + c_2 e^{ik\eta}. \tag{3.91}$$

However, we can make some considerations based on the equivalence principle and we state that, at very early times, $\eta \to -\infty$ and for very small scales $k/aH \gg 1$, which are much smaller than the Hubble radius, the spacetime can be approximated as flat. The mode function u_k takes the form of the vacuum on a Minkowski spacetime [36]

$$u_k(\eta) = \frac{1}{\sqrt{2k^3}}e^{-ik\eta}. \tag{3.92}$$

This is eventually the solution for (3.90), which is also known as the Bunch-Davies vacuum. Therefore, it is clear that, when deep inside the horizon, the mode functions manifest an oscillatory behavior which grows stronger as $-k\eta$ increases.

Super-horizon regime

In super-horizon regime, $k \ll aH$, the term z''/z dominates over k^2 and (3.85) becomes

$$u_k'' - \frac{z''}{z} u_k = 0. \tag{3.93}$$

The general solution of this equation $u_k(\eta) = B\, u_0^{(1)} + B\, u_0^{(2)}$ comprises a mode that grows during SR, $u_0^{(1)}$ and a decaying mode $u_0^{(2)}$, given by

$$u_0^{(1)} \propto z, \qquad u_0^{(2)} \propto z \int_\eta^0 \frac{d\eta'}{z^2(\eta')}. \tag{3.94}$$

The former will eventually dominate, since the latter becomes negligible after few e-folds after the beginning of inflation. The physical quantity is the curvature perturbation $\zeta = u/zM_{\mathrm{Pl}}$, and its solution in the super-horizon regime is $\zeta \propto$ const., which means that the amplitude of the scalar fluctuations remains frozen at super-Hubble regime.

The results in these two regimes makes the origin of primordial perturbations manifest: quantum fluctuations around the vacuum are stretched to classical perturbations, with a frozen amplitude, on super-horizon scales. This mechanism for the generation of perturbations, called gravitational amplification, is analogous to the creation of physical electrons and positrons from the vacuum, once an electric field is applied.

General solution

Going back to (3.89), its solution is a combination of Hankel functions of the first and second kind with index ν

$$u_k(\eta) = \sqrt{-\eta}\Big[c_1'(k) H_{\nu_s}^{(1)}(-k\eta) + c_2'(\eta) H_{\nu_s}^{(2)}(-k\eta)\Big], \qquad H_{\nu_s}^{(2)} = {H^{(1)}}^*_{\nu_s}. \tag{3.95}$$

The value of c_1' and c_2' needs to be fixed, and it can be done by imposing the Bunch-Davies vacuum condition. By matching (3.92) with the asymptotic expression of the Hankel functions [36]

$$H_{\nu_s}^{(1)}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}} e^{i(x - \pi\nu/2 - \pi/4)}, \qquad H_{\nu_s}^{(2)}(x \gg 1) \sim \sqrt{\frac{2}{\pi x}} e^{-i(x - \pi\nu/2 - \pi/4)} \tag{3.96}$$

with $x = -k\eta$, we find the following values for the c_1 and c_2 coefficients:

$$c_1'(k) = \sqrt{\frac{\pi}{4k}} e^{i\frac{\pi}{4}(2\nu+1)} \tag{3.97}$$

$$c_2'(k) = 0. \tag{3.98}$$

which also satisfy the condition (3.84).

3.7.3 Power spectrum

The power spectrum is a fundamental tool for cosmologists, allowing to compare theoretical models with observations. In the following we show how to compute the spectral power of scalar perturbations and which information can provide us.

We are interested in the asymptotic behavior of the solution when the mode is well outside the horizon, $-k\eta \ll 1$, and the asymptotic expansion of the Hankel function of the first kind is

$$H_{\nu_s}^{(1)}(x \ll 1) \sim \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{2}} 2^{\nu_s - \frac{3}{2}} \left(\frac{\Gamma(\nu_s)}{\Gamma(3/2)}\right) x^{-\nu_s} \tag{3.99}$$

where $x = -k\eta$ and Γ is the Euler gamma function. The mode functions associated to the Sasaki-Mukanov variable in this regime is

$$u_k \simeq \frac{e^{i(\nu_s-\frac{1}{2})\frac{\pi}{2}}}{\sqrt{2k}} 2^{\nu_s-\frac{3}{2}} \left(\frac{\Gamma(\nu_s)}{\Gamma(3/2)}\right) x^{\frac{1}{2}-\nu_s} \tag{3.100}$$

We can then switch to the variable $\zeta = v/zM_{\rm Pl}$, obtain the power spectrum by squaring its amplitude and keep only terms that are leading order in slow-roll. By doing so and restoring the dependence on the Planck mass $M_{\rm Pl}$, the dimensionless power spectrum of the curvature perturbation ζ is found to be

$$\Delta_\zeta(k) = \left(\frac{H^2}{\pi^2 \epsilon_1 M_{\rm Pl}^2}\right)\left(\frac{k}{aH}\right)^{n_s-1} \tag{3.101}$$

with $n_s - 1 = 3 - 2\nu_s$. The power spectrum exhibits a mild, but non vanishing, scale dependence entailed in the spectral index $n_s = 1 - 2\epsilon_1 - \epsilon_2$, causing it to be red tilted: the power decreases at increasing wavenumbers k.

3.8 Tensor power spectrum

3.8.1 Second order tensor action

We now want to find the expression for the power spectrum of gravitational waves. In order to do so, we need to find the quadratic action in γ. As there is no mixing between ζ and γ in the second order action, we can set the scalar perturbations to zero and the spatial metric metric takes the form

$$h_{ij} = a^2[\exp\gamma]_{ij} = a^2\left(\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{il}\gamma^l_j + \dots\right) \ , \qquad \gamma_{ii} = 0 \ , \qquad \partial_i\gamma_{ij} = 0 \tag{3.102}$$

with $\gamma_{ij}(\mathbf{x}, t)$ denoting the gravitational wave amplitude.

As in the previous section, we start by computing the curvature scalar calculated with the spatial metric h_{ij}. The spatial Ricci tensor is given by

$$\begin{aligned} {}^{(3)}R_{ij} &= {}^{(3)}R^l_{ilj} \\ &= \partial_l^{(3)}\Gamma^l_{ij} - \partial_j^{(3)}\Gamma^l_{il} + {}^{(3)}\Gamma^l_{lk}{}^{(3)}\Gamma^k_{ij} - {}^{(3)}\Gamma^l_{jk}{}^{(3)}\Gamma^k_{il} \end{aligned} \tag{3.103}$$

Taking the partial derivative with respect to the index l of the Christoffel symbol ${}^{(3)}\Gamma^l_{ij}$

$$\begin{aligned} {}^{(3)}\Gamma^l_{ij} &= \frac{1}{2}h^{lk}\left(\partial_i h_{jk} + \partial_j h_{ki} - \partial_k h_{ij}\right) \\ &= \frac{1}{2}\left(\delta^{lk} - \gamma^{lk} + \frac{1}{2}\gamma^{lm}\gamma^k_m\right)\left[\partial_i\left(\delta_{jk} + \gamma_{jk} + \frac{1}{2}\gamma_{jm}\gamma^m_k\right) + \partial_j\left(\delta_{ki} + \gamma_{ki} + \frac{1}{2}\gamma_{km}\gamma^m_i\right)\right. \\ &\left. - \partial_k\left(\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{im}\gamma^m_j\right)\right] \\ &\approx \frac{1}{2}\left[\partial_i\gamma^l_j + \partial_j\gamma^l_i - \partial^l\gamma_{ij} + \frac{1}{2}\partial_i\left(\gamma_{jm}\gamma^{ml}\right) + \frac{1}{2}\partial_j\left(\gamma^l_m\gamma^m_i\right) - \frac{1}{2}\partial^l\left(\gamma_{im}\gamma^m_j\right) - \gamma^{lk}\partial_i\gamma_{jk} - \gamma^{lk}\partial_j\gamma_{ki} + \gamma^{lk}\partial_k\gamma_{ij}\right] \end{aligned} \tag{3.104}$$

we get

$$\begin{aligned} \partial_l^{(3)}\Gamma^l_{ij} &= \frac{1}{2}\left[\partial_l\partial_i\gamma^l_j + \partial_l\partial_j\gamma^l_i - \partial_l\partial^l\gamma_{ij} + \frac{1}{2}\partial_l\partial_i\left(\gamma_{jm}\gamma^{ml}\right) + \frac{1}{2}\partial_l\partial_j\left(\gamma^{lm}\gamma_{mi}\right) - \frac{1}{2}\partial_l\partial^l\left(\gamma_{im}\gamma^m_j\right)\right. \\ &\left. - \partial_l\left(\gamma^{lk}\partial_i\gamma_{jk}\right) - \partial_l\left(\gamma^{lk}\partial_j\gamma_{ki}\right) + \partial_l\left(\gamma^{lk}\partial_k\gamma_{ij}\right)\right] \\ &= \partial_l A^l_{ij} - \frac{1}{2}\partial_l\partial^l\gamma_{ij} \end{aligned} \tag{3.105}$$

with A^l_{ij} being a second order quantity. In the previous equation we made use of $\partial_l \gamma^l_i = \partial_l(\gamma^{lj} h_{ij}) = \partial_l(a^2\gamma^{lj}\delta_{ij} + a^2\gamma^{lj}\gamma_{ij}) = \partial_l(a^2\gamma^{lj}\gamma_{ij})$.

Regarding the second term in (3.103), the expression for ${}^{(3)}\Gamma^l_{il}$ reads

$$\begin{aligned} {}^{(3)}\Gamma^l_{il} &= \frac{1}{2}h^{lk}\left[\partial_i h_{lk} + \partial_l h_{ki} - \partial_k h_{il}\right] \\ &\approx \frac{1}{2}(\delta^{lk} - \gamma^{lk})\left[\partial_i\gamma_{lk} + \frac{1}{2}\partial_i(\gamma_{lm}\gamma^m_k) + \partial_l\gamma_{ki} + \frac{1}{2}\partial_l(\gamma_{km}\gamma^m_i) - \partial_k\gamma_{il} - \frac{1}{2}\partial_k(\gamma_{im}\gamma^m_l)\right] \\ &\approx \frac{1}{2}\left[\frac{1}{2}\partial_i(\gamma^{km}\gamma_{mk}) + \frac{1}{2}\partial_l(\gamma^{lm}\gamma_{mi}) - \frac{1}{2}\partial_k(\gamma_{im}\gamma^{mk}) - \gamma^{lk}\partial_i\gamma_{lk} - \gamma^{lk}\partial_l\gamma_{ki} + \gamma^{lk}\partial_k\gamma_{il}\right] \end{aligned} \tag{3.106}$$

were we use the traceless property of the GW amplitude γ The partial derivative with respect to the index j is then given by the following expression:

$$\partial^{(3)}_j\Gamma^l_{il} = \frac{1}{2}\left[\frac{1}{2}\partial_j\partial_i(\gamma^{km}\gamma_{km}) + \frac{1}{2}\partial_j\partial_l(\gamma^{lm}\gamma_{mi} - \frac{1}{2}\partial_j\partial_k(\gamma_{im}\gamma^{mk}) - \partial_j(\gamma^{lk}\partial_i\gamma_{lk}) - \partial_j(\gamma^{lk}\partial_l\gamma_{ki}) + \partial_j(\gamma^{lk}\partial_k\gamma_{il})\right] \tag{3.107}$$

Rearranging the various terms and using the traceless property of γ it is possible to show that

$$\partial^{(3)}_j\Gamma^l_{il} = \partial_j B_i \tag{3.108}$$

with

$$B_i = \frac{1}{4}\left[\gamma_{km}\partial_i\gamma^{km} - \gamma^{km}\partial_i\gamma_{km} - \gamma_{im}\partial_k\gamma^{mk}\right] \tag{3.109}$$

which is a second order quantity.

The term ${}^{(3)}\Gamma^l_{lk}{}^{(3)}\Gamma^k_{ij}$ does not contribute at second order. In fact ${}^{(3)}\Gamma^l_{lk} \sim \mathcal{O}(\gamma^2)$ and ${}^{(3)}\Gamma^k_{ij} \sim \mathcal{O}(\gamma)$ so the product of the two will be a quantity of $\mathcal{O}(\gamma^3)$.

The last term in the expression of the spatial Ricci tensor is easier to compute and reads

$$\begin{aligned} {}^{(3)}\Gamma^l_{jk}{}^{(3)}\Gamma^k_{il} &\approx \frac{1}{4}(\partial_j\gamma^l_k + \partial_k\gamma^l_j - \partial^l\gamma_{jk})(\partial_i\gamma^k_l + \partial_l\gamma^k_i - \partial^k\gamma_{il}) \\ &= \frac{1}{4}(\partial_j\gamma^l_k\partial_i\gamma^k_l - \partial_k\gamma^l_j\partial^k\gamma_{il} - \partial^l\gamma_{jk}\partial_l\gamma^k_i) + (\text{total derivative}) \end{aligned} \tag{3.110}$$

where the total derivative terms are $\mathcal{O}(\gamma^2)$. We can now evaluate the scalar curvature ${}^{(3)}R$ evaluated using the spatial metric by contracting the Ricci tensor ${}^{(3)}R_{ij}$ with the inverse spatial metric h^{ij}. Since we are interested in expanding the action up to second order in γ and thus only in an expression of ${}^{(3)}R$ up to second order, we just need to expand the inverse metric up to first order. We then obtain the expression for the curvature scalar (3.111).

$$\begin{aligned} {}^{(3)}R &\approx \frac{1}{a^2}(\delta^{ij} - \gamma^{ij})\left[-\frac{1}{2}\partial^2\gamma_{ij} - \frac{1}{4}(\partial_j\gamma^l_k\partial_i\gamma^k_l - \partial_k\gamma^l_j\partial^k\gamma_{il} - \partial^l\gamma_{jk}\partial_l\gamma^k_i) + (\text{total derivative})\right] \\ &= \frac{1}{a^2}\left[\frac{1}{2}\gamma^{ij}\partial^2\gamma_{ij} + \frac{1}{4}\partial_i\gamma^{lk}\partial^i\gamma_{lk}\right] + (\text{total derivative}) \end{aligned} \tag{3.111}$$

Notice that the total derivative term, being already of second order, it is only multiplied by the Dirac delta insuring that it is still a total derivative.

We now compute the $E_{ij}E^{ij} - E^2$ in the action. Since we set to zero all the scalar perturbations, the extrinsic curvature[7] takes the form

$$E_{ij} = \frac{1}{2}\dot{h}_{ij} = \frac{1}{2}\left[2a\dot{a}\left(\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{il}\gamma^l_j\right) + a^2\left(\dot{\gamma}_{ij} + \frac{1}{2}\dot{\gamma}_{il}\gamma^l_j + \frac{1}{2}\gamma_{il}\dot{\gamma}^l_j\right)\right]; \tag{3.112}$$

[7] Since $E_{ij} = NK_{ij}$ and $N = 1$ at fist order and neglecting scalar perturbations, the extrinsic curvature and E_{ij} tensor are equivalent.

its fully covariant version is given by

$$E^{ij} = h^{il}h^{jm} = \Big(\frac{\dot a}{a^3}\Big)\Big(\delta^{ij} - \gamma^{ij} + \frac{1}{2}\gamma^{im}\gamma^{j}_{m}\Big) + \frac{1}{2}a^{-2}\dot\gamma^{ij} - \frac{1}{4}a^{-2}\dot\gamma^{i}_{m}\gamma^{jm} - \frac{1}{4}a^{-2}\gamma^{i}_{m}\dot\gamma^{jm}. \tag{3.113}$$

and the product of the two reads

$$\begin{aligned} E_{ij}E^{ij} &= \Big[a\dot a\Big(\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{il}\gamma^{l}_{j}\Big) + a^2\Big(\frac{1}{2}\dot\gamma_{ij} + \frac{1}{4}\dot\gamma_{il}\gamma^{l}_{j} + \frac{1}{4}\gamma_{il}\dot\gamma^{l}_{j}\Big)\Big] \\ &\times \Big[\Big(\frac{\dot a}{a^3}\Big)\Big(\delta^{ij} - \gamma^{ij} + \frac{1}{2}\gamma^{im}\gamma^{j}_{m}\Big) + a^{-2}\Big(\frac{1}{2}\dot\gamma^{ij} - \frac{1}{4}\dot\gamma^{i}_{m}\gamma^{jm} - \frac{1}{4}\gamma^{i}_{m}\dot\gamma^{jm}\Big)\Big] \\ &\approx 3H^2 + \frac{1}{4}\dot\gamma_{ij}\dot\gamma^{ij} \end{aligned} \tag{3.114}$$

By contracting (3.112) with the spatial metric we get

$$E = h^{ij}E_{ij} = a^{-2}\Big(\delta^{ij} - \gamma^{ij} + \frac{1}{2}\gamma^{il}\gamma^{j}_{l}\Big)\Big[a\dot a\Big(\delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{il}\gamma^{l}_{j}\Big) + a^2\Big(\frac{1}{2}\dot\gamma_{ij} + \frac{1}{4}\dot\gamma_{il}\gamma^{l}_{j} + \frac{1}{4}\gamma_{il}\dot\gamma^{l}_{j}\Big)\Big] \approx 3H \tag{3.115}$$

and finally

$$E_{ij}E^{ij} - E^2 = \frac{1}{4}\dot\gamma_{ij}\dot\gamma^{ij} - 6H^2. \tag{3.116}$$

We are now ready to compute the second order action for tensor perturbations. The tensor action up to second order is given by

$$\begin{aligned} S_\gamma &= \frac{1}{2}\int dtd^3x\sqrt{h}\Big[N^{(3)}R + N^{-1}\big(E_{ij}E^{ij} - E^2\big)\Big] \\ &= \frac{1}{2}\int dtd^3xa^3\Big[\frac{1}{2}a^{-2}\gamma^{ij}\partial^2\gamma_{ij} + \frac{1}{4}a^{-2}\partial_l\gamma^{ij}\partial^l\gamma_{ij} + \frac{1}{4}\dot\gamma_{ij}\dot\gamma^{ij} - 6H^2 + (\text{total derivative})\Big]. \end{aligned} \tag{3.117}$$

Keeping only the second order contribution, integrating by parts and neglecting the total derivative contributions, the second order tensor action can be recast as follow [7]:

$$S^{(2)}_\gamma = \frac{1}{8}\int \mathrm{d}t\mathrm{d}^3xa^3\Big[\dot\gamma_{ij}\dot\gamma^{ij} - a^{-2}\partial_l\gamma^{ij}\partial^l\gamma_{ij}\Big]. \tag{3.118}$$

3.8.2 Equation of motion of tensor perturbations

At this point, in analogy to what done in Sec. 3.7.2, we pass to Fourier space:

$$\gamma_{ij} = \sqrt{2}\sum_{\lambda=+,\times}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\phi_\lambda(\eta,\mathbf{k})\epsilon^{\lambda}_{ij}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \tag{3.119}$$

where ϕ_S are the mode function associated to each of the two polarization states. ϵ_{ij} is the polarization tensor, which is symmetric, $\epsilon_{ij} = \epsilon_{ji}$, transverse, $k_i\epsilon_{ij} = 0$, and traceless, $\epsilon^{i}_{i} = 0$. To canonically normalize the action, we define the auxiliary field $\mu_\mathbf{k} = a\phi_\mathbf{k}$ and we quantize it. Implementing this, for each of the two polarization states $\lambda = +, \times$, the mode functions μ_k of the field $\mu_\mathbf{k}$ satisfy the equation of motion (3.120).

$$\mu''_k + \Big(1 - \frac{a''}{a}\Big)\mu_k = 0 \tag{3.120}$$

This equation is analogous to (3.85) except for the effective mass term, which is exactly

$$\frac{a''}{a} = (aH)^2(2 - \epsilon_1). \tag{3.121}$$

Since $aH = -1/\eta(1 - \epsilon_1)$ up to first order in slow-roll approximation, we can express the effective mass term in terms of the conformal time η, as

$$\frac{a''}{a} = \frac{1}{\eta^2}(2 + 3\epsilon_1) = \frac{1}{\eta^2}\Big(\nu^2 - \frac{1}{4}\Big) \tag{3.122}$$

with $\nu_T = 3/2 + \delta_T = 3/2 + \epsilon_1$. Note that is different form the ν parameter introduced in the equation of motion for the Sasaki-Mukhanov variable. At this point, the e.o.m. for the variable μ_k is

$$\mu_k'' + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\eta^2}\right)\mu_k = 0 \tag{3.123}$$

The same consideration regarding the sub-horizon and super-horizon regime made for the scalar case hold also here, and the general solution for (3.123) is analogous to (3.95) with $c_2' = 0$ and $c_1' = (\pi/4k)^{1/2} e^{i\pi(2\nu+1)/4}$.

3.8.3 Power spectrum

Taking into account that $v^\lambda = \sqrt{2}\phi^\lambda/a$, with v^λ denoting the mode functions associated with one graviton polarization state, and restoring the Planck mass $M_{\rm Pl}$the dimensionless power spectrum for the single polarization reads [36]:

$$\Delta_{\gamma_{+,\times}}(k) = \frac{8}{\pi}\frac{H^2}{M_{\rm Pl}^2}\left(\frac{k}{aH}\right)^{n_T - 1} \tag{3.124}$$

with $n_T - 1 = 3 - 2\nu_T = 2\epsilon_1$. As well as the scalar power, the tensor power spectrum appears to be red tilted too, with a deviation from scale invariance of the order of the first slow-roll parameter. If we sum up the contribution form the two polarizations, we define the total tensor spectral power

$$\Delta_T = \frac{16}{\pi}\frac{H^2}{M_{\rm Pl}^2}\left(\frac{k}{aH}\right)^{n_T - 1} \tag{3.125}$$

Comparing the spectral amplitude of (3.125) to the one of (3.101), we see that the following relation holds:

$$r \equiv \frac{\Delta_T(k_*)}{\Delta_\zeta(k_*)} = 16\epsilon_1. \tag{3.126}$$

with k_* a fixed pivot scale. Since $n_T = 2\epsilon_1$, we can generalize it to $r = 8n_T$. This relation is extremely important and it is known as *consistency relation*: it relates the ratio between the amplitude of the tensor power to the one of the scalar power, both of them evaluated at the same pivot scale k_*, and it is a unique prediction of single-field. Once gravitational wave will be detected and the spectral power will be measured, the validity of such relation will establish the presence of an inflationary mechanism acting on the early universe, pointing towards single-field models of inflation.

Chapter 4

Ultra slow-roll Inflation

In the previous chapter, we discussed the simplest model of inflation, single field slow-roll inflation, and its observational imprints. In particular, this model is in very good agreement with observations: the Planck TT , TE, and EE spectra are accurately described by purely adiabatic spectrum of fluctuations with a spectral tilt $n_{\mathrm{s}} = 0.968 \pm 0.006$, consistent with the predictions of single-field inflationary model [8, 9, 10]. Moreover, the spectral amplitude is constrained to be $\mathcal{A}_{\mathrm{S}} \simeq 2 \times 10^{-9}$. However, these measures constrain the spectral power only at very large scales, and no data forbid it to deviate strongly from the scale invariant regime at smaller scales.

This means that we can consider models of inflation in which slow-roll conditions are violated for a short epoch. This is the case for USR inflation, during which the second slow-roll parameter becomes $\mathcal{O}(1)$, $\epsilon_2 \simeq -6$, invalidating the slow-roll conditions [12, 13, 14, 15]. A brief phase of USR can be realized by inflationary potentials admitting a point of inflection, and they are motivated by string theory and supergravity theories[18, 17]. The presence of a brief USR phase sandwiched between two SR epochs leads to distinctive observational imprints, the most striking being the enhancement of the scalar power, which reaches an amplitude at the peak $\mathcal{A}_{\mathrm{S}} \simeq \mathcal{O}(10^{-2} - 10^{-1})$ at certain scales [45, 17, 46, 18, 47]. These models are of significant interest among cosmologists since they are related to production of a significant amount of primordial black holes (PBH), which form as the enhanced scales reenter the Hubble horizon during radiation domination [21]. In addition, the scalar-induced gravitational waves (SIGW), tensor perturbations induced by second order scalar perturbations, generated by such models can have a strength which is much larger than the amplitude of primary GWs. For certain value for the onset and duration of USR, the dimensionless spectral density of secondary GWs falls inside the sensitivity region of different ongoing and forthcoming observatories, giving a possibility to test these inflationary scenarios [17].

In this chapter we shall briefly review the USR dynamics and its implications for the evolution of scalar perturbations.

4.1 Background dynamics

The dynamics of the inflaton field is governed by the Klein-Gordon equation [36]

$$\ddot{\phi}(t) + 3H\dot{\phi} + V_{,\phi} = 0. \tag{4.1}$$

Assume that the inflaton field traverses a region of the potential which is exactly flat, then $V_{,\phi} = 0$, implying $\ddot{\phi}(t) = -3H\dot{\phi}(t)$ and $\dot{\phi} \propto a^{-3}$. The first and second slow-roll parameters than read

$$\begin{aligned} &\epsilon_1 \propto a^{-6}, \\ &\epsilon_2 \propto 2\epsilon_1 + 2\frac{\ddot{\phi}}{H\dot{\phi}} = 2\epsilon_1 - 6 \approx -6 \end{aligned} \tag{4.2}$$

where in the second passage of the second equation we substituted $\ddot{\phi}$ using the Klein Gordon e.o.m. As we can notice, during USR the first slow-roll parameter ϵ_1 decreases extremely rapidly due to a rapid slowdown of the inflaton velocity, which causes the ϵ_2 to drop to -6 and to violate slow-roll conditions.

USR alone cannot be a plausible inflationary scenario since the perturbations that this model generates fail the data validation: in order to match the power spectrum at CMB scales, we need to require $H/M_{\rm Pl}$ of the order of the energy scale of the nucleosynthesis (MeV). This epoch alone does not have the required e-folds to solve the horizon problem. However, brief[1] phases of USR (one or more) sandwiched between SR epochs can be in agreement with CMB observations.

In the following we shall introduce the model of interest for this thesis and its consequences on scalar and tensor fluctuations. The model consists of a triparted inflationary scenario: two phases of slow-roll inflation are separated by an intermediate USR phase for $\eta_1 < \eta < \eta_2$, $\eta_1 < \eta_2 < 0$. Evidently, ϵ_1 and ϵ_2 denote the time at the beginning and at the end of USR. Instead of deriving the e.o.m. for perturbations from we work with background quantities: the effects of a potential is accounted in the slow-roll parameters. Certainly, slow-parameters are related to the shape of a specific potential. However, deriving the background dynamics from physical potential admitting a brief phase of violation of slow-roll condition necessitate a numerical approach once we want to study fluctuations and their correlators. Yet, we know exactly how slow-roll parameters behave in both SR and USR phases: we work only with background quantities and keep the potential implicit. To simplify the calculation further and perform calculations completely analytically, we approximate the SR/USR/SR transition as instantaneous. This approach has the advantage that calculations can be performed completely analytically.

From (4.2), the first slow-roll parameter evolve as

$$\epsilon_1(\eta) = \begin{cases} \epsilon_{1i} & \text{for } \eta < \eta_1 \\ \epsilon_{1i}(\eta/\eta_1)^6 & \text{for } \eta_1 < \eta < \eta_2 \\ \epsilon_{1i}(\eta_2/\eta_1)^6 = \epsilon_{1f} & \text{for } \eta > \eta_2 \ . \end{cases} \tag{4.3}$$

and $\epsilon_2 = -6$ during USR, while it much smaller than unity during SR phases.

4.2 Enhancement of scalar fluctuations

As we saw in the previous chapter, scalar perturbations are parametrized by the Mukhanov-Sasaki variable u_k since it is canonically normalized field and its evolution is governed by the Mukhanov-Sasaki equation [49]:

$$u_k'' + \left(k^2 - \frac{z''}{z}\right)u_k = 0, \tag{4.4}$$

whose solution satisfies the Wronskian condition

$$u_k'^* u_k - u_k' u_k^* = i. \tag{4.5}$$

In the previous equations $z = aM_{\rm Pl}\sqrt{2\epsilon_1}$. This variable is related to the gauge invariant curvature perturbation ζ as $f_k(\eta) = u_k(\eta)/z(\eta)$, where f_k is the mode function associated to ζ. The quantity z''/z can be expressed in terms of the slow-roll parameters in the following way:

$$\frac{z''}{z} = a^2H^2\left(2 - \epsilon_1 + \frac{3\epsilon_2}{2} + \frac{\epsilon_2^2}{4} - \frac{\epsilon_1\epsilon_2}{2} + \frac{\epsilon_2\epsilon_3}{2}\right). \tag{4.6}$$

Conversely to the SR phases, during USR the parameter $\epsilon_2 = -6$, which leads to the violation of slow-roll conditions. Due to the presence of three different phases during inflation, we solve the equation of motion for the Mukhanov-Sasaki variable separately at each stage, subsequently matching conditions on u_k and its derivative at transition times η_1 and η_2. Moreover, form now on, we will approximate

[1] From the perturbative bound on the power spectrum $\Delta_\zeta(k) < 1$, since $\Delta_\zeta \propto \exp(6\Delta N)$ (ΔN denoting the duration in e-folds), the duration is restricted to be less than 3 e-folds[48].

the universe and exact de Sitter and perturbations as massless. In slow-roll approximation and exact de Sitter, the effective mass is given by

$$\frac{z''}{z} = \frac{2}{\eta^2}. \tag{4.7}$$

The same holds also for USR because of Wands duality condition [50]. This duality relates two background conditions to the same cosmological perturbation spectrum. For both SR and USR, the solution to the e.o.m. (4.4) is proportional to combinations of Hankel function of the first and second kind with index $\nu = 3/2$. The solution of Mukhanov-Sasaki equation of motion in the first SR phase leads to (4.8), upon imposing Bunch-Davies vacuum conditions at very early times.

$$u_k^{\mathrm{I}} = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta} \tag{4.8}$$

In the USR and second SR phase, the vacuum condition is not imposed, so u_k retains the dependence on both positive and negative frequency modes:

$$u_k^{\mathrm{II}} = \frac{\gamma_k}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta} + \frac{\delta_k}{\sqrt{2k}}\left(1 + \frac{i}{k\eta}\right)e^{ik\eta}, \tag{4.9}$$

$$u_k^{\mathrm{III}} = \frac{\alpha_k}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta} + \frac{\beta_k}{\sqrt{2k}}\left(1 + \frac{i}{k\eta}\right)e^{ik\eta}, \tag{4.10}$$

$$\tag{4.11}$$

with γ_k, δ_k, α_k and β_k Bogoljubov coefficients to be determined by matching conditions at transition times.

At the transition times η_1 and η_2 the Mukhanov-Sasaki variable must be continuous:

$$u_k^{\mathrm{II}}(\eta_1) = u_k^{\mathrm{I}}(\eta_1), \qquad u_k^{\mathrm{III}}(\eta_2) = u_k^{\mathrm{II}}(\eta_2). \tag{4.12}$$

However, this is not the case for the first derivative. The effective mass term (4.6) at the transition is dominated by the $\epsilon_2\epsilon_3$ term since the second slow-roll parameter changes abruptly: ϵ_2 becomes -6 in USR, though being much smaller than one in the SR phases. Thus, assuming an instantaneous transition (for the first transition)[46]:

$$\frac{z''}{z} = -\frac{\epsilon_2^{\mathrm{II}} - \epsilon_2^{\mathrm{I}}}{2\eta}\delta(\eta - \eta_1). \tag{4.13}$$

Defining $\Delta\epsilon_2 = \epsilon_2^{\mathrm{II}} - \epsilon_2^{\mathrm{I}} = -6 = \epsilon_2^{\mathrm{II}} - \epsilon_2^{\mathrm{III}}$, from the Mukhanov-Sasaki equation the second derivative of u_k is

$$u_k'' = \frac{\Delta\epsilon_2}{2\eta}\delta(\eta - \eta_1)u_k \tag{4.14}$$

since the Dirac delta term is dominating over k^2 sufficiently close to transition times. By integrating both sides over the time interval around η_1, we find conditions on the first time derivative of the Mukhanov-Sasaki variable:

$${u^{\mathrm{II}}}'_k(\eta_1) = {u^{\mathrm{I}}}'_k(\eta_1) + \frac{\Delta\epsilon_2}{2\eta_1}u_k^{\mathrm{I}}(\eta_1), \qquad {u^{\mathrm{III}}}'_k(\eta_2) = {u^{\mathrm{II}}}'_k(\eta_2) - \frac{\Delta\epsilon_2}{2\eta_2}u_k^{\mathrm{II}}(\eta_2). \tag{4.15}$$

Solving equations (4.12) and (4.15) together leads to the expression for the Bogoljubov coefficients, that read

$$\gamma_k = 1 + \frac{3i}{2k\eta_1} + \frac{3i}{2k^3\eta_1^3}, \tag{4.16}$$

$$\delta_k = \left(-\frac{3i}{2k\eta_1} - \frac{3}{k^2\eta_1^2} + \frac{3i}{2k^3\eta_1^3}\right)e^{-2ik\eta_1}, \tag{4.17}$$

$$\alpha_k = \left(1 - \frac{3i}{2k\eta_2} - \frac{3i}{2k^3\eta_2^3}\right)\gamma_k - \left(\frac{3i}{2k\eta_2} - \frac{3}{k^2\eta_2^2} - \frac{3i}{2k^3\eta_2^3}\right)\delta_k e^{2ik\eta_2}, \tag{4.18}$$

$$\beta_k = \left(1 + \frac{3i}{2k\eta_2} + \frac{3i}{2k^3\eta_2^3}\right)\delta_k + \left(\frac{3i}{2k\eta_2} + \frac{3}{k^2\eta_2^2} - \frac{3i}{2k^3\eta_2^3}\right)\gamma_k e^{-2ik\eta_2}. \tag{4.19}$$

In this section we will show how a briefepoch of ultra slow-roll sliced between two slow roll inflationary phases can produce an enhancement of the power spectrum at scales larger than those constrained by CMB data. Following [46], we model the different inflationary phases using the behavior of the first slow-roll parameter. In this way, the argument is completely general and independent from a specific inflationary potential. Moreover, this approach allows us to carry out computations analytically.

Recall that ϵ_1 evolves as (4.2) and that parameters η_1 and η_2 denote the conformal times at the beginning and end of the USR phase respectively. Notice that during the SR epochs the parameter ϵ_1 small and constant at first order in slow-roll. After USR, ϵ_1 is still constant over time, but its value is significantly smaller than what it was during the first SR phase.

$$\left(\frac{\eta_2}{\eta_1}\right)^6 = \left(\frac{a_1}{a_2}\right)^6 = e^{-6\Delta N}$$

with ΔN the duration of USR in number of e-folds, $\Delta N = \int_{\eta_1}^{\eta_2} H dt$. Since in the power spectrum of ζ an inverse power of ϵ_1 appears, it is clear that its sharp decrease will cause the spectral power to be amplified. We can see this in more detail. According to the relation between the Sasaki-Mukhanov variable u and the curvature perturbation ζ, and given that the former can be written as (4.8), (4.9) and (4.10) in the various inflationary phases, the mode function for the curvature perturbation ζ is

$$f_p(\eta) = \frac{iH}{2M_{pl}\sqrt{p^3\epsilon_{1i}}} \times F_p(\eta) \tag{4.20}$$

with

$$F_p(\eta) = \begin{cases} (1+ip\eta)e^{-ip\eta} & \text{for } \eta < \eta_1 \\ (\eta_1/\eta)^3\big[\gamma_p(1+ip\eta)e^{-ip\eta} - \delta_p(1-ip\eta)e^{ip\eta}\big] & \text{for } \eta_1 < \eta < \eta_2\ . \\ (\eta_1/\eta_2)^3\big[\alpha_p(1+ip\eta)e^{-ip\eta} - \beta_p(1-ip\eta)e^{ip\eta}\big] & \text{for } \eta > \eta_2 \end{cases} \tag{4.21}$$

From the definition of adimensional (3.24) and dimensionful (3.30) power spectrum, in case a brief phase of USR occurs between two SR epochs,

$$\Delta_\zeta \equiv \frac{k^3}{2\pi^2}|f_k|^2_{\eta\to 0} = \frac{H^2}{8\pi^2 M_{\rm Pl}^2 \epsilon_{1_f}}|\alpha_k - \beta_k|^2 \tag{4.22}$$

and we analyze both its large and the small-scale regime.

The large-scale limit is achieved when the wavenumber k associated to a given mode function f_k, is much smaller than $k_1 = -1/\eta_1$. This is indeed equal to require that the physical length-scales are much larger than the one associated to modes exiting the horizon at the transition, since $\lambda_{phys} = a\lambda_{com} = 2\pi a/k$. Since $k \ll k_1 < k_2$, if $k/k_1 \to 0$ is satisfied, than also $k/k_2 \to 0$ holds. Taking the large-scale limit of the Bogoljubov coefficients α_k and β_k leads to divergencies:

$$\alpha_k = \left(-\frac{3}{5}ie^{3\Delta N} + \frac{3}{5}ie^{-2\Delta N}\right)\frac{1}{x} + \frac{e^{-3\Delta N}}{2} + \frac{e^{3\Delta N}}{2} + O(x), \tag{4.23}$$

$$\beta_k = \left(-\frac{3}{5}ie^{3\Delta N} + \frac{3}{5}ie^{-2\Delta N}\right)\frac{1}{x} - \frac{e^{-3\Delta N}}{2} + \frac{e^{3\Delta N}}{2} + O(x) \tag{4.24}$$

where $x = k/k_1$ and $k/k_2 = e^{\Delta N}x$. However, the power depends on the difference $\alpha_k - \beta_k$ which is finite and independent on the wavenumber k at leading order in x:

$$\alpha_k - \beta_k = e^{-3\Delta N} + O(x^2). \tag{4.25}$$

In this regime the adimensional power reduces to

$$\Delta_\zeta(k) \simeq \frac{H^2}{8\pi^2 M_{\rm Pl}^2 \epsilon_{1i}} \tag{4.26}$$

once we neglect higher order terms. This is exactly a scale invariant regime and in this limit the spectral power is analogous to that derived in case of slow-roll inflation.

The opposite regime, $x \to \infty$, corresponds to consider the behavior of modes whose wavenumber is much larger than the ones associated to the transition times η_1 and η_2, $k \gg k_2 > k_1$. In this limit the α_k and β_k coefficients reduce to unity and zero respectively. Therefore,

$$\Delta_\zeta(k) \simeq \frac{H^2}{8\pi^2 M_{\rm Pl}^2 \epsilon_{1f}} \tag{4.27}$$

which is amplified by a factor of $e^{6\Delta N}$ w.r.t. the spectral power at large scales. These behaviours suggest that these scales are largerly not affected by USR dynamics, expected for the change in the value of ϵ_1. Very large scales correspond to modes which left the horizon deep during the first SR epoch: the scalar power spectrum in that limit is exactly the one obtained in slow-roll inflation. On large scales, the power spectrum reaches scale invariant contribution again, since the contributors to the power spectra is obtained by the modes that are leaving the horizon during the second SR phase, when the scale invariance is restored again. The power spectrum is portrayed in its full glory in Fig.4.1: for scales $k \lesssim k_1 < k_2$, initially it gentle decreases as k increases, then, it features a dip associated to modes that leave horizon still during the first SR era. The presence of such a dip is a characteristic feature of USR model of inflation it has been shown to be present in *all* the primordial correlators at the same wavenumber $k_{dip} \simeq \sqrt{3}k_1 \exp(-3\Delta N/2)$[46], and can lead to interesting observational consequences [51]. The dip occurs because the mode function describing curvature perturbation corresponding to wavenumber k_{dip} goes to zero at late time close to the end of inflation. This is caused by a disruptive interference between the growing and decaying modes of the curvature fluctuation at super-horizon scale, since violations of SR conditions yields the decaying mode to dominate temporarily, eventually leading to a super-horizon evolution of ζ [52, 47]. For $k > k_{dip}$ the spectral power steadily increases as k^4 reaches its the peak and gradually settles to a scale invariant form again. The amplitude of the peaks depends exclusively on the duration of the USR phase, its location on the onset. Following the peak, the spectrum contains tiny oscillations persisting over small scales. These oscillations ar not physical, they rather an artifact of the sharp transition.

The tensor power spectrum is left untouched, as if inflation was purely slow-roll. Canonically normalized tensor modes μ_k satisfy an equation of the same type of (4.4), although with a different effective mass term:

$$\mu_k'' + \left(1 - \frac{a''}{a}\right)\mu_k = 0, \qquad \frac{a''}{a} = (aH)^2(2-\epsilon_1). \tag{4.28}$$

We work in exact de Sitter approximation, thus $a''/a = -2/\eta^2$ Since there is no term proportional to the derivative of slow-roll parameters, which are abruptly changing during the transitions, the mode function remains continuous across the three phases. Moreover, the amplification factor $e^{6\Delta N}$ is related to the negative power of ϵ_1 which do not appear in the expression for tensor modes.

A brief comment on the SR-USR-SR model that are considering: the sharp transition approximation is non-physical, since no potential arising from any physically reasonable high energy theory can generate an instantaneous transition between the two regimes. However, we employ it since it can significantly simplify the already tedious calculations. Moreover, the amplification of the spectral power does not depend on the smoothness of the transition. numerical studies show that an amplification of seven to eight orders of magnitude is achieved also if the transition is sufficiently smooth[17, 47]: as said before, it depends on the duration of USR ΔN. This approximation is often employed in the literature to get first estimates spectral power, non-Gaussianities and loop contribution in SR-USR-SR [46, 53].

We remark that in our analysis we fixed $\epsilon_2^{\rm III} = 0$. However, as noted in [54], in more realistic models of inflation $\epsilon_2^{\rm III} \neq 0$ and negative, so that the power spectrum decreases for modes $k \ll k_1$. This is indeed necessary if we want to connect USR with a phase of SR leading to the end of the inflationary phase. In fact, as $\epsilon_2^{\rm III} \approx 0$ is restored, $\epsilon_1^{\rm III}$ is forced to remain approximately constant and extremely small, leading the end of inflation impossible to reach[2].

[2]The reader should remember from Sec.2.2 that $\epsilon_1 \neq 0$ is a necessary condition to end of inflation. It is equivalent to impose that the kinetic energy of the inflaton is non-negligible w.r.t. the potential energy, a scenario in which an accelerated expansion is not possible to be realized.

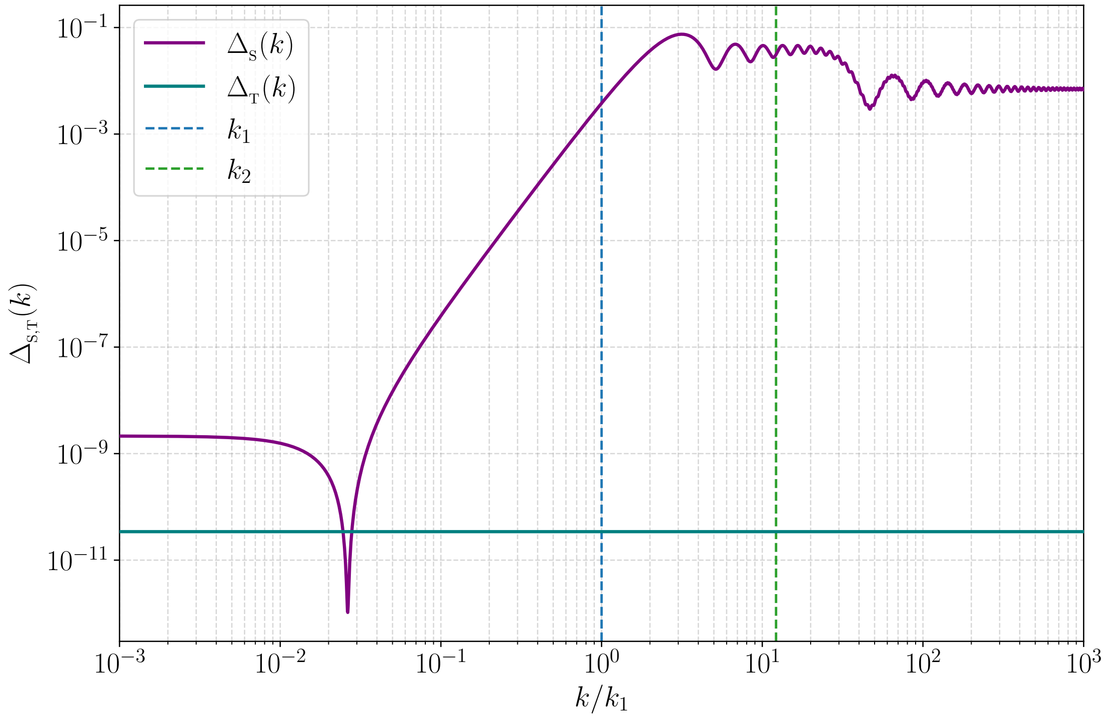

Figure 4.1: Tree level scalar and tensor power spectrum is displayed for the following values of parameters: $H = 1.5 \times 10^{-5} M_{\rm Pl}$, $\epsilon_{1,i} = 10^{-3}$, $\Delta N = 2.5$ and $k_1 = 10^{-4}{\rm Mpc}^{-1}$.

Chapter 5

Modifying general relativity

In general relativity, spacetime is regarded as a dynamical object rather than an arena in which the physics is played. For this reason an action for gravity is needed. Interaction among gravity and matter field are introduced as the actions for the various particle are modified according to the *principle of general covariance*: $\partial_\mu \to \nabla_\mu$. By doing so we still miss a kinetic term for the metric. This is provided by the Einstein-Hilbert action (5.1), which is built the *only* independent scalar constructed from the metric, which is no higher tha second order in its derivatives.

$$\mathcal{L}_{\mathrm{EH}} = -\frac{M_{\mathrm{Pl}}}{2}\sqrt{-g}R, \qquad\qquad g = \det\left[g_{\mu\nu}\right] \tag{5.1}$$

If we assume GR to be the actual description of gravity, even at very early times, then, if inflation is driven by only one scalar d.o.f ϕ, the Lagrangian of the very early universe reads:

$$\mathcal{L} = \sqrt{-g}\left[-\frac{M_{\mathrm{Pl}}}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{5.2}$$

Within this theory, the strength of primordial fluctuations, as observed in the CMB, indicates that the cosmic expansion rate H and the physical wavenumber k/a at horizon exit have the value $H = k/a \approx \sqrt{\epsilon_1} \times 10^{14}\,\mathrm{GeV}$, and therefore are likely to be much smaller than the Planck mass $M_{\mathrm{Pl}} \simeq 2.4 \times 10^{18}\,\mathrm{GeV}$, and the grand-unification scale $\approx 10^{16}\,\mathrm{GeV}$. However, H and k/a at horizon exit might not be completely negligible w.r.t. whatever fundamental energy scale M characterizing the physics of inflation. Moreover k/a at very early times is exponentially larger than at horizon exit. We can assume that (5.2) is the first term in a generic effective field theory, in which terms with higher derivatives are suppressed by negative powers of the characteristic mass M, with coefficients roughly of order unity [55]. The considerations above emphasize the necessity of going beyond leading order terms and seek for corrections of (5.2) with higher number of derivatives.

The first correction to the Lagrangian (5.1) is obtained by accounting for the most general covariant terms containing four derivatives of the metric. We can construct these term by contracting two tensors or scalars involving two derivatives: the Riemann tensor $R_{\mu\nu\rho\sigma}$, the Ricci tensor $R_{\mu\nu}$, and Ricci curvature R. Then, the most general expression for the higher derivative Lagrangian is [56]:

$$\Delta\mathcal{L}_{\mathrm{EH}} = \sqrt{-g}\left(f_1 R^2 + f_2 R_{\mu\nu}R^{\mu\nu} + f_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right) + f_4 \epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\tau\kappa}R^{\tau\kappa}_{\rho\sigma} \tag{5.3}$$

with $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor density with $\epsilon^{1230} \equiv +1$, and f_n are dimensionless coefficients.

If we also introduce a scalar field ϕ which plays the role of the inflaton field we should include also covariant terms up to four derivatives of the inflaton field itself. Including these corrections to (5.2) is:

$$
\begin{aligned}
\mathcal{L} = \sqrt{-g}\Bigg[&\frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\\
&+ f_1(\phi)\big(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\big)^2 + f_2(\phi)g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\,\Box\phi + f_3(\phi)\big(\Box\phi\big)^2\\
&+ f_4(\phi)R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + f_5(\phi)R\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + f_6(\phi)\,R\Box\phi\\
&+ f_7(\phi)R^2 + f_8(\phi)R^{\mu\nu}R_{\mu\nu} + f_9(\phi)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Bigg]\\
&+ f_{10}(\phi)\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\tau\kappa}R^{\tau\kappa}_{\rho\sigma}.
\end{aligned}
\tag{5.4}
$$

However, if we were to take (5.4) as the full Lagrangian describing inflation, we would have a proliferation of dynamical auxiliary fields. Thus, we must eliminate all second time derivatives and time derivative of the auxiliary field in the first correction of the EFT action by using the field equations derived from (5.2). Doing so, we end up with

$$
\begin{aligned}
\mathcal{L} = \sqrt{-g}\Bigg[&\frac{M_{\rm Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\\
&+ f_1(\phi)\big(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\big)^2 + f_7(\phi)R^2 + f_8(\phi)R^{\mu\nu}R_{\mu\nu} + f_9(\phi)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\Bigg] + f_{10}(\phi)\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\tau\kappa}R^{\tau\kappa}_{\rho\sigma}
\end{aligned}
\tag{5.5}
$$

The term multiplying $f_1(\phi)$ is of the type encountered in k-inflation [57] and must be included in the Lagrangian as a counterterm to the UV divergencies encountered once the leading term in (5.2) are used in loop orders. The second term $f_7(\phi)R^2 + f_8(\phi)R^{\mu\nu}R_{\mu\nu} + f_9(\phi)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ in the second line is a Gauss-Bonnet term which affects the propagation of scalar modes. The last term, $f_{10}(\phi)\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\tau\kappa}R^{\tau\kappa}_{\rho\sigma}$ is of great interest for us, since it is responsible to violation of parity symmetry in the gravitational sector: it has the form of a gravitational Chern-Simons term. The coefficients of the terms above need to be fixed by observations.

5.1 Chern-Simons gravity

Chern-Simons (CS) modified gravity is a four-dimensional deformation of GR postulated by Jackiw and Pi [58] that modifies conventional dynamics and kinematics in a Lorentz invariant and CPT violating way. Not only it is justified from an EFT perspective of inflation, but also arises from Green-Schwarz anomaly canceling mechanism in heterotic string theory [59].

In the case that Chern-Simons coupling is the only[1] one modification to standard GR, the action during inflation is then given by

$$S = S_{\rm EH} + S_\phi + S_{\rm matter} + \frac{1}{16}\int {\rm d}^4x\sqrt{-g}\omega(\phi)R\tilde{R}. \tag{5.6}$$

The quantity $R\tilde{R}$ is the Pontryagin density and is defined by

$$R\tilde{R} \equiv \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}R_{\alpha\beta\rho\sigma}R_{\gamma\delta}{}^{\rho\sigma} \tag{5.7}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the 4-dimensional Levi-Civita tensor related to the Levi-Civita symbol by $\epsilon^{0ijk} = \tilde{\epsilon}^{ijk}/\sqrt{-g}$. The coupling $\omega(\phi)$ is a function of the inflaton field; if ω is a constant, then CS gravity reduces to GR. This is because the Potryagin term can be expressed as the divergence

$$\nabla_\mu K^\mu = \frac{1}{2}R\tilde{R} \tag{5.8}$$

[1]As explained above, Chern-Simons is not expected to be the only one correction to (5.2), since it arises from an EFT approach together with the Gauss-Bonnet term. However, we are interested in studying parity violating effects in cosmology which are sourced by the CS term. For this reason we consider only that modification to GR.

of the Chern-Simons topological current

$$K^\mu = \epsilon^{\mu\nu\alpha\beta}\Gamma^\sigma_{\nu\rho}\left(\partial_\alpha\Gamma^\rho_{\beta\sigma} + \frac{2}{3}\Gamma^\rho_{\alpha\epsilon}\Gamma^\epsilon_{\beta\sigma}\right). \tag{5.9}$$

The Chern-Simons action then becomes

$$\begin{aligned} S_{\rm CS} &= \frac{1}{8}\int \mathrm{d}^4x\sqrt{-g}\omega(\phi)\nabla_\mu K^\mu \\ &= \frac{1}{8}\int \mathrm{d}^4x\omega(\phi)\partial_\mu\big(\sqrt{-g}K^\mu\big) \\ &= -\frac{1}{8}\int \mathrm{d}^4x\sqrt{-g}\big(\partial_\mu\omega(\phi)\big)K^\mu + (\text{surface terms}). \end{aligned} \tag{5.10}$$

If ω is constant then its partial derivative is zero and $S_{\rm CS}$ is reduce to a surface term that is equal to zero under setting proper boundary conditions and the total action coincides with the standard GR one. Notice that Chern-Simons gravity could have left signatures only on the primordial perturbations. In fact, at the end of inflation the Inflaton decays and thus the Chern-Simons terms reduces to a surface term.

The additional CS term does not modify the background equations because FRW metric is invariant under parity transformation.However, it does effect primordial perturbations: at the linear order the CS term affects tensor perturbations only, whereas scalar perturbations are left untouched.Considering (5.10) and fixing the comoving gauge, the CS contribution to the total action is given by

$$\begin{aligned} S_{\rm CS} = -\frac{1}{8}\int \mathrm{d}^4x\sqrt{-g}\Big(\dot\omega(\phi)K^0 + \partial_i\omega(\phi)K^i\Big) &= -\frac{1}{8}\int \mathrm{d}^4x\sqrt{-g}\dot\omega(\phi)K^0 \\ &= -\frac{1}{8}\int \mathrm{d}^4x\sqrt{-g}\dot\omega(\phi)\epsilon^{0ijk}\Gamma^\sigma_{i\rho}\left(\partial_j\Gamma^\rho_{k\sigma} + \frac{2}{3}\Gamma^\rho_{j\lambda}\Gamma^\lambda_{k\sigma}\right) \end{aligned} \tag{5.11}$$

5.1.1 Constraint equation

Since the gravitational sector has been modified, it is needed to ask whether (3.49) and (3.50) get modified: it is crucial to determine if N and N^i become dynamical or not in order to understand if there are extra d.o.f. propagating. We firstly note that the solution at order zero is not modified: CS term vanishes when evaluated on the background metric due to its invariance under parity. Let us focus on first order solutions. To do so, we can proceed in two ways starting from (5.11). The first and more straightforward way is to explicitly compute the Chern-Simons topological current as a function of the ADM variables, variate the action with respect to the *lapse* function and the *shift* vector and sum the result to the constraint equation in GR. In this way we are able to find the modified-gravity constraints in a straightforward way, cumbersome though. However, we can proceed as in [31], noting that the Levi-Civita tensor multiplies the combination of Christoffel symbols and its derivatives. Since the ADM metric takes the form (3.39) and we are interested in solving Hamiltonian and momentum constraint up to first order, we should consider all the bilinears that can be constructed combining scalars N and ψ, solenoidal vector $\bar N^i$, tensor h_{ij} and their derivatives with the Levi-Civita tensor.

scalar-scalar bilinears

Scalar-scalar bilinears are built by taking two scalars multiplying the Levi-Civita tensor, whose indices are contracted with partial derivatives. Regarding the N scalar, the maximum number of derivatives that can appear is three; for ψ instead, it is six.[2] Since the indices of the Levi-Civita symbol cannot

[2] This is because K^0 involves two terms. The first consists of a partial derivative and two Christoffel symbols, the second of three Christoffels: the maximum number of derivatives appearing in bilinears built out of scalars is then three. However the ψ scalar is always accompanied by a partial derivative since $\partial_i\psi$ is the scalar part of the *shift* vector. It follows that for ψ, the maximum number of derivatives in a scalar bilinear involving it is doubled.

be contracted with themselves, the only possible terms are given by two scalars multiplying the Levi-Civita and three or five derivatives, three of them contracted with the totally antisymmetric symbol and the other two contracted among themselves. All the possible bilinears can be recast in this form[3]

$$S\partial_i\partial_j\partial_k S\tilde{\epsilon}^{ijk} \ , \qquad S\partial_i\partial_j\partial_k\partial^2 S\tilde{\epsilon}^{ijk} \tag{5.12}$$

where S denotes either N or ψ. These two terms are clearly zero: two symmetric indices (partial derivatives commute) are contracted in a totally antisymmetric way.

scalar-vector bilinears

Regarding scalar vector bilinears, we have two possibilities: either contracting the indices of the Levi-Civita symbol with the vector and two derivatives, everything multiplied by the scalar S

$$S\partial_i\partial_j\bar{N}_k\tilde{\epsilon}^{ijk}, \tag{5.13}$$

or having three derivatives contracted to the Levi-Civita multiplying the divergence of $\bar{N}_i$ and the scalar S

$$S\partial_i\partial_j\partial_k\partial_l\bar{N}^l\tilde{\epsilon}^{ijk}. \tag{5.14}$$

Both terms are zero because of the symmetry of the covariant indices and the fact that the vector $\bar{N}^i$ is solenoidal.

scalar-tensor bilinears

In this case we have three types of bilinears to be considered:

$$S\partial_i h_{jk}\tilde{\epsilon}^{ijk} \ , \qquad S\partial_i\partial_j\partial_l h^l_k\tilde{\epsilon}^{ijk} \ , \qquad S\partial_i\partial_j\partial_k\partial_l\partial_m h^{mn}\tilde{\epsilon}^{ijk}. \tag{5.15}$$

All of these terms above are zero either because of the symmetry of the spatial metric tensor h_{ij} or the commutativity of partial derivatives.

vector-vector bilinears

Since $\bar{N}^i$ is transversal, the only non vanishing vector-vector bilinear is given by

$$\partial_j\bar{N}_i\bar{N}_k\tilde{\epsilon}^{ijk}. \tag{5.16}$$

vector-tensor bilinears

Three types of bilinears can be written

$$\bar{N}_i h_{jk}\epsilon^{ijk} \ , \qquad h^l_i\partial_j\partial_k\bar{N}_l\tilde{\epsilon}^{ijk} \ , \qquad \partial_j\bar{N}_i\partial_l h^l_k\tilde{\epsilon}^{ijk} \tag{5.17}$$

all of them vanishing. The first one vanishes since h_{ij} is a symmetric tensor and it is contracted to the Levi-Civita symbol. The second one is zero since $h^l_i = \delta^l_i$ and the same is for the third one.

At this point it is clear that the e.o.m. for the lapse function N remains invariant with respect to the standard GR scenario. On the other hand, the e.o.m. for the irrotational part of the shift vector $\bar{N}_i$ gets modified and a linear term is added. Furthermore $\bar{N}_i$ is non dynamical since $\dot{\bar{N}}_i$ does not appear in (5.11) In fact

$$\begin{aligned} K^0 = \epsilon^{0ijk}\Bigg[&\Gamma^0_{i0}\big(\partial_j\Gamma^0_{k0} + \frac{2}{3}\Gamma^0_{i0}\Gamma^0_{k0} + \frac{2}{3}\Gamma^0_{im}\Gamma^m_{k0}\Big) + \Gamma^0_{in}\left(\partial_j\Gamma^n_{k0} + \frac{2}{3}\Gamma^n_{i0}\Gamma^0_{k0} + \frac{2}{3}\Gamma^n_{im}\Gamma^m_{k0}\right) \\ &+ \Gamma^m_{i0}\left(\partial_j\Gamma^0_{km} + \frac{2}{3}\Gamma^0_{i0}\Gamma^0_{km} + \frac{2}{3}\Gamma^0_{it}\Gamma^t_{km}\right) + \Gamma^m_{in}\left(\partial_j\Gamma^n_{km} + \frac{2}{3}\Gamma^n_{i0}\Gamma^0_{km} + \frac{2}{3}\Gamma^n_{it}\Gamma^t_{km}\right)\Bigg] \end{aligned} \tag{5.18}$$

and the only Christoffel symbol containing time derivatives of $\bar{N}_i$ is Γ^i_{00}.

[3]After multiple integration by part and neglecting total derivatives.

5.1.2 Tensor power spectrum

We are then interested in the computation of the tensor power spectrum in case GR is modified by the introduction of the CS correction. We work in a perturbed FRW spacetime using conformal time η

$$g_{\mu\nu} = a^2(\tau) \begin{pmatrix} -1 & 0 \\ 0 & \delta_{ij} + \gamma_{ij} \end{pmatrix} \tag{5.19}$$

with the graviton being both transverse and traceless. We follow [32] and we consider the metric written making use of the ADM formalism

$$\begin{aligned} ds^2 = g_{\mu\nu} dx^\mu dx^\nu &= -\big(N^2 - N^i N_i\big) dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j \\ &= -\big(N^2 - N^i N_i\big) a^2 d\eta^2 + 2N_i a dx^i d\eta + a^2 \tilde{h}_{ij} dx^i dx^j \end{aligned} \tag{5.20}$$

where we have factorized the scale factor a^2 out of the metric and then $\tilde{h}_{ij} = h_{ij}/a^2$. By defining $N_i = a\tilde{N}_i$ the metric becomes

$$ds^2 = -\big(N^2 - a^2 \tilde{N}^i \tilde{N}_i\big) a^2 d\eta^2 + 2\tilde{N}_i a^2 dx^i d\eta + a^2 \tilde{h}_{ij} dx^i dx^j \tag{5.21}$$

which is related to

$$ds^2 = g'_{\mu\nu} dx^\mu dx^\nu = -\big(N^2 - a^2 \tilde{N}^i \tilde{N}_i\big) d\eta^2 + 2\tilde{N}_i dx^i d\eta + \tilde{h}_{ij} dx^i dx^j \tag{5.22}$$

by the conformal transformation $g'_{\mu\nu} = a^{-2} g_{\mu\nu}$. Because of the fact that Pontryagin density is invariant under conformal transformation of the metric, we can evaluate it using the metric (5.22). As we saw in the previously, the Potryagin density is proportional to the divergence of the CS topological curvature (5.8) and the dynamical Chern-Simons action can be written as (5.11). In order to compute e.o.m. for tensor perturbations is then necessary to evaluate the zero component of the CS topological current K^μ. We recall that the CS current is schematically given by

$$K^\mu \sim \Gamma \partial \Gamma + \Gamma\Gamma\Gamma \tag{5.23}$$

and we observe that the zero*th* order Christoffel symbol $\Gamma^{(0)} = 0$ since the scale factor do not appear in the unperturbed $\tilde{g}_{\mu\nu}$ metric and it is equivalent to the Minkowski spacetime. For this reason, we just need to compute the Christoffel symbols only up to first order [32] in the conformally transformed metric. The first order Christoffel symbol is then generically given

$$\Gamma^{\mu(1)}_{\alpha\beta} = \frac{\big[g^{\mu\lambda}\big]^{(0)}}{2} \big[\big(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\lambda\alpha} - \partial_\lambda g_{\alpha\beta}\big)\big]^{(1)} + \text{(quadratic terms)} \tag{5.24}$$

where the superscripts refer to the order of each term in perturbation theory. As we noted in Sec.3.6, the first order solution of the constraints do not depend on γ. Thus, as we did in Sec.3.8, we set $N = 1$ and $N^i = 0$ if we are interested in the second order tensor action. We can now proceed to compute K^0. The first-order Christoffel symbols evaluated on the conformally transformed metric

$$g'_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & \delta_{ij} + \gamma_{ij} \end{pmatrix} \tag{5.25}$$

read

$$\begin{aligned} \Gamma^0_{00} &= 0 = \Gamma^0_{0i} = \Gamma^i_{00} \\ \Gamma^0_{ij} &= \frac{1}{2}\gamma'_{ij} \\ \Gamma^i_{j0} &= \frac{1}{2}\gamma'^i_j \\ \Gamma^i_{jk} &= \frac{1}{2}\big(\partial_j \gamma^i_k + \partial_k \gamma^i_j - \partial^i \gamma_{jk}\big) \end{aligned} \tag{5.26}$$

where ' represent the conformal time derivative. We can then derive the expression for K^0

$$
\begin{aligned}
K^0 &= \epsilon^{0ijk}\Gamma^{\sigma}_{i\rho}\left(\partial_j\Gamma^{\rho}_{k\sigma}+\frac{2}{3}\Gamma^{\rho}_{j\lambda}\Gamma^{\lambda}_{k\sigma}\right)=\epsilon^{0ijk}\Gamma^{\sigma}_{i\rho}\partial_j\Gamma^{\rho}_{k\sigma}+\text{(higher order)}\\
&\approx \epsilon^{0ijk}\left[\Gamma^{0}_{iz}\partial_j\Gamma^{z}_{k0}+\Gamma^{t}_{i0}\partial_j\Gamma^{0}_{kt}+\Gamma^{t}_{iz}\partial_j\Gamma^{z}_{kt}\right]\\
&= \epsilon^{0ijk}\left[\frac{1}{4}\gamma'_{in}\partial_j\gamma'^{n}_{k}+\frac{1}{4}\gamma'^{n}_{i}\partial_j\gamma'_{kn}+\frac{1}{4}\left(\partial_i\gamma^n_m+\partial_m\gamma^n_i-\partial^n\gamma_{im}\right)\left(\partial_j\partial_k\gamma^m_n+\partial_j\partial_n\gamma^m_k-\partial_j\partial^m\gamma_{kn}\right)\right]\\
&= \epsilon^{0ijk}\left[\frac{1}{2}\gamma'^{n}_{i}\partial_j\gamma'_{kn}-\frac{1}{2}\partial^n\gamma_{im}\partial_j\partial_n\gamma^m_k\right]
\end{aligned}
\tag{5.27}
$$

where we kept only terms up to second order, neglected total derivative terms and exploited the total anti-symmetry of the Levi-Civita tensor. The Gauss-Bonnet action then becomes

$$
S_{\rm CS}=-\frac{1}{8}\int \mathrm{d}\eta\mathrm{d}^3x\sqrt{-g}\omega'(\phi)\epsilon^{0ijk}\left[\frac{1}{2}\gamma'^{n}_{i}\partial_j\gamma'_{kn}-\frac{1}{2}\partial^n\gamma_{im}\partial_j\partial_n\gamma^m_k\right]
\tag{5.28}
$$

and we can further manipulate it by integrating by parts and writing the Levi-Civita tensor as $\epsilon^{0ijk}=\tilde{\epsilon}^{ijk}/\sqrt{-g}$:

$$
S_{\rm CS}=\frac{1}{8}\int \mathrm{d}\eta\mathrm{d}^3x\omega'(\phi)\tilde{\epsilon}^{ijk}\left[\frac{1}{2}\gamma'_{kn}\partial_j\gamma'^{n}_{i}-\frac{1}{2}\gamma_{im}\partial_j\partial^2\gamma^m_n\right].
\tag{5.29}
$$

In Fourier space the tensor perturbation can be decomposed as

$$
\gamma_{ij}=\frac{\sqrt{2}}{M_{\rm pl}}\sum_{S=R,L}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\phi_s(\eta,\mathbf{k})\epsilon^s_{ij}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}.
\tag{5.30}
$$

For our purposes we express tensor perturbation using the chiral (helicity) polarization basis, which is better suited for investigating parity violation effects, defined as

$$
\begin{aligned}
\epsilon^{R,L}_{ij} &= \frac{1}{\sqrt{2}}\left(\epsilon^{+}_{ij}\pm i\epsilon^{\times}_{ij}\right)\\
\phi_{R,L} &= \frac{1}{\sqrt{2}}\left(\phi_+\mp i\phi_\times\right)
\end{aligned}
\tag{5.31}
$$

where $\phi_{+,\times}$ and $\epsilon^{+,\times}_{ij}$ denote the linear polarizations of tensor modes. The chiral polarization tensor satisfies the following normalization and symmetry properties

$$
\begin{aligned}
\epsilon^R_{ij}(\mathbf{k})\epsilon^{ij}_R(\mathbf{k}) &= \epsilon^L_{ij}(\mathbf{k})\epsilon^{ij}_L(\mathbf{k})=0\\
\epsilon^R_{ij}(\mathbf{k})\epsilon^{ij}_L(\mathbf{k}) &= 2\\
\epsilon^L_{ij}(-\mathbf{k}) &= \epsilon^R_{ij}(\mathbf{k})\\
\epsilon^{R\star}_{ij}(\mathbf{k}) &= \epsilon^L_{ij}(\mathbf{k})\\
\phi^\star_R(\eta,\mathbf{k}) &= \phi_L(\eta,\mathbf{k})\\
k_l\tilde{\epsilon}^{mlj}(\epsilon^s)^i_j(\mathbf{k}) &= \mp i\alpha_s k\epsilon^{im}_s(\mathbf{k})
\end{aligned}
\tag{5.32}
$$

where $\alpha_R=+1$, $\alpha_L=-1$, $k=|\mathbf{k}|$ and the upper sign refers to $\theta<\pi/2$ and the lower one $\theta>\pi/2$. Substituting (5.30) in (5.29) we obtain

$$
\begin{aligned}
S_{\rm CS}=\frac{1}{8}\sum_{s,t=R,L}\int\mathrm{d}\eta\omega'(\phi)\int\mathrm{d}^3x\int\frac{\mathrm{d}^3k_1}{(2\pi)^3}\frac{\mathrm{d}^3k_2}{(2\pi)^3}e^{i(\mathbf{k}_1+\mathbf{k}_2)}\frac{\tilde{\epsilon}^{ijk}}{M^2_{\rm pl}}\Big[&i\phi'_s(\eta,\mathbf{k}_1)\epsilon^s_{kn}(\mathbf{k}_1)(k_2)_j\phi'_t(\eta,\mathbf{k}_2)(\epsilon^t)^n_i(\mathbf{k}_2)\\
&+i\phi_s(\eta,\mathbf{k}_1)\epsilon^s_{im}(\mathbf{k}_1)(k_2)_j(k_2)^2\phi^t(\eta,\mathbf{k}_2)(\epsilon^t)^m_n(\mathbf{k}_2)\Big]
\end{aligned}
\tag{5.33}
$$

and exploiting the last property in (5.32)

$$
\begin{aligned}
&i\tilde{\epsilon}^{ijk}(k_2)_j(\epsilon^t)_i^n(\mathbf{k}_2) = -i\tilde{\epsilon}^{kij}(k_2)_j(\epsilon^t)_i^n(\mathbf{k}_2) = \mp\alpha_t k_2 \epsilon_t^{kn} \\
&i\tilde{\epsilon}^{ijk}(k_2)_j(\epsilon^t)_n^m(\mathbf{k}_2) = -\big(\mp\alpha_t k_2 \epsilon_t^{mi}\big)
\end{aligned}
\tag{5.34}
$$

we can eliminate Levi-Civita symbol

$$
\begin{aligned}
S_{\rm CS} &= \frac{1}{8M_{\rm pl}^2}\sum_{s,t=R,L}\int {\rm d}\eta\,\omega'(\phi)\int\frac{{\rm d}^3k}{(2\pi)^3}\big[-k\phi'^s_{\mathbf{k}}\epsilon^s_{mn}(\mathbf{k})\alpha_t\phi'^t_{-\mathbf{k}}\epsilon_t^{mn}(-\mathbf{k}) + k^3\phi^s_{\mathbf{k}}\epsilon^s_{im}(\mathbf{k})\alpha_t\phi^t_{-\mathbf{k}}\epsilon_t^{im}(-\mathbf{k})\big] \\
&= \frac{1}{4M_{\rm pl}^2}\int {\rm d}\eta\,\omega'(\phi)\int\frac{{\rm d}^3k}{(2\pi)^3}\Big[k|\phi'^L_{-\mathbf{k}}|^2 - k|\phi'^R_{-\mathbf{k}}|^2 - k^2\big(k|\phi^L_{-\mathbf{k}}|^2 - k|\phi^R_{-\mathbf{k}}|^2\big)\Big] \\
&= \frac{1}{4M_{\rm pl}^2}\int {\rm d}\eta\,\omega'(\phi)\int\frac{{\rm d}^3k}{(2\pi)^3}\sum_{s=R,L}\big(\mp k\big)^s\Big[|\phi'^s_{-\mathbf{k}}|^2 - k^2|\phi^s_{-\mathbf{k}}|^2\Big] \\
&= \frac{1}{4M_{\rm pl}^2}\int {\rm d}\eta\,\omega'(\phi)\int\frac{{\rm d}^3k}{(2\pi)^3}\sum_{s=R,L}\big(\mp k\big)^s\Big[|\phi'^s_{\mathbf{k}}|^2 - k^2|\phi^s_{\mathbf{k}}|^2\Big].
\end{aligned}
\tag{5.35}
$$

In deriving the expression above we assumed $\theta \leq \pi/2$, even if the final result is insensitive to this choice, and we exploited the fact that the integral is invariant under the transformation $\mathbf{k} \to -\mathbf{k}$. The total action for second order tensor perturbation is given by the sum of the action in standard gravity and the Chern-Simons correction and can be recast in the following form

$$
\begin{aligned}
S_{\gamma\gamma} &= S_{\rm GR} + S_{\rm CS} \\
&= \sum_{s=R,L}\frac{1}{2}\int {\rm d}\eta\frac{{\rm d}^3k}{(2\pi)^3}a^2\Big[|\phi'^s_{\mathbf{k}}|^2 - k^2|\phi^s_{\mathbf{k}}|^2\Big] + \sum_{s=R,L}\frac{1}{4M_{\rm pl}^2}\int {\rm d}\eta\frac{{\rm d}^3k}{(2\pi)^3}\dot{\omega}(\phi)a\big(\mp k\big)^s\Big[|\phi'^s_{\mathbf{k}}|^2 - k^2|\phi^s_{\mathbf{k}}|^2\Big] \\
&= \sum_{s=R,L}\frac{1}{2}\int {\rm d}\eta\frac{{\rm d}^3k}{(2\pi)^3}A_{\rm T}^2\Big[|\phi'^s_{\mathbf{k}}|^2 - k^2|\phi^s_{\mathbf{k}}|^2\Big]
\end{aligned}
\tag{5.36}
$$

where

$$
A_{\rm T}^2 \equiv a^2\left(1 \mp \frac{1}{M_c}\frac{k}{a}\Omega\right) \ , \qquad \Omega \equiv \frac{1}{2}\frac{M_c}{M_{\rm pl}}\frac{\dot{\omega}}{M_{\rm pl}}
\tag{5.37}
$$

with M_c being the UV cut-off scale of our model. Notice that we have omitted the s superscript for simplicity but the expression of $A_{\rm T}^2$ for the right helicity state comes with the minus in the second term, whereas the on for the left helicity state comes with the plus sign. From (5.37) it is possible to notice that $A_{\rm T}^2$ takes negative values in $|\Omega| > 1$ for one helicity mode at the cut-off scale $k/a = M_c$. This implies that the kinetic term in the action becomes negative and the modes become ghost-like in nature. Since we do not know how to deal properly with ghost modes, we impose $|\Omega| < 1$. We can always treat Chern-Simons contribution $k\Omega/M_c a$ as a small correction since $|\Omega| < 1$ and $k/a < M_c$ [60]. The ratio Ω/M_c is defined as the inverse of a new mass parameter $M_{\rm CS}$ and it is explicitly given by:

$$
M_{\rm CS}^{-1} = \frac{\Omega}{M_c} = \frac{1}{2}\frac{M_c}{M_{pl}}\frac{\dot{\omega}}{M_{pl}}\frac{1}{M_c} = \frac{1}{2}\frac{\partial_\phi\omega(\phi)\dot{\phi}}{M_{pl}^2} = \frac{\sqrt{2\epsilon}}{2}\frac{H}{M_{pl}}\partial_\phi\omega(\phi) \ .
\tag{5.38}
$$

In terms of $M_{\rm CS}$, it is the adimensional combination $H/M_{\rm CS}$ which is required to be smaller than unity in order to have physical modes for which the theory is not spoiled within the horizon [32]. Therefore, we can use $H/M_{\rm CS}$ as a perturbative parameter used to perform expansions in CS corrections. In the literature it is common to approximate $M_{\rm CS}$ as constant parameter [60, 32], which is justified if the inflaton is slowly rolling down its potential and the coupling function is sufficiently smooth. To see this, we quantify the logarithmic variation of the CS mass during an Hubble time and we require it to be much smaller than unity:

$$
\frac{\dot{M}_{\rm CS}}{HM_{\rm CS}} = \frac{1}{H}\frac{\partial^2_\phi\omega(\phi)\dot{\phi}^2 + \partial_\phi\omega(\phi)\ddot{\phi}}{\partial_\phi\omega(\phi)\dot{\phi}} \approx \frac{\dot{\phi}}{H}\frac{\partial^2_\phi\omega(\phi)}{\partial_\phi\omega(\phi)} \ll 1
\tag{5.39}
$$

since $\ddot{\phi} \ll H\dot{\phi}$ from the second slow-roll condition. This is equivalent to ask for the variation of coupling ω w.r.t. ϕ, to change slowly during the characteristic time-scale of expansion, $1/H$

$$\frac{1}{H}\frac{d\ln\omega_{,\phi}}{dt} = \frac{\dot{\phi}}{H}\frac{\partial^2_\phi\omega}{\partial_\phi\omega} \ll 1. \tag{5.40}$$

Therefore, if the $\omega(\phi)$ satisfies the smoothness condition (5.40), $M_{\rm CS} \approx$ const. if slow-roll condition hold. Violations of slow-roll conditions break this approximation, inducing a time dependence on $M_{\rm CS}$. Introducing the new variable $\mu_{\mathbf{k}} \equiv A_{\rm T}\phi^s_{\mathbf{k}}$, the action can be rewritten as

$$S_{\gamma\gamma} = \sum_{s=R,L}\frac{1}{2}\int \mathrm{d}\eta \frac{\mathrm{d}^3k}{(2\pi)^3}\left[\mu'^2_{\mathbf{k}} + \mu^2_{\mathbf{k}}\frac{A'^2_{\rm T}}{A^2_{\rm T}} - 2\mu'_{\mathbf{k}}\mu_{\mathbf{k}}\frac{A_{\rm T}}{A_{\rm T}} - k^2\mu^2_{\mathbf{k}}\right] \tag{5.41}$$

and then, by variating it with respect to $\mu_{\mathbf{k}}$, we obtain the following e.o.m.:

$$\mu''_{\mathbf{k}} + \left(k^2 - \frac{A''_{\rm T}}{A_{\rm T}}\right)\mu_{\mathbf{k}} = 0. \tag{5.42}$$

Notice that the term $A''_{\rm T}/A_{\rm T}$ in the e.o.m. for tensor perturbations plays the role of an effective mass term. Let us notice that at first order in slow roll parameters and $H/M_{\rm CS}$ we have

$$\frac{A''_{\rm T}}{A_{\rm T}} = \frac{1}{2}\frac{\left(A^2_{\rm T}\right)'}{A^2_{\rm T}} = \frac{a}{2}\frac{\left(\dot{A^2_{\rm T}}\right)}{A^2_{\rm T}} = \frac{a}{2}\left(2H \pm \frac{H}{M_{\rm CS}}\frac{k}{a}\right) = aH \pm \frac{k}{2}\frac{H}{M_{\rm CS}} = \frac{1+\epsilon}{-\eta} \pm \frac{k}{2}\frac{H}{M_{\rm CS}} \tag{5.43}$$

where we used that in a quasi de-Sitter universe $\eta = -1/aH(1-\epsilon)$, and so

$$\frac{A''_{\rm T}}{A_{\rm T}} = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{A'_{\rm T}}{A_{\rm T}}\right) + \left(\frac{A'_{\rm T}}{A_{\rm T}}\right)^2 = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1+\epsilon}{-\eta} \pm \frac{k}{2}\frac{H}{M_{\rm CS}}\right) + \left(\frac{1+\epsilon}{-\eta} \pm \frac{k}{2}\frac{H}{M_{\rm CS}}\right)^2 \approx \frac{2+3\epsilon}{\eta^2} \pm \frac{k}{-\eta}\frac{H}{M_{\rm CS}}. \tag{5.44}$$

At this point (5.42) becomes

$$\mu''_{\mathbf{k}} + \left(k^2 - \frac{\nu^2_{\rm T} - 1/4}{\eta^2} \mp \frac{k}{-\eta}\frac{H}{M_{\rm CS}}\right)\mu_{\mathbf{k}} = 0 \tag{5.45}$$

with $\nu_{\rm T}$ being defined as

$$\nu_{\rm T} \equiv \frac{3}{2} + \epsilon_1. \tag{5.46}$$

As we did in the GR case, we promote $\mu_{\mathbf{k}}$ to an operator $\hat{\mu}_{\mathbf{k}}$ and we write it as linear combination of creation and annihilation operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}_{-\mathbf{k}}$:

$$\hat{\mu}_{\mathbf{k}} = u_{\mathbf{k}}(\eta)\hat{a}_{\mathbf{k}} + u^*_{-\mathbf{k}}(\eta)\hat{a}_{-\mathbf{k}} \tag{5.47}$$

with the ladder operators satisfying the usual canonical commutation relation. The equation of motion for the mode function $u_{\mathbf{k}}(\eta)$ is the same as for $\mu_{\mathbf{k}}$ and thus reads:

$$u''_{\mathbf{k}} + \left(k^2 - \frac{\nu^2_{\rm T} - 1/4}{\eta^2} \mp \frac{k}{-\eta}\frac{H}{M_{\rm CS}}\right)u_{\mathbf{k}} = 0. \tag{5.48}$$

Following [59] and [61], by making the redefinitions

$$\tau \equiv 2ik\eta\,, \qquad c \equiv \pm\frac{i}{2}\frac{H}{M_{\rm CS}}\,, \qquad \xi = \nu_{\rm T} \tag{5.49}$$

(5.48) can be rewritten as

$$\frac{\mathrm{d}^2u_{\mathbf{k}}}{\mathrm{d}\tau^2} + \left[-\frac{1}{4} + \frac{c}{\tau} + \frac{(1/4 - \xi^2)}{\tau^2}\right]u_{\mathbf{k}} = 0 \tag{5.50}$$

which is the Whittaker equation. The general solution to this equation is given in terms of the Whittaker functions

$$u_{\mathbf{k}} = C_1 W_{c,\xi}(\tau) + C_2 W_{-c,\xi}(-\tau) \tag{5.51}$$

where the constants C_1 and C_2 need to be fixed by initial conditions. Note that the two solutions $W_{c,\xi}(\tau)$ and $W_{-c,\xi}(-\tau)$ are one the complex conjugate of the other. In fact, both τ and c are the only complex coefficients appearing in (5.50). Thus, we can use the Bunch-Davies vacuum as the initial condition for (5.48). By imposing this vacuum, the mode function $u_{\mathbf{k}}$ is found to be given by [60], [56]:

$$u_{\mathbf{k}} = e^{-ik\eta}\big(-2k\eta\big)^{\nu_\tau}\sqrt{-\eta}e^{-i(\pi/4+\pi\nu_{\mathrm{T}}/2)}U\left(\frac{1}{2}+\nu_{\mathrm{T}}\mp\frac{i}{2}\frac{H}{M_{\mathrm{CS}}},1+2\nu_{\mathrm{T}},2ik\eta\right)\exp\left(\pm\frac{\pi}{4}\frac{H}{M_{\mathrm{CS}}}\right) \tag{5.52}$$

where $U\left(\frac{1}{2}+\nu_{\mathrm{T}}\mp\frac{i}{2}\frac{H}{M_{\mathrm{CS}}},1+2\nu_{\mathrm{T}},2ik\eta\right)$ is the confluent hypergeometric function of the second kind. The asymptotic form in the super-horizon regime reads:

$$\begin{aligned} u_k(\eta) &= \sqrt{\frac{-\eta}{2(-k\eta)^3}}e^{i(-\pi/4+\pi\nu_{\mathrm{T}})}\frac{\Gamma(\nu_{\mathrm{T}})}{\Gamma(3/2)}\left(\frac{-k\eta}{2}\right)^{3/2-\nu_{\mathrm{T}}}\exp\left(\pm\frac{\pi}{4}\frac{H}{M_{\mathrm{CS}}}\right) \\ &= \sqrt{\frac{1}{2k^3\eta^2}}e^{i(-\pi/4+\pi\nu_{\mathrm{T}})}\frac{\Gamma(\nu_{\mathrm{T}})}{\Gamma(3/2)}\left(\frac{-k\eta}{2}\right)^{3/2-\nu_{\mathrm{T}}}\exp\left(\pm\frac{\pi}{4}\frac{H}{M_{\mathrm{CS}}}\right). \end{aligned} \tag{5.53}$$

We can then evaluate the tensor power spectrum for each polarization mode $P_{\mathrm{T}}^{\pm}$ and the total power spectrum P_{T}:

$$\langle 0|\hat{h}_{ij}\hat{h}^{ij}|0\rangle = \sum_{\pm}\int \mathrm{d}(\log k)\frac{1}{M_{pl}^2}\frac{2}{\pi^2}k^3\frac{|u_{\mathbf{k}}|^2}{A_{\mathrm{T}}^2} = \sum_{\pm}\int \mathrm{d}(\log k)P_{\mathrm{T}}^{\pm} = \int \mathrm{d}(\log k)P_{\mathrm{T}} \tag{5.54}$$

In super-horizon regime the chiral adimensional power spectrum is given by:

$$\begin{aligned} P_{\mathrm{T}}^{\pm} &= \frac{1}{M_{pl}^2}\frac{1}{\pi^2}\frac{(2k^3)}{A_{\mathrm{T}}^2}\left|\sqrt{\frac{1}{2k^3\eta^2}}e^{i(-\pi/4+\pi\nu_{\mathrm{T}}/2)}\frac{\Gamma(\nu_{\mathrm{T}}}{\Gamma(3/2)}\left(\frac{-k\eta}{2}\right)^{3/2-\nu_{\mathrm{T}}}\exp\left(\pm\frac{\pi}{4}\frac{H}{M_{\mathrm{CS}}}\right)\right|^2 \\ &= \frac{1}{M_{pl}^2}\frac{1}{\pi^2}\frac{1}{A_{\mathrm{T}}^2}\frac{1}{\eta^2}\left(\frac{\Gamma(\nu_{\mathrm{T}})}{\Gamma(3/2)}\right)^2\left(\frac{-k\eta}{2}\right)^{3-2\nu_{\mathrm{T}}}\exp\left(\pm\frac{\pi}{2}\frac{H}{M_{\mathrm{CS}}}\right) \\ &\approx \frac{1}{\pi^2}\left(\frac{H}{M_{pl}}\right)^2\left(\frac{-k\eta}{2}\right)^{3-\nu_{\mathrm{T}}}\left(1\pm\frac{\pi}{2}\frac{H}{M_{\mathrm{CS}}}\right) \end{aligned} \tag{5.55}$$

where, in the last passage we have taken the lowest order in slow-roll parameters and in H/M_{CS}. Moreover, the term

$$A_{\mathrm{T}}^2 = a^2\left(1\mp\frac{1}{M_{\mathrm{CS}}}\frac{k}{a}\right) \tag{5.56}$$

is taken to be equal to $a^2(\eta)$ because at the beginning of inflation (conformal time η_s) $k/a(\eta_s) \ll M_c$ and so

$$\frac{k}{a(\eta_e)} = \frac{k}{a(\eta_e)}\frac{a(\eta_s)}{a(\eta_s)} = \frac{k}{a(\eta_s)}e^{-N} \tag{5.57}$$

with η_s and η_e defining the time at the beginning and end of inflation and N being the number of e-folds. Since the number of e-folds necessary to solve the flatness and horizon problem is around 60, the term k/aM_{CS} is completely washed away due to the exponentially accelerated expansion of the inflationary phase. The total adimensional power spectrum is given by:

$$P_{\mathrm{T}} = \sum_{\pm}P_{\mathrm{T}}^{\pm} = \frac{2}{\pi^2}\left(\frac{H}{M_{pl}}\right)^2\left(\frac{k}{2aH}\right)^{3-2\nu_{\mathrm{T}}}. \tag{5.58}$$

In the $H/M_{\mathrm{CS}} \ll 1$ limit, we can evaluate the amount of circular polarization and it is predicted to be

$$\Pi = \frac{P_{\mathrm{T}}^{+} - P_{\mathrm{T}}^{-}}{P_{\mathrm{T}}^{+} + P_{\mathrm{T}}^{-}} = \frac{\pi}{2}\frac{H}{M_{\mathrm{CS}}}. \tag{5.59}$$

This quantity takes naturally a small value because of the smallness of H/M_{CS}However we can still expect Π to be of order of few percentage, which makes it observable in future experiments [60].

Chapter 6

Cosmological correlators

Inflation provides initial conditions for the universe. Indeed, quantum vacuum fluctuations of the inflaton, or other scalar fields, are stretched to cosmological scales, seeding CMB radiation temperature anisotropies and polarization, as well as the LSS. As we discussed in the previous chapters, the power spectrum of fluctuations is a powerful tool that allows us to compare theoretical models of the early universe with observations. However, it is difficult to differentiate among the plethora of inflationary models based only on it. In fact the power spectrum predicted by two or more models of inflation can be degenerate for given cosmological scales, or their differences can be too little to be detectable with present techniques. As an example, take models that allow for a violation of slow-roll condition, such as those admitting a brief phase of USR, whose corresponding spectral power matches, at large scales, that of the simpler single-field slow-roll inflation. Even in cases in which the power spectrum can already be informative on interaction taking place in the early universe, its magnitude might not be big enough to be detectable in the immediate future. This is the case of dynamical Chern-Simons coupling investigated in the previous chapter: although the spectral power of tensor perturbations develops chiral properties, with left-handed spectral power differing from the right-handed power spectrum, this effect is extremely suppressed and unlikely to be observed and characterized in the next following years.

The statistics of classical fluctuations retains the memory of its genesis, providing a glimpse of the first instants of the universe. If Gaussian, the only correlator that we need in order to get all the statistical properties of the early universe are encoded in the two-point function (i.e. power spectrum in Fourier space). The reason is simple and it is rooted in Wick's theorem, stating that

$$\langle 0|\chi_1(\eta)\dots\chi_n(\eta)|0\rangle = \begin{cases}\text{exactly vanishing for an odd number of fields}\\ \text{product of two point correlators for even number of fileds}\end{cases}$$

with χ being a Gaussian random field. However, in case the statistics is non-Gaussian, the two-point correlator is not the only correlator bearing information on the statistics, and the dynamics of inflation, as well as its particle content, can be unveiled by the study of N-point correlators. The correlators of classical fluctuations (temperature fluctuations, matter density contrast, galaxies, ...) can be related to the primordial correlator of adiabatic curvature fluctuations ζ. These primordial correlations live on the future boundary of the inflationary spacetime or, equivalently, the past boundary of the hot Big Bang universe, defined as $\eta \to 0^-$. Time does not appear explicitly in the observed correlators, but is encoded in the scale dependence since modes of different wave length exit horizon and freeze out at different times during inflation. Studying the scale dependence of late-time cosmological correlators can provide us with information on the physics of inflationary era. The standard approach to make predictions for the inflationary correlations is to follow the evolution of expectation values in a fixed initial time state. This makes the problem of computation of quantum correlators different from the computation of scattering-matrix elements relevant in particle physics. Conditions on the fields are not imposed both at very early times and very late times, as in the calculation of S-matrix elements, but only at very early times, when the modes are deep inside the horizon. In that case equivalence principle holds and the interaction picture fields and states should have the same form as in Minkowski spacetime. Moreover, in cosmology Poincarè symmetry is explicitly broken by the evolving background metric. The inflationary action has an explicit time dependence which is that encoded in the scale factor $a(t)$.

The formalism used in such computation is the Schwinger-Keldysh (also known as *in-in*), which was introduced by Schwinger [62] and developed by Keldysh [63] in the context of condensed matter physics. It has been then

Concerning this thesis work, we are interested in the computation and characterization of primordial correlators, computed at the end of inflation, which are the primordial source of non-Gaussianity. In the following we introduce the in-in formalism, a technique which allows the computation of expectation values of operators during inflation [64]. Moreover, as an useful exercise, we focus on the computation of the graviton-mediated scalar trispectrum in single-field slow-roll inflation by means of that formalism.

6.1 In-in formalism

Consider a dynamical system whose Hamiltonian depends on the canonical variable $\chi_a(\mathbf{x},\eta)$ and its conjugate $\pi_a(\mathbf{x},\eta)$ satisfying the commutation relations

$$\begin{aligned} \big[\chi_a(\mathbf{x},\eta),\pi_b(\mathbf{x},\eta)\big] &= i\delta_{ab}\delta^3(\mathbf{x}-\mathbf{y}),\\ \big[\chi_a(\mathbf{x},\eta),\chi_b(\mathbf{x},\eta)\big] &= 0 = \big[\pi_a(\mathbf{x},\eta),\pi_b(\mathbf{x},\eta)\big] \end{aligned} \tag{6.1}$$

with the Latin index labeling particular fields and their spin components. The canonical fields and their conjugate are operators, thus they satisfy the Heisenberg equation

$$\frac{\mathrm{d}\chi_a(\mathbf{x},\eta)}{\mathrm{d}\eta} = i\big[H(\chi(\eta),\pi(\eta)),\chi_a(\mathbf{x},\eta)\big], \qquad \frac{\mathrm{d}\pi_a(\mathbf{x},\eta)}{\mathrm{d}\eta} = i\big[H(\chi(\eta),\pi(\eta)),\pi_a(\mathbf{x},\eta)\big]. \tag{6.2}$$

We assume the existence of a classical solution $\bar{\chi}_a(\mathbf{x},\eta)$ and $\bar{\pi}_a(\mathbf{x},\eta)$, which is a c-number that satisfies the classical equation of motion

$$\frac{\mathrm{d}\bar{\chi}_a(\mathbf{x},\eta)}{\mathrm{d}\eta} = \frac{\delta H(\bar{\chi}_a(\eta),\bar{\pi}_a(\eta))}{\delta\bar{\pi}_a(\mathbf{x},\eta)}, \qquad \frac{\mathrm{d}\bar{\pi}_a(\mathbf{x},\eta)}{\mathrm{d}\eta} = -\frac{\delta H(\bar{\chi}_a(\eta),\bar{\pi}_a(\eta))}{\delta\bar{\chi}_a(\mathbf{x},\eta)}. \tag{6.3}$$

The solution $\bar{\chi}_a$ corresponds to the Robertson-Walker metric and the expectation value of the various scalar filed present at early times. Assuming the classical solution to be the dominant part of the total filed and conjugate momentum variables, we can Taylor expand around the classical solution finding

$$\chi_a(\mathbf{x},\eta) = \bar{\chi}_a(\mathbf{x},\eta) + \delta\chi_a(\mathbf{x},\eta), \qquad \pi_a = \bar{\pi}_a(\mathbf{x},\eta) + \delta\pi_a(\mathbf{x},\eta) \tag{6.4}$$

and since the classical solution is a c-number, the fluctuations satisfy the same commutation rules as the total variables:

$$\begin{aligned} \big[\delta\chi_a(\mathbf{x},\eta),\delta\pi_b(\mathbf{x},\eta)\big] &= i\delta_{ab}\delta^3(\mathbf{x}-\mathbf{y}),\\ \big[\delta\chi_a(\mathbf{x},\eta),\delta\chi_b(\mathbf{x},\eta)\big] &= 0 = \big[\delta\pi_a(\mathbf{x},\eta),\delta\pi_b(\mathbf{x},\eta)\big]. \end{aligned} \tag{6.5}$$

As the Hamiltonian is expanded in terms of fluctuations, it can be written as

$$\begin{aligned} H(\chi(\eta),\pi(\eta)) = H(\bar{\chi}(\eta),\bar{\pi}(\eta)) &+ \sum_a \int d^3x \frac{\delta H(\bar{\chi}(\eta),\bar{\pi}(\eta))}{\delta\bar{\chi}_a(\mathbf{x},\eta)}\delta\chi_a(\mathbf{x},\eta) + \sum_a \int d^3x \frac{\delta H(\bar{\chi}(\eta),\bar{\pi}(\eta))}{\delta\bar{\pi}_a(\mathbf{x},\eta)}\delta\pi_a(\mathbf{x},\eta)\\ &+ \tilde{H}(\delta\chi(\eta),\delta\pi(\eta);\eta) \end{aligned} \tag{6.6}$$

where $\tilde{H}(\delta\chi(\eta),\delta\pi(\eta);\eta)$ is the sum of all terms in $H(\chi(\eta),\pi(\eta))$ of second and higher order in fluctuations. From Eqs. (6.2), accounting for the fields expanded as quantum fluctuations on top of a classical background, we obtain

$$\dot{\bar{\chi}}_a(\mathbf{x},\eta) + \delta\dot{\chi}_a = i\big[H(\chi(\eta),\pi(\eta)),\delta\chi_a(\mathbf{x},\eta)\big], \qquad \dot{\bar{\pi}}_a(\mathbf{x},\eta) + \delta\dot{\pi}_a = i\big[H(\pi(\eta),\pi(\eta)),\delta\pi_a(\mathbf{x},\eta)\big]; \tag{6.7}$$

moreover, combining the EOMs for the classical background with the commutation relations satisfied by fluctuations, it is easy to show that:

$$
\begin{aligned}
i\Big[\sum_b \int d^3y \frac{\delta H(\bar\chi(\eta),\bar\pi(\eta))}{\delta\bar\chi_b(\mathbf{y},\eta)}\delta\chi_b(\mathbf{y},\eta) + \sum_b \int d^3y \frac{\delta H(\bar\chi(\eta),\bar\pi(\eta))}{\delta\bar\pi_b(\mathbf{y},\eta)}\delta\pi_b(\mathbf{y},\eta), \delta\chi_a(\mathbf{x},\eta)\Big] &= \dot{\bar\chi}_a(\mathbf{x},\eta)\\
i\Big[\sum_b \int d^3y \frac{\delta H(\bar\chi(\eta),\bar\pi(\eta))}{\delta\bar\chi_b(\mathbf{y},\eta)}\delta\chi_b(\mathbf{y},\eta) + \sum_b \int d^3y \frac{\delta H(\bar\chi(\eta),\bar\pi(\eta))}{\delta\bar\pi_b(\mathbf{y},\eta)}\delta\pi_b(\mathbf{y},\eta), \delta\pi_a(\mathbf{x},\eta)\Big] &= \dot{\bar\pi}_a(\mathbf{x},\eta).
\end{aligned}
\tag{6.8}
$$

Subtracting Eqs. (6.8) to the EOMs of the total quantities (6.7), we find:

$$
\delta\dot\chi_a = i\big[\tilde H(\chi(\eta),\pi(\eta);\eta),\delta\chi_a(\mathbf{x},\eta)\big], \qquad \delta\dot\pi_a = i\big[\tilde H(\chi(\eta),\pi(\eta);\eta),\delta\pi_a(\mathbf{x},\eta)\big]. \tag{6.9}
$$

The equations above describe the evolution of the field fluctuation $\delta\chi$ and its conjugate $\delta\pi$, and show that it depends on $\tilde H(\delta\chi(\eta),\delta\pi(\eta);\eta)$, the part of the Hamiltonian which is at least quadratic in perturbations.

This argument clarifies the prescription to follow in order to identify the Hamiltonian $\tilde H$ governing the time dependence of fluctuations:

1. Expand the Hamiltonian in powers of the field fluctuations and its conjugate.
2. Keep only terms at least of second order in $\delta\chi$ or $\delta\pi$

The fluctuations at time t can be expressed in terms of the same operators at some very early time t_0 through a unitary transformation

$$
\delta\chi_a(\eta) = U^{-1}(\eta,\eta_0)\delta\chi_a(\eta_0)U(\eta,\eta_0), \qquad \delta\pi_a(\eta) = U^{-1}(\eta,\eta_0)\delta\pi_a(\eta_0)U(\eta,\eta_0) \tag{6.10}
$$

where the time evolution operator satisfies the differential equation

$$
\frac{\mathrm{d}}{\mathrm{d}t}U(\eta,\eta_0) = -i\tilde H(\delta\chi(\eta),\delta\pi(\eta);\eta)U(\eta,\eta_0) \tag{6.11}
$$

with initial condition

$$
U(\eta_0,\eta_0) = 1. \tag{6.12}
$$

In the inflationary contest, $\eta_0 = -\infty$, since this is equivalent to consider any time early enough so that the wavelengths of interest are deep inside the horizon. We can further decompose the Hamiltonian $\tilde H$ as

$$
\tilde H(\delta\chi(\eta),\delta\pi(\eta);\eta) = H_0(\delta\chi(\eta),\delta\pi(\eta);\eta) + H_I(\delta\chi(\eta),\delta\pi(\eta);\eta)
$$

with H_0 denoting the Hamiltonian quadratic in $\delta\chi$ and $\delta\pi$ ans H_I the higher order Hamiltonian: the quadratic components govern the free fluctuation fields, whereas the higher order part governs its interactions. As introduced above, we make use of the interaction picture, in which both state vectors and operators carry parts of the time dependence of observables. Moreover, the time evolution of the operators is governed by the free Hamiltonian H_0 and the one of the state vectors is captured by the interaction Hamiltonian H_I. This picture is particularly useful since the free operators preserve the properties of the free fields satisfying linear wave equations. Therefore,

$$
\delta\dot\chi^I_a(\eta) = i\big[H_0(\delta\chi^I(\eta),\delta\pi(\eta)^I;\eta),\delta\chi^I_a(\eta)\big], \qquad \delta\dot\pi^I_a(\eta) = i\big[H_0(\delta\chi(\eta)^I,\delta\pi(\eta)^I;\eta),\delta\pi^I_a(\eta)\big], \tag{6.13}
$$

with initial conditions

$$
\delta\chi^I_a(\eta_0) = \delta\chi_a(\eta_0), \qquad \delta\pi^I_a(\eta_0) = \delta\pi_a(\eta_0). \tag{6.14}
$$

We note that in evaluating $H_0(\delta\chi^I,\delta\pi^I;\eta)$, we can take the fluctuations evaluated at any time t: for convenience, we choose to evaluate them at η_0 so that

$$
H_0(\delta\chi^I(\eta),\delta\pi^I(\eta);\eta) = H_0(\delta\chi(\eta_0),\delta\pi(\eta_0);\eta) \tag{6.15}
$$

and the intrinsic time-dependence of the Hamiltonian remains untouched. Then, the fields can be written as

$$
\delta\chi^I_a = U_0^{-1}(\eta,\eta_0)\delta\chi_a(\eta_0)U_0(\eta,\eta_0), \qquad \delta\pi^I_a = U_0^{-1}(\eta,\eta_0)\delta\pi_a(\eta_0)U_0(\eta,\eta_0) \tag{6.16}
$$

and the time-evolution operator satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\eta}U_0(\eta,\eta_0) = -iH_0(\delta\chi(\eta_0),\delta\pi(\eta_0);\eta)U_0(\eta,\eta_0) \tag{6.17}$$

with initial condition

$$U_0(\eta_0,\eta_0) = 1. \tag{6.18}$$

Consider now the product $U_0^{-1}(\eta,\eta_0)U(\eta,\eta_0)$ and take its time derivative

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\eta}(U_0^{-1}(\eta,\eta_0)U(\eta,\eta_0)) &= \frac{\mathrm{d}U_0^{-1}(\eta,\eta_0)}{\mathrm{d}\eta}U(\eta,\eta_0) + U_0^{-1}(\eta,\eta_0)\frac{\mathrm{d}U(\eta,\eta_0)}{\mathrm{d}\eta}\\ &= iU_0^{-1}(\eta,\eta_0)H_0(\delta\chi(\eta_0),\delta\pi(\eta_0);\eta)U(\eta,\eta_0) - iU_0^{-1}(\eta,\eta_0)\tilde{H}(\delta\chi(\eta),\delta\chi(\eta);\eta)U(\eta,\eta_0)\\ &= -iU_0^{-1}(\eta,\eta_0)H_I(\delta\chi(\eta_0),\delta\pi(\eta_0);\eta)U(\eta,\eta_0),\end{aligned} \tag{6.19}$$

we can define the composite operator $F(\eta,\eta_0) = U_0^{-1}(\eta,\eta_0)U(\eta,\eta_0)$ obtaining

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\eta}F(\eta,\eta_0) &= -iU_0^{-1}(\eta,\eta_0)H_I(\delta\chi(\eta_0),\delta\pi(\eta_0);\eta)U_0(\eta,\eta_0)F(\eta,\eta_0)\\ &= -iH_I(\delta\chi^I(\eta),\delta\pi^I(\eta);\eta)F(\eta,\eta_0)\end{aligned} \tag{6.20}$$

and

$$F(\eta_0,\eta_0) = 1. \tag{6.21}$$

The solution of the equation (6.20) is found iteratively to be

$$F(\eta,\eta_0) = T\exp\Big(-i\int_{\eta_0}^{\eta}H_I(\eta)d\eta\Big) \tag{6.22}$$

with the T denoting that the product of H_Is in the expansion of the equation is taken to be time ordered. The solution for the fluctuations in terms of the free fields in the interaction picture reads

$$\begin{aligned}Q(\eta) &= F^{-1}(\eta,\eta_0)Q^I(\eta)F(\eta,\eta_0)\\ &= \Big[\bar{T}\exp\Big(\int_{\eta_0}^{\eta}H_I(\eta)d\eta\Big)\Big]Q^I(\eta)\Big[T\exp\Big(\int_{\eta_0}^{\eta}H_I(\eta)d\eta\Big)\Big]\end{aligned} \tag{6.23}$$

where $Q(\eta)$ is any product of field fluctuations $\delta\chi_a$ or/and its con $\delta\pi_a$ and $Q^I(\eta)$ is the same quantity evaluated in the interaction picture, $\bar{T}$ denotes the anti-time ordering operation.

6.1.1 Time evolution of the vacuum

The time evolution operator F can be used to relate the interacting vacuum at an arbitrary time $|\Omega(t)\rangle$ with the Bunch-Davies vacuum $|0\rangle$. We can expand $|0\rangle$ in eigenstates of the full Hamiltonian ($H|n\rangle = E_n|n\rangle$)

$$|0\rangle = \sum_n |n\rangle\langle n|0\rangle = |\Omega\rangle\langle\Omega|0\rangle + \sum_{n\geq 1}|n\rangle\langle n|0\rangle \tag{6.24}$$

where $|\Omega\rangle$ denotes the eigenstate with eigenvalue given by the lowest energy level. We can then evolve it in time by means of the time-evolution operator $U(\eta,\eta_0)$

$$\begin{aligned}|0\rangle &= U(\eta,\eta_0)|0\rangle = U(\eta,\eta_0)\sum_n|n\rangle\langle n|0\rangle\\ &= \exp\Big(-i\int_{\eta_0}^{\eta}d\eta' H(\eta')\Big)\sum_n|n\rangle\langle n|0\rangle\\ &= \exp\Big(-i\int_{\eta_0}^{\eta}d\eta' H(\eta')\Big)\Big[|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\geq 1}|n\rangle\langle n|0\rangle\Big]\\ &= e^{-iE_\Omega(\eta-\eta_0)}|\Omega\rangle\langle\Omega|0\rangle + e^{-iE_n(\eta-\eta_0)}|n\rangle\langle n|0\rangle.\end{aligned} \tag{6.25}$$

We now add a small imaginary part to time:

$$\eta \longrightarrow \tilde{\eta} = \eta(1 - i\epsilon). \tag{6.26}$$

In this way, for $\eta_0 \to -\infty$

Take now the time η_0 to lie in the far past $\eta_0 \to -(1-i\epsilon)\infty$: in this case the only contribution to the time evolution of the Bunch-Davies vacuum is the term with the exponential of the energy E_Ω In fact, the $i\epsilon$ prescription, needed to ensure convergence of correlation function at early times, inserts a damping factor $e^{-E_n\infty}$ which makes quantum states with high energy more suppressed w.r.t. the low energy ones. From Eq. (6.25) we obtain the equality

$$e^{-iH(\eta-\eta_0)}|\Omega\rangle = \frac{1}{\langle\Omega|0\rangle} e^{-iH(\eta-\eta_0)}|0\rangle \tag{6.27}$$

which implies [65]

$$F(\eta,\eta_0)|\Omega\rangle = \frac{1}{\langle\Omega|0\rangle} F(\eta,\eta_0)|0\rangle \tag{6.28}$$

since $F(\eta,\eta_0) = U_0^{-1}(\eta,\eta_0)U(\eta,\eta_0)$. From now on we neglect the tilde on η: conformal times in the later sections should be understood with a small imaginary part.

6.2 In-in expectation values

If we combine (6.23) and (6.28), the time dependent expectation value of an operator can be written as [65]:

$$\langle\Omega|Q(\eta)|\Omega\rangle = \frac{\langle 0|\left[\bar{T}e^{i\int_{\eta_0}^{\eta} H_I(\eta')d\eta'}\right] Q^I(\eta)\left[Te^{-i\int_{\eta_0}^{\eta} H_I(\eta')d\eta'}\right]}{|\langle 0|\Omega\rangle|^2}. \tag{6.29}$$

Notice that in the anti-time ordering $\bar{T}$, the $i\epsilon$ prescription comes with the opposite sign w.r.t. (6.26), because the $\bar{T}$ part is the Hermitian conjugate of the T part. Regarding the denominator, if we take $Q(\eta)$ to be the identity matrix we can write $|\langle 0|\Omega\rangle|^2$ as

$$|\langle 0|\Omega\rangle|^2 = \frac{\langle 0|\left[\bar{T}e^{i\int_{\eta_0}^{\eta} H_I(\eta')d\eta'}\right]\left[Te^{-i\int_{\eta_0}^{\eta} H_I(\eta')d\eta'}\right]}{\langle\Omega|\Omega\rangle} = 1. \tag{6.30}$$

To get the second equality we required that both $|\Omega\rangle$ and $|0\rangle$ are normalized. Therefore we arrive to the master formula for cosmological correlators:

$$\langle\Omega|Q(\eta)|\Omega\rangle = \langle 0|\left[\bar{T}e^{i\int_{\eta_0}^{\eta} H_I(\eta')d\eta'}\right] Q^I(\eta)\left[Te^{-i\int_{\eta_0}^{\eta} H_I(\eta')d\eta'}\right]|0\rangle. \tag{6.31}$$

In order to obtain the n-th order correlator from (6.31), we expand the time-evolution operators in powers of H_I[1], keeping the terms associated to the order in perturbation theory in which we are interested. For example, if the interaction Hamiltonian entails a cubic intraction among three scalars and we are interested in the exchange trispectrum, i.e. correlator of four scalar perturbation with one exchanging scalar, we need to expand F to the second order in H_I. If instead H_I is a quartic interaction and we are interested in the contact tripsectrum, i.e. no particle is exchanged, we can stop the expansion at the first order in the interaction Hamiltonian. After the expansion of F and $\bar{F}$, we insert them in the master formula evaluating: this generates a sum of expectation values In a more straightforward way, the number of vertices sets the order of the expansion of (6.31) that we should consider.

[1]Perturbation theory holds if the couplings inside H_I are small. This means that we can expand in powers of interaction Hamiltonian H_I.

Let us derive the contribution to an n-*th* correlator at second order in H_I, i.e. two vertices, from the master formula. The operator $F(\eta,\eta_0)$ takes the form

$$F(\eta,\eta_0) = 1 - i\int_{\eta_0}^{\eta} d\eta' H_I(\eta') + \frac{(-i)^2}{2}\int_{\eta_0}^{\eta} d\eta' H_I(\eta')\int_{\eta_0}^{\eta} d\eta'' H_I(\eta'') + \dots, \tag{6.32}$$

and the master formula becomes

$$\begin{aligned}\langle\Omega|Q(\eta)|\Omega\rangle = \Big\langle O(\eta) &+ i\int_{\eta_0}^{\eta} d\eta' [H_I(\eta'), O(\eta)] - \int_{\eta_0}^{\eta} d\eta' \int_{\eta_0}^{\eta} d\eta'' H_I(\eta')H_I(\eta'')O(\eta) \\ &+ \int_{\eta_0}^{\eta} d\eta' \int_{\eta_0}^{\eta} d\eta'' H_I(\eta')O(\eta)H_I(\eta'') - \int_{\eta_0}^{\eta} d\eta' \int_{\eta_0}^{\eta} d\eta'' O(\eta)H_I(\eta')H_I(\eta'') + \dots\Big\rangle.\end{aligned} \tag{6.33}$$

Therefore, keeping only the terms quadratic in the interaction Hamiltonian, we obtain

$$\begin{aligned}\langle O(\eta)\rangle^{(2)} = &\int_{\eta_0}^{\eta} d\eta' \int_{\eta_0}^{\eta} d\eta'' \langle H_I(\eta')O(\eta)H_I(\eta'')\rangle - \int_{\eta_0}^{\eta} d\eta' \int_{\eta_0}^{\eta} d\eta'' \langle O(\eta)H_I(\eta')H_I(\eta'')\rangle \\ &- \int_{\eta_0}^{\eta} d\eta' \int_{\eta_0}^{\eta} d\eta'' \langle H_I(\eta')H_I(\eta'')O(\eta)\rangle.\end{aligned} \tag{6.34}$$

6.2.1 In-in Trispectrum

We apply the formalism introduced above in the computation of the equal time four point correlation function of inflaton field fluctuations coming from the exchange of a graviton. This warm-up example will be our guide line for the computations carried out in this thesis. The interaction Hamiltonian in Fourier space is

$$H_I = -\frac{1}{2}a^2(\eta)\sum_{s=+,\times}\int\left(\prod_{i=1}^{3}\frac{d^3k_i}{(2\pi)^3}\right)k_1^i k_2^j \epsilon_{ij}^s(\mathbf{k}_3)\delta\phi_{\mathbf{k}_1}(\eta)\delta\phi_{\mathbf{k}_2}(\eta)\gamma_{\mathbf{k}_3}^s(\eta)(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3) \tag{6.35}$$

and in this section we will take it as granted (for its derivation see Sec. 7.3.1). In order to find the four point correlation function arising from cubic interaction Hamiltonian, we start from the in-in master formula expanded at second order (6.34). Thus, the correlator will be given by the sum of three parts:

$$\langle O(\eta)\rangle = \langle O(\eta)\rangle^{\dagger}_{(0,2)} + \langle O(\eta)\rangle_{(1,1)} + \langle O(\eta)\rangle_{(0,2)} \tag{6.36}$$

with

$$\langle O(\eta)\rangle_{(1,1)} = \int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta} d\eta'' \,\langle H_I(\eta')O(\eta)H_I(\eta'')\rangle\ , \tag{6.37}$$

$$\langle O(\eta)\rangle_{(0,2)} = -\int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta'} d\eta'' \,\langle O(\eta)H_I(\eta')H_I(\eta'')\rangle\ , \tag{6.38}$$

$$O(\eta) = \delta\phi(\mathbf{p}_1,\eta)\delta\phi(\mathbf{p}_2,\eta)\delta\phi(\mathbf{p}_3,\eta)\delta\phi(\mathbf{p}_4,\eta)\ . \tag{6.39}$$

The $\langle O(\eta)\rangle_{(0,2)}$ part takes the form:

$$\begin{aligned}&-\langle\delta\phi_{\mathbf{p}_1}(\eta)\delta\phi_{\mathbf{p}_2}(\eta)\delta\phi_{\mathbf{p}_3}(\eta)\delta\phi_{\mathbf{p}_4}(\eta)\rangle_{(0,2)} = -\frac{1}{2}\int\prod_{a=1}^{3}\left(\frac{d^3k_a}{(2\pi)^3}\right)\times\left(-\frac{1}{2}\right)\int\prod_{b=4}^{6}\left(\frac{d^3k_b}{(2\pi)^3}\right)(2\pi)^3(2\pi)^3 \\ &= \delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\delta^3(\mathbf{k}_4+\mathbf{k}_5+\mathbf{k}_6)\sum_{s,l=+,\times} k_1^i k_2^j\epsilon_{ij}^s(\mathbf{k}_3)\,k_4^m k_5^n\epsilon_{mn}^l(\mathbf{k}_6)\int_{-\infty}^{\eta} d\eta'\,a^2(\eta')\int_{-\infty}^{\eta'} d\eta'' a^2(\eta'') \\ &\langle 0|\delta\phi_{\mathbf{p}_1}(\eta)\delta\phi_{\mathbf{p}_2}(\eta)\delta\phi_{\mathbf{p}_3}(\eta)\delta\phi_{\mathbf{p}_4}(\eta)\delta\phi_{\mathbf{k}_1}(\eta')\delta\phi_{\mathbf{k}_2}(\eta')\gamma_{\mathbf{k}_3}^s\delta\phi_{\mathbf{k}_4}(\eta'')\delta\phi_{\mathbf{k}_5}(\eta'')\gamma_{\mathbf{k}_6}^l(\eta'')|0\rangle\end{aligned} \tag{6.40}$$

where the red and blue contributions comes from the two H_I insertions. We evaluate the expectation value using the Wick theorem [66] and taking into account that the gravitons will be contracted only among themselves and that inflaton perturbation at time η' and η'' will be contracted only to those living in the boundary. There are 4! ways to contract the four inflaton fluctuations. An example of these contractions is the following:

$$\langle \delta\phi_{\mathbf{p}_1}(\eta)\delta\phi_{\mathbf{p}_2}(\eta)\delta\phi_{\mathbf{p}_3}(\eta)\phi_{\mathbf{p}_4}(\eta)\delta\phi_{\mathbf{k}_1}(\eta')\delta\phi_{\mathbf{k}_2}(\eta')\gamma^s_{\mathbf{k}_3}(\eta')\delta\phi_{\mathbf{k}_4}(\eta'')\delta\phi_{\mathbf{k}_5}(\eta'')\gamma^l_{\mathbf{k}_6}(\eta'')\rangle .$$

Inflaton fluctuations and gravitons are promoted to quantum fields: they are expressed as linear combination of the creation and annihilation operator satisfying the canonical commutation relations:

$$\delta\phi_{\mathbf{k}}(\eta) = u_k(\eta)a_{\mathbf{k}} + u^*_k(\eta)a^\dagger_{-\mathbf{k}} \ , \tag{6.41}$$

$$\gamma^s_{\mathbf{k}}(\eta) = v^s_k(\eta)b^s_{\mathbf{k}} + v^{s*}_k(\eta)b^{s\dagger}_{-\mathbf{k}} \ , \tag{6.42}$$

$$\left[a_{\mathbf{k}_1}, a^\dagger_{-\mathbf{k}_2}\right] = (2\pi)^3\delta^3(\mathbf{k}_1 - \mathbf{k}_2) \ , \qquad \left[b^s_{\mathbf{k}_1}, b^{s'\dagger}_{-\mathbf{k}_2}\right] = (2\pi)^3\delta^3(\mathbf{k}_1 - \mathbf{k}_2)\delta^{ss'} . \tag{6.43}$$

Considering that the annihilation operator $a_{\mathbf{k}}$ applied to the vacuum vanishes, using the relations above it is possible to find that the expectation value reads

$$\begin{aligned}
&\langle 0|\delta\phi_{\mathbf{p}_1}(\eta)\delta\phi_{\mathbf{p}_2}(\eta)\delta\phi_{\mathbf{p}_3}(\eta)\delta\phi_{\mathbf{p}_4}(\eta)\delta\phi_{\mathbf{k}_1}(\eta')\delta\phi_{\mathbf{k}_2}(\eta')\gamma^s_{\mathbf{k}_3}(\eta')\delta\phi_{\mathbf{k}_4}(\eta'')\delta\phi_{\mathbf{k}_4}(\eta'')\gamma^l_{\mathbf{k}_6}(\eta'')|0\rangle \\
&= u_{p_1}(\eta)u^*_{k_1}(\eta')(2\pi)^3\delta^3(\mathbf{p}_1+\mathbf{k}_1)u_{p_2}(\eta)u^*_{k_2}(\eta')(2\pi)^3\delta^3(\mathbf{p}_2+\mathbf{k}_2)u_{p_3}(\eta)u^*_{k_4}(\eta'')(2\pi)^3\delta^3(\mathbf{p}_3+\mathbf{k}_4) \\
&\times u_{p_4}(\eta)u^*_{k_5}(2\pi)^3\delta^3(\mathbf{p}_4+\mathbf{k}_5)v^s_{k_3}(\eta')v^{l*}_{k_6}(\eta'')\delta^3(\mathbf{k}_3+\mathbf{k}_6)\delta^{sl} + 23 \text{ perm.}
\end{aligned} \tag{6.44}$$

Inserting 6.44 into 6.40 and using Dirac deltas to remove momenta integrations, we get

$$\begin{aligned}
\langle \delta\phi_{\mathbf{p}_1}(\eta)\delta\phi_{\mathbf{p}_2}(\eta)\delta\phi_{\mathbf{p}_3}(\eta)\delta\phi_{\mathbf{p}_4}(\eta)\rangle_{(0,2)} &= -\frac{1}{4}(2\pi)^3\delta^3(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)\sum_{s=+,\times} p^i_1 p^j_2 \epsilon^s_{ij}(\mathbf{p}_1+\mathbf{p}_2)\times \\
\times\, p^m_3 p^n_4 \epsilon^s_{mn}(\mathbf{p}_3+\mathbf{p}_4)\int_{-\infty}^{\eta} d\eta'\, a(\eta')\int_{-\infty}^{\eta'} d\eta''\, a(\eta'')\, &\mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}_{p_4}(\eta,\eta'')\mathcal{F}_p(\eta',\eta'') + 23 \text{ perm.}
\end{aligned} \tag{6.45}$$

where $\mathcal{G}_p(\eta,\eta') = u_p(\eta)u^*_p(\eta')$, $\mathcal{G}^s_p(\eta',\eta'') = v^s_p(\eta')v^{s*}_p(\eta'')$ and $p_{12} = |\mathbf{p}_1 + \mathbf{p}_2| = |\mathbf{p}_3 + \mathbf{p}_4|$. In single slow roll inflation, de-Sitter approximation and Bunch-Davies vacuum, the modes take the following form:

$$u_p(\eta) = \frac{iH}{\sqrt{2p^3}}(1+ip\eta)e^{-ip\eta} \ , \qquad v^s_p(\eta) = \frac{iH}{M_{\rm Pl}\sqrt{p^3}}(1+ip\eta)e^{-ip\eta} \tag{6.46}$$

so that we can define the propagators function as

$$\mathcal{G}_p(\eta,\eta') = \frac{H^2}{2p^3}(1+ip\eta)(1-ip\eta')e^{ip(\eta'-\eta)} \ , \qquad \mathcal{G}^s_p(\eta',\eta'') = \frac{H^2}{M^2_{pl}p^3}(1+ip\eta')(1-ip\eta'')e^{ip(\eta''-\eta')} . \tag{6.47}$$

Notice that exchanging the order of η and η' in $\mathcal{G}$ is equivalent to taking its complex conjugate.

The $(2,0)$ contribution can be easily derived from $(0,2)$ because $\langle O(\eta)\rangle_{(2,0)} = \langle O(\eta)\rangle^\dagger_{(0,2)}$. The $(2,0)$ contribution then reads

$$\begin{aligned}
\langle \delta\phi_{\mathbf{p}_1}(\eta)\delta\phi_{\mathbf{p}_2}(\eta)\delta\phi_{\mathbf{p}_3}(\eta)\delta\phi_{\mathbf{p}_4}(\eta)\rangle_{(2,0)} &= -\frac{1}{4}(2\pi)^3\delta^3(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)\sum_{s=+,\times} p^i_1 p^j_2 \epsilon^s_{ij}(\mathbf{p}_1+\mathbf{p}_2)\times \\
\times\, p^m_3 p^n_4 \epsilon^s_{mn}(\mathbf{p}_3+\mathbf{p}_4)\int_{-\infty}^{\eta} d\eta'\, a(\eta')\int_{-\infty}^{\eta'} d\eta''\, a(\eta'')\, &\mathcal{G}^*_{p_1}(\eta,\eta')\mathcal{G}^*_{p_2}(\eta,\eta')\mathcal{G}^*_{p_3}(\eta,\eta'')\mathcal{G}^*_{p_4}(\eta,\eta'')\mathcal{G}^{s*}_{p_{12}}(\eta',\eta'') + 23 \text{ perm.}
\end{aligned} \tag{6.48}$$

where, in deriving this expression, we have used that $\mathcal{G}_p(\eta,\eta') = \mathcal{G}^*_p(\eta',\eta)$.

We can now focus on the last contribution $\langle O(\eta)\rangle_{(1,1)}$. Repeating the very same steps we get

$$\langle \delta\phi_{\mathbf{p}_1}(\eta)\delta\phi_{\mathbf{p}_2}(\eta)\delta\phi_{\mathbf{p}_3}(\eta)\delta\phi_{\mathbf{p}_4}(\eta)\rangle_{(1,1)} = \frac{1}{4}(2\pi)^3\delta^3(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)\sum_{s=+,\times} p_1^i p_2^j \epsilon_{ij}^s(\mathbf{p}_1+\mathbf{p}_2)\times$$
$$\times\, p_3^m p_4^n \epsilon_{mn}^s(\mathbf{p}_3+\mathbf{p}_4)\int_{-\infty}^{\eta} d\eta'\, a(\eta')\int_{-\infty}^{\eta} d\eta''\, a(\eta'')\, \mathcal{G}_{p_1}^*(\eta,\eta')\mathcal{G}_{p_2}^*(\eta,\eta')\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}_{p_4}(\eta,\eta'')\mathcal{G}_{p_{12}}^s(\eta',\eta'') + 23 \text{ perm.} \tag{6.49}$$

where we notice that there is no nested time integral. In order to write the total 4PCF in a more compact way, we can recast the last $(1,1)$ in two contributions with a nested time integration. In fact, given $f = f(\eta,\eta',\eta'')$ a (complex) function of the time variable, it is possible to express its double integration over η' and η'' as the sum of two nested integrations:

$$\begin{aligned}\int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta} d\eta''\, f(\eta,\eta',\eta'') &= \\ &= \left(\int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta'} d\eta'' + \int_{-\infty}^{\eta} d\eta'' \int_{-\infty}^{\eta''} d\eta'\right) f(\eta,\eta',\eta'') \\ &= \int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta'} d\eta'' f(\eta,\eta',\eta'') + \int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta'} d\eta'' f(\eta' \leftrightarrow \eta'')\ .\end{aligned} \tag{6.50}$$

Applying this identity, the $(1,1)$-type contribution can be recast as:

$$\langle \delta\phi_{\mathbf{p}_1}(\eta)\delta\phi_{\mathbf{p}_2}(\eta)\delta\phi_{\mathbf{p}_3}(\eta)\delta\phi_{\mathbf{p}_4}(\eta)\rangle_{(1,1)} = \frac{1}{4}(2\pi)^3\delta^3(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)$$
$$\sum_{s=+,\times} p_1^i p_2^j \epsilon_{ij}^s(\mathbf{p}_1+\mathbf{p}_2) p_3^m p_4^n \epsilon_{mn}^s(\mathbf{p}_3+\mathbf{p}_4)\int_{-\infty}^{\eta} d\eta'\, a(\eta')\int_{-\infty}^{\eta} d\eta''\, a(\eta'')$$
$$\left[\mathcal{G}_{p_1}^*(\eta,\eta')\mathcal{G}_{p_2}^*(\eta,\eta')\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}_{p_4}(\eta,\eta'')\mathcal{G}_{p_{12}}^s(\eta',\eta'') + \mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')\mathcal{G}_{p_3}^*(\eta,\eta'')\mathcal{G}_{p_4}^*(\eta,\eta'')\mathcal{G}^{s*}{}_{p}(\eta',\eta'')\right] + 23 \text{ perm.} \tag{6.51}$$

Collecting everything together, the total graviton mediated trispectrum is given by [67]

$$\langle \delta\phi_{\mathbf{p}_1}(\eta)\delta\phi_{\mathbf{p}_2}(\eta)\delta\phi_{\mathbf{p}_3}(\eta)\delta\phi_{\mathbf{p}_4}(\eta)\rangle = \frac{1}{4}(2\pi)^3\delta^3(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)\sum_{s=+,\times} p_1^i p_2^j \epsilon_{ij}^s(\mathbf{p}_1+\mathbf{p}_2) p_3^m p_4^n \epsilon_{mn}^s(\mathbf{p}_3+\mathbf{p}_4)$$
$$\int_{-\infty}^{\eta} d\eta'\, a(\eta')\int_{-\infty}^{\eta} d\eta''\, a(\eta'')\Big[\mathcal{G}_{p_1}^*(\eta,\eta')\mathcal{G}_{p_2}^*(\eta,\eta')\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}_{p_4}(\eta,\eta'')\mathcal{G}_{p_{12}}^s(\eta',\eta'') + \mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')$$
$$\mathcal{G}_{p_3}^*(\eta,\eta'')\mathcal{G}_{p_4}^*(\eta,\eta'')\mathcal{G}^{s*}{}_{p_{12}}(\eta',\eta'') - \mathcal{G}_{p_1}^*(\eta,\eta')\mathcal{G}_{p_2}^*(\eta,\eta')\mathcal{G}_{p_3}^*(\eta,\eta'')\mathcal{G}_{p_4}^*(\eta,\eta'')\mathcal{G}^{s*}{}_{p}(\eta',\eta'') - \mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')$$
$$\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}_{p_4}(\eta,\eta'')\mathcal{G}_{p}^s(\eta',\eta'')\Big] + 23 \text{ perm.}$$
$$= (2\pi)^3\delta\left(\sum_{a=1}^{4}\mathbf{p}_a\right)\mathcal{P}_{\delta\phi}(p_1)\mathcal{P}_{\delta\phi}(p_3)\mathcal{P}_{\gamma}(p_{12})\frac{1}{4p_2^3p_4^3}\sum_{s=+,\times}\epsilon_{ij}^s(\mathbf{p}_1+\mathbf{p}_2)\epsilon_{mn}^s(\mathbf{p}_3+\mathbf{p}_4)p_1^i p_2^j p_3^m p_4^n\, I_{1234} + 23 \text{ perm.} \tag{6.52}$$

with

$$I_{1234} = \int_{-\infty}^{\eta}\frac{d\eta'}{\eta'^2}\int_{-\infty}^{\eta'}\frac{d\eta''}{\eta''^2}\,\mathrm{Im}\Big[(1+ip_1\eta)(1-ip_1\eta')(1+ip_2\eta)(1-ip_2\eta')e^{i(p_1+p_2)(\eta'-\eta)}\Big]\times$$
$$\times\,\mathrm{Im}\Big[(1+ip_3\eta)(1-ip_3\eta'')(1+ip_4\eta)(1-ip_4\eta'')(1+ip_{12}\eta')(1-ip_{12}\eta'')e^{i(p_3+p_4)(\eta''-\eta)+ip_{12}(\eta''-\eta')}\Big] \tag{6.53}$$

and $\mathcal{P}_{\delta\phi}(p_1)$, $\mathcal{P}_{\delta\phi}(p_3)$ and $\mathcal{P}_{\gamma}(p_{12})$ being the dimensionful power spectra for $\delta\phi$ and γ in de-Sitter

$$\mathcal{P}_{\delta\phi}(p) = \frac{H^2}{2p^3}\,, \qquad \mathcal{P}_{\gamma}(p) = \frac{H}{M_{pl}^2\, p^3}\,. \tag{6.54}$$

In the last passage of 6.52 we have substituted the explicit form of $\mathcal{G}_p$ and used the identity $\mathrm{Im}(z) = (z - z^*)/2i$. In the last passage of (6.52) we have factorize the prefactor in such a way to write it as the product of three power spectra and some combination of the momenta for convenience. Moreover, at this stage, we chose to keep power spectra depending on momenta p_1, p_3 and p_{12} since they provide information regarding both vertices and the exchanged graviton. However, it would have been equivalent to keep any other combination of power spectra which entails information of both the vertices.

Let us now work out permutations. We notice that $\langle O(\eta)\rangle$ is invariant under permutation of two legs attached to the same vertex. Therefore, the total number of permutations gets decreased from 24 to six:

$$\begin{aligned}\langle\delta\phi_{\mathbf{p}_1}\delta\phi_{\mathbf{p}_2}\delta\phi_{\mathbf{p}_3}\delta\phi_{\mathbf{p}_4}\rangle' = \sum_{s=+,\times} 4\Bigg[&\mathcal{P}_{\delta\phi}(p_1)\mathcal{P}_{\delta\phi}(p_3)\mathcal{P}_{\gamma}(p_{12})\frac{1}{4p_2^3p_4^3}\epsilon^s_{ij}(\mathbf{p}_{12})\epsilon^s_{mn}(\mathbf{p}_{34})\, p_1^ip_2^jp_3^mp_4^n\cdot(I_{1234}+I_{3412})\\
&+\mathcal{P}_{\delta\phi}(p_1)\mathcal{P}_{\delta\phi}(p_2)\mathcal{P}_{\gamma}(p_{13})\frac{1}{4p_3^3p_4^3}\epsilon^s_{ij}(\mathbf{p}_{13})\epsilon^s_{mn}(\mathbf{p}_{24})\, p_1^ip_3^jp_2^mp_4^n\cdot(I_{1324}+I_{2413})\\
&+\mathcal{P}_{\delta\phi}(p_1)\mathcal{P}_{\delta\phi}(p_2)\mathcal{P}_{\gamma}(p_{14})\frac{1}{p_4^3p_3^3}\epsilon^s_{ij}(\mathbf{p}_{14})\epsilon^s_{mn}(\mathbf{p}_{23})\, p_1^ip_4^jp_2^mp_3^n\cdot(I_{1423}+I_{2314}).\end{aligned} \tag{6.55}$$

In the previous equation $\langle\delta\phi_{\mathbf{p}_1}\delta\phi_{\mathbf{p}_2}\delta\phi_{\mathbf{p}_3}\delta\phi_{\mathbf{p}_4}\rangle' = \langle\delta\phi_{\mathbf{p}_1}\delta\phi_{\mathbf{p}_2}\delta\phi_{\mathbf{p}_3}\delta\phi_{\mathbf{p}_4}\rangle(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)$.

Polarization sum

Before working out the integrations, it is convenient to make the polarization sum more explicit. Take for example $\epsilon^s_{ij}(\mathbf{p}_{12})$, it can be rewritten as

$$\epsilon^+_{ij} = \mathbf{e}_i\mathbf{e}_j - \bar{\mathbf{e}}_j\bar{\mathbf{e}}_j \ , \qquad \epsilon^\times_{ij} = \mathbf{e}_i\bar{\mathbf{e}}_j + \bar{\mathbf{e}}_i\bar{\mathbf{e}}_j \tag{6.56}$$

with $\mathbf{e}$ and $\bar{\mathbf{e}}$ being two orthogonal unit vectors perpendicular to $\mathbf{p}_{12}$. In order to evaluate terms of the type $\epsilon^s_{ij}p^ip^j$ it is useful to write the momenta of the inflaton perturbation in spherical coordinates with $\mathbf{e}$, $\bar{\mathbf{e}}$ and $\hat{\mathbf{p}}_{12}$ identifying the three axes, with $\hat{\mathbf{p}} = \mathbf{p}/p$. Identifying $\cos\theta_a = \hat{\mathbf{p}}_a\cdot\hat{\mathbf{p}}_{12}$ and $\cos\phi_a = \hat{\mathbf{p}}_a\cdot\mathbf{e}$, the angles θ_a and ϕ_a denote the polar and azimuthal coordinates respectively and $\mathbf{p}_a = p_a\big(\sin\theta_a\cos\phi_a, \sin\theta_a\sin\phi_a, \cos\theta_a\big)$. The projection of $\mathbf{p}_a$ onto the direction identified by $\mathbf{e}$ and $\bar{\mathbf{e}}$ is then given by

$$\mathbf{e}\cdot\mathbf{p}_a = p_a\sin\theta_a\cos\phi_a \ , \qquad \bar{\mathbf{e}}\cdot\mathbf{p}_a = p_a\sin\theta_a\sin\phi_a \ . \tag{6.57}$$

Given the expressions of ϵ^+_{ij} and $\epsilon^\times_{ij}$ we find

$$\begin{aligned}\epsilon^+_{ij}(\mathbf{p}_{12})\, p_1^ip_2^j = \big(\mathbf{e}_i\mathbf{e}_j - \bar{\mathbf{e}}_i\bar{\mathbf{e}}_j\big)p_1^ip_2^j &= p_1p_2\sin\theta_1\sin\theta_2\big(\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2\big)\\ &= p_1p_2\sin\theta_1\sin\theta_2\cos\left(\phi_1+\phi_2\right)\end{aligned} \tag{6.58}$$

$$\begin{aligned}\epsilon^\times_{ij}(\mathbf{p}_{12})\, p_1^ip_2^j = \big(\mathbf{e}_i\bar{\mathbf{e}}_j - \bar{\mathbf{e}}_i\mathbf{e}_j\big)p_1^ip_2^j &= p_1p_2\sin\theta_1\sin\theta_2\big(\cos\phi_1\sin\phi_2 - \sin\phi_1\cos\phi_2\big)\\ &= p_1p_2\sin\theta_1\sin\theta_2\sin\left(\phi_1+\phi_2\right)\end{aligned} \tag{6.59}$$

and the same relations hold for $\epsilon^+_{ij}\, p_3^ip_4^j$ and $\epsilon^\times_{ij}\, p_3^ip_4^j$. For further simplification, it is useful to notice that momenta $\mathbf{p}_{1(3)}$ and $\mathbf{p}_{2(4)}$ have the same measure but opposite direction with respect to the plane perpendicular to $\mathbf{p}_{12}$. This implies the following relations between the polar and azimuthal angle of the two momenta: $p_1\sin\theta_1 = p_2\sin\theta_2$ and $\phi_2 = \phi_1+\pi$. The polarization sum can be then explicitly written as

$$\begin{aligned}\sum_{s=+,\times}\epsilon^s_{ij}\epsilon^s_{mn}\, p_1^ip_2^jp_3^mp_4^n =& p_1^2p_3^2\sin^2\theta_1\sin^2\theta_3\cos\left(2\phi_1-2\phi_3\right)\\ &= p_1^2p_3^2\sin^2\theta_1\sin^2\theta_3\cos 2\chi_{12,34}\end{aligned} \tag{6.60}$$

and the graviton mediated trispectrum reads

$$\begin{aligned}\langle \delta\phi_{\mathbf{p}_1}\delta\phi_{\mathbf{p}_2}\delta\phi_{\mathbf{p}_3}\delta\phi_{\mathbf{p}_4}\rangle' = &\mathcal{P}_{\delta\phi}(p_1)\mathcal{P}_{\delta\phi}(p_3)\mathcal{P}_{\gamma}(p_{12})\frac{p_1^2p_3^2}{p_2^3p_4^3}\sin^2\theta_1\sin^2\theta_3\cos 2\chi_{12,34}\cdot(I_{1234}+I_{3412})\\ &+\mathcal{P}_{\delta\phi}(p_1)\mathcal{P}_{\delta\phi}(p_2)\mathcal{P}_{\gamma}(p_{13})\frac{p_1^2p_2^2}{p_3^3p_4^3}\sin^2\theta_1\sin^2\theta_2\cos 2\chi_{13,24}\cdot(I_{1324}+I_{2413})\\ &+\mathcal{P}_{\delta\phi}(p_1)\mathcal{P}_{\delta\phi}(p_2)\mathcal{P}_{\gamma}(p_{14})\frac{p_1^2p_2^2}{p_4^3p_3^3}\sin^2\theta_1\sin^2\theta_2\cos 2\chi_{14,23}\cdot(I_{1423}+I_{2314})\Bigg].\end{aligned} \tag{6.61}$$

Time integration

We are now in the position to solve time integrals. It can be useful to make a change of variable such that the integration variable becomes dimensionless. In this way, the dimensionful part can be factorized outside the integral and the integration would return some dimensionless combination of momenta. This allows to give a first estimate of the trispectrum even before an explicit computation of integration is carried out. Keeping in mind that $[p] = E$ and $[\eta] = E^{-1}$ we see that the combination $p \cdot \eta$ is dimensionless. Thus, we can define the new integration variable as $x = P_T \cdot \eta$ with $P_T = p_1 + p_2 + p_3 + p_4$. In this way, the trispectrum becomes

$$\langle \delta\phi_{\mathbf{p}_1}\delta\phi_{\mathbf{p}_2}\delta\phi_{\mathbf{p}_3}\delta\phi_{\mathbf{p}_4}\rangle' = \mathcal{P}_{\delta\phi}(p_1)\mathcal{P}_{\delta\phi}(p_3)\mathcal{P}_{\gamma}(p_{12})\frac{p_1^2p_3^2p_{\mathrm{T}}^2}{p_2^3p_4^3}\sin^2\theta_1\sin^2\theta_3\cos 2\chi_{12,34}\cdot I_{1234}^{\mathrm{adim}} + \text{perm.} \tag{6.62}$$

with

$$\begin{aligned}I_{1234}^{\mathrm{adim}} = &\int_{-\infty}^{x}\frac{dx'}{x'^2}\int_{-\infty}^{x'}\frac{dx''}{x''^2}\,\mathrm{Im}\Big[(1+ik_1x)(1-ik_1x')(1+ik_2x)(1-ik_2x')e^{i(k_1+k_2)(x'-x)}\Big]\times\\ &\times\mathrm{Im}\Big[(1+ik_3x)(1-ik_3x'')(1+ik_4x)(1-ik_4x'')(1+ik_{12}x')(1-ik_{12}x'')e^{i(k_3+k_4)(x''-x)+ik_{12}(x''-x')}\Big]\end{aligned} \tag{6.63}$$

where $k_i = p_i/p_{\mathrm{T}}$ is a dimensionless parameter. The integration returns a complicated function of k_i (A.26'). Finally, the trispectrum can be recast in the following way:

$$\langle \delta\phi_{\mathbf{p}_1}\delta\phi_{\mathbf{p}_2}\delta\phi_{\mathbf{p}_3}\delta\phi_{\mathbf{p}_4}\rangle' = \mathcal{P}_{\delta\phi}(p_1)\mathcal{P}_{\delta\phi}(p_3)\mathcal{P}_{\gamma}(p_{12})S(p_1,p_2,p_3,p_4)\sin^2\theta_1\sin^2\theta_3\cos 2\chi_{12,34} + 5\text{ perm.} \tag{6.64}$$

with $S(p_1,p_2,p_3,p_4) = p_1^2p_3^2p_T^2/p_2^3p_4^3 \times I_{1234}^{\mathrm{adim}}$ being the shape function. This function collapses to a number for a specific momentum configuration.

Chapter 7

Parity violation in slow-roll inflation

As we saw in Ch.5, the presence of a gravitational Chern-Simons to GR is associated with the production of chiral gravitational waves, leading to parity violating signatures already at the level of the tensor power. However, the amount of circular polarization generated by gravitational Chern-Simons interaction in the early universe is proportional to the parameter $H/M_{\rm CS}$, which is set to be much smaller than one to avoid propagation of ghost modes, making this effect unlikely to be detected in the upcoming years. This observation makes the study of alternative observables bearing information on the violation of parity symmetry compelling. Specifically, in [56] a complete analysis on the parity violating signatures entailed in the mixed bispectrum[1] was carried out, pointing out the presence of a relatively large parity breaking in the bispectrum $\langle\gamma\gamma\zeta\rangle$. Observationally, these correlators can be detected considering the mixed CMB bispectra among temperature T and E/B polarization modes: while T anisotropies are sourced mainly by scalar perturbations, B polarization modes are seeded only by primordial gravitational waves [68]. A similar analysis has been carried out in [69] in chiral scalar tensor (CST) theories of gravity, a generalization of CS interaction that introduces gravitational parity-breaking operators with higher number of time derivatives.

However, bispectra are not the only observables capturing parity violation: going to the next order in perturbation theory, the trispectrum carries information about the breaking of P symmetry as well. Moreover, in case we are interested in *scalar* correlators, the trispectrum is the lowest order statistics capturing effects of parity violation. The reason why this happens is rooted in the geometrical configuration of momenta, associated to the mode functions, in Fourier space, and will be clarified in this chapter. Since the graviton is chiral, we expect that the only parity violating contribution comes from the graviton-mediated trispectrum. This trispectrum has been investigated in the literature in the setting of slow-roll inflation and CS gravity [30, 32] and in case of CST theories of gravity and slow-roll [31]. From these studies, it does not seem possible for slow-roll models of inflation to produce a primordial parity-odd (PO) trispectrum capable of leaving imprints in the large distribution of galaxies: the signal is simply too suppressed. In the case of CST theories of gravity, there is room for a less suppressed signal, which can be detectable. This is due to the fact that the CST Lagrangian contains couplings that do not need to be suppressed to avoid ghost modes.In our case, we stick to CS gravity: since scalar modes are amplified by the USR dynamics, we can hope for an enhancement of the PO signal as well.

In this chapter we will briefly review the results for the graviton-mediated PO trispectrum in slow-roll inflation and CS gravity. In particular we will focus on the derivation of the *scalar-scalar-tensor* vertices which are the building blocks for our calculation. To simplify, we will approximate the spacetime during inflation as de Sitter, and scalar perturbations will be taken as massless. We will conclude with some considerations on relaxing these approximations.

[1]The bispectrum is defined as the Fourier transform of the three point correlation function. It can be computed using the in-in formalism in a way which is analogous to the one showed in Sec.6.2.1 for the trispectrum.

7.1 Why the trispectrum?

A parity (P) transformation is a spacetime transformation under which the sign of space coordinates is flipped:

$$\mathbf{P}: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}.$$

Despite scalars being invariant under parity transformation, their statistics may not be. In particular, it is possible to detect whether or not there had been parity violating interactions in the early universe by studying the scalar trispectrum, namely $\langle\zeta(k_1,\eta)\zeta(k_2,\eta)\zeta(k_3,\eta)\zeta(k_4,\eta)\rangle$. It is defined as the Fourier transform of the four-point correlation function (4PCF).

Since the universe is homogeneous and isotropic, cosmological correlators are invariant under spatial translation and spatial rotation. This invariance under spatial symmetries constraints the way in which correlators can be sensitive to parity. In fact it can be shown that performing a parity transformation on a power spectrum or a bispectrum is equivalent to a rotation. Thus, parity transformation cannot leave any imprints on them. For a trispectrum instead, this is possible: there is no way in which the parity transformed trispectrum can be obtained by implementing rotations and translations.

This very simple argument can be made more quantitative. Let us consider the general form of the power spectrum, bispectrum and trispectrum of real fields. Due to homogeneity (invariance under translation) of spacetime, two-, three- and four-point correlation function of real fields X, Y, Z, W, must be written in the following form [70]

$$\begin{aligned}
\langle X(\mathbf{k}_1)Y(\mathbf{k}_2)\rangle &\equiv P_{XY}(\mathbf{k}_1)(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2)\\
\langle X(\mathbf{k}_1)Y(\mathbf{k}_2)Z(\mathbf{k}_3)\rangle &\equiv B_{XYZ}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\\
\langle X(\mathbf{k}_1)Y(\mathbf{k}_2)Z(\mathbf{k}_3)W(\mathbf{k}_4)\rangle &\equiv T_{XYZW}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4)(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)
\end{aligned} \tag{7.1}$$

Imposing rotational invariance, the correlator can only depend on scalar function constructed out of the wavenumbers $\mathbf{k}_i$. In three dimensions, the only scalar functions we can build are $k_i = |\mathbf{k}_i|$, $\mu_{ij} = \hat{\mathbf{k}}_i \cdot \hat{\mathbf{k}}_j$, and the pseudo-scalar $\hat{\mathbf{k}}_i \cdot \hat{\mathbf{k}}_j \times \hat{\mathbf{k}}_k$. Since the momentum conservation imposes the momenta to close in geometrical configurations, the first and the second scalar quantities can be geometrically interpreted as the length of one side, the cosine of the angle comprised between two sides and the folding angle since $\cos\theta_{ij} \equiv \hat{\mathbf{k}}_i \cdot \hat{\mathbf{k}}_j$. The pseudo-scalar triple product can be related to the sine of the dihedral angle χ comprised between the plane $\{\mathbf{k}_1,\mathbf{k}_2\}$ and $\{\mathbf{k}_3,\mathbf{k}_4\}$[2].

Since the only scalar that can be constructed out of one vector is its modulus, the power spectrum will depend just on k_1: $P_{XY}(\mathbf{k}_1) = P_{XY}(k_1)$. The bispectrum cannot contain the pseudo-scalar triple product, since the three momenta are not all independent: conservation of momentum implies $\mathbf{k}_1 = -\mathbf{k}_2 - \mathbf{k}_3$ for example. So $\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2 \times \hat{\mathbf{k}}_3 = (-\hat{\mathbf{k}}_2 - \hat{\mathbf{k}}_3)\cdot \hat{\mathbf{k}}_2 \times \hat{\mathbf{k}}_3 = 0$, since the vector product $\hat{\mathbf{k}}_2 \times \hat{\mathbf{k}}_3$ returns a vector which is perpendicular to both $\mathbf{k}_2$ and $\mathbf{k}_3$. This is exactly consistent to the fact that conservation of momenta forces the vectors $\mathbf{k}_i$ to close in a triangle, a planar figure. Moreover, since triangles are completely specified by three parameters among amplitudes and angles, $B_{XYZ}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = B_{XYZ}(k_1,k_2,k_3,\mu_{12},\mu_{13},\mu_{23},\mathbf{k}_1\cdot\mathbf{k}_2\times\mathbf{k}_3) = B_{XYZ}(k_1,k_2,k_3)$. In the trispectrum and higher order correlators the triple product can be non vanishing, since the momentum conservation makes only one momentum independent out of four. We can explicitly show that the pseudo-scalar triple product is proportional to the sine of the dihedral angle $\sin\chi \propto (\hat{\mathbf{k}}_1 \times \hat{\mathbf{k}}_2)\times(\hat{\mathbf{k}}_3\times\hat{\mathbf{k}}_4)$. We can exploit the property $(A\times B)\times(C\times D) = C(A,B,D) - D(A,B,C)$ with $(A,B,C) = A\cdot B\times C$ and write

$$\sin\chi \propto \hat{\mathbf{k}}_3(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_4) - \hat{\mathbf{k}}_4(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_3) = -\hat{\mathbf{k}}_3(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_3) - \hat{\mathbf{k}}_4(\hat{\mathbf{k}}_1,\hat{\mathbf{k}}_2,\hat{\mathbf{k}}_3) \propto \hat{\mathbf{k}}_1\cdot(\hat{\mathbf{k}}_2\times\hat{\mathbf{k}}_3)$$

after employing the momentum conservation condition $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4 = 0$. This condition also forces the momenta to close in a tetrahedron, parameterized by six parameters. In this thesis we

[2]This is only one of the four possibilities. Without loss of generality we could have chosen any other the angle spanned between any other couple of planes identified by pairs of the external momenta $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ and $\mathbf{k}_4$.

choose to work with the parameters k_i, one diagonal $k_{12} = |\mathbf{k}_1 + \mathbf{k}_2|$ and the pseudo-scalar triple product $\hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2 \times \hat{\mathbf{k}}_3$: $T_{XYZW}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = T_{XYZW}(k_1, k_2, k_3, k_4, |\mathbf{k}_1 + \mathbf{k}_2|, \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2 \times \hat{\mathbf{k}}_3)$.

The trispectrum is the lowest order statistics statistics exhibiting the pseudo-scalar triple product, which makes it sensitive to parity symmetry. We can write the trispectrum as given by two contributions: a parity even (PE) contribution which do not change sign under parity transformation and a parity odd (PO) term, which, instead, changes sign:

$$T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = T^{\mathrm{PE}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) + T^{\mathrm{PO}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4).$$

The parity odd component turns out to be purely imaginary. Indeed,

$$T^{\mathrm{PO}}(-\mathbf{k}_1, -\mathbf{k}_2, -\mathbf{k}_3, -\mathbf{k}_4) = T^{\mathrm{PO}^*}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = -T^{\mathrm{PO}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$$

which means that $T^{\mathrm{PO}} = i\mathcal{T}$, with $\mathcal{T}$ a purely real quantity. On the contrary, the PE trispectrum is real. Clearly, only the PO trispectrum is proportional to the pseudo-scalar triple product, even if we have to pay attention on the momenta configuration: some momenta configurations, such as the collapsed (counter-collinear limit) lead the trispectrum to collapse on a planar figure and, therefore, the PO contribution in this configuration is expected to be vanishing.

Since in our model of interest parity symmetry is violated in the tensor sector, it is the graviton mediated trispectrum that develops a PO contribution.

7.2 Gauge choice

We are interested in the deriving the expression for the scalar trispectrum from the exchange of a graviton in slow-roll inflation. As we saw in Sec.2.2, this means that all slow-roll parameters ϵ_n are much smaller than unity, and we can use them as expansion parameters. Before proceeding with the evaluation of the interaction Hamiltonian H_I, we fix the gauge to avoid the propagation of unphysical degrees of freedom. Even if the goal of this chapter is to compute the correlator $\langle\zeta\zeta\zeta\zeta\rangle$, it is more suitable to work in the spatially flat gauge [7]:

$$\phi(\mathbf{x}, t) = \phi_0(t) + \delta\phi(\mathbf{x}, t) \quad h_{ij} = a^2(\delta_{ij} + \gamma_{ij} + \dots), \quad \gamma^i_i = 0, \quad \partial_i \gamma_{ii} = 0. \tag{7.2}$$

This gauge is useful to immediately identify the terms that are leading order in slow-roll expansion. Being interested in the 4PCF of curvature perturbations ζ, it is possible to switch to the comoving gauge (3.46) by means of the well known non linear relation [7]:

$$\begin{aligned}
\zeta =& \zeta_1 + \frac{1}{2}\frac{\ddot{\phi}}{\dot{\phi}H}\zeta_1^2 + \frac{1}{4}\frac{\dot{\phi}^2}{H^2}\zeta_1^2 \\
&+ \frac{1}{H}\dot{\zeta}_1\zeta_1 - \frac{1}{4}\frac{a^{-2}}{H^2}\left(\partial\zeta_1\right)^2 + \frac{1}{4}\frac{a^{-2}}{H^2}\partial^{-2}\partial_i\partial_j\left(\partial_i\zeta_1\partial_j\zeta_1\right) + \frac{1}{2}\frac{1}{H}\partial_i\chi\partial_i\zeta_1 \\
&- \frac{1}{2}\frac{1}{H}\partial^{-2}\partial_i\partial_j\left(\partial_i\chi\partial_j\zeta_1\right) - \frac{1}{4}\frac{1}{H}\dot{\gamma}_{ij}\partial_i\partial_j\zeta_1
\end{aligned} \tag{7.3}$$

where ζ_1 is the first order value of ζ

$$\zeta_1 = -\frac{H}{\dot{\phi}}\delta\phi. \tag{7.4}$$

7.3 Computation of the cubic interaction vertices

Since the goal of this chapter is to compute the PO and PE trispectra, we need to compute the total Lagrangian up to third order. In particular we are interested in the computation of the graviton-mediated trispectrum which requires *scalar-scalar-tensor* terms. Such interactions arise both from the slow-roll inflationary action (2.12), in particular from the inflaton kinetic term of the inflaton field, and the Chern-Simons action (5.11).

7.3.1 Standard gravity action

Let us consider the single-field inflationary action in case gravity is described by GR. We expect that, in the spatially flat gauge, the action will be approximately the one of a massless scalar field ϕ at leading order in slow roll. Indeed, recalling that the single field inflationary action can be written as function of the ADM variables as 3.41, by following exactly the same passages done in 3.6, it is possible to obtain the values for the non-dynamical variables as function of the inflaton fluctuations at first order in slow-roll approximation[3] [7]:

$$N = \sqrt{\frac{\epsilon}{2}}\delta\phi \;\; , \quad \bar{N}_i = \epsilon\partial_i\partial^{-2}\frac{d}{d\eta}\Big[-\frac{H}{\dot{\phi}}\delta\phi\Big] \simeq -\sqrt{\frac{\epsilon}{2}}\big(\partial_i\partial^{-2}\delta\phi'\big). \tag{7.5}$$

The leading-order term of the inflationary action is then the kinetic term

$$\frac{1}{2}\sqrt{-g}g^{\mu\nu}\partial_\mu\delta\phi\partial_\nu\delta\phi = \frac{1}{2}a^2\eta^{\mu\nu}\partial_\mu\delta\phi\partial_\nu\delta\phi + \frac{1}{2}a^2\gamma^{ij}\partial_i\delta\phi\partial_j\delta\phi \tag{7.6}$$

and the scalar-scalar-tensor action is:

$$S^{sst}_{\mathrm{GR}} = -\frac{1}{2}\int d^3x d\eta\, a^2(\eta)\gamma^{ij}\partial_i\delta\phi\partial_j\delta\phi. \tag{7.7}$$

It is useful to write the action in Fourier space. Keeping in mind that the Fourier modes of the inflaton field perturbations and tensor perturbation are expressed as

$$\begin{aligned} \delta\phi(\eta,\mathbf{x}) &= \int\frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\delta\phi_{\mathbf{k}}(\eta) \;\; , \\ \gamma_{ij}(\eta,\mathbf{x}) &= \int\frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\sum_{s=+,\times}\epsilon^s_{ij}(\mathbf{k})\gamma^s_{\mathbf{k}}(\eta) \end{aligned} \tag{7.8}$$

where the polarization tensors are normalized so that $\sum_{s,l}\epsilon^s_{ij}\epsilon^{ij,l} = 2\delta^{sl}$, the Lagrangian takes the form

$$\mathcal{L}^{sst}_{\mathrm{GR}} = \frac{1}{2}\,a^2(\eta)\int\Big(\prod_{a=1}^{3}\frac{d^3k_a}{(2\pi)^3}\Big)(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\sum_{s=+,\times}k_1^ik_2^j\epsilon^s_{ij}(\mathbf{k}_3)\delta\phi_{\mathbf{k}_1}(\eta)\delta\phi_{\mathbf{k}_2}(\eta)\gamma^s_{\mathbf{k}_3}(\eta) \tag{7.9}$$

and the correspondent interaction Hamiltonian is $H_{\mathrm{GR}} = -\mathcal{L}^{sst}_{\mathrm{GR}}$.

7.3.2 Dynamical Chern-Simons action

As we saw in Sec.5.1.2, the introduction of the gravitational CS correction affects the evolution of tensor modes, causing the tensor power spectrum to take two different values for the two helicity modes. In order to investigate parity violating effects at the level of scalar trispectrum, we need to compute the trilinear *scalar-scalar-tensor* vertex arising from that correction. In the followin calculation, we shall set $M_{\mathrm{Pl}} = 1$ and we will restore it at the end.

In CS gravity, the *sst* cubic action in the spatially flat gauge arises from the expansion of the coupling in 5.11, leading to [32]:

$$S_{\mathrm{CS}} = -\frac{1}{8}(\partial_\phi\omega)\int d^4x\sqrt{-g}\,\big[\delta\phi' K^0 + (\partial_i\delta\phi)K^i\big]. \tag{7.10}$$

Notice that, as $\omega(\phi)$ is taken to satisfy (5.40), $\partial_\phi\omega(\phi)$ can be approximated as constant and taken outside the integral. As we did in Sec.5.1.2, we exploit the property of conformal invariance of the Potryagin density, switching to the conformally transformed metric 5.22 for our computation. Since in the action there is already the first-order quantity $\delta\phi$ appearing and the CS topological current is

[3] Slow-roll approximation is employed since slow-roll parameters are treated as constants and can be brought outside time derivatives.

schematically given by $K \sim \Gamma\partial\Gamma + \Gamma\Gamma\Gamma$, we only need to compute the CS four-current up quadratic order. Furthermore, given that 5.22 has no dependence on the scale factor in the zero order quantities, $\Gamma^{(0)}$ vanishes. As noticed in Sec.5.1.2, the first order Christoffel symbols are given by (5.24) and so the Chern-Simons topological current can be written as follows:

$$K^{\mu\,(2)} = \epsilon^{\mu\nu\alpha\beta}\,\Gamma^{\sigma\,(1)}_{\nu\rho}\partial_\alpha\Gamma^{\rho\,(1)}_{\beta\sigma}. \tag{7.11}$$

Let us work out the time and spatial component of the current K. Starting with the time component of the CS current, it is explicitly written as:

$$K^0 = \epsilon^{0ijk}\,\Gamma^\sigma_{i\rho}\partial_j\Gamma^\rho_{k\sigma} = \epsilon^{0ijk}\left[\Gamma^0_{i0}\partial_j\Gamma^0_{k0} + \Gamma^0_{in}\partial_j\Gamma^n_{k0} + \Gamma^m_{i0}\partial_j\Gamma^0_{km} + \Gamma^m_{in}\partial_j\Gamma^n_{km}\right] \tag{7.12}$$

where we have left implicit the superscript (2) and (1) indicating the order in perturbation theory of K and the Christoffel symbols respectively. In order to compute it, we cannot use the Christoffel symbols listed in (5.26). They were used in Sec.3.8 to compute the second-order tensorial action: since at linear order scalar and tensor perturbation do not mix, we neglect the scalar contributions coming from the Christoffel symbols. In this case however, we need to keep track of the dependence on scalar quantities in the Christoffel symbols since our aim is to go beyond linear perturbation theory. Given that the transformed metric components up to second order are

$$g_{00} = -1 - 2N + \ldots, \qquad g_{0i} = g_{i0} = \tilde{N}_i, \quad g_{ij} = \delta_{ij} + h_{ij} + \ldots \tag{7.13}$$

the Christoffel symbols up to quadratic order read:

$$\begin{aligned}
\Gamma^0_{00} &= N' + \ldots\\
\Gamma^0_{0i} &= \partial_i N + \ldots\\
\Gamma^0_{ij} &= -\frac{1}{2}\partial_i\tilde{N}_j - \frac{1}{2}\partial_j\tilde{N}_i + \frac{1}{2}\gamma'_{ij} + \ldots\\
\Gamma^i_{00} &= \partial_i N + \tilde{N}'_i + \ldots\\
\Gamma^i_{0j} &= \frac{1}{2}\partial_j\tilde{N}_i - \frac{1}{2}\partial_i\tilde{N}_j + \frac{1}{2}\gamma'_{ij} + \ldots\\
\Gamma^i_{jk} &= \frac{1}{2}\partial_k\gamma_{ij} + \frac{1}{2}\partial_j\gamma_{ik} - \frac{1}{2}\partial_i\gamma_{jk} + \ldots.
\end{aligned} \tag{7.14}$$

The zeroth component of the CS topological at second order takes the form:

$$\begin{aligned}
K^{0\,(2)} = \epsilon^{0ijk}\Big[&\frac{1}{2}\big(\partial_n\tilde{N}_i\big)\big(\partial_j\partial_m\tilde{N}_k\big) - \frac{1}{2}\big(\partial_n\tilde{N}_i\big)\big(\partial_j\gamma'_{kn}\big) - \frac{1}{2}\gamma'_{in}\big(\partial_j\partial_n\tilde{N}_k\big) + \frac{1}{2}\gamma'_{in}\big(\partial_j\gamma'_{kn}\big)\\
&+ \frac{1}{2}\big(\partial_n\gamma_{im}\big)\big(\partial_j\partial_m\gamma_{kn}\big) - \frac{1}{2}\big(\partial_n\gamma_{im}\big)\big(\partial_j\partial_n\gamma_{km}\big)\Big].
\end{aligned} \tag{7.15}$$

We proceed to computing the spatial component of K:

$$\begin{aligned}
K^i = \epsilon^{0ijk}\Big[&\frac{1}{2}\big(\partial_m\tilde{N}_j\big)\big(\partial_m\tilde{N}'_k\big) - \frac{1}{2}\big(\partial_m\tilde{N}_j\big)\gamma''_{km} - \frac{1}{2}\gamma'_{mj}\big(\partial_m\tilde{N}'_k\big) + \frac{1}{2}\big(\partial_n\gamma_{mj}\big)\big(\partial_m\gamma'_{kn}\big)\\
&- \frac{1}{2}\big(\partial_n\gamma_{mj}\big)\big(\partial_n\gamma'_{km}\big) + \frac{1}{2}\gamma'_{mj}\gamma''_{km} + 2\big(\partial_m N\big)\big(\partial_j\partial_m\tilde{N}_k\big) - 2\big(\partial_m N\big)\big(\partial_j\gamma'_{km}\big)\\
&+ \tilde{N}'_m\big(\partial_j\partial_m\tilde{N}_k\big) - \big(\partial_n\tilde{N}_m\big)\big(\partial_j\partial_m\gamma_{kn}\big) + \big(\partial_n\tilde{N}_m\big)\big(\partial_j\partial_n\gamma_{km}\big) - \tilde{N}'_m\big(\partial_j\gamma'_{km}\big) + \frac{1}{2}\gamma'_{mj}\gamma''_{km}\Big]
\end{aligned} \tag{7.16}$$

The third order Chern-Simosn action is explicitly given by

$$
\begin{aligned}
S_{CS}^{(3)} = -\frac{1}{8}(\partial_\phi \omega)\int d^4x\,\tilde{\epsilon}^{ijk}\Bigg\{ \Bigg[& \frac{1}{2}(\partial_m \tilde{N}_i)(\partial_j\partial_m \tilde{N}_k) - \frac{1}{2}(\partial_m \tilde{N}_i)(\partial_j \gamma'_{mk}) - \frac{1}{2}\gamma'_{im}(\partial_j\partial_m \tilde{N}_k) + \\
& \frac{1}{2}\gamma'_{im}(\partial_j\gamma'_{mk}) + \frac{1}{2}(\partial_n\gamma_{im})(\partial_j\partial_m\gamma_{kn}) - \frac{1}{2}(\partial_n\gamma_{im})(\partial_j\partial_n\gamma_{km})\Bigg]\delta\phi' \\
& + \Bigg[\frac{1}{2}(\partial_m\tilde{N}_j)(\partial_m\tilde{N}'_k) - \frac{1}{2}(\partial_m\tilde{N}_j)\gamma''_{km} - \frac{1}{2}\gamma'_{mj}(\partial_m\tilde{N}'_k) + \frac{1}{2}(\partial_n\gamma_{mj})(\partial_m\gamma'_{kn}) \\
& - \frac{1}{2}(\partial_n\gamma_{mj})(\partial_n\gamma'_{km}) + \frac{1}{2}\gamma'_{mj}\gamma''_{km} + 2(\partial_m N)(\partial_j\partial_m\tilde{N}_k) - 2(\partial_m N)(\partial_j\gamma'_{km}) \\
& + \tilde{N}'_m(\partial_j\partial_m\tilde{N}_k) - (\partial_n\tilde{N}_m)(\partial_j\partial_m\gamma_{kn}) + (\partial_n\tilde{N}_m)(\partial_j\partial_n\gamma_{km}) - \tilde{N}'_m(\partial_j\gamma'_{km}) \\
& + \frac{1}{2}\gamma'_{mj}\gamma''_{km}\Bigg]\partial_i\delta\phi\Bigg\}
\end{aligned} \tag{7.17}
$$

where blue terms are the ones contributing to the *scalar-scalar-tensor* interaction that we are interested in. Out of the *sst* contributions, some of them trivially vanish after substituting N and $\tilde{N}$ with their expressions (7.5). In fact, the third term of the first term vanishes

$$
-\tilde{\epsilon}^{ijk}\frac{1}{2}\gamma'_{im}(\partial_j\partial_m\tilde{N}_k) \simeq \frac{1}{2}\sqrt{\frac{\epsilon}{2}}\tilde{\epsilon}^{ijk}\gamma'_{im}(\partial_j\partial_m\partial_k\partial^{-2}\delta\phi') = 0
$$

since partial derivatives commute and are contracted to the Levi-Civita tensor. The first and second terms in the fifth line of (7.17) are equal. To see this, it is sufficient to take the first term (for example), substitute the expression for $\tilde{N}$ and rename the contracted indices $l \leftrightarrow m$. Therefore, the *scalar-scalar-tensor* third order Chern-Simons action reads:

$$
\begin{aligned}
S_{\mathrm{CS}}^{sst\,(3)} = -\frac{1}{8}(\partial_\phi\omega)\int d^4x\,\tilde{\epsilon}^{ijk}\Bigg[& -\frac{1}{2}\delta\phi'(\partial_m\tilde{N}_i)(\partial_j\gamma'_{mk}) - \frac{1}{2}(\partial_i\delta\phi)(\partial_m\tilde{N}_j)\gamma''_{km} - \frac{1}{2}(\partial_i\delta\phi)\gamma'_{mj}(\partial_m\tilde{N}'_k) \\
& -2(\partial_i\delta\phi)(\partial_m N)(\partial_j\gamma'_{km}) - (\partial_i\delta\phi)\tilde{N}'_m(\partial_j\gamma'_{km})\Bigg].
\end{aligned} \tag{7.18}
$$

Let us now work out the terms appearing in the action. After substituting the explicit expressions of the shift and the lapse functions and integrating by parts, the various terms reads:

$$
\begin{aligned}
\text{first integral} &= -\frac{1}{2}\int d^4x\,\tilde{\epsilon}^{ijk}(\delta\phi')(\partial_m\tilde{N}_i)(\partial_j\gamma'_{mk}) \simeq \frac{1}{2}\sqrt{\frac{\epsilon}{2}}\int d^4x\,\tilde{\epsilon}^{ijk}(\delta\phi)(\partial_m\partial_i\partial^{-2}\delta\phi')(\partial_j\gamma'_{mk}) \\
&= -\frac{1}{2}\sqrt{\frac{\epsilon}{2}}\int d^4x\,\tilde{\epsilon}^{ijk}\Big[(\delta\phi)(\partial_m\partial_i\partial^{-2}\delta\phi'')(\partial_j\gamma'_{mk}) + (\delta\phi)(\partial_m\partial_i\partial^{-2}\delta\phi')(\partial_j\gamma''_{mk})\Big] \\
&= \frac{1}{2}\sqrt{\frac{\epsilon}{2}}\int d^4x\,\epsilon^{ijk}\Big[(\partial_i\delta\phi)(\partial_m\partial^{-2}\delta\phi'')(\partial_j\gamma'_{mk}) + (\partial_i\delta\phi)(\partial_m\partial^{-2}\delta\phi')(\partial_j\gamma''_{mk})\Big] \\
&= \frac{1}{2}\sqrt{\frac{\epsilon}{2}}\int d^4x\,\tilde{\epsilon}^{ijk}\Big[-4aH(\partial_i\delta\phi)(\partial_m\partial^{-2}\delta\phi')(\partial_j\gamma'_{mk}) + (\partial_i\delta\phi)(\partial_m\delta\phi)(\partial_j\gamma'_{mk}) \\
&\quad + (\partial_i\delta\phi)(\partial_m\partial^{-2}\delta\phi')(\partial_j\partial^2\gamma_{mk})\Big]
\end{aligned} \tag{7.19}
$$

$$
\begin{aligned}
\text{second integral} &= -\frac{1}{2}\int d^4x\,\tilde{\epsilon}^{ijk}(\partial_i\delta\phi)(\partial_m\tilde{N}_j)\gamma''_{km} \simeq \frac{1}{2}\sqrt{\frac{\epsilon}{2}}\int d^4x\,\tilde{\epsilon}^{ijk}(\partial_i\delta\phi\partial)(\partial_m\partial_j\partial^{-2}\delta\phi')\gamma''_{mk} \\
&= \frac{1}{2}\sqrt{\frac{\epsilon}{2}}\int d^4x\,\tilde{\epsilon}^{ijk}\Big[-2aH(\partial_i\delta\phi)(\partial_j\partial_m\partial^{-2}\delta\phi')\gamma'_{mk} + (\partial_i\delta\phi)(\partial_j\partial_m\partial^{-2}\delta\phi')(\partial^2\gamma_{mk})\Big] \\
&= \frac{1}{2}\sqrt{\frac{\epsilon}{2}}\int d^4x\,\tilde{\epsilon}^{ijk}\Big[2aH(\partial_i\delta\phi)(\partial_m\partial^{-2}\delta\phi')(\partial_j\gamma'_{mk}) - (\partial_i\delta\phi)(\partial_m\partial^{-2}\delta\phi')(\partial_j\partial^2\gamma_{mk})\Big]
\end{aligned} \tag{7.20}
$$

$$
\begin{aligned}
\text{third integral} &= -\frac{1}{2}\int d^4x\, \tilde{\epsilon}^{ijk}\big(\partial_i\delta\phi\big)\gamma'_{mj}\big(\partial_m\tilde{N}'_m\big) \simeq \frac{1}{2}\sqrt{\frac{\epsilon}{2}}\int d^4x\, \tilde{\epsilon}^{ijk}\big(\partial_i\delta\phi\big)\gamma'_{mj}\big(\partial_m\partial_k\partial^{-2}\delta\phi''\big) \\
&= \sqrt{\frac{\epsilon}{2}}\int d^4x\, \tilde{\epsilon}^{ijk}\Big[-aH\big(\partial_i\delta\phi\big)\big(\partial_m\partial^{-2}\delta\phi'\big)\big(\partial_j\gamma'_{mk}\big) + \frac{1}{2}\big(\partial_i\delta\phi\big)\big(\partial_m\delta\phi\big)\big(\partial_j\gamma'_{mk}\big)\Big]
\end{aligned} \tag{7.21}
$$

$$
\begin{aligned}
\text{fourth integral} &= -\int d^4x\, \tilde{\epsilon}^{ijk}\big(\partial_i\delta\phi\big)\tilde{N}'_m\big(\partial_j\gamma'_{mk}\big) \simeq \sqrt{\frac{\epsilon}{2}}\int d^4x\, \tilde{\epsilon}^{ijk}\big(\partial_i\delta\phi\big)\big(\partial_m\partial^{-2}\delta\phi'\big)\big(\partial_j\gamma'_{mk}\big) \\
&= -\sqrt{\frac{\epsilon}{2}}\int d^4x\, \tilde{\epsilon}^{ijk}\big(\partial_i\delta\phi\big)\gamma'_{mk}\big(\partial_j\partial_m\partial^{-2}\delta\phi'\big) = \sqrt{\frac{\epsilon}{2}}\int d^4x\tilde{\epsilon}^{ijk}\,\big(\partial_i\delta\phi\big)\gamma'_{mj}\big(\partial_k\partial_m\partial^{-2}\delta\phi''\big) \\
&= 2\times\text{third integral}
\end{aligned} \tag{7.22}
$$

$$
\begin{aligned}
\text{fifth integral} &= -2\int d^4x\, \tilde{\epsilon}^{ijk}\big(\partial_i\delta\phi\big)\big(\partial_m N\big)\big(\partial_j\gamma'_{mk}\big) = -2\sqrt{\frac{\epsilon}{2}}\int d^4x\, \tilde{\epsilon}^{ijk}\big(\partial_i\delta\phi\big)\big(\partial_m\delta\phi\big)\big(\partial_j\gamma'_{mk}\big) \\
&= 2\sqrt{\frac{\epsilon}{2}}\int d^4x\, \tilde{\epsilon}^{ijk}\big(\partial_i\delta\phi\big)\big(\partial_j\partial_m\delta\phi\big)\gamma'_{mk} = -2\sqrt{\frac{\epsilon}{2}}\int d^4x\, \tilde{\epsilon}^{ijk}\big(\partial_m\partial_i\delta\phi\big)\big(\partial_j\delta\phi\big)\gamma'_{mk}
\end{aligned} \tag{7.23}
$$

where, we made use of the equation of motion for both the graviton and the scalar perturbation to remove terms with double time derivative. In the computations above we made use of the antisymmetry of the Levi-Civita tensor and the transversality of the graviton as well. Summing together all the five contributions and doing some integration by parts, we obtain the final expression of the third order *scalar-scalar-tensor* gravitational CS action:

$$
S^{sst\,(3)}_{\rm CS} = \frac{\sqrt{2\epsilon}}{4}\big(\partial_\phi\omega\big)\int d^4x\, aH\tilde{\epsilon}^{ijk}\big(\partial_i\delta\phi\big)\big(\partial_m\partial^{-2}\delta\phi'\big)\big(\partial_j\gamma'_{mk}\big)\ . \tag{7.24}
$$

We can now restore the $M_{\rm Pl}$ factors that we have neglected so far. To do so, we simply estimate the dimensions of (7.24) and we will add to it the power of $M_{\rm Pl}$ so that the action turns to be dimensionless. Since inflaton fluctuations have mass dimension 1[4], $\big[\partial_i\delta\phi\big] = 2$ and $\big[\partial_m\partial^{-2}\delta\phi'\big] = 1$. The tensor perturbations are dimesnionless After reintroducing the $M_{\rm Pl}$, the action becomes

$$
\begin{aligned}
S^{sst\,(3)}_{\rm CS} &= \frac{\sqrt{2\epsilon}}{4}\frac{\big(\partial_\phi\omega\big)}{M_{\rm Pl}}\int d^4x\, aH\tilde{\epsilon}^{ijk}\big(\partial_i\delta\phi\big)\big(\partial_m\partial^{-2}\delta\phi'\big)\big(\partial_j\gamma'_{mk}\big) \\
&= \frac{1}{2M_{\rm CS}}\int d^4x\, a\tilde{\epsilon}^{ijk}\big(\partial_i\delta\phi\big)\big(\partial_m\partial^{-2}\delta\phi'\big)\big(\partial_j\gamma'_{mk}\big)\ .
\end{aligned} \tag{7.25}
$$

The interacting dCS Lagrangian in Fourier space takes the form

$$
\mathcal{L}^{sst}_{\rm CS} = -\frac{1}{2H}\frac{H}{M_{\rm CS}}a(\eta)\int\Big(\prod_{a=1}^{3}\frac{d^3k_a}{(2\pi)^3}\Big)(2\pi)^3\delta^3(\mathbf{k}_1{+}\mathbf{k}_2{+}\mathbf{k}_3)\frac{k_3}{k_2^2}\sum_{s=L,R}\alpha_s\, k_1^i k_2^j\epsilon^s_{ij}(\mathbf{k}_3)\delta\phi_{\mathbf{k}_1}(\eta)\delta\phi_{\mathbf{k}_2}(\eta)\gamma^s_{\mathbf{k}_3}(\eta) \tag{7.26}
$$

and the corresponding interacting Lagrangian is $H_{\rm CS} = -\mathcal{L}^{sst}_{\rm CS}$. Notice that we have multiplied and divided the Lagrangian by H, so that the small parameter $H/M_{\rm CS}$ could be isolated. This leads the CS vertex to be suppressed w.r.t. the GR one.

Notice that this interaction Hamiltonian is valid if we assume that slow-roll conditions are guaranteed throughout the entire duration of the inflationary period. Specifically, the expression for the auxiliary fields that has been substituted inside (7.18) is valid only in the slow-roll approximation.

7.4 Propagators

In the previous sections we saw that the introduction of the CS term leads to the presence of an extra $\delta\phi\delta\phi\gamma$ vertex. We are ultimately interested in computing the possible graviton-mediated trispectra

[4] The Inflaton in our perturbative setting is written as $\phi(\mathbf{x},t) = \phi(t) + \delta\phi(\mathbf{x},t)$. Since a scalar field in four spacetime dimensions has mass dimensions equal to 1, the field perturbations do the same.

arising from the introduction of the gravitational CS term coupled to the inflaton filed. As done in Sec.(6.2.1), we approximate the inflationary phase as de Sitter, and the inflaton perturbations as massless. In this setup, the mode functions related to the scalars and tensor perturbations are [32]:

$$u_{\delta\phi}(k,\eta) = \frac{H}{\sqrt{2k^3}}(1+ik\eta)e^{-ik\eta}, \tag{7.27}$$

$$v^s_\gamma(k,\eta) = \left[1 \mp \frac{k}{a}\frac{1}{M_{\rm CS}}\right]^{-1/2} \frac{H}{M_{\rm Pl}\sqrt{k^3}}(1+ik\eta)e^{-ik\eta}\left[\frac{U(2\mp\frac{i}{2}\frac{H}{M_{\rm CS}},4,2ik\eta)}{U(2,4,2ik\eta)}\right]\exp\left(\pm\frac{\pi}{4}\frac{H}{M_{\rm CS}}\Omega\right). \tag{7.28}$$

Since the CS correction is assumed to be small and it already enters in $H_{\rm CS}$, we can expand the tensor mode with respect to $k/aM_{\rm CS}$ and $H/M_{\rm CS}$, as done in Sec.(5.1.2), retaining only the zeroth order contribution. In this way, we compute the graviton-mediated trispectrum at the lowest order in CS correction[5]. The expression for the tensor modes in de Sitter and at zeroth order in Chern-Simons correction matches the one expected in the case of gravity being modeled by GR:

$$u^s_\gamma(k,\eta) = \frac{H}{M_{pl}\sqrt{k^3}}(1+ik\eta)e^{-ik\eta}. \tag{7.29}$$

Since the scalar-scalar-tensor Hamiltonian introduces time derivatives of both scalar and tensor modes, it is useful to introduce all the possible Green functions appearing in the in-in time integrals.

Since we will consider two kinds of graviton-mediated trispectra, one with two CS vertices and the other with one GR and one CS, there are two kinds of (scalar) propagators appearing in the time integrals:

$$\mathcal{G}_k(\eta,\eta') = u_{\delta\phi}(k,\eta)u^*_{\delta\phi}(k,\eta') = \frac{H^2}{2k^3}(1+ik\eta)(1-ik\eta')e^{-ik(\eta-\eta')} \tag{7.30}$$

$$\mathcal{G}^d_k(\eta,\eta') = u_{\delta\phi}(k,\eta){u^*}'_{\delta\phi}(k,\eta') = \frac{H^2}{2k}\eta'(1+ik\eta)e^{-i(\eta-\eta')} \tag{7.31}$$

with η' being the time over which we perform one of the two in-in integrations, and the prime on $u^*_{\delta\phi}$ on the second equation denoting the derivative over the integration time η'.

The CS vertex carries a time derivative of the graviton as well, leading to the presence of the time derivative of tensor modes in the in-in integration:

$$v^{s\prime}(k,\eta') = \frac{iH}{M_{\rm Pl}\sqrt{k^3}}k^2\eta' e^{-ik\eta'}. \tag{7.32}$$

7.5 Graviton-mediated scalar trispectrum

We now focus on the scalar graviton-mediated trispectrum. The only vertex which can violate parity is the CS one, since it distinguishes between the two helicities of the graviton. In fact, in $\mathcal{L}^{sst}_{\rm CS}$ (7.26) there appear the parameter α_s, which takes value $+1$ for right-handed gravitons, and -1 for left-handed ones. The vertex coming from the inflaton kinetic term couples to both the graviton helicities with the same strength, therefore it does not introduce any source of parity violation. We then have three possible graviton-mediated trispectra: two trispectra with both vertices of the same type and a "mixed" one with one CS vertex and the other coming from GR. We notice that trispectra with two vertices of the same kind are real. As a result, they do not contain any information about parity violating physics in the early universe. The only trispectrum which can provide us such information is the mixed trispectrum.

[5]This is not the only way to obtain the graviton mediated trispectra at first order in CS correction. As done in [30], in the in-in computation, we could have analogously expanded the modes function at first order in $k/aM_{\rm CS}$ and $H/M_{\rm CS}$ while neglecting the scalar-scalar-tensor vertex arising from the expansion of the Chern-Simons action.

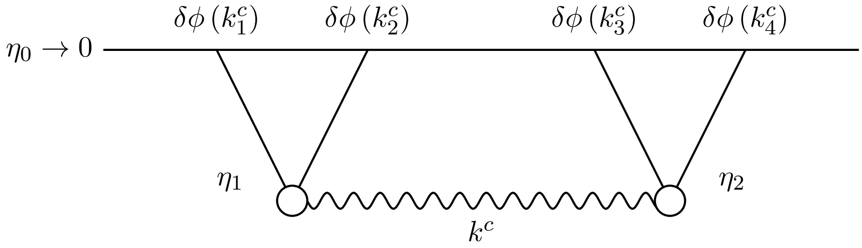

Figure 7.1: Diagram of the scalar PE trispectrum mediated by graviton exchange. The horizontal line represent the boundary $\eta_0 \to 0$, the external legs are scalar bulk-to-boundary propagators and the wavy line is the graviton bulk propagator. The white vertex comes from the expansion of the standard kinetic term of the inflaton.

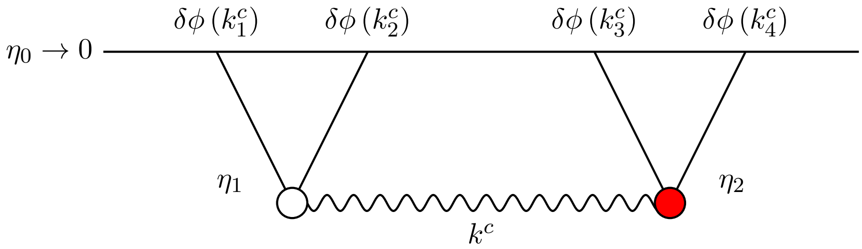

Figure 7.2: Diagram of the scalar trispectrum mediated by graviton exchange. The horizontal line represent the boundary $\eta_0 \to 0$, the external legs are scalar bulk-to-boundary propagators and the wavy line is the graviton bulk propagator. The white vertex comes from the standard kinetic term of the inflaton whereas the red one is introduced by the Chern-Simons gravity.

In the following, we will compute the three types of trispectra arising from the interaction Hamiltonian $H_I = H_{\rm GR} + H_{\rm CS}$. From this Hamiltonian we see the appearance the types of trispectra introduced before by substituting the expression of H_I inside 6.36. For instance

$$\begin{aligned}\langle O(\eta)\rangle_{(1,1)} = \int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta} d\eta'' \Big[&\langle H_{\rm CS}(\eta')O(\eta)H_{\rm CS}(\eta'')\rangle + \langle H_{\rm CS}(\eta')O(\eta)H_{\rm GR}(\eta'')\rangle \\ &+ \langle H_{\rm GR}(\eta')O(\eta)H_{\rm CS}(\eta'')\rangle + \langle H_{\rm GR}(\eta')O(\eta)H_{\rm GR}(\eta'')\rangle\Big]\end{aligned} \tag{7.33}$$

and the same holds for contribution $(0,2)$ and $(2,0)$. However, since the CS interaction Hamiltonian is suppressed by a power of $H/M_{\rm CS}$ parameter, the contribution to the PE trispectrum coming from two CS vertices is strongly suppressed w.r.t. the one with two GR vertices. Therefore, we consider only the trispectrum built from two CS vertices (Fig.7.1), the PE one, and the mixed trispectrum (Fig.7.2), which changes sign under parity transformation. We neglect the PE even contribution from two CS vertices.

7.5.1 Parity-even graviton-mediated scalar trispectrum

The PE contribution to the graviton-mediated trispectrum has been computed in Sec.6.2.1. The trispectrum is non-vanishing and depends non trivially on the external momenta $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$. It is written as

$$T^{\rm PE} = \mathcal{P}_{\delta\phi}(p_1)\mathcal{P}_{\delta\phi}(p_3)\mathcal{P}_{\gamma}(p_{12})S(p_1,p_2,p_3,p_4)\sin^2\theta_1\sin^2\theta_3\cos 2\chi_{12,34} + 5 \text{ perm.} \tag{7.34}$$

where $S(p_1,p_2,p_3,p_4)$ is an adimensional function of the momenta

$$S(p_1,p_2,p_3,p_4) = p_1^2 p_3^2 p_T^2/p_2^3 p_4^3 \times I_{1234} \tag{7.35}$$

with I_{1234} given by (A.26).

7.5.2 Parity-odd graviton-mediated scalar trispectrum in slow-roll

Let us now focus on the PO trispectrum diagrammatically represented in Fig.7.2. The trispectrum receives two contributions from the $(1,1)$, $(0,2)$ and $(2,0)$ terms:

$$\langle O(\eta)\rangle^{\rm PO}_{(1,1)} = \int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta} d\eta'' \Big(\langle H_{\rm GR}(\eta') O(\eta) H_{\rm CS}(\eta'')\rangle + \langle H_{\rm CS}(\eta') O(\eta) H_{\rm GR}(\eta'')\rangle \Big) \tag{7.36}$$

$$\langle O(\eta)\rangle^{\rm PO}_{(0,2)} = -\int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta'} d\eta'' \Big(\langle O(\eta) H_{\rm GR}(\eta') H_{\rm CS}(\eta'')\rangle + \langle O(\eta) H_{\rm CS}(\eta') H_{\rm GR}(\eta'')\rangle \Big) \tag{7.37}$$

$$\langle O(\eta)\rangle^{\rm PO}_{(2,0)} = -\int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta'} d\eta'' \Big(\langle H_{\rm GR}(\eta'') H_{\rm CS}(\eta') O(\eta)\rangle + \langle H_{\rm CS}(\eta'') H_{\rm GR}(\eta') O(\eta)\rangle \Big) \tag{7.38}$$

with $O(\eta) = \langle \delta\phi_{\mathbf{p}_1}(\eta)\delta\phi_{\mathbf{p}_2}(\eta)\delta\phi_{\mathbf{p}_3}(\eta)\delta\phi_{\mathbf{p}_4}(\eta)\rangle$.

Let us work out all the contributions listed above. In particular, we need to write them in their explicit form by performing the Wick contractions, and see if the two terms can be written as one the complex conjugate of the other such that we can factorize them as the real or imaginary part of one of the two. Starting from $\langle O(\eta)\rangle^{\rm PO}_{(1,1)}$, the first term in (7.36) is explicitly given by

$$\begin{aligned}
\int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta} d\eta'' \langle H_{\rm GR}(\eta') O(\eta) H_{\rm CS}(\eta'')\rangle &= -\frac{1}{4H}\frac{H}{M_{\rm CS}} \int \left(\prod_{a=1}^{6} \frac{d^3 k_a}{(2\pi)^3}\right) (2\pi)^6 \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\delta(\mathbf{k}_4+\mathbf{k}_5+\mathbf{k}_6)\\
&\times \sum_{\lambda,\mu} k_1^i k_2^j \epsilon^{\lambda}_{ij}(\mathbf{k}_3) \frac{k_6}{k_5^2} k_4^m k_5^n \epsilon^{\mu}_{mn}(\mathbf{k}_6) \int_{-\infty}^{\eta} d\eta'\, a^2(\eta') \int d\eta''\, a(\eta'') \langle \delta\phi_{k_1}(\eta')\delta\phi_{k_2}(\eta')\gamma^{\lambda}_{k_3}(\eta')\\
&\times \delta\phi_{p_1}(\eta)\delta\phi_{p_2}(\eta)\delta\phi_{p_3}(\eta)\delta\phi_{p_4}(\eta)\delta\phi_{k_4}(\eta'')\delta\phi_{k_5}(\eta'')\gamma'^{\mu}_{k_6}(\eta'')\rangle\\
&= -\frac{1}{4H}\frac{H}{M_{\rm CS}}(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)\sum_{\lambda} \alpha_\lambda p_1^i p_2^j \epsilon^{\lambda}_{ij}(\mathbf{p}_1+\mathbf{p}_2) \frac{p_3+p_4}{p_4^2} p_3^m p_4^n \epsilon^{\mu}_{mn}(\mathbf{p}_3+\mathbf{p}_4)\\
&\times \int_{-\infty}^{\eta} d\eta'\, a^2(\eta') \int_{-\infty}^{\eta} d\eta''\, a(\eta'')\, \mathcal{G}^*_{p_1}(\eta,\eta')\mathcal{G}^*_{p_2}(\eta,\eta')\mathcal{G}^*_{p_3}(\eta,\eta'')\mathcal{G}^{d\,*}_{p_4}(\eta,\eta'') v^{\lambda}_{p_{12}}(\eta') v'^{\lambda\,*}_{p_{12}}(\eta'')\\
&+ 23 \text{ permutations}
\end{aligned} \tag{7.39}$$

where, in the second passage, Wick contractions has been performed. By doing the same with the second term we find:

$$\begin{aligned}
\int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta} d\eta'' \langle H_{CS}(\eta') O(\eta) H_{GR}(\eta'')\rangle &= -\frac{1}{4H}\frac{H}{M_{\rm CS}}(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)\sum_{\lambda} \alpha_\lambda\, p_1^m p_2^n \epsilon^{\lambda}_{mn}(\mathbf{p}_1+\mathbf{p}_2)\\
\times \frac{p_3+p_4}{p_4^2} p_3^i p_4^j \epsilon^{\lambda}_{ij}(\mathbf{p}_3+\mathbf{p}_4) &\int_{-\infty}^{\eta} d\eta'\, a^2(\eta') \int_{-\infty}^{\eta} d\eta''\, a(\eta'')\, \mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')\mathcal{G}^*_{p_3}(\eta,\eta'')\mathcal{G}^{d\,*}_{p_4}(\eta,\eta') v^*_{p_{12}}(\eta') v'_{p_{12}}(\eta'')\\
&+ 23 \text{ permutations}
\end{aligned} \tag{7.40}$$

which is exactly the complex conjugate of the first term. Therefore, (7.36) can be written as follows:

$$\begin{aligned}
\langle O(\eta)\rangle^{\rm PO}_{(1,1)} &= -\frac{1}{4H}\frac{H}{M_{\rm CS}}(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)\sum_{\lambda} \alpha_\lambda\, p_1^m p_2^n \epsilon^{\lambda}_{mn}(\mathbf{p}_1+\mathbf{p}_2) \frac{p_3+p_4}{p_4^2} p_3^i p_4^j \epsilon^{\lambda}_{ij}(\mathbf{p}_3+\mathbf{p}_4)\\
&\times 2\mathrm{Re}\Big[\int_{-\infty}^{\eta} d\eta'\, a^2(\eta') \int_{-\infty}^{\eta} d\eta''\, a(\eta'')\, \mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')\mathcal{G}^*_{p_3}(\eta,\eta'')\mathcal{G}^{d\,*}_{p_4}(\eta,\eta') v^*_{p_{12}}(\eta') v'_{p_{12}}(\eta'')\Big]\\
&+ 23 \text{ permutations}
\end{aligned} \tag{7.41}$$

We can do the same with the remaining terms (7.37) and (7.38), and one is the complex conjugate of the other:

$$
\begin{aligned}
&\langle O(\eta)\rangle^{\rm PO}_{(0,2)} + \langle O(\eta)\rangle^{\rm PO}_{(2,0)} = \frac{1}{4H}\frac{H}{M_{\rm CS}}(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)\sum_\lambda \alpha_\lambda\, p_1^i p_2^j \epsilon_{ij}^\lambda(\mathbf{p}_1+\mathbf{p}_2)\frac{p_3+p_4}{p_4^2}p_3^m p_4^n \epsilon^\lambda_{mn}(\mathbf{p}_3+\mathbf{p}_4) \\
&\times\Bigg\{2\mathrm{Re}\Big[\int_{-\infty}^{\eta} d\eta'\, a^2(\eta')\int_{-\infty}^{\eta'} d\eta''\, a(\eta'')\mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}^d_{p_4}(\eta,\eta'')v^\lambda_{p_{12}}(\eta')v'^{\lambda *}_{p_{12}}(\eta'')\Big] \\
&+2\mathrm{Re}\Big[\int_{-\infty}^{\eta} d\eta''\, a(\eta'')\int_{-\infty}^{\eta''} d\eta'\, a^2(\eta')\,\mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}^d_{p_4}(\eta,\eta'')v^{\lambda *}_{p_{12}}(\eta')v'^{\lambda}_{p_{12}}(\eta'')\Big]\Bigg\} \\
&+ 23 \text{ permutations}
\end{aligned}
\tag{7.42}
$$

The total PO trispectrum,$T^{\rm PO}$ defined as $\langle\zeta_{\mathbf{p}_1}\zeta_{\mathbf{p}_2}\zeta_{\mathbf{p}_3}\zeta_{\mathbf{p}_4}\rangle_c^{\rm PO} = (2\pi)^3 T^{\rm PO}\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)$, is given by [32]:

$$
\begin{aligned}
T^{\rm PO} =& \frac{1}{2H}\frac{H}{M_{\rm CS}}\sum_\lambda \alpha_\lambda \frac{(p_3+p_4)}{p_4^2} p_1^i p_2^j p_3^m p_4^n \epsilon^\lambda_{ij}(\mathbf{p}_1+\mathbf{p}_2)\epsilon^\lambda_{mn}(\mathbf{p}_3+\mathbf{p}_4)\,\mathrm{Re}\big[I_1+I_2-I_3\big] \\
&+ 23 \text{ permutations}
\end{aligned}
\tag{7.43}
$$

with

$$I_1 = \int_{-\infty}^{\eta} d\eta'\, a^2(\eta')\int_{-\infty}^{\eta'} d\eta''\, a(\eta'')\mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}^d_{p_4}(\eta,\eta'')v^\lambda_{p_{12}}(\eta')v'^{\lambda *}_{p_{12}}(\eta'')\ , \tag{7.44}$$

$$I_2 = \int_{-\infty}^{\eta} d\eta''\, a(\eta'')\int_{-\infty}^{\eta''} d\eta'\, a^2(\eta')\,\mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}^d_{p_4}(\eta,\eta'')v^{\lambda *}_{p_{12}}(\eta')v'^{\lambda}_{p_{12}}(\eta'')\ , \tag{7.45}$$

$$I_3 = \int_{-\infty}^{\eta} d\eta'\, a^2(\eta')\int_{-\infty}^{\eta} d\eta''\, a(\eta'')\,\mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')\mathcal{G}^*_{p_3}(\eta,\eta'')\mathcal{G}^{d*}_{p_4}(\eta,\eta')v^{*\lambda}_{p_{12}}(\eta')v'^{\lambda}_{p_{12}}(\eta'')\ . \tag{7.46}$$

Polarization sum

Before proceeding with the evaluation of the time integrals, it is useful to make the polarization sum more explicit. The polarization tensors are written in the helicity basis, which is related to the plus-cross basis by the relation

$$\epsilon^{R,L}_{ij} = \frac{1}{\sqrt{2}}\big(\epsilon^+_{ij} \pm i\epsilon^\times_{ij}\big)\ . \tag{7.47}$$

The polarization sum is explicitly given by

$$
\begin{aligned}
&\sum_{\lambda=L,R}\alpha_\lambda p_1^i p_2^j p_3^m p_4^n \epsilon^\lambda_{ij}(\mathbf{p}_{12})\epsilon^\lambda_{mn}(\mathbf{p}_{34}) \\
&\quad = p_1^i p_2^j p_3^m p_4^n \epsilon^R_{ij}(\mathbf{p}_{12})\epsilon^R_{mn}(\mathbf{p}_{34}) - p_1^i p_2^j p_3^m p_4^n \epsilon^L_{ij}(\mathbf{p}_{12})\epsilon^L_{mn}(\mathbf{p}_{34})
\end{aligned}
\tag{7.48}
$$

where the minus sign in the second line is given by the fact that $\alpha_L = -1$. We recall that $\mathbf{p}_{12} = \mathbf{p}_1+\mathbf{p}_2$, $\mathbf{p}_{34} = \mathbf{p}_3+\mathbf{p}_4$, and,from the conservation of momenta ensured by the Dirac delta, follows that $\mathbf{p}_{34} = -\mathbf{p}_{12}$. From (7.47) and the results obtained in Sec.6.2.1 we find:

$$\epsilon^R_{ij}(\mathbf{p}_{12})p_1^i p_j^2 = -\frac{1}{\sqrt{2}}p_1^2\sin^2\theta_1\big[\cos(2\phi_1)+i\sin(2\phi_1)\big] = -\frac{1}{\sqrt{2}}p_1^2\sin^2\theta_1 e^{2i\phi_1}, \tag{7.49}$$

$$\epsilon^L_{ij}(\mathbf{p}_{12})p_1^i p_2^j = -\frac{1}{\sqrt{2}}p_1^2\sin^2\theta_1\big[\cos(2\phi_1)-i\sin(2\phi_1)\big] = -\frac{1}{\sqrt{2}}p_1^2\sin^2\theta_1 e^{-2i\phi_1}, \tag{7.50}$$

$$\epsilon^R_{ij}(\mathbf{p}_{34})p_3^i p_4^j = -\frac{1}{\sqrt{2}}p_3^2\sin^2\theta_3\big[\cos(2\phi_3)-i\sin(2\phi_3)\big] = -\frac{1}{\sqrt{2}}p_3^2\sin^2\theta_3 e^{-2i\phi_3}, \tag{7.51}$$

$$\epsilon^L_{ij}(\mathbf{p}_{34})p_3^i p_4^j = -\frac{1}{\sqrt{2}}p_3^2\sin\theta_3\big[\cos(2\phi_3)+i\sin(2\phi_3)\big] = -\frac{1}{\sqrt{2}}p_3^2\sin^2\theta_3 e^{2i\phi_3}. \tag{7.52}$$

Notice that we made use of the property $\epsilon_{ij}(\mathbf{p}_{34}) = \epsilon^*_{ij}(\mathbf{p}_{12})$ in deriving the last two relations above. By combining these results together, it is easy to show that

$$p_1^i p_2^j p_3^m p_4^n \epsilon^R_{ij}(\mathbf{p}_{12})\epsilon^R_{mn}(\mathbf{p}_{34}) = \frac{1}{2}p_1^2 p_3^2 \sin^2\theta_1 \sin^2\theta_3 e^{2i(\phi_1-\phi_3)}, \tag{7.53}$$

$$p_1^i p_2^j p_3^m p_4^n \epsilon^L_{ij}(\mathbf{p}_{12})\epsilon^L_{mn}(\mathbf{p}_{34}) = \frac{1}{2}p_1^2 p_3^2 \sin^2\theta_1 \sin^2\theta_3 e^{-2i(\phi_1-\phi_3)}. \tag{7.54}$$

Working out the polarization sum, we obtain

$$\sum_{\lambda=L,R} \alpha_\lambda p_1^i p_2^j p_3^m p_4^n \epsilon^\lambda_{ij}(\mathbf{p}_{12})\epsilon^\lambda_{mn}(\mathbf{p}_{34}) = i\, p_1^2 p_3^2 \sin^2\theta_1 \sin^2\theta_3 \sin(2\chi_{12,34}) \tag{7.55}$$

with $\chi_{12,34} = \phi_1 - \phi_3$ being the angle separating the planes identified by $\{\mathbf{p}_1, \mathbf{p}_2\}$ and $\{\mathbf{p}_3, \mathbf{p}_4\}$. Notice that the polarization sum is imaginary. Since all the other terms contributing to the PO trispectrum are real, the PO trispectrum itself is purely imaginary. This is what we expected from a naive analysis of parity violating vertices and trispectra.

Time integrals

We focus now on the solution of time integrals. First of all, we substitute the expression for the scalar Green functions and tensor mode function (7.30), (8.14) and their derivatives with respect to the internal time, in (7.43). Moreover, analogously to what was done in the computation of the parity-even trispectrum, it is useful to make a change of variable in such a way that the integration variable is dimensionless. After this change of variable, the trispectrum reduces to:

$$\begin{aligned} T^{\rm PO} = &= -\frac{iH}{2M_{\rm CS}}\mathcal{P}_{\delta\phi}(p_1)\mathcal{P}_{\delta\phi}(p_3)\mathcal{P}_\gamma(p_{12})\frac{(p_3+p_4)p_1^2p_3^2p_{12}^2}{4p_2^3p_4^3}\sin^2\theta_1\sin^2\theta_3\sin(2\xi_{12,34})\,\mathrm{Re}\big[\bar I_1+\bar I_2-\bar I_3\big] \\ &\quad + 23\text{ perm.}, \end{aligned} \tag{7.56}$$

with the integrals given by

$$\begin{aligned} \bar I_1 = &(1+ip_1\eta)(1+ip_2\eta)(1+ip_3\eta)(1+ip_4\eta)\, e^{-i(p_1+p_2+p_3+p_4)\eta} \\ &\int_{-\infty}^{\eta}\frac{d\eta'}{\eta'^2}(1-ip_1\eta')(1-ip_2\eta')(1+ip_{12}\eta')e^{i(p_1+p_2-p_{12})\eta'}\int_{-\infty}^{\eta'} d\eta''\eta''(1-ip_3\eta'')e^{i(p_3+p_4+p_{12})\eta''}, \end{aligned} \tag{7.57}$$

$$\begin{aligned} \bar I_2 = &(1+ip_1\eta)(1+ip_2\eta)(1+ip_3\eta)(1+ip_4\eta)\, e^{-i(p_1+p_2+p_3+p_4)\eta} \\ &\int_{-\infty}^{\eta} d\eta''\eta''(1-ip_3\eta'')e^{i(p_3+p_4-p_{12})\eta''}\int_{-\infty}^{\eta''}\frac{d\eta'}{\eta'^2}(1-ip_1\eta')(1-ip_2\eta')(1-ip_{12}\eta')e^{i(p_1+p_2+p_{12})\eta'}, \end{aligned} \tag{7.58}$$

$$\begin{aligned} \bar I_3 = &(1+ip_1\eta)(1+ip_2\eta)(1-ip_3\eta)(1-ip_4\eta)e^{-i(p_1+p_2-p_3-p_4)\eta} \\ &\int_{-\infty}^{\eta}\frac{d\eta'^2}{\eta'}(1-ip_1\eta')(1-ip_2\eta')(1-ip_{12}\eta')e^{i(p_1+p_2+p_{12})\eta'}\int_{-\infty}^{\eta} d\eta''\eta''(1+ip_3\eta'')e^{-i(p_3+p_4+p_{12})\eta''}. \end{aligned} \tag{7.59}$$

Solving these integrals (see Sec.A.1.2) and extracting the real part, leads to

$$\bar I_1 = -\frac{1}{p_T}\frac{3k_3+k_4+k_{12}}{(k_3+k_4+k_{12})^3}\frac{1}{x} \tag{7.60}$$

$$\bar I_2 = 0 \tag{7.61}$$

$$\bar I_3 = -\frac{1}{p_T}\frac{3k_3+k_4+k_{12}}{(k_3+k_4+k_{12})^3}\frac{1}{x} \tag{7.62}$$

where x and k_i, $i = 3, 4, 12$ are dimensionless parameters defined as $x = p_T\eta$ and $k_i = p_i/p_T$, where $p_T = p_1 + p_2 + p_3 + p_4$.

Parity-odd graviton mediated trispectrum

Collecting all the terms together we can write the final expression for the scalar PO trispectrum from graviton-exchange:

$$\begin{aligned} T^{\rm PO} &= = \frac{iH}{2M_{\rm CS}}\mathcal{P}_{\delta\phi}(p_1)\mathcal{P}_{\delta\phi}(p_3)\mathcal{P}_{\gamma}(p_{12})\frac{(p_3+p_4)p_1^2p_3^2p_{12}^2}{4p_Tp_2^3p_4^3}\sin^2\theta_1\sin^2\theta_3\sin\left(2\xi_{12,34}\right) \\ &\times\left\{\frac{3k_3+k_4+k_{12}}{(k_3+k_4+k_{12})^3}\frac{1}{x}+0-\frac{3k_3+k_4+k_{12}}{(k_3+k_4+k_{12})^3}\frac{1}{x}\right\}+23 \text{ permutation } = 0 \end{aligned} \tag{7.63}$$

which is zero, since all permutations vanish separately. This is in agreement with the result obtained by [32].

7.6 No-go and yes-go conditions

The vanishing of the parity-odd trispectrum is of no surprise: a series of no-go theorems prevents it from being different from zero at tree level if certain conditions are satisfied. In particular, in [33] it is stated that:

Exchange contributions to the quartic wavefunction coefficient of massless scalars with Bunch-Davies vacuum conditions cannot contribute to the parity-odd trispectrum if the exchanged field has massless or conformally-coupled mode functions, Bunch-Davies vacuum conditions and if the constituent cubic wavefunction coefficients are IR-finite.

In our setup, these assumptions are all fulfilled: Bunch-Davies vacuum condition has been imposed, we approximated the inflationary universe as a de Sitter space-time and there are no net infra-red (logarithmic) divergencies arising from the in-in integrations for $\eta \to 0$. In the same paper the authors demonstrate that the breaking of scale invariance, either through the presence of IR divergencies or of time dependent coupling constants at the level of the action, leads to a non vanishing parity-violating signal at the level of the trispectrum. However, perfect scale invariance is broken if we do not assume the inflationary phase to be a de Sitter stage: in quasi de Sitter inflation the power spectrum for scalar perturbations is slightly red-tilted with spectral index $n_s - 1 = -2\epsilon_1 - \epsilon_2 + O(\epsilon_1^2, \epsilon_2^2, \dots)$. This suggests that, moving away from the perfect scale-invariant regime, the parity-odd trispectrum should not be vanishing, even if its amplitude would be suppressed by powers of the slow-roll parameters.Regarding this, we can give a rough estimate of the amplitude of the trispectrum in case of quasi de Sitter inflationary phase.

Mode functions related to the curvature perturbation and tensor perturbations are proportional to the Hankel function of the first kind after BD vacuum conditions are imposed:

$$f_k(\eta) = \left(\frac{H}{M_{\rm pl}k\sqrt{2\epsilon_1}}\right)_\star\sqrt{\frac{\pi}{4}}(-k\eta)^{-(\frac{1}{2}-\nu_{\rm S})}e^{i\frac{\pi}{4}(2\nu_{\rm S}+1)}\sqrt{-\eta}H^{(1)}_{\nu_{\rm S}}(-k\eta) \tag{7.64}$$

$$v_k^s(\eta) = \left(\frac{2H}{M_{\rm pl}k}\right)_\star\sqrt{\frac{\pi}{4}}(-k\eta)^{-(\frac{1}{2}-\epsilon_1)}e^{i\frac{\pi}{4}(2\nu_{\rm T}+1)}\sqrt{-\eta}H^{(1)}_{\nu_{\rm T}}(-k\eta) \tag{7.65}$$

where subscript $\star$ indicates that the quantities are computed at horizon crossing. For simplicity, we take the asymptotic expression of the Hankel function in the super-Hubble regime:

$$\lim_{-k\eta\to 0} H^{(1)}_{\nu_i}(-k\eta) = -\frac{i}{\pi\nu_i}\Gamma(1+\nu_i)\left(\frac{-k\eta}{2}\right)^{-\nu_i}. \tag{7.66}$$

The modes in sub-Hubble regime are highly oscillatory and the he biggest contribution to the in-in integrals comes from the super-Hubble regime, in which the mode function freezes to a constant value. To understand the possible contribution to the PO trispectrum, we consider a Taylor expansion of the modes around $\nu_i = 3/2$, with $i = \rm S, T$. Moreover, since the in-in integrals involving $H^{(1)}_{3/2}$ have been shown to lead to a vanishing result, we focus on expanding just the Hankel function:

$$\begin{aligned} H^{(1)}_{\nu_i} &= H^{(1)}_{3/2}(-k\eta) + \delta\nu_i \frac{\partial H^{(1)}_{\nu_i}(-k\eta)}{\partial\nu}\bigg|_{\nu_i=3/2} + \dots \\ &= H^{(1)}_{3/2}\left\{1 + \delta\nu_i\left[2 - \gamma_{\mathrm{E}} - 2\ln 2 - \ln\left(\frac{-k\eta}{2}\right)\right]\right\} + \dots \end{aligned} \tag{7.67}$$

Substituting the expanded modes in the parity-odd trispectrum (7.43), it is very easy to identify the corrections arising in the quasi de Sitter regime. The zeroth order expression matches exactly with the de Sitter one, leading to a vanishing contribution to the PO trispectrum. Higher order corrections introduce terms proportional to $(\epsilon_1 + 1/2\epsilon_2)^n(\epsilon_1)^m$ with $l = m + n$ the order in slow-roll expansion. Specifically, the leading order PO trispectrum in quasi de Sitter slow-roll inflation can be recast as

$$T^{\mathrm{PO}} \propto \left(\frac{H}{M_{\mathrm{CS}}}\right)\mathcal{P}^2_\zeta\mathcal{P}_\gamma \sin^2\theta_1 \sin^2\theta_3 \sin\left(2\chi_{12,34}\right)\left[\left(\epsilon_1 + \frac{1}{2}\epsilon_2\right)S_1(p_i) + \epsilon_1 S_2(p_i)\right] + \text{permutations}, \tag{7.68}$$

where the angular factor comes from the polarization sum, the spectral powers from arranging the prefactors in ζ_{p_i} and $v_{p_{12}}$, and $S_1(p_i)$ and $S_2(p_i)$ are functions of the external and exchanged momenta arising from performing the time integrals.

The derivation above shows in a sketchy ways that both the PE and PO trispectra receive corrections at least of $O(\epsilon_1)$ or $O(\epsilon_1 + \epsilon_2/2)$ if we account for inflationary epoch being described by quasi de Sitter expansion phase. This is of great interest especially for the PO scalar trispectrum since it is identically vanishing in de Sitter with BD vacuum, according to no-go theorems. Specifically, although suppressed by at least one power of the slow parameter, this signal would be present even without introducing any deviation from single-field slow-roll inflation

Chapter 8

Signatures of parity violation in ultra slow-roll inflation

We are now interested in investigating whether the inflationary model introduced in Chapter 4 may lead to parity violating signatures once the gravitational sector have been modified by the introduction of the parity-violating CS interaction term. In particular, we are interested in calculating the PO graviton-mediated scalar trispectrum to understand whether SR-USR-SR models of inflation, once the inflaton is coupled to gravity via CS interaction term, can source a PO signal at the level of the scalar trispectrum which is not suppressed.

To do so, one could be tempted to keep the vertices introduced in the previous chapter, with the model of interest affecting the integrals (7.44), (7.45) and (7.46): the domain of integration would be triparted and the modes function for the various inflationary epochs I, II and III, substituted. However, this approach would lead us to neglect the contribution to the trispectrum coming from dominant terms in the SR-USR-SR inflationary model with sharp transition. In fact, both the GR and CS vertices derived in the previous chapter, come from the third order action at leading order in slow-roll. This is for sure a good approximation if we want to work just with SR inflation. Nevertheless, if we consider an inflationary model which explicitly violates slow-roll conditions for a given period of time, then there are other terms appearing in the action which might be of the same order of the terms considered before, or even dominant. These considerations had already been taken into account in [53] and [46] to deal with the cubic and quartic scalar action while studying non-Gaussianities and loop corrections in SR-USR-SR inflation.

In the following, we will consider the complete expression of the *sst* action from both the CS and the inflaton action (with gravity described by GR), analyzing which are the dominant contributions in our model of interest. Then, we will proceed to compute both the PO graviton-mediated trispectrum in search of parity violating signatures for such a model. We shall complete the analysis with the computation the dominant PE graviton-mediated trispectrum, and the estimation of the one sourced by ζ^3 vertices in the counter-collinear limit. At the end, some discussions on perturbativity and the effect of smoothening the SR/USR/SR transition.

8.1 Cubic action

Since in the SR-USR-SR inflationary model the slow-roll parameter have a non negligible time dependence, and we are ultimately interested in the 4PCF in terms of ζ it is useful to work in the comoving gauge.

In the following, we shall derive the expression for the scalar-scalar-tensor vertices in that gauge, without doing an expansion in slow-roll parameters. Then, similarly to [53, 46], we can isolate the larger contributions to the action, that we will use for the computation of the PO and PE trispectra.

8.1.1 Standard gravity cubic action

The *scalar-scalar-tensor* action in case of gravity being described by GR and in comoving gauge is given by [7]:

$$S = \int d^3x dt \left[a\epsilon_1 \gamma_{ij}(\partial_i\zeta)(\partial_j\zeta) + \frac{1}{4}a^3(\partial^2\gamma_{ij})(\partial_i\chi)(\partial_j\chi) + \frac{1}{2}a^3\epsilon_1\dot{\gamma}_{ij}(\partial_i\zeta)(\partial_j\chi) \right.$$
$$\left. + \hat{f}(\zeta,\gamma)\frac{\delta L}{\delta\zeta}\bigg|_1 + \hat{f}_{ij}(\zeta)\frac{\delta L}{\delta\gamma_{ij}}\bigg|_1 \right] \tag{8.1}$$

where the auxiliary field χ satisfies $\partial^2\chi = \epsilon_1\dot{\zeta}$. The terms proportional to the equations of motion can be removed by field redefinitions, which turn out not to be important after horizon crossing, and can be neglected. The first term of (8.1) is proportional to ϵ_1, the other two terms are higher order in slow-roll. By replacing χ with its expression and moving to conformal time $d\eta = dt/a$, the cubic *sst* action becomes

$$S = \int d^3x d\eta \left[a^2\epsilon_1\gamma_{ij}(\partial_i\zeta)(\partial_j\zeta) + \frac{1}{4}a^2\epsilon_1^2(\partial^2\gamma_{ij})(\partial_i\partial^{-2}\zeta')(\partial_j\partial^{-2}\zeta') + \frac{1}{2}a^2\epsilon_1^2\gamma'_{ij}(\partial_i\zeta)(\partial_j\partial^{-2}\zeta') \right] \ . \tag{8.2}$$

Notice that in the latter expression there is no appearance of the second slow roll parameter or its derivatives.

We might be tempted to move time derivatives either on ζ or on γ by performing integration by parts. In doing so, the parameter ϵ_2 and its derivative appear in both the manipulations of the second and third term. However, these terms turn out to be equal and opposite, canceling each other.

Since also in the SR-USR-SR model the first slow-roll parameter remains small throughout the entire inflationary phase, the leading order vertex is provided by the first term in the action (8.2). Therefore, the leading order $\gamma\zeta\zeta$ vertex in SR-USR-SR is the same as the one at leading order in slow-roll. Remembering that the curvature perturbation ζ is a dimensionless quantity, we restore the dependence on $M_{\rm pl}$ yielding to the action

$$S_{\rm GR} = M_{\rm pl}^2 \int d^3x d\eta\, a^2\epsilon_1\gamma_{ij}(\partial_i\zeta)(\partial_j\zeta) \ . \tag{8.3}$$

8.1.2 Chern-Simons cubic action

We start from the action (7.18) and we switch to the comoving gauge by means of the relation $\zeta = -(H/\dot{\phi})\delta\phi = -\delta\phi/\sqrt{2\epsilon_1}$. The shift and lapse vector are then expressed by

$$N = -\epsilon_1\zeta \ , \qquad \tilde{N}_i = \epsilon_1\partial_i\partial^{-2}\zeta' \tag{8.4}$$

and we substitute those expressions inside (7.18). If we do not treat the slow-roll parameters as constants, the expression of the Chern-Simons action in terms of the field γ and ζ is given by

$$\begin{aligned}
S_{CS}^{sst\,\text{comoving}} &= -\frac{1}{8}(\partial_\phi\omega)\int d^3x d\eta\, \tilde{\epsilon}^{ijk}\bigg[-\frac{1}{2}(-\sqrt{2\epsilon_1})'\epsilon_1\zeta(\partial_l\partial_i\partial^{-2}\zeta')(\partial_j\gamma'_{lk}) + \frac{1}{2}\sqrt{-2\epsilon_1}\epsilon_1\zeta'(\partial_l\partial_i\partial^{-2}\zeta)\gamma''_{lk} \\
&+ \frac{1}{2}\sqrt{2\epsilon_1}\epsilon_1(\partial_i\zeta)(\partial_l\partial_k\partial^{-2}\zeta')\gamma''_{lj} + \frac{1}{2}\sqrt{2\epsilon_1}\epsilon'_1(\partial_i\zeta)\gamma'_{lj}(\partial_l\partial_k\partial^{-2}\zeta') + \frac{1}{2}\sqrt{2\epsilon_1}\epsilon_1(\partial_i\zeta)\gamma'_{lj}(\partial_l\partial_k\partial^{-2}\zeta'') \\
&+ \sqrt{2\epsilon_1}\epsilon'_1(\partial_i\zeta)(\partial_l\partial^{-2}\zeta')(\partial_j\gamma'_{lk}) + \sqrt{2\epsilon_1}(\partial_i\zeta)(\partial_l\partial^{-2}\zeta'')(\partial_j\gamma'_{lk}) - 2\sqrt{2\epsilon_1}\epsilon_1(\partial_i\zeta)(\partial_l\zeta)(\partial_j\gamma'_{lk})\bigg] \\
&= -\frac{1}{8}(\partial_\phi\omega)\int d^3x d\eta\, \tilde{\epsilon}^{ijk}\bigg[-2\sqrt{2\epsilon_1}\epsilon'_1(\partial_i\zeta)(\partial_j\partial_l\partial^{-2}\zeta')\gamma'_{lk} + 4(aH)\sqrt{2\epsilon_1}\epsilon_1(\partial_i\zeta)(\partial_j\partial_l\partial^{-2}\zeta')\gamma'_{lk}\bigg]
\end{aligned} \tag{8.5}$$

We notice that the action contains also a term proportional to the derivative of the first slow-roll parameter that would have been higher order if SR approximation held. If we move the derivative

in the tensor and scalar perturbation by performing integration by parts, we can generate multiple terms, some of them may contain terms proportional to ϵ_2' which is dominating in our model due to the sharp nature of the SR-USR transition. In this way we can identify the interaction vertices which are dominating for our model of interested.
First of all, it is useful to rewrite the term inside squared brackets of eq. (8.5) using

$$\epsilon_1' = (aH)\,\epsilon_1\epsilon_2;$$

in this way:

$$\begin{aligned}-2\sqrt{2\epsilon_1}\epsilon_1'(\partial_i\zeta)(\partial_j\partial_l\partial^{-2}\zeta')\gamma_{lk}' + 4(aH)\sqrt{2\epsilon_1}\epsilon_1(\partial_i\zeta)(\partial_j\partial_l\partial^{-2}\zeta')\gamma_{lk}'\\ = 2(aH)\sqrt{2\epsilon_1}(\partial_i\zeta)(\partial_j\partial_l\partial^{-2}\zeta')\gamma_{lk}'\,(2\epsilon_1-\epsilon_1\epsilon_2).\end{aligned} \tag{8.6}$$

We can then proceed with the integration by part, giving us the following action:

$$\begin{aligned}&\int d^3x d\eta\, 2(aH)\sqrt{2\epsilon_1}(\partial_i\zeta)(\partial_j\partial_l\partial^{-2}\zeta')\gamma_{lk}'\,(2\epsilon_1-\epsilon_1\epsilon_2)\\ &=\int d^3x d\eta\,\sqrt{2\epsilon_1}\Big[\frac{4}{\eta^2}\epsilon_1(\partial_i\zeta)(\partial_j\partial_l\partial^{-2}\zeta)\gamma_{lk}' - \frac{8}{\eta^2}\epsilon_1\epsilon_2(\partial_i\zeta)(\partial_j\partial_l\partial^{-2}\zeta)\gamma_{lk}'\\ &+\frac{3}{\eta^2}\epsilon_1\epsilon_2^2(\partial_i\zeta)(\partial_j\partial_l\partial^{-2}\zeta)\gamma_{lk}' + \frac{4}{\eta}\epsilon_1(\partial_i\zeta')(\partial_j\partial_l\partial^{-2}\zeta)\gamma_{lk}' - \frac{2}{\eta}\epsilon_1\epsilon_2(\partial_i\zeta')(\partial_j\partial_l\partial^{-2}\zeta)\gamma_{lk}'\\ &-\frac{2}{\eta}\epsilon_1\epsilon_2'(\partial_i\zeta)(\partial_j\partial_l\partial^{-2}\zeta)\gamma_{lk}' + \frac{4}{\eta}\epsilon_1(\partial_i\zeta)(\partial_j\partial_l\partial^{-2}\zeta)(\partial^2\gamma_{lk}) - \frac{2}{\eta}\epsilon_1\epsilon_2(\partial_i\zeta)(\partial_j\partial_l\partial^{-2}\zeta)(\partial^2\gamma_{lk})\Big]\end{aligned} \tag{8.7}$$

and we can point out the terms that contribute the most in the inflationary model of interest. In the case of SR-USR-SR inflation, on one hand slow roll parameter ϵ_1 remains small throughout the entire inflationary phase, on the other hand ϵ_2 parameter, though much smaller than unity in the two SR phases, is of $O(1)$ during USR. Furthermore, in this work, we stick to the simplest SR-USR-SR model, characterized by a sharp transition between the two different phases. This means that, among all the terms appearing in (8.7), the dominant one is

$$\frac{2}{\eta}\sqrt{2\epsilon_1}\epsilon_1\epsilon_2'(\partial_i\zeta)(\partial_j\partial_l\partial^{-2}\zeta)\gamma_{lk}'$$

since it leads to the strongest possible divergence. Therefore, in the calculation of the PO trispectrum can be carried out by using the expression for the CS vertex taken from the cubic scalar-scalar-tensor action

$$\begin{aligned}S_{\rm CS}^{\gamma\zeta\zeta} &\simeq \frac{1}{8}(\partial_\phi\omega)M_{\rm Pl}\int d^3x d\eta\,\frac{2}{\eta}\sqrt{2\epsilon_1}\epsilon_1\epsilon_2'(\partial_i\zeta)(\partial_j\partial_l\partial^{-2}\zeta)\gamma_{lk}'\\ &= -\frac{1}{8}(\partial_\phi\omega)M_{\rm Pl}\int d^3x d\eta\,\frac{2}{\eta}\sqrt{2\epsilon_1}\epsilon_1\epsilon_2'(\partial_i\zeta)(\partial_l\partial^{-2}\zeta)(\partial_j\gamma_{lk}')\end{aligned} \tag{8.8}$$

when we also have reintroduced the dependence on the Planck mass to make the action dimensionless.

8.2 Propagators

Before proceeding with the evaluation of the integrals, it is necessary to focus on the explicit expression for the bulk-to-boundary propagators, which are defined as $\mathcal{G}_p(\eta,\eta') = f_p(\eta)f_p^*(\eta')$; recall that the expression for the mode function f_p in SR-USR-SR inflation is given by:

$$f_p(\eta) = \frac{iH}{2M_{\rm Pl}\sqrt{p^3\epsilon_{1i}}}\times F_p(\eta) \tag{8.9}$$

with

$$F_p(\eta) = \begin{cases}(1+ip\eta)e^{-ip\eta} & \text{for } \eta<\eta_1\\ (\eta_1/\eta)^3\big[\gamma_p(1+ip\eta)e^{-ip\eta} - \delta_p(1-ip\eta)e^{ip\eta}\big] & \text{for } \eta_1<\eta<\eta_2\ .\\ (\eta_1/\eta_2)^3\big[\alpha_p(1+ip\eta)e^{-ip\eta} - \beta_p(1-ip\eta)e^{ip\eta}\big] & \text{for } \eta>\eta_2\end{cases} \tag{8.10}$$

In the previous expression, the Bogoliubov coefficients take the form provided by (4.16), (4.17), (4.18) and (4.19). Accounting for the fact that we are computing the trispectrum at the end of inflation, i.e. when the inflaton has already undergone the second SR inflationary phase, the bulk-to-boundary propagator takes the following form:

$$\mathcal{G}_p(\eta,\eta') = \frac{H^2}{4M_{\rm Pl}p^3\epsilon_{1i}}\left(\frac{\eta_1}{\eta_2}\right)^3\left[\alpha_p(1+ip\eta)e^{-ip\eta} - \beta_p(1-ip\eta)e^{ip\eta}\right] \times F(\eta'). \tag{8.11}$$

We also recall that, in case of instantaneous transition SR to USR transition, the time evolution of the second slow-roll parameter is modeled by step-functions

$$\epsilon_2(\eta) = -\Delta\epsilon_2\left[\Theta(\eta-\eta_2) - \Theta(\eta-\eta_1)\right], \tag{8.12}$$

thus, its time derivative is modeled as Dirac-delta function

$$\epsilon_2'(\eta) = \Delta\epsilon_2\left[\delta(\eta-\eta_2) - \delta(\eta-\eta_1)\right]. \tag{8.13}$$

As discussed in Ch.4, tensor modes are not affected by USR dynamics in the model we are considering. Considering that we work at zero*th* order approximation in $H/M_{\rm CS}$ in the modes, they are written as

$$v_p^s(\eta) = \frac{H}{M_{\rm Pl}\sqrt{k^3}} \times F_p^s(\eta), \qquad F_p^s(\eta) = (1+ik\eta)e^{-ik\eta}. \tag{8.14}$$

Since in the CS vertex the graviton enters with a time derivative applied on it, we report the form of the corresponding mode

$$v_p^{s\prime}(\eta) = \frac{iH}{M_{\rm Pl}\sqrt{p^3}}p^2\eta e^{-ip\eta}. \tag{8.15}$$

To make the notation less cumbersome, from now on we will drop the superscript indicating the spin and that the mode is tensorial. Since tensor modes are associated to exchanged momentum p_{12}, whenever $v_{p_{12}}$ and $F_{p_{12}}$ are introduced, we refer to $v^s_{p_{12}}$ and $F^s_{p_{12}}$. Moreover, $F^s_{p_{12}}$ is equivalent to F_p (8.10) for $\eta < \eta_1$.

8.3 Parity-odd scalar trispectrum

As we saw in the previous chapter, the scalar PO trispectrum evaluated in slow-roll inflation vanishes. As analyzed, this result is expected since during slow-roll the theory is scale invariant [33]. However, in the case of SR-USR-SR inflation, there is an explicit breaking of the scale invariance, generated by the dynamics of USR. In this case, hypotheses of the no-go theorem are broken, and we can expect a non vanishing PO signal at the level of the trispectrum. In this section we are shall compute the PO trispectrum in SR-USR-SR inflation taking as CS vertex the one that provides the larger contribution during inflation due to the abrupt SR/USR/SR transition (8.8).

The interaction Hamiltonian of interest in the comoving gauge reads:

$$\begin{aligned}
H_{int} &= H_{\rm GR} + H_{\rm CS} = -a^2M_{\rm Pl}^2(\eta)\epsilon_1(\eta)(\partial_i\zeta)(\partial_j\zeta)\gamma_{ij} - a(\eta)\frac{\epsilon_1(\eta)\epsilon_2'(\eta)}{2M_{\rm CS}(\eta)}M_{\rm Pl}^2(\partial_i\zeta)(\partial_l\partial^{-2}\zeta)(\partial_j\gamma_{lk}') \\
&= M_{\rm Pl}^2\Bigg[a^2(\eta)\epsilon_1(\eta)\int\left(\prod_{a=1}^3\frac{d^2k_a}{(2\pi)^3}\right)(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\sum_{s=L,R}k_1^ik_2^j\epsilon_{ij}^s(\mathbf{k}_3)\,\zeta_{\mathbf{k}_1}(\eta)\zeta_{\mathbf{k}_2}(\eta)\gamma^s_{\mathbf{k}_3}(\eta) \\
&+ \frac{\epsilon_1(\eta)\epsilon_2'(\eta)}{2M_{\rm CS}(\eta)}a(\eta)\int\left(\prod_{a=1}^3\frac{d^2k_a}{(2\pi)^3}\right)(2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{k_3}{k_2^2}\sum_{s=L,R}\alpha_sk_1^ik_2^j\epsilon_{ij}^s(\mathbf{k}_3)\,\zeta_{\mathbf{k}_1}(\eta)\zeta_{\mathbf{k}_2}(\eta)\gamma'^s_{\mathbf{k}_3}(\eta)\Bigg].
\end{aligned} \tag{8.16}$$

By doing the very same steps carried out in Sec.7.5.2 we evaluate the (1,1), (2,0) and (0,2) contributions to the PO trispectrum. For simplicity we set $M_{\rm Pl} = 1$; we will resume it when computing the in-in time integrations. Thus, we find:

$$\langle\zeta_{\mathbf{p}_1}(\eta)\zeta_{\mathbf{p}_2}(\eta)\zeta_{\mathbf{p}_3}(\eta)\zeta_{\mathbf{p}_4}(\eta)\rangle'_{(1,1)} = \frac{p_3+p_4}{p_4^2}\sum_{s=L,R} p_1^i p_2^j p_3^m p_4^n \epsilon_{ij}^s(\mathbf{p}_1+\mathbf{p}_2)\epsilon_{mn}^s(\mathbf{p}_3+\mathbf{p}_4)$$
$$\times \operatorname{Re}\left[\int_{-\infty}^{\eta} d\eta' a^2(\eta')\epsilon_1(\eta')\mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')v_{p_{12}}^*(\eta')\int_{-\infty}^{\eta} d\eta'' a(\eta'')\frac{\epsilon_1(\eta'')\epsilon_2'(\eta'')}{M_{\text{CS}}(\eta'')}\mathcal{G}_{p_3}^*(\eta,\eta'')\mathcal{G}_{p_4}^*(\eta,\eta'')v'_{p_{12}}(\eta'')\right] \tag{8.17}$$

$$\langle\zeta_{\mathbf{p}_1}(\eta)\zeta_{\mathbf{p}_2}(\eta)\zeta_{\mathbf{p}_3}(\eta)\zeta_{\mathbf{p}_4}(\eta)\rangle'_{(0,2)} + \langle\zeta_{\mathbf{p}_1}(\eta)\zeta_{\mathbf{p}_2}(\eta)\zeta_{\mathbf{p}_3}(\eta)\zeta_{\mathbf{p}_4}(\eta)\rangle'_{(2,0)} =$$
$$-\frac{p_3+p_4}{p_4^2}\sum_{s=L,R}\alpha_s p_1^i p_2^j p_3^m p_4^n \epsilon_{ij}^s(\mathbf{p}_1+\mathbf{p}_2)\epsilon_{lm}^s(\mathbf{p}_3+\mathbf{p}_4)\times$$
$$\times \operatorname{Re}\left[\int_{-\infty}^{\eta} d\eta' a^2(\eta')\epsilon_1(\eta')\mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')v_{p_{12}}(\eta')\int_{-\infty}^{\eta'} d\eta'' a(\eta'')\frac{\epsilon_1(\eta'')\epsilon_2'(\eta'')}{M_{\text{CS}}(\eta'')}\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}_{p_4}(\eta,\eta'')v_{p_{12}}'^*(\eta'')\right.$$
$$\left.+\int_{-\infty}^{\eta} d\eta'' a(\eta'')\frac{\epsilon_1(\eta'')\epsilon_2'(\eta'')}{M_{\text{CS}}(\eta'')}\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}_{p_4}(\eta,\eta'')v'_{p_{12}}(\eta'')\int_{-\infty}^{\eta''} d\eta' a^2(\eta')\epsilon_1(\eta')\mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')v_{p_{12}}^*(\eta')\right] \tag{8.18}$$

where the apostrophe on the expectation value $\langle\ldots\rangle'$ indicates that we have absorbed the term $(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)$. Summing (8.17) and (8.18) together, we get exactly the PO trispectrum, since $\langle\zeta_{\mathbf{p}_1}\zeta_{\mathbf{p}_2}\zeta_{\mathbf{p}_3}\zeta_{\mathbf{p}_4}\rangle_c^{\text{PO}} = (2\pi)^3 T^{\text{PO}}\delta^{(3)}(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\mathbf{p}_4)$:

$$T^{\text{PO}} = -\frac{p_3+p_4}{p_4^2}\sum_{s=L,R}\alpha_s p_1^i p_2^j p_3^m p_4^n \epsilon_{ij}^s(\mathbf{p}_1+\mathbf{p}_2)\epsilon_{mn}^s(\mathbf{p}_3+\mathbf{p}_4)\times\operatorname{Re}\big[I_1+I_2-I_3\big]+\text{perms.} \tag{8.19}$$

with

$$I_1 = \int_{-\infty}^{\eta} d\eta' a^2(\eta')\epsilon(\eta')\mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')v_{p_{12}}(\eta')\int_{-\infty}^{\eta'} d\eta'' a(\eta'')\frac{\epsilon_1(\eta'')\epsilon_2'(\eta'')}{M_{\text{CS}}(\eta'')}\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}_{p_4}(\eta,\eta'')v^{*\prime}{}_{p_{12}}(\eta''), \tag{8.20}$$

$$I_2 = \int_{-\infty}^{\eta} d\eta'' a(\eta'')\frac{\epsilon_1(\eta'')\epsilon_2'(\eta'')}{M_{\text{CS}}(\eta'')}\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}_{p_4}(\eta,\eta'')v'_{p_{12}}(\eta'')\int_{-\infty}^{\eta''} d\eta' a^2(\eta')\epsilon_1(\eta')\mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')v_{p_{12}}^*(\eta'), \tag{8.21}$$

$$I_3 = \int_{-\infty}^{\eta} d\eta' a^2(\eta')\epsilon_1(\eta')\mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')v_{p_{12}}^*(\eta')\int_{-\infty}^{\eta} d\eta'' a(\eta'')\frac{\epsilon_1(\eta'')\epsilon_2'(\eta'')}{M_{\text{CS}}(\eta'')}\mathcal{G}_{p_3}^*(\eta,\eta'')\mathcal{G}_{p_4}^*(\eta,\eta'')v'_{p_{12}}(\eta'') \tag{8.22}$$

Let us focus on the time integrations for now.

8.3.1 Time integrations

Substituting the propagator (8.11) inside the integrals (8.20), (8.21) and (8.22), we obtain

$$I_1 = -\frac{\Delta\epsilon_2}{16}\frac{p_{12}^2}{p_2^3p_4^3}\frac{1}{{\epsilon_i}^2}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{p_2}-\beta_{p_2})(\alpha_{p_4}-\beta_{p_4})}{(\alpha_{p_1}-\beta_{p_1})^*(\alpha_{p_3}-\beta_{p_3})^*}\times$$
$$\int_{-\infty}^{\eta}\frac{d\eta'}{{\eta'}^2}\epsilon_1(\eta')F_{p_1}^*(\eta')F_{p_2}^*(\eta')F_{p_{12}}(\eta')\int_{-\infty}^{\eta''} d\eta''\frac{H}{M_{\text{CS}}(\eta'')}\epsilon_1(\eta'')\,F_{p_3}^*(\eta'')F_{p_4}^*(\eta'')e^{ip_{12}\eta''}\left[\delta(\eta''-\eta_2)-\delta(\eta''-\eta_1)\right] \tag{8.23}$$

$$I_2 = -\frac{\Delta\epsilon_2}{16}\frac{p_{12}^2}{p_2^3p_4^3}\frac{1}{{\epsilon_i}^2}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{p_2}-\beta_{p_2})(\alpha_{p_4}-\beta_{p_4})}{(\alpha_{p_1}-\beta_{p_1})^*(\alpha_{p_3}-\beta_{p_3})^*}\times$$
$$\int_{-\infty}^{\eta} d\eta''\frac{H}{M_{\text{CS}}(\eta'')}\epsilon_1(\eta'')F_{p_3}^*(\eta'')F_{p_4}^*(\eta'')e^{-ip_{12}\eta''}\left[\delta(\eta''-\eta_2)-\delta(\eta''-\eta_2)\right]\int_{-\infty}^{\eta'}\frac{d\eta'}{\eta'}\epsilon_1(\eta')F_{p_1}^*(\eta')F_{p_2}^*(\eta')F_{p_{12}}^*(\eta') \tag{8.24}$$

$$I_3 = -\frac{\Delta\epsilon_2}{16}\frac{p_{12}^2}{p_2^3 p_4^3}\frac{1}{\epsilon_i{}^2}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{p_2}-\beta_{p_2})(\alpha_{p_4}-\beta_{p_4})^*}{(\alpha_{p_1}-\beta_{p_1})^*(\alpha_{p_3}-\beta_{p_3})}\times$$
$$\int_{-\infty}^{\eta}\frac{d\eta'}{\eta'^2}\epsilon_1(\eta')F^*_{p_1}(\eta')F^*_{p_2}(\eta')F^s_{p_{12}}{}^*(\eta')\int_{-\infty}^{\eta}d\eta''\frac{H}{M_{\rm CS}(\eta'')}\epsilon_1(\eta'')\,F_{p_3}(\eta'')F_{p_4}(\eta'')e^{-ip_{12}\eta''}\left[\delta(\eta''-\eta_2)-\delta(\eta''-\eta_1)\right] \tag{8.25}$$

where the scalar power spectra $\mathcal{P}_\zeta$ are the ones derived in Ch.4

$$\mathcal{P}_\zeta(p) = \frac{H^2}{4M_{\rm Pl}^2 p^3\,\epsilon_{1_f}}\left|\alpha_p-\beta_p\right|^2. \tag{8.26}$$

Moreover, for simplicity of notation, we have already taken the expression for the scalar modes $f_p(\eta)$ for $\eta\to 0$ without setting this limit in the integration limits.

In dealing with the time integrals it is useful to make them dimensionless. This can be achieved by changing the integration variable to a dimensionless parameter. In this way, the dimensionful part of the integral can be factored out and the dimensions of the trispectrum are made manifest. The trispectrum depends on four momenta whose magnitudes can be taken to satisfy the relation $p_1\geq p_2\geq p_3\geq p_4$ without loss of generality. For each momentum p_i with $i=1,\dots,4$, we can define the ratio $r_i=p_i/p_1$, $0<r_i\leq 1$. We then introduce the dimensionless parameter $x=p_1\eta$, and focus on the term of the propagator that depends on the internal time η' over which we are integrating. This parametrization will be particularly useful for future observations. Notice that $\eta=\eta_1$ and $\eta=\eta_2$ the parameter x is $x_1=p_1\eta_1$ and $x_2=p_1\eta_2$, satisfying $x_1<x_2<0$. The F functions with this parametrization become

$$F_{r_i}(x) = \begin{cases}(1+ir_ix)e^{-ir_ix} & \text{for } x<x_1\\ (x_1/x)^3\left[\gamma_{r_i}(1+ir_ix)e^{-ir_ix}-\delta_{r_i}(1-ir_ix)e^{ir_ix}\right] & \text{for } x_1<x<x_2\\ (x_1/x_2)^3\left[\alpha_{r_i}(1+ir_ix)e^{-ir_ix}-\beta_{r_i}(1-ir_ix)e^{ir_ix}\right] & \text{for } x>x_2\end{cases}. \tag{8.27}$$

and $F_{r_{12}}(x)=(1+ir_{12}x)e^{-ir_{12}x}$.

The integrals above are given by

$$I_1 = -\frac{\Delta\epsilon_2}{16}\frac{p_{12}^2p_1}{p_2^3p_4^3}\frac{1}{\epsilon_i{}^2}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})^*}\int_{-\infty}^{x}dx'\epsilon_1(x')F^*_{r_1}(x')F^*_{r_2}(x')F_{r_{12}}(x')$$
$$\times\int_{-\infty}^{x'}dx''\frac{H}{M_{\rm CS}(x'')}\epsilon_1(x'')F^*_{r_3}(x'')F^*_{r_4}(x'')e^{ir_{12}x''}\left[\delta(x''-x_2)-\delta(x''-x_1)\right] \tag{8.28}$$

$$I_2 = -\frac{\Delta\epsilon_2}{16}\frac{p_{12}^2p_1}{p_2^3p_4^3}\frac{1}{\epsilon_i{}^2}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})^*}$$
$$\int_{-\infty}^{x}dx''\frac{H}{M_{\rm CS}(x'')}\epsilon_1(x'')F^*_{r_3}(x'')F^*_{r_4}(x'')e^{-ir_{12}}\left[\delta(x''-x_2)-\delta(x''-x_1)\right]\int_{-\infty}^{x''}dx'\epsilon_1(x')F^*_{r_1}(x')F^*_{r_2}(x')F^*_{r_{12}}(x') \tag{8.29}$$

$$I_3 = -\frac{\Delta\epsilon_2}{16}\frac{p_{12}^2p_1}{p_2^3p_4^3}\frac{1}{\epsilon_i^2}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})^*}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})}$$
$$\int_{-\infty}^{x}dx'\epsilon_1(x')F^*_{r_1}(x')F^*_{r_2}(x')F^*_{r_{12}}(x')\int_{-\infty}^{x}dx''\frac{H}{M_{CS}(x'')}\epsilon_1(x'')F_{r_3}(x'')F_{r_4}(x'')e^{-ir_{12}x''}\left[\delta(x''-x_2)-\delta(x''-x_1)\right] \tag{8.30}$$

where we used $a(\eta)=-1/H\eta$ since we are approximating the universe during inflation as a pure de Sitter spacetime.

These integrals are very much simplified by the Dirac deltas contained in the term corresponding to the integration of the CS vertex. However, the fact that they appear to be nested makes the integration domain of the remaining one worth a few more considerations.

Integral I_1

Consider the case in which a Dirac delta in the variable y'' is included within a nested integral in the integration variables y' and y''. This leads to:

$$\int_{-\infty}^{y} dy' f(y') \int_{-\infty}^{y'} dy'' g(y'')\delta(y''-y_*) = g(y_*) \int_{-\infty}^{y} dy' f(y)\Theta(y'-y_*) \ . \tag{8.31}$$

The appearance of the Heaviside function is due to the fact that the integral over η' is null if η_* falls outside its integration domain. This allows us to restrict the integration domain to $y' \in [y_*, y]$ and to remove the Heaviside theta function from the integral.

Recognizing that (8.31) is exactly the integral structure of (8.28), it becomes

$$\begin{aligned} I_1 = &-\frac{\Delta\epsilon_2}{16}\frac{p_{12}^2 p_1}{p_2^3 p_4^3}\frac{1}{\epsilon_i{}^2}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})^*} \\ &\left\{\left[\frac{H}{M_{\rm CS}(x_2)}\epsilon_1(x_2)e^{6\Delta N}\mathcal{F}^*_{r_3}(\alpha,\beta,x_2)\mathcal{F}^*_{r_4}(\alpha,\beta,x_2)e^{ir_{12}x_2} - \frac{H}{M_{\rm CS}(x_1)}\epsilon_1(x_1)\mathcal{F}^*_{r_3}(\alpha,\beta,x_1)\mathcal{F}^*_{r_4}(\alpha,\beta,x_1)e^{ir_{12}x_1}\right]\right. \\ &\times\int_{x_2}^{x} dx'\epsilon_1(x')F^*_{r_1}(x')F^*_{r_2}(x')F_{r_{12}}(x') \\ &\left.-\frac{H}{M_{\rm CS}(x_1)}\epsilon_1(x_1)\mathcal{F}^*_{r_3}(\alpha,\beta,x_1)\mathcal{F}^*_{r_4}(\alpha,\beta,x_1)e^{ir_{12}x_1}\int_{x_1}^{x_2} dx'\epsilon_1(x')F^*_{r_1}(x')F^*_{r_2}(x')F_{r_{12}}(x')\right\} \end{aligned} \tag{8.32}$$

where the function $\mathcal{F}_k(A,B,x)$ is defined for convenience as

$$\mathcal{F}_r(A,B,x) = A_r(1+irx)e^{-irx} - B_r(1-irx)e^{irx} \tag{8.33}$$

with $A_k = 1, \gamma_k, \alpha_k$ and $B_k = 0, \delta_k, \beta_k$.

Recalling that ϵ_1 evolves as (4.3) during the entire inflationary period, and the functional form of the F functions (8.10), the time dependence of the first slow-roll parameter inside the integral is exactly canceled by the product $F^*_{r_1}F^*_{r_2}$. Moreover, the Chern-Simons mass $M_{\rm CS} \propto 1/\sqrt{\epsilon_1}$ acquires a dependence on time as $M_{\rm CS} \propto (\eta/\eta_1)^3$ during USR,[1] and thus

$$\frac{H}{M_{\rm CS}(x_2)} = e^{-3\Delta N}\frac{H}{M_{\rm CS}(x_1)} = e^{-3\Delta N}\frac{H}{M_{{\rm CS},i}}. \tag{8.36}$$

Therefore, we can rewrite the term I_1 to be

$$\begin{aligned} I_1 = &-\frac{\Delta\epsilon_2}{16}\frac{H}{M_{{\rm CS},i}}\frac{p_{12}^2 p_1}{p_2^3 p_4^3}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})^*} \\ &\left\{\left[e^{-3\Delta N}\mathcal{F}^*_{r_3}(\alpha,\beta,x_2)\mathcal{F}^*_{r_4}(\alpha,\beta,x_2)e^{ir_{12}x_2} - \mathcal{F}^*_{r_3}(1,0,x_1)\mathcal{F}^*_{r_4}(1,0,x_1)e^{ir_{12}x_1}\right]\right. \\ &\times\int_{x_2}^{x} dx'\mathcal{F}^*_{r_1}(\alpha,\beta,x')\mathcal{F}^*_{r_2}(\alpha,\beta,x')\mathcal{F}_{r_{12}}(1,0,x') \\ &\left.-\mathcal{F}^*_{r_3}(1,0,x_1)\mathcal{F}^*_{r_4}(1,0,x_1)e^{ir_{12}x_1}\int_{x_1}^{x_2} dx'\mathcal{F}^*_{r_1}(\alpha,\beta,x')\mathcal{F}^*_{r_2}(\alpha,\beta,x')\mathcal{F}_{r_{12}}(\alpha,\beta,x')\right\}. \end{aligned} \tag{8.37}$$

[1]This indeed the correct scaling of the CS mass. In fact, its logarithmic variation during the characteristic time of expansion $1/H$ gives

$$\frac{\dot M_{\rm CS}}{HM_{\rm CS}} = \frac{1}{H}\frac{\partial^2_\phi\omega(\phi)\dot\phi^2 + \partial_\phi\omega(\phi)\ddot\phi}{\partial_\phi\omega(\phi)\dot\phi} = \frac{\dot\phi}{H}\frac{\partial^2_\phi\omega(\phi)}{\partial_\phi\omega(\phi)} - 3 \approx -3 \tag{8.34}$$

in case the coupling is a smooth function of the inflaton

$$\left|\frac{\dot\phi}{H}\frac{\partial^2_\phi\omega(\phi)}{\partial_\phi\omega(\phi)}\right| = \left|\frac{1}{H}\frac{d\ln\omega_{,\phi}(\phi)}{dt}\right| \ll 1, \qquad \Longrightarrow \qquad |\partial^2_\phi\omega(\phi)| \ll \left|\frac{\sqrt{2\epsilon_1}\,\partial_\phi\,\omega(\phi)}{M_{\rm Pl}}\right|^2 \tag{8.35}$$

which is equivalent to require that the variation of the coupling with respect to the field in logarithmic scales changes slowly during an Hubble time.

Integral I_2

In the second term I_2, another nested integral with Dirac delta appears:

$$\int_{-\infty}^{y} dy'' f(y'')\delta(y''-y_*)\int_{-\infty}^{y''} dy' g(y') = f(y_*)\int_{-\infty}^{y_*} dy' g(y') \ .$$

Implementing this to (8.29) and accounting for the time dependence of ϵ_1 and $M_{\rm CS}$ and the functional form of the mode functions in the various inflationary epochs, we arrive to:

$$\begin{aligned} I_2 = &-\frac{\Delta\epsilon_2}{16}\frac{H}{M_{{\rm CS},i}}\frac{p_{12}^2 p_1}{p_2^3 p_4^3}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})^*} \\ &\left\{\left[e^{-3\Delta N}\mathcal{F}^*_{r_3}(\alpha,\beta,x_2)\mathcal{F}^*_{r_4}(\alpha,\beta,x_2)e^{-ir_{12}x_2} - \mathcal{F}^*_{r_3}(1,0,x_1)\mathcal{F}^*_{r_4}(1,0,x_1)e^{-ir_{12}x_1}\right]\right. \\ &\quad\times\int_{-\infty}^{x_1} dx'\mathcal{F}^*_{r_1}(\alpha,\beta,x')\mathcal{F}^*_{r_2}(\alpha,\beta,x')\mathcal{F}^*_{r_{12}}(1,0,x') \\ &\left.+\,e^{-3\Delta N}\mathcal{F}^*_{r_3}(\alpha,\beta,x_2)\mathcal{F}^*_{r_4}(\alpha,\beta,x_2)e^{-ir_{12}x_2}\int_{x_1}^{x_2} dx'\mathcal{F}^*_{r_1}(\gamma,\delta,x')\mathcal{F}^*_{r_4}(\gamma,\delta,x')\mathcal{F}^*_{r_{12}}(\gamma,\delta,x')\right\}. \end{aligned} \tag{8.38}$$

Integral I_3

In the third term I_3 there is no nested integration appearing. This makes it the easiest integration to perform, without further considerations on the integration limit in the remaining operation.

By performing immediately the integration over x'' and accounting for the time dependence of the first slow-roll parameter and $M_{\rm CS}$, we get:

$$\begin{aligned} I_3 = &-\frac{\Delta\epsilon_2}{16}\frac{H}{M_{{\rm CS},i}}\frac{p_{12}^2 p_1}{p_2^3 p_4^3}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})^*}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})} \\ &\times\left[e^{-3\Delta N}\mathcal{F}_{r_3}(\alpha,\beta,x_2)\mathcal{F}_{r_4}(\alpha,\beta,x_2)e^{-ir_{12}x_2} - \mathcal{F}_{r_3}(1,0,x_1)\mathcal{F}_{r_4}(1,0,x_1)e^{-ir_{12}x_1}\right] \\ &\times\left[\int_{-\infty}^{x_1} dx'\mathcal{F}^*_{r_1}(1,0,x')\mathcal{F}^*_{r_2}(1,0,x')\mathcal{F}^*_{r_{12}}(1,0,x') + \int_{x_1}^{x_2} dx'\mathcal{F}^*_{r_1}(\gamma,\delta,x')\mathcal{F}^*_{r_2}(\gamma,\delta,x')\mathcal{F}^*_{r_{12}}(1,0,x')\right. \\ &\quad\left.+\int_{x_2}^{x} dx'\mathcal{F}^*_{r_1}(\alpha,\beta,x')\mathcal{F}^*_{r_2}(\alpha,\beta,x')\mathcal{F}^*_{r_{12}}(1,0,x')\right]. \end{aligned} \tag{8.39}$$

Let us now focus on the remaining time integration. We notice that there are only two types of integrals appearing in (8.37), (8.38) and (8.39):

$$\int dx'\mathcal{F}^*_{r_1}(A,B,x')\mathcal{F}^*_{p_2}(A,B,x')\mathcal{F}^*_{r_{12}}(1,0,x') \quad\text{and}\quad \int dx'\mathcal{F}^*_{r_1}(A,B,x')\mathcal{F}^*_{r_2}(A,B,x')\mathcal{F}_{r_{12}}(1,0,x'). \tag{8.40}$$

Since the complex conjugation on the function $\mathcal{F}$ corresponds to flip the sign of the momentum in it if the Bogoljubov coefficients A and B are respectively equal to the unity and zero, $\mathcal{F}^*_r(1,0,x) = (1-irx)e^{irx} = \mathcal{F}_{-r}(1,0,x)$, it is sufficient to solve just one of the two integral above, and we will obtain the other by reversing the sign of the dimensionless combination r_{12}. Therefore, we can consider only the first integral and its solution is given by (for the solution, see Sec. A.2.1 in the appendix)

$$\begin{aligned}
\bar{I}(r_1,r_2,r_{12},x_a,x_b) &= \int_{x_a}^{x_b} dx' \mathcal{F}^*_{r_1}(A,B,x')\mathcal{F}^*_{r_2}(A,B,x')\mathcal{F}^*_{r_{12}}(1,0,x') \\
&= A^*_{r_1}A^*_{r_2}\Big(\mathcal{I}(r_1,r_2,r_{12},x_b)-\mathcal{I}(r_1,r_2,r_{12},x_a)\Big) - A^*_{r_1}B^*_{r_2}\Big(\mathcal{I}(r_1,-r_2,r_{12},x_b)-\mathcal{I}(r_1,-r_2,r_{12},x_a)\Big) \\
&- B^*_{r_1}A^*_{r_2}\Big(\mathcal{I}(-r_1,r_2,r_{12},x_b)-\mathcal{I}(-r_1,r_2,r_{12},x_a)\Big) + B^*_{r_1}B^*_{r_2}\Big(\mathcal{I}(-r_1,-r_2,r_{12},x_b)-\mathcal{I}(-r_1,-r_2,r_{12},x_a)\Big)
\end{aligned} \tag{8.41}$$

with

$$\mathcal{I}(r_1,r_2,r_{12},x) = \Big(-\frac{1}{x} - i\frac{\beta_1\beta_2+\beta_3}{\beta_1^2} - \frac{\beta_3}{\beta_1}x\Big)e^{i\beta_1 x}. \tag{8.42}$$

with the beta functions given by:

$$\begin{cases} \beta_1 = r_1+r_2+r_{12} \\ \beta_2 = -(r_1+r_2)r_{12} - r_1 r_2 \\ \beta_3 = -r_1 r_2 r_{12} \end{cases} . \tag{8.43}$$

The integral (8.41), if $x_a = -\infty$ and $x_b = x_1$, reduces to

$$\bar{I} = \int_{-\infty}^{x_1} dx' \mathcal{F}^*_{r_1}(1,0,x')\mathcal{F}^*_{r_2}(1,0,x')\mathcal{F}^*_{r_{12}}(1,0,x') = \mathcal{I}(r_1,r_2,r_{12},x_1) \tag{8.44}$$

once we account for the $i\epsilon$ prescription.

If we evaluate it during the second slow-roll phase, between x_2 and 0, the contribution linearly diverges. In fact,

$$\begin{aligned}
\bar{I} &= \lim_{x\to 0^-}\int_{x_2}^{x} dx' \mathcal{F}^*_{r_1}(\alpha,\beta,x')\mathcal{F}^*_{r_2}(\alpha,\beta,x')\mathcal{F}^*_{r_{12}}(1,0,x') \\
&= \alpha^*_{r_1}\alpha^*_{r_2}\Big(\mathcal{I}(r_1,r_2,r_{12},0^-)-\mathcal{I}(r_1,r_2,r_{12},x_2)\Big) - \alpha^*_{r_1}\beta^*_{r_2}\Big(\mathcal{I}(r_1,-r_2,r_{12},0^-)-\mathcal{I}(r_1,-r_2,r_{12},x_2)\Big) \\
&- \beta^*_{r_1}\alpha^*_{r_2}\Big(\mathcal{I}(-r_1,r_2,r_{12},0^-)-\mathcal{I}(-r_1,r_2,r_{12},x_2)\Big) + \beta^*_{r_1}\beta^*_{r_2}\Big(\mathcal{I}(-r_1,-r_2,r_{12},0^-)-\mathcal{I}(-r_1,-r_2,r_{12},x_2)\Big)
\end{aligned} \tag{8.45}$$

and

$$\mathcal{I}(r_1,r_2,r_{12},x\to 0^-) = -\frac{1}{x} - i\frac{\beta_1\beta_2+\beta_3}{\beta_1^2} \tag{8.46}$$

where we have kept only terms that are not vanishing at the end of inflation. Notice that the contribution is divergent and the divergent part is purely real. This will be important in the following.

Then I_1, I_2 and I_3 are given by:

$$\begin{aligned}
I_1 = \text{pref} \times &\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})^*} \\
\times &\Big\{\big[A'(r_3,r_4,-r_{12},x_1,\Delta N) - A''(r_3,r_4,r_{12},x_1)\big]\bar{I}(r_1,r_2,r_{12},x_2,0) \\
&- A''(r_3,r_4,r_{12},x_1)\bar{I}(r_1,r_2,r_{12},x_1,x_2)\Big\},
\end{aligned} \tag{8.47}$$

$$\begin{aligned}
I_2 = \text{pref} \times &\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})^*} \\
&\Big\{\big[A'(r_1,r_2,-r_{12},x_1,\Delta N) - A''(r_1,r_2,-r_{12},x_1)\big]\bar{I}(r_1,r_2,-r_{12},-\infty,x_1) \\
&+ A'(r_1,r_2,-r_{12},x_1,\Delta N)\bar{I}(r_1,r_2,-r_{12},x_1,x_2)\Big\},
\end{aligned} \tag{8.48}$$

$$\begin{aligned}I_3 = \text{pref} \times &\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})^*}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})} \\ &\times \left[A'(r_1,r_2,r_{12},x_1,\Delta N) - A''(r_1,r_2,r_{12},x_1)\right]^* \\ &\times \left[\bar{I}(r_1,r_2,r_{12},-\infty,x_1) + \bar{I}(r_1,r_2,r_{12},x_1,x_2) + \bar{I}(r_1,r_2,r_{12},x_2,0)\right],\end{aligned} \tag{8.49}$$

with the functions

$$\text{pref} = -\frac{\Delta\epsilon_2}{16}\frac{H}{M_{\text{CS},i}}\frac{p_{12}^2 p_1}{p_2^3 p_4^3}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12}), \tag{8.50}$$

$$A'(r_1,r_2,r_{12},x_1,\Delta N) = e^{-3\Delta N}\mathcal{F}^*_{r_3}(\alpha,\beta,x_2)\mathcal{F}^*_{r_4}(\alpha,\beta,x_2)e^{ir_{12}x_2}, \tag{8.51}$$

$$A''(r_1,r_2,r_{12},x_1) = \mathcal{F}^*_{r_3}(1,0,x_1)\mathcal{F}^*_{r_4}(1,0,x_1)e^{ir_{12}x_1}. \tag{8.52}$$

8.3.2 Total parity-odd trispectrum

Turning to the total PO trispectrum, we perform the polarization sum:

$$\sum_{\lambda=L,R} \alpha_\lambda p_1^i p_2^j p_3^m p_4^n \epsilon^\lambda_{ij}(\mathbf{p}_{12})\epsilon^\lambda_{mn}(\mathbf{p}_{34}) = i\, p_1^2 p_3^2 \sin^2\theta_1 \sin^2\theta_3 \sin(2\chi_{12,34}) \tag{8.53}$$

were, following the same conventions defined in Sec.6.2.1, the θ_i angles are defined as the angles comprised between the momentum vector $\mathbf{p}_i$ and $\mathbf{p}_{12}$ which lies in the $\hat{\mathbf{z}}$ axis. Moreover, the angle χ is the angle between the two planes identified by couples of momenta vectors $\{\mathbf{p}_1,\mathbf{p}_2\}$ and $\{\mathbf{p}_3,\mathbf{p}_4\}$. Putting all pieces together, we find that the total parity odd trispectrum takes the form:

$$\begin{aligned}T^{\text{PO}} &= i\frac{\Delta\epsilon_2}{16}\frac{H}{M_{\text{CS},i}}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\sin^2\theta_1\sin^2\theta_3\sin 2\chi_{12,34} \\ &\times \frac{p_1^3 p_{12}^3 p_3^2}{p_4^5 p_2^3}\text{Re}\Bigg\{\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})^*}\bigg[\Big(A'(r_1,r_2,-r_{12},x_1,\Delta N) - A''(r_1,r_2,-r_{12},x_1)\Big) \\ &\times \bar{I}(r_1,r_2,-r_{12},-\infty,x_1) + \Big(A'(r_1,r_2,r_{12},x_1,\Delta N) - A''(r_1,r_2,r_{12},x_1)\Big)\bar{I}(r-1,r_2,r_{12},x_2,0) \\ &+ A'(r_1,r_2,-r_{12},x_1,\Delta N)\bar{I}(r_1,r_2,-r_{12},x_1,x_2) - A''(r_1,r_2,r_{12},x_1)\bar{I}(r_1,r_2,r_{12},x_1,x_2)\bigg] \\ &- \frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})^*}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})}\left[A'(r_1,r_2,r_{12},x_1,\Delta N) - A''(r_1,r_2,r_{12},x_1,\Delta N)\right]^* \\ &\times \left[\bar{I}(r_1,r_2,r_{12},-\infty,x_1) + \bar{I}(r_1,r_2,r_{12},x_1,x_2) + \bar{I}(r_1,r_2,r_{12},x_2,0)\right]\Bigg\} + \text{perms.}\end{aligned} \tag{8.54}$$

Notice that the PO trispectrum does not have any late-time divergences, even if $\lim_{x\to 0^-}\mathcal{I}(r_1,r_2,r_{12},x)$ linearly diverges. Let us consider only the divergent part of $\mathcal{I}(r_1,r_2,r_{12},x)$ (8.42) if we send x to zero from the left. Then

$$\lim_{x\to x_0}\mathcal{I}(r_1,r_2,r_{12},x) = -\frac{1}{x_0} - i\frac{\beta_1\beta_2+\beta_3}{\beta_1^2} \tag{8.55}$$

where we have kept only the terms that are non vanishing by the end of inflation and the divergent term is implicit. Of course $x_0 = 0^-$. Considering only the divergent term, it appears in two blocks of the trispectrum:

$$\begin{aligned}\bar{I}_1 &= C(\mathbf{p}_i,\eta_1)\text{Re}\bigg\{\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})^*}A'(r_3,r_4,r_{12},x_1,\Delta N) \\ &\times \left(\alpha^*_{r_1}\alpha^*_{r_2} - \alpha^*_{r_1}\beta^*_{r_2} - \beta^*_{r_1}\alpha^*_{r_2} + \beta^*_{r_1}\beta^*_{r_2}\right)\left(-\frac{1}{x_0}\right)\bigg\} \\ &= C(\mathbf{p}_i,\eta_1)\text{Re}\bigg\{-\frac{1}{x_0}\frac{\left|\alpha_{r_2}-\beta_{r_2}\right|^2(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_3}-\beta_{r_3})^*}A'(r_3,r_4,r_{12},x_1,\Delta N)\bigg\}\end{aligned} \tag{8.56}$$

and

$$\begin{aligned}\bar{I}_2 &= C(\mathbf{p}_i,\eta_1)\mathrm{Re}\bigg\{-\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})^*}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})}A'^*(r_3,r_4,r_{12},x_1,\Delta N)\\ &\times\left(\alpha^*_{r_1}\alpha^*_{r_2}-\alpha^*_{r_1}\beta^*_{r_2}-\beta^*_{r_1}\alpha^*_{r_2}+\beta^*_{r_1}\beta^*_{r_2}\right)\left(-\frac{1}{x_0}\right)\bigg\}\\ &= C(\mathbf{p}_i,\eta_1)\mathrm{Re}\bigg\{\frac{1}{x_0}\frac{\left|\alpha_{p_2}-\beta_{p_2}\right|^2(\alpha_{r_4}-\beta_{r_4})^*}{(\alpha_{r_3}-\beta_{r_3})}A'^*(r_3,r_4,r_{12},x_1,x_2)\bigg\}\end{aligned}\tag{8.57}$$

with

$$C(\mathbf{p}_i,\eta_1)=-i\frac{\Delta\epsilon_2}{16}\frac{H}{M_{\mathrm{CS},i}}\frac{p_1^3p_{12}^3p_3^2}{p_4^5p_2^3}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\sin^2\theta_1\sin^2\theta_3\sin(2\chi_{12,34})\tag{8.58}$$

and A' defined as (8.51). When these two blocks are summed together, we get terms of the type

$$T^{\mathrm{PO}}\supset C(\mathbf{p}_i,\eta_1)\mathrm{Re}\bigg\{\frac{1}{x_0}\left|\alpha_{r_2}-\beta_{r_2}\right|^2\left(X-X^*\right)\bigg\}\tag{8.59}$$

with $X=(\alpha_{r_4}-\beta_{r_4})A'(r_3,r_4,r_{12},x_1,x_2)/(\alpha_{r_3}-\beta_{r_3})^*$. Since $X-X^*=2i\,\mathrm{Im}X$, the term inside curly brackets in (8.59) is purely imaginary: in fact η_0, $\left|\alpha_p-\beta_p\right|^2$ and $\mathrm{Im}X$ are real quantities and are multiplied by the imaginary unit i. Therefore, the PO trispectrum does not manifest any late-time divergence since it is a purely imaginary contribution and only the real part contributes to the trispectrum.

We notice that, as we set $\eta_1=\eta_2$, which correspond to $x_1=x_2$ in our dimensionless parametrization, the Bogoljubov coefficients reduce to $\alpha=1$ and $\beta=0$, and the parity odd trispectrum (8.54) vanishes. In fact, $(A'-A'')$ vanish in this limit, and terms which do not contain this combination involve contributions of the type $\mathcal{I}(r_1,r_2,r_{12},x_2)-\mathcal{I}(r_1,r_2,r_{12},x_1)$ vanish once we set $x_1=x_2$. This is indeed a sanity check for the expression for parity odd trispectrum in SR-USR-SR inflation, since setting $\eta_1=\eta_2$ is equivalent to having no USR phase at all, and no-go theorems hold.

8.3.3 Large-scale limit

Let us focus on the large-scale limit $(p_1,p_2,p_3,p_4,p_{12}\ll k_1,k_2)$ of the expression for the PO trispectrum in SR-USR-SR inflation. Recall that k_1 and k_2 are the wavenumbers associated to modes crossing the horizon at conformal time η_1 and η_2. As implemented previously, without loss of generality, we can assume that the amplitudes of the momenta satisfy $p_1\geq p_2\geq p_3\geq p_4$. Here it is clear why the parameterization introduced in Sec.8.3.1 is advantageous. The large-scale limit corresponds to p_i/k_1, $p_i/k_2\to 0$, that is, the wavenumber of fluctuations involved in the correlator must be much smaller than that associated to the SR/USR/SR transitions. This limit results in simply taking the parameter x_1 in the trispectrum to satisfy $x_1\to 0$. In fact, $p_i/k_1=-p_i\eta_1=-r_ix_1$, with $x_1=p_1\eta_1$ and $r_i=p_i/p_1$, $i=1,\dots,4$, $0<r_i\leq 1$. Moreover, $p_i/k_2=e^{\Delta N}p_i/k_1$.

From Eq. 8.54, we see that the parameter x_1 appears only inside $\mathrm{Re}\{\dots\}$. Thus, we proceed expanding the PO trispectrum for $x_1\to 0$ as

$$T^{\mathrm{PO}}=Ax_1^{-n}+Bx_1^{-n+1}+\cdots+Cx_1^{-1}+Dx_1^0+Ex_1+\dots,\qquad x_1\to 0$$

where the coefficients $A,B,\dots$ are required to be real.

Proceeding evaluating the Laurent expansion of the PO trispectrum with Mathematica, we find that it is vanishing at large scales. The coefficients associated with negative powers of x_1 cancel out once all terms inside one single permutation are summed up, and the coefficient D in front of the zeroth power is purely imaginary, thus it does not contribute to the final expression of the PO which results to be $T^{\mathrm{PO}}\propto O(x_1)$. This makes T^{PO} identically vanishing in the large-scale limit.

This result is in agreement with no-go theorems and provides another sanity check of the expression 8.54. Indeed, when taking this limit, the trispectrum receives contributions from modes that have left horizon at very early times, when the inflaton is undergoing the first slow-roll evolution: during this phase the hypotheses of no-go theorem are satisfied[33]. For this reason, the vanishing of the PO trispectrum is expected in that limit, for any configuration of momenta, and it is a confirmation of the self-consistency of the result (8.54).

8.3.4 Small-scale limit

As the PO trispectrum vanishes in the large-scale limit, we expect it to be zero in the small-scale limit as well. In that regime, the trispectrum receives contributions from small wavenumber modes exiting the horizon during the second slow-roll phase, when scale invariance has been restored again.

Thus, we compute the large-scale limit, which corresponds to $p_i/k_1 = -r_i x_1 \to \infty$ and $p_i/k_2 \to \infty$. Actually, the limit does not exist: the PO trispectrum reduces to a combination terms of the type of $e^{i(r_1+r_2\pm r_{12})x_1}$, $e^{i(r_1+r_2\pm r_{12})x_2}$, $e^{i(r_3+r_4\pm r_{12})x_1}$ and $e^{i(r_3+r_4\pm r_{12})x_2}$, which heavily oscillate once $x_1, x_2 \to -\infty$. However, the presence of these terms is due to the sharp SR/USR/SR transition. These contributions arise from the fact that the sharp nature of the transition demands a tripartition of the in-in integration, causing the oscillatory terms of the type $e^{i\alpha x}$, $\alpha = \alpha(r_i)$, to be evaluated at x_1 and x_2, associated to the transition times η_1 and η_2, $\eta_1 < \eta_2 < 0$. If the transition were smooth, those contribution would be evaluated at the beginning and the end of inflation, solving the problem of the non-existence of the limit.

We can then expect the PO trispectrum to be vanishing in this limit due to the previous consideration and, in the instantaneous transition regime we chose to work in, the trispectrum at large values of p_i not to be physically informative.

8.4 Parity-odd trispectrum in the equilateral configuration

Having found that the SR-USR-SR inflationary model with CS gravity is able to generate a non vanishing PO trispectrum, we wonder which is its shape and which are the kinematic configurations such as the signal is maximized. In this section we took inspiration by the works [71, 72].

Consider a general tridimensional coordinate system and four momenta $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and $\mathbf{p}_4$ satisfying $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = 0$. Then, rotate the coordinate system such that $\mathbf{p}_{12} \parallel \hat{\mathbf{z}}$ and $\mathbf{p}_1$ and $\mathbf{p}_2$ lie in the $\{\hat{\mathbf{x}}, \hat{\mathbf{z}}\}$ plane; momenta $\mathbf{p}_3$ and $\mathbf{p}_4$ will lie on a different plane rotated by an angle χ with respect to the $\{\hat{\mathbf{x}}, \hat{\mathbf{z}}\}$ one. This coordinate system is exactly the same as the one we chose to make the polarization sum explicit. In this reference system, the trispectrum can be visualized as the tetrahedron built from two triangles, $\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_{12} = 0$ and $\mathbf{p}_3 + \mathbf{p}_4 + \mathbf{p}_{12} = 0$, which are rotated by an angle χ one with respect to each other.

In case the exchanged momentum $\mathbf{p}_{12} \to 0$, the configuration reduces to be planar: all the $\mathbf{p}_i$ vectors reduce to a triangle in $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$. Since the structure is planar, the PO trispectrum identically vanishes, considering that it is not possible to capture parity violating physics from a planar figure. In this case $\mathbf{p}_1 = -\mathbf{p}_2$ and $\mathbf{p}_3 = -\mathbf{p}_4$ and the momentum conservation condition can be expressed as

$$\delta^3(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) = \int d^3p_{13}\, \delta^3(\mathbf{p}_{13} - \mathbf{p}_1 - \mathbf{p}_3)\delta^3(\mathbf{p}_{13} + \mathbf{p}_2 + \mathbf{p}_4) = \int d^3p_{13}\, \delta^3(\mathbf{p}_{13} - \mathbf{p}_1 - \mathbf{p}_3)$$

if we eliminate the redundant Dirac delta. The momentum configuration reduces to a triangle, and, as analyzed in Sec.7.1, no non-vanishing pseudo-scalar can be constructed out of this configuration. In the same way, in the case in which one or more momenta collapse to zero, the configuration is no more tridimensional, and the PO trispectrum is expected to be identically vanishing. This points out the fact that the configurations of momenta maximizing the PO signal are those in which all the momenta have approximately the same size $p_1 \simeq p_2 \simeq p_3 \simeq p_4$. For this reason, we turn to inspect the the parity-odd trispectrum in the exact equilateral configuration. In this limit, all momenta magnitudes are equal $p_1 = p_2 = p_3 = p_4 = p$, and

$$\begin{aligned} p_1 &= p(\sin\theta, 0, \cos\chi) \\ p_2 &= p(-\sin\theta, 0, \cos\chi) \\ p_3 &= p(\sin\theta\cos\chi, \sin\theta\sin\chi, -\cos\theta) \\ p_4 &= p(-\sin\theta\cos\chi, -\sin\theta\sin\chi, -\cos\theta) \end{aligned}$$

with θ and χ being the polar and azimuthal angles respectively. Without loss of generality, we can set $0 < \theta < \pi/2$, since $\pi/2 < \theta < \pi$ generates the same momenta configuration as $0 < \theta < \pi/2$ just rotated by an angle π. Moreover, $0 < \theta < \pi$ correspond to exchange the two couple of momenta $\{\mathbf{p}_1, \mathbf{p}_2\}$ and $\{\mathbf{p}_3, \mathbf{p}_4\}$ with each other. Using this parameterization, the trispectrum is completely characterized by the wavenumber p and two angles (θ, χ): $T^{\mathrm{PO}}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = T^{\mathrm{PO}}(p, \theta, \chi)$. Moreover, we are interested in studying the behavior of the dimensionless trispectrum. The trispectrum is dimensionally equal to a power of three of spectral power

$$T^{\mathrm{PO}} = \mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})S(\mathbf{p}_i) \tag{8.60}$$

where S is the shape function. To recover the dimensionless trispectrum $\mathcal{T}^{\mathrm{PO}}$, we multiply and divide by $p_1^3 p_3^3 p_{12}^3/(2\pi^2)^3$ (8.63) and find

$$\mathcal{T}^{\mathrm{PO}} = p_1^3 p_3^3 p_{12}^3 T^{\mathrm{PO}} = (2\pi^2)^3 \Delta_\zeta(p_1)\Delta_\zeta(p_3)\Delta_\gamma(p_{12})S(\mathbf{p}_i) \tag{8.61}$$

We begin analyzing the scale-behavior of the trispectrum, shown in Fig. 8.1. The trispectrum displays a dip on scales $p_{dip} \simeq 2.65k_1 \times 10^{-2}$.The presence of a sharp dip implies the vanishing of the mode function f_p at those scales, and it is a characteristics of all inflationary models featuring a phase of USR [51, 17, 45]. Soon after the dip, the trispectrum steadily increases as p grows until it reaches its maximum, presumably at scales $k_1 < p < k_2$. The reason why the maximum is not easily identified is the highly oscillatory behavior of the trispectrum for $p \gtrsim 9k_1$. This case is indeed analogous to what noticed and analyzed in [73], in which the bispectrum of scalar perturbation in Starobinsky model of inflation was studied. Analogously to the PO trispectrum in SR-USR-SR inflation, the bispectrum grows indefinitely with the wavenumber. The authors identified the cause of this suspect behavior in the presence of sharp features in the potential. Analogously, we can safely consider the oscillatory behavior at those scales as non-physical, thus being an artifact of the sharp transition. For scales smaller than the dip, the amplitude of the trispectrum decreases for decreasing wavenumbers. This is in agreement with expectations since in the deep large-scale limit, $p/k_1 \to 0$, the PO trispectrum is expected to vanish.

We investigate the angular behavior of the trispectrum in proximity of its maximum, located approximately at wavelength $p \simeq 2.6k_1$. The dependence of the PO trispectrum on the angular parameters, θ and χ separately, is shown in Fig. 8.2. The dimensionless trispectrum across θ is symmetric under $\theta \to \pi - \theta$, as for $\pi/2 < \theta < \pi$ the momenta configuration is the same as for $0 < \theta < \pi/2$ a part for a rotation of π around the x-axis. We notice that the trispectrum is maximum around $\theta \simeq \pi/5$.We then show the dimensionless PO trispectrum against the aperture angle χ, for fixed values of $p = 2.6k_1$ and $\theta = \pi/5$. Notice that the maximum does not occur exactly for $\chi = \pi/4$, but at a slightly larger value. Even if this might seem counterintuitive since the folding integral enters the trispectrum inside the $\sin 2\chi$ term, this is only true for four permutations corresponding to the s-channel. For this case, with the choice of reference system illustrated above, $\chi = \chi_{12,34}$. However, the folding angle is channel dependent and the folding angles $\chi_{13,24}$ and $\chi_{14,23}$ are functions of the parameters θ and χ. Thus, since we are plotting everything as a function of $\chi = \chi_{12,34}$, which corresponds to the azimuthal angle of the $\mathbf{p}_3$ and $\mathbf{p}_4$ vectors in the chosen coordinate system, the trispectrum can be expected to reach its maximum for a value of the angle χ different from the one that maximizes the single channels.

In Fig.8.3 we show the dependence of the trispectrum on both θ and χ. As expected, the trispectrum vanishes for $\theta = \pi/2$, which corresponds to $|\mathbf{p}_1 + \mathbf{p}_2| \to 0$. It is identically zero also for $\chi = 0$ and $\chi = \pi/2$, since the configuration reduces to be planar. Moreover, the PO trispectrum is anti-symmetric w.r.t. $\chi = \pi/2$. This is a sign of the parity odd nature of the observable: $\chi \to \pi - \chi$ corresponds to swapping the handness of the tetrahedron (exchanging the two planes $\{\mathbf{p}_1, \mathbf{p}_2\}$ and $\{\mathbf{p}_3, \mathbf{p}_4\}$).

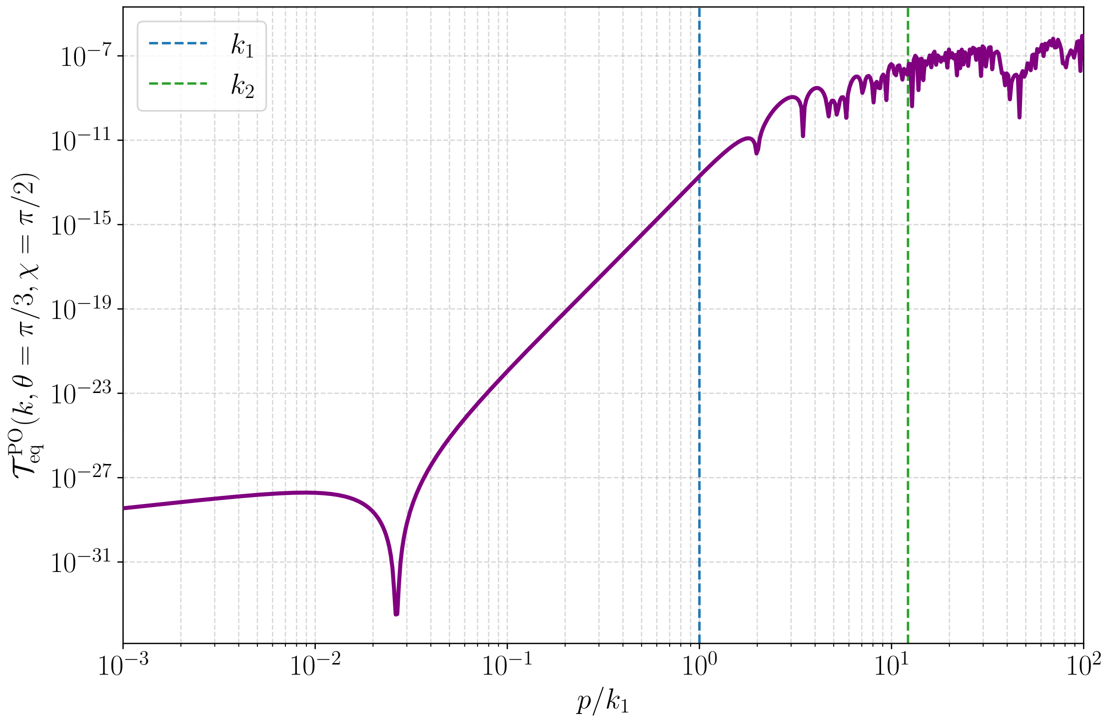

Figure 8.1: Behavior of the PO trispectrum in the equilateral configuration for fixed angles $\theta = \pi/3$ and $\chi = \pi/2$. The values of the cosmological parameters are: $H = 1.3 \times 10^{-5} M_{\rm Pl}$, $H/M_{\rm CS} = 10^{-2}$, $\epsilon_{1,i} = 10^{-3}$, $\Delta\epsilon_2 = -6$, $\Delta N = 2.5$ and $k_1 = 10^4 {\rm Mpc}^{-1}$. We retain the imaginary unit i.

8.5 Parity-even scalar trispectrum

Together with the PO trispectrum, we evaluate also the PE contribution. In principle we would have two trispectra contributing to the PE signal from graviton-exchange: one built from two GR vertices and one constructed out of two CS vertices. However, only the CS one contributes significantly in the inflationary model under consideration. In fact, even if suppressed by the first slow-roll parameter ϵ_1, it contains the derivative of the second slow-roll parameter ϵ_2' which is divergent in the approximation of instantaneous SR/USR/SR transition.

Let us now focus on evaluating this trispectrum. As explicitly shown in Sec.6.1, the expectation value of an operator O, which, in the case of $O = \zeta\zeta\zeta\zeta$, leads to the evaluation of the trispectrum, is given by three contributions

$$\langle O(\eta)\rangle = \langle O(\eta)\rangle_{(1,1)} + \langle O(\eta)\rangle_{(0,2)} + \langle O(\eta)\rangle_{(2,0)} \tag{8.62}$$

with the $(1,1)$, $(0,2)$ and $(2,0)$ terms given by (6.37), (6.38) and the hermitian conjugate of the last one since $\langle O(\eta)\rangle_{(2,0)} = \langle O(\eta)\rangle^\dagger_{(0,2)}$. Therefore, the equation above can be written as

$$\begin{aligned}\langle O(\eta)\rangle = &\int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta} d\eta'' \langle H_{\rm CS}(\eta') O(\eta) H_{\rm CS}(\eta'')\rangle \\ &- 2\,{\rm Re}\left[\int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta'} d\eta'' \langle O(\eta) H_{\rm CS}(\eta') H_{\rm CS}(\eta'')\rangle\right]\end{aligned} \tag{8.63}$$

Since the trispectrum built from two CS vertices is parity-even, it is convenient to write also the first term in 8.63 as the real part of an expectation value. The computation of this expectation value leads to a prefactor depending on momenta combination and angular part (coming from the polarization sum), and a time integration: since the prefactor and angular part will be real, and only the real part of the time integration contributes to the trispectrum. Written in this form, the PE trispectrum will be manifestly real.

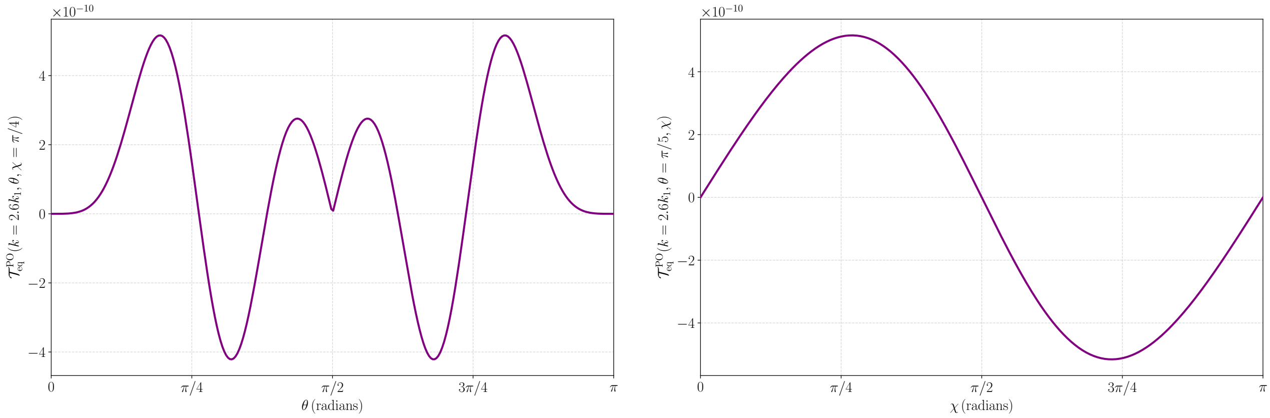

Figure 8.2: On the left, the adimensional PO trispectrum in the equilateral limit across θ, for $k = 2.6\,k_1$ and $\chi = \pi/2$. On the right, the same quantity in the equilateral limit across χ with $\theta = \pi/5$ and $k = 2.6\,k_1$. We retain the imaginary unit i.

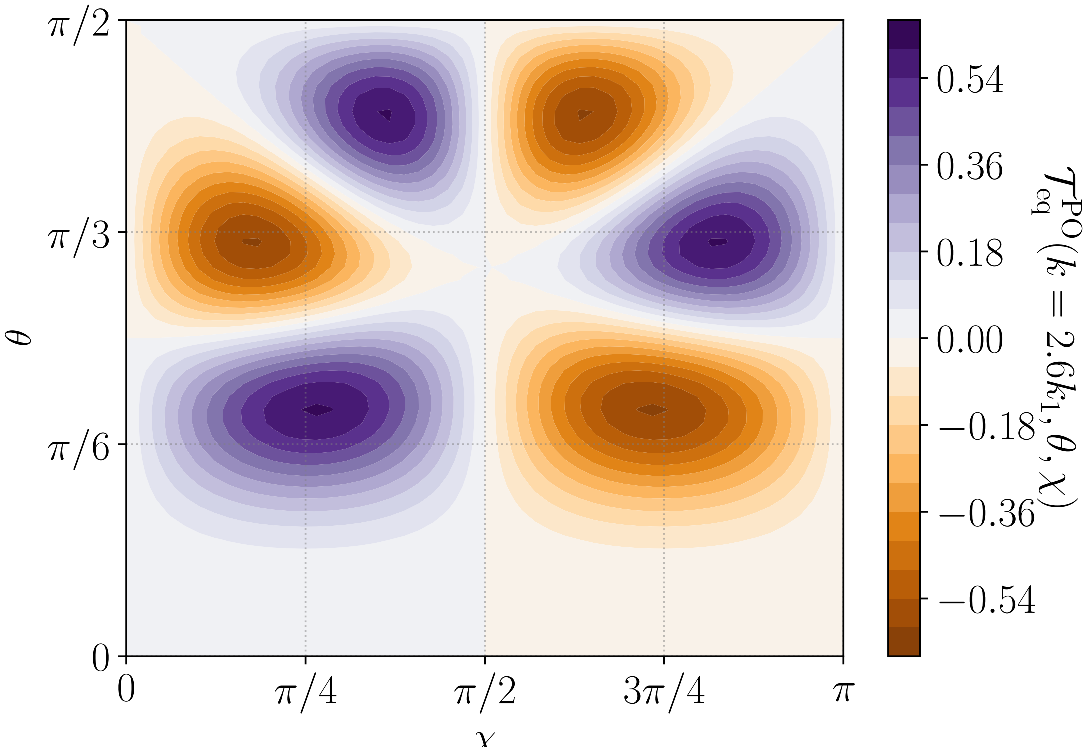

Figure 8.3: Contour plot of the dimensionless PO trispectrum multiplied by 10^{-9}. The color denotes the amplitude at each value of θ and χ.

We start with the $(1,1)$ contribution:

$$\begin{aligned}\langle O(\eta)\rangle_{(1,1)} &= \int_{-\infty}^{\eta} d\eta' \int_{-\infty}^{\eta} d\eta'' \langle H_{\rm CS}(\eta')O(\eta)H_{\rm CS}(\eta'')\rangle \\ &= \frac{1}{4}\int_{-\infty}^{\eta} d\eta' a(\eta')\frac{\epsilon_1(\eta')\epsilon_2'(\eta')}{M_{\rm CS}(\eta')}\int_{-\infty}^{\eta} d\eta'' a(\eta'')\frac{\epsilon_1(\eta'')\epsilon_2'(\eta'')}{M_{\rm CS}(\eta'')}\times \\ &\times \sum_{s,l=L,R}\alpha_s\alpha_l \int \left(\prod_{a=1}^{6}\frac{d^2k_a}{(2\pi)^3}\right)(2\pi)^6\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\delta(\mathbf{k}_4+\mathbf{k}_5+\mathbf{k}_6)\times \\ &\times \frac{k_3}{k_2^2}\frac{k_6}{k_5^2}k_1^i k_2^j k_4^m k_5^n \epsilon^s_{ij}(\mathbf{k}_3)\epsilon^l_{mn}(\mathbf{k}_6)\,\langle\zeta(\mathbf{k}_1,\eta')\zeta(\mathbf{k}_2,\eta')\gamma'(\mathbf{k}_3,\eta')\zeta(\mathbf{p}_1,\eta)\times \\ &\times \zeta(\mathbf{p}_2,\eta)\zeta(\mathbf{p}_3,\eta)\zeta(\mathbf{p}_4,\eta)\zeta(\mathbf{k}_4,\eta'')\zeta(\mathbf{k}_5,\eta'')\gamma'(\mathbf{k}_6,\eta'')\rangle\end{aligned} \tag{8.64}$$

We now have to perform the Wick contraction as usual. However, this time we perform a trick that will enable us to write the final result as a prefactor and the real part of the combination of bulk-to-boundary propagators. We split the expectation value in two identical contributions and we take two different permutations in such a way that the two contribution to the $(1,1)$ term are one the complex conjugate of the other:

$$
\begin{aligned}
&\langle\zeta(\mathbf{k}_1,\eta')\zeta(\mathbf{k}_2,\eta')\gamma'(\mathbf{k}_3,\eta')\zeta(\mathbf{p}_1,\eta)\zeta(\mathbf{p}_2,\eta)\zeta(\mathbf{p}_3,\eta)\zeta(\mathbf{p}_4,\eta)\zeta(\mathbf{k}_4,\eta'')\zeta(\mathbf{k}_5,\eta'')\gamma'(\mathbf{k}_6,\eta'')\rangle \\
&=\frac{1}{2}\Big[\langle\zeta(\mathbf{k}_1,\eta')\zeta(\mathbf{k}_2,\eta')\gamma'(\mathbf{k}_3,\eta')\zeta(\mathbf{p}_1,\eta)\zeta(\mathbf{p}_2,\eta)\zeta(\mathbf{p}_3,\eta)\zeta(\mathbf{p}_4,\eta)\zeta(\mathbf{k}_4,\eta'')\zeta(\mathbf{k}_5,\eta'')\gamma'(\mathbf{k}_6,\eta'')\rangle+\text{perm.} \\
&+\langle\zeta(\mathbf{k}_1,\eta')\zeta(\mathbf{k}_2,\eta')\gamma'(\mathbf{k}_3,\eta')\zeta(\mathbf{p}_1,\eta)\zeta(\mathbf{p}_2,\eta)\zeta(\mathbf{p}_3,\eta)\zeta(\mathbf{p}_4,\eta)\zeta(\mathbf{k}_4,\eta'')\zeta(\mathbf{k}_5,\eta'')\gamma'(\mathbf{k}_6,\eta'')\rangle+\text{perm.}\Big]
\end{aligned}
\tag{8.65}
$$

The $(1,1)$ contribution is then

$$
\begin{aligned}
\langle O(\eta)\rangle'_{(1,1)}&=\frac{1}{8}\frac{p_{12}^2}{p_2^2p_4^2}\sum_{s=L,R}\alpha_s^2\,p_1^ip_2^jp_3^mp_4^n\epsilon_{ij}^s(\mathbf{p}_{12})\epsilon_{mn}^s(\mathbf{p}_{34})\times \\
&\times\Big[\int_{-\infty}^{\eta}d\eta' a(\eta')\lambda(\eta')\mathcal{G}^*_{p_1}(\eta,\eta')\mathcal{G}^*_{p_2}(\eta,\eta')v'_{p_{12}}(\eta')\int_{-\infty}^{\eta}d\eta'' a(\eta'')\lambda(\eta'')\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}_{p_4}(\eta,\eta'')v^{*\prime}_{p_{12}}(\eta'') \\
&+\int_{-\infty}^{\eta}d\eta' a(\eta')\lambda(\eta')\mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')v'^{*}_{p_{12}}(\eta')\int_{-\infty}^{\eta}d\eta'' a(\eta'')\lambda(\eta'')\mathcal{G}^*_{p_3}(\eta,\eta'')\mathcal{G}^*_{p_4}(\eta,\eta'')v'_{p_{12}}(\eta'')\Big] \\
&=\frac{1}{4}\frac{p_{12}^2}{p_2^2p_4^2}\sum_{s=L,R}\alpha_s^2\,p_1^ip_2^jp_3^mp_4^n\epsilon_{ij}^s(\mathbf{p}_{12})\epsilon_{mn}^s(\mathbf{p}_{34})\times \\
&\times\operatorname{Re}\Big[\int_{-\infty}^{\eta}d\eta' a(\eta')\lambda(\eta')\mathcal{G}^*_{p_1}(\eta,\eta')\mathcal{G}^*_{p_2}(\eta,\eta')v'_{p_{12}}(\eta')\int_{-\infty}^{\eta}d\eta'' a(\eta'')\lambda(\eta'')\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}_{p_4}(\eta,\eta'')v^{*\prime}_{p_{12}}(\eta'')\Big]
\end{aligned}
\tag{8.66}
$$

with

$$
\lambda(\eta)=\frac{\epsilon_1(\eta)\epsilon_2'(\eta)}{M_{\rm CS}(\eta)}\quad . \tag{8.67}
$$

We work out the polarization sum:

$$
\begin{aligned}
&\sum_{s=R,L}\alpha_s^2p_1^ip_2^jp_3^mp_4^n\epsilon_{ij}^s(\mathbf{p}_{12})\epsilon_{mn}^s(\mathbf{p}_{34}) \\
&\qquad=p_1^ip_2^jp_3^mp_4^n\epsilon_{ij}^R(\mathbf{p}_{12})\epsilon_{mn}^R(\mathbf{p}_{34})+p_1^ip_2^jp_3^mp_4^n\epsilon_{ij}^L(\mathbf{p}_{12})\epsilon_{mn}^L(\mathbf{p}_{34}) \\
&\qquad=p_1^2p_3^2\sin^2\theta_1\sin^2\theta_3\big[\cos(2\chi_1)\cos(2\chi_3)-\sin(2\chi_1)\sin(2\chi_3)\big] \\
&\qquad=p_1^2p_3^2\sin^2\theta_1\sin^2\theta_3\cos(2\xi_{12,34})
\end{aligned}
\tag{8.68}
$$

where $\alpha_{R,L}^2=0$, and in the last equality we exploited the trigonometric identity $\cos(x_1+x_2)=\cos x_1\cos x_2-\sin x_1\sin x_2$. The angle $\xi_{12,34}$ is defined as the difference of χ_1 and χ_3, as before.

The $(1,1)$ contribution is then given by

$$
\begin{aligned}
\langle O(\eta)\rangle'_{(1,1)}&=\frac{1}{4}\frac{p_{12}^2p_1^2p_3^2}{p_2^2p_4^2}\sin^2\theta_1\sin^2\theta_3\cos(2\xi_{12,34})\times \\
&\times\operatorname{Re}\Big[\int_{-\infty}^{\eta}d\eta' a(\eta')\lambda(\eta')\mathcal{G}^*_{p_1}(\eta,\eta')\mathcal{G}^*_{p_2}(\eta,\eta')v'_{p_{12}}(\eta')\int_{-\infty}^{\eta}d\eta'' a(\eta'')\lambda(\eta'')\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}_{p_4}(\eta,\eta'')v^{*\prime}_{p_{12}}(\eta'')\Big] \\
&+\text{permutations}
\end{aligned}
\tag{8.69}
$$

Regarding the second term in (8.63), it becomes

$$2\,\mathrm{Re}\big[\langle 0(\eta)\rangle_{(0,2)}\big] = \frac{1}{4}\frac{p_{12}^2p_1^2p_3^2}{p_2^2p_4^2}\sin^2\theta_1\sin^2\theta_3\cos(2\xi_{12,34})\times$$
$$\times\,\mathrm{Re}\Bigg[2\int_{-\infty}^{\eta}d\eta' a(\eta')\lambda(\eta')\,\mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')v'_{p_{12}}(\eta')\int_{-\infty}^{\eta'}d\eta'' a(\eta'')\lambda(\eta'')\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}_{p_4}(\eta,\eta'')v^{*\prime}{}_{p_{12}}(\eta'')\Bigg]$$
$$+\text{ permutations} \tag{8.70}$$

with λ being defined as (8.67).

Putting these terms together, we find that the PE trispectrum can be written as

$$T^{\mathrm{PE}} = \frac{1}{4}\frac{p_{12}^2p_1^2p_3^2}{p_2^2p_4^2}\sin^2\theta_1\sin^2\theta_3\cos(2\xi_{12,34})\,\mathrm{Re}\big[\tilde{I}_1-2\tilde{I}_2\big]+\text{permutations}\ . \tag{8.71}$$

Notice that the form of the equation above makes manifest its reality.with

$$\tilde{I}_1 = \int_{-\infty}^{\eta}d\eta' a(\eta')\lambda(\eta')\mathcal{G}^*_{p_1}(\eta,\eta')\mathcal{G}^*_{p_2}(\eta,\eta')u'^s_{p_{12}}(\eta')\int_{-\infty}^{\eta}d\eta'' a(\eta'')\lambda(\eta'')\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}_{p_4}(\eta,\eta'')u^{*\prime s}{}_{p_{12}}(\eta'') \tag{8.72}$$

$$\tilde{I}_2 = \int_{-\infty}^{\eta}d\eta' a(\eta')\lambda(\eta')\mathcal{G}_{p_1}(\eta,\eta')\mathcal{G}_{p_2}(\eta,\eta')u'^s_{p_{12}}(\eta')\int_{-\infty}^{\eta'}d\eta'' a(\eta'')\lambda(\eta'')\mathcal{G}_{p_3}(\eta,\eta'')\mathcal{G}_{p_4}(\eta,\eta'')u^{*\prime s}{}_{p_{12}}(\eta'') \tag{8.73}$$

8.5.1 Time integrations

We can now substitute the expression for the coupling constant λ, the scalar bulk-to-boundary propagators and the tensor mode functions in $\tilde{I}_1$ and $\tilde{I}_2$ and solve the integrations. After doing that and taking the $\eta \to 0^-$, we find:

$$\begin{aligned}\tilde{I}_1 = {}& \frac{\Delta\epsilon_2}{64}\frac{p_{12}^4}{p_2^3p_4^3}\frac{1}{\epsilon_i^2}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{p_2}-\beta_{p_2})^*(\alpha_{p_4}-\beta_{p_4})}{(\alpha_{p_1}-\beta_{p_1})(\alpha_{p_3}-\beta_{p_3})^*}\\ &\times\int_{-\infty}^{\eta}d\eta'\frac{H}{M_{\mathrm{CS}}(\eta')}\epsilon_1(\eta')F_{p_1}(\eta')F_{p_2}(\eta')e^{-ip_{12}\eta'}\big[\delta(\eta'-\eta_2)-\delta(\eta''-\eta_1)\big]\\ &\times\int_{-\infty}^{\eta}d\eta''\frac{H}{M_{\mathrm{CS}}(\eta'')}\epsilon_1(\eta'')F^*_{p_3}(\eta'')F^*_{p_4}(\eta'')e^{ip_{12}\eta''}\big[\delta(\eta''-\eta_2)-\delta(\eta''-\eta_1)\big],\end{aligned} \tag{8.74}$$

$$\begin{aligned}\tilde{I}_2 = {}& \frac{\Delta\epsilon_2}{64}\frac{p_{12}^4}{p_2^3p_4^3}\frac{1}{\epsilon_i^2}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{p_2}-\beta_{p_2})(\alpha_{p_4}-\beta_{p_4})}{(\alpha_{p_1}-\beta_{p_1})^*(\alpha_{p_3}-\beta_{p_3})^*}\\ &\times\int_{-\infty}^{\eta}d\eta'\frac{H}{M_{\mathrm{CS}}(\eta')}\epsilon_1(\eta')F^*_{p_1}(\eta')F^*_{p_2}(\eta')e^{-ip_{12}\eta'}\big[\delta(\eta'-\eta_2)-\delta(\eta'-\eta_1)\big]\\ &\times\int_{-\infty}^{\eta'}d\eta''\frac{H}{M_{\mathrm{CS}}(\eta'')}\epsilon_1(\eta'')F^*_{p_3}(\eta'')F^*_{p_4}(\eta'')e^{ip_{12}\eta''}\big[\delta(\eta''-\eta_2)-\delta(\eta''-\eta_1)\big]\end{aligned} \tag{8.75}$$

where we accounted for the fact that we are approximating the space-time as pure de Sitter during inflation, $a(\eta) = -1/H\eta$.

Analogously to what we have been doing in the previous sections, we make the integrals dimensionless. To do so, we employ the parameterization introduced in 8.3.1. In this way, terms $\tilde{I}_1$ and $\tilde{I}_2$ become

$$\begin{aligned}\tilde{I}_1 = {}& \frac{\Delta\epsilon_2}{64}\frac{p_{12}^4}{p_2^3p_4^3}\frac{1}{\epsilon_i^2}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{r_2}-\beta_{r_2})^*(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})(\alpha_{r_3}-\beta_{r_3})^*}\\ &\times\int_{-\infty}^{x}dx'\frac{H}{M_{\mathrm{CS}}(x')}\epsilon_1(x')F_{r_1}(x')F_{r_2}(x')e^{-ir_{12}x'}\big[\delta(x'-x_2)-\delta(x''-x_1)\big]\\ &\times\int_{-\infty}^{x}dx''\frac{H}{M_{\mathrm{CS}}(x'')}\epsilon_1(x'')F^*_{r_3}(x'')F^*_{r_4}(x'')e^{ir_{12}x''}\big[\delta(x''-x_2)-\delta(x''-x_1)\big],\end{aligned} \tag{8.76}$$

$$\begin{aligned}\tilde{I}_2 = &\frac{\Delta\epsilon_2}{64}\frac{p_{12}^4}{p_2^3 p_4^3}\frac{1}{\epsilon_i^2}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})^*}\\ &\times\int_{-\infty}^{x}dx'\frac{H}{M_{\rm CS}(x')}\epsilon_1(x')F_{r_1}^*(x')F_{r_2}^*(x')e^{-ir_{12}x'}\left[\delta(x'-x_2)-\delta(x'-x_1)\right]\\ &\times\int_{-\infty}^{x'}dx''\frac{H}{M_{\rm CS}(x'')}\epsilon_1(x'')F_{r_3}^*(x'')F_{r_4}^*(x'')e^{ir_{12}x''}\left[\delta(x''-x_2)-\delta(x''-x_1)\right]\end{aligned} \tag{8.77}$$

where the adimensional F are of the form (8.27).
The integrals inside $\tilde{I}_1$ can be solved immediately, and gives

$$\begin{aligned}\tilde{I}_1 = &\frac{\Delta\epsilon_2}{64}\left(\frac{H}{M_{{\rm CS},i}}\right)^2\frac{p_{12}^4}{p_2^3 p_4^3}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{r_2}-\beta_{r_2})^*(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})(\alpha_{r_3}-\beta_{r_3})^*}\\ &\times\left[e^{-3\Delta N}\mathcal{F}_{r_1}(\alpha,\beta,x_2)\mathcal{F}_{r_2}(\alpha,\beta,x_2)e^{-ir_{12}x_2}-\mathcal{F}_{r_1}(1,0,x_1)\mathcal{F}_{r_2}(1,0,x_1)e^{-ir_{12}x_1}\right]\\ &\times\left[e^{-3\Delta N}\mathcal{F}_{r_3}^*(\alpha,\beta,x_2)\mathcal{F}_{r_4}^*(\alpha,\beta,x_2)e^{ir_{12}x_2}-\mathcal{F}_{r_3}^*(1,0,x_1)\mathcal{F}_{r_4}^*(1,0,x_1)e^{ir_{12}x_1}\right].\end{aligned} \tag{8.78}$$

In the expression above we have exploited that

$$\frac{H}{M_{\rm CS}(x_2)} = e^{-3\Delta N}\frac{H}{M_{\rm CS}(x_1)} = e^{-3\Delta N}\frac{H}{M_{{\rm CS},i}} \tag{8.79}$$

and the $\mathcal{F}$ functions are defined as (8.33).
Regarding the second term, it is also immediately solvable due to the presence of the Dirac deltas, but it needs to be considered with some care due to the presence of the nested integration. In fact,

$$\begin{aligned}\tilde{I}_2 = &\frac{\Delta\epsilon_1}{64}\frac{H}{M_{{\rm CS},i}}\frac{p_{12}^4}{p_2^3 p_4^3}\frac{1}{\epsilon_i}\mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})^*}\\ &\times\Bigg\{e^{-3\Delta N}\mathcal{F}_{r_3}^*(\alpha,\beta,x_2)\mathcal{F}_{r_4}^*(\alpha,\beta,x_2)e^{ir_{12}x_2}\\ &\quad\times\underbrace{\int_{-\infty}^{x}dx'\frac{H}{M_{\rm CS}(x')}\epsilon_1(x')F_{r_1}^*(x')F_{r_2}^*(x')e^{-ir_{12}x'}\theta(x'-x_2)\left[\delta(x'-x_2)-\delta(x'-x_1)\right]}_{\text{first term}}\\ &-\mathcal{F}_{r_3}^*(1,0,x_2)\mathcal{F}_{r_4}^*(1,0,x_1)e^{ir_{12}x_1}\\ &\quad\times\underbrace{\int_{-\infty}^{x}dx'\frac{H}{M_{\rm CS}(x')}\epsilon_1(x')F_{r_1}^*)(x')F_{r_2}^*(x')e^{-ir_{12}x'}\theta(x'-x_1)\left[\delta(x'-x_2)-\delta(x'-x_1)\right]}_{\text{second term}}\Bigg\}\end{aligned} \tag{8.80}$$

where the Heaviside theta functions are due to the fact that if the Dirac delta is not peaked at a value of x'' within the integration domain $x''\in[-\infty,x']$, it does not contribute to the integral. Their presence affect the first term and second term. In fact,

$$\begin{aligned}\text{first term} &= \epsilon_i\frac{H}{M_{{\rm CS},i}}\left[e^{-3\Delta N}\mathcal{F}_{r_1}^*(\alpha,\beta,x_2)\mathcal{F}_{r_2}^*(\alpha,\beta,x_2)e^{-ir_{12}x_2}\theta(0)-\mathcal{F}_{r_1}^*(1,0,x_1)\mathcal{F}_{r_2}^*(1,0,x_1)e^{-ir_{12}x_1}\theta(x_1-x_2)\right]\\ &= \frac{\epsilon_i}{2}\frac{H}{M_{{\rm CS},i}}e^{-3\Delta N}\mathcal{F}_{r_1}^*(\alpha,\beta,x_2)\mathcal{F}_{r_2}^*(\alpha,\beta,x_2)e^{-ir_{12}x_2}\end{aligned} \tag{8.81}$$

and the second contribution in the first line vanishes because $x_2 > x_1$: $x_1 = p_1\eta_1$ and η_1 corresponds to the beginning of the USR phase, instead $x_2 = p_1\eta_2$ with η_2 corresponding to the end of the USR stage. Moreover, $\theta(0) = 1/2$, which explains the difference in the prefactor between the first and second line.

The second term yields to

$$
\begin{aligned}
\text{second term} &= \epsilon_i \frac{H}{M_{\text{CS},i}} \left[e^{-3\Delta N} \mathcal{F}^*_{r_1}(\alpha,\beta,x_2)\mathcal{F}^*_{r_2}(\alpha,\beta,x_2) e^{-ir_{12}x_2}\theta(x_2-x_1) - \mathcal{F}^*_{r_1}(1,0,x_1)\mathcal{F}^*_{r_2}(1,0,x_1)e^{-ir_{12}x_1}\theta(0) \right] \\
&= \epsilon_i \frac{H}{M_{\text{CS},i}} \left[e^{-3\Delta N} \mathcal{F}^*_{r_1}(\alpha,\beta,x_2)\mathcal{F}^*_{r_2}(\alpha,\beta,x_2) e^{-ir_{12}x_2} - \frac{1}{2}\mathcal{F}^*_{r_1}(1,0,x_1)\mathcal{F}^*_{r_2}(1,0,x_1)e^{-ir_{12}x_1} \right]
\end{aligned} \tag{8.82}
$$

in this case, the Heaviside step-function $\theta(x_2 - x_1)$ is equal to one, since the condition $x_2 > x_1$ is always satisfied. Then, the $\tilde{I}_2$ contribution becomes

$$
\begin{aligned}
\tilde{I}_2 = &\frac{\Delta\epsilon_2}{64}\left(\frac{H}{M_{\text{CS},i}}\right)^2 \frac{p_{12}^4}{p_2^3 p_4^3} \mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12}) \frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})^*} \\
&\times \left\{ \frac{e^{-6\Delta N}}{2} \mathcal{F}^*_{r_1}(\alpha,\beta,x_2)\mathcal{F}^*_{r_2}(\alpha,\beta,x_2)\mathcal{F}^*_{r_3}(\alpha,\beta,x_2)\mathcal{F}^*_{r_4}(\alpha,\beta,x_2) e^{ir_{12}x_2} \right. \\
&- e^{-3\Delta N}\mathcal{F}^*_{r_1}(\alpha,\beta,x_2)\mathcal{F}^*_{r_2}(\alpha,\beta,x_2)\mathcal{F}^*_{r_3}(1,0,x_2)\mathcal{F}^*_{r_4}(1,0,x_1)e^{-ir_{12}x_2} \\
&\left. + \frac{e^{-3\Delta N}}{2}\mathcal{F}^*_{r_1}(1,0,x_1)\mathcal{F}^*_{r_2}(1,0,x_1)\mathcal{F}^*_{r_3}(1,0,r_3)\mathcal{F}^*_{r_4}(1,0,x_1)e^{-ir_{12}x_1} \right\}
\end{aligned} \tag{8.83}
$$

8.5.2 Total parity-even trispectrum

Putting the terms together, we obtain the complete expression for the PE trispectrum

$$
\begin{aligned}
T^{\text{PE}} = &\frac{\Delta\epsilon_2}{64}\left(\frac{H}{M_{\text{CS},i}}\right)^2 \mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12}) \frac{p_{12}^6 p_1^2 p_3^2}{p_2^5 p_4^5} \sin^2\theta_1 \sin^2\theta_3 \cos 2\chi_{12,34} \text{Re}\left\{ \frac{(\alpha_{r_2}-\beta_{r_2})^*(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})(\alpha_{r_3}-\beta_{r_3})^*} \right. \\
&\times \Big[\mathcal{A}'(r_1,r_2,r_{12},x_1,\Delta N) - \mathcal{A}''(r_1,r_2,r_{12},x_1)\Big]^* \Big[\mathcal{A}'(r_3,r_4,r_{12},x_1,\Delta N) - \mathcal{A}''(r_3,r_4,r_{12},x_1)\Big] \\
&- \frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})^*} \Big[\mathcal{A}'(r_3,r_4,r_{12},x_1,\Delta N)\mathcal{A}'(r_1,r_2,-r_{12},x_1,\Delta N) \\
&\left. - 2\mathcal{A}''(r_3,r_4,r_{12},x_1)\Big(\mathcal{A}'(r_1,r_2,-r_{12},x_1,\Delta N) - \mathcal{A}''(r_1,r_2,-r_{12},x_1)\Big)\Big] \right\} + \text{perms.}
\end{aligned} \tag{8.84}
$$

This trispectrum also vanishes once $\eta_2 = \eta_1$: even if setting $x_1 = x_2$, $\alpha = 1$ and $\beta = 0$ does not lead the term inside $\text{Re}\{\dots\}$ to be zero, the conditions are equivalent to impose that USR does not take place, and, therefore, that there is no discontinuity in ϵ_2, $\Delta\epsilon_2 = 0$. This is expected: the CS vertex which is dominant in SR-USR-SR scenario is localized at the transitions. Once the USR phase is removed, this vertex vanishes. If we are interested in studying this PE trispectrum in the $\Delta N = 0$ limit, we need to take the CS vertex as (7.26)

8.5.3 Large and small-scale limit

We analyze the asymptotic behavior of the PE trispectrum at large-scale. This limit in quite easy so can be solved analytically. First, we recall that the difference of Bogoljubov coefficients α and β in the large-scale limit is finite, despite the two coefficient diverge separately in this regime. Moreover,

$$
\alpha_r - \beta_r = e^{-3\Delta N} + O(x^2), \tag{8.85}
$$

making it equal to its complex conjugate at leading order in $x \to 0$ limit. Therefore,

$$
\frac{(\alpha_{r_2}-\beta_{r_2})(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_3}-\beta_{r_3})^*} = 1 + O(x^2) = \frac{(\alpha_{r_2}-\beta_{r_2})^*(\alpha_{r_4}-\beta_{r_4})}{(\alpha_{r_1}-\beta_{r_1})(\alpha_{r_3}-\beta_{r_3})^*}. \tag{8.86}
$$

At the same time,

$$\mathcal{A}'(r_1,r_2,\pm r_{12},x_1,\Delta N) = e^{-3\Delta N}(\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_2}-\beta_{r_2})^* + O(x) \tag{8.87}$$
$$\mathcal{A}'(r_3,r_4,\pm r_{12},x_1,\Delta N) = e^{-3\Delta N}(\alpha_{r_3}-\beta_{r_3})^*(\alpha_{r_4}-\beta_{r_4})^* + O(x) \tag{8.88}$$
$$\mathcal{A}''(r_1,r_2,\pm r_{12},x_1) = (\alpha_{r_1}-\beta_{r_1})^*(\alpha_{r_2}-\beta_{r_2})^* + O(x) \tag{8.89}$$
$$\mathcal{A}''(r_3,r_3,\pm r_{12},x_1) = (\alpha_{r_3}-\beta_{r_3})^*(\alpha_{r_4}-\beta_{r_4})^* + O(x) \tag{8.90}$$
$$\tag{8.91}$$

Substituting these expressions in (8.84) we obtain

$$\lim_{x\to 0} T^{\rm PE} = -\frac{\Delta\epsilon_2}{64}\left(\frac{H}{M_{{\rm CS},i}}\right)^2 \mathcal{P}_\zeta(p_1)\mathcal{P}_\zeta(p_3)\mathcal{P}_\gamma(p_{12})\frac{p_{12}^6 p_1^2 p_3^2}{p_2^5 p_4^5}\sin^2\theta_1\sin^2\theta_3\cos 2\chi_{12,34}e^{-6\Delta N}+O(x^2)+\text{perms.} \tag{8.92}$$

and we see that the PE trispectrum is constant at very large scales. This behavior is expected, since no theorem enforces the real part of the trispectrum to vanish under certain conditions.

The small-scale limit presents the same complications encountered in case of PO trispectrum: the limit does not actually exist due to the presence of oscillatory terms that do not cancel out. Again, this is due to the instantaneous nature of the SR/USR/SR transition, and can be solved by smoothing it. We can expect the PE trispectrum to set on a constant value, since the restoration of scale invariance during the second SR phase does not require the PE trispectrum to vanish. Moreover, a rise in the amplitude is not expected since will lead it to become $\mathcal{O}(1)$ at some scale, causing the breakdown of perturbation theory.

8.6 Parity-even trispectrum in the equilateral configuration

Analogously to the case of the PO trispectrum, we analyze the scale dependence of its PE counterpart in the equilateral configuration. Fixing the value of the angles, in Fig.8.4 we show the behavior of the PE trispectrum w.r.t. p/k_1, noting that it appears extremely similar to the PO trispectrum. In fact, it is featured by a dip for $p \simeq 2.65\times 10^{-2}k_1$, a characteristic of all correlators in models of inflation admitting a phase of USR, and a rise of the signal right after. The signal reaches its peak for $p \gtrsim 3k_1$, and For larger scales, the signal becomes heavily oscillating around zero. As discussed before the PE trispectrum is not informative at small scales, since a non physical oscillating behavior is depicted. However, we can expect it to stabilize itself to the maximum value $\mathcal{T}^{\rm PE}\simeq\mathcal{O}(10^{-11})$ and so be constant at small scales. This is because we expect a similar behavior to the large-scale one since the contributors to the signal in that regime are large wavenumber modes exiting the horizon once the SR regime is restored. For this reason, the peak is expected to be $k_1 < p_{peak} < k_2$, and choose $p \simeq 3.3k_1$, which correspond to the scale in which a distinguishable bump is located, in a regime in which oscillation have not became dominating yet. Notice that in the large-scale regime the constant behaviour of the PE trispectrum is manifest. This is again a confirmation of the sanity of the result.

For that value of the momenta, we show the contour plot of the PE trispectrum 8.5 for $\theta\in[0,\pi/2]$ and $\chi\in[0,\pi]$[2]. As expected, the trispectrum is symmetric under $\theta\to\pi-\theta$ since it is even under parity transformation. Note that the PE trispectrum is maximized for planar configuration of the momenta ($\chi=0$, $\chi=\pi/2$ and $\chi=\pi$, and $\theta=\pi/2$)

8.6.1 Counter-collinear limit

Previously we have stated that the dominant PE contribution to the scalar trispectrum in SR-USR-SR is the one built from two CS vertices. This statement is correct only if we account for graviton-mediated trispectra. However, this is not the only contribution to the PE scalar trispectrum: we also have to consider scalar trispectra from an exchange of scalar perturbation, so with ζ^3 vertices, and the trispectrum from contact ζ^4 interaction. These vertices are amplified as well by USR dynamics, and

[2]The detailed discussion on the angular intervals can be found in Sec. 8.4.

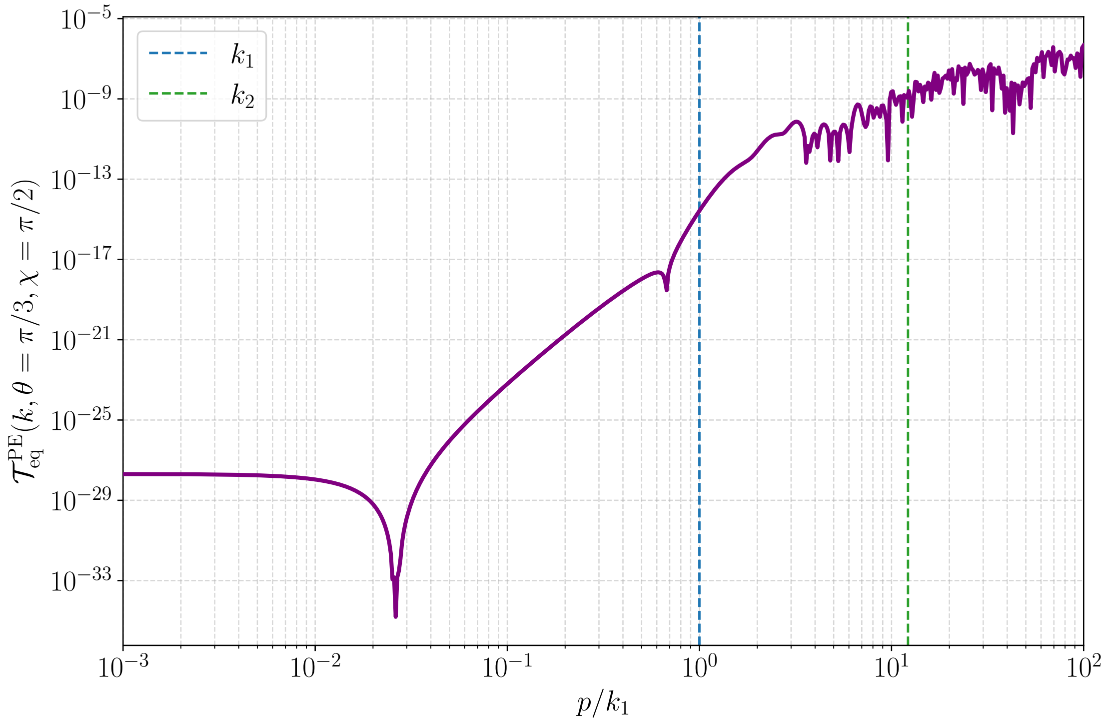

Figure 8.4: Behavior of the PE trispectrum in the equilateral configuration for fixed angles $\theta = \pi/3$ and $\chi = \pi/2$. The values of the cosmological parameters are: $H = 1.3 \times 10^{-5} M_{\rm Pl}$, $H/M_{\rm CS} = 10^{-2}$, $\epsilon_{1,i} = 10^{-3}$, $\Delta\epsilon_2 = -6$, $\Delta N = 2.5$ and $k_1 = 10^4 \mathrm{Mpc}^{-1}$. We retain an overall imaginary unit i.

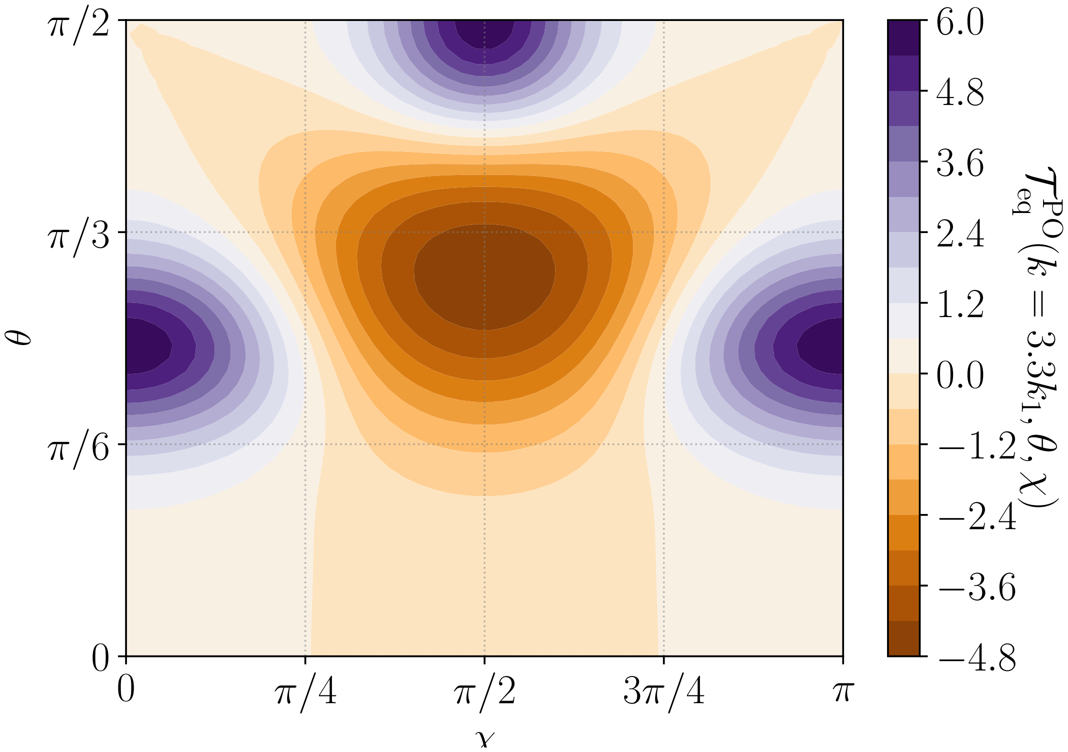

Figure 8.5: Contour plot of the normalized dimensionless PE trispectrum multiplied by 10^{11}. The color denotes the amplitude at each value of θ and χ. We retain an overall imaginary unit i.

the bispectrum and loop corrections to the spectral power from those interactions has been widely studied in the literature [46, 17, 18, 74]. Without having to perform any further calculation, it is very simple to estimate the contribution from the exchange PE trispectrum with ζ^3 vertices, to understand which is the dominant exchange triapectrum.

Non-Gaussianities entailed in higher order correlators, such as the bispectrum and the trispectrum, are expressed in terms of non-linearity parameters $f_{\rm NL}$, $\tau_{\rm NL}$ and $g_{\rm NL}$ as:

$$B_\zeta \equiv \frac{5}{6} f_{\rm NL} \sum_{a<b} \mathcal{P}_\zeta(p_a)\mathcal{P}_\zeta(p_b) \tag{8.93}$$

$$T_\zeta \equiv \tau_{\rm NL} \sum_{\substack{b<c\\ a\neq b,c}} \mathcal{P}_\zeta(p_{ab})\mathcal{P}_\zeta(p_b)\mathcal{P}_\zeta(p_c) + \frac{54}{25} g_{\rm NL} \sum_{a<b<c} \mathcal{P}_\zeta(p_a)\mathcal{P}_\zeta(p_b)\mathcal{P}_\zeta(p_c). \tag{8.94}$$

These parameters express the deviation of ζ from a Gaussian random field[3]. In the counter-collinear limit, the trispectrum T_ζ simplifies and can be written as [67]

$$T_\zeta(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = 4\tau_{\rm NL}^{\rm local} P_\zeta(p_{12})P_\zeta(p_1)P_\zeta(p_3). \tag{8.95}$$

In case inflation is sourced by a single field, relation (8.96) among local[4] $\tau_{\rm NL}$ and $f_{\rm NL}$ holds [75].

$$\tau_{\rm NL}^{\rm local} = \left(\frac{6}{5} f_{\rm NL}^{\rm local}\right)^2. \tag{8.96}$$

This makes clear the fact that, by knowing the non linearity parameter $f_{\rm NL}$, the amplitude of the exchange trispectrum in also known. That parameter enters in the squeezed limit of the trispectrum, which has been widely studied in slow-roll violating scenarios.

It is established that, in models admitting SR/USR transitions sufficiently long-wavelength modes $p/k_1 \ll 1$, exiting the horizon before the onset of the USR phase, satisfy Maldacena's consistency relation [74]

$$f_{\rm NL}^{\rm local}(p_{\rm L}, p_{\rm S}) = \frac{5}{12}\big(1 - n_s(p_{\rm L})\big). \tag{8.97}$$

This means that, if p_L takes value in a wavelength interval for which the spectral is very well known, we can infer the value of $f_{\rm NL}^{\rm local}$, extract $\tau_{\rm NL}^{\rm local}$ and give an estimate of the PE trispectrum with ζ^3 vertices where the momentum of the exchanged particle is $p_{12} \equiv p_{\rm S}$. In particular, from Fig.4.1, the power spectrum in the interval of scales $4.5 \times 10^{-2}\, k_1 \lesssim p \lesssim 1.5\, k_1$ increases steadily and the spectral index is well known to be $n_s = 4$. Therefore we can say that, for the long wavelength in that range of values, $f_{\rm NL}^{\rm local} = -5/4$ and, from 8.96, $\tau_{\rm NL}^{\rm local} = 9/4$.

To evaluate the trispectrum we consider the equilateral configuration for simplicity, fixing the external momenta to be $p = 0.005\, k_1$, whereas the exchanged momentum $p_{12} = 5\, k_1$, in such a way that we are allowed to distinguish between long wavelength modes, $p = p_{\rm L}$ and short wavelength modes, $p_{12} = p_{\rm S}$.

In terms of adimensional quantities, the relation is

$$\mathcal{T}_\zeta(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = 4\tilde{\tau}_{\rm NL}^{\rm local} \Delta_\zeta(p_{12})\Delta_\zeta(p_1)\Delta_\zeta(p_3) \tag{8.98}$$

where $\tilde{\tau}_{\rm NL}^{\rm local} = (2\pi^2)^3 \tau_{\rm NL}^{\rm local}$

$$\mathcal{T}_{\zeta^3}^{\rm PE} = 5.598 \times 10^{-5} \left(\frac{H}{10^{-5}}\right)\left(\frac{\epsilon_i}{10^{-3}}\right)^{-2}\left(\frac{H/M_{\rm CS}}{10^{-2}}\right)^2 \tag{8.99}$$

The PE trispectrum arising from CS vertices for the values of external and exchanged momenta reported above leads to

$$\mathcal{T}_{\zeta^2\gamma}^{\rm PE} = 1.201 \times 10^{-11} \left(\frac{H}{10^{-5}}\right)^2\left(\frac{\epsilon_i}{10^{-3}}\right)^{-2}\left(\frac{H/M_{\rm CS}}{10^{-2}}\right)^2 \tag{8.100}$$

[3]As we introduced in Ch.6, non-Gaussianities cannot arise if a field is Gaussian distributed, i.e. is a free field without any interaction term appearing in its Lagrangian

[4]The adjective local referring to the non-linearity parameters $f_{\rm NL}$ and $\tau_{\rm NL}$, means that these parameters are evaluated in case $p_{\rm L} = p_1 \ll p_2 \approx p_3 = p_S$ and $p_{\rm L} = p_{12} \ll p_1 \approx p_2 \approx p_3 \approx p_4 = p_S$ respectively.

This shows that the dominant contribution to the PE trispectrum comes from the cubic scalar vertex dominates over the PE trispectrum built from two CS vertices.

This estimate shows that, for the value of momenta assigned above, the dominant contribution to the PE trispectrum from exchange of particles is the one with ζ^3 vertices. This result is expected since the trispectrum with cubic scalar vertices is proportional to three power spectra, that experience a significant enhancement. The PE trispectrum form CS vertices is suppressed in two ways. First, it contains a power of two of $H/M_{\rm CS}$, which is the parameter used to expand in CS corrections. This parameter is set to have a value of 10^{-2}. Moreover, since the trispectrum is graviton-mediated, the tensor power enters in the expression. Since it is significantly smaller than the scalar power, it contributes to suppress the PE trispectrum from CS vertices as well. We extract the local non-linearity parameter $\tau_{\rm NL}$ for the CS interaction:

$$\tau_{\rm NL}^{\rm CS,\, local} = 3.899 \times 10^{-2} \left(\frac{H/M_{\rm CS}}{10^{-2}} \right)^2 \tag{8.101}$$

which is in agreement with the constraint from Planck data [76]: $\tau_{\rm NL}^{\rm local} < 1700$ at 95% CL.

8.7 Comments on perturbativity

When dealing with phases that violate slow-roll we need to be extremely careful with what give for granted in the slow-roll inflationary scenario. Boundary terms, that usually can discarded, may become important and perturbativity might be affected as well, resulting in loop corrections that may explode in SR-USR-SR inflation.

The study of loop corrections in inflationary model featuring a phase of USR [77, 53] triggered an intense debate about ruling out or not the mechanism of PBH formation via USR in single field inflation. In fact, from the calculation of one loop corrections to the spectral power the following bound is derived

$$\Delta_\zeta(k_{peak}) \ll \frac{1}{(\Delta\epsilon_2)^2} \simeq 0.03 \tag{8.102}$$

where $\Delta_\zeta(k_{peak})$ corresponds to the power spectrum at the the scales in which the peak is located, which are important for the formation of PBH. This is found by requiring the loop corrections of the scalar power spectrum at scales relevant for CMB observation ($k = O(k_\star)$, with $k_\star = 0.05\,{\rm Mpc}^{-1}$) to be much smaller then the tree level value to ensure the validity of perturbation theory.

It is well known that, on scales scales k_{peak} which are relevant for PBH formation, curvature perturbations with large amplitude are needed in order to produce an appreciable amount of those object [21]. This translates into requiring that models of SR-USR-SR inflation needs to be able to source $\Delta_\zeta(k_{peak}) \sim O(10^{-1} - 10^{-2})$ to be considered as a possible mechanism of PBH production. However, [77] employs a model featuring sharp SR/USR and USR/SR transitions, which are approximate models. Indeed, it has been shown that, once we account for a smooth transition in the ϵ_2 parameter, loop corrections are harmless [78, 79, 80, 81, 54], being of the level of few percentage [54] for models. Given that in USR models the size of loop contributions is of the order of few percent to the tree-level expressions, this indicates that they have been significantly amplified by USR dynamics, however perturbation theory remains unspoiled. As noticed in [54], even insisting in keeping a sharp transition, perturbativity is not yet threatened. This difference arises in the fact that the size of the loop correction needs to be compared with the PBH abundance rather than with the order-of-magnitude estimate of the enhancement of the power spectrum, as done in [77].

These considerations are the ones derived from the literature. Turning to the computation of the trispectrum, perturbativity seems to be broken at very small scales ($k > k_2$) by the trispectrum as it continues to grow indefinitely. As demonstrated in [73], this is an unphysical behaviour at very small scales and can be associated to the smoothness of the transition. Therefore, in order to assess the precise value of the enhancement of the trispectrum, extract $\tau_{\rm NL}^{\rm PO}$ and compare it with present and forthcoming data, an analysis accounting for the smoothness of the transition is mandatory. However, based on previous numerical studies of non-Gaussianities obtained in this regime [17], and from physical argument based on no-go theorems [33, 82], we can expect the PO trispectrum to be non vanishing and significantly enhanced to a peak at small scales, without the indefinite growth post the peak leading to the breaking of perturbativity.

8.8 Effects of smoothening the transition

We can still make some few considerations on the role of the smoothness of the transition starting from the expression for the PO trispectrum that we derived

We model the transition between phases as [46]

$$\epsilon_2' = \begin{cases} \frac{\epsilon_2^{\mathrm{II}}}{\sqrt{\pi}\Delta\eta}\,\mathrm{e}^{-\frac{(\eta-\eta_1)^2}{\Delta\eta^2}} & \text{around } \eta_1, \\ -\frac{\epsilon_2^{\mathrm{II}}}{\sqrt{\pi}\Delta\eta}\,\mathrm{e}^{-\frac{(\eta-\eta_2)^2}{\Delta\eta^2}} & \text{around } \eta_2, \end{cases} \tag{8.103}$$

where $\Delta\eta$ is the duration of the transition: if $\Delta\eta \to 0$, the transition is sharp, leading to $\epsilon_2' \to \pm\epsilon_2^{\mathrm{II}}\delta(\eta - \eta_{1,2})$.

Consider an integral involving the derivative of ϵ_2':

$$I = \int_{-\infty}^{\eta} d\eta' f(\eta)\epsilon_2' = \epsilon_2^{\mathrm{II}}\big[f(\eta_2)\theta(\eta-\eta_1) - f(\eta_1)\theta(\eta-\eta_2)\big] \tag{8.104}$$

if the transition is sharp. However, we are interested in investigating what happens once we do not send $\Delta\eta \to 0$. For simplicity, we focus on studying the behaviour of the integral around the first transition η_1. Once we substitute (8.103) inside the integral, we obtain:

$$\begin{aligned} I &= \int_{-\infty}^{\eta} d\eta' \frac{\epsilon_2^{\mathrm{II}}}{\sqrt{\pi}\Delta\eta}\,\mathrm{e}^{-\frac{(\eta-\eta_1)^2}{\Delta\eta^2}} f(\eta) \\ &= \frac{\epsilon_2^{\mathrm{II}}}{\sqrt{\pi}} \int_{-\infty}^{0} dx e^{-x^2}\left[f(\eta_1) + x\Delta\eta f'(\eta_1) + \frac{x^2\Delta\eta^2}{2} f''(\eta_1) + \dots\right] \end{aligned} \tag{8.105}$$

where we have defined the variable $x = \eta - \eta_1/\Delta\eta$, and we have Taylor expanded the function f around η_1. About the integration domain, integration variable x is evaluated in the interval $x \in [-\infty, 0]$, since they correspond to integrate over all possible values of the duration of the transition, $\Delta\eta$. Indeed, $x \to 0$ corresponds to $\Delta\eta \to 0$, while $x \to -\infty$ to $\Delta\eta \to 0$, an infinitely smooth transition.

We are interested to see whether terms such $x\Delta\eta f'(\eta_1)$, $x^2\Delta\eta^2 f''(\eta_1)/2$ can become large leading to deviations from the sharp transition case (8.104). The integral over the parameter x can be easily solved, once we consider that:

$$\int_{-\infty}^{0} dx e^{-x^2} = \frac{\sqrt{\pi}}{2} \tag{8.106}$$

$$\int_{-\infty}^{0} dx\, x e^{-x^2} = -\frac{1}{2} \tag{8.107}$$

$$\int_{-\infty}^{0} dx\, x^2 e^{-x^2} = \frac{\sqrt{\pi}}{4}. \tag{8.108}$$

This reduces the integral I to the form:

$$I(\eta \simeq \eta_1) = \frac{\epsilon_2^{\mathrm{II}}}{2} f(\eta_1) - \frac{\epsilon_2^{\mathrm{II}}}{2\sqrt{\pi}}\Delta\eta f'(\eta) + \frac{\epsilon_2^{\mathrm{II}}\Delta\eta^2}{8} f''(\eta_1) + \dots. \tag{8.109}$$

In order to see when smoothening the transition makes the integral deviate from (8.104), which is the form taken for an infinitely sharp transition, we focus on the first correction:

$$I(\eta \simeq \eta_1) = \frac{\epsilon_2^{\mathrm{II}}}{2} f(\eta_1)\left[1 - \frac{\Delta\eta}{\sqrt{\pi}}\frac{f'(\eta_1)}{f(\eta_1)} + \dots\right]. \tag{8.110}$$

In the PO trispectrum the integrands associated to the derivative of the second slow-roll parameter are of the form

$$f_1(\eta) = \frac{H}{M_{\rm CS}}\epsilon_1(\eta)\mathcal{F}_{p_3}(\eta)\mathcal{F}_{p_4}(\eta)e^{\pm ip_{12}\eta} \tag{8.111}$$

$$f_2(\eta) = \frac{H}{M_{\rm CS}}\epsilon_1(\eta)\mathcal{F}^*_{p_3}(\eta)\mathcal{F}^*_{p_4}(\eta)e^{\pm ip_{12}\eta} \tag{8.112}$$

$$\tag{8.113}$$

We consider

$$f(\eta) = \frac{H}{M_{\rm CS}}\epsilon_1(\eta)\mathcal{F}^*_{p_3}(\eta)\mathcal{F}^*_{p_4}(\eta)e^{-ip_{12}\eta} \tag{8.114}$$

but the same conclusions can be extended to all the others. We compute the correction

$$\left.\frac{f'(\eta)}{f(\eta)}\right|_{\eta=\eta_1} = \left.\frac{d\ln f(\eta)}{d\eta}\right|_{\eta=\eta_1} = \frac{d\ln\left(H/M_{\rm CS}\right)}{d\eta} + \frac{d\ln\epsilon_1}{d\eta} + \left.\frac{d\ln\left(\mathcal{F}^*_{p_3}(\eta)\mathcal{F}^*_{p_4}(\eta)e^{-ip_{12}\eta}\right)}{d\eta}\right|_{\eta=\eta_1}. \tag{8.115}$$

Accounting for the fact that $1/M_{\rm CS} \propto \sqrt{\epsilon_1}$, the first two terms read

$$\frac{d\ln\left(H/M_{\rm CS}\right)}{d\eta} + \left.\frac{d\ln\epsilon_1}{d\eta}\right|_{\eta=\eta_1} = \left.\frac{3}{2}\frac{d\ln\epsilon_1'}{d\eta}\right|_{\eta=\eta_1} = \frac{9}{\eta_1} \tag{8.116}$$

where we made use of the definition of the second slow-roll parameter ϵ_2 and that we approximate the universe as perfect de Sitter, $\eta = -1/aH$.

Regarding the third term, we recall that $\mathcal{F}^*_{p_3}(\eta)\mathcal{F}^*_{p_4}(\eta)e^{-ip_{12}\eta}$ consists on a sum of terms of the type $B_1B_2(1\pm ip_3\eta)(1\pm ip_4)e^{-i(\pm p_3\pm p_4+p_{12})}$, where B_1 and B_2 are the Bogoljubov coefficients. They can be both of the same kind, for example γ or δ, or separate. We choose $\mathcal{F}^*_{p_3}(\eta)\mathcal{F}^*_{p_4}(\eta)e^{-ip_{12}\eta} = \gamma_{p_3}\gamma_{p_4}(1+ip_3\eta)(1+ip_4\eta)e^{-i(p_3+p_4+p_{12})}$, but, again, the results can be extended to all the other terms. The third term in (8.115) is

$$\left.\frac{d\ln\left(\gamma_{p_3}\gamma_{p_4}(1+ip_3\eta)(1+ip_4\eta)e^{-i(p_3+p_4+p_{12})}\right)}{d\eta}\right|_{\eta=\eta_1} = \frac{d\ln\gamma_{p_3}}{d\eta}+\frac{d\ln\gamma_{p_4}}{d\eta}+\left.\left[\frac{p_3^2\eta}{1+ip_1\eta}+\frac{p_4^2\eta}{1+ip_2\eta}-ip_{12}\right]\right|_{\eta=\eta_1}. \tag{8.117}$$

We see the appearing of the logarithmic derivative of the Bogoljubov coefficients. They have been derived from matching conditions at η_1 in case of *sharp* SR/USR transition. This makes the Bogoljubov coefficients changing abruptly during the transition, from the value 1 to (4.16). We assume that, if the transition is smooth, these coefficients will change sufficiently smoothly during the transition, so that $\Delta\eta\, d\ln\gamma/d\eta$ is a small quantity.

Collecting everything, and setting $p_i = r_i x_1$, $x_1 = p_1/\eta_1$, $r_i = p_i/p_1$ with $i = 3, 4$, we obtain that the integral (8.110) around the transition reads

$$I(\eta\simeq\eta_1) = \frac{\epsilon_2^{\rm II}}{2}f(\eta_1)\left[1-\frac{9}{\sqrt{\pi}}\frac{\Delta\eta}{\eta_1}-\Delta\eta\, p_1\left(\frac{r_3x_1}{1+ir_3x_1}+\frac{r_4x_1}{1+ir_4x_1}-ir_{12}\right)+\dots\right]. \tag{8.118}$$

We remember that the trispectrum involves four wavenumbers p_i associated to each of the four external legs. We assume that p_1 is the largest wavenumber. The correction is small if three conditions are satisfied

1. The duration of the transition $\Delta\eta$ is much smaller than the time at which occurs.
2. The combination $\Delta\eta\, p_1$ is small, which is equivalent to ask that the time associated with horizon crossing of p_1, the larger wavenumber, needs to be much larger than the duration of the transition. In fact, the wavenumber is proportional to the inverse of the wavelength of the oscillations of the mode function $p_1 = 2\pi/\lambda_1$. Requiring $\Delta\eta\, p_1 \ll 1$ means that we require that the mode function associated to p_1 almost no oscillation during a time interval $\Delta\eta$. In this way, the modes cannot resolve whether the transition is infinitely sharp or smooth.

3. The wavenumbers p_3 and p_4 need to be smaller or at least of the same order of the those associated to the transitions.

If these conditions are satisfied, the sharp transition approximation is describing correctly the physics. Notice that the third condition means that the sharp transition approximation fails to provide the correct behavior of correlator functions at very small scales ($k > k_2$). This is in agreement with our argument in the previous section and the large-scale regime is well-approximated by the instantaneous regime.

Chapter 9

Conclusions and future prospects

The discovery of parity violation in cosmological observables will lead to major breakthroughs, pointing towards new interactions, either in gravitational or within the electromagnetic sector. A part from recent hints in the CMB in the form of cosmic birefringence [25, 83, 26], there is the possibility that new interaction left their imprint on the LSS, causing quartets of galaxies to look different on the mirror. Recent analysis of the 4PCF of galaxies from the BOSS and DESI catalog pointed towards a possible detection of violation of parity symmetry, with detection significance ranging from 2-7 σ [27, 28] Even if these results may be biased by systematics or covariance misestimation, it is impelling to study theoretical models capable of sourcing PO cosmological signals, given that one of the simplest models of inflation breaking P symmetry, single-field slow-roll inflation with CS gravity, is not able to seed any detectable effect on LSS [30, 32].

With this purpose, we studied for the first time the possibility of generating an enhanced PO signal at the level of the scalar trispectrum, once the standard slow-roll conditions are violated for a brief epoch. To introduce a source of chirality, GR is modified by introducing a CS coupling between the inflaton and gravity. This model is rooted in high-energy theory, with CS correction being justified from a EFT perspective [55] and arises also from anomaly cancellation in heterotic string theory [59]. In addition, phases of violation of slow-roll condition have been widely considered in the literature [13, 15, 16, 52, 17, 18, 47], since they can generate a copious amount of PBH, renowned for being a possible dark matter candidate. As shown and discussed in this thesis (Ch.4), a model of brief phase of USR amplifies scalar perturbations. Therefore, the model that we consider consists of single-field inflation, with one brief USR epoch sandwiched between two SR phases, accounting for the gravitational CS correction.

With this being said, we began our journey towards the calculation of the primordial PO graviton-mediated scalar trispectrum, expecting the PO signal to be significantly enhanced by USR dynamics and peak at small scales. In fact, previous studies of non-Gaussianities from scalar cubic interactions enlightened the capability of these models to amplify deviations from the Gaussian statistics [77, 74, 17]. In our analysis, the explicit form of the inflaton potential, which contains features enabling the inflaton to violate slow-roll conditions, i.e. a plateau, is kept implicit: we work accounting for a the non trivial time-evolution of background quantities, the slow-roll parameters. Moreover, to simplify calculations and solve everything analytically, we approximated the spacetime during inflation as pure de Sitter, consider scalar perturbations as massless, and an instantaneous SR/USR/SR transition. These approximations have extensively been employed in the literature so far: the first two in slow-roll in studying non-Gaussian correlators of primordial fluctuations [56, 32, 31], and all of them in [46, 74, 77], is SR-USR-SR models. Moreover, CS correction to GR naturally introduces a dimensionless parameter H/M_{CS} which has to be small to avoid the propagation of ghost modes [60]. This parameter enters both the tensor mode functions and the vertices involving the graviton. We can use it as a perturbative parameter to expand in CS corrections at the desired order.

After recovering the results for the PE and PO graviton-mediated scalar trispectrum in SR+CS for the aforementioned approximations [67, 32] (Ch.7), with the PO component being identically vanishing since exact de Sitter is a scale invariant regime, and a series of no-go theorems preventing the PO trispectrum to be non vanishing relates heavily on that condition [33]. Therefore, in Sec.7.6 we relax de Sitter approximation, to obtain an order-of-magnitude estimate of the PO trispectrum in SR+CS. The trispectrum is found to be extremely suppressed, with the leading-order contribution in slow-roll expansion being of the order of second power of the first slow-roll parameter or a combination of the first two.

We proceed with diving into the calculation of both PO and PE trispectrum in SR-USR-SR+CS, after a careful evaluation of the scalar-scalar-tensor vertices and identification of the dominant contributions. Having done that, the PO trispectrum is found to be non vanishing and the analysis of its scale dependence carried out in the equilateral configuration (Sec.8.4) shows that it peaks within the interval of wavenumbers associated to modes exiting the horizon during the USR phase. In addition, it depends only on few parameters characterizing the physics of inflation: the value of the Hubble parameter H, the duration of USR phase in number of e-folds ΔN, the value of the second slow-roll parameter during USR $\Delta\epsilon_2$, and the Chern-Simons correction $H/M_{\rm CS}$. The SR-USR-SR+CS circumvents no-go theorems for PO trispectrum since scale invariance is heavily violated.

In the large scale regime $p_i/k_1, p_i/k_2 \to 0$, $k_1 = -1/\eta_1$ and $k_2 = -1/\eta_2$, the PO trispectrum vanishes. The contributors of the PO trispectrum in this limit are modes exiting the horizon deep in the first SR phase, when no-go theorem holds. The other condition for it to vanish is to consider a ΔN to be zero, which corresponds for having no USR in the first place. Again, this happens because we are back in the scenario discussed in Sec.7.5.2: it is again expected from no-go theorems. Even if there is no such theorems applying to the PE trispectrum, it vanishes as well, but for different reasons. The PE trispectrum in SR-USR-SR+CS is constructed out of two CS vertices, which are localized at the transitions (Sec.8.1.2). If $\Delta N = 0$, no transition occurs, leading the correspondent trispectrum to vanish. In the small-scale limit, $p_i/k_1, p_i/k_2 \to \infty$, both trispectra display strong oscillations whose amplitude grows for increasing value of the wavenumbers. In analogy with [73], we state that this behaviour is due to the sharp SR/USR/SR transition employed in the calculations. However, sharp transition are not believed to exist in nature (Natura non facit saltum), and we can expect the PO trispectrum to vanish in the small-scale limit once we account for the smoothness of the transition. In such a regime, the major contributors to the trispectrum are modes exiting the horizon during the second SR phase, in which scale-invariance is restored again. Both the trispectra experience a sharp rise for wavenumbers $p \gtrsim 2.65 \times 10^{-2}\, k_1$, which corresponds to the scale in which the null is located, eventually reaching its maximum in an interval of scales exiting the Hubble horizon during the USR phase. It is of great interest that the PO signal reaches an amplitude of $\mathcal{O}(10^{-9})$ for scales in which we expect the peak to be located. The amplitude of the dominant contribution to the PE trispectrum from graviton-exchange is suppressed w.r.t. the PO of about two orders of magnitude. The angular behaviour around the peak has been investigated as well, allowing us to visually confirm the parity-odd and -even nature of the observables we have computed. The PO signal from the trispectrum is null in case it reduces to a planar configuration of momenta, i.e. for $\chi = \pi/2$ and all possible values of θ, and for $\theta = 0, \pi/2$ and unspecified χ. On the contrary, the collapsed configuration is the that maximize its PE counterpart. This implies that the counter-collinear limit $\theta = \pi/2$, which is usually considered even in dealing with PO trispectra [30], is totally uninformative.

To compare our model with observations, we take the counter-collinear limit of the PE trispectrum to extract $\tau_{\rm NL}^{\rm CS,local}$. We obtain

$$\tau_{\rm NL}^{\rm CS,\, local} = 3.899 \times 10^{-2} \left(\frac{H/M_{\rm CS}}{10^{-2}} \right)^2 \tag{9.1}$$

which is in agreement with the constraint from Planck data [76]: $\tau_{\rm NL}^{\rm local} < 1700$ at 95% CL. Moreover, we extract the same non-linearity parameter of cubic interaction ζ^3. This is done by extracting the non-linearity parameter $f_{\rm NL}$ from the Maldacena's consistency relation, which has been proved to be valid even in models of inflation admitting a brief phase of USR for sufficiently long modes $k_{\rm L}$ exiting the horizon much before the onset of the SR violating phase [74]. The parameter$\tau_{\rm NL}^{\rm local}$ in its local form is related to $f_{\rm NL}^{\rm local}$, and this makes the extraction of the PE trispectrum from ζ^3 an easy task. In this way, we obtain an estimate for the PE trispectrum mediated by a scalar perturbation in the counter-collinear limit, which turns out to be dominating over the graviton-exchanged. We have conclude with comments on the sharpness of the transition and how it contributes to the trispectrum.

This result is scientifically valuable since it provides the first calculation of the PO and PE scalar trispectrum in SR-USR-SR+CS model of inflation. Given the primordial signal we have furnished, there can be several ways to extend the study of parity violation in these models, looking for a way to test them. Such a primordial PO trispectrum is expected to seed non-Gaussianities in the LSS. However, its shape function will be inevitably distorted by the non linear gravitational clustering. To compare data to the mode, and possibly extract information on parity violation on PBH formation, we need to study the non-linear evolution of the trispectrum, which is left for future developments. On the tensor sector, the recent study [84] evidenced how a sufficiently peaked PO trispectrum is able to induce chiral gravitational waves whose amplitude peaks inside the interval of ongoing and upcoming gravitational wave experiments (LISA, SKAO, nanoGRAV). There is hope for inflationary models featuring a brief phase of USR and CS gravity can be constrained in the following years. This shall shed light on important open questions in high-energy physics and cosmology, with possible implications for mechanisms sourcing baryogenesis and the nature of dark matter.

Appendix A

In-In time integrations

A.1 SR Inflation

A.1.1 Parity-even trispectrum

In this section we are interested in the evaluation of the sum of the integrals $I^{\rm adim}_{1234} + I^{\rm adim}_{3412}$ appearing in the first two permutations of (6.62).

Consider the first integral $I^{\rm adim}_{1234}$, it can be written as the sum of four distinct integrals:

$$I^{\rm adim}_{1234} = -\frac{1}{4}(I_A - I_B - I_C + I_D) \tag{A.1}$$

with

$$I_A \equiv \int_{-\infty}^{x} \frac{dx'}{x'^2} XY \int_{-\infty}^{x'} \frac{dx''}{x''^2} Z \ , \tag{A.2}$$

$$I_B \equiv \int_{-\infty}^{x} \frac{dx'}{x'^2} XY^* \int_{-\infty}^{x'} \frac{dx''}{x''^2} Z^* \ , \tag{A.3}$$

$$I_C \equiv \int_{-\infty}^{x} \frac{dx'}{x'^2} X^*Y \int_{-\infty}^{x'} \frac{dx''}{x''^2} Z \ , \tag{A.4}$$

$$I_D = \int_{-\infty}^{x} \frac{dx'}{x'^2} X^*Y^* \int_{-\infty}^{x'} \frac{dx''}{x''^2} Z^* \tag{A.5}$$

where X, Y and Z have been defined as

$$X(x,x') = (1+ik_1x)(1+ik_2x)(1-ik_1x')(1-ik_1x')e^{i(x'-x)(k_1+k_2)} \tag{A.6}$$

$$Y(x,x') = (1+ik_{12}x')e^{-ix'k_{12}} \tag{A.7}$$

$$Z(x,x'') = (1+ik_3x)(1+ik_4x)(1-k_3x'')(1-ik_4x'')(1-ik_{12}x'')e^{i(x''-x)(k_3+k_4)+ix''k_{12}}. \tag{A.8}$$

In deriving the expression for eq.A.26 we made use of the identities $\mathrm{Im}(X) = (X - X^*)/2i$ and $\mathrm{Im}(YZ) = (YZ - Y^*Z^*)/2i$. We notice that integrals I_B, I_C and I_D can be directly related to I_A by simply inverting the momenta or their combination. In fact, $I_B = I_A(-k_1, -k_2, k_3, k_4, k_{12})$, $I_C = I_A(k_1, k_2, -k_3, -k_4, -k_{12})$ and $I_D = I_A(-k_1, -k_2, -k_3, -k_4, -k_{12})$. This noticeably simplifies calculations since we only need to compute I_A and then change signs to the momenta. The integral we have to compute reads

$$\begin{aligned} I_A &= (1+ik_1x)(1+ik_2x)(1+ik_3x)(1+ik_4x)e^{-ix(k_1+k_2+k_3+k_4)} \\ &\times \int_{-\infty}^{x} \frac{dx'}{x'^2}(1-ik_1x')(1-ik_2x')(1+ik_{12}x')e^{ix'(k_1+k_2-k_{12})} \\ &\times \int_{-\infty}^{x''} \frac{dx''}{x''^2}(1-ik_3x'')(1-ik_4x'')(1-ik_{12}x'')e^{ix''(k_3+k_4+k_{12})} \ . \end{aligned} \tag{A.9}$$

Remember that the integration variable x is related to the conformal time via the relation $x = p_{\mathrm{T}} \cdot \eta$, thus, $x = -\infty$ corresponds to the beginning of the inflationary epoch and $x = 0$ corresponds to the conformal time associated to the end of Inflation. We immediately notice that the exponential of the type e^{ikx} oscillates rapidly for $x \to -\infty$. In order to obtain something which is manifestly convergent we make use of the well known $i\epsilon$ prescription. We then send $x \to x(1-i\epsilon)$, corresponding to $\eta \to \eta(1-i\epsilon)$ which means that we slightly rotate the time axis in the complex plane, allowing the conformal time to acquire a small imaginary component. This prescription guaranties convergence in the limit for $x \to -\infty$. In fact $\lim_{x\to-\infty(1-i\epsilon)} e^{ikx} = e^{i\infty}e^{-\epsilon\infty} = 0$, since an oscillating function multiplies an exponentially decaying one. Before proceeding with the calculation, it is useful to expand the products in (A.9). After doing that, it can be written as

$$\begin{aligned} I_A = &\left(1 + i\delta x - \frac{1}{2}\delta^2 x^2 + \frac{1}{2}\sum_a k_a^2 x^2 + \dots\right) e^{-i\delta x} \\ &\times \int_{-\infty}^{x} dx' \left(-\frac{1}{x'^2} - i\frac{\alpha_1}{x'} + \alpha_2 - i\alpha_3 x'\right) e^{i\alpha_1 x'} \\ &\times \int_{-\infty}^{x'} \left(\frac{1}{x''^2} - i\frac{\beta_1}{x''} + \beta_2 - i\beta_3 x''\right) e^{i\beta_1 x''} \end{aligned} \tag{A.10}$$

with the αs and βs given by:

$$\begin{cases} \alpha_1 = k_1 + k_2 - k_{12} \\ \alpha_2 = (k_1 + k_2)k_{12} - k_1 k_2 \\ \alpha_3 = k_1 k_2 k_{12} \ , \end{cases} \qquad \begin{cases} \beta_1 = k_3 + k_4 + k_{12} \\ \beta_2 = -(k_3 + k_4)k_{12} - k_3 k_4 \\ \beta_3 = -k_3 k_4 k_{12} \ . \end{cases} \tag{A.11}$$

In (A.10), the term outside the integration, the prefactor, has been obtained also by considering terms up to second order in x. In fact, we are interested in the trispectrum at the end of inflation, which is identified by $\eta = 0$.

Let us start computing the inner integral I^{inner}

$$I^{\mathrm{inner}} = \int_{-\infty}^{x'} \left(\frac{1}{x''^2} - i\frac{\beta_1}{x''} + \beta_2 - i\beta_3 x''\right) e^{i\beta_1 x''} \ . \tag{A.12}$$

The integral can be split into four, each of them solved separately:

$$I^{\mathrm{inner}} = \underbrace{\int_{-\infty}^{x'} dx'' \frac{e^{i\beta_1 x''}}{x''^2}}_{①} - i\underbrace{\int_{-\infty}^{x'} dx'' \frac{\beta_1}{x''} e^{i\beta_1 x''}}_{②} + \underbrace{\int_{-\infty}^{x'} dx''\, \beta_2 e^{i\beta_1 x''}}_{③} - i\underbrace{\int_{-\infty}^{x'} dx''\, \beta_3 x'' e^{i\beta_1 x''}}_{④} \ . \tag{A.13}$$

Solving the terms one by one we get:

$$\begin{aligned} ① = &-\frac{1}{x''}e^{i\beta_1 x''}\Big]_{-\infty(1-i\epsilon)}^{x'} + \int_{-\infty}^{x'} dx'' \frac{1}{x''}(i\beta_1)e^{i\beta_1 x''} = -\frac{1}{x''}e^{i\beta_1 x''} + i\beta_1 \int_{-\infty}^{x'} dx'' \frac{e^{i\beta_1 x''}}{x''} \\ = &-\frac{1}{x''}e^{i\beta_1 x''} - i\beta_1 E_1(-i\beta_1 x'') \ , \end{aligned} \tag{A.14}$$

$$② = -i\int_{-\infty}^{x'} dx'' \frac{\beta_1}{x''} e^{i\beta_1 x''} = i\beta_1 E_1(-i\beta_1 x'') \ , \tag{A.15}$$

$$③ = -i\frac{\beta_2}{\beta_1}\int_{-\infty}^{x'} dx''\,(i\beta_1)e^{i\beta_1 x''} = -i\frac{\beta_2}{\beta_1}e^{i\beta_1 x''}\Big]_{-\infty(1-i\epsilon)}^{x'} = -i\frac{\beta_2}{\beta_1}e^{i\beta_1 x'} \ , \tag{A.16}$$

$$④ = -\frac{\beta_3}{\beta_1}\left(x'' e^{i\beta_1 x''}\Big]_{-\infty(1-i\epsilon)}^{x'} - \int_{-\infty}^{x'} dx''\, e^{i\beta_1 x''}\right) = -\frac{\beta_3}{\beta_1}x' e^{i\beta_1 x'} - i\frac{\beta_3}{\beta_1^2}e^{i\beta_1 x''} \ , \tag{A.17}$$

notice that the second term appearing in (A.14) is the exponential integral or incomplete Γ function

$$E_1(z) = \int_z^{\infty} \frac{e^{-t}}{t}\, dt = \gamma + \ln z + \sum_{k=1}^{\infty} \frac{(-z)^k}{k - k!} \tag{A.18}$$

which gives a logarithmically divergent contribution in the case of $z = -i\beta_1 x'$ going to zero, so at the end of inflation. However, this divergent contribution is exactly canceled by the term ②. Putting all four terms together leads to:

$$\int_{-\infty}^{x''} \frac{dx''}{x''^2}\left[1 - i\beta_1 x'' + \beta_2 x''^2 - i\beta_3 x''^3\right] e^{ix''(k_3+k_4+k_{12})} = \left(-\frac{1}{x'} - i\frac{\beta_2\beta_1 + \beta_3}{\beta_1^2} - \frac{\beta_3}{\beta_1}x'\right) e^{i\beta_1 x'} \ . \tag{A.19}$$

which depends on the integration variable x', so it falls inside the first integration.

At this point, by expanding the product in the second line of (A.9) and adding the contribution coming from the nested integral, (A.9) is schematically given by

$$I_A = \text{prefactor} \times I \tag{A.20}$$

with

$$\text{prefactor} = \left(1 + i\delta x - \frac{1}{2}\delta^2 x^2 + \frac{1}{2}\sum_a k_a^2 x^2 + \dots\right) e^{-i\delta x} \ , \tag{A.21}$$

$$I = \int_{-\infty}^{x} dx \left(-\frac{1}{x'^3} + i\frac{\gamma_1}{x'^2} + \frac{\gamma_2}{x'} + i\gamma_3 + \gamma_4 x' + i\gamma_5 x'^2\right) e^{i\delta x} \ , \tag{A.22}$$

and the parameters γ_i being related to α_i and β_i by the relations:

$$\begin{cases} \gamma_1 = \alpha_1 - \frac{\beta_1\beta_2+\beta_3}{\beta_1^2} \\ \gamma_2 = -\frac{\beta_3}{\beta_1} - \frac{\alpha_1}{\beta_1^2}(\beta_1\beta_2 + \beta_3) - \alpha_2 \\ \gamma_3 = \alpha_3 - \frac{\alpha_2}{\beta_1^2}(\beta_1\beta_2 + \beta_3) + \alpha_1\frac{\beta_3}{\beta_1} \\ \gamma_5 = \frac{\alpha_3\beta_3}{\beta_1} \ . \end{cases} \tag{A.23}$$

The integral A.22 is solved analogously to (A.12) an the final result reads:

$$\begin{aligned} I = \Bigg[& \frac{1}{2x^2} + i\left(\frac{\delta}{2x} - \frac{\gamma_1}{x}\right) - \left(\gamma_2 - \gamma_1\delta + \frac{\delta^2}{2}\right) E_i(-ix\delta) \\ & + \left(\frac{\gamma_3}{\delta} + \frac{\gamma_4}{\delta^2} - \frac{2\gamma_5}{\delta^3}\right) + \left(\frac{2\gamma_5}{\delta^2} - \frac{\gamma_4}{\delta}\right) x + \frac{\gamma_5}{\delta}x^2 \Bigg] e^{i\delta x} \ . \end{aligned} \tag{A.24}$$

After multiplying this expression by the prefactor and keeping only terms that do not vanish in the $x \to 0^-$ limit we obtain:

$$\begin{aligned} I_A = & \delta\left(\gamma_1 - \frac{3}{4}\delta\right) + \frac{\gamma_3}{\delta} + \frac{\gamma_4}{\delta^2} - 2\frac{\gamma_5}{\delta^3} \\ & + \frac{1}{4}\sum_a k_a^2 + \frac{1}{2x^2} + \frac{i}{x}(\delta - \gamma_1) - \left(\gamma_2 - \gamma_1\delta + \frac{\delta^2}{2}\right) E_i(i\delta x) \end{aligned} \tag{A.25}$$

$$
\begin{aligned}
I_{1234} =& \frac{(k_1+k_2)}{(k_{12}+k_3+k_4)^2}\frac{1}{2}\Big[2k_{12}^3+3k_{12}^2(k_3+k_4)+2k_{12}(k_3^2+k_4^2)+(k_3+k_4)(k_3^2+k_4^2)\Big] \\
&-\frac{1}{4(k_{12}+k_3+k_4)^2}\Bigg\{\frac{4k_1k_{12}^2k_2k_3k_4(k_{12}+k_3+k_4)}{(k_1+k_2-k_3-k_4)^3}+\frac{4k_1k_{12}^2k_2k_3k_4(k_{12}+k_3+k_4)}{(k1+k2+k3+k4)^3}+ \\
&\frac{2k_{12}^2}{(k_1+k_2+k_3+k_4)^2}\Big[k_2k_3k_4\big(k_{12}+k_3+k_4\big)+k_1\big(k_2k_3(k_{12}+k_3)+k_{12}(k_2+k_3)k_4 \\
&+k_3(3k_2+k_3)k_4+(k_2+k_3)k_4^2\big)\Big]+\frac{2k_{12}^2}{(k_1+k_2-k_3-k_4)^2}\Big[k_2k_3k_4\big(k_{12}+k_3+k_4\big) \\
&+k_1\big(k_{12}k_3k_4-k_{12}k_2(k_3+k_4)+k_3k_4(k_3+k_4)-k_2(k_3^2+3k_3k_4+k_4^2)\big)\Big] \\
&-\frac{1}{k_1+k_2+k_3+k_4}\Big[k_{12}\big(k_1-k_{12}+k_2\big)k_3k_4\big(k_{12}+k_3+k_4\big)-k_1k_{12}k_2\big(k_{12}+k_3+k_4\big)^2 \\
&+\big(k_1k_2-k_{12}k_1+k_{12}k_2\big)\big(k_{12}k_3k_4+(k_{12}+k_3+k_4)(k_3k_4+k_{12}k_3+k_{12}k_4)\big)\Big] \\
&-\frac{1}{k_1+k_2+k_3+k_4}\Big[k_{12}\big(k_1-k_{12}+k_2\big)k_3k_4\big(k_{12}+k_3+k_4\big)-k_1k_{12}k_2\big(k_{12}+k_3+k_4\big)^2 \\
&+\big(-k_{12}k_2+k_1(-k_{12}+k_2)\big)\big(k_{12}k_3k_4+(k_{12}+k_3+k_4)(k_3k_4+k_12k_3+k_{12}k_4)\big)\Big] \\
&+\frac{1}{k_1+k_2-k_3-k_4}\Big[k_{12}\big(k_1+k_{12}+k_2\big)k_3k_4\big(k_{12}+k_3+k_4\big)+k_1k_{12}k_2\big(k_{12}+k_3+k_4\big)^2 \\
&+\big(-k_{12}k_2-k_1(k_{12}+k_2)\big)\big(k_{12}k_3k_4+(k_{12}+k_3+k_4)(k_3k_4+k_{12}k3+k_{12}k_4)\big)\Big] \\
&+\frac{1}{k_1+k_2-k_3-k_4}\Big[k_{12}\big(k_1+k_{12}+k_2\big)k_3k_4\big(k_{12}+k_3+k_4\big)+k_1k_{12}k_2\big(k_{12}+k_3+k_4\big)^2 \\
&-\big(k_{12}k_2+k_1k_{12}+k_1k_2\big)\big(k_{12}k_3k_4+(k_{12}+k_3+k_4)(k_3k_4+k_{12}k_3+k_{12}k_4)\big)\Big]\Bigg\}
\end{aligned} \tag{A.26}
$$

A.1.2 Parity-odd trispectrum

In the evaluation of the *parity-odd* trispectrum we have three integrals to perform:

$$
\begin{aligned}
\bar{I}_1 =& (1+ip_1\eta)(1+ip_2\eta)(1+ip_3\eta)(1+ip_4\eta)\,e^{-i(p_1+p_2+p_3+p_4)\eta} \\
&\int_{-\infty}^{\eta}\frac{d\eta'}{\eta'^2}(1-ip_1\eta')(1-ip_2\eta')(1+ip_{12}\eta')e^{i(p_1+p_2-p_{12})\eta'}\int_{-\infty}^{\eta'}d\eta''\eta''(1-ip_3\eta'')e^{i(p_3+p_4+p_{12})\eta''},
\end{aligned} \tag{A.27}
$$

$$
\begin{aligned}
\bar{I}_2 =& (1+ip_1\eta)(1+ip_2\eta)(1+ip_3\eta)(1+ip_4\eta)\,e^{-i(p_1+p_2+p_3+p_4)\eta} \\
&\int_{-\infty}^{\eta}d\eta''\eta''(1-ip_3\eta'')e^{i(p_3+p_4-p_{12})\eta''}\int_{-\infty}^{\eta''}\frac{d\eta'}{\eta'^2}(1-ip_1\eta')(1-ip_2\eta')(1-ip_{12}\eta')e^{i(p_1+p_2+p_{12})\eta'},
\end{aligned} \tag{A.28}
$$

$$
\begin{aligned}
\bar{I}_3 =& (1+ip_1\eta)(1+ip_2\eta)(1-ip_3\eta)(1-ip_4\eta)e^{-i(p_1+p_2-p_3-p_4)\eta} \\
&\int_{-\infty}^{\eta}\frac{d\eta'}{\eta'}^2(1-ip_1\eta')(1-ip_2\eta')(1-ip_{12}\eta')e^{i(p_1+p_2+p_{12})\eta'}\int_{-\infty}^{\eta}d\eta''\eta''(1+ip_3\eta'')e^{-i(p_3+p_4+p_{12})\eta''}
\end{aligned} \tag{A.29}
$$

As we did for the computation of the *even-even* integral, it is useful to perform a change of variable to isolate its dimensionful part outside the integration. Accordingly to what has been done in the previous section, the integrals are rewritten as $\bar{I}_a = \bar{I}_a^{adim}/p_T$ with $a = 1,2,3$ and the adimensional integrals

$$\bar{I}_1^{adim} = (1+ik_1x)(1+ik_2x)(1+ik_3x)(1+ik_4x)\, e^{-i(k_1+k_2+k_3+k_4)x} \int_{-\infty}^{x} \frac{dx'}{x'^2}\,(1-ik_1x')(1-ik_2x')(1+ik_{12}x')e^{i(k_1+k_2-k_{12})x} \int_{-\infty}^{x'} dx''x''(1-ik_3x'')e^{i(k_3+k_4+k_{12})x''}, \tag{A.30}$$

$$\bar{I}_2^{adim} = (1+ik_1x)(1+ik_2x)(1+ik_3x)(1+ik_4x)\, e^{-i(k_1+k_2+k_3+k_4)x} \int_{-\infty}^{\eta} dx''x''(1-ik_3x'')e^{i(k_3+k_4-k_{12})x''} \int_{-\infty}^{x''} \frac{dx'}{x'^2}(1-ik_1x')(1-ik_2x')(1-ik_{12}x')e^{i(k_1+k_2+k_{12})x'}, \tag{A.31}$$

$$\bar{I}_3^{adim} = (1+ik_1x)(1+ik_2x)(1-ik_3x)(1-ik_4x)e^{-i(k_1+k_2-k_3-k_4)x} \int_{-\infty}^{x} \frac{dx'}{x'}^2\,(1-ik_1x')(1-ik_2x')(1-ik_{12}x')e^{i(k_1+k_2+k_{12})x'} \int_{-\infty}^{x} dx''x''(1+ik_3x'')e^{-i(k_3+k_4+k_{12})x''}. \tag{A.32}$$

with $x, x', x'' = p_T \cdot \eta, \eta', \eta''$.

Computation of $\bar{I}_1^{adim}$

Let us start with the computation of the first integral $\bar{I}_1^{adim}$, which is a nested integration in the variables x' and x''. We first focus on the solution of the integral with respect to x'', which leads to

$$\begin{aligned}&\int_{-\infty}^{x'} dx''x''(1-ik_3x'')e^{i(k_3+k_4+k_{12})x''} \\ &\quad= \int_{-\infty}^{x'} dx''x''(1-ik_3x'')e^{i\beta_1x''} = \left[\frac{1}{\beta_1^2}\left(1+\frac{2k_3}{\beta_1}\right) - \frac{i}{\beta_1}\left(1+\frac{2k_3}{\beta_1}\right)x' - \frac{k_3}{\beta_1}x'^2\right]e^{i\beta_1x'}\end{aligned} \tag{A.33}$$

with $\beta_1 = k_3 + k_4 + k_{12}$. In this way, the nested integral is reduced to a single integration over x', and the integral $\bar{I}_1^{adim}$ is

$$\begin{aligned}\bar{I}_1^{adim} =& \left(1 + i\delta x - \frac{1}{2}\delta^2x^2 + \frac{1}{2}\sum_a k_a^2x^2 + \dots\right)e^{-i\delta x}\times \\ &\times \int_{-\infty}^{x} dx'\left(\underbrace{\frac{\gamma_1}{x'^2}}_{①} - \underbrace{i\frac{\gamma_2}{x'}}_{②} - \underbrace{\gamma_3}_{③} + \underbrace{i\gamma_4x'}_{④} + \underbrace{\gamma_5x'^2}_{⑤} - \underbrace{i\gamma_6x'^3}_{⑥}\right)e^{i\delta x'}\end{aligned} \tag{A.34}$$

with

$$\begin{cases} \gamma_1 = \frac{1}{\beta_1^2}\left(1+\frac{2k_3}{\beta_1}\right) \\ \gamma_2 = \frac{\alpha_1+\beta_1}{\beta_1^2}\left(1+\frac{2k_3}{\beta_1}\right) \\ \gamma_3 = \frac{k_3}{\beta_1} + \frac{\alpha_1\beta_1+\alpha_2}{\beta_1^2}\left(1+\frac{2k_3}{\beta_1}\right) \\ \gamma_4 = \frac{\alpha_1k_3}{\beta_1} + \frac{\alpha_2\beta_1+\alpha_3}{\beta_1^2}\left(1+\frac{2k_3}{\beta_1}\right) \\ \gamma_5 = \frac{\alpha_2k_3}{\beta_1} + \frac{\alpha_3}{\beta_1}\left(1+\frac{2k_3}{\beta_1}\right) \\ \gamma_6 = \frac{\alpha_3k_3}{\beta_1} \end{cases} \tag{A.35}$$

and $\delta = \alpha_1 + \beta_1$. Before proceeding with the evaluation of the integral, it is useful to analyze the structure of $\bar{I}_1^{adim}$. Preliminarily, we point out that only the real part of $\bar{I}_1^{adim} + \bar{I}_2^{adim} - \bar{I}_3^{adim}$ contributes to the parity-odd trispectrum (7.43). Moreover, we are ultimately interested in estimating it at the end of inflation, namely for $\eta \to 0$. Having this in mind, we point out that for the last four terms inside the integral in (A.34), their contribution comes from the term which is proportional to the zeroth power of x if it is real. For all the other terms, there is no way they can survive since they will contain a positive power of x. The first two terms in the integral in (A.34) instead will lead to terms having a pole or logarithmic divergencies for $x \to 0$ and must be taken into account with more care.

We start with the first two integrations, which yield to the contributions

$$① = \int_{-\infty(1-i\epsilon)}^{x} dx' \frac{\gamma_1}{x'^2} e^{i\delta x'} = -\frac{\gamma_1}{x} e^{i\delta x} - i\gamma_1 E_i(-i\delta x') \tag{A.36}$$

$$② = -i\gamma_1\delta \int_{-\infty(1-i\epsilon)}^{x} \frac{1}{x'} e^{i\delta x'} = i\gamma_1 \delta E_i(-i\delta x') \ . \tag{A.37}$$

These terms diverge in the limit $x \to 0$, however the logarithmic divergence entailed in $E_i(-i\delta x')$ is canceled when summing the two terms together

$$① + ② = -\frac{\gamma_1}{x} e^{i\delta x} \ . \tag{A.38}$$

The exponential integral cancels out with the one appearing in the prefactor, and the term above becomes completely real: it can contribute to the real part of $\bar{I}_1^{adim}$ only if it is multiplied by the first term in the prefactor, 1.

Regarding the other terms, the integration by parts yield only to one term that will survive the $x \to 0$ limit once multiplied by the prefactor. These contributions can be either real or imaginary and this will depend on the number of integration by parts needed to completely solve the integral. In fact, we have integrations of the type

$$\int_{-\infty(1-i\epsilon)}^{x} dx' (x')^n e^{ikx'}, \qquad n \geq 0$$

that are easily solved if we rewrite the exponential term in the integral as its derivative with respect to x'. In this way, for each integration by part a factor of $-i$ will be introduced since

$$\begin{aligned} \int_{-\infty(1-i\epsilon)}^{x} dx'\, (x')^n e^{ikx'} &= \frac{-i}{k} \int_{-\infty(1-i\epsilon)}^{x} dx'\, (x')^n\, ik e^{ikx'} \\ &= -\frac{i}{k}\left\{ \Big[(x')^n e^{ikx'}\Big]_{-\infty(1-i\epsilon)}^{x} - n \int_{-\infty(1-i\epsilon)}^{x} dx'\, (x')^{n-1} e^{ikx'} \right\} \ . \end{aligned} \tag{A.39}$$

The term ③ requires only one integration by part to be solved, thus it is imaginary once the exponential has been simplified. Regarding ④, its contribution proportional to the zeroth power of x is obtained after two integrations by parts, making it a contribution to the integral $\bar{I}_1^{adim}$ evaluated at the end of inflation. However, this term carries a factor of i and the non vanishing term turns out to be imaginary. The same applies also to ⑤ and ⑥, since in the first case, the only non vanishing term is multiplied by $(-i)^3$ and, in the second, the multiplicative factor is $(-i)^5$ with four powers of the imaginary unit coming from the integration by part procedure and the other being already present.

Therefore, the real part of the first adimensional integral evaluated at the end of inflation is:

$$\mathrm{Re}\big[\bar{I}_1^{adim}\big] = -\frac{\gamma_1}{x} = -\frac{1}{\beta_1^2}\left(\frac{\beta_1 + 2k_3}{\beta_1}\right)\frac{1}{x} = -\frac{3k_3 + k_4 + k_{12}}{(k_3 + k_4 + k_{12})^3}\frac{1}{x} \ . \tag{A.40}$$

Computation of $\bar{I}_2^{adim}$

The integral $\bar{I}_2^{adim}$ is computed in a similar way as $\bar{I}_2^{adim}$, the difference is that this time the integration over x' is nested with respected that over x''. We thus start computing the inner integration yielding to

$$\int_{-\infty(1-i\epsilon)}^{x''} \frac{dx'}{x'^2}(1-ik_1x')(1-ik_2x')(1-ik_{12}x')e^{i(k_1+k_2+k_{12})x'}$$
$$= \int_{-\infty(1-i\epsilon)}^{x''} \frac{dx'}{x'^2}\big(1-i\beta_1x'+\beta_2x'^2+i\beta_3x'^3\big)e^{i\beta_1x'} = \left[-\frac{1}{x''}-\frac{i}{\beta_1}\left(\beta_2-\frac{\beta_3}{\beta_1}\right)+\frac{\beta_3}{\beta_1^2}\right]e^{i\beta_1x''} \tag{A.41}$$

with the coefficients $\beta_{1,2,3}$ given by:

$$\begin{cases}\beta_1 = k_1+k_2+k_{12}\\ \beta_2 = -(k_1+k_2)k_{12}-k_1k_2\\ \beta_3 = k_1k_2k_{12}\end{cases} \tag{A.42}$$

Notice that no logarithmically divergent term is present: they are generated in equal and opposite amount by the first and the second term in (A.41), canceling each other. The integral $\bar{I}_2^{adim}$ then becomes

$$\begin{aligned}\bar{I}_2^{adim} =& \left(1+i\delta x-\frac{1}{2}\delta^2x^2+\frac{1}{2}\sum_a k_a^2x^2+\dots\right)e^{-i\delta x}\times\\ &\times\int_{-\infty(1-i\epsilon)}^{x} dx''\big(-1+i\gamma_1x''-\gamma_2x''^2-i\gamma_3x''^3\big)e^{i\delta x''}\end{aligned} \tag{A.43}$$

with the gamma coefficient defined by the following equalities

$$\begin{cases}\gamma_1 = k_3-\frac{1}{\beta_1}\left(\beta_2-\frac{\beta_3}{\beta_1}\right)\\ \gamma_2 = \frac{k_3}{\beta_1}\left(\beta_2-\frac{\beta_3}{\beta_1}\right)-\frac{\beta_3}{\beta_1^2}\\ \gamma_3 = k_3\frac{\beta_3}{\beta_1^2}\end{cases} \tag{A.44}$$

By applying the same logic illustrated above, we can immediately infer that the integral does not have any real contribution, therefore $\mathrm{Re}\big[\bar{I}_2^{adim}\big]=0$.

Computation of $\bar{I}_3^{adim}$

In the last integral, $\bar{I}_3^{adim}$, the double integration is not nested, moreover the contributions depending on the two integration variables x' and x'' are factorized, so we can evaluate the integral with respect to the two variables separately. The third in-in integration then reduces to the computation of the product of three factors and retain the real part

$$\bar{I}_3^{adim} = \text{prefactor}\times I_1(k_1,k_2,k_{12})\times I_2(k_3,k_4,k_{12}) \ . \tag{A.45}$$

In the expression above, the prefactor is not the same as that appearing in (A.34) and (A.43). In fact, the combination of propagators related to the four momenta involved differs from the other two integrals: differently from (7.44) and (7.45), the complex conjugate of the propagator related to the modes with momenta p_3 and p_4 enters in (7.46). For this reason, the signs of p_3 and p_4 in (A.32) are swapped with respect to those appearing in the previous two integrations. The prefactor than reads

$$\text{prefactor} = \left(1+i\delta x-\frac{\delta^2}{2}x^2+\frac{1}{2}\sum_a k_a^2x^2+\dots\right)e^{i\delta x}, \tag{A.46}$$

with $\delta = k_3+k_4-k_1-k_2$.

The other two factors I_1 and I_2 are given by:

$$I_1 = \left(-\frac{1}{x}+i\frac{\beta_2}{\beta_1}-\frac{\beta_3}{\beta_1}-i\frac{\beta_3}{\beta_1^2}\right)e^{i\beta_1x} \tag{A.47}$$

$$I_2 = \left[\left(\frac{1}{\alpha_1^2} + \frac{2k_3}{\alpha_1^3}\right) + i\left(\frac{1}{\alpha_1} + \frac{2k_3}{\alpha_1^3}\right)x - \frac{k_3}{\alpha_1}\right]e^{-i\alpha_1 x}, \tag{A.48}$$

with the beta coefficients given by (A.42) and $\alpha_1 = k_3 + k_4 + k_{12}$.

Substituting (A.46), (A.47) and (A.48) in (A.45) and keeping only the terms that do not go to zero for $x \to 0$, we get

$$\bar{I}_3^{adim} = -\left(\frac{1}{\alpha_1^2} + \frac{2k_3}{\alpha_1^3}\right)\frac{1}{x} + i\left(1 - \frac{\delta}{\alpha_1} + \frac{\beta_2}{\alpha_1\beta_1} - \frac{\beta_3}{\alpha_1\beta_1^2}\right)\left(\frac{1}{\alpha_1} + \frac{2k_3}{\alpha_1^2}\right)\ . \tag{A.49}$$

Being interested to its real part, we find that

$$\mathrm{Re}\left[\bar{I}_3^{adim}\right] = -\left(\frac{\alpha_1 + 2k_3}{\alpha_1^3}\right)\frac{1}{x} = -\frac{3k_3 + k_4 + k_{12}}{(k_3 + k_4 + k_{12})^3}\frac{1}{x}\ . \tag{A.50}$$

A.2 SR-USR-SR Inflation

A.2.1 Parity-odd trispectrum

In the parity-odd trispectrum in SR-USR-SR inflation, there are only two time-integrals appearing, (8.40), and one of them can be reduced to the other by simply reversing the sign of the exchanged momentum p_{12}. Therefore, it is sufficient to solve one of the two

$$\begin{aligned}\bar{I} &= \int_{x_a}^{x_b} dx' \mathcal{F}_{r_1}^*(A, B, x')\mathcal{F}_{r_2}^*(A, B, x')\mathcal{F}_{r_{12}}^*(1, 0, x') \\ &= \int_{x_a}^{x_b} dx' \left[A_{r_1}^*(1 - ir_1x')e^{ir_1x'} - B_{r_1}^*(1 + ir_1x')e^{-ir_1x'}\right]\left[A_{r_2}^*(1 - ir_2x')e^{ir_2x'} - B_{r_2}^*(1 + ir_2x')e^{-ir_2x'}\right]\times \\ &\quad \times (1 - ir_{12}x')e^{ir_{12}x'}\end{aligned} \tag{A.51}$$

where $A = 1, \gamma, \alpha$ and $B = 0, \delta, \beta$. A brief comment about the integration extrema: in principle x_1 can be either $-\infty$ or x_1, whereas x_b is either x_1 or $x = 0$. Therefore, integrations along the entire inflationary history are still appearing. However, since the transition is sharp, the integration operation can always be divided into integrations over the single phases (either SR or USR). In the following, when considering x_a and x_b we will account as this procedure has already been carried out.

This integral can be decomposed into four, each of them related to the others by a reversing of the sign of r_1 or r_2, or both:

$$\bar{I} = A_{r_1}^* A_{p_2}^* \tilde{I}(r_1, r_2, r_{12}) - A_{r_1}^* B_{r_2}^* \tilde{I}(r_1, -r_2, r_{12}) - B_{r_1}^* A_{r_2}^* \tilde{I}(-r_1, r_2, r_{12}) + B_{p_1}^* B_{p_2}^* \tilde{I}(-r_1, -r_2, r_{12}) \tag{A.52}$$

with

$$\tilde{I} = \int_{x_a}^{x_b} dx'(1 - ir_1x')(1 - ir_2x')(1 - ir_{12}x')e^{i(r_1+r_2+r_{12})x'}. \tag{A.53}$$

In fact, the Bogoljubov coefficients are not explicitly time dependent within a given inflationary phase, so they can be taken outside the integral.

The integral of the type of $\tilde{I}$ has already been solved in Sec.A.1.1, A.19. We borrow the result and find that $\tilde{I}$

$$\begin{aligned}\tilde{I} &= \left(-\frac{1}{x_b} - i\frac{\beta_1\beta_2 + \beta_3}{\beta_1^2} - \frac{\beta_3}{\beta_1}x_b\right)e^{i\beta_1 x_b} - \left(-\frac{1}{x_a} - i\frac{\beta_1\beta_2 + \beta_3}{\beta_1^2} - \frac{\beta_3}{\beta_1}x_a\right)e^{i\beta_1 x_a} \\ &= \mathcal{I}(r_1, r_2, r_{12}, x_b) - \mathcal{I}(r_1, r_2, r_{12}, x_a)\end{aligned} \tag{A.54}$$

where the beta factors are defined as follows:

$$\begin{cases}\beta_1 = r_1 + r_2 + r_{12} \\ \beta_2 = -(r_1 + r_2)r_{12} - r_1r_2 \\ \beta_3 = -r_1r_2r_3\end{cases}\ . \tag{A.55}$$

Appendix B

Absence of boundary term contributions

In slow-roll inflation it is standard procedure to manipulate the action by performing integration by parts and setting the boundary terms to be zero. However, in USR, this cannot be done so naively, and one needs to check that boundary terms do not generate a contribution which cancel the effect that we are eventually interested in [85, 86].

Given the CS interaction Hamiltonian

$$H_{\rm CS}^{(3,\text{ bulk})} \propto -\frac{1}{\eta}\left(\frac{H}{M_{CS}}\right)\epsilon_1\epsilon_2'\zeta^2\gamma' \tag{B.1}$$

employing the sharp transition approximation, $\epsilon_2'(\eta) \propto \delta(\eta-\eta_2)+\delta(\eta-\eta_1)$, we assume that there exists a temporal boundary term at cubic order that leads to the Hamiltonian

$$H_{\rm CS}^{(3,\text{ boundary})}(\eta,\mathbf{x}) = \frac{\mathrm{d}}{\mathrm{d}\eta}B(\eta,\mathbf{x}) \tag{B.2}$$

with $B(\eta,\mathbf{x}) \propto (1/\eta)(H/M_{\rm CS})\epsilon_1\epsilon_2\zeta^2\gamma'$. Therefore, we can evaluate the trispectrum obtained by combining the GR vertex with the one introduced by the boundary Hamiltonian. The double integration inside $(1,1)$, $(0,2)$ and $(2,0)$ contributions simplifies, leading to:

$$\begin{aligned}\langle O(\eta)\rangle_{(1,1)} &= 2\mathrm{Re}\left\{\int_{-\infty}^{\eta}d\eta'\int_{-\infty}^{\eta}d\eta''\langle H_{\rm GR}(\eta')O(\eta)H_{\rm CS}^{(3,\text{ boundary})}(\eta'')\rangle\right\}\\ &= 2\mathrm{Re}\left\{\int_{-\infty}^{\eta}d\eta'\langle H_{\rm GR}(\eta')\hat{O}(\eta)B(\eta)\rangle\right\}\end{aligned} \tag{B.3}$$

$$\begin{aligned}\langle O(\eta)\rangle_{(0,2)+(2,0)} &= 2\mathrm{Re}\left\{\int_{-\infty}^{\eta}d\eta'\int_{-\infty}^{\eta'}d\eta''(\eta)H_{\rm GR}(\eta')H_{\rm CS}^{(3,\text{ boundary})}(\eta'')\rangle + \langle O(\eta)H_{\rm CS}^{(3,\text{ boundary})}(\eta')H_{\rm GR}(\eta'')\rangle\right\}\\ &= 2\mathrm{Re}\left\{\int_{-\infty}^{\eta}d\eta'\langle O(\eta)H_{\rm GR}(\eta')B(\eta')\rangle + \langle O(\eta)B(\eta)H_{\rm GR}(\eta')\rangle\right\}\end{aligned} \tag{B.4}$$

Since in (B.3) and the second term of (B.4) B is evaluated at time η, which is sent to zero, those contributions are extremely suppressed. In fact, $\epsilon_1(0^-) = e^{-6\Delta N}\epsilon_{1,i}$, $(H/M_{CS}(0^-)) \propto \sqrt{\epsilon_{1,f}} = e^{-3\Delta N}\sqrt{\epsilon_{1,i}}$, and ϵ_2 is extremely small. These contributions, due to the fact that the second slow-roll parameter at the end of the second SR epoch is a first order quantity, are suppressed with respect to terms of the trispectrum featuring the bulk vertex, since it is evaluated at transition times where $\epsilon_2 \sim \mathcal{O}(1)$.

The Remaining contribution that can be of the same order in slow-roll is the first term in (B.4)

$$\int_{-\infty}^{\eta}d\eta'\langle O(\eta)H_{\rm GR}(\eta')B(\eta')\rangle = \left(\int_{-\infty}^{\eta_1}d\eta' + \int_{\eta_1}^{\eta_2}d\eta' + \int_{\eta_2}^{\eta}d\eta'\right)\langle O(\eta)H_{\rm GR}(\eta')B(\eta')\rangle$$

. However, it is not able to generate a term which is exactly equal and opposite to (8.54), or even to cancel some of its contribution, due to differences in the functional form. In fact, in contributions $(1,1)$, $(0,2)$ and $(2,0)$, the CS interaction Hamiltonian is integrated immediately due to the presence of the Dirac delta function.

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Bibliography

[1] Alan H. Guth. “Inflationary universe: A possible solution to the horizon and flatness problems”. In: Phys. Rev. D 23 (2 Jan. 1981), pp. 347–356. DOI: 10.1103/PhysRevD.23.347. URL: https://link.aps.org/doi/10.1103/PhysRevD.23.347.

[2] A. D. Linde. “A new inflationary universe scenario: A possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems”. In: Physics Letters B 108.6 (Feb. 1982), pp. 389–393. DOI: 10.1016/0370-2693(82)91219-9.

[3] Viatcheslav F. Mukhanov and G. V. Chibisov. “Quantum Fluctuations and a Nonsingular Universe”. In: JETP Lett. 33 (1981), pp. 532–535.

[4] Andrei D. Linde. “Scalar Field Fluctuations in Expanding Universe and the New Inflationary Universe Scenario”. In: Phys. Lett. B 116 (1982), pp. 335–339. DOI: 10.1016/0370-2693(82)90293-3.

[5] Alan H. Guth and So-Young Pi. “The Quantum Mechanics of the Scalar Field in the New Inflationary Universe”. In: Phys. Rev. D 32 (1985), pp. 1899–1920. DOI: 10.1103/PhysRevD.32.1899.

[6] Viviana Acquaviva et al. “Gauge-invariant second-order perturbations and non-Gaussianity from inflation”. In: Nuclear Physics B 667.1-2 (Sept. 2003), pp. 119–148. DOI: 10.1016/S0550-3213(03)00550-9. arXiv: astro-ph/0209156 [astro-ph].

[7] Juan Maldacena. “Non-gaussian features of primordial fluctuations in single field inflationary models”. In: Journal of High Energy Physics 2003.05 (May 2003), pp. 013–013. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2003/05/013. URL: http://dx.doi.org/10.1088/1126-6708/2003/05/013.

[8] P. A. R. Ade et al. “Planck2013 results. XXII. Constraints on inflation”. In: Astronomy amp; Astrophysics 571 (Oct. 2014), A22. ISSN: 1432-0746. DOI: 10.1051/0004-6361/201321569. URL: http://dx.doi.org/10.1051/0004-6361/201321569.

[9] P. A. R. Ade et al. “Planck2015 results: XVII. Constraints on primordial non-Gaussianity”. In: Astronomy amp; Astrophysics 594 (Sept. 2016), A17. ISSN: 1432-0746. DOI: 10.1051/0004-6361/201525836. URL: http://dx.doi.org/10.1051/0004-6361/201525836.

[10] N. Aghanim et al. “Planck 2018 results: VI. Cosmological parameters”. In: Astronomy amp; Astrophysics 641 (Sept. 2020), A6. ISSN: 1432-0746. DOI: 10.1051/0004-6361/201833910. URL: http://dx.doi.org/10.1051/0004-6361/201833910.

[11] Ryo Saito, Jun’ichi Yokoyama, and Ryo Nagata. “Single-field inflation, anomalous enhancement of superhorizon fluctuations and non-Gaussianity in primordial black hole formation”. In: Journal of Cosmology and Astroparticle Physics 2008.06 (2008), p. 024. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2008/06/024. URL: http://dx.doi.org/10.1088/1475-7516/2008/06/024.

[12] N. C. Tsamis and R. P. Woodard. “Improved estimates of cosmological perturbations”. In: Physical Review D 69.8 (Apr. 2004). ISSN: 1550-2368. DOI: 10.1103/physrevd.69.084005. URL: http://dx.doi.org/10.1103/PhysRevD.69.084005.

[13] William H. Kinney. “Horizon crossing and inflation with large ¡mml:math xmlns:mml=“http://www.w3.org/ display=“inline”¿ ¡mml:mi¿¡/mml:mi¿ ¡/mml:math¿”. In: Physical Review D 72.2 (2005). ISSN: 1550-2368. DOI: 10.1103/physrevd.72.023515. URL: http://dx.doi.org/10.1103/PhysRevD.72.023515.

[14] Mohammad Hossein Namjoo, Hassan Firouzjahi, and Misao Sasaki. "Violation of non-Gaussianity consistency relation in a single-field inflationary model". In: EPL (Europhysics Letters) 101.3 (Feb. 2013), p. 39001. ISSN: 1286-4854. DOI: 10.1209/0295-5075/101/39001. URL: http://dx.doi.org/10.1209/0295-5075/101/39001.

[15] Jérôme Martin, Hayato Motohashi, and Teruaki Suyama. "Ultra slow-roll inflation and the non-Gaussianity consistency relation". In: Physical Review D 87.2 (Jan. 2013). ISSN: 1550-2368. DOI: 10.1103/physrevd.87.023514. URL: http://dx.doi.org/10.1103/PhysRevD.87.023514.

[16] Hayato Motohashi, Alexei A. Starobinsky, and Jun'ichi Yokoyama. "Inflation with a constant rate of roll". In: Journal of Cosmology and Astroparticle Physics 2015.09 (2015), pp. 018–018. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2015/09/018. URL: http://dx.doi.org/10.1088/1475-7516/2015/09/018.

[17] H. V. Ragavendra and L. Sriramkumar. "Observational Imprints of Enhanced Scalar Power on Small Scales in Ultra Slow Roll Inflation and Associated Non-Gaussianities". In: Galaxies 11.1 (Feb. 2023), p. 34. ISSN: 2075-4434. DOI: 10.3390/galaxies11010034. URL: http://dx.doi.org/10.3390/galaxies11010034.

[18] Jason Kristiano and Jun'ichi Yokoyama. "Generating Large Primordial Fluctuations in Single-Field Inflation for Primordial Black Hole Formation". In: Primordial Black Holes. Springer Nature Singapore, 2025, pp. 53–91. ISBN: 9789819788873. DOI: 10.1007/978-981-97-8887-3_3. URL: http://dx.doi.org/10.1007/978-981-97-8887-3_3.

[19] Simeon Bird et al. "Did LIGO Detect Dark Matter?" In: Physical Review Letters 116.20 (May 2016). ISSN: 1079-7114. DOI: 10.1103/physrevlett.116.201301. URL: http://dx.doi.org/10.1103/PhysRevLett.116.201301.

[20] Misao Sasaki et al. "Primordial black holes—perspectives in gravitational wave astronomy". In: Classical and Quantum Gravity 35.6 (Feb. 2018), p. 063001. ISSN: 1361-6382. DOI: 10.1088/1361-6382/aaa7b4. URL: http://dx.doi.org/10.1088/1361-6382/aaa7b4.

[21] Bernard Carr et al. "Primordial black holes: constraints, potential evidence and prospects". In: La Rivista del Nuovo Cimento 49.5 (Mar. 2026), pp. 225–274. ISSN: 1826-9850. DOI: 10.1007/s40766-026-00080-z. URL: http://dx.doi.org/10.1007/s40766-026-00080-z.

[22] Anne M. Green. Primordial Black Holes as a dark matter candidate – a brief overview. 2024. arXiv: 2402.15211 [astro-ph.CO]. URL: https://arxiv.org/abs/2402.15211.

[23] A. D. Sakharov. "Violation of CP Invariance, C asymmetry, and baryon asymmetry of the universe". In: Pisma Zh. Eksp. Teor. Fiz. 5 (1967), pp. 32–35. DOI: 10.1070/PU1991v034n05ABEH002497.

[24] Arthur Lue, Limin Wang, and Marc Kamionkowski. "Cosmological Signature of New Parity-Violating Interactions". In: Physical Review Letters 83.8 (Aug. 1999), pp. 1506–1509. ISSN: 1079-7114. DOI: 10.1103/physrevlett.83.1506. URL: http://dx.doi.org/10.1103/PhysRevLett.83.1506.

[25] Yuto Minami and Eiichiro Komatsu. "New Extraction of the Cosmic Birefringence from the Planck 2018 Polarization Data". In: Physical Review Letters 125.22 (Nov. 2020). ISSN: 1079-7114. DOI: 10.1103/physrevlett.125.221301. URL: http://dx.doi.org/10.1103/PhysRevLett.125.221301.

[26] Johannes R. Eskilt and Eiichiro Komatsu. "Improved constraints on cosmic birefringence from the WMAP and ¡i¿Planck¡/i¿ cosmic microwave background polarization data". In: Physical Review D 106.6 (2022). ISSN: 2470-0029. DOI: 10.1103/physrevd.106.063503. URL: http://dx.doi.org/10.1103/PhysRevD.106.063503.

[27] Oliver H. E. Philcox. "Probing parity violation with the four-point correlation function of BOSS galaxies". In: Physical Review D 106.6 (2022). ISSN: 2470-0029. DOI: 10.1103/physrevd.106.063501. URL: http://dx.doi.org/10.1103/PhysRevD.106.063501.

[28] Jiamin Hou, Zachary Slepian, and Robert N Cahn. "Measurement of parity-odd modes in the large-scale 4-point correlation function of Sloan Digital Sky Survey Baryon Oscillation Spectroscopic Survey twelfth data release CMASS and LOWZ galaxies". In: Monthly Notices of the Royal Astronon 522.4 (May 2023), pp. 5701–5739. ISSN: 1365-2966. DOI: 10.1093/mnras/stad1062. URL: http://dx.doi.org/10.1093/mnras/stad1062.

[29] Pavel Motloch, Ue-Li Pen, and Hao-Ran Yu. "Observational search for primordial chirality violations using galaxy angular momenta". In: Physical Review D 105.8 (Apr. 2022). ISSN: 2470-0029. DOI: 10.1103/physrevd.105.083512. URL: http://dx.doi.org/10.1103/PhysRevD.105.083512.

[30] Cyril Creque-Sarbinowski et al. Parity-Violating Trispectrum from Chern-Simons Gravity. 2023. arXiv: 2303.04815 [astro-ph.CO]. URL: https://arxiv.org/abs/2303.04815.

[31] Tommaso Moretti, Nicola Bartolo, and Alessandro Greco. "Breaking parity: the case of the trispectrum from chiral scalar-tensor theories of gravity". In: Journal of Cosmology and Astroparticle Physi 2025.07 (July 2025), p. 051. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2025/07/051. URL: http://dx.doi.org/10.1088/1475-7516/2025/07/051.

[32] Giorgio Orlando, Shingo Akama, and Chunshan Lin. Scalar parity-odd trispectrum from gravitational Cher 2025. arXiv: 2505.16920 [hep-th]. URL: https://arxiv.org/abs/2505.16920.

[33] Giovanni Cabass et al. "Parity violation in the scalar trispectrum: no-go theorems and yes-go examples". In: Journal of High Energy Physics 2023.2 (Feb. 2023). ISSN: 1029-8479. DOI: 10.1007/jhep02(2023)021. URL: http://dx.doi.org/10.1007/JHEP02(2023)021.

[34] Edwin Hubble. "A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae". In: Proceedings of the National Academy of Science 15.3 (Mar. 1929), pp. 168–173. DOI: 10.1073/pnas.15.3.168.

[35] Alain Coc and Elisabeth Vangioni. "Primordial nucleosynthesis". In: International Journal of Modern Physi 26.08 (Aug. 2017), p. 1741002. ISSN: 1793-6608. DOI: 10.1142/s0218301317410026. URL: http://dx.doi.org/10.1142/S0218301317410026.

[36] N. Bartolo et al. "Non-Gaussianity from inflation: theory and observations". In: Physics Reports 402.3–4 (Nov. 2004), pp. 103–266. ISSN: 0370-1573. DOI: 10.1016/j.physrep.2004.08.022. URL: http://dx.doi.org/10.1016/j.physrep.2004.08.022.

[37] S. SHANKARANARAYANAN. "WHAT IF THE INFLATON IS A SPINOR CONDENSATE?" In: International Journal of Modern Physics D 18.14 (Dec. 2009), pp. 2173–2179. ISSN: 1793-6594. DOI: 10.1142/s0218271809016181. URL: http://dx.doi.org/10.1142/S0218271809016181.

[38] Andrew R. Liddle, Paul Parsons, and John D. Barrow. "Formalizing the slow roll approximation in inflation". In: Phys. Rev. D 50 (1994), pp. 7222–7232. DOI: 10.1103/PhysRevD.50.7222. arXiv: astro-ph/9408015.

[39] Robert M. Wald. General relativity / Robert M. Wald. eng. Chicago London: University of Chicago Press, 1984. ISBN: 0226870332.

[40] Sabino Matarrese, Silvia Mollerach, and Marco Bruni. "Relativistic second-order perturbations of the Einstein–de Sitter universe". In: Physical Review D 58.4 (1998). ISSN: 1089-4918. DOI: 10.1103/physrevd.58.043504. URL: http://dx.doi.org/10.1103/PhysRevD.58.043504.

[41] Karim A. Malik and David Wands. "Cosmological perturbations". In: Phys. Rept. 475 (2009), pp. 1–51. DOI: 10.1016/j.physrep.2009.03.001. arXiv: 0809.4944 [astro-ph].

[42] Richard L. Arnowitt, Stanley Deser, and Charles W. Misner. "The Dynamics of general relativity". In: Gen. Rel. Grav. 40 (2008), pp. 1997–2027. DOI: 10.1007/s10714-008-0661-1. arXiv: gr-qc/0405109.

[43] Jérôme Martin and L Sriramkumar. "The scalar bi-spectrum in the Starobinsky model: the equilateral case". In: Journal of Cosmology and Astroparticle Physics 2012.01 (Jan. 2012), pp. 008–008. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2012/01/008. URL: http://dx.doi.org/10.1088/1475-7516/2012/01/008.

[44] V.F. Mukhanov, H.A. Feldman, and R.H. Brandenberger. "Theory of cosmological perturbations". In: Physics Reports 215.5 (1992), pp. 203–333. ISSN: 0370-1573. DOI: https://doi.org/10.1016/0370-1573(92)90044-Z. URL: https://www.sciencedirect.com/science/article/pii/037015739290044Z.

[45] Gianmassimo Tasinato. "Analytic approach to non-slow-roll inflation". In: Physical Review D 103.2 (Jan. 2021). ISSN: 2470-0029. DOI: 10.1103/physrevd.103.023535. URL: http://dx.doi.org/10.1103/PhysRevD.103.023535.

[46] Suvashis Maity et al. "Loop contributions to the scalar power spectrum due to quartic order action in ultra slow roll inflation". In: Journal of Cosmology and Astroparticle Physics 2024.05

(May 2024), p. 046. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2024/05/046. URL: http://dx.doi.org/10.1088/1475-7516/2024/05/046.

[47] Alexandros Karam et al. "Anatomy of single-field inflationary models for primordial black holes". In: Journal of Cosmology and Astroparticle Physics 2023.03 (Mar. 2023), p. 013. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2023/03/013. URL: http://dx.doi.org/10.1088/1475-7516/2023/03/013.

[48] H.V. Ragavendra, Anjan Kumar Sarkar, and Shiv K. Sethi. "Constraining ultra slow roll inflation using cosmological datasets". In: Journal of Cosmology and Astroparticle Physics 2024.07 (July 2024), p. 088. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2024/07/088. URL: http://dx.doi.org/10.1088/1475-7516/2024/07/088.

[49] Viatcheslav F. Mukhanov, H. A. Feldman, and Robert H. Brandenberger. "Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions". In: Phys. Rept. 215 (1992), pp. 203–333. DOI: 10.1016/0370-1573(92)90044-Z.

[50] David Wands. "Duality invariance of cosmological perturbation spectra". In: Physical Review D 60.2 (1999). ISSN: 1089-4918. DOI: 10.1103/physrevd.60.023507. URL: http://dx.doi.org/10.1103/PhysRevD.60.023507.

[51] Shyam Balaji et al. "Observing Nulling of Primordial Correlations via the 21-cm Signal". In: Physical Review Letters 129.26 (Dec. 2022). ISSN: 1079-7114. DOI: 10.1103/physrevlett.129.261301. URL: http://dx.doi.org/10.1103/PhysRevLett.129.261301.

[52] Gianmassimo Tasinato. "Large $|\eta|$ approach to single field inflation". In: Phys. Rev. D 108 (4 Aug. 2023), p. 043526. DOI: 10.1103/PhysRevD.108.043526. URL: https://link.aps.org/doi/10.1103/PhysRevD.108.043526.

[53] Jason Kristiano and Jun'ichi Yokoyama. "Constraining Primordial Black Hole Formation from Single-Field Inflation". In: Physical Review Letters 132.22 (May 2024). ISSN: 1079-7114. DOI: 10.1103/physrevlett.132.221003. URL: http://dx.doi.org/10.1103/PhysRevLett.132.221003.

[54] Gabriele Franciolini et al. "Perturbativity in the presence of ultraslow-roll dynamics". In: Physical Review D 109.12 (2024). ISSN: 2470-0029. DOI: 10.1103/physrevd.109.123550. URL: http://dx.doi.org/10.1103/PhysRevD.109.123550.

[55] Steven Weinberg. "Effective field theory for inflation". In: Physical Review D 77.12 (2008). ISSN: 1550-2368. DOI: 10.1103/physrevd.77.123541. URL: http://dx.doi.org/10.1103/PhysRevD.77.123541.

[56] Nicola Bartolo and Giorgio Orlando. "Parity breaking signatures from a Chern-Simons coupling during inflation: the case of non-Gaussian gravitational waves". In: Journal of Cosmology and Astroparticle 2017.07 (July 2017), pp. 034–034. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2017/07/034. URL: http://dx.doi.org/10.1088/1475-7516/2017/07/034.

[57] C. Armendáriz-Picón, T. Damour, and V. Mukhanov. "k-Inflation". In: Physics Letters B 458.2-3 (1999), pp. 209–218. ISSN: 0370-2693. DOI: 10.1016/s0370-2693(99)00603-6. URL: http://dx.doi.org/10.1016/S0370-2693(99)00603-6.

[58] R. Jackiw and S.-Y. Pi. "Chern-Simons modification of general relativity". In: Physical Review D 68.10 (Nov. 2003). ISSN: 1089-4918. DOI: 10.1103/physrevd.68.104012. URL: http://dx.doi.org/10.1103/PhysRevD.68.104012.

[59] Stephon Alexander and Nicolás Yunes. "Chern–Simons modified general relativity". In: Physics Reports 480.1–2 (Aug. 2009), pp. 1–55. ISSN: 0370-1573. DOI: 10.1016/j.physrep.2009.07.002. URL: http://dx.doi.org/10.1016/j.physrep.2009.07.002.

[60] Masaki Satoh. "Slow-roll inflation with the Gauss-Bonnet and Chern-Simons corrections". In: Journal of Cosmology and Astroparticle Physics 2010.11 (Nov. 2010), pp. 024–024. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2010/11/024. URL: http://dx.doi.org/10.1088/1475-7516/2010/11/024.

[61] Jérôme Martin and Dominik J. Schwarz. "New exact solutions for inflationary cosmological perturbations". In: Physics Letters B 500.1–2 (Feb. 2001), pp. 1–7. ISSN: 0370-2693. DOI: 10.

1016/s0370-2693(01)00071-5. URL: http://dx.doi.org/10.1016/S0370-2693(01)00071-5.

[62] Julian Seymour Schwinger. "On gauge invariance and vacuum polarization". In: Physical Review 82 (1951), pp. 664–679. URL: https://api.semanticscholar.org/CorpusID:122592895.

[63] L. V. Keldysh. "Diagram Technique for Nonequilibrium Processes". In: Sov. Phys. JETP 20 (1965), pp. 1018–1026. DOI: 10.1142/9789811279461_0007.

[64] Steven Weinberg. "Quantum contributions to cosmological correlations". In: Physical Review D 72.4 (Aug. 2005). ISSN: 1550-2368. DOI: 10.1103/physrevd.72.043514. URL: http://dx.doi.org/10.1103/PhysRevD.72.043514.

[65] Yi Wang. "Inflation, Cosmic Perturbations and Non-Gaussianities". In: Communications in Theoretical Phy 62.1 (2014), pp. 109–166. ISSN: 0253-6102. DOI: 10.1088/0253-6102/62/1/19. URL: http://dx.doi.org/10.1088/0253-6102/62/1/19.

[66] G. C. Wick. "The Evaluation of the Collision Matrix". In: Phys. Rev. 80 (2 Oct. 1950), pp. 268–272. DOI: 10.1103/PhysRev.80.268. URL: https://link.aps.org/doi/10.1103/PhysRev.80.268.

[67] David Seery, Martin S Sloth, and Filippo Vernizzi. "Inflationary trispectrum from graviton exchange". In: Journal of Cosmology and Astroparticle Physics 2009.03 (Mar. 2009), pp. 018–018. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2009/03/018. URL: http://dx.doi.org/10.1088/1475-7516/2009/03/018.

[68] Marc Kamionkowski and Tarun Souradeep. "Odd-parity cosmic microwave background bispectrum". In: Physical Review D 83.2 (Jan. 2011). ISSN: 1550-2368. DOI: 10.1103/physrevd.83.027301. URL: http://dx.doi.org/10.1103/PhysRevD.83.027301.

[69] Nicola Bartolo et al. "Tensor non-Gaussianity in chiral scalar-tensor theories of gravity". In: Journal of Cosmology and Astroparticle Physics 2021.03 (Mar. 2021), p. 073. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2021/03/073. URL: http://dx.doi.org/10.1088/1475-7516/2021/03/073.

[70] William R. Coulton, Oliver H. E. Philcox, and Francisco Villaescusa-Navarro. Signatures of a Parity-Violati 2023. arXiv: 2306.11782 [astro-ph.CO]. URL: https://arxiv.org/abs/2306.11782.

[71] Tomohiro Fujita et al. "Parity-violating scalar trispectrum from a rolling axion during inflation". In: Journal of Cosmology and Astroparticle Physics 2024.05 (May 2024), p. 127. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2024/05/127. URL: http://dx.doi.org/10.1088/1475-7516/2024/05/127.

[72] Kaito Yura et al. Parity-violating scalar trispectrum from helical primordial magnetic fields. 2025. arXiv: 2505.16844 [astro-ph.CO]. URL: https://arxiv.org/abs/2505.16844.

[73] Jérôme Martin, L. Sriramkumar, and Dhiraj Kumar Hazra. "Sharp inflaton potentials and bispectra: effects of smoothening the discontinuity". In: Journal of Cosmology and Astroparticle Physics 2014.09 (2014), pp. 039–039. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2014/09/039. URL: http://dx.doi.org/10.1088/1475-7516/2014/09/039.

[74] Hayato Motohashi and Yuichiro Tada. "Squeezed bispectrum and one-loop corrections in transient constant-roll inflation". In: Journal of Cosmology and Astroparticle Physics 2023.08 (Aug. 2023), p. 069. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2023/08/069. URL: http://dx.doi.org/10.1088/1475-7516/2023/08/069.

[75] Christian T. Byrnes, Misao Sasaki, and David Wands. "Primordial trispectrum from inflation". In: Physical Review D 74.12 (Dec. 2006). ISSN: 1550-2368. DOI: 10.1103/physrevd.74.123519. URL: http://dx.doi.org/10.1103/PhysRevD.74.123519.

[76] Kareem Marzouk, Antony Lewis, and Julien Carron. "Constraints on τ_{NL} from Planck temperature and polarization". In: JCAP 08.08 (2022), p. 015. DOI: 10.1088/1475-7516/2022/08/015. arXiv: 2205.14408 [astro-ph.CO].

[77] Jason Kristiano and Jun'ichi Yokoyama. "Perturbative region on non-Gaussian parameter space in single-field inflation". In: Journal of Cosmology and Astroparticle Physics 2022.07 (July 2022), p. 007. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2022/07/007. URL: http://dx.doi.org/10.1088/1475-7516/2022/07/007.

[78] Antonio Riotto. The Primordial Black Hole Formation from Single-Field Inflation is Not Ruled Out. 2023. arXiv: 2301.00599 [astro-ph.CO]. URL: https://arxiv.org/abs/2301.00599.

[79] A. Riotto. The Primordial Black Hole Formation from Single-Field Inflation is Still Not Ruled Out. 2023. arXiv: 2303.01727 [astro-ph.CO]. URL: https://arxiv.org/abs/2303.01727.

[80] Hassan Firouzjahi. One-loop Corrections in Power Spectrum in Single Field Inflation. 2023. arXiv: 2303.12025 [astro-ph.CO]. URL: https://arxiv.org/abs/2303.12025.

[81] Hassan Firouzjahi and Antonio Riotto. Primordial Black Holes and Loops in Single-Field Inflation. 2023. arXiv: 2304.07801 [astro-ph.CO]. URL: https://arxiv.org/abs/2304.07801.

[82] Ayngaran Thavanesan. No-go Theorem for Cosmological Parity Violation. 2025. arXiv: 2501.06383 [hep-th]. URL: https://arxiv.org/abs/2501.06383.

[83] P. Diego-Palazuelos et al. "Cosmic Birefringence from the¡i¿Planck¡/i¿Data Release 4". In: Physical Review Letters 128.9 (Mar. 2022). ISSN: 1079-7114. DOI: 10.1103/physrevlett.128.091302. URL: http://dx.doi.org/10.1103/PhysRevLett.128.091302.

[84] H. V. Ragavendra and Nicola Bartolo. Twisted echoes of an odd quartet: Scalar-induced gravitational wave 2025. arXiv: 2507.02733 [astro-ph.CO]. URL: https://arxiv.org/abs/2507.02733.

[85] Jacopo Fumagalli. "Absence of one-loop effects on large scales from small scales in non-slow-roll dynamics". In: Journal of High Energy Physics 2025.5 (May 2025). ISSN: 1029-8479. DOI: 10.1007/jhep05(2025)162. URL: http://dx.doi.org/10.1007/JHEP05(2025)162.

[86] Yuichiro Tada, Takahiro Terada, and Junsei Tokuda. "Cancellation of quantum corrections on the soft curvature perturbations". In: Journal of High Energy Physics 2024.1 (Jan. 2024). ISSN: 1029-8479. DOI: 10.1007/jhep01(2024)105. URL: http://dx.doi.org/10.1007/JHEP01(2024)105.

[87] James M. Bardeen. "Gauge-invariant cosmological perturbations". In: Phys. Rev. D 22 (8 Oct. 1980), pp. 1882–1905. DOI: 10.1103/PhysRevD.22.1882. URL: https://link.aps.org/doi/10.1103/PhysRevD.22.1882.

[88] Enrico Pajer, Guilherme L. Pimentel, and Jaap V.S. van Wijck. "The conformal limit of inflation in the era of CMB polarimetry". In: Journal of Cosmology and Astroparticle Physics 2017.06 (June 2017), pp. 009–009. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2017/06/009. URL: http://dx.doi.org/10.1088/1475-7516/2017/06/009.

[89] Lotfi Boubekeur et al. "Action approach to cosmological perturbations: the 2nd order metric in matter dominance". In: JCAP 08 (2008), p. 028. DOI: 10.1088/1475-7516/2008/08/028. arXiv: 0806.1016 [astro-ph].

[90] David Seery and James E Lidsey. "Primordial non-Gaussianities in single-field inflation". In: Journal of Cosmology and Astroparticle Physics 2005.06 (June 2005), pp. 003–003. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2005/06/003. URL: http://dx.doi.org/10.1088/1475-7516/2005/06/003.

[91] Stephon Alexander and Jérôme Martin. "Birefringent gravitational waves and the consistency check of inflation". In: Physical Review D 71.6 (Mar. 2005). ISSN: 1550-2368. DOI: 10.1103/physrevd.71.063526. URL: http://dx.doi.org/10.1103/PhysRevD.71.063526.

[92] Giorgio Orlando. Chern-Simons gravitational term coupled to an isocurvature field. 2025. arXiv: 2501.09630 [astro-ph.CO]. URL: https://arxiv.org/abs/2501.09630.

[93] Xingang Chen, Yi Wang, and Zhong-Zhi Xianyu. "Schwinger-Keldysh diagrammatics for primordial perturbations". In: Journal of Cosmology and Astroparticle Physics 2017.12 (Dec. 2017), pp. 006–006. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2017/12/006. URL: http://dx.doi.org/10.1088/1475-7516/2017/12/006.

[94] M. Srednicki. Quantum field theory. Cambridge University Press, Jan. 2007. ISBN: 978-0-521-86449-7, 978-0-511-26720-8. DOI: 10.1017/CBO9780511813917.

[95] Peter Adshead, Richard Easther, and Eugene A. Lim. "The 'in-in' Formalism and Cosmological Perturbations". In: Phys. Rev. D 80 (2009), p. 083521. DOI: 10.1103/PhysRevD.80.083521. arXiv: 0904.4207 [hep-th].

[96] Daniel Baumann. TASI Lectures on Inflation. 2012. arXiv: 0907.5424 [hep-th]. URL: https://arxiv.org/abs/0907.5424.

[97] Daniel Baumann. TASI Lectures on Primordial Cosmology. 2018. arXiv: 1807.03098 [hep-th]. URL: https://arxiv.org/abs/1807.03098.

[98] Francis Bernardeau, Tristan Brunier, and Jean-Philippe Uzan. "High order correlation functions for a self-interacting scalar field in de Sitter space". In: Physical Review D 69.6 (Mar. 2004). ISSN: 1550-2368. DOI: 10.1103/physrevd.69.063520. URL: http://dx.doi.org/10.1103/PhysRevD.69.063520.

[99] Nicola Bartolo, Emanuela Dimastrogiovanni, and Alberto Vallinotto. "One-loop corrections to the power spectrum in general single-field inflation". In: Journal of Cosmology and Astroparticle Physics 2010.11 (Nov. 2010), pp. 003–003. ISSN: 1475-7516. DOI: 10.1088/1475-7516/2010/11/003. URL: http://dx.doi.org/10.1088/1475-7516/2010/11/003.

[100] Scott. Dodelson. Modern cosmology. eng. San Diego, Calif: Academic Press, 2003. ISBN: 1-281-03624-2.

[101] Nicola Bartolo and Giorgio Orlando. "Parity breaking signatures from a Chern-Simons coupling during inflation: the case of non-Gaussian gravitational waves". In: JCAP 07 (2017), p. 034. DOI: 10.1088/1475-7516/2017/07/034. arXiv: 1706.04627 [astro-ph.CO].

[102] Jiro Soda, Hideo Kodama, and Masato Nozawa. "Parity violation in graviton non-gaussianity". In: Journal of High Energy Physics 2011.8 (Aug. 2011). ISSN: 1029-8479. DOI: 10.1007/jhep08(2011)067. URL: http://dx.doi.org/10.1007/JHEP08(2011)067.

[103] Sebastian Garcia-Saenz et al. "No-go theorem for scalar-trispectrum-induced gravitational waves". In: JCAP 03 (2023), p. 057. DOI: 10.1088/1475-7516/2023/03/057. arXiv: 2207.14267 [astro-ph.CO].

[104] Sha Azyzy et al. Nonlinear Evolution of the Matter Trispectrum with Primordial Parity Violation. 2025. arXiv: 2510.06164 [astro-ph.CO]. URL: https://arxiv.org/abs/2510.06164.

[105] Drew Jamieson et al. Parity-Odd Power Spectra: Concise Statistics for Cosmological Parity Violation. 2024. arXiv: 2406.15683 [astro-ph.CO]. URL: https://arxiv.org/abs/2406.15683.

[106] Marco Celoria and Sabino Matarrese. Primordial Non-Gaussianity. 2018. arXiv: 1812.08197 [astro-ph.CO]. URL: https://arxiv.org/abs/1812.08197.

[107] Eiichiro Komatsu. "New physics from the polarized light of the cosmic microwave background". In: Nature Reviews Physics 4.7 (May 2022), pp. 452–469. ISSN: 2522-5820. DOI: 10.1038/s42254-022-00452-4. URL: http://dx.doi.org/10.1038/s42254-022-00452-4.

[108] P. Ivanov, P. Naselsky, and I. Novikov. "Inflation and primordial black holes as dark matter". In: Phys. Rev. D 50 (1994), pp. 7173–7178. DOI: 10.1103/PhysRevD.50.7173.

[109] Jason Kristiano and Jun'ichi Yokoyama. "Comparing sharp and smooth transitions of the second slow-roll parameter in single-field inflation". In: JCAP 10 (2024), p. 036. DOI: 10.1088/1475-7516/2024/10/036. arXiv: 2405.12145 [astro-ph.CO].