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Wandering in the cosmological consistency relations: a critical assessment

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To my grandfather Luigi (1935-2019), the first to believe in me

Abstract

The thesis project deals with the so-called cosmological consistency relation, as predicted by inflationary models of the Early Universe. Such relation links the primordial power spectrum of the so-called comoving curvature perturbation (a gauge invariant measure of primordial density perturbations) or the tensor perturbations (i. e. primordial gravitational waves) with the corresponding bispectrum (the Fourier counterpart of the 3-point function), in the so-called squeezed limit, which means sending one momentum to zero in the 3-point correlator. This property is phenomenologically very important, since it holds in *any* single-field model of inflation, but many other models of inflation (such as multi-field inflation) do not respect it. This implies that a measurement of a deviation from the consistency relation would authomatically rule out all single-field models of inflation. However, some works have recently claimed that this relation is trivial and non-physical, because according to them the squeezed bispectrum can be set to zero through an appropriate gauge transformation. In this work we introduce all the tools necessary to discuss this issue, but we also provide some arguments that this cancellation argument is not valid and this relation indeed remains observable and phenomenologically very interesting. More specifically, the impossibility to cancel mixed bispectra involving the scalar curvature and the tensor perturbations of the metric is presented for the first time in this work.

Contents

Introduction

The thesis project deals with the so-called cosmological consistency relations holding in inflationary models. More specifically, this relation links the primordial power spectrum of the comoving curvature perturbation (in single-field models of inflation) or the tensor perturbations (i.e. primordial gravitational waves) with the corresponding bispectrum (which is the Fourier counterpart of the 3-point function), in the so-called squeezed limit, which means sending one momentum to zero in the correlator.
After having briefly reviewed standard cosmology and cosmological perturbations, we introduce Weinberg theorem. Weinberg showed that in the superhorizon limit, whatever the constituents of the Universe and under very general assumptions, there are always two independent scalar solutions of the linearized Einstein equations in the Newtonian gauge for which the curvature perturbation is time independent and there is one tensor mode for which its amplitude is time independent in this limit. The modes having constant curvature perturbations are usually called adiabatic. We use a geometrical definition for the comoving curvature which is related to the spatial metric in Arnowitt-Deser-Misner formalism. After that, we introduce inflation, which is a period in which the Universe expanded accelerating at early stages, able to provide the correct initial conditions for the evolution of the Universe and also to solve some shortcomings of the standard hot Big Bang model. We focus only on the simplest model of inflation, which is single-field inflation: in this case the mechanism is driven by a scalar field with an almost flat potential in order to fullfill the so-called slow-roll conditions. To test the different inflationary models one usually computes correlators: the 2-point function in Fourier space is related to the power spectrum of the cosmological perturbations; the 3-point function in Fourier space is related to the bispectrum. In general, as usual, a non-vanishing three point function is related to non-Gaussianity, which is due to interactions in the inflaton Lagrangian. To compute these correlators in Fourier space, a very powerful technique is the *in-in* formalism. In case of single-field inflation, in 2002 Maldacena was able to compute all the bispectra for this class of models, involving both curvature and tensor perturbations, starting from the action written in Arnowitt-Deser-Misner form.
From Maldacena's results the consistency relation emerges naturally taking the squeezed limit, which means to send a momentum to zero, so that the other two are equal (given that any bispectrum is multiplied by a Dirac delta of the sum of the momenta, the three momenta form a triangle in momentum space). In the following years, many other proofs of the consistency relation have been discussed, using different approaches, such as a "background wave" argument based on Weinberg theorem, path integration, Ward identities, BRST symmetry and holography. There are however models in which the consistency relation is violated, such as multi-field inflation or solid inflation, for which these demonstrations do not work.
Literature includes also many extensions involving an increasing number of fields in the correlators. The consistency relation linking the bispectrum with the power spectrum is very interesting, linking observables to be hopefully measured in the future. Since all single-field inflationary models must obey it, any deviation would unavoidably rule out all the single-field models, making this result a crucial key in order to understand Early Universe physics.
However, some works have apparently shown that the bispectrum could be cancelled using a coordinate transformation, implying that this relation is not observable. Their demonstrations are based on moving on a particular frame of reference, called conformal Fermi coordinates (CFC), and on the splitting of the comoving curvature into a long-wave and a short-wave part. Each part transforms differently under the coordinate transformation. The infinitesimal diffeomorphism relating the change from one frame to the other turns out to be a deformed dilatation. On the other hand, a recent

paper by Matarrese, Pilo and Rollo poses a critical assessment of this result in the case of a scalar bispectrum, for example the one involving the correlation among three curvature perturbations. In particular, it highlights that such a cancellation is true only in the exact squeezed limit in which one momentum is exactly null, which implies a perturbation mode to be infinitely long, so unphysical.
Our work aims at extending this result to all the bispectra involving both the curvature and the tensor perturbations (which are gravitational waves). We found that the splitting into a long-wave and a short-wave part of both the comoving curvature and the tensor perturbations is problematic. We have analyzed carefully the scalar-vector-tensor decomposition of the infinitesimal coordinate transformation used to pass from conformal frame to CFC: we have found that the transformation rules are not the ones used in the demonstrations. We also developed a Mathematica code, based on *in-in* formalism, showing that the action varies as a boundary term at third order in perturbations under the deformed dilatation. This implies that all the bispectra are uneffected from the deformed dilatation, so the transformations rules used to cancel the consistency relation are wrong.
Our results imply that the all the consistency relations can be gauged away only in the exact squeezed limit, which is non-physical, so they remain a crucial tool to test Early Universe Physics: any deviation would exclude all single-field models of inflation. Our results are about to appear as preprint in ArXiV in the near future and it will be submitted to an international journal.

The project is organized as follows. In chapter 1 we review the basics of standard cosmology, with the best model we have today: the Standard Hot Big Bang model. In chapter 2 we introduce the perturbative approach to general relativity, which generates a non-trivial gauge issue; we also review the equations for the evolution of the metric at linear order in perturbations. In chapter 3 we introduce an important result, Weinberg theorem, which ensures the conservation of some specific quantities in superhorizon limit. In chapter 4 we are ready to introduce inflation: we focus in particular on single-field slow-roll inflation, until the calculations of three point correlators, which are related to the primordial non-Gaussianity. Thanks to these results, we can finally introduce the Maldacena consistency relation in chapter 5, discussing a proof and some generalizations. Chapter 6 contains some arguments in favour to the fact that the consistency relation is physical, i. e. it can be really measured in an experiment, for all the types of bispectra. This is the main result of this project, appearing for the first time. We conclude with a brief discussion about how single field inflation can be experimentally tested in chapter 7.

Notation and conventions

In this work we use natural units to measure length, time, mass, energy and temperature. In these units one has $c = \hbar = k_B = 1$, so the mass dimensions are

$$[m] = [E] = [T] = 1 \ , \qquad [l] = [t] = -1 \ .$$

The (reduced) Planck mass is defined $M_P = \sqrt{\frac{\hbar c}{8\pi G}}$, or in natural units $M_P = \frac{1}{\sqrt{(8\pi G)}}$.
We use also the mostly plus convention for the metric tensor, so that in flat space it reads

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ .$$

The Einstein notation for the repeated indices is massively used. A Lorentz index is indicated with Greek letters (ex. x^μ), while spacetime points are indicated without the vector symbol (ex. x). Space coordinates are indicated with with Latin indices (ex. x^i) and a 3-vector is indicated with the vector symbol (ex. $\vec{x}$). The same convention is used for momenta (ex. k^μ, $\vec{k}$, k^i), but for sake of shortness the norm of the space component of the momentum is indicated without any symbol ($k = |\vec{k}|$).
The notation for the symmetrization or the antisymmetrization of two indices of a generic tensor $T_{\mu\nu}$ is the following:

$$T_{[\mu\nu]} = \frac{T_{\mu\nu} - T_{\nu\mu}}{2} \qquad T_{\{\mu\nu\}} = \frac{T_{\mu\nu} + T_{\nu\mu}}{2} \ .$$

Requested prerequisites to read this work

This project is self-consistent, since it introduces all the tools necessary to understand the final results obtained. However, a first introduction to general relativity and quantum field theory is necessary as a starting prerequisite.

Abbreviations

Since in this project there are many recurrent terms, we often abbreviate them as follows.

ADM	Arnowitt-Deser-Misner (formalism)
BRST	Becchi-Rouet-Stora-Tyutin (symmetry)
BT	Boundary term
CFC	Conformal Fermi coordinates
CL	Confidence level
CMB	Cosmic microwave background
CR	Consistency relation
EMT	Energy-momentum tensor/stress-energy tensor
eq.	equation
ex.	example
FLRW	Friedmann-Lemaître-Robertson-Walker (metric/spacetime)
FNC	Fermi normal coordinates
GR	General relativity
GUT	Great unification theory
GW	Gravitational wave
LIF	Local inertial frame
LHS	Left hand side
ODE	Ordinary differential equation
PDE	Partial differential equation
QFT	Quantum field theory
PDF	Probability density function/functional
PNG	Primordial non-Gaussianity
RHS	Right hand side
SCT	Special conformal transformation
SSB	Spontaneous Symmetry Breaking
SVT	Scalar-vector-tensor (decomposition)
VEV	Vacuum expectation value
ΛCDM	Standard hot Bang Model

Chapter 1

Review of standard cosmology

In this introductory chapter we briefly review the standard cosmological model, based on general relativity (GR). After its introduction in 1916 ([1]), GR is the best theory we have nowadays to describe gravity, both on astrophysical scales and on cosmological scales. The modern cosmology started from the work by Friedmann ([2]), introducing the correct metric to describe a Universe in accordance with the cosmological principle. The contemporary mode explaining in a quite exhaustive way the cosmological data is the standard Hot Big Bang model, also called ΛCDM model.
This chapter must be intended as a starting point to review what we need in the project and to fix the notation, so it is very far to be exhaustive on some huge and long studied topics. Since its content is very standard, it is based on classical references ([3–5]).

1.1 General relativity

The contemporary theory which describes gravity is GR, introduced by Albert Einstein in order to avoid the problems one typically encounters trying to covariantize the Newton's law in the contest of special relativity. The basic principle of GR is the equivalence principle, which in the Einstein formulation is based on the following postulates:

- equivalence of inertial mass and gravitational mass (weak equivalence principle);
- impossibility to detect gravity doing a local non-gravitational experiment (local Lorentz invariance);
- independence of the results of a local non-gravitational experiment on the position (local position invariance).

The mathematical formulation of GR is based on the fact that the spacetime is described by means of a 4-dimensional manifold, having in general a non-trivial intrinsic curvature. The matter distribution (or better, the energy, since in a relativistic contest matter is nothing more than a form of energy) establishes how the spacetime curves and at the same time this curvature is responsible to the presence of gravity (the motion of a particle in a gravitational field can be non-linear since the spacetime itself is curved). The distances between couples of points on this manifold are measured by using the metric tensor $g_{\mu\nu}$; in case of a flat space $g_{\mu\nu}$ is the Minkowski metric tensor $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, but in general it is a non-diagonal symmetric matrix, whose entries can depend on the spacetime point. In GR, in accordance with the equivalence principle, all the frames of reference are allowed to describe the spacetime. Mathematically, they are different charts covering the same portion of the spacetime manifold. This means that the physics described by GR must be invariant under diffeomorphisms. We remind that a general vector and covector change as follows under a diffeomorphism of type $x' = x'(x)$:

$$V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}(x) \ , \quad V'_{\mu}(x') = \frac{\partial x^{\nu}}{\partial x'^{\mu}} V_{\nu}(x) \ .$$

A generic tensor field with more indices transforms the same way, each index carrying a Jacobian as above.

The local Lorentz invariance imposed by the equivalence principle is analogous to impose the existence of a reference of frame x' such that the metric tensor, which changes as above under a diffeomorphism, can be written at a point P as

$$\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu}g_{\rho\sigma}\bigg|_P = g'_{\mu\nu}(x'_P) = \eta_{\mu\nu} \ .$$

Indeed, one can show[1] that there is always a diffeomorphism such that the metric is transformed into

$$g'_{\mu\nu}(x') = \eta_{\mu\nu} + \mathcal{O}\left((x' - x'_P)^2\right) \ , \tag{1.1}$$

in accordance with the equivalence principle.
Another important point is that in a theory invariant under diffeomorphisms one cannot use the standard derivative acting on a tensor field, since the result would be diffeomorphism dependent. On the contrary, one has to use the covariant derivative, acting on a generic tensor field F as follows

$$D_\mu F^{\alpha_1\dots\alpha_n}_{\beta_1\dots\beta_m} = \partial_\mu F^{\alpha_1\dots\alpha_n}_{\beta_1\dots\beta_m} + \Gamma^{\alpha_1}_{\mu\gamma}F^{\gamma\alpha_2\dots\alpha_n}_{\beta_1\dots\beta_m} + \dots + \Gamma^{\alpha_n}_{\mu\gamma}F^{\alpha_1\dots\alpha_{n-1}\gamma}_{\beta_1\dots\beta_m} - \Gamma^{\gamma}_{\mu\beta_1}F^{\alpha_1\dots\alpha_n}_{\gamma\dots\beta_m} - \dots - \Gamma^{\gamma}_{\mu\beta_m}F^{\alpha_1\dots\alpha_n}_{\beta_1\dots\beta_{m-1}\gamma} \ . \tag{1.2}$$

This expression can be shown to be diffeomorphism invariant: in a mathematical language, this expression is independent of the chart used to cover the spacetime manifold. The Γ's are called connection coefficients or Christoffel's symbols. In case of GR, one chooses the covariant derivative to be the Levi-Civita connection: in this case the Christoffel's symbols are uniquely fixed by the metric and their expression reads

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}) \ . \tag{1.3}$$

We remark that despite having 3 indices, the Christoffel's symbols are not tensors, that is to say that they do not transform as tensors under a generic diffeomorphism.
The equation describing the motion of a particle in the spacetime manifold in absence of external forces is the analogous of the second law of dynamics. Given a particle with trajectory $x^\mu(s)$, where s is the (affine) parameter, one expects that in absence of external forces the acceleration is null; in this case we have to derive the velocity $\frac{dx^\mu}{ds}$, with respect to s, but to be consistent with what we have just said we have to use the covariant derivative along the trajectory:

$$\frac{dx^\mu}{ds}D_\mu\left(\frac{dx^\nu}{ds}\right) = 0 \ .$$

The result is the geodesic equation[2]

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0 \ .$$

It is very important to notice that in case of a flat space, where the metric $g_{\mu\nu}$ is constant, all the Christoffel's symbols are null, so one gets back the standard expressions one is used to know.

Let us briefly argue how to get the Einstein equations. In contemporary language, general relativity is a field theory, whose field is the metric $g_{\mu\nu}$ itself, whose dynamics can be obtained by varying the Einstein-Hilbert action

$$S_{EH} = \frac{M_P^2}{2}\int d^4x\sqrt{-g}R$$

[1] See [3], pag. 50.

[2] This equation can be obtained by varying the following action

$$S_p = -m\int d^4x\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$$

with respect to the coordinate x^μ.

where R is the Ricci scalar of the 4-dimensional spacetime manifold and $g = \det g_{\mu\nu}$. The external source of matter is added as an action of type

$$S_m = \int d^4x\sqrt{-g}\mathcal{L}_m \ ,$$

where $\mathcal{L}_m$ is the Lagrangian of the particles content of the particular model under scrutiny. By varying the total action with respect to the metric, one finds the equations describing the evolution of the fields in terms of the matter-energy distribution, the famous Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{M_P^2}T_{\mu\nu} \ , \tag{1.4}$$

where $R_{\mu\nu}$ is the Ricci tensor of the spacetime manifold and $T_{\mu\nu}$ is the stress-energy tensor (EMT), which describes the distribution of matter and energy of the system, related to the matter Lagrangian by means of the following expression involving a functional derivative

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} \ . \tag{1.5}$$

The LHS of the Einstein equations is usually called Einstein tensor, which is indicated as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \ .$$

Notice that the Einstein-Hilbert action is invariant under a general diffeomorphism acting on the coordinates. Finally, since one can show that the Einstein tensor is covariantly conserved, in other words

$$D_\mu G^{\mu\nu} = 0 \ ,$$

from Einstein equations one gets immediately that also the stress-energy tensor is covariantly conserved

$$D_\mu T^{\mu\nu} = 0 \ , \tag{1.6}$$

which is the stress-energy tensor conservation in a covariant form. We finally remind that the Ricci curvature and scalar are defined as follows

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} \qquad R = g^{\mu\nu}R_{\mu\nu} \ , \tag{1.7}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor of the spacetime manifold, defined as

$$R^\mu{}_{\nu\rho\sigma} = 2(\partial_{[\mu}\Gamma^\rho_{\nu]\sigma} - \Gamma^\rho_{[\mu\tau}\Gamma^\tau_{\nu]\sigma}) \ . \tag{1.8}$$

Since R is a scalar, it is invariant under diffeomorphism: this means that is univocally associated to a manifold. Indeed, R can be used to measure the curvature of the manifold: if it is 0 the spacetime is flat, otherwise not.

1.2 Standard hot big bang model

The basis of modern cosmology is the cosmological principle, stating that every comoving observer sees the Universe at fixed time as homogeneous (invariant under translation) and isotropic (invariant under rotation), on a suitable large scale (experimentally $> 10^8$ Parsec). The main experimental proof that cosmological principle is plausible guiding principle to describe the Universe at large scale is the cosmic microwave background (CMB), which is a radiation appearing almost isotropic in the sky, remnant of the last scattering surface: after its formation, the Universe cooled until its mean energy was so low that the electrons were not able to leave the atomic nuclei, so the radiation produced in the whole Universe by Thomson scattering was not produced anymore and its relic still arrives to us today by means of CMB (the radiation was redshifted in microwave spectrum since the Universe expanded meanwhile). Its average temperature is $T_0 = 2.72548 \pm 0.00057$ K ([7]), with anisotropies of

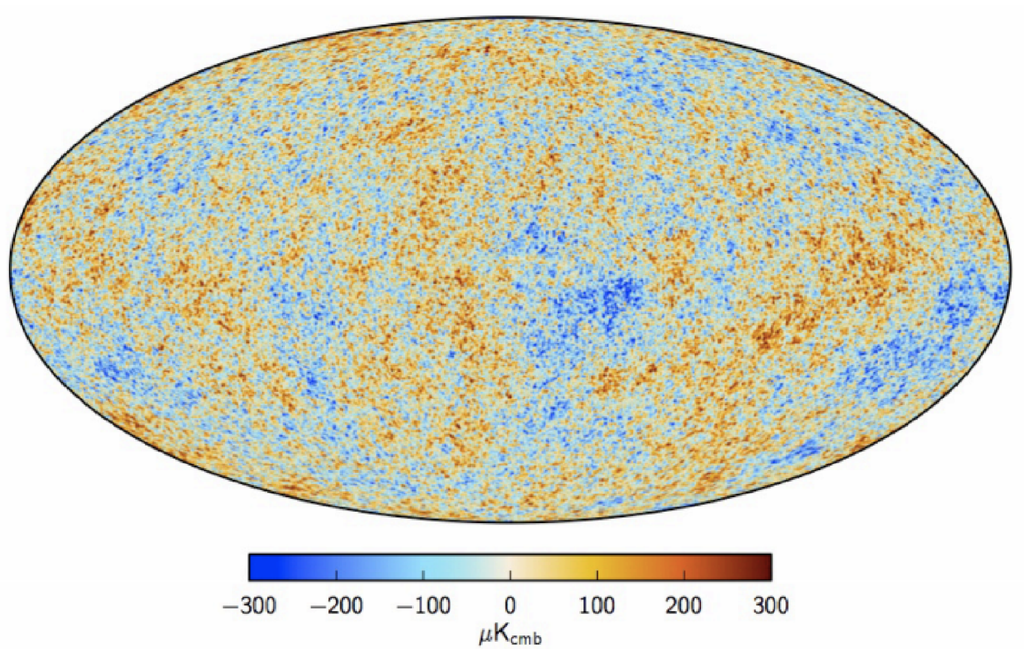

Figure 1.1: Full-sky map of CMB temperature anisotropies from the Planck satellite ([6]). The angular resolution is almost 5 arcminutes.

order $\frac{\delta T}{T_0} = 10^{-5}$, so its temperature is almost the same in the whole sky (see figure 1.1).
To relate the cosmological principle to GR, from a geometrical point of view this means to find the most general metric spatially translationally and rotationally invariant, but no conditions are strictly required on time coordinates transformations. Indeed it has been shown[3] that the maximum number of symmetries that a spacetime can have is 10 (in 3+1 dimensions). There are only 3 spacetimes having all these symmetries at the same time, called maximally symmetric spacetimes. One of them is obviously Minkowski spacetime $ds^2 = -dt^2 + \delta_{ij}dx^i dx^j$; the other two are de Sitter space, which in Lemaître coordinates reads

$$ds^2 = -dt^2 + e^{2Ht}\delta_{ij}dx^i dx^j \ ,$$

and anti-de Sitter space. However, in order to find the most general line element respecting the cosmological principle we have to relax the time translation invariance. As we have said in the introduction, it has been shown by Friedmann ([2]) that assuming the cosmological principle the metric describing the Universe is Friedmann-Lemaître-Robertson-Walker, which in spherical coordinates reads:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{1}{1-\kappa r^2}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)\right) \ , \tag{1.9}$$

where $a(t)$, called scale factor, is only function of time and $\kappa = 0, \pm 1$, which is a discrete parameter describing if the Universe is flat or not: if $\kappa = 0$ the Universe is flat, if $\kappa = 1$ it is closed and if $\kappa = -1$ it is open. The current measurements are in accordance with a small content of curvature $\kappa \simeq 0$ in energy density, so we will consider $\kappa = 0$ in the rest of the project.
The time dependent scale factor a describes how the relative distance between the two comoving observers changes over time and its dynamics is described by solving Einstein equations themselves, which in this case are called Friedmann equations. However, to solve the problem one has to consider an explicit expression for the stress-energy tensor: the simplest choice is to consider a Universe filled with a perfect fluid, whose EMT (in single fluid approximation) reads

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu} \ , \tag{1.10}$$

where u_μ is the 4-velocity vector field of the fluid

$$u^\mu = \frac{dx^\mu}{ds} \ , \quad u^\mu u_\mu = -1 \ ,$$

[3]See [3], eq. 5.44 in the case $n = 4$.

ρ can be identified with the energy density of the fluid and P is called isotropic pressure. This implies that in case of a static fluid, such that $u^\mu = (1, \vec{0})$,

$$T^0_0 = -\rho \ , \quad T^i_j = P\delta^i_j \ . \tag{1.11}$$

Substituting this stress-energy tensor and the metric 1.9 in 1.4, the Einstein equations reduce to

$$\begin{cases} H^2 = \frac{\rho}{3M_P^2} - \frac{k}{a^2} \\ \frac{\ddot{a}}{a} = -\frac{1}{6M_P^2}(\rho + 3P) \ , \end{cases} \tag{1.12}$$

where $H = \frac{\dot{a}}{a}$ is called Hubble parameter, which can be considered the expansion rate of the Universe. From the conservation of the stress-energy tensor one gets the so-called continuity equation

$$\dot{\rho} = -3H(\rho + P) \ . \tag{1.13}$$

The three coupled differential equations 1.12 and 1.13 we have got are called Friedmann equations. There are 3 relevant physical quantities (a, ρ, P), which are determined by these equations. However, the three equations are not linearly independent, since one can easily show that 1.13 can be obtained as a linear combination of 1.12: this is expected, since we have seen that conservation of stress-energy tensor comes directly from the structure of the Einstein tensor. This means that the system is not closed and we have to introduce another equation. The simplest choice is a barotropic fluid, which is described by an equation of state of type

$$P = w\rho \ ,$$

where w is a real parameter which descibes the typology of fluid ($w = 1/3$ radiation; $w = 0$ non-relativistic matter; $w = -1$ vacuum energy or equivalently the cosmological constant). This way, for $\kappa = 0$ the system of equation has solutions

$$\begin{cases} a(t) = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \\ \rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} \end{cases} , \tag{1.14}$$

where t_0, a_0 and ρ_0 are the initial conditions to be imposed. Notice that in a radiation or matter dominated epoch, one easily sees that a is growing with time. Depending on the epoch (i. e. on the value of a) the energy density ρ is dominated by a different typology of fluid. More in details, in the case of radiation ($w = 1/3$) one gets $\rho \propto a^{-4}$, which dominates for early times but decays later; in the case of incompressible matter ($w = 0$) one gets $\rho \propto a^{-3}$, which dominates in a intermediate era, but it still decays for late times; in the case of cosmological constant ($w = -1$) the solution 1.14 is not valid, however solving the system of equation in this particular case one gets ρ constant, dominating for late times.

In general instead of working with the metric 1.9 in the standard coordinates, in many situation it is more useful to put the scale factor a as an overall constant of the metric: this can be done redefining the time coordinate as

$$dt = a(\tau)d\tau \ .$$

τ is called conformal time and it is the time measured in comoving distances. On the contrary, we will refer to the time coordinate as in FLRW metric 1.9 as cosmic time. This way the line element (for a FLRW spacetime with $\kappa = 0$) becomes

$$ds^2 = a^2(\tau)(-d\tau^2 + \delta_{ij}dx^i dx^j) \ .$$

Notice that in this set of coordinates FLRW metric tensor is proportional to the Minkowski one. This means that FLRW is conformally flat. In the following we will need the relations between the first and the second derivative with respect to the cosmic time and with respect to the conformal time of a generic function f, indicated respectively with $\dot{f}$ and f'. Using the previous equation one gets

$$\begin{aligned}\dot{f}(t) &= \frac{f'(\tau)}{a(\tau)} \\ \ddot{f}(t) &= \frac{1}{a(\tau)}\left(\frac{f'(\tau)}{a(\tau)}\right)' = \frac{1}{a(\tau)}\left(\frac{f''(\tau)}{a(\tau)} - \frac{a'(\tau)f'(\tau)}{a^2(\tau)}\right) = \frac{f''(\tau)}{a^2(\tau)} - \frac{a'(\tau)}{a(\tau)}\frac{f'(\tau)}{a^2(\tau)}\ .\end{aligned} \tag{1.15}$$

To fix the notation we will call $H = \frac{\dot{a}}{a}$ the Hubble parameter in standard coordinates, while $\mathcal{H} = \frac{a'}{a}$ the Hubble parameter in conformal time. Using the first of 1.15, they are related by

$$H = \frac{\mathcal{H}}{a}\ . \tag{1.16}$$

Chapter 2

Cosmological perturbations

As it is clear from their expression 1.4, the Einstein equations are second-order non-linear differential equations which are very difficult to solve exactly. Exact solutions are well known (such as Schwarzschild solution for a spherical distribution or Kerr-Newman solution for a spherical rotating distribution), but they are very few. However, we know that FLRW metric 1.9 is the best analytic solution we have to describe a Universe compatible with the cosmological principle. This implies that the dynamics of the gravity field can be obtained perturbatively by splitting the metric tensor in background, respecting the cosmological principle so it is FLRW, and in a perturbation, an idea firstly introduced in [8]. In general, however, one can consider perturbations to a general order, even though calculations become rapidly challenging.
The splitting into a background is well-justified from an observational point of view: we have seen that the CMB appear to be very isotropic, with very small anisotropies. Moreover, the modern approach is to consider these anisotropies as the seeds of the generation of the cosmological structures, which formed through gravitational instability.
In this chapter we will see that the naive splitting $g_{\mu\nu} = g^0_{\mu\nu} + \delta g_{\mu\nu}$ is very subtle in general relativity, since $g_{\mu\nu}$ is deeply tied to the geometry of the space. This will introduce the possibility of having different solutions describing the same physics, related by a gauge transformation for the perturbations. After discussing how this gauge issue arises, we will perturb FLRW, we will derive how the first-order perturbations change under a gauge transformation and we will introduce some useful gauges for this work. Then, we will discuss the perturbed Einstein equations around FLRW and we will conclude by introducing some very important gauge invariant quantities.
The guidelines of this chapter are [8–10], with many adaptation.

2.1 The perturbative approach to Einstein equations and the gauge issue

Consider the Einstein equations 1.4. Suppose that we know a source term $T^0_{\mu\nu}$ for which the solution of the Einstein equations is well-known: an example is the stress-energy tensor 1.11, which admits FLRW as solution. The related Einstein equations are

$$G^0_{\mu\nu} = \frac{1}{M_P^2} T^0_{\mu\nu} \tag{2.1}$$

which is called background equation. The idea of the perturbative approach is that, adding in a consistent manner a perturbation term to the source term, $\delta T_{\mu\nu}$, we can split the Einstein equations the following way

$$G^0_{\mu\nu} + \delta G_{\mu\nu} = \frac{1}{M_P^2} T^0_{\mu\nu} + \frac{1}{M_P^2} \delta T_{\mu\nu} \ .$$

In order to have sense, we have to require that the perturbations are much smaller with respect to the background, i. e. given a generic quantity split into a background and a perturbation as

$f = f_0 + \delta f + \dots$ one has

$$\frac{\delta f}{f_0} \ll 1 \ .$$

Any equation needs to be satisfied order by order, and using 2.1

$$\delta G_{\mu\nu} = \frac{1}{M_P^2}\delta T_{\mu\nu} \ . \tag{2.2}$$

This procedure underlines that the metric can be split as $g_{\mu\nu} = g^0_{\mu\nu} + \delta g_{\mu\nu}$, where $g^0_{\mu\nu}$ is FLRW 1.9, and $\delta g_{\mu\nu}$ are the perturbations. However, this splitting is more subtle than it appears. In flat space the perturbative approach does not give any problem, but in GR we know that from the metric one can calculate the curvature of the space: this implies that perturbing the metric, the spacetime changes and the definition of the perturbation

$$\delta g_{\mu\nu} = g_{\mu\nu} - g^0_{\mu\nu}$$

loses meaning. This is essentially due to the fact that, although GR is invariant under diffeomorphisms, in other words manifestly coordinate choice independent, splitting the variable as before is not a covariant procedure.

In order to treat the situation properly ([11]), we assume the existence of family of solutions of the field equations parametrized by $\lambda \in \mathbb{R}$; this way we consider a family of metric tensors denoted by g^λ depending smoothly on λ and satisfying the Einstein equations

$$G^\lambda_{\mu\nu} = \frac{1}{M_P^2} T^\lambda_{\mu\nu} \ .$$

We also postulate that when $\lambda = 0$ we get the background solution. Mathematically, this means to have the space $\mathcal{F} = \mathcal{M} \times \mathbb{R}$, with a foliation $\mathcal{M}_\lambda$ parametrized by λ and the tensor fields $g^\lambda_{\mu\nu}$ and $T^\lambda_{\mu\nu}$ living in the space $\mathcal{F}$ (but in the same way, if one included other fields, for example the scalar inflaton, one would have ϕ^λ). The space with $\lambda = 0$ is generally called background space, while an element of the foliation is called physical space.
The naive definition of the perturbation δT of a tensor field T is

$$\delta T_\lambda = T_\lambda - T_0 \ .$$

However, as we have seen, this expression is not well-defined since to be subtracted T_λ and T_0 must be evaluated at the same point in spacetime; but they are defined in two different manifolds, $\mathcal{M}_\lambda$ and $\mathcal{M}_0$, so this is not the case. This implies that the definition above must be modified. In this direction, we define a 1-parameter family of diffeomorphisms $\varphi_\lambda : \mathcal{F} \longrightarrow \mathcal{F}$ such that

$$\varphi_\lambda|_{\mathcal{M}_0} : \mathcal{M}_0 \longrightarrow \mathcal{M}_\lambda \quad \text{and} \quad \varphi_0 = \text{id} \ .$$

However, this choice is not unique, but there are infinite equivalent choices: for future convenience, we indicate another 1-parameter family of diffeomorphisms with ψ_λ. The difference between these two maps stands in the way they connect points: specifically, if $p, q \in \mathcal{M}_0$ with $p \neq q$ and $r \in \mathcal{M}_\lambda$, we have two distinct points of the background mapped in same one of the "physical" manifold (this definition does not imply the use of any local charts):

$$\varphi_\lambda(p) = \psi_\lambda(q) = r \ . \tag{2.3}$$

In general a diffeomorphism induces the following differential map between cotangent spaces, called pull-back[1]

$$\varphi^*_\lambda|_{\varphi_\lambda(p)} : T^*_{\varphi_\lambda(p)}\mathcal{M}_\lambda \longrightarrow T^*_p\mathcal{M}_0 \ .$$

[1]We remind that mathematically the pull-back is defined in the following way: given two manifolds X and Y, a diffeomorphism f from Y to a tensor space and $F : X \to Y$ the pullback is the function F^* which associates $f \to F^*(f) = f \circ F$.

After having defined this object, we can define the perturbations of a generic tensor field T_λ in a consistent manner in general relativity:

$$\delta T_\lambda(p) := \left(\varphi_\lambda^* T\right)(p) - T_0(p) \ , \tag{2.4}$$

where $p \in \mathcal{M}_0$. Now, if we Taylor expand in λ the pulled-back tensor field $(\varphi_\lambda^* T)$, we get

$$\left(\varphi_\lambda^* T\right)(p) = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!}\left[\frac{d^k}{d\lambda^k}\left(\varphi_\lambda^* T\right)(p)\right]_{\lambda=0} = T_0(p) + \underbrace{\sum_{k=1}^{+\infty} \frac{\lambda^k}{k!}\left[\frac{d^k}{d\lambda^k}\left(\varphi_\lambda^* T\right)(p)\right]_{\lambda=0}}_{\lambda\delta T_\lambda(p)} . \tag{2.5}$$

Notice that at first-order the term reconstructs the Lie derivative of the tensor T_0 (see appendix A, eq. A.3):

$$\frac{d}{d\lambda}\left[\left(\varphi_\lambda^* T\right)(p)\right]_{\lambda=0} = \mathcal{L}_\xi T_0(p) \ .$$

Considering all the orders, one can write the following formal expansion

$$\left(\varphi_\lambda^* T\right)(p) = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!}\mathcal{L}_\xi^k T(p) = e^{\lambda \mathcal{L}_\xi} T(p) \ .$$

The proof of this fact can be performed by induction, but since we are interested to stop at linear order we do not give any detail[2]. This construction is called Lie dragging. The 4-vector field ξ^μ in this Lie derivative must be intended as the vector field generating the transformation; in other words, fixing a coordinate system x^μ a coordinate system on $\mathcal{M}_0$, the solution to the system of ODEs

$$\frac{dx^\mu}{d\lambda} = \xi^\mu$$

is given by a series of curves $x^\mu(\lambda)$, which for local unicity of the solution of the ODEs will never meet: each point is associated to only one integral line; the point p will always lie on one of these curves and it is not restrictive to put p at $\lambda = 0$. The coordinate of the point q (in the same curve) will be $x_q^\mu(\lambda)$. This implies that using the expansion we have found, applying it to to the tensor which is simply the coordinate functon x^μ, one has at first-order

$$x_q^\mu = x_p^\mu + \lambda\xi^\mu + \mathcal{O}\left(\lambda^2\right) \ .$$

Notice that this way choosing a ξ^μ is like choosing a 1-parameter diffeomorphism.
Now, remembering that there are infinite equivalent choices of the 1-parameter family of diffeomorphisms, we consider another one which is ψ_λ, with the attempt to connect $\mathcal{M}_0$ and $\mathcal{M}_\lambda$; using 2.3, one obtains a map moving the points of the background $\mathcal{M}_0 \longrightarrow \mathcal{M}_0$:

$$q = \phi_\lambda^{-1}\left(\psi_\lambda(p)\right) := \Theta_\lambda(p) \ .$$

The map Θ_λ is called gauge transformation. More specifically this approach to gauge transformation moving points is called active approach, opposed to another approach, we are not going to discuss, called passive approach, moving the coordinates. These two approaches are equivalent.
At this point we can show how tensor perturbation δT transforms under the gauge transformation Θ_λ, at first-order: considering the pulled-back tensor $\left(\psi_\lambda^* T\right)(p)$ and inserting the identity map $\left(\varphi_\lambda^*\right)^{-1}\varphi_\lambda^*$ one has simply

$$\left(\psi_\lambda^* T\right)(p) = \left(\psi_\lambda^*\left(\varphi_\lambda^*\right)^{-1}\varphi_\lambda^* T\right)(p) = \Theta_\lambda^*\left(\varphi_\lambda^* T\right)(p) \ .$$

At this point we are interested in expressions at first-order in λ, so we perform an expansion. The result at first-order is

$$\begin{aligned}\Theta_\lambda^*\left(\varphi_\lambda^* T\right)(p) = \Theta_\lambda^*\left(T_0 + \lambda\delta T\right)(p) &= \\ &= T_0(p) + \lambda\delta T(p) + \lambda\left(\mathcal{L}_\xi\left(T_0(p) + \lambda\delta T(p)\right)\right) + \mathcal{O}\left(\lambda^2\right) = \\ &= T_0(p) + \lambda\left(\delta T(p) + \mathcal{L}_\xi\left(T_0(p)\right)\right) + \mathcal{O}\left(\lambda^2\right) \ ,\end{aligned}$$

[2]The interested reader can look at Lemma 1 in [11].

where in the first line we have used $\varphi_\lambda^* T$ from the definition 2.4, while in the second we have used the expansion 2.5. This implies that the trasformation property of the perturbation δT under gauge transformation at first-order is finally:

$$\tilde{\delta T}(p) = \delta T(p) + \mathcal{L}_\xi \left(T_0(p) \right) \ . \tag{2.6}$$

Practically, in dealing with gauge transformations at first-order one simply starts from a generic 4-vector ξ^μ depending on the spacetime point, applying this rule.

2.2 Perturbed FLRW metric

Consider FLRW metric in conformal time, in the case $\kappa = 0$ (which is the interesting case to describe our Universe, since we have seen that today the curvature is negligible):

$$ds^2 = a^2(\tau)(-d\tau^2 + \delta_{ij} dx^i dx^j) \ .$$

We want to write the most general form of this metric considering the perturbations to all orders. A standard choice is

$$g_{\mu\nu} = a^2(\tau) \begin{pmatrix} -1 + 2\sum_{r=1}^{+\infty} \frac{\Phi^{(r)}}{r!} & \sum_{r=1}^{+\infty} \frac{\omega_j{}^{(r)}}{r!} \\ \sum_{r=1}^{+\infty} \frac{\omega_i^{(r)}}{r!} & \left(1 - 2\sum_{r=1}^{+\infty} \frac{\Psi^{(r)}}{r!}\right)\delta_{ij} + \sum_{r=1}^{+\infty} \frac{\chi_{ij}^{(r)}}{r!} \end{pmatrix} ,$$

where (r) indicates the order of the perturbation. Here, $\Phi^{(r)}$ and $\Psi^{(r)}$ are scalar perturbations, but $\omega_i^{(r)}$ and $\chi_{ij}^{(r)}$ are not really vector and tensor perturbations, since they must be scalar-vector-tensor (SVT) decomposed[3] as

$$\begin{aligned} \omega_i{}^{(r)} &= \omega_{i\perp}^{(r)} + \partial_i \omega_\parallel^{(r)} \\ \chi_{ij}^{(r)} &= \hat{D}_{ij} \chi_\parallel^{(r)} + \partial_i \chi_j^{\perp(r)} + \partial_j \chi_i^{\perp(r)} + \chi_{ijT}^{(r)} \ , \end{aligned}$$

where $\hat{D}_{ij}$ is the traceless operator $\hat{D}_{ij} = \partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial_k\partial_k$. This way, at each order, we have two scalars more, $\omega_\parallel$ and $\chi_\parallel$, two vectors, $\omega^{i\perp}$ and $\chi^{i\perp}$, which are divergence free ($\partial_i \omega^{\perp i} = 0$ and $\partial_i \chi^{\perp i} = 0$) and a tensor χ_{ij}^T, which is transverse and traceless ($\partial_j \chi_{ij}^T = 0$ and $\chi_i^{iT} = 0$). This implies that the ij components have been decomposed into a trace part $\left(1 - 2\sum_{r=1}^{+\infty} \frac{\Psi^{(r)}}{r!}\right)\delta_{ij}$ and a traceless part (the one containing the χ's). However, there is a little remark about this decomposition in trace and traceless part: in literature it can happen that the operator $\hat{D}_{ij}$ is replaced by $\partial_i\partial_j$. This simply means that the term containing the χ's is no more traceless; however, the degrees of freedom are still the same and the SVT decomposition is still valid. The only difference is that in the convention with $\hat{D}_{ij}$ one simply reabsorbs the trace part of $\hat{D}_{ij}\chi_\parallel^{(r)}$ into $\Psi^{(r)}$.
This way we obtain:

$$g_{\mu\nu} = a^2(\tau) \begin{pmatrix} -\left(1 + 2\sum_{r=1}^{+\infty} \frac{\Phi^{(r)}}{r!}\right) & \sum_{r=1}^{+\infty} \frac{\omega_{j\perp}^{(r)} + \partial_j \omega_\parallel^{(r)}}{r!} \\ \sum_{r=1}^{+\infty} \frac{\omega_{i\perp}^{(r)} + \partial_i \omega_\parallel^{(r)}}{r!} & \left(1 - 2\sum_{r=1}^{+\infty} \frac{\Psi^{(r)}}{r!}\right)\delta_{ij} + \sum_{r=1}^{+\infty} \frac{\hat{D}_{ij}\chi_\parallel^{(r)} + \partial_i \chi_j^{\perp(r)} + \partial_j \chi_i^{\perp(r)} + \chi_{ijT}^{(r)}}{r!} \end{pmatrix} . \tag{2.7}$$

From now on, since we will deal with first-order perturbations only, the first-order perturbations will be indicated without the index $^{(1)}$ for shortness.

[3]We remind that the scalar-vector-tensor decomposition is valid because of Helmholtz's theorem, which states that under the differentiability condition a general vector field $\vec{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and vanishing at infinity can be split into a conservative part and a solenoidal part, as

$$\vec{F} = \vec{\nabla}\phi + \vec{\nabla} \times \vec{A} \ ,$$

where $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, which are unique (modulo an integration constant). This can be also generalised to a rank 2 tensor as shown in the main text. It is worth to underline that such a decomposition has sense in $\mathbb{R}^3$ only if the field $\vec{F}$ vanishes at infinity: indeed, if this is not the case, the theorem does not strictly hold, since the decomposition could be not unique.

We will also need to perturb the RHS of the Einstein equations, i. e. the stress-energy tensor, which is 1.10. In the perturbed case we can consider a more general case, adding an anisotropic stress

$$T_{\mu\nu} = u_\mu u_\nu(\rho + P) + P g_{\mu\nu} + \pi_{\mu\nu} \ , \tag{2.8}$$

where in general $\pi_{\mu\nu}$ can be decomposed in scalar, vector and tensor modes as

$$\pi_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \hat{D}_{ij}\pi^{\|} + \partial_i \pi_j^{\perp} + \partial_j \pi_i^{\perp} + \pi_{ij}^T \ . \end{pmatrix} \ ,$$

The perturbed energy density and pressure are

$$\rho = \rho_0(\tau) + \sum_{r=1}^{+\infty} \frac{\delta\rho^{(r)}(\tau, x)}{r!} \ ,$$

$$P = P_0(\tau) + \sum_{r=1}^{+\infty} \frac{\delta P^{(r)}(\tau, x)}{r!} \ ,$$

while the perturbed pressure can be split into adiabatic and non-adiabatic:

$$\delta P = \left.\frac{\partial P}{\partial \rho}\right|_S \delta\rho + \left.\frac{\partial P}{\partial S}\right|_\rho \delta P = c_s^2 \delta\rho + \delta P_{NA} \ , \tag{2.9}$$

where $c_s^2 := \left.\frac{\partial P}{\partial \rho}\right|_S$ is the speed of sound in the fluid and δP_{NA} is the non-adiabatic component of the pressure. Finally the perturbed 4-velocity is

$$u^\mu = \frac{1}{a(\tau)} \left(\delta_0^\mu + \sum_{r=1}^{+\infty} \frac{v^{(r)\mu}(\tau, x)}{r!} \right) \ , \tag{2.10}$$

since the unperturbed 4-velocity for a fixed observer is $u^\mu = \frac{1}{a(\tau)}(1, \vec{0}) = \frac{\delta_0^\mu}{a(\tau)}$ (normalized so that in FLRW in conformal time $u^\mu u_\mu = -1$). Also in this case the first-order perturbation can be decomposed in scalar and vector parts as

$$v^\mu = \begin{pmatrix} v^0 \\ v_\perp^i + \partial^i v^{\|} \end{pmatrix} \ , \tag{2.11}$$

with $v^{\|}$ the scalar potential of the velocity.
The above definitions imply that the first-order stress-energy tensor 2.8 reads

$$T^\mu_\nu = \begin{pmatrix} -\rho_0 - \delta\rho & (\rho_0 + P_0)(v_j + \omega_j) \\ -(\rho_0 + P_0)v^i & \delta^i_j(P_0 + \delta P) + \pi^i_j \end{pmatrix} \ .$$

We have obviously to SVT decompose v_i, ω_i and $\pi_{\mu\nu}$ the way we have seen, obtaining finally

$$T^\mu_\nu = \begin{pmatrix} -\rho_0 - \delta\rho & (\rho_0 + P_0)(\partial_j v^{\|} + \partial_j \omega^{\|} + v_j^{\perp} + \omega_j^{\perp}) \\ -(\rho_0 + P_0)(\partial^i v^{\|} + v_\perp^i) & \delta^i_j(P_0 + \delta P) + \hat{D}^i_j \pi^{\|} + \partial^i \pi_j^{\perp} + \partial_j \pi^{\perp i} + \pi_j^{Ti} \end{pmatrix} \ . \tag{2.12}$$

2.3 Gauge transformations

As we have seen in section 2.1, a generic first-order perturbation of a tensor field T transforms under an infinitesimal diffeomorphism ξ^μ, i. e. under a gauge transformation, the following way[4]:

$$\tilde{\delta T}(x) = \delta T(x) + \mathcal{L}_\xi T_0(x) \ .$$

[4]We denote the gauge transformed quantity with a tilde and not with a prime to avoid confusion with the derivative with respect to the conformal time.

Firstly, notice that since ξ is totally arbitrary, the transformation is defined with the minus sign

$$\tilde{\delta T}(x) = \delta T(x) - \mathcal{L}_\xi T_0(x) \ ,$$

being sufficient to send $\xi \to -\xi$. For future convenience we will choose this convention. T is a generic tensor field which can carry an arbitrary number of upper and lower indices. The explicit expressions of $\mathcal{L}_\xi T$ we are going to use are derived in the appendix A. Since we want to have all the scalar, vector and tensor modes separated, we SVT decompose ξ^μ as

$$\xi^\mu = \begin{pmatrix} \alpha \\ d^i + \partial^i \beta \end{pmatrix} \ ,$$

which are two scalars (α, β) and a vector (d^i).

Let us start with the metric perturbations. The explicit expression of the Lie derivative gives

$$\tilde{\delta g}_{\mu\nu} = \delta g_{\mu\nu} - \mathcal{L}_\xi g^0_{\mu\nu} = \delta g_{\mu\nu} - \partial_\lambda g^0_{\mu\nu}\xi^\lambda - \partial_\mu \xi^\lambda g^0_{\lambda\nu} - \partial_\nu \xi^\lambda g^0_{\mu\lambda} \ .$$

We have to consider independently the 00, $0i$ and ij components, remembering that $g^0_{00} = -a^2(\tau)$, $g^0_{0i} = 0$ and $g^0_{ij} = a^2(\tau)\delta_{ij}$ and the expressions of the perturbations at first-order seen at the previous section.
The 00 component gives (using the fact that the $0i$ components of the unperturbed metric tensor are null, as well as its derivatives with respect to space components):

$$\tilde{\delta g}_{00} = \delta g_{00} - \partial_\tau g^0_{00}\xi^0 - 2\partial_\tau \xi^0 g^0_{00} \ .$$

Using the explicit expressions, one finds

$$-2a^2\tilde{\Phi} = -2a^2\Phi + \partial_\tau(a^2)\alpha + 2(\partial_\tau \alpha)a^2 = -2a^2\Phi + 2aa'\alpha + 2\alpha' a^2 \ ,$$

giving finally

$$\tilde{\Phi} = \Phi - \frac{a'}{a}\alpha - \alpha' = \Phi - \mathcal{H}\alpha - \alpha' \ .$$

The $0i$ components give, using a similar procedure

$$\tilde{\delta g}_{0i} = \delta g_{00} - \partial_i \xi^0 g^0_{00} - \partial_\tau \xi^j g^0_{ij} \ .$$

Using the explicit expressions, one finds

$$\tilde{\omega}_i = \omega_i + \partial_i \alpha - \partial_i \beta' - d'_i \ .$$

Decomposing ω_i in its scalar and vector components as $\omega_i = \partial_i \omega^{\parallel} + \omega_i^{\perp}$, one gets the transformation rules

$$\begin{cases} \tilde{\omega}^{\parallel} = \omega^{\parallel} + \alpha - \beta' \\ \tilde{\omega}^{i\perp} = \omega^{i\perp} - (d^i)' \end{cases} .$$

The ij components give, in the same way

$$\tilde{\delta g}_{ij} = \delta g_{ij} - \partial_\lambda g^0_{ij}\xi^\lambda - \partial_i \xi^\lambda g^0_{\lambda j} - \partial_j \xi^\lambda g^0_{i\lambda} = \delta g_{ij} - \partial_\tau g^0_{ij}\xi^0 - \partial_i \xi^k g^0_{kj} - \partial_j \xi^k g^0_{ik} \ ,$$

which opening the components reads

$$-2\tilde{\Psi}\delta_{ij} + \tilde{\chi}_{ij} = -2\Psi\delta_{ij} + \chi_{ij} - 2\frac{a'}{a}\delta_{ij}\alpha - \frac{2}{3}\delta_{ij}\nabla^2\beta - 2\hat{D}_{ij}\beta - (\partial_j d_i + \partial_j d_i) \ .$$

From the terms proportional to δ_{ij} we get

$$-2\tilde{\Psi} = -2\Psi - 2\frac{a'}{a}\alpha - \frac{2}{3}\nabla^2\beta \longrightarrow \tilde{\Psi} = \Psi + \mathcal{H}\alpha + \frac{1}{3}\nabla^2\beta \ ,$$

while the other terms give

$$\tilde{\chi}_{ij} = \chi_{ij} - 2\hat{D}_{ij}\beta - \partial_j d_i - \partial_j d_i \ ,$$

which we want to decompose in its SVT components as $\chi_{ij} = \hat{D}_{ij}\chi^{\parallel} + \partial_i\chi_j^{\perp} + \partial_j\chi_i^{\perp} + \chi_{ij}^{T}$, with the results

$$\begin{cases} \tilde{\chi}^{\parallel} = \chi^{\parallel} - 2\beta \\ \tilde{\chi}^{i\perp} = \chi^{i\perp} - d^i \\ \tilde{\chi}_{ij}^{T} = \chi_{ij}^{T} \end{cases} .$$

Notice that since ξ^μ does not contain any tensor perturbation the tensor part of the metric is gauge invariant. However, this fact remains true only at first-order.

Now we can switch to the stress-energy tensor perturbations. Since $\delta\rho$ is a scalar, its transformation is trivial:

$$\tilde{\delta\rho} = \delta\rho - \mathcal{L}_\xi\rho_0 = \delta\rho - \xi^\lambda\partial_\lambda\rho_0 = \delta\rho - \alpha\partial_\tau\rho_0 - \xi^i\partial_i\rho_0 = \delta\rho - \alpha\rho_0' \ . \tag{2.13}$$

Analogously, since δP is a scalar, one has $\tilde{\delta P} = \delta P - \alpha P_0'$.
The 4-velocity u^μ is a 4-vector, so it is less trivial. The transformation rule gives $\widetilde{\delta u^\mu} = \delta u^\mu - \mathcal{L}_\xi u_0^\mu$, which using the definition of the perturbations of u^μ becomes

$$\frac{1}{a}\tilde{v}^\mu = \frac{1}{a}v^\mu - \partial_\lambda u_0^\mu\xi^\lambda + \partial_\lambda\xi^\mu u_0^\lambda = \frac{1}{a}v^\mu + \frac{a'}{a^2}\alpha\delta_0^\mu + \frac{1}{a}\partial_0\xi^\mu$$

and finally $\tilde{v}^\mu = v^\mu + \mathcal{H}\alpha\delta_0^\mu + \partial_\tau\xi^\mu$. The $\mu = 0$ component gives

$$\tilde{v}^0 = v^0 + \mathcal{H}\alpha - \alpha' \ ,$$

while choosing $\mu = i$

$$\tilde{v}^i = v^i - \partial^i\beta' + \left(d^i\right)' \ .$$

We can now decompose v^i in scalar and vector components obtaining

$$\tilde{v}^i = \tilde{v}^{i\perp} + \partial^i\tilde{v}^{\parallel} = v^{i\perp} + \partial^i v^{\parallel} - \partial^i\beta' + \left(d^i\right)' \ ,$$

from which we get the following transormation rules

$$\begin{cases} \tilde{v}^{\parallel} = v^{\parallel} + \beta' \\ \tilde{v}^{i\perp} = v^{i\perp} + (d^i)' \end{cases} .$$

Finally, one can find also how the $\pi^{\mu\nu}$ components transform, but since we will not use this we skip.

2.4 Gauge invariant vs. scalar quantities

It is fundamental to make clear the distinction between gauge invariant and scalar quantities. From a formal point of view, given an infinitesimal coordinate transformation $x'^\mu = x^\mu + \epsilon^\mu(x)$, a quantity f is scalar if

$$f'(x') = f(x) \ .$$

Let us see some implications of these definition. In particular we would like to see how a general quantity transforms under the diffeomorphism $x'^\mu = x^\mu + \epsilon^\mu(x)$. Taylor expanding one has

$$\begin{aligned} f'(x) - f(x) &= f'(x' - \epsilon) - f(x) = f'(x') - \epsilon^\mu\partial_\mu f'(x') + \ldots - f(x) = \\ &= f(x) - \epsilon^\mu\partial_\mu f(x) + \ldots - f(x) = -\mathcal{L}_\epsilon f(x) + \ldots \ . \end{aligned} \tag{2.14}$$

This procedure can be repeated for every quantity changing non-trivially under a diffeomorphism. For example, in the case of a 2-covariant tensor one finds in the same way, using the proper transformation rule,

$$\begin{aligned}
T'_{\mu\nu}(x) - T_{\mu\nu}(x) &= T'_{\mu\nu}(x' - \epsilon) - T_{\mu\nu}(x) = T'_{\mu\nu}(x') - \epsilon^{\alpha}\partial_{\alpha}T'_{\mu\nu}(x') - T_{\mu\nu}(x) = \\
&= \frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}T_{\rho\sigma}(x) - \epsilon^{\alpha}\partial_{\alpha}\left[\frac{\partial x^{\rho}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}T_{\rho\sigma}(x)\right] - T_{\mu\nu}(x) = \\
&= T_{\mu\nu}(x) - \partial_{\mu}\epsilon^{\rho}T_{\rho\nu}(x) - \partial_{\nu}\epsilon^{\rho}T_{\mu\rho}(x) - \epsilon^{\rho}\partial_{\rho}T_{\mu\nu}(x) + \ldots - T_{\mu\nu}(x) = \\
&= -\mathcal{L}_{\epsilon}T_{\mu\nu}(x) + \ldots \ ,
\end{aligned}$$

having used $\frac{\partial x^{\rho}}{\partial x'^{\mu}} = \delta^{\rho}_{\mu} - \frac{\partial \epsilon^{\rho}}{\partial x^{\mu}}$, keeping only first-order terms in ϵ. This is expected, since Lie derivative measures how a quantity (scalar, vector or tensor) changes along the integral curves of a diffeomorphism, which is exactly what we explain in appendix A.
Notice that these results can be generalized to a generic tensor field T, which under a diffeomorphism is mapped to

$$T'(x) = T(x) - \mathcal{L}_{\epsilon}T(x) \ . \tag{2.15}$$

This equation reduces to 2.6 if one expands perturbatively $T = T^0 + \lambda\delta T + \ldots$ and $\epsilon = -\lambda\xi + \ldots$, taking the first-order terms.

The idea of gauge invariance is different. In particular, a quantity is gauge invariant if

$$f'(x) = f(x) \ .$$

Differently to the previous case, a gauge invariant quantity is related to a splitting of a quantity in a background and a perturbation, as we have seen. This way, suppose to split f into a background and a first-order term

$$f(x) = f^0(t) + f^{(1)}(x) + \ldots \ .$$

This implies that for a scalar

$$f'(x') = f(x) \longrightarrow f'^0(t') + f'^{(1)}(x') + \ldots = f_0(t) + f^{(1)}(x) + \ldots \ .$$

Expanding in Taylor series the LHS around x one has

$$\begin{aligned}
f'^0(t + \epsilon^0) + f'^{(1)}(x + \epsilon) + \ldots &= f^0(t) + f^{(1)}(x) + \ldots \\
f'^0(t) + \epsilon^0\partial_t f'^0(t) + f'^{(1)}(x) + \ldots &= f^0(t) + f^{(1)}(x) + \ldots \ .
\end{aligned}$$

This implies that to have a scalar quantity featured by first-order gauge invariance we need

$$f^0 = 0 \ .$$

This explains, for example, why perturbations of scalar quantities are usually affected by gauge transformations: for example, the energy density perturbation $\delta\rho$ transforms as 2.13, since the background $\rho_0 \neq 0$.

2.5 Gauge fixing and remarkable gauges

In this section we discuss the gauge fixing for the perturbations at first-order. In the previous section we have seen that the ξ^{μ} 4-vector of the infinitesimal diffeomorphism can be split in 2 scalar and 1 vector: this means that to completely fix a gauge we have to fix fix two scalars and a 3-vector. The result is a system of three equations having α, β and d_i as unknown.
However, this is true in case we assume the perturbations and ξ vanishing at infinity. If they do not, we have seen that the Helmholtz theorem is not valid, so it is possible that the SVT decomposition is not unique, or that the gauge is not totally fixed, that is to have a residual gauge freedom.
In literature, many gauges have been used but here we focus on particular cases which are relevant for future developments.

Poisson and Newtonian gauge

In Poisson gauge one sets to zero the following perturbations.

$$\omega_{\|} = 0 \ , \quad \chi_{\|} = 0 \ , \quad \chi^i_{\perp} = 0 \ .$$

In the case in which one is interested only in scalar perturbations, this gauge is called Newtonian gauge. This means that tensor and vector perturbations are completely neglected, so to gauge-fix the perturbations one simply has to choose 2 scalars, which are

$$\omega_{\|} = 0 \ , \quad \chi_{\|} = 0 \ .$$

FLRW metric in conformal time reads

$$g_{\mu\nu} = a^2(\tau) \begin{pmatrix} -(1+2\Phi) & 0 \\ 0 & (1-2\Psi)\,\delta_{ij} \end{pmatrix} \tag{2.16}$$

and the gauge is generally called conformal Newtonian ([12]). As underlined in [13], the conformal Newtonian gauge is a restricted gauge since the metric is applicable only for the scalar mode of the metric perturbations; the vector and the tensor degrees of freedom are eliminated from the beginning.

Synchronous gauge

Synchronous gauge is a class of gauges which put the 00 perturbation equal to zero, i. e. $\Phi = 0$. This implies directly its name, since the line element for a static observer ($dx^i = 0$) reads

$$ds^2 = a^2(\tau) d\tau^2 \ ,$$

which is independent of the perturbation. This means also that in this gauge the proper time of the observer coincides with unperturbed one. At this point one has to choose another scalar and a vector. The most useful choice is to kill the $0i$ elements of the metric, by choosing $\omega^{\|} = 0$ and $\omega^{i\perp} = 0$. This way, the line element reads

$$ds^2 = a^2(\tau)(d\tau^2 + (\delta_{ij} + h_{ij}) dx^i dx^j) \ .$$

where h_{ij} is the first-order perturbation of the ij elements and it contains both scalars, vectors and tensors. It can be decomposed in the usual way as

$$h_{ij} = -2\Psi\delta_{ij} + \hat{D}_{ij}\chi_{\|} + \partial_i\chi_j + \partial_j\chi_i + \chi^T_{ij} \ .$$

Comoving gauge

Comoving gauge is a class of gauges in which the peculiar velocities are set to 0. This implies that

$$v_{\|} = 0 \ , \quad v^i_{\perp} = 0 \ .$$

There is still a scalar degree of freedom to be fixed: one usually chooses $\omega_{\|} = 0$ or $\chi_{\|} = 0$.

Uniform density gauge

Uniform density gauge is a class of gauges in which the perturbation of the scalar energy density is null:

$$\delta\rho = 0 \ .$$

One is still free to set another scalar and a vector to 0.

Spatially flat gauge

In appendix B we define the spatial curvature $R^{(3)}$, finding that $R^{(3)} = \frac{4}{a^2}\nabla^2\left(\Psi + \frac{1}{6}\nabla^2\chi^{\|}\right)$ (for $\kappa = 0$). This implies that choosing

$$\Psi = 0 \ , \quad \chi^{\|} = 0 \ ,$$

one gets a spatially flat spacetime $R^{(3)} = 0$. The remaining degree of freedom to fix is a 3-vector and one usually chooses $\chi^{\perp}_i = 0$.

2.6 Perturbed Einstein equations around FLRW at first-order

In this section we want to give the evolution equations for first-order perturbations of FLRW metric, arising from the perturbed Einstein equations. As it is clear from the GR formulas, explicit calculations are very cumbersome and full of indices contractions so we limit ourselves to outline the procedure and give the references for the full detailed calculations.
To perturb the Einstein equations 1.4 one has to follow the procedure presented in the first section of the chapter, in particular one has to perturb the Einstein tensor $G_{\mu\nu}$. The passages are the following.

1. Consider FLRW perturbed metric 2.7, which in the case of only scalar perturbation (for simplicity) reads
$$g_{\mu\nu} = a^2(\tau)\begin{pmatrix} -1-2\Phi & \partial_j\omega_{\|} \\ \partial_i\omega_{\|} & (1-2\Psi)\delta_{ij} + \hat{D}_{ij}\chi^{\|} \end{pmatrix} ,$$
and invert it at first-order by imposing $g^{\mu\alpha}g_{\alpha\nu} = \delta^\mu_\nu$. The result is
$$g_{\mu\nu} = a^2(\tau)\begin{pmatrix} -1+2\Phi & \partial_j\omega_{\|} \\ \partial_i\omega_{\|} & (1-2\Psi)\delta_{ij} - \hat{D}_{ij}\chi^{\|} \end{pmatrix} . \tag{2.17}$$
From this one reads the background and the perturbation of the metric and of the inverse metric (at first-order, only for scalars)
$$g^{(0)}_{\mu\nu} = a^2(\tau)\begin{pmatrix} -1 & 0 \\ 0 & \delta_{ij} \end{pmatrix} \qquad \delta g_{\mu\nu} = a^2(\tau)\begin{pmatrix} -2\Phi & \partial_j\omega_{\|} \\ \partial_i\omega_{\|} & -2\Psi\delta_{ij} + \hat{D}_{ij}\chi^{\|} \end{pmatrix} \tag{2.18}$$
$$g^{\mu\nu}_{(0)} = \frac{1}{a^2(\tau)}\begin{pmatrix} -1 & 0 \\ 0 & \delta_{ij} \end{pmatrix} \qquad \delta g^{\mu\nu} = \frac{1}{a^2(\tau)}\begin{pmatrix} 2\Phi & \partial_j\omega_{\|} \\ \partial_i\omega_{\|} & 2\Psi\delta_{ij} - \hat{D}_{ij}\chi^{\|} \end{pmatrix} . \tag{2.19}$$

2. Perturb the Christoffel's symbols at first-order, as
$$\delta\Gamma^\mu_{\nu\rho} = \frac{1}{2}\delta g^{\mu\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}) + \frac{1}{2}g^{\mu\sigma}(\partial_\nu \delta g_{\rho\sigma} + \partial_\rho \delta g_{\sigma\nu} - \partial_\sigma \delta g_{\nu\rho})$$
and using the previous equations compute all the perturbed Christoffel's symbols.

3. Perturb the Ricci curvature, obtaining
$$\delta R_{\mu\nu} = \partial_\alpha\delta\Gamma^\alpha_{\mu\nu} - \partial_\mu\delta\Gamma^\alpha_{\nu\alpha} + \delta\Gamma^\alpha_{\sigma\alpha}\Gamma^\sigma_{\mu\nu} + +\Gamma^\alpha_{\sigma\alpha}\delta\Gamma^\sigma_{\mu\nu} - \delta\Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\mu\alpha} - \Gamma^\alpha_{\sigma\nu}\delta\Gamma^\sigma_{\mu\alpha}$$
and compute R_{00}, R_{0i} and R_{ij} at first-order using the perturbed Christoffel's symbols.

4. Perturb the Ricci scalar, obtaining
$$\delta R = \delta g^{\mu\alpha}R_{\alpha\mu} + g^{\mu\alpha}\delta R_{\alpha\mu}$$
and compute it at first-order using the perturbed using the perturbed inverse metric and the Ricci curvature perturbed components.

5. Perturb Einstein tensor as
$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2}\delta g_{\mu\nu}R - \frac{1}{2}g_{\mu\nu}\delta R$$
and calculate its elements.

To do all these calculations it is useful to remind that for FLRW (unperturbed) metric in conformal time the Ricci tensor and scalar are given by
$$R_{\mu\nu} = \begin{pmatrix} -3\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2 & 0 \\ 0 & \left(\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right)\delta_{ij} \end{pmatrix} , \qquad R = 6\frac{a''}{a} ,$$

so the Einstein tensor is

$$G_{\mu\nu} = \begin{pmatrix} 3\left(\frac{a'}{a}\right)^2 & 0 \\ 0 & \left(-2\frac{a''}{a} + \left(\frac{a'}{a}\right)^2\right)\delta_{ij} \end{pmatrix} .$$

For a full detailed list of all the expressions[5] of the various Γ's, the $R_{\mu\nu}$, R and $G_{\mu\nu}$ to second-order see [14].
Once the $G_{\mu\nu}$ elements have been obtained, one uses the perturbed Einstein equations 2.2 to relate it with the stress-energy tensor 2.12. Since we are interested in the evolution equations at linear order, it is not possible to have terms mixing the type of perturbation (scalar, vector or tensor): this means that the resulting equations can be split into a scalar, a vector and a tensor part and there is no interference between these sectors.
Another important remark deals with the true evolution equations contained in the Einstein equations. As we will see discussing ADM formalism in appendix B, not all the 10 Einstein equations are true dynamical equations (that is equations containing derivative with respect to time of the perturbations of the metric): some of them are constraints. This is expected, since it is a well-known fact that the metric tensor has only 2 physical degrees of freedom, despite having in principle 10 independent entries in 4 dimensions.
In the same way, one can perturb the stress-energy tensor conservation equation 1.6.

2.6.1 Equations for scalars

The dynamical equation for scalars comes from the ij components of the Einstein equations and it reads

$$\begin{aligned} &\left[\Psi'' + 2\mathcal{H}\Psi' + \mathcal{H}\Phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi\right]\delta^i_j + \frac{1}{2}\partial^i\partial_j\left[(\chi'_\parallel - \omega_\parallel)' + 2\mathcal{H}(\chi'_\parallel - \omega_\parallel) + \Psi - \Phi\right] = \\ &= \frac{1}{2M_P^2}a^2\left(\delta P\delta^i_j + \frac{2}{3}\nabla^2\pi^\parallel\delta^i_j + \partial^i\partial_j\pi^\parallel\right) . \end{aligned}$$

The Laplacian is $\nabla^2 = \delta^{ij}\partial_i\partial_j$. This way, one can split this equation into the part proportional to δ^i_j and the argument of $\partial^i\partial_j$:

$$\begin{aligned} \Psi'' + 2\mathcal{H}\Psi' + \mathcal{H}\Phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi &= \frac{1}{2M_P^2}a^2\left(\delta P + \frac{2}{3}\nabla^2\pi^\parallel\right) , \\ (\chi'_\parallel - \omega_\parallel)' + 2\mathcal{H}(\chi'_\parallel - \omega_\parallel) + \Psi - \Phi &= \frac{1}{M_P^2}a^2\pi^\parallel . \end{aligned} \tag{2.20}$$

We underline that the second equation comes from the argument of $\partial^i\partial_j$, which in Fourier space is a multiplication by k^ik_j. This will be useful when we will analyze the limit $k \to 0$.
On the contrary, from the 00 and the ij components one gets two constraints (called energy and momentum constraints):

$$\begin{aligned} 3\mathcal{H}\left(\Psi' + \mathcal{H}\Phi\right) - \nabla^2[\Psi + \mathcal{H}(\chi'_\parallel - \omega_\parallel)] &= -\frac{1}{2M_P^2}a^2\delta\rho , \\ \Psi' + \mathcal{H}\Phi &= -\frac{1}{2M_P^2}a^2(\rho_0 + P_0)(v_\parallel + \omega_\parallel) . \end{aligned} \tag{2.21}$$

Finally, there are two equations coming from the perturbations of the stress-energy tensor:

$$\begin{aligned} \delta\rho' + 3\mathcal{H}(\delta\rho + \delta P) - 3(\rho_0 + P_0)\Psi' + (\rho_0 + P_0)\nabla^2\left(v_\parallel + \chi'_\parallel\right) &= 0 , \\ (v + \omega_\parallel)' + \left(1 - 3c_s^2\right)\mathcal{H}(v + \omega_\parallel) + \Phi + \frac{1}{\rho_0 + P_0}\left(\delta P + \frac{2}{3}\nabla^2\pi^\parallel\right) &= 0 , \end{aligned} \tag{2.22}$$

[5] These computations can be more easily performed by using Mathematica.

where c_s is the adiabatic speed of sound, defined in 2.9.

In these equations we have not fixed any gauge. It is interesting to understand what happens in Poisson/Newtonian gauge (where $\chi_{\parallel} = \omega_{\parallel} = 0$), where the four equations above are:

$$\begin{aligned}
\Psi'' + 2\mathcal{H}\Psi' + \mathcal{H}\Phi' + \left(2\mathcal{H}' + \mathcal{H}^2\right)\Phi &= \frac{1}{2M_P^2}a^2\left(\delta P + \frac{2}{3}\nabla^2\pi^{\parallel}\right) \\
\Psi - \Phi &= \frac{1}{M_P^2}a^2\pi^{\parallel} \\
3\mathcal{H}\left(\Psi' + \mathcal{H}\Phi\right) - \nabla^2\Psi &= -\frac{1}{2M_P^2}a^2\delta\rho \\
\Psi' + \mathcal{H}\Phi &= -\frac{1}{2M_P^2}a^2(\rho_0 + P_0)v_{\parallel} \ .
\end{aligned} \tag{2.23}$$

If there is not anisotropic stress $\pi^{\parallel} = 0$, the second equation gives simply $\Phi = \Psi$.

2.6.2 Equations for vectors

Vector perturbations undergo simply a momentum constraint

$$\nabla^2(\chi_i^{\perp} - \omega_i^{\perp}) = -\frac{2a^2}{M_P^2}(\rho_0 + P_0)(v_i + \omega_i^{\perp}) \ .$$

From the conservation of the stress-energy tensor one gets the following equation

$$\left[(\rho_0 + P_0)(v_i + \omega_i^{\perp})\right]' + 4\mathcal{H}(\rho_0 + P_0)(v_i + \omega_i^{\perp}) = -P_0\partial_k(\partial_i\pi_k^{\perp} + \partial_k\pi_i^{\perp}) \ .$$

Vector perturbations are often not discussed in many cases, such as many inflationary models. The reason is that for a wide class of models vector perturbations do not produce sizeable effects, since they have an amplitude decaying very fastly in time. For example, for models where $\pi_i^T = 0$, the second equation becomes

$$\left[(\rho_0 + P_0)(v_i + \omega_i^{\perp})\right]' + 4\mathcal{H}(\rho_0 + P_0)(v_i + \omega_i^{\perp}) = 0$$

and substituting the momentum constraint $(\rho_0 + P_0)(v_i + \omega_i^{\perp}) = -\frac{M_P^2}{2a^2}\nabla^2(\chi_i^{\perp} - \omega_i^{\perp})$ one gets

$$\nabla^2(\chi_i^{\perp} - \omega_i^{\perp})' + 3\mathcal{H}\nabla^2(\chi_i^{\perp} - \omega_i^{\perp}) = 0 \ ,$$

which implies (the Laplacian is an invertible operator)

$$(\chi_i^{\perp} - \omega_i^{\perp})' + 3\mathcal{H}(\chi_i^{\perp} - \omega_i^{\perp}) = 0 \ .$$

This way one has

$$\chi_i^{\perp} - \omega_i^{\perp} \propto a^{-3} \ ,$$

so tensor modes decay as the Universe expands. We will see that in an inflationary model generally a grows extremely fast so vector perturbations are diluted very quickly, meaning that they can be safely neglected. Moreover, in this project we are interested in single-field inflation for which vector perturbations are null, so they are not considered here.

2.6.3 Equation for tensors

The tensor sector undergoes the following evolution equation

$$\chi_{ij}^{T\prime\prime} + 2\mathcal{H}\chi_{ij}^{T\prime} - \nabla^2\chi_{ij}^T = \frac{a^2}{M_P^2}\pi_{ij}^T \ . \tag{2.24}$$

In this case there are not constraint equations. In cosmic time the previous equation reads

$$\ddot{\chi}_{ij}^T + 3H\dot{\chi}_{ij}^T - \nabla^2\chi_{ij}^T = \frac{1}{M_P^2}\pi_{ij}^T \ . \tag{2.25}$$

2.7 Gauge invariant quantities

In this section we want to introduce some particular gauge invariant quantities, which are useful when dealing with perturbations ([8]). We consider only first-order quantities. At this level they appear to be a mere linear combination of first-order perturbations, but in section 4.6.2 we will see that they turn out to be related to the spatial curvatire curvature of a hypersurface, justifying their names.

2.7.1 Curvature perturbation on uniform density hypersurfaces

The curvature perturbation on uniform density hypersurfaces is defined as[6]

$$\mathcal{R} = -\Psi - \frac{1}{6}\nabla^2\chi^{\parallel} - \mathcal{H}\frac{\delta\rho}{\rho_0'} \ . \tag{2.26}$$

It is very easy to show the gauge invariance, since from the relation we derived in section 2.3 we get:

$$\tilde{\mathcal{R}} = -\left(\Psi + \frac{1}{3}\nabla^2\beta + \mathcal{H}\alpha\right) - \frac{1}{6}\nabla^2(\chi^{\parallel} - 2\beta) - \mathcal{H}\frac{\delta\rho - \alpha\rho_0'}{\rho_0'} = -\Psi - \frac{1}{6}\nabla^2\chi^{\parallel} - \mathcal{H}\frac{\delta\rho}{\rho_0'} = \mathcal{R} \ .$$

In Newtonian gauge this quantity reads (in cosmic time[7])

$$\mathcal{R} = -\Psi - H\frac{\delta\rho}{\dot{\rho}_0} \ .$$

The importance of this quantity resides in the fact that it is constant in the so-called superhorizon scales limit[8] $k \to 0$ for single-field inflationary models, as one can show using the perturbed stress-energy tensor conservation equation 2.22 ([16]). Indeed, working in the uniform density gauge where $\delta\rho = 0$ and $\chi^{\parallel} = 0$, this equation reads (the terms inside the Laplacian disappear in the limit $k \to 0$):

$$\mathcal{H}\delta P - (\rho_0 + P_0)\,\Psi = 0 \ .$$

Since in this gauge $\mathcal{R} = -\Psi$ and $\delta P = \delta P_{NA}$, we have

$$\mathcal{R}' = -\frac{\mathcal{H}}{\rho_0 + P_0}\delta P_{NA} \ .$$

This implies that $\mathcal{R}$ is conserved in case of only adiabatic perturbations ($\delta P_{NA} = 0$), which we will see to be the case for single-field inflationary models (in section 4.6.1).
The fact that $\mathcal{R}$ is constant is very important, since it can be used to connect the properties of the cosmological fluctuations produced during inflation to the properties of fluctuations closer to the present. In inflationary cosmologies the era of inflation is followed by a period when the energy in scalar fields was converted into matter and radiation, but the physical process leading to this is still unknown. Therefore, in relating the cosmological fluctuations produced during inflation with those observed in the cosmic microwave background or in large-scale cosmic structures, it is essential to employ this conservation law that is valid at large wavelengths independently of the details of cosmic evolution. We will introduce in the next chapter a theorem ensuring the existence of such conserved quantities, avoiding explicitly our ignorance about reheating.

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[6] Usually one defines

$$\hat{\Psi} = \Psi + \frac{1}{6}\nabla^2\chi^{\parallel} \ .$$

[7] Using 1.15 and 1.16

$$\mathcal{H}\frac{\delta\rho}{\rho_0'} = aH\frac{\delta\rho}{a\dot{\rho}_0} = H\frac{\delta\rho}{\dot{\rho}_0} \ .$$

[8] As explained in [15], this limit is due to the fact that k is always accompanied by a factor of $\frac{1}{a(t)}$, because it is only $\frac{k}{a}$ that is independent of the units chosen for the comoving spatial coordinates x^i. In this sense, the real superhorizon limit is $\frac{k}{a} \to 0$. During inflation, we will see that a grows extremely fast, so this limit is obtained by taking $a \to \infty$ or $k \to 0$.

2.7.2 Comoving curvature perturbation

We define the comoving curvature as

$$\zeta = -\Psi - \frac{1}{6}\nabla^2\chi^{\|} + \mathcal{H}(v^{\|} + \omega^{\|}) \ . \tag{2.27}$$

Also in this case it is very easy to show the gauge invariance from the relation we derived in section 2.3:

$$\tilde{\zeta} = -\left(\Psi + \frac{1}{3}\nabla^2\beta + \mathcal{H}\alpha\right) - \frac{1}{6}\nabla^2(\chi^{\|} - 2\beta) + \mathcal{H}(v^{\|} + \beta' + \omega^{\|} + \alpha - \beta') = -\Psi - \frac{1}{6}\nabla^2\chi^{\|} + \mathcal{H}(v^{\|} + \omega^{\|}) = \zeta \ .$$

In cosmic time one has[9]

$$\zeta = -\Psi - \frac{1}{6}\nabla^2\chi^{\|} + H(v^{\|} + \omega^{\|}) = -\Psi - \frac{1}{6}\nabla^2\chi^{\|} + H\delta u \ ,$$

having introduced the quantity $\delta u = v^{\|} + \omega^{\|}$. In Newtonian gauge one has $\delta u = v_{\|}$ and

$$\zeta = -\Psi + Hv^{\|} = -\Psi + H\delta u \ .$$

A final remark is necessary: the definition 2.27 is not the one we are going to use in this project. Indeed, in section 4.6.2, we will see that recently physicists prefer to define ζ in a geometrical way, leading to a slightly different definition from the one given in this section.

2.7.3 Their relation in the limit $k \longrightarrow 0$

In the limit $k \to 0$, the two quantities we have defined before coincide ([15]) in Newtonian gauge. To see this, consider the first two Einstein equations 2.21 in Newtonian gauge, when $\chi_{\|} = \omega_{\|} = 0$:

$$\begin{aligned} 3\mathcal{H}\left(\Psi' + \mathcal{H}\Phi\right) - \nabla^2\Psi &= -\frac{1}{2M_P^2}a^2\delta\rho \ , \\ \Psi' + \mathcal{H}\Phi &= -\frac{1}{2M_P^2}a^2(\rho_0 + P_0)v_{\|} \ . \end{aligned}$$

Combining the first with the second one gets (since in Newtonian gauge $\Phi = \Psi$)

$$\nabla^2\Phi = \frac{a^2}{2M_P^2}\left[\delta\rho - 3\mathcal{H}(\rho_0 + P_0)v^{\|}\right] \ , \tag{2.28}$$

which using the unperturbed Friedmann equation 1.13 in conformal time, $\rho_0' = -3\mathcal{H}(\rho_0 + P_0)$, becomes

$$\nabla^2\Phi = 4\pi G a^2(\delta\rho + \rho_0' v^{\|}) \ .$$

This equation in Fourier space reads

$$-k^2\Phi = 4\pi G a^2(\delta\rho + \rho_0' v^{\|}) \ . \tag{2.29}$$

But now, using the expressions of ζ and $\mathcal{R}$ in Newtonian gauge (where $\chi^{\|} = \omega^{\|} = 0$),

$$\zeta - \mathcal{R} = \mathcal{H}\left(v^{\|} + \frac{\delta\rho}{\rho_0'}\right) = -\frac{k^2\Phi}{4\pi G\rho_0' a^2} \ ,$$

which in the limit $k \to 0$ vanishes, so $\mathcal{R} = \zeta$.

The outcome of this argument is that since in the limit $k \to 0$ ζ coincides with $\mathcal{R}$, which we have shown to be constant in the case of an inflationary model, we expect ζ to be consant if $k \to 0$, so to have a crucial role to relate quantities at the beginning of inflation with the ones observed today. This is an important point we are going to discuss in the next chapter, introducing Weinberg theorem, which ensures the existence and the constancy of ζ for $k \to 0$.

[9] Passing from conformal to cosmic time in 2.7, one has $\omega^{\|}_{\text{cosmic}} = a\omega^{\|}_{\text{conformal}}$. Moreover, conventionally $v^{\|}_{\text{cosmic}} = av^{\|}_{\text{conformal}}$.

Chapter 3

Construction of adiabatic modes: the Weinberg theorem

In this section we introduce a very important result by Weinberg, which allows to construct adiabatic modes in the superhorizon limit $k \to 0$. Firstly, we will show the original argument by Weinberg, explicitly demonstrating the existence of conserved curvature perturbations in Newtonian gauge in [15]. After that, we will present and prove the so-called Weinberg theorem in its final version including tensor modes, as it is shown in [10]. One of the main important points is that these results are independent of the composition of the Universe.
The construction of adiabatic modes is crucial for the derivation of the consistency relations we are going to introduce in 5, because it allows us to trade modes with small wavenumbers $k \to 0$ as a change of coordinates and in this way to relate the squeezed limit of an $(N+1)$-point function to some variation of the corresponding N-point function, which is a "modern" version of the consistency relation.

3.1 Adiabatic and isocurvature perturbations

As we have seen in equation 2.9, a perturbation A of the matter content of the Universe is usually split into an adiabatic part and a non-adiabatic part, in a "thermodynamical" vision:

$$\delta A = \frac{\partial A}{\partial \rho}\bigg|_S \delta\rho + \frac{\partial A}{\partial S}\bigg|_\rho \delta A = c_s^2 \delta\rho + \delta A_{NA} \ .$$

Let us explain better this idea ([17, 18]). Consider a Universe composed by more than 1 fluid. The total stress-energy tensor will be $T_{\mu\nu} = \sum_f T^f_{\mu\nu}$, where f is the index for the various fluids. Each perturbation is adiabatic if each individual energy density is a unique function of ρ, which is the total energy density of the Universe:

$$\rho_f = \rho_f(\rho) \ .$$

This implies that $\delta A_{NA} = 0$ in the previous equation. This means that adiabatic perturbations are perturbations in the total energy density of the system and using Einstein equations this implies that there is a perturbation in the curvature (so they are also called curvature perturbations).
Alternatively, since $\delta\rho_a = \dot{\rho}_{0a}\delta t$ and δt is the same for all the fluids, one gets the condition

$$\frac{\delta\rho_a}{\dot{\rho}_{0a}} = \frac{\delta\rho_b}{\dot{\rho}_{0b}} \ .$$

Using the Friedmann equation 1.13 and the equation of state one has $\dot{\rho}_{0a} = -3H(1+w_a)\rho_{0a}$, giving the adiabaticity condition

$$\frac{1}{3(1+w_a)}\frac{\delta\rho_a}{\rho_{0a}} = \frac{1}{3(1+w_b)}\frac{\delta\rho_b}{\rho_{0b}} \ .$$

On the contrary, isocurvature (or entropic) perturbations leave the total energy density unperturbed by a relative fluctuation between the different components of the system, that is to perturb the matter components without perturbing the geometry. In this case the variation in the relative particle number densities between two species can be quantified by the so-called entropy perturbation

$$S_{a,b} = \frac{1}{1+w_a}\frac{\delta\rho_a}{\rho_{0a}} - \frac{1}{1+w_b}\frac{\delta\rho_b}{\rho_{0b}} \ .$$

Notice that when the Universe is dominated by a single type of fluid the adiabaticity condition is satisfied only when the pressure P is function only of ρ. This means that

$$\frac{\delta P}{\dot{P}_0} - \frac{\delta\rho}{\dot{\rho}_0} = 0 \ .$$

As underlined in [15], one can show that in the limit $k \to 0$, the quantity $\zeta = -\Psi + H\delta u$ (in Newtonian gauge) evolves according to the equation

$$\dot{\zeta} = \frac{\dot{\rho}_0 \delta P - \dot{P}_0 \delta\rho}{3(\rho_0 + P_0)^2} \ .$$

This means that ζ is constant in case of adiabatic modes only, since

$$\dot{\rho}_0 \delta P - \dot{P}_0 \delta\rho = \dot{\rho}_0 \dot{P}_0 \left(\frac{\delta P}{\dot{P}_0} - \frac{\delta\rho}{\dot{\rho}_0} \right) = 0 \ .$$

This is the reason why usually perturbations which are solutions to the evolution equation having constant comoving curvature ζ are called adiabatic (in the limit $k \to 0$).

3.2 A first result

Before presenting the Weinberg theorem in a detailed way, in this section we show the first argument ([15]) about the existence in Newtonian gauge of at least two independent solutions, called modes, such that in the limit $k \to 0$ there are:

- one solution with $\zeta \neq 0$ but constant (this is the most relevant one; indeed it can be used to connect observed cosmological fluctuations in this mode with those at very early times);
- one solution with $\zeta = 0$, also if $\frac{\delta\epsilon}{s} = \epsilon$ for any scalar quantity s.

For what we have seen in the previous section these solutions are adiabatic. Moreover, there can be other non-adiabatic modes.

The Weinberg argument is based on the fact that in the limit $k \to 0$ in Newtonian gauge there are still residual gauge modes. To see this, we have to find a general spacetime transformation of purely scalar perturbations that preserves the condition $k = 0$, in Newtonian gauge. Consider the diffeomorphism induced by an infinitesimal translation of the time coordinate and a rescaling of the space coordinate, explicitly

$$\begin{pmatrix} t \\ x^i \end{pmatrix} \longrightarrow \begin{pmatrix} t + \epsilon(t) \\ x^i(1-\lambda) \end{pmatrix} \ , \tag{3.1}$$

with $\epsilon(t)$ an arbitrary function of cosmic time and λ an arbitrary infinitesimal constant[1]. Notice that this diffeomorphism does not vanish at infinity, since it is a residual gauge degree of freedom. The origin of this functional form will be clear in a while; however, notice that it contains only scalars, since $-\lambda x^i = -\frac{\lambda}{2}\partial_i(x^j x_j)$. We know that under the gauge transformation above, the perturbation of the metric changes as

$$\tilde{\delta g}_{\mu\nu} = \delta g_{\mu\nu} - \mathcal{L}_\xi g^0_{\mu\nu} \ .$$

[1] In the previous section notation $\xi^\mu = (\epsilon(t), -\lambda x^i)$.

In Newtonian gauge the perturbed metric is 2.16, but in cosmic time:

$$g_{\mu\nu} = \begin{pmatrix} -(1+2\Phi) & 0 \\ 0 & a^2\,(1-2\Psi)\,\delta_{ij} \end{pmatrix} .$$

The time-time component gives

$$\tilde{\delta g}_{00} = \delta g_{00} - \partial_t g^{(0)}_{00}\xi^0 - \partial_t \xi^0 g^{(0)}_{00} - \partial_t \xi^0 g^{(0)}_{00} \ ,$$

which explicitly gives

$$\tilde{\Phi} = \Phi - \dot{\epsilon} \ .$$

The space-space components give

$$\tilde{\delta g}_{ij} = \delta g_{ij} - \partial_t g^{(0)}_{ij}\xi^0 - \partial_i \xi^k g^{(0)}_{kj} - \partial_j \xi^k g^{(0)}_{ik} \ ,$$

which explicitly gives

$$\tilde{\Psi} = \Psi + H\epsilon - \lambda \ . \tag{3.2}$$

Choosing a rescaling of the spatial components allows us to remain in Newtonian gauge: choosing other forms for the diffeomorphism would introduce unavoidably terms not proportional to δ_{ij}, generating perturbations out of Newtonian gauge. Moreover in Newtonian gauge we have to put $\omega_{\|}$ and $\chi_{\|}$ equal to 0. But in the limit $k \to 0$ this is not necessary, since these two perturbations appear in the metric with a derivative: $\partial_i \omega_{\|}$ and $\hat{D}_{ij}\chi_{\|}$ in Fourier space are proportional to k, which in the limit $k \to 0$ authomatically vanishes. So there are no other conditions to impose to ϵ and λ to preserve Newtonian gauge. This implies that 3.1 is a residual gauge freedom in the limit $k \to 0$.

Under this gauge transformation the stress-energy tensor components transform as we have seen in section 2.3:

$$\tilde{\delta\rho} = \delta\rho - \mathcal{L}_\xi \rho_0 = \delta\rho - \xi^\lambda \partial_\lambda \rho_0 = \delta\rho - \epsilon\dot{\rho}_0$$

and similarly the pressure, since it is a scalar, $\tilde{\delta P} = P - \epsilon \dot{P}_0$.

The previous results imply that there is always a solution in Newtonian gauge to the field equation for $k \to 0$ such that

$$\Phi = -\dot{\epsilon} \ , \quad \Psi = H\epsilon - \lambda \ , \quad \delta\rho = -\epsilon\dot{\rho}_0 \ , \quad \delta P = -\epsilon\dot{P}_0 \ , \tag{3.3}$$

with $\epsilon(t)$ arbitrary function of the cosmic time and λ arbitrary constant[2]. At this point one would be tempted to say that this construction allowed us to find a way to set the perturbations Φ and Ψ to zero, so they are purely a gauge mode in $k \to 0$ limit. However, this is not totally true, since in taking the limit $k \to 0$ one is reducing two of the Einstein equations for scalars to be trivial. In section 2.23 we have derived the equations (2.23): two of them in cosmic time are

$$\begin{aligned} \dot{\Psi} + H\Phi &= -\frac{1}{2M_P^2}(\rho_0 + P_0)v_{\|} \\ \Psi - \Phi &= \frac{1}{M_P^2}\pi^{\|} \ . \end{aligned} \tag{3.4}$$

These equations arises as arguments of spatial derivatives: this implies that in Fourier space in the limit $k \to 0$ they disappear. This means that these equations must be imposed by hand in the limit $k \to 0$. Imposing these conditions, one obtains solutions having physical relevance, extendable to the case of non-zero wave number. In other words, this subset of transformations can be thought as the $k \to 0$ limit of transformations valid in the case $k \neq 0$ and therefore they generate new physical

[2]The apparent discrepancy in the sign between 3.3 and the previous transformations is solved thinking that ϵ and λ are arbitrary, so the sign can be reabsorbed by a simple redefinition. This is done to conform with the convention used in [15].

solutions; on the contrary, all the other transformations are purely residual gauge transformations which appear only in the exact $k \to 0$ limit and they can be safely gauged away, by reabsorbing them into the background. Imposing the choice 3.3, from the second of 3.4 we get the ODE in ϵ,

$$\dot{\epsilon}_0 + H\epsilon = \lambda + \frac{1}{M_P^2}\pi^{\|} \ . \tag{3.5}$$

For what concerns the first equation of 3.4, we can combine two unperturbed Friedmann equations 1.12 and 1.13

$$\begin{cases} H^2 = \frac{\rho_0}{3M_P^2} \\ \dot{\rho} = -3H(\rho_0 + P_0) \end{cases}$$

to get a simpler expression for the right hand side. Deriving the first with respect to the cosmic time and using the second one gets

$$H\dot{H} = \frac{1}{6M_P^2}\dot{\rho}_0 = \frac{1}{6M_P^2}(-3H)(\rho_0 + P_0) = -\frac{1}{2M_P^2}H(\rho_0 + P_0) \ ,$$

which implies

$$\dot{H} = -\frac{1}{2M_P^2}(\rho_0 + P_0) \ . \tag{3.6}$$

Using this in the first of 3.4, we get

$$\dot{\Psi} + H\Phi = \dot{H}v_{\|} \ .$$

Imposing the choice 3.3 one finds (since λ is a constant)

$$\dot{H}v_{\|} = \frac{d}{dt}(H\epsilon - \lambda) - H\dot{\epsilon} = \dot{H}\epsilon \ ,$$

which imposes

$$\epsilon = v_{\|} \ . \tag{3.7}$$

Moreover, in the limit $k \to 0$, in section 2.7.3 we have seen that $\mathcal{R} = \zeta$. But in Newtonian gauge $\zeta = -\Psi + Hv_{\|}$ and we want to impose 3.3. Using the previous equation one gets $\zeta = -\Psi + H\epsilon$ and using 3.3 finally

$$\zeta = \mathcal{R} = -\Psi + H\epsilon = -(H\epsilon - \lambda) + H\epsilon = \lambda \ .$$

This implies that in the limit $k \to 0$ ζ and $\mathcal{R}$ are equal, constant and different from 0 as long as $\lambda \neq 0$.

As pointed out by Weinberg, to ensure that this solution is the limit as $k \to 0$ of a solution with $k \neq 0$ we have to impose a technical but simple condition. In general, any set of linear ODEs can be written in the form

$$\dot{\vec{y}} = A(t)\vec{y} \ ,$$

where $\vec{y} : \mathbb{R} \to \mathbb{R}^n$ and $A(t) : \mathbb{R} \to M_{n\times n}(\mathbb{R})$: this is always possible since it is a well known fact that a system of ODEs of order higher than one can be always put in a form with all the ODEs of first-order, introducing some auxiliary variables. The initial condition can be constrained as

$$\sum_n c_n y_n(t_0) = 0 \ .$$

If now A and c_n depends continuosly on an external parameter k in a neighborhood of $k = 0$, any solution $\vec{y}$ of the system above will be continuous in this neighborhood, so that it can be extended to a solution for $k \neq 0$ in this neighborhood, simply choosing the solution with $k \neq 0$.
This is exactly what happens with the couple of equations 3.4: asking that the coefficients are continuous in the neighborhood of $k = 0$, we can extend the solution also for $k \neq 0$. In inflationary models in general this condition is easily expected to hold.

In many theories the anisotropic stress coefficient $\pi^{\parallel}$ is expected to be a linear combination of δu, $\delta\rho$, and δP. But in 3.3 and 3.7 we have seen that these three quantities are proportional to ϵ, so that we can write $\pi^{\parallel} = \Sigma\epsilon$, with Σ the proportionality constant. This implies that 3.5 reads

$$\dot{\epsilon} + H\epsilon = \lambda + \frac{1}{M_P^2}\Sigma\epsilon \ ,$$

which has solution[3]

$$\epsilon(t) = \frac{\lambda}{\alpha(t)}\int^t \alpha\left(t'\right) dt' \quad \text{with} \quad \alpha(t) = a(t)e^{-\frac{1}{M_P^2}\int^t \Sigma(t')dt'} \ ,$$

where the lower extremum on the integrals is arbitrary. In the special case of vanishing anisotropic stress we have seen that $\Phi = \Psi$ and since $\Sigma = 0$ we have $\alpha(t) = a(t)$, i. e. α is the Robertson-Walker scale factor; the general solution becomes

$$\epsilon(t) = \frac{\lambda}{a(t)}\int^t a\left(t'\right) dt' \ .$$

However, in general, there is also a second mode, corresponding to the possibility of having $\zeta = 0$. Indeed, in the case $\zeta = \lambda = 0$, the equation has the solution

$$\epsilon(t) = \frac{1}{a(t)} e^{\frac{1}{M_P^2}\int^t \Sigma(t')dt'} \ ,$$

which is independent of λ (and it does not correspond to the previous mode, since it would be trivially null for $\lambda = 0$).

This concludes our proof, since we have shown the general existence of a pair of adiabatic solutions of these field equations in Newtonian gauge in the limit $k \to 0$: one with $\zeta \neq 0$ and constant in the limit, and the other with $\zeta = 0$. We have also seen that not all these solutions are physical, since we need that they are the limit for $k \to 0$ of a solution of the field equations for $k \neq 0$ in at least a neighborhood of $k = 0$.

The other important consequence of this result, as we have said, is that there are however modes which are non-adiabatic, so they can be reabsorbed into the background. Equation 3.3 allows to find the values of ϵ and λ in which the perturbations are null: but, with the same procedure, one can imagine to start from an unperturbed metric ($\Psi = \Phi = 0$) and through the dilatation above generate a perturbed one. This means that we can generate a long-wavelength adiabatic mode starting from an unperturbed FLRW metric, just performing a gauge transformation. However, we remark that this is possible only because $k = 0$; when $k \neq 0$ this is not true anymore.

In the following section we are going to present the Weinberg theorem in its final version, which is a generalization of this result including also tensor modes.

3.3 Weinberg theorem: statement

In the superhorizon limit $k \to 0$, whatever the constituents of the Universe, supposing that in this limit the anisotropic stress vanishes, there are always two independent scalar solutions of the linearized Einstein equations in the Newtonian gauge for which the quantity $\zeta(\vec{k}, t) = -\Psi(\vec{k}, t) + H\delta u(\vec{k}, t)$ is time independent and there is one tensor mode for which its amplitude χ_{ij}^T is time independent in this limit. In the case of the constant scalar mode $\zeta(k, t) = \zeta(k)$, the perturbations Φ in the Newtonian gauge has the following functional form:

$$\Phi(\vec{k}, t) = \zeta(\vec{k})\left[\frac{H(t)}{a(t)}\int_\tau^t a\left(t'\right) dt' - 1\right] \ ,$$

[3]As one can prove by explicitly substituting in the equation, remembering that $\partial_t \int^t f(t')dt' = f(t)$.

where τ is an arbitrary initial time. Called generally $s(x) = s_0(t) + \delta s$ a 4-scalar, its perturbation turns out to be:

$$\delta s(\vec{k}, t) = -\frac{\zeta(\vec{k})\dot{s_0}(t)}{a(t)} \int_\tau^t a\left(t'\right) dt' \ .$$

Instead the second solution for the scalar mode is $\zeta(\vec{k}) = 0$, with the following solutions:

$$\Phi(\vec{k}, t) = C(\vec{k})\frac{H(t)}{a(t)}, \quad \delta s(\vec{k}, t) = -C(\vec{k})\frac{\dot{s_0}(t)}{a(t)} \ ,$$

where $C(\vec{k})$ is a time independent coefficient (this implies that as a grows δs is decaying).

3.4 Weinberg theorem: proof

In the limit $k \to 0$ the perturbations have long wavelength, so we expect to apply the cosmological principle. This implies that we expect to have homogeneity at large scales. This way, we have to consider a perturbed FLRW line element but still spatially homogeneous. In this case the line element in Newtonian gauge including tensor modes (all the scalars inside gradients vanish in the limit $k \to 0$ and vectors are neglected) is

$$ds^2 = -[1 + 2\Phi(t)]dt^2 + a^2(t)[(1 - 2\Psi(t))\delta_{ij} + \chi_{ij}^T(t)] \ , \tag{3.8}$$

with χ_{ij}^T traceless by construction: $g^{ij}\chi_{ij}^T = 0$, implying at first-order in perturbations $\chi_{ii}^T = 0$. All the perturbations are spatially homogeneous, so we have Φ and χ_{ij}^T depending only on time.
In the previous chapter we have seen that under a general gauge transformation the perturbations changes as $\Delta\delta g_{\mu\nu} = -\mathcal{L}_\epsilon g_{\mu\nu}^{(0)}$, where $\epsilon^\mu = (\epsilon^0, \epsilon^i)$ (we are not decomposing ϵ into its scalar and vector parts). This rule explicitly gives (paying attention to rising and lowering the indices contracting with the metric tensor: $\epsilon^0 = -\epsilon_0$, $\epsilon^i = a^{-2}\epsilon_i$)

$$\begin{aligned}
\Delta\delta g_{00} &= 2\dot{\epsilon}^0 = -2\dot{\epsilon}_0 \\
\Delta\delta g_{0i} &= \partial_i\epsilon^0 - a^2\dot{\epsilon}^i = -\partial_i\epsilon_0 - a^2\partial_t(a^{-2}\epsilon_i) = -\partial_i\epsilon_0 + 2\frac{\dot{a}}{a}\epsilon_i - \partial_t\epsilon_i \\
\Delta\delta g_{ij} &= -2a\dot{a}\epsilon^0\delta_{ij} - a^2\partial_j\epsilon^i - a^2\partial_i\epsilon^j = \\
&= 2a\dot{a}\epsilon_0\delta_{ij} - a^2\partial_j(a^{-2}\epsilon_i) - a^2\partial_i(a^{-2}\epsilon_j) = 2a\dot{a}\epsilon_0\delta_{ij} - \partial_j\epsilon_i - \partial_i\epsilon_j \ .
\end{aligned} \tag{3.9}$$

Analogously to what we have seen in the previous chapter, we have also

$$\Delta\delta\rho = -\dot{\rho}_0\epsilon^0 = \dot{\rho}_0\epsilon_0 \ , \qquad \Delta\delta P = -\dot{P}_0\epsilon^0 = \dot{P}_0\epsilon_0 \ , \qquad \Delta\delta u = \epsilon^0 = -\epsilon_0 \ .$$

The Einstein equations associated to the metric in 3.8 are necessarily invariant under transformations preserving the condition of Newtonian gauge and spatial homogeneity. From the variation $\Delta\delta g_{00} = -2\dot{\epsilon}_0$, in order to have spatial homogeneity for δg_{00} one must have

$$\epsilon_0(x, t) = \epsilon(t) + \chi(x) \ , \tag{3.10}$$

which gives also $\Delta\delta g_{00} = -2\dot{\epsilon} = -2\Delta\Phi$, so $\Delta\Phi = \dot{\epsilon}$ (and so Φ depends always on t only). To preserve the form of the metric in Newtonian gauge we must have that no $0i$ terms are generated, so we have to impose that $\Delta\delta g_{0i} = -\partial_i\epsilon_0 + 2\frac{\dot{a}}{a}\epsilon_i - \partial_t\epsilon_i = 0$. Imposing 3.10 it is easy to show by a direct substitution that

$$\epsilon_i = -a^2 f_i(x) - a^2\partial_i\chi(x)\int^t \frac{dt'}{a^2(t')} \tag{3.11}$$

is the right choice, where f_i is a 3-vector of functions of the space coordinates. This way the remaining variation we have found becomes

$$\begin{aligned}
\Delta\delta g_{ij} &= 2a\dot{a}(\epsilon + \chi)\delta_{ij} - a^2\partial_j f_i - a^2\partial_i\partial_j\chi\int^t \frac{dt'}{a^2(t')} - a^2\partial_i f_j - a^2\partial_j\partial_i\chi\int^t \frac{dt'}{a^2(t')} = \\
&= -a^2(\partial_i f_j + \partial_j f_i) + 2a\dot{a}(\epsilon + \chi)\delta_{ij} - 2a^2\partial_i\partial_j\chi\int^t \frac{dt'}{a^2(t')} \ .
\end{aligned}$$

Similarly to the previous case, we do not want any x dependence in δg_{ij} for homogeneity and the only possible solution is χ constant, so the last term is 0. Moreover a constant χ can be simply reabsorbed in a redefinition of ϵ, so also in the second summand χ disappears. From 3.11, the constancy of χ implies also that $\epsilon_i = -a^2 f_i(x)$ and uppering the index one gets

$$\epsilon^i = -f_i(x) \ . \tag{3.12}$$

Finally, since f depends on the space coordinates, to have δg_{ij} independent of them it must be linear in x^i, so generically $f_i = \omega_{ij} x^j$ with ω_{ij} a constant 3×3 matrix[4]. Finally, $\Delta \delta g_{ij}$ is restricted to the form

$$\Delta \delta g_{ij} = -a^2(\omega_{ij} + \omega_{ji}) + 2a\dot{a}\epsilon\delta_{ij} \ .$$

Since $\Delta \delta g_{ij} = -2a^2 \delta_{ij} \Delta\Psi + a^2 \Delta \chi^T_{ij}$, we can extract the trace and the traceless part of the previous expression in order to match with the variations $\Delta\Psi$ and $\Delta\chi^T_{ij}$. The result is

$$\Delta\Psi = \frac{1}{3}\omega_{ii} - H\epsilon \ , \qquad \Delta\chi^T_{ij} = -\omega_{ij} - \omega_{ji} + \frac{2}{3}\delta_{ij}\omega_{kk} \ . \tag{3.13}$$

Since both $\delta g_{\mu\nu}$ and $\delta g_{\mu\nu} + \Delta \delta g_{\mu\nu}$ are solutions of the Einstein field equations, also the difference $\Delta \delta g_{\mu\nu}$ is a solution of the equations. Using what we have found before, this implies that there is always a spatially homogeneous solution in the Newtonian gauge with the following scalar perturbations:

$$\Psi = H\epsilon - \frac{\omega_{ii}}{3} \ , \quad \Phi = -\dot{\epsilon} \ , \quad \delta\rho = -\dot{\rho}_0\epsilon \ , \quad \delta P = -\dot{P}_0\epsilon \ , \quad \delta u = \epsilon \ . \tag{3.14}$$

In general for a scalar s one has

$$\delta s = -\dot{s}_0 \epsilon \ .$$

There is also a spatially homogeneous solution with a tensor perturbation

$$\chi^T_{ij} = \omega_{ij} + \omega_{ji} - \frac{2}{3}\delta_{ij}\omega_{kk} \ .$$

At this point we study the evolution of this perturbations using linearized Einstein equations in Fourier space in the limit $k \to 0$. For tensor modes the equations are easier, since there is only the equation 2.25, which in Fourier space and in absence of anistropic stess reads

$$\ddot{\chi}^T_{ij} + 3H\dot{\chi}^T_{ij} + k^2\chi^T_{ij} = 0 \ ,$$

which is trivially satisfied by $\chi^T_{ij} = \omega_{ij} + \omega_{ji} - \frac{2}{3}\delta_{ij}\omega_{kk}$, since it is constant in time and in the limit $k \to 0$ the last term drops out.

Regarding the scalars, the discussion is more complicated. As we have noticed, equations 3.4 disappears in the limit $k \to 0$, but they must be imposed by hand. Without anisotropic stress, the second gives $\Phi = \Psi$. Substituting 3.14 gives:

$$H\epsilon - \frac{\omega_{ii}}{3} = -\dot{\epsilon} \ .$$

It is worth to underline that this equation has to be imposed only in the exact $k = 0$ limit. This means that there is a residual gauge freedom, which is not present if $k \neq 0$. This does not happen for the tensor mode χ^T_{ij} since there are no equations disappearing in the limit $k \neq 0$. The solution to the previous equation is

$$\epsilon(t) = \frac{\omega_{kk}}{3a(t)} \int_\tau^t a(t')dt' \ , \tag{3.15}$$

with τ an arbitrary initial time. The value of ζ using 3.14 becomes:

$$\zeta = -\Psi + H\delta u = \frac{\omega_{ii}}{3} \ ,$$

[4]Be careful that ω_{ij} is simply a matrix multiplying the coordinate vector x^i and it is not contracted to it by means of the metric tensor: so one can equally write $\omega^{ij}x^j$ and $\omega^i_j x^j$.

which is constant. Putting all together, one gets

$$\Psi = \Phi = H\epsilon - \frac{\omega_{ii}}{3} = H\left(\frac{\omega_{ii}}{3a(t)}\int_{\tau}^{t} a(t')dt'\right) - \frac{\omega_{ii}}{3} = \zeta\left(\frac{H(t)}{a(t)}\int_{\tau}^{t} a(t')dt' - 1\right) .$$

Moreover from equation 3.14 we have

$$\frac{\delta\rho}{\dot{\rho}_0} = \frac{\delta P}{\dot{P}_0} = \frac{\delta s}{\dot{s}_0} = -\delta u = \epsilon = \frac{\omega_{kk}}{3a(t)}\int_{\tau}^{t} a(t')dt' .$$

On the contrary, in the case $\omega_{ii} = 0$, which implies $\zeta = 0$, it is easy to show that there is also the following solution to 3.15, independent of ζ, with

$$\epsilon(t) = \frac{C}{a(t)} ,$$

where C is constant. This way we obtain

$$\Psi = \Phi = C\frac{H(t)}{a(t)} , \quad \frac{\delta\rho}{\dot{\rho}_0} = \frac{\delta P}{\dot{P}_0} = \frac{\delta s}{\dot{s}_0} = -\frac{C}{a(t)} , \quad \delta u = \frac{C}{a(t)} .$$

Since usually $a(t)$ increases with time, this is a decaying mode, so at late times it is negligible.

Notice that the result $\frac{\delta\rho}{\dot{\rho}_0} = \frac{\delta P}{\dot{P}_0}$ coincides with the adiabaticity condition we have seen in section 3.1, so both the modes we have obtained are adiabatic.

This concludes the proof since all the three modes (2 scalars and 1 vector) in the statement have been explicitly derived.

Let us conclude with a final comment useful for the future. From 3.12 and the following arguments, the transformation introduced is $\epsilon^i = \omega^{ij}x^j$, with ω_{ij} a constant 3×3 matrix; however, one usually splits the matrix into a trace and a traceless part as follows

$$\epsilon^i = \lambda x^i + \omega^{ij}x^j , \tag{3.16}$$

with $\delta^{ij}\omega_{ij} = 0$. Notice that using this convention $\frac{\omega_{ii}}{3} = \lambda$ and the transformation rules 3.13 become

$$\Delta\Psi = \lambda - H\epsilon , \qquad \Delta\chi^T_{ij} = -\omega_{ij} - \omega_{ji} . \tag{3.17}$$

Notice also that result 3.2 of section 3.2 is recovered (the opposite sign is due to the transformation 3.1, which is defined with an opposite sign).

Chapter 4

Inflation

In this chapter we introduce the inflationary paradigm, which is a postulated period of accelerated expansion of the Universe able both to explain the shortcomings of the standard ΛCDM model and to provide a production mechanism for the cosmological perturbations. This idea was firstly proposed by Starobinski ([19]) and Guth ([20]) in 1980. More in particular, we will focus on single-field inflation, in which this expansion is driven only by one scalar field. Inflationary models driven by more than one field are well known in literature ([21, 22]), but in this project we want to discuss only single-field inflation. We will argue that primordial deviations from homogeneity and isotropy during the radiation phase can be traced back to the quantum fluctuations of a scalar field (the inflaton) around its vacuum expectation value (VEV).
From a phenomenological point of view, a crucial role can be ascribed to the inflaton N-point correlation functions, in particular for the $N = 2, 3$-point functions defining the famous power spectrum and bispectrum respectively. Signatures of these objects are in principle detectable, so they are golden channels in order to study the physics of the Early Universe. Indeed, it is well-known that deviations from a Gaussian statistics are a direct sign of "phase-correlations", thus a measurement of a non-trivial bispectrum could unveil the type of interactions in the inflaton sector constituting a smoking gun to discriminate different inflationary models. The last part of the chapter will be devoted to discuss the bispectra of a single-field inflationary model. The main references to this chapter are [5, 23–26].

4.1 Motivations

As anticipated, inflation gives a solution for the shortcomings of the standard Big bang model and at the same time gives a mechanism to reproduce the observed cosmological perturbations. Let us start from the shortcomings by defining the concept of the cosmological horizon: we define the comoving Hubble radius

$$r_H = \frac{1}{\dot{a}} = \frac{1}{aH} \ , \tag{4.1}$$

which represents physically the area around a point which is causally connected over one Hubble time $t_H = H^{-1}$, in comoving distances (legth divided by a). Using the solutions 1.14, it is easy to see that

$$r_H \propto t^{\frac{1+3w}{3(1+w)}} \ ,$$

which implies that for matter $r_H \propto t^{\frac{1}{3}}$ and for radiation $r_H \propto t^{\frac{1}{2}}$. In both cases, the comoving Hubble radius grows, which means that the causally connected region grows as time flows. But this also means that in the past two distant points in the sky must be causally disconnected for sure and this is in contrast with today observation of the Cosmic Microwave Background (CMB), which is almost homogeneous and isotropic in temperature (the anisotropies are of order $\delta T/T \sim 10^{-5}$, but as we are going to see they are crucial for fully studying inflation). This is called horizon problem, but actually the standard cosmological model has the following shortcomings.

- Horizon problem: as we said, we can observe regions that share the same statistical properties in CMB, without having been in causal connection ever before, since they are separated by distances that are much larger than the largest distance travelled by light in all the history of the Universe; in particular, one can estimate the causal connected region for CMB to subtend an angle of 2°, while all the sky appears isotropic.

- Flatness problem: the first Friedmann equation 1.12 can be rewritten as

$$\Omega - 1 = \frac{\kappa}{a^2 H^2} = \kappa r_H^2$$

  where $\Omega := \frac{\rho M_P^2}{3H^2}$ is called density parameter. The today data impose the bound[1] $|\Omega - 1| < 10^{-3}$ ([27]), but this leads to an unnatural assumption. Indeed we know that in standard cosmological model r_H grows, so to achieve the today value, one has to tremendously fine tune the initial condition (one can show this fine tuning to be at Planck time $|\Omega - 1|_{t_P} < 10^{-60}$).

- Monopole problem: this problem was the first origin to introduce inflation, but since to be appreciated it requires a lot of technical details of non-perturbative quantum field theories in the contest of Early Universe, its treatment is beyond the scope of this project; we limit ourselves to describe it qualitatively, the interested readers can find details in [5] or [28]. Loosely speaking, whenever a theory has a spontaneous symmetry breaking (SSB), one can show that there could exist some solutions to the field equation which are present when the coset group has non-trivial topology[2], corresponding to the production of new particles, which are in general very stable. This is exactly the case of the hypothetical great unification theory (GUT), based for example on the gauge group $SU(5)$ ([29]), broken at the temperature $T \sim 10^{15}$ GeV. Since one can show these relics to have mass of the same order of the scale of energy at which they are produced, this would determine a huge value for Ω associated to these particles (in other words, an overclosure of the Universe), which is in contrast with the observational data (these unwanted relics have observationally $\Omega \sim 0$).

However, all these problems find a clever solution if one admits that before the standard radiation dominated epoch the Universe undergoes a phase of accelerated expansion, which means that $w < -\frac{1}{3}$. Focusing on a quasi de-Sitter phase ($w \simeq -1$) and solving the Friedmann equations 1.12 with this equation of state one has $a \propto e^{Ht}$ (notice that this case is not contained in the general solutions 1.14, since for $w = -1$ the solution is singular). Notice also that in this case $H \sim constant$. This way, all the three problems above have a nice solution.

- Horizon problem: since in this case we have H almost constant and $a \propto e^{Ht}$, this means that $r_H = \frac{1}{\dot{a}} = \frac{H}{a} \propto e^{-Ht}$, so it decreases with time; this way, the inconsistency is solved since two points in the sky appearing to be causally disconnected according to standard cosmology at early times were connected, but they exit the horizon because of inflation (see figure 4.1).

- Flatness problem: in the same way, in this contest $\Omega - 1 = \kappa r_H^2 \propto e^{-2Ht}$ states that the deviation from 1 of the density parameter is pushed towards 0 by inflation itself, so today one can expect to still measure a negligible deviation.

- Monopole problem: in a simple way, since the density of these relics scales as $\rho \propto a^{-3}$ (the density is inversely proportional to the volume, proportional to the scale parameter to the cube), one has $\rho \propto e^{-3Ht}$, which is in the same way pushed towards 0 by inflation.

There is another reason why we need physics beyond the standard cosmological model. As we said previously, the CMB has observationally anisotropies in temperature: inflationary models provide a way to predict this result, which in standard cosmology would be impossible to describe. In this sense,

[1] We will consider always $\kappa = 0$ in FLRW metric as usual.

[2] More clearly, given the gauge group G broken to H, the presence of these relics is possible if

$$\pi_d(G/H) \neq 1$$

where π_d is the d dimensional homotopy group, with d dimension of the boundary of the field space.

inflation not only solves the problems present in standard ΛCDM, but it provides a mechanism to predict perturbations generation related to the initial conditions of the Universe.
By now, we have described what an inflationary model must predict, but we have said nothing about how this mechanism could work. In the following section we will give a simple example of inflationary model, driven by a single scalar field in slow-roll regime. However, this is only one example over a wide range of different models, which, for example, can be based over a different number of fields. Nevertheless, the consistency relation we are going to introduce is valid in single-field inflation, so this is interesting case to investigate now.

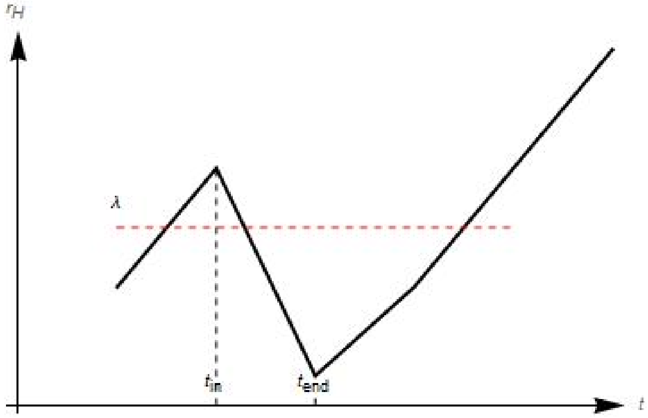

Figure 4.1: A schematic representation of the behavior comoving Hubble radius r_H as function of time: inflation starts at $t_{\rm in}$ and ends at $t_{\rm end}$. During inflation, the comoving Hubble radius decreases with time, so a given scale of legth λ rapidly exits the horizon; however, thanks to the expansion of the Universe during the standard cosmological period it reenters the horizon and this solves the horizon problem.

4.2 Slow-roll inflation from a single scalar field

At this point the natural problem is to establish which is the mechanism providing such quasi-de Sitter expansion. The purpose is to have a medium with equation of state $P \simeq -\rho$, as the case of cosmological constant: a general way to do this, is to enlarge the Lagrangian of the theory adding a new sector contributing to the stress-energy tensor (which contains P and ρ). Since it is necessary to respect the cosmological principle, the most natural choice one can do is to use a scalar field. Models using fields transforming differently under Lorentz group are also possible, but they must obey very stringent observational constraints. The Lagrangian for a scalar field (notice that we are obliged to consider a non-trivial metric) is

$$\mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \ .$$

Notice the minus sign before the kinetic term, due to the mostly plus choice for the metric (which is usually the opposite convention in the standard QFT books) and also the standard derivative instead of the covariant derivative (since ϕ is scalar). The field ϕ is called traditionally *inflaton*. The simplest form one can imagine for the potential is a mass term $V(\phi) = \frac{1}{2}\phi^2$, which means that the field is free. However, in the modern perspective one adds also self-interactions. Understanding which is the correct shape for $V(\phi)$ is one of the most important observational problem of the contemporary cosmology and, in some way, one of the motivations to produce this work.
In principle one can also add some interaction terms between the scalar field ϕ and the metric, obtaining a scalar-to-tensor theory[3], but here we do not deal with this case. This way, the full action reads

$$S_{TOT} = S_{EH} + S_\phi = \int d^4x \sqrt{-g} \left(\frac{M_P}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) \ . \tag{4.2}$$

[3] See [30] for a full review of the subject.

At this point one can compute the stress-energy tensor (from eq. 1.5):

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}} = \frac{-2}{\sqrt{-g}}\left[\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} - \partial_\alpha \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\partial_\alpha g^{\mu\nu}} + \dots\right] .$$

Firstly, notice that there are not derivatives of the metric in the Lagrangian, so all the terms in the sum in the square brakets are null with the exception of the first, which gives

$$\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}} = \frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}\mathcal{L} + \sqrt{-g}\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} = -\frac{\sqrt{-g}}{2}g_{\mu\nu}\mathcal{L} + \sqrt{-g}\partial_\mu\phi\partial_\nu\phi ,$$

where we have used the formal identity $\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{\sqrt{-g}}{2}g_{\mu\nu}$, coming from the matricial identity $\mathrm{tr}[\log g_{\mu\nu}] = \log[\det g_{\mu\nu}]$. Finally, substituting this result in the $T_{\mu\nu}$ expression one finds

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\mathcal{L} , \tag{4.3}$$

which is simply the generalisation for the expression one finds tensor for a scalar field in flat[4] space, computing it as a Nöther current for the Lorentz symmetry.

Now, to extract some information about the dynamics of the field one should solve the equation of motion which can be obtained varying the Lagrangian of the inflaton with respect to ϕ, but analytical results are impossible to obtain even in the simplest models. However, because of the cosmological principle, one easily realizes to expect that the inflaton dynamics will be driven almost totally by background following FLRW, which will depend only on time due to the rototranslational symmetry in space components. This way, one decomposes the scalar field as follows:

$$\phi(x,t) = \phi_0(t) + \delta\phi(x,t) ,$$

with $\phi_0(t)$ depending only on time, while $\delta\phi(x,t)$ can depend both on time and space. This construction is natural also because ϕ should be considered a quantum field so dynamics should take into account quantum field theory effects, which behave effectively as fluctuations (this will be analyzed in the section 4.4).
At this point, one can calculate the stress-energy tensor for the background, so to relate the results with ρ, P and w. We now want to upper an index in 4.3 to match it with 1.11, getting

$$T^\mu_\nu = \frac{1}{2}\partial^\mu\phi\partial_\nu\phi + \delta^\mu_\nu\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\nu\phi + \delta^\mu_\nu\left(-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right) . \tag{4.4}$$

This way, considering only the background ϕ_0 (dependent only on time), the components are (remembering that the indices must be uppered and lowered using FLRW metric)

$$\begin{aligned}
T^0_0 &= \partial^0\phi_0\partial_0\phi_0 + \left(-\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi_0\partial_\sigma\phi_0 - V(\phi_0)\right) = -\partial_0\phi_0\partial_0\phi_0 + \left(-\frac{1}{2}g^{00}\partial_0\phi_0\partial_0\phi_0 - V(\phi_0)\right) = \\
&= -\dot{\phi}_0^2 + \left(-\frac{1}{2}(-1)\dot{\phi}_0^2 - V(\phi_0)\right) = -\frac{1}{2}\dot{\phi}_0^2 - V(\phi_0) \\
T^i_j &= \partial^i\phi_0\partial_j\phi_0 + \delta^i_j\left(-\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi_0\partial_\sigma\phi_0 - V(\phi_0)\right) = 0 + \delta_{ij}\left(-\frac{1}{2}g^{00}\partial_0\phi_0\partial_0\phi_0 - V(\phi_0)\right) = \\
&= a^2\delta_{ij}\left(-\frac{1}{2}(-1)\dot{\phi}_0^2 - V(\phi_0)\right) = \delta_{ij}\left(\frac{1}{2}\dot{\phi}_0^2 - V(\phi_0)\right) .
\end{aligned} \tag{4.5}$$

Comparing with 1.11, we have

$$\rho = \frac{1}{2}\dot{\phi}_0^2 + V(\phi_0) , \quad P = \frac{1}{2}\dot{\phi}_0^2 - V(\phi_0) .$$

[4]To be correct, an almost flat space is needed, since in the always-flat case inflation will never end.

Notice that the energy density is correctly always positive, while the pressure can be also negative. This way, since we want $w \simeq -1$, i. e. $P \simeq -\rho$, we see that if $\dot{\phi}_0^2 \ll V(\phi_0)$ the condition is fulfilled.

At this point one can also compute the equation of motion for the background ϕ_0. Varying the action with respect to ϕ the result is the Klein-Gordon equation

$$\Box\phi = \frac{\partial V}{\partial \phi} \; .$$

The problem here is that the Laplacian is not trivial, since in curved space it is defined as $\Box = D_\mu(g^{\mu\nu}D_\nu)$, being D_μ the covariant derivative 1.2. However we can use the following property, which gives the covariant derivative of a 4-current avoiding to compute all the Christoffel symbols:

$$D_\mu j^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}j_\nu) \; . \tag{4.6}$$

In case of FLRW $\sqrt{-g} = a^3$ and $D_\nu\phi = \partial_\nu\phi$, since ϕ is scalar, so applying the formula one has

$$\Box\phi = D_\mu(g^{\mu\nu}D_\nu\phi) = D_\mu(g^{\mu\nu}\partial_\nu\phi) = \frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\phi) = \frac{1}{a^3}\partial_\mu(g^{\mu\nu}a^3\partial_\nu\phi) \tag{4.7}$$

and the result reads

$$\Box\phi = \frac{1}{a^3}\partial_\mu(g^{\mu\nu}a^3\partial_\nu\phi) = \frac{1}{a^3}(\partial_\mu a^3)g^{\mu\nu}\partial_\nu\phi + (\partial_\mu g^{\mu\nu})\partial_\nu\phi + g^{\mu\nu}\partial_\mu\partial_\nu\phi \; ,$$

so opening the indices[5]

$$\Box\phi = -3\frac{\dot{a}}{a}\dot{\phi} - \ddot{\phi} + \frac{1}{a^2}\nabla^2\phi$$

and the equation of motion simply reads

$$\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi = -\frac{\partial V}{\partial\phi} \; . \tag{4.8}$$

In the case of the background

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 = -\frac{\partial V}{\partial\phi} \; , \tag{4.9}$$

which must be coupled with the Friedmann equation ($\kappa = 0$ as usual)

$$H^2 = \frac{\rho}{3M_P^2} = \frac{1}{3M_P^2}\left(\frac{\dot{\phi}_0^2}{2} + V(\phi_0)\right) .$$

Notice that in the equation of motion for ϕ the term $3H\dot{\phi}$ behaves like a friction term: since we want the inflation to occur for a sufficient amount of time, we require to have $\ddot{\phi} \ll 3H\dot{\phi}$. The two conditions

$$\begin{cases} \dot{\phi}_0^2 \ll V(\phi_0) \\ \ddot{\phi}_0 \ll 3H\dot{\phi}_0 \end{cases}$$

are called slow-roll conditions and they are fulfilled in case of a flat V. This way, the two equations of motion read

$$\begin{cases} H^2 \simeq \frac{1}{3M_P^2}V(\phi_0) \\ 3H\dot{\phi}_0 \simeq -\frac{\partial V}{\partial\phi} \end{cases} \; . \tag{4.10}$$

Now we need a way to quantify the slow-roll regime dynamics in order to give predictions of specific models, to compare to model with others and with observations. This can be done by means of the two slow-roll parameters ϵ and η, defined as

$$\begin{cases} \epsilon = -\frac{\dot{H}}{H^2} \\ \eta = -\frac{\ddot{\phi}_0}{H\dot{\phi}_0} \end{cases} \; . \tag{4.11}$$

[5]As before, $\nabla^2 = \delta^{ij}\partial_i\partial_j$.

The physical meaning of these parameters is the following. Firstly, since

$$\epsilon = -\frac{d}{dt}\left(\frac{\dot{a}}{a}\right)^2\left(\frac{a}{\dot{a}}\right)^2 = -\frac{\ddot{a}a - \dot{a}^2}{a^2}\frac{a^2}{\dot{a}^2} = -\frac{\ddot{a}a - \dot{a}^2}{\dot{a}^2} \ ,$$

we have

$$\ddot{a} = (1-\epsilon)\frac{\dot{a}^2}{a} \ , \tag{4.12}$$

so if ϵ is small, the acceleration of a is positive, so the Universe is inflating. On the contrary, when $\epsilon \sim 1$ the accelerated phase ends.
We want to rewrite ϵ in such a way that it depends only on V, since this relation will be useful in future. Deriving the first equation of motion 4.10 one gets

$$2H\dot{H} = \frac{1}{3M_P^2}\left(\dot{\phi}_0\ddot{\phi}_0 + V'(\phi_0)\dot{\phi}_0\right) \ .$$

However, we have that $\ddot{\phi}_0 + 3H\dot{\phi}_0 = -V'$, so this equation can be rewritten as

$$\dot{H} = -\frac{1}{2M_P^2}\dot{\phi}_0^2 \ , \tag{4.13}$$

which is still an exact relation. Thus, using the definition of ϵ one gets

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{1}{2M_P^2}\frac{\dot{\phi_0}^2}{H^2} = \frac{1}{2M_P^2}\frac{{\phi_0'}^2}{\mathcal{H}^2} \ . \tag{4.14}$$

Using the first equation of motion in slow-roll regime 4.10 one gets

$$\epsilon \simeq \frac{3}{2}\frac{\dot{\phi_0}^2}{V} \ ,$$

which is only valid in slow-roll regime. Finally, exploiting $H\dot{\phi}_0 \simeq -V'$ and again the first equation of motion in slow-roll regime 4.10, we can write

$$\epsilon \simeq \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2 \ . \tag{4.15}$$

This relation allows to link the slow-roll parameter ϵ to the shape of the potential. It also implies that if $\epsilon \sim 0$ V' is small and the potential is flat: in this sense ϵ quantifies the flatness of the potential.
Similarly to the previous case, when second slow-roll parameter η is small it follows by its definition that $\ddot{\phi} \sim 0$, so the $\dot{\phi} \sim$ *constant*; but also, since we have seen that $\epsilon \sim \frac{\dot{\phi}^2}{H^2}$, this implies that ϵ remains constant for long. If it is very small, it remains small for long and inflation can take place. This way, the meaning of η is that it allows inflation to last for a sufficient amount of time.
Usually, instead of η, one uses another slow-roll parameter called η_V, which is more useful since it depends only on the shape of the potential V, as ϵ, defined as follows:

$$\eta_V = \frac{V''}{3H^2} \simeq M_P^2\frac{V''}{V} \ , \tag{4.16}$$

where the second equality is valid in slow-roll regime. However, it is not a third slow-roll parameter, since it is dependent on the other two, as we are going to prove. From the second equation of motion in slow-roll regime 4.10 one gets $\dot{\phi}_0 = -\frac{V'}{3H}$ and deriving both members with respect to the time the result is

$$\ddot{\phi}_0 = -\frac{V''\dot{\phi}_0}{3H} + \frac{\dot{H}V'}{3H^2} \longrightarrow -\frac{\ddot{\phi}_0}{H\dot{\phi}_0} = \frac{V''}{3H^2} - \frac{\dot{H}}{H^2}\frac{V'}{3H\dot{\phi}_0} \ . \tag{4.17}$$

But now one can use the definitions of ϵ, η and η_V, while the last fraction is -1 because of the second equation of motion in slow-roll regime, so one has finally

$$\eta \sim \eta_V - \epsilon \ .$$

This implies that the three slow-roll parameters are dependent, so one can freely choose η or η_V.

From a phenomenological point of view, inflation is a very robust model since it survived through years a series of observational tests. However, the mechanism we have described in this section has never been proved, since actual experiments does not have the sufficient sensibility to disentangle different inflationary models. In general we will see that different observables are functions of the slow-roll parameters, but different inflationary models predict a different functional form. These observables are related to the N-point correlation function of the inflaton. As we will see, this is related to the primordial non-Gaussianity.
Before proceeding to quantize the inflaton field, we review some basic concepts of quantum field theory, since they will be useful in future.

4.3 Review of quantum field theory

In this section we present some very basilar concepts of quantum field theories (QFT), in particular in the operator formalism. The topics presented in this chapter are very basilar in QFT and are ubiquitous in QFT books (we can cite for example [31–35]).

Consider a system composed by a particle with mass m and coordinate position q; the motion is described the following action (and Lagrangian)

$$S = \int dtL = \int dt \left(\frac{m}{2}\dot{q}^2 - V \right) ,$$

where V is the potential describing the external forces acting on the particle. The classical trajectory of the particle can be found by looking for stationary points of the action, $\frac{\delta S}{\delta q} = 0$, which means to find the solution to the Euler-Lagrange equations:

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 .$$

Defining the conjugate momentum $p = \frac{\partial L}{\partial \dot{q}}$, one can find the Hamiltonian by Legendre transforming

$$H = p\dot{q} - L = \frac{p^2}{2m} - V .$$

In this case the equation of motions are given by the Hamilton equations

$$\begin{cases} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{cases} ,$$

which can be rewritten using the Poisson brackets $\{f, g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$ as

$$\begin{cases} \dot{q} = \{q, H\} \\ \dot{p} = \{p, H\} \end{cases} .$$

Using the canonical quantization prescription (first quantization) one promotes q and p to the operators $\hat{P}$ and $\hat{Q}$ and substitutes the Poisson bracket with the quantum commutator $\{\} \to -i[]$, so that $\hat{P}$ and $\hat{Q}$ satisfy the Heisenberg evolution equations

$$\begin{cases} i\dot{\hat{Q}} = [\hat{Q}, \hat{H}] \\ i\dot{\hat{P}} = [\hat{P}, \hat{H}] \end{cases} .$$

Since considering a system with N degrees of freedom, classically one has $\{q_i, q_j\} = \{p_i, p_j\} = 0$ and $\{q_i, p_j\} = \delta_{ij}$ $(i = 1, l \ldots, N)$, the canonical quantization gives $[\hat{X}_i, \hat{X}_j] = [\hat{P}_i, \hat{P}_j] = 0$ and

$[\hat{X}_i, \hat{P}_j] = i\delta_{ij}$.

Consider now a system composed by a field, i. e. a function of the spacetime point $\phi(t,x)$ (which we consider to be real). Since we want to treat this field relativistically, the Lagrangian is no more the proper object to describe the dynamic of this field, rather the Lagrangian density $\mathcal{L}$ defined by $L = \int d^3x \mathcal{L}$, so that one has

$$S = \int dt L = \int dt d^3x \mathcal{L} \ .$$

so that the spacetime measure $d^4x = dtd^3x$ has appeared. The analogous to the previous action is the following

$$S = \int d^4x \mathcal{L} = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - V \right) \ .$$

The classical trajectory of the field is obtained by computing $\frac{\delta S}{\delta \phi} = 0$ as before, giving the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0 \ .$$

In this case one can define the Hamiltonian density by $H = \int d^3x \mathcal{H}$, giving

$$\mathcal{H} = \pi \dot{\phi} - \mathcal{L} = \frac{\pi^2}{2} + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{m^2}{2} \phi^2 + V \ ,$$

where $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$ is the density of conjugate momentum. However, in field theory literature one usually forgets to indicate the quantity as densities, so one refers to $\mathcal{L}$ simply as Lagrangian, π as momentum and similar. In this case the Hamilton equations are (notice the presence of the Hamiltonian density rather than the Hamiltonian)

$$\begin{cases} \dot{\phi} = \frac{\partial \mathcal{H}}{\partial \pi} \\ \dot{\pi} = -\frac{\partial \mathcal{H}}{\partial \phi} \end{cases}$$

and introducing the Poisson brackets, this time defined as $\{f, g\} = \int d^3x \left(\frac{\delta f}{\delta \phi(x)} \frac{\delta g}{\delta \pi(x)} - \frac{\delta f}{\delta \pi(x)} \frac{\delta g}{\delta \phi(x)} \right)$, the Hamilton equations are

$$\begin{cases} \dot{\phi} = \{\phi, H\} \\ \dot{\pi} = \{\pi, H\} \end{cases} \ .$$

The canonical quantization prescription (second quantization) follows the previous idea: one promotes ϕ and π to the operators $\hat{\phi}$ and $\hat{\pi}$ and substitutes the Poisson bracket with the quantum commutator $\{\} \to -i[]$, so that $\hat{\phi}$ and $\hat{\pi}$ satisfy the Heisenberg evolution equations

$$\begin{cases} i\dot{\hat{\phi}} = [\hat{\phi}, \hat{H}] \\ i\dot{\hat{\pi}} = [\hat{\pi}, \hat{H}] \end{cases} \ . \tag{4.18}$$

In a field theory one has infinite degrees of freedom (since the i index defined before is the continuous coordinate now) and classically one has $\{\phi(\vec{x}), \phi(\vec{y})\} = \{\pi(\vec{x}), \pi(\vec{y})\} = 0$ and $\{\phi(\vec{x}), \pi(\vec{y})\} = \delta^{(3)}(\vec{x} - \vec{y})$ ($x, y \in \mathbb{R}^3$), so the canonical quantization gives

$$[\hat{\phi}(\vec{x}), \hat{\phi}(\vec{y})] = [\hat{\pi}(\vec{x}), \hat{\pi}(\vec{y})] = 0 \ , \quad [\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \ . \tag{4.19}$$

Notice now that this poses an important technical problem: from this last result, quantum fields must be distributions: in the Lagrangian the fields are multiplied among them, but a product of distributions is ill-defined. This is substantially the problem of defining a quantum field theory in a mathematical rigorous manner.

To construct the Hilbert space describing the system in the non-interacting case $V = 0$, one solves the equation of motion in Fourier space, which is nothing else than the Klein-Gordon equation $\Box\phi + m^2\phi = 0$, obtaining the modes[6]

$$\hat{\phi}(t,\vec{x}) = \frac{1}{(2\pi)^3}\int d^3k \left[u_k(t)\hat{a}_{\vec{k}}e^{-i\vec{k}\cdot\vec{x}} + u_k^*(t)\hat{a}^*_{\vec{k}}e^{i\vec{k}\cdot\vec{x}}\right] , \tag{4.20}$$

with $u(k)$ is called mode and it satisfies the following normalization condition

$$u_k^*(t)u_k'(t) - u_k(\tau)u_k^{*\prime}(t) = -i \ .$$

There is a basis of 2 solutions since the Klein-Gordon equation is a second-order equation of motion. For example, in the flat space one finds

$$u_k(t) = \frac{1}{\sqrt{2\omega_k}}e^{i\omega_k t} , \tag{4.21}$$

where $\omega_k = \sqrt{k^2 + m^2}$. It is trivial to verify that this choice satisfies the normalization condition:

$$u_k^*(t)u_k'(t) - u_k(t)u_k^{*\prime}(t) = \frac{e^{i\omega_k t}(-i\omega_k)e^{-i\omega_k t} - e^{-i\omega_k t}i\omega_k e^{i\omega_k t}}{2\omega_k} = -i \ .$$

At this point one promotes $a_{\vec{k}}$ and $a^*_{\vec{k}}$ to operators $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$. Using the canonical quantization conditions 4.19, it follows that these operators have to satisfy the following equal-time commutation relations (in case of a bosonic field[7])

$$[\hat{a}_{\vec{k}_1}, \hat{a}_{\vec{k}_2}] = 0 \qquad [\hat{a}^\dagger_{\vec{k}_1}, \hat{a}^\dagger_{\vec{k}_2}] = 0 \qquad [\hat{a}_{\vec{k}_1}, \hat{a}^\dagger_{\vec{k}_2}] = \delta^{(3)}(\vec{k}_1 - \vec{k}_2) \ . \tag{4.22}$$

However, if one considers an interacting theory (i. e. $V \neq 0$), the equation of motion is not solvable in Fourier modes even in the simplest cases, so this procedure appears to break down. We are going to deal with this issue in the following subsection.
In the rest of this project, we will omit the hat to indicate the operators for shortness, since it will be always clear from the contest if an object is an operator or not.

To conclude the section, as we said in the previous one, we will be very interested in computing correlators. In QFT the correlation function is defined the following way:

$$C(x_1, x_2, \ldots) = \langle\phi(x_1)\phi(x_2)\ldots\rangle = \langle\Omega|T\phi(x_1)\phi(x_2)\ldots|\Omega\rangle ,$$

where T is the T-product[8] and $|\Omega\rangle$ is the vacuum state of the interacting theory. In particle physics this quantity is absolutely central since it can be related to the S-matrix elements by the Lehmann-Symanzyk-Zimmerman formula:

$$\langle f|i\rangle = i^{n_i+n_f}\prod_{j=1}^{n_i} d^4x_j e^{-ik_jx_j}(\Box_j + m^2)\prod_{k=1}^{n_f} d^4x_k e^{-ik'_kx'_k}(\Box'_k + m^2)C(x_1, \ldots, x_{n_i}, x'_i, \ldots, x'_{n_f}) ,$$

[6] In the rest of this project, we will indicate the momentum vector with $\vec{k}$, while its norm is simply k, as said in the introduction.

[7] In case of a fermionic field one must substitute the commutators with anticommutators. However, this is not the case for the inflaton, which is a scalar field so for the spin-statistics theorem is a boson.

[8] The T-product, or time ordered product, is defined as

$$T\phi(x)\phi(y) = \theta(x_0 - y_0)\phi(x)\phi(y) - \theta(y_0 - x_0)\phi(y)\phi(x) \ .$$

One can also define the anti-time ordered product as

$$\bar{T}\phi(x)\phi(y) = \theta(x_0 - y_0)\phi(y)\phi(x) - \theta(y_0 - x_0)\phi(x)\phi(y) \ .$$

where $|i\rangle$ and $|f\rangle$ are the initial and the final state. However, for reasons which will be clear, in cosmology one is interested in computing Wightman functions, which are defined without the T-product[9]:

$$W(x_1, x_2, \ldots) = \langle\Omega|\phi(x_1)\phi(x_2)\ldots|\Omega\rangle \ . \tag{4.23}$$

To conclude this section we want to underline what is the main disadvantage in using the operator approach to quantize a field theory. Since to obtain the Heisenberg evolution equation one has to pass through Hamiltonian formalism, this approach is not covariant, since Hamiltonian formalism is not. But since we are interested in relativistic field theory mainly, this can become a problem, since in breaking covariance some technical complications can arise.

Interacting field and interaction picture

Since it is almost prohibitive to solve the equations of motion of an interacting theory even in flat space, the canonical quantization in the operator formalism is impossible to obtain, since one would have to solve the equation of motion in Fourier space, in order to extract the a and a^* coefficients to promote to operators a and $a^\dagger$. This problem can be overcome[10] by using the perturbative approach in interaction picture, as we are going to illustrate now.

As done usually, we consider a theory described by a Lagrangian density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$ which for simplicity contains only one field ϕ. Since in this project we are thinking ϕ to be the inflaton, we consider it as a real scalar field (however nothing would change in case of different behavior under the Lorentz group). The two terms are $\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{2}\phi^2$, which is the Lagrangian for the free field (kinetic and mass term) and $\mathcal{L}_{\text{int}}$ is the interaction term. By Legendre transform, the Hamiltonian density can be similarly written as $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}$, which implies, integrating in the space coordinates, that the Hamiltonian is

$$H = H_0 + H_{\text{int}} \ .$$

From this, one defines the evolution operator in Schrödinger picture as

$$U(t, t_0) = e^{-iH(t-t_0)}$$

and the free evolution operator as

$$U_0(t, t_0) = e^{-iH_0(t-t_0)} \ .$$

Now we define the interaction picture with respect to the Schrödinger picture in the following way with respect to states and operators (we denote by I and S the quantities in interaction and Schrödinger picture respectively)

$$\begin{cases} |\psi(t)\rangle_I = U_0^\dagger(t,0)|\psi(t)\rangle_S \\ O_I(t) = U_0^\dagger(t,0)O_S(t)U_0(t,0) \end{cases} \tag{4.24}$$

[9] The 2-point Wightman functions are the building-blocks to compute the quantum propagators. For example, the scalar propagator for a free field is

$$D(x-y) = \langle\phi(x_1)\phi(x_2)\rangle = i\oint_{\mathcal{C}} \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2-m^2}$$

and depending on the integration path $\mathcal{C}$ chosen (with respect to the two poles $k = \pm m$) we obtain one of the following four type of propagators, which are built from 2-point Wightman functions:

- Feynman or time ordered propagator: $\theta(x_0 - y_0)\langle 0|\phi(x)\phi(y)|0\rangle - \theta(y_0 - x_0)\langle 0|\phi(y)\phi(x)|0\rangle$;
- anticipated propagator: $-\theta(y_0 - x_0)\langle 0|\phi(x)\phi(y)|0\rangle - \theta(y_0 - x_0)\langle 0|\phi(y)\phi(x)|0\rangle$;
- retarded propagator: $\theta(x_0 - y_0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(x_0 - y_0)\langle 0|\phi(y)\phi(x)|0\rangle$;
- anti-time ordered propagator: $-\theta(y_0 - x_0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(x_0 - y_0)\langle 0|\phi(y)\phi(x)|0\rangle$.

[10] From a technical point of view, this problem is not totally solved since a famous theorem by Haag ([36]) proved that interaction picture is mathematically not well defined. This pushed mathematical physicists to look for a well-defined procedure to define correlators, which is called reconstruction, but nonetheless this procedure has sense only in few simple cases. However, as done by physicists, we skip this technical details since we are interested in the numerical results of the calculations, which are predictive.

We remind that in QFT fields are operators so they evolve using the second law. This way, we can calculate how states evolve in interaction picture:

$$\begin{aligned}
i\frac{d}{dt}|\psi(t)\rangle_I = i\frac{d}{dt}\left(U_0^+(t,0)|\psi(t)\rangle_S\right) &= i\frac{d}{dt}\left(U_0^+(t,0)U\left(t,t_0\right)|\psi\left(t_0\right)\rangle_S\right) \\
&= \left[-U_0^+(t,0)H_0U\left(t,t_0\right)+U_0^+(t,0)HU\left(t,t_0\right)\right]|\psi\left(t_0\right)\rangle_S \\
&= U_0^+(t,0)\left(H-H_0\right)U\left(t,t_0\right)|\psi\left(t_0\right)\rangle_S \\
&= U_0^+(t,0)H_{\text{int}}U\left(t,t_0\right)|\psi\left(t_0\right)\rangle_S = U_0^+(t,0)H_{\text{int}}|\psi(t)\rangle_S \ ,
\end{aligned}$$

where we have used the fact that in Schrödinger picture the states evolve as $|\psi(t)\rangle_S = U\left(t,t_0\right)|\psi\left(t_0\right)\rangle_S$ and the derivative of the exponential operator $\frac{d}{dt}e^{iA} = i\frac{dA}{dt}e^{iA} = ie^{iA}\frac{dA}{dt}$. But then, from the definition of interaction picture

$$|\psi(t)\rangle_I = U_0^+(t,0)|\psi(t)\rangle_S \rightarrow U_0(t,0)|\psi(t)\rangle_I = |\psi(t)\rangle_S$$

from which

$$i\frac{d}{dt}|\psi(t)\rangle_I = U_0^+(t,0)H_{\text{int}}U_0(t,0)|\psi(t)\rangle_I \ .$$

Defining $H_{\text{int}}^I(t) \equiv U_0^+(t,0)H_{\text{int}}U_0(t,0)$, which is the interaction Hamiltonian in interaction picture, one gets the evolution of the states in interaction picture:

$$i\frac{d}{dt}|\psi(t)\rangle_I = H_{\text{int}}^I(t)|\psi(t)\rangle_I \ . \tag{4.25}$$

The same way, one can compute how operators (and so the fields) evolve:

$$\begin{aligned}
i\frac{d}{dt}O_I(t) = i\frac{d}{dt}\left(U_0^+(t,0)O_SU_0(t,0)\right) &= \\
&= -U_0^+(t,0)H_0O_SU_0(t,0)+U_0^+(t,0)O_SH_0U_0(t,0) \\
&= -H_0U_0^+(t,0)O_SU_0(t,0)+U_0^+(t,0)O_SU_0(t,0)H_0 \\
&= \left[U_0^+(t,0)O_SU_0(t,0),H_0\right] = \left[O_I(t),H_0\right] \ ,
\end{aligned}$$

where as before we have used the derivative of the exponential operators. This is exactly the Heisenberg evolution equation, but with the respect to the free Hamiltonian. This fact is crucial, since thanks to this the fields can be trated as free, so their evolution follows the free one. This implies that they can be written in Fourier modes, so they can be quantized using the standard procedure in operator formalism, i. e. promoting to operators a and a^*. What becomes non-trivial in interaction picture is the evolution of the states.

At this point we would like to write the expression of the correlator 4.23 in interaction picture. The important point is that we want to compute this correlator as it evolves in time. We start in Schrödinger picture (the vacuum state $|\Omega\rangle_S$ evolves in time while the fields not) and then we go to interaction picture using 4.24:

$$\begin{aligned}
&\langle\Omega(t)|_S\phi_S(t,x_1)\phi_S(t,x_2)\ldots|\Omega(t)\rangle_S = \langle\Omega(t_0)|_SU^\dagger(t,t_0)\phi_S(t_0,x_1)\phi_S(t_0,x_2)\ldots U(t,t_0)|\Omega(t_0)\rangle_S = \\
&= \langle\Omega(t_0)|_IU_0^\dagger(t,0)U^\dagger(t,t_0)U_0(t,0)\phi_S(t_0,x_1)U_0^\dagger(t,0)U_0(t,0)\phi_S(t_0,x_2)\ldots U_0^\dagger(t,0)U(t,t_0)U_0(t,0)|\Omega(t_0)\rangle_I = \\
&= \langle\Omega(t_0)|_IU_0^\dagger(t,0)U^\dagger(t,t_0)U_0(t,0)\phi_S(t_0,x_1)\phi_S(t_0,x_2)\ldots U_0^\dagger(t,0)U(t,t_0)U_0(t,0)|\Omega(t_0)\rangle_I = \\
&= \langle\Omega(t_0)|_IU_I^\dagger(t,t_0)\phi_S(t_0,x_1)\phi_S(t_0,x_2)\ldots U_I(t,t_0)|\Omega(t_0)\rangle_I \ ,
\end{aligned}$$

where we have defined

$$U_I(t,t_0) = U_0^\dagger(t,0)U(t,t_0)U_0(t,0) \ .$$

This operator is nothing else than the evolution operator for states in interaction picture, as one can prove by applying the definitions:

$$|\psi(t)\rangle_I = U_0^\dagger(t,0)|\psi(t)\rangle_S = U_0^\dagger(t,0)U(t,t_0)|\psi(0)\rangle_S = U_0^\dagger(t,0)U(t,t_0)U_0(t_0,0)|\psi(t_0)\rangle_I = U_I(t,t_0)|\psi(t_0)\rangle_I \ .$$

To find an explicit expression for $U_I(t, t_0)$ we substitute this into equation 4.25, giving

$$i\frac{d}{dt}U_I(t, t_0) = H^I_{int}(t)U_I(t, t_0) \ ,$$

which is a Schrödinger-like equation, with solution

$$U_I(t, t_0) = T\left[e^{-i\int_{t_0}^{t} H^I_{int}(t')dt'}\right] \ ,$$

where T is the T-product, present since U is a potential infinite matrix (and in the interesting cases it is).

4.4 Semiclassical quantization of the inflaton in de Sitter space

In section 4.2 we have analyzed the background dynamics for ϕ_0, eq. 4.9: in this one we want to consider the fluctuations $\delta\phi$, depending both on t and $\vec{x}$. Starting from the full equation of motion for the inflaton 4.8, we get the equation of motion for the perturbation:

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{1}{a^2}\nabla^2\delta\phi = -\frac{\partial^2 V}{\partial\phi^2}\delta\phi \ . \tag{4.26}$$

For future results, notice that the RHS of this equation can be obtained as variation of the following action

$$S_{SM} = \int d^4x \left[\frac{a^3}{2}\dot{\delta\phi}^2 - \frac{a}{2}(\partial_i\delta\phi)^2\right] \ , \tag{4.27}$$

which is sometimes called Sasaki-Mukhanov action. Using 1.15, we can rewrite the equation 4.4 in conformal time as

$$\delta\phi'' + 2\mathcal{H}\delta\phi' - \nabla^2\delta\phi = -a^2\frac{\partial^2 V}{\partial\phi^2}\delta\phi$$

and the Sasaki-Mukhanov action becomes

$$S_{SM} = \int d^4x \frac{a^2}{2}\left[(\delta\phi)'^2 - (\partial_i\delta\phi)^2\right] \ .$$

In order to simplify the computation, one can perform the following field redefinition

$$\bar{\delta\phi} = a(t)\delta\phi \ , \tag{4.28}$$

giving the following equation of motion

$$\bar{\delta\phi}'' - \frac{a''}{a}\bar{\delta\phi} - \nabla^2\bar{\delta\phi} = -a^2\frac{\partial^2 V}{\partial\phi^2}\bar{\delta\phi} \ , \tag{4.29}$$

whose RHS can be obtained by varying the following action

$$S_{SM} = \int d^4x \frac{1}{2}\left(\bar{\delta\phi}'^2 - (\partial_i\bar{\delta\phi})^2 + \frac{a''}{a}\bar{\delta\phi}^2\right) \ .$$

As we have said in the previous section, we are assuming that, contrarily to the background, these fluctuations have a quantum nature. This way, the purpose is to treat $\delta\phi$ (so also $\bar{\delta\phi}$) as a quantum field: in the operatorial formalism we have seen that this means that we want to write the solutions to the equation of motion and then promoting a and a^* to operators a and $a^\dagger$. It is worth to underline that this procedure is semiclassical, since we do not have a consistent theory for quantum gravity.
The equation 4.29 can be seen as a Klein-Gordon equation with effective mass $m_{\bar{\delta\phi}} = \frac{a''}{a}$, so the solutions in Fourier space are of type 4.20. In Fourier space 4.29 reads

$$u_k'' + \left(k^2 - \frac{a''}{a} + a^2\frac{\partial^2 V}{\partial\phi^2}\right)u_k(\tau) = 0 \ .$$

In the following we will consider a massless inflaton, so $\frac{\partial^2 V}{\partial \phi^2} = m_\phi^2 = 0$. In the case of a de Sitter expansion, when $a(t) \propto e^{Ht}$, the conformal time reads $d\tau \propto \frac{dt}{e^{Ht}}$, which integrated gives $\tau \propto -\frac{1}{H}e^{-Ht} = -\frac{1}{aH}$ and finally $a = -\frac{1}{H\tau}$. This way, we can rewrite the term containing the scale factor in terms of the independent variable τ

$$\frac{a''}{a} = \left(-\frac{1}{H\tau}\right)'' \left(-\frac{1}{H\tau}\right)^{-1} = \frac{2}{\tau^2} \,, \tag{4.30}$$

so the equation reads

$$u_k'' + \left(k^2 - \frac{2}{\tau^2}\right) u_k(\tau) = 0 \,. \tag{4.31}$$

This equation is a second-order ODE, with two independent solutions

$$u_k = \frac{1}{\sqrt{2k}} e^{\pm ik\tau} \left(i \mp \frac{1}{k\tau}\right) \,. \tag{4.32}$$

Notice that in both these solutions on small scales $\lambda \propto \frac{1}{k}$, so for large k, we get that $u_k \sim \frac{1}{\sqrt{k}} e^{ik\tau}$, which is exactly the solution 4.21 in the flat space (here the inflaton is massless, so $\omega_k = k$): this is not a case, since we want that at small scales the solution behaves like the flat space one, for the equivalence principle. This is called Bunch-Davis vacuum choice.
Combining the two independent mode solutions we have found, we get a family of solutions of type:

$$u_k(\tau) = C_1 \frac{1}{\sqrt{2k}} e^{-ik\tau} \left(i + \frac{1}{k\tau}\right) + C_2 \frac{1}{\sqrt{2k}} e^{ik\tau} \left(i - \frac{1}{k\tau}\right)$$

for C_1 and C_2 general complex numbers. In literature this solution can be written in different ways. For example, using $\tau \propto -\frac{1}{aH}$, redefining the arbitrary constant one gets

$$C_{1/2} \frac{1}{2\sqrt{k}} e^{\pm ik\tau} \left(i \mp \frac{1}{2k\tau}\right) = C_{1/2} \frac{\pm 1}{2\sqrt{k}k\tau} e^{\pm ik\tau} \left(\mp ik\tau + 1\right) = C'_{1/2} \frac{aH}{\sqrt{2k^3}} e^{\pm ik\tau} \left(\mp ik\tau + 1\right) \,,$$

so the solution is rewritten as[11]

$$u_k(\tau) = C'_1 \frac{aH}{\sqrt{2k^3}} e^{-ik\tau} \left(1 + ik\tau\right) + C'_2 \frac{aH}{\sqrt{2k^3}} e^{ik\tau} \left(1 - ik\tau\right) \,. \tag{4.33}$$

Returning back to 4.32, we would like to examine the behaviour of the solution in the subhorizon (i. e. $\lambda_{\text{physical}} \sim \frac{a}{k} \ll H^{-1} \to k \gg aH$) and superhorizon ($k \ll aH$)) limit.

- Subhorizon $k \gg aH$: in this case

  $$u_k = \frac{1}{\sqrt{2k}} e^{-ik\tau} \,,$$

  so the unscaled field (4.28) in Fourier space has amplitude

  $$|\delta\phi_{\vec{k}}| = \frac{|u_k|}{a} = \frac{1}{a\sqrt{2k}} \,.$$

  As one can see from the imaginary exponential, the field oscillates but its amplitude decreases extremely fast since $a \propto e^{Ht}$.

- Superhorizon $k \ll aH$:

  $$u_k = \frac{1}{\sqrt{2k}k\tau} e^{-ik\tau} \,,$$

  which implies as before

  $$|\delta\phi_{\vec{k}}| = \frac{|u_k|}{a} = \frac{1}{\sqrt{2k^3}a\tau} = \frac{H}{\sqrt{2k^3}} \,, \tag{4.34}$$

  where we have used the fact that $H = -\frac{1}{a\tau}$.

[11] This is equation 2.22 of [24] (the a of difference is due to the fact that in this paper the solution is not rescaled as 4.28).

If we want to quantize the field $\delta\phi$ (so $\bar{\delta\phi}$), as we said we have to promote $a_{\vec{k}}$ and $a^*_{\vec{k}}$ to operators $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$, satisfying the commutation relations 4.22. Finally, if we want to consider a quasi de Sitter spacetime, we need to recalculate 4.30 as an expansion in the slow-roll parameters. Firstly, we have

$$\frac{a''}{a} = a^2H^2\left(2+\frac{\dot{H}}{H^2}\right) \ ,$$

where we have used 1.15 to pass to conformal time. Then from 4.12 one has

$$a = (1-\epsilon)\frac{\dot{a}^2}{\ddot{a}} \ ,$$

so one gets

$$d\tau = \frac{dt}{a} = -\frac{1}{(1-\epsilon)Ha} \ ,$$

implying

$$H = -\frac{1}{(1-\epsilon)\tau a} \ . \tag{4.35}$$

This way one gets the generalization[12] of equation 4.30:

$$\frac{a''}{a} = a^2H^2\left(2+\frac{\dot{H}}{H^2}\right) \simeq \frac{2}{\tau^2}\left(1+\frac{3}{2}\epsilon\right) \ ,$$

so the equation 4.31 becomes

$$u''_k + \left(k^2 - \frac{2+3\epsilon}{\tau^2}\right)u_k = 0$$

and defining $\nu^2 = \frac{9}{4}+3\epsilon$, one gets, at linear order in ϵ, $\nu^2-\frac{1}{4} = 2+3\epsilon$, which gives the equation

$$u''_k + \left(k^2 - \frac{\nu^2-\frac{1}{4}}{\tau^2}\right)u_k(\tau) = 0 \ .$$

This still admits explicit solutions in terms of the Hankel functions[13], which in the two limits above become:

- subhorizon $k \gg aH$:

$$u_k = \frac{1}{\sqrt{2k}}e^{-ik\tau} \ ,$$

  as in de Sitter case;

- superhorizon scale

$$u_k = \frac{1}{\sqrt{2k}}(-k\tau)^{\frac{1}{2}-\nu}e^{i\left(\nu-\frac{1}{2}\right)\frac{\pi}{2}}2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma\left(\frac{3}{2}\right)}$$

  where Γ is the Euler gamma function, so that (using $a = \frac{1}{\tau H}$)

$$|\delta\phi_{\vec{k}}| = 2^{\nu-\frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma\left(\frac{3}{2}\right)}\frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} = \mathcal{O}(1)\times\frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} \ . \tag{4.36}$$

Notice also that introducing a mass for the inflaton (potential $V = \frac{m^2}{2}\phi^2$) means modifying the mass term, which implies to modify the value of ν. Indeed, the equation of motion becomes

$$u''_k + \left(k^2 - \frac{2+3\epsilon}{\tau^2} + m^2_\phi a^2\right)u_k = 0 \ ,$$

[12] See [23], eq. 70.
[13] See [23], section 2.3.

but using $a = \frac{1}{\tau H}$ and the definition of $\eta_V = \frac{V''}{3H^2} = \frac{m_\phi^2}{3H^2}$ one has

$$u_k'' + \left(k^2 - \frac{2 + 3\epsilon - 3\eta_V}{\tau^2}\right) u_k = 0 \ ,$$

which has exactly the same solution as before, imposing $\nu^2 - \frac{1}{4} = 2 + 3\epsilon - 3\eta_V$, giving $\frac{3}{2} - \nu = \eta_V - \epsilon$.

However, this is not fully consistent yet, since we have solved the equation perturbatively only in the inflaton field, but without perturbing the gravity sector. But this is required, since a perturbation in the inflaton certainly modifies the stress-energy tensor (see the definition 1.5), which is the source of the Einstein equations 1.4. This implies that also the gravity field, which is the metric $g_{\mu\nu}$ must be treated perturbatively. However, differently to the perturbation of a scalar field, the perturbations in GR are not so easy to treat in a fully consistent way, as we have seen in chapter 2. This is what we are going to analyze in a future section.

4.5 Power spectrum

A very important concept to introduce is power spectrum. It is a quantity which is used in a wide range of fields in physics in order to study the statistics of a stochastic process. In the context of cosmology, it is a very important tool to study the statistics of the CMB fluctuations and so to relate theory with experimental data ([37]). It is also a very important object useful to understand non-Gaussianities, as we will see in future, and it is crucial to understand the consistency relation.

Let $\delta(x)$ be a generical cosmological perturbation. Usually it is expanded in Fourier series as

$$\delta(\vec{x}) = \frac{1}{(2\pi)^3} \int d^3k e^{i\vec{k}\cdot\vec{x}} \delta(\vec{k}) \ . \tag{4.37}$$

The $\vec{k}$'s are always 3-vectors, unless differently indicated. As we have seen, the fact that the integration is performed over the spatial components of $\vec{k}$ is due to the cosmological principle. We will always consider real fields, for which[14]

$$\delta(\vec{x}) = \delta^*(\vec{x}) \longrightarrow \delta(\vec{k}) = \delta(-\vec{k}) \ .$$

The reason why we prefer to work in the momentum space rather than the coordinates space is that gravitational force usually spoils the independence between two different positions. On the contrary, it is a much better idea to think of the perturbation as a superposition of plane waves, which have the advantage that they evolve independently while the fluctuations are still linear. We now introduce the correlation function, which is by definition

$$\xi(\vec{r}) = \langle \delta(\vec{x})\delta(\vec{x}+\vec{r})\rangle = \frac{1}{(2\pi)^6} \int d^3k_1 d^3k_2 e^{i(\vec{k}_1\cdot\vec{x}+\vec{k}_2\cdot(\vec{x}+\vec{r}))} \langle \delta(\vec{k}_1)\delta(\vec{k}_2)\rangle \ . \tag{4.38}$$

This quantity is extremely important in cosmology. As one can see, it is expected that ξ depends only on r: this is due to rotational invariance (isotropy) of the space, which we assume for the cosmological principle. From this considerations, we can write

$$\langle \delta(\vec{k}_1)\delta(\vec{k}_2)\rangle = (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2) P(k_1) \ , \tag{4.39}$$

[14] Usually also one indicates the perturbation in Fourier space with a tilde ($\tilde{\delta}(\vec{k})$), but we will omit this for shortness, since the space of definition of the field will be always clear from the contest.

where P is called power spectrum and this expression is its definition. Notice that it depends only on the norm of $\vec{k}$. If one inserts this expression in the one of the correlation, the result is

$$\begin{aligned}\xi(\vec{r}) &= \frac{1}{(2\pi)^6}\int d^3k_1 d^3k_2 e^{i\left(\vec{k}_1\cdot\vec{x}+\vec{k}_2\cdot(\vec{x}+\vec{r})\right)}(2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2)P(k_1) = \\ &= \frac{1}{(2\pi)^3}\int d^3k_1 e^{i\left(\vec{k}_1\cdot\vec{x}-\vec{k}_1\cdot(\vec{x}+\vec{r})\right)}P(k_1) = \frac{1}{(2\pi)^3}\int d^3k_1 e^{-i\vec{k}_1\cdot\vec{r}}P(k_1) = \\ &= \frac{1}{(2\pi)^3}\int d^3k_1 e^{i\vec{k}_1\cdot\vec{r}}P(k_1)\ .\end{aligned} \tag{4.40}$$

which is called Wiener-Kintchine theorem (in last passage we changed the integration variable $\vec{k}\to-\vec{k}$ and then we used the symmetry of the domain to invert the integration extrema to compensate the minus sign coming from the measure). Given that P is function of the norm of $\vec{k}_1$ only, in cosmology this expression can be further simplified passing to spherical coordinates. Since we are interested in the variance we put $r=0$ and we find

$$\begin{aligned}\sigma^2 = \xi(0) &= \frac{1}{(2\pi)^3}\int d^3k_1 P(k_1) = \frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk_1 k_1^2 P(k_1) = \\ &= \frac{1}{2\pi^2}\int_0^{\infty} dk_1 k_1^2 P(k_1)\ .\end{aligned}$$

This expression can be rewritten introducing the adimensional power spectrum $\Delta(k)=\frac{k^3}{2\pi^2}P(k)$ as

$$\sigma^2 = \int_{-\infty}^{\infty} d\log k\Delta(k)\ ,$$

since $d\log k\Delta(k) = \frac{dk}{k}\frac{k^3}{2\pi^2}P(k) = dk\frac{k^2}{2\pi^2}P(k)$. To proceed, one has to fix a power spectrum and in order to do so we have to understand its origin. In the 1970s the form of the spectrum was chosen as the best to explain structure formation: the simplest assumption is

$$\Delta(k) = Ak^{n_s-1}\ , \tag{4.41}$$

where n_s is called spectral index[15]. This choice has the scaling property $\Delta(\alpha k)=\alpha^{n_s-1}\Delta(k)$. The most natural choice for initial fluctuations according to various physical arguments appeared to be the case $n_s=1$, suggested independently by Peebles and Yu ([38]) and Harrison ([39]) in 1970 and Zel'dovich ([40]) in 1972, which is now usually known as the Harrison–Zel'dovich spectrum: in this case, the power spectrum is scale invariant.

Thanks to what we obtained in section 4.4, we can calculate the power spectrum of quantum perturbations of the inflaton field (in case of a single-field inflationary model). Working in momentum space, from the solution of the equation of motion 4.20 one has (the second mode is obtained by the substitution of the integration variable $\vec{k}\to-\vec{k}$)

$$\delta\phi(\tau,\vec{k}) = \frac{1}{a}\left(u(\tau,k)a_{\vec{k}} + u^*(\tau,k)a^\dagger_{-\vec{k}}\right) = \delta\phi^+(\tau,\vec{k}) + \delta\phi^-(\tau,\vec{k})\ .$$

To compute the power spectrum we use the definition 4.39, taking into account the fact that the correlator is a quantum correlator. Since the annihilation operator gives 0 when contracted with the vacuum state, there is only one summand surviving in the contraction once the explicit expression of $\delta\phi$ is substituted:

$$\langle\delta\phi(\tau,\vec{k}_1)\delta\phi(\tau,\vec{k}_2)\rangle = \langle 0|\delta\phi(\tau,\vec{k}_1)\delta\phi(\tau,\vec{k}_2)|0\rangle = \langle 0|\delta\phi^+(\tau,\vec{k}_1)\delta\phi^-(\tau,\vec{k}_2)|0\rangle := \overline{\delta\phi(\tau,\vec{k}_1)\delta\phi(\tau,\vec{k}_2)}\ ,$$

[15]Equivalently, one can define

$$n_s = 1 + \frac{d\log\Delta(k)}{d\log k}\ . \tag{4.42}$$

which is explicitly

$$
\begin{aligned}
\overline{\delta\phi(\tau,\vec{k}_1)\delta\phi(\tau,\vec{k}_2)} &= \langle 0|\delta\phi^+(\tau,\vec{k}_1)\delta\phi^-(\tau,\vec{k}_2)|0\rangle = \frac{1}{a^2}\langle 0|u(\tau,k_1)a_I(\vec{k}_1)u^*(\tau,k_2)a_I^\dagger(\vec{k}_2)|0\rangle = \\
&= \frac{1}{a^2}u(\tau,k_1)u^*(\tau,k_2)\langle 0|[a_I(\vec{k}_1)a_I^\dagger(\vec{k}_2)]|0\rangle = (2\pi)^3\frac{|u(\tau,k_1)|^2}{a^2}\delta^{(3)}(\vec{k}_1-\vec{k}_2) \ ,
\end{aligned} \tag{4.43}
$$

where to pass from the first to the second line we have summed a term $\frac{1}{a^2}u(k_1)u^*(k_2)\langle 0|a_I^\dagger(\vec{k}_2)a_I(\vec{k}_1)|0\rangle$, which is 0 for the action of the creation and annihilation operators. Using 4.39 one easily gets $P(k) = \frac{|u(k)|^2}{a^2} = \frac{H^2}{2k^3}$, where we have used 4.34 since the measurements we made today are on superhorizon scales. This implies that the adimensional power spectrum is

$$
\Delta_{\delta\phi}(k) = \frac{H^2}{(2\pi)^2} \ ,
$$

so the spectral index is $n_s = 1$, a perfect Harrison-Zel'dovich spectrum. However, this computation has been done in exact de Sitter space, while inflation is quasi-de Sitter. In this case, we have to substitute the equation 4.36, so that

$$
P_{\delta\phi}(k) = \mathcal{O}(1)\times\frac{H^2}{2k^3}\left(\frac{k}{aH}\right)^{3-2\nu} \ ,
$$

and the adimensional power spectrum is

$$
\Delta_{\delta\phi}(k) = \mathcal{O}(1)\times\left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} \ , \tag{4.44}
$$

so that $n_s = 4-2\nu$. Since $\nu = \frac{3}{2}+\epsilon-\eta_V$, one has $n_s = 1-2\epsilon+2\eta_V$, so the deviation from Harrison-Zel'dovich is of the order of the slow-roll parameters, so very small. Indeed, experimentally, $n_s = 0.965\pm 0.004$ ([27]). However this expression of n_s is not totally correct, since we will see that there is a correction due to the perturbation of gravity sector of the action, while here we have imposed to be in quasi-de Sitter *a priori*.

4.6 Perturbations theory of single-field inflationary models

In section 4.4 we have studied the inflaton perturbations assuming a fixed background; however, as we have remarked, this is not totally correct since the gravity sector of the inflaton action produces perturbations which are of the same order in slow-roll parameters so both the sectors must be perturbed together. Here we study a single field inflationary model in a fully consistent perturbative manner, perturbing both the inflaton and the metric tensor as shown in the previous chapter.
Firstly we perturb the inflaton equation; then we see that inflation naturally produces a stochastic background of gravitational waves, with power spectrum related in a precise way to the one of the inflaton perturbations, a useful relation to test inflationary models in future; finally, for future convenience, we introduce the so-called ζ-gauge, which we will use in future.

4.6.1 The perturbed stress-energy tensor

Firstly, we have to compute the EMT for the single field inflation at first-order in perturbations ([26]). We have already computed its backround values in eq. 4.5. The same computation starting from a FLRW metric written in conformal time gives

$$
\begin{aligned}
T^0_0 &= -\frac{1}{2}\phi_0'^2 - V(\phi_0)a^2 \\
T^i_j &= \delta^i_j\left(\frac{1}{2}\phi_0'^2 - V(\phi_0)a^2\right) \\
T^i_0 &= T^0_j = 0 \ ,
\end{aligned}
$$

giving $\rho_0 = \frac{1}{2}\phi_0'^2 + V(\phi_0)$ and $P_0 = \frac{1}{2}\phi_0'^2 - V(\phi_0)$. To find the corrections which are first-order in perturbations we perturb the equation 4.4, giving

$$\begin{aligned}\delta T_{\mu\nu} =& \partial_\mu \delta\phi \partial_\nu \phi_0 + \partial_\mu \phi_0 \partial_\nu \delta\phi - \delta g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi_0\partial_\beta\phi_0 + V(\phi)\right) \\ & - g_{\mu\nu}\left(\frac{1}{2}\delta g^{\alpha\beta}\partial_\alpha\phi_0\partial_\beta\phi_0 + g^{\alpha\beta}\partial_\alpha\delta\phi\partial_\beta\phi_0 + \frac{\partial V}{\partial\phi}\delta\phi + \frac{\partial V}{\partial\phi}\delta\phi\right) ,\end{aligned}$$

then we upper the first index by using $\delta T^\mu_\nu = \delta\left(g^{\mu\alpha}T_{\alpha\nu}\right) = \delta g^{\mu\alpha}T_{\alpha\nu} + g^{\mu\alpha}\delta T_{\alpha\nu}$. Substituting the expression for the perturbed metric 2.7 and its inverse 2.17 (we consider only scalar perturbations in the metric for simplicity) the result is

$$\begin{aligned}\delta T^0_0 &= \Phi\phi_0'^2 - \delta\phi'\phi_0' - \delta\phi\frac{\partial V}{\partial\phi}a^2 \\ \delta T^0_j &= -\partial_j\omega_\| \phi_0'^2 - \partial_j\delta\phi\phi_0' \\ \delta T^i_0 &= \partial^i\delta\phi\phi_0' \\ \delta T^i_j &= \left(-\Phi\phi_0'^2 + \delta\phi'\phi_0' - \delta\phi\frac{\partial V}{\partial\phi}a^2\right)\delta^i_j \ .\end{aligned}$$

Comparing this result with 2.12, from δT^i_j one immediately sees that for a single-field inflationary model the anisotropic stress is null. Moreover, from the comparison of the δT^i_0 one has, using the background expressions for ρ_0 and P_0,

$$\delta T^i_0 = -(\rho_0 + P_0)\partial^i v^\| = -\phi_0'^2\partial^i v^\| = \partial^i\delta\phi\phi_0' \ ,$$

implying

$$v^\| = -\frac{\delta\phi}{\phi_0'} \ . \tag{4.45}$$

Moreover, $v_i^T = 0$ so there are no vector perturbations.

4.6.2 The curvature perturbation as a geometrical quantity

The definition 2.27 we have given for the comoving curvature is defined choosing a linear combination *ad hoc* to be gauge invariant. This is correct, but if one tries to extend this procedure to higher order perturbations the definitions become rapidly involving. This is the reason why recently physicists prefer to define it using a geometrical quantity. Introducing ADM formalism (see appendix B), we will introduce the curvature R_S induced into a hypersuface defined by the equation $S(x) = constant$, where S is a generic scalar. In the case of a perturbed FLRW, splitting S into a homogeneous and isotropic background and a first-order perturbation $S = S_0(t) + S^{(1)}(x) + \ldots$ one has at first-order in perturbations:

$$R_S = \frac{4}{a^2}\nabla^2\left(\frac{\mathcal{H}}{S_0'}S^{(1)} + \Psi + \frac{1}{6}\nabla^2\chi_\|\right) \ . \tag{4.46}$$

This quantity is gauge invariant, since, exactly as 2.13, a first-order perturbation of scalar quantity transforms as

$$\tilde{S^{(1)}} = S^{(1)} - \mathcal{L}_\xi S_0 = S^{(1)} - \alpha S_0' \ ,$$

while Ψ and $\chi_\|$ transform according to the relations derived in section 2.3, so

$$\frac{H}{S_0'}\tilde{S^{(1)}} + \tilde{\Psi} + \frac{1}{6}\nabla^2\tilde{\chi_\|} = \frac{H}{S_0'}(S^{(1)} - \alpha S_0') + \left(\Psi + \frac{1}{3}\nabla^2\beta + H\alpha\right) + \frac{1}{6}\nabla^2(\chi_\| - 2\beta) = \frac{H}{S_0'}S^{(1)} + \Psi + \frac{1}{6}\nabla^2\chi_\|$$

and gauge invariance is proved.

As we have seen in section 2.3, one can define the perturbations Ψ and $\chi_\|$ in such a way that $\hat{D}_{ij}$ is replaced by $\partial_i\partial_j$: in that case, it is easy to realize that R_S simplifies to

$$R_S = \frac{4}{a^2}\nabla^2\left(\frac{\mathcal{H}}{S_0'}S^{(1)} + \Psi\right) \ .$$

The argument of the Laplacian in the first-order expression of R_S resembles the definition we gave for the comoving curvature in 2.7.2. Indeed, we can use it to define a comoving curvature ζ in a geometrical way:

$$R_S = -\frac{4}{a^2}\nabla^2\zeta \ .$$

Since R_S is a 3-scalar, ζ is a 3-scalar the same way. At first-order, this new definition of ζ is gauge invariant, but this is no more true at higher orders. This a natural consequence of what we have said in section 2.4: as one can see from the value of R_S, eq. 4.46, which has null background, so at first-order it is naturally gauge invariant.

From now, on we will choose this approach to define ζ. In the language of subsection 2.4, this means that the comoving curvature is such that

$$\tilde{\zeta}(\tilde{x}) = \zeta(x)$$

and not $\tilde{\zeta}(x) = \zeta(x)$; this is true only at first-order:

$$\tilde{\zeta}^{(1)}(x) = \zeta^{(1)}(x) \ .$$

In case of a single-field inflationary model, we can choose the hypersurfaces such that $\phi = \phi_0 + \delta\phi + \ldots = 0$, which implies at first-order

$$\zeta = -H\frac{\delta\phi}{\phi_0'} - \Psi = Hv_{\parallel} - \Psi \ ,$$

where we have used 4.45.
Finally, notice that choosing the hypersurfaces $\rho = \rho_0 + \delta\rho + \ldots = 0$ we get exactly

$$\zeta_\rho = -H\frac{\delta\rho}{\rho_0'} - \Psi \ ,$$

which coincides with the curvature on a uniform density hypersurfaces 2.26, $\zeta_\rho = \mathcal{R}$. These final considerations justify *a posteriori* the names given to ζ and $\mathcal{R}$.

4.6.3 The Sasaki-Mukhanov equation

As we have said, we would like to write the equation of motion for the inflaton perturbations 4.4, considering also the perturbations of the metric we have introduced in chapter 2. The background equation of motion is 4.9, which in conformal time becomes

$$\phi_0'' + 2\frac{a'}{a}\phi_0' = -\frac{\partial V}{\partial\phi}a^2 \ .$$

Following [26], we want to perturb the inflaton equation of motion, which is

$$\Box\phi = \frac{\partial V}{\partial\phi} \ ,$$

where (see eq. 4.7)

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\partial_\nu\phi) \ .$$

In this case perturbing the box operator we have to perturb also the metric. The result of the perturbation is, considering the metric in conformal time with only scalar perturbations,

$$\delta\Box\phi = \frac{1}{a^2}\left[-\delta\phi'' - 2\frac{a'}{a}\delta\phi' + \nabla^2\delta\phi + 2\Phi\phi_0'' + 4\frac{a'}{a}\Phi\phi_0' + \Phi'\phi_0' + 3\Psi'\phi_0' + \nabla^2\omega_{\parallel}\phi_0' - a\phi_0'\nabla^2\chi_{\parallel}\right] \ ,$$

while the LHS of the Klein-Gordon equation is

$$\delta\frac{\partial V}{\partial\phi} = \delta\phi\frac{\partial^2 V}{\partial\phi^2} \ .$$

Now, multiplying both the members of the background equation of motion by 2Φ, one gets

$$2\Phi\phi_0'' + 4\frac{a'}{a}\Phi\phi_0' = -2\Phi\frac{\partial V}{\partial\phi}a^2 \ ,$$

which can be used to simplify the perturbation equation of motion[16] $\delta\Box\phi = \delta\phi\frac{\partial^2 V}{\partial\phi^2}$, finding

$$\delta\phi'' + 2\frac{a'}{a}\delta\phi' - \nabla^2\delta\phi - \Phi'\phi_0' - 3\Psi'\phi_0' - \nabla^2\omega_\parallel\phi_0' + a\phi_0'\nabla^2\chi_\parallel = -\delta\phi\frac{\partial^2 V}{\partial\phi^2}a^2 - 2\Phi\frac{\partial V}{\partial\phi} \ .$$

At this point one can pass to Fourier space and work in spatially flat gauge, where $\Psi = 0$ and $\chi_\parallel = 0$. Using the constraint equation of motion for the perturbations in this gauge, one can substitute the values of the metric perturbations, arriving to an equation containing only the inflaton fluctuations[17]

$$(a\delta\phi)'' + \left(k^2 - \frac{a''}{a} + a^2\mathcal{M}^2\right)a\delta\phi = 0 \ ,$$

where $\mathcal{M}$ is an effective mass term which in function of the slow-roll parameter is

$$\mathcal{M}^2 \simeq \mathcal{H}^2\left(3\eta - 6\epsilon\right) \ .$$

We define now the Sasaki-Mukhanov variable

$$Q := \delta\phi + \frac{\phi_0'}{\mathcal{H}}\Psi$$

and we notice that that in spatially flat gauge in which $\Psi = 0$ it coincides with the inflaton perturbation. This means that in this gauge the previous equation can be rewritten as

$$(aQ)'' + \left(k^2 - \frac{a''}{a} + a^2\frac{\mathcal{M}^2}{\mathcal{H}^2}\right)aQ = 0 \ . \tag{4.47}$$

Repeating the computations done in case of the quasi-de Sitter background, one finds the equation

$$(aQ)'' + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right)aQ = 0 \ , \tag{4.48}$$

where this time one has $\nu^2 - \frac{1}{4} = 9\epsilon - 3\eta_V = 6\epsilon - 3\eta$, i. e. $\nu \simeq \frac{3}{2} + 3\epsilon - \eta_V$. Since in this gauge $Q = \delta\phi$, taking into account the metric perturbations has simply modified the value of ν at slow-roll level. This means that we can simply redo all the previous computation, substituing the new value of ν, so we obtain the adimensional power spectrum

$$\Delta_{\delta\phi} = \mathcal{O}(1) \times \left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu}$$

and the spectral index is now $n_s = 1 - 6\epsilon + 2\eta_V$.
Finally, notice that in spatially flat gauge one has

$$\zeta = -\mathcal{H}\frac{\delta\phi}{\phi_0'} = -\frac{H}{\dot\phi_0}Q \ .$$

This implies (using 4.14)

$$\Delta_\zeta(k) = \frac{H^2}{\dot\phi_0^2}\Delta_Q(k) = \mathcal{O}(1) \times \left(\frac{H^2}{2\pi\dot\phi_0}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} = \mathcal{O}(1) \times \frac{1}{2\epsilon}\left(\frac{H}{2\pi M_P}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} \ . \tag{4.49}$$

In order to be compared to the statistics of the experimental data, the power spectrum, but also all the other correlators, must be evaluated at the end of inflation. However, one can evaluate them at horizon

[16] Cfr. [14], eq. B.2.
[17] See [23], eq. 208-210.

crossing, since ζ and D must be constant on superhorizon scales, thanks to Weinberg theorem[18]. Indicating with H_* the value of H at horizon crossing, when $k = aH_*$ one has $\Delta_\zeta = \mathcal{O}(1) \times \frac{1}{2\epsilon}\left(\frac{H_*}{2\pi M_P}\right)^2$ and the corresponding power spectrum is

$$P_\zeta(k) = \frac{2\pi^2}{k^3}\Delta_\zeta(k) = \mathcal{O}(1) \times \frac{H_*^2}{4\epsilon M_P^2 k^3} \ . \tag{4.50}$$

4.6.4 Gravitational waves from inflation

The gravitational waves are tensor perturbations of FLRW, which obey equation 2.24

$$(\chi_{ij}^T)'' + 2\mathcal{H}(\chi_{ij}^T)' - \nabla^2 \chi_{ij}^T = 0 \ , \tag{4.51}$$

since for a single field inflationary model $\pi_{ij}^T = 0$ (we have seen in 4.6.1 that in single-field inflation there is no anisotropic stress). Substituting $\chi_{ij}^T \longrightarrow \sqrt{32\pi G}\delta\phi$ one gets exactly the evolution equation for the inflaton perturbations, with no mass (the factor $\sqrt{32\pi G}$ is a convention). This means that we expect to have exactly the same result for adimensional power spectrum of $\delta\phi$ eq. 4.44, considering the extra normalization factor and with $\nu = \frac{3}{2} + \epsilon$, so that

$$\Delta_T(k) = \mathcal{O}(1) \times 32\pi G \left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{-2\epsilon} = \mathcal{O}(1) \times 4 \left(\frac{H}{2\pi M_P}\right)^2 \left(\frac{k}{aH}\right)^{-2\epsilon} \ .$$

In this case we define the tensor spectral index as (notice that in this case there is not -1 with respect to the definition 4.42 of n_s)

$$n_T = \frac{d \log \Delta_T(k)}{d \log k} \ ,$$

giving $n_T = -2\epsilon$. At horizon-crossing one has $k = aH_*$, so that $\Delta_T(k) = \mathcal{O}(1) \times 4\left(\frac{H_*}{2\pi M_P}\right)^2$ and the power spectrum is

$$P_T(k) = \frac{2\pi^2}{k^3}\Delta_T(k) = \mathcal{O}(1) \times \frac{2H_*^2}{M_P^2 k^3} \ . \tag{4.52}$$

Finally, we define the tensor-to-scalar ratio $r := \frac{2\Delta_T}{\Delta_\zeta}$, where the factor 2 takes into account the two polarization states of the gravitational waves. At horizon crossing one has

$$r = 2\frac{4\left(\frac{H_*}{2\pi M_P}\right)^2}{\frac{1}{2\epsilon}\left(\frac{H_*}{2\pi M_P}\right)^2} = 16\epsilon = -8n_T \ .$$

This relation is called consistency relation for inflation. Unfortunately, this is not the consistency relation for correlators we have mentioned before as the focus of the project. However, this relation is an experimentally testable relation, which can reveal if inflation is the correct mechanism describing the Early Universe (the problem is that GWs from inflation have never been seen, so n_T cannot be measured, but its value can only be limited).

4.6.5 ζ-gauge

As a final issue, we want to introduce a very important gauge in dealing with perturbations in inflationary models. This gauge has a wide range of names in literature (uniform inflaton gauge, comoving gauge), however we choose to call it ζ-gauge in order to avoid confusion with other gauges.

[18]Another way to see this is the following. During inflation H is not strictly constant (although it is approximately), since the expansion is quasi-de Sitter: however, we can expand it around the instant t^* of horizon crossing as $H = H_* + \dot{H}(t - t^*) + \mathcal{O}((t - t^*)^2)$ and, noting that $\dot{H} \propto \epsilon$, this means that all the higher orders are subleading in the slow-roll parameters expansion.

In this gauge we focus on the ij components of the metric and we kill $\chi_\|$ and the vector perturbations; we are also free to fix another scalar and we set the inflaton perturbation to zero. We get finally:

$$\delta\phi = 0 \ , \quad \chi_\| = 0 \ , \quad \chi^i_\perp = 0 \ .$$

This implies that the ij component of the metric perturbation becomes

$$\delta g_{ij} = a^2[(1-2\Psi)\delta_{ij} + \chi^T_{ij}] \ ,$$

where $\chi^{Ti}_i = 0$ and $\partial^i\chi^T_{ij} = 0$, as we know. Since in this gauge $\zeta = -\Psi$ we have that

$$\delta g_{ij} = a^2[(1+2\zeta)\delta_{ij} + \chi^T_{ij}] \ .$$

This gauge is very important since it is the most convenient to use in order to compute primordial bispectra using the ADM formalism.

4.6.6 Weinberg theorem in ζ-gauge

Since dealing with primordial non-Gaussianities one usually uses the ζ-gauge instead of synchronous gauge, we have to be sure that the results we have derived hold also in ζ-gauge. The statement of the theorem proved 3.4 relies on the Newtonian gauge, as in the original Weinberg's article. However, it is easy to realize that an analogous, even more simpler result, holds also in ζ-gauge. Indeed, the condition that $\delta\phi = 0$ imposes that

$$\tilde{\delta\phi} = \delta\phi - \dot{\phi}_0\epsilon = 0 = \delta\phi \ ,$$

which means that $\epsilon_0 = 0$. On the contrary, the conditions to impose on Δg_{0i} and Δg_{ij} remain the same (with $\epsilon_0 = 0$), implying that the more general transformation respecting the gauge fixing in $k \to 0$ limit one can write is, as we have seen in equation 3.16,

$$\xi^i = x^i + \lambda x^i + \omega^i_j x^j \ ,$$

with λ constant and ω_{ij} constant matrix such that $\omega_{ij}\delta^{ij} = 0$. An analogous result holds also in spatially flat gauge.

4.7 Primordial non-Gaussianity

In this section we introduce the concept of non-Gaussianity, firstly reviewing the Gaussian distributions ([41]). The starting point is the following fundamental integral

$$\prod_{i=1}^n \int_{-\infty}^{\infty} dx_i \mathrm{exp}\left(-\frac{1}{2}\vec{x}^T A\vec{x} + \vec{b}^T\vec{x}\right) = \sqrt{\frac{(2\pi)^n}{\det A}} e^{\frac{1}{2}\vec{b}^T A^{-1}\vec{b}} \ , \tag{4.53}$$

where $\vec{x} = (x_1, \ldots, x_n)$, A is an $n \times n$ positive definite matrix, $\vec{b}$ is a vector of n numbers, $\vec{x}^T A\vec{x} = \sum_{i,j=1}^n = x_i A_{ij} x_j$ and $\vec{b}^T\vec{x} = \sum_{i=1}^n b_i x_i$. This result can be proved firstly by changing the variable $\vec{x} \to \vec{x} + A^{-1}\vec{b}$ and then performing another change of variables which diagonalises the matrix (which has all the eigenvalues positive by hypothesis), so that we obtain n disentangled Gaussian integrals, for which we can apply the well-known result $\int_{\mathbb{R}} dx e^{-\frac{\lambda_i x^2}{2}} = \sqrt{\frac{2\pi}{\lambda_i}}$, with λ_i eigenvalue of A, in this case. Then $\det A$ is the product of the eigenvalues of the matrix, so one concludes.

To study the statistics of a system one wants to calculate the N-point correlation functions, also called N-point functions, which is the analogue to the quantum N-point correlation function 4.23. If the probability density function (PDF) is $p(x_1, \ldots, x_n)$, the N-point function is defined as

$$\langle x_{k_1} \ldots x_{k_N}\rangle = \mathcal{N}\int_{\mathbb{R}^n} d^n x x_{k_1} \ldots x_{k_N} p(x_1, \ldots, x_n) \ ,$$

where the normalisation is simply the 0-point function, i. e. $\mathcal{N} = \int d^n x x_{k_1} \dots x_{k_n} p(x_1, \dots, x_n)$. As one can easily check, introducing an external source $\vec{b}$ (which is exactly the vector defined before), the definition N-point function can be rewritten as

$$\langle x_{k_1} \dots x_{k_N} \rangle = \mathcal{N} \frac{\partial}{\partial b_{k_1}} \cdots \frac{\partial}{\partial b_{k_N}} \int_{\mathbb{R}^n} d^n x p(x_1, \dots, x_n) e^{\vec{b}^T \vec{x}} \Big|_{b=0} .$$

In this context, a very important result is (classical) Wick theorem, stating that any even correlation function for a Gaussian distribution can be written as the sum of products of 2-point functions, contracted in all the possible ways:

$$\langle x_{k_1} \dots x_{k_N} \rangle = \sum_{\text{permutations}} \prod_{(j,k)= \text{ a couple from } k_1 \dots k_l} \langle x_j x_k \rangle \ ,$$

It is crucial to underline that this result is valid only for Gaussian PDF: if this is not the case, for example, one can have surely a 3-point function different from 0. This means that its measurement (which will translate into the measurement of the bispectrum, as we will see) is a key channel to unveil non-Gaussianity.

In a cosmological contest, one is interested in examining the statistics for example of the CMB temperature anisotropies or of the primordial GWs background, which can contain useful information in order to understand inflation. A quantum field alone is not observable, so to understand what fields are involved in the inflationary mechanism we have to construct objects which are classical functions. We have already seen an example of this, which is the power spectrum, related to the 2-point correlation function. However, for what we have just said, this is not enough to examine the presence of a primordial non-Gaussianity: it is necessary to look at least at the 3-point function.

In principle, each N-point function potentially contains unique information that is not found in any of the others. This way, given an inflationary model, it is very important to compute both the power spectrum and the bispectrum, so to compare it to the experimental data in order to test it. However, from a phenomenological point of view, increasing the number of fields in an N-point function translates into a growing uncertainity, which makes the higher-N-correlation functions useless to test the model. Today experiments have measured quite precisely the power spectrum, but they are very far to constrain the bispectrum in such a way to disentagle the different inflationary models.

In the next section we are going to introduce a very useful procedure to compute higher-order correlators in inflationary models, which is called the *in-in* formalism. Then, we define the analogue to the power spectrum corresponding to the 3 and 4 point functions in Fourier space: bispectrum and trispectrum.

4.8 *In-in* formalism

The *in-in* formalism was developed firstly by Schwinger ([42]) and then by Keldysh ([43]) in physics of condensed matter. This formalism was useful, for example, to treat critical phenomena and non-equilibrium systems. In cosmology, it was firstly applied by Jordan ([44]) and Calzetta and B. L. Hu ([45]). In 2002 this formalism was used by Maldacena to compute all the three-point correlators expected in the single-field slow-roll model of inflation, in the seminal paper [46] which we will encounter also in the future. It was finally completely formalized by Weinberg a couple of years later ([47]). A review of the topic is given by [24].
The *in-in* formalism provides a method to calculate the N-point functions of cosmological perturbations. The main purpose of this section is to derive an explicit formula to compute the correlator

$$\langle \Omega | \phi(x_1) \dots \phi(x_N) | \Omega \rangle \ ,$$

which is nothing else than an N-point Wightman function. This is because, contrarily to the particle physics case in which we have *in* states at time $t \to -\infty$ and *out* states at time $t \to \infty$, here we want to compute the correlation only for states at very early times, so *in* states only. This explains the name given. In other words, this formalism is used to calculate expectation values of operators from only the initial Cauchy data without having to know about the final states of the system. We will comment on this later.

4.8.1 Background field method

When dealing with perturbations, we have to split the Hamiltonian as follows

$$H = H_b + H_0 + H_{\text{int}} \ . \tag{4.54}$$

The Hamiltonian is split into three parts: H_b made out of the classical background fields; H_0 made out of perturbations without interactions; H_{int} which is the interaction Hamiltonian for perturbations. This is the analogous of the splitting we made in section 4.4, in which we divided the field into a classical background and a quantum fluctuation.

Let us be more precise. In the section 4.3 we have seen that quantum fields evolve according to the Heisenberg equations: in conformal time they read

$$\begin{cases} i\phi' = [\phi, H] \\ i\pi' = [\pi, H] \end{cases} .$$

Here ϕ can be any sort of field (the scalar inflaton, the metric $g_{\mu\nu}$, ...). We want to split these equations into a classical background and a quantum fluctuation, $\phi = \phi_0 + \delta\phi$ and $\pi = \pi_0 + \delta\pi$ (in this case we consider the background dependent also on x, since we are not assuming *a priori* isotropicity and homogeneity). For simplicity, we consider a single-field theory. For what concerns the Hamiltonian, we Taylor expand in functional sense[19]

$$H[\phi(\vec{x},\tau), \pi(\vec{x},\tau)] = H_b + \int d^3x \frac{\partial \mathcal{H}}{\partial \phi_0(\vec{x},\tau)} \delta\phi(\vec{x},\tau) + \int d^3x \frac{\partial \mathcal{H}}{\partial \pi_0(\vec{x},\tau)} \delta\pi(\vec{x},\tau) + H_{\text{int}} \ ,$$

where H_{int} contains second-order terms of higher[20]

$$\begin{aligned} H_{\text{int}}[\delta\phi(\vec{x},\tau), \delta\pi(\vec{x},\tau); \tau] =& \frac{1}{2} \left[\frac{\delta^2 H}{\delta\phi_0(\vec{x},\tau)\delta\phi_0(\vec{y},\tau)} \delta\phi(\vec{x},\tau)\delta\phi(\vec{y},\tau) + \right. \\ & \left. + \frac{\delta^2 H}{\delta\pi_0(\vec{x},\tau)\delta\pi_0(\vec{y},\tau)} \delta\pi(\vec{x},\tau)\delta\pi(\vec{y},\tau) + 2\frac{\delta^2 H}{\delta\phi_0(\vec{x},\tau)\delta\pi_0(\vec{y},\tau)} \delta\phi(\vec{x},\tau)\delta\pi(\vec{y},\tau) \right] + \dots \ . \end{aligned}$$

Substituting this expansion into the Heisenberg equation for ϕ one gets

$$i\frac{d}{d\tau}(\phi_0(\vec{x},\tau) + \delta\phi(\vec{x},\tau)) = \left[\phi_0(\vec{x},\tau) + \delta\phi(\vec{x},\tau), H_b + \frac{\delta H}{\delta\phi_0(\vec{y},\tau)}\delta\phi(\vec{y},\tau) + \frac{\delta H}{\pi_0(\vec{y},\tau)}\delta\pi(\vec{y},\tau) + H_{\text{int}} \right] \ .$$

At this point we impose that the background is classical, so ϕ_0 and π_0 commute with everything while the perturbations $\delta\phi$ and $\delta\pi$ satisfy canonical commutation relation. Since the background commutes, one is left with

$$i\phi_0'(\vec{x},\tau) + i\delta\phi'(\vec{x},\tau) = \left[\delta\phi(\vec{x},\tau), \frac{\delta H}{\delta\phi_0(\vec{y},\tau)}\delta\phi(\vec{y},\tau) + \frac{\delta H}{\delta\pi_0(y,t)}\delta\pi(\vec{y},\tau) + H_{\text{int}} \right] \ ,$$

but since $[\delta\phi(\vec{x},\tau), \delta\phi(\vec{y},\tau)] = 0$ the first summand in the commutator vanishes. Moreover, since $[\delta\phi(\vec{x}), \delta\pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$, using the Hamilton equation for the background one has

$$\left[\delta\phi(\vec{x},\tau), \frac{\delta H}{\delta\pi_0(\vec{y},\tau)}\delta\pi(\vec{y},\tau) \right] = i\frac{\partial \mathcal{H}}{\partial \pi_0(\vec{x},\tau)} = i\phi_0'(\vec{x},\tau) \ ,$$

[19] In the Taylor expansion, all the $\mathcal{H}$ are evaluated at the background value of the fields $\phi_0(t), \pi_0(t)$.

[20] For sake of notation we indicate $\int \frac{\partial \mathcal{H}}{\partial g(\vec{x})}\delta g(\vec{x}) d^3x \to \frac{\delta H}{\delta g(\vec{x})}\delta g(\vec{x})$ and similarly for higher order cases.

these two terms cancels and one is left with (neglecting higher orders)

$$i\delta\phi'(\vec{x},\tau) = [\delta\phi(\vec{x},\tau), H_{\rm int}[\delta\phi(\tau),\delta\pi(\tau);\tau]] \ .$$

Repeating exactly the same computations for the second Heisenberg equation one gets

$$i\delta\pi'(\vec{x},\tau) = [\delta\pi(\vec{x},\tau), H_{\rm int}[\delta\phi(\tau),\delta\pi(\tau);\tau]] \ .$$

These last two equations are exactly two Heisenberg equations for the perturbations.
This implies that quantum perturbation evolve through Heiseberg equations 4.18, where the Hamiltonian is given by the interaction Hamiltonian in 4.54. This means that to compute the correlators between perturbations we can proceed exactly as we did in section 4.3, since they behave as quantum fields with Hamiltonian $H_{\rm int}$.

4.8.2 The *in-in* formula

Let $Q(\tau,\vec{x}_1,\ldots)$ be an equal-time product of quantum perturbations: in section 4.3 we have seen that in interaction picture the correlator becomes

$$\langle Q(\tau,\vec{x}_1,\ldots)\rangle = \langle\Omega(\tau_0)|_I U_I^\dagger(\tau,\tau_0) Q(\tau,\vec{x}_1,\ldots)\ldots U_I(\tau,\tau_0)|\Omega(\tau_0)\rangle_I$$

where[21]

$$U_I(\tau,\tau_0) = T\left[e^{-i\int_{\tau_0}^{\tau} H^I_{\rm int}(\tau')d\tau'}\right] \ . \tag{4.55}$$

This implies that the correlator is

$$\langle Q(\tau,\vec{x}_1,\ldots)\rangle = \langle\Omega(\tau_0)|_I \bar{T}\left[e^{i\int_{\tau_0}^{\tau} H^I_{\rm int}(\tau')d\tau'}\right] Q(\tau,\vec{x}_1,\ldots) T\left[e^{-i\int_{\tau_0}^{\tau} H^I_{\rm int}(\tau')d\tau'}\right]|\Omega(\tau_0)\rangle_I \ , \tag{4.56}$$

which is called *in-in* formula. Notice that the initial time is t_0 but one usually is interested in epochs when the modes are well inside the horizon, so $\tau_0 \to -\infty$. However this formula is still very hard to apply, for two reasons. Firstly, the evolution equation 4.25 is difficult to solve exactly, i. e. the solution 4.55 is difficult to handle: it contains an infinite series of operators. This problem can be overcome if one imagines to treat the problem perturbatively, i.e one can truncate the expansion to a chosen order. This is what we are going to explain in the next section. However, there is a second problem: in perturbative expansions one deals with creation and annihilation operators acting on the vacuum state of the free theory, but in the formula there is the vacuum of the interacting theory. So, before passing to treat the *in-in* equation perturbatively, we have to find a form of it containing the vacuum of the free theory $|0\rangle_I$ insted of the vacuum of the interacting theory $|\Omega\rangle_I$. The main idea behind this is to obtain a free theory for $\tau \to -\infty$: indeed, the vacuum state of the free theory $|0\rangle$ evolves in Schrödinger picture thanks to the action of the evolution operator U

$$|0(\tau)\rangle_S = U_S(\tau,\tau_0)\,|0(\tau_0)\rangle_S = e^{-iH(\tau-\tau_0)}|0(\tau_0)\rangle_S \ .$$

This can be related to $|\Omega\rangle_S$ using a complete set of energy eigenstates of the interacting theory (for simplicity we consider a discrete set and not a continuos one), giving (from now on for sake of notation we will omit to indicate τ_0 in the vacuum states and the S indicating the Schrödinger picture)

$$e^{-iH(\tau-\tau_0)}|0\rangle = \sum_n e^{-E_n(\tau-\tau_0)}|n\rangle\langle n|0\rangle = e^{-iE_\Omega(\tau-\tau_0)}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq\Omega} e^{-iE_n(\tau-\tau_0)}|n\rangle\langle n|0\rangle \ ,$$

where $E_\Omega = \langle\Omega|H|\Omega\rangle$ is the energy of the ground state of the interacting theory. If one Wick rotates τ by a small angle ϵ in the complex plane

$$\tau \to \tilde{\tau} = \tau(1-i\epsilon) \ ,$$

[21] Remember also that in interaction picture the S-matrix is defined as

$$S := \lim_{\substack{\tau\to+\infty \\ \tau_0\to-\infty}} U_I(\tau,\tau_0) \ .$$

This is the *in-out* case of particle physics, while here we work only in the limit $\tau_0 \to -\infty$.

in the limit $\tau_0 \to -\infty$ the $i\epsilon$ factor kills the exponential. Then the above equation can be solved as

$$e^{-iE_\Omega(\tilde{\tau}-\tilde{\tau}_0)}|\Omega\rangle = \frac{e^{-iH(\tilde{\tau}-\tilde{\tau}_0)}|0\rangle}{\langle\Omega|0\rangle} - \frac{1}{\langle\Omega|0\rangle}\sum_{n\neq\Omega} e^{-iE_n(\tilde{\tau}-\tilde{\tau}_0)}|n\rangle\langle n|0\rangle \ .$$

At this point we want to go back to interaction picture, so we use 4.24 and we multiply both sides by $U_0^\dagger(\tilde{\tau}_0, 0)$. Then we take the limit $\tau_0 \to -\infty$ and $\epsilon \to 0$, which implies that the second summand on the RHS is null, since E_Ω is the lowest value among all the energy eigenvalues, so the damping effect is slower for the ground state. The result is

$$\lim_{\substack{\tau_0\to-\infty\\ \epsilon\to 0}} U_I(\tau,\tau_0)|\Omega\rangle_I = \lim_{\substack{\tau_0\to-\infty\\ \epsilon\to 0}} \frac{U_I(\tau,\tau_0)|0\rangle_I}{\langle\Omega|0\rangle} \ .$$

Then we can use this result into the *in-in* formula derived above to get finally

$$\langle Q(\tau,\vec{x}_1,\ldots)\rangle = \frac{1}{|\langle 0|\Omega\rangle|^2}\langle 0(\tau_0)|_I \bar{T}\left[e^{i\int_{\tau_0}^{\tau}\tilde{H}^I(\tau')d\tau'}\right] Q(\tau,\vec{x}_1,\ldots) T\left[e^{-i\int_{\tau_0}^{\tau}\tilde{H}^I(\tau')d\tau'}\right]|0(\tau_0)\rangle_I \ . \qquad (4.57)$$

The denominator can be furtherly simplified, since it is equal to one. Indeed, we can equate the two *in-in* formulas we have derived 4.56 and 4.57 and setting $Q(\tau, x_1,\ldots) = 1$ one has (the limit $\tau_0 \to -\infty$, $\epsilon \to 0$ is underlined)

$$\begin{aligned}\langle 1\rangle &= \langle\Omega(\tau_0)|_I \bar{T}\left[e^{i\int_{\tau_0}^{t}\tilde{H}^I(\tau')d\tau'}\right] T\left[e^{-i\int_{\tau_0}^{\tau}\tilde{H}^I(\tau')d\tau'}\right]|\Omega(t_0)\rangle_I = \\ &= \frac{1}{|\langle 0|\Omega\rangle|^2}\langle 0(\tau_0)|_I \bar{T}\left[e^{i\int_{\tau_0}^{t}\tilde{H}^I(\tau')d\tau'}\right] T\left[e^{-i\int_{\tau_0}^{\tau}\tilde{H}^I(\tau')d\tau'}\right]|0(\tau_0)\rangle_I \ ,\end{aligned}$$

which implies

$$\langle\Omega(\tau_0)|_I\Omega(\tau_0)\rangle_I = \frac{1}{|\langle 0|\Omega\rangle|^2}\langle 0(\tau_0)|_I 0(\tau_0)\rangle_I$$

so, assuming that the two vacua (free and interacting one) are normalized

$$\langle 0(\tau_0)|0(\tau_0)\rangle_I = \langle\Omega(\tau_0)|\Omega(\tau_0)\rangle_I = 1 \ ,$$

one gets $|\langle 0|\Omega\rangle|^2 = 1$.

Notice finally that the free theory vacuum in interaction picture $|0\rangle_I$ is the one which is annihilated by the operator a_I, the one obtained by solving the equation of motion for the fields $\delta\phi_I$ and $\delta\pi_I$ in interaction picture, which can be explicitly found since in interaction picture they evolve as free fields.

4.8.3 The *in-in* perturbative formula

We assume now the perturbative hypothesis, i.e.

$$H_{\text{int}} \ll H_0 \ ,$$

which can be imposed for example assuming that the interaction constant $g \propto H^I_{\text{int}}$ is such that $g \ll 1$. Now, the explicit expression of U_I contains an exponential of an operator, which is defined as an infinite Taylor series. Since each power of the Hamiltonian will be negligible with respect to the following power, we are allowed to stop at a given order. For example, at first-order the T product is not necessary and using the fact that U_I is self-adjoint by construction the result is

$$\langle Q(\tau,\vec{x}_1,\ldots)\rangle = \langle 0(\tau_0)|_I Q(\tau,\vec{x}_1,\ldots)|0_I(\tau_0)\rangle + i\int_{-\infty(1-i\epsilon)}^{\tau}\langle 0_I(\tau_0)|[H^I_{int}(\tau'), Q(\tau,\vec{x}_1,\ldots)]|0(\tau_0)\rangle_I d\tau' + \ldots \ . \qquad (4.58)$$

Generally the first term of the series can be null, since usually one has in mind to compute a bispectrum (see the following section), which means that Q is made of three fields: since the first term is given

by free fields, from Wick theorem we will see that the odd-point functions are null[22].
Proceeding by induction, one can show that the n-th order term has the following expression[23]

$$i^n\int_{\tau_0}^{\tau} d\tau_1\int_{\tau_0}^{\tau_1} d\tau_2\ldots\int_{\tau_0}^{\tau_{n-1}} d\tau_n\langle 0(\tau_0)|[H^I_{int}(\tau_n)[H^I_{int}(\tau_{n-1})\cdots[H^I_{int}(\tau_1),Q(\tau,\vec{x}_1,\ldots)]\cdots]]|0(\tau_0)\rangle\ .$$

This expansion can be also expressed using Feynman diagrams to guide calculations; however, since this is beyond the scope of this project we will not discuss this matter. As we have seen, in the end one has to take the limit $\tau_0\to-\infty$, but, in general the integrals appearing in the expansion are ill-defined, since the integrand is usually oscillatory at infinity. This bad behaviour usually disappears if one uses $-\infty(1-i\epsilon)$ (taking then $\epsilon\to 0$) in the lower extremum; alternatively one can keep $-\infty$ and after the computation performing the Wick rotation $\tau\to -i\tau$ in the integration variable. For example, two common integrals which can be evaluated using the Wick rotation are the following:

$$\int_{-\infty}^{0} d\tau(i\tau)^n e^{ia\tau}=i(-1)^n\frac{n!}{a^{n+1}}\ ,\quad \int_{-\infty}^{0}\frac{d\tau}{t}e^{ia\tau}=-\frac{1}{a}\ . \tag{4.59}$$

4.8.4 Wick theorem for *in-in* formalism

In order to compute each term of the perturbative series one generally consider the solution to the equation of motion in Fourier space, which is given by 4.20

$$\begin{aligned}\delta\phi(\tau,\vec{x})&=\int\frac{d^3k}{(2\pi)^3}\left[a^I_{\vec{k}}u(\tau,k)e^{-i\vec{k}\cdot\vec{x}}+a^{I\dagger}_{\vec{k}}u^*(\tau,k)e^{i\vec{k}\cdot\vec{x}}\right]=\\&=\int\frac{d^3k}{(2\pi)^3}\left[\delta\phi_+(\tau,\vec{k})e^{-i\vec{k}\cdot\vec{x}}+\delta\phi_-(\tau,\vec{k})e^{i\vec{k}\cdot\vec{x}}\right]=\delta\phi_+(\tau,\vec{x})+\delta\phi_-(\tau,\vec{x})\ ,\end{aligned}$$

then one uses the algebra of the creation and annihilation operators 4.22 and their action on the vacuum state $a|0\rangle=0$ and $\langle 0|a^\dagger=0$, in order to simplify the results. Unfortunately, this procedure can be really long since the interaction Hamiltonian and Q contains at least three fields $\delta\phi$. However, there is a standard QFT result, called Wick theorem, which allows to simplify the computations.
In *in-out* QFT, Wick theorem states that

$$\begin{aligned}T[\phi(x_1)\ldots\phi(x_n)]=&N[\phi(x_1)\ldots\phi(x_n)]+\sum_{\text{1 contraction}}N[\phi(x_1)\ldots\phi(x_n)]+\ldots+\\&+\sum_{\text{all contractions}}N[\phi(x_1)\ldots\phi(x_n)]\ ,\end{aligned}$$

where a contraction is equal to the Feynman propagator $\overline{\phi(x_1)\phi(x_2)}=D_F(x-y)$ and N is the normal ordered product.
In *in-in* case there is no the T product inside the expectation value, since one wants to compute a Wightman function, so one expects not to have a Feynman propagator (which contains a T product), but instead a contraction of type 4.43 (the a in the denominator is an overall normalization which is usually not present):

$$\overline{\delta\phi^+(\tau,\vec{k}_1)\delta\phi^-(\tau,\vec{k}_2)}=(2\pi)^3\frac{|u(\tau,k_1)|^2}{a^2}\delta^{(3)}(\vec{k}_1-\vec{k}_2)\ .$$

This is exactly what happens, as one can show, repeating the demonstration of the *in-out* Wick theorem, which can be performed by induction (see [35]). One finds that Wick thoerem in this case is still valid without T, but using this result for the contractions.
Finally, if in Wick contractions the operators depend on time one usually encounters derivative terms such as $\overline{\delta\phi(\vec{k}_1,\tau)\partial_\tau\delta\phi(\vec{k}_2,\tau)}$. This gives simply the result

$$\overline{\delta\phi(\tau,\vec{k}_1)\partial_\tau\delta\phi'(\tau,\vec{k}_2)}=(2\pi)^3u(k_1)\frac{du(\tau,k_2)}{d\tau}\delta^{(3)}(\vec{k}_1-\vec{k}_2)\ .$$

[22] This is true both in classical field theory and in quantum field theory, see the following subsection.
[23] See [24], eq. 3.41.

4.9 Bispectrum and trispectrum

In section 4.5 we have introduced the power spectrum, which is related to the 2-point function. However, this is not sufficient to infer a non-Gaussianity, as we have seen in section 4.7. This way, we define the bispectrum B from the 3-point function as the analogous to the power spectrum from the 2-point function:

$$\langle \delta(\vec{k}_1)\delta(\vec{k}_2)\delta(\vec{k}_3)\rangle = (2\pi)^3 \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) B(k_1, k_2, k_3) \ . \tag{4.60}$$

As before we will omit to indicate that B depends only on the norm of the three vectors. As in the case of the power spectrum, we have the Dirac delta which implies the momentum conservation (for homogenity) and B depends only on the norm of the $\vec{k}$'s for isotropy. In general the bispectrum can be rewritten as

$$B(k_1, k_2, k_3) = f_{NL} F(k_1, k_2, k_3) \ ,$$

where f_{NL} is called amplitude. We also define the shape function S as follows

$$S(k_1, k_2, k_3) = \frac{1}{N}(k_1 k_2 k_3)^2 B(k_1, k_2, k_3) \tag{4.61}$$

where N is normalization factor (introduced to reabsorb all the numerical coefficients).

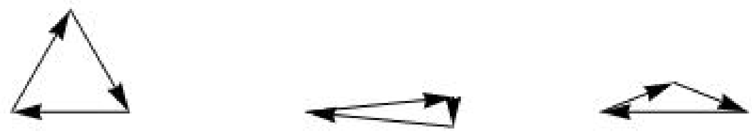

Figure 4.2: The three limits in the triangle of the momenta. From left to right: equilateral, squeezed, folded.

Even assuming statistical isotropy, there is still an infinite freedom in the functional form of F. However, there are shapes which are physically more relevant, since they appear many times in different reasonable models of inflation. In this sense, inflationary models can be classified by means of the shape of F they produce. The advantage of the this method is that it provides a direct link between observations and theory: for reasons regarding the weakness of the CMB signal to measure f_{NL} one has to assume a specific shape. In general, one tries to find some "linear combinations" of shape functions, which can encompass a large class of possible shapes. This means that in order to compare theory and data, a set of theoretically motivated ansatzes for the bispectrum form must be provided, named *templates*. These templates are tied with a particular configuration of the three $\vec{k}$'s in the momentum space. Indeed, the various $\vec{k}$ are blocked to sum up to 0 for the presence of the Dirac delta, so they must form a triangle in momentum space.
Having said this, the principal templates are the following ([24]).

- Equilateral shape:

$$S_{\text{equilateral}} = 6\left(\frac{k_1^2}{k_2 k_3} + \text{sym}\right) + 6\left(\frac{k_1}{k_2} + \text{sym}\right) - 12 \ .$$

  It peaks at $k_1 = k_2 = k_3$, so in the case of an equilateral triangle in momentum space. This shape is typical for inflationary models in which the Lagrangian includes non-canonical kinetic terms.

- Local or squeezed shape:

$$S_{\text{local}} = \frac{k_1^2}{k_2 k_3} + \text{sym} \ .$$

  It peaks in the squeezed limit $k_1 = 0$ and $k_2 = k_3$. This shape is expected in standard single-field or multi-field inflationary models.

- Folded shape:

$$S_{\text{folded}} = 6\left(\frac{k_1^2}{k_2 k_3} + \text{sym}\right) - 6\left(\frac{k_1}{k_2} + \text{sym}\right) + 18 \ .$$

It peaks for $k_1 = 2k_2$ and $k_2 = k_3$. This shape is typical of inflationary models with TransPlanckian theories, i. e. theories where the initial state is not Bunch-Davies but an excited state ([24]).

- Orthogonal shape:

$$S_{\text{orthogonal}} = -18\left(\frac{k_1^2}{k_2k_3} + \text{sym}\right) + 18\left(\frac{k_1}{k_2} + \text{sym}\right) - 48\ .$$

As the equilateral form, this ansatz emerges in models characterized by derivative interactions.

It is worth to stress that these are only four examples of a vast variety of different ansatzes proposed.

Finally, one can also define the trispectrum T as expected,

$$\langle \delta(\vec{k}_1)\delta(\vec{k}_2)\delta(\vec{k}_3)\delta(\vec{k}_4)\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4)T(k_1, k_2, k_3, k_4)$$

and all the higher order spectra. However, it is clear that increasing N in the N-point function implies that calculations become rapidly very long and the final results cumbersome. Moreover, still nowadays one in interested in the inflationary 3-point functions, since the sensibility of the experiments is not enough to measure primordial non-Gaussianities.
The focus of this project is not to compute the bispectrum of a specific inflationary model, but to study a specific relation between power spectra and bispectra, the Maldacena consistency relation, which we will introduce in the next chapter. This means that we are not going to compute explicitly different bispectra provided by different models, which can be really complicated. However, in the following section we are going to show an example of this computation in the simplest case of the single-field inflation we have introduced, since it was the starting point that lead physicists to the consistency relation.

4.10 The primordial non-Gaussianity from a single field inflationary model

In principle one could apply the *in-in* formula to compute bispectra and trispectra of the inflation perturbation theory we considered in 4.4. In this case we would have only pertubations of the inflaton field so we have the correlators $\langle\delta\phi(x_1)\delta\phi(x_2)\rangle$, giving the power spectrum, $\langle\delta\phi(x_1)\delta\phi(x_2)\delta\phi(x_3)\rangle$, giving the trispectrum and so on and so forth. However, for a fully consistent computation one has to consider the perturbations of the metric as we saw in chapter 2. In this type of computations, as we will see, in general one is interested in the correlators involving the curvature perturbation ζ and the tensor modes D_{ij}, in Fourier space: $\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle$, $\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)D_{ij}(\vec{k}_3)\rangle$, $\langle\zeta(\vec{k}_1)D_{ij}(\vec{k}_2)D_{kl}(\vec{k}_3)\rangle$, $\langle D_{ij}(\vec{k}_1)D_{kl}(\vec{k}_2)D_{mn}(\vec{k}_3)\rangle$ (as we have seen, correlators including vector perturbations are generally neglected, since they have no role in single-field inflation). This is because for Weinberg theorem ζ and D are constant on superhorizon scales, so they provide a natural link to inflation. This is also the reason why ζ-gauge is the privileged one to compute such bispectra.
All these correlators are equal time correlators, evaluated at the end of inflation, in order to compare theory to data. For this reasons, in the following we will always omit to indicate the instant of time of evaluation.
This is what we are going to present in this section. The first idea of this computation is contained in [14], then a complete analysis was performed by Maldacena in [46]. We follow also [48] which is a pedagogical review of the original computation.

4.10.1 second-order action

Since we would like to apply *in-in* formalism to compute the bispectrum arising from the action 4.2, as we have seen we have to split the fields into a background and a fluctuation, isolate the fluctuation Lagrangian and then do the Legendre transform to get H_{int}. This operation is trivial if one considers the inflaton in a fixed background (which could be (quasi) de Sitter), but here we want to consider the perturbations also of the metric, as we have seen in chapter 2. This poses from the beginning a

problem, since one would like to work in a Hamiltonian formulation of general relativity. A solution to this is given by the ADM formalism, which is discussed in details in appendix B. Here we limit ourselves to write down the results obtained in the appendix. Moreover, calculations are very lengthy and since they are not so crucial for the purpose of this project in certain cases we will limit to report directly the results without doing all the explicit algebraic passages.
The metric in ADM parametrization is

$$ds^2 = -\left(N^2 - N_i N^i\right) dt^2 + 2N_i dx^i dt + \gamma_{ij} dx^i dx^j \tag{4.62}$$

where N and N_i have the role of Lagrange multiplier and they are called *lapse* and *shift*. Using this decomposition, one can show (see the appendix) that the Einstein-Hilbert action is equivalent to (see the appendix)

$$S_{ADM} = \int dt d^3x \sqrt{\gamma} N \left(R^{(3)} + K^{ij} K_{ij} - K^i_i K^j_j\right) \ , \tag{4.63}$$

where $R^{(3)}$ is the 3-dimensional Ricci curvature (defined in the appendix) and

$$K_{ij} = -N\Gamma^0_{ij} = \frac{1}{2N}(D^{(3)}_i N_j + D^{(3)}_j N_i - \dot{\gamma}_{ij}) \ ,$$

with $D^{(3)}_i$ spatial covariant derivative, defined in the appendix (we put $M_P = 1$ for sake of shortness and we restore it in the end of the computation). Notice that in ADM formalism, since the spacetime is foliated in costant time hypersurfaces, one usually splits the spacetime element as $d^4x = dt d^3x$. Since one has $\sqrt{-g} = N\sqrt{\gamma}$ with $\gamma = \det \gamma_{ij}$, the action 4.2 coupling the inflaton to the metric in ADM formalism becomes

$$S = \frac{1}{2} \int dt d^3x \sqrt{\gamma} \left(NR^{(3)} + N^{-1}(E_{ij}E^{ij} - E^2) + N^{-1}(\dot{\phi} - N^i \partial_i \phi)^2 - N\gamma^{ij}\partial_i\phi\partial_j\phi - 2NV\right) \ ,$$

where we have defined $E_{ij} = NK_{ij}$ and $E = \gamma^{ij}E_{ij}$ (defined this way to be consistent with [46]). As explained in the appendix, N and N_i are Lagrange multipliers, as one can see from the action, having no time derivatives. Their meaning is that the variation of the action with respect to N and N_i gives four equations which are not dynamical, but they are constraints. This comes from the fact that not all the ten elements of the Einstein equations are dynamical, but there are 4 constraints: indeed the real degrees of freedom of the metric this way are $10 - 4 - 4 = 2$ (10 since the metric is a symmetric matrix; first 4 since 4 equations are not dynamical; second 4 since they are 4 constraints), which is a well known result. The ADM formalism has the advantage to make this fact immediately clear. These two constraints are the following. The variation with respect to N is trivial and means simply to calculate $\frac{\partial \mathcal{L}}{\partial N} = 0$. The result is

$$R^{(3)} - 2V - \frac{1}{N^2}\left(E_{ij}E^{ij} - E^2 - (\dot{\phi} - N^i\partial_i\phi)^2\right) = 0 \ . \tag{4.64}$$

The variation with respect to N_i is slightly more difficult since E_{ij} contains $D^{(3)}_i N_j$ into itself and the result[24] is

$$D^{(3)}_i \left(N^{-1}(E^i_j - E\delta^i_j)\right) = 0 \ . \tag{4.65}$$

At this point the strategy is the following. As deriving 2.23, since GR calculations are cumbersome, we limit to report the sketch of the calculations.

[24]One firstly computes

$$\begin{aligned} \frac{\partial E_{ab}}{\partial D^{(3)}_i N_j} &= \frac{1}{2}\frac{\partial}{\partial D^{(3)}_i N_j}\left(D^{(3)}_i N_j + D^{(3)}_j N_i - \dot{\gamma}_{ij}\right) = \delta_{ja}\delta_{bi} + \delta_{bj}\delta_{ai} \\ \frac{\partial E}{\partial D^{(3)}_i N_j} &= \gamma^{ab}\frac{\partial E_{ab}}{\partial D^{(3)}_i N_j} = 2\delta_{ja}\delta_{ai} \end{aligned} \ ,$$

and then using the Euler-Lagrange equation $\frac{\partial \mathcal{L}}{\partial N} - D^{(3)}_i N_j \frac{\partial E_{ab}}{\partial D^{(3)}_i N_j} = 0$ gets the result.

- We make the computations in ζ-gauge, defined in section 4.6.5. This way, the spatial part of the metric is
$$\delta g_{ij} = a^2(\tau)\left((1+2\zeta)\delta_{ij} + D_{ij}\right) \ . \tag{4.66}$$
From now on, in the rest of this project we call the tensor perturbation $D_{ij} = \chi^T_{ij}$. However, in this section we will not compute correlators involving tensors, so we neglect D_{ij} by now. This expression is usually rewritten including an exponential matrix
$$\delta g_{ij} = a^2(\tau) e^{2\zeta\delta_{ij} + D_{ij}} \ .$$
This can be simple to manipulate doing computation and, despite containing perturbations at all orders, it must be always truncated at first-order in the end.

- Using the ADM expansion of the metric in this gauge, we compute the relevant quantities in the metric. For simplicity we consider only scalar modes. The ij component of the metric is $\gamma_{ij} = a^2 e^{2\zeta}\delta_{ij}$, so $\gamma^{ij} = a^{-2}e^{-2\zeta}\delta^{ij}$. This way
$$\dot{\gamma}_{ij} = \partial_t(a^2 e^{2\zeta}\delta_{ij}) = (2a\dot{a}e^{2\zeta} + a^2 2\dot{\zeta}e^{2\zeta})\delta_{ij} = 2a^2(H+\dot{\zeta})e^{2\zeta}\delta_{ij}$$
and similarly $\dot{\gamma}^{ij} = -2a^{-2}(H+\dot{\zeta})\delta^{ij}$. This way one can compute the connection coefficients which are
$$\Gamma^{(3)k}_{ij} = \frac{1}{2}\gamma^{kl}\left(\partial_j\gamma_{il} + \partial_i\gamma_{lj} - \partial_l\gamma_{ij}\right) = \delta^{kl}\left(\partial_j\zeta\delta_{il} + \partial_i\zeta\delta_{jl} - \partial_l\zeta\delta_{ij}\right) \ .$$
From this one can compute the spatial curvature
$$R^{(3)} = \gamma^{ik}\gamma^{jk}\left(\gamma_{im}\partial_k\Gamma^{(3)m}_{jl} + \Gamma^{(3)}_{ikm}\Gamma^{(3)m}_{lj} - (k\leftrightarrow l)\right) = -2a^{-2}e^{-2\zeta}\left(2\nabla^2\zeta + (\partial_k\zeta)^2\right)$$
and the extrinsic curvature
$$\begin{aligned} E_{ij} &= a^2 e^{2\zeta}\delta_{ij} - \partial_{\{i}N_{j\}} + \left(2N_{\{i}\partial_{j\}}\zeta - N_k\partial_k\zeta\delta_{ij}\right) \\ E^{ij} &= \gamma^{ik}\gamma^{jl}E_{kl}a^{-4} = a^{-4}e^{-4\zeta}\delta^{ik}\delta^{jl}E_{kl} \ , \end{aligned}$$
so that
$$E = h^{ij}E_{ij} = 3(H+\dot{\zeta}) - a^{-2}e^{-2\zeta}\left(\partial_k N_k + N_k\partial_k\zeta\right)$$
and thus one has
$$\begin{aligned} E^{ij}E_{ij} - E^2 = &- 6(H+\dot{\zeta})^2 + \frac{4e^{-2\zeta}}{a^2}(H+\dot{\zeta})\left(\partial_i N_i + N_i\partial_i\zeta\right) + \\ &- \frac{e^{-4\zeta}}{a^4}\left[(\partial_i N_i)^2 + 2\left(\partial_i N_i\zeta\right)^2 - \left(\partial_{\{i}N_{j\}} - (\partial_i N_i + N_i\partial_i\zeta)\right)^2\right] \ . \end{aligned}$$
The values obtained for $R^{(3)}$ and $E^{ij}E_{ij} - E^2$ can be substituted in the action.

- We expand the shift and the lapse as
$$\begin{aligned} N &= N^{(0)} + N^{(1)} + \dots \\ N_i &= N_i^{(0)} + N_i^{(1)} + \dots \end{aligned}$$
where the shift can be SVT expanded as $N_i = \partial_i\alpha + \beta_i$. Then, since we are considering only the scalar sector, β_i is neglected.

- We solve the constraint equations 4.64 and 4.65 at zeroth order in the shift and in the lapse; the result is
$$N^{(0)} = 1 \ , \quad N_i^{(0)} = 0 \ .$$

- Using this result, we solve the constraint equations 4.64 and 4.65 at first-order in the lapse; the result is

$$N^{(1)} = \frac{\dot{\zeta}}{H} \ , \quad N_i^{(1)} = \partial_i \alpha \ , \quad \alpha = -\frac{\zeta}{H} + a^2 \frac{\dot{\phi}_0^2}{2H^2} \nabla^{-2} \dot{\zeta} \ , \tag{4.67}$$

where ∇^{-2} is the inverse of the Laplacian operator ∇^2.

- At this point one has to insert these results in the action in ADM formalism. The result at second-order is

$$\begin{aligned} S_2 =& \frac{1}{2} \int dt d^3x \left[a e^{\zeta} \left(1 + \frac{\dot{\zeta}}{H} \right) \left(-4\nabla^2 \zeta - 2(\partial_k \zeta)^2 - 2V a^2 e^{2\zeta} \right) \right. \\ & \left. + e^{3\zeta} a^3 \left(1 - \frac{\dot{\zeta}}{H} + \frac{\dot{\zeta}^2}{H^2} \right) \left[-6(H + \dot{\zeta})^2 + \dot{\phi}^2 + 4a^{-2} e^{-2\zeta} (H + \dot{\zeta}) \left(\partial_i \psi \partial_i \zeta + \partial^2 \psi \right) \right] \right] \ . \end{aligned}$$

- Finally, integrating by parts, many pieces in the Lagrangian result to be boundary terms, which can be erased. Using the background equation of motion $H^2 = \frac{1}{3M_P^2} \left(\frac{\dot{\phi}_0^2}{2} + V \right)$ and $\frac{\ddot{a}}{a} = -\frac{1}{6M_P^2} \left(\dot{\phi}_0^2 - V \right) = \frac{V}{M_P^2} - 2H^2$, one gets the simple result

$$S_2 = \int dt d^3x \frac{\dot{\phi}_0^2}{H^2} \left[\frac{a^3}{2} \dot{\zeta}^2 - \frac{a}{2} (\partial_k \zeta)^2 \right] \ . \tag{4.68}$$

  The first-order turns out to be a total derivative term[25].

Remembering that the slow-roll parameter ϵ can be written as $\epsilon = \frac{\dot{\phi}_0^2}{2H^2}$ (since $M_P = 1$), we have that

$$S_2 = \epsilon \int dt d^3x \left[a^3 \dot{\zeta}^2 - a (\partial_k \zeta)^2 \right] \ .$$

This way, the second-order action of the perturbations is suppressed, since proportional to a slow-roll parameter. Notice that this action is exactly the Sasaki-Mukhanov action 4.27 (up to the suppression parameter ϵ).

A second way to derive Sasaki-Mukhanov equation

In this section we are going to show that from the second-order action S_2 we can derive the Sasaki-Mukhanov equation 4.47 ([48]). This is a consistency check of the result obtained in 4.6.3. However, to perform this we have to switch to spatially flat gauge, where $\Psi = \chi_{\parallel} = 0$ and $\chi_i^{\perp} = 0$. This way, the inflaton and the ij components of the metric become

$$\phi = \phi_0 + \delta\phi \ , \quad g_{ij} = a^2(\tau)(1 + D_{ij}) \ .$$

We indicate with ζ_n the comoving curvature in this gauge. We know that $\zeta_n = -H \frac{\delta\phi}{\dot{\phi}_0}$; we have also to take into account the rescaling 4.28 $\bar{\delta\phi} = a\delta\phi$, so that $\zeta_n = -\frac{H}{a} \frac{\bar{\delta\phi}}{\dot{\phi}_0}$. In appendix A of his paper, Maldacena has shown that ζ_n and ζ are related perturbatively by the following relation

$$\begin{aligned} \zeta =& \zeta_n + \frac{1}{2} \frac{\ddot{\phi}_0}{\dot{\phi}_0 H} \zeta_n^2 + \frac{1}{4} \frac{\dot{\phi}_0^2}{H^2} \zeta_n^2 + \frac{1}{H} \dot{\zeta}_n \zeta_n - \frac{1}{4} \frac{a^{-2}}{H^2} \nabla^{-2} \partial_i \partial_j \left(\partial_i \zeta_n \partial_j \zeta_1 \right) + \\ & + \frac{1}{2H} \partial_i \alpha \partial_i \zeta_n - \frac{1}{2H} \nabla^{-2} \partial_i \partial_j \left(\partial_i \alpha \partial_j \zeta_n \right) - \frac{1}{4H} \dot{D}_{ij} \partial_i \partial_j \zeta_n + \dots \ , \end{aligned} \tag{4.69}$$

where D_{ij} is the tensor perturbation. Since we want to consider only an action expanded up to quadratical order in the fields, substituting this into 4.68 we can simply use $\zeta = \zeta_n$. The derivative with respect to time of ζ_n is

$$\dot{\zeta}_n = \frac{H}{\dot{\phi}_0 a} \left(\dot{\bar{\delta\phi}} + \bar{\delta\phi} \left(\frac{\dot{H}}{H} - \frac{\ddot{\phi}_0}{\dot{\phi}_0} - \frac{\dot{a}}{a} \right) \right) = \frac{H}{\dot{\phi}_0 a} \left(\dot{\bar{\delta\phi}} - H \bar{\delta\phi} \left(1 - \eta + \epsilon \right) \right) \ ,$$

[25] For more information see [49], appendix D.

where we have the definition of the slow-roll parameters 4.11. Performing the field redefinition, the action 4.68 becomes

$$S_2 = \int dtd^3x \left[\frac{a}{2}(\dot{\bar{\delta\phi}})^2 + \frac{a}{2}H^2(1-2\eta+2\epsilon)(\bar{\delta\phi})^2 - aH(1-\eta+\epsilon)\dot{\bar{\delta\phi}}\bar{\delta\phi} - \frac{1}{2a}(\partial_k\bar{\delta\phi})^2\right] ,$$

where we have obviously neglected terms quadratic in the slow-roll parameters. One can show[26] that the two summands in the middle can be put into the standard form $\frac{a}{2}H^2(2-3\eta+2\epsilon)(\bar{\delta\phi})^2$, which behaves like an effective mass term, but time dependent. At this point we pass to conformal time using the definition $d\tau = adt$, giving

$$S_2 = \int d\tau d^3x \left[\frac{1}{2}(\bar{\delta\phi})'^2 + \frac{1}{2}a^2H^2(2-3\eta+2\epsilon)(\bar{\delta\phi})^2 - \frac{1}{2}(\partial_k\bar{\delta\phi})^2\right] .$$

Since from 4.35 one has $H = -\frac{1+\epsilon}{\tau a}$, the effective mass can be expressed as an expansion in the slow-roll parameter as

$$m^2 = -a^2H^2(2-3\eta+2\epsilon) = -\frac{1}{\tau^2}(2-3\eta+6\epsilon) .$$

The equation of motion associated to the action above in Fourier space reads simply

$$\bar{\delta\phi}'' + (k^2+m^2)\bar{\delta\phi} = 0 .$$

Since $\bar{\delta\phi} = a\delta\phi = aQ$ and $m^2 = -\frac{1}{\tau^2}(2-3\eta+6\epsilon) = -\frac{\nu^2-\frac{1}{4}}{\tau^2}$, this is exactly equation 4.48.

The power spectrum

Thanks to the results we have found, we can easily rederive the expression for the power spectrum of ζ. As done in section 4.4, we want to get $u^{\delta\phi}(k,\tau)$ from the solution to the equation of motion at second-order, which is Sasaki-Mukhanov. The mode of $\bar{\delta\phi}$ are given by 4.33, $u_k^{\bar{\delta\phi}}(\tau) = \frac{aH}{\sqrt{2k^3}}e^{-ik\tau}(1+ik\tau)$; to get the mode of ζ_n it is sufficient to remember that $\zeta_n = -\frac{H}{a}\frac{\bar{\delta\phi}}{\dot{\phi}_0}$, so that using eq. 4.14 one gets[27]

$$u_k^{\zeta_n}(\tau) = \frac{H}{\sqrt{4\epsilon k^3}}(1+ik\tau)e^{-ik\tau} . \tag{4.70}$$

The power spectrum scales analogously, so that using $\zeta_n = -H\frac{\delta\phi}{\dot{\phi}_0}$ and 4.14 one gets

$$P_{\zeta_n} = \frac{H^2}{\dot{\phi}_0^2}P_{\delta\phi} = \frac{H^2}{4M_P^2\epsilon k^3} , \tag{4.71}$$

having restored the Planck mass. This coincides with 4.50: at first-order in the fields $\zeta = \zeta_n$; $H = H^*$ up to higher order in slow-roll expansion (see footnote 18); the different numerical coefficient is simply a different normalization.

[26] Indeed:

$$\begin{aligned}
&\frac{a}{2}H^2(1-2\eta+2\epsilon)(\bar{\delta\phi})^2 - aH(1-\eta+\epsilon)\dot{\bar{\delta\phi}}\bar{\delta\phi} + \frac{1}{2}\partial_t(aH(1+\epsilon-\eta)(\bar{\delta\phi})^2) = \\
&\frac{a}{2}H^2(1-2\eta+2\epsilon)(\bar{\delta\phi})^2 + \frac{1}{2}\dot{a}H(1+\epsilon-\eta)(\bar{\delta\phi})^2 + \frac{1}{2}a\dot{H}(1+\epsilon-\eta)(\bar{\delta\phi})^2 = \\
&\frac{a}{2}H^2(1-2\eta+2\epsilon)(\bar{\delta\phi})^2 + \frac{a}{2}H^2(1+\epsilon-\eta)(\bar{\delta\phi})^2 - \frac{a}{2}\epsilon H^2(1+\epsilon-\eta)(\bar{\delta\phi})^2 = \\
&\frac{a}{2}H^2(2-3\eta+2\epsilon)(\bar{\delta\phi})^2 ,
\end{aligned}$$

where we have neglected derivative with respect to time of the slow-roll parameters since they are higher order.

[27] The minus sign can be reabsorbed in the coefficients $C'_{1/2}$ in 4.33.

4.10.2 Third order action

As we have seen, in order to compute the bispectrum one needs the action expanded at least at third order in the field perturbations. The procedure is essentially the same we have presented in 4.10.1, but keeping also the third order terms: the algebra is really involving and as we have done before we limit ourselves to report the result firstly obtained by Maldacena ([46]). In this case it is simpler to work directly in spatially flat gauge. After having obtained the expansion and having erased all the boundary terms by integrating by parts, the third order action reads[28]:

$$S = \int dt d^3x \epsilon^2 \left[a^3 \zeta_n \dot{\zeta}_n^2 + a\zeta_n \partial_i \zeta_n \partial_i \zeta_n - 2a^3 \dot{\zeta}_n \partial_i \nabla^{-2} \dot{\zeta}_n \partial_i \zeta_n \right] + \dots ,$$

where the missing terms are higher order in slow-roll parameters, so we neglect them. This implies that the interaction Hamiltonian at third order is

$$H_{\text{int}} = -\int d^3x \epsilon^2 \left[a^3 \zeta_n \dot{\zeta}_n^2 + a\zeta_n \partial_i \zeta_n \partial_i \zeta_n - 2a^3 \dot{\zeta}_n \partial_i \nabla^{-2} \dot{\zeta}_n \partial_i \zeta_n \right] .$$

To perform the computation we prefer to use cosmic time as time coordinate. This way, using $d\tau = adt$ and 4.14 the result is

$$H_{\text{int}} = -\int d^3x \epsilon^2 \left[a\zeta_n \zeta_n'^2 + a\zeta_n \partial_i \zeta_n \partial_i \zeta_n - 2a\zeta_n' \partial_i \nabla^{-2} \zeta_n' \partial_i \zeta_n \right] .$$

Since the bispectrum is defined in Fourier space, we would also like to rewrite this in Fourier space, using the usual Fourier expansion $\zeta_n(\vec{x},\tau) = \int \frac{d^3k}{(2\pi)^3} \zeta_n(\vec{k},\tau) e^{i\vec{k}\cdot\vec{x}}$. The result is

$$H_{\text{int}} = -\int d^3k d^3p d^3q \frac{1}{(2\pi)^6} \delta^{(3)}(\vec{k}+\vec{p}+\vec{q}) \epsilon^2 \left[a\zeta_n(\vec{k})\zeta_n'(\vec{p})\zeta_n'(\vec{q}) - a(\vec{p}\cdot\vec{q})\zeta_n(\vec{k})\zeta_n(\vec{p})\zeta_n(\vec{q}) + \right.$$
$$\left. -2a\frac{(\vec{p}\cdot\vec{q})}{p^2}\zeta_n'(\vec{k})\zeta_n'(\vec{p})\zeta_n(\vec{q}) \right] .$$

4.10.3 Scalar bispectrum

At this point we have the cubic action, which is what is required to use the *in-in* perturbative formula 4.58. Notice that in H_{int} one could in principle include also a second-order term, but it is null since in the commutator we would have an odd number of creation and annihilation operators, which is zero when contracted with the vacua. We use conformal time and we evaluate the correlator at the instant $\tau = 0$ when inflation ends (this is super-horizon limit). Inserting the third-order Hamiltonian we find three terms to evaluate separately:

$$\langle \zeta_n(\vec{k}_1)\zeta_n(\vec{k}_2)\zeta_n(\vec{k}_3)\rangle = i\epsilon^2 \int d^3k d^3p d^3q \delta^{(3)}(\vec{k}+\vec{p}+\vec{q}) \int_{-\infty}^0 d\tau' a \left[\mathcal{A}_1(\tau') + \mathcal{A}_2(\tau') + \mathcal{A}_3(\tau') \right] .$$

where

$$\mathcal{A}_1(\tau') = a\left\langle 0 \middle| \left[\zeta_n\left(\vec{k}_1,0\right)\zeta_n\left(\vec{k}_2,0\right)\zeta_n\left(\vec{k}_3,0\right), \zeta_n\left(\vec{k},\tau'\right)\zeta_n'\left(\vec{p},\tau'\right)\zeta_n'\left(\vec{q},\tau'\right) \right] \middle| 0 \right\rangle$$
$$\mathcal{A}_2(\tau') = -a(\vec{p}\cdot\vec{q})\left\langle 0 \middle| \left[\zeta_n\left(\vec{k}_1,0\right)\zeta_n\left(\vec{k}_2,0\right)\zeta_n\left(\vec{k}_3,0\right), \zeta_n\left(\vec{k},\tau'\right)\zeta_n\left(\vec{p},\tau'\right)\zeta_n\left(\vec{q},\tau'\right) \right] \middle| 0 \right\rangle$$
$$\mathcal{A}_3(\tau') = -2a\frac{(\vec{p}\cdot\vec{q})}{p^2}\left\langle 0 \middle| \left[\zeta_n\left(\vec{k}_1,0\right)\zeta_n\left(\vec{k}_2,0\right)\zeta_n\left(\vec{k}_3,0\right), \zeta_n'\left(\vec{k},\tau'\right)\zeta_n'\left(\vec{p},\tau'\right)\zeta_n\left(\vec{q},\tau'\right) \right] \middle| 0 \right\rangle .$$

Notice that in the *in-in* perturbative formula we have not inserted the $i\infty$ term in the lower bound of integration simply because we still have to Wick rotate the result. To evaluate these quantum

[28]Cfr. [46], eq. 3.8, using $\zeta_n \sim H\frac{\delta\phi}{\dot{\phi}_0}$ and the definition of the slow-roll parameters.

correlators we use Wick theorem introduced in section 4.8.4; the contractions give the results:

$$
\begin{aligned}
\mathcal{A}_1 =&(2\pi)^9 a \left[u\left(k_1,0\right) u\left(k_2,0\right) u\left(k_3,0\right) \left(\frac{d}{d\tau}u^*\left(k_1,\tau'\right)\right) \left(\frac{d}{d\tau}u^*\left(k_2,\tau'\right)\right) u^*\left(k_3,\tau'\right) - \text{ c.c. } \right] + \\
&+ \text{sym}\left(k_i\right) \\
\mathcal{A}_2 =&-(2\pi)^9 a \left(\vec{k_1}\cdot\vec{k_2}\right) \left[u\left(k_1,0\right) u\left(k_2,0\right) u\left(k_3,0\right) u^*\left(k_1,\tau'\right) u^*\left(k_2,\tau'\right) u^*\left(k_3,\tau'\right) - \text{ c.c. } \right] + \\
&+ \text{sym}\left(k_i\right) \\
\mathcal{A}_3 =&(2\pi)^9(-2a)\frac{\left(\vec{k_1}\cdot\vec{k_2}\right)}{k_1^2} \left[u\left(k_1,0\right) u\left(k_2,0\right) u\left(k_3,0\right) \left(\frac{d}{d\tau}u^*\left(k_1,\tau'\right)\right) u^*\left(k_2,\tau'\right) \left(\frac{d}{d\tau}u^*\left(k_3,\tau'\right)\right) - \text{c.c.} \right] + \\
&+ \text{sym}\left(k_i\right) \ ,
\end{aligned}
$$

where c.c. is the complex conjugate and $\text{sym}\left(k_i\right)$ is a sum over all the possibile combinations of k_i's taking into account all the possibile Wick contractions. Putting all together one finds that the *in-in* formula gives

$$
\begin{aligned}
\left\langle \zeta_n\left(\vec{k_1}\right)\zeta_n(\vec{k_2})\,\zeta_n\left(\vec{k_3}\right)\right\rangle =&(2\pi)^3\delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3})\epsilon^2\times \\
&\text{Im}\left[I_1 - \left(\vec{k_1}\cdot\vec{k_2}\right) I_2 - 2\frac{\left(\vec{k_1}\cdot\vec{k_2}\right)}{k_1^2} I_3 - \text{ c.c. } \right] + \text{sym}\left(k_i\right) \ ,
\end{aligned}
\tag{4.72}
$$

where the I's are 3 integrals in τ':

$$
\begin{aligned}
I_1 &= u\left(k_1,0\right) u\left(k_2,0\right) u\left(k_3,0\right) \int_{-\infty}^{0} d\tau' a^2 \left[\left(\frac{d}{d\tau}u^*\left(k_1,\tau'\right)\right) \left(\frac{d}{d\tau}u^*\left(k_2,\tau'\right)\right) u^*\left(k_3,\tau'\right) \right] \\
I_2 &= u\left(k_1,0\right) u\left(k_2,0\right) u\left(k_3,0\right) \int_{-\infty}^{0} d\tau' a^2 \left[u^*\left(k_1,\tau'\right) u^*\left(k_2,\tau'\right) u^*\left(k_3,\tau'\right) \right] \\
I_3 &= u\left(k_1,0\right) u\left(k_2,0\right) u\left(k_3,0\right) \int_{-\infty}^{0} d\tau' a^2 \left[\left(\frac{d}{d\tau}u^*\left(k_1,\tau'\right)\right) u^*\left(k_2,\tau'\right) \left(\frac{d}{d\tau}u^*\left(k_3,\tau'\right)\right) \right] \ .
\end{aligned}
$$

This way, the problem is reduced in computing these integrals. The function $u(k,\tau)$ is the solution to the background equation, which is 4.70, so that

$$
u_k(\tau) = \frac{H}{\sqrt{4\epsilon k^3}}(1+ik\tau)e^{-ik\tau} \ , \quad \frac{d}{d\tau}u_k(\tau) = \frac{H}{\sqrt{4\epsilon k^3}}k^2 e^{-ik\tau} \ .
$$

Furthermore, in section 4.4 we saw that $a(\tau) = -\frac{1}{H\tau}$ and in footnote 18 $H = H_*$, both at zeroth-order in slow-roll parameters (remember that $H = H_*$ is the value of the Hubble parameter at horizon crossing). However, in evaluating the integrals one could be worried about the fact that substituting these expressions the value of H_* can be different from mode to mode; however, one can choose a given time instant, corresponding for example to the time when $\vec{K} = \vec{k_1}+\vec{k_2}+\vec{k_3}$ crossed the horizon: this way, one is sure that all the three modes are superhorizon.
Using these three ingredients, as we said introducing the *in-in* formula, the resulting integrands have usually a bad oscillatory behaviour at infinity so one has to give sense to them by performing a Wick rotation. I_1 and I_2 can be computed by using the formulas 4.59, which include the Wick rotation. The results are:

$$
I_1 = -H_*^4 k_1^2 k_2^2 \left(\prod_{i=1}^{3}\frac{1}{4\epsilon k_i^3}\right)\int_{-\infty}^{0} d\tau' \left(1-ik_3\tau'\right)e^{iK\tau'} = iH_*^4 k_1^2 k_2^2 \left(\prod_{i=1}^{3}\frac{1}{4\epsilon k_i^3}\right)\left(\frac{1}{K}+\frac{k_3}{K^2}\right) \ ,
$$

$$
I_3 = -H_*^4 k_1^2 k_3^2 \left(\prod_{i=1}^{3}\frac{1}{4\epsilon k_i^3}\right)\int_{-\infty}^{0} d\tau' \left(1-ik_2\tau'\right)e^{iK\tau'} = iH_*^4 k_1^2 k_3^2 \left(\prod_{i=1}^{3}\frac{1}{4\epsilon k_i^3}\right)\left(\frac{1}{K}+\frac{k_2}{K^2}\right) \ .
$$

To compute I_2 we have to compute an integral which is divergent; however, in 4.72 we need only its imaginary part, which is convergent. Explicitly:

$$
\begin{aligned}
\mathrm{Im} \int_{-\infty}^{0} \frac{dx}{x^2}(1 - iKx)e^{iKx} &= \mathrm{Im}\left[\int_{-\infty}^{0} \frac{dx}{x^2} e^{iKx} - iK \int_{-\infty}^{0} \frac{dx}{x} e^{iKx}\right] = \\
&= \mathrm{Im}\left[- \left.\frac{e^{Kx}}{x}\right|_{-\infty}^{0} + iK \int_{-\infty}^{0} \frac{dx}{x} e^{iKx} - iK \int_{-\infty}^{0} \frac{dx}{x} e^{iKx}\right] = \\
&= \mathrm{Im}\left[- \left.\frac{\cos(Kx)}{x}\right|_{-\infty}^{0} - i \left.\frac{\sin(Kx)}{x}\right|_{-\infty}^{0}\right] = -K \ .
\end{aligned}
$$

Using this and 4.59, the result of the integral is:

$$
\begin{aligned}
\mathrm{Im} I_2 = &-H_*^4 \left(\prod_{i=1}^{3} \frac{1}{4\epsilon k_i^3}\right) \mathrm{Im} \int_{-\infty}^{0} \frac{d\tau'}{\tau'^2} \left(1 - ik_1\tau'\right)\left(1 - ik_2\tau'\right)\left(1 - ik_3\tau'\right) e^{iK\tau'} = \\
= &\, H_*^4 \left(\prod_{i=1}^{3} \frac{1}{4\epsilon k_i^3}\right) \left(K - \frac{k_1k_2 + k_2k_3 + k_1k_3}{K} - \frac{k_1k_2k_3}{K^2}\right) \ .
\end{aligned}
$$

At this point we have to substitute these results in the expression of the three point function 4.72. The scalar products in the expression can be simplified as follows. Using the fact that $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = \vec{0}$ (thanks to the Dirac delta), squaring this expression one finds

$$
\left(\vec{k}_1 + \vec{k}_2 + \vec{k}_3\right)^2 = k_1^2 + k_2^2 + k_3^2 + 2\vec{k}_1 \cdot \vec{k}_2 + 2\vec{k}_1 \cdot \vec{k}_3 + 2\vec{k}_2 \cdot \vec{k}_3 = 0
$$

which then implies

$$
\vec{k}_1 \cdot \vec{k}_2 + \vec{k}_1 \cdot \vec{k}_3 + \vec{k}_2 \cdot \vec{k}_3 = -\frac{1}{2}\left[k_1^2 + k_2^2 + k_3^2\right] \ .
$$

Moreover, the scalar product $\vec{k}_1 \cdot \vec{k}_2$, can be written as

$$
\vec{k}_1 \cdot \vec{k}_2 = \frac{1}{2}\left[k_1^2 + 2\vec{k}_1 \cdot \vec{k}_2 + k_2^2\right] - \frac{1}{2}k_1^2 - \frac{1}{2}k_2^2 = \frac{1}{2}\left(\vec{k}_1 + \vec{k}_2\right)^2 - \frac{1}{2}k_1^2 - \frac{1}{2}k_2^2 \ .
$$

Using the conservation of momentum given by the Dirac delta $\vec{k}_1 + \vec{k}_2 = -\vec{k}_3$. Thus

$$
\vec{k}_1 \cdot \vec{k}_2 = \frac{1}{2}\left[k_3^2 - k_1^2 - k_2^2\right]
$$

and similarly for the others:

$$
\vec{k}_1 \cdot \vec{k}_3 = \frac{1}{2}\left[k_2^2 - k_1^2 - k_3^2\right] \ , \quad \vec{k}_2 \cdot \vec{k}_3 = \frac{1}{2}\left[k_1^2 - k_2^2 - k_3^2\right] .
$$

The final result, after having symmetrized over all the moments is[29]:

$$
\langle\zeta_n(\vec{k}_1)\zeta_n(\vec{k}_2)\zeta_n(\vec{k}_3)\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)\frac{2H_*^4}{\epsilon M_P^4}\left(\prod_{i=1}^{3}\frac{1}{4k_i^3}\right)\left[-\sum k_i^3 + \sum_{i\neq j} k_i k_j^2 + \frac{8}{K}\sum_{i>j} k_i^2 k_j^2\right] \ . \tag{4.73}
$$

We restored the Planck mass initially set to 1. However, we want to find the bispectrum for three ζ's and not for ζ_n: the transformation law is 4.69. Thanks to Weinberg theorem we know that on superhorizon scales ζ and D_{ij} are constant in time, while gradients can be neglected, so the result simplifies to

$$
\zeta = \zeta_n + \frac{1}{2}\frac{\ddot{\phi}_0}{\dot{\phi}_0 H}\zeta_n^2 + \frac{1}{4}\frac{\dot{\phi}_0^{\,2}}{H^2}\zeta_n^2 = \zeta_n + \frac{1}{2}\left(-\eta + \epsilon\right)\zeta_n^2 \ .
$$

[29] The explicit symmetrization procedure is in [48], sections V.D-V.E.

This scaling at leading order in slow-roll expansion sends the correlator in coordinate space to (using Wick theorem to compute the 4-point functions and 4.39)

$$\langle\zeta(\vec{x}_1)\zeta(\vec{x}_2)\zeta(\vec{x}_3)\rangle = \langle\zeta_n(\vec{x}_1)\zeta_n(\vec{x}_2)\zeta_n(\vec{x}_3)\rangle + (-\eta+\epsilon)\left[\langle\zeta_n(\vec{x}_1)\zeta_n(\vec{x}_2)\rangle\langle\zeta_n(\vec{x}_1)\zeta_n(\vec{x}_3)\rangle + \text{sym}\right] .$$

Passing to Fourier space one gets

$$\begin{aligned}\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle =& \langle\zeta_n(\vec{k}_1)\zeta_n(\vec{k}_2)\zeta_n(\vec{k}_3)\rangle + (-\eta+\epsilon)\,(2\pi)^3\delta^{(3)}\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right)\times\\ &\left[P_{\zeta_n}(k_1)\,P_{\zeta_n}(k_2) + P_{\zeta_n}(k_2)\,P_{\zeta_n}(k_3) + P_{\zeta_n}(k_1)\,P_{\zeta_n}(k_3)\right] .\end{aligned}$$

Using 4.71 and 4.73 the final result is:

$$\begin{aligned}\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle =& (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\times\\ &\frac{H_*^4}{M_P^4}\frac{1}{(k_1k_2k_3)^3}\frac{1}{4\epsilon^2}\left[\frac{\epsilon-\eta}{4}\sum k_i^3 + \frac{\epsilon}{8}\left(-\sum k_i^3 + \sum_{i\neq j}k_ik_j^2 + \frac{8}{K}\sum_{i>j}k_i^2k_j^2\right)\right] .\end{aligned} \tag{4.74}$$

From its definition, the second line of this result is exactly the bispectrum $B_{\zeta\zeta\zeta}$, at leading order in slow-roll approximation. Notice that in this approximation the 3-point function does not depend on the type of interaction in V: the dependence is encoded in the slow-roll parameters.

4.10.4 Tensor and mixed bispectra

In his paper [46] Maldacena computed also the tensor bispectrum B_{DDD} between three tensor perturbations and the mixed bispectra $B_{\zeta DD}$ and $B_{\zeta\zeta D}$. As it is clear from the size of the computation we have done to derive $B_{\zeta\zeta\zeta}$, reproducing the computations leading to these results is very lengthy, so we limit ourselves to report the results[30]:

$$\begin{aligned}B_{DDD}(k_1,s_1,k_2,s_2,k_3,s_3) &= \frac{H_*^4}{M_P^4}\frac{2}{k_1^3k_2^3k_3^3}\epsilon^{s_1}_{ii'}\epsilon^{s_2}_{jj'}\epsilon^{s_3}_{ll'}t_{ijl}t_{i'j'l'}I\\ B_{\zeta DD}(k_1,k_2,s_2,k_3,s_3) &= \frac{H_*^4}{M_P^4}\frac{1}{16k_1^3k_2^3k_3^3}\epsilon^{s_2}_{ij}\epsilon^{s_3}_{ij}\left[-k_1^3 + 2k_1(k_2^2+k_3^2) + 16\frac{k_2^2k_3^2}{K}\right]\\ B_{\zeta\zeta D}(k_1,k_2,k_3,s_3) &= \frac{H_*^4}{\epsilon M_P^4}\frac{1}{2k_1^3k_2^3k_3^3}\epsilon^{s_3}_{ij}k_2^ik_3^jI ,\end{aligned} \tag{4.75}$$

where each s_i is the spin associated to the tensor mode, I is the result of an integral over τ born from the use of *in-in* formalism

$$I = K - \frac{\sum_{i>j}k_ik_j}{K} - \frac{k_1k_2k_3}{K^2}$$

and

$$t_{ijl} = k_1^l\delta_{ij} + k_2^i\delta_{jl} + k_3^j\delta_{il} .$$

Moreover, ϵ^s_{ij} is the spin-2 polarization tensor, such that

$$D_{ij} = \sum_{s=\pm2}\epsilon^s_{ij}D^s(k) . \tag{4.76}$$

Notice that D^s depends only on the norm of $\vec{k}$. A possible explicit representation of the ϵ_{ij} is given by the ±2 polarizations

$$\epsilon^s_{ij} = \frac{(e^1_i+ie^2_i)(e^1_j+ie^2_j)}{2}\delta^s_1 + \frac{(e^1_i-ie^2_i)(e^1_j-ie^2_j)}{2}\delta^s_2 , \tag{4.77}$$

[30] In the following there are some extra factors in the normalization, due to the fact that in [46] the power spectrum has a different normalization with respect to our conventions. More specifically, they assume

$$P_T(k) = \frac{H_*^2}{M_P^2k_3^2} ,$$

which has an extra factor of 2 with respect to 4.52.

being e^1 and e^2 two vectors constituting the base of the subspace perpendicular to the direction of the tensor modes $\vec{k}_l$:

$$e_i^a k_{lj} g^{ij} = 0 \ .$$

Another representation is given by the plus and cross polarizations:

$$\epsilon_{ij}^s = (e_i^1 e_j^1 - e_i^2 e_j^2)\delta_+^s + (e_i^1 e_j^2 + e_i^2 e_j^1)\delta_\times^s \ ,$$

The spin tensor has also the property

$$\epsilon_{s_1}^{ij} \epsilon_{ij}^{s_2} = \delta_{s_1}^{s_2} \ . \tag{4.78}$$

Chapter 5

The consistency relation

After having introduced cosmological perturbations and the inflationary mechanism, we can finally derive and discuss the consistency relation. Loosely speaking, a consistency relation links the bispectrum with the power spectrum in the limit when one momentum is sent to zero, that is the squeezed limit. This relation naturally arises from the computations of single-field inflation bispectra as Maldacena himself pointed out. However, the issue turned out to be very general, since immediately after the Maldacena work it was shown that this relation is valid in every single-field model without relying on slow-roll approximation. But this is not the only possibile generalization: the consistency relation turned out to be strictly related to the structure of the symmetries of de Sitter space; starting from this, one can prove that there exists an infinite number of consistency relations, each one relating the $N+1$-point function with the N-point function. In this case the Maldacena results appear to be only the $N=2$ case. In this framework, consistency relations including also tensor perturbations have been proposed, as we are going to discuss.
In this chapter we briefly discuss the Maldacena argument both in the case of the scalar bispectrum and in the case of bispectra involving tensor perturbations. We will quote the various formal proofs for the consistency relation, then presenting a very important one, according to which the consistency relations are Ward identities associated to conformal symmetries. These proofs provide also a way to generalize the Maldacena result for generic N-point functions. We will conclude by discussing when this relation is not valid, listing the possible models which are alternatives to the single-field models of inflation.

5.1 A first hint for the consistency relation

In his seminal paper [46], after having obtained the bispectra, Maldacena pointed also out a very important fact. He considered the 3-point functions and took the limit $k_1 \to 0$ (for symmetry one could take k_2 or k_3 as well). Since the bispectra is multiplied by a Dirac delta, we have $\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3) \to \delta^{(3)}(\vec{k}_2+\vec{k}_3)$, so in this limit we have $\vec{k}_2 \to -\vec{k}_3$. Graphically this has a clear interpretation, since in momentum space the Dirac delta implies that the sum of the momenta is 0, so three momenta close in a triangle. In the limit $k_1 \to 0$ this means that the triangle squeezes and $k_2 = k_3$.

Consider now the Maldacena result for $B_{\zeta\zeta\zeta}$, eq. 4.74. We want to take the squeezed limit $k_1 \to 0$ and $k_2 \to k_3$ (remember that these are the norms of the vectors). This way the second summand between brackets in 4.74 becomes

$$\frac{\epsilon}{8}\left(-\sum k_i^3 + \sum_{i\neq j} k_i k_j^2 + \frac{8}{K}\sum_{i>j} k_i^2 k_j^2\right) \longrightarrow \frac{2\epsilon}{8}\sum_i k_i^3 \; .$$

This implies that in the squeezed limit the correlator is

$$\lim_{k_1\to 0}\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{H_*^4}{M_p^4}\frac{1}{(k_1k_2k_3)^3}\frac{1}{4\epsilon^2}\frac{2\epsilon-\eta}{4}\sum k_i^3 \ .$$

Notice that computing the shape function S using 4.61 one gets exactly the local form form $S_{\rm local}$. From this emerges the point: we have computed the power spectrum in 4.10.1, which is $P_{\zeta_n} = P_\zeta = \frac{H_*^2}{4M_P^2\epsilon k^3}$ (remember that $\zeta = \zeta_n$ at first-order in the fields) and in section 4.6.3 we have seen that

$$n_s - 1 = -6\epsilon + 2\eta_V = -4\epsilon + 2\eta \ ,$$

where we have used 4.17. These imply that, since in this limit $\sum_i k_i^3 \simeq 2k_3^3$,

$$\begin{aligned}\lim_{k_1\to 0}\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle &= (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{H_*^2}{4M_P^2\epsilon k_1^3}\frac{H_*^2}{4M_P^2\epsilon k_2^3}\frac{1}{k_3^3}\frac{4\epsilon-2\eta}{2}2k_3^3 = \\ &= -(2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)(n_s-1)P_\zeta(k_1)P_\zeta(k_2) \ .\end{aligned} \tag{5.1}$$

So in the squeezed limit the bispectrum is the product of the two power spectra, with a coefficient which is the deviation of n_s from 1. The previous result can be also rewritten as[1]

$$\lim_{k_1\to 0}\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle = -(2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)P_\zeta(k_1)P_\zeta(k_2)\frac{\partial\log\left[k_2^3P_\zeta(k_2)\right]}{\partial\log k_2} \ , \tag{5.2}$$

using $\Delta(k) = \frac{k^3}{2\pi^2}P_\zeta(k)$ and the definition of the spectral index n_s.
As pointed out by Maldacena himself, this result has a peculiar physical meaning. In the limit $k_1 \to 0$, the corresponding mode $\zeta(\vec{k}_1)$ has large wavelength: this means that the fluctuation of $\zeta(\vec{k}_1)$ is already frozen by the time the other two momenta, such that $k_{2/3} \gg k_1$ (so they are short), cross the horizon. So its only effect is to rescale the other two momenta so that we get a contribution proportional to the violation in scale invariance of the two point function of the two fluctuations with large momenta. This is exactly what happens. It is not a proof, but simply a semiqualitative argument. However, this consistency relation has been explicitly proved and generalized in years, as we are going to see in a future section.
The phenomenological importance of this consistency relation can be already appreciated. Since it is valid in single-fields inflationary models, a future measurement of the bispectrum can test this relation. This means that if a violation was measured, this would immediately rule out all the single-field inflationary models.

5.2 Consistency relation for tensors

In this section we derive the consistency relations associated to the bispectra 4.75 involving tensor perturbations D ([46, 50, 51]). The most used is the one arising from the correlator $\langle\zeta\zeta D\rangle$, squeezing the momentum of the tensor perturbation. In the mixed bispectra one usually squeezes the momentum associated to the field in the correlator which appears one time, in this case D. One has

$$\begin{aligned}\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)D(\vec{k}_3,s_3)\rangle &= (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)B_{\zeta\zeta D}(k_1,k_2,k_3,s_3) = \\ &= (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{H_*^4}{\epsilon M_P^4}\frac{1}{2k_1^3k_2^3k_3^3}\epsilon_{ij}^{s_3}k_2^ik_3^jI \ ,\end{aligned}$$

so taking the limit $k_3 \to 0$ and $k_1 \to k_2$ one has (since $K = \sum k_i \to 2k_2$)

$$I = K - \frac{\sum_{i>j}k_ik_j}{K} - \frac{k_1k_2k_3}{K^2} \to 2k_2 - \frac{k_2^2}{2k_2} = \frac{3}{2}k_2 \ ,$$

[1] To be mathematically consistent, the correct limit to take would be

$$\lim_{k_1\to 0}\frac{1}{P_\zeta(k_1)}\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\zeta(\vec{k}_3)\rangle$$

and similarly in the other consistency relations presented in this chapter. We have chosen to conform with the original Maldacena convention.

so that

$$\lim_{k_3\to 0}\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)D(\vec{k}_3,s_3)\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{2H_*^2}{M_P^2k_3^3}\frac{H_*^2}{4\epsilon M_P^2k_2^3}\frac{1}{k_2^3}\epsilon_{ij}^{s_3}k_2^ik_2^j\frac{3}{2}k_2 =$$
$$= (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{3}{2}P_T(k_3)P_\zeta(k_2)\epsilon_{ij}^{s_3}\frac{k_2^ik_2^j}{k_2^2}\ , \tag{5.3}$$

having used 4.52 and 4.71.

Other tensor consistency relations involve the squeezed limit of the other two bispectra in 4.75, $\langle\zeta DD\rangle$ and $\langle DDD\rangle$. The first one is associated to the bispectrum

$$\langle\zeta(\vec{k}_1)D(\vec{k}_2,s_2)D(\vec{k}_3,s_3)\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)B_{\zeta DD}(k_1,k_2,s_2,k_3,s_3) =$$
$$= (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{H_*^4}{M_P^4}\frac{1}{16k_1^3k_2^3k_3^3}\epsilon_{ij}^{s_2}\epsilon_{ij}^{s_3}\left[-k_1^3+2k_1(k_2^2+k_3^2)+16\frac{k_2^2k_3^2}{K}\right]\ .$$

In this case we have to squeeze the momentum associated to ζ. We can use 4.78 and in the limit $k_2\simeq k_3\gg k_1$ one has

$$-k_1^3+2k_1(k_2^2+k_3^2)+16\frac{k_2^2k_3^2}{K}\to 8k_2^3\ ,$$

so that

$$\lim_{k_3\to 0}\langle\zeta(\vec{k}_1)D(\vec{k}_2,s_2)D(\vec{k}_3,s_3)\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\epsilon\frac{H_*^2}{4\epsilon M_P^2k_1^3}\frac{2H_*^2}{M_P^2k_2^3}\frac{1}{8}\delta_{s_2s_3}8k_2^3 =$$
$$= (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\epsilon P_\zeta(k_1)P_T(k_2)\delta_{s_2s_3}\ .$$

Finally, we want to squeeze a momentum in the $\langle DDD\rangle$ correlator in 4.75:

$$\langle D(\vec{k}_1,s_1)D(\vec{k}_2,s_2)D(\vec{k}_3,s_3)\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)B_{DDD}(k_1,k_2,k_3,s_1,s_2,s_3) =$$
$$= (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{H_*^4}{M_P^4}\frac{2}{k_1^3k_2^3k_3^3}\epsilon_{ii'}^{s_1}\epsilon_{jj'}^{s_2}\epsilon_{ll'}^{s_3}t_{ijl}t_{i'j'l'}I\ .$$

As before, in the limit $k_2\simeq k_3\gg k_1$ one has $I=\frac{3}{2}k_2$ and

$$t_{ijl}\to k_2^i\delta_{jl}+k_2^j\delta_{il}\ .$$

Thanks to the transversality condition of ϵ one has $\epsilon_{ij}^{s_2}k_2^i=0$ and using 4.78 the result is

$$\epsilon_{ii'}^{s_1}\epsilon_{jj'}^{s_2}\epsilon_{ll'}^{s_3}t_{ijl}t_{i'j'l'}\to\epsilon_{ii'}^{s_1}k_2^ik_2^{i'}\delta_{s_2s_3}\ .$$

In the end, in the squeezed limit one gets the following consistency relation

$$\lim_{k_1\to 0}\langle D(k_1,s_1)D(k_2,s_2)D(k_3,s_3)\rangle = (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{2H_*^2}{M_P^2k_1^3}\frac{2H_*^2}{M_P^2k_2^3}\frac{1}{k_2^3}\epsilon_{ii'}^{s_1}k_2^ik_2^{i'}\delta_{s_2s_3}\frac{3}{2}k_2 =$$
$$= (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)P_T(k_1)P_T(k_2)\frac{3}{2}\epsilon_{ii'}^{s_1}\frac{k_2^ik_2^{i'}}{k_2^2}\delta_{s_2s_3}\ .$$

It is worth to underline that these last two relations, involving more than one tensor mode, are less interesting from a phenomenological point of view, as we will see in chapter 7.

Up to now, these consistency relations involving tensor perturbations are only a peculiar limit of the Maldacena results 4.75. However, as in the case of the correlator $\langle\zeta\zeta\zeta\rangle$, it can be proved rigorously that they are valid in any single-field model of inflation. Moreover, all these 4 consistency relations for the 3-point functions of the single-field inflationary models can be generalized in many ways, as we are going to discuss in the following section.

5.3 An overview about a proof and generalizations

Historically, the first attempt to get a formal proof the consistency relations was proposed in 2004 by Creminelli and Zaldarriaga ([52]), better presented by [49] and [53]. This proof is based on the so-called "background wave" method, based on the fact that the long mode can be considered as a background of the two fast modes (see figure 5.1). The assumption is that the inflaton is the only dynamical field during inflation.
However, this approach turned out to be difficulty justifiable, since the cosmological correlators have a quantum nature, fact that makes the "background wave" method quite unsatisfactory from the formal point of view. However, through years many other demonstrations appeared, based on very different techniques. The original consistency relation can be derived from the Ward identity for spontaneously broken spatial dilatations: this was already pointed out by Maldacena himself in [46] and then presented formally by Hui *et al.* in [54] and [50]. This approach emphasizes the non-perturbative nature of the consistency relations. Moreover, there is another proof in [55], which uses standard quantum field theory techniques such as the path integral formalism and the introduction of the proper vertex functions. Other common QFT techniques used to prove the consistency relations are the wave functional ([56, 57]) and the BRST symmetry ([58, 59]). Finally, Maldacena pointed out also the relation between the consistency relation and the AdS/CFT correspondence, leading to demonstrations based on holographic arguments ([60, 61]).
The importance of these results is that they naturally provide generalizations for the consistency relation we have proved for the correlator $\langle\zeta\zeta\zeta\rangle$: they can contain tensor perturbations, as we have seen in the previous section, but they also involve a generic number of fields in the correlators, or even include other completely new relations among correlators in the limit $k \to 0$. This section has two different purposes: on one hand, to give an idea of a formal demonstration for the consistency relation, underlining that it is not simply a mere coincidence, but it is strictly related to the structure of the spacetime in which inflation takes place; on the other hand, to introduce briefly the simplest possible generalizations (without entering too much in technical details which are beyond the scope of this project).
Indeed, many of the different demonstrations unveil the strict connection between the consistency relation and the symmetry breaking pattern of the de Sitter space, which we will discuss. The consistency relation is also related to the adiabatic modes arising in the limit $k \to 0$; we will represent them using the different approach of [50]. Then we will focus on the consistency relations seen as Ward identity and finally we will comment on possible further generalizations.

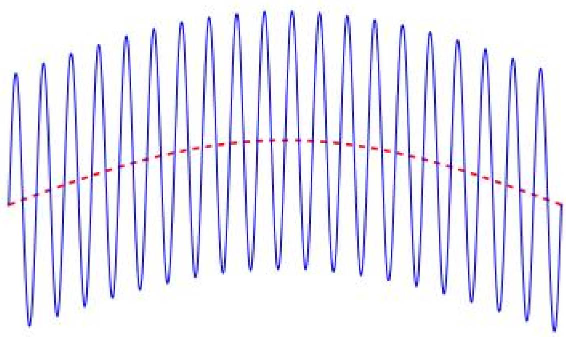

Figure 5.1: A shematic representation of the basic idea for the "background wave" argument: the red dashed line represents a mode with $k \to 0$, so its wavelength is long; on the contrary, the blue line represents a short mode and the red dashed line behaves like a background for this mode. A first idea of this argument was introduced also by Maldacena in [46], as we have said.

5.3.1 SSB pattern in inflation

In this section we discuss the symmetry breaking pattern in single-field inflation ([50, 62, 63]). As we have seen, inflation takes place in a quasi-de Sitter space ([62]), which in Lemaître coordinates reads

$$ds^2 = -dt^2 + e^{2Ht}\delta_{ij}dx^i dx^j \ ,$$

with H constant. Being a maximally symmetric space, it has 10 symmetries (in $(3+1)$ dimensions), each one associated to a Killing vector. As in Minkowski, we have 3 spatial translations and 3 spatial rotations. On the contrary, the remaining 4 symmetries are very peculiar. A first symmetry is the following

$$t' = t + c \ , \quad x'^i = e^{-Hc} x_i$$

which is exactly a rescaling of the coordinates[2]. The proof that de Sitter space is invariant under this transformation is almost trivial:

$$ds'^2 = -dt'^2 + e^{Ht} \delta_{ij} dx'^i dx'^j = -dt^2 + e^{2H(t+c)} e^{-2Hc} \delta_{ij} dx^i dx^j = ds^2 \ .$$

The remaining 3 symmetries are the conformal dilatations, acting infinitesimally on the coordinates as

$$t' = t - 2H b_i x^i \quad x'^i = x^i - b^i(-H^{-2} e^{-2Ht} + (x^j)^2) + 2x^i (b_j x^j) \ ,$$

where $\vec{b}$ is a 3-vector parametrizing the 3 directions of dilatation. These 4 symmetries (rescaling + 3 conformal dilatations) are the generators of the conformal group, which is the basis of the (A)dS/CFT correspondence ([64, 65]).

However, we are going to show explicitly that considering a perturbation for the inflaton breaks some of these generators. Since the background is FLRW, we expect that the conformal group generators are broken, leading to a symmetry breaking pattern:

$$\begin{gathered} dS \to FLRW \\ SO(4,1) \to \text{spatial rotations + translations} \ . \end{gathered}$$

This is because the inflaton establishes the end of inflation, so perturbations along the inflaton trajectory are sensitive to departures from de Sitter space. As we have seen, we can slice de Sitter spacetime by taking $\phi(t) = constant$, spontaneously breaking $SO(1,4)$. This implies that inflaton correlation functions cannot be invariant under the full $SO(1,4)$ symmetries. Notice that we are making the assumption that the spacetime can be sliced, which is possible in single-field inflation, but not in multi-field inflation.

To better explain this, consider now an infinitesimal diffeomorphism of parameter $\xi^\mu = (\xi, \xi^i)$. The metric written in ADM form, which can be read from the full line element 4.62, transforms according to 2.15, so $g'_{\mu\nu}(x) = g_{\mu\nu}(x) - \mathcal{L}_\xi g_{\mu\nu}(x)$. From this one finds that the lapse, the shift and the spatial metric are sent to

$$\begin{aligned} \delta N^i &= \xi^j \partial_j N^i - \partial_j \xi^i N^j + \partial_t \left(\xi N^i\right) + \dot{\xi}^i - \left(N^2 \gamma^{ij} + N^i N^j\right) \partial_j \xi \\ \delta N &= \xi^i \partial_i N + \partial_t (\xi N) - N N^i \partial_i \xi \\ \delta \gamma_{ij} &= D_i \xi_j + D_j \xi_i + \xi \dot{\gamma}_{ij} + N_i \partial_j \xi + N_j \partial_i \xi \ . \end{aligned} \tag{5.4}$$

Now, if all the fields and the elements of ξ are null at infinity, there are no residual gauge symmetries. Working in ζ-gauge, we have seen that the spatial metric is

$$g_{ij} = \gamma_{ij} = a^2(t) e^{2\zeta} e^{D_{ij}} \ .$$

[2]To better see this, one has to change coordinates

$$t = -\frac{\log \eta}{H} \ , \quad x = \frac{y}{H} \ ,$$

in which the metric becomes

$$ds^2 = \frac{1}{H^2 \eta^2} (-d\eta^2 + \delta_{ij} dy^i dy^j)$$

and the symmetry

$$\eta' = e^{-Hc} \eta \ , \quad y'^i = e^{-Hc} y^i \ ,$$

which is exactly a dilatation with parameter $\lambda = e^{-Hc}$.

Under a diffeomorphism, thanks to 2.6, the variation of the spatial metric is

$$\delta\left[a^2(t)e^{2\zeta}e^{D_{ij}}\right]=\mathcal{L}_\xi\left[a^2(t)e^{2\zeta}e^{D_{ij}}\right]\ . \tag{5.5}$$

We want to find transformations respecting the ζ-gauge mapping this spatial metric into iteself, so that they appear to be residual symmetries of the metric. As we have commented in chapter 2, the only possible residual gauge freedom is given by infinitesimal diffeomorphisms which do not decay at infinity.
In order to preserve the ζ-gauge definition, $\delta\phi=0$ always, so we have to choose $\xi=0$. At this point we want to solve the previous equation order by order as follows

$$\begin{aligned}\delta\zeta&=\delta\zeta^{(0)}+\delta\zeta^{(1)}+\dots\\ \delta D_{ij}&=\delta D_{ij}^{(0)}+\delta D_{ij}^{(1)}+\dots\\ \xi_i&=\xi_0^{(0)}+\delta\xi^{(1)}+\dots\ .\end{aligned}$$

We limit ourselves to the lowest order in perturbations, being this sufficient to derive the simplest consistency relations. The discussion for the higher order terms can be found in [50]. At zeroth order, the expansion of 5.5 gives

$$2\delta\zeta^{(0)}\delta_{ij}+\delta D_{ij}^{(0)}=2\xi_k^{(0)}\partial^k\zeta\delta_{ij}+\partial_i\xi_j^{(0)}+\partial_j\xi_i^{(0)}\ . \tag{5.6}$$

Since D_{ij} is traceless, taking the trace allows us to solve for $\delta\zeta^{(0)}$,

$$\delta\zeta^{(0)}=\frac{1}{3}\partial^i\xi_i^{(0)}+\xi_i^{(0)}\partial^i\zeta^{(0)}\ . \tag{5.7}$$

Plugging back this into 5.6, solving for $\delta D_{ij}^{(0)}$ one gets

$$\delta D_{ij}^{(0)}=\partial_i\xi_j^{(0)}+\partial_j\xi_i^{(0)}-\frac{2}{3}\partial^k\xi_k^{(0)}\delta_{ij}\ . \tag{5.8}$$

Taking the divergence of this expression gives an equation for $\xi_i^{(0)}$,

$$\nabla^2\xi_i^{(0)}+\frac{1}{3}\partial_i\partial^j\xi_j^{(0)}=0\ . \tag{5.9}$$

Any $\xi_i^{(0)}$ (possibly time dependent) satisfying this equation preserves ζ-gauge up to zeroth-order. This means that this equation describes the residual diffeomorphism. Notice that taking its divergence one gets

$$\nabla^2\partial^i\xi_i^{(0)}=0$$

and assuming fields vanishing at infinity one has only $\partial^i\xi_i^{(0)}=0$. Substituting this into 5.9 gives $\nabla^2\xi_i^{(0)}=0$, which then implies $\xi_i^{(0)}=0$. This implies that $\xi_i^{(0)}$ must be non-zero at infinity, as expected.
Notice that the (time-dependent) dilatation and the special conformal transformations (SCT) are solutions to 5.9:

$$\begin{aligned}\xi_i^{\text{dilatation}}&=\lambda(t)x_i\\ \xi_i^{\text{SCT}}&=2b^j(t)x_jx_i-x^jx_jb_i(t)\ .\end{aligned} \tag{5.10}$$

These are the generators of conformal transformations on spatial slices, which are exactly the broken generators: fixing a gauge for fields vanishing at infinity, we are automathically excluding these generators.

To sum up, this symmetry breaking is associated to having fixed ϕ to a constant. In an analogous manner to what happens in the usual SSB mechanism of particle physics, we naturally split $\phi=\phi_0+\delta\phi$:

ϕ_0 can be interpreted as the vacuum expectation value and $\delta\phi$ becomes the Goldstone boson[3]. Since (without having fixed a gauge) $\zeta \sim \delta\phi$, one can interpret ζ as the Goldstone boson associated to the symmetry breaking. Under these symmetries it transforms at zeroth-order according to 5.7

$$\begin{aligned} \delta^{\text{dilatation}}\, \zeta^{(0)} &= \lambda(1 + x_i \partial_i \zeta) \\ \delta^{\text{SCT}}\, \zeta^{(0)} &= 2b_i x_i + (2b_i x_i x_j - x^i x^i b_j)\partial_j \zeta \ . \end{aligned}$$

5.3.2 Adiabatic modes once again

As we have seen, the residual transformations ξ_i have a field-independent part which does not fall off at spatial infinity. This implies that they map field configurations which fall off at infinity into those which do not. However, we have seen in section 3.2 that that a subset of these transformations can be thought of as the $k \to 0$ limit of transformations which do fall off at infinity and therefore generate new physically acceptable solutions. These are called adiabatic modes.
Proving Weinberg theorem, we have seen that a solution which does not fall at infinity can be seen as the $k \to 0$ limit of a physical configuration which is 0 at infinity, only if satisfies the Einstein equations which become trivial in this limit. In the ADM formalism, they coincide with the Hamiltonian and momentum constraints, which in single-field models are perturbatively solved by 4.67,

$$N^{(1)} = \frac{\dot\zeta}{H} \ , \quad N_i^{(1)} = \partial_i \alpha \ , \quad \alpha = -\frac{\zeta}{H} + a^2 \frac{\dot\phi_0^2}{2H} \nabla^{-2} N^{(1)} \ .$$

We then consider a profile which locally looks pure gauge: to linear order we only need the nonlinear part of the transformation laws 5.4 and 5.7

$$\delta N^i = \dot\xi^i \ , \quad \delta N = 0 \ , \quad \delta\zeta^{(0)} = \frac{1}{3}\partial^i \xi_i^{(0)} \ .$$

To be extendible to a physical field configuration we have to impose the constraints 4.67. One finds

$$0 = \delta N^{(1)} = \frac{\delta\dot\zeta}{H} = \frac{1}{3H}\partial_i \dot\xi^{(0)i} \longrightarrow \partial_i \dot\xi^{(0)i} = 0 \tag{5.11}$$

and similarly

$$\dot\xi_i^{(0)} = \delta N_i^{(1)} = -\frac{\partial_i \delta\zeta}{H} + a^2 \frac{\dot\phi_0^2}{2H} \nabla^{-2} \partial_i \delta N^{(1)} = -\frac{\partial_i \partial_j \xi^{(0)j}}{3H} \ . \tag{5.12}$$

The physical solutions to 5.9 must respect these two constraints 5.11 and 5.12. In order to find the solutions we SVT decompose ξ^i as follows

$$\xi^{(0)i} = \bar\xi^{(0)i} + \xi_{\text{V}}^{(0)i} \ ,$$

where $\bar\xi^{(0)i} = \partial^i \xi_S^{(0)}$ and $\partial^i \xi_i^{\text{V}} = 0$ as usual. The first constraint 5.11 gives

$$\partial_i \dot{\bar\xi}^{(0)i} = 0 \ . \tag{5.13}$$

This means that any time-dependent contribution to $\bar\xi^i$ must be divergence-free, so it can be reabsorbed into a redefinition of ξ_{V}^i. Therefore, we can assume $\bar\xi^i$ time-independent. This means that the second constraint 5.12 gives

$$\dot\xi_{\text{V}}^{(0)i} = -\frac{1}{3H}\partial_i \partial_j \bar\xi^{(0)j} \ ,$$

which is solved by

$$\xi_{\text{V}}^{(0)i} = -\frac{1}{3}\int^t \frac{\mathrm{d}t'}{H(t')} \partial_i \partial_j \bar\xi^{(0)j} \ . \tag{5.14}$$

[3]As pointed out in [63], one would expect having 4 Goldstone bosons, since 4 generators are broken. On the contrary, this is actually false, since it has been proved ([66]) that standard Goldstone theorem is not valid for spacetime symmetries such as diffeomorphism invariance.

Moreover, since 5.9 is respected by the scalar and the vector sector separately, $\bar{\xi}^{(0)i}$ must satisfy

$$\nabla^2 \bar{\xi}_i^{(0)} + \frac{1}{3}\partial_i \partial^j \bar{\xi}_j^{(0)} = 0 \ . \tag{5.15}$$

The physically allowed diffeomorphisms are therefore given by

$$\xi^{(0)i} = \bar{\xi}_i^{(0)} + \xi_{\mathrm{V}i}^{(0)} = \left(1 + \int^t \frac{\mathrm{d}t'}{H(t')}\nabla^2\right)\bar{\xi}^{(0)i} \ . \tag{5.16}$$

The second summand can be interpreted as a correction to the transformation which makes a general diffeomorphism physically allowed. It is interesting to notice that this result completely agrees with the demonstration of the Weinberg theorem. Indeed, taking a transformation of the type $\xi^i = \lambda x^i + \omega^{ij}x^j$ with λ and ω exactly constant, the second term is null, so there is no correction. This implies that this transformation is associated to an adiabatic mode. Notice on the contrary that the generator of the special conformal transformation ξ_i^{SCT} does receive a correction, so it is not adiabatic.

Finally, we derive the transformation law for ζ and D. Substituting the SVT decomposition of $\xi^{(0)i}$ in 5.7 one gets

$$\delta\zeta^{(0)} = \frac{1}{3}\partial_i\left(\bar{\xi}^{(0)i} + \xi_{\mathrm{V}}^{(0)i}\right) + \xi_i\partial^i\zeta = \frac{1}{3}\partial_i\bar{\xi}^{(0)i} + \xi_i\partial^i\zeta \ . \tag{5.17}$$

Similarly for $\delta D^{(0)}$ one finds from 5.8

$$\delta D_{ij}^{(0)} = \partial_i\bar{\xi}_j^{(0)} + \partial_j\bar{\xi}_i^{(0)} - \frac{2}{3}\partial^k\bar{\xi}_k^{(0)}\delta_{ij} + \partial_i\xi_j^V + \partial_j\xi_i^V \ . \tag{5.18}$$

We can rename $\delta\bar{D}_{ij}^{(0)} := \partial_i\bar{\xi}_j^{(0)} + \partial_j\bar{\xi}_i^{(0)} - \frac{2}{3}\partial^k\bar{\xi}_k^{(0)}\delta_{ij}$ and using 5.14 and 5.15, as well as 5.13 stating that $\partial_i\bar{\xi}^{(0)i}$ is independent of time, one gets

$$\begin{aligned}
\delta D_{ij}^{(0)} &= \delta\bar{D}_{ij}^{(0)} - \frac{1}{3}\partial_i\int^t \frac{dt'}{H(t')}\partial_j\partial_k\bar{\xi}^{(0)k} - \frac{1}{3}\partial_j\int^t \frac{dt'}{H(t')}\partial_i\partial_k\bar{\xi}^{(0)k} = \\
&= \delta\bar{D}_{ij}^{(0)} + \partial_i\int^t \frac{dt'}{H(t')}\nabla^2\bar{\xi}_j^{(0)} + \partial_j\int^t \frac{dt'}{H(t')}\nabla^2\bar{\xi}_i^{(0)} = \\
&= \delta\bar{D}_{ij}^{(0)} + \nabla^2\left(\partial_i\bar{\xi}_j^{(0)} + \partial_j\bar{\xi}_i^{(0)}\right)\int^t \frac{dt'}{H(t')} \ .
\end{aligned} \tag{5.19}$$

5.3.3 Physical symmetries Taylor expansion

At this point one is interested in finding the diffeomorphisms $\bar{\xi}^{(0)i}$ satisfying 5.15, which gives all the physical diffeomorphisms 5.16. One can try to solve the equation using a power-series

$$\bar{\xi}_i = \sum_{n=0}^{\infty}\frac{1}{(n+1)!}M_{i\ell_0\ldots\ell_n}x^{\ell_0}\cdots x^{\ell_n} \ ,$$

where the $M_{i\ell_0\cdots\ell_n}$ is constant and symmetric in its last $n+1$ indices (any antisimmetric part would vanish). The equation 5.15 gives the condition on the M's:

$$M_{i\ell\ell_2\ldots\ell_n} = -\frac{1}{3}M_{\ell i\ell\ell_2\ldots\ell_n} \quad (n \geq 1) \ .$$

Moreover, we have to require D_{ij} to be transverse, that is $q^i\delta D_{ij} = 0$: to achieve this, we assume M to be $\hat{q}$ dependent. This implies, through 5.18 (ξ_i^V is authomatically transverse), that

$$\hat{q}^i\left(M_{i\ell_0\ell_1\ldots\ell_n}(\hat{q}) + M_{\ell_0 i\ell_1\ldots\ell_n}(\hat{q}) - \frac{2}{3}\delta_{i\ell_0}M_{\ell\ell\ell_1\ldots\ell_n}(\hat{q})\right) = 0 \ . \tag{5.20}$$

Each term of the series gives rise to a different diffeomorphism. We are interested in the lowest order term $n = 0$; we will see that all the other terms $n \geq 1$ would give rise to new consistency relations, which we are not going to discuss.

In the case $n = 0$ the diffeomorphism is $\bar{\xi}_i = M_{i\ell_0} x^{\ell_0}$. This diffeomorphism is adiabatic, which implies $\xi^i = \bar{\xi}^i$, so that the transformation rules 5.17 and 5.19 become

$$\begin{aligned} \delta\zeta^{(0)} &= \frac{1}{3}\partial_i \bar{\xi}^{(0)i} + \bar{\xi}_i \partial^i \zeta = \frac{1}{3} M_{ii} + M_{i\ell_0} x^{\ell_0} \partial_i \zeta \\ \delta D_{ij}^{(0)} &= \partial_i \bar{\xi}_j^{(0)} + \partial_j \bar{\xi}_i^{(0)} - \frac{2}{3}\partial^k \bar{\xi}_k^{(0)} \delta_{ij} = M_{ij} + M_{ji} - \frac{2}{3} M_{kk}\delta_{ij} \ . \end{aligned}$$

Passing to Fourier space these become

$$\begin{aligned} \delta\zeta^{(0)}(\vec{k}) &= \frac{1}{3} M_{ii} (2\pi)^3 \delta^{(3)}(\vec{k}) - M_{i\ell_0} k^i \frac{\partial \zeta(\vec{k})}{\partial k_{\ell_0}} \\ \delta D_{ij}^{(0)}(\vec{k}) &= \left(M_{ij} + M_{ji} - \frac{2}{3} M_{kk}\delta_{ij} \right) (2\pi)^3 \delta^{(3)}(\vec{k}) \ . \end{aligned}$$

Moreover, in the $n = 0$ case, there is only one index ℓ_0 so there is no constraint due to antisymmetry. The matrix can be generally decomposed as

$$M_{i\ell_0} = \lambda \delta_{i\ell_0} + S_{i\ell_0} + A_{i\ell_0} \ ,$$

where S is symmetric and traceless $S_{ii} = 0$ and A is antisymmetric (naturally traceless). In other words, we have split M into an antisymmetric and a symmetric part, which can be furtherly divided into a trace and a traceless part. We have also to impose $M_{i\ell_0}$ to be transverse in Fourier space (for eq. 5.20), which is authomatically true for $\lambda\delta_{i\ell_0}$ and $A_{i\ell_0}$ (thanks to their property), but must be imposed to S:

$$\hat{q}^i S_{i\ell_0}(\vec{q}) = 0 \ .$$

Let us analyze how these different terms contribute to the transformations of ζ and D. In the case of $M_{i\ell_0} = \lambda\delta_{i\ell_0}$, which is a dilatation, one gets as before

$$\delta\zeta^{(0)} = \lambda(1 + x^i \partial_i \zeta) \ , \quad \delta D_{ij}^{(0)} = 0 \ .$$

The case $M_{i\ell_0} = S_{i\ell_0}$ corresponds to a volume-preserving anisotropic rescaling of coordinates, under which the fields transfom as

$$\delta\zeta^{(0)} = \lambda x^i \partial_i \zeta \ , \quad \delta D_{ij}^{(0)} = 2 S_{ij} \ .$$

Finally, $M_{i\ell_0} = A_{i\ell_0}$ is a rotation, under which the fields transfom as

$$\delta\zeta^{(0)} = A_{ij} x^j \partial^i \zeta \ , \quad \delta D_{ij}^{(0)} = 0 \ .$$

Notice that the transformation we have found coincide with the Weinberg transformation 3.16, with the exception of the antisymmetric matrix A, which is however irrelevent. Indeed in the limit $k \to 0$ one can neglect gradients[4], so that the result is given by the Weinberg transformation rules 3.17 with $\epsilon = 0$, with the identification[5] $\lambda \to -\lambda$ and $2S_{ij} \to \omega_{ij} + \omega_{ji}$, while A does not contribute.

[4]We will see in the next chapter that neglecting gradients is dangerous, since it can lead to wrong results. In this case, however, this procedure leads to the expected results. As we will see, this is because the transformation is an exact Weinberg transformation, that is λ and ω are exactly constant.

[5]This difference in the sign of λ is due to the fact that in ζ-gauge $\zeta = -\Psi$.

For sake of completeness, we will report the result from [50] at all orders, including the orders we have not discuss in section 5.3.1. The variations in Fourier space are shown to be:

$$
\begin{aligned}
\delta\zeta^{(n)}(\vec{k}) =& \frac{(-i)^n}{3n!} M_{ii\ell_1\ldots\ell_n} \frac{\partial^n}{\partial k_{\ell_1}\cdots\partial k_{\ell_n}} \left((2\pi)^3\delta^{(3)}(\vec{k})\right) \\
&- \frac{(-i)^n}{n!} M_{i\ell_0\ldots\ell_n} \left(\delta^{i\ell_0}\frac{\partial^n}{\partial k_{\ell_1}\cdots\partial k_{\ell_n}} + \frac{k^i}{n+1}\frac{\partial^{n+1}}{\partial k_{\ell_0}\cdots\partial k_{\ell_n}}\right)\zeta(\vec{k}) \\
&+ \frac{(-i)^n}{n!} M_{\ell\ell_0\ldots\ell_n} \Upsilon^{\ell\ell_0 ij}(\hat{k}) \frac{\partial^n}{\partial k_{\ell_1}\cdots\partial k_{\ell_n}} D_{ij}(\vec{k}) + \ldots \\
\delta D^{(n)}_{ij}(\vec{k}) =& \frac{(-i)^n}{n!}\left(M_{ij\ell_1\ldots\ell_n} + M_{ji\ell_1\ldots\ell_n} - \frac{2}{3}\delta_{ij}M_{\ell\ell\ell_1\ldots\ell_n}\right)\frac{\partial^n}{\partial k_{\ell_1}\cdots\partial k_{\ell_n}}\left((2\pi)^3\delta^3(\vec{k})\right) \\
&- \frac{(-i)^n}{n!} M_{\ell\ell_0\ldots\ell_n}\left(\delta^{\ell\ell_0}\frac{\partial^n}{\partial k_{\ell_1}\cdots\partial k_{\ell_n}} + \frac{k^\ell}{n+1}\frac{\partial^{n+1}}{\partial k_{\ell_0}\cdots\partial k_{\ell_n}}\right) D_{ij}(\vec{k}) \\
&+ \frac{(-i)^n}{n!} M_{ab\ell_1\ldots\ell_n}\Gamma^{ab}_{ijk\ell}(\hat{k})\frac{\partial^n}{\partial k_{\ell_1}\cdots\partial k_{\ell_n}}\gamma^{k\ell}(\vec{k}) + \ldots
\end{aligned}
\tag{5.21}
$$

where

$$
\begin{aligned}
\Upsilon_{abcd}(\hat{k}) &\equiv \frac{1}{4}\delta_{ab}\hat{k}_c\hat{k}_d - \frac{1}{8}\delta_{ac}\hat{k}_b\hat{k}_d - \frac{1}{8}\delta_{ad}\hat{k}_b\hat{k}_c \\
\Gamma_{abijk\ell}(\hat{k}) &= -\frac{1}{2}\left(\delta_{ij} + \hat{k}_i\hat{k}_j\right)\left(\delta_{ab}\hat{k}_k\hat{k}_\ell - \frac{1}{2}\delta_{ak}\hat{k}_\ell\hat{k}_b - \frac{1}{2}\delta_{a\ell}\hat{k}_k\hat{k}_b\right) + \delta_{b\{i}\delta_{j\}\{k}\delta_{\ell\}a} - \delta_{a\{i}\delta_{j\}\{k}\delta_{\ell\}b} \\
&\quad - \delta_{b\{i}\hat{k}_{j\}}\delta_{a\{k}\hat{k}_{\ell\}} + \delta_{a\{i}\hat{k}_{j\}}\delta_{b\{k}\hat{k}_{\ell\}} - \delta_{a\{k}\delta_{\ell\}\{i}\hat{k}_{j\}}\hat{k}_b - \delta_{b\{k}\delta_{\ell\}\{i}\hat{k}_{j\}}\hat{k}_a + 2\delta_{ab}\hat{k}_{\{i}\delta_{j\}\{k}\hat{k}_{\ell\}}
\end{aligned}
$$

with $\hat{k}^i = \frac{k^i}{k}$. In the variations we have neglected higher order terms in the fields, of type $\sim\gamma^2$, $\sim\zeta\gamma$ and similar[6].

5.3.4 Nöther charges and the consistency relation as a Ward identity

At this point, given a symmetry of the action, we can associate a conserved current via Nöther theorem ([67]), which is

$$
j^\mu = \mathcal{L}\delta x^\mu + \sum_r \frac{\delta\mathcal{L}}{\delta\dot{\varphi}_r}\delta\varphi_r \ ,
$$

where φ_r are the different fields in the Lagrangian and in this case $\delta x^\mu = (0, \xi^i)$, so that its 0-component is

$$
j^0 = \frac{\delta\mathcal{L}}{\delta\dot{\zeta}}\delta\zeta + \frac{\delta\mathcal{L}}{\delta\dot{D}_{ij}}\delta D_{ij} = \frac{1}{2}\{\pi^\zeta, \delta\zeta\} + \frac{1}{2}\{\pi^D_{ij}, \delta D_{ij}\} \ ,
$$

with $\pi^\zeta = \frac{\delta\mathcal{L}}{\delta\dot{\zeta}}$ and $\pi^D_{ij} = \frac{\delta\mathcal{L}}{\delta\dot{D}_{ij}}$ the conjugate momenta. The anticommutators are irrelevant at classical level, but necessary at quantum level to have a Hermitian charge operator. The Nöther charge is

$$
Q = \int d^3x j^0 = \frac{1}{2}\int d^3x \left[\{\pi^\zeta, \delta\zeta\} + \{\pi^D_{ij}, \delta D_{ij}\}\right] .
$$

At quantum level, we remind that the operator Q is the generator of the symmetry, which means that the fields transform under the symmetry as $\varphi'(x) = e^{i\alpha Q}\hat{\varphi}e^{-i\alpha Q}$ ($\alpha\in\mathbb{R}$), implying that

$$
\delta\varphi = i[Q, \varphi] \ .
$$

The Ward identities are obtained by taking the *in-in* vacuum expectation value of the action of the charges on the operator $\mathcal{O}$

$$
\langle\Omega|[Q, \mathcal{O}]|\Omega\rangle = -i\langle\Omega|\delta\mathcal{O}|\Omega\rangle \ ,
$$

[6] As we will see, these terms can be seen as the squeezing of more than one field.

where $|\Omega\rangle$ is the *in*-vacuum of the interacting theory in Heisenberg picture and $\mathcal{O}$ denotes an equal-time product of N scalar and tensor fields:

$$\mathcal{O}\left(\vec{k}_1,\ldots,\vec{k}_N\right)=\mathcal{O}^{\zeta}\left(\vec{k}_1,\ldots,\vec{k}_M\right)\cdot\mathcal{O}^{D}_{i_{M+1}j_{M+1},\ldots,i_Nj_N}\left(\vec{k}_{M+1},\ldots,\vec{k}_N\right)$$

where $0\le M\le N$, with $\mathcal{O}^{\zeta}:=\prod_{a=1}^{M}\zeta\left(\vec{k}_a,t\right)$ and $\mathcal{O}^{D}_{i_{M+1}j_{M+1},\ldots,i_Nj_N}:=\prod_{b=M+1}^{N}D_{i_bj_b}\left(\vec{k}_b,t\right)$. Notice that the order of the fields is irrelevant, since being an equal-time product they commute. Moreover, it is possible that a subset of the tensor indices are contracted among themselves or not.

Using standard QFT techniques, one can show ([50]) that the Ward identity associated to the generic transformation 5.21 is given by

$$\begin{aligned}
&\lim_{q\to0}M_{i\ell_0\ldots\ell_n}\frac{\partial^n}{\partial q_{\ell_1}\cdots\partial q_{\ell_n}}\left(\frac{1}{P_D(q)}\left\langle D^{i\ell_0}(\vec{q})\mathcal{O}\left(\vec{k}_1,\ldots,\vec{k}_N\right)\right\rangle_c'+\frac{\delta^{i\ell_0}}{3P_\zeta(q)}\left\langle\zeta(\vec{q})\mathcal{O}\left(\vec{k}_1,\ldots,\vec{k}_N\right)\right\rangle_c'\right)\\
&=-M_{i\ell_0\ldots\ell_n}\left\{\sum_{a=1}^{N}\left(\delta^{i\ell_0}\frac{\partial^n}{\partial k^a_{\ell_1}\cdots\partial k^a_{\ell_n}}-\frac{\delta_{n0}}{N}\delta^{i\ell_0}+\frac{k^i_a}{n+1}\frac{\partial^{n+1}}{\partial k^a_{\ell_0}\cdots\partial k^a_{\ell_n}}\right)\left\langle\mathcal{O}\left(\vec{k}_1,\ldots,\vec{k}_N\right)\right\rangle_c'\right.\\
&\quad-\sum_{a=1}^{M}\Upsilon^{i\ell_0i_aj_a}\left(\hat{k}_a\right)\frac{\partial^n}{\partial k^a_{\ell_1}\cdots\partial k^a_{\ell_n}}\left\langle\mathcal{O}^{\zeta}\left(\vec{k}_1,\ldots,\vec{k}_{a-1},\vec{k}_{a+1},\ldots\vec{k}_M\right)D_{i_aj_a}\left(\vec{k}_a\right)\mathcal{O}^{D}\left(\vec{k}_{M+1},\ldots,\vec{k}_N\right)\right\rangle_c'\\
&\quad\left.-\sum_{b=M+1}^{N}\Gamma^{i\ell_0}_{i_bj_b}{}_{k_b\ell_b}\left(\hat{k}_b\right)\frac{\partial^n}{\partial k^b_{\ell_1}\cdots\partial k^b_{\ell_n}}\left\langle\mathcal{O}^{\zeta}\left(\vec{k}_1,\ldots,\vec{k}_M\right)\mathcal{O}^{D}_{i_{M+1}j_{M+1},\ldots,k_b\ell_b,\ldots i_Nj_N}\left(\vec{k}_{M+1},\ldots,\vec{k}_N\right)\right\rangle_c'\right\}\\
&\quad+\ldots\,,
\end{aligned}\tag{5.22}$$

where $'$ indicates the N-point function in momentum space without the momentum conserving Dirac delta, C indicates connected correlators and the dots stand for higher order terms in perturbations (see footnote 6). The squeezed momentum $q\to0$ is usually called soft-momentum.
In the next section we are going to show that this general result reduces to some generalizations of the standard consistency relations involving a generic N-point function.

5.3.5 The simplest generalizations

The standard consistency relations we have found in sections 5.1 and 5.2 arise taking the $n=0$ diffeomorphism we have discussed in section 5.3.3.

More in particular, taking $M_{i\ell_0}=\lambda\delta_{i\ell_0}$ gives a generalization of the relation for $\langle\zeta\zeta\zeta\rangle$. Indeed, this way, the first line of 5.22 is given by the second summand only, given that the first is null being D traceless; in the second line the first summand is not there, as the entire third and fourth lines. Taking $\mathcal{O}=\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_N}$, the final result is

$$\lim_{q\to0}\langle\zeta_{\vec{q}}\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_N}\rangle_c=-\delta^{(N+1)}(\vec{q}+\vec{k}_1+\ldots+\vec{k}_N)P_\zeta(q)(3(N-1)+\mathcal{D})\langle\zeta_{\vec{k}_1}\ldots\zeta_{\vec{k}_N}\rangle_c'\,,\tag{5.23}$$

where $\mathcal{D}$ is the operator $\mathcal{D}=k^i_a\frac{\partial}{\partial k^i_a}$ (with the convention that repeated indices are summed: the index a runs over the different k's so $a=1,\ldots N$, while i runs over the spatial components). We want to see that this generalization is consistent with the case $N=2$ we have treated before, i. e. the squeezed limit of the 3-point function. In the case $N=2$ the connected correlators coincide with the correlators so we can safely forget about this fact. Then, since $\langle\zeta_{k_1}\zeta_{k_2}\rangle'=(2\pi)^3P_\zeta(k_1)$, using the chain rule one has[7]

$$\mathcal{D}P_\zeta(k_1) = k_1^i \frac{\partial}{\partial k_1^i} P_\zeta(k_1) = k_1 \frac{\partial}{\partial k_1} P_\zeta(k_1) \ ,$$

and moreover

$$P_\zeta(k_1) \frac{\partial \log(k_1^3 P(k_1))}{\partial \log k_1} = 3P_\zeta(k_1) + k_1 \frac{\partial P_\zeta(k_1)}{\partial k_1} \ ,$$

so that one gets exactly 5.2:

$$\begin{aligned}\lim_{q\to 0} \langle \zeta_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle &= -\delta^{(3)}(\vec{q} + \vec{k}_1 + \vec{k}_2) P_\zeta(q)(3 + \mathcal{D}) \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle' = \\ &= -(2\pi)^3 \delta^{(3)}(\vec{q} + \vec{k}_1 + \vec{k}_2) P_\zeta(q) P_\zeta(k_1) \frac{\partial \log(k_1^3 P_\zeta(k_1))}{\partial \log k_1} \ .\end{aligned}$$

On the contrary, the tensor consistency relation of section 5.2 comes from the Ward identity 5.22 for the $n = 0$ diffeomorphism $M_{i\ell^0} = S_{i\ell^0}$, with $S_{i\ell^0}$ symmetric and traceless $S_{ii} = 0$. Indeed, this way the second summand in the second line is null since $M_{i\ell^0}$ is traceless; for the same reason only the last summand in the second line survives; the last two lines are null. Then the matrix $S^{i\ell_0}$ appears in both members and can be simplified. Finally, projecting both sides into the helicity basis one gets

$$\lim_{q\to 0} \langle D^s_{\vec{q}} \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_N} \rangle_c = -\delta^{(3)}(\vec{q} + \vec{k}_1 + \ldots + \vec{k}_N) \frac{1}{2} P_T(q) \epsilon^s_{il} \sum_a k^i_a \frac{\partial}{\partial k^l_a} \langle \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_N} \rangle'_c$$

As before, we want to see that this reduces to 5.3 in the case $N = 2$:

$$\lim_{q\to 0} \langle D^s_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle = -\delta^{(3)}(\vec{q} + \vec{k}_1 + \vec{k}_2) \frac{1}{2} P_T(q) \epsilon^s_{il} \sum_a k^i_a \frac{\partial}{\partial k^l_a} \langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle' \tag{5.24}$$

To see that this constitutes a generalization of the previous result, we proceed as before[8]

$$\epsilon^s_{il} \sum_a k^i_a \frac{\partial}{\partial k^l_a} \langle \zeta_{k_1} \zeta_{k_2} \rangle' = (2\pi)^3 \epsilon^s_{il} k^i_1 \frac{\partial}{\partial k^l_1} P_\zeta(k_1) = (2\pi)^3 \epsilon^s_{il} \frac{k^i_1 k^l_1}{k_1} \frac{\partial}{\partial k_1} P_\zeta(k_1) \ . \tag{5.25}$$

To get back the previous result now we have to use the explicit form for $P_\zeta(k_1)$, which is 4.71, finding

$$\epsilon^s_{il} \sum_a k^i_a \frac{\partial}{\partial k^l_a} \langle \zeta_{k_1} \zeta_{k_2} \rangle' = (2\pi)^3 \epsilon^s_{il} \frac{k^i_1 k^l_1}{k_1} (-3) \frac{H^2_*}{4\epsilon M^2_P k^4} = -3(2\pi)^3 \epsilon^s_{il} \frac{k^i_1 k^l_1}{k^2_1} P_\zeta(k_1) \ ,$$

so that plugging this back to 5.24 the result is exactly 5.3:

$$\lim_{q\to 0} \langle D^s_{\vec{q}} \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle = (2\pi)^3 \delta^{(3)}(\vec{q} + \vec{k}_1 + \vec{k}_2) \frac{3}{2} P_T(q) P_\zeta(k_1) \epsilon^s_{il} \frac{k^i_1 k^l_1}{k^2_1} \ .$$

Finally, notice that the consistency relations 5.23 and 5.24 are more general than 5.1 and 5.3, since they do not rely on slow-roll approximation.

As it is clear, the two generalizations we have discussed in details are only the simplest among a wide range of results one can derive from eq. 5.22, which produces an infinite set of Ward identities.

[7] Indeed:

$$\begin{aligned}k^i_1 \frac{\partial}{\partial k^l_1} P_\zeta(k_1) &= k^i_1 \frac{\partial k_1}{\partial k^l_1} \frac{\partial}{\partial k_1} P_\zeta(k_1) = k^i_1 \frac{\partial \sqrt{k^j_1 k^j_1}}{\partial k^l_1} \frac{\partial}{\partial k_1} P_\zeta(k_1) = \\ &= k^i_1 \frac{k^l_1}{\sqrt{k^j_1 k^j_1}} \frac{\partial}{\partial k_1} P_\zeta(k_1) = \frac{k^i_1 k^l_1}{k_1} \frac{\partial}{\partial k_1} P_\zeta(k_1) \ .\end{aligned}$$

Contracting i and l one gets the result.

[8] See footnote 7.

In particular, the most important ones are the so-called *conformal consistency relations*, which were introduced in [62]. Indeed, the results presented in the previous sections arise from choosing the $n = 0$ diffeomorphisms: notice that looking at the conformal group generators 5.10 the dilatation is an $n = 0$ diffeomorphism, while the special conformal transformations not. Indeed, in [50] it is shown that they are exactly a subgroup of the diffeomorphisms in the $n = 1$ case. In this case there are 3 conformal consistency relations associated to the 3 generators of the special conformal transformations. We leave the details to the interested reader, but we limit ourselves to say that these relations have exactly the same expression of 5.23, with the exception that the derivative operator $\mathcal{D}$ has a different (and more complicated) form.

All the other relations arising from the Ward identities from $n > 1$ diffeomorphisms constitute new results.

Finally, a very recent generalization of these results is contained in [68], where an early-late time consistency relation is presented, relating correlation functions with an initial time soft insertion (i. e. a $\zeta(\vec{q})$ with $q \to 0$) to the symmetry-trasformed late-time correlators.

5.4 Violating the consistency relation

As we have seen in section 5.3, the consistency relation can be proved starting from the assumption that the inflation is driven by a single degree of freedom. However, it is natural to wonder which inflationary models violate it. This is very important, since the consistency relation is very interesting from an experimental point of view: in the future we hope that experiments are able to measure the bispectra, so if the experimental results violate it, physicists are forced to think to alternative models, such as the ones we are going to discuss now briefly.
To understand how to violate the consistency relation, one can carefully examine which are the underlined assumptions used in the proof.

- A first strong assumption made is that the spacetime can be sliced in order to write the metric in the ADM form. As it is well-known in literature, in single-field inflation models the inflaton behaves like a clock measuring how the time passes in different portion of the Universe. Technically, this is because one can imagine to slice the spacetime by choosing the slices such that $\delta\phi = 0$ (this is ζ-gauge; see also appendix B). However, if one considers a model with more than one inflaton, this approach breaks down. In general, multi-field inflationary models ([21]), involving isocurvature perturbations, violate the Maldacena consistency relation.
- Another important assumption was inflation to be an attractor, so that ζ quickly converges to a constant and N and N_i to their unperturbed values. There are models with a non-attractor phase, for example fast-roll ([69]) and ultra-slow-roll inflation ([70, 71]).
- We made also the hypothesis of a stable background. For example, there are models with the vacuum state which is not Bunch-Davies ([72]) or in which the cosmic fluid has a sound speed which varies in time ([73]).
- Finally, we assumed space diffeomorphism invariance. Some models are based on the breaking of this symmetry: probably, the most famous one is solid inflation ([74]). Breaking also time diffeomorphisms one gets a generalization called supersolid inflation ([75, 76]).

Chapter 6

Is the consistency relation physical?

In last ten years, some papers ([77–80]) claimed that the consistency relation and tensor squeezed correlators can be considered as gauge artifacts, i. e. unphysical. The argument is based on a peculiar gauge transformation, sending FLRW in conformal time to the so-called conformal Fermi coordinates (CFC) frame, in which the bispectrum vanishes. This result has been cited and used in other papers (for example [81, 82]).
This discussion has then focused on the effects on large scale structures, in the so-called halo bias ([83–86]), and tensor fossils[1] ([72, 88–90]). In [78] it is claimed that CFC frame transformations encodes projection effects of the late time Universe, however as we will see this believing has origins in an erroneous gradient expansion. Thus, in this chapter we focus only on the Early Universe consistency relation, which is the one we have discussed in the previous chapter.

Recently, the arguments leading to the cancellation of the consistency relation have been criticized. The idea that it was observable was already present in [50], deriving the consistency relation as a conformal Ward identity. A more explicit argument is contained in a very recent paper by Matarrese, Pilo and Rollo ([91]), which focuses on the case of the bispectrum of curvature perturbation $B_{\zeta\zeta\zeta}$.

In this chapter we are going to discuss briefly the arguments used to cancel the consistency relation. Then we present the results of [91], showing explicitly that these procedures are not correct, including not only the bispectrum $B_{\zeta\zeta\zeta}$, but also the ones including tensor modes for the first time in literature. The latter is the main result of this project.

6.1 The argument used to cancel the bispectrum: Conformal Fermi coordinates

In this section we present briefly the argument used to cancel the consistency relation, based on arguing the existence of a gauge transformation which applied to squeezed bispectrum makes it vanish,

$$\lim_{k_L \to 0} B(k_L, k_1, k_2) = 0 \ .$$

We have indicated with k_L the long wavelength mode ($k_L \to 0$) and with k_1 and k_2 the two modes remaining short; we remind that for the momentum conservation in the squeezed limit $k_1 = k_2 + \mathcal{O}(k_L)$.

[1]Many inflationary models introduce new fields that may couple to the inflaton responsible for generating primordial density perturbations, interacting very weakly, or without interacting at all, at late times. The effects of these fields can be revealed only by looking at CMB and they might consist in non-Gaussian signatures (for example non-trivial 4-point correlation functions, i. e. trispectra in Fourier space), since they can produce non-trivial trispectrum. The term *tensor fossil* was introduced in [87] to indicate possible signatures in the non-linear power spectrum or the bispectrum of density perturbation due to a new tensor field.

The first paper proposing the cancellation was [77] by Tanaka and Urakawa. Their work is the last of a series of papers ([92–94]) in which they proposed an alternative approach to cosmological perturbations vanishing at infinity: since the observable Universe is finite, their idea is to consider boundary conditions at a finite, not at infinity. This modifies the way in which gauge invariant quantities are defined. We do not go through the details of these papers, since developing this formalism is very technical but the result is essentially the same of [78], which is widely known in literature.
Here we are going to sketch the original argument by Pajer, Schmidt and Zaldarriaga ([78]). The argument was repeated also in [79] and [80], which slightly generalized the results at higher order in k_L or to non-attractor models. However, we will be the more elementary as possible, being this sufficient for our purposes.

The frame of reference used to cancel the consistency relation is given by conformal Fermi coordinates (CFC). The equivalence principle of GR introduced in section 1.1 imposes local Lorentz invariance: this implies the possibility to construct a local set of coordinates around an arbitrary point of the spacetime, where the metric looks like Minkowski in the neighborhood of the point chosen. Moreover, in some circumstances, it is useful to go into a reference frame where the metric looks like Minkowski in the neighborhood of an entire worldline. This frame of reference is called Fermi normal coordinates (FNC). In appendix C we formulate the problem in a mathematically precise way and we explicitly construct the FNC frame.
However, in some cases, the FNC are still not enough, for the following reason. Since equations describing the transformation from a general frame to the FNC frame are usually impossible to solve exactly, one relies on a perturbative approach, where the expansion parameter is linked to the distance from the general point to the central geodesic. This is natural, being interested in describing the spacetime in the neighborhood of the geodesic itself. However, there are cases where this becomes a problem and a clear example is the Universe during a quasi de Sitter phase (such as inflation). As explained in the appendix, taking a starting point from a FLRW Universe, the distance from the geodesic at which the transformation from FLRW to the FNC frame breaks down is of order r_H (the comoving Hubble radius 4.1). This is not a problem during the standard eras of cosmology, but during inflation is, since the comoving Hubble radius gets smaller and smaller, so the region of validity of the FNC frame is very tiny, so that the FNC frame is essentially useless to describes physics in a sufficient large portion of the Universe. Indeed, we have to remember that we want to compute 3-point functions, so we have to be sure that the patch of coordinates covers these three points.
This problem can be solved by the introduction of the Conformal Fermi Coordinates (CFC). Loosely speaking, for the FNC frame we required the metric to become of the form $ds^2 = \eta_{\mu\nu}$ up to higher orders; in this case we relax this by requiring that $ds^2 = a_F^2(\tau)\eta_{\mu\nu}$. In other words, we are asking the metric to become conformally flat. In the appendix C we explicitly construct this frame of reference, giving the transformation rules from a general frame to the CFC one. More specifically, since we have always used a perturbed FLRW Universe in conformal or cosmic time, we derive the coordinate transformation from this frame to CFC.

What we have said helps us to explain why CFC have been used to argue that the bispectrum is cancelled in the squeezed limit. Every observer in the Universe follows its own geodesic in the spacetime. Going to the FNC adapted to that geodesic, one can understand if the space is flat or not, indeed the higher order corrections to the metric $ds^2 = \eta_{\mu\nu}$ are related to the intrinsic curvature of the spacetime. But this construction does not work for inflation so one is forced to use CFC, which are defined in a patch of spacetime which stays always sufficiently large in order to include the region we need. However, we have to remark that this is still not enough to relate the primordial bispectra with the observations, since one has to take into account all the evolution of the Universe until today, which is a very difficult issue which we are not going to treat.

As we have said, the details about FNC and CFC frames are in appendix C. Now we want to comment

on a very important point: the coordinate transformation in a FLRW Universe in cosmic time (or equivalently in conformal time) into the CFC frame can be seen as a Weinberg transformation 3.16, with λ and ω dependent on the spacetime point.
The coordinate transformation, including both scalar and tensor first-order perturbations, up to third order in the old coordinates Δx, is given by the sum of the contributions of equation C.6 and C.7 in the appendix C:

$$\begin{aligned}\Delta x_F^k =& \Delta x^k - \bar{\Delta} x^k \Psi\Big|_P - \frac{1}{2}\bar{\Delta} x^i \bar{\Delta} x^j (\delta_j^k \partial_i \Psi + \delta_i^k \partial_j \Psi - \delta_{ij}\partial^k \Psi)\Big|_P + \\ & - \frac{1}{6}\bar{\Delta} x^i \bar{\Delta} x^j \bar{\Delta} x^l (\delta_j^k \partial_l \partial_i \Psi + \delta_i^k \partial_l \partial_j \Psi - \delta_{ij}\partial_l \partial^k \Psi)\Big|_P + \\ & + \frac{1}{2} D_i^k \bar{\Delta} x^i\Big|_P + \frac{1}{4}\bar{\Delta} x^i \bar{\Delta} x^j (\partial_i D_j^k + \partial_j D_i^k - \partial^k D_{ij})\Big|_P + \\ & + \frac{1}{12}\bar{\Delta} x^i \bar{\Delta} x^j \bar{\Delta} x^l (\partial_l \partial_i D_j^k + \partial_l \partial_j D_i^k - \partial_l \partial^k D_{ij})\Big|_P ,\end{aligned} \tag{6.1}$$

where $\Delta x^\mu = x^\mu(Q) - x^\mu(P)$, which is the deviation of the chosen point Q from the point P on the central geodesics and $\bar{\Delta} x^\mu$ is its unperturbed value. We have set $x^\mu(Q) = x^\mu$ and $x^\mu(P) = p^\mu(\tau)$, which is the central worldline.

If $k = 0$, thanks to Weinberg theorem we know that ζ and D are constant. In ζ-gauge, where $\zeta = -\Psi$, this implies that Ψ is constant. This means that in the exact $k = 0$ limit the transformation is simply given by

$$\Delta x_F^k = \Delta x^k - \bar{\Delta} x^k \Psi + \frac{1}{2} D_i^k \bar{\Delta} x^i , \tag{6.2}$$

which can be exactly written as a Weinberg transformation 3.16, with $\lambda = -\Psi$ and $\omega_i^k = D_i^k$.

All the demonstrations used to gauge away the consistency relations are based on a splitting in short and long wavelength contributions of the cosmological perturbations. The main example is the curvature perturbations ([79, 80]), which is split as

$$\zeta = \zeta_S + \zeta_L .$$

The two terms must be intended in Fourier space as

$$\zeta_S = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{i\vec{k}\cdot\vec{x}} (1 - W(k))\zeta(k) , \quad \zeta_L = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{i\vec{k}\cdot\vec{x}} W(k)\zeta(k) ,$$

where $W(k)$ is the filter (k_c is the cutoff scale)

$$W(k) = \theta\left(\frac{k_c - k}{H}\right) . \tag{6.3}$$

Working in ζ-gauge, one has

$$g_{ij} = (1 + 2\zeta)\delta_{ij} + D_{ij} .$$

At this point, one has to see how $\zeta_{L/S}$ behaves under a gauge transformation induced by the infinitesimal diffeomorphism obtained from 6.1. Notice that it does not change the gauge, since in the exact $k \to 0$ limit it becomes a Weinberg transformation, which is a residual gauge freedom of the ζ-gauge as we have seen in chapters 3 and 5. Moreover, according to [78], ζ is not a scalar, but only ζ_S. This is due to the fact that the authors, even without providing a clear definition for ζ in their paper, have in mind a definition which is different from the one used in section 4.6.2, but the following

$$\zeta^{(\text{alt})} = \frac{1}{6} \log \det \left(\frac{g_{ii}}{a^2}\right) , \tag{6.4}$$

as explicitly explained in the following papers [79,80]. In case of a perturbed FLRW, at first-order in perturbation one gets $\zeta^{(\text{alt})} = -\Psi + \frac{1}{6}\nabla^2\chi_\parallel$. Notice that in ζ-gauge $\zeta = -\Psi = \zeta^{(\text{alt})}$, but this is valid only in this specific gauge. Moreover, with this definition $\zeta^{(\text{alt})}$ is not gauge invariant since under the rules found in section 2.3 one has

$$\Delta\zeta^{(\text{alt})} = -\Delta\Psi + \frac{1}{6}\nabla^2\Delta\chi_\parallel = -\mathcal{H}\alpha \neq 0 \ .$$

Now, their claim is that in CFC all the long part of the perturbations are null; more specifically, CFC remove all the long part up to second-order derivatives, which are null in the limit $k \to 0$, so that denoting with a prime the CFC frame one has

$$\zeta'_L(x') = \mathcal{O}(\partial_i\partial_j\zeta_L) \ . \tag{6.5}$$

On the contrary, the CFC transformation leaves the short part untouched, so

$$\zeta'_S(x') = \zeta_S(x) \ , \tag{6.6}$$

i. e. it is considered a scalar.

6.1.1 Bispectrum cancellation

We can now briefly review the original argument leading to the bispectrum cancellation in [78]. The idea is to study a general three-point function involving two types of fields: X, which is the perturbation with squeezed mometum (it can be both a scalar and a tensor, but here it does not matter), and another perturbation δ. In [78] it is shown that the squeezed bispectrum in Fourier space transforms as follows under a gauge transformation:

$$\begin{aligned} B^{\text{squeezed}}_{X'\delta'\delta'}(k_L, k_1, k_2, \tau') =& P_{Xa}(k_L)P_\delta(k_S)\left(3 + \frac{d\log P_\delta(k_S)}{d\log k_S}\right) + P_{Xa^T_{ij}}\frac{k^i_S k^j_S}{k^2_S}\frac{dP_\delta(k_S)}{d\log k_S} + \\ &+ P_{X\Delta\tau}(k_L)\frac{\partial}{\partial\tau}P_\delta(k_S,\tau) + B^{\text{squeezed}}_{X\delta\delta}(k_L, k_1, k_2, \tau) \ . \end{aligned} \tag{6.7}$$

The squeezed limit is assumed to be $k_L \gg k_1$, k_2 and the short momentum is defined to be $\vec{k}_S = \frac{\vec{k}_1+\vec{k}_2}{2}$, so that

$$B^{\text{squeezed}}_{X\delta\delta}(k_L, k_1, k_2, \tau) := \lim_{k_L\to 0,\, k_1\to k_2} B_{X\delta\delta}(k_L, k_1, k_2, \tau) \ .$$

P_{xy} is the power spectrum coming from the correlator of the two fields x and y in Fourier space ($P_x := P_{xx}$). The quantities a and a^T_{ij} are defined as follows: a_{ij} is the following matrix related to the space components of the gauge transformation as

$$a_{ij} = \delta_{ij} - \frac{\partial x_i}{\partial x'^j} \ ,$$

then a and a^T_{ij} are constructed from a_{ij} as follows

$$a = \frac{a^i_i}{3} \ , \quad a^T_{ij} = a_{ij} - a\delta_{ij} \ ,$$

which means to have decomposed a_{ij} into its trace and traceless part. Finally, $\Delta\tau$ is linked to the time component of the gauge transformation as

$$\Delta\tau = \tau - \tau' \ .$$

Let us specialize this to the transformation from a FLRW to a CFC frame. In this case a_{ij} can be read from the inverse of 6.1, which is trivial at first-order. The result is

$$a_{ij} = -\Psi\delta_{ij} + \frac{1}{2}D_{ij} = \zeta_S\delta_{ij} + \frac{1}{2}D_{ij} \ ,$$

where we are assuming to work in ζ-gauge and in CFC frame $\zeta = \zeta_S$. In this case we are assuming no time transformation[2], $\Delta\tau = 0$. The trace and traceless parts of a_{ij} are trivially given by

$$a = \zeta_S \ , \quad a_{ij}^T = \frac{1}{2} D_{ij} \ ,$$

so that the transformation rule becomes for linearity

$$B^{\text{squeezed}}_{X'\delta'\delta'}(k_L, k_1, k_2) = P_{X\zeta_S}(k_L) P_\delta(k_S) \left(3 + \frac{d\log P_\delta(k_S)}{d\log k_S}\right) + \frac{1}{2} P_{XD_{ij}} \frac{k_S^i k_S^j}{k_S^2} \frac{dP_\delta(k_S)}{d\log k_S} + B^{\text{squeezed}}_{X\delta\delta}(\vec{k}_L, \vec{k}_1, \vec{k}_2) \ . \tag{6.8}$$

Taking now $\delta = X = \zeta_S$, given that $P_{\zeta_S D_{ij}} = 0$, the bispectrum transformation is

$$B^{\text{squeezed}}_{\zeta_S'\zeta_S'\zeta_S'}(k_L, k_1, k_2) = P_{\zeta_S}(k_L) P_{\zeta_S}(k_S) \left(3 + \frac{d\log P_{\zeta_S}(k_S)}{d\log k_S}\right) + B^{\text{squeezed}}_{\zeta_S\zeta_S\zeta_S}(\vec{k}_L, \vec{k}_1, \vec{k}_2) \ .$$

However, from equation 5.2 one knows that in the squeezed limit

$$B^{\text{squeezed}}_{\zeta_S\zeta_S\zeta_S}(k_L, k_1, k_2) = -P_{\zeta_S}(k_L) P_{\zeta_S}(k_S) \frac{d\log\left[k_S^3 P_{\zeta_S}(k_S)\right]}{d\log k_S} \ ,$$

which exactly cancels the first term. So, apparently, thanks to equation 6.7, in the CFC frame the squeezed bispectrum cancels.

Let us pass to the choice $X = D_{ij} = \sum_s \epsilon_{ij}^s D^s$ and $\delta = \zeta_S$. In this case 6.8 gives

$$B^{\text{squeezed}}_{D'\zeta_S'\zeta_S'}(k_L, s, k_1, k_2) = \frac{1}{2} P_T(k_L) \epsilon_{ij}^s \frac{k_S^i k_S^j}{k_S^2} \frac{dP_{\zeta_S}(k_S)}{d\log k_S} + B^{\text{squeezed}}_{D\zeta_S\zeta_S}(\vec{k}_L, \vec{k}_1, \vec{k}_2) \ .$$

As before, in the squeezed limit we have found that (combining 5.24 and 5.25)

$$B^{\text{squeezed}}_{D\zeta_S\zeta_S}(k_L, k_1, k_2) = -\frac{1}{2} P_T(k_L) \epsilon_{il}^s \frac{k_S^i k_S^l}{k_S} \frac{\partial}{\partial k_S} P_{\zeta_S}(k_S) \ ,$$

cancelling exactly the bispectrum in the squeezed limit.

In the next section we are going to present some explicit proofs that such type of arguments are not correct, but they remain valid only in the exact $k = 0$, which is unphysical. The key will be to analyze the transformation to the CFC frame (eq. 6.1) at non-linear order. This idea seems to be still underlined in [50], where however it is not examined in details.

6.2 CFC at non-linear order

As a first step to understand the problem, let us analyze carufelly the transformation for a FLRW spacetime from conformal to CFC frame, which is given by equation 6.1. We have the freedom to choose the value of the coordinates of the central worldline at the initial proper time τ_i, so we can set

$$p^i(\tau_i) = 0 \ , \quad p^i(\tau) = \int_{\tau_i}^{\tau} v^i(\tau', 0) d\tau' \ ,$$

where v^i is the velocity of the observer along the geodesic. To obtain a metric of the form $g^{\mu\nu} = \eta^{\mu\nu} + \mathcal{O}(\bar{\Delta}x)$ on the central geodesic it is necessary to consider the full expansion up to third order in

[2] This is because we are working only with spatial coordinates as in equation 6.1. In any case, one can rigorously prove ([78]) that the term with $\Delta\tau$ is subleading with respect to the others in the squeezed limit.

$\bar{\Delta}x^k$.
Since in single-field inflation[3] $v^i = \partial^i v_{\parallel}$, one has

$$p^k(\tau, P) = \int_{\tau_i}^{\tau} v^k(\tau')\Big|_P d\tau' = \int_{\tau_i}^{\tau} \partial^k v_{\parallel}(\tau')\Big|_P d\tau' = \partial^k \int_{\tau_i}^{\tau} v_{\parallel}(\tau')\Big|_P d\tau' \ .$$

We call the argument of the gradient V. Since, as we will see, the scalar fields will depend only on $|\vec{x}|$, we expect v to depend only on $|\vec{x}|$. Renaming the argument of the gradient $V(|\vec{x}|)$, using the chain rule[4] we find

$$\partial^k V(|\vec{x}|) = \frac{x^k}{|\vec{x}|} V'(|\vec{x}|) = x^k \mathcal{V}(|\vec{x}|) \ , \tag{6.10}$$

where $\mathcal{V}(|\vec{x}|) = \frac{V'(|\vec{x}|)}{|\vec{x}|}$. This implies that

$$\Delta x^k = x^k - p^k(\tau) = x^k - \partial^k V(|\vec{x}|) = x^k - x^k \mathcal{V}(|\vec{x}|) \ .$$

From this we see that $\bar{\Delta}x^k = x^k$ (being the background value).
Let us now focus on the first two lines of 6.1. Since we are neglecting second-order perturbations, we can set $\Psi(x) = \Psi(|\bar{\Delta}x|)$ and $\partial_j \Psi|_P = \partial_j \Psi|_{\bar{\Delta}x=0}$. This way the transformation becomes

$$\begin{aligned}\Delta x^k_F =& x^k - x^k \mathcal{V}(|\vec{x}|) - x^k \Psi\Big|_0 - \frac{1}{2} x^i x^j (\delta^k_j \partial_i \Psi + \delta^k_i \partial_j \Psi - \delta_{ij} \partial^k \Psi)\Big|_0 + \\ & - \frac{1}{6} x^i x^j x^l (\delta^k_j \partial_l \partial_i \Psi + \delta^k_i \partial_l \partial_j \Psi - \delta_{ij} \partial_l \partial^k \Psi)\Big|_0 \ .\end{aligned}$$

In order to write this as a deformed dilatation[5] we have to Taylor expand $\Psi|_0$ around $|\vec{x}|$, at third order in x^i, as explained in [91]:

$$\Psi|_0(x) = \Psi|_0(|\vec{x}|) - \partial_i \Psi|_0(|\vec{x}|) x^i - \frac{1}{2} \partial_i \partial_j \Psi|_0(|\vec{x}|) x^i x^j - \frac{1}{6} \partial_i \partial_j \partial_l \Psi|_0(|\vec{x}|) x^i x^j x^l + \dots \ .$$

We have to substitute this into 6.1, expanding up to third order terms in x^i. The result is ([91])

$$\Delta x^k_F = x^k \left(1 - \mathcal{V}(|\vec{x}|) - \Psi|_0(|\vec{x}|) + \frac{1}{2}|\vec{x}| \Psi'|_0(|\vec{x}|) - \frac{1}{6}|\vec{x}|^2 \Psi''|_0(|\vec{x}|) + \frac{1}{12}|\vec{x}|^3 \Psi'''|_0(|\vec{x}|)\right) + \dots \ .$$

As we have seen, in ζ-gauge one has $\zeta = -\Psi$, so that

$$\Delta x^k_F = x^k \left(1 - \mathcal{V}(|\vec{x}|) + \zeta(|\vec{x}|) - \frac{1}{2}|\vec{x}| \zeta'|_0(|\vec{x}|) + \frac{1}{6}|\vec{x}|^2 \zeta''|_0(|\vec{x}|) - \frac{1}{12}|\vec{x}|^3 \zeta'''|_0(|\vec{x}|)\right) + \dots \ .$$

This implies that the transformation between spatial CFC and spatial comoving coordinates matches a spacetime dilatation with

$$\lambda = -\mathcal{V}(|\vec{x}|) + \zeta(|\vec{x}|) - \frac{1}{2}|\vec{x}| \zeta'|_0(|\vec{x}|) + \frac{1}{6}|\vec{x}|^2 \zeta''|_0(|\vec{x}|) - \frac{1}{12}|\vec{x}|^3 \zeta'''|_0(|\vec{x}|) \ . \tag{6.11}$$

On the contrary, the third and the fourth line of 6.1 can be rewritten as $A^k_i x^i$, where the matrix A is given by:

$$A^k_i = \frac{1}{2} D^k_i\Big|_0 + \frac{1}{4} \bar{\Delta}x^j (\partial_i D^k_j + \partial_j D^k_i - \partial^k D_{ij})\Big|_0 + \frac{1}{12} \bar{\Delta}x^j \bar{\Delta}x^l (\partial_l \partial_i D^k_j + \partial_l \partial_j D^k_i - \partial_l \partial^k D_{ij})\Big|_0 \ . \tag{6.12}$$

[3] See section 4.6.1; we are assuming a comoving observer as in [78].
[4] One has:

$$\partial^k f(|\vec{x}|) = \frac{\partial |\vec{x}|}{\partial x_k} f'(|\vec{x}|) = \frac{\partial \sqrt{x^i x_i}}{\partial x_k} f'(|\vec{x}|) = \frac{x^k}{\sqrt{x^i x_i}} f'(|\vec{x}|) = \frac{x^k}{|\vec{x}|} f'(|\vec{x}|) \ , \tag{6.9}$$

where $'$ denotes a derivative with respect to $|\vec{x}|$.
[5] A standard dilatation is a transformation such that $\delta x^i = \lambda x^i$; we will call *deformed dilatation* a dilatation in which the parameter λ becomes a function of the space(time) point: $\delta x^i = \lambda(x) x^i$.

It is very easy to see that A is traceless, $A^k_i \delta^i_k = 0$. For sake of shortness from now on we call $x^i := \bar{\Delta} x^i$. We would like to write the transformation as

$$A^k_i x^i = \partial^k \epsilon + \epsilon^k_V \ , \tag{6.13}$$

where $\partial_i \epsilon^i_V = 0$, in order to separate explicitly the scalar and the vector part. To do so, we take the 3-divergence of the matrix A and we get

$$\partial_k A^k_i = -\frac{1}{12} x^j \nabla^2 D_{ij} \ .$$

This terms comes from the last summand in 6.12. We can call

$$B^k_i = \frac{1}{2} D^k_i \bigg|_0 + \frac{1}{4} \bar{\Delta} x^j (\partial_i D^k_j + \partial_j D^k_i - \partial^k D_{ij}) \bigg|_0 + \frac{1}{12} \bar{\Delta} x^j \bar{\Delta} x^l (\partial_l \partial_i D^k_j + \partial_l \partial_j D^k_i) \bigg|_0 \ ,$$

which is transverse and traceless. As such $B^k_i x^i$ contributes authomatically to ϵ^i_V, since

$$\partial_k (B^k_i x^i) = \partial_k B^k_i x^i + B^k_i \delta^i_k = 0 \ .$$

The last summand in 6.12 contributes both to the scalar part and to the vector part of the transformation. Let it be $A'^k_i = -\frac{1}{12} x^j x^l \partial_l \partial^k D_{ij}$, so the variation is given by

$$\delta x^k := A'^k_i x^i = -\frac{1}{12} x^j x^l x^i \partial_l \partial_k D_{ij} \bigg|_0 \ . \tag{6.14}$$

Decomposing this result in scalar and vector part is not trivial in coordinate spacel since it would be required to invert a Laplacian. So we pass to Fourier space, where the decomposition is explicit. At first-order we can substitute $D_{ij}\Big|_0 \to D_{ij}(x)$, so that

$$\delta x^k = A'^k_i x^i = -\frac{1}{12} x^j x^l x^i \partial_l \partial^k D_{ij}(x) \ .$$

This allows to simplify computations in Fourier space, avoiding to use distributions. The matrix A' in Fourier space reads

$$A'^k_i(k) = -\frac{1}{12} \left(4 D^k_i + 5 k^k \partial_{k^j} D_{ij} + k^m \partial_{k^m} D^k_i + k^m k^k \partial_{k^m} \partial_{k^j} D_{ij} \right) \ ,$$

so that from 6.14 one has

$$\delta x^k = i \partial_{k^i} A'^k_i(k) = -\frac{i}{12} \left(10 \partial_{k^i} D^k_i + 2 k_m \partial_{k_m} \partial_{k_j} D^k_j + k^k \partial_{k_i} \partial_{k_j} D_{ij} + k^k k^m \partial_{k_i} \partial_{k_j} \partial_{k^m} D_{ij} \right) \ .$$

The first two summands are the vector part since their 3-divergence is null: indeed, in Fourier space a 3-divergence corresponds to multiplying by the vector ik^k; using the property of D, which is transverse and traceless by definition, one gets 0. This way, one has

$$\epsilon^k_V = -\frac{i}{12} \left(10 \partial_{k_i} D^k_i + 2 k_m \partial_{k_m} \partial_{k_j} D_{kj} \right) \ .$$

On the contrary the terms proportional to ik^k give the scalar part

$$\epsilon = -\frac{1}{12} \left(\partial_{k_i} \partial_{k_j} D_{ij} + k^m \partial_{k_i} \partial_{k_j} \partial_{k^m} D_{ij} \right) \ .$$

This expression contains $\partial_{k_i} \partial_{k_j} D_{ij}(k)$, which can be explicitly computed writing the tensor perturbation as 4.76. Thus

$$\partial_{k_j} D_{ij}(k) = \partial_{k_j} \sum_{s=\pm 2} \epsilon^s_{ij} D^s(k) \quad = \sum_{s=\pm 2} \left[\partial_{k_j} \epsilon^s_{ij} D^s(k) + \epsilon^s_{ij} \frac{k_j}{k} D'^s(k) \right] = \sum_{s=\pm 2} \partial_{k^j} \epsilon^s_{ij} D^s(k) \ .$$

where we have used the chain rule 6.9 and then the transversality of $\epsilon^s_{ij}(k)$. This way,

$$\epsilon = -\frac{1}{12}\left(1 + k^m \partial_{k^m}\right) \sum_{s=\pm 2} \partial_{k_i}\partial_{k_j}\epsilon^s_{ij} D^s(k) \ .$$

Using the explicit expression for the polarization tensors one can show that[6]

$$\partial_{k_i}\partial_{k_j}\epsilon^{s=\pm 2}_{ij} = \frac{1}{k^2} \ ,$$

so that using once again the chain rule 6.9 one gets

$$\begin{aligned}\epsilon &= -\frac{1}{12}\left(1 + k^m \partial_{k^m}\right)\left(\frac{1}{k^2}\sum_{s=\pm 2} D^s(k)\right) = -\frac{1}{12k^2}\sum_{s=\pm 2}\left(D^s(k) + k^3\left(\frac{D^s(k)}{k^2}\right)'\right) = \\ &= -\frac{1}{12k^2}\sum_{s=\pm 2}\left(-D^s(k) + kD^{s\prime}(k)\right) \ .\end{aligned} \tag{6.15}$$

Finally, concerning the vector part, one can show (see the appendix D.2) that a 3 dimensional vector can be always written in the form

$$\epsilon^k_V = M^k_i x^i \ ,$$

with M a 3×3, point dependent, transverse and traceless matrix. This way, summing up all the results one gets that the tensor part of the CFC transformation can be written as

$$x'^k = A^k_i x^i = B^k_i x^i + A'^k_i x^i = B^k_i x^i + M^k_i x^i + \partial^k \epsilon = (B^k_i + M^k_i)x^i + \partial^k \epsilon \ .$$

From now on, we call $\omega^k_i = B^k_i + M^k_i$, which is transverse and traceless since B and M are. Surprisingly, the gravitational waves perturbations give a scalar contribution once the CFC frame is considered. This term can be encompassed by an effective λ-deformed dilatation contributing to eq. 6.11 as

$$\lambda_{\rm tot} = \lambda + \lambda_g$$

where λ_g is defined such that[7]

$$\lambda_g = \frac{\epsilon'(|\vec{x}|)}{|\vec{x}|} \ ,$$

generalizing the result presented in [91].

To summarize, the transformation for a FLRW spacetime from conformal to CFC frame can be written as a Weinberg deformed dilatation of type

$$x'^i = x^i + \lambda(|\vec{x}|)x^i + \omega^i_k(\vec{x})x^k \ , \tag{6.16}$$

where λ and ω are dependent on the spacetime point and ω transverse and traceless. In the exact limit $k = 0$, in the previous section we have seen that they became exactly constant (thanks to Weinberg theorem).

[6] For details see the appendix D.1.

[7] Given that 6.15 depends only on the norm of $\vec{k}$, its Fourier transform depends only on $|\vec{x}|$. In the following section, we will see that in this case the transformation can be always rewritten as a deformed dilatation (see eq. 6.32), with λ given by the expression in the main text.

6.3 Large wavelength perturbations

In this section we reexamine the gauge transformations we have introduced in section 2.3, but we focus in particular on the Weinberg transformation 3.16, which is a long-wavelength perturbation, in the sense that this ξ does not decay at infinity. As we have seen in the previous subsection, the infinitesimal diffeomorphism from conformal frame to CFC is a peculiar case of this transformation.

First of all, to conform with [91], we consider a perturbed metric around a flat FLRW spacetime of the following form

$$ds^2 = (g^0_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu = -dt^2 + a^2\delta_{ij}dx^i dx^j + h_{\mu\nu}dx^\mu dx^\nu$$

and we want to rederive the transformation rules we have found in 2.3 for an infinitesimal diffeomorphism ϵ^μ. Notice that in this case the metric is written in cosmic time. The transformation rules[8] are exactly 3.9:

$$\begin{aligned}
\Delta h_{00} &= 2\dot\epsilon^0\\
\Delta h_{0i} &= \partial_i\epsilon^0 - a^2\dot\epsilon^i\\
\Delta h_{ij} &= -2a\dot a\epsilon^0\delta_{ij} - a^2\partial_j\epsilon^i - a^2\partial_i\epsilon^j\ .
\end{aligned}\tag{6.17}$$

At this point we parametrize the first-order perturbations similarly to what we have seen in section 2.2

$$\begin{aligned}
h_{00} &= -2\Phi\\
h_{0i} &= a(\partial_i\omega_\parallel + \omega_{i\perp})\\
h_{ij} &= a^2\left(-2\Psi\delta_{ij} + \partial_i\partial_j B + \partial_j C_i + \partial_i C_j + D_{ij}\right)\ .
\end{aligned}$$

Notice that in this case we choose the convention that does not split the ij-components in trace and traceless part (we did not put the operator $\hat D_{ij} = \partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial_k\partial_k$).
At this point, using these results, we want to find the transformation rules for the split perturbations, as usual. The paper [91] focuses only on scalar perturbations, while we want to extend it also to tensor perturbations, so we find all the transformation rules required; moreover, we will see that vector perturbations in this case do matter, so we find also the transformation rules for them. We decompose ϵ^μ as usual, so $\epsilon^\mu = (\epsilon^0, \partial^i\epsilon + \epsilon^i_V)$. This way, combining our results, considering only scalars, one gets the transformation rules for the perturbations. The 00-component gives

$$-2\Delta\Phi = \Delta h_{00} = 2\dot\epsilon^0 \longrightarrow \Delta\Phi = -\dot\epsilon^0\ .$$

The $0i$-components give

$$a(\partial_i\Delta\omega_\parallel + \Delta\omega_i^\perp) = \Delta h_{0i} = \partial_i\epsilon^0 - a^2\partial_i\dot\epsilon - a^2\dot\epsilon^i_V \longrightarrow \Delta\omega_\parallel = \frac{1}{a}\epsilon^0 - a\dot\epsilon\ ,\quad \Delta\omega_i^\perp = -a\dot\epsilon^i_V$$

and the ij-components give

$$\begin{aligned}
&a^2\left(-2\Psi\delta_{ij} + \partial_i\partial_j B + \partial_j C_i + \partial_i C_j + D_{ij}\right) = \Delta h_{ij} = -2a\dot a\epsilon^0\delta_{ij} - 2\partial_i\partial_j\epsilon a^2 - \partial_i\epsilon^j_V a^2 - \partial_j\epsilon^i_V a^2\\
&\longrightarrow \Delta\Psi = H\epsilon^0\ ,\quad \Delta B = -2\epsilon\ ,\quad \Delta C_i = -\epsilon^i_V\ ,\quad \Delta D_{ij} = 0\ .
\end{aligned}\tag{6.18}$$

However, the case of very large-wavelength perturbations needs to be analyzed carefully. We have to consider an infinitesimal diffeomorphism of type 3.16

$$\epsilon^i = \lambda x^i + \omega^i_j x^j\ .\tag{6.19}$$

Through the Weinberg argument, we have seen that λ and ω must be constant in time and space in the exact $k \to 0$ limit. Since we are considering transformations only in spatial coordinates, we set $\epsilon^0 = 0$. However, in this case we want to study also the case in which λ and ω_{ij} are spacetime

[8] Having rised the indices of ϵ in 3.9 produces equalities where the LHS and the RHS have the indices in different positions: this is the reason why in this section this happens.

dependent, since we are considering very large wavelength perturbations, but not in the exact $k \to 0$ limit. Applying to this case the rules 6.17 one has

$$
\begin{aligned}
\Delta h_{00} =&0 \\
\Delta h_{0i} =&-a^2 \dot{\omega}^i_j x^j \\
\Delta h_{ij} =&-\partial_i(\lambda x^j)a^2 - \partial_j(\lambda x^i)a^2 - \partial_i(\omega^j_k x^k)a^2 - \partial_j(\omega^i_k x^k)a^2 = \\
=&-a^2(2\lambda\delta_{ij} + \omega_{ij} + \omega_{ji} + x^k\partial_i\omega^j_k + x^k\partial_j\omega^i_k + x^i\partial_j\lambda + x^j\partial_i\lambda) \ .
\end{aligned}
\tag{6.20}
$$

However, let us start from the case of $k \to 0$ and see that it is really peculiar. In this exact limit, λ and ω_{ij} are pure constants, so that the last equation of 6.20 reads

$$
\Delta h_{ij} = -a^2\left(2\lambda\delta_{ij} + \omega_{ij} + \omega_{ji}\right) \ . \tag{6.21}
$$

Notice that this case is pathological, since trying to SVT decompose the result, the decomposition is not unique. Indeed, 6.21 is exactly reproduced also by taking

$$
\Delta\Psi = \alpha\lambda \ , \quad \Delta B = \lambda(\alpha-1)x^j x_j + \gamma\omega_{ij}x^i x^j \ , \quad \Delta C_i = (\beta-1)\omega_{ik}x^k \ , \quad \Delta D_{ij} = -(\beta+\gamma)(\omega_{ij}+\omega_{ji}) \ . \tag{6.22}
$$

with α, β, $\gamma \in \mathbb{R}$. This means also that there is a 3-parameter degeneracy in the usual SVT decomposition. Notice that ΔC_i is divergence-free, as a vector must be, since ω_{ij} is traceless: $\partial_i(\omega^i_k x^k) = \omega^i_k\delta^k_i = 0$. On the contrary, ΔD_{ij} is a tensor since it is trivially traceless and transverse. To show Weinberg theorem, however, we used the case $\alpha = \beta = 1$ and $\gamma = 0$, as we have seen in equation 3.17 (here $\epsilon^0 = 0$, so there is not the transformation of Ψ):

$$
\Delta\Psi = \lambda \ , \qquad \Delta D_{ij} = -\omega_{ij} - \omega_{ji} \ .
$$

Notice that if ω_{ij} is a constant, it is a tensor since it is transverse $\partial_i\omega^{ij} = 0$ and traceless $\omega_{ij}\delta^{ij} = 0$.

We pass now to the long-wavelenght perturbations, that is the case in which k is very small but not exactly 0. In this case the transformation has λ and ω dependent on spacetime point and the transformation rules are fully 6.20. We would like to study the decomposition of the transformation 6.19 into its scalar and vector part. For linearity, we consider separately the two summands.
Let us start with the transformation $\epsilon^i = \lambda x^i$. In this case it can be written as a gradient of a scalar, i. e. $\epsilon^i = \partial_i\epsilon$ only if a derivative condition on λ is fulfilled. Indeed taking the divergence of the relation $\epsilon^i = \partial_i\epsilon = \lambda x^i$, one gets

$$
\partial_j\partial_i\epsilon = \partial_j(\lambda x^i) = \partial_j\lambda x^i + \lambda\delta^i_j
$$

and antisymmetrizing over i and j the LHS and the second summand on the RHS drop since they are symmetric, so one gets

$$
x^i\partial_j\lambda - x^j\partial_i\lambda = 0 \ , \tag{6.23}
$$

which is a condition on λ. The case in which λ is constant obviously fullfils this condition, even if it is pathological. Another important case is when λ depends only on $|\vec{x}|$: indeed, by using the chain rule 6.9 on $\lambda(t, |\vec{x}|)$ one has

$$
x^i\partial_j\lambda(t,|\vec{x}|) - x^j\partial_i\lambda(t,|\vec{x}|) = x^i\frac{x^j}{|\vec{x}|}\lambda'(t,|\vec{x}|) - x^j\frac{x^i}{|\vec{x}|}\lambda'(t,|\vec{x}|) = 0 \ .
$$

This implies also that in this case we can extract λ as follows (assuming that also ϵ depends only on $|\vec{x}|$):

$$
\lambda(t,|\vec{x}|)x^i = \partial_i\epsilon = \partial_i|\vec{x}|\epsilon' = \frac{x^i}{|\vec{x}|}\epsilon' \longrightarrow \lambda(t,|\vec{x}|) = \frac{\epsilon'}{|\vec{x}|} \ .
$$

Notice that this is exactly the case of the deformed dilatation 6.11.
When λ does not respect the condition 6.23, it is not trivial to derive explicitly the SVT decomposition of ϵ^i but we expect to have both a scalar and a vector term, thanks to Helmholtz theorem.

Let us pass to the term $\epsilon^i = \omega^i_j x^j$. In this case we consider ω_{ij} is spacetime dependent, transverse $\partial_i \omega^{ij} = 0$ and traceless, since in the previous section we have constructed it in such a way that these conditions are fulfilled. In this case, it is easy to show that the whole transformation is transverse, thanks to the traceless condition on ω^{ij}:

$$\partial_i \epsilon^i = \partial_i \omega^i_j x^j + \omega^i_j \delta^j_i = 0 \ .$$

This implies that in case ω^{ij} is transverse and traceless the transformation is a pure vector.

Before ending the section, let us comment on what happens to the degeneracy, which is present in the case of λ and ω exactly constant, when they become spacetime independent. The simplest case we can consider is the following:

$$\lambda = \lambda_0 + \lambda_1 x^i n_i \ , \quad \omega_{ij} = \omega^0_{ij} + \omega^1_{ij} x^k n_k \ ,$$

with λ_0, λ_1, ω_0 and ω_1 constant. For sake of shortness, we consider separately the two contributions. From 6.20 one gets from the first

$$\Delta h_{ij} = -a^2 \Big(2\lambda\delta_{ij} + x^i \partial_j (\lambda_0 + \lambda_1 x^k n_k) + x^j \partial_i (\lambda_0 + \lambda_1 x^k n_k) \Big) = -a^2 (2\lambda\delta_{ij} + \lambda_1 x^i n_j + \lambda_1 x^j n_i) \ .$$

Notice that this is reproduced by the following transformation of the scalar perturbations

$$\Delta\Phi = 0 \ , \quad \Delta\omega_{||} = 0 \ , \quad \Delta\Psi = \frac{\lambda}{2} \ , \quad \Delta B = -\frac{\lambda}{2} x^i x_i \ , \tag{6.24}$$

indeed from the definition of the perturbations of the ij components one has

$$\Delta h_{ij} = a^2(-2\Delta\Psi\delta_{ij} + \partial_i\partial_j \Delta B) = -a^2 \left(2\lambda\delta_{ij} + \lambda_1 n_i x^j + \lambda_1 n_j x^i\right) \ ,$$

which coincides exactly with the previous result. Comparing 6.22 with 6.24, one easily sees that that choosing $\lambda = \lambda_0 + \lambda_1 x^i n_i$ is like choosing $\lambda = \lambda_0$ and $\alpha = \frac{1}{2}$, which means that the degeneracy is lifted. However, the gauge transformation has a very different result on the scalar part of the spatial metric, since, differently to the previous case, the metric is no longer diagonal. Indeed, the scalar part before was changed to

$$\Delta h_{ij} = a^2(-2\Delta\Psi\delta_{ij} + \partial_{ij}\Delta B + \ldots) \ = a^2(-2\alpha\lambda_0\delta_{ij} + \lambda_0(\alpha - 1)2\delta_{ij} + \ldots) \propto \delta_{ij} \ ,$$

but now $\Delta h_{ij} = -a^2 \left(2\lambda\delta_{ij} + \lambda_1 n_i x^j + \lambda_1 n_j x^i\right)$, which is not diagonal.
Furthermore, considering the part of the transformation involving ω, from 6.20 one gets

$$\begin{aligned} \Delta h_{ij} &= -a^2 \Big(\omega_{ij} + \omega_{ji} + x^k \partial_i (\omega^0_{jk} + \omega^1_{jk} x^l n_l) + x^k \partial_j (\omega^0_{ik} + \omega^1_{ik} x^l n_l) \Big) = \\ &= -a^2 \Big(\omega_{ij} + \omega_{ji} + x^k \omega^1_{jk} n_i + x^k \omega^1_{ik} n_j \Big) \ . \end{aligned}$$

This case in reproduced taking a pure vector transformation

$$\Delta B = 0 \ , \quad \Delta C_i = -\omega_{ik} x^k \ , \quad \Delta D_{ij} = 0 \ .$$

This is not a case, since we have seen that when ω_{ij} is spacetime dependent, it must contribute as a vector. Comparing with 6.22, this corresponds to the case $\alpha = \beta = 0$ and the degeneracy is lifted. Indeed, in this case

$$\Delta h_{ij} = a^2 \left(\partial_i\partial_j \Delta B + \partial_j \Delta C_i + \partial_i \Delta C_j + \Delta D_{ij}\right) = -a^2 \Big(\omega_{ij} + \omega_{ji} + x^k \omega^1_{ik} n_j + x^k \omega^1_{jk} n_i \Big) \ .$$

Finally, if one considers more complicated expressions for λ and ω not only the degeneracy is lifted, but also it is impossible to reconduct the transformation to the case 6.22. For example, the simplest generalization for the λ considered in the previous case is

$$\lambda = \lambda_0 + \lambda_1 x^i n_i + M_{ij} x^i x^j \ ,$$

with M constant symmetric matrix. By a direct computation, in this case one can easily show that the extra term Δh_{ij} cannot be reproduced by any variation of type 6.22.

The outcome of this discussion is that the degeneracy is present only for λ and ω exactly constant, which is the exact limit $k \to 0$. There are no transformations with λ and ω almost constant able to reduce to continuity to the constant case: this is because the limit and the smallness of the gradients $x^k \partial_i \omega^{jk}$ and $x^i \partial_j \lambda$ do not commute, giving rise to different transformation rules. As we will see in the following section, this implies that these gradients cannot be neglected merely in a gradient expansion.

The transformation rules 6.5 and 6.6 are saying that any very long wavelength transformation can be gauged away. Being $\zeta = -\Psi$ in ζ-gauge, the splitting $\zeta = \zeta_L + \zeta_S$ is analogous to say $\Psi = \Psi_L + \Psi_S$, with $\Psi_{S/L}$ containg only short/long (high/low frequency) modes. However, under the usual gauge transformation 3.16, from 6.18 it is easy to see that $\Delta\Psi = 0$; this variation is different from 0 only in the pathological case of λ and ω exactly constant, as one can see from 6.22, which is the exact $k \to 0$ limit, or in case λ is linear in the coordinates, as one can see from 6.24. If $k = 0$ the transformation preserves also the ζ-gauge: as we discussed in subsection 5.3.1, the residual gauge transformations preserving the ζ-gauge in the $k \to 0$ limit are the dilatation and the special conformal transformations, but not 3.16 with λ and ω dependent on the spacetime point.
On the contrary, in the case of a deformed dilatation, this is no more valid: since $\zeta = -\Psi - H\frac{\delta\phi}{\dot{\phi}_0}$, under a gauge transformation it is sent to $\Delta\zeta = -\Delta\Psi - H\frac{\Delta\delta\phi}{\dot{\phi}_0}$. However, in this case the transformation rules are 6.18, having two important consequences. Since $\Delta\Psi = 0$ and $\Delta\delta\phi = 0$ (having considered $\epsilon_0 = 0$), one cannot gauge away any long wavelength part of ζ. Moreover, since $\Delta B \neq 0$, these rules imply to exit the ζ-gauge (indeed a deformed dilatation is not a residual gauge freedom).
The case $\alpha = \beta = 1$ and $\gamma = 0$ was also used in [50] and [63] to prove the consistency relation. However, here the procedure is different since the scalar variation ΔB is put to 0 by hand (choosing $\alpha = 1$ in 6.22), a procedure which is impossible to reproduce by means of a standard gauge transformation.

The conclusion to this argument is that in ζ cannot transform as 6.5 and 6.6 remaining in ζ-gauge for k large but not exactly null.

6.4 Deformed dilatations and gradient expansion

In this section we show that a transformation of type 6.19

$$x'^i = x^i + \lambda(|\vec{x}|)x^i + \omega^i_j(x)x^j \ , \tag{6.25}$$

which we have already called *deformed dilatation*, can be always described via standard gauge transformation rules; the only exception is the case in which λ and ω are constant as we have seen in the previous section. As before, for linearity we consider separately the two pieces contributing to the deformed dilatation and we consider ω_{ij} transverse and traceless.

Let us start with the λ-piece: the deformed dilatation is $x'^i = x^i + \lambda(|\vec{x}|)x^i$. In order to proceed it is better to work in Fourier space, where $\lambda(|\vec{x}|)$ reads

$$\lambda(|\vec{x}|) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k e^{i\vec{k}\cdot\vec{x}} \lambda(k) \ . \tag{6.26}$$

From 6.11, we see that at first-order in $|\vec{x}|$, the standard expression for λ in Fourier space is $\lambda(k) = W(k)\zeta(k)$, where $W(k)$ is a filter selecting only the long modes of ζ as in 6.3,

$$W(k) = \theta\left(\frac{1}{H}(k_c - k)\right) \ , \tag{6.27}$$

with $k_c \ll H$.
Under such a dilatation the metric transforms according to 6.20, which contains many terms of type $x^i\partial_j\lambda$ or $x^i\partial_j\omega$. These off-diagonal terms can be apparently neglected in the limit $k \to 0$, indeed the gradients are proportional to k in Fourier space but we want to take the limit $k \to 0$. However, this is not true, as we are going to show now.
Consider a term of type $x^i\partial_j\lambda$ in Fourier space:

$$
\begin{aligned}
x^i\partial_j\lambda = x^i\partial_j\left(\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k e^{i\vec{k}\cdot\vec{x}}\lambda(k)\right) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k i x^i k_j e^{i\vec{k}\cdot\vec{x}}\lambda(k) = \\
= \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k k_j \partial_{k_i} e^{i\vec{k}\cdot\vec{x}}\lambda(k) &= -\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k \partial_{k_i}(k_j\lambda(k))e^{i\vec{k}\cdot\vec{x}} + \mathrm{BT}\ ,
\end{aligned}
$$

where in the last passage we have integrated by parts over k; the boundary term BT generated can be set to zero using Gauss theorem since the window function W selects only little k. This implies that in Fourier space the λ-part of 6.20 reads

$$
\begin{aligned}
\Delta h_{ij}(k) &= -a^2\left(2\lambda(k)\delta_{ij} - \partial_{k_j}(k_i\lambda(k)) - \partial_{k_i}(k_j\lambda(k))\right) = \\
&= -a^2\left(2\lambda(k)\delta_{ij} - \delta_{ij}\lambda(k) - k_i\partial_{k_j}\lambda(k) - \delta_{ij}\lambda(k) - k_j\partial_{k_i}\lambda(k)\right) = \\
&= a^2\left(k_i\partial_{k_j}\lambda(k) + k_j\partial_{k_i}\lambda(k)\right)\ .
\end{aligned}
$$

If we take W dependent only on $k = |\vec{k}|$, we have that in coordinate space λ depends only on $|\vec{x}|$: the discussion in the previous section implies that the condition 6.23 is fulfilled, so we expect to have a pure scalar perturbation. Using the chain rule 6.9 as before one has

$$
\begin{aligned}
\Delta h_{ij}(k) &= a^2\left(k_i\partial_{k_j}\lambda(k) + k_j\partial_{k_i}\lambda(k)\right) = a^2\left(k_i\partial_{k_j}k\lambda'(k) + k_j\partial_{k_i}k\lambda'(k)\right) = \\
&= a^2\left(k_i\frac{k_j}{k}\lambda'(k) + k_j\frac{k_i}{k}k\lambda'(k)\right) = 2a^2\frac{k_ik_j}{k}\lambda'(k)\ .
\end{aligned}
$$

If we now compare this to the scalar perturbations of the ij component of the metric in Fourier space

$$
\Delta h_{ij}(k) = a^2(-2\Delta\Psi\delta_{ij} - k_ik_j\Delta B + ik_j\Delta C_i + ik_i\Delta C_j + \Delta D_{ij})\ , \tag{6.28}
$$

we find the following variation under the deformed dilatation

$$
\Delta\Psi = 0\ , \quad \Delta B = -\frac{2}{k}\lambda'(k)\ , \quad \Delta C_i = 0\ , \quad \Delta D_{ij} = 0\ .
$$

These are exactly the transformation rules 6.18 with $\epsilon^0 = 0$ and

$$
\epsilon(k) = \frac{\lambda'(k)}{k}\ .
$$

Let us pass to the ω-piece: the deformed dilatation is $x'^i = x^i + \omega^i_j(x)x^j$, with ω^{ij} traceless and transverse. In Fourier space we can write

$$
\omega_{ij}(\vec{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k e^{i\vec{k}\cdot\vec{x}}\omega_{ij}(\vec{k})\ , \tag{6.29}
$$

where, as before, $\omega(k)$ is usually a product containing a filter dropping the short modes. Transversality in Fourier space reads $k^i\omega^{ij} = 0$. Then, as before, we consider a term of type $x^k\partial_i\omega^j_k$ in Fourier space, which can be treated the same way:

$$
\begin{aligned}
x^k\partial_i\omega^j_k = x^k\partial_i\left(\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k e^{i\vec{k}\cdot\vec{x}}\omega^j_k(\vec{k})\right) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k i x^k k_i e^{i\vec{k}\cdot\vec{x}}\omega^j_k(\vec{k}) = \\
= \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k k_i\partial_{k^k} e^{i\vec{k}\cdot\vec{x}}\omega^{jk}(\vec{k}) &= -\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k \partial_{k^k}(k_i\omega^{jk}(\vec{k}))e^{i\vec{k}\cdot\vec{x}} + \mathrm{BT}\ .
\end{aligned}
$$

As before, we have obtained a boundary term, which drops at infinity. Finally, considering the ω-part of 6.20 one finds

$$\begin{aligned}\Delta h_{ij}(k) &= -a^2 \left(\omega_{ij}(k) + \omega_{ji}(\vec{k}) - \partial_{k^k}(k_i\omega^{jk}(\vec{k})) - \partial_{k^k}(k_j\omega^{ik}(\vec{k}))\right) = \\ &= -a^2 \left(\omega_{ij}(\vec{k}) + \omega_{ji}(\vec{k}) - \delta_{ki}\omega^{jk}(\vec{k}) - k_i\partial_{k^k}\omega^{jk}(\vec{k}) - \delta_{kj}\omega^{ik}(\vec{k}) - k_j\partial_{k^k}\omega^{ik}(\vec{k})\right) = \\ &= a^2 \left(k_i\partial_{k^k}\omega^{jk}(\vec{k}) + k_j\partial_{k^k}\omega^{ik}(\vec{k})\right) \ .\end{aligned}$$

Comparing this with 6.28, one gets that

$$\Delta\Psi = 0 \ , \quad \Delta B = 0 \ , \quad \Delta C_i = \partial_{k^k}\omega^{ik}(\vec{k}) \ , \quad \Delta D_{ij} = 0 \ ,$$

which compared to the transformation rules 6.18 gives

$$\epsilon^i_V(k) = -\partial_{k^k}\omega^{ik}(\vec{k}) \ .$$

But for this to be consistent with the SVT decomposition, it must be a vector and indeed it is:

$$k_i\partial_{k^k}\omega^{ik}(\vec{k}) = k_i\partial_{k^k}\omega^{ik}(\vec{k}) + \delta_{ik}\omega^{ik}(\vec{k}) = k_i\partial_{k^k}\omega^{ik}(\vec{k}) + \partial_{k^k}k_i\omega^{ik}(\vec{k}) = \partial_{k^k}\left(k_i\omega^{ik}(\vec{k})\right) = 0 \ ,$$

where at the first passage we have added $\delta_{ik}\omega^{ik}$ which is null and in the last one we have used the transversality condition. This implies that the deformed dilatation does not modify the tensor sector of the metric perturbation, but only the vector one.

This discussion implies that despite the filtering procedure dropping the short modes, the transformation rules are exactly the same we found in the previous section. This means that transformation rules such as 6.6 and 6.5 used in [78], [79] and [80] are not right.

Another problem which arises from this discussion concerns the definition of ζ ([79], [80]): we have said that it does not concide with the gauge invariant quantity defined in a geometrical way in section 4.6.2, but it is assumed to be 6.4

$$\zeta^{(alt)} = -\Psi + \frac{1}{6}\nabla^2 B \ .$$

In ζ-gauge $\zeta^{(alt)} = \zeta = -\Psi$, which is assumed to be always valid in [79] and [80], also after the gauge transformation. This is because in Fourier space $\zeta^{(alt)} = -\Psi + \frac{1}{6}\nabla^2 B$, so in the $k \to 0$ limit apparently $\lim_{k\to 0}\zeta^{(alt)}(k) = -\Psi$. However, as we have seen, we cannot neglect gradients simply because they are proportional to k, since this procedure leads to wrong results. This is because this definition of ζ is not gauge invariant, so it is problematic.

6.5 Bispectrum gauge transformation under deformed dilatation

In this section we present a confutation to the bispectrum transformation rule used in [78]: we explicitly prove that the bispectrum is left unchanged by the gauge transformation 3.16, that is

$$\Delta B(x_1, x_2, x_3) = \tilde{B}(x_1, x_2, x_3) - B(x_1, x_2, x_3) = 0 \ .$$

This explicitly contradicts 6.7. We provide a direct computation of this based on *in-in* formalism and we also give an argument which holds in case of a correlator composed only by ζ's.

6.5.1 A direct computation

This direct computation is based on *in-in* formalism we have discussed in section 4.8. Consider the action for the single field inflation in the ADM form (see appendix B):

$$S = \int dtd^3x\sqrt{\gamma}N\left(R^{(3)} + K^{ij}K_{ij} - K^i_iK^j_j + \mathcal{L}_m\right) \ ,$$

where $\mathcal{L}_m$ is the inflaton Lagrangian. We can write

$$S = \int dt d^3x \sqrt{\gamma} \mathcal{S}(x) \ ,$$

where $\mathcal{S}(x) = N\left(R^{(3)} + K^{ij}K_{ij} - K^i_i K^j_j + \mathcal{L}_m\right)$, which is a 3-scalar. We define its variation under a gauge transformation

$$\Delta_{\mathcal{S}} = \sqrt{\tilde{\gamma}(x)}\tilde{\mathcal{S}}(x) - \sqrt{\gamma(x)}\mathcal{S}(x) \ , \tag{6.30}$$

so that $H_{\text{int}} = -L_{\text{int}} = -\int d^3x \sqrt{\gamma}\mathcal{S}$. Let $b(t,x)$ be generic product of three fields (in particular we have in mind ζ^3, $\zeta^2 D$, ζD^2 and D^3). The *in-in* formula 4.58 (the first summand is null since we are computing an odd-point function) gives

$$\begin{aligned}
\Delta\langle b(t,x)\rangle &= \langle\tilde{b}(t,x)\rangle - \langle b(t,x)\rangle = \\
&= i\int_{t_0}^{t}\langle[\tilde{H}^I_{\text{int}}(t',x),\tilde{b}(t,x)]\rangle dt' - i\int_{t_0}^{t}\langle[H^I_{\text{int}}(t',x),b(t,x)]\rangle dt' = \\
&= -i\int_{t_0}^{t}\langle[\int d^3x\left(\sqrt{\tilde{\gamma}}\tilde{\mathcal{S}} - \sqrt{\gamma}\mathcal{S}\right)(t',x),\tilde{b}(t,x)]\rangle dt' = i\int_{t_0}^{t}\langle[\tilde{b}(t,x),\int d^3x\Delta_{\mathcal{S}}(t',x)]\rangle dt' \ .
\end{aligned} \tag{6.31}$$

At this point we consider the gauge transformation 3.16

$$x'^i = x^i + \lambda(|\vec{x}|)x^i + \omega^i_j(\vec{x})x^j$$

and we have to understand how $\mathcal{S}$ changes under this infinitesimal diffeomorphism, more specifically we want $\Delta_{\mathcal{S}}$. Firstly, we need $\tilde{\mathcal{S}}(x)$. From what we have seen in section 2.4, defining $x'^i - x^i = \delta x^i = \epsilon^i$ one gets thanks to equation 2.14

$$\tilde{\mathcal{S}}(x) = \mathcal{S}(x) - \mathcal{L}_\epsilon\mathcal{S}(x) + \ldots = \mathcal{S}(x) - \delta x^i\partial_i\mathcal{S}(x) + \ldots \ .$$

Since one usually splits the 3-scalar into a background, a first and a second-order perturbation $\mathcal{S}(x) = \mathcal{S}_0(t) + \mathcal{S}^1(x) + \mathcal{S}^2(x) + S^3(x) + \ldots$, the result is

$$\tilde{\mathcal{S}}(x) = \mathcal{S}_0(t) + \mathcal{S}^{(1)}(x) + \mathcal{S}^{(2)}(x) + S^{(3)}(x) - \delta x^i\partial_i\left(\mathcal{S}^{(1)}(x) + \mathcal{S}^{(2)}(x)\right) + \ldots \ .$$

It is very important to notice that since λ and ω are referred to the long mode and since in the consistency relation there is only one mode which is squeezed: terms containing $\lambda\omega$ or a higher power of λ or ω would imply vertices with two or three squeezed momenta so they are not relevant in the squeezed limit. For the same reason, since $\delta x^i = \epsilon^i = \lambda x^i + \omega^i_j x^j$, terms containing more than one δx^i can be neglected. This is exactly what happens in the previous equation, where $\ldots$ contains an incresing number of Lie derivatives, each one carrying one δx^i inside. Moreover, we stop at third order in cubic action since in 6.31 we have a 3-point function.
Next, we have to derive the transformation rule for $\sqrt{\gamma} = \sqrt{\det\gamma_{ij}}$. The expression for γ_{ij} in ζ-gauge, up to third order in perturbations, is given by 4.66,

$$\gamma_{ij} = a^2(t)\left(e^{2\zeta}\delta_{ij} + D_{ij} + \frac{1}{2}D_{ik}D^k_j + \frac{1}{6}D_{ik}D^k_l D^l_j\right) \ .$$

The expansion of its determinant up to third order in perturbations is

$$\sqrt{\gamma}(x) = a^3(t)\left(1 + 3\zeta + \frac{9}{2}\zeta^2 + \frac{9}{2}\zeta^3 + \ldots\right) \ .$$

Notice that D_{ij} does not contribute to $\sqrt{\gamma}$ up to third order. We now want to find the expression for $\sqrt{\tilde{\gamma}(x)}$, which is non-trivial. We have implemented a Mathematica code to get the result, which turns out to be

$$\begin{aligned}\sqrt{\tilde{h}}(x) =& a^3 + a^3\left[3\zeta - \partial_i\left(x^i\lambda\right)\right] + \frac{1}{2}a^3\left[9\zeta^2 - 6\zeta\partial_j\left(x^j\lambda\right) - 6\lambda x^j\partial_j\zeta\right] \\ &+ \frac{1}{2}a^3\left[9\zeta^3 - 9\zeta^2\partial_j\left(x^j\lambda\right) - 18\zeta\lambda x^j\partial_j\zeta\right] + \\ &- 3a^3 x_j\partial_i\zeta\omega^{ij} - 18a^3\zeta x_j\partial_i\zeta\omega^{ij} + \\ &+ \mathcal{O}\left(\lambda^2, \lambda\omega, \omega^2, \lambda^3, \lambda^2\omega, \lambda\omega^2, \omega^3\right) \ .\end{aligned}$$

The first and the second line contain the unperturbed value (a^3) and the contribution of the λ-piece up to third order perturbations; the third line is the contribution of the ω-piece up to third order perturbations (notice that it starts at second-order in perturbations); the last line is remnant of the fact that we have neglected all the terms containing more than one squeezed vertex, as we said. Finally, from 6.30 one finds the transformation rules $\Delta_{\mathcal{S}}$ up to third order[9]

$$\begin{cases} \bar{\Delta}_S = 0 \\ \Delta_{\mathcal{S}}^{(1)} = a^3\,\bar{\mathcal{S}}\,\partial_i\left(\lambda\,x^i\right) \\ \Delta_{\mathcal{S}}^{(2)} = -a^3\,\partial_i\left[\left(\mathcal{S}^{(1)} + 3\,\bar{\mathcal{S}}\zeta\right)\,\left(\lambda\,x^i + \omega_{ij}\,x^j\right)\right] \\ \Delta_{\mathcal{S}}^{(3)} = -a^3\,\partial_i\left[\left(\frac{9}{2}\,\bar{\mathcal{S}}\,\zeta^2 + 3\,\mathcal{S}^{(1)}\,\zeta + \mathcal{S}^{(2)}\right)\,\left(\lambda\,x^i + \omega^i_j\,x^j\right)\right] \ . \end{cases}$$

This implies that also the variation of the cubic Lagrangian is just a boundary term, so for 6.30 and 6.31 one has that the gauge variation

$$\left\langle\tilde{\mathcal{O}}\left(\vec{x}_1,\ldots\vec{x}_N\right)\right\rangle - \left\langle\mathcal{O}\left(\vec{x}_1,\ldots\vec{x}_N\right)\right\rangle \tag{6.32}$$

is identically zero, and not $-\left\langle\mathcal{O}\left(\vec{x}_1,\ldots\vec{x}_N\right)\right\rangle$. The operator $\mathcal{O}$ can be considered a general combination of ζ and D fields

$$\mathcal{O}\left(\vec{x}_1,\ldots\vec{x}_N\right) = \zeta\left(\vec{x}_1\right)\ldots\zeta\left(\vec{x}_M\right)D\left(\vec{x}_{M+1}\right)\ldots D\left(\vec{x}_N\right) \ .$$

This approach is very powerful since it is valid whatever is the product of fields $b(x)$: it must be in particular valid for $\langle\zeta^3\rangle$ and $\langle\zeta^2 D\rangle$, which are the bispectra cancelled in [78].

6.5.2 An analytical proof

There is another simple proof of the fact that

$$\Delta B(x_1, x_2, x_3) = \tilde{B}(x_1, x_2, x_3) - B(x_1, x_2, x_3) = 0 \ ,$$

which is valid in the case of a N-point function composed only by ζ's:

$$\langle\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_N}\rangle \ .$$

From section 4.6.2 we know that ζ is a scalar $\tilde{\zeta}(\tilde{x}) = \zeta(x)$ and so using the result in section 2.4 one has

$$\tilde{\zeta}(x) = \zeta(x) - \mathcal{L}_\epsilon\zeta(x) = \zeta(x) - (\lambda x^i + \omega^i_k x^k)\partial_i\zeta(x) \ ,$$

where ϵ^i is given by 6.19. For sake of shortness we set

$$\tilde{\zeta}(x) = \zeta(x) + \delta\zeta(x) = \zeta(x) + \delta_1\zeta(x) + \delta_2\zeta(x) \ .$$

At leading order, the variation of the N-point function under the gauge transformation is given by

$$\Delta\langle\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_N}\rangle = \langle\delta_1\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_N}\rangle + \langle\delta_2\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_N}\rangle + \ldots + \langle\zeta_{\vec{k}_1}\cdots\delta_1\zeta_{\vec{k}_N}\rangle + \langle\zeta_{\vec{k}_1}\cdots\delta_2\zeta_{\vec{k}_N}\rangle \ . \tag{6.33}$$

As one can easily realize, terms having more than one variation correspond to taking more than one squeezed limit, which is not the case considering the consistency relation. The rest of the demonstration consists in showing that each of these terms can be seen as a boundary term in Fourier space. The

[9] Also this result can be found using Mathematica.

expression of λ and ω in Fourier space are 6.26 and 6.29, but we are dealing with quantum correlators, so ζ, λ and ω must be seen as operators: as usual

$$\zeta_{\vec{k}} = v_k a_{\vec{k}} + v_k^* a_{\vec{k}}^\dagger \ , \quad \lambda_{\vec{k}} = v_k W_k a_{\vec{k}} + v_k^* W_k a_{\vec{k}}^\dagger \ , \quad \omega_{ij}^{s,\vec{k}} = v_k \epsilon_{ij}^s W_k a_{\vec{k}} + v_k^* \epsilon_{ij}^s W_k a_{\vec{k}}^\dagger \ .$$

The variation of the N-point function gives rise to many terms to be summed; fortunately they are very similar and they can be treated at the same time. We consider them separately.

Let us start with $\delta_1 \zeta = -\lambda x^i \partial_i \zeta$. For what we have seen before, in Fourier space one has

$$(\lambda x^i \partial_i \zeta)_{\vec{k}} = \int \frac{d^3x}{(2\pi)^9} e^{-i\vec{k}\cdot\vec{x}} \left(\int d^3p_1 d^3p_2 e^{i\vec{p}_1\cdot\vec{x}} x^i \partial_i (e^{i\vec{p}_2\cdot\vec{x}}) \lambda_{\vec{p}_1} \zeta_{\vec{p}_2} \right) \ .$$

The contribution to the variation of the N-point function is given by

$$\begin{aligned}
\langle \delta_1 \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_N} \rangle &= - \langle (\lambda x^i \partial_i \zeta)_{\vec{k}_1} \cdots \zeta_{\vec{k}_N} \rangle = \\
&= - \int \frac{d^3x}{(2\pi)^9} e^{-i\vec{k}_1\cdot\vec{x}} \int d^3p_1 d^3p_2 e^{i\vec{p}_1\cdot\vec{x}} x^i \partial_i (e^{i\vec{p}_2\cdot\vec{x}}) \langle \lambda_{\vec{p}_1} \zeta_{\vec{p}_2} \zeta_{\vec{k}_2} \cdots \zeta_{\vec{k}_N} \rangle = \\
&= - \int \frac{d^3p_1 d^3p_2}{(2\pi)^6} \int \frac{d^3x}{(2\pi)^3} e^{-\vec{k}_1\cdot\vec{x}} e^{i\vec{p}_1\cdot\vec{x}} p_2^i \partial_{p_2^i} e^{i\vec{p}_2\cdot\vec{x}} \langle \lambda_{\vec{p}_1} \zeta_{\vec{p}_2} \zeta_{\vec{k}_2} \cdots \zeta_{\vec{k}_N} \rangle = \\
&= \int \frac{d^3p_1 d^3p_2}{(2\pi)^6} \int \frac{d^3x}{(2\pi)^3} e^{-\vec{k}_1\cdot\vec{x}} e^{i\vec{p}_1\cdot\vec{x}} e^{i\vec{p}_2\cdot\vec{x}} \partial_{p_2^i} \left(p_2^i \langle \lambda_{\vec{p}_1} \zeta_{\vec{p}_2} \zeta_{\vec{k}_2} \cdots \zeta_{\vec{k}_N} \rangle \right) = \\
&= \int \frac{d^3p_2}{(2\pi)^6} \partial_{p_2^i} \left(p_2^i \langle \lambda_{\vec{k}_1 - \vec{p}_2} \zeta_{\vec{p}_2} \zeta_{\vec{k}_2} \cdots \zeta_{\vec{k}_N} \rangle \right) \ ,
\end{aligned}$$

where we have integrated by parts and used the Fourier transform of the Dirac delta. This is a boundary term, which vanishes for the presence of filters.

The term $\delta_2 \zeta = -\omega^{ik} x^k \partial_i \zeta$ is very similar to the previous. In Fourier space

$$(\omega^{ik} x^k \partial_i \zeta)_{\vec{k}} = \int \frac{d^3x}{(2\pi)^9} e^{-i\vec{k}\cdot\vec{x}} \left(\int d^3p_1 d^3p_2 e^{i\vec{p}_1\cdot\vec{x}} x^k \partial_i (e^{i\vec{p}_2\cdot\vec{x}}) \omega_{\vec{p}_1}^{ik} \zeta_{\vec{p}_2} \right)$$

and repeating the previous computations

$$\begin{aligned}
\langle \delta_2 \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_N} \rangle &= - \langle (\omega^{ik} x^k \partial_i \zeta)_{\vec{k}_1} \cdots \zeta_{\vec{k}_N} \rangle = \\
&= - \int \frac{d^3x}{(2\pi)^9} e^{-i\vec{k}_1\cdot\vec{x}} \int d^3p_1 d^3p_2 e^{i\vec{p}_1\cdot\vec{x}} x^k \partial_i (e^{i\vec{p}_2\cdot\vec{x}}) \langle \omega_{\vec{p}_1}^{ik} \zeta_{\vec{p}_2} \zeta_{\vec{k}_2} \cdots \zeta_{\vec{k}_N} \rangle = \\
&= \int \frac{d^3p_2}{(2\pi)^6} \partial_{p_2^k} \left(p_{2i} \langle \omega_{\vec{k}_1 - \vec{p}_2}^{ik} \zeta_{\vec{p}_2} \zeta_{\vec{k}_2} \cdots \zeta_{\vec{k}_N} \rangle \right) \ ,
\end{aligned}$$

which is also a boundary term.

The same results hold for the other terms in 6.33: this means that all the variations are boundary terms in Fourier space, which vanish for the presence of the filter W, making the integrand vanishing at infinity.

One could in principle consider a mixed correlator of type

$$\langle \zeta_{\vec{k}_1} \cdots \zeta_{\vec{k}_N} D_{i_1 j_1}^{\vec{q}_1} \cdots D_{i_M j_M}^{\vec{q}_M} \rangle \ .$$

This case includes the bispectra $B_{\zeta\zeta D}$, $B_{\zeta DD}$ and B_{DDD}. In this case one would have to consider the variation of the tensor perturbation D under the gauge transformation 3.16. To get the result one has to consider the variation of the spatial metric according to the result in section 4.6.2

$$\Delta g_{ij} = -\mathcal{L}_\epsilon g_{ij} \ .$$

Starting from ζ-gauge one has

$$\Delta g_{ij} = -(\lambda x^k + \omega_l^k x^l)\partial_k(2\zeta\delta_{ij} + D_{ij}) + \partial_i(\lambda x^k + \omega_l^k x^l)(2\zeta\delta_{kj} + D_{kj}) + \partial_j(\lambda x^k + \omega_l^k x^l)(2\zeta\delta_{ki} + D_{ki}) \ .$$

The problem here is that to extract the tensor perturbation one should SVT decompose the RHS, which in this non-linear case is highly non-trivial. A possible way to proceed is to use a projector on the tensor sector, for example the one defined in [95], but its highly non-trivial expression makes the SVT decomposition very cumbersome. This could be an issue to examine in a future work.

6.6 Results

We have explicitly confuted the results presented in [77], [78], [79] and [80]. We summarize here the arguments.

- In $k \to 0$ limit, the transformation to a CFC frame 6.1 seems to be a Weinberg transformation of type 6.2. However, in subsection 6.2 we have seen that at non-linear order it is not, but it can be written as 6.16, which is a deformed dilatation.
- By a careful analysis of the SVT decomposition in sections 6.3 and 6.4, we have seen that there is no shift in Ψ (so also in ζ) under a gauge transformation of type 6.16. This is explicit in contrast with the transformation rules 6.6 and 6.5 advocated in [78] and following papers. Our analysis also implies that the limit $k \to 0$ does not commute with the smallness of the gradients.
- Under a gauge transformation of type 6.25, the bispectrum is not altered in the squeezed limit, as shown in section 6.5. This implies that the transformation rule 6.8 is wrong. Indeed, it is based on the splitting $\Psi = \Psi_L + \Psi_S$ producing the transformation rules 6.6 and 6.5, which we have explicitly shown not to hold for a deformed dilatation.

The cancellation presented in [78] and following papers remains present only in the exact $k = 0$ limit. Indeed, in section 6.3 we have seen that for a transformation of type 3.16, which actually coincides with 3.16 where ω and λ are exactly constant, the SVT decomposition is ambiguous, so that also Ψ and D can shift. This fact was used by Weinberg to show his theorem, but we underline that it is valid only in the exact $k = 0$ limit. In this case the transformation in Fourier space has no more the θ filter, but λ becomes easily (thanks to 6.26)

$$\lambda(k) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3x e^{-i\vec{k}\cdot\vec{x}} \lambda = \lambda\delta^{(3)}(k) \ .$$

However, $k = 0$ implies an infinitely long wave mode, which is unphysical, given that the observable Universe is finite.

The result of our arguments is that the squeezed limit of the bispectrum remains observable. We remind that the consistency relation is fundamental to test the Early Universe, since it is valid only in single-field models of inflation, which means that measuring the power spectra and the bispectra of the cosmological perturbations one can test the inflationary model used. The squeezed limit remains a key channel to test the Early Universe physics. In the next chapter we are going to examine shortly how this could be possible.

Chapter 7

Observational prospects

As we have seen through this project, the consistency relation remains a key result to study Early Universe physics. One of the main experimental goals in the future will be to measure accurately the primordial non-Gaussianity, in order to test the inflationary model. In this conclusive chapter, we are going to discuss briefly what the actual results are and the perspectives for the future.

7.1 Planck results

Planck satellite has been the last satellite mission for CMB measurements ([96]) and has provided the tightest constraints on a variety of cosmological parameters (defining the standard ΛCDM model, [27]). The Planck satellite can be regarded essentially as the ultimate experiment in terms of the cosmological information that can be extracted from temperature data, even though it has also efficiently exploited CMB polarization data ([97]). Its measurements have also allowed to set the tightest constraints on the possible deviations from a pure Gaussian distribution of primordial density perturbations, that we are going to recall now.

As we have seen the statistics of the CMB anisotropies is related to the primordial correlators of ζ, so one can use these measurements to test the various inflationary models. In section 4.9 we have introduced the amplitude parameter f_{NL}, but also the different templates. In general, its definition is not univocal, but it depends on the templates used.

The amplitude parameter is defined as follows[1]

$$f_{\rm NL} = \frac{5}{12}\frac{B_{\zeta\zeta\zeta}(k,k,k)}{P_\zeta^2(k)} \ .$$

In the case of single-field models of inflation, from the Maldacena consistency relation 5.1 we expect that the local contribution to the bispectra is given by

$$f_{\rm NL}^{\rm local} = -\frac{5}{12}(n_s - 1) = \frac{5}{12}(6\epsilon - 2\eta_V) \ .$$

This relation gives a tool to test single-field inflation, since both $f_{\rm NL}^{\rm local}$ and n_s have been measured by Planck. We see that since we have $n_s - 1$ very small, we expect $f_{\rm NL}^{\rm local}$ very small.

The latest results published in 2018 ([98]) provide the following estimations for the three amplitudes

$$\begin{aligned} f_{\rm NL}^{\rm local} &= -\,0.9 \pm 5.1\,(68\%\,CL) \\ f_{\rm NL}^{\rm equilateral} &= -\,26 \pm 47\,(68\%\,CL) \\ f_{\rm NL}^{\rm orthogonal} &= -\,38 \pm 24\,(68\%\,CL) \ , \end{aligned}$$

[1] For a detailed definition the interested reader can lok at [98].

while for the spectral index the result ([27]) provides

$$n_s = 0.965 \pm 0.004 \, (68\% \, CL) \ .$$

We see that the data are well compatible with the consistency relation. Despite the fact that Planck has set the tightest constraints on primordial non-Gaussianity, there is still a large window before being able to confirm or discard observationally the validity of the consistency relation (essentially because the error bar for $f_{\rm NL}^{\rm local}$ is still too big).

After Planck, the next generation of satellite devoted to the measurement of the CMB anisotropies is LiteBIRD ([99–101]), planned for 2028. Another proposed satellite able to map CMB anisotropies improving the resolution is CORE ([102]): the 1σ error bar $f_{\rm NL}^{\rm local}$ is forecast to be $\Delta f_{\rm NL}^{\rm local} = 3.6$ (while Planck has provided $\Delta f_{\rm NL}^{\rm local} = 5.1$).

The Planck measurements can only pose bounds on the bispectrum $B_{\zeta\zeta\zeta}$; in order to measure the bispectra involving the tensor perturbation D one should measure the stochastic GWs background and cross-correlate the results with the Planck data. As we will see in section 7.3, today's instruments able to reveal GWs (interferometers) cannot measure any GW from inflation. By now, in absence of direct detection, the current bound on non-Gaussianities can be only posed in an indirect way, using CMB data themselves. Some bounds have been obtained in [103]; the amplitude f^{ttt} is related[2] to the bispectrum B_{DDD}, while the amplitude f^{tss} is related to the bispectrum $B_{D\zeta\zeta}$. The results can be summarized in the following table, where the different rows indicate the estimation from the different datasets: WMAP temperature data, the Planck temperature data, the Planck E-mode polarization data and the combination of Planck temperature and E-mode data.

	$f_{\rm NL}^{ttt,\rm equilateral}$	$f_{\rm NL}^{ttt,\rm squeezed}$	$f_{\rm NL}^{tss,\rm squeezed}$
WMAP T only	600 ± 1500	220 ± 170	84 ± 49
Planck T only	600 ± 1600	290 ± 180	–
Planck E only	2900 ± 6700	–	–
Planck $T + E$	800 ± 1100	–	–

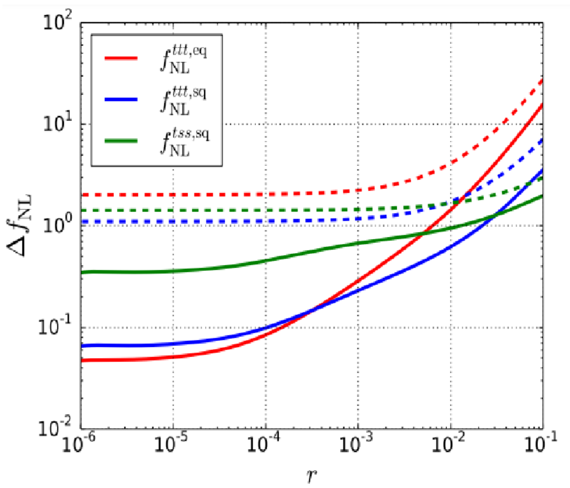

Figure 7.1: Figure taken from [103], showing the expected 1σ errors for $\Delta f_{\rm NL}^{ttt,\rm eq}$ (red lines) and $\Delta f_{\rm NL}^{ttt,\rm sq}$ (blue lines) from BBB, and $\Delta f_{\rm NL}^{tss,}$ (green lines) from BTT, as a function of the tensor-to-scalar ratio r. The two linestyles discriminate the cleanliness level of the B-mode data: non-delensed and noiseless full-sky one (solid lines); LiteBIRD-like realistic one (dashed lines).

[2]We refer the reader to [103] for the technical definitions of $f_{\rm NL}^{ttt,\rm equilateral}$, $f_{\rm NL}^{ttt,\rm squeezed}$ and $f_{\rm NL}^{tss,\rm squeezed}$.

7.2 Constraints from large scale structures

A second manner to constrain the primordial non-Gaussianity comes from the large scale structures of the Universe. In particular, some works ([104, 105]) have shown that the primordial non-Gaussianity affects the clustering of dark matter halos inducing (in the case of local non-Gaussianity) a relationship between the spatial distribution of galaxies and the underlying dark matter density field. This effect goes under the name of non-Gaussian halo bias.
Mapping with high precision the large scale structure one can hopefully extract some information about the non-Gaussianity parameters ([106]). However, the current measurements are not able to produce an estimation for $f_{\rm NL}$ that is competitive with CMB measurements.
The satellite Euclid ([107, 108]), with launch planned in 2022, is expected to improve the sensibility to the same level of CMB constraints. Another important mission is SPHEREx ([109–112]), with launch planned for 2024, aiming at measuring $f_{\rm NL} \sim 1$ at 2σ. This is very promising, since in a not-so-far future it is expected to obtain some contraints competitive with the ones from CMB data.

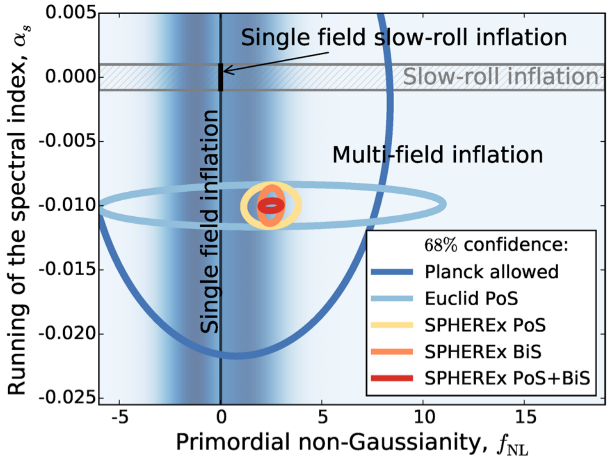

Figure 7.2: Figure taken from [113], showing that SPHEREx and Euclid establish powerful constraints on $f_{\rm NL}$ as compared to Planck data. Ellipses correspond to observational constraints while the shaded regions identify families of models. SPHEREX is supposed to be able to discriminate between classes of inflationary models: SPHEREx and Euclid forecasts are centered arbitrarily, while the dimension of the allowed regions is the expected one.

7.3 Future expectations from interferometers

After the first gravitational event named GW150914, measured by the interferometer LIGO ([114,115]) on 14 September 2015, many other events have been reported, giving rise to the era of gravitational waves astronomy. These events are in general due to the mergers of very massive black holes. However, to relate the correlator invoving gravitational waves to measurable quantities, one needs to measure the gravitational waves background generated by inflation, which is impossible to do in nowadays interferometers.
In general, an interferometer has different sensitivities depending on the frequency. To understand which the range of frequencies are, one usually constructs sensitivity curves ([116]). For example these curves are shown in figure 7.3 where the fractional energy density of gravitational waves per logarithmic wave-number interval is plotted as a function of the frequency f (or equivalently the wave number k). It is defined as

$$\Omega_{\rm GW} = \frac{1}{\rho_C}\frac{d\rho_{\rm GW}}{d\log f} \ ,$$

where $\rho_{\rm GW}$ is the energy density of GWs and $\rho_C = 3M_P^2 H^2$ is the critical density of the Universe. The curves for a specific interferometer correspond to the points in which the signal has a fixed signal-to-noise ratio (usually greater than 1). This implies that if the curve of a predicted signal lies above the interferometer curve in some frequency band, then the signal becomes conventionally easily detectable.

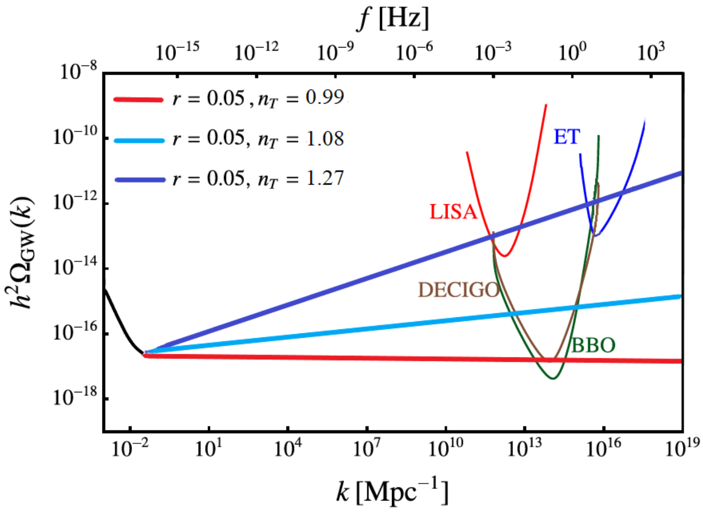

Figure 7.3: Sensitivity curves for 4 future interferometers. The three straight lines correspond to the theory prediction for different values of the spectral index and the signal-to-noise ratio. As it is possible to see, these new interferometers will be hopefully able to detect the stochastic background of gravitational waves.

The main interferometers used today are ground based and they are LIGO and VIRGO ([117, 118]). Other similar ground based interferometers planned for the future are KAGRA ([118, 119]), Einstein Telescope ([120, 121]) and Cosmic explorer ([122, 123]). All these interferometers (with the exception of Einstein Telescope) have an L shape, as the Michelson-Morley interferometer, and the frequency band of sensitivity is $1 - 10^4$ Hz. They can measure GWs from astrophysical events.

As we have said, the interferometers which are operative today are not able to detect any gravitational wave from inflation. In the next decades, the idea is to build interferometers able to detect signals in smaller frequency bands. For example, LISA ([124]) will be an interferometer to be located in space: it will be composed by three satellites placed in an equilater triangle configuration; this way the length of the arms will be much longer than the Earth-placed interferometers of the Michelson-Morley type. The frequency to which LISA is planned to be sensitive is in the range $10^{-5} - 1$ Hz, which are much smaller than the ones for the ground-based interferometers. The main target is the GWs signal from massive black hole binaries, but some works have shown that LISA could be in principle able to measure the stochastic background from the Early Universe, depending on the inflationary model and on the parameter space ([125, 126]).
Other future experiments to be highlighted are DECIGO ([127]) and BBO ([128]), both satellite experiments, similar to LISA. The target of the first one, with launch scheduled for 2027, are the frequencies in the range $0.1 - 10$ Hz, between LIGO and LISA. On the contrary, BBO is the successor of LISA and its launch is still to be decided. Its primary goal will be the measurements of the GWs from inflation. In figure 7.3 there are the sensitivity curves for these three future experiments. However, to test the consistency relation one should find a way to experimentally measure the correlation functions of GWs using the interferometers and the procedure to do so is still not clear at all nowadays, both theoretically and experimentally (and in any case this would require an incredible precision in mapping the stochastic GWs background).

Chapter 8

Conclusions

In this project we have discussed the inflationary consistency relations, which are valid in single-field models of inflation, introducing all the tools necessary to understand how they were derived. Inflation was introduced in order to solve the shortcomings of the standard hot Big Bang model (see chapter 1) and to provide some initial conditions for the evolution of the Universe. The simplest model involves the introduction of a single scalar field, called inflaton (introduced in chapter 4): in order to study the dynamics of such types of models one proceeds perturbatively, splitting the fields in a background (homogeneous and isotropic for the cosmological principle) and a perturbation, which is treated as a quantum field. However, in the case of the gravitational field, this procedure is subtle since this splitting generates a gauge freedom due to the diffeomorphism invariance of general relativity (as explained in chapter 2). In order to test the different inflationary models one has firstly to find quantities able to produce observables. The most important quantities are the curvature perturbation ζ, defined through the spatial curvature in Arnowitt-Deser-Misner formalism and related to the CMB temperature anisotropies map, and the tensor perturbation of the metric, which is related to the primordial gravitational wave background. These quantities are very important, since under adiabatic conditions they are constant (see chapter 3) on superhorizon scales (that is the limit $k \to 0$), so that the physics at the end of inflation can be directly connected with the physics at the beginning of inflation.

Then, to test the different inflationary models one has to find proper quantities to observationally probe and the most important ones are quantum correlators between these. Indeed, quantum correlators can be directly linked with the statistics of CMB data (and in future, hopefully, with the statistics of the primordial GWs background). The most important objects in this context are the power spectrum, related to the Fourier transform of the 2-point correlation function, and the bispectrum, related to the Fourier transform of the 3-point correlation function. A non-zero bispectrum can be related to the presence of a primordial non-Gaussianity due to the interaction of the inflaton with itself (or with other fields).

From the Maldacena computation of bispectra ([46]) we have seen that taking the squeezed limit (that is sending one momentum in the correlator to zero) the bispectrum becomes a product of two power spectra. This result is intimately related to the symmetries of the spacetime during inflation (in chapter 5), since the consistency relations can be seen as Ward identities, as shown in [50], associated to the broken generators of the SSB pattern of inflation. The consistency relation can be generalized for higher order correlators, obtaining an infinite set of consistency relations linking the $N+1$-point function to the N-point functions.

However, the physical existence and consequently the measurability of the consistency relations has been recently criticized (as explained in chapter 6), since it has been claimed that in the squeezed limit the bispectrum can be set to zero through an appropriate gauge transformation ([77–80]), which, more specifically corresponds to the transformation to conformal Fermi coordinates ([81]).

We have explicitly shown that this is not the case, since the transformation rules used to transform the bispectrum have not been correctly implemented and interpreted. We summarize here the main reasons we have discussed in chapter 6.

- In the exact $k \to 0$ limit, the transformation to a CFC frame is a Weinberg transformation and in this case the cancellation can happen; however, in realistic cases k cannot be exactly null (the observable Universe is finite), so the transformation becomes a deformed dilatation.
- In this case, a careful analysis of the scalar-vector-tensor (SVT) decomposition implies that the tranaformation rules of ζ, based on the splitting in long and short part, are not correct.
- Gradients cannot be neglected in the limit $k \to 0$, only because in Fourier space they are proportional to k^i: we have seen that the limit and the gradients do not commute.
- The definition of ζ used in the papers claiming the cancellation of the bispectra is problematic: in [78] it is not very clear which definition is used, but it is simply assumed that ζ is not a scalar, while we have seen that ζ is a scalar; in [79, 80], on the contrary, a different definition is introduced, which is not gauge invariant. Given that a deformed dilatation is not a residual gauge freedom (it is only if $k = 0$ exactly), this quantity changes under a gauge transformation, so it is not a good observable.
- Finally, we have explicitly computed that under a deformed dilatation the bispectrum is not altered in the squeezed limit, which is in contrast with the transformation rule derived in [78], based on the splitting in long and short wavelength modes.

This happens both in the bispectrum involving only the curvature perturbation ζ, which was recently discussed in [91], but also in mixed bispectra, which involve also the tensor perturbations. The extension of this result to the tensor sector is the main crucial improvement achieved through this project. The results presented for the first time are about to appear as preprint in ArXiV in the near future ([129]), to be submitted to an international journal.
We remark once again the importance of these results. Consider e.g. the inflationary consistency relation involving the bispectrum of the curvature perturbation ζ. It is valid only in single-field inflation, so measuring a violation of this relation would prove that all single-field models are not enough to explain the inflationary mechanism. On the other hand, testing such a consistency relation (that we have proved to be physical and in principle observable) would be an incredibly strong evidence of the single-field inflationary mechanism. Finally we have discussed what are the prospects in the future to improve the sensitivity to the primordial non-Gaussianity both through CMB or large scale structure observables or via gravitational wave interferometers (see chapter 7).

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Appendix A

Lie derivative

In this section we define Lie derivative in a formal way and we derive the formulas we have used to calculate the transformations of perturbations under a gauge transformation. We use a mathematical approach ([130]), but we skip very formal details which are unnecessary in this discussion.
Lie derivative is a differential operator measuring how much a generic tensor field changes along the integral curves of a given vector field. In mathematics, vector fields are seen as derivations, i. e. there is a bijection between a vector field and the following derivative operator:

$$V^\mu \leftrightarrow V = V^\mu \partial_\mu \ .$$

We remind that ∂_μ is defined with lower indices, this means that the sum over the repeated indices is not contracted with the metric tensor, but it is a simple sum. One also defines the Lie brackets between the two derivation as the commutator between them, computed using Leibniz rule:

$$\begin{aligned}
[X, Y] &= X^\mu \partial_\mu (Y^\nu \partial_\nu) - Y^\nu \partial_\nu (X^\mu \partial_\mu) = \\
&= X^\mu \partial_\mu Y^\nu \partial_\nu + X^\mu Y^\nu \partial_\mu \partial_\nu - Y^\nu \partial_\nu X^\mu \partial_\mu - Y^\nu X^\mu \partial_\nu \partial_\mu = \\
&= X^\mu \partial_\mu Y^\nu \partial_\nu - Y^\nu \partial_\nu X^\mu \partial_\mu = X^\mu \partial_\mu Y^\nu \partial_\nu - Y^\mu \partial_\mu X^\nu \partial_\nu = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) \partial_\nu \ .
\end{aligned}$$

In particular we see that thanks to Schwartz theorem the Lie bracket remains a first-order operator.
At this point we define the Lie derivative along $X = X^\mu \partial_\mu$ of a scalar function as

$$\mathcal{L}_X f = X^\mu \partial_\mu f \ .$$

This is natural, since we want to derive f along X.
For what concerns the Lie derivative of a vector field we define

$$\mathcal{L}_X Y = [X, Y] = (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) \partial_\nu \ ,$$

which implies

$$\mathcal{L}_X Y^\nu = X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu \ .$$

This is also natural, since assuming an infinitesimal transformation of the coordinates along the vector field X^μ of type $x'^\mu = x^\mu + \lambda V^\mu$ with λ infinitesimal parameter, the vector field Y changes as

$$\begin{aligned}
Y'^\mu(x) - Y^\mu(x) &= Y'^\mu(x' - \lambda X) - Y^\mu(x) = Y'^\mu(x') - \lambda X^\alpha \partial_\alpha Y'^\mu(x') + \ldots - Y^\mu(x) = \\
&= \frac{\partial x'^\mu}{\partial x^\rho} Y^\rho(x) - \lambda X^\alpha \partial_\alpha \left(\frac{\partial x'^\mu}{\partial x^\rho} Y^\rho(x) \right) + \ldots - Y^\mu(x) = \\
&= \left(\delta^\mu_\rho + \lambda \partial_\rho X^\mu \right) Y^\rho(x) - \lambda X^\alpha \partial_\alpha Y^\mu(x) + \ldots - Y^\mu(x) = \\
&= Y^\mu(x) + \lambda \partial_\rho X^\mu Y^\rho(x) - \lambda X^\alpha \partial_\alpha Y^\mu(x) + \ldots - Y^\mu(x) = \\
&= -\lambda \mathcal{L}_X Y^\mu(x) + \ldots \ .
\end{aligned} \tag{A.1}$$

This means that at first-order the Lie derivative measures[1] how much the vector field Y changes along (the integral lines of) X. From this, one can see that the Lie derivative of a vector field can be defined as

$$\mathcal{L}_X Y = \lim_{\lambda \to 0} \frac{Y^\mu(x) - Y'^\mu(x)}{\lambda} = \frac{d}{d\lambda}[\varphi^*_{\lambda X} Y(x)]\bigg|_{\lambda=0} , \tag{A.2}$$

where $\varphi^*_{\lambda X}$ is the pullback along X. To define the Lie derivative of a covariant vector ω (in mathematics it is a 1-form) we put

$$\mathcal{L}_X \omega(Y) = X(\omega(Y)) - \omega([X,Y]) ,$$

where we have put $\omega(Y) = \omega_\mu Y^\mu$, since they couple, being one the dual of the other. Opening the coordinates one has

$$\begin{aligned} \omega([X,Y]) &= \omega_\nu (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) \\ X(\omega(Y)) &= X^\mu \partial_\mu (\omega_\nu Y^\nu) = X^\mu \partial_\mu \omega_\nu Y^\nu + X^\mu \omega_\nu \partial_\mu Y^\nu , \end{aligned}$$

so one finds explicitly

$$\begin{aligned} \mathcal{L}_X \omega(Y) &= X^\mu \partial_\mu \omega_\nu Y^\nu + X^\mu \omega_\nu \partial_\mu Y^\nu - \omega_\nu (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) = \\ &= X^\mu \partial_\mu \omega_\nu Y^\nu + Y^\mu \partial_\mu X^\nu \omega_\nu = (X^\mu \partial_\mu \omega_\nu + \partial_\nu X^\mu \omega_\mu) Y^\nu , \end{aligned}$$

giving

$$\mathcal{L}_X \omega_\nu = X^\mu \partial_\mu \omega_\nu + \partial_\nu X^\mu \omega_\mu .$$

Finally one can generalize these definitions to get the Lie derivative of a generic tensor field T with p controvariant indices and q covariant indices as

$$\begin{aligned} \mathcal{L}_X T = \mathcal{L}_X (T(Y^1, \ldots, Y^q, \omega_1, \ldots, \omega_p)) + \\ - T([X, Y^1], Y^2, \ldots, Y^q, \omega_1, \ldots, \omega_p) - \ldots - T(Y^1, Y^2, \ldots, [X, Y^q], \omega_1, \ldots, \omega_p) + \\ - T(Y^1, Y^2, \ldots, Y^q, \mathcal{L}_X \omega_1, \ldots, \omega_p) - \ldots - T(Y^1, Y^2, \ldots, Y^q, \omega_1, \ldots, \mathcal{L}_X \omega_p) , \end{aligned}$$

which can be seen as a generalization of the Leibniz rule. $Y^1, \ldots, Y^q$, $\omega_1, \ldots, \omega_p$ are the basis vectors which in the end have to be put out of the expression, in order to have an expression of the Lie derivative in the $p+q$ indices of T: these fields have been put to perform a coordinate-independent calculation. Notice that the all summands on the RHS have already been defined: the first is the Lie derivative of a scalar, the others contain Lie derivatives of controvariant or covariant vectors. Repeating the calculations done before, one finds that the Lie derivative of the degree-two tensors are

$$\begin{aligned} \mathcal{L}_X T^{\mu\nu} &= \partial_\lambda T^{\mu\nu} X^\lambda - \partial_\lambda X^\mu T^{\lambda\nu} - \partial_\lambda X^\nu T^{\lambda\mu} \\ \mathcal{L}_X T_{\mu\nu} &= \partial_\lambda T_{\mu\nu} X^\lambda + \partial_\mu X^\lambda T_{\lambda\nu} + \partial_\nu X^\lambda T_{\lambda\mu} \\ \mathcal{L}_X T^\mu_\nu &= \partial_\lambda T^\mu_\nu X^\lambda + \partial_\nu X^\lambda T^\mu_\lambda - \partial_\lambda X^\mu T^\lambda_\nu , \end{aligned}$$

which are the formulas we have widely used to understand how a perturbation changes under an infinitesimal gauge transformation. These formulas can also be easily obtained using the generalization of definition A.2 for a generic tensor field T

$$\mathcal{L}_X T = \lim_{\lambda \to 0} \frac{T^\mu(x) - T'^\mu(x)}{\lambda} = \frac{d}{d\lambda}[\varphi^*_{\lambda X} T(x)]\bigg|_{\lambda=0} , \tag{A.3}$$

expanding as we did in A.1.
As a final comment, notice that in all the formulas above one can replace the partial derivative with the covariant derivative, since this reduces to sum and subtract terms containing Christoffel's symbols,

[1] More formally, one can show (See [130], proposition 3.4.6) that

$$\mathcal{L}_X Y = \lim_{\lambda \to 0} \frac{\mathrm{d}\,(\theta_{-\lambda})_{\theta_\lambda}(Y) - Y}{\lambda} ,$$

where θ is the local flux of the vector field X and d the differential.

using the property that $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$. In case of a scalar this is trivial, since $D_\mu = \partial_\mu$. In case of a controvariant vector, for example, we have $D_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha$, so one has

$$\begin{aligned}\mathcal{L}_X V^\mu &= X^\mu \partial_\mu V^\nu - V^\mu \partial_\mu X^\nu = X^\mu \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} X^\mu V^\alpha - \Gamma^\nu_{\mu\alpha} X^\mu V^\alpha - V^\mu \partial_\mu X^\nu = \\ &= X^\mu (\partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha) - V^\mu (\partial_\mu X^\nu + \Gamma^\nu_{\mu\alpha} X^\alpha) = X^\mu D_\mu V^\nu - V^\mu D_\mu X^\nu \ ,\end{aligned}$$

where we have used the symmetry of Γ in the lower indices and we have renamed the mute indices $\alpha \leftrightarrow \mu$. This procedure can be repeated also for the Lie derivative of a covariant vector and also of a generic tensor with an arbitrary number of indices.
This implies that Lorentz indices in Lie derivative transform controvariantly or covariantly.

Appendix B

ADM formalism

The Arnowitt-Deser-Misner formalism (ADM) is a peculiar decomposition of the metric which allows a Hamiltonian formulation of general relativity. It was firstly introduced by the authors in [131] and in other following papers and it has been very influent on a wide class of research topics, such as quantum gravity or modified theories of gravity. Indeed, in this project we have seen that a way to quantize a theory knowing its Lagrangian is to compute the conjugate momenta and the Hamiltonian density and then to impose the canonical commutation relations. This is not trivial at all starting from the Einstein-Hilbert action, but ADM formalism provides a natural way to proceed. For sake of simplicity, in this section we put $M_P = 1$.
The ADM formalism arises naturally using the extrinsic geometry of hypersurfaces ([132]), but since here we want to remain at an elementary level, we will only comment about this in the end. By now, we limit ourselves to report the main idea that lead to the formalism, giving an explicit derivation of the equations we needed in this project, based on [133]. These results are rarely presented in literature in a such an explicit way.
The starting point is the fact that it is possible to demonstrate that a wide class of spacetime manifold can be foliated into hypersurfaces Σ labeled by a time parameter t, which can be assumed, for example, to be the cosmic time. This means that the spatial hypersurfaces of this foliation of the spacetime are hypersurfaces of "constant time", i. e. they are the level sets of some time function

$$\Sigma_{t_0} = \{x^\mu : t(x^\alpha) = t_0\} \ ,$$

with future-oriented (timelike) normal vector N^i. Notice that since the gradient is orthogonal to the level sets one has $N_i \propto \partial_i t$. The spacetime manifold is thus locally omeomorphic to the product $\mathbb{R} \times \Sigma$, where Σ is an evolving structure in time, called Cauchy surface. In literature the foliation is usually called *slicing*, while the choice of the spatial coordinates on Σ *threading*. Notice that in general a gauge transformation changes both slicing and threading.

B.1 The ADM decomposition of the metric

The main idea underlying the ADM formalism is that in Einstein equations not all the 10 equations are dynamical, but 4 of them are constraints. This is analogous to what happens in electromagnetic theory, where the equations of motion of the Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ are not all really dynamical: the field is A^μ which is a 4-vector, but the true degrees of freedom of the photon are 2, since only 3 equations of motions obtained by varying the Lagrangian are all dynamical (in the sense that they contain a derivative with respect to time) and 1 is a constraint. In general, one can show this fact to be strictly related to the presence of a gauge degree of freedom, which is what happens also in Einstein equations (the gauge symmetry is the invariance under diffeomorphisms).

In order to enforce the slicing, Arnowitt-Deser-Misner introduced the following $(3 + 1)$-decomposition of the metric

$$ds^2 = -\left(N^2 - N_i N^i\right) dt^2 + 2N_i dx^i dt + \gamma_{ij} dx^i dx^j \ ,$$

where N and N_i are commonly called lapse and shifts respectively. It is important to underline from the beginning that the spatial indices are raised and lowered by the spatial metric γ_{ij}, so we have $N_i = \gamma_{ij} N^j$, $N^i = \gamma^{ij} N_j$ and $N_i N^i = N^k N^i \gamma_{ik}$, with $\gamma^{ij} = \gamma_{ij}^{-1}$. Be careful also that $N^2 = N \cdot N \neq N_i N^i = N^k N^i \gamma_{ik}$. The metric components are

$$g_{00} = -\left(N^2 - N_i N^i\right) \ , \quad g_{0i} = N_i \ , \quad g_{ij} = \gamma_{ij} \ .$$

In the following we will need the expression of the inverse metric, which can be found by imposing $g^{\mu\alpha} g_{\alpha_\nu} = \delta^\mu_\nu$. The result is

$$g^{00} = -\frac{1}{N^2} \ , \quad g^{0i} = \frac{N^i}{N^2} \ , \quad g^{ij} = \gamma^{ij} - \frac{N^i N^j}{N^2} \ ,$$

as one can prove by a direct computation:

$$\begin{aligned}
g^{0\nu} g_{\nu 0} &= g^{00} g_{00} + g^{0i} g_{i0} = \frac{1}{N^2}\left(N^2 - N_i N^i\right) + \frac{N^i}{N^2} N_i = 1 = \delta^0_0 \\
g^{0\nu} g_{\nu i} &= g^{00} g_{0i} + g^{0j} g_{ji} = -\frac{1}{N^2} N_i + \frac{N^j}{N^2} \gamma_{ij} = \frac{-N_i + N_i}{N^2} = 0 = \delta^0_i \\
g^{i\nu} g_{\nu 0} &= g^{i0} g_{00} + g^{ij} g_{j0} = -\frac{N^i}{N^2}\left(N^2 - N_i N^i\right) + \left(\gamma^{ij} - \frac{N^i N^j}{N^2}\right) N_j = \\
&= -N^i + N^i \frac{N_j N^j}{N^2} + N^i - N^i \frac{N_j N^j}{N^2} = 0 = \delta^i_0 \\
g^{i\nu} g_{\nu j} &= g^{i0} g_{0j} + g^{ik} g_{kj} = \frac{N^i}{N^2} N_j + \left(\gamma^{ik} - \frac{N^i N^k}{N^2}\right) \gamma_{kj} = \frac{N^i}{N^2} N_j + \gamma^{ik}\gamma_{kj} - \frac{N^i}{N^2} N_j = \gamma^{ik}\gamma_{kj} = \delta^i_j \ .
\end{aligned}$$

Notice also that the ADM metric tensor can be decomposed in the following matrix product:

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^j N^k \gamma_{ij} & N_j \\ N_i & \gamma_{ij} \end{pmatrix} = \begin{pmatrix} 1 & N_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -N^2 & 0 \\ 0 & \gamma_{ij} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ N_j & 1 \end{pmatrix} \ .$$

Since the first and the third matrices have determinant equal to 1, while the second has determinant $-N^2 \det \gamma_{ij}$, this implies that $g := \det g_{\mu\nu} = -N^2 \gamma$ (where $\gamma := \det \gamma_{ij}$), so one gets $\sqrt{-g} = N\sqrt{\gamma}$. Let us now define the unit normal vector $n^\mu \equiv \left(\frac{1}{N}, -\frac{N^i}{N}\right)$, normalized using the ADM metric since

$$\begin{aligned}
n^\mu n_\mu = n^\mu n^\nu g_{\mu\nu} &= (n^0)^2 g_{00} + 2n^0 n^i g_{0i} + n^i n^j g_{ij} = \\
&= -\frac{1}{N^2}\left(N^2 - N_i N^i\right) - 2\frac{1}{N}\frac{N^i}{N} N_i + \frac{N^i N^j}{N^2}\gamma_{ij} = \\
&= -1 + \frac{N_i N^i}{N^2} - 2\frac{N_i N^i}{N^2} + \frac{N_i N^i}{N^2} = -1 \ .
\end{aligned}$$

Its covariant couterpart can be found by lowering the indices using the ADM metric:

$$\begin{aligned}
n_0 &= g_{0\mu} n^\mu = g_{00} n^0 + g_{0i} n^i = -\left(N^2 - N_i N^i\right)\frac{1}{N} - N_i \frac{N^i}{N} = -N \\
n_k &= g_{k\mu} n^\mu = g_{k0} n^0 + g_{ki} n^i = N_k \frac{1}{N} - \gamma_{ki}\frac{N^i}{N} = \frac{N_k}{N} - \frac{N_k}{N} = 0 \ ,
\end{aligned}$$

this way $n_\mu = (-N, \vec{0})$. Since these objects are not covariantly constant, one can define the extrinsic curvatures of the spatial slices $t = \textit{constant}$ as

$$K_{ij} := D_i n_j = \partial_i n_j + \Gamma^\mu_{ij} n_\mu = 0 + \Gamma^\mu_{ij} n_\mu = \Gamma^0_{ij} n_0 + \Gamma^k_{ij} n_k = -N\Gamma^0_{ij} \ . \tag{B.1}$$

Notice also that the inverse metric can be written in terms of n^μ as follows

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^j}{N^2} \\ \frac{N^i}{N^2} & \gamma^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{pmatrix} - \begin{pmatrix} \frac{1}{N^2} & -\frac{N^j}{N^2} \\ -\frac{N^i}{N^2} & \frac{N^i N^j}{N^2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{pmatrix} - n^\mu n^\nu \ , \tag{B.2}$$

which is an expression useful in the future.

B.2 Einstein-Hilbert action in ADM decomposition

In order to proceed we have to compute all the Christoffel's symbols of the ADM metric, through equation 1.3. Focusing on B.1, we want to compute Γ^0_{ij}:

$$
\begin{aligned}
\Gamma^0_{ij} &= \frac{1}{2}g^{0\mu}(\partial_i g_{\mu j} + \partial_j g_{\mu i} - \partial_\mu g_{ij}) = -\frac{1}{2N^2}(\partial_i g_{0j} + \partial_j g_{0i} - \partial_0 \gamma_{ij}) + \frac{g^{0k}}{2}(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}) = \\
&= -\frac{1}{2N^2}(\partial_i N_j + \partial_j N_i - \dot{\gamma}_{ij}) + \frac{N^k}{N^2}\Gamma^{(3)}_{ijk} \\
&= -\frac{1}{2N^2}(\partial_i N_j + \partial_j N_i - \dot{\gamma}_{ij}) + \frac{1}{2N^2}\Gamma^{(3)}_{ijk}N^k + \frac{1}{2N^2}\Gamma^{(3)}_{jik}N^k \\
&= -\frac{1}{2N^2}(\partial_i N_j - \Gamma^{(3)}_{ijk}N^k + \partial_j N_i - \Gamma^{(3)}_{jik}N^k - \dot{\gamma}_{ij}) = \\
&= -\frac{1}{2N^2}(D^{(3)}_i N_j + D^{(3)}_j N_i - \dot{\gamma}_{ij}) \ ,
\end{aligned}
$$

where we have defined the spatial Christoffel's symbol $\Gamma^{(3)}_{ijk} := \Gamma_{ijk} = \frac{1}{2}(\partial_j g_{ki} + \partial_k g_{ji} - \partial_i g_{jk})$, which automathically defines a spatial covariant-derivative $D^{(3)}_i N_j := \partial_i N_j - \Gamma^{(3)}_{ijk}N^k$. It must be underlined that in this convention the first index is the one to raise to obtain a Christoffel's symbol with an upper and two lower indices. The previous result also implies that the extrinsic curvature is

$$
K_{ij} = -N\Gamma^0_{ij} = \frac{1}{2N}(D^{(3)}_i N_j + D^{(3)}_j N_i - \dot{\gamma}_{ij}) \ . \tag{B.3}
$$

Notice that K_{ij} is symmmetric in its indices. Using this result, one can compute all the others Christoffel's symbols, using 1.3. We give directly the results, which are

$$
\begin{aligned}
\Gamma^0_{00} &= \frac{1}{N}\left(\dot{N} + N^i\partial_i N - N^i N^j K_{ij}\right) \\
\Gamma^0_{ij} &= -\frac{1}{N}K_{ij} \\
\Gamma^0_{0i} = \Gamma^0_{i0} &= \frac{1}{N}\left(\partial_i N - N^j K_{ij}\right) \\
\Gamma^i_{0j} = \Gamma^i_{j0} &= -\frac{N^i\partial_j N}{N} - N\left(\gamma^{ik} - \frac{N^i N^k}{N^2}\right)K_{kj} + D^{(3)}_j N^i \\
\Gamma^i_{jk} &= \Gamma^{(3)i}_{jk} + \frac{N^i}{N}K_{jk} \ .
\end{aligned}
$$

One can also compute the Christoffel's symbols with all the indices lowered. Some useful expressions are:

$$
\begin{aligned}
\Gamma_{0ij} &= \frac{1}{2}(\partial_i N_j + \partial_j N_i - \dot{\gamma}_{ij}) \\
\Gamma_{ij0} = \Gamma_{i0j} &= -NK_{ij} + D^{(3)}_j N_i \\
\Gamma_{ijk} &= \Gamma^{(3)}_{ijk} \ .
\end{aligned}
$$

At this point one wants to compute the Ricci scalar of the ADM metric, defined in 1.7. This calculation is quite involving and we use a trick in order to simplify it, based on the decomposition B.2. Indeed, from the definitions of Ricci tensor and Ricci curvature, using B.2, we have

$$
\begin{aligned}
R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}g^{\alpha\beta}R_{\mu\alpha\nu\beta} &= \gamma^{ik}\gamma^{jk}R_{ijkl} - \gamma^{ik}n^\mu n^\nu R_{i\mu k\nu} - \gamma^{ik}n^\mu n^\nu R_{\mu i\nu k} - n^\mu n^\alpha n^\nu n^\beta R_{\mu\alpha\nu\beta} = \\
&= \gamma^{ik}\gamma^{jk}R_{ijkl} - 2\gamma^{ik}n^\mu n^\nu R_{\mu i\nu k} \ ,
\end{aligned}
\tag{B.4}
$$

where we have also used the symmetry of the Riemann tensor $R_{\mu\alpha\nu\beta} = R_{\alpha\mu\beta\nu}$ and the fact that the term $n^\mu n^\alpha n^\nu n^\beta R_{\mu\alpha\nu\beta}$ is null, since the Riemann tensor, which is antisymmetric in some indices, is contracted with a completely symmetric tensor. So we need to compute explicitly these two pieces.

Let us start with R_{ijkl}. Using the definition of the Riemann tensor 1.8 in terms of the Christoffel's symbols one finds:

$$\begin{aligned}
R_{ijkl} =& g_{i\rho}\partial_k\Gamma^\rho_{lj} - g_{i\rho}\partial_l\Gamma^\rho_{kj} + \Gamma_{ik\rho}\Gamma^\rho_{lj} - \Gamma_{il\rho}\Gamma^\rho_{kj} = \\
=& g_{i0}\partial_k\Gamma^0_{lj} + g_{im}\partial_k\Gamma^m_{lj} + \Gamma_{ik0}\Gamma^0_{lj} + \Gamma_{ikm}\Gamma^m_{lj} - (k \leftrightarrow l) = \\
=& - N_i\partial_k\left(\frac{1}{N}K_{jl}\right) + \gamma_{im}\partial_k\left(\Gamma^{(3)m}_{jl} + \frac{N^m}{N}K_{jl}\right) - \frac{1}{N}K_{jl}\left(-NK_{ik} + D^{(3)}_k N_i\right) \\
&+ \Gamma^{(3)}_{ikm}\left(\Gamma^{(3)m}_{lj} + \frac{N^m}{N}K_{lj}\right) - (k \leftrightarrow l) = \\
=& - \frac{N_i}{N}\partial_k K_{jl} + N_i\frac{\partial_k N}{N^2}K_{jl} + \gamma_{im}\partial_k\Gamma^{(3)m}_{jl} + \gamma_{im}\partial_k\left(\frac{N^m}{N}K_{jl}\right) + K_{jl}K_{ik} - \frac{1}{N}K_{jl}D^{(3)}_k N_i + \\
&+ \Gamma^{(3)}_{ikm}\Gamma^{(3)m}_{lj} + \Gamma^{(3)}_{ikm}\frac{N^m}{N}K_{lj} - (k \leftrightarrow l) = \\
=& - \frac{N_i}{N}\partial_k K_{jl} + N_i\frac{\partial_k N}{N^2}K_{jl} + \gamma_{im}\partial_k\Gamma^{(3)m}_{jl} + \gamma_{im}\frac{\partial_k N^m}{N}K_{jl} + \gamma_{im}\frac{N^m}{N}\partial_k K_{jl} - \gamma_{im}\frac{\partial_k N N^m}{N^2}K_{jl} + \\
&+ K_{jl}K_{ik} - \frac{1}{N}K_{jl}\left(\partial_k N_i - \Gamma^{(3)m}_{ik}N_m\right) + \Gamma^{(3)}_{ikm}\Gamma^{(3)m}_{lj} + \Gamma^{(3)}_{ikm}\frac{N^m}{N}K_{lj} - (k \leftrightarrow l) = \\
=& - \frac{N_i}{N}\partial_k K_{jl} + N_i\frac{\partial_k N}{N^2}K_{jl} + \gamma_{im}\partial_k\Gamma^{(3)m}_{jl} + \frac{\partial_k N_i}{N}K_{jl} + \frac{N_i}{N}\partial_k K_{jl} - \frac{\partial_k N N_i}{N^2}K_{jl} + \\
&+ K_{jl}K_{ik} - \frac{1}{N}K_{jl}\partial_k N_i + \frac{1}{N}K_{jl}\Gamma^{(3)m}_{ik}N_m + \Gamma^{(3)}_{ikm}\Gamma^{(3)m}_{lj} + \Gamma^{(3)}_{ikm}\frac{N^m}{N}K_{lj} - (k \leftrightarrow l) = \\
=& \gamma_{im}\partial_k\Gamma^{(3)m}_{jl} + \Gamma^{(3)}_{ikm}\Gamma^{(3)m}_{lj} + K_{jl}K_{ik} + 2\Gamma^{(3)}_{ikm}\frac{N^m}{N}K_{lj} - (k \leftrightarrow l) \ .
\end{aligned}$$

Now we can define the spatial Riemann tensor $R^{(3)}_{ijkl} := \gamma_{im}\partial_k\Gamma^{(3)m}_{jl} + \Gamma^{(3)}_{ikm}\Gamma^{(3)m}_{lj} - (k \leftrightarrow l)$; notice also that the last term is symmetric under the exchange $k \leftrightarrow l$, since K_{kl} is symmetric, so it cancels out after the antisymmetrization. We are left with:

$$R_{ijkl} = R^{(3)}_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk} \ . \tag{B.5}$$

Now we pass to the second term required to compute the Ricci scalar B.4. Let us start from the following quantity:

$$\begin{aligned}
n_\mu R^\mu_{ijk} =& n_0 R^0_{ijk} + n_l R^l_{ijk} = -NR^0_{ijk} + 0 = -N(\partial_j\Gamma^0_{ki} + \Gamma^0_{j\rho}\Gamma^\rho_{ki}) + (j \leftrightarrow k) = \\
=& - N\partial_j\left(-\frac{1}{N}K_{ki}\right) - N(\Gamma^0_{j0}\Gamma^0_{ki} + \Gamma^0_{jl}\Gamma^l_{ki}) + (j \leftrightarrow k) = \\
=& \partial_j K_{ki} - \frac{\partial_j N}{N}K_{ki} - N\left(\frac{1}{N}(\partial_j N - N^l K_{jl})\left(-\frac{1}{N}\right)K_{ki} + \right. \\
&\left. + \left(-\frac{1}{N}\right)K_{jl}\left(\Gamma^{(3)l}_{ki} + \frac{N^l}{N}K_{ki}\right)\right) + (j \leftrightarrow k) = \\
=& \partial_j K_{ki} - \frac{\partial_j N}{N}K_{ki} + \frac{\partial_j N}{N}K_{ki} - \frac{N^l}{N}K_{jl}K_{ki} + \Gamma^{(3)l}_{ki}K_{jl} + \frac{N^l}{N}K_{jl}K_{ki} + (j \leftrightarrow k) = \\
=& \partial_j K_{ki} + \Gamma^{(3)l}_{ki}K_{jl} + (j \leftrightarrow k) \ ,
\end{aligned}$$

where we have used $n_\mu = (-N, \vec{0})$, the definition of the Riemann tensor 1.8 and the explicit expression of the Christoffel's symbols. Noting that in the result the two summands reconstruct the definition of the 3-dimensional covariant derivative, so in the end we get

$$n_\mu R^\mu_{ilj} = D^{(3)}_l K_{ji} - D^{(3)}_j K_{li} \ . \tag{B.6}$$

We compute now the similar quantity:

$$\begin{aligned}
n_\mu R^\mu_{i0j} &= n_0 R^0_{i0j} + n_k R^k_{i0j} = -N(\partial_0\Gamma^0_{ij} + \Gamma^0_{0\rho}\Gamma^\rho_{ij}) + (0 \leftrightarrow j) = \\
&= -N(\partial_0\Gamma^0_{ij} + \Gamma^0_{00}\Gamma^0_{ij} + \Gamma^0_{0k}\Gamma^k_{ij}) + (0 \leftrightarrow j) \ .
\end{aligned}$$

Inserting here the exact expressions of the Christoffel's symbols requires a longer but analogous calculations, which we skip for shortness. The final result can be nicely rewritten in terms of the spatial covariant derivatives as:

$$n_\mu R^\mu_{i0j} = \dot{K}_{ij} + D^{(3)}_i D^{(3)}_j N + N K^k_i K_{kj} - D^{(3)}_j (K_{ik} N^k) - K_{kj} D^{(3)}_i N^k \; . \tag{B.7}$$

Combining B.7 and B.6 we get

$$\begin{aligned}
&n^\mu n^\nu R_{\mu i \nu j} = n^\nu n_\mu R^\mu_{i\nu j} = n^0 n_\mu R^\mu_{i0j} + n^l n_\mu R^\mu_{ilj} = \\
&= \frac{1}{N}\left(\dot{K}_{ij} + D^{(3)}_i D^{(3)}_j N + N K^k_i K_{kj} - D^{(3)}_j (K_{ik} N^k) - K_{kj} D^{(3)}_i N^k\right) - \frac{N^l}{N}\left(D^{(3)}_l K_{ji} - D^{(3)}_j K_{li}\right) = \\
&= \frac{1}{N}\left(\dot{K}_{ij} + D^{(3)}_i D^{(3)}_j N + N K^k_i K_{kj}\right) - \frac{1}{N}\left(D^{(3)}_j (K_{ik} N^k) + K_{kj} D^{(3)}_i N^k + N^l D^{(3)}_l K_{ji} - N^l D^{(3)}_j K_{li}\right) \; ,
\end{aligned}$$

but the last term inside brackets can be rewritten as

$$\begin{aligned}
&D^{(3)}_j (K_{ik} N^k) + K_{kj} D^{(3)}_i N^k + N^l D^{(3)}_l K_{ji} - N^l D^{(3)}_j K_{li} = \\
&= D^{(3)}_j K_{ik} N^k + K_{ik} D^{(3)}_j N^k + K_{kj} D^{(3)}_i N^k + N^l D^{(3)}_l K_{ji} - N^l D^{(3)}_j K_{li} = \\
&= K_{ik} D^{(3)}_j N^k + K_{kj} D^{(3)}_i N^k + N^l D^{(3)}_l K_{ji} \; .
\end{aligned}$$

Moreover, remembering the expression of Lie derivative in the space coordinates of a rank 2 tensor along the spatial vector $\vec{N}$ (that we derived in appendix A) one has

$$\mathcal{L}_{\vec{N}} K_{ij} = N^l \partial_l K_{ij} + K_{ik} \partial_j N^k + K_{kj} \partial_i N^k = N^l D^{(3)}_l K_{ij} + K_{ik} D^{(3)}_j N^k + K_{kj} D^{(3)}_i N^k \; , \tag{B.8}$$

since in the previous chapter we have seen that all the partial derivatives in its explicit expression can be always replaced by covariant derivative (in this case we consider only the spatial coordinates). This implies that

$$n^\mu n^\nu R_{\mu i \nu j} = \frac{1}{N}\left(\dot{K}_{ij} + D^{(3)}_i D^{(3)}_j N + N K^k_i K_{kj} - \mathcal{L}_{\vec{N}} K_{ij}\right) \; .$$

Using this result and B.5 in the expression for the Ricci scalar B.4 one finds

$$\begin{aligned}
R &= \gamma^{ik}\gamma^{jl} R_{ijkl} - 2\gamma^{ij} n^\mu n^\nu R_{\mu i \nu j} = \\
&= \gamma^{ik}\gamma^{jl}(R^{(3)}_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk}) - 2\gamma^{ij}\frac{1}{N}\left(\dot{K}_{ij} + D^{(3)}_i D^{(3)}_j N + N K^k_i K_{kj} - \mathcal{L}_{\vec{N}} K_{ij}\right) = \\
&= R^{(3)} + K^i_i K^j_j - K^{ij}K_{ij} - \frac{2}{N}\gamma^{ij}\dot{K}_{ij} - \frac{2}{N}\Delta^{(3)} N - 2K^{ij}K_{ij} + \frac{2}{N}\gamma^{ij}\mathcal{L}_{\vec{N}} K_{ij} \; ,
\end{aligned}$$

where we have also defined the spatial scalar curvature $R^{(3)} := \gamma^{ik}\gamma^{jl} R^{(3)}_{ijkl}$ and the spatial covariant Laplacian $\Delta^{(3)} := \gamma^{ij} D^{(3)}_i D^{(3)}_j$. Using the definition B.8 one has

$$\begin{aligned}
\frac{2}{N}\gamma^{ij}\mathcal{L}_{\vec{N}} K_{ij} &= \frac{2}{N}\gamma^{ij}\left(N^l D^{(3)}_l K_{ij} + K_{ik} D^{(3)}_j N^k + K_{kj} D^{(3)}_i N^k\right) = \\
&= \frac{2}{N}\left(N^l D^{(3)}_l K^i_i + 2K^{ij} D^{(3)}_j N_i\right) \; ,
\end{aligned}$$

which gives

$$\begin{aligned}
R &= R^{(3)} + K^i_i K^j_j - 3K^{ij}K_{ij} - \frac{2}{N}\gamma^{ij}\dot{K}_{ij} - \frac{2}{N}\Delta^{(3)} N + \frac{2}{N}\left(N^l D^{(3)}_l K^i_i + 2K^{ij} D^{(3)}_j N_i\right) = \\
&= R^{(3)} + K^{ij}K_{ij} + 2\frac{N^j}{N} D^{(3)}_j K^i_i - \frac{2}{N}\Delta^{(3)} N - \frac{2}{N}\gamma^{ij}\dot{K}_{ij} - 3K^{ij}K_{ij} + \frac{4}{N}K^{ij} D^{(3)}_j N_i \; .
\end{aligned}$$

Moreover the last two summands can be simplified as follows

$$
\begin{aligned}
&-\frac{2}{N}\gamma^{ij}\dot{K}_{ij}-3K^{ij}K_{ij}+\frac{4}{N}K^{ij}D^{(3)}_jN_i=\\
&=-\frac{2}{N}\partial_0(\gamma^{ij}K_{ij})+\frac{2}{N}\dot{\gamma}^{ij}K_{ij}+K^{ij}K_{ij}-4K^{ij}\left(K_{ij}-\frac{1}{N}D^{(3)}_jN_i\right)=\\
&=-\frac{2}{N}\dot{K}^i_i+K^{ij}K_{ij}-4K^{ij}\left(K_{ij}-\frac{1}{2N}D^{(3)}_iN_j-\frac{1}{2N}D^{(3)}_jN_i+\frac{\dot{\gamma}_{ij}}{2N}\right)=\\
&=-\frac{2}{N}\dot{K}^i_i+K^{ij}K_{ij}-4K^{ij}\left(K_{ij}-\frac{1}{2N}(D^{(3)}_iN_j+D^{(3)}_jN_i-\dot{\gamma}_{ij})\right)=\\
&=-\frac{2}{N}\dot{K}^i_i+K^{ij}K_{ij}-4K^{ij}\left(K_{ij}-K_{ij}\right)=-\frac{2}{N}\dot{K}^i_i+K^{ij}K_{ij}\ ,
\end{aligned}
$$

where we have used the symmetry of K_{ij} and its value B.3. The final result is finally

$$
R=R^{(3)}+K^{ij}K_{ij}+K^i_iK^j_j-\frac{2}{N}\dot{K}^i_i+2\frac{N^j}{N}D^{(3)}_jK^i_i-\frac{2}{N}\Delta^{(3)}N\ .
$$

This is not all, since we want to derive the Einstein-Hilbert density $\sqrt{-g}R$ and fully simplify it. We know that $\sqrt{-g}=N\sqrt{\gamma}$ and we split the result as follows:

$$
\begin{aligned}
\sqrt{-g}R=&N\sqrt{\gamma}\left(R+K^{ij}K_{ij}+K^i_iK^j_j-\frac{2}{N}\dot{K}^i_i+2\frac{N^j}{N}D^{(3)}_jK^i_i-\frac{2}{N}\Delta^{(3)}N\right)=\\
=&N\sqrt{\gamma}\left(R+K^{ij}K_{ij}-K^i_iK^j_j\right)+\\
&+N\sqrt{\gamma}\left(2K^i_iK^j_j-\frac{2}{N}\dot{K}^i_i+2\frac{N^j}{N}D^{(3)}_jK^i_i\right)-2\sqrt{\gamma}\Delta^{(3)}N\ .
\end{aligned}
\tag{B.9}
$$

We remind now that boundary terms in the Lagrangian are of type $\partial_\mu j^\mu$ and they do not contribute to equations of motion, so they can be ruled out after the spacetime integration. Moreover, using 4.6, terms of type $\sqrt{-g}D_\mu j^\mu$ are boundary terms, since for 4.6

$$
\sqrt{-g}D_\mu j^\mu=\sqrt{-g}\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}j^\mu)=\partial_\mu(\sqrt{-g}j^\mu)
\tag{B.10}
$$

and for space coordinates only one has similarly

$$
\sqrt{\gamma}D^{(3)}_ij^i=\sqrt{\gamma}\frac{1}{\sqrt{\gamma}}\partial_i(\sqrt{\gamma}j^i)=\partial_i(\sqrt{\gamma}j^i)\ .
\tag{B.11}
$$

The last summand in B.9 is a boundary term for this last case. We now want to show that also the second summand is a boundary term. Indeed

$$
\begin{aligned}
&N\sqrt{\gamma}\left(2K^i_iK^j_j-\frac{2}{N}\dot{K}^i_i+2\frac{N^j}{N}D^{(3)}_jK^i_i\right)=-2\sqrt{\gamma}\dot{K}^i_i+2\sqrt{\gamma}N^jD^{(3)}_jK^i_i+2\sqrt{\gamma}NK^l_lK^i_i=\\
&=-2\sqrt{\gamma}\dot{K}^i_i+2\sqrt{\gamma}N^jD^{(3)}_jK^i_i+2\sqrt{\gamma}NK^l_l\gamma^{ij}K_{ij}=\\
&=-2\sqrt{\gamma}\dot{K}^i_i+2\sqrt{\gamma}N^jD^{(3)}_jK^i_i+2\sqrt{\gamma}NK^l_l\gamma^{ij}\frac{1}{2N}(D^{(3)}_iN_j+D^{(3)}_jN_i-\dot{\gamma}_{ij})\\
&=-2\sqrt{\gamma}\dot{K}^i_i+2\sqrt{\gamma}N^jD^{(3)}_jK^i_i+\sqrt{\gamma}K^l_l\gamma^{ij}(2D^{(3)}_iN_j-\dot{\gamma}_{ij})=\\
&=-2\sqrt{\gamma}\dot{K}^i_i-\sqrt{\gamma}K^l_l\gamma^{ij}\dot{\gamma}_{ij}+2\sqrt{\gamma}N^jD^{(3)}_jK^i_i+2\sqrt{\gamma}K^l_l\gamma^{ij}D^{(3)}_iN_j=\\
&=-2\sqrt{\gamma}\dot{K}^i_i-2K^l_l\partial_0\sqrt{\gamma}+2\sqrt{\gamma}D^{(3)}_j(K^i_iN^j)=-2\partial_0(\sqrt{\gamma}K^i_i)+2\sqrt{\gamma}D^{(3)}_j(K^i_iN^j)
\end{aligned}
$$

where we have used B.3 and the formula $\partial_0\sqrt{\gamma} = \frac{\sqrt{\gamma}}{2}\gamma^{ij}\dot{\gamma}_{ij}$. Using 4.6, B.11, $\sqrt{-g} = N\sqrt{\gamma}$ and $n_\mu = (-N, \vec{0})$, one has

$$\begin{aligned}
-2\sqrt{-g}D_\mu\left(K^i_i n^\mu\right) &= -2\sqrt{-g}\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}K^i_i n^\mu) = -2\partial_\mu(N\sqrt{\gamma}K^i_i g^{\mu\nu}n_\nu) = \\
&= -2\partial_\mu(N\sqrt{\gamma}K^i_i g^{\mu 0}n_0) + 0 = -2\partial_0(N\sqrt{\gamma}K^i_i g^{00}n_0) - 2\partial_j(N\sqrt{\gamma}K^i_i g^{j0}n_0) = \\
&= -2\partial_0(N\sqrt{\gamma}K^i_i\frac{-1}{N^2}(-N)) - 2\partial_j\left(N\sqrt{\gamma}K^i_i\frac{N^j}{N^2}(-N)\right) = \\
&= -2\partial_0(\sqrt{\gamma}K^i_i) + 2\partial_j(\sqrt{\gamma}K^i_i N^j) = -2\partial_0(\sqrt{\gamma}K^i_i) + 2\sqrt{\gamma}\frac{1}{\sqrt{\gamma}}\partial_j(\sqrt{\gamma}K^i_i N^j) = \\
&= -2\partial_0(\sqrt{\gamma}K^i_i) + 2\sqrt{\gamma}D^{(3)}_j(K^i_i N^j)\ ,
\end{aligned}$$

which is exactly the second summand in the Lagrangian, which for B.10 is a boundary term. This way, integrating over the spacetime elements one can rule out all the boundary terms, getting finally the simple result

$$S_{ADM} = \int dtd^3x\sqrt{\gamma}N\left(R^{(3)} + K^{ij}K_{ij} - K^i_i K^j_j\right)\ ,$$

which is equation 4.63.
From this action we see that only the time derivatives of the six elements of γ_{ij} (inside K_{ij}) appear in the Lagrangian, so they are dynamical. On the contrary, the lapse N and the shift N_i play the role of Lagrange multipliers. Calculating the equation of motion relative to N and N_i one does not find any dynamical equation, but four constraints corresponding to the diffeomorphism invariance of the theory. This implies that the total number of physical degrees of freedom is 2, which is a well known result, usually discussed in gravitational wave physics.

B.3 A general slicing

In the previous construction we have explicitly chosen $n_\mu = (-N, \vec{0})$. This is because we decided to slice the spacetime along the hypersurfaces at t constant. However, one could in principle choose to slice the spacetime using the set of hypersurfaces

$$S(x) = \text{constant}\ ,$$

where S is a general scalar. In this case the versor orthogonal to the hypersurface is given by

$$n_\mu = \frac{\partial_\mu S}{\sqrt{-g^{\alpha\beta}\partial_\alpha S\partial_\beta S}}\ .$$

In case $S(x) = x^\mu v_\mu$ in the reference frame where $v_\mu = (-1, \vec{0})$, one gest back the previous results since $S(x) = -t$ so the slicing is given by $t = constant$ and one has

$$n_0 = -\frac{\partial_0 t}{\sqrt{-g^{00}\partial_0 t\partial_0 t}} = -N\ , \quad n_i = 0\ .$$

In the general case the instrinsic curvature is $K_{ij} = D_i n_j$ and using the differential geometry of the hypersurfaces one can show that the Ricci scalar can be rewritten as[1]

$$R_n = R = R^{(3)} + K^{ij}K_{ij} - K^i_i K^j_j - 2D_\alpha(n^\beta D_\beta n^\alpha - n^\alpha D_\beta n^\beta)\ .$$

Notice that the last summand is a boundary term in the Einstein-Hilbert action.

[1] See [132], eq. 3.5.7, where $\epsilon = -1$.

B.4 Perturbed FLRW

In case of a FLRW metric with $\kappa = 0$, computing all the Christoffel's symbols one can show[2] that the spatial curvature is zero, $R^{(3)} = 0$. This is not valid in the case of a perturbed FLRW. The result is

$$R^{(3)} = \frac{4}{a^2}\nabla^2\left(\Psi + \frac{1}{6}\nabla^2\chi^{\parallel}\right) .$$

In the case of a general slicing, assuming to split the scalar S into a background and a first-order perturbation

$$S = S_0(t) + S^{(1)}(x) + \dots$$

one can compute that at first-order in perturbations one has

$$R_n = \frac{4}{a^2}\nabla^2\left(\frac{\mathcal{H}}{S_0'}S^{(1)} + \Psi + \frac{1}{6}\nabla^2\chi_{\parallel}\right) .$$

A little remark is about the case in which the perturbed metric is defined with $\partial_i\partial_j\chi^{\parallel}$ instead $\hat{D}_{ij}\chi^{\parallel}$, as we have commented in section 2.2. In this case one has simply ([91])

$$\begin{aligned} R^{(3)} &= \frac{4}{a^2}\nabla^2\Psi \\ R_n &= \frac{4}{a^2}\nabla^2\left(\frac{\mathcal{H}}{S_0'}S^{(1)} + \Psi\right) . \end{aligned}$$

[2]For example using a computer program such as Mathematica.

Appendix C

CFC coordinates

In this appendix we introduce the conformal Fermi coordinates discussed in this project, deriving also the results we have used.
As we have introduced in section 1.1, the basic principle of GR is the equivalence principle, which imposes to assume local Lorentz invariance, that is the impossibility to detect gravity doing a local non-gravitational experiment. This implies to construct a local set of coordinates around an arbitrary point P, where the metric looks like Minkowski in the neighborhood of the point chosen. This is called local intertial frame (LIF). As we have seen, mathematically this translates into imposing that at the point P in this set of coordinates the metric is such that

$$g_{\mu\nu} = \eta_{\mu\nu} \ , \quad \partial_\rho g_{\mu\nu} = 0 \ , \tag{C.1}$$

which implies that its Taylor expansion is 1.1, $g_{\mu\nu}(x) = \eta_{\mu\nu} + \mathcal{O}((x - x_P)^2)$. This means that the space is flat up to higher order corrections in the neighborhood of the point P chosen. If such a frame existed, it would be always possible to go in this frame, as a consequence of the invariance under diffeomorphisms of GR, which ensures that physics is invariant, no matter the frame of reference chosen. However, one can show that this frame always exists: the demonstration reduces in counting the degrees of freedom in the metric tensor and in a general diffeomorphism ([3]). This system of coordinates is called Riemann normal coordinates.
Moreover, in some circumstances, it is useful to go into a reference frame where the two conditions above are respected into an entire worldline. This is for example what happens in all the experiments where gravitational effects are negligible. This frame of reference is called Fermi normal coordinates (FNC), which we are going to introduce in the next section and which will be the starting point to introduce the conformal Fermi coordinates (CFC).

Before going into the discussion, we remind some technicalities about Cartan formalism (or tetrad formalism), which we are going to use. Basing on what we have just said, it is always possible to find a set of local coordinates in each point in which the metric is Minkowski. We parametrize the change as e^a_μ, called *vielbein* or *tetrad*, such that $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, so the metric is locally flat.

C.1 Fermi normal coordinates

In this section we introduce Fermi normal coordinates ([83]). The starting point is a free falling observer moving along a timelike geodesic $h(\gamma)$, parametrized by γ affine parameter: we have seen that we can introduce the orthonormal set $e^\mu_a = (e^\mu_0, e^\mu_1, e^\mu_2, e^\mu_3)$ point by point, with e_0 timelike and e_i spacelike; moreover, we can parallely transport it along h. We impose e_0 to be tangent to the geodesic at its origin, so parallely transporting it will remain tangent for all values of the affine parameter γ.
The global coordinate frame is the one covering the whole spacetime manifold (or at least a sufficiently large neighborhood of a given point P): we will indicate it by means of x^μ_G; it is not restrictive to assume $x^i_G = 0$, that is to say that the tangent to the geodesic coincide everywhere to the coordinate

axis x_G^0 (however, in this reference frame it is not ensured that the space is flat along the geodesic). We fix now a point on the geodesics, choosing the affine parameter $P = h(\gamma_0)$: let $\mathfrak{U}$ be a neighborhood of all the spacetime points around P (not necessarily along the geodesics) and Q an arbitrary point in $\mathfrak{U}$. Our purpose is to describe the spacetime in $\mathfrak{U}$: Q can be connected to P with a geodesic $g(\lambda)$ that is perpendicular to the tangent vector of h in P. This implies that the tangent vector $\vec{v}$ to $g(\lambda)$ at P is a linear combination of the e_i.
Given this construction we can define the Fermi coordinates x_L^μ: the time component is chosen to be the proper time τ of the observer moving along h, while the space components are the coefficients of the linear combination of e_i. We can always rescale the coordinates in such a way that Q corresponds to $\lambda = 1$ along the geodesic g.
For what we have said, the initial conditions for the geodesic connecting P and Q are

$$x_G^i(\lambda = 0) = 0 \ , \quad \tau(\lambda = 0) = t_L \ , \quad \left.\frac{dx_G^i}{d\lambda}\right|_{\lambda=0} = x_L^i e_i\left(t_0\right) \ , \quad \left.\frac{dt_G}{d\lambda}\right|_{\lambda=0} = 0 \ .$$

The point $Q \in \mathfrak{U}$ with Fermi coordinates x_L^μ is then found by propagating along $g(\lambda)$ until $\lambda = 1$.
At this point we want to find the mapping between arbitrary coordinates which we assume to be the "global" ones x_G^μ and the Fermi coordinates by explicitly finding the geodesic $g(\lambda)$. This means that we have to solve the geodesic equation for $g(\lambda)$

$$\frac{d^2 x_G^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx_G^\alpha}{d\lambda}\frac{dx_G^\beta}{d\lambda} = 0 \ .$$

This equation is obviously very difficult to be solved exactly, but we can solve it perturbatively using a power law as

$$x_G^\mu(\lambda) = \alpha_0^\mu + \alpha_1^\mu\lambda + \alpha_2^\mu\lambda^2 + \alpha_3^\mu\lambda^3 + \dots \ .$$

However, this expansion has meaning only in a neighborhood of the central geodesic h, where $\lambda < 1$. To find explicitly the first values of the various α's, from the initial conditions one gets trivially the first two:

$$\begin{aligned}\alpha_0^\mu &= \left(t_0, \vec{0}\right) \ , \\ \alpha_1^\mu &= \left.\frac{dx_G^\mu}{d\lambda}\right|_{\lambda=0} = x_L^i e_i^\mu \ ,\end{aligned}$$

where t_0 is the coordinate time corresponding to γ_0. The coefficients of the second-order term can be derived directly from the geodesic equation:

$$\alpha_2^\mu = \left.\frac{1}{2!}\frac{d^2 x_G^\mu}{d\lambda^2}\right|_{\lambda=0} = -\left.\frac{1}{2}\Gamma^\mu_{\gamma\nu}\frac{dx_G^\gamma}{d\lambda}\frac{dx_G^\nu}{d\lambda}\right|_{\lambda=0} = -\frac{1}{2}\Gamma^\mu_{\gamma\nu}\alpha_1^\gamma\alpha_1^\nu \ ,$$

where we used the initial conditions. Finally, the third order term follows deriving the geodesic equation and using the initial conditions exactly in the same way:

$$\alpha_3^\mu = \left.\frac{1}{6!}\frac{d^3 x_G^\mu}{d\lambda^3}\right|_{\lambda=0} = -\frac{1}{6}\left(\frac{\partial\Gamma^\mu_{\gamma\nu}}{\partial x^\kappa}\alpha_1^\gamma\alpha_1^\nu\alpha_1^\kappa + 4\Gamma^\mu_{\gamma\nu}\alpha_1^\gamma\alpha_2^\nu\right) \ .$$

Before passing to the CFC, let us comment the reason why they have been introduced. In the case of FLRW spacetime with $\kappa = 1$

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j \ ,$$

one can show[1] that the metric in Fermi normal coordinates becomes

$$ds^2 = -\left[1 - \left(\dot{H}\left(t_L\right) + H^2\left(t_L\right)\right)\vec{x}_L^2\right]dt_L^2 + \left[1 - H^2\left(t_L\right)\frac{\vec{x}_L^2}{2}\right]d\vec{x}_L^2 \ .$$

with $H = \frac{1}{a}\frac{da}{dt}$ the Hubble parameter. However, as we have seen, these coordinates are valid around the central geodesic; more specifically, they are only valid on scales that are much smaller than the horizon ($r_H \propto H^{-1}$, see eq. 4.1), since they appear as a perturbative expansion in Hx_F^i: when $x_F^i \sim r_H \sim H^{-1}$, this quantity becomes order one and the perturbative description breaks down. We will see that this problem disappears using conformal Fermi coordinates.

[1]See [83], appendix B.

C.2 Conformal Fermi coordinates

Conformal Fermi coordinates were introduced firstly in [78], then formalized in [81]. As before, we want to construct a system of coordinates in the vicinity of the central geodesics, but we relax the hypothesis for the local spacetime to be Minkowski; on the contrary, we require the space to be locally homogeneous in time, that is we want the metric to be of type

$$g^F_{\mu\nu} = a_F^2(\tau_F)(\eta_{\mu\nu} + h^F_{\mu\nu}(\tau_F, x^i_F)) \ , \quad h^F_{\mu\nu} = \mathcal{O}[(x^i_F)^2] \ .$$

This means that we are substituting the condition C.1 with the less restricting one that the metric is Minkowski up to a conformal factor $a_F^2(\tau_F)$; notice that corrections to the conformally flat part start at quadratic order in x^i_F, as in FNC.
Then, one can proceed as in the case of FNC, but some subtleties arise. Firstly, the CFC time τ_F should be some suitable conformal time rather than the observer's proper time. There are also problems in defining correctly the slicing of the spacetime at constant τ_F. Moreover, a suitable local scale factor $a_F(\tau_F)$ should be defined in a physical way: in particular, if $g_{\mu\nu}$ describes an unperturbed FLRW metric (but given in some unusual coordinates), then for consistency the CFC construction should yield the metric in the canonical FLRW form.

As before, we indicate as global coordinates the ones valid in the whole region surrounding the geodesic considered. The geometrical situation is the same of the one considered for the FNC: we have a geodesic h, which the point P belong to, and another point Q in its neighborhood (see figure C.1). The starting point to construct CFC is the same tetrad e^μ_P as in the construction of the FNC; by now, we parametrize the central geodesic with the proper time t_L as before. Introducing a spacetime scalar $a_F(x)$, which we require to be positive at least in a finite region around the central geodesic (playing the role of the local scale factor), we can define a "conformal proper time" τ_F through

$$d\tau_F = a_F^{-1}\left(P\left(t_F\right)\right) dt_F \ , \tag{C.2}$$

where $P\left(t_F\right)$ is the point along the central geodesic at proper time t_F. Integrating this equation in t_F one gets a unique relation $\tau_F\left(t_F\right)$ (up to an integration constant which can be reabsorbed into a redifinition of a_F). As we have said, we choose τ_F as time coordinate. For sake of shortness, we will often write $a_F\left(\tau_F\right)$ instead of $a_F\left(P\left(\tau_F\right)\right)$. However, a_F and $\tau_F\left(t_F\right)$ depend on the specific geodesic under consideration. Now we need to define the slices of constant τ_F, but as we have anticipated this requires some care. Indeed, proceeding as in the FNC construction, one has to trace out spatial geodesics orthogonal to the central geodesic, but these are not geodesics for a homogeneous flat FLRW spacetime. On the contrary, they are geodesics with respect to the conformal transformed metric $\eta_{\mu\nu} = a_F^{-2}(x)g_{\mu\nu}$. As a consequence, the hypersurfaces of constant τ_F in CFC, which we call Σ_F, should be spanned by space-like conformal geodesics, namely geodesics with respect to the conformal transformed metric

$$\tilde{g}_{\mu\nu}(x) = a_F^{-2}(x)g_{\mu\nu}(x) \ . \tag{C.3}$$

From now on, a tilde denotes quantities defined with respect to this conformal transformed metric. Notice that for a perturbed FLRW metric (in conformal time) $g_{\mu\nu} = a^2\left(\eta_{\mu\nu} + h_{\mu\nu}\right)$, in general $a_F \neq a$ so $\tilde{g}_{\mu\nu} \neq \eta_{\mu\nu} + h_{\mu\nu}$.

To fix ideas, we repeat schematically the procedure to establish the CFC coordinates $\left(\tau_F, x^i_F\right)$.

1. Consider the central geodesic h and find the point P corresponding to the CFC time τ_F using C.2. This point thus has CFC coordinates $\left(\tau_F, \vec{0}\right)$.

2. We introduce a family of geodesics $\tilde{g}\left(\tau_F; \alpha^i; \lambda\right)$, geodesics with respect to the conformal metric $\tilde{g}_{\mu\nu}$ defined through C.3, passing through P. The affine parameter of this geodesics is λ, with $\lambda = 0$ corresponding to the point in P. As in the case of FNC, we impose that the tangent

vector at P is given by a linear combination of e_i up to the conformal factor,

$$\left.\frac{dx_G^\mu}{d\lambda}\right|_{\lambda=0} = a_F(P)\alpha^i\,(e_i)_P^\mu \ ,$$

with α^i constants specifying the initial direction of the geodesic and λ measures the geodesic distance with respect to the conformal metric (up to a constant factor). As before the α's constitute the spatial CFC: $\alpha^i = x_F^i$.

3. Consider a point Q in the neighborhood of h: let Q be located on the conformal geodesic $\tilde{g}\left(\tau_F; a_F(P)\beta^i; \lambda\right)$. Choosing $\lambda = \sqrt{\delta_{ij}x_F^i x_F^j}$ and $\beta^i = x_F^i/\sqrt{\delta_{ij}x_F^i x_F^j}$ ensures that the proper distance squared from P to Q is $a_F^2\delta_{ij}x_F^i x_F^j$ at lowest order.

This uniquely specifies the CFC coordinates. As in the case of FNC, this set of coordinates is guaranteed to be regular in a finite tubelike region around the central geodesic (note that $a_F > 0$ is a necessary condition), bounded by a hypersurface which we call B.

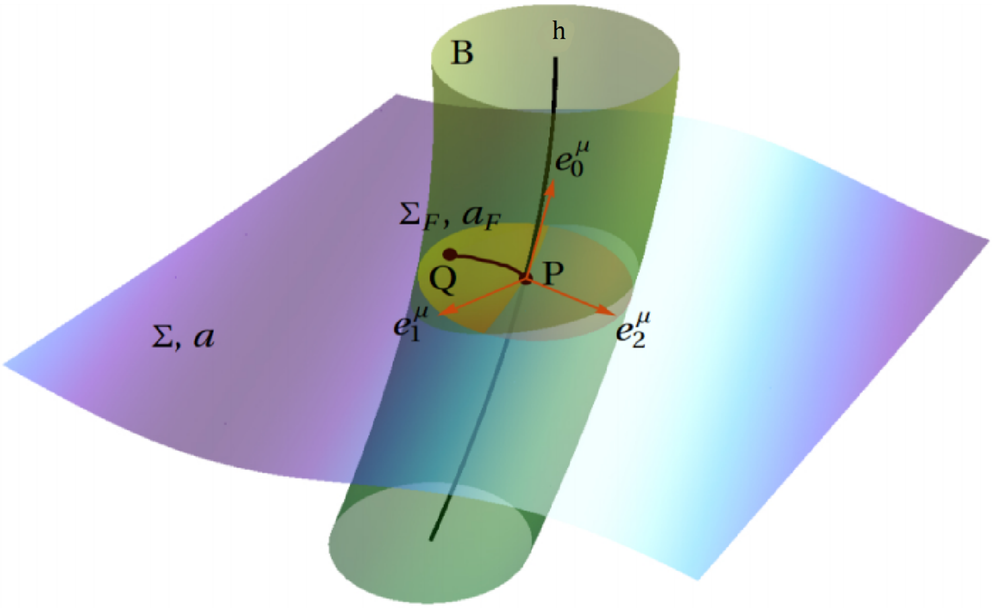

Figure C.1: Figure taken from [81], depicting schematically the CFC construction. Σ is the spatial hypersurface having constant conformal time τ and scale factor $a(\tau)$, intersecting the observer's geodesics h at point P; the same way, the spatial hypersurface Σ_F, having constant CFC conformal τ_F and CFC scale factor $a_F(\tau_F)$ intersects h at P too. However, as we have said in the main text, these two hypersurfaces do not coincide in general.

As before, to find explicitly the coordinates we have to solve the geodesic equation, but this time we have to consider the Christoffel's symbols of the conformal transformed metric:

$$\frac{d^2x_G^\mu}{d\lambda^2} + \tilde{\Gamma}^\mu_{\alpha\beta}\frac{dx_G^\alpha}{d\lambda}\frac{dx_G^\beta}{d\lambda} = 0 \ .$$

As a consequence, we have to find an explicit expression for the various $\tilde{\Gamma}$'s. Christoffel's symbols are obtained through the formula 1.3, where the metric is $\tilde{g} = a_F^{-2}g$. This implies that

$$\begin{aligned}
\tilde{\Gamma}^\mu_{\nu\rho} &= \frac{1}{2}\tilde{g}^{\mu\sigma}(\partial_\nu\tilde{g}_{\rho\sigma} + \partial_\rho\tilde{g}_{\sigma\nu} - \partial_\sigma\tilde{g}_{\nu\rho}) = \\
&= \frac{1}{2}g^{\mu\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}) - g^{\mu\sigma}\left(\frac{\partial_\nu a_F}{a_F}g_{\rho\sigma} + \frac{\partial_\rho a_F}{a_F}g_{\sigma\nu} - \frac{\partial_\sigma a_F}{a_F}g_{\nu\rho}\right) = \\
&= \Gamma^\mu_{\nu\rho} - \left(\delta^\mu_\rho\partial_\nu\log a_F + \delta^\mu_\nu\partial_\rho\log a_F - g^{\mu\sigma}g_{\nu\rho}\partial_\sigma\log a_F\right) \ .
\end{aligned}$$

This is the relation between the Christoffel's symbols of $g_{\mu\nu}$ and the ones of the conformal transformed metric $\tilde{g}_{\mu\nu}$. At this point, as in the case of FNC, we would like to solve the geodesic equation perturbatively as

$$x^\mu_G(\lambda) = \sum_{n=0}^{\infty} \alpha^\mu_n \lambda^n \ .$$

This curve connects point P, having CFC $x^\mu_F(P) = \left(\tau_F, \vec{0}\right)$ as we said, with point Q having CFC $x^\mu_F(Q) = \left(\tau_F, x^i_F\right)$. Since P is chosen as the spatial origin, we have

$$\alpha^\mu_0 = x^\mu(P) = \left(\tau_F, \vec{0}\right) \ .$$

As in the case of FNC, rescaling λ so that it runs from $\lambda = 0$ at P to $\lambda = 1$ at Q, the tangent vector at $\lambda = 0$ is specified by x^i_F through

$$\alpha^\mu_1 = \left.\frac{dx^\mu_G}{d\lambda}\right|_{\lambda=0} = a_F(P)\,(e_i)^\mu_P\, x^i_F \ .$$

Furthermore, higher-order coefficients α^μ_n can then be recursively computed using the geodesic equation, as in case of FNC. The second-order is

$$\alpha^\mu_2 = \frac{1}{2}\left.\frac{d^2 x^\mu_G}{d\lambda^2}\right|_P = -\frac{1}{2}\tilde{\Gamma}^\mu_{\alpha\beta}\left.\frac{dx^\alpha_G}{d\lambda}\frac{dx^\beta_G}{d\lambda}\right|_P = -\frac{1}{2}a^2_F(P)\left(\tilde{\Gamma}^\mu_{\alpha\beta}\right)_P (e_i)^\alpha_P\,(e_j)^\beta_P\, x^i_F x^j_F \ .$$

The third order, as before, can be obtained through a derivation:

$$\begin{aligned}
\alpha^\mu_3 &= \frac{1}{6}\left.\frac{d^3 x^\mu_G}{d\lambda^3}\right|_P = -\frac{1}{6}\frac{d}{d\lambda}\left(\tilde{\Gamma}^\mu_{\alpha\beta}\frac{dx^\alpha_G}{d\lambda}\frac{dx^\beta_G}{d\lambda}\right)_P \\
&= -\frac{1}{6}a^3_F(P)\left(\partial_\gamma\tilde{\Gamma}^\mu_{\alpha\beta} - 2\tilde{\Gamma}^\mu_{\sigma\alpha}\tilde{\Gamma}^\sigma_{\beta\gamma}\right)_P (e_i)^\alpha_P\,(e_j)^\beta_P\,(e_k)^\gamma_P\, x^i_F x^j_F x^k_F \ .
\end{aligned}$$

Finally, the transformation from the CFC frame to the global coordinates, expanded to third order in x^i_F, is then given by

$$\begin{aligned}
\Delta x^\mu :=& x^\mu_G(Q) - x^\mu_G(P) = a_F(P)\,(e_i)^\mu_P\, x^i_F - \frac{1}{2}\left(\tilde{\Gamma}^\mu_{\alpha\beta}\right)_P a^2_F(P)\,(e_i)^\alpha_P\,(e_j)^\beta_P\, x^i_F x^j_F \\
&- \frac{1}{6}\left(\partial_\gamma\tilde{\Gamma}^\mu_{\alpha\beta} - 2\tilde{\Gamma}^\mu_{\sigma\alpha}\tilde{\Gamma}^\sigma_{\beta\gamma}\right)_P a^3_F(P)\,(e_i)^\alpha_P\,(e_j)^\beta_P\,(e_k)^\gamma_P\, x^i_F x^j_F x^k_F + \mathcal{O}\left[\left(x^i_F\right)^4\right] \ .
\end{aligned} \tag{C.4}$$

C.3 Perturbed FLRW

In this section we want to apply the results obtained to the case of a perturbed FLRW, with the final purpose to find the relations used in the project. We have to consider the perturbed FLRW metric 2.7 at first-order, which reads

$$ds^2 = a^2(\tau)\left[-(1+2\Phi)d\tau^2 + 2V_i dx^i d\tau + (1-2\Psi)\delta_{ij}dx^i dx^j + T_{ij}dx^i dx^j\right] \ ,$$

where we have set $V_i = \partial_i\omega_\parallel + \omega^\perp_i$ and $T_{ij} = \partial_i\partial_j\chi_\parallel + \partial_i\chi^\perp_j + \partial_j\chi^\perp_i + D_{ij}$. Since we work in ζ-gauge, we set $\chi_\parallel = 0$ and $\chi^\perp_i = 0$, so that $T_{ij} = D_{ij}$.
We work at first-order in perturbations, so scalar, vector and tensor perturbations are always decoupled. This way, to avoid writing cumbersome expressions, we give separately the results firstly including only scalars and then including only tensors (we are not interested in vectors as usual).

Including scalars

At this level we are not interested in tensor perturbations so we will neglect D_{ij}. The metric is reproduced by the following vielbeins (neglecting non-linear order in the perturbations as usual)

$$e^0_\mu = a(\tau)\,(1+\Phi, -V_j) \ , \quad e^i_\mu = a(\tau)\left(0, 1-\Psi\delta^i_j\right) \ ,$$

as one can prove by a direct verification:

$$\begin{aligned}
g_{00} &= e^a_0 e^b_0 \eta_{ab} = e^0_0 e^0_0(-1) + e^i_0 e^j_0 \delta_{ij} = -a^2(\tau)(1+\Phi)^2 = -a^2(\tau)(1+2\Phi) \\
g_{0k} &= e^a_0 e^b_k \eta_{ab} = e^0_0 e^0_k(-1) + e^i_0 e^j_k \delta_{ij} = -a^2(\tau)(1+\Phi)(-V_k) = a^2(\tau) V_k \\
g_{kl} &= e^a_k e^b_l \eta_{ab} = e^0_k e^0_l(-1) + e^i_k e^j_l \delta_{ij} = -a^2(\tau) V_k V_l + a^2(\tau)(1-\Psi)^2 \delta_{kl} = a^2(\tau)(1-2\Psi)\delta_{kl} \ .
\end{aligned}$$

In the expression C.4 one needs the inverse vielbeins e^μ_a. Since we know that $e^a_\mu e^\mu_b = \delta^a_b$, this implies to invert (perturbatively) the matrix e^a_μ. It is trivial to show that the right choice is

$$e^\mu_0 = \frac{1}{a(\tau)}\,(1-\Phi, V_j) \ , \quad e^\mu_i = \frac{1}{a(\tau)}\left(0, (1+\Psi)\delta^i_j\right) \ .$$

For sake of shortness, we indicate with $\mathcal{O}(p)$ terms which are linear in the perturbations, so $\mathcal{O}(p) := \mathcal{O}(\Phi, \Psi, V_i)$. We will need also the relation $a_F(\tau_F)$ and $a(\tau)$. It can be shown[2] that

$$\frac{a_F(\tau_F)}{a(\tau)} = 1 + \mathcal{O}(p) \ .$$

This is expected since for null perturbations one gets the unperturbed FLRW and CFC are constructed in such a way FLRW metric is untouched by the transformation (the construction must regive FLRW itself): if we set to zero the perturbations, we must have $a_F = a$. We will need also $\tilde{\Gamma}$, which are the Christoffel's symbols of the conformally transformed metric, which in this case is simply (neglecting tensor perturbations)

$$d\tilde{s}^2 = -(1+2\Phi)d\tau^2 + 2V_i dx^i d\tau + (1-2\Psi)\delta_{ij}dx^i dx^j + D_{ij}dx^i dx^j \ .$$

A direct computation at first-order gives

$$\tilde{\Gamma}^k_{ij} = \frac{1}{2}\tilde{g}^{k\mu}(\partial_i \tilde{g}_{\mu j} + \partial_j \tilde{g}_{\mu i} - \partial_\mu \tilde{g}_{ij}) = -(\delta^k_j \partial_i \Psi + \delta^k_i \partial_j \Psi - \delta_{ij}\partial^k \Psi) \ . \tag{C.5}$$

Notice that since $\tilde{g}_{ij} = \delta_{ij}$, spatial indices can be uppered and lowered freely in dealing with quantities in the CFC frame. To conform with conventions of [91], we call $\Delta x^\mu = x^\mu_G(Q) - x^\mu_G(P)$, that is the deviation from the central wordline, and $\Delta x_F = x_F$. At this point we are ready to compute the various terms appearing C.4.

$$a_F(P)\,(e_i)^k_P\, x^i_F = \frac{a_F}{a}(1+\Psi)\bigg|_P \delta^k_i \Delta x^i_F = \Delta x^k_F + \Psi\bigg|_P \Delta x^k_F \ ,$$

$$\begin{aligned}
&\left(\tilde{\Gamma}^k_{\alpha\beta}\right)_P a^2_F(P)\,(e_i)^\alpha_P\,(e_j)^\beta_P\, x^i_F x^j_F = \\
&= -(\delta^k_l \partial_m \Psi + \delta^k_m \partial_l \Psi - \delta_{ml}\partial^k \Psi)\bigg|_P \frac{a^2_F}{a^2}\,(\delta^m_i + \mathcal{O}(p))\,(\delta^l_j + \mathcal{O}(p))\Delta x^i_F \Delta x^j_F = \\
&= -(\delta^k_j \partial_i \Psi + \delta^k_i \partial_j \Psi - \delta_{ij}\partial^k \Psi)\bigg|_P \Delta x^i_F \Delta x^j_F
\end{aligned}$$

and similarly

$$\begin{aligned}
&\left(\partial_\gamma \tilde{\Gamma}^k_{\alpha\beta} - 2\tilde{\Gamma}^k_{\sigma\alpha}\tilde{\Gamma}^\sigma_{\beta\gamma}\right)_P a^3_F(P)\,(e_i)^\alpha_P\,(e_j)^\beta_P\,(e_m)^\gamma_P\, x^i_F x^j_F x^m_F = \\
&= \left(\partial_g \tilde{\Gamma}^k_{ab} + \mathcal{O}(p)\right)\bigg|_P \frac{a^3_F(P)}{a^3}\,(\delta^a_i + \mathcal{O}(p))\,(\delta^b_j + \mathcal{O}(p))\,(\delta^g_m + \mathcal{O}(p))\,\Delta x^i_F \Delta x^j_F \Delta x^m_F = \\
&= \partial_m \tilde{\Gamma}^k_{ij}\bigg|_P \Delta x^i \Delta x^j \Delta x^m = -(\delta^k_j \partial_m \partial_i \Psi + \delta^k_i \partial_m \partial_j \Psi - \delta_{ij}\partial_m \partial^k \Psi)\bigg|_P \Delta x^i_F \Delta x^j_F \Delta x^m_F \ .
\end{aligned}$$

[2] See [81], section 4.

Putting all together in C.4 one gets

$$\begin{aligned}\Delta x^k =& \Delta x_F^k + \Delta x_F^k \Psi\Big|_P + \frac{1}{2}\Delta x_F^i \Delta x_F^j (\delta_j^k \partial_i \Psi + \delta_i^k \partial_j \Psi - \delta_{ij}\partial^k \Psi)\Big|_P + \\ &+ \frac{1}{6}\Delta x_F^i \Delta x_F^j \Delta x_F^l (\delta_j^k \partial_l \partial_i \Psi + \delta_i^k \partial_l \partial_j \Psi - \delta_{ij}\partial_l \partial^k \Psi)\Big|_P \ .\end{aligned}$$

However, we will need $x_F^k = \Delta x_F^k$ as a function of Δx^k, so we have to invert the previous expansion perturbatively. It is easy to realize that the correct choice is

$$\begin{aligned}\Delta x_F^k =& \Delta x^k - \Delta x^k \Psi\Big|_P - \frac{1}{2}\Delta x^i \Delta x^j (\delta_j^k \partial_i \Psi + \delta_i^k \partial_j \Psi - \delta_{ij}\partial^k \Psi)\Big|_P + \\ &- \frac{1}{6}\Delta x^i \Delta x^j \Delta x^l (\delta_j^k \partial_l \partial_i \Psi + \delta_i^k \partial_l \partial_j \Psi - \delta_{ij}\partial_l \partial^k \Psi)\Big|_P \ .\end{aligned}$$

This can be shown by composing the two expressions: one must find $\Delta x_F(\Delta x) = x_F + \ldots$, where dots are terms which are fourth order in x_F or second-order in perturbations. Moreover, if we indicate with $\bar{\Delta} x^k$ the unperturbed value of Δx^k, that is the value Δx^k assumes in the limit of null perturbations, as in the case of a_F and a, we expect that

$$\Delta x^k = \bar{\Delta} x^k + \mathcal{O}(p) \ .$$

This implies that in the previous expression we can replace each Δx^k mutiplied by a perturbations with $\bar{\Delta} x^k$, since corrections to this would be unavoidably of second-order in perturbations. This way, one gets finally

$$\begin{aligned}\Delta x_F^k =& \Delta x^k - \bar{\Delta} x^k \Psi\Big|_P - \frac{1}{2}\bar{\Delta} x^i \bar{\Delta} x^j (\delta_j^k \partial_i \Psi + \delta_i^k \partial_j \Psi - \delta_{ij}\partial^k \Psi)\Big|_P + \\ &- \frac{1}{6}\bar{\Delta} x^i \bar{\Delta} x^j \bar{\Delta} x^l (\delta_j^k \partial_l \partial_i \Psi + \delta_i^k \partial_l \partial_j \Psi - \delta_{ij}\partial_l \partial^k \Psi)\Big|_P \ ,\end{aligned} \tag{C.6}$$

that is the equation we have used in this project.

Including tensors

Including tensor perturbations, the metric turns out to be described by the following vielbeins

$$e_\mu^0 = a(\tau)\,(1+\Phi, -V_j) \ , \quad e_\mu^i = a(\tau)\left(0, (1-\Psi)\delta_j^i + \frac{1}{2}D_j^i\right) \ ,$$

as one can easily see:

$$\begin{aligned} g_{kl} &= e_k^a e_l^b \eta_{ab} = e_k^0 e_l^0(-1) + e_k^i e_l^j \delta_{ij} = \\ &= -a^2(\tau)V_k V_l + a^2(\tau)\left[(1-\Psi)^2\delta_k^i + \frac{1}{2}D_k^i\right]\left[(1-\Psi)^2\delta_l^j + \frac{1}{2}D_l^j\right]\delta_{ij} = \\ &= a^2(\tau)(1-2\Psi)\delta_{kl} + D_{kl} \ .\end{aligned}$$

As before, the inverse vielbeins are

$$e_0^\mu = \frac{1}{a(\tau)}\,(1-\Phi, V_j) \ , \quad e_i^\mu = \frac{1}{a(\tau)}\left(0, (1+\Psi)\delta_j^i - \frac{1}{2}D_j^i\right) \ .$$

Since the new term is linear in the metric, in the Christoffel's symbols it produces simply a new summand

$$\tilde{\Gamma}_{ij}^k = \tilde{\Gamma}_{Pij}^k + \frac{1}{2}\left(\partial_i D_j^k + \partial_j D_i^k - \partial^k D_{ij}\right) \ ,$$

where $\tilde{\Gamma}_P$ is the previous Christoffel's symbol including only the Ψ perturbation (eq. C.5). The result is that computing the CFC transformation one gets simply some extra summands, which are precisely

$$\begin{aligned}\Delta x^k\Big|_{\text{tensors}} =& -\frac{1}{2}D^k_i\Delta x^i_F\Big|_P - \frac{1}{4}\Delta x^i_F\Delta x^j_F(\partial_i D^k_j + \partial_j D^k_i - \partial^k D_{ij})\Big|_P + \\ & -\frac{1}{12}\Delta x^i_F\Delta x^j_F\Delta x^l_F(\partial_l\partial_i D^k_j + \partial_l\partial_j D^k_i - \partial_l\partial^k D_{ij})\Big|_P .\end{aligned}$$

As before, we want Δx^k_F as a function of Δx^k; given tha all is linear, the result is analogous to what we found in the case of the scalar part:

$$\begin{aligned}\Delta x^k_F =& \frac{1}{2}D^k_i\Delta x^i\Big|_P + \frac{1}{4}\Delta x^i\Delta x^j(\partial_i D^k_j + \partial_j D^k_i - \partial^k D_{ij})\Big|_P + \\ & + \frac{1}{12}\Delta x^i\Delta x^j\Delta x^l(\partial_l\partial_i D^k_j + \partial_l\partial_j D^k_i - \partial_l\partial^k D_{ij})\Big|_P .\end{aligned}$$

The same way, we can replace Δx with $\bar{\Delta}x$, since we are interested in a first-order expansion in perturbations, so that

$$\begin{aligned}\Delta x^k_F =& \frac{1}{2}D^k_i\bar{\Delta}x^i\Big|_P + \frac{1}{4}\bar{\Delta}x^i\bar{\Delta}x^j(\partial_i D^k_j + \partial_j D^k_i - \partial^k D_{ij})\Big|_P + \\ & + \frac{1}{12}\bar{\Delta}x^i\bar{\Delta}x^j\bar{\Delta}x^l(\partial_l\partial_i D^k_j + \partial_l\partial_j D^k_i - \partial_l\partial^k D_{ij})\Big|_P .\end{aligned} \tag{C.7}$$

Appendix D

Two useful results

D.1 Derivatives of the polarization tensor

Consider the polarization tensors 4.77 in Fourier space:

$$\begin{aligned} \epsilon^1_{ij} &= \frac{(e^1_i + ie^2_i)(e^1_j + ie^2_j)}{2} \\ \epsilon^2_{ij} &= \frac{(e^1_i - ie^2_i)(e^1_j - ie^2_j)}{2} \ . \end{aligned} \tag{D.1}$$

$e^{1/2}$ are two versors generating the subspace perpendicular[1] to the direction of the gravitational waves, which is

$$\vec{k} = (k_1, k_2, k_3) \ .$$

To construct explicitly $\epsilon^{1/2}_{ij}$ we have to find explicitly the versors $e^{1/2}$. A possible choice is given by

$$\begin{aligned} e^1 &= \frac{1}{\sqrt{k_1^2 + k_2^2}} (-k_2, k_1, 0) \\ e^2 &= \frac{1}{\sqrt{k_1^2 k_3^2 + k_2^2 k_3^2 + (k_1^2 + k_2^2)^2}} \left(k_1 k_3, k_2 k_3, -k_1^2 - k_2^2\right) \ . \end{aligned}$$

Notice that in this case we are assuming that k_1 and k_2 are not null at the same time. This couple of versors allows to construct explicitly the polarization tensors $\epsilon^{1/2}_{ij}$ through D.1. Since the explicit expressions are very cumbersome, we say only that they can be easily derived using a computer program such as Mathematica.
Using Mathematica one can immediately compute the gradients of the polarization tensors, which are

$$\partial_{k_i} \epsilon^1_{ij}(k) = \partial_{k_i} \epsilon^2_{ij}(k) = \begin{pmatrix} \frac{k_1^3 k_3^2 + k_1 k_2^2 k_3^2 + i k_2 k_3 (k_1^2 + k_2^2)\sqrt{k_1^2 + k_2^2 + k_3^2}}{(k_1^2 + k_2^2)(k_1^2 + k_2^2 + k_3^2)} \\ \frac{-k_1^2 k_2 k_3^2 - k_2^3 k_3^2 + i k_1 k_3 (k_1^2 + k_2^2)\sqrt{k_1^2 + k_2^2 + k_3^2}}{(k_1^2 + k_2^2)(k_1^2 + k_2^2 + k_3^2)} \\ \frac{k_3}{k_1^2 + k_2^2 + k_3^2} \end{pmatrix} \ .$$

This result diverges in the limit $(k_1, k_2) \to (0, 0)$, which coincides with the case in which our choice of versors is not well-defined[2]. However, this singularity is fictitious, since it is linked to the choice of the versors $e^{1/2}_{ij}$; choosing a different couple, this singularity vanishes. Finally, one gets

$$\partial_{k_i} \partial_{k_j} \epsilon^1_{ij}(k) = \partial_{k_i} \partial_{k_j} \epsilon^2_{ij}(k) = \frac{1}{k_1^2 + k_2^2 + k_3^2} \ .$$

This result, differently to the previous, is not divergent in the limit $(k_1, k_2) \to (0, 0)$. This is due to the fact that it is a scalar, so it cannot show any divergence due to the coordinate choice.

[1] Perpendicular means $e^j k^i g_{ij} = 0$. However, in this case we can consider a Euclidean scalar product since we want to work at zeroth order in perturbations. Indeed, ϵ_{ij} is always multiplied by $D(k)$, which is first-order.

[2] And of course in the limit $(k_1, k_2, k_3) \to (0, 0, 0)$, when the direction of the gravitational wave is not well-defined.

D.2 Writing a vector transformation in matrix form

We want to show that given a vector ϵ^i_V, it is always possible to write it as

$$\epsilon^i_V = M^i_j x^j \ , \tag{D.2}$$

where M is a spacetime-dependent 3×3 matrix, transverse ($\partial_i M^i_j = 0$) and traceless ($\delta^j_i M^i_j = 0$), but not necessarily symmetric. This way $M^i_j x^j$ is transverse, as a vector must be.

Let us consider a matrix of type

$$M^{ij} = \begin{pmatrix} 0 & B & C \\ D & 0 & F \\ G & H & 0 \end{pmatrix} \ ,$$

where B, C, D, F, G and H are functions of the spacetime. This matrix is traceless. We have to impose the 3 conditions $D.2$

$$\begin{cases} By + Cz = \epsilon^1_V \\ Dx + Fz = \epsilon^2_V \\ Gx + Hy = \epsilon^3_V \end{cases} \tag{D.3}$$

and the 3 transversality conditions

$$\begin{cases} \partial_y D + \partial_z G = 0 \\ \partial_x B + \partial_z H = 0 \\ \partial_x C + \partial_y F = 0 \end{cases} \ . \tag{D.4}$$

Isolating C, D and H in D.3 one has

$$\begin{cases} C = \frac{1}{z}(\epsilon^1_V - By) \\ D = \frac{1}{x}(\epsilon^2_V - Fz) \\ H = \frac{1}{y}(\epsilon^3_V - Gx) \end{cases} \ , \tag{D.5}$$

which substituted into D.4 gives a system of equations into the parameters B, F and G, which can be rewritten in matricial form as

$$\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} y\partial_x B \\ z\partial_y F \\ x\partial_z G \end{pmatrix} = \begin{pmatrix} -\partial_y \epsilon^2_V \\ -\partial_z \epsilon^3_V \\ -\partial_x \epsilon^1_V \end{pmatrix} \ . \tag{D.6}$$

The constant matrix is singular, so $y\partial_x B$, $z\partial_y F$ and $x\partial_z G$ can be isolated only if the last equation is a linear combination of the first two. Indeed, combining the first two equations one gets

$$z\partial_y F - y\partial_x B = \partial_y \epsilon^2_V + \partial_z \epsilon^3_V \ ,$$

which compared to the third equation $z\partial_y F - y\partial_x B = -\partial_x \epsilon^1_V$ gives

$$\partial_x \epsilon^1_V + \partial_y \epsilon^2_V + \partial_z \epsilon^3_V = \partial_i \epsilon^i_V = 0 \ , \tag{D.7}$$

which is true. This way, the system D.6 can be always solved in terms of $y\partial_x B$, $z\partial_y F$ and $x\partial_z G$. Since there are 2 linear independent equations and 3 variables, we are free to set one of the variables to a constant, so that the system simplifies. Choosing for example $G = constant$, the first two equations give

$$\begin{cases} z\partial_y F = -\partial_y \epsilon^2_V \\ y\partial_x B = -\partial_z \epsilon^3_V \end{cases} \ . \tag{D.8}$$

This system can be solved very easily in terms of F and B.

To summarize, the entries of M^i_j can be found as follows: G can be set to a constant; then, F and B are derived from D.8; finally D.5 gives C, D and H.

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