SCUOLA GALILEIANA DI STUDI SUPERIORI

Università degli Studi di Padova

Scuola Galileiana di Studi Superiori

Classe di Scienze Naturali

Tesi di Diploma Galileiano

Hölder continuity and Harnack's inequality for solutions of elliptic partial differential equation

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Preface

One of the most influential theories for elliptic equations of the following form $-\text{div}(\mathbb{A}(x)\nabla u(x)) = 0$ is the De Giorgi-Nash-Moser theorem on the continuity of weak solutions to uniformly elliptic equations. Ennio De Giorgi first proved the theorem in 1957 [1] and indipendently was also discovered by Jonh Nash [2] in 1958. Later J. Moser [3] in 1960 gave a proof using the powers of the solution and John-Nirenberg inequality. De Giorgi used the theorem to solve Hilbert's 19th problem and his proof has far reaching influence on the whole field of the analysis of partial differential equations.
In the first chapter we will exhibit a more generalized version of the De Giorgi method combined with the use of Harnack's inequality to show the iteration of Moser. Applying the De Giorgi-Nash-Moser theorem we obtain that $Dw \in C^{0,\alpha}$. Then, with a bootstrap argument, we will conclude the solution to the Hilbert problem with the Calderon–Zygmund result.
After exhibiting some applications of Harnack's inequality, in the last chapter we will shift our focus on proving the Harnack's inequality on general second-order elliptic equation of the form $-\triangle u + b \cdot \nabla u + au = 0$. In the previous elliptic equation we set $\mathbb{A} = \mathbb{1}$ to concentrate on the coefficients a, b of inferior order. The qualitative properties of solutions to elliptic and parabolic equations in divergence form with low regularity of the coefficients have been studied extensively, starting with the classical papers of De Giorgi [1], Nash [2], and Moser [3]. We are mostly interested in the improved regularity for divergence free drifts b, which arise in fluid dynamics models [12].

1 Elliptic equations

In this section, we consider the following linear equation, set on some domain (i.e. nonempty, open, connected set) of $\mathbb{R}^n$, where $n \geq 2$:

$$-\mathrm{div}(\mathbb{A}(x)\nabla u(x)) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\Big(a^{ij}(x)\frac{\partial}{\partial x^j}u(x)\Big) = 0, \tag{1}$$

where the matrix $\mathbb{A}(x) = [a^{ij}(x)]_{1\leq i,j\leq n}$ has measurable entries. If u is such that (1) holds we will write $L_{\mathbb{A}}u = 0$ to abbreviate the future notations. We will also need the boundedness and ellipticity of $\mathbb{A}$, therefore there exist $\Lambda \geq 1$ such that

$$\Lambda^{-1}\mathbb{I} \leq \mathbb{A}(x) \leq \Lambda\mathbb{I} \tag{2}$$

for almost all x, where the previous inequality is meant in the following sense: for almost all x and for all $\xi \in \mathbb{R}^n$ with modulus 1, we require

$$\Lambda^{-1} \leq \langle \xi, \mathbb{A}(x)\xi \rangle \leq \Lambda,$$

for some $\Lambda \geq 1$. If the matrix $\mathbb{A}$ is diagonalizable and $(\lambda_i(x))_{1\leq i\leq n}$ denote the eigenvalues of $\mathbb{A}(x)$, this condition is equivalent to $\Lambda^{-1} \leq \lambda_i(x) \leq \Lambda$ for almost every x. This also means that $\mathbb{A}$ is positive definite.

Definition 1.0.1 (Weak solution)**.** *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. We say that $u \in W^{1,2}(\Omega)$ is a weak solution of* (1) *if the equation*

$$\int_\Omega \langle \nabla\phi(x), \mathbb{A}(x)\nabla u(x)\rangle dx = 0$$

holds for all test functions $\phi \in C_c^\infty(\Omega)$.

Theorem 1.0.2 (**De Giorgi, Nash, Moser**)**.** *Let $u \in W^{1,2}(\Omega)$ be a weak solution of* (1) *where $\mathbb{A}$ fulfils the condition* (2)*. Then there exists $0 < \alpha(n,\Lambda) \leq 1$ such that $u \in C^{0,\alpha}(\Omega')$ for all Ω' relatively compact in Ω. Moreover, there exists $C(n,\Lambda,\Omega,\Omega') > 0$ such that the following norm estimate holds:*

$$\|u\|_{C^{0,\alpha}(\Omega')} \leq C\|u\|_{L^2(\Omega)}.$$

The method of De Giorgi

In the following subchapter we will prove *Theorem* 1.0.2 for a general matrix $\mathbb{A}$ which satisfies (2), contrarily to the work of De Giorgi (see [1]), who confined to consider $\mathbb{A}$ symmetric. Anyway, we will consider a general matrix and do not require $\mathbb{A}$ to be symmetric.

Remark 1.0.2.1. *Let $u \in W^{1,2}$ be a weak solution of* (3.1) *on some domain Ω, where $\mathbb{A}$ fulfils* (2)*. Scalings and translations of u solve a similar equation, for which* (2) *still holds. In more detail let $\lambda > 0$, $x_0 \in \Omega$ and $\epsilon > 0$ and define*

$$v(y) := \lambda u(x_0 + \epsilon y),$$

for $y \in \tilde{\Omega} \subset \{y \in \mathbb{R}^n : x_0 + \epsilon y \in \Omega\}$. Let $\psi \in C_c^\infty(\tilde{\Omega})$ and define $\phi := y \to \psi(x_0 + \epsilon y), \mathbb{B} := y \to \mathbb{A}(x_0 + \epsilon y)$. We then have

$$\begin{aligned}\int_{\tilde{\Omega}} \langle \nabla_y v(y), \mathbb{B}(y) \nabla_y \psi(y) \rangle dy &= \int_{\tilde{\Omega}} \langle \epsilon\lambda \nabla_x u(x_0 + \epsilon y), \mathbb{A}(x_0 + \epsilon y) \epsilon \nabla_x \phi(x_0 + \epsilon y) \rangle dy, \\ &= \epsilon^{2-n} \lambda \int_\Lambda \langle \nabla_x u(x), \mathbb{A}(x) \nabla_x \phi(x) \rangle dx \\ &= 0,\end{aligned}$$

due to the fact that $\phi \in C_c^\infty(\Omega)$. From the previous equation it follows that v is a weak solution on $L_{\mathbb{B}}$ on $\tilde{\Omega}$, where $\mathbb{B}$ satisfies (2).

Remark 1.0.2.2. *Without loss of generality it is enough to prove the theorem on the balls B_1 and $B_{1/2}$ since by translation and scaling one can get the desired result for any ball as follows: let $d = dist(\Omega', \partial\Omega)$. For any $x_0 \in \Omega'$, we define*

$$f(x) = u(x_0 + dx),$$

where $x \in B_1$, so that by the previous remark, f is a weak solution $L_{\mathbb{B}_d} = 0$ on B_1, where $\mathbb{B}_d := y \to \mathbb{A}(x_0 + dy)$ fulfils (2)*. Then $v \in C^{0,\alpha}(B_{1/2})$ where α does not depend on x_0 or d, therefore $u \in C^{0,\alpha}(\Omega')$.*

The proof can be split in two steps: the first one consists in estimating the supremum of u using the energy. For the second one we can use the estimate in $L^\infty(\Omega')$, to prove the fact that u is in $C^{0,\alpha}(\Omega')$.

First step: the supremum bound

Lemma 1.0.3. *There exists $\delta > 0$, which depends only on n and Λ, such that for any $u \in W^{1,2}(B_1)$ which is a weak solution of* (1) *in B_1, where $\mathbb{A}$ fulfils* (2)*, the following proposition is true: if*

$$\|u_+\|_{L^2(B_1)} \le \delta,$$

then

$$\|u_+\|_{L^\infty(B_{\frac{1}{2}})} \le \frac{1}{2}.$$

First of all we introduce some notation. We initially define the following family of balls centered in the origin:

$$B'_k := B_{\frac{1}{2(1+2^{-k})}},$$

which by definition satisfy $B'_0 = B_1$ and $\lim_{k\to\infty} B'_k = B_{1/2}$. We will also define $u_k := (u - \frac{1-2^{-k}}{2})_+$ which is the positive part of the function $u - \frac{1-2^{-k}}{2}$ and we define as well

$$\mathcal{U}_k = \int_{B'_k} |u_k(x)|^2 dx.$$

We would like to derive an estimate of the following form

$$\mathcal{U}_{k+1} \leq C^k \mathcal{U}_k^\beta, \tag{3}$$

where $C > 1$ and $\beta > 1$. Using that by definition $\mathcal{U}_0 = \|u_+\|^2_{L^2(B_1)}$ and $\lim_{k\to\infty} C^k = +\infty$, since $\beta > 1$ if $\mathcal{U}_0$ (the L^2- norm of u_+ on B_1) is small enough, this factor will overwhelm the factor C^k and the sequence $\mathcal{U}_k$ converges to 0, which means that by dominated convergence

$$\int_{B_{\frac{1}{2}}} \Big(u(x) - \frac{1}{2}\Big)_+^2 dx = 0,$$

which implies that $u(x) \leq \frac{1}{2}$ for almost every x in $B_{\frac{1}{2}}$, therefore $u \in L^\infty(B_{\frac{1}{2}})$. In the previous propositions we introduced the key ideas to the proof of the *Lemma* 1.0.3 which consisted in proving the estimate (3) and using this inequality to prove $u \in L^\infty(B_{\frac{1}{2}})$ in a rigorous way. To prove the estimate (3) we have to apply the three following well-known inequalities:

Theorem 1.0.4 (Sobolev' s inequality). *Assume $n \geq 3$. Let $2^\star = \frac{2n}{n-2}$. For any smooth, bounded domain $\Omega \subset \mathbb{R}^n$ there exist a constant S such that for any $u \in H^1_0(\Omega)$ we have the following inequality:*

$$\|u\|_{L^{2^\star}(\Omega)} \leq S\|\nabla u\|^2_{L^2(\Omega)}. \tag{4}$$

Proof. We initially remark that is enough to prove the theorem in case $u \in \mathbb{C}^\infty_0(\Omega)$ by density. We firstly assume that $p = 1$, then $p^\star = \frac{n}{n-1}$. Due to the fact that u has compact support we obtain that

$$\begin{aligned}|u(x)| &\leq \int_{-\infty}^{x_i} |u_{x_i}(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)| dy_i \\ &\leq \int_{-\infty}^{+\infty} |Du(x_1, \dots, , x_{i-1}, y_i, x_{i+1}, \dots, x_n)| dy_i\end{aligned}$$

for all $i \in \{1, \dots, n\}$ therefore taking the product we obtain

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^n \Big(\int_{-\infty}^{+\infty} |Du(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)| dy_i\Big)^{\frac{1}{n-1}}.$$

Integrating the previous inequality with respect to x_i we obtain:

$$\begin{aligned}\int_{-\infty}^{+\infty} |u|^{\frac{n}{n-1}} dx_1 &\leq \int_{-\infty}^{+\infty} \prod_{i=1}^{n} \Big(\int_{-\infty}^{+\infty} |Du| dy_i\Big)^{\frac{1}{n-1}} dx_1 \\ &= \Big(\int_{-\infty}^{+\infty} |Du| dy_1\Big)^{\frac{1}{n-1}} \int_{-\infty}^{+\infty} \prod_{i=2}^{n} \Big(\int_{-\infty}^{+\infty} |Du| dy_i\Big)^{\frac{1}{n-1}} dx_1 \\ &\leq \Big(\int_{-\infty}^{+\infty} |Du| dy_1\Big)^{\frac{1}{n-1}} \Big(\prod_{i=2}^{n} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |Du| dx_1 dy_i\Big)^{\frac{1}{n-1}}\end{aligned}$$

where the last inequality follows from Hölder' s inequality. Iterating the procedure with respect to the other variables we obtain

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \leq \Big(\int_{\mathbb{R}^n} |Du|^{\frac{n}{n-1}} dx\Big)^{\frac{n}{n-1}}$$

as desired. Now we will consider the case $1 < p < n$. We apply the previous estimate on $v := |u|^{\alpha}$. Then

$$\begin{aligned}\Big(\int_{\mathbb{R}^n} |u|^{\frac{\alpha n}{n-1}} dx\Big)^{\frac{n-1}{n}} &\leq \int_{\mathbb{R}^n} |D|u|^{\alpha}| dx = \alpha \int_{\mathbb{R}^n} |u|^{\alpha-1} |Du| dx \\ &\leq \alpha\Big(\int_{\mathbb{R}^n} |u|^{(\alpha-1)\frac{p}{p-1}} dx\Big)^{\frac{p-1}{p}} \Big(\int_{\mathbb{R}^n} |Du|^p dx\Big)\frac{1}{p}\end{aligned}$$

where in the last inequality we applied the general Hölder' s inequality and if we choose α such that $\frac{\alpha n}{n-1} = (\alpha - 1)\frac{p}{p-1}$, therefore $\alpha := \frac{p(n-1)}{n-p} > 1$, we obtain that $\frac{\alpha n}{n-1} = \frac{np}{n-1} = p^{\star}$ from which it follows that

$$\|u\|_{L^{p^{\star}}(\mathbb{R}^n)} \leq \frac{p(n-1)}{n-p} \|Du\|_{L^p(\mathbb{R}^n)},$$

which proves the general theorem with $S_p = \frac{p(n-1)}{n-p}$ and assigning $p = 2$ we obtain the thesis. □

Theorem 1.0.5 (Markov's inequality)**.** *Let f be a positive and measurable function. Then we have the following inequality for any $a > 0$:*

$$|\{x : f(x) \geq a\}| \leq \frac{\|f\|_{L^1}}{a}.$$

Proof. Since f is positive the theorem follows trivially from:

$$\|f\|_{L^1} = \int_{\mathbb{R}^n} f(x) dx \geq \int_{\{x: f(x) \geq a\}} f(x) dx \geq \int_{\{x: f(x) \geq a\}} a dx.$$ □

Theorem 1.0.6 (Caccioppoli inequality)**.** *Let $u \in W^{1,2}(B_r)$ be a weak solution of $L_{\mathbb{A}} u = 0$, where $\mathbb{A}$ fulfils (2), and let $\phi \in C_c^{\infty}(B_r)$. Then there exists $C > 0$ independent of u such that the following inequality holds:*

$$\int_{B_r} |\nabla(\phi u_+)|^2 dx \leq C \|\nabla \phi\|_{L^{\infty}}^2 \int_{B_r \cap supp\, \phi} u_+^2 dx.$$

Moreover $C = \Lambda^2$ if the matrix $\mathbb{A}$ is symmetric.

Proof. Since u is a weak solution of $L_{\mathbb{A}} u = 0$, testing with $\phi^2 u_+$ we obtain

$$\int_{B_r} \langle \nabla(\phi^2 u_+), \mathbb{A}\nabla u_+ \rangle dx = 0.$$

And we do the following computations to use the fact that the matrix $\mathbb{A}$ satisfies (2):

$$\begin{aligned}
0 &= \int_{B_r} \langle \nabla(\phi^2 u_+), \mathbb{A}\nabla u_+ \rangle dx \\
&= \int_{B_r} \langle \phi\nabla(\phi u_+), \mathbb{A}\nabla u_+ \rangle dx + \int_{B_r} \langle (\phi u_+)\nabla(\phi), \mathbb{A}\nabla u_+ \rangle dx \\
&= \int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}\nabla(\phi u_+) \rangle - \int_{B_r} \langle \nabla(\phi u_+), \mathbb{A} u_+ \nabla\phi \rangle dx + \int_{B_r} \langle (\phi u_+)\nabla\phi, \mathbb{A}\nabla u_+ \rangle dx \\
&= \int_{B_r} \langle (\phi u_+), \mathbb{A}(\phi u_+) \rangle dx - \int_{B_r} \langle \nabla(\phi u_+), (\mathbb{A} - \mathbb{A}^T) u_+ \nabla\phi \rangle dx \\
&- \int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}^T u_+ \nabla\phi \rangle dx + \int_{B_r} \langle (\phi u_+)\nabla\phi, \mathbb{A}\nabla u_+ \rangle dx \\
&= \int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}\nabla(\phi u_+) \rangle dx - \int_{B_r} \langle \nabla(\phi u_+), (\mathbb{A} - \mathbb{A}^T) u_+ \nabla\phi \rangle dx \\
&- \int_{B_r} \langle u_+^2 \nabla\phi, \mathbb{A}^T \nabla\phi \rangle dx - \int_{B_r} \langle (\phi u_+)\nabla u_+, \mathbb{A}^T \nabla\phi \rangle dx + \int_{B_r} \langle (\phi u_+)\nabla\phi, \mathbb{A}\nabla u_+ \rangle dx
\end{aligned}$$

And if the matrix $\mathbb{A}$ is symmetric the second term vanishes and we obtain

$$\int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}\nabla(\phi u_+) \rangle dx = \int_{B_r} u_+^2 \langle \nabla\phi, \mathbb{A}^T \nabla\phi \rangle dx,$$

and using the ellipticity (2) of the matrix $\mathbb{A}$ we obtain

$$\int_{B_r} |\nabla(\phi u_+)|^2 dx \leq \Lambda^2 \int_{B_r} u_+^2 |\nabla\phi|^2 dx \leq \Lambda^2 \|\nabla\phi\|_{L^\infty}^2 \int_{B_r \cap supp\, \phi} u_+^2 dx.$$

If $\mathbb{A}$ is not symmetric we use the following estimate:

$$\begin{aligned}
\left| \int_{B_r} \langle \nabla(\phi u_+), (\mathbb{A} - \mathbb{A}^T) u_+ \nabla\phi \rangle dx \right| &\leq \int_{B_r} \Big(|\langle \nabla(\phi u_+), \mathbb{A} u_+ \nabla\phi \rangle| + |\langle \mathbb{A}\nabla(\phi u_+), u_+ \nabla\phi \rangle| \Big) dx \\
&\leq 2\Lambda \|\nabla(\phi u_+)\|_{L^2} \|u_+ \nabla\phi\|_{L^2} \\
&\leq 2\Lambda^{\frac{3}{2}} \Big(\int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}\nabla(\phi u_+) \rangle dx \Big)^{\frac{1}{2}} \|u_+ \nabla\phi\|_{L^2} \\
&\leq \frac{1}{2} \int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}\nabla(\phi u_+) \rangle dx + 2\Lambda^3 \int_{B_r} u_+^2 |\nabla\phi|^2 dx
\end{aligned}$$

where in the penultimate line we used the ellipticity condition and in the last line we used $a^2+b^2 \geq 2ab$. Plugging this inequality in the expression above we obtain

$$\frac{1}{2}\int_{B_r} \langle \nabla(\phi u_+), \mathbb{A}\nabla(\phi u_+)\rangle dx - \Lambda(1+2\Lambda^2)\int_{B_r} u_+^2 |\nabla\phi|^2 dx \leq 0,$$

and using the ellipticity again we obtain:

$$\begin{aligned}
&\int_{B_r} |\nabla(\phi u_+)|^2 dx \leq 2\Lambda^2(1+2\Lambda^2)\int_{B_r} u_+^2|\nabla\phi|^2 dx \\
&\leq 2\Lambda^2(1+\Lambda^2)\|\nabla\phi\|_{L^\infty}^2 \int_{B_r\cap supp\,\phi} u_+^2 dx,
\end{aligned}$$

which concludes the proof.

□

Once showed the previous theorems we can proceed to the proof of $\mathcal{U}_{k+1} \leq C^k \mathcal{U}_k^\beta$ by defining the following family of cut-off functions ϕ_k such that: $\phi_k \in C_c^\infty(B'_{k-1})$ and $\phi_k = 1$ in B'_k with $\|\nabla\phi_k\|_{L^\infty} \leq C\cdot 2^k$. By definition we have

$$\mathcal{U}_k = \int_{B'_k} |u_k|^2 dx \leq \int_{B_1} \phi_k^2 u_k^2 dx$$

and also $\mathbb{1}_{B'_{k+1}} \leq \phi_k \leq \mathbb{1}_{B'_k}$ and $u_{k+1} \leq u_k$, therefore applying Sobolev' s inequality on $\phi_{k+1}u_{k+1}$ in B'_k and taking C big enough (> 16) we obtain

$$\Big(\int_{B'_k} (\phi_{k+1}u_{k+1})^{2^\star} dx\Big)^{\frac{2}{2^\star}} \leq S\int_{B'_k} |\nabla(\phi_{k+1}u_{k+1})|^2 dx,$$

and using Cacciopoli inequality (*Theorem* 1.0.6) we obtain

$$\Big(\int_{B'_k} (\phi_{k+1}u_{k+1})^{2^\star} dx\Big)^{\frac{2}{2^\star}} \leq C2^{2k}\int_{B'_k} |u_{k+1}|^2 dx \leq C2^{2k}\int_{B'_k} |u_k|^2 dx \leq C^k\mathcal{U}_k.$$

Using Markov' s inequality (*Theorem* 1.0.5), Hölder' s inequality with exponents $(\frac{n}{n-2}, \frac{n}{2})$ and previous inequality we obtain

$$\begin{aligned}
\mathcal{U}_{k+1} &\leq \int_{B'_k} \phi_{k+1}^2 u_{k+1}^2 dx \\
&\leq \Big(\int_{B'_k} (\phi_{k+1}u_{k+1})^{2^\star} dx\Big)^{\frac{2}{2^\star}} \big|\{\phi_{k+1}u_{k+1} > 0\}\big|^{\frac{2}{n}} \\
&\leq C^k\mathcal{U}_k|\{\phi_k u_k > 2^{-(k+2)}\}|^{\frac{2}{n}} = C^k\mathcal{U}_k\big|\big\{(\phi_k u_k)^2 > 2^{-2(k+2)}\big\}\big|^{\frac{2}{n}} \\
&\leq \frac{C^k}{2^{\frac{-4(k+2)}{n}}}\mathcal{U}_k^{1+\frac{2}{n}} \leq 2^{\frac{8}{n}}\big(2^{\frac{4}{n}}C\big)^k\mathcal{U}_k^{1+\frac{2}{n}}.
\end{aligned}$$

which proves (3) with $\beta = 1 + \frac{2}{n}$ and $k \geq 2$.
Proven this inequality we can conclude the proof of the *Lemma* 1.0.3 by looking for δ such that if $\|u_+\|_{L^2(B_1)} = \mathcal{U}_0 \leq \delta$, then $\|u\|_{L^\infty(B_{\frac{1}{2}})} \leq \frac{1}{2}$. The idea is to show that $\mathcal{U}_k$ converges to zero. To do so let $E(k)$ the following inequality:

$$E(k): \qquad C^k \mathcal{U}_k^{\beta-1} \leq \frac{1}{(2C)^{\frac{1}{\beta-1}}}. \tag{5}$$

We will prove that if $\mathcal{U}_0$ is small enough $E(k)$ holds for all k. Choosing k_0 such that $\frac{1}{2^{k_0}} \leq \frac{1}{(2C)^{\frac{1}{\beta-1}}}$, for δ small enough $E(k)$ holds for any $k \leq k_0$ since $\mathcal{U}_{\|+\infty} \leq \mathcal{U}_k$ by definition. Using strong induction on k we will prove that $E(k)$ holds for all k. Using (3) we have that $\mathcal{U}_{k+1} \leq \frac{1}{(2C)^{\frac{k+1}{\beta-1}}}$, which implies that

$$C^{k+1}\mathcal{U}_{k+1}^{\beta-1} \leq \frac{1}{2^{k+1}} \leq \frac{1}{2^{k_0}} \leq \frac{1}{(2C)^{\frac{1}{\beta-1}}},$$

which proves $E(k+1)$. Using this result we obtain that

$$\int_{B_{\frac{1}{2}}} (u - \frac{1}{2})_+^2 \, dx = \lim_{k \to +\infty} \mathcal{U}_k = 0,$$

which implies that $\|u_+\|_{L^\infty(B_{\frac{1}{2}})} \leq \frac{1}{2}$, which concludes the proof.

Corollary 1.0.6.1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Let $u \in W^{1,2}(\Omega)$ be a weak solution of* (1) *in Ω where $\mathbb{A}$ fulfils the ellipticity condition* (2)*. Then for any $\Omega' \Subset \Omega$ we have that $u \in L^\infty(\Omega')$.*

Proof. Let $d := dist(\Omega', \partial\Omega)$. For any $x_0 \in \Omega'$, we define v on B_1 as

$$v(y) := \delta \frac{d^{\frac{n}{2}}}{\|u\|_{L^2(\Omega)}} u(x_0 + dy)$$

where δ is the constant in *Lemma* 1.0.3. By *Remark* 1.0.2.1 v is a weak solution of $L_{\mathbb{B}} = 0$ on B_1 for some $\mathbb{B}$ fulfilling condition (2). Moreover we have that $\|v\|_{L^2(B_1)} \leq \delta$, which implies by *Lemma* 1.0.3 that $v(y) \leq \frac{1}{2}$ for almost every y in B_1. Using same procedure for $-v$ we obtain $|v(y)| \leq \frac{1}{2}$ for almost every y in B_1 which implies that

$$\|u\|_{L^\infty(\Omega')} \leq \frac{\|u\|_{L^2(\Omega)}}{2\delta \cdot d^{\frac{n}{2}}}.$$

□

Second step: the Oscillation Lemma

In this section we will conclude the proof of *Theorem* 1.0.2 by proving the Oscillation Lemma below.

Definition 1.0.7 (Oscillation)**.** *For any open set A and any real-valued function f on A, the oscillation of f on A is defined as*

$$osc_A f = \sup_A f - \inf_A f.$$

Lemma 1.0.8 (Oscillation Lemma)**.** *Let $u \in W^{1,2}(B_2)$ be a weak solution of* (1) *on B_2 where $\mathbb{A}$ fulfils the ellipticity condition* (2)*. Then there exists a constant $\lambda(\Omega, n) < 1$ such that the following inequality holds:*

$$osc_{B_{\frac{1}{2}}} u \leq \lambda \, osc_{B_2} u.$$

The De Giorgi-Nash-Moser Theorem follows as a conseguence of the Oscillation Lemma:

Proof of Theorem 1.0.2. Take $x_0 \in \Omega' \Subset \Omega$ and let $d := dist(\Omega', \partial\Omega)$ as before. We introduce the following family of functions v_k on B_2: $v_1(y) = u(x_0 + \frac{d}{2}y)$,

$$v_k(y) = v_{k-1}\Big(\frac{y}{4}\Big) = u\Big(x_0 + \frac{1}{4^{k-1}}\frac{d}{2}y\Big),$$

which, as proven by *Remark* 1.0.2.1 and *Remark* 1.0.2.2, the functions v_k are weak solutions of $L_{\mathbb{B}_k} v_k = 0$, where $\mathbb{B}_k(y) := \mathbb{A}\big(x_0 + \frac{1}{4^{k-1}}\frac{d}{2}y\big)$ fulfils the ellipticity condition (2) with the same constant Λ of $\mathbb{A}$. Applying the Oscillation Lemma:

$$osc_{B_{\frac{1}{2}}} v_{k+1} \leq \lambda osc_{B_2} v_{k+1} \leq \lambda osc_{B_{\frac{1}{2}}} v_k \leq \lambda^k osc_{B_{\frac{1}{2}}} v_1 \leq 2\lambda^k \|u\|_{L^\infty(\Omega')},$$

where we used the definition of $v_{k+1} : y \to v_k(\frac{y}{4})$. Using the previous estimate we obtain:

$$\sup_{|x_0 - x| \leq 4^{-k-1}d} |u(x_0) - u(x)| \leq osc_{B_{\frac{1}{2}}} v_{k+1} \leq 2\lambda^k \|u\|_{L^\infty(\Omega)},$$

which don' t depend on d, x_0. Let I_k be the following interval $I_k := [4^{-k-1}d, 4^{-k}d]$, then

$$\begin{aligned}\sup_{|x_0 - x| \leq d} \frac{|u(x_0) - u(x)|}{|x_0 - x|^\alpha} &= \sup_{k \in \mathbb{N}} \sup_{|x_0 - x| \in I_k} \frac{|u(x_0) - u(x)|}{|x_0 - x|^\alpha} \\ &\leq \sup_{k \in \mathbb{N}} \frac{2\lambda^{k-1}\|u\|_{L^\infty(\Omega)}}{4^{-\alpha(k+1)}} = \sup_{k \in \mathbb{N}} \frac{2\lambda^{-1}\|u\|_{L^\infty(\Omega)}}{4^{-\alpha}} (\lambda 4^\alpha)^k.\end{aligned}$$

Picking $\alpha = -\frac{\ln \lambda}{\ln 4}$ the right-hand side is limited and $u \in C^{0,\alpha}(\Omega')$. $\square$

To prove the Oscillation Lemma (*Lemma* 1.0.8) we will prove the equivalent proposition:

Proposition 1.0.9. *Let $v \leq 1$ be a weak solution of $L_{\mathbb{A}} v = 0$ on B_2, with $\mathbb{A}$ fulfiling* (2)*. If there exists $\mu > 0$ such that $|B_1 \cap \{v \leq 0\}| \geq \mu$, then there exists a constant λ depending only on n, μ and Λ such that the following estimate holds:*

$$\sup_{B_{\frac{1}{2}}} v \leq 1 - \lambda.$$

We will prove below how this proposition implies the Oscillation lemma:

Proof. Let u be a function which satisfies the hypothesis of *Lemma* 1.0.8 (Oscillation lemma). We rescale the function u by putting its image between -1 and 1 by introducing the function v as

$$v(x) := \frac{2}{osc_{B_2} u}\Big(u(x) - \frac{\sup_{B_2} u + \inf_{B_2} u}{2}\Big).$$

Without loss of generality we can assume $v \leq 0$ on at least half of B_1 (otherwise we can replace v with $-v$). Applying *Proposition* 1.0.9 we obtain

$$osc_{B_{\frac{1}{2}}} v = \sup_{B_{\frac{1}{2}}} v - \inf_{B_{\frac{1}{2}}} v \leq 1 - \lambda - (-1) = 2 - \lambda,$$

from which it follows by definition of v that

$$osc_{B_{\frac{1}{2}}} u = \frac{osc_{B_2} u}{2} osc_{B_{\frac{1}{2}}} v \leq (1 - \frac{\lambda}{2}) osc_{B_2} u.$$ □

To prove *Proposition* 1.0.9 we will introduce the following notations: let w a measurable function defined on B_1, we define the following subsets of B_1: $S^0_w := B_1 \cap \{w \leq 0\}$, $S^{\frac{1}{2}}_{0,w} := B_1 \cap \{0 < w < \frac{1}{2}\}$, $S_{\frac{1}{2},w} := B_1 \cap \{\frac{1}{2} \leq w\}$.

Lemma 1.0.10 (De Giorgi's isoperimetric inequality)**.** *There exists a constant $C_n > 0$, only depending on n, such that the following holds: if $w \in W^{1,2}(B_1)$ is such that $\int_{B_1} |\nabla w_+|^2 dx \leq C_0$, then we have*

$$C_n \Big(|S_{\frac{1}{2},w}| |S^0_w|^{1-\frac{1}{n}} \Big)^2 \leq C_0 |S_{0,w}|^{\frac{1}{2}}.$$

Proof. We start by setting the function $\overline{w}$ as $\overline{w}(x) := \max\{0, \min(w(x), \frac{1}{2})\}$. Note that by definition $\nabla \overline{w} = \mathbb{1}_{\{0 \leq w \leq \frac{1}{2}\}} \nabla w_+$. For any $x \in S^0_w, y \in S_{\frac{1}{2},w}$, we have

$$\begin{aligned} \frac{1}{2} \leq \overline{w}(y) - \overline{w}(x) &= \int_0^1 \frac{d}{dt} \overline{w}(x + t(y-x)) dt \\ &= \int_0^1 (y - x) \cdot \nabla \overline{w}(x + t(y - x)) dt \\ &\leq \int_0^{|x-y|} \Big| \nabla \overline{w}\Big(x + s\frac{x-y}{|x-y|}\Big) \Big| ds. \end{aligned}$$

Extending the last integral we obtain $\frac{1}{2} \le \int_0^\infty \Big|\nabla \overline{w}\Big(x+s\frac{x-y}{|x-y|}\Big)\Big|ds$, by which integrating in $y \in S_{\frac{1}{2},w}$ we obtain

$$\frac{|S_{\frac{1}{2},w}|}{2} \le \int_{S_{\frac{1}{2},w}} \Big(\int_0^\infty \Big|\nabla \overline{w}\Big(x+s\frac{x-y}{|x-y|}\Big)\Big|ds\Big)dy \le \int_{B_1} \Big(\int_0^\infty \Big|\nabla \overline{w}\Big(x+s\frac{x-y}{|x-y|}\Big)\Big|ds\Big)dy$$

and changing to the polar coordinates we obtain

$$\begin{aligned}\frac{|S_{\frac{1}{2},w}|}{2} &\le \int_0^1 r^{n-1}\int_{\mathbb{S}^{n-1}}\Big(\int_0^\infty \Big|\nabla \overline{w}\Big(x+s\frac{x-y}{|x-y|}\Big)\Big|ds\Big)d\sigma dr\\ &\le \int_{\mathbb{S}^{n-1}}\int_0^\infty \Big|\nabla \overline{w}\Big(x+s\frac{x-y}{|x-y|}\Big)\Big|dsd\sigma\\ &\le \int_{\mathbb{S}^{n-1}}\int_0^\infty \frac{s^{n-1}}{s^{n-1}}\Big|\nabla \overline{w}\Big(x+s\frac{x-y}{|x-y|}\Big)\Big|dsd\sigma\\ &= \int_{B_1}\frac{\nabla \overline{w}(y)|}{|x-y|^{n-1}}dy,\end{aligned}$$

and integrating now with respect to $x \in S_w^0$ we obtain:

$$\frac{|S_w^0||S_{\frac{1}{2},w}|}{2} \le \int_{B_1}|\nabla \overline{w}(y)|\Big(\int_{S_w^0}\frac{dx}{|y-x|^{n-1}}\Big)dy.$$

Changing the variable we obtain that

$$\int_{S_w^0}\frac{dx}{|y-x|^{n-1}} \le \int_{\mathbb{S}^{n-1}}d\theta\int_0^{\left(\frac{n|S_w^0|}{|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n}}}\frac{r^{n-1}}{r^{n-1}}dr \le c_n|S_w^0|^{\frac{1}{n}},$$

where we used the fact that $|y-x|$ is a non-increasing function which maximized the integral in a ball of radius $\left(\frac{n|S_w^0|}{|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n}}$ centered on y and $c_n = |\mathbb{S}^{n-1}|^{1-\frac{1}{n}}\Big(n|S_w^0|\Big)^{\frac{1}{n}}$. By definition of $\overline{w}$, $\nabla\overline{w}$ is zero outside of B_1, therefore applying Cauchy-Schwarz inequality we obtain

$$\frac{|S_w^0||S_{\frac{1}{2},w}|}{2} \le c_n|S_w^0|^{\frac{1}{n}}\Big(\int_{S_{0,w}^{\frac{1}{2}}}|\nabla w_+|^2dx\Big)^{\frac{1}{2}}|S_{0,w}^{\frac{1}{2}}|^{\frac{1}{2}}.$$

Using by hypothesis that $\int_{S_{0,w}^{\frac{1}{2}}}|\nabla w_+|^2dx \le C_0$ we have completed the proof since $C_n = \frac{1}{2c_n}$ is a constant depending on n. □

Proof of Proposition 1.0.9. We consider the family of functions: $w_k := 2^k(v-(1-2^{-k})) = 2^k(v-1)+1$. By definition $w_k \le 1$ and $w_{k+1} = 2w_k - 1$. Using

Caccioppoli inequality or energy inequality (Theorem 1.0.6) with $r = 2$ and $\mathbb{1}_{B_1} \leq \phi \leq \mathbb{1}_{B_2}$, we have

$$\int_{B_1} |\nabla(w_k)_+|^2 dx \leq \int_{B_2} |\nabla(\phi w_k)_+|^2 \leq C \int_{B_1} |(w_k)_+|^2 dx \leq C_0.$$

And we can apply De Giorgi' s isoperimetric inequality (*Lemma* 1.0.10) on w_k with the constant δ of *Lemma* 1.0.3 if

$$\int_{B_1} (w_{k+1})_+^2 dx \geq \delta^2. \tag{6}$$

and from this we can deduce that

$$|B_1 \cap \{w_{k+1} \geq 0\}| = |B_1 \cap \{2w_k \geq 1\}| \geq \int_{B_1} (w_{k+1})_+^2 dx \geq \delta^2.$$

So applying De Giorgi' s isoperimetric inequality (*Lemma* 1.0.10) $\exists C_n > 0$ such that

$$\Big|B_1 \cap \{0 < w_k < \frac{1}{2}\}\Big| \geq \frac{C_n}{C_0}\Big(|S_{\frac{1}{2},w}||S_w^0|^{1-\frac{1}{n}}\Big)^2,$$

and combining with

$$|S^0_{w_k}| = |B_1 \cap \{w_k \leq 0\}| \geq |S^0_{w_{k-1}}| \geq |S^0_{w_0}| = |B_1 \cap \{v \leq 0\}| \geq \mu > 0,$$

we obtain that

$$|B_1 \cap \{0 < w_k < \frac{1}{2}\}| \geq \gamma$$

for some $\gamma > 0$. Then by definition of w_k we obtain that $|S^0_{w_k}| \geq |S^0_{w_{k-1}}| + \gamma \geq \mu + k\gamma$ which is absurd as k is large enough, therefore (6) is false for $k = k_0$:

$$\int_{B_1} (w_{k_0+1})_+^2 dx \leq \delta^2.$$

By *Lemma* 1.0.3 $w_{k_0+1} \leq \frac{1}{2}$, which by definition of w_k concludes the proof of *Proposition* 1.0.9 with $\lambda = 2^{-(k_0+2)}$. □

The iteration of Moser

In this section we will present the alternative approach by Moser, published in [3], in which we will derive the proof by proving the Harnack's inequality. To do so we will remind some definitions:

Definition 1.0.11 (Sub and super-solutions)**.** *We say that* $u \in W^{1,2}(\Omega)$ *is a weak sub-solution (supersolution) of* (1) *if*

$$\int_\Omega \langle \nabla\phi(x), \mathbb{A}(x)\nabla u(x)\rangle dx \leq (\geq) 0$$

holds for all $\phi \in C_c^\infty(\Omega)$.

Theorem 1.0.12 (Harnack's inequality)**.** *Let $u \in W^{1,2}(\Omega)$ be a non-negative, weak sub-solution of (1), where $\mathbb{A}$ is symmetric and satisfies (2). Then there is a constant $c(n,\Lambda) > 0$ such that for every ball $B_r(y) \subset \Omega$ we have*

$$\sup_{B_{\frac{r}{2}}(y)} u \leq c \inf_{B_{\frac{r}{2}}(y)} u.$$

Remark 1.0.12.1. *Note that in this section we assumed $\mathbb{A}$ symmetric, contrary to the previous one.*

As a consequence of the Harnack's inequality we have the following theorem:

Theorem 1.0.13. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of (1) where $\mathbb{A}$ is symmetric and fulfils (2). Then there is $0 < \alpha(n,\Lambda) \leq 1$, such that $u \in C^{0,\alpha}(\Omega)$. Moreover, for every ball $B_R(y) \subset \Omega$ and all $0 < r \leq R < \infty$, we have*

$$osc_{B_r(y)} u \leq 2^\alpha \left(\frac{r}{R}\right)^\alpha osc_{B_R(y)} u.$$

The proof is again divided in two parts: one for the sup and one for the inf.

Harnack's inequality: sup

We will start by proving the fact that the weak solutions are bounded locally.

Lemma 1.0.14. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of (1) where $\mathbb{A}$ is symmetric and fulfils (2). Then $u \in L^\infty_{loc}(\Omega)$. Moreover, for every ball $B_r(y) \subset \Omega$, we have that there exists a constant $c = c(n,\Lambda)$ such that*

$$\sup_{B_{\frac{r}{2}}(y)} u \leq c \left(\fint_{B_r(y)} |u|^2 dx \right)^{\frac{1}{2}}.$$

As before we will prove by iteration using Caccioppoli inequality and Sobolev's inequality. We will start by proving the following inequality:

Lemma 1.0.15. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of (1) where $\mathbb{A}$ is symmetric and satisfies (2). Then for any $\alpha \geq 0$ such that $u \in L^{\alpha+2}_{loc}(\Omega)$, we have*

$$\int_\Omega |u|^\alpha |\nabla u|^2 \eta^2 dx \leq c \int_\Omega |u|^{\alpha+2} |\nabla \eta|^2 dx,$$

where $c = c(\Lambda) > 0$.

Proof. Fix $\eta \in C^\infty_c(\Lambda)$ and let $t \geq 0$. Define $v = (u-t)_+$. We test the equation (1) with $\phi = v\eta^2 \in H^1_0(\Lambda)$ and we obtain:

$$\begin{aligned} 0 &= \int_\Omega \langle \nabla \phi, \mathbb{A} \nabla u \rangle dx \\ &= \int_\Omega \langle \nabla (u-t)_+, \mathbb{A}\nabla u \rangle \eta^2 dx + 2 \int_\Omega \langle \nabla \eta, \mathbb{A} \nabla u \rangle (u-t)_+ \eta dx. \end{aligned}$$

To estimate the last integral we use the following Cauchy-Schwarz inequality combined with the fact that the matrix $\mathbb{A}$ is symmetric:

$$|\langle \nabla \eta, \mathbb{A} \nabla u \rangle| \leq \langle \nabla u, \mathbb{A} \nabla u \rangle^{\frac{1}{2}} \langle \nabla \eta, \mathbb{A} \nabla \eta \rangle^{\frac{1}{2}}$$

combined with Hölder inequality. After squaring, we obtain

$$\int_{\{u>t\}} \langle \nabla u, \mathbb{A} \nabla u \rangle \eta^2 \leq 4 \int_{\{u>t\}} \langle \nabla \eta, \mathbb{A} \nabla \eta \rangle |(u-t)_+|^2 dx, \tag{7}$$

and using the ellipticity condition (2) we get

$$\int_{\{u>t\}} |\nabla u|^2 \eta^2 dx \leq 4\Lambda^2 \int_{\{u>t\}} |(u-t)_+|^2 |\nabla \eta|^2 dx \leq 4\Lambda^2 \int_{\{u>t\}} |u_+|^2 |\nabla \eta|^2 dx.$$

Since the above inequality holds for any $t \geq 0$, we can can multiply both sides by $\alpha t^{\alpha-1}$ and integrating with respect to t over the interval $(0, +\infty)$ to obtain

$$\int_0^{+\infty} \alpha t^{\alpha-1} \left(\int_{\{u>t\}} |\nabla u|^2 \eta^2 dx \right) dt \leq 4\Lambda^2 \int_0^{+\infty} \alpha t^{\alpha-1} \left(\int_{\{u>t\}} |u_+|^2 |\nabla \eta|^2 dx \right) dt,$$

and using Fubini-Tonelli's theorem we obtain:

$$\int_\Omega |\nabla u_+|^2 \eta^2 \left(\int_0^\infty \alpha t^{\alpha-1} \mathbb{1}^{(x)}_{\{u>t\}} dt \right) dx \leq 4\Lambda^2 \int_\Omega |u_+|^2 |\nabla \eta|^2 \left(\int_0^\infty \alpha t^{\alpha-1} \mathbb{1}^{(x)}_{\{u>t\}} dt \right) dx.$$

Noting that $\{u > t\} = \{u_+ > t\}$ and that $\int_0^{+\infty} \alpha t^{\alpha-1} \mathbb{1}_{\{u>t\}}(x) dt = |u|^\alpha$ almost everywhere we obtain:

$$\int_\Omega |u|^\alpha |\nabla u_+|^2 \eta^2 dx \leq 4\Lambda^2 \int_\Omega |u_+|^{\alpha+2} |\nabla \eta|^2 dx.$$

Similarly we can get the same estimate for u_- and we can conclude the result by summing the two estimates. □

From this we can use Sobolev' s inequality to obtain

Lemma 1.0.16. *Assume $n \geq 3$ and let 2^* be the following Sobolev exponent $2^* = \frac{2n}{n-2}$. Let $u \in W^{1,2}(\Omega)$ be a weak solution of* (1) *where $\mathbb{A}$ is symmetric and satisfies* (2). *Then for any $\alpha \geq 0$, $u \in L_{loc}^{\frac{2^*(\alpha+2)}{2}}(\Omega)$ if $u \in L^{\alpha+2}_{loc}(\Omega)$. Moreover, for any $\eta \in C_c^\infty(\Omega)$,*

$$\left(\int_\Omega |u|^{\frac{2^*(\alpha+2)}{2}} \eta^{2^*} dx \right)^{\frac{2}{2^*}} \leq c(\alpha+2)^2 \int_\Omega |u|^{\alpha+2} |\nabla \eta|^2 dx, \tag{8}$$

where $c = c(n, \Lambda) > 0$.

Proof. We define $v := |u|^{\frac{\alpha}{2}} u\eta$, therefore $\nabla v = (\frac{\alpha}{2}+1)|u|^{\frac{\alpha}{2}}\eta\nabla u + |u|^{\frac{\alpha}{2}} u\nabla\eta$.

To estimate the L^2-norm of ∇v, we use Young's inequality and Lemma 1.0.15 to get

$$\int_\Omega |\nabla v|^2 dx \leq c(\alpha+2)^2 \int_\Omega |u|^{\alpha+2}|\nabla\eta|^2 dx,$$

We can use Sobolev's inequality for v due to the fact that $n \geq 3$, which implies that

$$\|v\|^{\frac{2}{2^*}}_{L^{2^*}(\Omega)} = \left(\int_\Omega |u|^{\frac{2^*(\alpha+2)}{2}} \eta^{2^*} dx\right)^{\frac{2}{2^*}} \leq C \int_\Omega |u|^{\alpha+2}|\nabla\eta|^2 dx,$$

which concludes the proof. □

As a consequence we have the following corollary:

Corollary 1.0.16.1. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of* (1) *where $\mathbb{A}$ is symmetric and fulfils* (2)*. Then $u \in L^q_{loc}(\Omega)$ for every $q \geq 1$. Moreover, for every $\alpha \geq 0$, every ball $B_r(y) \subset \Omega$ and every $0 < r' < r$, we have the following reverse inequality:*

$$\left(\int_{B_{r'}(y)} |u|^{\frac{2^*(\alpha+2)}{2}} dx\right)^{\frac{2}{2^*(\alpha+2)}} \leq \frac{c^{\frac{1}{\alpha+2}}(\alpha+2)^{\frac{2}{\alpha+2}}}{(r-r')^{\frac{2}{\alpha+2}}} \left(\int_{B_r(y)} |u|^{\alpha+2} dx\right)^{\frac{1}{\alpha+2}}, \quad (9)$$

where $c = c(n,\Lambda) > 0$.

Proof. For any compact subset of Ω, there exists $\eta_K \in C^\infty_c(\Omega)$ with $\eta_K\restriction_K \equiv 1$. Starting with $\alpha = 0$, one can iterate (8) with $\eta = \eta_K$ to get $u \in L^{q_k}_{loc}(\Omega)$ for $q_k := 2^*k, k \in \mathbb{N}$. Since $L^p_{loc}(\Omega) \subset L^q_{loc}(\Omega)$, integrability holds for all $q \geq 1$. To derive (9) we take a cut-off function $\eta \in C^\infty_c(\Omega)$ such that $\mathbb{1}_{B_{r'}(y)} \leq \eta \leq \mathbb{1}_{B_r(y)}$ with $|\nabla\eta| < \frac{2}{r-r'}$, and apply Lemma 1.0.16. □

We can now prove *Lemma* 1.0.14 by iterating the previous corollary:

Proof of Lemma 1.0.14. We fix a ball $B_r(y) \subset \Omega$ and define $(\alpha_i)_{\{i\in\mathbb{N}\}}$ as $\alpha_i := 2\left(\frac{2^*}{2}\right)^i - 2$. We also define $(r_i)_{\{i\in\mathbb{N}\}}$ as $r_i := \frac{r}{2} + \frac{r}{2^{i+1}}$. By defining $\beta_{i+1} := \alpha_{i+1} + 2 = \frac{2^*}{2}\beta_i$ and

$$M_i := \left(\int_{B(y,r_i)} |u|^{\beta_i} dx\right)^{\frac{1}{\beta_i}},$$

and applying *Corollary* 1.0.16.1 with $r = r_i$, $r' = r_{i+1}$ and $\alpha = \alpha_i$ we get

$$M_{i+1} \leq c^{\frac{1}{\beta_i}} \beta_i^{\frac{2}{\beta_i}} \left(\frac{r}{2^{i+2}}\right)^{-\frac{2}{\beta_i}} M_i.$$

By iterating *Corollary* 1.0.16.1 we have $M_{i+1} \leq c_i M_0$, where $\lim_{i\to\infty} c_i = c_\infty < +\infty$. Therefore

$$\sup_{B_{\frac{r}{2}}(y)} |u| = \lim_{i\to+\infty}\left(\int_{B_{\frac{r}{2}}(y)} |u|^i dx\right)^{\frac{1}{i}} \leq \lim_{i\to+\infty} M_i \leq c_\infty M_0 \leq c\left(\fint_{B_r(y)} |u|^2 dx\right)^{\frac{1}{2}}.$$

□

Lemma 1.0.14 can be generalized to non-negative weak sub-solutions:

Lemma 1.0.17. *Let $u \in W^{1,2}(\Omega)$ be a non-negative weak sub-solution of equation* (1)*. Then $u \in L^\infty_{loc}(\Omega)$. Moreover, for every ball $B_r(y) \subset \Omega$ and $0 < \sigma < 1$, we have*

$$\sup_{B_{\sigma r}(y)} u \leq \frac{c}{(1-\sigma)^{\frac{n}{2}}} \left(\fint_{B_r(y)} u^2 dx \right)^{\frac{1}{2}},$$

where $c = c(n, \Lambda) > 0$.

By iterating *Lemma* 1.0.17, one can strengthen the result by lowering the exponent on the right-hand side:

Lemma 1.0.18. *Let $u \in W^{1,2}(\Omega)$ be a non-negative weak sub-solution of equation* (1)*. Then $u \in L^\infty_{loc}(\Omega)$. Moreover, for every ball $B_r(y) \subset \Omega$ and $0 < \sigma < 1$, we have*

$$\sup_{B_{\sigma r}(y)} u \leq \frac{c}{(1-\sigma)^{\frac{n}{q}}} \left(\fint_{B_r(y)} u^q dx \right)^{\frac{1}{q}} \quad \textit{for} \ \ 0 < q \leq 2,$$

$$\sup_{B_{\sigma r}(y)} u \leq c \left(\fint_{B_r(y)} u^q dx \right)^{\frac{1}{q}} \quad \textit{for} \ \ q > 2,$$

where $c = c(n, \Lambda, q) > 0$.

Proof. We proceed in two steps. We first assume $q \leq 2$. Take $B_r(y) \subset \Omega$ and some $0 < \sigma < 1$. Define $(\sigma_i)_{i \in \mathbb{N}}$ as $\sigma_i := 1 - \frac{1-\sigma}{2^i}$, so that σ_i varies monotonically from σ to 1 as i varies from 0 to ∞. We can now use *Lemma* 1.0.17 with $r = \sigma_{i+1} r$ and $\sigma = \frac{\sigma_i}{\sigma_{i+1}}$ and get

$$\begin{aligned} M_i := \sup_{B_{\sigma_i r}(y)} u &\leq \frac{c}{(1-\frac{\sigma_i}{\sigma_{i+1}})^{\frac{n}{2}}} \left(\fint_{B_{\sigma_i r}(y)} u^2 dx \right)^{\frac{1}{2}} \\ &\leq \frac{c}{(1-\frac{\sigma_i}{\sigma_{i+1}})^{\frac{n}{2}}} \left(\fint_{B_{\sigma_i r}(y)} u^q dx \right)^{\frac{1}{2}} \left(\sup_{B_{\sigma_{i+1} r}(y)} u \right)^{\frac{2-q}{2}} \\ &\leq \frac{c}{(1-\frac{\sigma_i}{\sigma_{i+1}})^{\frac{n}{2}}} \left(\fint_{B_{\sigma_i r}(y)} u^q dx \right)^{\frac{1}{2}} M_{i+1}^{\frac{2-q}{q}}. \end{aligned}$$

Iterating this inequality we obtain the inequality for $q \leq 2$.
Now we will assume $q > 2$. Using the above result with $q = 2$:

$$\sup_{B_{\sigma r}(y)} u \leq c \left(\fint_{B_r(y)} u^2 dx \right)^{\frac{1}{2}} = c \left(\fint_{B_r(y)} u^2 dx \right)^{\frac{1}{q} \frac{q}{2}}$$

and using Jensen's inequality on the convex function $x \to x^{\frac{q}{2}}$ $(q > 2)$ we obtain:

$$\sup_{B_{\sigma r}(y)} u \leq c\left(\fint_{B_r(y)} u^q dx \right)^{\frac{1}{q}}$$

which completes the proof. $\square$

Harnack's inequality: inf

Lemma 1.0.19. *Let $u \in W^{1,2}(\Omega)$ be a non-negative weak supersolution of (1), where $\mathbb{A}$ fulfils (2). Then there are $q = q(n, \Lambda) > 0$ such that, for every ball $B_{2r}(y) \subset \Omega$, we have*

$$\inf_{B_{\frac{r}{2}}(y)} u \geq c\left(\fint_{B_r(y)} u^q dx \right)^{\frac{1}{q}}. \tag{10}$$

Remark 1.0.19.1. *Given $\epsilon > 0$, we can assume $u \geq \epsilon$ in Ω by replacing u with $u + \epsilon$.*

The main idea of the proof is the fact that $\log u$ is a function of bounded mean oscillation.

Lemma 1.0.20. *Let $u \in W^{1,2}(\Omega)$ be a non-negative weak solution of (1), where $\mathbb{A}$ fulfils (2). Suppose that $u \geq \epsilon$ in Ω for some $\epsilon > 0$. Then, for any $q > 0$, there is a constant $c = c(n, \Lambda, q) > 0$ such that the following holds:*

$$\inf_{B_{\frac{r}{2}}(y)} u \geq c\left(\fint_{B_r(y)} u^{-q} dx \right)^{-\frac{1}{q}}.$$

Proof. We claim that u^{-1} is a sub-solution of (1): indeed $u^{-1} \in H^1_{loc}(\Omega)$ and for any non-negative $\eta \in C^\infty_c(\Omega)$, define $\phi := \eta u^{-2}$. We test (1) with ϕ to get

$$\begin{aligned} 0 &= \int_\Omega \langle \nabla \phi, \mathbb{A}(x) \nabla u \rangle dx \\ &= \int_\Omega u^{-2} \langle \nabla \eta, \mathbb{A}(x) \nabla u \rangle dx - 2 \int_\Omega \eta u^{-3} \langle \nabla u, \mathbb{A} \nabla u \rangle dx. \end{aligned}$$

The last term is non-positive, therefore we have

$$\int_\Omega \langle \nabla \eta, \mathbb{A}(x) \nabla u^{-1} \rangle = - \int_\Omega u^{-2} \langle \nabla \eta, \mathbb{A}(x) \nabla u \rangle dx \leq 0,$$

and so u^{-1} is a sub-solution. We can then apply *Lemma* 1.0.18 with $\sigma = \frac{1}{2}$ and get

$$\inf_{B_{\frac{r}{2}}(y)} u \geq c\left(\fint_{B_r(y)} u^{-q} dx \right)^{-\frac{1}{q}}.$$

$\square$

We can now show that $\log u$ is a function of bounded mean oscillation:

Lemma 1.0.21. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of* (1)*, where $\mathbb{A}$ fulfils* (2)*. Suppose that $u \geq \epsilon$ in Ω for some $\epsilon > 0$. Then, for every ball $B_{2r}(y) \subset \Omega$, we have*

$$\int_{B_r(y)} |\nabla v|^2 dx \leq c r^{n-2},$$

where $v = \log u$ and $c = c(n, \Lambda) > 0$.

Proof. Fix $\eta \in C_c(\Omega)$ and let $\phi = \eta^2 u^{-1}$. We again test (1) with ϕ and get that

$$\begin{aligned} 0 &= \int_\Omega \langle \nabla \phi, \mathbb{A}(x) \nabla u \rangle dx \\ &= -\int_\Omega \eta^2 u^{-2} \langle \nabla u, \mathbb{A} \nabla u \rangle dx + 2 \int_\Omega \eta u^{-1} \langle \nabla \eta, \mathbb{A}(x) \nabla u \rangle dx \end{aligned}$$

which using the ellipticity and continuity of $\mathbb{A}$ as well as the Cauchy-Schwarz inequality like before, gives

$$\int_\Omega |\nabla v|^2 \eta^2 dx \leq 4\Lambda^2 \int_\Omega |\nabla \eta|^2 dx.$$

The result follows by choosing $\eta \in C_c^\infty(\Omega)$ such that $\mathbb{1}_{B_{2r}(y)} \leq \eta \leq \mathbb{1}_{B_r(y)}$ and $|\nabla \eta| \leq \frac{2}{r}$. ☐

Proof of Lemma 1.0.19. We define v_{B_r} as the function $v_{B_r(y)} := \fint_{B_r(y)} v(x) dx$. On any ball $B_{2r}(y) \subset \Omega$, we can use the Poincaré-Wirtinger inequality and *Lemma* 1.0.21 to obtain

$$\int_{B_r(y)} |v - v_{B_r(y)}|^2 dx \leq c(n) r^2 \int_{B_r(y)} |\nabla v|^2 \leq c(n, \Lambda) r^2 r^{n-2},$$

which yields

$$\fint_{B_r(y)} |v - v_{B_r(y)}|^2 dx \leq c(n, \Lambda).$$

In particular $v = \log u$ is a function of bounded mean oscillation. We can then use the John-Nirenberg inequality [5, *Theorem* 7.21] which yields

$$\fint_{B_r(y)} \exp(c_1 |v - v_{B_r(y)}|) dx \leq c_2$$

for $c_1 = c_1(n, \Lambda) > 0$ and $c_2 = c_2(n, \Lambda) > 0$. Then we have

$$\begin{aligned} \fint_{B_r(y)} u^{c_1} dx \fint_{B_r(y)} u^{-c_1} dx &= \fint_{B_r(y)} \exp(c_1(v - v_{B_r(y)})) dx \fint_{B_r(y)} \exp(c_1(v_{B_r(y)} - v)) dx \\ &\leq \left(\fint_{B_r(y)} \exp(c_1 |v - v_{B_r(y)}|) dx \right)^2 \leq (c_2)^2. \end{aligned}$$

This, together with *Lemma* 1.0.20, proves (10) with $q = c_1$ which concludes the proof. ☐

Combining *Lemma* 1.0.18 and 1.0.19 one concludes the proof of the Harnack's inequality *Theorem* 1.0.12. We can now be able to conclude the Hölder continuity of solutions due to the iteration of Moser:

Proof of Theorem 1.0.13. Thanks to *Lemma* 1.0.14, we know that $\sup u$ and $\inf u$ are locally bounded, so we only need an estimate for the Hölder semi-norm. We take $B_R(x_0) \Subset \Omega$ and define $m(x_0, R) := \inf_{B_R(x_0)} u$ and $M(x_0, R) := \sup_{B_R(x_0)} u$. Next we can apply *Harnack* inequality *Theorem* 1.0.12 to the non-negative functions $M(x_0, R) - u$ and $u - m(x_0, R)$, and get

$$\begin{aligned} M(x_0, R) - m(x_0, \frac{R}{2}) &\leq c(n, \Lambda)(M(x_0, R) - M(x_0, \frac{R}{2})), \\ M(x_0, \frac{R}{2}) - m(x_0, R) &\leq c(n, \Lambda)(m(x_0, \frac{R}{2}) - m(x_0, R)), \end{aligned}$$

where the constants are the same on both lines and summing we obtain

$$osc_{B_R(x_0)} u + osc_{B_{\frac{R}{2}}(x_0)} \leq c(n, \Lambda)\Big(osc_{B_R(x_0)} u - osc_{B_{\frac{R}{2}}(x_0)} u\Big).$$

Hence, we have

$$osc_{B_{\frac{R}{2}}(x_0)} u \leq 2^{-\alpha} osc_{B_R(x_0)} u,$$

for some $\alpha \in (0, 1]$ satisfying

$$2^{-\alpha} \geq \frac{c(n, \Lambda) - 1}{c(n, \Lambda) + 1}.$$

Note that α does not depend on x_0. We can iterate this estimate and find

$$osc_{B_{2^{-j}R}(x_0)} u \leq 2^{-j\alpha} osc_{B_R(x_0)} u \quad \text{for all } j \in \mathbb{N}.$$

For $r \in (0, R]$, there is a unique $j_0 \in \mathbb{N}$ such that $2^{-j_0-1}R < r \leq 2^{-j_0}R$, from which we get

$$osc_{B_r(x_0)} u \leq osc_{B_{2^{j_0}R}(x_0)} u \leq 2^{-j_0\alpha} osc_{B_R(x_0)} u \leq 2^{\alpha}\Big(\frac{r}{R}\Big)^{\alpha} osc_{B_R(x_0)} u. \tag{11}$$

which gives the result with the same argument of the proof of *Theorem* 1.0.2 at the end of page 9 and start of page 10. □

2 Applications of Moser-Harnack inequality

So we have shown that for any $u \in W^{1,2}(\Omega)$ weak solution (1) of the elliptic equation

$$L_{\mathbb{A}} u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left[a^{ij}(x)\frac{\partial}{\partial x_j}u(x)\right] = 0$$

where the coefficients of the matrix $\mathbb{A} = \{a^{ij}\}_{\{i.j\in\{1,\ldots,n\}\}}$ are measurable and satisfy the ellipticity condition (2), then u is Hölder continuous for some $\alpha \in (0,1)$. This regularity result has significant applications, the first of which is a stronger version of the maximum principle and the second of which is an analogue of $Liouville's$ theorem.

Theorem 2.0.1. *Let $u \in W^{1,2}(\Omega)$ be a weak sub-solution of* (1)*, i.e. $L_{\mathbb{A}} \geq 0$, where $\mathbb{A}$ fulfils* (2)*. Suppose that for some open ball $B_R(y_0) \subset\subset \Omega$ we have*

$$\sup_{B_R(y_0)} u = \sup_{\Omega} u. \tag{12}$$

Then we have that u is constant on the whole domain of Ω.

Proof. Due to the fact that (12) holds, then there is another ball $B_{r_0}(x_0)$ with $r \in (0,R)$ and $B_{4r_0}(x_0) \subset \Omega$ such that $\sup_{B_{r_0}(x_0)} u = \sup_{\Omega} u$. Moreover by Theorem 1.0.13 $\sup_{\Omega} u < M$, i.e. must be finite, then $M - u$ is a positive super-solution to $L_{\mathbb{A}}$, and hence applying *Lemma* 1.0.19 and taking the limit we obtain $M = \sup_{\Omega} u$. Again applying *Lemma* 1.0.19 we obtain

$$\fint_{B_{2r_0}(x_0)} (M-u) \leq c \inf_{B_{x_0}(r_0)} (M-u) = 0.$$

Since M is equal to the *supremum* of u, we obtain that $u \equiv M$ on $B_{2r_0}(x_0)$. To extend this result let $y \in \Omega$ be arbitrary. The idea is to construct a series of balls that connect x and y where in every ball u is constant, therefore $u(x) = u(y)$. Then there exists a sequence of ball $\{B_i\}_{\{i=1,\ldots,n\}}$ such that $B_i := B_{r_i}(x_i)$, $B_{4r_i}(x_i) \subset \Omega$ and $B_{i-1} \cap B_i \neq \emptyset$ for $i = 0, \ldots, N-1$. With the same argument as before u is constant on every ball, therefore on every point $y \in \Omega$ chosen arbitrarily which implies $u \equiv M$ on Ω. □

This result can be improved with stronger assumptions:

Theorem 2.0.2. *Let $u \in W^{1,2}(\mathbb{R}^n)$ be a weak solution of* (1)*, i.e. $L_{\mathbb{A}} = 0$, where $\mathbb{A}$ fulfils* (2)*. Then we have that u is a constant function.*

Proof. By the assumption (2) we know that there exists α such that $\alpha < \inf_{\mathbb{R}^n} u$. Then we know that $u - \alpha$ is a positive sub-solution to $L_{\mathbb{A}} = 0$ on all of $\mathbb{R}^n$. Thus applying Harnack's inequality we obtain

$$0 \leq \sup_{B_R(0)} u - \alpha \leq c\Big[\inf_{B_R(0)} u - \alpha\Big],$$

and taking the limit as $R \to \infty$ we obtain

$$0 \le \sup_{\mathbb{R}^n} u - \alpha \le c\Big[\inf_{\mathbb{R}^n} u - \alpha\Big] = 0,$$

which implies that u is constant on all of $\mathbb{R}^n$. □

3 Regularity

Theorem 3.0.1. *Let $\mathbb{A}$ a matrix with regular and bounded entries which fulfils (2). If $u \in W^{1,2}(\Omega)$ is a weak solution to $L_{\mathbb{A}} = 0$ in Ω, i.e., for all $\phi \in W_0^{1,2}(\Omega)$*

$$\int_\Omega \langle \nabla\phi(x), \mathbb{A}(x)\nabla u(x)\rangle dx = 0. \tag{13}$$

then $u \in C^\infty(\Omega)$.

To prove this theorem we will use the Hölder continuity of weak solutions to the elliptic equation $L_{\mathbb{A}}u = 0$ proved with both methods of *De Giorgi* and *Moser* combined and with a few lemmas:

Lemma 3.0.2. *Let $\mathbb{A}$ a matrix with regular and bounded entries which fulfils (2). Let $u \in W^{1,2}(\Omega)$ be a weak solution to $L_{\mathbb{A}} = 0$ in Ω. For any $X \subset\subset \Omega$ we have that $u \in W^{2,2}(X)$, and the inequality*

$$\|u\|_{W^{2,2}(X)} \le c\|u\|_{W^{1,2}(\Omega)}$$

holds, where $c = c(\lambda, \Lambda, dist(X, \partial\Omega))$.

Proof. Let $A^j : \mathbb{R}^n \to \mathbb{R}$ be the $j-th$ row of the matrix $\mathbb{A}$. Let h be such that $|h| < dist(supp\phi, \partial\Omega)$ so that $\phi_{k,-h}(x) := \phi(x - he_k) \in W_0^{1,2}(\Omega)$. We then have that

$$\begin{aligned} 0 &= \int_\Omega \sum_{i=1}^n A^i(Du(x))D_i\phi_{k,-h}(x)dx = \int_\Omega \sum_{i=1}^n A^i(Du(x))D_i\phi(x - he_k)dx \\ &= \int_\Omega \sum_{i=1}^n A^i(Du(y + he_k))D_i\phi(y)dy \\ &= \int_\Omega \sum_{i=1}^n A^i((Du)_{k,h})D_i\phi. \end{aligned}$$

By subtracting from the above (13), which can be rewritten as

$$\int_\Omega \langle \nabla\phi(x), \mathbb{A}(x)\nabla u(x)\rangle dx = \int_\Omega \sum_{i=1}^n A^i(Du)D_i\phi = 0,$$

we obtain that

$$\int_\Omega \sum_{i=1}^n \big[A^i(Du(x + he_k)) - A^i(Du(x))\big]D_i\phi(x) = 0. \tag{14}$$

And using that for a.e. $x \in \Omega$ by *Lagrange* theorem:

$$A^i(Du(x+he_k)) - A^i(Du(x)) = \int_0^1 \frac{d}{dt} A^i(tDu(x+he_k) + (1-t)Du(x))dt$$
$$= \int_0^1 \left[\sum_{i=1}^n A^i_{y_j}(tDu(x+he_k) + (1-t)Du(x))D_j(u(x+he_k) - u(x)) \right] dt$$

where $A^i_{y_j} := \frac{\partial}{\partial y_j} A^i$. We can now define a_h^{ij} and $\triangle_k^h u(x)$ as the following:

$$a_h^{ij}(x) := \int_0^1 A^i_{yj}[tDu(x+he_k) + (1-t)Du(x))]dt$$
$$\triangle_k^h u(x) := \frac{u(x+he_k) - u(x)}{h}.$$

So (14) can be written as

$$\int_\Omega \sum_{i,j=1}^n a_h^{ij}(x) D_j(\triangle_k^h u(x)) D_i \phi(x) dx = 0. \tag{15}$$

Notice that the coefficients a_h^{ij} also satisfy the ellipticity conditions. Then let $\eta \in C_0^1(X')$ where $X \subset\subset X' \subset\subset \Omega$ with $max\{dist(X', \partial\Omega), dist(X, \partial X')\} > \frac{1}{4} dist(X, \partial\Omega)$, such that η is bounded as follows: $0 \le \eta \le 1$, $\eta(x) = 1$ for all $x \in X$, $|D\eta| \le \frac{8}{dist(X,\partial\Omega)}$ and $|2h| < dist(X', \partial\Omega)$. Using the ellipticity conditions on (15) we deduce that

$$\lambda \int_\Omega |D\triangle_k^h u|^2 \eta^2 \le \int_\Omega \sum_{i,j=1}^n a_h^{ij} (D_j \triangle_k^h u)(D_i \triangle_k^h u)\eta^2$$
$$= -\int_\Omega \sum_{i,j=1}^n a_h^{ij} D_j \triangle_k^h u \cdot 2\eta (D_i \eta) \triangle_k^h u.$$

But from here we can apply $Young's$ inequality to get that for any $\epsilon > 0$:

$$\lambda \int_\Omega |D\triangle_k^h u|^2 \eta^2 \le \epsilon \Lambda \int_\Omega |D\triangle_k^h u|^2 + \frac{\Lambda}{\epsilon} \int_\Omega |\triangle_k^h u|^2 |D\eta|^2$$

and if we take $\epsilon = \frac{\lambda}{2\Lambda}$ we obtain that

$$\int_\Omega |D\triangle_k^h u|^2 \eta^2 \le c \int_{X'} |\triangle_k^h u|^2 \le c \int_\Omega |Du|^2.$$

Therefore we have shown that $\|D\triangle_k^h u\|_{L^2(X)} \le c\|Du\|_{L^2(\Omega)}$. We thus deduce that $D^2 u \in L^2(X)$, and hence

$$\|D^2 u\|_{L^2(X)} \le c\|Du\|_{L^2(\Omega)}, \tag{16}$$

from which it follows that $u \in W^{2,2}(X)$, as was to be shown.

$\square$

We note that $\triangle_k^h$ is an approximation to the derivative, so that in the limit as $h \to 0$, if we let a^{ij} and v be defined as $a^{ij} := A^i_{y_j}(Du(x))$, $v = D_k u$. Then we obtain that

$$\int_\Omega \sum_{i,j=1}^n a^{ij}(x) D_j v D_i \phi = 0 \quad \text{for all } \phi \in H_0^{1,2}(\Omega).$$

Therefore applying the fact that a weak solution of an elliptic partial differential equation is Hölder continuous (*Theorem* 1.0.2), we deduce trivially the following lemma.

Lemma 3.0.3. *Let $u \in W^{1,2}(\Omega)$ be a weak solution of* (1), *where $\mathbb{A}$ fulfils* (2). *Using Theorem* 1.0.2 *we have that $Du \in C^\alpha(\Omega)$ for some Hölder exponent $\alpha \in (0,1)$. In particular, this means that $u \in C^{1,\alpha}(\Omega)$ for some $0 < \alpha < 1$.*

Therefore for each $k = 1, \dots, n$ we have that $v = D_k u$ is a solution to the divergence-type equation

$$\sum_{i,j=1}^n D_i(a^{ij}(x) D_j v) = 0 \tag{17}$$

where the coefficients $a^{ij}(x)$ satisfy the ellipticity requirements (2). In order to prove Theorem 3.0.1 we need to prove some lemmas, the first of which is a different form of *Caccioppoli inequality* then the one seen before:

Lemma 3.0.4. *Let $u \in W^{1,2}(\Omega)$ be a weak solution to the differential equation*

$$\sum_{i,j=1}^n D_j(A^{ij} D_i u) = 0 \tag{18}$$

in Ω where the matrix $\{A^{ij}\}_{\{i,j=1,\dots,n\}}$ fulfils (2). *Then for any $x_0 \in \Omega$ and radius r with $r < R < dist(x_0, \partial\Omega)$, we have that*

$$\int_{B_r(x_0)} |Du|^2 \leq \frac{c}{(R-r)^2} \int_{B_R(x_0)\setminus B_r(x_0)} |u-k|^2 \tag{19}$$

for any $k \in \mathbb{R}$.

Proof. We define a cut-off function $\eta \in W_0^{1,2}(B_R(x_0))$ by confining $0 \leq \eta \leq 1$ with the following conditions: $\eta = 1$ on $B_R(x_0)$ and $|D\eta| \leq \frac{2}{R-r}$. Then let ϕ be a test function defined by $\phi = (u-\mu)\eta^2$ so that we can get

$$\begin{aligned} 0 &= \int_\Omega \sum_{i,j=1}^n A^{ij} D_i u D_j((u-\mu)\eta^2) \\ &= \int_\Omega \sum_{i,j=1}^n A^{ij} D_i u D_j u \eta^2 + 2\int_\Omega \sum_{i,j=1}^n A^{ij} D_i u (u-\mu)\eta D_j \eta. \end{aligned}$$

From this we can use the ellipticity of the matrix and the fact that $D\eta = 0$ on the ball $B_r(x_0)$, since η is constant on $B_R(x_0)$, to deduce from $Young's$ inequality that

$$\begin{aligned}\lambda \int_{B_R(x_0)} |Du|^2\eta^2 &\leq \int_{B_R(x_0)} \sum_{i,j=1}^{n} A^{ij} D_i u D_j u \eta^2 \\ &\leq \epsilon\Lambda n \int_{B_R(x_0)} |Du|^2\eta^2 + \frac{\Lambda}{\epsilon} n \int_{B_R(x_0)\setminus B_r(x_0)} |D\eta|^2|u-\mu|^2\end{aligned}$$

for any $\epsilon > 0$. In particular, taking $\epsilon = \frac{\lambda}{2\Lambda n}$ we obtain that

$$\int_{B_R(x_0)} |Du|^2\eta^2 \leq \frac{c}{(R-r)^2} \int_{B_R(x_0)\setminus B_r(x_0)} |u-\mu|^2.$$

Using the fact that (by definition of η) $\int_{B_r(x_0)} |Du|^2 \leq \int_{B_R(x_0)} |Du|^2\eta^2$, the lemma is now proved. $\square$

Lemma 3.0.5 (*Campanato* inequalities)**.** *Let $u \in W^{1,2}(\Omega)$ be a weak solution to the differential equation*

$$\sum_{i,j=1}^{n} D_j(A^{ij} D_i u) = 0 \tag{20}$$

in Ω where the matrix $\{A^{ij}\}_{\{i,j=1,\dots,n\}}$ fulfils (2)*. Then for any $x_0 \in \Omega$ and radius r with $r < R < dist(x_0, \partial\Omega)$, we have the following two inequalities:*

$$\int_{B_r(x_0)} |u|^2 \leq c_3\left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |u|^2, \tag{21}$$

$$\int_{B_r(x_0)} \left|u - \fint_{B_R(x_0)} u\right|^2 \leq c_4\left(\frac{r}{R}\right)^{n+2} \left|u - \fint_{B_R(x_0)} u\right|^2. \tag{22}$$

Proof. Without loss of generality we assume that $r < \frac{R}{2}$. So choose $k > n$; by the *Sobolev* embedding theorem we have that $W^{k,2}(B_R(x_0)) \subset C^0(B_R(x_0))$. So $u \in W^{k,2}(B_{\frac{R}{2}})$, and hence we have

$$\begin{aligned}\int_{B_r(x_0)} |u|^2 &\leq c_5 r^n \sup_{B_r(x_0)} |u|^2 \leq c_6 \frac{r^n}{R^{n-2k}} \|u\|_{W^{k,2}(B_{R/2}(x_0))} \\ &\leq c_3 \frac{r^n}{R^n} \int_{B_R(x_0)} |u|^2.\end{aligned}$$

Hence we have proved (21). Testing (21) with Du we get that

$$\int_{B_r(x_0)} |Du|^2 \leq c_3\left(\frac{r}{R}\right)^n \int_{B_{\frac{R}{2}}(x_0)} |Du|^2 \tag{23}$$

and so by the *Poincarè* inequality, we get that

$$\int_{B_r(x_0)} \Big| u - \fint_{B_R(x_0)} u \Big|^2 \leq c_8 r^2 \int_{B_r(x_0)} |Du|^2. \tag{24}$$

Lemma 3.0.4 implies that

$$\int_{B_{\frac{R}{2}}(x_0)} |Du|^2 \leq \frac{c_9}{R^2} \int_{B_R(x_0)} \Big| u - \fint_{B_R(x_0)} u \Big|^2. \tag{25}$$

Combining the inequalities (23), (24), (25) together proves the lemma. □

Using *Campanato's* inequalities, we can derive the desired regularity result.

Theorem 3.0.6. *Let the matrix $\mathbb{A}$ fulfils* (2) *with entries $a^{ij}(x) \in C^\alpha(\Omega)$ $\forall i, j \in \{1, \ldots, n\}$. We then have that any weak solution v to*

$$\sum_{i,j}^{n} D_j(a^{ij}(x) D_i v) = 0 \tag{26}$$

is a $C^{1,\alpha'}(\Omega)$ function for any $\alpha' \in (0, \alpha)$.

Proof. For an arbitrary $x_0 \in \Omega$ we rewrite a^{ij} as $a^{ij} = a^{ij}(x_0) + (a^{ij}(x) - a^{ij}(x_0))$. Then if we define $A^{ij} := a^{ij}(x_0)$, equation (26) turns into

$$\sum_{i,j=1}^{n} D_j(A^{ij} D_i v) = \sum_{i,j=1}^{n} D_j((a^{ij}(x_0) - a^{ij}(x)) D_i v) = \sum_{j=1}^{n} D_j(f^j(x))$$

where we defined f^j as the following sum

$$f^j(x) := \sum_{i=1}^{n} ((a^{ij}(x_0) - a^{ij}(x)) D_i v).$$

We therefore have the following equality for each $\phi \in W_0^{1,2}(\Omega)$:

$$\int_\Omega \sum_{i,j=1}^{n} A^{ij} D_i v D_j \phi = \int_\Omega \sum_{j=1}^{n} f^j D_j \phi. \tag{27}$$

From here taking some ball $B_R(x_0) \subset \Omega$, and letting $w \in W^{1,2}(\Omega)$ be the weak solution inside the ball to

$$\sum_{i,j=1}^{n} D_j(A^{ij} D_i w) = 0 \text{ inside } B_R(x_0);\ w \equiv v \text{ on } \partial B_R(x_0). \tag{28}$$

Such a function exists by the Lax-Milgram lemma. Then we know that w is the solution to the differential equation for all $\phi \in W_0^{1,2}(B_R(x_0))$ inside the ball:

$$\int_{B_R(x_0)} \sum_{i,j=1}^{n} A^{ij} D_i w D_j \phi = 0. \tag{29}$$

Noting that (28) is a linear differential equation with constant coefficients, we know that w is a solution implies that $D_k w$ is as well for each $k \in \{1, \ldots, n\}$. Thus we get that

$$\int_{B_r(x_0)} |Dw|^2 \leq c_{10}\Big(\frac{r}{R}\Big)^n \int_{B_R(x_0)} |Dw|^2, \tag{30}$$

and since the functions w and v are equal on the boundary of the ball $B_R(x_0)$, we can set $\phi = v - w$ to be a test function in (29) to obtain that

$$\int_{B_R(x_0)} \sum_{i,j=1}^n A^{ij} D_i w D_j w = \int_{B_R(x_0)} \sum_{i,j=1}^n A^{ij} D_i w D_j v. \tag{31}$$

We then use *Cauchy-Schwarz* inequality together with (31) and (2) to get that

$$\int_{B_R(x_0)} |Dw|^2 \leq \Big(\frac{n\Lambda}{\lambda}\Big)^2 \int_{B_R(x_0)} |Dv|^2. \tag{32}$$

So then (27) and (29) give us that for any $\phi \in W_0^{1,2}(B_R(x_0))$, we have

$$\int_{B_R(x_0)} \sum_{i,j=1}^n A^{ij} D_i(v-w) D_j \phi = \int_{B_R(x_0)} \sum_{i,j=1}^n f^j D_j \phi.$$

Since this holds for any $\phi \in W_0^{1,2}(B_R(x_0))$, we can test $\phi = v - w$ to obtain that

$$\begin{aligned}
\int_{B_R(x_0)} |D(v-w)|^2 &\leq \frac{1}{\lambda} \int_{B_R(x_0)} \sum_{i,j=1}^n A^{ij} D_i(v-w) D_j(v-w) \\
&= \frac{1}{\lambda} \int_{B_R(x_0)} \sum_{j=1}^n f^j D_j(v-w) \\
&\leq \frac{1}{\lambda} \Big[\int_{B_R(x_0)} |D(v-w)|^2\Big]^{\frac{1}{2}} \Big[\int_{B_R(x_0)} \sum_{j=1}^n |f^j|^2\Big]^{\frac{1}{2}},
\end{aligned}$$

where in the last inequality we used *Cauchy-Schwarz* inequality. We thus deduce that

$$\int_{B_R(x_0)} |D(v-w)|^2 \leq \frac{1}{\lambda^2} \int_{B_R(x_0)} \sum_{j=1}^n |f^j|^2. \tag{33}$$

Putting all of the previous inequalities together, we have by (30) and (32) that for any $0 < r \leq R$,

$$\begin{aligned}
\int_{B_r(x_0)} |Dv|^2 &\leq 2 \int_{B_R(x_0)} |Dw|^2 + 2 \int_{B_R(x_0)} |D(v-w)|^2 \\
&\leq c_{11}\Big(\frac{r}{R}\Big)^n \int_{B_R(x_0)} |Dv|^2 + 2 \int_{B_R(x_0)} |D(v-w)|^2.
\end{aligned}$$

Therefore, applying (33) and $a^{ij}(x) \in C^{\alpha}(\Omega)\ \forall i,j \in \{1,\dots,n\}$, we obtain that

$$\begin{aligned}\int_{B_r(x_0)} |D(v-w)|^2 &\le \int_{B_R(x_0)} |D(v-w)|^2 \\ &\le \frac{1}{\lambda^2}\int_{B_R(x_0)} \sum_{j=1}^{n} |f^j|^2 \\ &\le \frac{1}{\lambda^2} \sup_{i,j} |a^{ij}(x_0) - a^{ij}(x)|^2 \int_{B_R(x_0)} |Dv|^2 \\ &\le C\,R^{2\alpha} \int_{B_R(x_0)} |Dv|^2,\end{aligned}$$

from which we can deduce the following estimate

$$\int_{B_R(x_0)} |Dv|^2 \le \gamma\Big[\Big(\frac{r}{R}\Big)^n + R^{2\alpha}\Big] \int_{B_R(x_0)} |Dv|^2. \tag{34}$$

We can then bound the $R^{2\alpha}$ term in the above with the following lemma.

Lemma 3.0.7. *Let $\sigma(r)$ be a positive increasing function such that for any $0 < r \le R \le R_0$ with $\mu > \nu$ and $\delta \le \delta_0(\gamma,\mu,\nu)$,*

$$\sigma(r) \le \gamma\Big(\Big(\frac{r}{R}\Big)^{\mu} + \delta\Big)\sigma(R) + \kappa R^{\nu}.$$

If δ_0 is small enough, then again for $0 < r \le R \le R_0$ we have that

$$\sigma(r) \le \gamma_1\Big(\frac{r}{R}\Big)^{\nu}\sigma(R) + \kappa_1 r^{\nu}$$

where $\gamma_1 = \gamma_1(\gamma,\mu,\nu)$ and $\kappa_1 = \kappa_1(\gamma,\mu,\nu,\kappa)$.

Proof. Let $t \in (0,1)$ and $R < R_0$. By assumption, we thus have that

$$\sigma(tR) \le \gamma t^{\mu}(1 + \delta t^{-\mu})\sigma(R) + \kappa R^{\nu}.$$

So let t be such that $t^{\lambda} = 2\gamma t^{\mu}$, with $\nu < \lambda < \mu$, and assume that $\delta_0 t^{-\mu} \le 1$. We then have that

$$\sigma(tR) \le t^{\lambda}\sigma(R) + \kappa R^{\nu}.$$

We can continue this inequality iteratively to get for any $m \in \mathbb{N}$,

$$\begin{aligned}\sigma(t^{m+1}(R)) &\le t^{\lambda}\sigma(t^m R) + \kappa t^{m\nu} R^{\nu} \\ &\le t^{\lambda(m+1)}\sigma(R) + \kappa t^{m\nu} R^{\nu} \sum_{j=0}^{m} t^{j(\lambda-\nu)} \\ &\le \gamma_0 t^{\nu(m+1)}[\sigma(R) + \kappa R^{\nu}].\end{aligned}$$

So let $m \in \mathbb{N}$ be large enough such that $t^{m+2}R < r \leq t^{m+1}R$, and then we get the desired inequality:

$$\sigma(r) \leq \sigma(t^{m+1}(R)) \leq \gamma_1 \left(\frac{r}{R}\right)^{\nu} \sigma(R) + \kappa_1 r^{\nu}.$$

□

This lemma will allow us to deal with the $R^{2\alpha}$ term in (34), but we will prove one last lemma before concluding the proof.

Lemma 3.0.8. *Let $f \in L^2(B_R(x_0))$. Then if we denote f_{avg} as the average of f over the ball $B_R(x_0)$, then we have that*

$$\int_{B_R(x_0)} |f - f_{avg}|^2 = \inf_{\beta \in \mathbb{R}} \int_{B_R(x_0)} |f - \beta|^2.$$

Proof. The function $F(\beta) := \int_{B_R(x_0)} |f - \beta|^2$ is convex and differentiable since $f \in L^2(B_R(x_0))$. Its derivative is given by

$$F'(\beta) = 2 \int_{B_R(x_0)} (\beta - f)$$

and so $F'(0) = 0$ when $\beta = \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} f = f_{avg}$. Since F is convex, this critical point is a minimizer of the functional. □

Finally we return to the proof of *Theorem* 3.0.6. Let us use *Lemma* 3.0.7 in equation (34) for $0 < r \leq R \leq R_0$ and $R_0^{2\alpha} \leq \delta_0$ to get that for any $\epsilon > 0$,

$$\int_{B_R(x_0)} |Dv|^2 \leq c_3 \left(\frac{r}{R}\right)^{n-\epsilon} \int_{B_R(x_0)} |Dv|^2. \tag{35}$$

Repeating this procedure, we obtain that

$$\int_{B_R(x_0)} |Dw - (Dw)_{avg}|^2 \leq c_4 \left(\frac{r}{R}\right)^{n+2} \int_{B_R(x_0)} |Dw - (Dw)_{avg}|^2 \tag{36}$$

where the average is taken over the ball $B_R(x_0)$ as defined in *Lemma* 3.0.8. Now using again *Lemma* 3.0.8 we obtain that

$$\int_{B_R(x_0)} |Dw - (Dw)_{avg}|^2 \leq \int_{B_R(x_0)} |Dw - (Dv)_{avg}|^2.$$

By (31), this means that

$$\begin{aligned}\int_{B_R(x_0)} |Dw - (Dv)_{av}|^2 &\leq \frac{1}{\lambda} \int_{B_R(x_0)} \sum_{i,j=1}^{n} A^{ij} (D_i w - (D_i v)_{av})(D_j w - (D_j v)_{av}) \\ &= \frac{1}{\lambda} \int_{B_R(x_0)} \sum_{i,j=1}^{n} A^{ij} (D_i w - (D_i v)_{av})(D_j v - (D_j v)_{av}) \\ &+ \frac{1}{\lambda} \int_{B_R(x_0)} \sum_{i,j=1}^{n} A^{ij} (D_i v)_{av} (D_j v - D_j w).\end{aligned}$$

Since $u - v \in W_0^{1,2}(B_R(x_0))$ and $A^{ij}(D_j v)_{av}$ is constant the last term is zero, and so by *Cauchy-Schwarz* we obtain that

$$\int_{B_R(x_0)} |Dw - (Dw)_{av}|^2 \leq \frac{\Lambda^2}{\lambda^2} n^2 \int_{B_R(x_0)} |Dv - (Dv)_{av}|^2. \tag{37}$$

So by Hölder inequality and (34), we obtain that

$$\begin{aligned}\int_{B_r(x_0)} |Dv - (Dv)_{av}|^2 &\leq 3 \int_{B_r(x_0)} |Dw - (Dw)_{av}|^2 \\ &\quad + 3 \int_{B_r(x_0)} |Dv - Dw|^2 + 3 \int_{B_r(x_0)} |(Dv)_{av} - (Dw)_{av}|^2 \\ &\leq 3 \int_{B_r(x_0)} |Dw - (Dw)_{av}|^2 + 6 \int_{B_r(x_0)} |Dv - Dw|^2 \\ &\leq 3 \int_{B_r(x_0)} |Dw - (Dw)_{av}|^2 + c_5 R^{2\alpha} \int_{B_r(x_0)} |Dv|^2,\end{aligned}$$

where all the averages here are taken over the ball $B_r(x_0)$. Putting the previous inequality together with (35), (36) and (37) give us that

$$\begin{aligned}\int_{B_r(x_0)} |Dv - (Dv)_{av}|^2 &\leq c_6 \Big(\frac{r}{R}\Big)^{n+2} \int_{B_r(x_0)} |Dv - (Dv)_{av}|^2 + c_7 R^{2\alpha} \int_{B_R(x_0)} |Dv|^2 \\ &\leq c_6 \Big(\frac{r}{R}\Big)^{n+2} \int_{B_r(x_0)} |Dv - (Dv)_{av}|^2 + c_8 R^{n-\epsilon+2\alpha}.\end{aligned}$$

We then use *Lemma* 3.0.7 to finally obtain that

$$\int_{B_r(x_0)} |Dv - (Dv)_{av}|^2 \leq c_9 \Big(\frac{r}{R}\Big)^{n-\epsilon+2\alpha} \int_{B_R(x_0)} |Dv - (Dv)_{av}|^2 + c' r^{n-\epsilon+2\alpha}, \tag{38}$$

and *Campanato's* theorem thus proves the theorem. □

We can now finally complete the proof of Theorem 3.0.1.

Proof. Let $v = Du$ and use *Theorem* 3.0.6 to deduce that $v \in C^{1,\alpha'}$ for any $\alpha' < \alpha$ non-zero. Therefore we have that $u \in C^{2,\alpha'}$ for any $0 < \alpha' < \alpha$. We can then differentiate with respect to x_k and use that each of the derivatives

$$D_i D_k u \quad j, k = 1, \ldots, n$$

satisfy the same equation, so that we can apply the theorem again to deduce that $D^2 u \in C^{1,\alpha''}$, and so that $u \in C^{3,\alpha''}$. Evidently we can iterate this process to deduce that $u \in C^{k,\alpha_k}$ for each natural number k, with $\alpha_k \in (0,1)$ for all k. This means $u \in C^{\infty}$. □

4 The generalized Harnack inequality

Our main result is the generalized *Harnack inequality* for the following second-order elliptic equation which arise in fluid dynamics models:

Theorem 4.0.1 (Harnack inequality)**.** *Let u be a non-negative Lipschitz solution to the following elliptic equation*

$$-\triangle u + b\cdot\nabla u + au = 0 \tag{39}$$

in a usual domain $\Omega\subset\mathbb{R}^n$. Assume that $a\in L^q(\Omega)$, $b\in L^{\overline{q}}(\Omega)$ for $\frac{n}{2}<q,\overline{q}\le n$ and $\overline{q}\ge 2$, and that div $b=0$ in the sense of distributions. Then for any $B_R\subset\Omega$ we have

$$\sup_{B_R} u \le C\inf_{B_R} u. \tag{40}$$

Here C is a constant depending on $n,q,\overline{q},R$, and $M_1=1+\|a\|_{L^q}+\|b\|_{L^2}^2+\|b\|_{L^{\overline{q}}}$.

Remark 4.0.1.1. *From the proof it follows that*

$$C = C(n,q,\overline{q})\Big(R^{-1}+(R^{-1}\|a\|_{L^q})^{\frac{1}{1-\frac{n}{2q}}}+(R^{-1}\|b\|_{L^{\overline{q}}})^{\frac{1}{1-\frac{n}{2\overline{q}}}}\Big)^{C(n)R^{-1}M_1}, \tag{41}$$

where $M_1=1+\|a\|_{L^q}+\|b\|_{L^2}^2+\|b\|_{L^{\overline{q}}}$.

Theorem 4.0.1 has the following consequence when $\Omega=\mathbb{R}^n$.

Theorem 4.0.2 (One-sided Lioville's theorem)**.** *Let $a(x)\equiv 0$ and $b(x)$ as in theorem* 4.0.1*. Then any non-negative Lipschitz solution u to the elliptic equation* (39) *in $\mathbb{R}^n$ is equal to a constant.*

Proof. Without loss of generality, we may assume that $\inf_{\mathbb{R}^n} u = 0$. Then for every $\epsilon>0$, we have $\inf_{B_R} u\le\epsilon$ for any sufficiently large ball B_R. By *Theorem* 4.0.1, $\sup_{B_R} u\le C\inf_{B_R} u\le C\epsilon$ for all sufficiently large $R>0$. Observe that the constant C given explicitly by (41) depends on R but remains bounded as $R\to\infty$. Therefore, the assertion is established. □

Theorem 4.0.1 is an immediate consequence of the following two lemmas that compare $\sup_{B_{\theta R}} u$ and $\inf_{B_\theta R} u$ to $\|u\|_{L^p(B_{\tau R})}$ with some small $p>0$ and $0<\theta<\tau<1$.

Lemma 4.0.3. *Assume that u is a non-negative Lipschitz sub-solution to the equation*

$$-\triangle u + b\cdot\nabla u + au = 0 \tag{42}$$

with $a\in L^q(\Omega)$, $b\in L^{\overline{q}}(\Omega)$ for $\frac{n}{2}<q,\overline{q}\le n$ and div $b\le 0$ in the sense of distributions. Then for any $B_R\subset\Omega, p>0$, and $0<\theta<\tau<1$

$$\sup_{B_{\theta R}} u \le C\Bigg(R^{-\frac{n}{p}}+\Big(R^{-\frac{1}{2-\frac{n}{q}}}\|a\|_{L^q(\Omega)}^{\frac{1}{2-\frac{n}{q}}}\Big)^{\frac{n}{p}}+\Big(R^{-\frac{1}{2-\frac{n}{\overline{q}}}}\|b\|_{L^{\overline{q}}(\Omega)}^{\frac{1}{2-\frac{n}{\overline{q}}}}\Big)^{\frac{n}{p}}\Bigg)\|u\|_{L^p(B_{\tau R})}, \tag{43}$$

where $C=C(n,p,\overline{q},\theta,\tau)$ is a positive constant.

Lemma 4.0.4. *Assume that u is a non-negative Lipschitz super-solution to (39) satisfying the assumptions of Theorem 4.0.1. Then for any $B_R \subset \Omega$ and $0 < \theta < \tau < 1$ there exists a small positive number $p_0 = p_0(n, q, \overline{q}, \theta, \tau, R, M_1)$ such that*

$$\inf_{B_{\theta R}} u \geq C\left(\int_{B_{\tau R}} u^{p_0}\right)^{\frac{1}{p_0}} \tag{44}$$

where $C = C(n, q, \overline{q}, \theta, \tau, R, M_1)$ is a positive constant and $M_1 = 1 + \|a\|_{L^q} + \|b\|_{L^2}^2 + \|b\|_{L^{\overline{q}}}$.

So to prove the generalized *Harnack* inequality we will need to prove Lemmas 4.0.3 and 4.0.4. To do so we will use the *Moser* iteration, with the general strategy based on the proof of the *Harnack* inequality in [12].

The proof of Lemma 4.0.3. Let u be a non-negative *Lipschitz* sub-solution of (42) in Ω, that is,

$$\int_\Omega (\partial_j u)(\partial_j \phi) + \int_\Omega b_j (\partial_j u)\phi + \int_\Omega a u \phi \leq 0 \tag{45}$$

for any *Lipschitz* function $\phi \geq 0$ in Ω such that $\phi = 0$ in Ω^c. For simplicity of the presentation of the proof we assume that $a = 0$. The first part of the proof consists in obtaining an a priori bound on the $L^{p_1} - norm$ of u on a smaller ball B_{r_1}, in terms of an L^{p_2}−norm of u on a larger ball B_{r_2} with $r_1 < r_2$ but $p_1 > p_2$. Then an iterative procedure is used to bring the gap between r_1 and r_2 to zero and simultaneously send p_1 to infinity.

Let $\beta \geq 0$ and $\eta(x)$ be a *Lipschitz* cut-off function in the ball $B_{\tau R}$ such that $0 \leq \tau(x) \leq 1$. We use $(\frac{\beta}{2} + 1)u^{\beta+1}\eta^{2\gamma}$ as a test function in (45) to obtain that

$$\left(\frac{\beta}{2}+1\right)\Big[\int_\Omega (\partial_j u)\partial_j (u^{\beta+1})\eta^{2\gamma} + \int_\Omega u^{\beta+1}(\partial_j u)\partial_j(\eta^{2\gamma}) + \int_\Omega b_j u^{\beta+1}(\partial_j u)\eta^{2\gamma}\Big] \leq 0. \tag{46}$$

Let $w = u^{\frac{\beta}{2}+1}$ so that $\partial_j w = \left(\frac{\beta}{2} + 1\right)u^{\frac{\beta}{2}}\partial_j u$. By (46), we get that

$$\frac{\beta+1}{\frac{\beta}{2}+1}\int_\Omega |\partial_j w|^2 \eta^{2\gamma} \leq -2\gamma \int_\Omega w(\partial_j w)\eta^{2\gamma-1}(\partial_j \eta) - \int_\Omega b_j w(\partial_j w)\eta^{2\gamma}. \tag{47}$$

For the first term in the right side we have

$$-2\gamma \int_\Omega w(\partial_j w)\eta^{2\gamma-1}\partial_j \eta = \gamma \int_\Omega w^2(\eta^{2\gamma-1}\triangle\eta + (2\gamma - 1)\eta^{2\gamma-2}|\partial_j \eta|^2), \tag{48}$$

while for the second term

$$-\int_\Omega b_j w(\partial_j w)\eta^{2\gamma} = \frac{1}{2}\int_\Omega (\partial_j b_j)w^2\eta^{2\gamma} + \gamma\int_\Omega b_j w^2 \eta^{2\gamma-1}\partial_j \eta \leq \gamma \int_\Omega b_j w^2 \eta^{2\gamma-1}\partial_j \eta, \tag{49}$$

as div $b \le 0$. Next set $\gamma_0 = \frac{n}{\overline{q}}$. Then, as $\overline{q} > \frac{n}{2}$, we have $\gamma_0 \in (0,2)$ and, in addition

$$\frac{1}{\overline{q}} + \frac{\gamma_0}{2^*} + \frac{2-\gamma_0}{2} = 1 \tag{50}$$

for $n \ge 3$. Note that if $n = 2$ then γ_0 can be also chosen so that (50) is satisfied.

Assume also that γ is sufficiently large so that $\gamma\gamma_0 \le 2\gamma - 1$. Then, by Hölder's inequality we have, using (50)

$$\int_\Omega b_j w^2 \eta^{2\gamma-1} \partial_j \eta \le \int_\Omega |b_j||w\eta^\gamma|^{\gamma_0}|w|^{2-\gamma_0}|\partial_j\eta| \le \|b\|_{L^{\overline{q}}}\|w\eta^\gamma\|_{L^{2^*}}^{\gamma_0}\,\|w|\nabla\eta|^{\frac{1}{2-\gamma_0}}\|_{L^2}^{2-\gamma_0}, \tag{51}$$

as $0 \le \eta \le 1$. By Young's and the Gagliardo-Nirenberg inequalities, this leads to

$$\int_\Omega b_j w^2 \eta^{2\gamma-1}\partial_j\eta \le \frac{1}{2}\|\nabla(w\eta^\gamma)\|_{L^2}^2 + C\|b\|_{L^q}^{\frac{1}{1-\frac{n}{2\overline{q}}}}\|w|\nabla\eta|^{\frac{1}{2-\frac{n}{\overline{q}}}}\|_{L^2}^2. \tag{52}$$

By (47), (48), and (52), we obtain

$$\int_\Omega |\nabla(u^{\frac{\beta}{2}+1}\eta^\gamma)|^2 \le C\int_\Omega u^{\beta+2}\eta^{2\gamma-1}|\triangle\eta| \tag{53}$$

$$+ C\int_\Omega u^{\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2 + C\|b\|_{L^q}^{\frac{1}{1-\frac{n}{2\overline{q}}}}\|u^{\frac{\beta}{2}+1}(\nabla\eta)^{\frac{1}{2-\frac{n}{\overline{q}}}}\|_{L^2}^2. \tag{54}$$

By *Sobolev* embedding used in the right side of (53), we obtain that

$$\|u^{\frac{\beta}{2}+1}\eta^\gamma\|_{L^{2\chi}} \le C\Big(\int_\Omega u^{\beta+2}\eta^{2\gamma-1}|\triangle\eta|\Big)^{\frac{1}{2}} + C\Big(\int_\Omega u^{\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2\Big)^{\frac{1}{2}} \tag{55}$$

$$+ C\|b\|_{L^{\overline{q}}}^{\frac{1}{2-\frac{n}{\overline{q}}}}\|u^{\frac{\beta}{2}+1}(\nabla\eta)^{\frac{1}{2-\frac{n}{\overline{q}}}}\|_{L^2} \tag{56}$$

where $\chi = \frac{n}{n-2}$ if $n \ge 3$ and $\chi > 2$ is arbitrary for $n = 2$. Now, let $\eta \in C_0^\infty(\Omega)$ be such that $\eta \equiv 1$ in $B_{\theta R}$, $\eta \equiv 0$ in $B_{\tau R}^c$, $|\nabla\eta| \le \frac{C}{R(\tau-\theta)}$ and $|\triangle\eta| \le \frac{C}{R^2(\tau-\theta)^2}$. Then, we obtain that

$$\|u^{\frac{\beta}{2}+1}\|_{L^{2\chi}(B_{\theta R})} \le \frac{C}{R(\tau-\theta)}\Big(\int_{B_{\tau R}} u^{\beta+2}\Big)^{\frac{1}{2}} + \frac{C}{(R(\tau-\theta))^{\frac{1}{2-\frac{n}{\overline{q}}}}}\|b\|_{L^{\overline{q}}(B_{\tau R})}^{\frac{1}{2-\frac{n}{\overline{q}}}}\|u^{\frac{\beta}{2}+1}\|_{L^2(B_{\tau R})}. \tag{57}$$

The main point of (57) is that, since $\chi > 1$, we have a bound on a higher norm of u on a smaller ball in terms of the lower norm of u on a larger ball. We now apply the estimate (57) iteratively on pairs of balls $B_{r_{i+1}} \subset B_{r_i}$, and also let $\beta_i \to +\infty$. More precisely, we choose $\beta_i = 2(\chi^i - 1)$ and $r_i = \theta R + (\tau-\theta)R2^{-i}$ for $i = 0, 1, 2, \ldots$, so that $r_i - r_{i+1} = (\tau-\theta)R2^{-(i+1)}$. We obtain

$$\|u\|_{L^{2\chi^{i+1}}(B_{r_{i+1}})} \le C^{\frac{1}{\chi^i}}2^{\frac{i}{\chi^i}}(R(\tau-\theta))^{-\frac{1}{\chi^i}}\|u\|_{L^{2\chi^i}(B_{r_i})} \tag{58}$$

$$+ \Big(C2^{\frac{i}{2-\frac{n}{\overline{q}}}}(R(\tau-\theta))^{-\frac{1}{2-\frac{n}{\overline{q}}}}\|b\|_{L^{\overline{q}}(B_{r_i})}^{\frac{1}{2-\frac{n}{\overline{q}}}}\Big)^{\frac{1}{\chi^i}}\|u\|_{L^{2\chi^i}(B_{r_i})}. \tag{59}$$

By iteration, letting $i \to +\infty$, we conclude that the estimate (43) holds for $p > 2$.

Now, let $p \in (0, 2)$. We have just shown that

$$\sup_{B_{\theta R}} u \le C\Big((R(\tau-\theta))^{-\frac{n}{2}} + \Big((R(\tau-\theta))^{-\frac{1}{2-\frac{n}{\overline{q}}}} \|b\|_{L^{\overline{q}}(B_{r_i})}^{\frac{1}{2-\frac{n}{\overline{q}}}}\Big)^{\frac{n}{2}}\Big)\|u\|_{L^2(B_{\tau R})} \quad (60)$$

$$\le C\Big((R(\tau-\theta))^{-\frac{n}{2}} + \Big((R(\tau-\theta))^{-\frac{1}{2-\frac{n}{\overline{q}}}} \|b\|_{L^{\overline{q}}(B_{r_i})}^{\frac{1}{2-\frac{n}{\overline{q}}}}\Big)^{\frac{n}{2}}\Big)\|u\|_{L^\infty(B_{\tau R})}^{1-\frac{p}{2}}\|u\|_{L^p(B_{\tau R})}^{\frac{p}{2}} \quad (61)$$

which implies

$$\sup_{B_{\theta R}} u \le \frac{1}{2}\|u\|_{L^\infty(B_{\tau R})} + C\Big((R(\tau-\theta))^{-\frac{n}{p}} + \Big((R(\tau-\theta))^{-\frac{1}{2-\frac{n}{\overline{q}}}} \|b\|_{L^{\overline{q}}(B_{r_i})}^{\frac{1}{2-\frac{n}{\overline{q}}}}\Big)^{\frac{n}{p}}\Big)\|u\|_{L^p(B_{\tau R})}$$

A standard iteration argument ([12, Lemma 4.3]) then implies that

$$\sup_{B_{\theta R}} u \le C\Big((R(\tau-\theta))^{-\frac{n}{p}} + \Big((R(\tau-\theta))^{-\frac{1}{2-\frac{n}{\overline{q}}}} \|b\|_{L^{\overline{q}}(B_{r_i})}^{\frac{1}{2-\frac{n}{\overline{q}}}}\Big)^{\frac{n}{p}}\Big)\|u\|_{L^p(B_{\tau R})} \quad (62)$$

and the proof of Lemma 4.0.3 is complete.

□

The proof of Lemma 4.0.4. We assume without loss of generality that $R = 1$. The idea of the proof is similar to that of Lemma 4.0.3: we obtain an a priori bound and use it iteratively. Assume that u is a non-negative Lipschitz supersolution to (39), and consider $v = \frac{1}{u}$. The function v satisfies

$$-\triangle v + b \cdot \nabla v - av \le 0 \text{ in } \Omega \quad (63)$$

or equivalently

$$\int_\Omega (\partial_j v)(\partial_j \phi) + \int_\Omega b_j(\partial_j v)\phi - \int_\Omega av\phi \le 0 \quad (64)$$

for any function $\phi \in C_0^\infty(\Omega)$ such that $\phi \ge 0$ in Ω. By Lemma 4.0.3, it follows that for any $0 < \theta < \tau < 1$ and $p > 0$, we have

$$\sup_{B_\theta} v \le C\|v\|_{L^p(B_\tau)} \quad (65)$$

with $C = C(n, p, q, \overline{q}, \tau, \theta, M_1)$. Therefore, we have

$$\inf_{B_\theta} u \ge \frac{1}{C}\Big(\int_{B_\tau} u^{-p} \int_{B_r} u^p\Big)^{-\frac{1}{p}}\Big(\int_{B_\tau} u^p\Big)^{\frac{1}{p}}. \quad (66)$$

We claim that there exists $p_0 > 0$ such that

$$\int_{B_\tau} u^{-p_0} \int_{B_\tau} u^{p_0} \le C \quad (67)$$

with a constant $C = C(n, q, \overline{q}, \tau, M_1)$, which would finish the proof of Lemma 4.0.4.

The first step of the proof consists in a reduction to an exponential bound: in order to prove (67) for some sufficiently small $p_0 > 0$, denote

$$(\log u)_{B_\tau} = \frac{1}{|B_\tau|}\int_{B_\tau} \log u,$$

and set

$$w = \log u - (\log u)_{B_\tau}. \tag{68}$$

We shall show that there exists $p_0 > 0$ such that

$$\int_{B_\tau} e^{p_0|w|} \leq C \tag{69}$$

where $C = C(\tau)$, which implies (67). Indeed, if we assume that (69) holds, then

$$\int_{B_\tau} e^{p_0(\log u-(\log u)_{B_\tau})} \leq C \tag{70}$$

and

$$\int_{B_\tau} e^{-p_0(\log u-(\log u)_{B_\tau})} \leq C. \tag{71}$$

Therefore, we have $e^{-p_0(\log u)_{B_\tau}}\int_{B_\tau} e^{p_0\log u} \leq C$ and $e^{p_0(\log u)_{B_\tau}}\int_{B_\tau} e^{-p_0\log u} \leq C$. Multiplying these two inequalities then leads to (67).

The second step of the proof consists in proving an L^2-bound for w. We now prove (69). Firstly, we establish bounds on the L^2-norm of w. The function w satisfies

$$|\nabla w|^2 \leq -\triangle w + b\cdot\nabla w + a \quad \text{in } B_1. \tag{72}$$

Fix $\tau \in (0,1)$, and let $\eta \in C_0^1(\Omega)$ with $0 \leq \eta \leq 1$ be a cutoff function such that $\eta \equiv 1$ on $B_{\frac{1+\tau}{2}}$, $\eta \equiv 0$ on B_1^c, and $|\nabla\eta| \leq \frac{C}{1-\tau}$. Multiplying (72) by η^2 and integrating over B_1, we obtain

$$\int_{B_1} |\nabla w|^2\eta^2 \leq 2\int_{B_1}(\partial_j w)\eta(\partial_j\eta) + \int_{B_1} b_j(\partial_j w)\eta^2 + \int_{B_1} a\eta^2 \tag{73}$$

$$\leq 2\|\eta\nabla w\|_{L^2}\|\nabla\eta\|_{L^2} + \|b\|_{L^2}\|\eta\nabla w\|_{L^2}\|\eta\|_{L^\infty} + \|a\|_{L^q}\|\eta^2\|_{L^{q'}} \tag{74}$$

where $\frac{1}{q}+\frac{1}{q'} = 1$. Absorbing the factors $\|\eta\nabla w\|_{L^2}$ on the right using the term on the left, we get

$$\int_{B_{\frac{1+\tau}{2}}} |\nabla w|^2 \leq C_\tau M_0 \tag{75}$$

where $M_0 = 1 + \|a\|_{L^q} + \|b\|_{L^2}^2$, and the constant C_τ may depend on $\tau \in (0,1)$. Also, since

$$\int_{B_\tau} w = 0,$$

and $\frac{1+\tau}{2} \geq \tau$, we have by the Poincaré inequality

$$\int_{B_{\frac{1+\tau}{2}}} w^2 \leq C \int_{B_{\frac{1+\tau}{2}}} |\nabla w|^2 \leq C_\tau M_0. \tag{76}$$

The third step of the proof consists on making bounds on the higher norms of w. Next, we need to estimate $\int_{B_\tau} |w|^\beta$ for all $\beta \geq 1$. As in the proof of Lemma 2.4 the idea is to bound first the higher norms of w on smaller balls in terms of the lower norms of w on larger balls and then use the iteration process.

We multiply (72) by $|w|^{2\beta}\eta^{2\gamma}$ and integrate over B_1 in order to obtain

$$\int_{B_1} |w|^{2\beta}|\nabla w|^2\eta^{2\gamma} \leq 2\beta \int_{B_1} |w|^{2\beta-2}w|\nabla w|^2\eta^{2\gamma} + 2\gamma \int_{B_1} |w|^{2\beta}(\partial_j w)\eta^{2\gamma-1}(\partial_j \eta) \tag{77}$$

$$- \frac{2\gamma}{2\beta+1} \int_{B_1} b_j|w|^{2\beta}w\eta^{2\gamma-1}(\partial_j\eta) + \int_{B_1} a|w|^{2\beta}\eta^{2\gamma}. \tag{78}$$

Here we utilized div $b = 0$ and $\partial_j|w| = w\frac{\partial_j w}{|w|}$. For the first term in the right side of (77) we use

$$2\beta|w|^{2\beta-1} \leq \frac{1}{4}|w|^{2\beta} + (8\beta)^{2\beta}, \tag{79}$$

while for the second

$$2\gamma \int_{B_1} |w|^{2\beta}(\partial_j w)\eta^{2\gamma-1}(\partial_j\eta) \leq \frac{1}{4}\int_{B_1} |w|^{2\beta}|\nabla w|^2\eta^{2\gamma} + C\gamma^2 \int_{B_1} |w|^{2\beta}\eta^{2\gamma-2}|\nabla\eta|^2. \tag{80}$$

This leads to

$$\int_{B_1} |w|^{2\beta}|\nabla w|^2\eta^{2\gamma} \leq C(8\beta)^{2\beta}\int_{B_1} |\nabla w|^2\eta^{2\gamma} + C\gamma^2 \int_{B_1} |w|^{2\beta}\eta^{2\gamma-2}|\nabla\eta|^2 \tag{81}$$

$$+ \frac{C\gamma}{\beta+1}\int_{B_1} |b||w|^{2\beta+1}\eta^{2\gamma-1}|\nabla\eta| + C\int_{B_1} |a||w|^{2\beta}\eta^{2\gamma}. \tag{82}$$

Let $\tau \leq r \leq R \leq \frac{1+\tau}{2}$. We now choose a cutoff $\eta \in C_0^1(\Omega)$ with $0 \leq \eta \leq 1$ such that $\eta \equiv 1$ on B_r, $\eta \equiv 0$ on B_R^c, and $|\nabla\eta| \leq \frac{C}{R-r}$. By (73), for the first term in the right side of (81) we have

$$(8\beta)^{2\beta}\int_{B_1} |\nabla w|^2\eta^{2\gamma} \leq (8\beta)^{2\beta}\int_{B_{\frac{1+\tau}{2}}} |\nabla w|^2 \leq C_\tau(8\beta)^{2\beta}M_0. \tag{83}$$

On the other hand, for the left side of (73), we use

$$\left|\nabla(|w|^{\beta+1}\eta^\gamma)\right|^2 \leq 2\gamma^2|w|^{2\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2 + 2(\beta+1)^2|w|^{2\beta}|\nabla w|^2\eta^{2\gamma}. \tag{84}$$

Hence, we obtain

$$\int_{B_1}\Big|\nabla(|w|^{\beta+1}\eta^{\gamma})\Big|^2 \le C\gamma^2\int_{B_1}|w|^{2\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2 + C(\beta+1)^2(8\beta)^{2\beta}M_0 \quad (85)$$

$$+\,C\gamma^2(\beta+1)^2\int_{B_1}|w|^{2\beta}\eta^{2\gamma-2}|\nabla\eta|^2 \quad (86)$$

$$+\,C\gamma(\beta+1)\int_{B_1}|b||w|^{2\beta+1}\eta^{2\gamma-1}|\nabla\eta| + C(\beta+1)^2\int_{B_1}|a||w|^{2\beta}\eta^{2\gamma}. \quad (87)$$

For the third term in the right hand side we utilize

$$(\beta+1)^2|w|^{2\beta} \le \frac{(\beta+1)^{2\beta+2}}{\beta+1} + \frac{\beta(|w|^{2\beta})^{\frac{\beta+1}{\beta}}}{\beta+1} \le (8\beta)^{2\beta} + |w|^{2\beta+2} \quad (88)$$

which gives

$$C\gamma^2(\beta+1)^2\int_{B_1}|w|^{2\beta}\eta^{2\gamma-2}|\nabla\eta|^2 \le C(8\beta)^{2\beta}\gamma^2\int_{B_1}\eta^{2\gamma-2}|\nabla\eta|^2 \quad (89)$$

$$+\,C\gamma^2\int_{B_1}|w|^{2\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2 \le \frac{C(8\beta)^{2\beta}\gamma^2 M_0}{(R-r)^2} + C\gamma^2\int_{B_1}|w|^{2\beta+2}\eta^{2\gamma-2}|\nabla\eta|^2, \quad (90)$$

as $M_0 \ge 1$. The equations (86) and (87) are estimated as follows. First, we have

$$\int_{B_1}|a||w|^{2\beta}\eta^{2\gamma} = \int_{B_1}|a|(|w|^{\beta+1}\eta^{\gamma})^{\frac{2\beta}{\beta+1}}\eta^{\frac{2\gamma}{\beta+1}} \le \|a\|_{L^q}\||w|^{\beta+1}\eta^{\gamma}\|^{\frac{2\beta}{\beta+1}}_{L^{\frac{2\beta q'}{\beta+1}}} \quad (91)$$

where $\frac{1}{q}+\frac{1}{q'}=1$. Now, we use the Gagliardo-Nirenberg inequality

$$\||w|^{\beta+1}\eta^{\gamma}\|_{L^{\frac{2\beta q'}{\beta+1}}} \le C\||w|^{\beta+1}\eta^{\gamma}\|^{1-\alpha}_{L^2}\|\nabla(|w|^{\beta+1}\eta^{\gamma})\|^{\alpha}_{L^2} \quad (92)$$

with $\alpha = \frac{n}{2} - \frac{n(\beta+1)}{2\beta q'}$ if $\frac{2\beta q'}{\beta+1} \ge 2$, and $\alpha = 0$ otherwise. By Young's inequality, we obtain

$$\int_{B_1}|a||w|^{2\beta}\eta^{2\gamma} \le C\|a\|_{L^q}\||w|^{\beta+1}\eta^{\gamma}\|^{\frac{2\beta(1-\alpha)}{\beta+1}}_{L^2}\|\nabla(|w|^{\beta+1}\eta^{\gamma})\|^{\frac{2\alpha\beta}{\beta+1}}_{L^2} \quad (93)$$

$$\le \left(\frac{1}{(2(\beta+1))^{\frac{2\alpha\beta}{\beta+1}}}\|\nabla(|w|^{\beta+1}\eta^{\gamma})\|^{\frac{2\alpha\beta}{\beta+1}}_{L^2}\right)^{\frac{\beta+1}{\alpha\beta}} \quad (94)$$

$$+\,C\Big((2(\beta+1))^{\frac{2\alpha\beta}{\beta+1}}\|a\|_{L^q}\||w|^{\beta+1}\eta^{\gamma}\|^{\frac{2(1-\alpha)\beta}{\beta+1}}_{L^2}\Big)^{\frac{\beta+1}{\beta(1-\alpha)+1}}. \quad (95)$$

As $\alpha \in (0,1)$, this implies

$$\int_{B_1}|a||w|^{2\beta}\eta^{2\gamma} \le \frac{1}{(2(\beta+1))^2}\|\nabla(|w|^{\beta+1}\eta^{\gamma})\|^2_{L^2} + C(\beta+1)^{2\alpha_1}\|a\|^{\alpha_1}_{L^q}\||w|^{\beta+1}\eta^{\gamma}\|^{\alpha_2}_{L^2}. \quad (96)$$

Here we denoted $\alpha_1 = \frac{\beta+1}{\beta(1-\alpha)+1}$ and $\alpha_2 = \frac{2\beta(1-\alpha)}{\beta(1-\alpha)+1}$. Observe that $\alpha \geq 1$ and α_1 is smaller than a constant independent of β, while $0 < \alpha_2 < 2$ with $\alpha \to 2$ as $\beta \to \infty$.

For the term in (87), we have

$$C\gamma(\beta+1)\int_{B_1} |b||w|^{2\beta+1}\eta^{2\gamma-1}|\nabla\eta| = C\gamma(\beta+1)\int_{B_1} |b|\Big(|w|^{\beta+1}\eta^{\gamma}\Big)^{\frac{2\beta+1}{\beta+1}}\eta^{\frac{\gamma}{\beta+1}-1}|\nabla\eta|. \tag{97}$$

Let us choose $\gamma = \beta+1$. Then, the above expression becomes

$$C\gamma(\beta+1)\int_{B_1} |b|\Big(|w|^{\beta+1}\eta^{\gamma}\Big)^{\frac{2\beta+1}{\beta+1}}|\nabla\eta| \leq C(\beta+1)^2\|b\|_{L^{\overline{q}}}\||w|^{\beta+1}\eta^{\gamma}\|^{\frac{2\beta+1}{\beta+1}}_{L^{\frac{(2\beta+1)\overline{q}'}{\beta+1}}}\|\nabla\eta\|_{L^\infty} \tag{98}$$

where $\frac{1}{\overline{q}} + \frac{1}{\overline{q}'} = 1$. Once again we apply the Gagliardo-Nirenberg inequality

$$\||w|^{\beta+1}\eta^{\gamma}\|_{L^{\frac{\overline{q}'(2\beta+1)}{\beta+1}}} \leq C\||w|^{\beta+1}\eta^{\gamma}\|^{1-\overline{\alpha}}_{L^2}\|\nabla(|w|^{\beta+1}\eta^{\gamma})\|^{\overline{\alpha}}_{L^2} \tag{99}$$

with $\overline{\alpha} = \frac{n}{2} - \frac{n(\beta+1)}{\overline{q}'(2\beta+1)}$ if $\frac{\overline{q}'(2\beta+1)}{\beta+1} \geq 2$ and $\overline{\alpha} = 0$ otherwise.

Thus, by Young's inequality, we have

$$C\gamma(\beta+1)\int_{B_1} |b|\Big(|w|^{\beta+1}\eta^{\gamma}\Big)^{\frac{2\beta+1}{\beta+1}}|\nabla\eta| \tag{100}$$

$$\leq C(\beta+1)^2\|b\|_{L^{\overline{q}}}\||w|^{\beta+1}\eta^{\gamma}\|^{\frac{(1-\overline{\alpha})(2\beta+1)}{\beta+1}}_{L^2}\|\nabla(|w|^{\beta+1}\eta^{\gamma})\|^{\frac{\overline{\alpha}(2\beta+1)}{\beta+1}}_{L^2}\|\nabla\eta\|_{L^\infty} \tag{101}$$

$$\leq \frac{1}{4}\|\nabla(|w|^{\beta+1}\eta^{\gamma})\|^2_{L^2} + \frac{C(\beta+1)^{2\overline{\alpha}_1}}{(R-r)^{\overline{\alpha}_1}}\|b\|^{\overline{\alpha}_1}_{L^{\overline{q}}}\||w|^{\beta+1}\eta^{\gamma}\|^{\overline{\alpha}_2}_{L^2}. \tag{102}$$

Here we denoted $\overline{\alpha}_1 = \frac{2\beta+2}{2\beta(1-\overline{\alpha})+2-\overline{\alpha}}$ and $\overline{\alpha}_2 = \frac{2(2\beta+1)(1-\overline{\alpha})}{2\beta(1-\overline{\alpha})+2-\overline{\alpha}}$. Note that, as in (92), we have $\overline{\alpha}_1 \geq 1$ and $\overline{\alpha}_1$ is less than a constant independent of β, while $0 < \overline{\alpha}_2 < 2$, and $\overline{\alpha}_2 \to 2$ when $\beta \to \infty$.

Putting together (85), (88), (92) and (102), we obtain

$$\|\nabla(|w|^{\beta+1}\eta^{\gamma})\|_{L^2(B_r)^2} \leq \frac{C(\beta+1)^2}{(R-r)^2}\||w|^{\beta+1}\|^2_{L^2(B_R)} + \frac{C(\beta+1)^2(8\beta)^{2\beta}M_0}{(R-r)^2} \tag{103}$$

$$+ C(\beta+1)^{2\alpha_1+2}\|a\|^{\alpha_1}_{L^q(B_R)}\||w|^{\beta+1}\|^{\alpha_2}_{L^2(B_R)} + \frac{C(\beta+1)^{2\overline{\alpha}_1}}{(R-r)^{\overline{\alpha}_1}}\|b\|^{\overline{\alpha}_1}_{L^{\overline{q}}(B_R)}\||w|^{\beta+1}\|^{\overline{\alpha}_2}_{L^2(B_R)}. \tag{104}$$

Using Sobolev embedding, we may rewrite (103) in the form

$$\||w|^{\beta+1}\|^2_{L^{2\chi}(B_r)} \leq \frac{C(\beta+1)^{2\kappa}}{(R-r)^{\overline{\alpha}_1+2}}\Big(\||w|^{\beta+1}\|^2_{L^2(B_R)} + (8\beta)^{2\beta}M_0 \tag{105}$$

$$+ \|a\|^{\alpha_1}_{L^q(B_R)}\||w|^{\beta+1}\|^{\alpha_2}_{L^2(B_R)} + \|b\|^{\overline{\alpha}_1}_{L^{\overline{q}}(B_R)}\||w|^{\beta+1}\|^{\overline{\alpha}_2}_{L^2(B_R)}\Big) \tag{106}$$

where κ =max$\{\alpha_1+1, \overline{\alpha}_1\}$ and $\chi = \frac{n}{n-2}$ if $n \geq 3$ and $\chi > 2$ if $n = 2$. Estimate (105) is analogous to (57): a higher norm of w on a smaller ball is bounded in terms of a lower norm of w on a larger ball.

The fourth and last step on the proof is the iteration process. Let $\beta_i = \chi^i - 1$ and $r_i = \tau + \frac{1+\tau}{2^{i+1}}$ for $i = 0, 1, 2, \ldots$. From (105), we get

$$\||w|^{\chi^i}\|^2_{L^{2\chi}(B_{r_{i+1}})} \leq C\chi^{2\kappa i} 2^{(\overline{\alpha}_1+2)(i+2)} \Big(\||w|^{\chi^i}\|^2_{L^2(B_{r_i})} + (8\chi^i)^{2\chi^i} M_0 \tag{107}$$

$$+ \|a\|^{\alpha_1}_{L^q(B_{r_i})} \||w|^{\chi^i}\|^{\alpha_2}_{L^2(B_{r_i})} + \|b\|^{\overline{\alpha}_1}_{L^{\overline{q}}(B_{r_i})} \||w|^{\chi^i}\|^{\overline{\alpha}_2}_{L^2(B_{r_i})} \tag{108}$$

for all $i = 0, 1, 2, \ldots$. Taking $\frac{1}{(2\chi^i)}$ power on both sides of (107) gives

$$\|w\|_{L^{2\chi^{i+1}}(B_{r_{i+1}})} \leq C^{\frac{1}{2\chi^i}} \chi^{\frac{\kappa i}{\chi^i}} 2^{\frac{(\overline{\alpha}_1+2)(i+2)}{2\chi^i}} \Big(\|w\|_{L^{2\chi^i}(B_{r_i})} + 8\chi^i M_0^{\frac{1}{2\chi^i}} \tag{109}$$

$$+ \|a\|^{\frac{\alpha_1}{2\chi^i}}_{L^q(B_{r_i})} \|w\|^{\frac{\alpha_2}{2}}_{L^{2\chi^i}(B_{r_i})} + \|b\|^{\frac{\overline{\alpha}_1}{2\chi^i}}_{L^{\overline{q}}(B_{r_i})} \|w\|^{\frac{\overline{\alpha}_2}{2}}_{L^{2\chi^i}(B_{r_i})}\Big). \tag{110}$$

This leads to the inequality

$$\|w\|_{L^{2\chi^{i+1}}(B_{r_{i+1}})} \leq (CM_1)^{\frac{\tilde{\alpha}}{2\chi^i}} (2\chi)^{\frac{\kappa i}{\chi^i}} \Big(\|w\|_{L^{2\chi^i}(B_{r_i})} + 8\chi^i + \|w\|^{\frac{\alpha_2}{2}}_{L^{2\chi^i}(B_{r_i})} \tag{111}$$

$$+ \|w\|^{\frac{\overline{\alpha}_2}{2}}_{L^{2\chi^i}(B_{r_i})}\Big) \leq (CM_1)^{\frac{\tilde{\alpha}}{2\chi^i}} (2\chi)^{\frac{\kappa i}{\chi^i}} \Big(\|w\|_{L^{2\chi^i}(B_{r_i})} + 8\chi^i\Big), \tag{112}$$

for all $i = 0, 1, 2, \ldots$, with $\tilde{\alpha}$ =max$\{\alpha_1, \overline{\alpha}_1\}$ and $M_1 = 1 + \|a\|_{L^q} + \|b\|^2_{L^2} + \|b\|_{L^{\overline{q}}}$. For the inequality in (112) we also used $\alpha_2, \overline{\alpha}_2 \leq 2$, so that $\|w\|^{\frac{\alpha_2}{2}}_{L^p} \leq 1 + \|w\|^2_{L^p}$ and $\|w\|^{\frac{\overline{\alpha}_2}{2}}_{L^p} \leq 1 + \|w\|^2_{L^p}$.

Note that if a sequence Y_i satisfies $Y_{i+1} \leq C_i(Y_i + \chi^i)$ with $C_i \geq 1$ and $\prod_{i=1}^\infty C_i \leq \overline{K}$, then by induction we have

$$Y_i \leq C\overline{K}(Y_0 + \sum_{j=0}^{i} \chi^{j-1}) \leq C(Y_0 + \chi^i), \tag{113}$$

for all $i = 0, 1, 2, \cdots$. Thus, iterating (111), we obtain

$$\|w\|_{L^{2\chi^{i+1}}(B_{r_{i+1}})} \leq CM_1^{C(n)}(CM_1 + \chi^{i+1}) \leq CM_1^{C(n)}\chi^{i+1},$$

for all $i = 0, 1, 2, \ldots$, as $\sum_{j=1}^i \frac{j}{\chi^j} \leq C$ and $\sum_{j=1}^i \chi^j \leq \chi^{i+1}$ for $\chi > 1$.

Finally, for any $\beta \geq 1$ there exists $i = 0, 1, 2, \ldots$ such that

$$2\chi^i \leq \beta + 1 \leq 2\chi^{i+1}.$$

Thus, in particular, we have

$$\Big(\int_{B_r} |w|^{\beta+1}\Big)^{\frac{1}{\beta+1}} \leq C\|w\|_{L^{2\chi^{i+1}}(B_{r_{i+1}})} \leq CM_1^{C(n)}(\beta+1).$$

Therefore, for all $\beta \geq 1$, we obtain

$$\int_{B_\tau} \frac{(p_0|w|)^{(\beta+1)}}{(\beta+1)!} \leq p_0^{\beta+1}\left(CM_1^{C(n)}e\right)^{(\beta+1)} \leq \frac{1}{2^{(\beta+1)}} \tag{114}$$

by taking

$$p_0 = \frac{1}{CM_1^{C(n)}e}$$

sufficiently small. By (76), we also have

$$\int_{B_\tau} |w| \leq C \int_{B_\tau} w^2 \leq CM_0$$

which gives (114) for $\beta = 0$ as well. It follows from (114) that (69) holds, and therefore the proof of the lemma is complete.

□

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