UNIVERSITÀ DEGLI STUDI DI PADOVA
FACOLTÀ DI INGEGNERIA

Dipartimento di Ingegneria dell'Informazione

DIPARTIMENTO DI INGEGNERIA DELL'INFORMAZIONE

Corso di Laurea Magistrale in Ingegneria dell'Automazione

SPECTRAL FACTORIZATION OF RATIONAL MATRIX VALUED FUNCTIONS

Laureando: GIACOMO BAGGIO

Matricola: 1036318

Relatore: PROF. AUGUSTO FERRANTE

Controrelatore: PROF. GIORGIO PICCI

ANNO ACCADEMICO 2013-2014

Contents

Abstract

The spectral factorization problem is a cornerstone of many areas of systems, control and prediction theory. Starting from the seminal studies of Kolmogorov and Wiener, much work has been done on the subject and many approaches have been proposed to solve this classical problem. More recently, a natural extension of the spectral factorization problem, the so-called J-spectral factorization problem, has been investigated in several papers. Interestingly, the latter problem plays a crucial role in $\mathscr{H}_\infty$ control and estimation theory.

In this thesis, we address the two aforementioned problems. We first review the multivariate spectral factorization method devised by Youla in his celebrated paper Youla [1961] focusing, in particular, on some of its remarkable features. Then, in the spirit of Youla's work, we present a technique which provide a solution to the multivariate spectral factorization problem in discrete-time. Finally, a J-spectral extension of the proposed factorization approach is discussed.

0. General notation

We fix here some general terminology and notation which will be used throughout the thesis. Additional nomenclature, when needed, will be introduced at the beginning or within the following Chapters.

$\varnothing$	empty set
$\mathbb{N}$	set of natural numbers $\{0,1,2,3,\dots\}$
$\mathbb{Z}$	ring of integers $\{\dots,-2,-1,0,1,2,\dots\}$
$\mathbb{Q}$	field of rational numbers
$\mathbb{R}$	field of real numbers
$\mathbb{C}$	field of complex numbers
$\mathbb{R}^{n\times m}$	set of real $n\times m$ matrices
$\mathbb{R}[z]$	ring of real polynomials
$\mathbb{R}[z,z^{-1}]$	ring of real Laurent polynomials (L-polynomials, for short)
$\mathbb{R}(z)$	field of real rational functions
$\mathbb{R}[z]^{n\times m}$	set of real polynomial $n\times m$ matrices
$\mathbb{R}[z,z^{-1}]^{n\times m}$	set of real L-polynomial $n\times m$ matrices
$\mathbb{R}(z)^{n\times m}$	set of real rational $n\times m$ matrices
$\Re\mathrm{e}\, a$	real part of $a\in\mathbb{C}$
$\Im\mathrm{m}\, a$	imaginary part of $a\in\mathbb{C}$
$\overline{a}$	complex conjugate of $a\in\mathbb{C}$
$\mathbf{0}_{n,m}$	$n\times m$ zero matrix
$\mathbf{0}_n$	$n\times n$ zero matrix

I_n	$n \times n$ identity matrix
$\ker(A)$	kernel of matrix A
$\det(A)$	determinant of matrix A
$\mathrm{rank}(A)$	rank of matrix A
$\mathrm{in}(A)$	inertia of A, *i.e.*, triple (ν_p, ν_0, ν_n) denoting the number of positive, zero, negative eigenvalues of A, in the order shown
$\mathrm{diag}[a_1, \dots, a_n]$	diagonal matrix with diagonal elements $a_1, \dots, a_n$
$[A]_{ij}$	entry at (i, j) of matrix A
$[A]_{i:j,k:h}$	sub-matrix of A obtained by extracting the rows from index i to index j $(i \leq j)$ of A and the columns from index k to index h $(k \leq h)$ of A
$\overline{A}$	complex conjugate of matrix A
$A^\top$	transpose of matrix A
A^{-1}	inverse of matrix A
A^{-R}	right inverse of matrix A
A^{-L}	left inverse of matrix A
$\mathrm{rk}(A)$	normal rank of polynomial/rational matrix A (see also p.14)

As a final remark, we recall that a *minor* of a matrix A is the determinant of some smaller square matrix, cut down from A by removing one or more of its rows, indexed by tuple $\mathbf{i}$, or columns, indexed by tuple $\mathbf{j}$. If, furthermore, $\mathbf{i} = \mathbf{j}$ the minor is called a *principal minor* of A.

1. Introduction

The purpose of this Chapter is to provide a brief introduction to the two major topics we address in this work, namely the spectral and J-spectral factorization problem. We will briefly review the historical developments, some applications and the current state-of-the-art of these problems. In the final section, we will present the contributions and the organization of the work.

1.1 | Spectral factorization

Spectral factorization is a classical and extensively studied topic in control and systems theory. The origins of this mathematical tool can be traced back in the forties, when Kolmogorov and Wiener, independently each other, introduced it in order to obtain a frequency domain solution of optimal filtering problems for both the discrete-time scalar case Kolmogorov [1939] and continuous-time scalar case (the well-known *Wiener-Hopf technique*) Wiener [1949]. Since that time, spectral factorization has turned to be a crucial problem in many other areas beyond optimal filtering and prediction theory, such as circuit and network theory Oono [1956], Anderson and Vongpanitlerd [1973], Fornasini [1977], and linear-quadratic optimization Willems [1971].

Although extensions to the non-rational case have been handled in scientific literature, *e.g.*, Ferrante [1997b], in its most common continuous-time form, the multivariate spectral factorization problem can be stated as follows.

Problem 1.1 (Spectral factorization – continuous-time)
Consider a square real rational matrix $\Phi(s) \in \mathbb{R}(s)^{n\times n}$ *of normal rank* $\mathrm{rk}(\Phi) = r \leq n$*, satisfying:*

- $\Phi(s) = \Phi^\top(-s)$,
- $\Phi(j\omega) \geq 0$, $\forall \omega \in \mathbb{R}$, $s = j\omega$ *not a pole of* $\Phi(s)$.

Find a spectral factor $W(s) \in \mathbb{R}(s)^{r \times n}$, i.e., *a real rational matrix such that*

$$\Phi(s) = W^{\top}(-s)W(s), \tag{1.1}$$

with the following properties:

(i) $W(s)$ *is analytic in a right half-plane* $\{\Re\mathrm{e}\ s > \tau_1,\ \tau_1 < 0,\ s \in \mathbb{C}\}$,

(ii) *its (right) inverse* $W^{-R}(s)$ *is analytic in a right half-plane* $\{\Re\mathrm{e}\ s > \tau_2,\ \tau_2 < 0,\ s \in \mathbb{C}\}$.

The discrete-time version of Problem 1.1 is reported below.

Problem 1.2 (Spectral factorization – discrete-time)
Consider a square real rational matrix $\Phi(z) \in \mathbb{R}(z)^{n \times n}$ *of normal rank* $\mathrm{rk}(\Phi) = r \leq n$, *satisfying:*

- $\Phi(z) = \Phi^{\top}(1/z)$,
- $\Phi(e^{j\omega}) \geq 0$, $\forall \omega \in [0, 2\pi)$, $z = e^{j\omega}$ *not a pole of* $\Phi(z)$.

Find a spectral factor $W(z) \in \mathbb{R}(z)^{r \times n}$, i.e., *a real rational matrix such that*

$$\Phi(z) = W^{\top}(1/z)W(z), \tag{1.2}$$

with the following properties:

(i) $W(z)$ *is analytic in a region* $\{|z| > \tau_1,\ \tau_1 < 1,\ z \in \mathbb{C}\}$ *without any pole at infinity,*

(ii) *its (right) inverse* $W^{-R}(z)$ *is analytic in a region* $\{|z| > \tau_2,\ \tau_2 < 1,\ z \in \mathbb{C}\}$ *without any pole at infinity.*

It is worth noticing that, in the context of filtering and estimation theory, $\Phi(s)$ denotes the *spectral density* (or *spectrum*) of a multivariate continuous-time stationary stochastic process, while $\Phi(z)$ denotes its discrete-time counterpart.

The existence of a solution of Problem 1.1 was firstly proved by Youla in his celebrated work Youla [1961]. In this paper, Youla presents an ingenious technique, which exploits the *Smith-McMillan canonical form* of rational matrices, to compute a spectral factor that satisfies properties (i)-(ii) of Problem 1.1 (the so-called *minimum-phase stable* spectral factor). Subsequent to this fundamental work, many other algorithms have been proposed in order to solve the multivariate spectral factorization problem. By following the description given in Picci [2007, Ch.4], we can distinguish two general sets of techniques:

1. The first one uses the *Matrix Fraction Description* (MFD) theory in order to simplify the problem to that of a polynomial matrix factorization. Indeed, there exist efficient algorithms which are able to compute the minimum-phase polynomial factor, *viz.*

   (a) algorithms based on Cholesky factorization of the covariance matrix of the process Rissanen [1973] or on analogous techniques for equivalent state-space representations Anderson et al. [1974],

   (b) Newton-like iterative algorithms Ježek and Kučera [1985], Tunnicliffe-Wilson [1972].

2. The second set of techniques is closely related to the Kalman filtering theory. It provides a solution of the spectral factorization problem in terms of a solution of a suitable *Algebraic Riccati Equation* (ARE), *cf.* Picci [2007, Ch.10]. This kind of techniques has gathered momentum in the last decades and it has been used to provide complete parametrization and interesting characterization of *(stochastically) minimal* spectral factors Ferrante et al. [1993], Ferrante [1997a].

Far from being only of historical interest, the approach presented in Youla's paper can be regarded an interesting and useful tool when applied in other areas of control and systems theory. An example can be found in Ferrante and Pandolfi [2002]. In this paper, the method devised by Youla is used to compute a γ-spectral factor in order to weaken the standard assumption for the solvability of the classical *Positive Real Lemma* equations. Notably, unlike the other algorithms mentioned above, Youla's approach allows to easily modify the region of analyticity of both the obtained spectral factor and its (right) inverse. This can be considered one of the peculiar and relevant feature of Youla's method. Another relevant aspect of Youla's factorization approach is that it always leads to the computation of a (stochastically) minimal spectral factor, *i.e.*, a factor with the least possible *McMillan degree*[1], that is, in turn, the dimension of a minimal state-space realization of the spectral factor.

[1] Such McMillan degree is equal to one-half of the McMillan degree of the spectrum.

1.2 | J-spectral factorization

The J-spectral factorization problem can be considered a natural extension of the standard spectral factorization problem discussed above. In fact, the former problem encompasses the latter. More specifically, in the standard spectral factorization the rational spectrum Φ is required to be positive semi-definite on certain contours of the complex plane (namely, the imaginary axis in the continuous-time case, the unit circle in the discrete-time case). Instead, in the J-spectral factorization it is required that Φ has constant inertia (*i.e.*, constant number of positive, zero and negative eigenvalues) on certain contours of the complex plane.

The J-spectral factorization problem naturally arises in different areas of systems and control theory and it plays a prominent role in $\mathscr{H}_\infty$ estimation and control theory. A general survey on the subject can be found, for instance, in Stoorvogel [1992], while references based on the J-spectral factorization approach are, *e.g.*, Green et al. [1990], Colaneri and Ferrante [2002], Colaneri and Ferrante [2006].

In the continuous-time case the multivariate J-spectral factorization problem can be formulated as follows.

Problem 1.3 (J-spectral factorization – continuous-time)
Consider a square real rational matrix $\Phi(s) \in \mathbb{R}(s)^{n\times n}$ of normal rank $\mathrm{rk}(\Phi) = r \leq n$, satisfying:

- $\Phi(s) = \Phi^\top(-s)$,
- $\mathrm{in}(\Phi(j\omega)) = (\nu_p, \nu_0, \nu_n)$, $\forall \omega \in \mathbb{R}$, $s = j\omega$ *not a zero/pole of* $\Phi(s)$.

*Find a J-*spectral factor $W(s) \in \mathbb{R}(s)^{r\times n}$, i.e., *a real rational matrix such that*

$$\Phi(s) = W^\top(-s) \left[\begin{array}{c|c} I_{\nu_p} & \mathbf{0}_{\nu_p,\nu_n} \\ \hline \mathbf{0}_{\nu_n,\nu_p} & -I_{\nu_n} \end{array}\right] W(s), \tag{1.3}$$

with the following properties:

(i) $W(s)$ *is analytic in a right half-plane* $\{\Re\mathrm{e}\ s > \tau_1,\ \tau_1 < 0,\ s \in \mathbb{C}\}$,

(ii) *its (right) inverse* $W^{-R}(s)$ *is analytic in a right half-plane* $\{\Re\mathrm{e}\ s > \tau_2,\ \tau_2 < 0,\ s \in \mathbb{C}\}$.

In this work we will study the discrete-time version of the aforementioned problem, which is stated below.

Problem 1.4 (J-spectral factorization – discrete-time)
Consider a square real rational matrix $\Phi(z) \in \mathbb{R}(z)^{n\times n}$ *of normal rank* $\mathrm{rk}(\Phi) = r \leq n$*, satisfying:*

- $\Phi(z) = \Phi^\top(1/z)$,
- $\mathrm{in}(\Phi(e^{j\omega})) = (\nu_p, \nu_0, \nu_n)$, $\forall \omega \in [0, 2\pi)$, $z = e^{j\omega}$ *not a zero/pole of* $\Phi(z)$.

Find a J-spectral factor $W(z) \in \mathbb{R}(z)^{r\times n}$, i.e., *a real rational matrix such that*

$$\Phi(z) = W^\top(1/z) \left[\begin{array}{c|c} I_{\nu_p} & \mathbf{0}_{\nu_p,\nu_n} \\ \hline \mathbf{0}_{\nu_n,\nu_p} & -I_{\nu_n} \end{array}\right] W(z), \tag{1.4}$$

with the following properties:

(i) $W(z)$ *is analytic in a region* $\{|z| > \tau_1,\ \tau_1 < 1,\ z \in \mathbb{C}\}$ *without any pole at infinity,*

(ii) *its (right) inverse* $W^{-R}(z)$ *is analytic in a region* $\{|z| > \tau_2,\ \tau_2 < 1,\ z \in \mathbb{C}\}$ *without any pole at infinity.*

In analogy to standard spectral factorization, the rational matrix Φ is usually termed J*-spectrum*.

The J-spectral factorization problem is quite recent in literature. The most common approach for solving this problem hinges on the solution of suitable AREs (see, for instance, Colaneri and Ferrante [2006] and references therein). Other types of methods have been investigated, *e.g.*, in Kwakernaak and Šebek [1994] and in Trentelman and Rapisarda [1999], but the majority of them work only for the polynomial matrix case.

Moreover, conversely to the standard spectral factorization, a (stochastically) minimal J-spectral factor, *i.e.*, a J-spectral factor having the least possible McMillan degree, does not always exist, as shown in Colaneri and Ferrante [2006]. However, this rather counterintuitive fact has not been fully understood yet.

1.3 | Contributions and outline of the thesis

The contribution of this work is threefold.

1. We give a detailed, and, in the author's opinion, simplified with reference to certain steps, description of the Youla's factorization method. We focus on some peculiar features of the method, namely, on the minimality and on some properties of analyticity of the computed spectral factor, which are neither explicitly discussed in the original paper Youla [1961] nor in more recent works.

2. We present a discrete-time version of the Youla's factorization method. In particular, we draw the attention on the main differences between the continuous- and discrete-time case and on some special properties of the factorization.

3. We extend, under mild assumptions, the previous approach to the J-spectral factorization problem in discrete-time and we discuss certain issues which arise in this more general case.

The thesis is organized as follows. In Chapter 2, we review some mathematical notions and results on polynomial and rational matrices. In Chapter 3, we analyze in detail the factorization method proposed by Youla in his paper Youla [1961]. In particular, in section §3.3, we focus on some interesting by-products of this approach. By following the lines of Youla's approach, in Chapter 4, we present a method for solving the discrete-time spectral factorization problem. In Chapter 5, a "J-spectral" generalization of the previous method is proposed. Finally, in Chapter 6, we draw some final considerations and we describe a number of possible future developments.

2. Mathematical preliminaries

In this Chapter, we review the main mathematical notions and tools used in the development of Youla's approach and in its extensions. More specifically, we give a very short introduction to some abstract algebraic structures, polynomials, polynomial matrices and rational matrices. We will stress on those aspects which we consider more relevant in order to fully understand the rest of the work. We refer the interested reader to Jacobson [1985], Mac Lane and Birkhoff [1999], Fornasini [2011, Ch.1,3,4] and Kailath [1998, Ch.6] for an exhaustive and more rigorous dissertation on these subjects.

2.1 | Basic facts of algebra

This section is devoted to provide to the reader some background material on polynomial algebra over fields.

2.1.1 | Rings and fields

Throughout this Chapter a ring is intended to be a commutative ring with identity, unless otherwise indicated. We refer to Jacobson [1985, Vol.I, Ch.2, §1 and §17] for more general definitions of ring.

Definition 2.1.1 (Ring) A ring (commutative, with identity) is a set R equipped with two operations, addition "$+$" and multiplication "$\cdot$", and with a zero element 0 and an identity element 1, which obey the following axioms, for all $a,b,c \in R$:

A1) $(a+b)+c = a+(b+c)$

A2) $a+b = b+a$

A3) $a+0 = 0+a = a$

A4) for each a there is an inverse $-a$ such that $a+(-a)=0=-a+a$

M1) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

M2) $a \cdot b = b \cdot a$

M3) $a \cdot 1 = 1 \cdot a = a$

D1) $a \cdot (b+c) = a \cdot b + a \cdot c$

Definition 2.1.2 (Subring) A subring of a ring R is a subset of R that is itself a ring when binary operations of addition and multiplication on R are restricted to the subset, and which contains the multiplicative identity of R.

If a is any element of a ring R such that $a \cdot b = 1$ for some element b of R, then a is said to be a *unit* of R.

If a and b are elements of a ring R, then we say *a divides b*, $a \mid b$, if there exists an element q of R such that $b = a \cdot q$. In this case a is called a divisor (or factor) of b and b is called a multiple of a. If an element d of R divides both a and b, then d is a common divisor of a and b; if, furthermore, d is a multiple of every common divisor of a and b, then d is a *greatest common divisor* (g.c.d.) of a and b. If an element m of R is a multiple of both a and b, then it is called a common multiple of a and b; if, furthermore, m is a divisor of every common multiple of a and b, then m is a *least common multiple* (l.c.m.) of a and b.

We say that elements a and b are *coprime* or *relatively prime* in R if their greatest common divisor is a unit of R.

Definition 2.1.3 (Field) A (commutative) ring R in which every non-zero element is a unit is called a *field*.

Thus, in a field. the set of axioms M1)-M3) can be completed by:

M4) $a \cdot a^{-1} = 1,\ a \neq 0,\ a \in R.$

Two rings, or fields, are *isomorphic* if there is a one-to-one correspondence between their elements which preserves the operations of addition and multiplication.

To conclude this subsection, we observe that a well-known example of a ring is the set of integers $\mathbb{Z}$ with usual $+, \cdot, 0, 1$, while examples of a field are the rationals

$\mathbb{Q}$, the reals $\mathbb{R}$, and the complex numbers $\mathbb{C}$.

2.1.2 | Polynomials

A polynomial $p(z)$ over a field $\mathbb{F}$ is an expression

$$p(z) := p_n z^n + \cdots + p_1 z + p_0 \tag{2.1}$$

in which $p_0, p_1, \cdots, p_n$ belong to $\mathbb{F}$. In the sequel, for our purposes, we set $\mathbb{F} = \mathbb{R}$.

If $p_n \neq 0$, the non-negative integer n is the *degree* of $p(z)$, written $\deg p(z)$. If $p(z)$ is the zero polynomial, we use the convention $\deg p(z) = -\infty$. If $p_n = 1$, the polynomial is said to be *monic*.

We define the addition of two polynomials $p(z) = \sum_i p_i z^i$ and $q(z) = \sum_i q_i z^i$ as

$$p(z) + q(z) := \sum_i (p_i + q_i) z^i, \tag{2.2}$$

and the multiplication of the two polynomial as

$$p(z) \cdot q(z) := \sum_i \sum_h (p_{i-h} \cdot q_h) z^i. \tag{2.3}$$

With these two operations, the set of polynomials over $\mathbb{R}$ is a (commutative) ring, denoted by $\mathbb{R}[z]$. The units of $\mathbb{R}[z]$ are polynomials of zero degree.

At least one greatest common divisor and one least common multiple exist for any pair of polynomials in $\mathbb{R}[z]$. Moreover, any greatest common divisor, $d(z)$, of two polynomials $a(z)$ and $b(z)$ can be expressed in the form

$$d(z) = a(z) \cdot p(z) + b(z) \cdot q(z), \tag{2.4}$$

for some (relatively prime) polynomials $p(z)$ and $q(z)$ of $\mathbb{R}[z]$ and any least common multiple, $m(z)$, of $a(z)$ and $b(z)$ is given by

$$m(z) = a(z) \cdot r(z) = -b(z) \cdot s(z), \tag{2.5}$$

where $r(z)$ and $s(z)$ are relatively prime polynomials in $\mathbb{R}[z]$ that satisfy

$$a(z) \cdot r(z) + b(z) \cdot s(z) = 0. \tag{2.6}$$

Hence, $a(z)$ and $b(z)$ are relatively prime in $\mathbb{R}[z]$ if and only if there exist polynomials $p(z)$ and $q(z)$ in $\mathbb{R}[z]$ such that

$$a(z) \cdot p(z) + b(z) \cdot q(z) = 1. \tag{2.7}$$

This relationship is known as the *Bézout identity*. We notice also that g.c.d. and l.c.m. are unique up to multiplication by units.

We point out that a polynomial $p(z)$ may be viewed in two different ways. Either it is considered to be an element of $\mathbb{R}[z]$, in which case z is an indeterminate over $\mathbb{R}$ and no question arises of giving it a value in $\mathbb{R}$, or it is considered to be a function that associates to each element z of $\mathbb{R}$, another element $p(z)$, *i.e.*,

$$\begin{aligned} p\colon \mathbb{R} &\to \mathbb{R} \\ z &\mapsto p(z) \end{aligned}$$

Since different elements of $\mathbb{R}[z]$ identify different polynomial functions, no distinction need to be made between these two points of view. In general, if we deal with the ring $\mathbb{F}[z]$, $\mathbb{F}$ being an arbitrary field, this fact is not always true (see Fornasini [2011, Ch.1, §5] for further details).

The values of z for which a polynomial $p(z)$ takes the value zero in $\mathbb{R}$ are called the *roots* or *zeros* of $p(z)$. They reside in an *algebraic closure* of $\mathbb{R}$, which is the field of complex number $\mathbb{C}$. We say that a polynomial $p(z)$ is *Hurwitz* if it has no root z such that $\Re\mathrm{e}\, z > 0$ and it is *strictly Hurwitz* if it has no root z such that $\Re\mathrm{e}\, z \geq 0$. Furthermore, we say that a polynomial $p(z)$ is *Schur* if it has no root z such that $|z| > 1$ and it is *strictly Schur* if it has no root z such that $|z| \geq 1$.

2.1.3 | Laurent polynomials

A Laurent polynomial, or briefly L-polynomial, $p(z)$ over a field $\mathbb{F}$ is an expression

$$p(z) := p_{-m}z^{-m} + \cdots + p_{-1}z^{-1} + p_0 + p_1 z + \cdots + p_n z^n \tag{2.8}$$

in which $p_{-m}, \ldots, p_{-1}, p_0, p_1, \ldots, p_n$ belong to $\mathbb{F}$. The set of L-polynomials over $\mathbb{R}$ equipped with operations defined in (2.2) and (2.3) is a (commutative) ring, denoted by $\mathbb{R}[z, z^{-1}]$. In this case, the units of $\mathbb{R}[z, z^{-1}]$ are the monomials αz^k, $\alpha \in \mathbb{R}$, $\alpha \neq 0$, $k \in \mathbb{Z}$.

We define the *maximum-degree* of $p(z)$, denoted by $\max\deg p$, as the largest integer n such that $p_n \neq 0$, the *minimum-degree* of $p(z)$, denoted by $\min\deg p$, as the smallest integer m such that $p_m \neq 0$ and the *total degree*, or simply degree, of $p(z)$ as $\deg p := n+m$. If $p(z)$ is the zero L-polynomial, we set by convention $\max\deg p = \deg p = -\infty$ and $\min\deg p = +\infty$. Similarly to polynomials, we can define the greatest common divisor and the least common multiple for L-polynomials.

2.1.4 | Rational functions

In this subsection, we will construct the field of rational functions starting from the polynomial ring $\mathbb{R}[z]$ and we will study some of its elementary properties.

Let us consider the set $\mathbb{R}[z]\times(\mathbb{R}[z]\setminus\{0\})$, which consists of the ordered pairs $(p(z),q(z))$, $q(z)\neq 0$. We introduce in this set the equivalence relation $\sim$, requiring that

$$(p(z),q(z)) \sim (n(z),d(z)) \quad \Leftrightarrow \quad n(z)q(z) = p(z)d(z). \tag{2.9}$$

We denote the equivalence class identified by the pair $(p(z),q(z))$ as $p(z)/q(z)$. In the equivalence class set $\mathbb{R}[z]\times(\mathbb{R}[z]\setminus\{0\})/\sim$, the addition and multiplication operations are defined as follows

$$\frac{p(z)}{q(z)} + \frac{n(z)}{d(z)} := \frac{p(z)\cdot d(z) + n(z)\cdot q(z)}{q(z)\cdot d(z)}, \tag{2.10}$$

$$\frac{p(z)}{q(z)} \cdot \frac{n(z)}{d(z)} := \frac{p(z)\cdot n(z)}{q(z)\cdot d(z)}. \tag{2.11}$$

The set equipped with these operations takes the form of a field, the *field of real rational functions*. We denote it by symbol $\mathbb{R}(z)$, as stated in Chapter 0.

Let $f(z) = p(z)/q(z) \in \mathbb{R}(z)$ be a non-zero rational function. We can always write $f(z)$ in the form

$$f(z) = \frac{n(z)}{d(z)}(z-\alpha)^{\nu}, \quad \forall \alpha \in \mathbb{C}, \tag{2.12}$$

where ν is a integer and $n(z)$, $d(z) \in \mathbb{R}[z]$ are non-zero polynomials such that $n(\alpha)\neq 0$ and $d(\alpha)\neq 0$. The integer ν is called *valutation of f(z) at* α, we denote it by symbol $v_\alpha(f)$. The valutation of $f(z)$ at infinity is defined as $v_\infty(f) := \deg q(z) - \deg p(z)$. If $f(z)$ is the null function, by definition, $v_\alpha(f) = +\infty$ for every $\alpha \in \mathbb{C}$. If $v_\alpha(f) < 0$, then $\alpha \in \mathbb{C}$ is called a *pole of $f(z)$ of multiplicity* $-v_\alpha(f)$. If $v_\alpha(f) > 0$, then $\alpha \in \mathbb{C}$ is called a *zero of $f(z)$ of multiplicity $v_\alpha(f)$*. We can define the pole and zero at infinity in a similar way. A rational function $f(z)$ is said to be *proper* if $v_\infty(f) \geq 0$, *strictly proper* if $v_\infty(f) > 0$.

A real rational function $f(z) \in \mathbb{R}(z)$ is called *weakly Hurwitz stable* if it does not possess any pole in the open right half-plane $\{\Re\text{e}\ z > 0,\ z \in \mathbb{C}\}$ and *weakly Schur stable* if it does not possess any pole in the region $\{|z| > 1,\ z \in \mathbb{C}\}$. Furthermore, $f(z) \in \mathbb{R}(z)$ is called *strictly Hurwitz stable* if it does not possess any pole in the closed right half-plane $\{\Re\text{e}\ z \geq 0,\ z \in \mathbb{C}\}$ and *strictly Schur stable* if it does not possess any pole in the region $\{|z| \geq 1,\ z \in \mathbb{C}\}$.

As a final remark, we notice that the ring of Laurent polynomials is a subring of the rational functions.

2.2 | Polynomial matrices

We recall, from Chapter 0, that $\mathbb{R}[z]^{p\times m}$ denotes the set of $p \times m$ matrices with entries in $\mathbb{R}[z]$ and $\mathbb{R}[z, z^{-1}]^{p\times m}$ denotes the set of $p \times m$ matrices with entries in $\mathbb{R}[z, z^{-1}]$.

2.2.1 | Elementary matrices and canonical forms

We start by reviewing some basic definitions.

Definition 2.2.1 (Singular polynomial matrix) A matrix $M(z) \in \mathbb{R}[z]^{m\times m}$ is said to be *singular* if its determinant is the zero polynomial, *i.e.*, $\det M(z) = 0$. Otherwise, $M(z)$ is said to be *non-singular*.

Definition 2.2.2 (Normal rank) A matrix $M(z) \in \mathbb{R}[z]^{p\times m}$ has *normal rank* r, written $\text{rk}(M) = r$, if all its $(r+1)\times(r+1)$ minors are zero polynomials and there exists at least one non-zero $r \times r$ minor. Clearly, $\text{rk}(M) \leq \min\{p, m\}$. If $\text{rk}(M) = \min\{p, m\}$, then $M(z)$ is said to be of *full normal rank*.

It is worthwhile noticing that the normal rank of $G(z)$ coincides with the rank of $G(z)$ almost everywhere (*i.e.*, for all but finitely many points) in $z \in \mathbb{C}$.

A particularly relevant class of polynomial matrices is the class of *elementary matrices*. These are square matrices which can assume one of the following three structures:

$$\tilde{E}_1(z) := \begin{bmatrix} 1 & 0 & & \cdots & & & 0 \\ 0 & \ddots & & & & & \\ & & 1 & & & & \\ \vdots & & & \alpha & & & \vdots \\ & & & & 1 & & \\ & & & & & \ddots & 0 \\ 0 & & & \cdots & & 0 & 1 \end{bmatrix}, \quad \alpha \in \mathbb{R} \setminus \{0\},$$

$$\tilde{E}_2(z) := \begin{bmatrix} 1 & & & \cdots & & & 0 \\ 0 & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \vdots \\ \vdots & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & 0 \\ 0 & & & \cdots & & & 1 \end{bmatrix},$$

$$\tilde{E}_3(z) := \begin{bmatrix} 1 & 0 & & \cdots & & & 0 \\ 0 & \ddots & & & & & \\ & & 1 & \cdots & p(z) & & \\ \vdots & & & \ddots & \vdots & & \vdots \\ & & & & 1 & & \\ & & & & & \ddots & 0 \\ 0 & & & \cdots & & & 1 \end{bmatrix}, \quad p(z) \in \mathbb{R}[z].$$

An elementary matrix $\tilde{E}_i(z) \in \mathbb{R}[z]^{m\times m}$, $i = 1,2,3$, can be viewed as a linear operator acting on the columns of an arbitrary $p \times m$ polynomial matrix $M(z)$, *i.e.*,

$$\tilde{E}_i \colon \mathbb{R}[z]^{p\times m} \to \mathbb{R}[z]^{p\times m}, \quad M(z) \mapsto M(z)\tilde{E}_i(z),$$

More specifically, $\tilde{E}_1(z)$ corresponds to multiply a column of $M(z)$ by a non-zero real constant α, $\tilde{E}_2(z)$ to swap two columns of $M(z)$ and $\tilde{E}_3(z)$ to sum a column of $M(z)$ multiplied by $p(z) \in \mathbb{R}[z]$ to another column of $M(z)$. Clearly, if we consider the linear operator

$$\tilde{E}_i \colon \mathbb{R}[z]^{p\times m} \to \mathbb{R}[z]^{p\times m}, \quad M(z) \mapsto \tilde{E}_i(z)M(z),$$

then we have analogous elementary operations acting on the rows of $M(z)$.

We also notice that the determinant of an elementary matrix is a non-zero constant. Hence, an elementary matrix is invertible and its inverse also is an elementary matrix.

Definition 2.2.3 Let $M(z),\ N(z) \in \mathbb{R}[z]^{p\times m}$. We say that $M(z)$ is *equivalent* to $N(z)$, written $M(z) \sim N(z)$, if there exist elementary matrices $E_1(z), E_2(z), \dots, E_k(z) \in \mathbb{R}[z]^{p\times p}$ and $E'_1(z), E'_2(z), \dots, E'_h(z) \in \mathbb{R}[z]^{m\times m}$ such that

$$E_k(z)\cdots E_2(z)E_1(z)N(z)E'_1(z)E'_2(z)\cdots E'_h(z) = M(z). \tag{2.13}$$

The relation $\sim$ in $\mathbb{R}[z]^{p\times m}$ is an equivalence relation and, therefore, the set $\mathbb{R}[z]^{p\times m}$ can be partitioned in disjoint equivalence classes. The following Theorem states that every polynomial matrix is equivalent to a polynomial matrix with a peculiar structure, the *Smith canonical form* of $M(z)$.

Theorem 2.2.1 (Smith canonical form) *Let $M(z) \in \mathbb{R}[z]^{p\times m}$. There exists a finite sequence of elementary matrices which reduces $M(z)$ to the form*

$$\begin{aligned}\Gamma(z) &:= E_k(z)\cdots E_1(z)M(z)E'_1(z)\cdots E'_h(z)\\ &= \left[\begin{array}{ccc|c} \gamma_1(z) & & & \\ & \ddots & & \mathbf{0}_{r,m-r} \\ & & \gamma_r(z) & \\ \hline & \mathbf{0}_{p-r,r} & & \mathbf{0}_{p-r,m-r} \end{array}\right]\end{aligned} \tag{2.14}$$

where $r = \operatorname{rk}(M)$ and $\gamma_1(z), \gamma_2(z), \dots, \gamma_r(z) \in \mathbb{R}[z]$ are monic polynomials which satisfy $\gamma_i(z) \mid \gamma_{i+1}(z)$, for $i = 1, 2, \dots, r-1$. These polynomials are *uniquely determined by the previous conditions and they are termed* invariant polynomials *of* $M(z)$.

A proof of Theorem 2.2.1 can be found in Fornasini [2011, Ch.3, §1, Thm. 3.1.4]. We underline the fact that, although the Smith canonical form is unique, the sequence of elementary matrices used to obtain it is not so.

By using elementary operations acting only on the rows, we can convert a polynomial matrix to another well-known "standard" form, the *column Hermite form*.[1] In what follows, we restrict the analysis only to polynomial matrices whose normal rank is equal to the number of columns. For a complete discussion on the topic, as well as for a proof of the Theorem below, we refer to [Kailath, 1998, Ch.6, §3].

Theorem 2.2.2 (Column Hermite form) *Let* $M(z) \in \mathbb{R}[z]^{p\times m}$ *be a polynomial matrix of normal rank* $\mathrm{rk}(M) = m$. *There exists a family of elementary matrices* $E_1(z),\ E_2(z), \ldots,\ E_k(z) \in \mathbb{R}[z]^{p\times p}$ *such that*

$$H(z) := E_k(z)\cdots E_1(z)M(z) = \left[\begin{array}{cccc} h_{11}(z) & h_{12}(z) & \cdots & h_{1m}(z) \\ 0 & h_{22}(z) & \cdots & h_{2m}(z) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_{mm}(z) \\ \hline \multicolumn{4}{c}{\mathbf{0}_{p-m,m}} \end{array}\right] \tag{2.15}$$

with $h_{jj}(z) \in \mathbb{R}[z]$ *monic satisfying* $\deg h_{jj} > \deg h_{ij}$ *for* $j = 1, 2, \ldots, m$ *and* $i < j$.

By interchanging the roles of rows and columns, one can obtain a similar *row Hermite form*, the details of which we shall not spell out.

[1] It is worthwhile noticing that this form is obtained by using elementary operations acting only on the rows of the matrix and, therefore, does not represent a canonical form for the equivalence relation $\sim$ introduced before.

2.2.2 | Unimodular matrices and coprime matrices

Unimodular matrices are the units (*i.e.*, the invertible elements) of the (non-commutative) ring of square polynomial matrices $\mathbb{R}[z]^{m\times m}$. Obviously, the elementary matrices introduced in the previous subsection are also unimodular matrices. Actually, they are the building blocks of unimodular matrices, since any unimodular matrix can be written as a product of suitable elementary matrices.

Definition 2.2.4 (Unimodular matrix) A matrix $U(z)\in\mathbb{R}[z]^{m\times m}$ is *unimodular* if it is invertible in $\mathbb{R}[z]^{m\times m}$.

The following Proposition provides various characterizations of unimodular matrices. The proof can be found in Fornasini [2011, Ch.3, §2, Prop.3.2.2].

Proposition 2.2.1 *Let $U(z)\in\mathbb{R}[z]^{m\times m}$. The following statements are equivalent:*

1. *$U(z)$ is a unimodular matrix;*
2. $\det U(z)$ *is a non-zero real constant;*
3. *the Smith canonical form of $U(z)$ is the identity matrix I_m;*
4. *the (column and row) Hermite form of $U(z)$ is the identity matrix I_m;*
5. *$U(z)$ can be written as a product of elementary matrices.*

A direct consequence of Proposition 2.2.1 is that two polynomial matrices $N(z)$ and $M(z)$ are equivalent if and only if $N(z)=U(z)M(z)V(z)$, where $U(z)$ and $V(z)$ are suitable unimodular matrices. In particular, Theorem 2.2.1 can be reformulated as follows: let $M(z)\in\mathbb{R}[z]^{p\times m}$ be a polynomial matrix of normal rank $\mathrm{rk}(M)=r$, there exist a pair of unimodular matrices $U(z)\in\mathbb{R}[z]^{p\times p}$ and $V(z)\in\mathbb{R}[z]^{m\times m}$ such that

$$\Gamma(z)=U(z)M(z)V(z)=\left[\begin{array}{ccc|c}\gamma_1(z) & & & \\ & \ddots & & \mathbf{0}_{r,m-r} \\ & & \gamma_r(z) & \\ \hline & \mathbf{0}_{p-r,r} & & \mathbf{0}_{p-r,m-r}\end{array}\right], \tag{2.16}$$

where $r = \mathrm{rk}(M)$ and $\gamma_1(z), \gamma_2(z), \dots, \gamma_r(z) \in \mathbb{R}[z]$ are uniquely determined monic polynomials which satisfy $\gamma_i(z) \mid \gamma_{i+1}(z)$, for $i = 1, 2, \dots, r-1$.

A similar result holds for Theorem 2.2.2. Let $M(z) \in \mathbb{R}[z]^{p\times m}$ be a polynomial matrix of normal rank $\mathrm{rk}(M) = m$. Then, there exists an unimodular matrix $U(z) \in \mathbb{R}[z]^{p\times p}$ such that

$$H(z) := U(z)M(z) = \left[\begin{array}{cccc} h_{11}(z) & h_{12}(z) & \cdots & h_{1m}(z) \\ 0 & h_{22}(z) & \cdots & h_{2m}(z) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_{mm}(z) \\ \hline \multicolumn{4}{c}{\mathbf{0}_{p-m,m}} \end{array}\right] \tag{2.17}$$

with $h_{jj}(z) \in \mathbb{R}[z]$ monic satisfying $\deg h_{jj} > \deg h_{ij}$ for $j = 1, 2, \dots, m$ and $i < j$.

It is useful to extend the notion of unimodular matrices to the non-square case. In this case, a unimodular matrix $U(z) \in \mathbb{R}[z]^{m\times p}$ is intended to be a polynomial matrix possessing either a right or left polynomial inverse. By using the latter definition of unimodular matrices, the Smith decomposition (2.16) can be equivalently rewritten as

$$\Gamma'(z) = U'(z)M(z)V'(z) = \begin{bmatrix} \gamma_1(z) & & & \\ & \gamma_2(z) & & \\ & & \ddots & \\ & & & \gamma_r(z) \end{bmatrix}, \tag{2.18}$$

where the only difference is that $\Gamma'(z) \in \mathbb{R}[z]^{r\times r}$ is non-singular and $U'(z) \in \mathbb{R}[z]^{p\times r}$, $V'(z) \in \mathbb{R}[z]^{r\times m}$ are non-square unimodular matrices, which are related to $U(z)$ and $V(z)$ in (2.16) by

$$U'(z) := U(z)\left[\begin{array}{c} I_r \\ \hline \mathbf{0}_{p-r,r} \end{array}\right], \quad V'(z) := \left[\begin{array}{c|c} I_r & \mathbf{0}_{r,m-r} \end{array}\right] V(z). \tag{2.19}$$

With a partial abuse of notation, in the following, when we will refer to the Smith decomposition of $M(z)$ we intend a decomposition of the form (2.18). In particular, by using this convention, we have that also in the non-square case the Smith canonical form of a unimodular matrix is the identity matrix of suitable dimension.

The notion of divisor of a polynomial can be extended to the matrix case as described in the following Definition.

Definition 2.2.5 (Right and left matrix divisors) Let $M(z) \in \mathbb{R}[z]^{p\times m}$. A square matrix $\Delta \in \mathbb{R}[z]^{m\times m}$ is said to be a *right divisor of* $M(z)$ if there exists a matrix $\bar{M}(z) \in \mathbb{R}[z]^{p\times m}$ such that

$$M(z) = \bar{M}(z)\Delta(z). \tag{2.20}$$

Moreover, a square matrix $\nabla \in \mathbb{R}[z]^{p\times p}$ is said to be a *left divisor of* $M(z)$ if there exists a matrix $\tilde{M}(z) \in \mathbb{R}[z]^{p\times m}$ such that

$$M(z) = \nabla(z)\tilde{M}(z). \tag{2.21}$$

In particular, we have that if in every factorization (2.20) $\Delta(z)$ is unimodular, then $M(z)$ is said to be *right coprime*. Similiarly, if in every factorization (2.21) $\nabla(z)$ is unimodular, then $M(z)$ is said to be *left coprime*.

2.2.3 | Column-reduced and row-reduced matrices

Definition 2.2.6 (Degree of a polynomial vector) The *degree* of a non-zero polynomial column (or row) vector $\mathbf{v}(z) \in \mathbb{R}[z]^p$ is defined as the highest degree of its polynomial entries.

Definition 2.2.7 (External and internal degree) Let $M(z) \in \mathbb{R}[z]^{p\times m}$ be a polynomial matrix of normal rank $\mathrm{rk}(M) = m$. Let k_i, $i = 1,\dots,m$, be the degree of the i-th column of $M(z)$. The *external (column) degree* of $M(z)$ is defined as

$$\mathrm{ext_c}\deg M := \sum_{i=1}^{m} k_i. \tag{2.22}$$

The *internal degree* of $M(z)$, int deg M, is defined as the highest degree between all the $m \times m$ minors of $M(z)$.

It can be shown (see Fornasini [2011, Ch.3, §5]) that the external degree of $M(z)$ provides an upper bound on the internal degree of $M(z)$, *i.e.*,

$$\text{int deg } M \leq \text{ext}_\text{c} \text{ deg } M. \tag{2.23}$$

Inequality may hold because of possible cancellations. However, if $M(z)$ is such that equality holds in the above, we shall say that $M(z)$ is *column reduced*.

Definition 2.2.8 (Column-reduced matrix) A polynomial matrix $M(z) \in \mathbb{R}[z]^{p\times m}$ of normal rank $\text{rk}(M) = m$ is said to be *column-reduced* if

$$\text{int deg } M = \text{ext}_\text{c} \text{ deg } M. \tag{2.24}$$

In particular, a square column-reduced polynomial matrix $M(z) \in \mathbb{R}[z]^{m\times m}$ satisfies

$$\text{deg det } M = \text{ext}_\text{c} \text{ deg } M = \sum_{i=1}^{m} k_i. \tag{2.25}$$

Let $M(z) \in \mathbb{R}[z]^{p\times m}$ and let $\text{rk}(M) = m$. We denote by $M^\text{hc} \in \mathbb{R}^{p\times m}$ the *highest-column-degree coefficient matrix* of $M(z)$, *i.e.*, the matrix whose i-th column consists of the coefficients of the monomials z^{k_i}'s of the same column of $M(z)$. Then, we can write $M(z)$ in the form

$$M(z) = M^\text{hc} \begin{bmatrix} z^{k_1} & & & \\ & z^{k_2} & & \\ & & \ddots & \\ & & & z^{k_m} \end{bmatrix} + M_\text{rem}(z), \tag{2.26}$$

where the column degrees of the "remainder matrix" $M_\text{rem}(z)$ are strictly lower than the corresponding column degrees of $M(z)$. The next Proposition provides a simple test to verify if $M(z)$ is column-reduced. For a proof of the latter we refer to Fornasini [2011, Ch.3, §5, Prop.3.5.3].

Proposition 2.2.2 *A matrix $M(z) \in \mathbb{R}[z]^{p\times m}$ of normal rank $\mathrm{rk}(M) = m$ is column-reduced if and only if its highest-column-degree coefficient matrix M^{hc} has rank m.*

Let $M(z) \in \mathbb{R}[z]^{p\times m}$ be a polynomial matrix of normal rank $\mathrm{rk}(M) = p$ and let $h_i,\ i = 1,\dots,p$, be the degree of the i-th row of $M(z)$. We can define the *external (row) degree* of $M(z)$ as $\mathrm{ext_r}\deg := \sum_{i=1}^{p} h_i$. The Definition of *row-reduced matrix* is similar to Definition 2.2.8 of column-reduced matrix.

Definition 2.2.9 (Row-reduced matrix) A polynomial matrix $M(z) \in \mathbb{R}[z]^{p\times m}$ of normal rank $\mathrm{rk}(M) = p$ is said to be *row-reduced* if

$$\mathrm{int}\deg M = \mathrm{ext_r}\deg M. \tag{2.27}$$

In particular, a square row-reduced polynomial matrix $M(z) \in \mathbb{R}[z]^{p\times p}$ satisfies

$$\deg\ \det M = \mathrm{ext_r}\deg M = \sum_{i=1}^{p} h_i. \tag{2.28}$$

At this point, if we denote by $M^{\mathrm{hr}} \in \mathbb{R}^{p\times m}$ the *highest-row-degree coefficient matrix* of $M(z) \in \mathbb{R}[z]^{p\times m}$, *i.e.*, the matrix whose i-th row consists of the coefficients of the monomials z^{h_i}'s of the same row of $M(z)$, we have a "row-reduced" counterpart of Proposition 2.2.2.

Proposition 2.2.3 *A matrix $M(z) \in \mathbb{R}[z]^{p\times m}$ of normal rank $\mathrm{rk}(M) = p$ is row-reduced if and only if its highest-row-degree coefficient matrix M^{hr} has rank p.*

2.2.4 | Laurent polynomial matrices

The results on polynomial matrices contained in the previous sections can be adapted to Laurent polynomial matrices, or more briefly, L-polynomial matrices. The main difference lies in the fact that the units of the ring $\mathbb{R}[z,z^{-1}]$ are the monomials $\alpha z^k,\ \alpha \in \mathbb{R},\ \alpha \neq 0,\ k \in \mathbb{Z}$, as noticed before in §2.1.3.

The class of *Laurent elementary matrices*, or simply *L-elementary matrices*, is wider than that of elementary polynomial matrices introduced in §2.2.1. In fact, in this case, matrices of the form $\tilde{E}_1$ can have non-zero monomials on their diagonal elements and in matrices of the form $\tilde{E}_3$ the polynomial $p(z)$ is an element of the ring $\mathbb{R}[z,z^{-1}]$. In analogy to Definition 2.2.4, we define the *Laurent unimodular matrices* as the units of the (non-commutative) ring of square L-polynomial matrices $\mathbb{R}[z,z^{-1}]^{m\times m}$.

Definition 2.2.10 (L-unimodular matrix) A matrix $U(z) \in \mathbb{R}[z,z^{-1}]^{m\times m}$ is *Laurent unimodular*, or simply *L-unimodular*, if it is invertible in $\mathbb{R}[z,z^{-1}]^{m\times m}$.

A matrix $U(z) \in \mathbb{R}[z,z^{-1}]^{m\times m}$ is L-unimodular if and only if its determinant is a non-zero monomial. Moreover, any L-unimodular matrix $U(z)$ possesses a factorization in terms of L-elementary matrices. The notion of L-unimodular matrices can be extended to the non-square case as follows: the non-square L-polynomial matrix $U(z) \in \mathbb{R}[z,z^{-1}]^{p\times m}$ is L-unimodular if it has either a right or left L-polynomial inverse.

Let $M(z) \in \mathbb{R}[z,z^{-1}]^{p\times m}$ be a L-polynomial matrix of normal rank $\mathrm{rk}(M) = r$. By applying suitable L-unimodular transformations to $M(z)$, we can reduce $M(z)$ to its *Smith canonical form*

$$\Gamma(z) := \begin{bmatrix} \gamma_1(z) & & & \\ & \gamma_2(z) & & \\ & & \ddots & \\ & & & \gamma_r(z) \end{bmatrix}, \tag{2.29}$$

where the L-polynomials $\gamma_i(z)$'s are uniquely determined by the following conditions: (i) they are monic polynomials which belong to $\mathbb{R}[z]$, (ii) they have non-zero constant term and (iii) they satisfy $\gamma_i(z) \mid \gamma_{i+1}(z)$, for $i = 1,2,\dots,r-1$. It can be verified that the Smith canonical form of a L-unimodular matrices is the identity matrix of suitable dimension.

Moreover, by pre-multiplying $M(z)$ by a suitable square L-unimodular matrix, we can convert it to the *column Hermite form*. In particular, if $\mathrm{rk}(M) = m$, the latter is given by

$$H(z) := \left[\begin{array}{cccc} h_{11}(z) & h_{12}(z) & \cdots & h_{1m}(z) \\ 0 & h_{22}(z) & \cdots & h_{2m}(z) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_{mm}(z) \\ \hline \multicolumn{4}{c}{\mathbf{0}_{p-m,m}} \end{array}\right] \tag{2.30}$$

where $h_{jj}(z)$, $j = 1,2,\ldots,m$, satisfies the conditions: (i) it is a monic polynomial which belong to $\mathbb{R}[z]$, (ii) it has non-zero constant term and (iii) $\deg h_{jj} > \deg h_{ij}$ for $i < j$. For a generic L-unimodular matrix $U(z) \in \mathbb{R}[z,z^{-1}]^{p\times m}$, $p \geq m$, the column Hermite form is

$$H(z) := \left[\begin{array}{c} I_m \\ \hline \mathbf{0}_{p-m,m} \end{array}\right]. \tag{2.31}$$

We can also adapt the results about column-reduced and row-reduced polynomial matrices described in §2.2.3 to L-polynomial matrices. For the sake of brevity, we will restrict the analysis only to the column-reduced L-polynomial case. For details and proofs we refer to Fornasini [2011, Ch.3, §7].

Definition 2.2.11 (Degrees of a L-polynomial vector) Let $\mathbf{v}(z) \in \mathbb{R}[z,z^{-1}]^p$ be a non-zero L-polynomial column (or row) vector. We define the *maximum-degree* of $\mathbf{v}(z)$ as the highest maximum-degree of its L-polynomial entries and the *minimum-degree* of $\mathbf{v}(z)$ as the lowest minimum-degree of its L-polynomial entries.

Let $M(z) \in \mathbb{R}[z,z^{-1}]^{p\times m}$ and let $\mathrm{rk}(M) = m$. We denote by $K_1,\ldots,K_m$ and by $k_1,\ldots,k_m$ the maximum-degrees and the minimum-degrees of the columns of $M(z)$, respectively. Moreover, we define $M^{\mathrm{hc}} \in \mathbb{R}^{p\times m}$ as the *highest-column-degree coefficient matrix* of $M(z)$, *i.e.*, the matrix whose i-th column consists of the coefficients of the monomials z^{K_i}'s of the same column of $M(z)$, and $M^{\mathrm{lc}} \in \mathbb{R}^{p\times m}$ as the *lowest-column-degree coefficient matrix* of $M(z)$, *i.e.*, the matrix whose i-th column consists of the coefficients of the monomials z^{k_i}'s of the same column of

$M(z)$. We can now introduce the Definition of external (column) degree and internal degree of a L-polynomial matrix.

Definition 2.2.12 (External and internal degree) Let $M(z) \in \mathbb{R}[z,z^{-1}]^{p\times m}$ and let $\mathrm{rk}(M) = m$. The *external (column) degree* of $M(z)$ is defined as

$$\mathrm{ext_c}\ \deg M := \sum_{i=1}^{m} K_i - \sum_{i=1}^{m} k_i. \tag{2.32}$$

The *internal degree* of $M(z)$, int deg M, is defined as

$$\mathrm{int}\ \deg M := \max_{\mathbf{i}}\{\max\deg(\det M^{(\mathbf{i})})\} - \min_{\mathbf{i}}\{\min\deg(\det M^{(\mathbf{i})})\}, \tag{2.33}$$

where $M^{(\mathbf{i})}$ denotes the sub-matrix obtained by selecting the rows of $M(z)$ contained in the ordered m-tuple $\mathbf{i} = (i_1,\dots,i_m)$, $1 \le i_1 < \cdots < i_m \le p$.

The following Definition and the subsequent Proposition are the L-polynomial counterparts of Definition 2.2.8 and Proposition 2.2.2, respectively.

Definition 2.2.13 (Column-reduced matrix) A L-polynomial matrix $M(z) \in \mathbb{R}[z,z^{-1}]^{p\times m}$ of normal rank $\mathrm{rk}(M) = m$ is said to be *column-reduced* if

$$\mathrm{int}\ \deg M = \mathrm{ext_c}\ \deg M. \tag{2.34}$$

Notably, a square column-reduced L-polynomial matrix $M(z) \in \mathbb{R}[z,z^{-1}]^{m\times m}$ satisfies

$$\max\deg\,(\det\, M) - \min\deg\,(\det\, M) = \sum_{i=1}^{m} K_i - \sum_{i=1}^{m} k_i. \tag{2.35}$$

Actually, it can be verified by using the Leibniz formula for determinants (see Fornasini [2011, Ch.3, §7]), that equality

$$\max_{\mathbf{i}}\{\max\deg(\det M^{(\mathbf{i})})\} = \sum_{i=1}^{m} K_i \tag{2.36}$$

holds if and only if M^{hc} has rank m. Similarly,

$$\min_{\mathbf{i}}\{\min\deg(\det M^{(\mathbf{i})})\} = \sum_{i=1}^{m} k_i \tag{2.37}$$

holds if and only if M^{lc} has rank m. These considerations lead to the following Proposition.

Proposition 2.2.4 *A matrix $M(z) \in \mathbb{R}[z,z^{-1}]^{p\times m}$ of normal rank $\mathrm{rk}(M) = m$ is column-reduced if and only if both M^{hc} and M^{lc} have rank m.*

As a final remark, we notice that similar definitions and results can be derived if we consider, instead of columns, the rows of a L-polynomial matrix.

2.3 | Rational matrices

We recall, from Chapter 0, that $\mathbb{R}(z)^{p\times m}$ denotes the set of $p\times m$ matrices with real rational entries. Moreover, as in the polynomial case, we define the normal rank of a rational matrix $G(z) \in \mathbb{R}(z)^{p\times m}$ as the rank of $G(z)$ almost everywhere in $z \in \mathbb{C}$.

2.3.1 | Matrix fraction descriptions

Consider a pair of polynomial matrices $(N(z),D(z)) \in \mathbb{R}[z]^{p\times m} \times \mathbb{R}[z]^{m\times m}$, where $D(z)$ is non-singular. We can associate to the pair $(N(z),D(z))$ the *right matrix fraction* $N(z)D^{-1}(z)$. In a similar way, we can associate to a pair $(Q(z),P(z)) \in \mathbb{R}[z]^{p\times m} \times \mathbb{R}[z]^{m\times m}$, $Q(z)$ being non-singular, the *left matrix fraction* $Q^{-1}(z)P(z)$. Since $D^{-1}(z)$ and $Q^{-1}(z)$ are rational matrices, both the right matrix fraction and the left matrix fraction obtained above belong to $\mathbb{R}(z)^{p\times m}$. *Vice versa*, let us consider an arbitrary rational matrix $G(z) \in \mathbb{R}(z)^{p\times m}$, then there exist two pairs of matrices $(N(z),D(z))$ and $(Q(z),P(z))$ such that

$$G(z) = N(z)D^{-1}(z) = Q^{-1}(z)P(z). \tag{2.38}$$

In fact, if $d(z) \in \mathbb{R}[z]$ is the l.c.m. of all the denominators appearing in $G(z)$, we can write

$$G(z) = M(z)[d(z)I_m]^{-1} = [d(z)I_p]^{-1}M(z), \tag{2.39}$$

for a suitable polynomial matrix $M(z) \in \mathbb{R}[z]^{p\times m}$.

We will refer to $N(z)D^{-1}(z)$ and $Q^{-1}(z)P(z)$ as the right and left, respectively, *matrix fraction description* (MFD) of $G(z)$. Furthermore, in analogy to the scalar case, $N(z)$ and $P(z)$ will be called *numerator matrices*, while $D(z)$ and $Q(z)$ *denominator matrices*.

Given a rational matrix $G(z) \in \mathbb{R}(z)^{p\times m}$ there exist an infinite number of left an right MFDs of $G(z)$. A particular class of (right) MFDs is described in the following Definition.

Definition 2.3.1 Let $G(z) \in \mathbb{R}(z)^{p\times m}$. The (right) MFD $N_R(z)D_R^{-1}(z)$ is said to be an *irreducible (right) matrix fraction description* of $G(z)$ if

$$G(z) = N_R(z)D_R^{-1}(z) \tag{2.40}$$

and $N_R(z) \in \mathbb{R}[z]^{p\times m}$ and $D_R(z) \in \mathbb{R}[z]^{m\times m}$ are right coprime polynomial matrices.

2.3.2 | Smith-McMillan canonical form

Similarly to the Smith canonical form of polynomial matrices discussed in §2.2.1, we can also introduce a canonical form for rational matrices, the so-called *Smith-McMillan canonical form*.

Theorem 2.3.1 (Smith-McMillan canonical form) *Let $G(z) \in \mathbb{R}(z)^{p\times m}$ be a rational matrix of normal rank* $\mathrm{rk}(G) = r$. *There exist unimodular matrices $U(z) \in \mathbb{R}[z]^{p\times r}$ and $V(z) \in \mathbb{R}[z]^{r\times m}$ such that*

$$\begin{aligned} S(z) &:= U(z)G(z)V(z) \\ &= \mathrm{diag}\left[\frac{\varepsilon_1(z)}{\psi_1(z)}, \frac{\varepsilon_2(z)}{\psi_2(z)}, \dots, \frac{\varepsilon_r(z)}{\psi_r(z)}\right], \end{aligned} \tag{2.41}$$

where $\varepsilon_1(z), \varepsilon_2(z), \dots, \varepsilon_r(z), \psi_1(z), \psi_2(z), \dots, \psi_r(z) \in \mathbb{R}[z]$ are monic polynomials satisfying the conditions: (i) *$\varepsilon_i(z)$'s and $\psi_i(z)$'s are relatively prime, $i = 1, 2, \dots, r$,* (ii) *$\varepsilon_i(z) \mid \varepsilon_{i+1}(z)$ and $\psi_{i+1}(z) \mid \psi_i(z)$, $i = 1, 2, \dots, r-1$. The rational functions $\varepsilon_i(z)/\psi_i(z)$, $i = 1, \dots, r$, are termed* invariant factors *of $G(z)$.*

A proof of the previous Theorem and further properties of the Smith-McMillan canonical form may be found in Fornasini [2011, Ch.4, §3] and in Kailath [1998, Ch.6, §5]. We just mention an important Definition and a useful Lemma (we refer to Fornasini [2011, Ch.4, §4, Lemma 4.4.3] for a proof of the latter).

Definition 2.3.2 (McMillan degree) Let $G(z) \in \mathbb{R}(z)^{p\times m}$ be a proper rational matrix, *i.e.*, a rational matrix whose entries are proper rational functions, and let (2.41) be its Smith-McMillan canonical form. Then, we define the *McMillan degree* of $G(z)$, $\delta_M(G)$, as follows

$$\delta_M(G) := \sum_{i=1}^{r} \deg \psi_i(z). \tag{2.42}$$

The Definition above has an alternative and interesting system theoretic interpretation (for a comprehensive treatment on systems theory one may refer to the book Fornasini and Marchesini [2011]). In fact, let $\Sigma = (A,B,C,D)$ be a *state-space realization* of the proper transfer matrix $G(z)$, *i.e.*, a quadruple of matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, such that

$$G(z) = C(zI_n - A)^{-1}B + D, \tag{2.43}$$

then the McMillan degree of $G(z)$ coincides with the minimum *state-space dimension* (*i.e.*, the smallest dimension n of matrix A) of any realization of $G(z)$. Any such a realization is termed a *minimal realization* of $G(z)$.

Lemma 2.3.1 *Let (2.41) be the Smith-McMillan canonical form of $G(z) \in \mathbb{R}(z)^{p\times m}$ and denote by $g_{\mathbf{ij}}^{(\ell)}$ and $s_{\mathbf{ij}}^{(\ell)}$ the $\ell \times \ell$ minor ($1 \le \ell \le \mathrm{rk}(G)$) of the rational matrices $G(z)$ and $D(z)$, respectively, obtained by selecting those rows and columns whose indices appear in the ordered ℓ-tuples $\mathbf{i}$ and $\mathbf{j}$, respectively. Then, for every $\alpha \in \mathbb{C}$*

$$\tilde{v}_\alpha^{(\ell)} := \min_{\mathbf{ij}} v_\alpha(s_{\mathbf{ij}}^{(\ell)}) = \min_{\mathbf{ij}} v_\alpha(g_{\mathbf{ij}}^{(\ell)}) =: v_\alpha^{(\ell)}, \tag{2.44}$$

where $v_\alpha(\cdot)$ denotes the valuation at α of a rational function (see §2.1.4).

A noteworthy Corollary of the previous Lemma is reported below together with a proof.

Corollary 2.3.1 *Let $G(z) \in \mathbb{R}(z)^{n\times n}$ and let $\mathbb{T}$ be a region of the complex plane such that*

$$\mathbf{x}^\top G(\lambda)\mathbf{x} \ge 0, \tag{2.45}$$

for all $\mathbf{x} \in \mathbb{R}^n$ *and for all values* $\lambda \in \mathbb{T}$ *for which* $G(\lambda)$ *has finite entries. (Let us denote this subset of* $\mathbb{T}$ *by* $\bar{\mathbb{T}}$*). Then, for every* $\alpha \in \mathbb{T}$ *equation (2.44) can be equivalently replaced by*

$$\tilde{v}_\alpha^{(\ell)} := \min_{\mathbf{i}} v_\alpha(s_{\mathbf{ii}}^{(\ell)}) = \min_{\mathbf{i}} v_\alpha(g_{\mathbf{ii}}^{(\ell)}) =: v_\alpha^{(\ell)}. \tag{2.46}$$

PROOF. Since $G(z)$ is non-negative Hermitian in the region $\bar{\mathbb{T}}$, it admits a decomposition of the form $G(\lambda) = W(\lambda)W^*(\lambda)$ for all $\lambda \in \bar{\mathbb{T}}$. By applying the Binet-Cauchy Theorem (see Gantmacher [1959, Vol.I, Ch.1, §2]), we have[2]

$$g_{\mathbf{ij}}^{(\ell)}(\lambda) = \sum_{\mathbf{h}} w_{\mathbf{ih}}^{(\ell)}(\lambda)\overline{w_{\mathbf{jh}}^{(\ell)}(\lambda)}, \tag{2.47}$$

$$g_{\mathbf{ii}}^{(\ell)}(\lambda) = \sum_{\mathbf{h}} w_{\mathbf{ih}}^{(\ell)}(\lambda)\overline{w_{\mathbf{ih}}^{(\ell)}(\lambda)} = \sum_{\mathbf{h}} \left| w_{\mathbf{ih}}^{(\ell)}(\lambda) \right|^2, \tag{2.48}$$

where $g_{\mathbf{ij}}^{(\ell)}(\lambda)$ and $w_{\mathbf{ij}}^{(\ell)}(\lambda)$ denote the $\ell \times \ell$ minor of matrices $G(\lambda)$ and $W(\lambda)$, respectively, obtained by selecting those rows and columns whose indices appear in the ordered ℓ-tuples $\mathbf{i} := (i_1, \dots, i_\ell)$, $1 \le i_1 < \cdots < i_\ell \le n$, and $\mathbf{j} := (j_1, \dots, j_\ell)$, $1 \le j_1 < \cdots < j_\ell \le n$, respectively. Moreover, in both the summations (2.47) and (2.48), $\mathbf{h} := (h_1, \dots, h_\ell)$, $1 \le h_1 < \cdots < h_\ell \le n$, runs through all such multi-indices.

By using Cauchy-Schwarz inequality and equation (2.48), we have

$$\begin{aligned}
\left| g_{\mathbf{ij}}^{(\ell)}(\lambda) \right| &= \left| \sum_{\mathbf{h}} w_{\mathbf{ih}}^{(\ell)}(\lambda)\overline{w_{\mathbf{jh}}^{(\ell)}(\lambda)} \right| \\
&\le \sqrt{\sum_{\mathbf{h}} \left| w_{\mathbf{ih}}^{(\ell)}(\lambda) \right|^2 \sum_{\mathbf{h}} \left| w_{\mathbf{jh}}^{(\ell)}(\lambda) \right|^2} \\
&= \sqrt{g_{\mathbf{ii}}^{(\ell)}(\lambda) g_{\mathbf{jj}}^{(\ell)}(\lambda)}, \\
&\le \max\left\{ g_{\mathbf{ii}}^{(\ell)}(\lambda), g_{\mathbf{jj}}^{(\ell)}(\lambda) \right\}, \quad \forall \lambda \in \bar{\mathbb{T}}.
\end{aligned} \tag{2.49}$$

The latter inequality implies that for every zero $\alpha \in \mathbb{T}$ of multiplicity k of a minor of G, there exists at least one principal minor of G which has the same α either as a zero of multiplicity less than or equal to k or a pole of multiplicity greater than

[2] In deriving expressions (2.47) and (2.48), we implicitly use the fact that the determinant of the complex conjugate of a square matrix A is the complex conjugate of the determinant, *i.e.*, $\det(\overline{A}) = \overline{\det(A)}$.

or equal to 0. Similarly, inequality (2.49) implies also that for every pole $\alpha \in \mathbb{T}$ of multiplicity k of a minor of G, there exists at least one principal minor of G which has the same pole of multiplicity greater than or equal to k.

Therefore,

$$\min_{\mathbf{ij}} \nu_\alpha(g_{\mathbf{ij}}^{(\ell)}) = \min_{\mathbf{i}} \nu_\alpha(g_{\mathbf{ii}}^{(\ell)}), \quad \forall \alpha \in \mathbb{T}. \tag{2.50}$$

Since $\min_{\mathbf{ij}} \nu_\alpha(s_{\mathbf{ij}}^{(\ell)}) = \min_{\mathbf{i}} \nu_\alpha(s_{\mathbf{ii}}^{(\ell)})$, $\forall \alpha \in \mathbb{T}$, we finished. ■

2.3.3 | Poles and zeros of a rational matrix

The Smith-McMillan canonical form introduced in Theorem 2.3.1 is especially useful to define poles and zeros for rational matrices.

Definition 2.3.3 (Zeros and poles of a rational matrix) Consider a rational matrix $G(z) \in \mathbb{R}(z)^{p\times m}$ of normal rank $r = \mathrm{rk}(G)$ and let

$$S(z) = \mathrm{diag}\left[\frac{\varepsilon_1(z)}{\psi_1(z)}, \frac{\varepsilon_2(z)}{\psi_2(z)}, \dots, \frac{\varepsilon_r(z)}{\psi_r(z)}\right], \tag{2.51}$$

be its Smith-McMillan canonical form. The complex number α is a *(finite) zero* of $G(z)$ if it is a zero of at least one of the polynomials $\varepsilon_i(z)$, $i = 1,\dots,r$. The complex number α is a *(finite) pole* of $G(z)$ if it is a zero of at least one of the polynomials $\psi_i(z)$, $i = 1,\dots,r$.

From the conditions on $\varepsilon_i(z)$ and $\psi_i(z)$, $i = 1,\dots,r$, imposed in Theorem 2.3.1, it follows that the (finite) zeros of $G(z)$ coincides with the zeros of $\varepsilon_r(z)$ and the (finite) poles of $G(z)$ with the zeros of $\psi_1(z)$. Moreover, it is worthwhile noticing that, unlike what happens in the scalar case, the set of zeros and poles of a rational matrix may not be disjoint.

Let $\alpha_1, \alpha_2, \dots, \alpha_t$ be the (finite) zeros and (finite) poles of $G(z) \in \mathbb{R}(z)^{p\times m}$, $r = \mathrm{rk}(G)$, and let

$$S(z) = \mathrm{diag}\left[(z-\alpha_1)^{\nu_1^{(1)}} \cdots (z-\alpha_t)^{\nu_t^{(1)}}, \dots, (z-\alpha_1)^{\nu_1^{(r)}} \cdots (z-\alpha_t)^{\nu_t^{(r)}}\right], \tag{2.52}$$

be the Smith-McMillan canonical form of $G(z)$. Then the integer exponents

$$\nu_i^{(1)} \le \nu_i^{(2)} \le \cdots \le \nu_i^{(r)}, \tag{2.53}$$

are called the *structural indices* of $G(z)$ at α_i, $i = 1, \dots, t$.

We introduce now the notion of degree of a pole for rational matrices, which will be useful in the following.

Definition 2.3.4 (Degree of a pole) Let $G(z) \in \mathbb{R}(z)^{p \times m}$ be a rational matrix of normal rank $\mathrm{rk}(G) = r$ and let p_0 be a pole (not necessarily finite) of $G(z)$. We define the *degree* of $z = p_0$ as a pole of $G(z)$ as follows

$$\delta(G; p_0) := -\min_{\ell} v_{p_0}^{(\ell)}, \tag{2.54}$$

where $v_{p_0}^{(\ell)}$ denotes the minimum valuation at p_0 between all the $\ell \times \ell$ ($1 \leq \ell \leq r$) minors of $G(z)$ (*cf.* equation (2.44)).

In other words, the degree $\delta(G; p_0)$ of $z = p_0$ as a pole of $G(z)$ equals the largest multiplicity it possesses as a pole of *any minor* of $G(z)$. Moreover, if $S(z)$ is the Smith-McMillan form of $G(z)$, by Lemma 2.3.1, the degree of a finite pole $p_0 \in \mathbb{C}$ of $G(z)$ is equal to the degree of p_0 as a pole of $S(z)$ which, in turn, coincides with the sum, changed in sign, of all the negative structural indices of $G(z)$ at p_0. However, the latter result does not apply, in general, for the pole at infinity. Lastly, if $p_0 \in \mathbb{C} \cup \{\infty\}$ is a pole of $G(z)$ of degree n, then we will also say that $G(z)$ has n poles at p_0.

The following Propositions (see Fornasini [2011, Ch.4, §4] for the proofs) provide some additional characterization of the structure of the (finite) poles and (finite) zeros of a rational matrix.

Proposition 2.3.1 *Let $N_R(z)D_R^{-1}(z)$ be a irreducible right MFD of the rational matrix $G(z) \in \mathbb{R}(z)^{p \times m}$ and let α be a complex number. The following facts are equivalent:*

1. *α is a pole of $G(z)$;*

2. *α is a zero of $\det D_R(z)$;*

3. *α is a pole of some entry $[G(z)]_{ij}$ of $G(z)$.*

Proposition 2.3.2 *Let $N_R(z)D_R^{-1}(z)$ be a irreducible right MFD of the rational matrix $G(z) \in \mathbb{R}(z)^{p\times m}$ of normal rank $\mathrm{rk}(G) = r$ and let α be a complex number. The following facts are equivalent:*

1. *α is a zero of $G(z)$;*
2. *the rank of $N_R(\alpha)$ is lower than r,* i.e., $\mathrm{rank}(N_R(\alpha)) < r$;
3. *the smallest valuation at α, $v_\alpha^{(r)}$, of the $r \times r$ minors of $G(z)$ is strictly greater than the smallest valuation at α, $v_\alpha^{(r-1)}$, of the $(r-1)\times(r-1)$ minors of $G(z)$,* i.e., $v_\alpha^{(r)} > v_\alpha^{(r-1)}$.

It is important to remark that Definition 2.3.3, Proposition 2.3.1 and Proposition 2.3.2 apply only to poles and zeros at *finite* points in the complex plane, because the (highly non-unique) unimodular matrices used to get the unique Smith-McMillan form destroy information about the behaviour at infinity. In fact, unimodular matrices can have both poles and zeros at infinity. To obtain the zero-pole structure[3] at infinity of a rational matrix $G(z) \in \mathbb{R}(z)^{p\times m}$, we can proceed as described in Kailath [1998, Ch.6]. We make a change of variable, $z \to \lambda^{-1}$, and compute the Smith-McMillan form of $G(\lambda^{-1})$, then the zero-pole structure of $G(\lambda^{-1})$ at $\lambda = 0$ will give the zero-pole structure of $G(z)$ at $z = \infty$.

Finally, we notice that we can extend the definition of the McMillan degree of a rational matrix by taking into account its behaviour at infinity. In fact, we previously defined the McMillan degree of a proper rational matrix as the sum of the degrees of the denominator polynomials in its Smith-McMillan form, *cf.* equation (2.42). For a general rational matrix $G(z) \in \mathbb{R}(z)^{p\times m}$, we can decompose it as

$$G(z) = G_{\mathrm{sp}}(z) + P(z), \tag{2.55}$$

where $G_{\mathrm{sp}}(z)$ is strictly proper and $P(z)$ is polynomial, and define its McMillan degree as

$$\delta_M(G(z)) = \delta_M(G_{\mathrm{sp}}(z)) + \delta_M(P(\lambda^{-1})). \tag{2.56}$$

[3]The zero-pole structure at $\alpha \in \mathbb{C}$ of a rational matrix $G(z) \in \mathbb{R}(z)^{p\times m}$ of normal rank $\mathrm{rk}(G) = r$ is given by the set of structural indices $\{v_\alpha^{(1)}, \dots, v_\alpha^{(r)}\}$.

Moreover, by exploiting the Definition 2.3.4 of polar degree, we can restate in a equivalent and particularly simple way the Definition of McMillan degree given above. In fact, the McMillan degree of a (either proper or non-proper) rational matrix $G(z)$ equals the sum of the degrees of all its distinct poles, the pole at infinity included (see Kailath [1998, Ch.6]). Thus, if $p_1, \ldots, p_h$ are the distinct poles of $G(z)$ with associated degrees $\delta(G; p_i)$, $i = 1, \ldots, h$, then

$$\delta_M(G) = \sum_{i=1}^{h} \delta(G; p_i). \tag{2.57}$$

2.3.4 | Some special classes of rational matrices

We collect here a set of definitions involving some classes of rational matrices which share a particular structure. For each definition we distinguish the "continuous-time" case and the "discrete-time" case by using the variable s and z, respectively, as indeterminate of rational matrices.

Definition 2.3.5 (Para-Hermitian matrix) A rational matrix $G(s) \in \mathbb{R}(s)^{n\times n}$ is said to be *continuous-time (CT) para-Hermitian* if $G^\top(-s) = G(s)$. Similarly, a rational matrix $G(z) \in \mathbb{R}(z)^{n\times n}$ is said to be *discrete-time (DT) para-Hermitian* if $G^\top(1/z) = G(z)$.

We notice that, a CT para-Hermitian matrix $G(s)$ is Hermitian in the ordinary sense on the imaginary axis, *i.e.*, $G^*(j\omega) = G(j\omega)$. Similarly, a DT para-Hermitian matrix $G(z)$ is Hermitian in the ordinary sense on the unit circle, *i.e.*, $G^*(e^{j\omega}) = G(e^{j\omega})$.

When there is no risk of confusion, we will use a common notation for both the continuous-time and discrete-time para-Hermitianity operator, namely

$$G^*(s) := G^\top(-s), \tag{2.58}$$

$$G^*(z) := G^\top(1/z). \tag{2.59}$$

Moreover, for convenience, we define

$$G^{-*}(s) := [G^*(s)]^{-1}, \quad G^{-*}(z) := [G^*(z)]^{-1}, \tag{2.60}$$

$$G^{-R*}(s) := [G^*(s)]^{-R}, \quad G^{-R*}(z) := [G^*(z)]^{-R}, \tag{2.61}$$

$$G^{-L*}(s) := [G^*(s)]^{-L}, \quad G^{-L*}(z) := [G^*(z)]^{-L}. \tag{2.62}$$

Let $A(z), B(z) \in \mathbb{R}(z)^{m\times m}$, we observe that the relations

$$A^{**}(z) = A(z), \tag{2.63}$$
$$[AB]^{*}(z) = B^{*}(z)A^{*}(z) \tag{2.64}$$

hold in both the continuous-time and discrete-time case.

Definition 2.3.6 (Para-unitary matrix) A rational matrix $G(s) \in \mathbb{R}(s)^{n\times n}$ is said to be *continuous-time (CT) para-unitary* if

$$G^{*}(s)G(s) = G(s)G^{*}(s) = I_n. \tag{2.65}$$

Similarly, a rational matrix $G(z) \in \mathbb{R}(z)^{n\times n}$ is said to be *discrete-time (DT) para-unitary* if

$$G^{*}(z)G(z) = G(z)G^{*}(z) = I_n. \tag{2.66}$$

We notice that, a CT para-unitary matrix $G(s)$ is unitary in the ordinary sense on the imaginary axis, *i.e.*, $G^{*}(j\omega)G(j\omega) = G(j\omega)G^{*}(j\omega) = I_n$. Similarly, a DT para-Hermitian matrix $G(z)$ is unitary in the ordinary sense on the unit circle, *i.e.*, $G^{*}(e^{j\omega})G(e^{j\omega}) = G(e^{j\omega})G^{*}(e^{j\omega}) = I_n$.

Definition 2.3.7 ($J_{p,q}$-para-unitary matrix) Let us define

$$J_{p,q} := \left[\begin{array}{c|c} I_p & \mathbf{0}_{p,q} \\ \hline \mathbf{0}_{q,p} & -I_q \end{array}\right], \quad p+q =: n. \tag{2.67}$$

A rational matrix $G(s) \in \mathbb{R}(s)^{n\times n}$ is said to be *continuous-time (CT) $J_{p,q}$-para-unitary* if

$$G^{*}(s)J_{p,q}G(s) = G(s)J_{p,q}G^{*}(s) = J_{p,q}. \tag{2.68}$$

Similarly, a rational matrix $G(z) \in \mathbb{R}(z)^{m\times n}$ is said to be *discrete-time (DT) $J_{p,q}$-para-unitary* if

$$G^{*}(z)J_{p,q}G(z) = G(z)J_{p,q}G^{*}(z) = J_{p,q}. \tag{2.69}$$

A $J_{p,q}$-para-unitary matrix is para-unitary if $q = 0$. $J_{p,q}$-para-unitary matrices play an important role in the J-spectral factorization problem.

Definition 2.3.8 (Regular matrix) A rational matrix $G(s) \in \mathbb{R}(s)^{m\times n}$ is said to be *continuous-time (CT) regular* if it is analytic in the open right half-plane $\{\Re\mathrm{e}\ s > 0,\ s \in \mathbb{C}\}$. Similarly, a rational matrix $G(z) \in \mathbb{R}(z)^{n\times n}$ is said to be *discrete-time (DT) regular* if it is analytic outside the closed unit disk, *i.e.*, in $\{|z| > 1,\ z \in \mathbb{C}\}$.

Definition 2.3.9 (Anti-regular matrix) A rational matrix $G(s) \in \mathbb{R}(s)^{m\times n}$ is said to be *continuous-time (CT) anti-regular* if it is analytic in the open left half-plane $\{\Re\mathrm{e}\ s < 0,\ s \in \mathbb{C}\}$. Similarly, a rational matrix $G(z) \in \mathbb{R}(z)^{m\times n}$ is said to be *discrete-time (DT) anti-regular* if it is analytic in the open unit disk $\{|z| < 1,\ z \in \mathbb{C}\}$.

For the sake of brevity, we will often drop the prefix "continuous-time" and "discrete-time" in the following, since the meaning will be clear from the context.

$$\Phi(s) = W^*(s)W(s) = W^\top(-s)W(s)$$

3. Continuous-time spectral factorization

In this Chapter, we will describe and study in detail the rational matrix factorization method proposed by Youla in his classical paper Youla [1961]. In particular, this ingenious technique can be used to solve the multivariate *continuous-time* spectral factorization problem, presented as Problem 1.1 in Chapter 1. Unlike Youla's paper, in the sequel, we will deal with real coefficients rational matrices.

A remark on notation. A rational matrix $A(s)$ is said to be analytic in a region of the complex plane if all its entries are analytic in this region. Moreover, as in Youla [1961], with a slight abuse of notation, when we say that a rational function $f(s)$ is analytic in a closed region $\mathbb{T}$ of the complex plane we mean that $f(s)$ is analytic in an open region $\mathbb{T}_\varepsilon \supset \mathbb{T}$ which is "larger" than $\mathbb{T}$ of an arbitrarily small quantity. For example, if $f(s)$ is rational and has all its poles in the open left half complex plane, we say that $f(s)$ is analytic in the closed right half complex plane to mean that there exists $\varepsilon > 0$ s.t. $f(s)$ is analytic in $\{\Re\mathrm{e}\ s > -\varepsilon,\ s \in \mathbb{C}\}$. Similarly, we say that $f(s)$ is analytic on the imaginary axis in place of $f(s)$ is analytic on an open strip containing the imaginary axis. When dealing with rational functions that feature a finite number of poles, this abuse of notation does not cause any problem. Finally, we say that a rational matrix is canonic if it satisfies the properties of the Smith-McMillan Theorem 2.3.1. For other standard notation we refer to Chapter 0 and Chapter 2.

3.1 | Preliminary results

The auxiliary results reported in this section will be used to prove the main factorization Theorem of the next section.

Lemma 3.1.1 *A matrix $G(s) \in \mathbb{R}(s)^{m\times n}$ is analytic in the entire complex plane together with its inverse (either right, left or both) if and only if it is a unimodular polynomial matrix.*

PROOF. If $G(s)$ is a unimodular polynomial matrix, then, from §2.2.2, we know that $G(s)$ has an inverse (either right, left or both) which is also polynomial. Without loss of generality, we can suppose that $G(s)$ has a left polynomial inverse. Then $\mathrm{rk}(G) = n$ and the Smith-McMillan canonical form of $G(s)$, denoted by $D(s)$, is the identity matrix of dimension n, *i.e.*, $D(s) = I_n$. This implies that $G(s)$ does not have any finite pole and zero. Hence, $G(s)$ must be analytic together with its left inverse in $\mathbb{C}$.

Vice versa, suppose that $G(s)$ is analytic with its inverse (either right, left or both) in $\mathbb{C}$. First, we notice that the existence of a left or right inverse for $G(s)$ implies that the normal rank of $G(s)$ is either $r = n$ or $r = m$, respectively. Without loss of generality, we can suppose that $r = n$. By the Smith-McMillan Theorem 2.3.1, we can write $G(s) = C(s)D(s)F(s)$, where $C(s) \in \mathbb{R}[s]^{m\times n}$, $F(s) \in \mathbb{R}[s]^{n\times n}$ are unimodular polynomial matrices and $D(s) \in \mathbb{R}(s)^{n\times n}$ is diagonal, canonic of the form

$$D(s) = \mathrm{diag}\left[\frac{\varepsilon_1(s)}{\psi_1(s)}, \frac{\varepsilon_2(s)}{\psi_2(s)}, \dots, \frac{\varepsilon_n(s)}{\psi_n(s)}\right], \tag{3.1}$$

where $\varepsilon_k(z)$, $\psi_k(z)$, $k = 1, \dots, n$, are relatively prime monic polynomials such that $\varepsilon_k(s) \mid \varepsilon_{k+1}(s)$, $\psi_{k+1}(s) \mid \psi_k(s)$, $k = 1, \dots, n-1$. The analyticity of $G(s)$ in $\mathbb{C}$ implies that all ψ's are non-zero real constants. Moreover, the Smith-McMillan canonical form of the left inverse of $G(s)$, $G^{-L}(s)$, is given by

$$\mathrm{diag}\left[\frac{\psi_n(s)}{\varepsilon_n(s)}, \frac{\psi_{n-1}(s)}{\varepsilon_{n-1}(s)}, \dots, \frac{\psi_1(s)}{\varepsilon_1(s)}\right], \tag{3.2}$$

and so the analyticity of $G^{-L}(s)$ in $\mathbb{C}$ implies that all ε's are non-zero real constants. Hence, $D(s)$ in (3.1) is a diagonal constant matrix. Since $G(s)$ is the product of three unimodular polynomial matrices, $G(s)$ must also be a unimodular polynomial matrix. ■

Lemma 3.1.2 *The only CT regular para-unitary matrices with regular inverse are constant orthogonal matrices.*

PROOF. From Definition 2.3.6, we recall that a CT para-unitary matrix $G(s) \in \mathbb{R}(s)^{n\times n}$ satisfies

$$G^*(s)G(s) = G(s)G^*(s) = I_n. \tag{3.3}$$

The analyticity of the inverse of $G(s)$ in $\{\Re\mathrm{e}\ s > 0,\ s \in \mathbb{C}\}$ implies that of $G(-s)$ in the same region, and therefore that of $G(s)$ in $\{\Re\mathrm{e}\ s < 0,\ s \in \mathbb{C}\}$. We also notice that in the imaginary axis $\{j\omega,\ \omega \in \mathbb{R}\}$ we have $G^*(j\omega)G(j\omega) = I_n$ and we can write out the diagonal element of $G(j\omega)$ in expanded form as

$$\sum_{i=1}^{n} |[G(j\omega)]_{ik}|^2 = 1, \quad \forall k = 1,\ldots,n,\ \forall \omega \in \mathbb{R}. \tag{3.4}$$

The latter equation implies that

$$|[G(j\omega)]_{ik}| \leq 1, \quad \forall i,k = 1,\ldots,n,\ \forall \omega \in \mathbb{R}, \tag{3.5}$$

which, in turn, implies that $G(s)$ is bounded at infinity and analytic in the entire imaginary axis. Thus, $G(s)$ is analytic (together with its inverse $G^{-1}(s) = G^*(s)$) in the entire complex plane and bounded at infinity. We are in position to apply Liouville's Theorem (see Lang [1985, Ch.V, §1, Thm.1.4]) and conclude that $G(s)$ must be a constant orthogonal matrix. ■

Definition 3.1.1 (CT left-standard factorization) Let $G(s) \in \mathbb{R}(s)^{m\times n}$ and let $\mathrm{rk}(G) = r \leq \min\{m,n\}$. A decomposition of the form

$$G(s) = A(s)\Delta(s)B(s) \tag{3.6}$$

is called a *continuous-time (CT) left-standard factorization* if

1. $\Delta(s) \in \mathbb{R}(s)^{r\times r}$ is diagonal and analytic together with its inverse in the entire complex plane with the possible exception of a finite number of points on the imaginary axis $\{\Re\mathrm{e}\ s = 0,\ s \in \mathbb{C}\}$;

2. $A(s) \in \mathbb{R}(s)^{m\times r}$ is analytic together with its left inverse in $\{\Re\mathrm{e}\ s \leq 0,\ s \in \mathbb{C}\}$;

3. $B(s) \in \mathbb{R}(s)^{r\times n}$ is analytic together with its right inverse in $\{\Re\mathrm{e}\ s \geq 0,\ s \in \mathbb{C}\}$.

By interchanging $A(s)$ and $B(s)$ in Definition 3.1.1, we obtain a *CT right-standard factorization*. Any CT left-standard factorization of $G(s)$ gives rise to a CT right-standard factorization of $G^\top(s)$, $G^{-1}(s)$ (if $G(s)$ is non-singular), $G(-s)$, *e.g.*, in the first case we have

$$G^\top(s) = B^\top(s)\Delta(s)A^\top(s). \tag{3.7}$$

A CT left-standard factorization of a rational matrix $G(s)$ always exists, as stated in the following Lemma.

Lemma 3.1.3 *Any rational matrix $G(s) \in \mathbb{R}(s)^{m \times n}$ of normal rank $\mathrm{rk}(G) = r \leq \min\{m,n\}$ has a CT left-standard factorization.*

PROOF. By the Smith-McMillan Theorem 2.3.1, we can write $G(s) = C(s)D(s)F(s)$, where $C(s) \in \mathbb{R}[s]^{m \times r}$, $F(s) \in \mathbb{R}[s]^{r \times n}$ are unimodular polynomial matrices and $D(s) \in \mathbb{R}(s)^{r \times r}$ is diagonal and canonic of the form

$$D(s) = \mathrm{diag}\left[\frac{\varepsilon_1(s)}{\psi_1(s)}, \frac{\varepsilon_2(s)}{\psi_2(s)}, \ldots, \frac{\varepsilon_r(s)}{\psi_r(s)}\right], \tag{3.8}$$

where $\varepsilon_k(s)$, $\psi_k(s)$, $k = 1, \ldots, r$, are relatively prime monic polynomials such that $\varepsilon_k(s) \mid \varepsilon_{k+1}(s)$, $\psi_{k+1}(s) \mid \psi_k(s)$, $k = 1, \ldots, r-1$. We can factor $\varepsilon_i(s)$ and $\psi_i(s)$, $i = 1, \ldots, r$, in $D(s)$ into the product of three polynomials: the first without zeros in $\{\Re\mathrm{e}\, s \leq 0,\ s \in \mathbb{C}\}$, the second without zeros in $\{\Re\mathrm{e}\, s \neq 0,\ s \in \mathbb{C}\}$ and the third without zeros in $\{\Re\mathrm{e}\, s \geq 0,\ s \in \mathbb{C}\}$. Thus, it is possible to write

$$D(s) = D_-(s)\Delta(s)D_+(s), \tag{3.9}$$

where $D_-(s)$ and its inverse are analytic in $\{\Re\mathrm{e}\, s \leq 0,\ s \in \mathbb{C}\}$, $\Delta(s)$ and its inverse in $\{\Re\mathrm{e}\, s \neq 0,\ s \in \mathbb{C}\}$ and $D_+(s)$ and its inverse in $\{\Re\mathrm{e}\, s \geq 0,\ s \in \mathbb{C}\}$. Finally, by defining $A(s) := C(s)D_-(s)$ and $B(s) := D_+(s)F(s)$, we have that

$$G(s) = A(s)\Delta(s)B(s) \tag{3.10}$$

is a CT left-standard factorization of $G(s)$. ■

Now, consider two left-standard factorizations of $G(s)$, the following Theorem gives a characterization of the two decompositions.

Theorem 3.1.1 *Let $G(s) \in \mathbb{R}(s)^{m \times n}$ be a rational matrix of normal rank $\mathrm{rk}(G) = r \leq \min\{m,n\}$ and let $A(s)\Delta(s)B(s)$, $A_1(s)\Delta_1(s)B_1(s)$ be two CT left-standard factorizations of $G(s)$. Then,*

$$A_1(s) = A(s)M^{-1}(s), \quad B_1(s) = N(s)B(s) \tag{3.11}$$

where $M(s) \in \mathbb{R}[s]^{r\times r}$ and $N(s) \in \mathbb{R}[s]^{r\times r}$ are two unimodular polynomial matrices such that

$$M(s)\Delta(s)N^{-1}(s) = \Delta_1(s). \tag{3.12}$$

PROOF. By assumption,

$$G(s) = A(s)\Delta(s)B(s) = A_1(s)\Delta_1(s)B_1(s), \tag{3.13}$$

and, therefore,

$$\Delta_1^{-1}(s)A_1^{-L}(s)A(s)\Delta(s) = B_1(s)B^{-R}(s). \tag{3.14}$$

By Definition 3.1.1 of CT left-standard factorization, the right-hand side of (3.14) is analytic in $\{\Re\mathrm{e}\ s \geq 0,\ s \in \mathbb{C}\}$, while the left-hand side in $\{\Re\mathrm{e}\ s < 0,\ s \in \mathbb{C}\}$. Hence, $B_1(s)B^{-R}(s)$ is analytic in the entire complex plane. Moreover,

$$[B_1(s)B^{-R}(s)]^{-1} = \Delta^{-1}(s)[A_1^{-L}(s)A(s)]^{-1}\Delta_1(s) \tag{3.15}$$

is also analytic in the entire complex plane. Thus, for Lemma 3.1.1, $N(s) := B_1(s)B^{-R}(s)$ must be a $r \times r$ unimodular polynomial matrix. Similarly, $M(s) := A_1^{-L}(s)A(s)$ must be a $r \times r$ unimodular polynomial matrix. Finally, from (3.13) we have

$$M(s)\Delta(s)N^{-1}(s) = \Delta_1(s). \tag{3.16}$$

and we finished. ■

It is not difficult to derive a CT right-standard counterpart of Lemma 3.1.3 and Theorem 3.1.1.

Let $\Phi(s) \in \mathbb{R}(s)^{n\times n}$ be a para-Hermitian matrix of normal rank $\mathrm{rk}(\Phi) = r \leq n$ and let $\Phi(s) = A(s)\Delta(s)B(s)$ be a CT left-standard factorization of $\Phi(s)$. We have that

$$\Phi(s) = \Phi^*(s) = B^*(s)\Delta^*(s)A^*(s) \tag{3.17}$$

is also a CT left-standard factorization of $\Phi(s)$. In particular, $\Delta^*(s)$ is equal to $\Delta(s)$, except, perhaps, for the signs of its diagonal elements, *i.e.*,

$$\Delta^*(s) = \Sigma\Delta(s), \tag{3.18}$$

where

$$\Sigma = \mathrm{diag}\,[e_1, e_2, \ldots, e_r] \tag{3.19}$$

and $e_i = \pm 1,\ i = 1, \ldots, r$. By invoking Theorem 3.1.1, we can write

$$A^*(s) = N(s)B(s), \tag{3.20}$$

$$B^*(s) = A(s)M^{-1}(s), \tag{3.21}$$

where $N(s),\ M(s) \in \mathbb{R}(s)^{r\times r}$ are unimodular polynomial matrices.

If a para-Hermitian matrix is positive semi-definite on the imaginary axis then the result reported below holds.

Lemma 3.1.4 *Let $\Phi(s) \in \mathbb{R}(s)^{n\times n}$ be a CT para-Hermitian matrix of normal rank* $\mathrm{rk}(\Phi) = r \le n$ *which is positive semi-definite on the imaginary axis,* i.e., $\mathbf{x}^\top \Phi(j\omega)\mathbf{x} \ge 0$, $\forall \mathbf{x} \in \mathbb{R}^n$ *and* $\forall \omega \in \mathbb{R}$ *such that* $s = j\omega$ *is not a pole of* $\Phi(s)$. *Let*

$$\Phi(s) = C(s)D(s)F(s) \tag{3.22}$$

with $D(s) \in \mathbb{R}(s)^{r\times r}$ be the Smith-McMillan canonical form of $\Phi(s)$. Then, the zeros and poles on the imaginary axis of the diagonal elements of $D(s)$ must be of even multiplicity.

PROOF. First of all, we can suppose that the numerators and denominators of all entries in $\Phi(s)$ are coprime polynomials. Let

$$\alpha_1 = j\omega_1,\ \alpha_2 = j\omega_2, \ldots,\ \alpha_t = j\omega_t, \tag{3.23}$$

be the finite zeros/poles on the imaginary axis of $\Phi(s)$ and let

$$\nu_i^{(1)}, \nu_i^{(2)}, \ldots, \nu_i^{(r)}, \quad \nu_i^{(1)} \le \nu_i^{(2)} \le \cdots \le \nu_i^{(r)}, \tag{3.24}$$

be the structural indices of $\Phi(s)$ at $\alpha_i,\ i = 1, \ldots, t$, *i.e.*, the valuations at α_i of the diagonal terms of $D(s)$ (see §2.3.3). Since $\Phi(s)$ is non-negative on the imaginary axis, it is easy to verify that the zeros and poles on the imaginary axis of the principal minors of $\Phi(s)$ are of even multiplicity. Furthermore, by choosing $\mathbb{T} = \{\Re\mathrm{e}\ s = 0,\ s \in \mathbb{C}\}$, we are in position of applying Corollary 2.3.1. Thus, by considering the minors of order $\ell = 1$, it follows that

$$\nu_i^{(1)} \text{ is even},\ i = 1, 2, \ldots, t. \tag{3.25}$$

Similarly, by considering the minors of order $\ell = 2$ in Corollary 2.3.1, it follows that

$$\nu_i^{(1)} + \nu_i^{(2)} \text{ is even}, \ i = 1,2,\dots,t. \tag{3.26}$$

Since $\nu_i^{(1)}$ is even, then also $\nu_i^{(2)}$ must be even for all $i = 1,2,\dots,t$. By iterating the argument, we conclude that every zero/pole on the imaginary axis of the diagonal elements of $D(s)$ must be of even multiplicity. ■

Let $\Phi(s) \in \mathbb{R}(s)^{n\times n}$ be a para-Hermitian matrix of normal rank $\mathrm{rk}(\Phi) = r \leq n$ and let $D(s) \in \mathbb{R}(s)^{r\times r}$ be its Smith-McMillan canonical form. We have

$$\Phi(s) = C(s)D(s)F(s) = F^*(s)D^*(s)C^*(s) = \Phi^*(s), \tag{3.27}$$

and, by a previous argument,

$$D^*(s) = \Sigma' D(s), \tag{3.28}$$

where Σ' has the form (3.19). By (3.28), every zero/pole at α of the diagonal elements of $D(s)$ is accompanied by a zero/pole at $-\alpha$ and we can always write $D(s)$ in the form

$$D(s) = \Sigma_1 \Lambda^*(s)\Delta(s)\Lambda(s), \tag{3.29}$$

where $\Lambda(s) \in \mathbb{R}(s)^{r\times r}$ is diagonal, canonic and analytic with its inverse in $\{\Re\mathrm{e}\ s \geq 0,\ s \in \mathbb{C}\}$ and $\Sigma_1\Lambda^*(s) \in \mathbb{R}(s)^{r\times r}$ is diagonal and analytic with its inverse in $\{\Re\mathrm{e}\ s \leq 0,\ s \in \mathbb{C}\}$. The diagonal matrix $\Delta(s) \in \mathbb{R}(s)^{r\times r}$ is canonic and analytic with its inverse in $\{\Re\mathrm{e}\ s \neq 0,\ s \in \mathbb{C}\}$ and, by exploiting Lemma 3.1.4, can be written as

$$\Delta(s) = \Theta^2(s) = \Sigma_2\Theta^*(s)\Theta(s), \tag{3.30}$$

with $\Theta(s) \in \mathbb{R}(s)^{r\times r}$ diagonal and canonic. In conclusion, we can rearrange $D(s)$ in the form

$$D(s) = \Sigma_3\Lambda^*(s)\Theta^*(s)\Theta(s)\Lambda(s), \tag{3.31}$$

where Σ_1, Σ_2 and $\Sigma_3 := \Sigma_1\Sigma_2$ are of the form (3.19).

3.2 | The main theorem

We present here the main result of the Chapter due to Youla, Youla [1961, Thm. 2].

Theorem 3.2.1 *Let $\Phi(s) = \Phi^*(s) \in \mathbb{R}(s)^{n\times n}$ be a CT para-Hermitian matrix of normal rank* $\text{rk}(\Phi) = r \leq n$ *which is positive semi-definite on the imaginary axis $\{j\omega,\ \omega \in \mathbb{R}\}$. Then, there exists a rational matrix $W(s) \in \mathbb{R}(s)^{r\times n}$ such that*

(i) $\Phi(s) = W^*(s)W(s)$.

(ii) *$W(s)$ and its (right) inverse $W^{-R}(s)$ are both analytic in $\{\Re\text{e } s > 0,\ s \in \mathbb{C}\}$.*

(iii) *$W(s)$ is unique up to within a constant, orthogonal $r\times r$ matrix multiplier on the left,* i.e., *if $W_1(s)$ also satisfies points* (i) *and* (ii)*, then $W_1(s) = TW(s)$ where $T \in \mathbb{R}^{r\times r}$ is a constant orthogonal matrix.*

(iv) *Any factorization of the form $\Phi(s) = L^*(s)L(s)$ in which $L(s) \in \mathbb{R}(s)^{r\times n}$ is analytic in $\{\Re\text{e } s > 0,\ s \in \mathbb{C}\}$, is given by $L(s) = V(s)W(s)$, $V(s) \in \mathbb{R}(s)^{r\times r}$ being an arbitrary, CT regular para-unitary matrix.*

(v) *If $\Phi(s)$ is analytic on the imaginary axis, then $W(s)$ is analytic in a region $\{\Re\text{e } s > \tau,\ \tau < 0,\ s \in \mathbb{C}\}$.*

(vi) *If $\Phi(s)$ is analytic on the imaginary axis and the rank of $\Phi(s)$ is constant on the imaginary axis, then $W(s)$ and its (right) inverse $W^{-R}(s)$ are both analytic in a region $\{\Re\text{e } s > \tau_1,\ \tau_1 < 0,\ s \in \mathbb{C}\}$.*

PROOF. We first consider statement (iii). Let $W(s)$ and $W_1(s)$ be two matrices satisfying (i) and (ii). Then,

$$W^*(s)W(s) = W_1^*(s)W_1(s). \tag{3.32}$$

The latter equation implies

$$V^*(s)V(s) = I_r, \tag{3.33}$$

where we have defined $V(s) := W_1(s)W^{-R}(s)$ which is analytic in $\{\Re\text{e } s > 0,\ s \in \mathbb{C}\}$. Hence, $V(s) \in \mathbb{R}(s)^{r\times r}$ is a regular para-unitary matrix. But, from (3.32), we also have

$$V(s) = W_1^{-R*}(s)W^*(s) \tag{3.34}$$

and so $V^*(s) = V^{-1}(s) = W(s)W_1^{-R}(s)$ which is also regular. By applying Lemma 3.1.2, we conclude that $V(s)$ must be a constant orthogonal matrix $T \in \mathbb{R}^{r\times r}$, $T^\top T = TT^\top = I_r$.

Consider now statement (iv). Let $\Phi(s) = L^*(s)L(s)$ where $L(s) \in \mathbb{R}(s)^{n\times r}$ is analytic in $\{\Re\text{e } s > 0,\ s \in \mathbb{C}\}$. Notice that in this case we do not suppose that $L^{-R}(s)$ is analytic in $\{\Re\text{e } s > 0,\ s \in \mathbb{C}\}$. We can write

$$L^*(s)L(s) = W^*(s)W(s). \tag{3.35}$$

The latter equation implies

$$V^*(s)V(s) = I_r, \tag{3.36}$$

where $V(s) := L(s)W^{-R}(s)$ and $W(s) \in \mathbb{R}(s)^{r\times n}$ is a rational matrix satisfying (i) and (ii). Since $L(s)$ and $W^{-R}(s)$ are both analytic in $\{\Re\text{e } s > 0,\ s \in \mathbb{C}\}$, then $V(s) \in \mathbb{R}(s)^{r\times r}$ is a regular para-unitary matrix and we finished.

In order to prove the existence of a matrix $W(s)$ with the properties (i) and (ii) we use a constructive procedure which consists of the following four steps.

Step 1. Reduce $\Phi(s)$ to the Smith-McMillan canonical form. One possible technique to perform this reduction is described below.

- Assuming that all entries of $\Phi(s)$ are relatively prime, we write

$$\Phi(s) = \frac{1}{\varphi(s)}\tilde{\Phi}(s), \tag{3.37}$$

  where $\varphi(s)$ is the normalized l.c.m. of all denominators appearing in $\Phi(s)$ and $\tilde{\Phi}(s)$ is a polynomial matrix.

- The polynomial matrix $\tilde{\Phi}(s)$ can be reduced to its Smith canonical form by the standard technique described in Fornasini [2011, Ch.3] or in Kailath [1998, Ch.6]:

$$\tilde{\Phi}(s) = \tilde{C}(s)\tilde{E}(s)\tilde{F}(s), \tag{3.38}$$

where $\tilde{C}(s) \in \mathbb{R}[s]^{n\times n}$ and $\tilde{F}(s) \in \mathbb{R}[s]^{n\times n}$ are square unimodular polynomial matrices and

$$\tilde{E}(s) = \mathrm{diag}[\tilde{\varepsilon}_1(s), \tilde{\varepsilon}_2(s), \ldots, \tilde{\varepsilon}_r(s), 0, 0, \ldots, 0]. \tag{3.39}$$

The $\tilde{\varepsilon}_i(s)$, $i = 1, \ldots, r$, appearing in (3.39) are monic polynomials arranged so that $\tilde{\varepsilon}_i(s) \mid \tilde{\varepsilon}_{i+1}(s)$, $i = 1, 2, \ldots, r-1$.

- Let

$$J := \left[\begin{array}{c} I_r \\ \hline \mathbf{0}_{n-r,r} \end{array}\right] \in \mathbb{R}^{n\times r}. \tag{3.40}$$

Then $C(s) := \tilde{C}(s)J \in \mathbb{R}(s)^{n\times r}$ and $F(s) := J^\top \tilde{F}(s) \in \mathbb{R}(s)^{r\times n}$ are unimodular polynomial matrices. Moreover, we have

$$\tilde{\Phi}(s) = C(s)E(s)F(s), \tag{3.41}$$

where

$$E(s) := \mathrm{diag}[\tilde{\varepsilon}_1(s), \tilde{\varepsilon}_2(s), \ldots, \tilde{\varepsilon}_r(s)]. \tag{3.42}$$

- Finally, if we define

$$D(s) := \mathrm{diag}\left[\frac{\tilde{\varepsilon}_1(s)}{\varphi(s)}, \frac{\tilde{\varepsilon}_2(s)}{\varphi(s)}, \ldots, \frac{\tilde{\varepsilon}_r(s)}{\varphi(s)}\right] \tag{3.43}$$

each element being normalized in lowest terms, then the Smith-McMillan decomposition for $\Phi(s)$ is given by $\Phi(s) = C(s)D(s)F(s)$.

Step 2. According to (3.31), we can write $D(s)$ in the form

$$D(s) = \Sigma\Lambda^*(s)\tilde{\Delta}(s)\Lambda(s), \tag{3.44}$$

where:

1. $\Lambda(s) \in \mathbb{R}(s)^{r\times r}$ is diagonal, canonic and analytic together with $\Lambda^{-1}(s)$ in $\{\Re\mathrm{e}\ s \geq 0,\ s \in \mathbb{C}\}$;

2. $\tilde{\Delta}(s) := \Theta^*(s)\Theta(s) = \tilde{\Delta}^*(s)$, where $\Theta(s) \in \mathbb{R}(s)^{r\times r}$ is diagonal, canonic and analytic together with $\Theta^{-1}(s)$ in $\{\Re\mathrm{e}\ s \neq 0,\ s \in \mathbb{C}\}$;

3. $\Sigma \in \mathbb{R}^{r\times r}$ is diagonal with diagonal elements ± 1.

Let

$$A(s) := C(s)\Sigma\Lambda^*(s), \tag{3.45}$$
$$B(s) := \Lambda(s)F(s). \tag{3.46}$$

Then

$$\Phi(s) = A(s)\tilde{\Delta}(s)B(s) \tag{3.47}$$

is a CT left-standard factorization of $\Phi(s)$.

Step 3. Let

$$I(s) := B^{-R}(s)\Theta^{-1}(s). \tag{3.48}$$

Recall that, by equation (3.20), we have $A^*(s) = N(s)B(s)$ and so, by direct computation,

$$\begin{aligned} I^*(s)\Phi(s)I(s) &= I^*(s)\Phi^*(s)I(s) \\ &= \Theta^{-*}(s)B^{-R*}(s)B^*(s)\tilde{\Delta}^*(s)N(s)B(s)B^{-R}(s)\Theta^{-1}(s) \\ &= \Theta^{-*}(s)\Theta^*(s)\Theta(s)N(s)\Theta^{-1}(s) \\ &= \Theta(s)N(s)\Theta^{-1}(s), \end{aligned} \tag{3.49}$$

where $N(s) = A^*(s)B^{-R}(s) \in \mathbb{R}[s]^{r\times r}$ is a unimodular polynomial matrix. Let us define

$$\Psi(s) := \Theta(s)N(s)\Theta^{-1}(s). \tag{3.50}$$

By (3.49), $\Psi(s)$ is $r\times r$, para-Hermitian and non-negative on the imaginary axis. Actually, we notice that $A(s)\tilde{\Delta}(s)B(s)$ and $B^*(s)\tilde{\Delta}(s)A^*(s)$ are two CT left-standard factorizations of $\Phi(s)$. Hence, by replacing $\Delta_1(s)$ with $\tilde{\Delta}(s) = \tilde{\Delta}^*(s)$ in (3.12), we can write

$$\tilde{\Delta}(s)N(s)\tilde{\Delta}^{-1}(s) = M(s), \tag{3.51}$$

where $M(s) \in \mathbb{R}[s]^{r\times r}$ is unimodular. Since $\tilde{\Delta}(s) = \Theta^*(s)\Theta(s)$ is diagonal and $\Theta(s) := \mathrm{diag}[\theta_1(s),\dots,\theta_r(s)]$ canonic, (3.51) implies that $[N(s)]_{ij}$ is divisible by the polynomial $[\tilde{\Delta}(s)]_{jj}/[\tilde{\Delta}(s)]_{ii}$, $j \geq i$. But

$$\begin{aligned}[\tilde{\Delta}(s)]_{ii} &= \theta_i^*(s)\theta_i(s) \\ &= \theta_i(-s)\theta_i(s) \\ &= \pm\theta_i^2(s),\end{aligned} \tag{3.52}$$

for $i = 1, \ldots, r$. So, $[N(s)]_{ij}$ must be divisible by the polynomial

$$f_{ij}^2(s) := \frac{\theta_j^2(s)}{\theta_i^2(s)}, \quad j \geq i, \tag{3.53}$$

and, *a fortiori*, by

$$f_{ij}(s) = \frac{\theta_j(s)}{\theta_i(s)},\ j \geq i. \tag{3.54}$$

This suffices to establish that $\Psi(s)$ is polynomial. Finally, since by (3.49) $\det\Psi(s)$ is a non-zero positive constant, $\Psi(s)$ must be a para-Hermitian unimodular polynomial matrix which is positive definite on the imaginary axis. The problem is now reduced to that of finding a factorization of $\Psi(s)$ of the form

$$\Psi(s) = P^*(s)P(s), \tag{3.55}$$

where $P(s) \in \mathbb{R}[s]^{r\times r}$ is a unimodular polynomial matrix. After this is achieved, the desired factorization for $\Phi(s)$ is obtained as $\Phi(s) = W^*(s)W(s)$ with

$$\begin{aligned}W(s) &:= P(s)\Theta(s)B(s) \\ &= P(s)\Theta(s)\Lambda(s)F(s) \\ &= P(s)D_+(s)F(s),\end{aligned} \tag{3.56}$$

where we have defined $D_+(s) := \Theta(s)\Lambda(s)$. In fact, by straightforward algebra,

$$\begin{aligned}W^*(s)W(s) &= B^*(s)\Theta^*(s)P^*(s)P(s)\Theta(s)B(s) \\ &= B^*(s)\tilde{\Delta}(s)N(s)B(s) \\ &= B^*(s)\tilde{\Delta}(s)A^*(s) \\ &= \Phi^*(s) \\ &= \Phi(s).\end{aligned} \tag{3.57}$$

Step 4. An algorithm which provides a factorization of a unimodular polynomial matrix $\Psi(s) \in \mathbb{R}[s]^{r\times r}$ positive definite on the imaginary axis into the product $P^*(s)P(s)$, where $P(s)$ is a unimodular polynomial matrix, is due to Oono and Yasuura and first appeared in the paper Oono and Yasuura [1954]. This algorithm is the one adopted in the approach devised by Youla in his paper Youla [1961]. In what follows, we present a revised version of the latter algorithm which makes use of the mathematical machinery introduced in Chapter 2.

The proposed procedure is based on two steps. First, let us define $\Psi_1(s) := \Psi(s)$ and denote by $h \in \mathbb{N}$ the loop counter of the algorithm, which is initially set to $h := 1$.

I. By the positive nature of $\Psi_h(s)$, it follows that the degrees of the diagonal elements of $\Psi_h(s)$ are even non-negative integers. Let k_i, $i = 1,\dots,r$, be the half diagonal degrees (that is, half the degrees of the diagonal entries) of $\Psi_h(s)$. Since $\Psi_h(s)$ is positive definite on the imaginary axis, no element in $\Psi_h(s)$ has degree exceeding $2k_{\max}$, with

$$k_{\max} := \max\{k_1,\dots,k_r\}. \tag{3.58}$$

If $\Psi_h(s)$ is a constant positive definite matrix, we skip to step II. Otherwise, we construct a polynomial matrix $\Omega_h^{-1}(s)$ such that, by operating the transformation

$$\Psi_{h+1}(s) := \Omega_h^{-*}(s)\Psi_h(s)\Omega_h^{-1}(s), \tag{3.59}$$

we obtain a new positive matrix $\Psi_{h+1}(s)$ with the same determinant of $\Psi_h(s)$ but a lower diagonal degree. More specifically, the matrix $\Omega_h^{-1}(s)$ can be viewed as the product of three matrices, namely

$$\Omega_h^{-1}(s) := Q_h(s)T_hQ_h^{-1}(s), \tag{3.60}$$

where:

- the polynomial matrix

$$Q_h(s) := \operatorname{diag}[s^{k_{\max}-k_1},\dots,s^{k_{\max}-k_r}] \tag{3.61}$$

is such that

$$\Psi_h'(s) := Q_h^*(s)\Psi_h(s)Q_h(s) \tag{3.62}$$

has all the diagonal elements of the same degree $2k_{\max}$. This is always possible since, by the positive nature of $\Psi_h(s)$, $k_i \geq 0$ for all $i = 1, \ldots, r$. Hence, any diagonal element of $\Psi_h(s)$ cannot be identically zero.

- T_h is a constant matrix constructed in order to reduce the degree of a diagonal element of $\Psi'_h(s)$. The construction of this matrix requires a further explanation. First, we notice that the highest-column-degree coefficient matrix (recall the definitions given in §2.2.3) of $\Psi'_h(s)$, denoted by Ψ'^{hc}_h, is symmetric and equal to the highest-row-degree coefficient matrix, of $\Psi'_h(s)$, denoted by Ψ'^{hr}_h, since all the diagonal entries of $\Psi'_h(s)$ have the same maximum degree $2k_{\max}$ and $\Psi'_h(s) = \Psi'^{*}_h(s)$. Moreover, the external (column) degree[1] of $\Psi'_h(s)$ satisfies

$$\text{ext}_\text{c} \deg \Psi'_h(s) = 2rk_{\max}. \tag{3.63}$$

At this point, we note that, by (3.62),

$$\deg \det \Psi'_h(s) = 2 \deg \det Q_h(s) < 2rk_{\max}, \tag{3.64}$$

so, $\deg \det \Psi'_h(s) < \text{ext}_\text{c} \deg \Psi'_h$ and, in view of Proposition 2.2.2, we have that Ψ'^{hc}_h is singular. Thus, we can compute a non-zero vector $\mathbf{v}_h := [v_1, v_2, \ldots, v_r]^\top \in \mathbb{R}^r$ such that $\Psi'^{\text{hc}}_h \mathbf{v}_h = \mathbf{0}$. Let us define the *active index set*

$$\mathscr{I} := \{i \ : \ v_i \neq 0\} \tag{3.65}$$

and the *highest degree active index set*, $\mathscr{M} \subset \mathscr{I}$,

$$\mathscr{M} := \{i \in \mathscr{I} \ : \ k_i \geq k_j, \ \forall j \in \mathscr{I}\}. \tag{3.66}$$

We choose an index $p \in \mathscr{M}$. Then, T_h is constructed by replacing the p-th column of the $r \times r$ identity matrix with the vector

$$\mathbf{v}'_h := \left[\frac{v_1}{v_p}, \ldots, \frac{v_{p-1}}{v_p}, 1, \frac{v_{p+1}}{v_p}, \ldots, \frac{v_r}{v_p}\right]^\top \tag{3.67}$$

and it is such that

$$\Psi''_h(s) := T_h^\top \Psi'_h(s) T_h \tag{3.68}$$

[1] that is, the sum of all the column degrees of $\Psi'_h(s)$ (Definition 2.2.7 of §2.2.3).

has the diagonal entry at (p,p) of degree at least two less than $2k_{\max}$. In fact, since $\Psi_h'^{\mathrm{hc}}\mathbf{v}_h' = \mathbf{0}$, we have

$$\Psi_h'^{\mathrm{hc}} T_h = \left[\begin{array}{c|c|c} [\Psi_h'^{\mathrm{hc}}]_{1:r,1:p-1} & \mathbf{0}_{1,r} & [\Psi_h'^{\mathrm{hc}}]_{1:r,p+1:r} \end{array}\right]. \tag{3.69}$$

Analogously, since $\Psi_h'^{\mathrm{hc}}$ is symmetric, $\mathbf{v}_h^\top \Psi_h'^{\mathrm{hc}} = \mathbf{0}^\top$ and

$$T_h^\top \Psi_h'^{\mathrm{hc}} T_h = \left[\begin{array}{c|c|c} [\Psi_h'^{\mathrm{hc}}]_{1:p-1,1:p-1} & \mathbf{0}_{p-1,1} & [\Psi_h'^{\mathrm{hc}}]_{1:p-1,p+1:r} \\ \hline \mathbf{0}_{1,p-1} & 0 & \mathbf{0}_{1,r-p+1} \\ \hline [\Psi_h'^{\mathrm{hc}}]_{p+1:r,1:p-1} & \mathbf{0}_{r-p+1,1} & [\Psi_h'^{\mathrm{hc}}]_{p+1:r,p+1:r} \end{array}\right]. \tag{3.70}$$

By the positive nature of $\Psi_h''(s)$, the degrees of its diagonal entries of must be even non-negative integers, therefore, by (3.70), the diagonal element (p,p) of $\Psi_h''(s)$ must have degree at least two less than $2k_{\max}$.

- $Q_h^{-1}(s)$ is such that $\Psi_{h+1}(s) := Q_h^{-*}(s)\Psi_h''(s)Q_h^{-1}(s)$ possesses the original diagonal degrees except for the degree of the reduced diagonal entry at (p,p), which is at least two less than $2k_p$.

To sum up, $\Omega_h^{-1}(s)$ takes the form

$$\Omega_h^{-1}(s) := \begin{array}{c} \text{column } p \\ \begin{bmatrix} 1 & \cdots & 0 & \frac{v_1}{v_p} s^{k_p-k_1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots & & & 0 \\ \vdots & & 1 & \frac{v_{p-1}}{v_p} s^{k_p-k_{p-1}} & & & \vdots \\ \vdots & & & 1 & & & \vdots \\ \vdots & & & \frac{v_{p+1}}{v_p} s^{k_p-k_{p+1}} & 1 & & \vdots \\ 0 & & & \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{v_r}{v_p} s^{k_p-k_r} & 0 & \cdots & 1 \end{bmatrix} \end{array}, \tag{3.71}$$

and it is polynomial since $k_p \geq k_i$, for all $i = 1,\ldots,r$ such that $v_i \neq 0$. Actually, $\Omega_h^{-1}(s)$ is a polynomial unimodular matrix, since, from (3.71), we find that $\det \Omega_h^{-1}(s) = 1$.

Eventually, we update the index h by setting $h := h+1$ and return to step I.

II. Since $\Psi_h \in \mathbb{R}^{r\times r}$ is positive definite, we can always factorize it into the product $\Psi_h = C^\top C$ where $C \in \mathbb{R}^{r\times r}$, by using, for instance, the Cholesky decomposition (see Golub and Van Loan [1996, Ch.4]). Finally, we have constructed the desired polynomial unimodular matrix

$$P(s) = C\Omega_{h-1}(s)\Omega_{h-2}(s)\cdots\Omega_1(s). \tag{3.72}$$

such that $\Psi(s) = P^*(s)P(s)$.

It is worthwhile noticing that the procedure at step I is always brought to an end (after a maximum of $k_1 + \cdots + k_p$ iterations) since at the h-th iteration the degree of a diagonal element of $\Psi_h(s)$ is reduced, while the degree of all the other diagonal elements is not affected.

Finally, we notice that $W(s)$ in (3.56) is analytic with its (right) inverse in $\{\Re\mathrm{e}\, s > 0,\ s \in \mathbb{C}\}$ by construction, since $D_+(s)$ is so and $F(s)$, $P(s)$ are unimodular polynomial matrices. Hence, the proof of points (i)-(ii) is concluded.

Now consider statement (v). If $\Phi(s)$ is analytic on the imaginary axis, then $\Theta(s)$ does not have any finite pole. This, in turn, implies that $D_+(s) = \Theta(s)\Lambda(s)$ is analytic in a region $\{\Re\mathrm{e}\, s > \tau,\ \tau < 0,\ s \in \mathbb{C}\}$. Thus, $W(s)$, as defined in (3.56), is also analytic in the same region. It is worth noticing that this region is completely determined by the poles of $\Lambda(s)$.

The additional assumption that the rank of $\Phi(s)$ is constant on the imaginary axis implies that $\Theta(s)$ does not have any finite zero. Thus, $\Theta(s) = I_r$ and, by (3.56),

$$W^{-R}(s) = F^{-R}(s)\Lambda^{-1}(s)P^{-1}(s) \tag{3.73}$$

is analytic in a region $\{\Re\mathrm{e}\ s > \bar{\tau},\ \bar{\tau} < 0,\ s \in \mathbb{C}\}$. Hence, $W(s)$ is analytic together with its (right) inverse in a half-plane on the right of a vertical line $\{\Re\mathrm{e}\ s > \tau_1,\ \tau_1 < 0,\ s \in \mathbb{C}\}$ where $\tau_1 := \max\{\tau, \bar{\tau}\}$ is completely determined by the zeros and poles of $\Lambda(s)$. The proof of point (vi) along with that of the Theorem is concluded. ■

For the sake of completeness, we present below two simple Corollaries of Theorem 3.2.1.

Corollary 3.2.1 *Let $L(s) \in \mathbb{R}(s)^{m\times n}$, then $\Phi(s) = L^*(s)L(s)$ if and only if*

$$L(s) = V(s) \left[\begin{array}{c} I_r \\ \hline \mathbf{0}_{m-r,r} \end{array} \right] W(s), \tag{3.74}$$

where $V(s) \in \mathbb{R}(s)^{m\times m}$ is an arbitrary CT para-unitary matrix and $r = \mathrm{rk}(\Phi) \leq m$.

PROOF. By repeating an argument used to prove statements (iii) and (iv) of Theorem 3.2.1, we have that $L(s) = U(s)W(s)$, with $U(s) \in \mathbb{R}(s)^{m\times r}$ a rational matrix satisfying $U^*(s)U(s) = I_r$. If we choose $V(s) \in \mathbb{R}(s)^{m\times m}$ to be any para-unitary matrix with $U(s)$ incorporated into its first r columns, *i.e.*,

$$U(s) = V(s) \left[\begin{array}{c} I_r \\ \hline \mathbf{0}_{m-r,r} \end{array} \right], \tag{3.75}$$

we finished. ■

Corollary 3.2.2 *If $\Phi(s)$ is polynomial, then $W(s)$ is polynomial.*

PROOF. If $\Phi(s)$ is polynomial, then it does not have finite poles. By (3.56), also $W(s)$ does not have finite poles, therefore $W(s)$ is polynomial. ■

In Fig.3.1 is shown a scheme of the Youla's algorithm used in the constructive proof of Theorem 3.2.1.

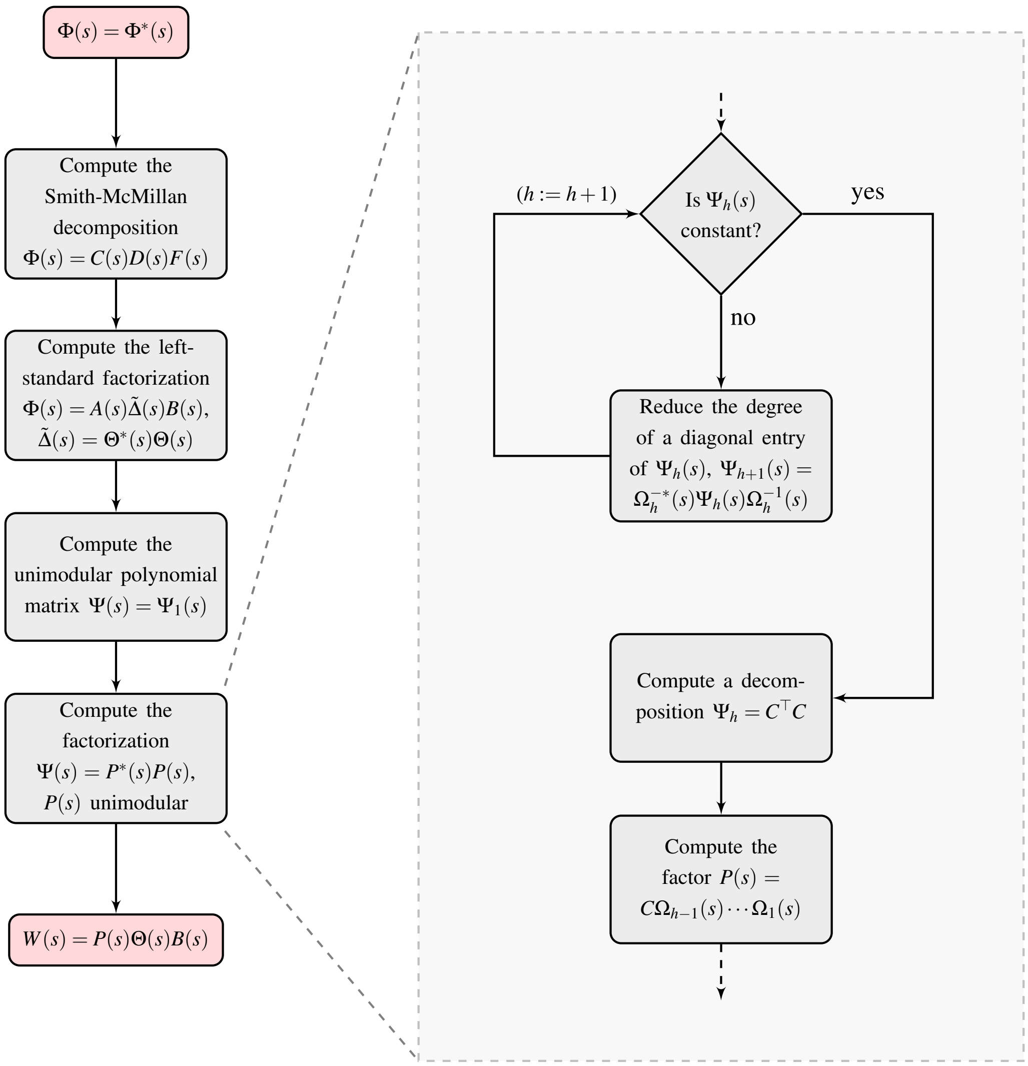

Figure 3.1: *Schematic representation of Youla's algorithm used for the construction of the factorization* $\Phi(s) = W^*(s)W(s)$.

3.3 | Some additional remarks

In this section, we want to point out two important facts regarding some properties of the factorization approach discussed before.

The first consideration regards the *stochastic minimality* of the factor $W(s)$ and it is stated in the Theorem below. (We refer to Lindquist and Picci [1991] for a detailed discussion on the minimality of spectral factors).

Theorem 3.3.1 *Let $\Phi(s) \in \mathbb{R}(s)^{n\times n}$ be a CT para-Hermitian matrix nonnegative on the imaginary axis and let $r = \mathrm{rk}(\Phi)$. Consider the factorization $\Phi(s) = W^*(s)W(s)$, where $W(s) \in \mathbb{R}(s)^{r\times n}$ is computed by following the procedure described in Theorem 3.2.1. Then, the McMillan degree of $W(s)$ satisfies*

$$\delta_M(W) = \frac{1}{2}\delta_M(\Phi). \tag{3.76}$$

Proof. First, we recall that, by equation (3.56), we have

$$W(s) = P(s)D_+(s)F(s), \tag{3.77}$$

where $P(s) \in \mathbb{R}[s]^{r\times r}$, $F(s) \in \mathbb{R}[s]^{r\times n}$ are unimodular polynomial matrices and $D_+(s) \in \mathbb{R}(s)^{r\times r}$ is diagonal, canonic and regular together with its inverse. Moreover, $D_+(s)$ satisfies

$$D(s) = \Sigma D_+^*(s)D_+(s), \tag{3.78}$$

where $D(s) \in \mathbb{R}(s)^{r\times r}$ is the Smith-McMillan canonical form of $\Phi(s)$ and $\Sigma \in \mathbb{R}^{r\times r}$ is a constant diagonal matrix with elements ± 1 on its diagonal. Let $p_1,\dots,p_h$ be the finite poles of $\Phi(s)$. By (3.77) and (3.78), it follows that[2]

$$\delta(\Phi;p_i) = \begin{cases} \delta(W;p_i) & \text{if } \Re\mathrm{e}\ p_i < 0,\\ 2\delta(W;p_i) & \text{if } \Re\mathrm{e}\ p_i = 0,\\ \delta(W;-p_i) & \text{if } \Re\mathrm{e}\ p_i > 0. \end{cases} \tag{3.79}$$

[2]For the definition of degree of a pole of a rational matrix we refer to §2.3.3, Definition 2.3.4.

Moreover, if p_i is a pole of $\Phi(s)$ then also $-p_i$ is a pole of $\Phi(s)$ and if p_i is not a pole of $\Phi(s)$ then neither p_i nor $-p_i$ are poles of $W(s)$. Thus, we have

$$\begin{aligned}\sum_{i=1}^{h}\delta(\Phi;p_i) &= \sum_{i\,:\,\Re\mathrm{e}\ p_i<0}\delta(W;p_i)+\sum_{i\,:\,\Re\mathrm{e}\ p_i>0}\delta(W;-p_i)+\sum_{i\,:\,\Re\mathrm{e}\ p_i=0}2\delta(W;p_i)\\ &= 2\sum_{i\,:\,\Re\mathrm{e}\ p_i\leq 0}\delta(W;p_i) \qquad (3.80)\end{aligned}$$

By equation (2.57) of section §2.3.3, the McMillan degree of a rational matrix equals the sum of the degrees of all its poles, the pole at infinity included. Hence, it remains to prove that

$$\delta(\Phi;\infty)=2\delta(W;\infty). \qquad (3.81)$$

The degree of the pole at $s=\infty$ of $\Phi(s)$ is equal to the degree of the pole at $\lambda=0$ of $\Phi(\lambda^{-1})$. The rational matrix $\Phi(\lambda^{-1})$ is para-Hermitian and positive semi-definite on the imaginary axis, so we can apply Theorem 3.2.1 and obtain

$$\Phi(\lambda^{-1})=\tilde{W}^*(\lambda)\tilde{W}(\lambda), \qquad (3.82)$$

Since $\Phi(\lambda^{-1})=W^*(\lambda^{-1})W(\lambda^{-1})$ is also a factorization satisfying the properties of Theorem 3.2.1, by statement (iii) of the same Theorem, we have

$$W(\lambda^{-1})=T\tilde{W}(\lambda), \qquad (3.83)$$

$T\in\mathbb{R}^{r\times r}$ being a constant orthogonal matrix. Therefore, by a previous argument,

$$\begin{aligned}\delta(\Phi(s);\infty) &= \delta(\Phi(\lambda^{-1});0)\\ &= 2\delta(\tilde{W}(\lambda);0)\\ &= 2\delta(W(\lambda^{-1});0)=2\delta(W(s);\infty). \qquad (3.84)\end{aligned}$$

In view of (3.80) and (3.84), we conclude that

$$\begin{aligned}\delta_M(\Phi) &= \sum_{i=1}^{h}\delta(\Phi;p_i)+\delta(\Phi;\infty)\\ &= 2\sum_{i\,:\,\Re\mathrm{e}\ p_i\leq 0}\delta(W;p_i)+2\delta(W;\infty)=2\delta_M(W), \qquad (3.85)\end{aligned}$$

and, hence, the thesis. ■

Another interesting by-product of Youla's method is that the procedure can be easily modified in order to change the region of analyticity of $W(s)$ and of its (right) inverse. First of all, let us introduce some nomenclature. We say that a region of the complex plane $\mathscr{A}$ is *continuous-time skew-symmetric* (for short, skew-symmetric), if it satisfies

$$\mathscr{A} \cup \mathscr{A}^* = \mathbb{C} \setminus \{\Re\mathrm{e}\, s = 0, s \in \mathbb{C}\} \quad \text{and} \quad \mathscr{A} \cap \mathscr{A}^* = \varnothing, \tag{3.86}$$

where $\mathscr{A}^* := \{s \in \mathbb{C} \,:\, -s \in \mathscr{A}\}$.

Now, suppose that we want to compute a factor $W_{\Omega,\Omega'}(s)$ analytic in a skew-symmetric region Ω (see Fig.3.2) and with (right) inverse $W_{\Omega,\Omega'}^{-R}(s)$ analytic in a skew-symmetric region Ω'.[3] To obtain such a factor, we only need to write the diagonal matrix $D(s)$ in (3.44) in a different form, namely

$$D(s) = \Sigma \Lambda_{\Omega,\Omega'}^*(s) \Theta^*(s) \Theta(s) \Lambda_{\Omega,\Omega'}(s), \tag{3.87}$$

where $\Lambda_{\Omega,\Omega'}(s)$ is diagonal, canonic and analytic in Ω with inverse $\Lambda_{\Omega,\Omega'}^{-1}(s)$ analytic in Ω', and apply Youla's factorization algorithm.

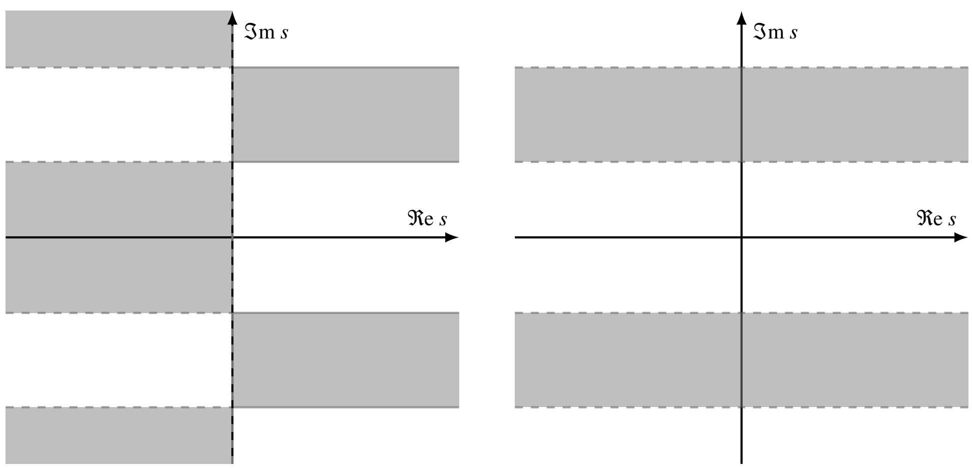

Figure 3.2: *Example of an admissible choice of the region Ω (filled in gray, on the left) and a not admissible choice of Ω (filled in gray, on the right). Here, dashed gray lines denote open boundaries, while solid gray lines denote closed boundaries.*

[3] In the following, we use the notation $A_{\Omega,\Omega'}$ to emphasize the dependence of matrix A on the choice of the regions Ω and Ω'. If $\Omega = \Omega'$, we write, for short, $A_{\Omega} = A_{\Omega,\Omega'}$.

To prove this fact, it is sufficient to show that the $r \times r$ matrix

$$\begin{aligned}\Psi_{\Omega,\Omega'}(s) &= \Theta(s)N_{\Omega,\Omega'}(s)\Theta^{-1}(s)\\ &= \Theta(s)A^*_{\Omega,\Omega'}(s)B^{-R}_{\Omega,\Omega'}(s)\Theta^{-1}(s)\\ &= \Theta(s)\Lambda_{\Omega,\Omega'}(s)\Sigma C^*(s)F^{-R}(s)\Lambda^{-1}_{\Omega,\Omega'}(s)\Theta^{-1}(s),\end{aligned} \tag{3.88}$$

defined in (3.50), still remains a polynomial unimodular matrices for any choice of Ω and Ω'. This is provided by the following Lemma.

Lemma 3.3.1 *Consider the $r \times r$ matrix $\Psi_{\Omega,\Omega'}(s)$ described by equation (3.88). For any choice of the skew-symmetric regions Ω and Ω', $\Psi_{\Omega,\Omega'}(s)$ is a unimodular polynomial matrix.*

PROOF. We can rewrite equation (3.88) in a more compact form as

$$\Psi_{\Omega,\Omega'}(s) = \Gamma_{\Omega,\Omega'}(s)R(s)\Gamma^{-1}_{\Omega,\Omega'}(s), \tag{3.89}$$

where $\Gamma_{\Omega,\Omega'}(s) := \Theta(s)\Lambda_{\Omega,\Omega'}(s) \in \mathbb{R}(s)^{r\times r}$ and $R(s) := \Sigma C^*(s)F^{-R}(s) \in \mathbb{R}[s]^{r\times r}$. We notice that $R(s)$ unimodular, since $C(s)$, $F(s)$ are so and Σ is constant, and does not depend on the choice of Ω and Ω'. Moreover, $\Gamma_{\Omega,\Omega'}(s)$ is diagonal and canonic for any choice of Ω and Ω'.

Consider first the standard choice $\Omega = \Omega' = \mathbb{C}^+ := \{\Re\text{e } s > 0,\ s \in \mathbb{C}\}$. By Theorem 3.1.1 and Theorem 3.2.1 (step 3), $\Psi_{\mathbb{C}^+}(s)$ is unimodular. Since $\Gamma_{\mathbb{C}^+}(s) \in \mathbb{R}(s)^{r\times r}$ is diagonal and canonic, by (3.89), it follows that $[R(s)]_{ij}$ must be divisible by the polynomial

$$p_{ij}(s) := \frac{[\Gamma_{\mathbb{C}^+}(s)]_{jj}}{[\Gamma_{\mathbb{C}^+}(s)]_{ii}}, \quad j \geq i. \tag{3.90}$$

On the other hand, consider the opposite choice $\Omega = \Omega' = \mathbb{C}^- := \{\Re\text{e } s < 0,\ s \in \mathbb{C}\}$. By using the right-standard counterpart of Theorem 3.1.1 and following the same argument used in the step 3 of Theorem 3.2.1, it can be proved that $\Psi_{\mathbb{C}^-}(s)$ is unimodular. Hence, by (3.89), $[R(s)]_{ij}$ must be also divisible by the polynomial $p_{ij}(-s)$, $j \geq i$.

Therefore, $[R(s)]_{ij}$ must be divisible by every factor which appears in the polynomial

$$q_{ij}(s) := p_{ij}(s)p_{ij}(-s), \quad j \geq i. \tag{3.91}$$

Since, for any choice of Ω and Ω', the factors of $[\Gamma_{\Omega,\Omega'}(s)]_{jj}/[\Gamma_{\Omega,\Omega'}(s)]_{ii}$ are contained in the ones of $q_{ij}(s)$, then $[R(s)]_{ij}$ must be divisible by the polynomial $[\Gamma_{\Omega,\Omega'}(s)]_{jj}/[\Gamma_{\Omega,\Omega'}(s)]_{ii}$, $j \geq i$, and so $\Psi_{\Omega,\Omega'}(s)$ must be a polynomial matrix for any choice of Ω and Ω'. Finally, by (3.89), $\det \Psi_{\Omega,\Omega'}(s) = \text{constant}$, for any choice of Ω and Ω', therefore we conclude that $\Psi_{\Omega,\Omega'}(s)$ is unimodular. ■

In particular, we can use Youla's method in order to compute some interesting "extremal" factors:

- the regular factor with regular (right) inverse, called the *minimum-phase regular factor*, $\underline{W}_-$ (whose existence was proved in Theorem 3.2.1);
- the anti-regular factor with regular (right) inverse, called the *minimum-phase anti-regular factor*, $\overline{W}_-$;
- the regular factor with anti-regular (right) inverse, called the *maximum-phase regular factor*, $\underline{W}_+$;
- the anti-regular factor with anti-regular (right) inverse, called the *maximum-phase anti-regular factor*, $\overline{W}_+$.

These four "extremal" factors are unique (modulo orthogonal transformations) and are related each other as shown in the commutative diagram of Fig.3.3.

$$\begin{tikzcd}
\underline{W}_- \arrow[r, "\underline{V}'"] & \underline{W} \arrow[r, "\underline{V}''"] & \underline{W}_+ \\
W_- \arrow[u, "K''_-"] \arrow[r, "V'"] & W \arrow[u, "K''"] \arrow[r, "V''"] & W_+ \arrow[u, "K''_+"] \\
\overline{W}_- \arrow[u, "K'_-"] \arrow[r, "\overline{V}'"] & \overline{W} \arrow[u, "K'"] \arrow[r, "\overline{V}''"] & \overline{W}_+ \arrow[u, "K'_+"]
\end{tikzcd}$$

Figure 3.3: *Relations between the "extremal" factors. Here arrows indicate pre-multiplication, V's and K's are para-unitary matrices. In particular, the meaning of symbols is as follows: underbar stands for regular, overbar for anti-regular, subscript $-$ for minimum-phase and subscript $+$ for maximum-phase.*

3.4 | An illustrative example

In this final section, we present an illustrative example of application of the Youla's method described in the constructive proof of Theorem 3.2.1.

Let us consider the following 2×2 rational matrix

$$\Phi(s) = \begin{bmatrix} \frac{1}{1-s^2} & \frac{1}{s(1-s^2)} \\ -\frac{1}{s(1-s^2)} & \frac{s^2-2}{s^2(1-s^2)} \end{bmatrix}. \tag{3.92}$$

It may be verified that $\Phi(s)$ is para-Hermitian and positive definite on the imaginary axis.

Step 1. The Smith-McMillan canonical form of $\Phi(s)$ is given by

$$D(s) = \begin{bmatrix} \frac{1}{s^2(s^2-1)} & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.93}$$

The rational matrix $\Phi(s)$ is related to its Smith-McMillan canonical form by

$$\Phi(s) = C(s)D(s)F(s), \tag{3.94}$$

where $C(s)$ and $F(s)$ are (non-unique) unimodular polynomial matrices, *e.g.*,

$$C(s) = \begin{bmatrix} -\frac{1}{2}s & 1 \\ 1-\frac{1}{2}s^2 & s \end{bmatrix}, \quad F(s) = \begin{bmatrix} s(s^2+1) & 2 \\ \frac{1}{2} & 0 \end{bmatrix}. \tag{3.95}$$

Step 2. With reference to the notation introduced in the corresponding step of Theorem 3.2.1, the regular rational matrix $\Lambda(s)$ is given by

$$\Lambda(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.96}$$

while the diagonal matrix $\Theta(s)$ by

$$\Theta(s) = \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.97}$$

The matrices $A(s)$ and $B(s)$, defined in (3.45) and (3.46), respectively, take the form

$$A(s) = \begin{bmatrix} \frac{s}{2(s-1)} & 1 \\ \frac{\frac{1}{2}s^2-1}{s-1} & s \end{bmatrix}, \tag{3.98}$$

$$B(s) = \begin{bmatrix} \frac{s(s^2+1)}{s+1} & \frac{2}{s+1} \\ \frac{1}{2} & 0 \end{bmatrix}. \tag{3.99}$$

Step 3. We have that $\Psi(s) = \Theta(s)N(s)\Theta^{-1}(s)$ has the form

$$\begin{aligned} \Psi(s) &= \Theta(s)A^*(s)B^{-1}(s)\Theta^{-1}(s) \\ &= \begin{bmatrix} -\frac{1}{4}s^2+\frac{1}{2} & \frac{1}{2}s^2(s-1) \\ -\frac{1}{2}s^2(s+1) & s^4+s^2+2 \end{bmatrix}. \end{aligned} \tag{3.100}$$

It may be checked directly that $\Psi(s)$ is a para-Hermitian, unimodular polynomial matrix which is positive definite on the imaginary axis.

Step 4. We make all diagonal entries in $\Psi_1(s) = \Psi(s)$ equidegree by operating the transformation

$$\Psi_1'(s) = Q_1^*(s)\Psi_1(s)Q_1(s) = \begin{bmatrix} \frac{1}{4}s^4-\frac{1}{2}s^2 & -\frac{1}{2}s^3(s-1) \\ -\frac{1}{2}s^3(s+1) & s^4+s^2+2 \end{bmatrix}, \tag{3.101}$$

where $Q_1(s) = \mathrm{diag}[s, 1]$. The highest-column-degree coefficient matrix $\Psi_1'^{\mathrm{hc}}$ is given by

$$\Psi_1'^{\mathrm{hc}} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}. \tag{3.102}$$

Since $\Psi_1'^{\mathrm{hc}}$ is singular, we calculate a vector $\mathbf{v}_1 \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ such that $\Psi_1'^{\mathrm{hc}}\mathbf{v}_1 = \mathbf{0}$. In our case such a vector is given, for example, by $\mathbf{v}_1 = [2,\ 1]^\top$. In order to reduce the degree of a diagonal element of $\Psi_1'(s)$, we construct the matrix T_1

$$T_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}. \tag{3.103}$$

By applying to $\Psi_1'(s)$ the transformation induced by T_1, we reduce the diagonal degree of the entry $(2,2)$,

$$\Psi_1''(s) = T_1^\top \Psi_1'(s) T_1 = \begin{bmatrix} \frac{1}{4}s^4 - \frac{1}{2}s^2 & \frac{1}{2}s^2(s-2) \\ -\frac{1}{2}s^2(s+2) & -s^2+2 \end{bmatrix}. \qquad (3.104)$$

Finally, by operating the inverse transformation $Q_1^{-1}(s)$ to $\Psi_1''(s)$, the first reduction cycle is concluded

$$\Psi_2(s) = Q_1^{-*}(s)\Psi_1''(s)Q_1^{-1}(s) = \begin{bmatrix} -\frac{1}{4}s^2 + \frac{1}{2} & -\frac{1}{2}s(s-2) \\ -\frac{1}{2}s(s+2) & -s^2+2 \end{bmatrix}. \qquad (3.105)$$

The overall transformation of the first reduction cycle is given by

$$\Omega_1^{-1}(s) = Q_1(s)T_1Q_1^{-1}(s) = \begin{bmatrix} 1 & 2s \\ 0 & 1 \end{bmatrix}. \qquad (3.106)$$

Since $\Psi_2(s)$ is polynomial, we repeat the reduction procedure. The diagonal entries in $\Psi_2(s)$ are equidegree, thus $Q_2(s) = I_2$ and $\Psi_2'(s) = \Psi_2(s)$. The highest-column-degree coefficient matrix $\Psi_2'^{\mathrm{hc}}$ is given by

$$\Psi_2'^{\mathrm{hc}} = \begin{bmatrix} -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & -1 \end{bmatrix} \qquad (3.107)$$

and is singular. Hence, we compute a vector $\mathbf{v}_2 \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ such that $\Psi_2'^{\mathrm{hc}}\mathbf{v}_2 = \mathbf{0}$. In our case such a vector is given, for example, by $\mathbf{v}_2 = [-2,\ 1]^\top$. In order to reduce the degree of a diagonal element of $\Psi_2'(s)$, we construct the matrix T_2

$$T_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}. \qquad (3.108)$$

By applying to $\Psi_2'(s)$ the transformation induced by T_2, we reduce the diagonal degree of the entry $(2,2)$,

$$\Psi_3(s) = T_2^\top \Psi_2'(s) T_2 = \begin{bmatrix} -\frac{1}{4}s^2 + \frac{1}{2} & s-1 \\ -s-1 & 4 \end{bmatrix}. \qquad (3.109)$$

The second reduction cycle is concluded and we can define the overall transformation of this reduction cycle as $\Omega_2^{-1}(s) = T_2$.

An additional reduction cycle is needed, since $\Psi_3(s)$ is polynomial. We make all diagonal entries in $\Psi_3(s)$ equidegree by operating the transformation

$$\Psi_3'(s) = Q_3^*(s)\Psi_3(s)Q_3(s) = \begin{bmatrix} -\frac{1}{4}s^2 + \frac{1}{2} & s(s-1) \\ s(s+1) & -4s^2 \end{bmatrix}, \tag{3.110}$$

where $Q_3(s) = \operatorname{diag}[1,\ s]$. The highest-column-degree coefficient matrix $\Psi_3'^{\text{hc}}$ is given by

$$\Psi_3'^{\text{hc}} = \begin{bmatrix} -\frac{1}{4} & 1 \\ 1 & -4 \end{bmatrix}. \tag{3.111}$$

Since $\Psi_3'^{\text{hc}}$ is singular, we calculate a vector $\mathbf{v}_3 \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ such that $\Psi_3'^{\text{hc}}\mathbf{v}_3 = \mathbf{0}$. In our case, such a vector is given, for example, by $\mathbf{v}_3 = [4,\ 1]^\top$. In order to reduce the degree of a diagonal element of $\Psi_3'(s)$, we construct the matrix

$$T_3 = \begin{bmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{bmatrix}. \tag{3.112}$$

By applying to $\Psi_3'(s)$ the transformation induced by T_3, we reduce the diagonal degree of the entry $(1,1)$,

$$\Psi_3''(s) = T_3^\top \Psi_3'(s) T_3 = \begin{bmatrix} \frac{1}{2} & -s \\ s & -4s^2 \end{bmatrix}. \tag{3.113}$$

Finally, by operating the transformation $Q_3^{-1}(s)$ to $\Psi_3''(s)$, the third reduction cycle is concluded

$$\Psi_4 = Q_3^{-*}(s)\Psi_3''(s)Q_3^{-1}(s) = \begin{bmatrix} \frac{1}{2} & -1 \\ -1 & 4 \end{bmatrix}. \tag{3.114}$$

The overall transformation of the third reduction cycle is given by

$$\Omega_3^{-1}(s) = Q_3(s)T_3Q_3^{-1}(s) = \begin{bmatrix} 1 & 0 \\ \frac{1}{4}s & 1 \end{bmatrix}. \tag{3.115}$$

Since Ψ_4 is a constant positive definite matrix, we can decompose it as $\Psi_4 = C^\top C$ by using the Cholesky factorization

$$C = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}. \tag{3.116}$$

In this way, we have found a factorization $\Psi(s) = P^*(s)P(s)$, where $P(s)$ is unimodular of the form

$$P(s) = C\Omega_3(s)\Omega_2(s)\Omega_1(s) = \sqrt{2}\begin{bmatrix} \frac{1}{4}s+\frac{1}{2} & -\frac{1}{2}s^2-\frac{1}{2}s \\ -\frac{1}{4}s & \frac{1}{2}s^2-\frac{1}{2}s+1 \end{bmatrix}. \tag{3.117}$$

At the end, the desired factorization of $\Phi(s)$ is given by $\Phi(s) = W^*(s)W(s)$, where

$$W(s) = P(s)\Theta(s)B(s) = \sqrt{2}\begin{bmatrix} \frac{1}{2(s+1)} & \frac{s+2}{2s(s+1)} \\ \frac{1}{2(s+1)} & \frac{-1}{2(s+1)} \end{bmatrix}. \tag{3.118}$$

The factor $W(s)$ is analytic together with its inverse in the open right half-plane $\{\Re\mathrm{e}\; s > 0,\ s \in \mathbb{C}\}$, as required.

$$\Phi(z) = W^*(z)W(z)$$
$$= W^\top(1/z)W(z)$$

4. Discrete-time spectral factorization

In this Chapter, we will present a modification of Youla's method, reviewed in Chapter 3, which can be applied to provide a solution to the multivariate *discrete-time* spectral factorization problem (*cf.* Problem 1.2 of Chapter 1). Even if some results follow almost *verbatim* from that of the continuous-time case, there are some aspects which are peculiar of the discrete-time case and significantly differ from the analysis carried out in Youla [1961]. Our attention will be focused on these aspects, in particular.

A remark on notation. A rational matrix $A(z)$ is said to be analytic in a region of the complex plane if all its entries are analytic in this region. Moreover, as in Chapter 3, with a slight abuse of notation, when we say that a rational function $f(z)$ is analytic in a closed region $\mathbb{T}$ of the complex plane we mean that $f(z)$ is analytic in an open region $\mathbb{T}_\varepsilon \supset \mathbb{T}$ which is "larger" than $\mathbb{T}$ of an arbitrarily small quantity. For example, if $f(z)$ is rational and has all its poles inside the open unit circle, we say that $f(z)$ is analytic outside the closed unit circle to mean that there exists $\varepsilon > 0$ s.t. $f(z)$ is analytic in $\{|z| > 1-\varepsilon,\ z \in \mathbb{C}\}$. Similarly, we say that $f(z)$ is analytic on the unit circle in place of $f(z)$ is analytic on an open annulus containing the unit circle. When dealing with rational functions that feature a finite number of poles, this abuse of notation does not cause any problem. Finally, we say that a rational matrix is canonic if it satisfies the properties of the Smith-McMillan Theorem 2.3.1. For other standard notation refer to Chapter 0 and 2.

4.1 | Preliminary results

As in Chapter 3, §3.1, we collect in this section some auxiliary results that we will exploit in the proof of the main Theorem, reported in the next section.

Lemma 4.1.1 *A matrix $G(z) \in \mathbb{R}(z)^{m\times n}$ is analytic in $\mathbb{C} \setminus \{0\}$ together with its inverse (either right, left or both) if and only if it is a L-unimodular polynomial matrix.*

PROOF. If $G(z)$ is a L-unimodular polynomial matrix, then, from §2.2.4, we know that $G(z)$ has an inverse (either right, left or both) which is also L-polynomial. Hence, the only possible finite zeros/poles of $G(z)$ are located at $z = 0$. This, in turn, implies that $G(z)$ must be analytic together with its inverse in $\mathbb{C}\setminus\{0\}$.

Vice versa, suppose that $G(z)$ is analytic with its inverse in $\mathbb{C}\setminus\{0\}$. First, we notice that the existence of a left or right inverse for $G(z)$ implies that the normal rank of $G(z)$ is either $r = n$ or $r = m$, respectively. Without loss of generality, we can suppose that $r = n$. By the Smith-McMillan Theorem 2.3.1, we can write $G(z) = C(z)D(z)F(z)$, where $C(z) \in \mathbb{R}[z]^{m\times r}$, $F(z) \in \mathbb{R}[z]^{r\times n}$ are unimodular (and, *a fortiori*, L-unimodular) polynomial matrices, respectively, and $D(z) \in \mathbb{R}(z)^{r\times r}$ is diagonal, canonic of the form

$$D(z) = \operatorname{diag}\left[\frac{\varepsilon_1(z)}{\psi_1(z)}, \frac{\varepsilon_2(z)}{\psi_2(z)}, \dots, \frac{\varepsilon_n(z)}{\psi_n(z)}\right], \tag{4.1}$$

where $\varepsilon_k(z)$, $\psi_k(z)$, $k = 1,\dots,r$, are relatively prime monic polynomials such that $\varepsilon_k(z) \mid \varepsilon_{k+1}(z)$, $\psi_{k+1}(z) \mid \psi_k(z)$, $k = 1,\dots,r-1$. The analyticity of $G(z)$ in $\mathbb{C}\setminus\{0\}$ implies that all ψ's are non-zero monomials. The Smith-McMillan canonical form of the left inverse of $G(z)$, $G^{-L}(z)$, is given by

$$\operatorname{diag}\left[\frac{\psi_n(z)}{\varepsilon_n(z)}, \frac{\psi_{n-1}(z)}{\varepsilon_{n-1}(z)}, \dots, \frac{\psi_1(z)}{\varepsilon_1(z)}\right]. \tag{4.2}$$

So, the analyticity of $G^{-L}(z)$ in $\mathbb{C}\setminus\{0\}$ implies that all ε's are non-zero monomials. Hence, $D(z)$ is a L-unimodular polynomial matrix. Since $G(z) = C(z)D(z)F(z)$ is the product of three L-unimodular polynomial matrices, $G(z)$ must be a L-unimodular polynomial matrix. ■

Lemma 4.1.2 *The only DT regular para-unitary matrices with regular inverse are DT para-unitary matrices analytic together with their inverse in* $\mathbb{C}\setminus\{0\}$.

PROOF. From Definition 2.3.6, we recall that a DT para-unitary matrix $G(z) \in \mathbb{R}(z)^{n\times n}$ satisfies

$$G^*(z)G(z) = G(z)G^*(z) = I_n. \tag{4.3}$$

The analyticity of the inverse of $G(z)$ in $\{|z| > 1,\ z \in \mathbb{C}\}$ implies that of $G(1/z)$ in the same region, and therefore that of $G(z)$ in $\{|z| < 1,\ z \in \mathbb{C}\setminus\{0\}\}$.[1] We also

[1] Notice that, unlike the continuous-time case, here $G(z)$ can be, in general, not bounded at infinity.

notice that in the unit circle $\{e^{j\omega},\ \omega \in [0,2\pi)\}$ we have $G^*(e^{j\omega})G(e^{j\omega}) = I_n$ and we can write out the diagonal element in expanded form as

$$\sum_{i=1}^{n} |[G(e^{j\omega})]_{ik}|^2 = 1, \quad \forall k = 1,\ldots,n,\ \forall \omega \in [0,2\pi). \tag{4.4}$$

The latter equation implies that

$$|[G(e^{j\omega})]_{ik}| \leq 1, \quad \forall i,k = 1,\ldots,n,\ \forall \omega \in [0,2\pi), \tag{4.5}$$

and, therefore, we proved the analyticity of $G(z)$ on the unit circle. We conclude that $G(z)$ is analytic together with its inverse $G^{-1}(z) = G^*(z)$ in $\mathbb{C} \setminus \{0\}$. ■

Corollary 4.1.1 *Let $G(z) \in \mathbb{R}(z)^{n\times n}$ be a DT regular para-unitary matrix without poles at infinity. Let the inverse of $G(z)$ be also regular without poles at infinity. Then $G(z)$ is a constant orthogonal matrix.*

PROOF. In this case, the additional assumption that $G(z)$ and its inverse do not possess poles at infinity, implies that $G(z)$ is bounded at infinity and analytic on the entire complex plane. Hence, we can apply Liouville's Theorem Lang [1985, Ch.V, §1, Thm.1.4] and conclude that $G(z)$ must be a constant orthogonal matrix. ■

The following Definition is specular to Definition 3.1.1, given in the continuous-time case.

Definition 4.1.1 (DT left-standard factorization) Let $G(z) \in \mathbb{R}(z)^{m\times n}$ and let $\mathrm{rk}(G) = r \leq \min\{m,n\}$. A decomposition of the form

$$G(z) = A(z)\Delta(z)B(z) \tag{4.6}$$

is called a *discrete-time (DT) left-standard factorization* if

1. $\Delta(z) \in \mathbb{R}(z)^{r\times r}$ is diagonal and analytic together with its inverse in $\mathbb{C} \setminus \{0\}$ with the possible exception of a finite number of points on the unit circle $\{|z| = 1,\ z \in \mathbb{C}\}$;

2. $A(z) \in \mathbb{R}(z)^{m\times r}$ is analytic together with its left inverse in $\{|z| \leq 1,\ z \in \mathbb{C} \setminus \{0\}\}$;

3. $B(z) \in \mathbb{R}(z)^{r\times n}$ and analytic together with its right inverse in $\{|z| \geq 1,\ z \in \mathbb{C}\}$.

If $A(z)$ and $B(z)$ are interchanged, we have a *DT right-standard factorization*. Hence, any DT left-standard factorization of $G(z)$ generates a DT right-standard factorization of $G^\top(z)$, $G^{-1}(z)$ (if $G(z)$ is non-singular), $G(1/z)$, *e.g.*, in the first case we have

$$G^\top(z) = B^\top(z)\Delta(z)A^\top(z). \tag{4.7}$$

The following Lemma ensures that a DT left-standard factorization of a rational matrix $G(z)$ always exists.

Lemma 4.1.3 *Any rational matrix $G(z) \in \mathbb{R}(z)^{m\times n}$ of normal rank* $\mathrm{rk}(G) = r \leq \min\{m,n\}$ *admits a DT left-standard factorization.*

PROOF. By the Smith-McMillan Theorem 2.3.1, we can write $G(z) = C(z)D(z)F(z)$, where $C(z) \in \mathbb{R}[z]^{m\times r}$, $F(z) \in \mathbb{R}[z]^{r\times n}$ are unimodular polynomial matrices and $D(z) \in \mathbb{R}(z)^{r\times r}$ is diagonal and canonic of the form

$$D(z) = \mathrm{diag}\left[\frac{\varepsilon_1(z)}{\psi_1(z)}, \frac{\varepsilon_2(z)}{\psi_2(z)}, \dots, \frac{\varepsilon_r(z)}{\psi_r(z)}\right], \tag{4.8}$$

where $\varepsilon_k(z)$, $\psi_k(z)$, $k = 1,\dots,r$, are relatively prime monic polynomials such that $\varepsilon_k(z) \mid \varepsilon_{k+1}(z)$, $\psi_{k+1}(z) \mid \psi_k(z)$, $k = 1,\dots,r-1$. We factor $\varepsilon_i(z)$ and $\psi_i(z)$, $i = 1,\dots,r$, in $D(z)$ into the product of three polynomials: the first without zeros in $\{|z| \leq 1,\ z \in \mathbb{C}\}$, the second without zeros in $\{|z| \neq 1,\ z \in \mathbb{C}\}$ and the third without zeros in $\{|z| \geq 1,\ z \in \mathbb{C}\}$. Thus, it is possible to write

$$D(z) = D_-(z)\Delta(z)D_+(z), \tag{4.9}$$

where $D_-(z)$ and its inverse are analytic in $\{|z| \leq 1,\ z \in \mathbb{C}\}$, $\Delta(z)$ and its inverse in $\{|z| \neq 1,\ z \in \mathbb{C}\}$ and $D_+(z)$ and its inverse in $\{|z| \geq 1,\ z \in \mathbb{C}\}$. Finally, by choosing $A(z) := C(z)D_-(z)$ and $B(z) := D_+(z)F(z)$, we have that

$$G(z) = A(z)\Delta(z)B(z) \tag{4.10}$$

is a DT left-standard factorization of $G(z)$. ■

Left-standard factorizations are not unique. Indeed, any two decompositions are connected as follows.

Theorem 4.1.1 *Let $G(z) \in \mathbb{R}(z)^{m\times n}$ be a rational matrix of normal rank* $\mathrm{rk}(G) = r \le \min\{m,n\}$ *and let $A(z)\Delta(z)B(z)$, $A_1(z)\Delta_1(z)B_1(z)$ be two DT left-standard factorizations of $G(z)$. Then,*

$$A_1(z) = A(z)M^{-1}(z), \quad B_1(z) = N(z)B(z), \tag{4.11}$$

where $M(z) \in \mathbb{R}[z,z^{-1}]^{r\times r}$ and $N(z) \in \mathbb{R}[z,z^{-1}]^{r\times r}$ are two L-unimodular polynomial matrices such that

$$M(z)\Delta(z)N^{-1}(z) = \Delta_1(z). \tag{4.12}$$

PROOF. By assumption,

$$G(z) = A(z)\Delta(z)B(z) = A_1(z)\Delta_1(z)B_1(z), \tag{4.13}$$

which, in turn, implies

$$\Delta_1^{-1}(z)A_1^{-L}(z)A(z)\Delta(z) = B_1(z)B^{-R}(z). \tag{4.14}$$

By Definition 4.1.1 of DT left-standard factorization, the right-hand side of (4.14) is analytic in $\{|z| \ge 1,\ z \in \mathbb{C}\}$, while the left-hand side in $\{|z| < 1,\ z \in \mathbb{C} \setminus \{0\}\}$. Therefore $B_1(z)B^{-R}(z)$ is analytic in $\mathbb{C} \setminus \{0\}$. The inverse of $B_1(z)B^{-R}(z)$ satisfies

$$[B_1(z)B^{-R}(z)]^{-1} = \Delta^{-1}(z)[A_1^{-L}(z)A(z)]^{-1}\Delta_1(z) \tag{4.15}$$

and is also analytic in $\mathbb{C} \setminus \{0\}$. Thus, $N(z) := B_1(z)B^{-R}(z)$ must be a $r \times r$ L-unimodular polynomial matrix (see Lemma 4.1.1). Similarly, $M(z) := A_1^{-L}(z)A(z)$ is a $r \times r$ L-unimodular polynomial matrix. Finally, from (4.13), we have

$$M(z)\Delta(z)N^{-1}(z) = \Delta_1(z). \tag{4.16}$$

and the proof is concluded. ■

We notice also that it is straightforward to derive a DT right-standard counterpart of Lemma 4.1.3 and Theorem 4.1.1.

Let $\Phi(z) \in \mathbb{R}(z)^{n\times n}$ be a para-Hermitian matrix of normal rank $\mathrm{rk}(\Phi) = r \le n$ and let $\Phi(z) = A(z)\Delta(z)B(z)$ be a DT left-standard factorization of $\Phi(z)$. We have that

$$\Phi(z) = \Phi^*(z) = B^*(z)\Delta^*(z)A^*(z) \tag{4.17}$$

is also a DT left-standard factorization of $\Phi(z)$. In particular, $\Delta^*(z)$ is equal to $\Delta(z)$, except for multiplication of suitable monomials of the form $\pm z^{k_i}$ in its diagonal elements, *i.e.*,

$$\Delta^*(z) = \Sigma(z)\Delta(z), \tag{4.18}$$

where

$$\Sigma(z) = \operatorname{diag}\left[e_1(z), e_2(z), \ldots, e_r(z)\right] \tag{4.19}$$

and $e_i(z) = \pm z^{k_i},\ k_i \in \mathbb{Z},\ i = 1, \ldots, r$. By invoking Theorem 4.1.1, we can write

$$A^*(z) = N(z)B(z), \tag{4.20}$$

$$B^*(z) = A(z)M^{-1}(z), \tag{4.21}$$

where $N(z),\ M(z) \in \mathbb{R}[z, z^{-1}]^{r\times r}$ are L-unimodular polynomial matrices.

The following Lemma provides a further characterization of a para-Hermitian matrix when it is non-negative definite on the unit circle.

Lemma 4.1.4 *Let $\Phi(z) \in \mathbb{R}(z)^{n\times n}$ be a DT para-Hermitian matrix of normal rank $\operatorname{rk}(\Phi) = r \leq n$ which is positive semi-definite on the unit circle,* i.e., $\mathbf{x}^\top \Phi(e^{j\omega})\mathbf{x} \geq 0,\ \forall \mathbf{x} \in \mathbb{R}^n$ *and* $\forall \omega \in [0, 2\pi)$ *such that* $z = e^{j\omega}$ *is not a pole of* $\Phi(z)$*. Let*

$$\Phi(z) = C(z)D(z)F(z) \tag{4.22}$$

with $D(z) \in \mathbb{R}(z)^{r\times r}$ be the Smith-McMillan canonical form of $\Phi(z)$. Then, the zeros and poles on the unit circle of the diagonal elements of $D(z)$ are of even multiplicity.

PROOF. We can assume that the numerators and denominators of all entries in $\Phi(z)$ are relatively prime polynomials. Let

$$\alpha_1 = e^{j\omega_1},\ \alpha_2 = e^{j\omega_2}, \ldots,\ \alpha_t = e^{j\omega_t}, \tag{4.23}$$

the zeros/poles on the unit circle of $\Phi(z)$ and let

$$\nu_i^{(1)}, \nu_i^{(2)}, \ldots, \nu_i^{(r)},\ \ (\nu_i^{(1)} \leq \nu_i^{(2)} \leq \cdots \leq \nu_i^{(r)}), \tag{4.24}$$

the structural indices of $\Phi(z)$ at α_i, $i=1,\dots,t$, *i.e.*, the valuations at α_i of the diagonal terms of $D(z)$ (see §2.3.3). Since $\Phi(z)$ is non-negative on the unit circle, it is easy to verify that the zeros and poles on the unit circle of the principal minors of $\Phi(z)$ must be of even multiplicity. Now, by setting $\mathbb{T}=\{|z|=1,\ z\in\mathbb{C}\}$, we are in position of applying Corollary 2.3.1. By considering the minors of order $\ell=1$, it follows that

$$\nu_i^{(1)} \text{ is even } \forall i=1,2,\dots,t. \tag{4.25}$$

Now, by considering the minors of order $\ell=2$ in Corollary 2.3.1, it follows that

$$\nu_i^{(1)}+\nu_i^{(2)} \text{ is even } \forall i=1,2,\dots,t. \tag{4.26}$$

Since $\nu_i^{(1)}$ is even, then also $\nu_i^{(2)}$ must be even for all $i=1,2,\dots,t$. By iterating the argument, we conclude that every zero/pole on the unit circle of the diagonal elements of $D(z)$ is of even multiplicity. ■

Let $\Phi(z)\in\mathbb{R}(z)^{n\times n}$ be a para-Hermitian matrix of normal rank $\mathrm{rk}(\Phi)=r\le n$ and let $D(z)\in\mathbb{R}(z)^{r\times r}$ be its Smith-McMillan canonical form. We have

$$\Phi(z)=C(z)D(z)F(z)=F^*(z)D^*(z)C^*(z)=\Phi^*(z), \tag{4.27}$$

and, similarly to a previous argument,

$$D^*(z)=\Sigma'(z)D(z), \tag{4.28}$$

where, in this case, $\Sigma'(z)$ has the form

$$\Sigma'(z)=\mathrm{diag}\left[e_1'(z),e_2'(z),\dots,e_r'(z)\right] \tag{4.29}$$

with $e_i'(z)=\alpha_i z^{k_i}$, $\alpha_i\in\mathbb{R}\setminus\{0\}$, $k_i\in\mathbb{Z}$, $i=1,\dots,r$. Moreover, since by (4.28) any zero/pole at $\alpha\neq 0$ in the diagonal terms of $D(z)$ is accompanied by a zero/pole at $1/\alpha$, we can always write $D(z)$ in the form

$$D(z)=\Sigma_1(z)\Lambda^*(z)\Delta(z)\Lambda(z), \tag{4.30}$$

where $\Lambda(z)\in\mathbb{R}(z)^{r\times r}$ is diagonal, canonic and analytic with its inverse in $\{|z|\ge 1,\ z\in\mathbb{C}\}$, $\Sigma_1(z)\Lambda^*(z)\in\mathbb{R}(z)^{r\times r}$ is diagonal and analytic with its inverse in $\{|z|\le 1,\ z\in\mathbb{C}\}$ and $\Delta(z)\in\mathbb{R}(z)^{r\times r}$ is diagonal, canonic and analytic with its inverse in $\{|z| \neq 1,\ z \in \mathbb{C}\}$. Consequently, $\Lambda(z)$ possesses the same structural indices at $z = 0$ of $D(z)$. By exploiting Lemma 4.1.4, $\Delta(z)$ can be written as

$$\Delta(z) = \Theta^2(z) = \Sigma_2(z)\Theta^*(z)\Theta(z), \tag{4.31}$$

with $\Theta(z) \in \mathbb{R}(z)^{r\times r}$ diagonal, canonic and analytic together with its inverse in $\{|z| \neq 1,\ z \in \mathbb{C}\}$. Finally, we can rearrange $D(z)$ in the form

$$D(z) = \Sigma_3(z)\Lambda^*(z)\Theta^*(z)\Theta(z)\Lambda(z), \tag{4.32}$$

where $\Sigma_2(z)$ has the form (4.19), while $\Sigma_1(z)$ and $\Sigma_3(z) := \Sigma_1(z)\Sigma_2(z)$ possess the form (4.29).

We report below another result which will be useful in the proof of the main Theorem of the next section.

Lemma 4.1.5 *Let $\Psi(z) = \Psi^*(z) \in \mathbb{R}[z, z^{-1}]^{r\times r}$ be a DT para-Hermitian L-unimodular matrix which is positive definite on the unit circle. Let $\Psi^{\mathrm{hc}} \in \mathbb{R}^{r\times r}$ denote the highest-column-degree coefficient matrix of $\Psi(z)$. Then, Ψ^{hc} is non-singular if and only if $\Psi(z)$ is a constant matrix.*

PROOF. If $\Psi(z)$ is a constant matrix then $\Psi^{\mathrm{hc}} = \Psi(z)$ is non-singular, by definition of L-unimodular matrix.

Conversely, assume that Ψ^{hc} is non-singular. Let us denote by $K_i \in \mathbb{Z}$, $i = 1, \ldots, r$, the maximum-degree of the i-th column of $\Psi(z)$ and by $k_i \in \mathbb{Z}$, $i = 1, \ldots, r$, the minimum-degree of the i-th row of $\Psi(z)$.[2] Since $\Psi(z) = \Psi^*(z)$, we have that $\det \Psi(z)$ is a non-zero real constant and

$$K_i = -k_i, \quad i = 1, \ldots, r. \tag{4.33}$$

Moreover, since $\Psi(z)$ is positive definite on the unit circle, the diagonal elements of $\Psi(z)$ cannot be equal to zero and, therefore, $K_i \geq 0$, $i = 1, \ldots, r$. Actually, the non-singularity of Ψ^{hc} implies that

$$K_i = 0, \quad i = 1, \ldots, r, \tag{4.34}$$

otherwise the maximum-degree of $\det \Psi(z)$ would be strictly positive (*cf.* equation (2.36) of §2.2.4). By (4.34), all the entries of $\Psi(z)$ must have maximum-degree

[2]Recall the Definition 2.2.11 of maximum- and minimum-degree of a L-polynomial vector.

less than or equal to zero. But, by (4.33), $k_i = -K_i$ for all $i = 1, \ldots, r$, and so (4.34) also implies that all the entries of $\Psi(z)$ must have minimum-degree greater than or equal to zero. We conclude that

$$\max\deg\,[\Psi(z)]_{ij} = \min\deg\,[\Psi(z)]_{ij} = 0, \quad i, j = 1, \ldots, r, \tag{4.35}$$

and, therefore, $\Psi(z)$ must be a constant matrix. ■

4.2 | The main theorem

In this section, we present the main result of the Chapter: the discrete-time counterpart of Youla's Theorem 3.2.1.

Theorem 4.2.1 *Let $\Phi(z) = \Phi^*(z) \in \mathbb{R}(z)^{n\times n}$ be a DT para-Hermitian matrix of normal rank* $\mathrm{rk}(\Phi) = r \leq n$ *which is positive semi-definite on the unit circle $\{e^{j\omega},\ \omega \in [0, 2\pi)\}$. Then, there exists a matrix $W(z) \in \mathbb{R}(z)^{r\times n}$ such that*

(i) $\Phi(z) = W^*(z)W(z)$.

(ii) *$W(z)$ and its (right) inverse $W^{-R}(z)$ are both analytic in $\{|z| > 1,\ z \in \mathbb{C}\}$ without poles at infinity.*

(iii) *$W(z)$ is unique up to within a constant, orthogonal $r \times r$ matrix multiplier on the left,* i.e., *if $W_1(z)$ also satisfies points* (i) *and* (ii)*, then $W_1(z) = TW(z)$ where $T \in \mathbb{R}^{r\times r}$ is a constant orthogonal matrix.*

(iv) *Any factorization of the form $\Phi(z) = L^*(z)L(z)$ in which $L(z) \in \mathbb{R}(z)^{r\times n}$ is analytic in $\{|z| > 1,\ z \in \mathbb{C}\}$ without poles at infinity, is given by $L(z) = V(z)W(z)$, $V(z) \in \mathbb{R}(z)^{r\times r}$ being an arbitrary, DT regular para-unitary matrix without poles at infinity.*

(v) *If $\Phi(z)$ is analytic on the unit circle, then $W(z)$ is analytic in a region $\{|z| > \tau,\ \tau < 1,\ z \in \mathbb{C}\}$ without poles at infinity.*

(vi) *If $\Phi(z)$ is analytic on the unit circle and the rank of $\Phi(z)$ is constant on the unit circle, then $W(z)$ and its (right) inverse $W^{-R}(z)$ are both analytic in a region $\{|z| > \tau_1,\ \tau_1 < 1,\ z \in \mathbb{C}\}$ without poles at infinity.*

PROOF. We first prove statement (iii). Let $W(z)$ and $W_1(z)$ be two matrices satisfying (i) and (ii). Then,

$$W^*(z)W(z) = W_1^*(z)W_1(z). \tag{4.36}$$

The latter equation implies

$$V^*(z)V(z) = I_r, \tag{4.37}$$

where $V(z) := W_1(z)W^{-R}(z)$ is analytic in $\{|z| > 1,\ z \in \mathbb{C}\}$ without poles at infinity. Thus, $V(z) \in \mathbb{R}(z)^{r\times r}$ is a regular para-unitary matrix without poles at infinity. Moreover, from (4.36), we also have

$$V(z) = W_1^{-R*}(z)W^*(z) \tag{4.38}$$

and so $V^*(z) = V^{-1}(z) = W(z)W_1^{-R}(z)$ is regular without poles at infinity. In view of Corollary 4.1.1, we conclude that $V(z)$ is a constant orthogonal matrix $T \in \mathbb{R}^{r\times r}$, $T^\top T = TT^\top = I_r$.

Consider now statement (iv) and let $\Phi(z) = L^*(z)L(z)$ where $L(z) \in \mathbb{R}(z)^{n\times r}$ is analytic in $\{|z| > 1,\ z \in \mathbb{C}\}$ without poles at infinity. We underline the fact that we do not suppose that $L^{-R}(z)$ is analytic in $\{|z| > 1,\ z \in \mathbb{C}\}$ and/or $L^{-R}(z)$ does not have poles at infinity. In this case, we can write

$$L^*(z)L(z) = W^*(z)W(z). \tag{4.39}$$

The latter equation implies

$$V^*(z)V(z) = I_r, \tag{4.40}$$

where $V(z) := L(z)W^{-R}(z)$ and $W(z) \in \mathbb{R}(z)^{r\times n}$ is a rational matrix satisfying (i) and (ii). Since $L(z)$ and $W(z)^{-R}$ are both analytic in $\{|z| > 1,\ z \in \mathbb{C}\}$ without poles at infinity, then $V(z) \in \mathbb{R}(z)^{r\times r}$ is a regular para-unitary matrix without poles at infinity and we finished.

Now, we provide a constructive proof of statements (i) and (ii), which represent the core of the Theorem. The procedure is divided in four steps.

Step 1. Reduce $\Phi(z)$ to the Smith-McMillan canonical form. By using the same standard procedure described in Theorem 3.2.1, we arrive at

$$\Phi(z) = C(z)D(z)F(z), \tag{4.41}$$

where $C(z) \in \mathbb{R}[z]^{n\times r}$, $F(z) \in \mathbb{R}[z]^{r\times n}$ are unimodular polynomial matrices and $D(z) \in \mathbb{R}(z)^{r\times r}$ is diagonal and canonic.

Step 2. According to (4.32), we can write $D(z)$ in the form

$$D(z) = \Sigma(z)\Lambda^*(z)\tilde{\Delta}(z)\Lambda(z), \tag{4.42}$$

where:

1. $\Lambda(z) \in \mathbb{R}(z)^{r\times r}$ is diagonal, canonic and analytic together with $\Lambda^{-1}(z)$ in $\{|z| \geq 1,\ z \in \mathbb{C}\}$ and possesses the same structural indices at $z = 0$ of $D(z)$;
2. $\tilde{\Delta}(z) := \Theta^*(z)\Theta(z) = \tilde{\Delta}^*(z)$, where $\Theta(z) \in \mathbb{R}(z)^{r\times r}$ is diagonal, canonic and analytic together with $\Theta^{-1}(z)$ in $\{|z| \neq 1,\ z \in \mathbb{C}\}$;
3. $\Sigma(z) \in \mathbb{R}(z)^{r\times r}$ is diagonal of the form

$$\Sigma(z) = \operatorname{diag}\left[e_1(z), e_2(z), \ldots, e_r(z)\right], \tag{4.43}$$

   where $e_i(z) = \alpha_i z^{k_i},\ \alpha_i \in \mathbb{R}\setminus\{0\},\ k_i \in \mathbb{Z},\ i = 1, \ldots, r$.

Let

$$A(z) := C(z)\Sigma(z)\Lambda^*(z), \tag{4.44}$$
$$B(z) := \Lambda(z)F(z). \tag{4.45}$$

We have that

$$\Phi(z) = A(z)\tilde{\Delta}(z)B(z) \tag{4.46}$$

is a DT left-standard factorization of $\Phi(z)$.

Step 3. Let

$$I(z) := B^{-R}(z)\Theta^{-1}(z). \tag{4.47}$$

By equation (4.20), we have $A^*(z) = N(z)B(z)$ and, therefore,

$$\begin{aligned} I^*(z)\Phi(z)I(z) &= I^*(z)\Phi^*(z)I(z) \\ &= \Theta^{-*}(z)B^{-R*}(z)B^*(z)\tilde{\Delta}^*(z)N(z)B(z)B^{-R}(z)\Theta^{-1}(z) \\ &= \Theta^{-*}(z)\Theta^*(z)\Theta(z)N(z)\Theta^{-1}(z) \\ &= \Theta(z)N(z)\Theta^{-1}(z), \end{aligned} \tag{4.48}$$

where $N(z) = A^*(z)B^{-R}(z) \in \mathbb{R}[z,z^{-1}]^{r\times r}$ is a L-unimodular polynomial matrix. Let us define

$$\Psi(z) := \Theta(z)N(z)\Theta^{-1}(z). \tag{4.49}$$

By (4.48), $\Psi(z)$ is a para-Hermitian matrix non-negative definite on the unit circle. Actually a good deal more is true. We notice that $A(z)\tilde{\Delta}(z)B(z)$ and $B^*(z)\tilde{\Delta}(z)A^*(z)$ are two DT left-standard factorizations of $\Phi(z)$. Hence, by replacing $\Delta_1(z)$ with $\tilde{\Delta}(z) = \tilde{\Delta}^*(z)$ in (4.12), we obtain

$$\tilde{\Delta}(z)N(z)\tilde{\Delta}^{-1}(z) = M(z), \tag{4.50}$$

where $M(z) \in \mathbb{R}[z,z^{-1}]$ is L-unimodular. Since $\tilde{\Delta}(z) = \Theta^*(z)\Theta(z)$ is diagonal and $\Theta(z) := \mathrm{diag}[\theta_1(z),\dots,\theta_r(z)]$ canonic, equation (4.50) implies that $[N(z)]_{ij}$ is divisible by the L-polynomial $[\tilde{\Delta}(z)]_{jj}/[\tilde{\Delta}(z)]_{ii},\ j \geq i$. But

$$\begin{aligned}[\tilde{\Delta}(z)]_{ii} &= \theta_i^*(z)\theta_i(z)\\ &= \theta_i(1/z)\theta_i(z)\\ &= \pm z^{k_i}\theta_i^2(z),\end{aligned} \tag{4.51}$$

where $k_i \in \mathbb{Z},\ i = 1,\dots,r$. So, $[N(z)]_{ij}$ must be divisible by the polynomial

$$f_{ij}^2(z) := \frac{\theta_j^2(z)}{\theta_i^2(z)}, \quad j \geq i, \tag{4.52}$$

and, *a fortiori*, by

$$f_{ij}(z) = \frac{\theta_j(z)}{\theta_i(z)}, \quad j \geq i. \tag{4.53}$$

This suffices to establish that $\Psi(z)$ is L-polynomial. Finally, since by (4.48) $\det\Psi(z)$ is a non-zero positive constant, $\Psi(z)$ is a para-Hermitian L-unimodular polynomial matrix which is positive definite on the unit circle. The problem is now reduced to that of finding a factorization of $\Psi(z)$ of the form

$$\Psi(z) = P^*(z)P(z), \tag{4.54}$$

where $P(z) \in \mathbb{R}[z]^{r\times r}$ is a unimodular polynomial matrix. After this is achieved, the desired factorization for $\Phi(z)$ is obtained as $\Phi(z) = W^*(z)W(z)$ with

$$\begin{aligned}W(z) &:= P(z)\Theta(z)B(z)\\ &= P(z)\Theta(z)\Lambda(z)F(z)\\ &= P(z)D_+(z)F(z),\end{aligned} \tag{4.55}$$

where we have defined $D_+(z) := \Theta(z)\Lambda(z)$. Indeed, by straightforward algebra,

$$\begin{aligned} W^*(z)W(z) &= B^*(z)\Theta^*(z)P^*(z)P(z)\Theta(z)B(z) \\ &= B^*(z)\tilde{\Delta}(z)N(z)B(z) \\ &= B^*(z)\tilde{\Delta}(z)A^*(z) \\ &= \Phi^*(z) \\ &= \Phi(z). \end{aligned} \tag{4.56}$$

Step 4. We illustrate an algorithm which provides a factorization of a para-Hermitian L-unimodular polynomial matrix $\Psi(z) = \Psi^*(z) \in \mathbb{R}[z, z^{-1}]^{r\times r}$ positive definite on the unit circle into the product $P^*(z)P(z)$, where $P(z)$ is a unimodular polynomial matrix. This algorithm can be regarded as the discrete-time counterpart of the technique described in the step 4 of Theorem 3.2.1.

The algorithm consists of the following two steps. First of all, we define $\Psi_1(z) := \Psi(z)$ and denote by $h \in \mathbb{N}$ the loop counter of the algorithm, which is initially set to $h := 1$.

I. Let $K_i \in \mathbb{Z}$, $i = 1, \ldots, r$, be the maximum-degree of the i-th column of $\Psi_h(z)$ and $k_i \in \mathbb{Z}$, $i = 1, \ldots, r$, be the minimum-degree of the i-th row of $\Psi_h(z)$. Consider the highest-column-degree coefficient matrix of $\Psi_h(z)$, denoted by Ψ_h^{hc}, and the lowest-row-degree coefficient matrix of $\Psi_h(z)$, denoted by Ψ_h^{lr} (recall the definitions given in §2.2.4). As noticed in the proof of Lemma 4.1.5, the positive nature of $\Psi_h(z)$ implies that $K_i \geq 0$ for all $i = 1, \ldots, r$. Moreover, the para-Hermitianity of $\Psi_h(z)$ implies that

$$\Psi_h^{\mathrm{hc}} = (\Psi_h^{\mathrm{lr}})^\top, \tag{4.57}$$

and, therefore, $K_i = -k_i$ for all $i = 1, \ldots, r$.

By Lemma 4.1.5, it follows that Ψ_h^{hc} is non-singular if and only if $\Psi_h(z)$ is a constant matrix. If $\Psi_h(z)$ is a constant matrix, we skip to step II. If this is not the case, we calculate a non-zero vector $\mathbf{v}_h = [v_1\ v_2\ \ldots\ v_r]^\top \in \mathbb{R}^r$ such that $\Psi_h^{\mathrm{hc}}\mathbf{v}_h = \mathbf{0}$. Let us define the *active index set*

$$\mathscr{I} := \{i \,:\, v_i \neq 0\} \tag{4.58}$$

and the *highest maximum-degree active index set*, $\mathscr{M} \subset \mathscr{I}$,

$$\mathscr{M} := \{i \in \mathscr{I} \,:\, K_i \geq K_j,\ \forall j \in \mathscr{I}\}. \tag{4.59}$$

We pick an index $p \in \mathscr{M}$. Then, we define the matrix

$$\Omega_h^{-1}(z) := \begin{array}{c} \text{column } p \\ \begin{bmatrix} 1 & \cdots & 0 & \frac{v_1}{v_p} z^{K_p - K_1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots & & & 0 \\ \vdots & & 1 & \frac{v_{p-1}}{v_p} z^{K_p - K_{p-1}} & & & \vdots \\ \vdots & & & 1 & & & \vdots \\ \vdots & & & \frac{v_{p+1}}{v_p} z^{K_p - K_{p+1}} & 1 & & \vdots \\ 0 & & & \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{v_r}{v_p} z^{K_p - K_r} & 0 & \cdots & 1 \end{bmatrix} \end{array}. \tag{4.60}$$

Notice that the entry at (i, p) of $\Omega_h^{-1}(z)$ has the form

$$\frac{v_i}{v_p} z^{K_p - K_i} = \alpha_i z^{\delta_i}, \quad i = 1, \ldots, r, \tag{4.61}$$

with $\alpha_i := v_i / v_p \in \mathbb{R}$ and $\delta_i := K_p - K_i \geq 0$. In fact, if $K_i > K_p$, then $v_i = 0$ and so $\alpha_i = 0$. By (4.60),

$$\det \Omega_h^{-1}(z) = 1 \tag{4.62}$$

and, therefore, $\Omega_h^{-1}(z) \in \mathbb{R}[z]^{r \times r}$ is a unimodular polynomial matrix. By operating the transformation

$$\Psi_{h+1}(z) := \Omega_h^{-*}(z) \Psi_h(z) \Omega_h^{-1}(z), \tag{4.63}$$

we obtain a new positive matrix $\Psi_{h+1}(z)$ with the same determinant of $\Psi_h(z)$. Furthermore, the maximum-degree of the p-th column of $\Psi_{h+1}(z)$ is lower than K_p, while the maximum-degree of the i-th column, $i \neq p$, is not greater than K_i.

This fact needs a detailed explanation. If we post-multiply $\Psi_h(z)$ by $\Omega_h^{-1}(z)$, we obtain a matrix of the form

$$\begin{aligned} \Psi_h'(z) &:= \Psi_h(z) \Omega_h^{-1}(z) \\ &= \left[\begin{array}{c|c|c} [\Psi_h(z)]_{1:r,1:p-1} & \psi_h(z) & [\Psi_h(z)]_{1:r,p+1:r} \end{array} \right], \end{aligned} \tag{4.64}$$

where all the L-polynomials in the p-th column vector

$$\psi_h(z) = [\Psi_h(z)]_{1:r,p:p} + \sum_{i \neq p} \alpha_i z^{\delta_i} [\Psi_h(z)]_{1:r,i:i} \tag{4.65}$$

have maximum-degree lower than K_p, since $\mathbf{v}_h \in \ker(\Psi_h^{\mathrm{hc}})$, and minimum-degree which satisfies

$$\min\deg\,[\psi_h(z)]_i \geq k_i = -K_i, \quad i = 1, \ldots, r, \tag{4.66}$$

since in (4.65) $\delta_i \geq 0$, for all i such that $\alpha_i \neq 0$ (*cf.* equation (4.61)).

Now, by pre-multiplying $\Psi'_h(z)$ by $\Omega_h^{-*}(z)$, the resulting matrix $\Psi_{h+1}(z)$ can be written in the form

$$\begin{aligned}\Psi_{h+1}(z) &= \Omega_h^{-*}(z)\Psi_h(z)\Omega_h^{-1}(z)\\ &= \left[\begin{array}{c|c|c} [\Psi_h(z)]_{1:p-1,1:p-1} & \psi_{h+1}(z) & [\Psi_h(z)]_{1:p-1,p+1:r} \\ \hline \psi_{h+1}^\top(1/z) & \psi'_{h+1}(z) & {\psi''_{h+1}}^\top(1/z) \\ \hline [\Psi_h(z)]_{p+1:r,1:p-1} & \psi''_{h+1}(z) & [\Psi_h(z)]_{p+1:r,p+1:r} \end{array}\right],\end{aligned} \tag{4.67}$$

where the p-th column vector $[\psi_{h+1}(z)|\psi'_{h+1}(z)|\psi''_{h+1}(z)]^\top$ differs from $\psi_h(z)$ only for the value of the p-th entry $\psi'_{h+1}(z)$. Moreover, the maximum-degree of $\psi'_{h+1}(z)$ cannot increase after the operation is performed, since

$$\psi'_{h+1}(z) = [\psi_h(z)]_p + \sum_{i \neq p} \alpha_i z^{-\delta_i} [\psi_h(z)]_i, \tag{4.68}$$

and, by (4.61), $\delta_i \geq 0$, for all i such that $\alpha_i \neq 0$.

We conclude that all the L-polynomials in the p-th column of $\Psi_{h+1}(z)$ have maximum-degree lower than K_p, while, by (4.66), the maximum-degree of all the other columns does not increase. We notice also that, since $\Psi_{h+1}(z) = \Psi_{h+1}^*(z)$, all the L-polynomials in the p-th row of $\Psi_{h+1}(z)$ have minimum-degree greater than $k_p = -K_p$, while the minimum-degree of the all other rows does not decrease.

Eventually, we update the value of the loop counter h by setting $h := h+1$ and return to step I.

II. Since $\Psi_h \in \mathbb{R}^{r\times r}$ is positive definite, we can always factorize it into the product $\Psi_h = C^\top C$ where $C \in \mathbb{R}^{r\times r}$, by using standard techniques such as the

Cholesky decomposition (see Golub and Van Loan [1996, Ch.4]). Finally, we have constructed a polynomial unimodular matrix

$$P(z) = C\Omega_{h-1}(z)\Omega_{h-2}(z)\cdots\Omega_1(z). \tag{4.69}$$

such that $\Psi(z) = P^*(z)P(z)$, with $P(z) \in \mathbb{R}[z]^{r\times r}$ unimodular.

It is worthwhile noticing that the iterative procedure of step I is always brought to an end (after a maximum of $K_1 + \cdots + K_p$ iterations) since at the h-th iteration the maximum-degree of a column of $\Psi_h(z)$ is reduced at least by one, while the maximum-degree of all the other columns does not increase.

To complete the proof of points (i) and (ii), we notice that, by construction, the rational matrix $W(z)$, as defined in (4.55), and its (right) inverse are analytic in $\{|z| > 1,\ z \in \mathbb{C}\}$. Moreover, we recall that $D_+(z)$ and $D(z)$ have the same zero-pole structure at $z = 0$ (*i.e.*, they have the same structural indices at $z = 0$). Now, suppose, by contradiction, that $W(z)$ has a pole at $z = \infty$. Then $W^*(z)$ has a pole at $z = 0$. But, since $\Phi(z) = W^*(z)W(z)$, it follows that

$$\begin{aligned} W^*(z) &= \Phi(z)W^{-R}(z) \\ &= C(z)D(z)F(z)F^{-R}(z)D_+^{-1}(z)P^{-1}(z) \\ &= C(z)D(z)D_+^{-1}(z)P^{-1}(z) \\ &= C(z)D_-(z)P^{-1}(z), \end{aligned} \tag{4.70}$$

where $D_-(z) := D(z)D_+^{-1}(z)$ has no pole at $z = 0$. Since $P^{-1}(z)$ and $C(z)$ are polynomial unimodular matrices, in view of (4.70), also $W^*(z)$ has no pole at $z = 0$. Hence the contradiction. We conclude that $W(z)$ has no pole at infinity. Finally, by following a similar argument, it can be verified that also $W^{-R}(z)$ has no pole at infinity.

Now consider statement (v). If $\Phi(z)$ is analytic on the unit circle, then $\Theta(z)$ does not possess any finite pole. This, in turn, implies that $D_+(z) = \Theta(z)\Lambda(z)$ is analytic in $\{|z| > \tau,\ \tau < 1,\ z \in \mathbb{C}\}$. Thus, $W(z)$, as defined in (4.55), is also analytic in the same region and, by a previous argument, it does not possess any pole at infinity (and the latter fact is also true for $W^{-R}(z)$). It is worth noticing that the above defined region is completely determined by the poles of $\Lambda(z)$.

The additional assumption that the rank of $\Phi(z)$ is constant on the unit circle implies that $\Theta(z)$ does not possess any finite zero. Thus, $\Theta(z) = I_r$ and, by (4.55),

$$W^{-R}(z) = F^{-R}(z)\Lambda^{-1}(z)P^{-1}(z) \tag{4.71}$$

is analytic in a region $\{|z| > \bar{\tau},\ \bar{\tau} < 1,\ z \in \mathbb{C}\}$. Hence, $W(z)$ and its (right) inverse $W^{-R}(z)$ are both analytic in a region $\{|z| > \tau_1,\ \tau_1 < 1,\ z \in \mathbb{C}\}$ without poles at infinity. Here, $\tau_1 := \max\{\tau, \bar{\tau}\}$ is completely determined by the zeros and poles of $\Lambda(z)$. This concludes the proof of point (vi), and, in turn, the proof of the entire Theorem. ■

As in Chapter 3, to conclude this section, we present two straightforward Corollaries of Theorem 4.2.1.

Corollary 4.2.1 *Let $L(z) \in \mathbb{R}(z)^{m\times n}$, then $\Phi(z) = L^*(z)L(z)$ if and only if*

$$L(z) = V(z)\left[\begin{array}{c} I_r \\ \hline \mathbf{0}_{m-r,r} \end{array}\right] W(z), \tag{4.72}$$

where $V(z) \in \mathbb{R}(z)^{m\times m}$ is an arbitrary DT para-unitary matrix and $r = \mathrm{rk}(\Phi) \leq m$.

PROOF. By repeating an argument used in points (iii) and (iv) of Theorem 4.2.1, we have that $L(z) = U(z)W(z)$, with $U(z) \in \mathbb{R}(z)^{m\times r}$ a rational matrix satisfying $U^*(z)U(z) = I_r$. If we choose $V(z) \in \mathbb{R}(z)^{m\times m}$ to be any para-unitary matrix with $U(z)$ incorporated into its first r columns, *i.e.*,

$$U(z) = V(z)\left[\begin{array}{c} I_r \\ \hline \mathbf{0}_{m-r,r} \end{array}\right], \tag{4.73}$$

we finished. ■

Corollary 4.2.2 *If $\Phi(z)$ is L-polynomial then $W(z)$ is polynomial in z^{-1} and $W^*(z)$ is polynomial (in z).*

PROOF. If $\Phi(z)$ is L-polynomial, then the only finite poles it may possess are located at $z = 0$. Since $W(z)$ does not have poles at infinity, $W(z)$ must be polynomial in z^{-1}. The latter fact, in turn, implies that $W^*(z)$ must be a polynomial matrix. ■

In Fig.4.1 is shown a schematic representation of the procedure used in the constructive proof of Theorem 4.2.1.

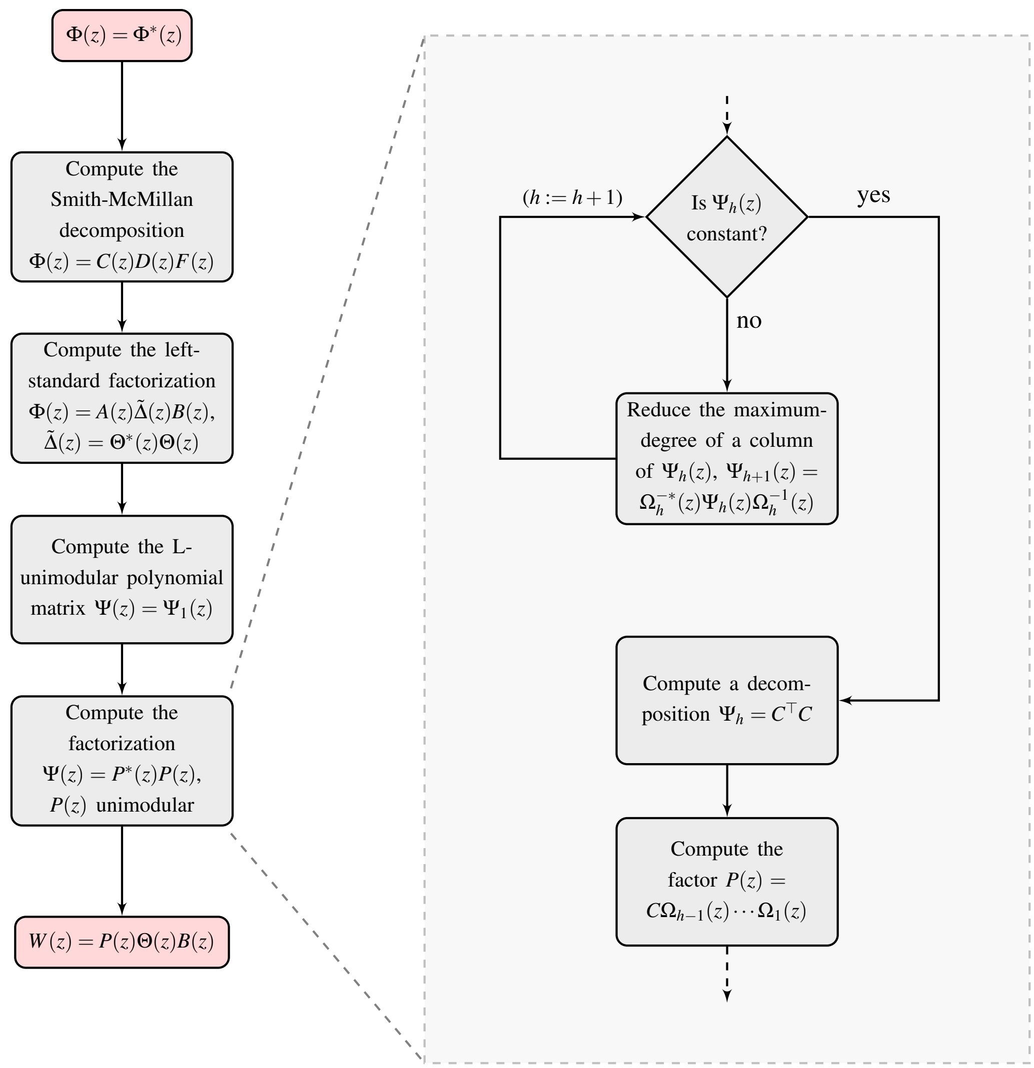

Figure 4.1: *Schematic representation of the procedure used for the construction of the factorization* $\Phi(z) = W^*(z)W(z)$.

4.3 | Some additional remarks

As in section §3.3 of the previous Chapter, here, we want to underline two important facts regarding the properties of the factorization approach discussed before.

The first remark concerns the *stochastic minimality* of the factor $W(z)$. More precisely, the McMillan degree of $W(z)$ satisfies $\delta_M(W) = \frac{1}{2}\delta_M(\Phi)$ (recall the definition given in §2.3.2 and its extension of §2.3.3) which is the minimum attainable value. This fact can be formally stated as follows.

Theorem 4.3.1 *Let $\Phi(z) \in \mathbb{R}(z)^{n\times n}$ be a DT para-Hermitian matrix nonnegative on the unit circle and let $r = \mathrm{rk}(\Phi)$. Consider the factorization $\Phi(z) = W^*(z)W(z)$ where $W(z) \in \mathbb{R}(z)^{r\times n}$ is computed by following the procedure described in the previous section. Then, the McMillan degree of $W(z)$ satisfies*

$$\delta_M(W) = \frac{1}{2}\delta_M(\Phi). \tag{4.74}$$

PROOF. We notice that, by equation (4.55), we have

$$W(z) = P(z)D_+(z)F(z), \tag{4.75}$$

where $P(z) \in \mathbb{R}[z]^{r\times r}$, $F(z) \in \mathbb{R}[z]^{r\times n}$ are unimodular polynomial matrices and $D_+(z) \in \mathbb{R}(z)^{r\times r}$ is diagonal, canonic, regular and satisfies

$$D(z) = \Sigma(z)D_+^*(z)D_+(z), \tag{4.76}$$

where $D(z) \in \mathbb{R}(z)^{r\times r}$ is the Smith-McMillan canonical form of $\Phi(z)$ and $\Sigma(z) \in \mathbb{R}(z)^{r\times r}$ is a diagonal matrix with elements $\alpha_i z^{k_i}$, $\alpha_i \neq 0$, $k_i \in \mathbb{Z}$, on its diagonal. Moreover, we recall that $W(z)$ and $\Phi(z)$ have the same structural indices at $z = 0$. Let $p_1, \dots, p_h$ be the non-zero finite poles of $\Phi(z)$. By (4.75) and (4.76), it follows that[3]

$$\delta(\Phi; p_i) = \begin{cases} \delta(W; p_i) & \text{if } |p_i| < 1, \\ 2\delta(W; p_i) & \text{if } |p_i| = 1, \\ \delta(W; 1/p_i) & \text{if } |p_i| > 1. \end{cases} \tag{4.77}$$

[3] For the definition of degree of a pole of a rational matrix we refer to §2.3.3, Definition 2.3.4.

Moreover, if $p_i \neq 0$ is a pole of $\Phi(z)$ then also $1/p_i$ is a pole of $\Phi(z)$ and if $p_i \neq 0$ is not a pole of $\Phi(z)$ then neither p_i nor $1/p_i$ are poles of $W(z)$. Thus, we have

$$\begin{aligned}\sum_{i=1}^{h} \delta(\Phi; p_i) &= \sum_{i\,:\,|p_i|<1} \delta(W; p_i) + \sum_{i\,:\,|p_i|>1} \delta(W; 1/p_i) + \sum_{i\,:\,|p_i|=1} 2\delta(W; p_i) \\ &= 2 \sum_{i\,:\,|p_i|\leq 1} \delta(W; p_i) \end{aligned} \tag{4.78}$$

By equation (2.57) of section §2.3.3, the McMillan degree of a rational matrix equals the sum of the degrees of all its poles, the pole at infinity included. Since $\Phi(z) = \Phi^*(z)$, if $\Phi(z)$ has a pole at $z = \infty$, then $\Phi(z)$ has also a pole at $z = 0$. In particular, from the Definition 2.3.4 of polar degree, we have

$$\delta(\Phi; \infty) = \delta(\Phi; 0). \tag{4.79}$$

Now, since $W(z)$ and $\Phi(z)$ have the same structural indices at $z = 0$ and $W(z)$ has no pole at $z = \infty$, it follows that

$$\delta(\Phi; 0) = \delta(W; 0) \quad \text{and} \quad \delta(W; \infty) = 0. \tag{4.80}$$

Therefore, by equations (4.78) and (4.80),

$$\begin{aligned}\delta_M(\Phi) &= \sum_{i=1}^{h} \delta(\Phi; p_i) + 2\delta(\Phi; 0) \\ &= 2 \sum_{i\,:\,|p_i|\leq 1} \delta(W; p_i) + 2\delta(W; 0) = 2\delta_M(W), \end{aligned} \tag{4.81}$$

and we finished. ■

The second remarkable feature of the proposed factorization method is that it can be easily modified in order to change the region of analyticity of $W(z)$ and of its (right) inverse. First of all, let us introduce some nomenclature. We say that a region of the extended complex plane $\mathscr{A}$ is *discrete-time skew-symmetric* (for short, skew-symmetric), if it satisfies

$$\mathscr{A} \cup \mathscr{A}^* = \overline{\mathbb{C}} \setminus \{|z| = 1, z \in \mathbb{C}\} \quad \text{and} \quad \mathscr{A} \cap \mathscr{A}^* = \varnothing, \tag{4.82}$$

where $\mathscr{A}^* := \{z \in \overline{\mathbb{C}} \,:\, 1/z \in \mathscr{A}\}$ and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denotes the extended complex plane.

Suppose that we want to compute a factor $W_{\Omega,\Omega'}(z)$ analytic in a skew-symmetric region Ω of the extended complex plane (see Fig.4.2) and with (right) inverse $W^{-R}_{\Omega,\Omega'}(z)$ analytic in another skew-symmetric region Ω' of the extended complex plane.[4] To obtain such a factor, we only need to rearrange the diagonal matrix $D(z)$ of equation (4.42) in a different form, namely,

$$D(z) = \Sigma(z)\Lambda^*_{\Omega,\Omega'}(z)\Theta^*(z)\Theta(z)\Lambda_{\Omega,\Omega'}(z), \tag{4.83}$$

where $\Lambda_{\Omega,\Omega'}(z)$ is diagonal analytic in Ω (with the possible exception of the point at infinity) and its inverse $\Lambda^{-1}_{\Omega,\Omega'}(z)$ is analytic in Ω' (with the possible exception of the point at infinity) and apply the same procedure described in the previous section. The proof of this fact follows, *mutata mutandis*, from that one adopted in §3.3 for the continuous-time case.

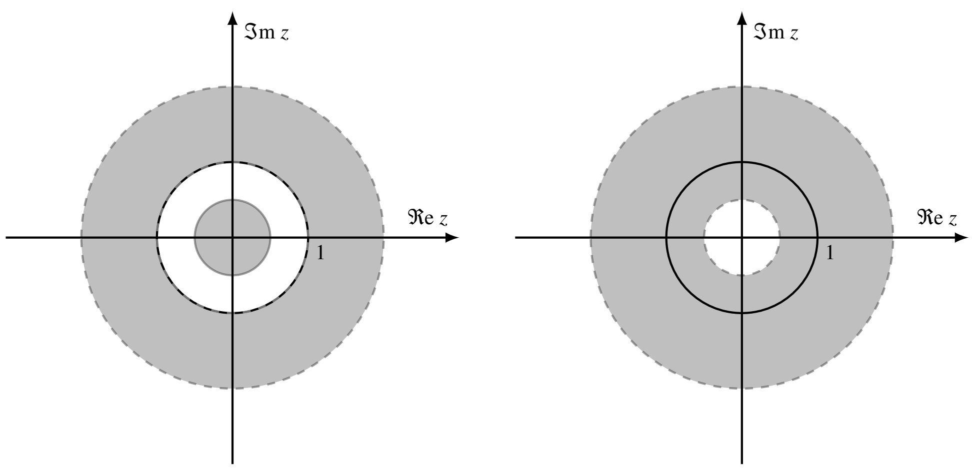

Figure 4.2: *Example of an admissible choice of the region Ω (filled in gray, on the left) and a not admissible choice of Ω (filled in gray, on the right). Here, dashed gray lines denote open boundaries, while solid gray lines denote closed boundaries.*

Similarly to what done in Chapter 3, also in this case we can compute some “extremal” factors of particular relevance:

[4]In the following, we use the notation $A_{\Omega,\Omega'}$ to emphasize the dependence of matrix A on the choice of the regions Ω and Ω'.

- the regular factor having no poles at infinity with regular (right) inverse having no poles at infinity, called the *minimum-phase regular factor*, $\underline{W}_-$ (whose existence was proved in Theorem 3.2.1);
- the anti-regular factor with regular (right) inverse having no poles at infinity, called the *minimum-phase anti-regular factor*, $\overline{W}_-$;
- the regular factor having no poles at infinity with anti-regular (right) inverse, called the *maximum-phase regular factor*, $\underline{W}_+$;
- the anti-regular factor with anti-regular (right) inverse, called the *maximum-phase anti-regular factor*, $\overline{W}_+$.

The four "extremal" factors are essentially unique (*i.e.*, unique up to pre-multiplication by orthogonal matrices) and are related each other in a manner similar to that shown in Fig.3.3.

4.4 | An illustrative example

In this final section, we present a simple example of application of the method described in Theorem 4.2.1.

Let us consider the following 2×2 rational spectrum

$$\Phi(z) = \begin{bmatrix} \frac{1}{(z-1/2)(z^{-1}-1/2)} + 1 & \frac{z^{-1}}{(z-1/2)(z^{-1}-1/2)} \\ \frac{z}{(z-1/2)(z^{-1}-1/2)} & \frac{1}{(z-1/2)(z^{-1}-1/2)} + 1 \end{bmatrix}. \tag{4.84}$$

It may be easily verified that $\Phi(z)$ is para-Hermitian and positive definite on the unit circle.

Step 1. The Smith-McMillan canonical form of $\Phi(z)$ is given by

$$D(z) = \begin{bmatrix} \frac{1}{(z-1/2)(z-2)} & 0 \\ 0 & \frac{(z-\alpha)(z-1/\alpha)(z-\beta)(z-1/\beta)}{(z-1/2)(z-2)} \end{bmatrix}, \tag{4.85}$$

where $\alpha = \frac{3-\sqrt{5}}{2} \simeq 0.3820$ and $\beta = \alpha^2 = \frac{7-3\sqrt{5}}{2} \simeq 0.1459$. The Smith-McMillan decomposition of the spectrum $\Phi(z)$ is given by

$$\Phi(z) = C(z)D(z)F(z), \tag{4.86}$$

where $C(z)$ and $F(z)$ are (non-unique) unimodular polynomial matrices, *e.g.*,

$$C(z) = \begin{bmatrix} -2 & 0 \\ z^2 - 5z + 1 & \frac{1}{2} \end{bmatrix}, \quad F(z) = \begin{bmatrix} -\frac{1}{2}z^2 + \frac{5}{2} - \frac{1}{2} & 1 \\ 1 & 0 \end{bmatrix}. \tag{4.87}$$

Step 2. In this case, the regular rational matrix $\Lambda(z)$ is given by

$$\Lambda(z) = \begin{bmatrix} \frac{1}{z-\frac{1}{2}} & 0 \\ 0 & \frac{(z-\alpha)(z-\beta)}{z-\frac{1}{2}} \end{bmatrix}. \tag{4.88}$$

Since $\Phi(z)$ does not possess any zero and pole on the unit circle, $\Theta(z) = I_2$ and, therefore, $D_+(z) := \Lambda(z)\Theta(z) = \Lambda(z)$. The matrices $A(z)$ and $B(z)$, defined in (4.44) and (5.25), respectively, take the form

$$A(z) = \begin{bmatrix} -\frac{2}{z-2} & 0 \\ \frac{z^2-5z+1}{z-2} & \frac{(z-1/\alpha)(z-1/\beta)}{2(z-2)} \end{bmatrix}, \tag{4.89}$$

$$B(z) = \begin{bmatrix} \frac{-\frac{1}{2}z^2+\frac{5}{2}z-\frac{1}{2}}{z-\frac{1}{2}} & \frac{1}{z-\frac{1}{2}} \\ \frac{(z-\alpha)(z-\beta)}{z-\frac{1}{2}} & 0 \end{bmatrix}. \tag{4.90}$$

Step 3. We have that $\Psi(z) = \Theta(z)N(z)\Theta^{-1}(z) = N(z)$ has the form

$$\begin{aligned} \Psi(z) &= N(z) = A^*(z)B^{-1}(z) \\ &= \begin{bmatrix} -\frac{1}{2}z + \frac{5}{2} - \frac{1}{2}z^{-1} & -\frac{1}{4}z^{-1}\left(z - \frac{1}{\alpha}\right)\left(z - \frac{1}{\beta}\right) \\ -\frac{1}{4}z\left(z^{-1} - \frac{1}{\alpha}\right)\left(z^{-1} - \frac{1}{\beta}\right) & \frac{1}{4\alpha\beta}\left(-\frac{1}{2}z + \frac{5}{2} - \frac{1}{2}z^{-1}\right) \end{bmatrix}. \end{aligned} \tag{4.91}$$

The L-polynomial matrix $\Psi(z)$ is para-Hermitian, L-unimodular with determinant a positive constant, *viz.* $\det\Psi(z) = \frac{1}{4\alpha\beta} = \frac{9+4\sqrt{5}}{4} \simeq 4.486$.

Step 4. The highest-column-degree coefficient matrix of $\Psi(z) = \Psi_1(z)$ is

$$\Psi_1^{\text{hc}} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4\alpha\beta} & -\frac{1}{8\alpha\beta} \end{bmatrix}. \tag{4.92}$$

Since Ψ_1^{hc} is singular, we calculate a vector $\mathbf{v}_1 \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ such that $\Psi_1^{\mathrm{hc}}\mathbf{v}_1 = \mathbf{0}$. In our case such a vector is given, for example, by $\mathbf{v}_1 = [1,\ -2]^\top$. In order to reduce the maximum-degree of the second column of $\Psi_1(z)$, we construct the matrix $\Omega_1^{-1}(z)$ whose structure is described in (4.60)

$$\Omega_1^{-1}(z) = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}. \tag{4.93}$$

By applying to $\Psi_1(z)$ the transformation induced by $\Omega_1^{-1}(z)$, we obtain a new para-Hermitian L-unimodular matrix $\Psi_2(z)$ with lower maximum-degree in the second column (and higher minimum-degree in the second row),

$$\begin{aligned} \Psi_2(z) &= \Omega_1^{-*}(z)\Psi_1(z)\Omega_1^{-1}(z) \\ &= \begin{bmatrix} -\frac{1}{2}z + \frac{5}{2} - \frac{1}{2}z^{-1} & -\frac{5\alpha^2-\alpha-1}{4\alpha^3} - \frac{1-\alpha^3}{4\alpha^3}z^{-1} \\ -\frac{5\alpha^2-\alpha-1}{4\alpha^3} - \frac{1-\alpha^3}{4\alpha^3}z & \frac{5}{8}\frac{\alpha^3+1}{\alpha^3} - \frac{1}{4}\frac{\alpha+1}{\alpha^2} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2}z + \frac{5}{2} - \frac{1}{2}z^{-1} & \frac{\sqrt{5}}{2} - (2+\sqrt{5})z^{-1} \\ \frac{\sqrt{5}}{2} - (2+\sqrt{5})z & 5+2\sqrt{5} \end{bmatrix} \end{aligned} \tag{4.94}$$

The highest-column-degree coefficient matrix of $\Psi_2(z)$ is

$$\Psi_2^{\mathrm{hc}} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{5}}{2} \\ -(2+\sqrt{5}) & 5+2\sqrt{5} \end{bmatrix}. \tag{4.95}$$

Since Ψ_2^{hc} is singular, there exists a vector $\mathbf{v}_2 \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ such that $\Psi_2^{\mathrm{hc}}\mathbf{v}_2 = \mathbf{0}$. In our case such a vector is given, for example, by $\mathbf{v}_2 = [1,\ \sqrt{5}]^\top$. As before, we construct a suitable unimodular matrix $\Omega_2^{-1}(z)$ in order to reduce the maximum-degree of the first column,

$$\Omega_2^{-1}(z) = \begin{bmatrix} 1 & 0 \\ \frac{1}{\sqrt{5}}z & 1 \end{bmatrix}. \tag{4.96}$$

Eventually, by applying to $\Psi_2(z)$ the transformation induced by $\Omega_2^{-1}(z)$, we obtain a positive definite constant matrix Ψ_3,

$$\Psi_3 = \Omega_2^{-*}(z)\Psi_2(z)\Omega_2^{-1}(z) = \begin{bmatrix} \frac{3}{2} - \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{2} \\ \frac{\sqrt{5}}{2} & 5+2\sqrt{5} \end{bmatrix} \tag{4.97}$$

The problem is now reduced to that of finding a matrix C such that $\Psi_3 = C^\top C$. This may be accomplished by using the Cholesky decomposition of Ψ_3,

$$\Psi_3 = C^\top C = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \tag{4.98}$$

with $a := \sqrt{\frac{3}{2} - \frac{2\sqrt{5}}{5}} \simeq 0.7782$, $b := \sqrt{\frac{75}{58} + \frac{10\sqrt{5}}{29}} \simeq 1.4367$ and $c := \sqrt{\frac{365}{58} + \frac{68\sqrt{5}}{29}} \simeq 2.7216$. To sum up, we have found a factorization $\Phi(z) = P^*(z)P(z)$, where $P(z)$ is unimodular of the form

$$P(z) = C\Omega_2(z)\Omega_1(z) = \begin{bmatrix} a - \frac{b}{\sqrt{5}}z & \frac{1}{2}a + b - \frac{b}{2\sqrt{5}}z \\ -\frac{c}{\sqrt{5}}z & c - \frac{c}{2\sqrt{5}}z \end{bmatrix}. \tag{4.99}$$

The desired factorization of $\Phi(z)$ is given by $\Phi(z) = W^*(z)W(z)$, where

$$\begin{aligned} W(z) = P(z)\Theta(z)B(z) &= \begin{bmatrix} \frac{\left(\frac{-15+6\sqrt{5}}{5}b+\sqrt{5}a\right)z+(4-2\sqrt{5})a+(9-4\sqrt{5})b}{z-\frac{1}{2}} & \frac{a-\frac{b}{\sqrt{5}}z}{z-\frac{1}{2}} \\ \frac{\frac{-15+6\sqrt{5}}{5}cz+(9-4\sqrt{5})c}{z-\frac{1}{2}} & \frac{-\frac{c}{\sqrt{5}}z}{z-\frac{1}{2}} \end{bmatrix} \\ &\simeq \begin{bmatrix} \frac{1.2851z-0.2874}{z-0.5} & \frac{-0.6425z+0.7782}{z-0.5} \\ \frac{-0.8620z+0.1517}{z-0.5} & \frac{-1.2171z}{z-0.5} \end{bmatrix}. \end{aligned} \tag{4.100}$$

Finally, one may check that $W(z)$ is analytic together with its inverse in $\{|z| \geq 1,\ z \in \mathbb{C}\}$ without any zero/pole at infinity.

5. Discrete-time J-spectral factorization

In this Chapter, we will present a J-spectral generalization (see Problem 1.4 of Chapter 1) of the factorization Theorem 4.2.1 of Chapter 4. Notably, in this general case, there are some critical issues which are characteristic of the J-spectral factorization and do not occur in the standard (positive semi-definite) spectral factorization. These issues concern, in particular, the existence of a J-spectral factor and its (stochastic) minimality. As in Chapter 4, in what follows, we will deal with the real rational case.

A remark on notation. We let $J_{p,q}$ denote a constant block matrix of the form

$$J_{p,q} := \left[\begin{array}{c|c} I_p & \mathbf{0}_{p,q} \\ \hline \mathbf{0}_{q,p} & -I_q \end{array}\right] = J_{p,q}^* = J_{p,q}^{-1}. \tag{5.1}$$

Moreover, in order to lighten the notation, we set $\mathbb{C}_0 := \mathbb{C} \setminus \{0\}$. Other conventions/notations remain unchanged from Chapters 0, 2 and 4.

5.1 | Preliminary results

The first result concerns the structure of a special class of $J_{p,q}$-para-unitary-matrices, *i.e.*, the class of matrices $G(z) \in \mathbb{R}(z)^{n\times n}$ satisfying

$$G^*(z)J_{p,q}G(z) = G(z)J_{p,q}G^*(z) = J_{p,q}, \quad p+q = n. \tag{5.2}$$

Lemma 5.1.1 *The only DT regular $J_{p,q}$-para-unitary matrices with regular inverse are DT $J_{p,q}$-para-unitary matrices analytic together with their inverse in $\mathbb{C}_0 \setminus \{e^{j\omega},\ \omega \in [0, 2\pi)\}$.*

PROOF. Let $G(z) \in \mathbb{R}(z)^{n\times n}$ be a DT regular $J_{p,q}$-para-unitary matrix. Since $J_{p,q}$ is a constant matrix, the analyticity of the inverse of $G(z)$,

$$G^{-1}(z) = J_{p,q}G^*(z)J_{p,q}, \tag{5.3}$$

in $\{|z|>1,\ z\in\mathbb{C}\}$ implies that of $G(1/z)$ in the same region, and therefore that of $G(z)$ in $\{|z|<1,\ z\in\mathbb{C}_0\}$. Thus, $G(z)$ is analytic together with its inverse $G^{-1}(z)$ in the region $\mathbb{C}_0\setminus\{e^{j\omega},\ \omega\in[0,2\pi)\}$. ■

From the previous Lemma, it follows that a DT regular $J_{p,q}$-para-unitary matrix with regular inverse may possess poles on the unit circle. In fact, for instance,

$$G(z)=\begin{bmatrix}\frac{2z}{1-z} & \frac{1+z}{1-z}\\ \frac{1+z}{1-z} & \frac{2}{1-z}\end{bmatrix} \tag{5.4}$$

is $J_{1,1}$-para-unitary and has a pole and a zero at $z=1$. It is worthwhile noticing that the latter fact cannot happen if $J_{p,q}=I_n$, *i.e.*, if $G(z)$ is simply a DT para-unitary matrix. Indeed, in this case, $G(z)$ is analytic together with its inverse in $\mathbb{C}_0$, as stated in Lemma 4.1.2.

Consider a para-Hermitian rational matrix $\Phi(z)\in\mathbb{R}(z)^{n\times n}$ of normal rank $\mathrm{rk}(\Phi)=r\le n$ and let $D(z)\in\mathbb{R}(z)^{r\times r}$ be its Smith-McMillan canonical form. In Chapter 4, we have seen that, if $\Phi(z)$ is positive semi-definite on the unit circle, then all the zeros/poles on the unit circle of the diagonal terms of $D(z)$ are of even multiplicity (Lemma 4.1.4). This fact, in general, does not hold if $\Phi(z)$ has constant inertia upon the unit circle, *i.e.*, if $\mathrm{in}(\Phi(e^{j\omega}))=(\nu_p,\nu_0,\nu_n)$ for all $\omega\in[0,2\pi)$ ($z=e^{j\omega}$ not a zero/pole of $\Phi(z)$).

Indeed, consider the following trivial counterexample

$$\Phi_{\mathrm{ex}}(z)=\begin{bmatrix}0 & \frac{z-1}{z+1}\\ -\frac{z-1}{z+1} & 0\end{bmatrix}. \tag{5.5}$$

The eigenvalues of $\Phi_{\mathrm{ex}}(e^{j\omega})$ are given by

$$\lambda_1(\omega):=\frac{\sin\omega}{1+\cos\omega}=\tan\frac{\omega}{2}, \tag{5.6}$$

$$\lambda_2(\omega):=-\frac{\sin\omega}{1+\cos\omega}=-\tan\frac{\omega}{2}. \tag{5.7}$$

Hence, $\mathrm{in}(\Phi_{\mathrm{ex}}(e^{j\omega}))=(1,0,1)$ for all $\omega\in[0,2\pi)$, $\omega\neq k\pi$, $k=0,1$. However, the Smith-McMillan canonical form of $\Phi_{\mathrm{ex}}(z)$ is

$$D_{\mathrm{ex}}(z)=\begin{bmatrix}\frac{z-1}{z+1} & 0\\ 0 & \frac{z-1}{z+1}\end{bmatrix} \tag{5.8}$$

and its diagonal terms have a zero at $z = 1$ and a pole at $z = -1$ of odd multiplicity.

In general, this pathological behaviour occurs when some of the eigenvalues of $\Phi(e^{j\omega})$ flip their sign (from positive to negative or *vice versa*) while $\mathrm{in}(\Phi(e^{j\omega}))$ remains constant. For the J-spectrum (5.5) of the previous example, the latter fact is graphically shown in Fig.5.1.

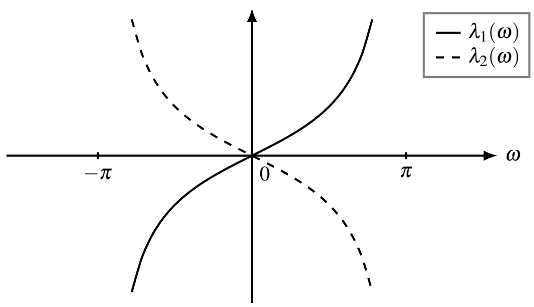

Figure 5.1: *Eigenvalues of* $\Phi_{\mathrm{ex}}(e^{j\omega})$, $\omega \in [-\pi, \pi)$, *in (5.5).*

For the sake of simplicity, in what follows, we make the additional hypothesis that the zeros and poles on the unit circle of the diagonal elements of the Smith-McMillan form of $\Phi(z)$ have even multiplicity. The latter is a sufficient, but not necessary, condition for the existence of a J-spectral factor. In fact, with reference to the previous example, we find that a J-spectral factorization actually exists, *e.g.*,

$$\Phi_{\mathrm{ex}}(s) = W_{\mathrm{ex}}^*(s) J_{1,1} W_{\mathrm{ex}}(s), \quad W_{\mathrm{ex}}(s) := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \frac{z-1}{z+1} \\ 1 & -\frac{z-1}{z+1} \end{bmatrix}. \tag{5.9}$$

The following two Lemmata will be useful in the proof of the main Theorem of the next section.

Lemma 5.1.2 *Let* $\Psi(z) = \Psi^*(z) \in \mathbb{R}[z, z^{-1}]^{r \times r}$ *be a non-constant L-unimodular matrix of normal rank* $\mathrm{rk}(\Psi) = r$. *Let* $\Psi^{\mathrm{hc}} \in \mathbb{R}^{r \times r}$ *denote the highest-column-degree coefficient matrix of* $\Psi(z)$. *If* Ψ^{hc} *is non-singular, then* $\Psi(z)$ *has at least a zero entry on its diagonal.*

PROOF. First of all, since $\Psi(z)$ is L-unimodular and para-Hermitian we have that $\det \Psi(z) = \det \Psi^*(z)$ is a non-zero real constant. Let us denote by $K_i \in \mathbb{Z}$, $i =$

$1,\ldots,r$, the maximum-degree of the i-th column of $\Psi(z)$.[1] Now, suppose that $\Psi(z)$ has no diagonal element equal to zero. Therefore, it follows that

$$K_i \geq 0, \quad i = 1,\ldots,r. \tag{5.10}$$

Moreover, by assumption, $\Psi(z)$ is non-constant, thus there exists at least one index $j \in \{1,\ldots,r\}$ such that $K_j > 0$. But then Ψ^{hc} is singular, because otherwise, by equation (2.36) of §2.2.4, it would be $\max\deg(\det\Psi(z)) > 0$. ■

Lemma 5.1.3 *Consider a para-Hermitian L-unimodular block matrix of the form*

$$\Xi(z) := \left[\begin{array}{c|c} \mathbf{0}_p & I_p \\ \hline I_p & \Xi_{22}(z) \end{array}\right] \in \mathbb{R}[z,z^{-1}]^{2p\times 2p}. \tag{5.11}$$

There exists a factorization

$$\Xi(z) = U^*(z) C U(z), \tag{5.12}$$

with $U(z) \in \mathbb{R}[z,z^{-1}]^{2p\times 2p}$ L-unimodular and $C \in \mathbb{R}^{2p\times 2p}$ constant symmetric.

PROOF. The proof is constructive. By direct computation, we have

$$\begin{aligned} \Xi(z) &= \left[\begin{array}{c|c} \mathbf{0}_p & I_p \\ \hline I_p & \Xi_{22}(z) \end{array}\right] \\ &= \left[\begin{array}{c|c} I_p & \frac{1}{2}\Xi_{22}(z) \\ \hline \mathbf{0}_p & I_p \end{array}\right]^* \left[\begin{array}{c|c} \mathbf{0}_p & I_p \\ \hline I_p & \mathbf{0}_p \end{array}\right] \left[\begin{array}{c|c} I_p & \frac{1}{2}\Xi_{22}(z) \\ \hline \mathbf{0}_p & I_p \end{array}\right]. \end{aligned} \tag{5.13}$$

Hence, by defining

$$U(z) := \left[\begin{array}{c|c} I_p & \frac{1}{2}\Xi_{22}(z) \\ \hline \mathbf{0}_p & I_p \end{array}\right], \quad C := \left[\begin{array}{c|c} \mathbf{0}_p & I_p \\ \hline I_p & \mathbf{0}_p \end{array}\right], \tag{5.14}$$

we have constructed a factorization of the form (5.12). ■

[1] Recall the Definition 2.2.11 of maximum-degree of a L-polynomial vector.

5.2 | The main theorem

The following Theorem may be considered the "J-spectral" generalization of Theorem 4.2.1 and is the main result of this Chapter.

Theorem 5.2.1 *Let $\Phi(z) = \Phi^*(z) \in \mathbb{R}(z)^{n\times n}$ be a DT para-Hermitian matrix of normal rank* $\mathrm{rk}(\Phi) = r \leq n$ *which has constant inertia on the unit circle,* i.e., $\mathrm{in}(\Phi(e^{j\omega})) = (\nu_p, \nu_0, \nu_n)$ *for all $\omega \in [0, 2\pi)$ such that $z = e^{j\omega}$ is not a zero/pole of $\Phi(z)$. Furthermore, assume that the zeros and poles on the unit circle of the diagonal entries of the Smith-McMillan canonical form of $\Phi(z)$ have even multiplicity. Then, there exists a rational matrix $W(z) \in \mathbb{R}(z)^{r\times n}$ such that*

(i) $\Phi(z) = W^*(z) J_{\nu_p,\nu_n} W(z)$.

(ii) *$W(z)$ and its (right) inverse $W^{-R}(z)$ are both analytic in $\{|z| > 1,\ z \in \mathbb{C}\}$.*

(iii) *If $W_1(z) \in \mathbb{R}(z)^{r\times n}$ also satisfies points (i) and (ii), then $W_1(z) = T(z)W(z)$, with $T(z) \in \mathbb{R}(z)^{r\times r}$ being a DT J_{ν_p,ν_n}-para-unitary matrices analytic together with its inverse in $\mathbb{C}_0 \setminus \{e^{j\omega},\ \omega \in [0, 2\pi)\}$.*

(iv) *Any factorization of the form $\Phi(z) = L^*(z) J_{\nu_p,\nu_n} L(z)$ in which $L(z) \in \mathbb{R}(z)^{r\times n}$ is analytic in $\{|z| > 1,\ z \in \mathbb{C}\}$, is given by $L(z) = V(z)W(z)$, $V(z) \in \mathbb{R}(z)^{r\times r}$ being an arbitrary, DT regular J_{ν_p,ν_n}-para-unitary matrix.*

(v) *If $\Phi(z)$ is analytic on the unit circle, then $W(z)$ is analytic in a region $\{|z| > \tau,\ \tau < 1,\ z \in \mathbb{C}\}$.*

(vi) *If $\Phi(z)$ is analytic on the unit circle and the rank of $\Phi(z)$ is constant on the unit circle, then $W(z)$ and its (right) inverse $W^{-R}(z)$ are both analytic in a region $\{|z| > \tau_1,\ \tau_1 < 1,\ z \in \mathbb{C}\}$.*

PROOF. We address first statement (iii). Let $W(z)$ and $W_1(z)$ be two matrices satisfying (i) and (ii). Then,

$$W^*(z) J_{\nu_p,\nu_n} W(z) = W_1^*(z) J_{\nu_p,\nu_n} W_1(z). \tag{5.15}$$

By defining $T(z) := W_1(z)W^{-R}(z)$, in view of the previous identity, we have

$$T^*(z)J_{\nu_p,\nu_n}T(z) = J_{\nu_p,\nu_n}. \tag{5.16}$$

By points (i) and (ii), we observe that $T(z)$ must be analytic in $\{|z| > 1,\ z \in \mathbb{C}\}$. Hence, $T(z) \in \mathbb{R}(z)^{r\times r}$ is a regular J_{ν_p,ν_n}-para-unitary matrix. Equation (5.15) also yields

$$T(z) = J_{\nu_p,\nu_n}W_1^{-R*}(z)W^*(z)J_{\nu_p,\nu_n} \tag{5.17}$$

so that $T^{-1}(z) = J_{\nu_p,\nu_n}T^*(z)J_{\nu_p,\nu_n} = W(z)W_1^{-R}(s)$ is also regular. By applying Lemma 5.1.1, we conclude that $T(z)$ must be a J_{ν_p,ν_n}-para-unitary matrix analytic together with its inverse in $\mathbb{C}_0 \setminus \{e^{j\omega},\ \omega \in [0,2\pi)\}$.

Consider now assertion (iv). Let $\Phi(z) = L^*(z)J_{\nu_p,\nu_n}L(z)$ where $L(z) \in \mathbb{R}(z)^{n\times r}$ is analytic in $\{|z| > 1,\ z \in \mathbb{C}\}$. (In this case we do not suppose that $L^{-R}(z)$ is analytic in $\{|z| > 1,\ z \in \mathbb{C}\}$). We have

$$L^*(z)J_{\nu_p,\nu_n}L(z) = W^*(z)J_{\nu_p,\nu_n}W(z). \tag{5.18}$$

The latter equation implies

$$V^*(z)J_{\nu_p,\nu_n}V(z) = I_r, \tag{5.19}$$

where $V(z) := L(z)W^{-R}(z)$ and $W(z) \in \mathbb{R}(z)^{r\times n}$ is a rational matrix satisfying points (i) and (ii). Since $L(z)$ and $W^{-R}(z)$ are both analytic in $\{|z| > 1,\ z \in \mathbb{C}\}$, then, by (5.19), $V(z) \in \mathbb{R}(z)^{r\times r}$ is a regular J_{ν_p,ν_n}-para-unitary matrix and we are done.

The proof of statement (i) and (ii) is constructive and is split in four steps, as in the standard (positive semi-definite) case of Theorem 4.2.1. Steps 1–3 are essentially the same of the corresponding steps of Theorem 4.2.1. Thus, we will very briefly review these steps and we refer the reader to Theorem 4.2.1 for details. A more in-depth analysis will be devoted to step 4, which represents the main difference between the present Theorem and the aforementioned one.

Step 1. Reduce $\Phi(z)$ to the Smith-McMillan form

$$\Phi(z) = C(z)D(z)F(z), \tag{5.20}$$

$C(z) \in \mathbb{R}[z]^{n\times r}$, $F(z) \in \mathbb{R}[z]^{r\times n}$ being unimodular and $D(z) \in \mathbb{R}(z)^{r\times r}$ of the form

$$D(z) = \operatorname{diag}\left[\frac{\varepsilon_1(z)}{\psi_1(z)}, \frac{\varepsilon_2(z)}{\psi_2(z)}, \dots, \frac{\varepsilon_r(z)}{\psi_r(z)}\right], \tag{5.21}$$

where $\varepsilon_k(z)$, $\psi_k(z)$, $k = 1, \dots, r$, are relatively prime monic polynomials such that $\varepsilon_k(z) \mid \varepsilon_{k+1}(z)$, $\psi_{k+1}(z) \mid \psi_k(z)$, $k = 1, \dots, r-1$.

Step 2. Since, by assumption, the zeros and poles of the diagonal elements of $D(z)$ are of even multiplicity, we can rearrange $D(z)$ in the form

$$D(z) = \Sigma(z)\Lambda^*(z)\tilde{\Delta}(z)\Lambda(z), \tag{5.22}$$

where:

1. $\Lambda(z) \in \mathbb{R}(z)^{r\times r}$ is diagonal, canonic and analytic together with $\Lambda^{-1}(z)$ in $\{|z| \geq 1,\ z \in \mathbb{C}\}$ and possesses the same structural indices at $z = 0$ of $D(z)$;
2. $\tilde{\Delta}(z) := \Theta^*(z)\Theta(z) = \tilde{\Delta}^*(z)$, where $\Theta(z) \in \mathbb{R}(z)^{r\times r}$ is diagonal, canonic and analytic together with $\Theta^{-1}(z)$ in $\{|z| \neq 1,\ z \in \mathbb{C}\}$;
3. $\Sigma(z) \in \mathbb{R}(z)^{r\times r}$ is diagonal of the form

$$\Sigma(z) = \operatorname{diag}[e_1(z), e_2(z), \dots, e_r(z)], \tag{5.23}$$

where $e_i(z) = \alpha_i z^{k_i}$, $\alpha_i \in \mathbb{R} \setminus \{0\}$, $k_i \in \mathbb{Z}$, $i = 1, \dots, r$.

Let

$$A(z) := C(z)\Sigma(z)\Lambda^*(z), \tag{5.24}$$

$$B(z) := \Lambda(z)F(z). \tag{5.25}$$

The decomposition

$$\Phi(z) = A(z)\tilde{\Delta}(z)B(z) \tag{5.26}$$

is a DT left-standard factorization of $\Phi(z)$.

Step 3. Let us define

$$I(z) := B^{-R}(z)\Theta^{-1}(z). \tag{5.27}$$

We have

$$I^*(z)\Phi(z)I(z) = \Theta(z)N(z)\Theta^{-1}(z), \tag{5.28}$$

where $N(z) := A^*(z)B^{-R}(z) \in \mathbb{R}[z,z^{-1}]^{r\times r}$ is a L-unimodular matrix. Let

$$\Psi(z) := \Theta(z)N(z)\Theta^{-1}(z). \tag{5.29}$$

By following the same argument used in the step 3 of Theorem 4.2.1, we find that $\Psi(z)$ is a para-Hermitian L-unimodular matrix which has constant inertia upon the unit circle, *viz.* $\text{in}(\Psi(e^{j\omega})) = (\nu_p, 0, \nu_n)$ for all $\omega \in [0,2\pi)$. Hence, the problem is reduced to that of factorizing $\Psi(z)$ in the form

$$\Psi(z) = P^*(z)J_{\nu_p,\nu_n}P(z), \tag{5.30}$$

where $P(z) \in \mathbb{R}[z,z^{-1}]^{r\times r}$ is a L-unimodular matrix. After this is achieved, the desired factorization for $\Phi(z)$ is obtained as $\Phi(z) = W^*(z)J_{\nu_p,\nu_n}W(z)$ with

$$\begin{aligned} W(z) &:= P(z)\Theta(z)B(z) \\ &= P(z)\Theta(z)\Lambda(z)F(z) \\ &= P(z)D_+(z)F(z), \end{aligned} \tag{5.31}$$

with $D_+(z) := \Theta(z)\Lambda(z)$. Indeed,

$$\begin{aligned} W^*(z)J_{\nu_p,\nu_n}W(z) &= B^*(z)\Theta^*(z)P^*(z)J_{\nu_p,\nu_n}P(z)\Theta(z)B(z) \\ &= B^*(z)\tilde{\Delta}(z)N(z)B(z) \\ &= B^*(z)\tilde{\Delta}(z)A^*(z) \\ &= \Phi^*(z) \\ &= \Phi(z). \end{aligned} \tag{5.32}$$

Step 4. In what follows, we will describe an algorithm which provides a factorization of a L-unimodular matrix $\Psi(z) \in \mathbb{R}[z,z^{-1}]^{r\times r}$ with constant inertia on the unit circle into the product $P^*(z)J_{\nu_p,\nu_n}P(z)$, where $P(z)$ is a L-unimodular polynomial matrix. This algorithm can be viewed as an extension of the procedure described in the step 4 of Theorem 4.2.1. Notice that in this case $P(z)$ is required to be L-unimodular rather than unimodular.

The proposed algorithm consists of the successive application of three types of steps. We define $\Psi_1(z) := \Psi(z)$ and denote by $h \in \mathbb{N}$ the loop counter of the algorithm, which is initially set to $h := 1$.

I. Let $K_i \in \mathbb{Z}$, $i = 1, \ldots, r$, be the maximum-degree of the i-th column of $\Psi_h(z)$. Consider the highest-column-degree coefficient matrix of $\Psi_h(z)$, denoted by Ψ_h^{hc} (recall the definition given in §2.2.4).

At this point, if $\Psi_h(z)$ is a constant matrix, we skip to step II; else if Ψ_h^{hc} is non-singular, we go to step III. Otherwise, we proceed as in step 4-I of Theorem 4.2.1. We compute a non-zero vector $\mathbf{v}_h = [v_1\ v_2\ \ldots\ v_r]^\top \in \mathbb{R}^r$ such that $\Psi_h^{\text{hc}} \mathbf{v}_h = \mathbf{0}$. We define the *active index set*

$$\mathscr{I} := \{i \ : \ v_i \neq 0\} \tag{5.33}$$

and the *highest maximum-degree active index set*, $\mathscr{M} \subset \mathscr{I}$,

$$\mathscr{M} := \{i \in \mathscr{I} \ : \ K_i \geq K_j, \ \forall j \in \mathscr{I}\}. \tag{5.34}$$

We choose an index $p \in \mathscr{M}$. Hence, we calculate the unimodular matrix

$$\Omega_h^{-1}(z) := \begin{array}{c} \text{column } p \\ \begin{bmatrix} 1 & \cdots & 0 & \frac{v_1}{v_p} z^{K_p - K_1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots & & & 0 \\ \vdots & & 1 & \frac{v_{p-1}}{v_p} z^{K_p - K_{p-1}} & & & \vdots \\ \vdots & & & 1 & & & \vdots \\ \vdots & & & \frac{v_{p+1}}{v_p} z^{K_p - K_{p+1}} & 1 & & \vdots \\ 0 & & & \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{v_r}{v_p} z^{K_p - K_r} & 0 & \cdots & 1 \end{bmatrix} \end{array}. \tag{5.35}$$

By operating the transformation

$$\Psi_{h+1}(z) := \Omega_h^{-*}(z) \Psi_h(z) \Omega_h^{-1}(z), \tag{5.36}$$

we obtain a new matrix $\Psi_{h+1}(z)$ with the same inertia on the unit circle of $\Psi_h(z)$. Moreover, the maximum-degree of the p-th column of $\Psi_{h+1}(z)$ is lower than K_p, while the maximum-degree of the i-th column, $i \neq p$, is not greater than K_i. One can verify the latter assertion by following the same lines of step 4-I of Theorem 4.2.1.

Then, we set $h := h + 1$ and return to step I.

II. Since $\Psi_h \in \mathbb{R}^{r\times r}$ is symmetric and $\mathrm{in}(\Psi_h) = (\nu_p, 0, \nu_n)$, we can always factorize it into the product $\Psi_h = C^\top J_{\nu_p,\nu_n} C$ where $C \in \mathbb{R}^{r\times r}$, *e.g.*, by means of a Schur decomposition (see Golub and Van Loan [1996, Ch.7]).

III. In this case, by Lemma 5.1.2, $\Psi_h(z)$ has at least one diagonal entry equal to zero. Hence, by suitable symmetric row and column permutations, we can bring $\Psi_h(z)$ into the block matrix form

$$\Psi'_h(z) := T^\top \Psi_h(z) T = \left[\begin{array}{c|c} \mathbf{0}_p & \Psi'^*_{h,21}(z) \\ \hline \Psi'_{h,21}(z) & \Psi'_{h,22}(z) \end{array}\right] \tag{5.37}$$

Notice that $\Psi'_{h,21}(z)$ is tall (*i.e.*, has at least as many rows as columns), otherwise $\Psi'_h(z)$ would be singular. By pre-multiplying $\Psi'_{h,21}(z)$ by a suitable L-unimodular matrix $V(z) \in \mathbb{R}[z,z^{-1}]^{(r-p)\times(r-p)}$, we can bring $\Psi'_{h,21}(z)$ into its column Hermite form (see §2.2.4),

$$H_{21}(z) := V(z)\Psi'_{h,21}(z) = \left[\begin{array}{c} H'_{21}(z) \\ \hline \mathbf{0}_{r-2p,p} \end{array}\right], \quad H'_{21}(z) \in \mathbb{R}[z,z^{-1}]^{p\times p}. \tag{5.38}$$

Let us define

$$V'(z) := \left[\begin{array}{c|c} I_p & \mathbf{0}_{p,r-p} \\ \hline \mathbf{0}_{r-p,p} & V^*(z) \end{array}\right] \in \mathbb{R}[z,z^{-1}]^{r\times r}. \tag{5.39}$$

Hence, we have

$$\begin{aligned} \Psi''_h(z) &:= V'^*(z)\Psi'_h(z)V'(z) \\ &= \left[\begin{array}{c|c} I_p & \mathbf{0}_{p,r-p} \\ \hline \mathbf{0}_{r-p,p} & V(z) \end{array}\right] \Psi'_h(z) \left[\begin{array}{c|c} I_p & \mathbf{0}_{p,r-p} \\ \hline \mathbf{0}_{r-p,p} & V^*(z) \end{array}\right] \\ &= \left[\begin{array}{c|c} \mathbf{0}_p & H^*_{21}(z) \\ \hline H_{21}(z) & V(z)\Psi'_{h,22}(z)V^*(z) \end{array}\right] \\ &= \left[\begin{array}{c|c|c} \mathbf{0}_p & H'^*_{21}(z) & \mathbf{0}_{p,r-2p} \\ \hline H'_{21}(z) & \Psi''_{h,22}(z) & \Psi''_{h,23}(z) \\ \hline \mathbf{0}_{r-2p,p} & \Psi''^*_{h,23}(z) & \Psi''_{h,33}(z) \end{array}\right], \end{aligned} \tag{5.40}$$

where we have defined

$$V(z)\Psi'_{h,22}(z)V^*(z) =: \left[\begin{array}{c|c} \Psi''_{h,22}(z) & \Psi''_{h,23}(z) \\ \hline \Psi''^{*}_{h,23}(z) & \Psi''_{h,33}(z) \end{array}\right]. \tag{5.41}$$

By equation (5.40), we find that[2]

$$\det \Psi''_h(z) = (-1)^p \det H'_{21}(z) \det \Psi''_{h,33}(z) \det H'^{*}_{21}(z). \tag{5.42}$$

Since $\Psi''_h(z)$ is L-unimodular, the latter equation implies that $H'_{21}(z)$ and $\Psi''_{h,33}(z)$ are also L-unimodular. Hence $H'_{21}(z) = I_p$.

By operating another symmetric L-unimodular transformation, we can remove the off-diagonal blocks $\Psi''_{h,23}(z)$ and $\Psi''^{*}_{h,23}(z)$ from (5.40). Thus, we obtain

$$\begin{aligned} \Psi'''_h(z) &:= Q^*(z)\Psi''_h(z)Q(z) \\ &= \left[\begin{array}{c|c|c} \mathbf{0}_p & I_p & \mathbf{0}_{p,r-2p} \\ \hline I_p & \Psi''_{h,22}(z) & \mathbf{0}_{p,r-2p} \\ \hline \mathbf{0}_{r-2p,p} & \mathbf{0}_{r-2p,p} & \Psi''_{h,33}(z) \end{array}\right], \end{aligned} \tag{5.43}$$

with

$$Q(z) := \left[\begin{array}{c|c|c} I_p & \mathbf{0}_p & -\Psi''_{h,23}(z) \\ \hline \mathbf{0}_p & I_p & \mathbf{0}_{p,r-2p} \\ \hline \mathbf{0}_{r-2p,p} & \mathbf{0}_{r-2p,p} & I_{r-2p} \end{array}\right] \in \mathbb{R}[z,z^{-1}]^{r\times r}. \tag{5.44}$$

Now the block square upper-left corner $[\Psi'''_h(z)]_{1:2p,1:2p}$ in (5.43) can be factorized as shown in Lemma 5.1.3, while, the factorization of the L-unimodular para-Hermitian matrix $\Psi''_{h,33}(z)$ follows recursively from step I.

At the end of the recursion, we find a L-unimodular $U^{-1}(z) \in \mathbb{R}[z,z^{-1}]^{r\times r}$ such that

$$\Psi_{h+1} := U^{-*}(z)\Psi_h(z)U^{-1}(z), \tag{5.45}$$

is a constant symmetric matrix. Eventually, we update the value of h by setting $h := h+1$ and go to step II.

[2]This fact may be seen by bringing $\Psi''_h(z)$ in (5.40) to triangular block form by suitable row-column permutations. Then, the determinant is given by the product of the determinants of the blocks on the diagonal (up to a plus-minus sign).

To sum up, we have constructed a L-unimodular matrix

$$P(z) = CU(z)\Omega_{h-1}(z)\Omega_{h-2}(z)\cdots\Omega_1(z) \tag{5.46}$$

such that

$$\Psi(z) = P^*(z)J_{\nu_p,\nu_n}P(z). \tag{5.47}$$

It is worthwhile noticing that, if it is not required to perform step III, then $U(z) = I_r$ and $P(z)$ is unimodular, since $\Omega_{h-1}(z),\ldots,\Omega_1(z)$ are so.

By introducing $P(z)$ in equation (5.31), we obtain the desired factor $W(z) = P(z)D_+(z)F(z)$. Since $F(z)$ is unimodular and $P(z)$ is, in general, L-unimodular, we conclude that $W(z)$ is analytic together with its (right) inverse in $\{|z| > 1,\ z \in \mathbb{C}\}$.[3] Hence, the proof of points (i)-(ii) is concluded.

The proof of assertions (v)-(vi) is similar to that of Theorem 4.2.1. Consider, first, statement (v). If $\Phi(z)$ is analytic on the unit circle, then $\Theta(z)$ does not possess any finite pole. This, in turn, implies that $D_+(z) = \Theta(z)\Lambda(z)$ is analytic in $\{|z| > \tau,\ \tau < 1,\ z \in \mathbb{C}\}$. Thus, $W(z)$, as defined in (5.31), is also analytic in the same region.

The additional assumption that the rank of $\Phi(z)$ is constant on the unit circle implies that $\Theta(z)$ does not possess any finite zero. Thus, $\Theta(z) = I_r$ and, by (5.31), $W^{-R}(z)$ is analytic in a region $\{|z| > \bar{\tau},\ \bar{\tau} < 1,\ z \in \mathbb{C}\}$. Therefore, $W(z)$ is analytic together with its (right) inverse in a region $\{|z| > \tau_1,\ \tau_1 < 1,\ z \in \mathbb{C}\}$, where $\tau_1 := \max\{\tau, \bar{\tau}\}$ is completely determined by the zeros and poles of $\Lambda(z)$. Hence the last statement (vi) is proved and the proof concluded. ■

A Corollary of the previous Theorem is presented below.

Corollary 5.2.1 *If $\Phi(z)$ is L-polynomial then $W(z)$ is L-polynomial.*

PROOF. If $\Phi(z)$ is L-polynomial, then also $D_+(z)$ is L-polynomial. Hence, according to (5.31), $W(z)$ must be a L-polynomial matrix. ■

In Fig.5.2 is shown a diagram of the algorithmic procedure used in the proof of points (i)-(ii) of Theorem 4.2.1.

[3] Notice that $W(z)$ (and/or $W^{-R}(z)$) may have a pole at $z = \infty$. In the next section, we will address this problem in more detail.

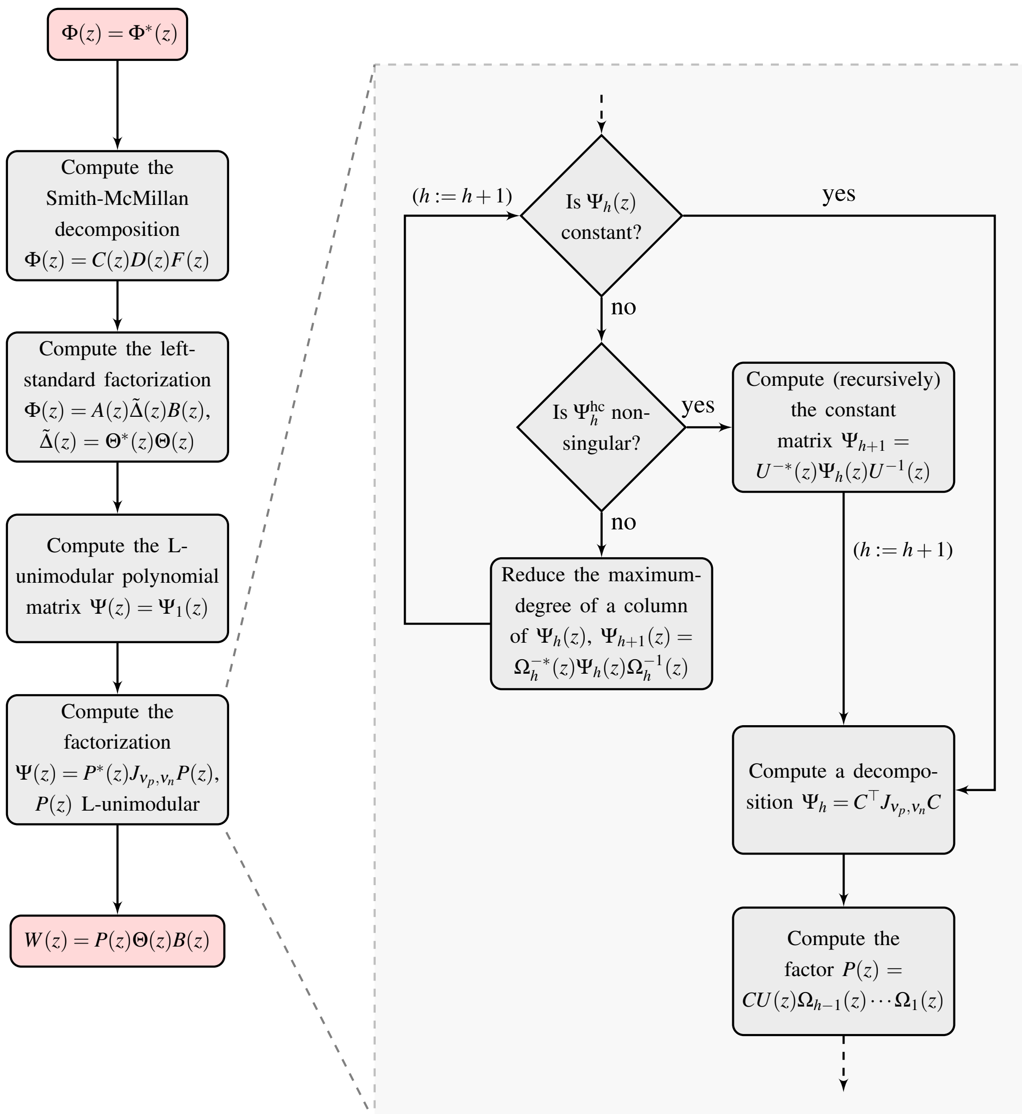

Figure 5.2: *Schematic representation of the procedure used for the construction of the factorization* $\Phi(z) = W^*(z)J_{\nu_p,\nu_n}W(z)$.

5.3 | Some additional remarks

The factor $W(z)$ computed by using the procedure described in the previous section and its (right) inverse $W^{-R}(z)$ may possess poles at infinity. Moreover, $W(z)$ is not, in general, (stochastically) minimal, since it may happen that $\delta_M(W) > \frac{1}{2}\delta_M(\Phi)$ for some particular J-spectra $\Phi(z)$ (see the example of the next section). However, as stated in the Theorem below, there exists a simple condition that ensures that both $W(z)$ and $W^{-R}(z)$ have no poles at infinity and $\delta_M(W) = \frac{1}{2}\delta_M(\Phi)$.

Theorem 5.3.1 *Let $\Phi(z) = \Phi^*(z) \in \mathbb{R}(z)^{n\times n}$ be a DT para-Hermitian matrix of normal rank* $\mathrm{rk}(\Phi) = r \leq n$ *which satisfies the conditions of Theorem 5.2.1. Consider the factorization $\Phi(z) = W^*(z)J_{\nu_p,\nu_n}W(z)$ where $W(z) \in \mathbb{R}(z)^{r\times n}$ is computed by following the procedure described in the proof of Theorem 5.2.1. If it is not necessary to perform step 4-III of the procedure, then*

(i) *$W(z)$ and its (right) inverse $W^{-R}(z)$ are both analytic in $\{|z| > 1,\ z \in \mathbb{C}\}$ without poles at infinity.*

(ii) *$W(z)$ is (stochastically) minimal,* i.e.,

$$\delta_M(W) = \frac{1}{2}\delta_M(\Phi). \tag{5.48}$$

PROOF. First of all, we recall that

$$\Phi(z) = C(z)D(z)F(z) \tag{5.49}$$

is the Smith-McMillan decomposition of $\Phi(z)$. Moreover, by equation (5.31), we have

$$W(z) = P(z)D_+(z)F(z), \tag{5.50}$$

where $D_+(z) \in \mathbb{R}(z)^{r\times r}$ is diagonal, canonic, regular and possesses the same structural indices at $z = 0$ of $D(z)$ and $F(z) \in \mathbb{R}[z]^{r\times n}$ is unimodular. If it is not necessary to perform step 4-III of the constructive procedure of Theorem 5.2.1, by equation (5.46), $P(z) \in \mathbb{R}[z]^{r\times r}$ is also unimodular (rather than L-unimodular).

Since $\Phi(z) = W^*(z)J_{\nu_p,\nu_n}W(z)$, by direct calculation, we have

$$
\begin{aligned}
W^*(z) &= \Phi(z)W^{-R}(z)J_{\nu_p,\nu_n} \\
&= C(z)D(z)F(z)F^{-R}(z)D_+^{-1}(z)P^{-1}(z)J_{\nu_p,\nu_n} \\
&= C(z)D(z)D_+^{-1}(z)P^{-1}(z)J_{\nu_p,\nu_n} \\
&= C(z)D_-(z)P^{-1}(z)J_{\nu_p,\nu_n},
\end{aligned}
\tag{5.51}
$$

where $D_-(z) := D(z)D_+^{-1}(z)$ has no pole at $z = 0$. Since $P^{-1}(z)$, $C(z)$ are unimodular and J_{ν_p,ν_n} is constant, in view of the previous equation, $W^*(z)$ has no pole at $z = 0$. The latter fact, in turn, implies that $W(z)$ has no pole at $z = \infty$. A similar argument can be used to prove that also $W^{-R}(z)$ has no pole at $z = \infty$. This concludes the proof of point (i).

The proof of statement (ii) is essentially the same of that of Theorem 4.3.1 of Chapter 4. ■

To conclude this section, we notice that is always possible to find, under the hypotheses of Theorem 5.2.1, a J-spectral factor which is analytic together with its (right) inverse in $\{|z| > 1,\ z \in \mathbb{C}\}$ and, in addition, has no pole at infinity. In fact, let $W(z)$ be a J-spectral factor of $\Phi(z)$ computed by using the procedure described in Theorem 5.2.1. Let us define

$$
\ell := -\min_{i,j} v_\infty\left([W(z)]_{ij}\right),
\tag{5.52}
$$

where $v_\infty(\cdot)$ denotes the valuation at infinity of a rational function (see §2.1.4). Then, we have that

$$
W'(z) := \frac{W(z)}{z^\ell}
\tag{5.53}
$$

satisfies points (i) and (ii) of Theorem 5.2.1 and has no pole at infinity.

5.4 | An illustrative example

In this section, we present a worked example of the factorization algorithm described in Theorem 5.2.1.

Let us consider the following J-spectrum

$$\Phi(z) = \begin{bmatrix} \frac{1}{(z-2)(z^{-1}-2)} + \varepsilon & \frac{z}{(z-2)(z^{-1}-2)} \\ \frac{z^{-1}}{(z-2)(z^{-1}-2)} & \frac{1}{(z-2)(z^{-1}-2)} - \varepsilon \end{bmatrix}, \quad \varepsilon \in \mathbb{R} \setminus \{0\}. \tag{5.54}$$

One can verify that $\text{in}(\Phi(e^{j\omega})) = (1,0,1)$ for all $\omega \in [0, 2\pi)$.

Step 1. The Smith-McMillan decomposition of $\Phi(z)$ is given by

$$\Phi(z) = C(z)D(z)F(z), \tag{5.55}$$

with $D(z)$ canonical of the form

$$D(z) = \begin{bmatrix} \frac{1}{(z-2)(z-1/2)} & 0 \\ 0 & (z-2)(z-1/2) \end{bmatrix}, \tag{5.56}$$

while the (highly non-unique) unimodular matrices $C(z)$ and $F(z)$ may be taken to be

$$C(z) = \begin{bmatrix} \varepsilon z^2 - \frac{1+5\varepsilon}{2} z + \varepsilon & 1 \\ -\frac{1}{2} & 0 \end{bmatrix}, \quad F(z) = \begin{bmatrix} 1 & 2\varepsilon z^2 + (1-5\varepsilon)z + 2\varepsilon \\ 0 & -2\varepsilon^2 \end{bmatrix}. \tag{5.57}$$

Step 2. With reference to the notation introduced in the corresponding step of Theorem 5.2.1, we have

$$\Lambda(z) = \begin{bmatrix} \frac{1}{z-\frac{1}{2}} & 0 \\ 0 & z-\frac{1}{2} \end{bmatrix}, \quad \Sigma(z) = \begin{bmatrix} -\frac{1}{2z} & 0 \\ 0 & -2z \end{bmatrix}, \quad \Theta(z) = I_2, \tag{5.58}$$

Thus, by computing

$$A(z) = C(z)\Sigma(z)\Lambda^*(z) = \begin{bmatrix} \frac{\varepsilon z^2 - \frac{1+5\varepsilon}{2} z + \varepsilon}{z-2} & z-2 \\ -\frac{1}{2(z-2)} & 0 \end{bmatrix}, \tag{5.59}$$

$$B(z) = \Lambda(z)F(z) = \begin{bmatrix} \frac{1}{z-\frac{1}{2}} & \frac{2\varepsilon z^2 + (1-5\varepsilon)z + 2\varepsilon}{z-\frac{1}{2}} \\ 0 & -2\varepsilon^2\left(z-\frac{1}{2}\right) \end{bmatrix}, \tag{5.60}$$

we find that $\Phi(z) = A(z)\tilde{\Delta}(z)B(z)$, with $\tilde{\Delta}(z) = \Theta^*(z)\Theta(z) = I_2$, is a DT left-standard factorization of $\Phi(z)$.

Step 3. Since $\Theta(z) = I_r$, we have

$$\Psi(z) = \Theta(z)N(z)\Theta^{-1}(z) = N(z) = A^*(z)B^{-1}(z). \tag{5.61}$$

Hence, by direct computation, we obtain

$$\Psi(z) = \begin{bmatrix} -\frac{1}{2}\varepsilon z + \frac{1+5\varepsilon}{4} - \frac{1}{2}\varepsilon z^{-1} & -2z^{-1} + 2 + \frac{1}{2}z \\ -2z + 2 - \frac{1}{2}z^{-1} & -\frac{2}{\varepsilon}z - \frac{1-5\varepsilon}{\varepsilon^2} - \frac{2}{\varepsilon}z^{-1} \end{bmatrix}, \quad \varepsilon \in \mathbb{R}\setminus\{0\}. \tag{5.62}$$

Step 4. Let $\Psi_1(z) := \Psi(z)$. The highest-column-degree coefficient matrix of $\Psi_1(z)$ is given by

$$\Psi_1^{\mathrm{hc}} = \begin{bmatrix} -\frac{1}{2}\varepsilon & -\frac{1}{2} \\ -2 & -\frac{2}{\varepsilon} \end{bmatrix} \tag{5.63}$$

and is singular for all $\varepsilon \in \mathbb{R}\setminus\{0\}$. Thus, we compute a non-zero vector $\mathbf{v}_1$ such that $\Psi_1^{\mathrm{hc}}\mathbf{v}_1 = \mathbf{0}$, *e.g.*, $\mathbf{v}_1 = [1,\ -\varepsilon]^\top$. Then, we construct the unimodular matrix

$$\Omega_1^{-1}(z) = \begin{bmatrix} 1 & 0 \\ \varepsilon & 1 \end{bmatrix}, \tag{5.64}$$

and we conclude the first reduction cycle by computing

$$\begin{aligned} \Psi_2(z) &= \Omega_1^{-*}(z)\Psi_1(z)\Omega_1^{-1}(z) \\ &= \begin{bmatrix} \frac{-3+9\varepsilon}{4} & \frac{3}{2}z + \frac{1-5\varepsilon}{\varepsilon} + 2 \\ \frac{3}{2}z^{-1} + \frac{1-5\varepsilon}{2} + 2 & -\frac{2}{\varepsilon}z - \frac{1-5\varepsilon}{\varepsilon^2} - \frac{2}{\varepsilon}z^{-1} \end{bmatrix}, \quad \varepsilon \in \mathbb{R}\setminus\{0\}. \end{aligned} \tag{5.65}$$

The highest-column-degree coefficient matrix of $\Psi_2(z)$ has the form

$$\Psi_2^{\mathrm{hc}} = \begin{bmatrix} \frac{-3+9\varepsilon}{4} & \frac{3}{2} \\ \frac{1-3\varepsilon}{\varepsilon} & -\frac{2}{\varepsilon} \end{bmatrix} \tag{5.66}$$

and is singular for all $\varepsilon \in \mathbb{R} \setminus \{0\}$ except for $\varepsilon = 1/3$. In fact, in the latter case, the first diagonal entry of $\Psi_2(z)$ is equal to zero and by (5.65)

$$\Psi_2^{\mathrm{hc}} = \begin{bmatrix} 0 & \frac{3}{2} \\ \frac{3}{2} & -6 \end{bmatrix} \tag{5.67}$$

which is clearly non-singular. Consider first the case $\varepsilon \neq 1/3$. As before, we calculate a non-zero vector $\mathbf{v}_2$ such that $\Psi_2^{\mathrm{hc}} \mathbf{v}_2 = \mathbf{0}$, *e.g.*, $\mathbf{v}_2 = [2/(1-3\varepsilon),\ 1]^\top$. Finally, we compute the unimodular matrix

$$\Omega_2^{-1}(z) = \begin{bmatrix} 1 & \frac{2}{1-3\varepsilon} z \\ 0 & 1 \end{bmatrix}. \tag{5.68}$$

and obtain the constant symmetric matrix

$$\begin{aligned} \Psi_3 &= \Omega_2^{-*}(z) \Psi_2(z) \Omega_2^{-1}(z) \\ &= \begin{bmatrix} \frac{-3+9\varepsilon}{4} & \frac{1-3\varepsilon}{\varepsilon} \\ \frac{1-3\varepsilon}{\varepsilon} & \frac{-12\varepsilon^2+8\varepsilon-1}{\varepsilon^2(1-3\varepsilon)} \end{bmatrix}, \quad \varepsilon \in \mathbb{R} \setminus \{0, 1/3\}. \end{aligned} \tag{5.69}$$

Since $\mathrm{in}(\Psi_3) = (1,0,1)$ for all $\varepsilon \in \mathbb{R} \setminus \{0, 1/3\}$, there exists a factorization of the form $\Psi_3 = C^\top J_{1,1} C$ for all $\varepsilon \in \mathbb{R} \setminus \{0, 1/3\}$ with

$$C = \begin{cases} \frac{1}{\sqrt{\varepsilon - \frac{1}{3}}} \begin{bmatrix} \frac{3}{2}\varepsilon - \frac{1}{2} & \frac{-6\varepsilon+2}{3\varepsilon} \\ 0 & \frac{1}{3\varepsilon} \end{bmatrix} & \text{if } \varepsilon > \frac{1}{3}, \\ \frac{1}{\sqrt{-\varepsilon + \frac{1}{3}}} \begin{bmatrix} 0 & \frac{1}{3\varepsilon} \\ -\frac{3}{2}\varepsilon + \frac{1}{2} & \frac{6\varepsilon-2}{3\varepsilon} \end{bmatrix} & \text{if } \varepsilon < \frac{1}{3},\ \varepsilon \neq 0. \end{cases} \tag{5.70}$$

Eventually, we compute the J-spectral factor

$$\begin{aligned} W(z) &= P(z)\Theta(z)B(z) \\ &= \begin{cases} \frac{1}{\sqrt{\varepsilon - \frac{1}{3}}} \begin{bmatrix} \frac{\varepsilon z - \frac{1}{2}\varepsilon + \frac{1}{6}}{z - \frac{1}{2}} & \frac{\frac{1}{6} z}{z - \frac{1}{2}} \\ \frac{\frac{1}{3}}{z - \frac{1}{2}} & \frac{\frac{1}{3} z - \varepsilon z + \frac{1}{2}\varepsilon}{z - \frac{1}{2}} \end{bmatrix} & \text{if } \varepsilon > \frac{1}{3}, \\ \frac{1}{\sqrt{-\varepsilon + \frac{1}{3}}} \begin{bmatrix} \frac{\frac{1}{3}}{z - \frac{1}{2}} & \frac{\frac{1}{3} z - \varepsilon z + \frac{1}{2}\varepsilon}{z - \frac{1}{2}} \\ \frac{-\varepsilon z + \frac{1}{2}\varepsilon - \frac{1}{6}}{z - \frac{1}{2}} & \frac{-\frac{1}{6} z}{z - \frac{1}{2}} \end{bmatrix} & \text{if } \varepsilon < \frac{1}{3},\ \varepsilon \neq 0. \end{cases} \end{aligned} \tag{5.71}$$

where $P(z) = C\Omega_2(z)\Omega_1(z)$. It may be verified that $W(z)$ has no zero/pole at infinity and it is (stochastically) minimal, *i.e.*, $\delta_M(W) = \frac{1}{2}\delta_M(\Phi)$.

Now consider the particular case $\varepsilon = 1/3$. By plugging this value into equation (5.65), we obtain

$$\Psi_2(z) = \begin{bmatrix} 0 & \frac{3}{2}z \\ \frac{3}{2}z^{-1} & -6z+6-6z^{-1} \end{bmatrix}. \tag{5.72}$$

As shown before, Ψ_2^{hc} is non-singular, so we have to apply step 4-III of Theorem 5.2.1. By using the same nomenclature introduced in Theorem 5.2.1, we have $T = I_r$ and so $\Psi_2'(z) = \Psi_2(z)$. Hence, we compute the matrix

$$V'(z) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{2}{3}z^{-1} \end{bmatrix} \tag{5.73}$$

such that

$$\begin{aligned} \Psi_2''(z) &= V'^*(z)\Psi_2'(z)V'(z) \\ &= \begin{bmatrix} 0 & 1 \\ 1 & -\frac{8}{3}z+\frac{8}{3}-\frac{8}{3}z^{-1} \end{bmatrix}. \end{aligned} \tag{5.74}$$

Then, as shown in Lemma 5.1.3, we reduce $\Psi_2''(z)$ to a constant symmetric matrix

$$\Psi_3 = \begin{bmatrix} 1 & -\frac{1}{2}p(z) \\ 0 & 1 \end{bmatrix}^* \Psi_2''(z) \begin{bmatrix} 1 & -\frac{1}{2}p(z) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{5.75}$$

with $p(z) := -\frac{8}{3}z+\frac{8}{3}-\frac{8}{3}z^{-1}$. To sum up, we find that the overall L-unimodular transformation of this cycle is given by

$$U^{-1}(z) = V'(z) \begin{bmatrix} 1 & -\frac{1}{2}p(z) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{4}{3}z-\frac{4}{3}+\frac{4}{3}z^{-1} \\ 0 & \frac{2}{3}z^{-1} \end{bmatrix}. \tag{5.76}$$

To conclude, we factorize Ψ_3 as $\Psi_3 = C^\top J_{1,1} C$, with

$$C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{5.77}$$

Finally, we find that the factor $W(z)$ has the form

$$
\begin{aligned}
W(z) &= P(z)\Theta(z)B(z) \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{-\frac{2}{3}z^2+\frac{7}{6}z+\frac{1}{3}}{z-\frac{1}{2}} & \frac{\frac{1}{3}z^2-\frac{1}{12}z+\frac{1}{3}}{z-\frac{1}{2}} \\ \frac{-\frac{2}{3}z^2+\frac{1}{6}z+\frac{1}{3}}{z-\frac{1}{2}} & \frac{\frac{1}{3}z^2-\frac{7}{12}z+\frac{1}{3}}{z-\frac{1}{2}} \end{bmatrix}.
\end{aligned} \tag{5.78}
$$

where $P(z) = CU(z)\Omega_1(z)$. Notably, in this case $W(z)$ has a pole at infinity and it is not (stochastically) minimal, since one may check that $\delta_M(W) = \delta_M(\Phi)$. By multiplying all the entries of $W(z)$ by $1/z$, we obtain the factor

$$
W'(z) := \frac{W(z)}{z} \tag{5.79}
$$

which has no pole at infinity, but the same McMillan degree of $W(z)$.

6. Conclusions

In this thesis, we dealt with two crucial problems in systems and control theory, namely, the spectral and the J-spectral factorization problem.

The continuous-time spectral factorization problem was solved by using a well-known factorization algorithm due to Youla. In Chapter 3, we described in detail this approach. In order to explain more clearly certain critical points and make the discussion more self-contained, we introduced some minor modifications of the original method (*cf.* the proof of Lemma 3.1.4 and the step 4 of the proof of Theorem 3.2.1). Moreover, in section §3.3, we pointed out two important features of the method. The first one concerns the (stochastic) minimality of the Youla's spectral factor, while the second one the possibility of modifying the region of analyticity of the spectral factor and that of its (right) inverse without significantly affecting the algorithm's structure.

In Chapter 4, we presented a discrete-time version of Youla's algorithm. In particular, the main difference between the continuous- and the discrete-time approach may be found in the step 4 of the constructive proof of Theorem 4.2.1. Here, we discussed in detail an *ad hoc* procedure to unimodularly factorize a para-Hermitian L-unimodular matrix. Furthermore, in analogy to the continuous-time case, we listed, in section §4.3, some remarkable properties of the factorization approach.

Finally, in Chapter 5, we proposed a "J-spectral" generalization of the factorization algorithm analyzed in Chapter 4. This method works under mild assumptions on the Smith-McMillan canonical form of the J-spectrum. Moreover, in section §5.3, we established a simple condition to ensure the (stochastic) minimality of the J-spectral factor. Unlike the most common approaches studied in literature which are based on state-space methods and on the solution of suitable AREs, the proposed J-spectral factorization approach allows, in the author's opinion, to investigate in more detail the internal matrix structure of the J-spectral factor.

There are a number of interesting directions in which our work could be extended; in particular, with reference to the J-spectral factorization method described in Chapter 5. Firstly, one may study a continuous-time counterpart of the latter approach. Secondly, the initial assumptions on the Smith-McMillan canonical form of the J-spectrum may be removed or replaced by weaker ones. Eventually, it would be interesting to exploit this method in order to derive some conditions on the (stochastic) minimality of J-spectral factor which rely on some "structural properties" of the J-spectrum.

Bibliography

B. D. O. Anderson and S. Vongpanitlerd. *Network Analysis And Synthesis: A Modern Systems Theory Approach*. Prentice-Hall electrical engineering series. Prentice-Hall, Englewood Cliffs, NJ, USA, 1973.

B. D. O. Anderson, K. Hitz, and N. Diem. Recursive algorithm for spectral factorization. *Circuits and Systems, IEEE Transactions on*, **21**(6):742–750, 1974.

P. Colaneri and A. Ferrante. A J-spectral factorization approach for $\mathscr{H}_\infty$ estimation problems in discrete-time. *Automatic Control, IEEE Transactions on*, **47** (12):2108–2113, 2002.

P. Colaneri and A. Ferrante. Algebraic Riccati equation and J-spectral factorization for $\mathscr{H}_\infty$ filtering and deconvolution. *SIAM Journal on Control and Optimization*, **45**(1):123–145, 2006.

A. Ferrante. A homeomorphic characterization of minimal spectral factors. *SIAM Journal on Control and Optimization*, **35**(5):1508–1523, 1997a.

A. Ferrante. A parametrization of the minimal square spectral factors of a nonrational spectral density. *Journal of Mathematical Systems, Estimation and Control*, **7**(2):197–226, 1997b.

A. Ferrante and L. Pandolfi. On the solvability of the positive real lemma equations. *Systems & Control Letters*, **47**(3):209–217, 2002.

A. Ferrante, G. Michaletzky, and M. Pavon. Parametrization of all minimal square spectral factors. *Systems & Control Letters*, **21**:249–254, 1993.

E. Fornasini. *Appunti di teoria delle reti elettriche*. CLEUP, Padova, 1977.

E. Fornasini. *Appunti del corso di sistemi multivariabili*. Available on-line at http://www.dei.unipd.it/~fornasini/, 2011.

E. Fornasini and G. Marchesini. *Appunti di teoria dei sistemi*. Edizioni Libreria Progetto, Padova, 2011.

F. Gantmacher. *The Theory of Matrices*, volume I-II of *AMS Chelsea Publishing Series*. AMS Chelsea, 1959.

G. H. Golub and C. F. Van Loan. *Matrix Computations (3rd Ed.)*. Johns Hopkins University Press, Baltimore, MD, USA, 1996.

M. Green, K. Glover, D. Limebeer, and J. Doyle. A J-spectral factorization approach to $\mathscr{H}_\infty$ control. *SIAM Journal on Control and Optimization*, **28**(6):1350–1371, 1990.

N. Jacobson. *Basic Algebra*, volume I-II. Freeman, 1985.

J. Ježek and V. Kučera. Efficient algorithm for matrix spectral factorization. *Automatica*, **21**(6):663–669, 1985.

T. Kailath. *Linear Systems*. Prentice-Hall information and system sciences series. Prentice Hall International, 1998.

A. N. Kolmogorov. Sur l'interpolation et extrapolation des suites stationnaires. *Comptes rendus de l'Académie des Sciences*, **208**:2043–2045, 1939.

H. Kwakernaak and M. Šebek. Polynomial J-spectral factorization. *Automatic Control, IEEE Transactions on*, **39**(2):315–328, 1994.

S. Lang. *Complex analysis*. Graduate texts in mathematics. Springer-Verlag, 1985.

A. Lindquist and G. Picci. A geometric approach to modelling and estimation of linear stochastic systems. *Journal of Mathematical Systems, Estimation and Control*, **1**:241–333, 1991.

S. Mac Lane and G. Birkhoff. *Algebra*. Chelsea Publishing Series. Chelsea Publishing Company, 1999.

Y. Oono. Application of scattering matrices to the synthesis of n ports. *Circuit Theory, IRE Transactions on*, **3**(2):111–120, 1956.

Y. Oono and K. Yasuura. Synthesis of finite passive $2n$-terminal networks with prescribed scattering matrices. *Mem. Kyushu Univ. (Engineering), Japan*, **14**(2):125–177, 1954.

G. Picci. *Filtraggio statistico (Wiener, Levinson, Kalman) e applicazioni*. Edizioni Libreria Progetto, Padova, 2007.

J. Rissanen. Algorithms for the triangular decomposition of block Hankel and Toeplitz matrices with application to factoring positive matrix polynomials. *Mathematics of Computation*, **27**:147–154, 1973.

A. Stoorvogel. *The $\mathscr{H}_\infty$ control problem*. Series in systems and control engineering. Prentice Hall, 1992.

H. L. Trentelman and P. Rapisarda. New algorithms for polynomial J-spectral factorization. *Mathematics of Control, Signals and Systems*, **12**(1):24–61, 1999.

G. Tunnicliffe-Wilson. The Factorization of Matricial Spectral Densities. *SIAM Journal on Applied Mathematics*, **23**(4):420–426, 1972.

N. Wiener. *Extrapolation, interpolation, and smoothing of stationary time series*. Wiley, New York, 1949.

J. C. Willems. Least squares stationary optimal control and the algebraic Riccati equation. *Automatic Control, IEEE Transactions on*, **16**(6):621–634, 1971.

D. C. Youla. On the factorization of rational matrices. *Information Theory, IRE Transactions on*, **7**(3):172–189, 1961.