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# Introduction to Mori Theory and the Minimal Model Program

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# Abstract

After introducing the concept of divisor and the generalities of Intersection Theory, we will see Mori's Cone Theorem. This result will be the starting point of the Minimal Model Program (MMP), whose goal is to find a "simple" birational model of any projective variety. We will analyze in detail the MMP for surfaces, and then we will try to understand the strategy for bigger dimentional cases.

# Contents

# Introduction

The purpose of this thesis is to present an introductory course to the Minimal Model Program, which is a relevant field of study in modern Mathematics: it suffices to say that in 2014, Caucher Birkar was awarded the Fields Medal "for the proof of the boundedness of Fano varieties and for contributions to the minimal model program". We are going to follow Mori's path, and use his Cone Theorem to walk towards the MMP: this cohomological approach is founded on the study of contractions of extremal rays of the effective cone which have negative intersection with the canonical divisor. Despite the fact that for curves and surfaces we have now a complete understanding on how to find a minimal model, in the general setting this remains an open problem.

Let us present more specifically the content of each chapter. In the first one, we will list some elemental definitions and present (almost always without proof) some basic propositions regarding those concepts. In the first section we will focus on sheaves and schemes with nice structures, while in the second we will use Čech cohomology in order to introduce the concept of the $n^{th}$ cohomology group and its primary properties.

In Chapter 2, we are going to define the concepts of Weil and Cartier divisor and study their correlation, see the connection between invertible sheaves and line bundles, look at the notion of Picard group, and analyze the correspondence between linear systems and morphisms to projective spaces. We will then study fundamental properties of divisors, such as when they are globally generated, ample, very ample; for each one of those, we will try to find cohomological consequences.

The Riemann-Roch Theorem will be the main result of the third Chapter, in which we will also study intersection theory, first of curves and then in general (by using the Hilber-Serre Theorem). We are also going to talk about blow-ups of points: this section is central

in the study of the MMP, and is almost sufficient to give the complete classification of minimal models for curves (that we will not treat in this thesis).
Chapter 4 will be centred on cones: we will see the elementary properties of cones (in particular of the ones that contain no lines) and of extremal subcones; then, we will introduce the notions of $N^1(X)_{\mathbb{R}}$ and $N_1(X)_{\mathbb{R}}$, and of the Picard number of $X$, which will lead to the concept of cone of curves and effective cone. We are going to see a numerical characterization of ampleness through Kleiman's criterion, study the order of dimension of some $n^{th}$ cohomology groups, introduce the property of being big for a divisor, define the relative cone of curves and present some examples, other than find the correlation between $\varphi$ and $\overline{NE}(\varphi)$ as a consequence of Stein factorization. Before all that, we will present Nakai-Moishezon ampleness criterion, and then talk about nef divisors and their characterization regarding intersection with curves.
In Chapter 5 we will initially introduce some concepts and results, often without a complete proof, such as the notion of canonical divisor $K_X$, the Adjunction formula, the Riemann-Roch Theorem for curves formulated using $K_X$ (for which we will use Serre's duality Theorem), and the genus formula for curves. Then, Tsen's Theorem will be the starting point to introduce ruled surfaces and their characterization. We are now able to talk about the extremal rays of the effective cone; after citing the Cone Theorem for surfaces, we will see Castelnuovo's Theorem. This result, together with the consequences of the Theorem of elimination of indeterminacies, will provide a complete understanding of the MMP for surfaces.
The dense last chapter will try to present the main results of the MMP in dimension higher than 3. At the beginning, we will define the quasi-projective variety that parametrizes morphisms, and make observations about its local dimension. Then, we are going to present (often without proof) the main results of Mori Theory through his "bend-and-break" lemmas, that will allow us to know in advance the presence of rational curves through certain varieties with "nice" canonical divisor: in this context, it is of great importance the Miyaoka-Mori Theorem. We are now ready to prove Mori's Cone Theorem in its full generality, which will allow us to study the properties of $K_X$-negative extremal rays

of the effective cone; the existence of a contraction of those is guaranteed by Kawamata's base-point-free Theorem. The central part of the chapter is dedicated to the study of the exceptional locus and its irreducible components, and will allow us to make a distinction between all possible type of contractions: they can be of fiber type, divisorial, or small. If the first two types guarantee nice properties for the codomain, small categories are much difficult to handle, so that we need the new concept of flip in order to manage them. Lastly, we are going to talk about minimal models: the origin of the name, a summary of all results we were able to prove during this thesis (and some more), and the problems that arise with this approach, some of which are still open in full generality.

# Chapter 1

# Preliminaries

In this whole thesis, we will always talk about commutative rings with unity, and $K$ will always be a field.

In this chapter, we will say (pre)sheaf for a (pre)sheaf of abelian groups.

## 1.1 Schemes

**Definition 1.1.1.** *Let $\mathcal{F}$ be a sheaf on $X$. A subsheaf of $\mathcal{F}$ is a sheaf $\mathcal{G}$ on $X$ s.t. $\mathcal{G}(U) \leqslant \mathcal{F}(U)$ for any open subset $U$ of $X$, and the restriction maps of $\mathcal{G}$ are induced by those of $\mathcal{F}$.*

**Definition 1.1.2.** *Let $\mathcal{G}$ be a subsheaf of a sheaf $\mathcal{F}$ on $X$. We define the quotient sheaf $\mathcal{F}/\mathcal{G}$ as the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U)/\mathcal{G}(U)$.*

**Remark 1.1.3.** $(\mathcal{F}/\mathcal{G})_x = \mathcal{F}_x/\mathcal{G}_x$ *for any $x \in X$.*

**Definition 1.1.4.** *Let $f : X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathcal{F}$ on $X$, we define the direct image sheaf $f_*\mathcal{F}$ on $Y$ by $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$.*

**Definition 1.1.5.** *If $\mathcal{F}$ and $\mathcal{G}$ are (pre)sheaves on $X$, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is a family of morphisms of abelian groups $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set $U$ of $X$, s.t. whenever $V \subseteq U$ we have that the diagram*

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ {\scriptstyle \rho_{U,V}}\downarrow & & \downarrow{\scriptstyle \rho'_{U,V}} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

*is commutative.*

Let us recall some basic facts about sheaves.

**Proposition 1.1.6.** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on $X$. Then, $\varphi$ is an isomorphism $\iff$ $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for every $x \in X$.*

**Definition 1.1.7.** *Let $R$ be a ring, and $X = SpecR$. Taking $f \in R$, set $C_f = \{g \in R \mid V(g) \subseteq V(f)\}$. Then, we define $\mathcal{O}_X(X_f) = {C_f}^{-1}R$.*

**Remark 1.1.8.** *There is a natural isomorphism $R_f \cong \mathcal{O}_X(X_f)$.*

**Proposition 1.1.9.** *The previous Def. determines a sheaf on the base of distinguished open subsets of $X$, and thus on the whole $X$.*

**Definition 1.1.10.** *Let $R$ be a ring, and $X = SpecR$. $\mathcal{O}_X$ is called the structure sheaf of $X$. The ringed space $(X, \mathcal{O}_X)$ is said to be an affine scheme.*

**Definition 1.1.11.** *An affine scheme is a ringed space that is isomorphic to $(SpecR, \mathcal{O}_{SpecR})$ for some ring $R$. A ringed space $(X, \mathcal{O}_X)$ is a scheme if $\forall x \in X \;\; \exists U$ open neighborhood of $x$ s.t. $(U, \mathcal{O}_X|_U)$ is an affine scheme.*

**Definition 1.1.12.** *A morphism of schemes $\varphi : X \to Y$ is affine if the inverse image of every affine open subset of $Y$ is an affine open subset of $X$.*

**Definition 1.1.13.** *A ring is said to be reduced if it has no nonzero nilpotents. Moreover, a scheme $(X, \mathcal{O}_X)$ is said to be:*

- *reduced if $\mathcal{O}_X(U)$ is reduced for every open set $U$ of $X$;*
- *integral if $\mathcal{O}_X(U)$ is an integral domain for every nonempty open subset $U$ of $X$.*

**Remark 1.1.14.** *An affine scheme $SpecR$ is integral if and only if $R$ is an integral domain.*

**Proposition 1.1.15.** *A scheme is integral if and only if it is irreducible and reduced.*

**Definition 1.1.16.** *Let $R$ be a ring. A scheme $(X, \mathcal{O}_X)$ is an $R$-scheme, or a scheme over $R$, if $\mathcal{O}_X(U)$ is an $R$-algebra for any open subset $U$ of $X$, and all restriction maps are morphisms of $R$-algebras.*

**Definition 1.1.17.** *An $R$-scheme $(X, \mathcal{O}_X)$ is said to be locally of finite type if $X$ can be covered by a family of affine open sets $\{SpecA_i\}_{i\in I}$ s.t. $A_i$ is a f.g. $R$-algebra $\forall i \in I$.*
*If furthermore $X$ is quasi-compact, the scheme is said to be of finite type (over $R$).*

**Definition 1.1.18.** *An affine scheme that is reduced and of finite type over $K$ is said to be an affine variety (over $K$). A $K$-variety is an integral scheme of finite type over $K$.*

**Definition 1.1.19.** *Let $(X, \mathcal{O})$ be a ringed space. A sheaf $\mathcal{F}$ on $X$ is an $\mathcal{O}$-module if $\mathcal{F}(U)$ is an $\mathcal{O}(U)$-module for any open subset $U$ of $X$, and this structure behaves well with respect to restriction maps. Explicitely, if $U \subseteq V$, then*

$$\begin{tikzcd}
\mathcal{O}(V) \times \mathcal{F}(V) \arrow[r, "*_V"] \arrow[d, "\rho_{V,U} \times \rho'_{V,U}"'] & \mathcal{F}(V) \arrow[d, "\rho'_{V,U}"] \\
\mathcal{O}(U) \times \mathcal{F}(U) \arrow[r, "*_U"] & \mathcal{F}(U)
\end{tikzcd}$$

*is a commutative diagram, where $*_W$ is the operation associated to the multiplicative structure of $\mathcal{F}(W)$ as an $\mathcal{O}(W)$-module.*

**Definition 1.1.20.** *A (locally) free sheaf on a ringed space $(X, \mathcal{O})$ is an $\mathcal{O}$-module (locally) isomorphic to $\mathcal{O}^{(I)}$ for some index set $I$; it has rank $n$ if $|I| = n$.*
*A rank 1 locally free sheaf is called an invertible sheaf.*

**Definition 1.1.21.** *Given a topological space $X$, let $Y$ be an irreducible closed subset of $X$. $\eta \in Y$ is a generic point for $Y$ if $\overline{\{\eta\}} = Y$.*

**Proposition 1.1.22.** *Every nonempty irreducible closed subset of a scheme has a unique generic point.*

**Definition 1.1.23.** *A variety $X$ is normal at a point $x \in X$ if $\mathcal{O}_{X,x}$ is integrally closed in $K(X)$. If this holds for any $x \in X$, then $X$ is said to be normal.*

**Definition 1.1.24.** *A scheme is said to be locally noetherian if it can be covered by a family of affine open subsets $\{SpecA_i\}_{i \in I}$, where each $A_i$ is a noetherian ring.*
*A scheme is noetherian if it is locally noetherian and quasi-compact.*

**Proposition 1.1.25.** *A scheme is locally noetherian if and only if for every affine open subset SpecA, A is a noetherian ring.*
*In particular, an affine scheme SpecA is noetherian if and only if A is noetherian.*

**Definition 1.1.26.** *A local noetherian ring R with maximal ideal $\mathfrak{m}$ is called regular if $\mathfrak{m}$ is generated by dimR elements, i.e. if the tangent space has dimention*
$dim_{R/\mathfrak{m}}\mathfrak{m}/\mathfrak{m}^2 = dimR$.
*A noetherian ring R is said to be regular if $R_\mathfrak{p}$ is regular for any prime ideal $\mathfrak{p} \trianglelefteq R$.*

**Remark 1.1.27.** *A regular local ring is integral.*

**Definition 1.1.28.** *A scheme $(X, \mathcal{O}_X)$ is said to be regular (or nonsingular) in codimension one if every local ring $\mathcal{O}_{X,x}$ of dimension 1 is regular.*

**Remark 1.1.29.** *Nonsingular varieties over a field and noetherian normal schemes are both regular in codimension 1.*

**Definition 1.1.30.** *A scheme $(X, \mathcal{O}_X)$ is said to be locally factorial if every local ring $\mathcal{O}_{X,x}$ is a UFD.*

**Definition 1.1.31.** *Given a scheme $(X, \mathcal{O}_X)$ and an open subset $U$ of $X$, let us denote by $S(U)$ the set of elements of $\mathcal{O}_X(U)$ which are not zero divisors in any local ring $\mathcal{O}_{X,u}$ for $u \in U$. Then, the presheaf $\mathcal{F}$ defined by $\mathcal{F}(U) = S(U)^{-1}\mathcal{O}_X(U)$ can be extended to a sheaf $\mathcal{K}_X$, called the sheaf of (total) quotient rings of $\mathcal{O}_X$.*

**Definition 1.1.32.** *Given a ringed space $(X, \mathcal{O})$, let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}$-modules. We define $\mathcal{F} \otimes \mathcal{G}$ as the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)$.*

**Proposition 1.1.33.** *Given a ringed space $(X, \mathcal{O})$, let $\mathcal{F}, \mathcal{G}$ be sheaves of $\mathcal{O}$-modules. Then, for any $x \in X$ we have that $(\mathcal{F} \otimes \mathcal{G})_x = \mathcal{F}_x \otimes \mathcal{G}_x$.*

**Definition 1.1.34.** *Given a ring $R$, let $M$ be an $R$-module. We define the sheaf $\widetilde{M}$ as* $\widetilde{M}(U) = \{s : U \to \bigsqcup_{\mathfrak{p}\in U} M_{\mathfrak{p}} \mid s(\mathfrak{p}) \in M_{\mathfrak{p}}\ \forall \mathfrak{p} \in U$*, and $s$ is locally of the type $m/r$ for some* $m \in M, r \in R\}$ *for any open set $U \subseteq SpecR$.*

**Proposition 1.1.35.** *Given a ring $R$ and an $R$-module $M$, let $\widetilde{M}$ be the sheaf on $X = SpecR$ associated to $M$. Then, $\widetilde{M}$ is an $\mathcal{O}_X$-module s.t. $\widetilde{M}(X) = M$, and $\widetilde{M}(X_f) = M_f$ for any $f \in K[X]$.*

**Definition 1.1.36.** *Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is said to be quasi-coherent if $X$ can be covered by open affine subsets $U_i = SpecA_i$ such that for each $i$ we have $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for some $A_i$-module $M_i$. We say that $\mathcal{F}$ is coherent if furthermore each $M_i$ can be taken to be f.g.*

**Remark 1.1.37.** *The structure sheaf $\mathcal{O}_X$ is coherent.*

**Proposition 1.1.38.** *Let $(X, \mathcal{O}_X)$ be a scheme. Then, an $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent $\iff$ for every affine open subset $U = SpecA$ of $X$ there is an $A$-module $M$ s.t. $\mathcal{F}|_U \cong \widetilde{M}$.*

**Proposition 1.1.39.** *Let $(X, \mathcal{O}_X)$ be a noetherian scheme. Then, an $\mathcal{O}_X$-module $\mathcal{F}$ is coherent $\iff$ for every affine open subset $U = SpecA$ of $X$ there is a f.g. $A$-module $M$ s.t. $\mathcal{F}|_U \cong \widetilde{M}$.*

**Theorem 1.1.40.** *Let $R$ be a noetherian local domain of dim $= 1$, with maximal ideal $\mathfrak{m}$. Then, TFAE:*

- *$R$ is a DVR;*
- *$R$ is integrally closed;*
- *$R$ is a regular local ring;*
- *$\mathfrak{m}$ is a principal ideal.*

## 1.2 Čech cohomology

**Definition 1.2.1.** *Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of a topological space $X$. For any finite set of indexes $i_0, \ldots, i_n \in I$, we set $U_{i_0,\ldots,i_n} = \bigcap_{m=0}^{n} U_{i_m}$. Let $\mathcal{F}$ be a sheaf on $X$. After fixing a well-ordering on $I$, for any $n \geqslant 0$ we define $C^n(\mathcal{U},\mathcal{F}) = \prod_{i_0<\cdots<i_n} \mathcal{F}(U_{i_0,\ldots,i_n})$. Thus, each $\alpha \in C^n(\mathcal{U},\mathcal{F})$ is completely determined by a family of elements $\alpha_{i_0,\ldots,i_n} \in \mathcal{F}(U_{i_0,\ldots,i_n})$ for each $(n+1)$-tuple $i_0 < \ldots < i_n$ of indexes in $I$. Now, we define the $n^{th}$ coboundary map $d^n : C^n(\mathcal{U},\mathcal{F}) \to C^{n+1}(\mathcal{U},\mathcal{F})$ as $(d^n(\alpha))_{i_0,\ldots,i_{n+1}} = \sum_{m=0}^{n+1}(-1)^m \alpha_{i_0,\ldots,\widehat{i_m},\ldots,i_n}|_{U_{i_0,\ldots,i_{n+1}}}$, where the hat over an index means that it is omitted.*

**Proposition 1.2.2.** *Let $\mathcal{U}$ be an open cover of a topological space $X$, and let $\mathcal{F}$ be a sheaf on $X$. Then,*

$$0 \to C^0(\mathcal{U},\mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{U},\mathcal{F}) \xrightarrow{d^1} C^2(\mathcal{U},\mathcal{F}) \xrightarrow{d^2} \ldots$$

*is a cochain complex of abelian groups, called Čech complex.*

**Definition 1.2.3.** *Let $\mathcal{U}$ be an open cover of a topological space $X$. For any sheaf $\mathcal{F}$ on $X$, we define the $n^{th}$ Čech cohomology group of $\mathcal{F}$ with respect to $\mathcal{U}$ as $\check{H}^n(\mathcal{U},\mathcal{F}) := ker(d^n)/Im(d^{n-1})$.*

**Lemma 1.2.4.** *Let $\mathcal{U}$ be an open cover of a topological space $X$, and let $\mathcal{F}$ be a sheaf on $X$. Then, $\check{H}^0(\mathcal{U},\mathcal{F}) = \mathcal{F}(X)$.*

**Proof.** $\check{H}^0(\mathcal{U},\mathcal{F}) = ker(d^0)$. If $\mathcal{U} = \{U_i\}_{i\in I}$ and $\alpha \in C^0(\mathcal{U},\mathcal{F})$ is given by $\{\alpha_i \in \mathcal{F}(U_i)\}_{i\in I}$, then $(d^0(\alpha))_{i,j} = \alpha_j - \alpha_i$ for any $i < j$. Therefore, the condition $d^0(\alpha) = 0$ means that the sections $\alpha_i$ and $\alpha_j$ agree on $U_i \cap U_j$. By the sheaf axioms, the thesis follows. $\square$

**Theorem 1.2.5.** *Given a quasi-compact separated scheme $X$, let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. Then, $\check{H}^n(\mathcal{U},\mathcal{F})$ is independent of the choice of affine open cover $\mathcal{U}$.*

**Definition 1.2.6.** *Given a quasi-compact separated scheme $X$, let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. For any affine open cover $\mathcal{U}$ of $X$, we define its $n^{th}$ cohomology group as $H^n(X,\mathcal{F}) := \check{H}^n(\mathcal{U},\mathcal{F})$.*

**Proposition 1.2.7.** *Given a ring $R$ and a quasi-compact separated $R$-scheme $X$, let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. Then, $H^n(X,\mathcal{F})$ is an $R$-module. Moreover, $H^n(X, \_) : QCoh_X \to Mod_R$ is a covariant functor.*

**Definition 1.2.8.** *Given a quasi-compact separated $K$-scheme $X$, let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. Then, by the previous Proposition we know that $H^n(X,\mathcal{F})$ is a vector space over $K$, so we can define $h^n(X,\mathcal{F}) := dim_K H^n(X,\mathcal{F})$.*

**Proposition 1.2.9.** $h^s(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(m)) = \begin{cases} \binom{n+m}{m} & \text{if } s = 0 \text{ and } m \geqslant 0 \\ \binom{-m-1}{-n-m-1} & \text{if } s = n \text{ and } m \leqslant -n-1 \\ 0 & \text{otherwise} \end{cases}$.

**Remark 1.2.10.** *The fact that* $h^0(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(m)) = \begin{cases} \binom{n+m}{m} & \text{if } m \geqslant 0 \\ 0 & \text{if } m < 0 \end{cases}$ *will be a consequence of Proposition* 2.2.14.

**Corollary 1.2.11.** *The cohomology groups $H^s(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(m))$ are always finite dimensional $K$-v.s., and they vanish in degree above $n$.*

**Proposition 1.2.12.** *$H^n(X, \_)$ respects direct sums, i.e.* $H^n(X, \bigoplus_{i\in I} \mathcal{F}_i) = \bigoplus_{i\in I} H^n(X,\mathcal{F}_i)$.

**Theorem 1.2.13.** *Let $X$ be quasi-compact and separated. Then, every exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of quasi-coherent sheaves on $X$ induces a long exact sequence of cohomology groups $0 \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{G}) \to H^0(X,\mathcal{H}) \to H^1(X,\mathcal{F}) \to \dots$*

**Theorem 1.2.14.** *If $X$ can be covered by $n+1$ affine open sets, or $dimX = n$, then $H^m(X,\mathcal{F}) = \{0\}$ $\forall m > n$ for any quasi-coherent sheaf $\mathcal{F}$ on $X$.*

**Corollary 1.2.15.** *On an affine scheme, all higher (i.e. for $m > 0$) quasi-coherent cohomology groups vanish.*

**Definition 1.2.16.** *Given a projective $K$-scheme $X$, let $\mathcal{F}$ be a coherent sheaf on $X$. We define the Euler-Poincaré characteristic of $\mathcal{F}$ as* $\chi(X,\mathcal{F}) := \sum_{m=0}^{dimX} (-1)^m h^m(X,\mathcal{F})$.

**Proposition 1.2.17.** $\chi(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(m)) = \binom{n+m}{m}$.

**Proposition 1.2.18.** *Given a projective $K$-scheme $X$, let $0 \to \mathcal{F}_1 \to \dots \to \mathcal{F}_n \to 0$ be an exact sequence of coherent sheaves on $X$. Then,* $\sum_{i=1}^{n} (-1)^i \chi(X,\mathcal{F}_i) = 0$.

# Chapter 2

# Divisors and line bundles

In this chapter we will often consider schemes with the property of being

($\star$) noetherian integral separated schemes which are regular in codimension one.

## 2.1 Weil and Cartier divisors

**Definition 2.1.1.** *Let $(X, \mathcal{O}_X)$ be a scheme. A prime divisor on $X$ is a closed integral subscheme $Y$ of codimension 1. The free abelian group generated by prime divisors of $X$ is indicated as $Div(X)$, and its elements are called Weil divisors on $X$.*
*$D = \sum_{i=1}^{m} n_i Y_i \in Div(X)$ has support $Supp(D) := \bigcup_{n_i \neq 0} Y_i$, and it is said to be effective if $n_i \geqslant 0 \;\; \forall i = 1, \ldots, m$.*

**Remark 2.1.2.** *Given a scheme $(X, \mathcal{O}_X)$ satisfying ($\star$), let $Y$ be a prime divisor on $X$. If $\eta \in Y$ is its generic point, then the local ring $\mathcal{O}_{X,\eta}$ is regular of dimension 1, so it's a DVR thanks to Theorem 1.1.40. We call the corresponding discrete valuation $v_Y : K(X) \to \mathbb{Z} \cup \{\infty\}$ the valuation of $Y$. Taking $f \in K(X) \smallsetminus \{0\}$, we have two possibilities for $v_Y(f)$: if it is positive, we say $f$ has a zero along $Y$ of order $v_Y(f)$; if it is negative, we say $f$ has a pole along $Y$ of order $-v_Y(f)$.*

**Lemma 2.1.3.** *Let $f \in K(X) \smallsetminus \{0\}$. Then, $v_Y(f) = 0$ for almost all prime divisors $Y$ on $X$.*

**Definition 2.1.4.** *We define the divisor of $f \in K(X) \smallsetminus \{0\}$ as $div(f) = \sum_Y v_Y(f) Y$, where the sum is taken over all prime divisors $Y$ on $X$. Notice that $div(f) \in Div(X)$ thanks to* *the previous Lemma. Any Weil divisor which can be (locally) written as the divisor of a nonzero rational function is said to be (locally) principal.*

**Remark 2.1.5.** *Recall that prime ideals of height* 1 *in a UFD are principal. Therefore, if $X$ is locally factorial, then any hypersurface can be defined locally by one (regular) equation; hence, any Weil divisor is locally principal.*

**Definition 2.1.6.** *Let $(X, \mathcal{O}_X)$ be a scheme. A Cartier divisor $D$ on $X$ is a global section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$. It has support $Supp(D) := \{x \in X \mid D_x \neq 1\}$.*

**Remark 2.1.7.** *In other words, a Cartier divisor on $X$ is given by a family $\{(U_i, f_i)\}_{i\in I}$, where $\{U_i\}_{i\in I}$ is an open cover of $X$ and $f_i$ is an invertible element of $\mathcal{K}_X(U_i)$ s.t. $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$ for any $i, j \in I$.*

**Remark 2.1.8.** *If $U$ is integral, then $\mathcal{K}_X^*(U) = Frac(\mathcal{O}_X(U))$ as a multiplicative group. Therefore, if $X$ is integral we may take an open cover of integral sets $\{U_i\}_{i\in I}$, and $f_i$ is then a nonzero rational function on $U_i$ s.t. $f_i/f_j$ is a regular function on $U_i \cap U_j$ that does not vanish.*

**Proposition 2.1.9.** *Let $(X, \mathcal{O}_X)$ be a scheme satisfying $(\star)$. Then, any Cartier divisor on $X$ is associated to a Weil divisor on $X$.*

**Proof.** Let $\{(U_i, f_i)\}_{i\in I}$ be a Cartier divisor on $X$. Taken a prime divisor $Y$ on $X$, notice that $v_{Y\cap U_i}(f_i) = v_{Y\cap U_i\cap U_j}(f_j \cdot f_i/f_j) = v_{Y\cap U_j}(f_j) + v_{Y\cap U_i\cap U_j}(f_i/f_j) = v_{Y\cap U_j}(f_j)$ for any $i, j \in I$ s.t. $Y \cap U_i \neq \emptyset \neq Y \cap U_j$, because $X$ is integral by hypotesis, so the previous Remark holds. Therefore, we can associate to the initial Cartier divisor the Weil divisor $\sum_Y v_{Y\cap U_i}(f_i)Y$, where for any $Y$ we choose $i$ s.t. $Y \cap U_i \neq \emptyset$; notice that it is well-defined (i.e. it's independent on the choice of $i$) thanks to the previous reasoning. □

**Theorem 2.1.10.** *Let $(X, \mathcal{O}_X)$ be a locally factorial variety satisfying $(\star)$. Then, there is a bijection between the Cartier divisors on $X$ and $Div(X)$.*

**Proof.** By the previous Proposition, we know that we can associate a Weil divisor to any Cartier divisor on $X$. Using Remark 2.1.5, we have that any $D \in Div(X)$ is locally principal, thus it can be written as $D = \sum_Y v_{Y \cap U_i}(f_i)Y$, where $\{U_i\}_{i \in I}$ is an open cover of $X$. Then, we get a Cartier divisor $\{(U_i, f_i)\}_{i \in I}$, which is well-defined since every $f_i$ is a nonzero rational function. Since those constructions give maps that are mutual inverses, we can conclude. $\square$

**Definition 2.1.11.** *A Cartier divisor $D$ on $X$ is effective if it can be defined by a family $\{(U_i, f_i)\}_{i \in I}$ with $f_i \in \mathcal{O}_X(U_i)$; in this case, we write $D \geqslant 0$.*

**Remark 2.1.12.** *Every nonzero effective Cartier divisor $D = \{(U_i, f_i)\}_{i \in I}$ on $X$ defines a subscheme of $X$ of codimension 1 given by $V(f_i)$ on each $U_i$; we still denote it by $D$.*

**Definition 2.1.13.** *A Cartier divisor on $X$ is principal if it is in the image of the natural map $\mathcal{K}_X^*(X) \to (\mathcal{K}_X^*/\mathcal{O}_X^*)(X)$.*

**Remark 2.1.14.** *Let $(X, \mathcal{O}_X)$ be a scheme satisfying $(\star)$. A Cartier divisor on $X$ is principal if it can be defined by a global nonzero rational function.*

**Remark 2.1.15.** *Let $(X, \mathcal{O}_X)$ be a locally factorial variety satisfying $(\star)$. Then, the bijection between Cartier divisors on $X$ and $Div(X)$ respects principality.*

**Definition 2.1.16.** *Two Cartier divisors $D, D'$ on $X$ are linearly equivalent, and we write $D \sim D'$, if their difference is principal. Similarly, if $(X, \mathcal{O}_X)$ satisfies $(\star)$, two Weil divisors are linearly equivalent if their difference is the divisor of a nonzero rational function.*

**Remark 2.1.17.** *This clearly defines an equivalence relation.*

**Example 2.1.18.** *Let $X \subseteq \mathbb{A}_K^3$ be the quadric cone defined by the equation $xy = z^2$; notice that $X$ is normal. The line $\ell$ defined by $x = 0 = z$ is contained in $X$, thus it defines a Weil divisor on $X$, which cannot be defined near the origin by one equation since the ideal $(x, z)$ is not principal in $\mathcal{O}_{X,0}$. $X$ is therefore not locally principal, so the previous is not a Cartier divisor. However, $2\ell$ is a principal Cartier divisor, defined by $x$.*

**Example 2.1.19.** *On a smooth projective curve $X$ over $K$, a (Weil) divisor is just a finite formal linear combination of closed points, i.e. $D = \sum_{p \in X} n_p\{p\}$. We define its degree as*

*$deg(D) = \sum_{p \in X} n_p[K(p) : K]$; notice that if $K$ is algebraically closed, then $deg(D) = \sum_{p \in X} n_p$. One can prove that the degree of the divisor of a regular function is 0; this tells us that we have a well-defined map $deg : Div(X)/\sim \to \mathbb{Z}$ given by $deg([D]_\sim) = deg(D)$.*

## 2.2 Invertible sheaves

**Proposition 2.2.1.** *Given a ringed space $(X, \mathcal{O})$, let $\mathcal{L}, \mathcal{M}$ be invertible sheaves on $X$. Then, $\mathcal{L} \otimes \mathcal{M}$ is invertible on $X$. Moreover, if $\mathcal{L}^{-1} = Hom(\mathcal{L}, \mathcal{O})$ then $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}$.*

**Definition 2.2.2.** *Let $(X, \mathcal{O})$ be a ringed space. We define the Picard group of $X$ as the group of isomorphism classes of invertible sheaves on $X$ with operation $\otimes$; it is denoted as $Pic(X)$. By the previous Proposition, it is indeed a group, with identity $\mathcal{O}$.*

**Theorem 2.2.3.** *Let $(X, \mathcal{O}_X)$ be a noetherian separated scheme. Then,*
*$Pic(X) \cong\ H^1(X, \mathcal{O}_X^*)$.*

**Proof.** We have to show that $Pic(X) \cong\ \check{H}^1(\mathcal{U}, \mathcal{O}_X^*)$ for any affine open cover $\mathcal{U}$ of $X$. Taking $\mathcal{L}$ invertible sheaf on $X$, let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of $X$ by affine open subsets s.t. $\mathcal{L}|_{U_i} = \mathcal{O}_{U_i}$. Since $\mathcal{L}$ is invertible, we have an isomorphism $\varphi_{ij} : (\mathcal{L}|_{U_i})|_{U_j} \to (\mathcal{L}|_{U_j})|_{U_i}$ given by $1 \mapsto g_{ij}$ for some $g_{ij} \in \mathcal{O}^*_{U_i \cap U_j}$; notice that $g_{ji} = g_{ij}^{-1}$. Since the map $(\mathcal{L}|_{U_i})|_{U_j \cap U_k} \xrightarrow{\varphi_{ij}} (\mathcal{L}|_{U_j})|_{U_i \cap Uk} \xrightarrow{\varphi_{jk}} (\mathcal{L}|_{U_k})|_{U_j \cap U_i}$ given by $1 \mapsto g_{ij} \mapsto g_{ij}g_{jk}$ must be equal to $(\mathcal{L}|_{U_i})|_{U_k \cap U_j} \xrightarrow{\varphi_{ik}} (\mathcal{L}|_{U_k})|_{U_i \cap U_j}$ defined by $1 \mapsto g_{ik}$, we get that $g_{ij}g_{jk} = g_{ik}$, i.e. the cocycle condition $g_{ij}g_{jk}g_{ki} = 1$ is satisfied. Now we can define a map $\varphi : Pic(X) \to \check{H}^1(\mathcal{U}, \mathcal{O}_X^*)$ as $[\mathcal{L}]_\sim \mapsto [\{g_{ij}\}_{i,j}]_\sim$. Let us prove that it is well-defined: indeed, if $\mathcal{L} \sim \mathcal{L}'$, i.e. there exists an isomorphism $h : \mathcal{L} \to \mathcal{L}'$, then for any $i < j$ we have the commutative diagram

$$\begin{array}{ccc} (\mathcal{L}|_{U_i})|_{U_j} & \xrightarrow{h_{ij}} & (\mathcal{L}'|_{U_i})|_{U_j} \\ {\scriptstyle \varphi_{ij}}\downarrow & & \downarrow{\scriptstyle \varphi'_{ij}} \\ (\mathcal{L}|_{U_j})|_{U_i} & \xrightarrow{h_{ji}} & (\mathcal{L}'|_{U_j})|_{U_i} \end{array}$$

This means that $\varphi'_{ij} \circ h_{ij} = h_{ji} \circ \varphi_{ij}$, which evaluated in 1 gives us $g'_{ij}h_{ij} = h_{ji}g_{ij}$, thus $g_{ij} = (h_{ij}/h_{ji})g'_{ij}$. But $h_{ij}/h_{ji} \in Im(d^0)$, so this implies that $\{g_{ij}\}_{i,j} \sim \{g'_{ij}\}_{i,j}$ as we wanted.

Since $\mathcal{U}$ is an open cover of $X$, the map $\psi : \check{H}^1(\mathcal{U}, \mathcal{O}_X^*) \to Pic(X)$ defined as $[\{g_{ij}\}_{i,j}]_\sim \mapsto [\mathcal{L}^g]_\sim$, where $\mathcal{L}^g$ is s.t. $\mathcal{L}^g|_{U_i} = \mathcal{O}_{U_i}$ and $1 \mapsto g_{ij}$ gives an isomorphism $\varphi_{ij} : (\mathcal{L}^g|_{U_i})|_{U_j} \to (\mathcal{L}^g|_{U_j})|_{U_i}$, is well-defined: it suffices to look at the previous computations in the inverse order.

Since $\varphi$ and $\psi$ are clearly mutually inverse, we are done. □

**Definition 2.2.4.** *Let $D = \{(U_i, f_i)\}_{i\in I}$ be a Cartier divisor on $(X, \mathcal{O}_X)$. We define the sheaf associated to $D$, and we denote it by $\mathcal{O}_X(D)$, as the sub-$\mathcal{O}_X$-module of $\mathcal{K}_X$ generated by $1/f_i$ on $U_i$ for each $i \in I$.*

**Remark 2.2.5.** *$\mathcal{O}_X(D)$ is well-defined: indeed, since $f_i/f_j$ is invertible on $U_i \cap U_j$, $1/f_i$ and $1/f_j$ generate the same $\mathcal{O}_X$-module.*

**Proposition 2.2.6.** *Let $(X, \mathcal{O}_X)$ be a scheme. Then:*

- *for any Cartier divisor $D$ on $X$, $\mathcal{O}_X(D)$ is an invertible sheaf on $X$;*
- *the map $D \mapsto \mathcal{O}_X(D)$ gives a one-to-one correspondence between Cartier divisors on $X$ and invertible subsheaves of $\mathcal{K}_X$;*
- *$\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2) \cong \mathcal{O}_X(D_1 + D_2)$;*
- *$D_1 \sim D_2 \iff \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$.*

**Proof.** Refer to [4], Proposition II.6.13. □

**Proposition 2.2.7.** *Let $(X, \mathcal{O}_X)$ be an integral $K$-scheme. Then, every invertible sheaf on $X$ is isomorphic to a subsheaf of $\mathcal{K}_X$.*

**Proof.** Look at [4], Proposition II.6.15. □

**Corollary 2.2.8.** *Let $(X, \mathcal{O}_X)$ be an integral scheme. Then, $Pic(X) \cong \{$Cartier divisors on $X\}/\sim$.*

**Proposition 2.2.9.** *Let $(X, \mathcal{O}_X)$ be a scheme satisfying $(\star)$. If $D$ is a Cartier divisor on $X$, then $\mathcal{O}_X(D)(X) \cong \{f \in \mathcal{K}_X(X) \mid f = 0 \ \vee \ div(f) + D \geqslant 0\}$.*

**Proof.** Let $D = \{(U_i, f_i)\}_{i\in I}$. Any global section of $\mathcal{O}_X(D)$ is a rational function $f$ on $X$ s.t. $f|_{U_i} f_i$ is regular on each $U_i$; thus, $div(f) + D$ is effective.
Conversely, if $f$ is a nonzero rational function on $X$ s.t. $div(f) + D$ is effective, then $f f_i$ is regular on $U_i$, and $f|_{U_i} = (f f_i)/f_i$ defines a section of $\mathcal{O}_X(D)$ over $U_i$. □

**Remark 2.2.10.** *Let $D$ be a nonzero effective Cartier divisor on $(X, \mathcal{O}_X)$. If we denote by $D$ also the subscheme of $X$ that it defines, then we have the exact sequence*
$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$

**Lemma 2.2.11.** *$Pic(\mathbb{P}^n_K) \cong \mathbb{Z}$.*

**Proof.** Any prime divisor $Y$ on $\mathbb{P}^n_K$ corresponds to a principal prime ideal of the factorial ring $K[x_0, \ldots, x_n]$ thanks to Remark 2.1.5. Hence, $Y$ is defined by one homogeneous irreducible polynomial $f$ of degree $d$ (called the degree of $Y$). This defines a surjective morphism $deg : Div(\mathbb{P}^n_K) \to \mathbb{Z}$.
Since any rational function on $\mathbb{P}^n_K$ is the quotient of two homogeneous polynomials of the same degree, the correspondent divisor has $deg = 0$; therefore, we get a well-defined surjective map $deg : Div(\mathbb{P}^n_K)/\sim \;\to \mathbb{Z}$. If we can show that it is injective, then by Corollary 2.2.8 we are done. Since $f/x_0^d$ is a rational function on $\mathbb{P}^n_K$ with associated divisor $Y - dH_0$, where $H_0$ is the hyperplane defined by $x_0 = 0$, we have that $Y \sim dH_0$, so that $deg(Y) = deg(dH_0)$. This shows that any $[D]_\sim \in Div(\mathbb{P}^n_K)/\sim$ has a representative of the type $mH_0$ for some $m \in \mathbb{Z}$, so the proof is complete. □

**Definition 2.2.12.** *Let $D \in Div(\mathbb{P}^n_K)$ be of degree $d$; then, we set $\mathcal{O}_{\mathbb{P}^n_K}(d) := \mathcal{O}_{\mathbb{P}^n_K}(D)$.*

**Remark 2.2.13.** *Thanks to the previous Lemma, this is a good definition.*

**Proposition 2.2.14.** $\mathcal{O}_{\mathbb{P}^n_K}(d)(\mathbb{P}^n_K) \cong \begin{cases} K[x_0, \ldots, x_n]_d & \textit{if } d \geqslant 0 \\ \{0\} & \textit{if } d < 0 \end{cases}.$

**Proof.** Since $\mathcal{O}_{\mathbb{P}^n_K}(d)$ is an invertible sheaf on $\mathbb{P}^n_K$, if $\{U_i\}_{i=0,\ldots,n}$ is the standard affine open cover of $\mathbb{P}^n_K$, then $\mathcal{O}_{\mathbb{P}^n_K}(d)|_{U_i} = \mathcal{O}_{U_i}$. With the notations of the previous proof, we have that $\mathcal{O}_{\mathbb{P}^n_K}(d) = \mathcal{O}_{\mathbb{P}^n_K}(dH_0)$ is generated by $1/x_0^d$; therefore, if we have coordinates $x$ on $U_i \cap U_j$, it follows that $\mathcal{O}_{U_i} = t_0^{-d} K[t_0, \ldots, \widehat{t_i}, \ldots, t_n]$, where $t_k = x_k/x_i \; \forall k \neq i$.

The latter can be embedded in $\mathcal{O}_{U_i}|_{U_i \cap U_j} = K[t_0, \dots, \widehat{t_i}, \dots, t_n, \frac{1}{t_j}]$, where clearly $\frac{1}{t_j} = \frac{x_i}{x_j}$. With the same reasoning for the index $j$, we get that $\mathcal{O}_{U_j} = s_0^{-d} K[s_0, \dots, \widehat{s_j}, \dots, s_n] \hookrightarrow \mathcal{O}_{U_j}|_{U_i \cap U_j} = K[s_0, \dots, \widehat{s_j}, \dots, s_n, \frac{1}{s_i}]$ with $s_k = x_k/x_j$ $\forall k \neq j$, and in particular $\frac{1}{s_i} = \frac{x_j}{x_i}$. By the fact that $\mathcal{O}_{\mathbb{P}^n_K}(d)$ is invertible, we have an isomorphism $\mathcal{O}_{U_i}|_{U_i \cap U_j} \to \mathcal{O}_{U_j}|_{U_i \cap U_j}$, i.e. $K[t_0, \dots, \widehat{t_i}, \dots, t_n, \frac{1}{t_j}] \cong K[s_0, \dots, \widehat{s_j}, \dots, s_n, \frac{1}{s_i}]$, that associates every $g_i = t_0^{-d} p(t_0, \dots, \hat{t_i}, \dots, t_n) \in \mathcal{O}_{U_i}$ to a $g_j = s_0^{-d} q(s_0, \dots, \hat{s_j}, \dots, s_n) \in \mathcal{O}_{U_j}$ when their variables are seen in $U_i \cap U_j$. Since $\frac{x_k}{x_j} = \frac{x_k}{x_i} \frac{x_i}{x_j}$, i.e. $s_k = t_k \frac{1}{t_j}$, we obtain the condition $t_0^{-d} p(t_0, \dots, \hat{t_i}, \dots, t_n) = (\frac{t_0}{t_j})^{-d} q(\frac{t_0}{t_j}, \dots, \hat{1}, \dots, \frac{t_n}{t_j})$, which tells us that the isomorphism is given by $p(t_0, \dots, \hat{t_i}, \dots, t_n) \mapsto t_j^d q(\frac{t_0}{t_j}, \dots, \hat{1}, \dots, \frac{t_n}{t_j})$.
If $d < 0$, then we do not have any polynomial that satisfies the condition; if instead $d \geqslant 0$, what we have seen implies that by gluing the local sections together we obtain $K[x_0, \dots, x_n]_d$. □

**Proposition 2.2.15.** *Let $(X, \mathcal{O}_X)$ be a scheme satisfying $(\star)$. Then,*
*$Pic(X \times \mathbb{P}^n_K) \cong Pic(X) \times \mathbb{Z}$.*

**Proof.** Proceed as in [4], following Proposition II.6.6 and Example II.6.6.1. □

**Remark 2.2.16.** *Let $\pi : Y \to X$ be a morphism between schemes, and let $D$ be a Cartier divisor on $X$. The pull-back $\pi^* \mathcal{O}_X(D)$ is an invertible subsheaf of $\mathcal{K}_Y$, hence by Proposition* 2.2.6 *it defines an equivalence class of Cartier divisors on $Y$, which is (improperly) denoted by $\pi^* D$. If $Y$ is reduced and the support of $D = \{(U_i, f_i)\}_{i \in I}$ does not contain the image of any irreducible component of $Y$, then the collection*
*$\{(\pi^{-1}(U_i), f_i \circ \pi)\}_{i \in I}$ defines a Cartier divisor in that class, which is also denoted by $\pi^* D$. This tells us that we can restrict an equivalence class of divisors to any subvariety, and that we can restrict a Cartier divisor to a subvariety not contained in its support.*

## 2.3 Line bundles

**Definition 2.3.1.** *A line bundle on a scheme $(X, \mathcal{O}_X)$ is a pair $(L, \pi)$, where $(L, \mathcal{O}_L)$ is a scheme and $\pi : L \to X$ is a morphism which is locally trivial, i.e. there is an open cover $\mathcal{U}$ of $X$ and a family of isomorphisms $\gamma_U : \pi^{-1}(U) \to \mathbb{A}^1_U$ for any $U \in \mathcal{U}$ s.t. the* *changes of trivializations* $\mathbb{A}^1_U \to \mathbb{A}^1_U$ *are linear, so given by* $(x,t) \to (x, \varphi(x)t)$ *for some* $\varphi \in \mathcal{O}^*_U(U)$. *A section of* $\pi$ *over an open subset* $V$ *of* $X$ *is a morphism* $s : V \to L$ *s.t.* $\pi \circ s = id_V$.

**Proposition 2.3.2.** *There is a one-to-one correspondence between line bundles on* $X$ *and invertible sheaves on* $X$*, and this map preserves the sections.*

**Proof.** Let $(L, \pi)$ be a line bundle on $X$. Then, the sheaf of sections of $\pi$, i.e. the one defined by $U \to \{\text{sections of } \pi \text{ over U}\}$, is an invertible sheaf on $X$.
Now, let $\mathcal{L}$ be an invertible sheaf on $X$. Then, we can suppose that $\{U_i\}_{i\in I}$ is an affine open cover of $X$ on which $\mathcal{L}$ is trivial, but we have isomorphisms $\varphi_{ij} : (\mathcal{L}|_{U_i})|_{U_j} \to (\mathcal{L}|_{U_j})|_{U_i}$ given by $1 \mapsto g_{ij}$ for some $g_{ij} \in \mathcal{O}^*_{U_i \cap U_j}$. Then, using $(x,t) \to (x, g_{ij}(x)t)$ as changes of trivializations, we obtain a line bundle.
From this construction, we immediately deduce that the sections of a line bundle are exactly the sections of the associated invertible sheaf and viceversa. □

**Definition 2.3.3.** *Given a scheme* $(X, \mathcal{O}_X)$ *satisfying* $(\star)$*, let* $(L, \pi)$ *be a line bundle on* $X$*. Then, every nonzero global section* $s : X \to L$ *of* $\pi$ *defines an effective Cartier divisor* $div(s)$ *on* $X$ *given by the equations* $s = 0$ *on each affne open subset of* $X$ *over which* $L$ *is trivial.*

**Remark 2.3.4.** *Let* $D$ *be a Cartier divisor on a scheme* $(X, \mathcal{O}_X)$ *satisfying* $(\star)$*. Thanks to Corollary* 2.2.8, $[D]_\sim$ *is associated to a unique (up to isomorphism) invertible sheaf* $\mathcal{O}_X(D)$ *of* $X$*, which by Proposition 2.3.2 corresponds to a line bundle* $(L, \pi)$ *with the same sections. Hence, each nonzero global section* $s : X \to L$ *of* $\pi$ *(i.e. of* $\mathcal{O}_X(D)$*) defines an effective Cartier divisor* $div(s) \in \mathcal{O}_X(D)(X)$*. Therefore, by Proposition* 2.2.9 *we get that* $div(s) = div(f) + D \geqslant 0$ *for some* $f \in K(X)$*.*

**Remark 2.3.5.** *If* $D$ *is an effective Cartier divisor on a scheme* $(X, \mathcal{O}_X)$ *satisfying* $(\star)$*, then* $f = 1$ *corresponds to a section of* $\mathcal{O}_X(D)$ *with divisor* $D$*. More generally, any nonzero rational function* $f$ *on* $X$ *can be seen as a nonzero section of* $\mathcal{O}_X(-div(f))$*.*

**Definition 2.3.6.** *The projective space associated to a* $K$*-v.s.* $V$ *is* $\mathbb{P}(V) = (V \setminus \{0\})/\sim$*, where* $v \sim w \iff \exists \alpha \in K$ *s.t.* $w = \alpha v$*.*

**Remark 2.3.7.** *$\mathbb{P}(V) \cong \{W \leqslant V \mid dim_K W = 1\}$ can be seen as the set of linear subspaces of $V$.*

**Example 2.3.8.** *Given a $K$-v.s. $V$ with basis $\{u_1, \dots, u_n\}$, let $U_i = \{u_i \neq 0\}$ be a standard open set of $\mathbb{P}(V)$. Then, $L = \{(\ell, v) \in \mathbb{P}(V) \times V \mid v \in \ell\}$ with the map $\pi : L \to \mathbb{P}(V)$ given by $\pi(\ell, v) = \ell$ defines a line bundle. Noticing that $\pi^{-1}(\ell) = \{(\ell, v) \mid v \in \ell\} \cong \ell$, we get that $L$ is defined in $\pi^{-1}(U_k) \times V \cong U_k \times V$ with coordinates $(x, v)$ by the condition $rk \begin{pmatrix} x_1 & \dots & x_n \\ v_1 & \dots & v_n \end{pmatrix} = 1$, i.e. by the equations $x_i v_j - x_j v_i = 0 \quad \forall i, j = 1, \dots, n$. Looking at $U_i$ as a subset of $\mathbb{A}^n_K$ with 1 in the $i^{th}$ component, we get the family of isomorphism $\pi^{-1}(U_i) \times V \to U_i \times V$ given by $([x_1, \dots, x_n], v) \mapsto ((\frac{x_1}{x_i}, \dots, 1, \dots, \frac{x_n}{x_i}), v)$. Therefore, we get the trivializations $U_i \times V \to U_j \times V$ defined as $(y, v) \mapsto (z, v)$ with $y_k = \frac{x_k}{x_i}$ and $z_k = \frac{x_k}{x_j}$ $\forall k = 1, \dots, n$, so that $z_k = \frac{x_i}{x_j} y_k$, hence $g_{ij}(x) = \frac{x_i}{x_j}$ for any $x \in U_i \cap U_j$. One can check that this line bundle is isomorphic to $\mathcal{O}_{\mathbb{P}(V)}(-1)$.*

**Example 2.3.9.** *Let $X$ be a complex manifold of dimension $n$. We define the canonical (line) bundle $\omega_X$ on $X$ as the one whose fiber at $x \in X$ is the $1-dim$ v.s. of $\mathbb{C}$-differential $n$-forms on $T_{X,x}$. Any divisor associated to a section of this bundle is called a canonical divisor, and is denoted by $K_X$.*
*If $X = \mathbb{P}^n_K$, any differential $n$-form over $x$ is a multiple of $dx_1 \wedge \dots \wedge dx_n$. If $U_i = \{x_i \neq 0\}$, on $U_0 \cap U_1$ we have that $(x_0, 1, x_2, \dots, x_n) = (1, \frac{1}{x_0}, \frac{x_2}{x_0}, \dots, \frac{x_n}{x_0})$, so $dx_1 \wedge \dots \wedge dx_n =$ $= d(\frac{1}{x_0}) \wedge d(\frac{x_2}{x_0}) \wedge \dots \wedge d(\frac{x_n}{x_0}) = -\frac{1}{x_0^{n+1}} dx_0 \wedge dx_2 \wedge \dots \wedge dx_n$. Therefore, any section of the canonical bundle is associated to the divisor $-(n+1)H_0$, where $H_0$ is the hyperplane of equation $x_0 = 0$. Hence, $\omega_{\mathbb{P}^n_K} \cong \mathcal{O}_{\mathbb{P}^n_K}(-n-1)$.*
*If instead $X \subseteq \mathbb{P}^n_K$ is a smooth hypersurface of $deg = d$ defined by a homogeneous polynomial $f \in K[x_0, \dots, x_n]_d$, the $(n-1)$-form defined on $X \cap U_0$ by $(-1)^i \frac{dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n}{(\partial f / \partial x_i)(x)}$ does not depend on $i$ and does not vanish. Similarly as before, with the choice $i = 2$, it can be written on $X \cap U_0 \cap U_1$ as $\frac{d(\frac{1}{x_0}) \wedge d(\frac{x_3}{x_0}) \wedge \dots \wedge d(\frac{x_n}{x_0})}{(\partial f / \partial x_i)(1, \frac{1}{x_0}, \frac{x_2}{x_0}, \dots, \frac{x_n}{x_0})} = -\frac{1}{x_0^{n-(d-1)}} \frac{dx_0 \wedge dx_3 \wedge \dots \wedge dx_n}{(x_0, 1, x_2, \dots, x_n)}$. Therefore, any section of the canonical bundle is associated to the divisor $-(n+1-d)(H_0 \cap X)$. Hence, $\omega_X \cong \mathcal{O}_X(-n-1+d)$.*

## 2.4 Linear systems and morphisms to projective spaces

**Definition 2.4.1.** *Given a normal $K$-variety $(X, \mathcal{O}_X)$ satisfying $(\star)$, let $\mathcal{L}$ be an invertible sheaf on $X$. We define the linear system associated to $\mathcal{L}$ as the set of effective divisors of nonzero global sections of $\mathcal{L}$, and we denote it by $|\mathcal{L}|$.*

**Remark 2.4.2.** *The quotient of two global sections which have the same divisor is a regular function on $X$ which does not vanish. Therefore, if $X$ is projective we have that the map $div : \mathbb{P}(\mathcal{L}(X)) \to |\mathcal{L}|$ is bijective.*

**Remark 2.4.3.** *Let $D$ be a Cartier divisor on $X$. Since each element of $|D| := |\mathcal{O}_X(D)|$ is of the type $div(f) + D$ for some $f \in K(X)$, $|D|$ is the set of effective divisors on $X$ which are linearly equivalent to $D$, i.e. $|D| = [D]_\sim$.*

**Theorem 2.4.4.** *Let $(X, \mathcal{O}_X)$ be a $K$-variety. Then:*

1. *any morphism from $X$ to a projective space is associated to a finite-dim vector space of sections of an invertible sheaf on $X$;*

2. *any finite-dim vector space of sections of an invertible sheaf on $X$ is associated to a rational map from $X$ to a projective space.*

**Proof.** *(1)* Let $V$ be a $K$-v.s. of finite *dim*, and consider a morphism $\varphi : X \to \mathbb{P}(V)$. Taking the invertible sheaf $\mathcal{L} = \varphi^*(\mathcal{O}_{\mathbb{P}(V)}(1))$ on $X$, we have a linear map $\varphi^* : V^* \cong \mathcal{O}_{\mathbb{P}(V)}(1)(\mathbb{P}(V)) \to \mathcal{L}(X)$ given by the pull-back, i.e. by $\varphi^*(s) = s \circ \varphi$. Then, its codomain is a finite-*dim* v.s. of sections of an invertible sheaf on $X$ as we wanted.
*(2)* Let $W$ be a finite-*dim* v.s. of sections of an invertible sheaf $\mathcal{L}$ on $X$. Then, the map $\psi : X \dashrightarrow \mathbb{P}(W)^*$ which sends $x$ to the hyperplane of sections of $\mathcal{L}$ that vanish at $x$ is a rational map, since it is not defined at the base-points of $W$, i.e. the $x \in X$ where all sections of $\mathcal{L}$ vanish. Notice that, chosen a basis $\{s_0, \dots, s_n\}$ of $W$, $\psi$ can be locally written as $\psi(x) = (s_0(x), \dots, s_n(x))$ (where "locally" means that the choice of the basis depends on the trivialization of $\mathcal{L}$ in a neighborhood of $x$). $\square$

**Corollary 2.4.5.** *Let $(X, \mathcal{O}_X)$ be a $K$-variety. Then, we have a one-to-one correspondence between morphisms from $X$ to a projective space whose image is not contained in any hyperplane, and finite-dim vector spaces of sections of an invertible sheaf on $X$ which are base-point-free.*

**Proof.** With the notations of the previous proof, notice that by definition a section of $\mathcal{O}_{\mathbb{P}(V)}(1)$ vanishes on a hyperplane, thus its image by $\varphi^*$ is zero if and only if $\varphi(X)$ is contained in this hyperplane; in particular, $\varphi^*$ is injective if and only if $\varphi(X)$ is not contained in any hyperplane. Moreover, if $W$ is base-point-free, then the map $\psi$ is a morphism. This way, the two constructions of the previous Theorem becomes inverse of one another. $\square$

**Example 2.4.6.** *We already know that, fixed $n \geqslant 0$, the v.s. $\mathcal{O}_{\mathbb{P}^1_K}(n)(\mathbb{P}^1_K) \cong K[x_0, x_1]_n$ has dim $= n+1$, with basis given by $\{s^n, s^{n-1}t, \ldots, t^n\}$. The corresponding linear system is base-point-free and induces a morphism $\mathbb{P}^1_K \to \mathbb{P}^n_K$ given by $[s,t] \mapsto [s^n, s^{n-1}t, \ldots, t^n]$. The image of the latter is called the rational normal curve, and it is defined by the condition $rk \begin{pmatrix} x_0 & x_1 & \ldots & x_{n-1} \\ x_1 & x_2 & \ldots & x_n \end{pmatrix} = 1$, i.e. by the equations $x_i x_{j+1} - x_{i+1} x_j = 0 \ \ \forall i, j = 0, \ldots, n-1$.*

**Example 2.4.7.** *The Cremona involution is the rational map $\zeta : \mathbb{P}^2_K \to \mathbb{P}^2_K$ given by $[x, y, z] \to [\frac{1}{x}, \frac{1}{y}, \frac{1}{z}] = [yz, xz, xy]$, which is well-defined everywhere except at the $3$ points $[1,0,0], [0,1,0], [0,0,1]$. It is associated to the v.s. $\langle yz, xz, xy \rangle$ of sections of $\mathcal{O}_{\mathbb{P}^2_K}(2)$, which is the space of all conics passing through these $3$ points.*

## 2.5 Globally generated sheaves

**Definition 2.5.1.** *Given a scheme $(X, \mathcal{O}_X)$, an $\mathcal{O}_X$-module $\mathcal{F}$ is globally generated (or generated by global sections) at $x \in X$ if the images of the global sections of $\mathcal{F}$ in $\mathcal{F}_x$ generate that stalk as an $\mathcal{O}_{X,x}$-module. We say that $\mathcal{F}$ is globally generated if it is globally generated at each point $x \in X$; this is equivalent to the surjectivity of the evaluation map $\nu : \mathcal{F}(X) \otimes_K \mathcal{O}_X \to \mathcal{F}$ given by $\nu_U(s \otimes \varphi) = \varphi(s|_U)$. Moreover, $\mathcal{F}$ is finitely globally generated (or generated by finitely many global sections) if we can replace $\mathcal{F}(X)$ in the domain of $\nu$ with a vector subspace generated by finitely many global sections.*

**Remark 2.5.2.** *Since $\mathcal{F}(X)$ is a $K$-v.s., we have that $\mathcal{F}(X) \cong K^{(I)}$ for some set $I$. Hence, $\mathcal{F}(X) \otimes_K \mathcal{O}_X \cong K^{(I)} \otimes_K \mathcal{O}_X \cong (K \otimes_K \mathcal{O}_X)^{(I)} \cong \mathcal{O}_X^{(I)}$. Therefore, $\mathcal{F}$ is globally generated $\iff$ it admits a surjection from a free sheaf on $X$, i.e. a map of the type $\mathcal{O}_X^{(I)} \twoheadrightarrow \mathcal{F}$. Moreover, $\mathcal{F}$ is finitely globally generated if $I$ can be taken to be finite. Also, $\mathcal{F}$ is globally generated at $x \in X$ $\iff$ $\mathcal{F}_x$ is generated by global sections of $\mathcal{F}$, i.e. if there exists a map $\varphi : \mathcal{O}_X^{(I)} \to \mathcal{F}$ that is surjective on stalks at $x$, so s.t. $\varphi_x : \mathcal{O}_{X,x}^{(I)} \twoheadrightarrow \mathcal{F}_x$. Equivalently, $\mathcal{F}$ is globally generated at $x$ if every germ at $x$ is a linear combination over $\mathcal{O}_{X,x}$ of germs of global sections.*

**Remark 2.5.3.** *The set of points at which $\mathcal{F}$ is globally generated is an open set.*

**Remark 2.5.4.** *Since closed points are dense in $X$, it is enough to check global generation at every closed point $x \in X$.*

**Remark 2.5.5.** *By Nakayama's Lemma, $\mathcal{F}$ is globally generated at $x \in X$ $\iff$ $\nu_x : \mathcal{F}(X) \to (\mathcal{F} \otimes K(x))(X)$ is surjective.*

**Remark 2.5.6.** *Since any $A$-module $M$ is the homomorphic image of a free module, we get that every quasi-coherent sheaf (of finite type) on any affine scheme is (finitely) globally generated.*

**Remark 2.5.7.** *Quotient, tensor product and restriction to a subscheme preserve the property of being globally generated.*

**Proposition 2.5.8.** *Given a scheme $(X, \mathcal{O}_X)$ and a point $x \in X$, let $\mathcal{F}$ be a finite type quasi-coherent sheaf on $X$. Then:*

- *$\mathcal{F}$ is globally generated at $x$ $\iff$ the fiber of $\mathcal{F}$ is generated by global sections at $x$;*
- *if $\mathcal{F}$ is globally generated at $x$, then $\mathcal{F}$ is globally generated near $x$, i.e. there exists an open neighborhood $U$ of $x$ s.t. $\mathcal{F}$ is globally generated at every point $u \in U$.*

**Proposition 2.5.9.** *An invertible sheaf $\mathcal{L}$ on $X$ is globally generated $\iff$ for any $x \in X$ there exists a global section of $\mathcal{L}$ not vanishing at $x$.*

**Remark 2.5.10.** *By the correspondence seen in Corollary* 2.4.5*, it follows that an invertible sheaf $\mathcal{L}$ is finitely globally generated $\iff$ there exists a morphism $\varphi : X \to \mathbb{P}^n_K$ s.t. $\mathcal{L} \cong \varphi^*(\mathcal{O}_{\mathbb{P}^n_K}(1))$.*

**Definition 2.5.11.** *We say that a Cartier divisor $D$ on $X$ is globally generated if $\mathcal{O}_X(D)$ is globally generated.*

**Proposition 2.5.12.** *A Cartier divisor $D$ on $X$ is globally generated $\iff$ for any $x \in X$ there exists a divisor in $[D]_\sim$ whose support does not contain $x$.*

**Proposition 2.5.13.** *$\mathcal{O}_{\mathbb{P}^n_K}(d)$ is globally generated $\iff$ $d > 0$.*

**Proof.** If $d \leqslant 0$, then either $\mathcal{O}_{\mathbb{P}^n_K}(d)(\mathbb{P}^n_K) = \{0\}$ (if $d < 0$) or $\mathcal{O}_{\mathbb{P}^n_K}(0)(\mathbb{P}^n_K) \cong K$, so any global section is constant, hence this sheaf is not globally generated.
If $d > 0$, then we know that $\mathcal{O}_{\mathbb{P}^n_K}(d)(\mathbb{P}^n_K) \cong K[x_0, \dots, x_n]_d$. At any point of $\mathbb{P}^n_K$, at least one of the homogeneous coordinates, say $x_i$, does not vanish, hence the section $x_i^d$ does not vanish either. Therefore, in this case the sheaf is globally generated. □

## 2.6 Ample divisors

**Definition 2.6.1.** *Let $(X, \mathcal{O}_X)$ be a noetherian scheme. A Cartier divisor $D$ on $X$ is ample if for every coherent sheaf $\mathcal{F}$ on $X$, the sheaf $\mathcal{F}(nD) := \mathcal{F} \otimes \mathcal{O}_X(nD)$ is globally generated for all $n$ large enough.*

**Remark 2.6.2.** *Any Cartier divisor on a noetherian affine scheme is ample.*

**Remark 2.6.3.** *Any sufficiently high multiple of an ample divisor is globally generated: it is enough to take $\mathcal{F} = \mathcal{O}_X(D)$ in the definition.*

**Proposition 2.6.4.** *Sum and restriction to closed subschemes preserve ampleness. Moreover, a sum of an ample and a globally generated Cartier divisor is ample.*

**Proof.** The first statement is immediate. To prove the second one, let $D, E$ be respectively an ample and a globally generated divisor on the same noetherian scheme $X$. Taken a coherent sheaf $\mathcal{F}$ on $X$, for $n$ large enough we have that both $\mathcal{F}(nD)$ and $\mathcal{O}_X(nE)$ are globally generated, hence $\mathcal{F}(n(D+E)) = \mathcal{F} \otimes \mathcal{O}_X(nD) \otimes \mathcal{O}_X(nE) = \mathcal{F}(nD) \otimes \mathcal{O}_X(nE)$ is globally generated thanks to Remark 2.5.7. □

**Proposition 2.6.5.** *Let $D$ be a Cartier divisor on a noetherian scheme. Then, TFAE:*

1. *$D$ is ample;*
2. *$nD$ is ample $\forall n > 0$;*
3. *$nD$ is ample for some $n > 0$.*

**Proof.** (*1* $\Rightarrow$ *2*) It comes directly from the definition.
(*2* $\Rightarrow$ *3*) Obvious.
(*3* $\Rightarrow$ *1*) If $\mathcal{F}(mnD)$ is globally generated for all $m$ large enough, then clearly $\mathcal{F}(pD)$ is globally generated for all $p$ large enough. □

**Proposition 2.6.6.** *Let $D, E$ be Cartier divisors on a noetherian scheme. If $D$ is ample, so is $nD + E$ for all $n$ large enough.*

**Proof.** Since $D$ is ample, with the choice $\mathcal{F} = \mathcal{O}_X(E)$ we get that $E + mD$ is globally generated for all $m$ large enough. Therefore, $D + (mD + E) = (m+1)D + E$ is ample thanks to Proposition 2.6.4. □

**Definition 2.6.7.** *Given a normal scheme $(X, \mathcal{O}_X)$, a $\mathbb{Q}$-divisor on $X$ is a $\mathbb{Q}$-linear combination of prime divisors of $X$. A $\mathbb{Q}$-divisor is said to be $\mathbb{Q}$-Cartier if some multiple of it has integral coefficients and is a Cartier divisor. A $\mathbb{Q}$-Cartier divisor on a noetherian scheme is ample if some (integral) positive multiple of it is ample.*

**Example 2.6.8.** *The line $\ell$ of Example* 2.1.18 *is associated to a $\mathbb{Q}$-divisor in $X$ which is not a Cartier divisor.*

**Remark 2.6.9.** *Proposition* 2.6.6 *can be rephrased in the following way: let $D, E$ be $\mathbb{Q}$-divisors on a noetherian normal scheme. If $D$ is ample, then $D + qE$ is ample for all positive $q \in \mathbb{Q}$ small enough.*

**Lemma 2.6.10.** *If* $\cdots \to U \xrightarrow{\varphi} V \xrightarrow{\psi} W \to \cdots$ *is an exact sequence in* $Vect_K$*, then* $dimV \leqslant dimU + dimW$*.*

**Proof.** From the split short exact sequence $0 \to Im(\varphi) \to V \to Im(\psi) \to 0$, one gets that $dimV = dim\, Im(\varphi) + dim\, Im(\psi) \leqslant dimU + dimW$. □

**Lemma 2.6.11.** *Given an affine scheme* $X = SpecA$*, let* $f \in A$*, and let* $\mathcal{F}$ *be a quasi-coherent sheaf on* $X$*. Then:*

1. *if* $s \in \mathcal{F}(X)$ *is s.t.* $s|_{X_f} = 0$*, then* $\exists n > 0$ *s.t.* $sf^n = 0$*;*
2. *if* $t \in \mathcal{F}(X_f)$*, then* $\exists m > 0$ *s.t.* $tf^m$ *extends to a global section of* $\mathcal{F}$ *over* $X$*.*

**Proof.** One can look at [4], Lemma II.5.3. □

**Theorem 2.6.12 (Serre's Thm.).** *For any coherent sheaf* $\mathcal{F}$ *on* $\mathbb{P}^n_K$*, the sheaf* $\mathcal{F}(m) := \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n_K}(m)$ *is finitely globally generated for all* $m$ *large enough. In particular, the hyperplane divisor on* $\mathbb{P}^n_K$ *is ample.*

**Proof.** Let $\{U_i\}_{i=0,\ldots,n}$ be the standard affine open cover of $\mathbb{P}^n_K$. Since for any $i = 0, \ldots, n$ the restriction $\mathcal{F}|_{U_i}$ is generated by finitely many sections of $\mathcal{F}(U_i)$, let $s$ be one of those; if we can show that $sx_i^m \in \mathcal{F}(m)(U_i)$ extends for $m$ large enough to a global section $t \in \mathcal{F}(m)(\mathbb{P}^n_K)$, then we are done. By Lemma 2.6.11 – (*2*), for each $j = 0, \ldots, n$ we have that $sx_i^p|_{U_i \cap U_j} \in \mathcal{F}(p)(U_i \cap U_j)$ extends to a section $t_j \in \mathcal{F}(p)(U_j)$ for $p$ large enough. In particular, for any $j, k = 0, \ldots, n$ we have that $t_j|_{U_i \cap U_j \cap U_k} = t_k|_{U_i \cap U_j \cap U_k}$, i.e. $(t_j - t_k)|_{U_i \cap U_j \cap U_k} = 0$. By Lemma 2.6.11 – (*1*), we get that $(t_j - t_k)x_i^q|_{U_j \cap U_k} = 0$ for $q$ large enough, i.e. $t_j x_i^q|_{U_j \cap U_k} = t_k x_i^q|_{U_j \cap U_k}$. This means that the $t_j x_i^q$ glue to a section $t \in \mathcal{F}(p+q)(\mathbb{P}^n_K)$ which extends $x_i^{p+q}s$, as we wanted. □

**Corollary 2.6.13.** *Given a closed subscheme* $X$ *of* $\mathbb{P}^n_K$*, let* $\mathcal{F}$ *be a coherent sheaf on* $X$*. Then:*

1. *the* $K$*-v.s.* $H^s(X, \mathcal{F})$ *have finite dimension for any* $s \geqslant 0$*;*
2. $H^s(X, \mathcal{F}(m)) = \{0\}$ *for all* $s > 0$ *and all* $m$ *large enough.*

**Proof.** Since any coherent sheaf on $X$ can be considered as a coherent sheaf on $\mathbb{P}^n_K$ with the same cohomology, we may assume $X = \mathbb{P}^n_K$.
If $s > n$, then by Theorem 1.2.14 we have that $H^s(\mathbb{P}^n_K, \mathcal{F}) = 0$, so in this case both theses are true. We proceed by descending induction on $s$. In other words, suppose the thesis is true for $s + 1$; we want to show it also holds for $s \leqslant n$.
By Serre's Theorem, there exist $r, p \in \mathbb{Z}$ and an exact sequence
$0 \to \mathcal{G} \to \mathcal{O}_{\mathbb{P}^n_K}(-p)^r \to \mathcal{F} \to 0$ of coherent sheaves on $\mathbb{P}^n_K$. By Theorem 1.2.13 we get the exact sequence $H^s(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(-p))^r \to H^s(\mathbb{P}^n_K, \mathcal{F}) \to H^{s+1}(\mathbb{P}^n_K, \mathcal{G})$. Since both left and right term are finite dimensional (the left thanks to Corollary 1.2.11, and the right by inductive hypothesis), by Lemma 2.6.10 also the middle one is: this proves (*1*).
Starting with the same short exact sequence, by applying the exact functor $\_ \otimes \mathcal{O}_{\mathbb{P}^n_K}(m)$ and again by Theorem 1.2.13, we get the exact sequence
$H^s(\mathbb{P}^n_K, \mathcal{O}_{\mathbb{P}^n_K}(m-p))^r \to H^s(\mathbb{P}^n_K, \mathcal{F}(m)) \to H^{s+1}(\mathbb{P}^n_K, \mathcal{G}(m))$. Since both left and right term vanish for all $m > p - n - 1$ if $s > 0$ (the left thanks to Corollary 1.2.11, and the right by inductive hypothesis), also the middle one is zero: this proves (*2*). $\square$

## 2.7 Very ample divisors

**Definition 2.7.1.** *Let $(X, \mathcal{O}_X)$ be a scheme of finite type over $K$. A Cartier divisor $D$ on $X$ is very ample if there exists an embedding $i : X \hookrightarrow \mathbb{P}^n_K$ s.t. $D \sim i^*H$, where $H$ is a hyperplane in $\mathbb{P}^n_K$.*

**Remark 2.7.2.** *A Cartier divisor on $X$ is very ample $\iff$ its sections define a morphism from $X$ to a projective space which induces an isomorphism between $X$ and a locally closed subscheme of the projective space.*

**Remark 2.7.3.** *Restriction to locally closed subschemes preserves very ampleness.*

**Proposition 2.7.4.** *Very ample $\Rightarrow$ finitely globally generated.*

**Remark 2.7.5.** *By Serre's Theorem* 2.6.12*, a very ample divisor on $\mathbb{P}^n_K$ is also ample.*

**Proposition 2.7.6.** *Let $H$ be a hyperplane in $\mathbb{P}^n_K$. Then, $D \sim dH \in Div(\mathbb{P}^n_K)$ is very ample $\iff d > 0 \iff D$ is ample.*

**Proof.** If $d > 0$, then $dH$ is the inverse image of a hyperplane by the Veronese embedding $v_{n,d} : \mathbb{P}^n_K \hookrightarrow \mathbb{P}_K^{\binom{n+d}{d}-1}$. If instead $d < 0$, then we know $dH$ is not ample. $\square$

**Proposition 2.7.7.** *$D \sim aH_1 + bH_2 \in Div(\mathbb{P}^n_K \times \mathbb{P}^m_K)$ is very ample $\iff a, b > 0 \iff$ $D$ is ample.*

**Proof.** By Proposition 2.2.15, any $D \in Div(\mathbb{P}^n_K \times \mathbb{P}^m_K)$ is linearly equivalent to a divisor of the type $aH_1 + bH_2$, where $H_1, H_2$ are the pull-backs of hyperplanes on each factor.
$H_1 + H_2$ is very ample because it is the inverse image of a hyperplane by the Segre embedding $s_{n,m} : \mathbb{P}^n_K \times \mathbb{P}^m_K \hookrightarrow \mathbb{P}_K^{(n+1)(m+1)-1}$. Similarly, if $a, b > 0$ then $aH_1 + bH_2$ is very ample, because it is the inverse image of a hyperplane by the embedding
$s_{\binom{n+a}{a}-1,\binom{m+b}{b}-1} \circ (v_{n,a}, v_{m,b}) : \mathbb{P}^n_K \times \mathbb{P}^m_K \hookrightarrow \mathbb{P}_K^{\binom{n+a}{a}\binom{m+b}{b}-1}$.
On the other hand, if $H_2 : y = 0$, then $aH_1 + bH_2$ restricts to $aH_1$ on $\mathbb{P}^n_K \times \{y\}$, hence it is not ample if $a < 0$; with a symmetric argument, it is not ample if $b < 0$. $\square$

**Proposition 2.7.8.** *Given a $K$-scheme $(X, \mathcal{O}_X)$ of finite type, let $D, E$ be Cartier divisors on $X$. If $D$ is very ample and $E$ is globally generated, then $D + E$ is very ample.
In particular, the sum of two very ample divisors is very ample.*

**Proof.** Since $D$ is very ample, there exists an embedding $i : X \hookrightarrow \mathbb{P}^n_K$ s.t. $D \sim i^*H$ for some hyperplane $H$ of $\mathbb{P}^n_K$. Since $E$ is globally generated and $X$ is noetherian, $E$ is finitely globally generated, so there exists a morphism $j : X \hookrightarrow \mathbb{P}^m_K$ s.t. $E \sim j^*H'$ for some hyperplane $H'$ in $\mathbb{P}^m_K$. Now, consider the morphism $(i, j) : X \to \mathbb{P}^n_K \times \mathbb{P}^m_K$: since its composition with the first projection is $i$, it is an embedding. Moreover, the composition $\varepsilon = s_{n,m} \circ (i, j) : X \hookrightarrow \mathbb{P}_K^{(n+1)(m+1)-1}$ is an embedding that satisfies $\varepsilon^*(H \times H') \sim D + E$. Therefore, $D + E$ is very ample. $\square$

**Corollary 2.7.9.** *Let $D, E$ be Cartier divisors on a noetherian $K$-scheme of finite type. If $D$ is very ample, so is $nD + E$ for all $n$ large enough.*

**Proof.** Since $D$ is ample, $mD + E$ is globally generated for all $m$ large enough. Hence, $D + (mD + E) = (m + 1)D + E$ is very ample by the previous Proposition. $\square$

**Theorem 2.7.10.** *Given a noetherian $K$-scheme $(X, \mathcal{O}_X)$ of finite type, let $D$ be a Cartier divisor on $X$. Then, $D$ is ample $\iff$ $nD$ is very ample for some (or all) $n$ large enough.*

**Proof.** ($\Longleftarrow$) If $nD$ is very ample, then it is ample, thus so is $D$.
($\Longrightarrow$) Take $x \in X$ and let $U$ be an affine open neighborhood of $x$ over which $\mathcal{O}_X(D)$ is trivial, i.e. $\mathcal{O}_X(D)|_U \cong \mathcal{O}_U$. Setting $Y$ as the complement of $U$ in $X$, let $\mathcal{F}_Y$ be the ideal sheaf of $\mathcal{O}_X$ associated to $Y$. Since $D$ is ample, there exists $m > 0$ s.t. $\mathcal{F}_Y(mD)$ is globally generated; notice that its sections can be seen as sections of $\mathcal{O}_X(mD)$ that vanish on $Y$. Therefore, there exists $s \in \mathcal{F}_Y(X)$ which does not vanish at $x$. Since $\mathcal{O}_X(D)$ is trivial on $U$, the section $s$ can be seen as a regular function on $U$; therefore, $X_s$ is a distinguished affine open subset of $U$ containing $x$. Since $X$ is noetherian, it can be covered by a finite number of open subsets of this form. Upon replacing $s$ with a power, we may assume that $m$ is the same for all these open subsets. Hence, we have a finite number of sections $s_1, \dots, s_p$ of $\mathcal{O}_X(mD)$ with no common zeroes s.t. the $X_{s_i}$ are an affine open cover of $X$. Now, let $f_{ij}$ be finitely many generators of the $K$-algebra $\mathcal{O}_{X_{s_i}}(X_{s_i})$; the same proof as that of Serre's Theorem 2.6.12 shows that there exists an integer $r$ such that $s_i^r f_{ij}$ extends to a global section $s_{ij} \in \mathcal{O}_X(rmD)(X)$. The global sections $s_i^r, s_{ij}$ of $\mathcal{O}_X(rmD)$ have no common zeroes, so they define a morphism $\psi : X \to \mathbb{P}^N_K$. Let $U_i \subseteq \mathbb{P}^N_K$ be the standard open subset corresponding to the coordinate $s_i^r$; then, $\psi^{-1}(U_i) = X_{s_i}$, and $\{U_i\}_{i=1,\dots,p}$ is an open cover of $Im(\psi)$. Moreover, the induced surjective morphism $\psi_i : X_{s_i} \to U_i$ corresponds to a surjection $\psi_i^* : \mathcal{O}_{U_i}(U_i) \to \mathcal{O}_{X_{s_i}}(X_{s_i})$; therefore, $\psi_i$ is also injective, so it induces an isomorphism between $X_{s_i}$ and its image. It follows that $\psi$ is an isomorphism onto its image, hence $rmD$ is very ample. $\square$

**Corollary 2.7.11.** *A proper scheme which is noetherian of finite type is projective $\iff$ it carries an ample divisor.*

**Proposition 2.7.12.** *Any Cartier divisor on a projective scheme is linearly equivalent to the difference of two effective Cartier divisors.*

**Proof.** Let $D$ be a Cartier divisor on a projective scheme $(X, \mathcal{O}_X)$, and take $H$ to be an effective ample divisor on $X$ (which exists thanks to the previous Corollary). Then, we know that for $n$ large enough $D + nH$ is globally generated; in particular, $\mathcal{O}_X(D + nH)$ has a nonzero global section $s$. Hence, we have that $div(s) \sim D+nH$, so $D \sim div(s)-nH$ and we are done. □

## 2.8 A cohomological characterization of ample divisors

**Theorem 2.8.1.** *Given a projective scheme $(X, \mathcal{O}_X)$ over $K$, let $D$ be a Cartier divisor on $X$. Then, TFAE:*

1. *$D$ is ample;*
2. *for each coherent sheaf $\mathcal{F}$ on $X$, we have that $H^n(X, \mathcal{F}(mD)) = 0$ for all $n > 0$ and all $m$ large enough;*
3. *for each coherent sheaf $\mathcal{F}$ on $X$, we have that $H^1(X, \mathcal{F}(mD)) = 0$ for all $m$ large enough.*

**Proof.** (*1* $\Rightarrow$ *2*) By Theorem 2.7.10, $aD$ is very ample for all $a$ large enough. Thanks to Corollary 2.6.13, we get that $H^n(X, (\mathcal{F}(aD))(bD)) = 0$ for all $n > 0$ and all $b$ large enough, so we are done.
(*2* $\Rightarrow$ *3*) Obvious.
(*3* $\Rightarrow$ *1*) Taken a closed point $x \in X$, consider the surjection $\mathcal{F} \to \mathcal{F} \otimes K(x)$, and let $\mathcal{G}$ be its kernel. Then, we get the exact sequence $0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{F} \otimes K(x) \to 0$, to which we can apply the exact functor $\_ \otimes \mathcal{O}_X(mD)$, obtaining the exact sequence $0 \to \mathcal{G}(mD) \to \mathcal{F}(mD) \to \mathcal{F}(mD) \otimes K(x) \to 0$. Since by hypothesis we have that $H^1(X, \mathcal{G}(mD)) = 0$ for all $m$ large enough, by Theorem 1.2.13 we get the exact sequence $H^0(X, \mathcal{F}(mD)) \to H^0(X, \mathcal{F}(mD) \otimes K(x)) \to 0$, i.e. the surjection $\mathcal{F}(mD)(X) \to (\mathcal{F}(mD) \otimes K(x))(X)$. This means that $\mathcal{F}(mD)$ is globally generated at $x$. Since this holds for any closed point $x \in X$, we get that $\mathcal{F}(mD)$ is globally generated for all $m$ large enough, proving that $D$ is ample. □

**Proposition 2.8.2.** *Let $\varphi : X \to Y$ be a projective morphism of schemes of finite type over $K$. If $\mathcal{F}$ is a coherent sheaf on $X$, then $\varphi_*\mathcal{F}$ is a coherent sheaf on $Y$.*

**Proof.** Look at [4], Corollary II.5.20. □

**Proposition 2.8.3.** *Given $X, Y$ projective schemes over $K$, let $\varphi : X \to Y$ be a morphism with finite fibers. Then, for any coherent sheaf $\mathcal{F}$ on $X$, we have a natural isomorphism $H^n(Y, \varphi_*\mathcal{F}) \cong H^n(X, \mathcal{F})$.*

**Proof.** By the previous Proposition, $\varphi_*\mathcal{F}$ is a coherent sheaf on $Y$. As a consequence of Zariski's Main Theorem, since $\varphi$ is a projective morphism with finite fibers, it is finite; in particular, it is affine. Therefore, if $\mathcal{U}$ is an affine open cover of $Y$, then $\varphi^{-1}(\mathcal{U})$ is an affine open cover of $X$. By definition of $\varphi_*\mathcal{F}$, we obtain that $\check{H}^n(\mathcal{U}, \varphi_*\mathcal{F}) \cong \check{H}^n(\varphi^{-1}(\mathcal{U}), \mathcal{F})$, which concludes the proof. □

**Lemma 2.8.4 (Projection formula).** *Let $\varphi : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{F}$ is an $\mathcal{O}_X$-module and $\mathcal{G}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then there is a natural isomorphism $\varphi_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{G} \cong \varphi_*(\mathcal{F} \otimes_{\mathcal{O}_X} \varphi^*\mathcal{G})$.*

**Corollary 2.8.5.** *Given $X, Y$ projective schemes over $K$, let $\varphi : X \to Y$ be a morphism with finite fibers. If $D$ is an ample $\mathbb{Q}$-Cartier divisor on $Y$, then the $\mathbb{Q}$-Cartier divisor $\varphi^*D$ on $X$ is ample.*

**Proof.** By hypotesis, $\exists n > 0$ s.t. $nD$ is an ample Cartier divisor. Let $\mathcal{F}$ be a coherent sheaf on $X$; then, since $\mathcal{O}_Y(mD)$ is a locally free $\mathcal{O}_Y$-module of finite rank for any $m \geqslant n$, by the Projection formula we get that $(\varphi_*\mathcal{F})(mD) \cong \varphi_*(\mathcal{F}(m\varphi^*D))$. Hence, by the previous Proposition we have that $H^1(Y, (\varphi_*\mathcal{F})(mD)) \cong H^1(Y, \varphi_*(\mathcal{F}(m\varphi^*D))) \cong$ $\cong H^1(X, \mathcal{F}(m\varphi^*D))$. Since $mD$ is ample, the left-hand side vanishes for all $m$ large enough thanks to Theorem 2.8.1, hence so does the right-hand side. By the same Theorem, it follows that $\varphi^*D$ is ample. □

**Proposition 2.8.6.** *Let $X$ be a projective scheme over $K$. Then, a Cartier divisor on $X$ is ample $\iff$ it is ample on each irreducible component of $X_{red}$.*

# Chapter 3

# Intersections of curves and divisors

## 3.1 Curves

**Definition 3.1.1.** *A curve is a projective integral scheme of dimension* 1 *over a field.*

**Definition 3.1.2.** *Let* $(X, \mathcal{O}_X)$ *be a curve. Then, we define its (arithmetic) genus as* $g(X) := h^1(X, \mathcal{O}_X)$.

**Example 3.1.3.** $g(\mathbb{P}^1_K) = 0$. *Indeed, let* $\mathcal{U} = \{U_0, U_1\}$ *be the standard affine open cover of* $\mathbb{P}^1_K$. *Then,* $C^0(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1_K}) = \mathcal{O}_{\mathbb{P}^1_K}(U_0) \oplus \mathcal{O}_{\mathbb{P}^1_K}(U_1) = K[t] \oplus K[t^{-1}]$ *and*
$C^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1_K}) = \mathcal{O}_{\mathbb{P}^1_K}(U_0 \cap U_1) = K[t, t^{-1}]$, *hence* $d^0 : K[t] \oplus K[t^{-1}] \to K[t, t^{-1}]$ *is defined as* $(p(t), q(t^{-1})) \mapsto p(t) - q(t^{-1})$. *Since every polynomial in* $K[t, t^{-1}]$ *can be written as* $\sum_{i=-m}^{n} a_i t^i = \sum_{i=-m}^{-1} a_i t^i + \sum_{i=0}^{n} a_i t^i$ *for some* $a_i \in K$, $d^0$ *is surjective, therefore*
$H^1(\mathbb{P}^1_K, \mathcal{O}_{\mathbb{P}^1_K}) = \{0\}$.

**Proposition 3.1.4.** *Let* $C \subseteq \mathbb{P}^2_K$ *be a plain curve of degree* $d$. *Then,* $g(C) = \binom{d-1}{2}$.

**Idea of Proof.** To shorten the notation, in this proof all rings of polynomials will contain only elements of degree $d$. Let $C = div(f)$ with $f \in K[x_0, x_1, x_2]_h$. Up to a change of coordinates, we can suppose that $[0, 0, 1] \notin Supp(C)$, i.e. that $f(x_0, x_1, x_2) = x_2^d + \dots$ . In this setting, if $\{U_0, U_1\}$ is the standard affine open cover of $\mathbb{P}^2_K$ and we set $V_i = U_i \cap C$, then $\mathcal{V} = \{V_0, V_1\}$ is an affine open cover of $C$.
Let us assume that $f(x_0, x_1, x_2) = x_2^d - x_0^{d-1}x_1$. Then, $\mathcal{O}_C(V_0) = \frac{K[x,y]}{(y^d - x)} \cong K[y]$ with $x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}$, and $\mathcal{O}_C(V_1) = \frac{K[s,t]}{(t^d - s^{d-1})}$ with $s = \frac{x_0}{x_1} = x^{-1}, t = \frac{x_2}{x_1} = yx^{-1}$. Hence,

$C^0(\mathcal{V}, \mathcal{O}_C) \cong K[y] \oplus \frac{K[s,t]}{(t^d - s^{d-1})}$, and $\mathcal{O}_C(V_0 \cap V_1) = \frac{K[x,y]_x}{(y^d - x)} = \frac{K[x,y,x^{-1}]}{(y^d - x)} \cong K[y, x^{-1}]$. Therefore, $d^0 : K[y] \oplus \frac{K[s,t]}{(t^d - s^{d-1})} \to K[y, x^{-1}]$ is defined as $(p(y), [q(s,t)]) \mapsto p(y) - q(x^{-1}, yx^{-1})$; we want to compute $H^1(C, \mathcal{O}_C) = \frac{K[y,x^{-1}]}{Im(d^0)}$. $Im(d^0)$ contains all (and only) polynomials of the form $\sum_{i=0}^{d} a_i y_i + \sum_{j+k=0}^{d} b_{jk}(x^{-1})^j (yx^{-1})^k =$ $= \sum_{i=0}^{d} a_i y_i + \sum_{l \geqslant k} c_{kl} y^k (x^{-1})^l$, i.e. those of the type $\sum_{i,j=0}^{d} \alpha_{ij} y^i (x^{-1})^j$ with $j = 0$ or $j \geqslant i$. Recalling that $y^d = x \Rightarrow y^d x^{-1} = 1$, if we take the quotient of $K[y, x^{-1}]$ by those we get all polynomials which can be written as $\sum_{i,j=0}^{d} \beta_{ij} y^i (x^{-1})^j$ with $0 < j < i < d$. This means that for any $i = 2, \ldots, d-1$ we have $i - 1$ choices for $j$: hence, we get a number of $\sum_{i=2}^{d-1}(i-1) = \sum_{i=1}^{d-2} i = \frac{(d-1)(d-2)}{2} = \binom{d-1}{2}$ linearly independent generators of $H^1(C, \mathcal{O}_C)$. □

**Remark 3.1.5.** *Let $(X, \mathcal{O}_X)$ be a smooth curve over $K$. If $D = \sum_{p \in X} n_p\{p\}$ is an effective divisor on $X$, we can view it as a 0-dimensional subscheme of $X$. On its support, i.e. the set of points $p \in X$ for which $n_p > 0$, it is defined by the ideal $\mathfrak{m}_{X,p}^{n_p}$. Notice that it holds* $\chi(D, \mathcal{O}_D) = h^0(D, \mathcal{O}_D) = \sum_{p \in X} dim_K(\mathcal{O}_{X,p}/\mathfrak{m}_{X,p}^{n_p}) = \sum_{p \in X} n_p \, dim_K(\mathcal{O}_{X,p}/\mathfrak{m}_{X,p}) =$ $= \sum_{p \in X} n_p \, [K(p) : K] = deg(D)$.

**Lemma 3.1.6.** *Given a ring $R$, let $A \xrightarrow{\varphi} B \xrightarrow{\psi} C \xrightarrow{\gamma} D$ be an exact sequence in $R-Mod$. If $\gamma$ is injective, then $\varphi$ is surjective.*

**Proof.** Since $Im(\psi) = ker(\gamma) = \{0\}$, $\psi = 0$, so $Im(\varphi) = ker(\psi) = B$. □

**Notation.** *Given a projective $K$-scheme $X$, let $D$ be a divisor on $X$. Then, following Definition 1.2.16 we set $\chi(X, D) := \chi(X, \mathcal{O}_X(D))$.*

**Theorem 3.1.7 (Riemann-Roch Thm.).** *Given a smooth curve $X$, let $D$ be a divisor on $X$. Then, $\chi(X, D) = deg(D) + \chi(X, \mathcal{O}_X) = deg(D) + 1 - g(X)$.*

**Proof.** Thanks to Proposition 2.7.12, we have that $D \sim E - F$ for some effective divisors $E, F$ on $X$. By Remark 2.2.10, we get the exact sequences
$0 \to \mathcal{O}_X(D) \cong \mathcal{O}_X(E - F) \to \mathcal{O}_X(E) \to \mathcal{O}_F(E) \cong \mathcal{O}_F \to 0$ and

$0 \to \mathcal{O}_X = \mathcal{O}_X(E-E) \to \mathcal{O}_X(E) \to \mathcal{O}_E(E) \cong \mathcal{O}_E \to 0$ , where $\mathcal{O}_F(E) \cong \mathcal{O}_F$ because $\mathcal{O}_X(E)$ is isomorphic to $\mathcal{O}_X$ in a neighborhood of the support of $F$, and $\mathcal{O}_E(E) \cong \mathcal{O}_E$ with a similar argument. By Proposition 1.2.18, from the first exact sequence we get that $\chi(X,D) = \chi(X,E) - \chi(X,\mathcal{O}_F) = \chi(X,E) - \chi(F,\mathcal{O}_F)$, while from the second one we have $\chi(X,E) = \chi(X,\mathcal{O}_X) + \chi(X,\mathcal{O}_E) = \chi(X,\mathcal{O}_X) + \chi(E,\mathcal{O}_E)$. Recalling that $\chi(F,\mathcal{O}_F) =$ $= deg(F)$ and $\chi(E,\mathcal{O}_E) = deg(E)$, we conclude that $\chi(X,D) = \chi(X,E) - \chi(F,\mathcal{O}_F) =$ $= \chi(X,\mathcal{O}_X) + \chi(E,\mathcal{O}_E) - deg(F) = \chi(X,\mathcal{O}_X) + deg(E) - deg(F) = \chi(X,\mathcal{O}_X) + deg(D)$.

□

**Corollary 3.1.8.** *Let $X$ be a smooth curve. Then, a divisor $D$ on $X$ is ample $\iff$ $deg(D) > 0$.*

**Proof.** ($\Longrightarrow$) Let $p$ be a closed point of $X$. Since $D$ is ample, $\exists m_2 > 0$ s.t. $m_2D$ is very ample and effective; hence, $-\{p\} + m_1(m_2D)$ is globally generated for any $m_1$ sufficiently large. Therefore, taken a nonzero global section $s \in \mathcal{O}_X(-\{p\} + mD)$, we have that $div(s) \sim -\{p\} + mD$ for any $m$ large enough. Being $div(s)$ effective, we get that $0 \leqslant deg(mD - \{p\}) = m\, deg(D) - deg(\{p\})$, which implies $deg(D) \geqslant deg(\{p\})/m > 0$.
($\Longleftarrow$) By the Riemann-Roch Thm. 3.1.7, we have that $h^0(X,\mathcal{O}_X(mD)) - h^1(X,\mathcal{O}_X(mD)) =$ $= \chi(X,D) = deg(mD) + 1 - g(X)$, which is strictly positive for $m$ large enough, thus $h^0(X,\mathcal{O}_X(mD)) > h^1(X,\mathcal{O}_X(mD)) \geqslant 0$, so $H^0(X,mD) \neq \{0\}$. Taking a nonzero global section $s \in H^0(X,mD)$, we have that $div(s) = div(f) + mD$ for some $f \in K(X)$, hence $mD \sim div(s)$; therefore, at most by replacing $D$ with a positive multiple, we can assume that it is effective. From Remark 2.2.10 we then get the exact sequence $0 \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_X(mD) \to \mathcal{O}_D \to 0$. Since $H^1(D,mD) = \{0\}$ because $D$ is a 0-dimensional scheme, from the previous sequence we get a surjection $H^1(X,(m-1)D)) \to H^1(X,mD)$; therefore, $\{h^1(X,mD)\}_{m>0}$ is a decreasing sequence which becomes stationary as $m$ goes to infinity (since at most it reaches 0). Hence, for $m$ large enough the previous map is a bijection; from Lemma 3.1.6 we get that $H^0(X,mD) \to H^0(D,\mathcal{O}_D)$ is a surjection. In particular, the evaluation map $\nu_x$ for the sheaf $\mathcal{O}_X(mD)$ is surjective at every point $x \in Supp(D)$; since the same map is trivially surjective for $x$ outside of this support, $\mathcal{O}_X(mD)$ is globally generated. Therefore,

its global sections define a morphism $\varphi : X \to \mathbb{P}^n_K$ s.t. $\mathcal{O}_X(mD) = \varphi^*\mathcal{O}_{\mathbb{P}^n_K}(1)$. Since $\mathcal{O}_X(mD)$ is non trivial, $\varphi$ is not constant, so it is finite because $X$ is a curve. Hence, $\mathcal{O}_X(mD) = \varphi^*\mathcal{O}_{\mathbb{P}^n_K}(1)$ is ample thanks to Corollary 2.8.5, so $D$ is ample. □

## 3.2 Surfaces

**Definition 3.2.1.** *A surface is a smooth connected projective scheme of dimension* 2 *over an algebraically closed field.*

**Definition 3.2.2.** *Given a surface $(X, \mathcal{O}_X)$ over $K$, let $C, D$ be two curves on $X$ with no common components. If $f, g$ are generators of the ideals of $C$ and $D$ respectively at a point $x \in C \cap D$, we define the intersection multiplicity of $C$ and $D$ at $x$ as $m_x(C \cap D) := dim_K \mathcal{O}_{C\cap D,x}$, where $\mathcal{O}_{C\cap D,x} = \mathcal{O}_{X,x}/(f, g)$ by looking at $C \cap D$ as a scheme-theoretic intersection. Moreover, we set $(C \cdot D) := \chi(X, \mathcal{O}_{C\cap D}) = h^0(X, \mathcal{O}_{C\cap D}) = = \sum_{x\in C\cap D} m_x(C \cap D)$.*

**Remark 3.2.3.** *By the Nullstellensatz, $(f, g)$ contains a power of the maximal ideal $\mathfrak{m}_{X,x}$, say $\mathfrak{m}^n_{X,x}$, hence $m_x(C \cap D) \leqslant dim_K \mathcal{O}_{X,x}/\mathfrak{m}^n_{X,x} = n[K(x) : K]$, so in particular the number $m_x(C \cap D)$ is finite. Moreover, $m_x(C \cap D) = 1 \iff f$ and $g$ generate $\mathfrak{m}_{X,x} \iff$ $C$ and $D$ meet transversally at $x$.*

**Theorem 3.2.4.** *Given a surface $(X, \mathcal{O}_X)$, let $C, D$ be two curves on $X$ with no common components. Then, $(C \cdot D) = \chi(X, -C - D) - \chi(X, -C) - \chi(X, -D) + \chi(X, \mathcal{O}_X)$.*

**Proof.** Let $s$ be a global section of $\mathcal{O}_X(C)$ with $div(s) = C$, and let $t$ be a global section of $\mathcal{O}_X(D)$ with $div(t) = D$. Then, we have the exact sequence $0 \to \mathcal{O}_X(-C - D) \xrightarrow{(t,-s)} \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-D) \xrightarrow{\binom{s}{t}} \mathcal{O}_X \xrightarrow{\pi} \mathcal{O}_{C\cap D} \to 0$. By Proposition 1.2.18, the thesis follows. □

We now generalize this result to any couple of divisors by the following:

**Definition 3.2.5.** *Given a surface $(X, \mathcal{O}_X)$, let $C, D$ be divisors on $X$. Then, we set $(C \cdot D) = \chi(X, -C - D) - \chi(X, -C) - \chi(X, -D) + \chi(X, \mathcal{O}_X)$.*

**Remark 3.2.6.** *By definition, the intersection of two divisors depends only on their linear equivalence classes. Therefore,* $(\_ \cdot \_)$ *defines a symmetric bilinear form on* $Pic(X)$.

**Lemma 3.2.7.** *Given a surface* $(X, \mathcal{O}_X)$*, let* $C$ *be a smooth curve on* $X$*. If* $D$ *is a divisor on* $X$*, then* $(C \cdot D) = deg(D|_C)$.

**Proof.** From the exact sequences $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ and $0 \to \mathcal{O}_X(-D-C) \to \mathcal{O}_X(-D) \to \mathcal{O}_C(-D|_C) \to 0$, we get that $\chi(X, -C) = \chi(X, \mathcal{O}_X) - \chi(C, \mathcal{O}_C)$ and $\chi(X, -D-C) = \chi(X, -D) - \chi(C, -D|_C)$ respectively. It follows that $(C \cdot D) = \chi(X, -C-D) - \chi(X, -C) - \chi(X, -D) + \chi(X, \mathcal{O}_X) = \cancel{\chi(X, -D)} - \chi(C, -D|_C) - \cancel{\chi(X, \mathcal{O}_X)} + \chi(C, \mathcal{O}_C) - \cancel{\chi(X, -D)} + \cancel{\chi(X, \mathcal{O}_X)} = -\chi(C, -D|_C) + \chi(C, \mathcal{O}_C)$.
By the Riemann-Roch Theorem 3.1.7, we have that $deg(D|_C) = -deg(-D|_C) = \chi(C, \mathcal{O}_C) - \chi(C, -D|_C)$. Therefore, we obtain that $(C \cdot D) = deg(D|_C)$. □

## 3.3 Blow-ups

In this section, $K$ will be an algebraically closed field.

### 3.3.1 Blow-up of a point in $\mathbb{P}^n_K$

Let $O$ be a point of $\mathbb{P}^n_K$, and let $H$ be a hyperplane in $\mathbb{P}^n_K$ which does not contain $O$. The projection $\pi : \mathbb{P}^n_K \dashrightarrow H$ from $O$ is a rational map with domain $\mathbb{P}^n_K \smallsetminus \{O\}$. Taking coordinates s.t. $O = [0, \ldots, 0, 1]$ and $H = V(x_n)$, we have that $\pi([x_0, \ldots, x_n]) = [x_0 \ldots, x_{n-1}]$. The graph of $\pi$ is $\Gamma(\pi) = \{(x, y) \in \mathbb{P}^n_K \times H | x \neq O \text{ and } x_i = y_i \ \ \forall i = 0, \ldots, n-1\}$. One can check that its closure $\tilde{\mathbb{P}}^n_K := \overline{\Gamma(\pi)} \subseteq \mathbb{P}^n_K \times H$, which is called the blow-up of $\mathbb{P}^n_K$ at $O$, is defined by the homogeneous equations $x_i y_j = x_j y_i \ \ \forall i, j = 0, \ldots, n-1$.
The first projection $\varepsilon : \tilde{\mathbb{P}}^n_K \to \mathbb{P}^n_K$ is called the blow-up morphism; its fiber is
$\varepsilon^{-1}(x) = \begin{cases} (x, \pi(x)) & \text{if } x \neq O \\ \{O\} \times H \cong H & \text{if } x = O \end{cases}$. Hence, $\varepsilon$ induces an isomorphism $\tilde{\mathbb{P}}^n_K \smallsetminus H \cong \mathbb{P}^n_K \smallsetminus \{O\}$; it is therefore a birational morphism. The fibers of the second projection $q : \tilde{\mathbb{P}}^n_K \to H$ are all isomorphic to $\mathbb{P}^1_K$; however, $\tilde{\mathbb{P}}^n_K$ is not isomorphic to $\mathbb{P}^1_K \times H$, although it is locally a product over each standard open subset $U_i$ of $H$ (we say that it is a projective bundle): indeed, we have an isomorphism $q^{-1}(U_i) = \tilde{\mathbb{P}}^n_K \cap (\mathbb{P}^n_K \times U_i) \to \mathbb{P}^1_K \times U_i$

given by $(x, y) \mapsto ([x_i, x_n], y)$.

One can see that this construction is local and intrinsic; in particular, it is independent of the choice of the hyperplane $H$.

Thinking of $H$ as the set of lines in $\mathbb{P}^n_K$ passing through $O$, we have the geometric description $\tilde{\mathbb{P}}^n_K = \{(x, \ell) \in \mathbb{P}^n_K \times H | x \in \ell\}$.

**Example 3.3.1.** *Consider the Cremona involution $\zeta : \mathbb{P}^2_K \dashrightarrow \mathbb{P}^2_K$, i.e. the rational map given by $\zeta([x_0, x_1, x_2]) = [x_1x_2, x_0x_2, x_0x_1]$, which is regular except at $[1, 0, 0], [0, 1, 0], [0, 0, 1]$. If $\tilde{\mathbb{P}}^2_K$ is the blow-up at $O = [0, 0, 1]$, let $\varepsilon : \tilde{\mathbb{P}}^2_K \to \mathbb{P}^2_K$ be the blow-up morphism. Then, on the open set $x_2 = 1 = y_0$ we have $x_1 = x_0y_1$, hence $\zeta \circ \varepsilon([x_0, x_1, 1], [1, y_1]) =$ $= [x_0y_1, x_0, x_0^2y_1]$, which can be extended to a regular map above $O$ by setting $\tilde{\zeta}([x_0, x_1, 1], [1, y_1]) = [y_1, 1, x_0y_1]$. Similarly, on the open set $x_2 = 1 = y_1$ we have $x_0 = x_1y_0$, hence $\zeta \circ \varepsilon([x_0, x_1, 1], [y_0, 1]) = [x_1, x_1y_0, x_1^2y_0]$, which can be extended above $O$ by $\tilde{\zeta}([x_0, x_1, 1], [y_0, 1]) = [1, y_0, x_1y_0]$. We can conclude that, if $\alpha : X \to \mathbb{P}^2_K$ is the blow-up at $O$, then there exists a regular map $\tilde{\zeta} : X \to \mathbb{P}^2_K$ s.t. $\tilde{\zeta} = \zeta \circ \alpha$.*

### 3.3.2 Blow-up of a point in a subvariety of $\mathbb{P}^n_K$

Given a subvariety $X$ of $\mathbb{P}^n_K$, take $O \in X$. If $\varepsilon : \tilde{\mathbb{P}}^n_K \to \mathbb{P}^n_K$ is the first projection, we define the blow-up of $X$ at $O$ as the closure $\tilde{X}$ of $\varepsilon^{-1}(X \setminus \{O\})$ in $\varepsilon^{-1}(X)$. This yields a birational morphism $\varepsilon : \tilde{X} \to X$. Again, this construction can be made local and intrinsic; in particular, it is independent of the embedding $X \hookrightarrow \mathbb{P}^n_K$.

If $X$ is smooth at a point $x$, the projective space $E := \varepsilon^{-1}(x)$ of dimension $dimX - 1$ is called exceptional divisor, and it is naturally isomorphic to $\mathbb{P}(T_{X,x})$.

**Example 3.3.2.** *Consider the plane cubic $C \subseteq \mathbb{P}^2_K$ with equation $x_1^2x_2 = x_0^2(x_2 - x_0)$, which has a singularity at $O = [0, 0, 1]$; we want to compute the equations associated to the blow-up $\tilde{C}$ at $O$. At a point $([x_0, x_1, x_2], [y_0, y_1])$ of $\varepsilon^{-1}(C \setminus \{O\})$ with $y_0 = 1$, we have $x_1 = x_0y_1$, hence (as $x_0 \neq 0$) we get the equation $x_2y_1^2 = x_2 - x_0$, which defines $\tilde{C}$ on the open set $\mathbb{P}^2_K \times U_0$. At a point with $y_1 = 1$, we have $x_0 = x_1y_0$, hence (as $x_1 \neq 0$) we get the equation $x_2 = y_0^2(x_2 - x_1y_0)$, which defines $\tilde{C}$ on the open set $\mathbb{P}^2_K \times U_1$. Moreover, both points of the fiber $\varepsilon^{-1}(O) = \{([0, 0, 1], [1, 1]), ([0, 0, 1], [1, -1])\}$ belong to both open sets.*

*Notice that with this process we have desingularized $C$.*

### 3.3.3 Blow-up of a point in a smooth surface

**Proposition 3.3.3.** *Given a smooth projective surface $X$ over an algebraically closed field, let $\varepsilon : \tilde{X} \to X$ be the blow-up morphism of a point $x \in X$, with exceptional divisor $E$. Then, for any divisors $C, D$ on $X$, we have that $(\varepsilon^* C \cdot \varepsilon^* D) = (C \cdot D)$, $(\varepsilon^* C \cdot E) = 0$ and $(E \cdot E) = -1$.*

**Proof.** Up to replacing $C$ and $D$ by linearly equivalent divisors whose supports do not contain $x$ (which is possible thanks to Proposition 2.7.12), the first two equalities are obvious.
Assume now that $C$ is a smooth curve in $X$ passing through $x$, and let $\tilde{C} = \overline{\varepsilon^{-1}(C \setminus \{x\})}$ be its strict transform in $\tilde{X}$. Since $\tilde{C}$ meets $E$ transversally at the point corresponding to the tangent direction to $C$ at $x$, we have that $\varepsilon^* C = \tilde{C} + E$. Therefore,
$0 = (\varepsilon^* C \cdot E) = (\tilde{C} \cdot E) + (E \cdot E) = 1 + (E \cdot E)$, which concludes the proof. □

**Corollary 3.3.4.** *Given a smooth projective surface $X$ over an algebraically closed field, let $\varepsilon : \tilde{X} \to X$ be the blow-up morphism of a point $x \in X$, with exceptional divisor $E$. Then, $Pic(\tilde{X}) \cong Pic(X) \oplus \mathbb{Z}[E]$.*

**Proof.** We want to show that the map $Pic(X) \oplus \mathbb{Z} \to Pic(\tilde{X})$ defined as
$([D]_\sim, n) \to [\varepsilon^* D + nE]_\sim$ is bijective.
Taken an irreducible curve $\tilde{C}$ on $\tilde{X}$ distinct from $E$, the pull-back $\varepsilon^*(\varepsilon(\tilde{C}))$ is the sum of $\tilde{C}$ with a certain number of copies of $E$: this shows that the previous map is surjective.
To prove injectivity, consider $\varepsilon^* D + nE \sim 0$; then, $0 = ((\varepsilon^* D + nE) \cdot E) = (\varepsilon^* D \cdot E) + +n(E \cdot E) = -n \Rightarrow n = 0$. Moreover, $\mathcal{O}_X \cong \varepsilon_* \mathcal{O}_{\tilde{X}} \cong \varepsilon_*(\mathcal{O}_{\tilde{X}}(\varepsilon^* D)) \cong \mathcal{O}_X(D)$, where the first and last isomorphisms hold thanks to Zariski's Main Theorem and the Projection formula 2.8.4 respectively. This implies that $D \sim 0$, concluding the proof. □

**Definition 3.3.5.** *Given a noetherian scheme $X$, let $\mathcal{I}$ be a coherent sheaf of ideals on $X$. Then, $\tilde{X} = Proj(\bigoplus_{d \in \mathbb{N}} \mathcal{I}^d)$ is called the blow-up of $X$ with respect to $\mathcal{I}$. If $Y$ is the closed subscheme of $X$ corresponding to $\mathcal{I}$, then we also call $\tilde{X}$ the blow-up of $X$ along $Y$.*

**Proposition 3.3.6** (**Universal property of the blow-up**). *Given a noetherian scheme $X$ and a coherent sheaf of ideals $\mathcal{I}$ on $X$, let $\varepsilon : \tilde{X} \to X$ be the blow-up with respect to $\mathcal{I}$. If $f : Y \to X$ is a morphism s.t. the sheaf of ideals $f^{-1}\mathcal{I}$ is invertible in $\mathcal{O}_Y$, then there exists a unique morphism $g : Y \to \tilde{X}$ factoring $f$, i.e. satisfying $f = \varepsilon \circ g$.*

**Proof.** See [4], Proposition II.7.14. □

## 3.4 General intersection numbers

**Theorem 3.4.1** (**Hilbert-Serre Thm.**). *Let $(X, \mathcal{O}_X)$ be a closed subscheme of $\mathbb{P}^N_K$ of dimension $n$. Then, for all $m$ large enough the function $m \mapsto \chi(X, \mathcal{O}_X(m))$ takes the same values on the integers as a uniquely determined polynomial of degree $n$ with rational coefficients, called the Hilbert polynomial of $X$ and denoted by $P_X$.*

**Proof.** See [4], Theorem I.7.5. □

**Definition 3.4.2.** *Let $(X, \mathcal{O}_X)$ be a closed subscheme of $\mathbb{P}^N_K$ of dimension $n$. We define $degX$ as $n!$ times the leading coefficient of $P_X$.*

**Proposition 3.4.3.** *If $\emptyset \neq Y \subseteq \mathbb{P}^n_K$, then $degY$ is a positive integer.*

**Proof.** One can look at [4], Proposition I.7.6-*(a)*. □

**Remark 3.4.4.** *Let $(X, \mathcal{O}_X)$ be a reduced closed subscheme of $\mathbb{P}^N_K$ of dimension $n$. If $K$ is algebraically closed, taken $H_1, \ldots, H_n$ general hyperplanes, then $degX$ is the number of points in $X \cap \bigcap_{i=1}^{n} H_i$. In other words, if $H_i^X$ is the Cartier divisor on $X$ defined by $H_i$, then $degX$ is the number of points in $\bigcap_{i=1}^{n} H_i^X$.*

**Lemma 3.4.5.** *Let $f : \mathbb{Z} \to \mathbb{Z}$ be s.t. the function $m \mapsto f(m) - f(m-1)$ is a rational polynomial of degree $d$. Then, $f$ takes the same values as a rational polynomial of degree $d + 1$.*

**Proof.** Look at [4], Proposition I.7.3-*(b)*. □

**Lemma 3.4.6.** *Let $f : \mathbb{Z}^r \to \mathbb{Z}$ be s.t. for each $(n_1, \dots, n_{r-1}) \in \mathbb{Z}^{r-1}$ the function $m \mapsto f(n_1, \dots, n_{i-1}, m, n_i, \dots, n_{r-1})$ is a rational polynomial of degree at most $d$. Then, $f$ takes the same values as a rational polynomial in $r$ indeterminates.*

**Proof.** We proceed by induction on $r$. The case $r = 1$ trivially holds. Assume that the thesis holds for $r - 1$; we want to prove it for $r$.
Surely, there exist $f_0, \dots, f_d : \mathbb{Z}^{r-1} \to \mathbb{Q}$ s.t. $f(m_1, \dots, m_r) = \sum_{j=0}^{d} f_j(m_1, \dots, m_{r-1}) m_r^j$.
Now, let $c_0, \dots, c_d$ be distinct integers; by inductive hypothesis, for any $i = 0, \dots, d$ there exists a polynomial $P_i \in \mathbb{Q}[x_1, \dots, x_{r-1}]$ s.t. $f(m_1, \dots, m_{r-1}, c_i) = \sum_{j=0}^{d} f_j(m_1, \dots, m_{r-1}) c_i^j =$

$= P_i(m_1, \dots, m_{r-1})$. Let us set $v = \begin{pmatrix} f(m_1, \dots, m_{r-1}, c_0) \\ \vdots \\ f(m_1, \dots, m_{r-1}, c_d) \end{pmatrix}$, $C = (c_i^j)_{i,j=0,\dots,d}$,

$u = \begin{pmatrix} f_0(m_1, \dots, m_{r-1}) \\ \vdots \\ f_d(m_1, \dots, m_{r-1}) \end{pmatrix}$ and $w = \begin{pmatrix} P_0(m_1, \dots, m_{r-1}) \\ \vdots \\ P_d(m_1, \dots, m_{r-1}) \end{pmatrix}$; then, we have that $v = Cu = w$.
Since $C$ is a Vandermonde matrix, it has $det(C) = \prod_{i<j} (c_j - c_i) \neq 0$ because the $c_i$ are all distinct, hence $C \in GL_{d+1}(\mathbb{Z})$. Therefore, $u = C^{-1} w$ with $C^{-1} \in GL_{d+1}(\mathbb{Q})$, which means that each $f_j$ is a linear combination of $P_0, \dots, P_d$ with rational coefficients. This concludes the proof. □

**Theorem 3.4.7.** *Given a projective $K$-scheme $(X, \mathcal{O}_X)$ of $dim = d$, let $D_1, \dots, D_r$ be Cartier divisors on $X$. Then, the function $(m_1, \dots, m_r) \mapsto \chi(X, \sum_{i=1}^{r} m_i D_i)$ takes the same values on $\mathbb{Z}^r$ as a polynomial with rational coefficients of degree at most $d$.*

**Proof.** First, consider the case $r = 1$; we proceed by induction on $d$. If $d = 0$, then $\chi(X, D) = h^0(X, \mathcal{O}_X) \in \mathbb{N}$, so the thesis trivially holds.
Assume that the thesis is true for any projective $K$-scheme of dimension strictly less then $d$. By Proposition 2.7.12, we can write $D = D_1 \sim E_1 - E_2$ with $E_1, E_2$ effective, which in particular tells us that $\mathcal{O}_X(mD - E_1) = \mathcal{O}_X((m-1)D - E_2)$. By applying Proposition 1.2.18 to the exact sequences $0 \to \mathcal{O}_X(mD - E_1) \to \mathcal{O}_X(mD) \to \mathcal{O}_{E_1}(mD) \to 0$ and $0 \to \mathcal{O}_X((m-1)D - E_2) \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_{E_2}((m-1)D) \to 0$, we get that

$\chi(X, mD) - \chi(E_1, mD) = \chi(X, mD - E_1) = \chi(X, (m-1)D - E_2) = \chi(X, (m-1)D) - -\chi(E_2, (m-1)D)$, so $\chi(X, mD) - \chi(X, (m-1)D) = \chi(E_1, mD) - \chi(E_2, (m-1)D)$. By inductive hypothesis, the right-hand side of this equality is a rational polynomial function in $m$ of degree $\delta < d$. By Lemma 3.4.5, $\chi(X, mD)$ is a rational polynomial function in $m$ of degree $\delta + 1 \leqslant d$. This concludes the case $r = 1$.

We now consider the general case. For any Cartier divisor $D_0$ on $X$, by the same proof as the case $r = 1$ after tensoring both the exact sequencies by $\mathcal{O}_X(D_0)$, we get that $m \mapsto \chi(X, D_0 + mD)$ is a rational polynomial function of degree strictly less than $d$. Therefore, by the previous Lemma there exists a polynomial $P \in \mathbb{Q}[x_1, \ldots, x_r]$ s.t. $\chi(X, \sum_{i=1}^{r} m_i D_i) = P(m_1, \ldots, m_r)$ for all $(m_1, \ldots, m_r) \in \mathbb{Z}^r$. If $degP = \tilde{\delta}$, let $n_1, \ldots, n_r \in \mathbb{Z}$ be s.t. the degree of $Q(y) = P(n_1 y, \ldots, n_r y)$ is still $\tilde{\delta}$. Since $Q(m) = \chi(X, m \sum_{i=1}^{r} n_i D_i)$, it follows from the case $r = 1$ that $\tilde{\delta} \leqslant d$. $\square$

**Definition 3.4.8.** *Given a projective $K$-scheme $(X, \mathcal{O}_X)$ of dim $\leqslant r$, let $D_1, \ldots, D_r$ be Cartier divisors on $X$. We define the intersection number $(D_1 \cdot \ldots \cdot D_r)$ as the coefficient of $\prod_{i=1}^{r} m_i$ in $\chi(X, \sum_{i=1}^{r} m_i D_i) \in \mathbb{Q}[m_1, \ldots, m_r]$.*

**Remark 3.4.9.** *Since the Euler-Poincaré characteristic is defined using the $\mathcal{O}_X(D_i)$, this number only depends on the linear equivalence classes of the divisors $D_i$. Moreover, by Theorem* 3.4.7 *we get that $(D_1 \cdot \ldots \cdot D_r)$ is an integer that vanish if $dimX < r$ since $\chi(X, \sum_{i=1}^{r} m_i D_i)$ has degree at most $dimX$.*

**Remark 3.4.10.** *Taken $\alpha \in \mathbb{Z}$, then $(\alpha D_1 \cdot D_2 \cdot \ldots \cdot D_r)$ is the coefficient of $\prod_{i=1}^{r} m_i$ in $\chi(X, m_1(\alpha D_1) + \sum_{i=2}^{r} m_i D_i)$, which is $\alpha$ times the coefficient of the same monomial in $\chi(X, \sum_{i=1}^{r} m_i D_i)$. Therefore, we have that $(\alpha D_1 \cdot D_2 \cdot \ldots \cdot D_r) = \alpha (D_1 \cdot \ldots \cdot D_r)$.*

**Remark 3.4.11.** *For any polynomial $P(x_1, \ldots, x_r)$ of degree at most $r$, the coefficient of $\prod_{i=1}^{r} x_i$ in $P$ is $\sum_{I \subseteq \{1, \ldots, n\}} \alpha_I \, P(-x^I)$, where $\alpha_I = (-1)^{|I|}$ and $x_i^I = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{otherwise} \end{cases}$.*

**Proposition 3.4.12.** *Given a projective $K$-scheme $(X, \mathcal{O}_X)$ of dim $\leqslant r$, let $D_1, \ldots, D_r$ be Cartier divisors on $X$. Then, $(D_1 \cdot \ldots \cdot D_r) = \sum_{I \subseteq \{1,\ldots,n\}} \alpha_I \, \chi(X, -\sum_{i \in I} D_i)$.*

**Proof.** It comes directly from the previous Remark. □

**Remark 3.4.13.** *The previous Proposition shows that this definition coincides with the one for surfaces given in Definition* 3.2.5. *Moreover, by the Riemann-Roch Theorem* 3.1.7 *we get that on a curve it holds $deg(D) = (D)$.*

**Remark 3.4.14.** *If $X$ has dimension $n$ and $H$ is a hyperplane section of $X$, then $(H^n) = degX$. More generally, if $K$ is algebraically closed and $D_1, \ldots, D_n$ are effective and meet properly in a finite number of points, then $(D_1 \cdot \ldots \cdot D_n)$ is the number of points in $\bigcap_{i=1}^{n} D_i$ counted with multiplicity, i.e. the length of $\bigcap_{i=1}^{n} D_i$ as a 0-dimensional scheme.*

**Definition 3.4.15.** *Given a morphism $f : C \to X$ from a projective curve to a quasi-projective scheme, let $D$ be a Cartier divisor on $X$. Then, we define $(C \cdot D) := deg(f^*D)$.*

**Proposition 3.4.16.** *Given a projective $K$-scheme $(X, \mathcal{O}_X)$ of dim $= n$, let $D$ be a Cartier divisor on $X$. Then, $\chi(X, mD) = \frac{(D^n)}{n!} m^n + O(m^{n-1})$.*

**Proof.** Since for $m$ large enough the function $m \mapsto \chi(X, mD)$ is a polynomial of the form $P(x) = \sum_{j=0}^{n} a_j x^j$, $(D^n)$ is the coefficient of $\prod_{k=1}^{n} m_k$ in $\chi(X, \sum_{i=1}^{n} m_i D) = P(\sum_{i=1}^{n} m_i) =$
$= \sum_{j=0}^{n} a_j (m_1 + \ldots + m_n)^j$, i.e. $a_n n!$ . Therefore, $\chi(X, mD) = P(m) = a_n m^n + O(m^{n-1}) =$
$= \frac{(D^n)}{n!} m^n + O(m^{n-1})$. □

**Proposition 3.4.17.** *Given a projective $K$-scheme $(X, \mathcal{O}_X)$ of dim $= n$, let $D_1, \ldots, D_n$ be Cartier divisors on $X$. Then:*

1. *the map $(D_1, \ldots, D_n) \mapsto (D_1 \cdot \ldots \cdot D_n)$ is $\mathbb{Z}$-multilinear, symmetric and takes integral values;*

2. *if $D_n$ is effective, then $(D_1 \cdot \ldots \cdot D_n) = (D_1|_{D_n} \cdot \ldots \cdot D_{n-1}|_{D_n})$.*

**Proof.** (*1*) We already observed in Remark 3.4.9 that the map above takes integral values; moreover, it is symmetric by definition. It remains to prove that it is $\mathbb{Z}$-multilinear.
Let $D_1, D_1', D_2, \ldots, D_n$ be Cartier divisors on $X$; we want to show that
$((D_1 + D_1') \cdot D_2 \cdot \ldots \cdot D_n) = (D_1 \cdot D_2 \cdot \ldots \cdot D_n) + (D_1' \cdot D_2 \cdot \ldots \cdot D_n)$. Writing
$((D_1 + D_1') \cdot D_2 \cdot \ldots \cdot D_n) = \sum_{I \subseteq \{2,\ldots,n\}} \alpha_I \Big( \chi(X, -\sum_{i \in I} D_i) - \chi(X, -D_1 - D_1' - \sum_{i \in I} D_i) \Big)$ and
$(D_1 \cdot D_2 \cdot \ldots \cdot D_n) + (D_1' \cdot D_2 \cdot \ldots \cdot D_n) = \sum_{I \subseteq \{2,\ldots,n\}} \alpha_I \Big( 2\chi(X, -\sum_{i \in I} D_i) - \chi(X, -D_1 - \sum_{i \in I} D_i) -$
$- \chi(X, -D_1' - \sum_{i \in I} D_i) \Big)$, by subtracting the former from the latter we obtain that
$(D_1 \cdot D_2 \cdot \ldots \cdot D_n) + (D_1' \cdot D_2 \cdot \ldots \cdot D_n) - ((D_1 + D_1') \cdot D_2 \cdot \ldots \cdot D_n) = \sum_{I \subseteq \{2,\ldots,n\}} \alpha_I \Big( \chi(X, -\sum_{i \in I} D_i) -$
$- \chi(X, -D_1 - \sum_{i \in I} D_i) - \chi(X, -D_1' - \sum_{i \in I} D_i) + \chi(X, -D_1 - D_1' - \sum_{i \in I} D_i) \Big) =$
$= (D_1 \cdot D_1' \cdot D_2 \cdot \ldots \cdot D_n) = 0$, where the last equality comes from Remark 3.4.9. This shows what we wanted.
Since by Remark 3.4.10 we know that $(aD_1 \cdot D_2 \cdot \ldots \cdot D_n) = a(D_1 \cdot D_2 \cdot \ldots \cdot D_n)$, we can conclude.
(*2*) We can write $(D_1 \cdot \ldots \cdot D_n) = \sum_{I \subseteq \{1,\ldots,n-1\}} \alpha_I \Big( \chi(X, -\sum_{i \in I} D_i) - \chi(X, -D_n - \sum_{i \in I} D_i) \Big)$.
Since $D_n$ is effective, we have the short exact sequence
$0 \to \mathcal{O}_X(-D_n - \sum_{i \in I} D_i) \to \mathcal{O}_X(-\sum_{i \in I} D_i) \to \mathcal{O}_{D_n}(-\sum_{i \in I} D_i) \to 0$. Therefore, we get that
$(D_1 \cdot \ldots \cdot D_n) = \sum_{I \subseteq \{1,\ldots,n-1\}} \alpha_I \, \chi(D_n, -\sum_{i \in I} D_i) = (D_1|_{D_n} \cdot \ldots \cdot D_{n-1}|_{D_n})$. □

**Proposition 3.4.18 (Pull-back formula).** *Given a surjective morphism $\varphi : X \to Y$ between projective varieties, let $D_1, \ldots, D_r$ be Cartier divisors on $Y$ with $r \geqslant dim X$. Then, we have that $(\varphi^* D_1 \cdot \ldots \cdot \varphi^* D_r) = deg(\varphi)(D_1 \cdot \ldots \cdot D_r)$.*

**Proof.** A sketch of the proof can be found in [1], Proposition 3.16. □

**Definition 3.4.19.** *Given a morphism $\varphi : X \to Y$ between projective varieties, let $C$ be a curve on $X$. We define $\varphi_* C := \begin{cases} 0 & \text{if } \varphi(C) \text{ is a point} \\ d\varphi(C) & \text{if } \varphi(C) \text{ is a curve} \end{cases}$, where in the second case $d$ is the degree of $\varphi|_C : C \to \varphi(C)$.*

**Proposition 3.4.20** (**Projection formula for curves**). *Given a morphism $\varphi : X \to Y$ between projective varieties, let $C$ be a curve on $X$ and $D$ a Cartier divisor on $Y$. Then, we have that $(\varphi_* C \cdot D) = (C \cdot \varphi^* D)$.*

**Proof.** It comes directly from the Pull-back formula. □

**Definition 3.4.21.** *Given a $K$-scheme $(X, \mathcal{O}_X)$, a point $p \in X$ is said to be a $K$-point if $K(p) \cong K$.*

**Corollary 3.4.22.** *Let $(X, \mathcal{O}_X)$ be a curve of genus $0$ over $K$. If $X$ has a $K$-point, then $X \cong \mathbb{P}^1_K$.*

**Proof.** Let $p$ be a $K$-point of $X$. Since $H^1(X, \mathcal{O}_X) = 0$, the long exact sequence in cohomology associated to the short exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}_{X,p} \to K(p) \to 0$ is $0 \to \mathcal{O}_X(X) \to \mathcal{O}_{X,p}(X) \to K_p \to 0$. In particular, $h^0(X, \mathcal{O}_{X,p}) = 2$, so the invertible sheaf $\mathcal{O}_{X,p}$ is generated by two global sections; those define a finite morphism $\varphi : X \to \mathbb{P}^1_K$ s.t. $\varphi^* \mathcal{O}_{\mathbb{P}^1_K}(1) \cong \mathcal{O}_{X,p}$. By the Projection formula for curves, we get that
$1 = deg(\mathcal{O}_{X,p}) = deg(\varphi)$, so $\varphi$ is an isomorphism. □

**Remark 3.4.23.** *One can define by linearity the intersection number of $\mathbb{Q}$-Cartier divisors.*

# Chapter 4

# Cones of curves

All results in this chapter will be stated for Cartier divisors, but they clearly hold also for $\mathbb{Q}$-Cartier divisors.

## 4.1 The Nakai-Moishezon ampleness criterion

**Theorem 4.1.1 (Nakai-Moishezon criterion).** *A Cartier divisor $D$ on a projective $K$-scheme $X$ is ample $\iff$ $((D|_Y)^r) > 0$ for every integral subscheme $Y$ of $X$ of dimension $r$.*

**Proof.** ($\Longrightarrow$) By hypothesis $\exists m > 0$ s.t. $mD$ is very ample, so there is an embedding $i : X \hookrightarrow \mathbb{P}^n_K$ s.t. s.t. $\mathcal{O}_X(mD) \cong i^*\mathcal{O}_{\mathbb{P}^n_K}(1)$. This tells us that for any (closed) subscheme $Y$ of $X$ of dimension $r$ we have $((mD|_Y)^r) = deg(i(Y))$, which is strictly positive thanks to Proposition 3.4.3.

($\Longleftarrow$) By Proposition 2.8.6, we may assume that $X$ is integral. We will show by induction on $dimX$ that $D$ is ample on $X$.

Written $D \sim E_1 - E_2$ as the difference of two effective divisors, we have the exact sequences $0 \to \mathcal{O}_X(mD - E_1) \to \mathcal{O}_X(mD) \to \mathcal{O}_{E_1}(mD) \to 0$ and $0 \to \mathcal{O}_X((m-1)D - E_2) \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_{E_2}((m-1)D) \to 0$. Since by induction $D$ is ample on both $E_1$ and $E_2$, we have that for all $m$ large enough $H^i(E_j, mD) = \{0\} \quad \forall j = 1, 2,\ i > 0$. By taking the long sequences of cohomology, we respectively get $H^i(X, mD - E_1) \cong H^i(X, mD)$ and $H^i(X, (m-1)D - E_2) \cong$ $\cong H^i(X, (m-1)D)$ for all $i \geqslant 2$. Since $\mathcal{O}_X(mD - E_1) = \mathcal{O}_X((m-1)D - E_2)$, it follows that

$H^i(X, mD) \cong H^i(X, (m-1)D)$ $\forall i > 0$, so in particular $h^i(X, mD) = h^i(X, (m-1)D)$; this tells us that $\chi(X, mD)$ is asymptotic to $h^0(X, mD) - h^1(X, mD)$ as $m \to +\infty$.
Now, if $dim X = n$, by Proposition 3.4.16 we have that $\chi(X, mD) =$
$= \frac{(D^n)}{n!} m^n + O(m^{n-1}) \xrightarrow[m \to +\infty]{} +\infty$ because $(D^n) > 0$ by hypothesis, thus
$h^0(X, mD) - h^1(X, mD) \xrightarrow[m \to +\infty]{} +\infty$, so the same surely happens to $h^0(X, mD)$.
To prove that $D$ is ample we can replace it with any positive multiple, so we may assume that $D$ is effective; therefore, we have the exact sequence
$0 \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_X(mD) \to \mathcal{O}_D(mD) \to 0$. Since $D$ is ample on $D$ by induction hypothesis, by Theorem 2.8.1 we get that $H^1(D, mD) = \{0\}$ for all $m$ large enough; from this, by taking the long sequence of cohomology one gets a surjection
$H^1(X(m-1)D) \to H^1(X, mD)$. This tells us that $\{h^1(X, mD)\}_{m>0}$ is a decreasing sequence which becomes stationary as $m$ goes to infinity, so for $m$ large enough the previous map is a bijection; then by Lemma 3.1.6 we get that the map
$H^0(X, mD) \to H^0(D, mD)$ is surjective. This implies the surjectivity of the evaluation map $\nu_x$ for the sheaf $\mathcal{O}_X(mD)$ at every point, thus $\mathcal{O}_X(mD)$ is globally generated for $m$ large enough. Therefore, its global sections define a morphism $\varphi : X \to \mathbb{P}^n_K$ s.t. $\mathcal{O}_X(mD) = \varphi^* \mathcal{O}_{\mathbb{P}^n_K}(1)$. Since $D$ has positive degree on every curve by hypothesis, $\varphi$ has finite fibers hence, being projective, is finite; Corollary 2.8.5 allows us to conclude. $\square$

**Remark 4.1.2.** *This Theorem generalizes Corollary 3.1.8.*

## 4.2 Nef divisors

**Definition 4.2.1.** *A Cartier divisor $D$ on a projective scheme $X$ is nef if it satisfies $((D|_Y)^r) \geqslant 0$ for every subscheme $Y$ of $X$ of dimension $r$.*

**Remark 4.2.2.** *The restriction of a nef divisor to a subscheme is again nef.*
*By the Pull-back formula 3.4.18, the pull-back of a nef divisor by any morphism between projective schemes is still nef.*
*Ample $\Rightarrow$ nef.*
*A divisor $D$ on a curve is nef $\iff$ $deg(D) \geqslant 0$.*

**Lemma 4.2.3.** *Let $X$ be a projective $K$-scheme of dim $= n$. If the Cartier divisors $D$ and $H$ on $X$ are respectively nef and ample, then $(D^r \cdot H^{n-r}) \geqslant 0$.*

**Proof.** We proceed by induction on $n$. By ampleness of $H$, $\exists m > 0$ s.t. $mH$ is very ample; then, let $E$ be an effective divisor in $|mH|$. If $r = n$, then the thesis comes directly from the fact that $D$ is nef. If $r < n$, using Proposition 3.4.17-*(2)* one gets that $(D^r \cdot H^{n-r}) = \frac{1}{m}(D^r \cdot H^{n-r-1} \cdot mH) = \frac{1}{m}((D|_E)^r \cdot (H|_E)^{n-r-1})$, which is nonnegative by induction hypothesis. □

**Proposition 4.2.4.** *Let $D$ and $H$ be respectively a nef and an ample Cartier divisor on the same projective $K$-scheme $X$. Then, $D + tH$ is ample for any $t > 0$.*

**Proof.** It suffices to show that $D + H$ is ample. Let $Y$ be an $r$-dimensional subscheme of $X$. Since $D|_Y$ is nef, by Lemma 4.2.3 we have that $((D|_Y)^s \cdot (H|_Y)^{r-s}) \geqslant 0$ for all $0 \leqslant s \leqslant r$, hence $((D|_Y + H|_Y)^r) = \sum_{s=0}^{r} \binom{r}{s}((D|_Y)^s \cdot (H|_Y)^{r-s}) \geqslant ((H|_Y)^r) > 0$ because $H|_Y$ is ample. By the Nakai-Moishezon criterion 4.1.1 we can conclude. □

**Proposition 4.2.5.** *On a projective $K$-scheme, a sum of nef divisors is nef.*

**Proof.** Let $D$ and $E$ be nef divisors on a projective $K$-scheme $X$, and let $H$ be an ample divisor on $X$. By the previous Proposition, $E+tH$ is ample $\forall t > 0$, and so is $D+(E+tH)$. For every subscheme $Y$ of $X$ of dimension $r$ we then have that $((D|_Y + E|_Y + tH|_Y)^r) > 0$ thanks to the Nakai-Moishezon criterion 4.1.1. By taking the limit as $t \to 0$, multilinearity assures us that $((D|_Y + E|_Y)^r) \geqslant 0$, i.e. $D + E$ is nef. □

**Proposition 4.2.6.** *Let $X$ be a projective $K$-scheme. Then, a Cartier divisor on $X$ is nef $\iff$ it is nef on each irreducible component of $X_{red}$.*

**Theorem 4.2.7.** *Let $X$ be a projective $K$-scheme. Then, a Cartier divisor $D$ on $X$ is nef $\iff$ it has nonnegative intersection with every curve on $X$.*

**Proof.** ($\Longrightarrow$) It comes directly from the definition.

($\Longleftarrow$) By the previous Proposition, we may assume that $X$ is integral. Proceeding by induction on $n = dimX$, it is enough to prove that $(D^n) \geqslant 0$. Let $H$ be an ample divisor on $X$, and set $D_t = D + tH$; then, consider the polynomial $P(t) = (D_t^n) = = \sum_{m=0}^{n} \binom{n}{m} (D^{n-m} \cdot H^m) t^m$ of $deg = n$. If we manage to show that $P(0) \geqslant 0$, then we are done. Assume by contradiction that $P(0) < 0$; since the leading coefficient of $P(t)$ is $(H^n) > 0$ (because $H$ is ample), then $\lim_{t \to +\infty} P(t) = +\infty$, so $P(t)$ has a largest positive real root $t_0$ and $P(t) > 0 \ \ \forall t > t_0$. Taken a subscheme $Y$ of $X$ of $dim = r < n$, surely $D|_Y$ is nef by inductive hypothesis. Since $H|_Y$ is still ample, it holds $((H|_Y)^r) > 0$; moreover, by Lemma 4.2.3 we have that $((D|_Y)^s \cdot (H|_Y)^{r-s}) \geqslant 0$ for all $0 \leqslant s \leqslant r$. This implies that for any $t > 0$ we have $((D_t|_Y)^r) = \sum_{s=0}^{r} \binom{r}{s} ((D|_Y)^{r-s} \cdot (H|_Y)^s) t^s \geqslant ((H|_Y)^r) > 0$ for all $t > 0$. Moreover, since $(D_t^n) = P(t) > 0 \ \ \forall t > t_0$, the Nakai-Moishezon criterion 4.1.1 implies that $D_t$ is ample for all $t > t_0$. Now, let us define the polynomials $Q(t) = (D_t^{n-1} \cdot D) = = \left( \sum_{m=0}^{n-1} \binom{n-1}{m} (D^{n-m-1} \cdot H^m) t^m \cdot D \right) = \sum_{m=0}^{n-1} \binom{n-1}{m} (D^{n-m} \cdot H^m) t^m$ and

$R(t) = t(D_t^{n-1} \cdot H) = \sum_{m=0}^{n-1} \binom{n-1}{m} (D^{n-m-1} \cdot H^{m+1}) t^{m+1} = \sum_{m=1}^{n} \binom{n-1}{m-1} (D^{n-m} \cdot H^m) t^m$;

notice that their sum is $Q(t)+R(t) = (D^n) + \sum_{m=1}^{n-1} \left( \binom{n-1}{m} + \binom{n-1}{m-1} \right) (D^{n-m} \cdot H^m) t^m + + (H^n) t^n = \sum_{m=0}^{n} \binom{n}{m} (D^{n-m} \cdot H^m) t^m = P(t)$. Since $D_t$ is ample for $t > t_0$ and $D$ has nonnegative degree on curves, by Lemma 4.2.3 we get that $Q(t) \geqslant 0 \ \ \forall t \geqslant t_0$; with the same Lemma and the induction hypothesis we have that $(D^r \cdot H^{n-r}) \geqslant 0$ for all $0 \leqslant r < n$, which implies that $R(t_0) \geqslant (H^n) t_0^n > 0$. From this we get the contradiction $0 = P(t_0) = Q(t_0) + R(t_0) \geqslant R(t_0) > 0$, and the proof is done. $\square$

## 4.3 Elementary properties of cones

**Definition 4.3.1.** *A subset $C$ of a vector space $V$ over an ordered field $(K, \leqslant)$ is a (linear) cone if $\alpha x \in C \ \ \forall x \in C, \alpha > 0$. A cone $C$ is a convex cone if $\alpha x + \beta y \in C \ \ \forall x, y \in C$, $\alpha, \beta > 0$, or equivalently if $C + C \subseteq C$.*

**Remark 4.3.2.** *A subset $C$ of a $K$-v.s. is a convex cone $\iff$ $\alpha C = C \ \ \forall \alpha > 0$ and $C + C = C$.*

**Definition 4.3.3.** *In an Euclidean space, the convex hull of a set $X$ is the minimal convex set containing $X$.*

**Remark 4.3.4.** *The convex hull of $X$ is the intersection of all convex sets containing $X$, or equivalently the set of all convex combinations of points in $X$.*

**Definition 4.3.5.** *Let $C$ be a cone in the $K$-v.s. $V$; we define its dual cone as $C^* = \{v^* \in V^* | v^*(x) \geqslant 0 \ \ \forall x \in C\}$.*

**Remark 4.3.6.** *The dual cone is a convex cone. Moreover, $W \subseteq V \Rightarrow V^* \subseteq W^*$.*

**Definition 4.3.7.** *Let $V$ be a cone in $\mathbb{R}^m$. A subcone $W$ of $V$ is extremal if it is closed and convex and if any two elements of $V$ whose sum is in $W$ are both in $W$. Given an extremal subcone $W$ of $V$, a nonzero linear form $\ell \in V^*$ is a supporting function of $W$ if it vanishes on it. An extremal subcone of dim $= 1$ is called an extremal ray.*

**Lemma 4.3.8.** *Let $V$ be a convex cone in $\mathbb{R}^m$. Then:*

1. *$V = V^{**}$;*
2. *$V$ contains no lines $\Longleftrightarrow \langle V^* \rangle = (\mathbb{R}^m)^*$;*
3. *$(\overset{\circ}{V^*}) = \{\ell \in (\mathbb{R}^m)^* | \ell > 0$ on $V \smallsetminus \{0\}\}$; in particular, for any $v \in \partial V$ there exists $0 \neq \ell \in V^*$ s.t. $\ell(v) = 0$;*
4. *given $\ell \in V^*$, any extremal ray in $ker(\ell) \cap V$ is still extremal in $V$;*
5. *if $V$ contains no lines, it is the convex hull of its extremal rays;*
6. *any proper extremal subcone of $V$ is contained in $\partial V$; in particular, it has a supporting function;*
7. *if $V$ contains no lines and $W$ is a proper subcone of $V$, there exists a linear form in $V^*$ which is positive on $W \smallsetminus \{0\}$ and vanishes on some extremal ray of $V$.*

**Proof.** *(1)* Since $V^{**} = \{v \in (\mathbb{R}^m)^{**} | v \geqslant 0 \text{ on } V^*\} = \{v \in \mathbb{R}^m | \ell(v) \geqslant 0 \ \forall \ell \in V^*\} \supseteq V$, it remains to show $V^{**} \subseteq V$. Choose a scalar product $\langle \_, \_ \rangle$ on $\mathbb{R}^m$; pick $z \notin V$, and let

$p_V(z)$ be its projection on $V$. Since $V$ is a cone, $z - p_V(z) \perp p_V(z)$; therefore, the linear form $\langle p_V(z) - z, \_\rangle$ is nonnegative on $V$ and negative at $z$, hence $z \notin V^{**}$.
*(2)* ($\Longleftarrow$) If $V$ contains a line $L$, for any $v^* \in V^*, \ell \in L$ and $\lambda \in \mathbb{R}$ we have that $\lambda v^*(\ell) = v^*(\lambda\ell) \geqslant 0$, so in particular both $v^*(\ell)$ and $-v^*(\ell)$ are nonnegative, hence it must be $v^*(\ell) = 0$. This tells us that $V^* \subseteq L^\perp$.
($\Longrightarrow$) If $\langle V^*\rangle \neq (\mathbb{R}^m)^*$, then $V^*$ is contained in a hyperplane $H$, hence the line $H^\perp$ is contained in $V^{**} = V$.
*(3)* Let $\ell \in (\mathring{V}^*)$; for any $0 \neq v \in V$, there exists a linear form $\ell'$ with $\ell'(v) > 0$ and small enough so that $\ell - \ell' \in V^*$. This implies $(\ell - \ell')(v) \geqslant 0$, hence $\ell(v) \geqslant \ell'(v) > 0$: this proves ($\subseteq$). Since the set $\{\ell \in (\mathbb{R}^m)^* | \ell > 0 \text{ on } V \setminus \{0\}\}$ is open, also ($\supseteq$) holds, so we are done (since the consequence is immediate).
*(4)* Consider the extremal ray $\mathbb{R}^+ r$ in $ker(\ell) \cap V$. If $r = x_1 + x_2$ with $x_i \in V \ \ \forall i = 1, 2$, since $\ell(x_i) \geqslant 0$ and $\ell(r) = 0$ we have that $\ell(x_i) = 0$, hence $x_i \in ker(\ell) \cap V$. By the fact that $\mathbb{R}^+ r$ is extremal in $ker(\ell) \cap V$, we get that $x_i \in \mathbb{R}^+ r$.
*(5)* We want to prove by induction on $m$ that any point of $V$ is in the linear span of $m$ extremal rays. Let $0 \neq \ell \in V^*$; by induction hypothesis, there exists an extremal ray $\mathbb{R}^+ r$ in $ker(\ell) \cap V$; by *(4)*, it is still extremal in $V$. Taken $v \in V$, the set $\{\lambda \in \mathbb{R}^+ | v - \lambda r \in V\}$ is a closed nonempty (since at least it contains 0) interval which is bounded above: otherwise, $-r = \lim_{\lambda \to +\infty} \frac{1}{\lambda}(v - \lambda r) \in V$, which contradicts the fact that $V$ contains no lines. If $\lambda_0$ is its maximum, then $v - \lambda_0 r \in \partial V$, so by *(3)* there exists $0 \neq \ell' \in V^*$ that vanishes at $v - \lambda_0 r$. Since $v = \lambda_0 r + (v - \lambda_0 r)$, from the induction hypothesis applied to the convex cone $ker(\ell') \cap V$ and *(4)* we can conclude.
*(6)* Let $W$ be a proper extremal subcone of $V$. If $W$ contains a point $v \in \mathring{V}$, then for any small $x \in V$ we have $v \pm x \in V$, hence $2v = (v + x) + (v - x) \in V$ implies $v \pm x \in W$. This shows that $\mathring{V} \subseteq W$, so by taking the closure we get (since $W$ is closed) that $V \subseteq W \subseteq V \Rightarrow W = V$, which is a contradiction. Therefore, $W \subseteq \partial V$. Now, take $0 \neq w \in W \subseteq \partial V$; by *(3)*, there exists $0 \neq \ell \in V^* \subseteq W^*$ s.t. $\ell(w) = 0$. Hence, $\ell \in W^* \setminus (\mathring{W}^*)$, i.e. $\ell = 0$ on $W$.
*(7)* Since W contains no lines, there exists by *(2)* a point in $(\mathring{W}^*) \setminus V^*$. The segment

connecting it to a point in $(\overset{\circ}{V^*})$ crosses $\partial(V^*)$ at a point which is contained in $(\overset{\circ}{W^*})$. This point corresponds to a linear form $\ell$ that is positive on $W \setminus \{0\}$ and vanishes at a nonzero point of $V$. By *(5)*, the cone $ker(\ell) \cap V$ has an extremal ray, which is still extremal in $V$ thanks to *(4)*: this allows us to conclude. $\square$

## 4.4 The cone of curves and the effective cone

**Definition 4.4.1.** *Given a projective $K$-scheme $X$, a real $1$-cycle on $X$ is a finite formal linear combinations with real coefficients of irreducible curves in $X$; it is said to be effective if all its coefficients are nonnegative.*

**Definition 4.4.2.** *Given a projective $K$-scheme $X$, we say that two Cartier divisors $D, D'$ on $X$ are numerically equivalent if they have the same degree on every curve $C$ on $X$, i.e. if $(D \cdot C) = (D' \cdot C)$ $\forall C$; in this case, we write $D \equiv D'$. Since this is clearly an equivalence relation, we can set $N^1(X)_{\mathbb{Z}} := \{\text{Cartier divisors on } X\}/\equiv$, $N^1(X)_{\mathbb{Q}} := N^1(X)_{\mathbb{Z}} \otimes \mathbb{Q}$ and $N^1(X)_{\mathbb{R}} := N^1(X)_{\mathbb{Z}} \otimes \mathbb{R}$. The last two are finite-dimensional v.s., and their dim is called the Picard number of $X$ and is denoted by $\rho_X$.*

**Definition 4.4.3.** *We say that a property of a divisor is numerical if it depends only on its numerical equivalence class, i.e. if it depends only of its intersection numbers with real $1$-cycles.*

**Definition 4.4.4.** *Given a projective $K$-scheme $X$, we say that two $1$-cycles $C, C'$ on $X$ are numerically equivalent if they have the same intersection number with every Cartier divisor; in this case, we write $C \equiv C'$. Since this is an equivalence relation, we can set $N_1(X)_{\mathbb{Z}} := \{\text{real } 1\text{-cycles on } X\}/\equiv$, $N_1(X)_{\mathbb{Q}} := N_1(X)_{\mathbb{Z}} \otimes \mathbb{Q}$ and $N_1(X)_{\mathbb{R}} := N_1(X)_{\mathbb{Z}} \otimes \mathbb{R}$.*

**Remark 4.4.5.** *The intersection pairing $(\_ \cdot \_) : N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R}$ is nondegenerate by definition. This implies that $dim N^1(X)_{\mathbb{R}} = dim N_1(X)_{\mathbb{R}}$, hence $N_1(X)_{\mathbb{R}}$ is a finite-dimensional $\mathbb{R}$-v.s.*

**Remark 4.4.6.** *Since $X$ is projective, no numerical class of curve is $0$ in $N_1(X)_{\mathbb{R}}$.*

**Definition 4.4.7.** *Let $X$ be a projective $K$-scheme. We define the cone of curves as* $NE(X) := \{[C]_\equiv \in N_1(X)_\mathbb{R} | C \text{ is effective}\}$, *and the effective cone as* $NE^1(X) := \{[D]_\equiv \in N^1(X)_\mathbb{R} | D \text{ is effective}\}$. *Their closures are denoted by* $\overline{NE}(X)$ *and* $\overline{NE}^1(X)$ *respectively; we will refer to* $\overline{NE}^1(X)$ *as the pseudo-effective cone on* $X$.

## 4.5 A numerical characterization of ampleness

**Notation.** *Let $D$ be a Cartier divisor on a projective variety $X$. Then, for any* $[C]_\equiv \in N_1(X)_\mathbb{R}$ *we set* $D \cdot [C]_\equiv := (D \cdot C)$. *Notice that this is well-defined, i.e. it does not depend on the choice of the representative, simply by definition of* $N_1(X)_\mathbb{R}$.

**Theorem 4.5.1 (Kleiman's criterion).** *Let $X$ be a projective variety. Then:*

1. *a Cartier divisor $D$ on $X$ is ample* $\iff$ $D \cdot z > 0 \ \ \forall 0 \neq z \in \overline{NE}(X)$;
2. *for any ample divisor $H$ on $X$ and any $k \in \mathbb{Z}$, the set* $\{z \in \overline{NE}(X) | H \cdot z \leqslant k\}$ *is compact, so in particular it contains only finitely many classes of curves.*

**Proof.** *(1)* ($\Longrightarrow$) Let $z \in \overline{NE}(X) \smallsetminus \{0\}$; since $D$ is nef, we have $D \cdot z \geqslant 0$. Assume by contradiction that $D \cdot z = 0$. Since the intersection pairing is nondegenerate (and $z \neq 0$), there exists a divisor $E$ such that $E \cdot z \neq 0$; at most by changing the sign of $E$, we can assume that $E \cdot z < 0$, from which $(D + tE) \cdot z < 0 \ \ \forall t > 0$. This tells us that $D + tE$ is not ample even if $D$ is, contradicting Remark 2.6.9.
($\Longleftarrow$) Choose a norm $\| \_ \|$ on $\overline{NE}(X)$ (notice that it exists: for example, one can choose $\| \_ \|_D$ defined as $\|z\|_D = D \cdot z$, which is a good definition since $D \cdot z \geqslant 0$ by hypothesis); then, the set $T = \| \_ \|^{-1}(\{1\})$ is compact. Since $T \subseteq \overline{NE}(X) \smallsetminus \{0\}$, the linear map $z \mapsto D \cdot z$ is positive on $T$, so by Weierstrass Thm. it is bounded from below by a positive rational number $a$. Let $H$ be an ample divisor on $X$; by the same Thm., the linear map $z \mapsto H \cdot z$ is bounded from above on $T$ by a positive rational number $b$. It follows that $D - \frac{a}{b}H$ is nonnegative on $T$, hence on the whole cone $\overline{NE}(X)$ (because every $0 \neq z \in \overline{NE}(X)$ is just a positive multiple of an element of $T$). This shows that $D - \frac{a}{b}H$ is nef, hence by Proposition 4.2.4 we conclude that $D = (D - \frac{a}{b}H) + \frac{a}{b}H$ is ample.
*(2)* Let $\mathfrak{B} = \{[D_1]_\equiv, \ldots, [D_n]_\equiv\}$ be a basis for $N^1(X)_\mathbb{R}$. By Proposition 2.6.9, there

exists $m > 0$ s.t. $mH \pm D_i$ is ample $\forall i = 1, \dots, n$. For any $z \in \overline{NE}(X)$ we then have $(mH \pm D_i) \cdot z \geqslant 0$, hence $|D_i \cdot z| \leqslant mH \cdot z$. If $H \cdot z \leqslant k$, this bounds the coordinates of $z$ in the dual basis $\mathfrak{B}^*$, therefore it defines a closed bounded set, i.e. a compact one. Since the classes of curves in this set have integral coordinates in $\mathfrak{B}^*$, so they form a discrete set in $N_1(X)_{\mathbb{R}}$, we conclude that they are at most finitely many in $\{z \in \overline{NE}(X) | H \cdot z \leqslant k\}$. □

## 4.6 Around the Riemann-Roch theorem

**Proposition 4.6.1.** *Let $D$ be a Cartier divisor on a projective $K$-scheme $X$ of dim $= n$. Then:*

1. $h^i(X, mD) = O(m^n)$ $\forall i \in \mathbb{N}$;
2. *if $D$ is nef, then* $h^i(X, mD) = O(m^{n-1})$ $\forall i > 0$ *and* $h^0(X, mD) = \frac{(D^n)}{n!} m^n + O(m^{n-1})$.

**Proof.** *(1)* Written $D \sim E_1 - E_2$ as the difference of two effective Cartier divisors, we have the exact sequences $0 \to \mathcal{O}_X(mD - E_1) \to \mathcal{O}_X(mD) \to \mathcal{O}_{E_1}(mD) \to 0$ and $0 \to \mathcal{O}_X((m-1)D - E_2) \to \mathcal{O}_X((m-1)D) \to \mathcal{O}_{E_2}((m-1)D) \to 0$. From the long exact sequences in cohomology and the previous Lemma, we obtain that
$h^i(X, mD) \leqslant h^i(X, mD - E_1) + h^i(E_1, mD) = h^i(X, (m-1)D - E_2) + h^i(E_1, mD) \leqslant$
$\leqslant h^i(X, (m-1)D) + h^i(E_2, (m-1)D) + h^i(E_1, mD)$ for all $i \in \mathbb{N}$. We proceed by induction on $n$; if $n = 0$ there is nothing to prove. From the previous inequality and the induction hypothesis, we obtain that $h^i(X, mD) \leqslant h^i(X, (m-1)D) + O(m^{n-1}) \leqslant$
$\leqslant h^i(X, (m-2)D) + 2O(m^{n-1}) \leqslant \dots \leqslant mO(m^{n-1}) = O(m^n)$, so we are done.
*(2)* If $D$ is nef, so are $D|_{E_1}$ and $D|_{E_2}$; in the same way as in *(1)*, we get that
$h^i(X, mD) \leqslant h^i(X, (m-1)D) + O(m^{n-2})$, which implies $h^i(X, mD) = O(m^{n-1})$ for all $i > 0$. Therefore, $h^0(X, mD) = \chi(X, mD) + O(m^{n-1}) = \frac{(D^n)}{n!} m^n + O(m^{n-1})$ by Proposition 3.4.16. □

**Definition 4.6.2.** *A Cartier divisor $D$ on a projective $K$-scheme $X$ is said to be big if* $\limsup\limits_{m \to +\infty} \dfrac{h^0(X, mD)}{m^n} > 0$.

**Remark 4.6.3.** *From the previous Proposition, a nef Cartier divisor $D$ on a projective scheme of dim $= n$ is big $\iff$ $(D^n) > 0$. In particular, ample $\Rightarrow$ nef and big.*

**Corollary 4.6.4.** *Let $D$ be a nef and big $\mathbb{Q}$-divisor on a projective variety $X$. Then, there exists an effective $\mathbb{Q}$-divisor $E$ on $X$ s.t. $D - tE$ is ample for all rationals $t$ in $]0,1]$.*

**Proof.** Up to considering a multiple of $D$, we may assume it has integral coefficients. Set $n = dim(X)$, and let $H$ be an effective ample divisor on $X$. Since $h^0(H, mD) = O(m^{n-1})$ and $h^0(X, mD) = \frac{(D^n)}{n!} m^n + O(m^{n-1})$ by Proposition 4.6.1, from the fact that $(D^n) > 0$ by the previous Remark we get that $H^0(X, mD - H) \neq \{0\}$ for all $m$ large enough. This tells us that $mD - H$ is linearly equivalent to an effective divisor $E'$; therefore, for any rational $t \in ]0,1]$ we have that $mD = m(tD + (1-t)D) \sim t(H + E') + m(1-t)D$, which implies $D \sim (\frac{t}{m} H + (1-t)D) + \frac{t}{m} E'$. Since $\frac{t}{m} H + (1-t)D$ is ample for any choice of $t \in ]0,1]$ thanks to Proposition 4.2.4, the proof is done with $E = \frac{1}{m} E'$. □

## 4.7 Relative cone of curves

**Definition 4.7.1.** *If $\varphi : X \to Y$ is a morphism between projective varieties, we define the induced morphisms $\varphi^* : N^1(Y)_{\mathbb{Z}} \to N^1(X)_{\mathbb{Z}}$ and $\varphi_* : N_1(Y)_{\mathbb{Z}} \to N_1(X)_{\mathbb{Z}}$ respectively by $\varphi^*([D]_\equiv) = [\varphi^*(D)]_\equiv$ and $\varphi_*([C]_\equiv) = [\varphi_*(C)]_\equiv = deg(C \xrightarrow{\varphi|_C} \varphi(C))[\varphi(C)]_\equiv$. Those can be naturally extended to $\mathbb{R}$-linear maps $\varphi^* : N^1(Y)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$ and $\varphi_* : N_1(Y)_{\mathbb{R}} \to N_1(X)_{\mathbb{R}}$.*

**Remark 4.7.2.** *From the Projection formula 3.4.20, we have that the previous morphisms satisfy $\varphi^*(d) \cdot c = d \cdot \varphi_*(c)$ for all $d \in N^1(Y)_{\mathbb{R}}, c \in N_1(Y)_{\mathbb{R}}$. This tells us that $ker(\varphi^*) \perp Im(\varphi_*)$ with respect to the intersection pairing of Remark 4.4.5; in particular, if $\varphi_*$ is surjective then $\varphi^*$ is injective.*

**Proposition 4.7.3.** *If $\varphi : X \to Y$ is a surjective morphism between projective varieties, then $\varphi_* : N_1(X)_{\mathbb{R}} \to N_1(Y)_{\mathbb{R}}$ is surjective.*

**Proof.** Taken a curve $C \subseteq Y$, by hypothesis there exists a curve $C' \subseteq X$ s.t. $\varphi(C') = C$. Therefore, we have that $\varphi_*([C']_\equiv) = m[C]_\equiv$ for some $m > 0$, and this concludes the proof. □

**Definition 4.7.4.** *Let $\varphi : X \to Y$ be a proper morphism of locally noetherian schemes. We say that a subset $Z$ of $X$ is contracted by $\varphi$ if $\varphi(Z)$ is a single point and $\varphi$ is an isomorphism elsewhere.*

**Definition 4.7.5.** *Let $\varphi : X \to Y$ be a morphism between projective varieties. Then, the relative cone of curves $NE(\varphi)$ is the convex subcone of $NE(X)$ generated by the classes of curves on $X$ contracted by $\varphi$.*

**Remark 4.7.6.** *Since $Y$ is projective, an irreducible curve $C$ on $X$ is contracted by $\varphi \iff \varphi_*([C]_\equiv) = 0$, or equivalently if $(\varphi^*H \cdot C) = 0$ for any ample divisor $H$ on $Y$. Notice that the first characterization tells us that being contracted is a numerical property.*

**Remark 4.7.7.** *From the previous Remark, for any ample divisor $H$ on $Y$ we have that $NE(\varphi) = NE(X) \cap (\varphi^*H)^\perp$. Hence, $NE(\varphi)$ is closed in $NE(X)$, and it holds $\overline{NE}(\varphi) \subseteq \overline{NE}(X) \cap (\varphi^*H)^\perp$.*

**Remark 4.7.8.** *The morphisms starting from $X$ given by the identity and the map to a point correspond respectively to the relative subcones $\{0\}$ and $NE(X)$.*

**Example 4.7.9.** *Given a curve $C$ and a hypersurface $H$ in $\mathbb{P}^N_K$, it holds $(H \cdot C) = deg(H)deg(C)$. From this we have that the map $N_1(\mathbb{P}^N_K) \to \mathbb{R}$ defined by $\sum_{i=1}^n \lambda_i[C_i]_\equiv \mapsto \sum_{i=1}^n \lambda_i deg(C_i)$ is an isomorphism. This implies that $NE(\mathbb{P}^N_K) \cong \mathbb{R}^+$.*

**Example 4.7.10.** *Consider $X = \mathbb{P}^n_K \times \mathbb{P}^m_K$. From Proposition 2.2.15, one gets that $dim\, N^1(X)_\mathbb{R} = 2$; hence, $N_1(X)_\mathbb{R}$ has $dim = 2$ as well, and it is generated by the class of a line $\ell$ in $\mathbb{P}^n_K$ and the class of a line $\ell'$ in $\mathbb{P}^m_K$. Therefore, $NE(X) = \mathbb{R}^+[\ell]_\equiv \bigoplus \mathbb{R}^+[\ell']_\equiv$. The relative subcones of $NE(X)$ corresponding to the two projections are $\mathbb{R}^+[\ell]_\equiv$ and $\mathbb{R}^+[\ell']_\equiv$.*

**Example 4.7.11.** *If $X$ is a smooth quadric in $\mathbb{P}^3_K$ and $C_1, C_2$ are lines in $X$ which meet, the relations $(C_1 \cdot C_2) = 1$ and $(C_1 \cdot C_1) = 0 = (C_2 \cdot C_2)$ imply that $[C_1]_\equiv \neq [C_2]_\equiv$. Therefore, $N_1(X) = \mathbb{R}[C_1]_\equiv \bigoplus \mathbb{R}[C_2]_\equiv$ and $NE(X) = \mathbb{R}^+[C_1]_\equiv \bigoplus \mathbb{R}^+[C_2]_\equiv$.*

**Example 4.7.12.** *If $X$ is a smooth cubic in $\mathbb{P}^3_K$, it contains $27$ lines $C_1, \ldots, C_{27}$, and one can show that exactly $6$ of those are pairwise disjoint, say $C_1, \ldots, C_6$. If $C$ is the smooth* *plane cubic obtained by cutting $X$ with a general plane, we have that* $N_1(X) = \mathbb{R}[C]_\equiv \bigoplus_{i=1}^{6} \mathbb{R}[C_i]_\equiv$ *and* $NE(X) = \mathbb{R}^+[C]_\equiv \bigoplus_{i=1}^{6} \mathbb{R}^+[C_i]_\equiv$.

**Definition 4.7.13.** *Given a projective morphism of $K$-schemes $\varphi : X \to Y$, we say that a Cartier divisor $D$ on $X$ is $\varphi$-ample if the restriction of $D$ to every fiber of $\varphi$ is ample.*

**Proposition 4.7.14.** *Given a projective morphism of $K$-schemes $\varphi : X \to Y$, let $D$ be a Cartier divisor on $X$. Then, $D$ is $\varphi$-ample $\iff$ $D \cdot z > 0$ $\forall 0 \neq z \in \overline{NE}(\varphi)$. Moreover, if $D$ is $\varphi$-ample and $H$ is ample on $Y$, then $m\varphi^* H + D$ is ample for all $m$ large enough.*

**Proof.** It comes directly from Kleiman's criterion 4.5.1. □

**Definition 4.7.15.** *A $K$-scheme $X$ of finite type is said to be geometrically integral if $X \times \overline{K}$ is integral over $K$.*

**Definition 4.7.16.** *Given a morphism of $K$-schemes $\varphi : X \to Y$, let $y \in Y$. We say that the fiber $\varphi^{-1}(y)$ is geometrically connected if $\varphi^{-1}(y) \otimes L$ is connected for every field extension $L$ of $K$.*

**Proposition 4.7.17.** *Let $\varphi : X \to Y$ be a projective morphism between integral schemes with $Y$ normal. Then, TFAE:*

- $\varphi_* \mathcal{O}_X \cong \mathcal{O}_Y$;
- $K(Y)$ *is algebraically closed in* $K(X)$;
- *the generic fiber of $\varphi$ is geometrically integral.*

*Moreover, if one of those holds, $\varphi$ is surjective and its fibers are geometrically connected.*

**Proof.** Look at [2], III, Corollaire (4.3.12). □

**Remark 4.7.18.** *Since the closure of the image of any morphism $f : X \to Y$ coincides with the ideal sheaf kernel of the canonical map $\mathcal{O}_Y \to f_* \mathcal{O}_X$, the surjectivity of $\varphi$ in the previous Proposition follows directly from the condition $\varphi_* \mathcal{O}_X \cong \mathcal{O}_Y$.*

**Proposition 4.7.19 (Stein Factorization).** *Let $\varphi : X \to Y$ be a projective morphism of noetherian schemes. Then, there exist a projective morphism with connected fibers $\varphi' : X \to Z$ and a finite morphism $\psi : Z \to Y$ s.t. $\varphi = \psi \circ \varphi'$.*

**Proof.** See [4], Corollary III.11.5. $\square$

**Remark 4.7.20.** *In the proof of the previous Proposition, one chooses $Z = Spec(\varphi_* \mathcal{O}_X)$; therefore, it holds $\varphi'_* \mathcal{O}_X \cong \mathcal{O}_Z$. If the fibers of $\varphi$ are connected, then $\psi$ is bijective; if moreover the characteristic of $X$ is $0$ and $Y$ is normal, then $\psi$ is an isomorphism and it holds $\varphi_* \mathcal{O}_X \cong \mathcal{O}_Y$.*

**Remark 4.7.21.** *For any projective morphism $\varphi : X \to Y$ with Stein factorization $X \xrightarrow{\varphi'} Z \xrightarrow{\psi} Y$, the curves contracted by $\varphi$ and $\varphi'$ are the same, hence $NE(\varphi) = NE(\varphi')$.*

**Lemma 4.7.22.** *Let $\varphi : X \to Y$ and $\varphi' : X \to Y'$ be projective morphisms between integral schemes s.t. $\varphi_* \mathcal{O}_X \cong \mathcal{O}_Y$.*

1. *if $\varphi'$ contracts one fiber $\varphi^{-1}(y_0)$ of $\varphi$, then there exists an open neighborhood $Y_0$ of $y_0$ in $Y$ s.t. there is a factorization $\varphi'|_{\varphi^{-1}(Y_0)} : \varphi^{-1}(Y_0) \xrightarrow{\varphi} Y_0 \to Y'$;*

2. *if $\varphi'$ contracts each fiber of $\varphi$, then it factors through $\varphi$.*

**Proof.** *(1)* Define $g = (\varphi, \varphi') : X \to Y \times Y'$, and set $Z = Im(g)$. If $p : Z \to Y$ and $p' : Z \to Y'$ are the two projections, by hypothesis we have that $\varphi^{-1}(y_0) = g^{-1}(p^{-1}(y_0))$ is contracted by $\varphi'$, hence by $g$. It follows that the fiber $p^{-1}(y_0) = g(g^{-1}(p^{-1}(y_0)))$ is a point, hence the proper surjective morphism $p$ is finite over an open affine neighborhood $Y_0$ of $y_0$ in $Y$. Now, set $X_0 = \varphi^{-1}(Y_0)$ and $Z_0 = p^{-1}(Y_0)$, and let $p_0 : Z_0 \to Y_0$ be the restriction of $p$. Then, from $\mathcal{O}_{Z_0} \subseteq g_* \mathcal{O}_{X_0}$ we have that $\mathcal{O}_{Y_0} \subseteq (p_0)_* \mathcal{O}_{Z_0} \subseteq (p_0)_* g_* \mathcal{O}_{X_0} = \varphi_* \mathcal{O}_{X_0} = \mathcal{O}_{Y_0}$, so the equal holds, hence $(p_0)_* \mathcal{O}_{Z_0} \cong \mathcal{O}_{Y_0}$. Since $p_0$, being finite, is affine, the previous condition tells us that $p_0$ induces an isomorphism between the coordinate rings of $Z_0$ and $Y_0$. Therefore, $p_0$ is an isomorphism, and $\varphi'|_{X_0} = p' \circ p_0^{-1} \circ \varphi|_{X_0}$, proving what we wanted.
*(2)* If $\varphi'$ contracts each fiber of $\varphi$, then the morphism $p$ above is finite, so one can take $Y_0 = Y$ and $\varphi'$ factors through $\varphi$. $\square$

**Proposition 4.7.23.** *Let $X, Y, Y'$ be projective varieties and let $\varphi : X \to Y$ be a morphism. Then:*

1. *the subcone $\overline{NE}(\varphi)$ of $\overline{NE}(X)$ is extremal and, if $H$ is an ample divisor on $Y$, it satisfies $\overline{NE}(\varphi) = \overline{NE}(X) \cap (\varphi^* H)^\perp$;*

2. *assume that $\varphi_* \mathcal{O}_X \cong \mathcal{O}_Y$, and let $\varphi' : X \to Y'$ be another morphism.*

   - *If $\overline{NE}(\varphi) \subseteq \overline{NE}(\varphi')$, then there is a unique morphism $f : Y \to Y'$ s.t. $\varphi' = f \circ \varphi$;*
   - *$\varphi$ is uniquely determined by $\overline{NE}(\varphi)$ up to isomorphism.*

**Proof.** *(1)* Since $\varphi^* H$ is nonnegative on $\overline{NE}(X)$, it defines a supporting hyperplane of this cone; from $\overline{NE}(\varphi) \subseteq \overline{NE}(X\cap) \cap (\varphi^* H)^\perp$ we then get that $\overline{NE}(\varphi)$ is extremal. It remains to show the equality in the previous inclusion. Suppose by contradiction that $\overline{NE}(\varphi) \subsetneq \overline{NE}(X) \cap (\varphi^* H)^\perp$; then by taking the dual we have (thanks to Lemma 4.3.8-*(3)*) that there exists a linear form $\ell$ which is positive on $\overline{NE}(\varphi) \smallsetminus \{0\}$, but is s.t. $\ell(z) < 0$ for some $z \in \overline{NE}(X) \cap (\varphi^* H)^\perp$. Without loss of generality, we can assume that $\ell$ is given by intersecting with a Cartier divisor $D$. Recalling that a morphism between projective varieties is always projective, by the relative version of Kleiman's criterion 4.7.14 we get that $D$ is $\varphi$-ample, and by the same Proposition $m\varphi^* H + D$ is ample for all $m$ large enough. But $(m\varphi^* H + D) \cdot z = D \cdot z < 0$, which gives us a contradiction.
*(2)* The assumption $\overline{NE}(\varphi) \subseteq \overline{NE}(\varphi')$ implies that every curve contracted by $\varphi$ is also contracted by $\varphi'$, hence every fiber of $\varphi$ is contracted by $\varphi'$. By item *(2)* of the previous Lemma, the existence of such $f$ is guaranteed. Following the notation of the previous Lemma, if $\exists f' : Y \to Y'$ s.t. $\varphi' = f' \circ \varphi$, then $(f' \circ p) \circ g = f' \circ \varphi = \varphi' = p' \circ g$, so by the surjectivity of $g$ we have that $f' \circ p = p'$, i.e. $f' = p' \circ p^{-1} = f$.
To prove the second item, assume that $\overline{NE}(\varphi) = \overline{NE}(\varphi')$; then, by what we have just shown $\exists! f : Y \to Y', f' : Y' \to Y$ s.t. $\varphi' = f \circ \varphi$ and $\varphi = f' \circ \varphi'$. It follows that $\begin{cases} \varphi = f' \circ (f \circ \varphi) \Rightarrow f' \circ f = id_Y \\ \varphi' = f \circ (f' \circ \varphi') \Rightarrow f \circ f' = id_{Y'} \end{cases}$, where the implications come from the uniqueness given by the fact that $\overline{NE}(\varphi) = \overline{NE}(\varphi)$ and analogously for $\varphi'$. This tells us that $f, f'$ are isomorphisms, so we are done. $\square$

**Example 4.7.24.** *We already know that the extremal subcones of* $NE(\mathbb{P}^N_K) \cong \mathbb{R}^+$ *are the trivial ones, i.e.* $\{0\}$ *and itself. By the previous Proposition, it follows that (up to isomorphism) the only morphisms* $\varphi : \mathbb{P}^N_K \to Y$ *satisfying* $\varphi_* \mathcal{O}_{\mathbb{P}^N_K} \cong \mathcal{O}_Y$ *are the identity and the map to a point.*

**Example 4.7.25.** *Given* $X = \mathbb{P}^n_K \times \mathbb{P}^m_K$*, we already know that the extremal subcones of* $NE(\mathbb{P}^N_K) \cong \mathbb{R}^+$ *are four: the trivial ones and the two associated to the projections. By the previous Proposition, it follows that a morphism* $\varphi : X \to Y$ *satisfying* $\varphi_* \mathcal{O}_X \cong \mathcal{O}_Y$ *is up to isomorphism either the identity, the map to a point or one of the two projections.*

**Proposition 4.7.26.** *Given a smooth projective variety* $X$*, let* $\varepsilon : \tilde{X} \to X$ *be the blow-up of a point, with exceptional divisor* $E$*. Then,* $Pic(\tilde{X}) \cong Pic(X) \oplus \mathbb{Z}[\mathcal{O}_{\tilde{X}}(E)]_\sim$ *and* $N^1(\tilde{X})_\mathbb{R} \cong N^1(X)_\mathbb{R} \oplus \mathbb{Z}[E]_\equiv$.

# Chapter 5

# Surfaces

## 5.1 Preliminary results

**Remark 5.1.1.** *The canonical divisor $K_X$ introduced over $\mathbb{C}$ in Example 2.3.9 is not uniquely defined, but its numerical equivalence class (which is said to be canonical and is denoted with the same symbol) is; moreover, the definition of $K_X$ can be given for any smooth projective variety $X$.*

**Proposition 5.1.2** (**Adjunction formula**)**.** *Given a smooth projective variety $X$, let $Y \subseteq X$ be a smooth hypersurface. Then, $K_Y = (K_X + Y)|_Y$.*

**Idea of Proof.** Let $\Omega_{X/K}$ be the (locally free) sheaf of differentials; over $\mathbb{C}$, this is just the dual of the sheaf of local sections of the tangent bundle $T_X$. If $f_i$ is a local equation for $Y$ in $X$ on an open set $U_i$, the sheaf $\Omega_{Y/K}$ is the quotient of the restriction of $\Omega_{X/K}$ to $Y$ by the ideal generated by $df_i$. Dually, over $\mathbb{C}$ we have that in local coordinates $x_1, \dots, x_n$ on $X$, the tangent space $T_{Y,p}$ at a point $p \in Y$ is defined by the equation $df_i(p)(t) = \nabla f_i(p) \cdot t = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(p) t_j = 0$. If $g = (g_{ij})_{i,j}$ gives the change of coordinates, then on $Y \cap U_{ij}$ we have that $f_i = g_{ij} f_j$, hence $df_i = dg_{ij} f_j + g_{ij} df_j = g_{ij} df_j$. Since $g$ defines the invertible sheaf $\mathcal{O}_X(-Y)$, we obtain the exact sequence of locally free sheaves $0 \to \mathcal{O}_Y(-Y) \to \Omega_{X/K} \otimes \mathcal{O}_Y \to \Omega_{Y/K} \to 0$. From the fact that $det(\Omega_{X/K}) = \mathcal{O}_X(K_X)$, by taking determinants we can conclude.
One can refer to [4], Proposition II.8.20 for a more detailed proof. $\square$

**Theorem 5.1.3 (Serre duality).** *Given a smooth projective variety $X$ of dim $= n$, let $D$ be a Cartier divisor on $X$. Then, for any $0 \leqslant i \leqslant n$, the natural pairing $H^i(X, D) \otimes H^{n-i}(X, K_X - D) \to H^n(X, K_X) \cong K$ is non-degenerate; in particular, $h^i(X, D) = h^{n-i}(X, K_X - D)$.*

**Definition 5.1.4.** *Let $X$ be a smooth projective surface. We define the geometric genus of $X$ and the irregularity of $X$ respectively as $p_g(X) = h^0(X, K_X) = h^2(X, \mathcal{O}_X)$ and $q(X) = h^1(X, K_X) = h^1(X, \mathcal{O}_X)$.*

**Remark 5.1.5.** *It holds $\chi(X, \mathcal{O}_X) = p_g(X) - q(X) + 1$.*

**Proposition 5.1.6.** *Let $X$ be a smooth projective curve. Then, $deg(K_X) = 2g(X) - 2$.*

**Proof.** By Serre duality, we have $g(X) = h^0(X, K_X)$. The Riemann-Roch Thm. 3.1.7 tells us that, for any Cartier divisor $D$ on $X$, $h^0(X, D) - h^0(X, K_X - D) = deg(D) + 1 - g(X)$. Using the previous observation and taking $D = K_X$, we can conclude. □

**Lemma 5.1.7.** *Any Cartier divisor on a smooth projective surface is linearly equivalent to the difference of two smooth curves.*

**Proposition 5.1.8 (Riemann-Roch Thm. for curves).** *Given a smooth projective surface $X$, let $D$ be a divisor on $X$. Then, $\chi(X, D) = \frac{1}{2}((D^2) - (K_X \cdot D)) + \chi(X, \mathcal{O}_X)$.*

**Proof.** By the previous Lemma, $D \sim C - C'$ with $C, C'$ smooth curves in $X$; hence, by Theorem 3.2.4 we have that $\chi(X, D) = -(C \cdot C') + \chi(X, C) + \chi(X; -C') - \chi(X, \mathcal{O}_X)$. From the exact sequences $0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0$ and $0 \to \mathcal{O}_X(-C') \to \mathcal{O}_X \to \mathcal{O}_{C'} \to 0$, we get that $\chi(X, D) = -(C \cdot C') + \chi(X, \mathcal{O}_X) + $ $+ \chi(C, C|_C) - \chi(C', \mathcal{O}_{C'})$. Then, by Riemann-Roch Thm. 3.1.7 on both $C$ and $C'$ we get $\chi(X, D) = -(C \cdot C') + \chi(X, \mathcal{O}_X) + (C^2) + 1 - g(C) - (1 - g(C'))$. From the previous Proposition and the Adjunction formula 5.1.2, we have that $2g(C) - 2 = deg(K_C) =$ $= deg(K_X + C)|_C = ((K_X + C) \cdot C)$, and analogously for $C'$. From this we obtain that $\chi(X, D) - \chi(X, \mathcal{O}_X) = -(C \cdot C') + (C^2) - \frac{1}{2}((K_X + C) \cdot C) + \frac{1}{2}((K_X + C') \cdot C') = -(C \cdot C') +$ $+ (C^2) + \frac{1}{2}\left((K_X \cdot C') + ((C')^2) - (K_X \cdot C) - (C^2)\right) = -(C \cdot C') + \frac{1}{2}((K_X \cdot C') + ((C')^2) -$

$-(K_X \cdot C) + (C^2)) = -(C \cdot C') + \frac{1}{2}((K_X \cdot (C' - C)) + ((C')^2) + (C^2)) =$
$= \frac{1}{2}(((C - C')^2) - (K_X \cdot D))$, which concludes the proof. $\square$

**Remark 5.1.9 (Genus formula for curves).** *Given a smooth projective variety $X$, let $C$ be an irreducible curve in $X$. Then, with the same calculations as in the previous Proof, one can see that $g(C) = 1 + \frac{1}{2}((C^2) + (K_X \cdot C))$. In particular, from Corollary 3.4.22 we have that $(C^2) + (K_X \cdot C) = -2 \iff C$ is smooth and rational.*

**Example 5.1.10.** *Given a smooth curve $C$ of genus $g$, let $X$ be the surface $C \times C$, and let $p_1, p_2 : X \to C$ be the two projections. Consider the numerical equivalence classes $x_1$ of $\{*\} \times C$, $x_2$ of $C \times \{*\}$, and $\Delta$ of the diagonal. The canonical class of $X$ is $K_X = p_1^* K_C + p_2^* K_C \equiv deg(K_C)(x_1 + x_2) = (2g-2)(x_1 + x_2)$. Since it holds $(\Delta \cdot x_i) = 1$, we have that $(K_X \cdot \Delta) = (2g-2)(1+1) = 4(g-1)$. From the Genus formula for curves and the fact that $\Delta$ has genus $g$, we get that $(\Delta^2) = 2g - 2 - (K_X \cdot \Delta) = -2(g-1)$.*

## 5.2 Ruled surfaces

**Lemma 5.2.1.** *Any geometrically integral curve $C$ of genus 0 over $K$ is isomorphic to a nondegenerate conic in $\mathbb{P}^2_K$.*

**Theorem 5.2.2 (Tsen's Thm.).** *Let $\varphi : X \to B$ be a surjective morphism from a projective surface onto a smooth curve over an algebraically closed field $K$, s.t. its generic fiber is a geometrically integral curve of genus 0. Then, $X$ is birational over $B$ to $B \times \mathbb{P}^1_K$.*

**Proof.** Let $C$ be a generic fiber of $\varphi$; by Corollary 3.4.22, it suffices to show that $C$ has a $K(B)$-point. From the previous Lemma, there exists a nondegenerate conic $\mathcal{C} \subseteq \mathbb{P}^2_{K(B)}$ s.t. $\mathcal{C} \cong C$; let $q(x_0, x_1, x_2) = \sum_{0 \leqslant i,j \leqslant 2} a_{ij} x_i x_j = 0$ be an equation for $\mathcal{C}$. Notice that all $a_{ij} \in K(B)$ can be viewed as sections of $\mathcal{O}_B(E)$ for some nonzero effective divisor $E$ on $B$. Consider now, for any $m > 0$, the map $f_m : H^0(B, mE)^3 \to H^0(B, (2m+1)E)$ defined by $(x_0, x_1, x_2) \mapsto \sum_{0 \leqslant i,j \leqslant 2} a_{ij} x_i x_j$. Since $E$ is ample, by Riemann-Roch Thm. 3.1.7 we have that for $m$ large enough the domain of $f_m$ has dimension $\alpha_m = 3(m\, deg(E) + 1 - g(B))$, while the dimension of its codomain is $\beta_m = (2m+1)\, deg(E) + 1 - g(B)$. A $K(B)$-point for

$\mathcal{C}$ would be a nonzero $(x_0, x_1, x_2) \in H^0(B, mE)^3$ s.t. $q(x_0, x_1, x_2) = 0$, hence an element in the intersection of $\beta_m$ quadrics in a $K$-projective space of $dim = \alpha_m - 1$. Since for $m$ large enough $\alpha_m - 1 \geqslant \beta_m$, such $(x_0, x_1, x_2)$ exists because $K$ is algebraically closed. $\square$

**Theorem 5.2.3** **(Base change Thm.).** *Given a projective morphism of noetherian schemes $f : X \to Y$, let $\mathcal{F}$ be a coherent sheaf on $X$ which is flat over $Y$. Taken a point $y \in Y$, then:*

1. *if the natural map $\varphi^i(y) : R^i f_*(\mathcal{F}) \otimes K(y) \to H^i(X_y, \mathcal{F}_y)$ is surjective, then it is an isomorphism, and the same is true for all $y'$ in a suitable neighborhood of $y$;*
2. *if $\varphi^i(y)$ is surjective, TFAE:*
   - *$\varphi^{i-1}(y)$ is surjective;*
   - *$R^i f_*(\mathcal{F})$ is locally free in a neighborhood of $y$.*

**Proof.** A reference can be found in [4], Theorem III.12.11. $\square$

**Proposition 5.2.4** **(Universal property of $\mathbb{P}(\mathcal{E})$).** *Given a noetherian scheme $X$, let $\mathcal{E}$ be a locally free coherent sheaf on $X$. Taken a morphism $g : Y \to X$, we have that $\{$morphisms $Y \to \mathbb{P}(\mathcal{E})$ over $X\} \leftrightarrow \{$surjective maps $g^*\mathcal{E} \to \mathcal{L}$ of sheaves on $Y | \mathcal{L}$ is invertible$\}$ is a one-to-one correspondence.*

**Proof.** See [4], Proposition II.7.12. $\square$

**Definition 5.2.5.** *A ruled surface is a projective surface $X$ with a surjective morphism $\varphi : X \to B$ onto a smooth projective curve s.t. the fiber of every closed point is isomorphic to $\mathbb{P}^1_K$.*

**Theorem 5.2.6.** *Let $X$ be a ruled surface over an algebraically closed field $K$. Then, there exists a locally free rank-2 sheaf $\mathcal{E}$ on $B$ s.t. $X \cong \mathbb{P}(\mathcal{E})$ over $B$.*

**Proof.** First, notice that the sheaf $\varphi_*\mathcal{O}_X$ is locally free on $B$. Since $\varphi$ is flat, and $h^0(X_b, \mathcal{O}_{X_b}) = 1$ for all closed points $b \in B$, the Base change Thm. 5.2.3 implies that $\varphi_*\mathcal{O}_X$ has rank 1, hence is locally isomorphic to $\mathcal{O}_B$. It follows from Proposition 4.7.17

that the generic fiber of $\varphi$ is geometrically integral. Since $h^1(X_b, \mathcal{O}_{X_b}) = 0$ for all closed points $b \in B$, again by the Base change Thm. we have that $R^1\varphi_*\mathcal{O}_X = 0$ and that the generic fiber of $\varphi$ has genus 0. It follows from Tsen's Thm. 5.2.2 that $\varphi$ has a rational section which, since $B$ is smooth, extends to a global section $\sigma : B \to X$. Setting $C = Im(\sigma)$, then $(C \cdot X_b) = 1$ for all $b \in B$, so from the Base change Thm. we get that $\mathcal{E} = \varphi_*(\mathcal{O}_X(C))$ is a locally free rank-2 sheaf on $B$. Furthermore, the canonical morphism $\varphi^*(\varphi_*(\mathcal{O}_X(C))) \to \mathcal{O}_X(C)$ is surjective, hence by Proposition 5.2.4 there exists a morphism $f : X \to \mathbb{P}(\mathcal{E})$ over $B$ s.t. $f^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{O}_X(C)$. Since $\mathcal{O}_X(C)$ is very ample on each fiber, $f$ is an isomorphism. $\square$

**Remark 5.2.7.** *By the proof above, since $\varphi_*\mathcal{O}_X = \mathcal{O}_B$ and $R^1\varphi_*\mathcal{O}_X = 0$, the direct image by $\varphi_*$ of the exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0$ is*
*$0 \to \mathcal{O}_B \to \mathcal{E} \to \sigma^*\mathcal{O}_C(C) \to 0$. In particular, $(C^2) = deg(det\mathcal{E})$.*

**Remark 5.2.8.** *Theorem 5.2.6 tells us also that any ruled surface is smooth.*
*Moreover, such a surface is of the form $\mathbb{P}(\mathcal{E})$, and if $\varphi$ is the associated morphism it holds $\varphi_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \cong \mathcal{E}$.*

**Proposition 5.2.9.** *Given the morphism $\varphi : X \to B$ associated to the ruled surface $X$, let $F$ be a fiber and $B \to C$ be a section. Then, the map $\theta : \mathbb{Z} \times Pic(B) \to Pic(X)$ defined by $(n, [D]_\sim) \mapsto [nC + \varphi^*D]_\sim$ is a group isomorphism, and $N^1(X) \cong \mathbb{Z}[C]_\equiv \oplus \mathbb{Z}[F]_\equiv$. Moreover, $(C \cdot F) = 1$ and $(F^2) = 0$.*

**Proof.** First, notice that $(C \cdot F) = 1$ because $C$ and $F$ meet transversally at a single point, and $(F^2) = 0$ because two distinct fibers do not meet.
Notice that the numerical equivalence class of $F$ does not depend on the choice of the fiber: this follows from the Projection formula 3.4.20. Therefore, $N^1(X) \cong \mathbb{Z}[C]_\equiv \oplus \mathbb{Z}[F]_\equiv$ immediately follows from the fact that $(C \cdot F) = 1$.
Since $\theta$ is clearly a group homomorphism, it remains to show that it is bijective.
Taken a divisor $E$ on $X$, set $n = (E \cdot F)$. By the Base change Thm. 5.2.3,
$\mathcal{M} = \varphi_*(\mathcal{O}_X(E - nC))$ is an invertible sheaf on $B$, and the canonical morphism
$\varphi^*(\varphi_*(\mathcal{O}_X(E - nC))) \to \mathcal{O}_X(E - nC)$ is bijective. Hence, $\mathcal{O}_X(nC) \otimes \varphi^*\mathcal{M} \cong \mathcal{O}_X(E)$,

which proves that $\theta$ is surjective.

To prove injectivity, note that $nC + \varphi^*D \sim 0 \Rightarrow 0 = ((nC + \varphi^*D) \cdot F) = n$, which implies $n = 0$ and thus $\varphi^*D \sim 0$. From the Projection formula 2.8.4, we then have that $\mathcal{O}_B \cong \varphi_*\mathcal{O}_X \cong \varphi_*\mathcal{O}_X(\varphi^*D) \cong \varphi_*\varphi^*\mathcal{O}_B(D) \cong \mathcal{O}_B(D) \otimes \varphi_*\mathcal{O}_X \cong \mathcal{O}_B(D)$, so $D \sim 0$. □

**Remark 5.2.10.** *In the setting of the previous Proposition, let $\mathcal{E}, \mathcal{E}'$ be two locally free rank-2 sheaves on $B$ s.t. there is an isomorphism $f : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E}')$ over $B$. Then, since $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and $f^*\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)$ both have intersection number 1 with a fiber of $\varphi$, there is an invertible sheaf $\mathcal{M}$ on $B$ s.t. $f^*\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \varphi^*\mathcal{M}$. By taking direct images, we get that $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{M}$.*

**Proposition 5.2.11.** *Let $\mathbb{P}(\mathcal{E})$ be a ruled surface. Then, $((\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^2) = deg(det\mathcal{E})$.*

**Proof.** If $\varphi : X \to B$ is the morphism associated to $\mathbb{P}(\mathcal{E})$, let $B \to C$ be a section. Then, by Remark 5.2.7 the thesis holds for $\mathcal{E} = \varphi_*\mathcal{O}_X(C)$. Taken another locally free rank-2 sheaf $\mathcal{E}'$ on $B$, by the previous Remark there exists an invertible sheaf $\mathcal{M}$ on $B$ s.t.
$\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{M}$, hence $\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \varphi^*\mathcal{M}$. Therefore, $deg(det\mathcal{E}') =$
$= deg((det\mathcal{E}) \otimes \mathcal{M}^2) = deg(det\mathcal{E}) + 2deg(\mathcal{M}) = (C^2) + 2deg(\mathcal{M})$, while $((\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1))^2) =$
$= ((\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \varphi^*\mathcal{M})^2) = ((C + deg(\mathcal{M})F)^2) = (C^2) + (F^2) + 2deg(\mathcal{M})(C \cdot F) =$
$= (C^2) + 2deg(\mathcal{M})$, and we are done. □

**Proposition 5.2.12.** *Let $X = \mathbb{P}(\mathcal{E})$ be a ruled surface with corresponding morphism $\varphi : \mathbb{P}(\mathcal{E}) \to B$. Then, there is a one-to-one correspondence*
*$\{$sections $\sigma : B \to \mathbb{P}(\mathcal{E})$ of $\varphi\} \leftrightarrow \{\mathcal{L}$ invertible sheaf on $B | \exists \mathcal{E} \to \mathcal{L}$ surjective morphisms$\}$ given by $\sigma \mapsto \sigma^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Moreover, the section $\sigma$ corresponding to the sheaf $\mathcal{L}$ satisfies $(\sigma(B))^2 = 2deg(\mathcal{L}) - deg(det\mathcal{E})$.*

**Proof.** Setting $C = \sigma(B)$ and $\mathcal{E}' = \varphi_*\mathcal{O}_X(C)$, we have that $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{M}$ for some invertible sheaf $\mathcal{M}$ on $B$, so $\mathcal{O}_X(C) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1) \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \varphi^*\mathcal{M}$. Applying $\sigma^*$, we obtain that $\sigma^*\mathcal{O}_X(C) \cong \mathcal{L} \otimes \mathcal{M}$, hence $(C^2) = deg(\mathcal{L}) + deg(\mathcal{M})$. It follows that
$(C^2) = deg(det\mathcal{E}') = deg(det\mathcal{E}) + 2deg(\mathcal{M}) = deg(det\mathcal{E}) + 2((C^2) - deg(\mathcal{L}))$, and this concludes the proof. □

**Example 5.2.13.** *It can be shown that any locally free rank-2 sheaf on $\mathbb{P}^1_K$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1_K}(a) \oplus \mathcal{O}_{\mathbb{P}^1_K}(b)$ for some $a, b \in \mathbb{Z}$. It follows that any ruled surface over $\mathbb{P}^1_K$ is isomorphic to one of the Hirzebruch surfaces $F_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1_K} \oplus \mathcal{O}_{\mathbb{P}^1_K}(n))$ for some $n \in \mathbb{N}$. Moreover, the surjection $\mathcal{O}_{\mathbb{P}^1_K} \oplus \mathcal{O}_{\mathbb{P}^1_K} \to \mathcal{O}_{\mathbb{P}^1_K}$ gives a section $C_n \subseteq F_n$ s.t. $(C_n^2) = -n$. One can also prove that, when $n < 0$, $C_n$ is the only (integral) curve on $F_n$ with negative self-intersection.*

## 5.3 Extremal rays

**Lemma 5.3.1.** *Let $D$ be a divisor on a smooth projective surface $X$. If there exists an ample divisor $H$ on $X$ s.t. $(D \cdot H) > 0$, then $H^2(X, mD) = \{0\}$ for any $m$ large enough.*

**Proof.** Since the divisor $K_X - mD$ has negative intersection with $H$ for any $m > \frac{(K_X \cdot H)}{(D \cdot H)}$, it cannot be equivalent to an effective divisor. It follows that $h^0(X, K_X - mD) = 0$ for $m$ large enough, hence $h^2(X, mD) = 0$ by Serre duality 5.1.3. □

**Proposition 5.3.2.** *Given a smooth projective surface $X$, let $C$ be an irreducible curve on $X$, and let $r \in \overline{NE}(X)$. Then:*

1. *if $(C^2) \leqslant 0$, then $[C]_\equiv \in \partial\overline{NE}(X)$;*
2. *if $(C^2) < 0$, then $[C]_\equiv$ spans an extremal ray of $\overline{NE}(X)$;*
3. *if $(C^2) = 0$ and $(K_X \cdot C) < 0$, then $X$ is a ruled surface over a smooth curve, $C$ is a fiber of the associated morphism and $\rho_X = 2$;*
4. *if $r$ spans an extremal ray of $\overline{NE}(X)$, either $r^2 \leqslant 0$ or $\rho_X = 1$;*
5. *if $r$ spans an extremal ray of $\overline{NE}(X)$ and $r^2 < 0$, such extremal ray is spanned by the class of an irreducible curve.*

**Proof.** *(1)* Let $H$ be an ample divisor on $X$. Assume by contradiction that $[C]_\equiv$ is in the interior of $\overline{NE}(X)$; then, so is $[C]_\equiv + t[H]_\equiv$ for all $t$ small enough. Since $[C]_\equiv$ has nonnegative intersection with the class of any effective divisor, hence with any element of $\overline{NE}(X)$, it follows that $0 \leqslant (C \cdot (C + tH)) = (C^2) + t(C \cdot H) \leqslant t(C \cdot H)$ for all $t$ small

enough. But with (sufficiently small) $t < 0$ we reach a contradiction, because $(C \cdot H) > 0$.
*(2)* We want to prove that, if $[C]_\equiv = z_1 + z_2$ with $z_i \in N_1(X)$ $\forall i = 1, 2$, then $z_i \in \mathbb{R}^+[C]_\equiv$. We can write $z_i = \alpha_i[C]_\equiv + z_i'$ for some $\alpha_i \geqslant 0$ and $z_i' \in N_1(X)$ s.t. $C \cdot z_i' \geqslant 0$. Hence, from $[C]_\equiv = (\alpha_1 + \alpha_2)[C]_\equiv + z_1' + z_2'$ we can take intersections with $C$, obtaining that $(C^2) \geqslant (\alpha_1 + \alpha_2)(C^2) \Rightarrow (C^2)(\alpha_1 + \alpha_2 - 1) \leqslant 0$, thus from $(C^2) \leqslant 0$ we get that $\alpha_1 + \alpha_2 - 1 \geqslant 0$. But $0 = (\alpha_1 + \alpha_2 - 1)[C]_\equiv + z_1' + z_2'$, so by Remark 4.4.6 we get that $z_1' = 0 = z_2'$ (since they are not multiples of $[C]_\equiv$, and they can't be opposites because of the condition $C \cdot z_i' \geqslant 0$). This shows that $z_i = \alpha_i[C]_\equiv$, concluding the proof.
*(3)* By the Genus formula for curves 5.1.9, we have that $(K_X \cdot C) = -2$ and $C$ is smooth and rational. If $H$ is an ample divisor on $X$, then $(C \cdot H) > 0$, hence by the previous Lemma $H^2(X, mC) = \{0\}$ for all $m$ large enough. Since $(C^2) = 0$ and $(K_X \cdot C) = -2$, by the Riemann-Roch Thm. for curves 5.1.8 we get that $h^0(X, mC) - h^1(X, mC) =$
$= m + \chi(X, \mathcal{O}_X)$; in particular, $h^0(X, (m-1)C) < h^0(X, mC)$ for $m$ large enough. From the exact sequence $0 \to \mathcal{O}_X((m-1)C) \to \mathcal{O}_X(mC) \to \mathcal{O}_C(mC) \cong \mathcal{O}_C \to 0$, we get the exact sequence $0 \to H^0(X, (m-1)C) \to H^0(X, mC) \xrightarrow{\psi} H^0(C, mC) \cong H^0(C, \mathcal{O}_C) \cong K$, and the restriction map $\psi$ is surjective by the previous observation about dimentions. It follows that $|mC|$ has no base-points: the only possible base-points are on $C$, but a section $s \in H^0(C, mC)$ s.t. $\psi(s) = 1$ does not vanish on $C$. Therefore, $|mC|$ defines a morphism from $X$ to a projective space whose image is a curve. Its Stein factorization yields a morphism from $X$ onto a smooth curve whose general fiber $F$ is numerically equivalent to some positive rational multiple of $C$. Since $(K_X \cdot C) = -2$, we have that $(K_X \cdot F) < 0$, so from the fact that $F$ is a curve s.t. $(F^2) = 0$ we obtain that $(K_X \cdot F) = -2 = (K_X \cdot C)$, hence $F$ is rational and s.t. $F \equiv C$. Since $\mathbb{R}^+[C]_\equiv$ is extremal and $[C]_\equiv$ is not divisible in $N_1(X)$, all fibers are integral, and this concludes the proof.
*(4)* Let $D, H$ be divisors on $X$ s.t. $(D^2) = 0$, $H$ is ample and $(D \cdot H) > 0$. By the previous Lemma, $H^2(X, mD) = \{0\}$ for $m$ large enough, so by Proposition 3.4.16 we have that $h^0(X, mD) \geqslant \frac{(D^2)}{2}m^2 + O(m)$. Since $(D^2) > 0$, this proves that $mD$ is linearly equivalent to an effective divisor for $m$ sufficiently large, hence $[D]_\equiv \in NE(X)$. Therefore, $\{z \in N_1(X)_\mathbb{R} | z^2 > 0, H \cdot z > 0\} \subseteq NE(X)$; since the set on the left is open, it is contained

in the interior of $NE(X)$, hence does not contain any extremal rays of $NE(X)$, except if $\rho_X = 1$.

*(5)* Since $r \in \overline{NE}(X)$, it is the limit of a sequence $\{z_i\}_{i\in\mathbb{N}}$ in $N_1(X)_{\mathbb{R}}$. Since $\exists j$ s.t. $r \cdot z_j < 0$, there exists an irreducible curve $C$ in $X$ s.t. $r \cdot C < 0$. We can write $z_i = \alpha_i[C]_{\equiv} + z_i'$ with $\alpha_i \geqslant 0$ and $C \cdot z_i' \geqslant 0$; surely, there exist $\alpha := \lim_{i\to+\infty} \alpha_i \geqslant 0$ and $r' := \lim_{i\to+\infty} z_i'$. By taking the limit, we have that $r = \alpha[C]_{\equiv} + r'$, so it holds $0 \leqslant C \cdot r' = C \cdot r - \alpha(C^2) < -\alpha(C^2)$, which implies that $\alpha > 0$ and $(C^2) < 0$. Since $\mathbb{R}^+ r$ is extremal by hypothesis and $r = \alpha[C]_{\equiv} + r'$, we conclude that $r$ must be a multiple of $[C]_{\equiv}$. □

**Theorem 5.3.3 (Hodge Index Thm.).** *Given a projective surface $X$, let $D, H$ be divisors on $X$ s.t. $(H^2) \geqslant 0$. Then, $(D \cdot H)^2 \geqslant (D^2)(H^2)$.*

**Definition 5.3.4.** *An abelian surface is a smooth projective surface which is an (abelian) algebraic group; the structure morphisms are regular maps.*

**Proposition 5.3.5.** *Let $X$ be an abelian surface. Then:*

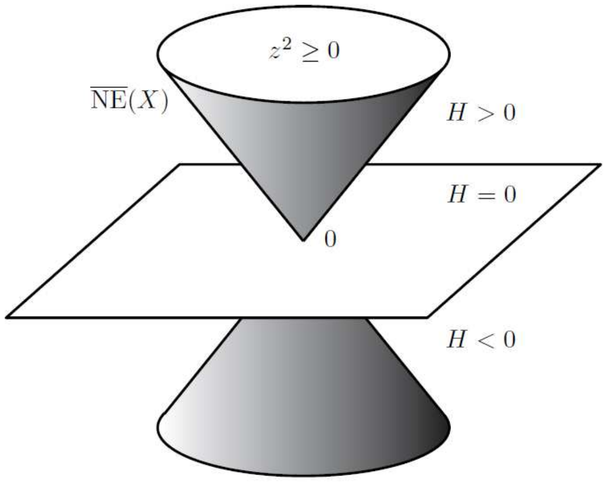
1. *any curve on $X$ has nonnegative self-intersection;*

2. *if $H$ is an ample divisor on $X$, we have $\overline{NE}(X) = \{z \in N_1(X)_{\mathbb{R}} | z^2 \geqslant 0, H \cdot z \geqslant 0\}$.*

**Proof.** *(1)* Taken a curve $C$ on $X$, then $(C^2) = (C \cdot (x + C)) \geqslant 0 \ \ \forall x \in X$.

*(2)* By point *(4)* of the previous Proof, $\{z \in N_1(X)_{\mathbb{R}} | z^2 > 0, H \cdot z > 0\} \subseteq NE(X)$, so $(\supseteq)$ holds. $(\subseteq)$ follows from *(1)*. □

**Example 5.3.6.** *Let $X$ be an abelian surface. By the Hodge Index Thm., the intersection form on $N_1(X)_{\mathbb{R}}$ has exactly one positive eigenvalue; therefore, when this vector space has dim $= 3$, $\overline{NE}(X)$ looks like the following picture.*




*In particular, $\overline{NE}(X)$ is not finitely generated: indeed, every boundary point generates an extremal ray, hence there are extremal rays whose only rational point is $0$, so they cannot be generated by the numerical class of a curve on $X$.*

**Example 5.3.7.** *Let $X$ be a ruled surface with associated morphism $\varphi$. By Proposition 5.2.9, $\overline{NE}(X)$ is a closed convex cone in $\mathbb{R}^2$, so it has two extremal rays. Let $F$ be a fiber of $\varphi$; since $(F^2) = 0$, by Proposition 5.3.2 we have that $[F]_\equiv \in \partial\overline{NE}(X)$ spans an extremal ray. Let $\xi$ be the generator of the other extremal ray; by point (4) of the same Proposition $\xi^2 \leqslant 0$, so we have two cases:*

- *if $\xi^2 < 0$, by point (5) of the same Proposition there exists an irreducible curve $C$ on $X$ s.t. $\xi = [C]_\equiv$, and $NE(X) = \mathbb{R}^+[C]_\equiv + \mathbb{R}^+[F]_\equiv$ is closed;*
- *if $\xi^2 = 0$, by decomposing $\xi$ in a basis $([F]_\equiv, z)$ for $N_1(X)_\mathbb{Q}$ as $\xi = az + b[F]_\equiv$ we have that $a/b \in \mathbb{Q}$, so we may assume that $\xi$ is rational. However, it may happen that no multiple of $\xi$ can be represented by an effective divisor, in which case $NE(X)$ is not closed.*

## 5.4 The cone theorem for surfaces

**Definition 5.4.1.** *Given a projective $K$-scheme $X$, let $D$ be a divisor on $X$. Then, for any $S \subseteq N_1(X)_\mathbb{R}$ we define $S_{D\geqslant 0} = \{z \in S | D \cdot z \geqslant 0\}$, and similarly for $S_{D\leqslant 0}$, $S_{D>0}$ and $S_{D<0}$.*

**Theorem 5.4.2** (**Cone Thm. for surfaces**). *Let $X$ be a smooth projective surface. Then, there exists a family of irreducible rational curves $\{C_i\}_{i\in\mathbb{N}}$ s.t. $-3 \leqslant (K_X \cdot C_i) < 0$ and $\overline{NE}(X) = \overline{NE}(X)_{K_X\geqslant 0} + \sum_{i\in\mathbb{N}} \mathbb{R}^+[C_i]_\equiv$. Moreover, the rays $\mathbb{R}^+[C_i]_\equiv$ are extremal, can be contracted, and can only accumulate on the hyperplane $(K_X)^\perp$.*

**Proof.** One can refer to [5], D.3.2. □

**Remark 5.4.3.** *The extremal rays $\mathbb{R}^+[C_i]_\equiv$ can be contracted in three different ways:*

- *if $\exists i$ s.t. $(C_i^2) > 0$, by Proposition 5.3.2-(4) we get that $\rho_X = 1$ and $-K_X$ is ample. The contraction of $\mathbb{R}^+[C_i]_\equiv$ is the map to a point, and $X \cong \mathbb{P}^2_K$;*

- *if $\exists i$ s.t. $(C_i^2) = 0$, by item (3) of the same Proposition we have that $X$ is a ruled surface, and $C_i$ is a fiber of the associated morphism $\varphi$. The contraction of $\mathbb{R}^+[C_i]_\equiv$ is $\varphi$;*

- *if $(C_i^2) < 0 \ \forall i$, it follows from the Genus formula 5.1.9 that $C_i$ is smooth and s.t. $(K_X \cdot C_i) = (C_i^2) = -1$ for all $i$.*

The study of the last case led to the following classical Theorem.

**Theorem 5.4.4** (**Castelnuovo's Thm.**). *Given a smooth projective surface $X$, let $C$ be a smooth rational curve on $X$ s.t. $(C^2) = -1$. Then, there exist a smooth projective surface $Y$, a point $p \in Y$ and a morphism $\varepsilon : X \to Y$ s.t. $\varepsilon(C) = \{p\}$; moreover, $\varepsilon$ is isomorphic to the blow-up of $Y$ at $p$.*

**Proof.** We will only prove the existence of such an $\varepsilon$; for the proof of the smoothness of $Y$, we refer the avid reader to [4], Theorem V.5.7.
Let $H$ be a very ample divisor on X. Upon replacing $H$ with $mH$ with $m$ large enough, we may assume that $H^1(X, H) = \{0\}$. Set $k = (H \cdot C) > 0$, and define $D = H + kC$.
Since $(D \cdot C) = 0$, $\mathcal{O}_X(D)$ is associated to a morphism to a projective space which contracts $C$ and no other curve to a point. From the exact sequence
$0 \to \mathcal{O}_X(H+(i-1)C) \to \mathcal{O}_X(H+iC) \to \mathcal{O}_C(k-i) \to 0$ and the fact that $H^1(X, H) = \{0\}$, we easily see by induction on $i \in \{0, \ldots, k\}$ that $H^1(X, H + iC) = \{0\}$. In particular,

for $i = k$ we get a surjection $H^0(X, D) \to H^0(C, \mathcal{O}_C) \cong K$: this tells us that the sheaf $\mathcal{O}_X(D)$ is generated by its global sections, hence it defines a morphism $\varepsilon : X \to \mathbb{P}^n_K$ which contracts the curve $C$ to a point $p$. Moreover, $\varepsilon$ induces an isomorphism between $X \smallsetminus C$ and $\varepsilon(X) \smallsetminus \{p\}$. $\square$

**Definition 5.4.5.** *A del Pezzo surface $X$ is a smooth projective surface s.t. $-K_X$ is ample.*

**Remark 5.4.6.** *If $X$ is a del Pezzo surface, then by Kleiman's criterion 4.5.1-*(1) *we have that $\overline{NE}(X) \smallsetminus \{0\} \subseteq \overline{NE}(X)_{K_X<0}$. From the Cone Thm. for surfaces 5.4.2, it follows that the set of extremal rays is discrete and compact, hence finite. Therefore, $\overline{NE}(X) = NE(X) = \sum_{i=1}^{m} \mathbb{R}^+[C_i]_\equiv$. One can also check that if $X$ is a ruled surface, then it can only be isomorphic to $F_0$ or $F_1$.*

**Example 5.4.7.** *$\mathbb{P}^2_K$ and a smooth cubic surface $X \subseteq \mathbb{P}^3_K$ are both del Pezzo surfaces. In this last case, we have that $\overline{NE}(X) = NE(X) = \sum_{i=1}^{27} \mathbb{R}^+[C_i]_\equiv \subseteq \mathbb{R}^7$, where the $C_i$ are the* 27 *lines on $X$.*

## 5.5 Rational maps between smooth surfaces

**Proposition 5.5.1.** *Given a rational map $\varphi : X \dashrightarrow Y$ between integral schemes with domain $U$, set $X' = \Gamma(\varphi)$ and let $p : X' \to X$ be the first projection. If $X$ is normal and $Y$ is proper, then:*

1. *$codim_X(X \smallsetminus U) \geqslant 2$;*

2. *$X \smallsetminus U$ is exactly the set of points of $X$ where $p$ has positive-dimensional fibers.*

**Proof.** *(1)* If $x$ is a point of codimension 1 in $X$, then $\mathcal{O}_{X,x}$ is an integrally closed noetherian local domain of $dim = 1$, hence is a DVR thanks to Theorem 1.1.40. By the local valuative criterion for properness, it follows that the generic point $Spec(K(X)) \to Y$ extends to $Spec(\mathcal{O}_{X,x}) \to Y$.
*(2)* By Zariski's Main Theorem, $p$ is proper and its fibers are connected. If a fiber $p^{-1}(x)$

is a single point, then $x$ has a neighborhood $V$ in $X$ s.t. the map $p^{-1}(V) \to V$ induced by $p$ is finite; since it is birational and $X$ is normal, it is an isomorphism. By *(1)*, the thesis follows. □

**Remark 5.5.2.** *A rational map from a smooth curve is actually a morphism, hence two smooth birational curves are isomorphic. Moreover, a rational map from a smooth surface is defined on the complement of a finite set.*

**Theorem 5.5.3** (**Elimination of indeterminacies**)**.** *Let $\varphi : X \dashrightarrow Y$ be a rational map, where $X$ is a smooth projective surface and $Y$ is projective. Then, there exists a birational morphism $\varepsilon : \tilde{X} \to X$ which is a composition of blow-ups of points, s.t. $\varphi \circ \varepsilon : \tilde{X} \to Y$ is a morphism.*

**Proof.** We can replace $Y$ with a projective space $\mathbb{P}^n_K$, so that $\varphi$ can be written as $\varphi(x) = (s_0(x), \dots, s_n(x))$, where $s_0, \dots, s_n$ are sections of the invertible sheaf $\varphi^*\mathcal{O}_{\mathbb{P}^n_K}(1)$. Since $\mathcal{O}_{\mathbb{P}^n_K}(1)$ is globally generated, so is $\varphi^*\mathcal{O}_{\mathbb{P}^n_K}(1)$ on the domain $U \subseteq X$ of $\varphi$. In particular, there exist two effective divisors $D, D'$ in the linear system $\varphi^*|\mathcal{O}_{\mathbb{P}^n_K}(1)|$ with no common component in $U$. Since by the previous Remark $X \smallsetminus U$ is just a finite set of points, $D$ and $D'$ have no common component on the whole $X$, hence $(D^2) = (D \cdot D') \geqslant 0$. If $\varphi$ is a morphism, there is nothing to prove. Otherwise, let $x$ be a point of $X$ where $s_0, \dots, s_n$ all vanish, and let $\varepsilon : \tilde{X} \to X$ be the blow-up of this point, with exceptional curve $E$. Since the sections $s_0 \circ \varepsilon, \dots, s_n \circ \varepsilon \in H^0(\tilde{X}, \varepsilon^* D)$ all vanish identically on $E$, let $m > 0$ be the largest integer s.t. they all vanish there at order $m$. Taken $s \in H^0(\tilde{X}, E)$ s.t. $div(s) = E$, we can write $s_i \circ \varepsilon = \tilde{s}_i s^m$, where $\tilde{s}_0, \dots, \tilde{s}_n$ do not all vanish identically on $E$. These sections define a morphism $\tilde{\varphi} := \varphi \circ \varepsilon : \tilde{X} \to \mathbb{P}^n_K$ which satisfies $\tilde{\varphi}^*\mathcal{O}_{\mathbb{P}^n_K}(1) \cong \mathcal{O}_{\tilde{X}}(\tilde{D})$ with $\tilde{D} = \varepsilon^* D - mE$. If $\tilde{\varphi}$ is a morphism we are done; otherwise, we iterate the previous process. Since $(\tilde{D}^2) = (D^2) - m^2 < (D^2)$ and $(\tilde{D}^2)$ must remain nonnegative for the same reason that $(D^2)$ was, the process must stop after at most $(D^2)$ steps. □

**Remark 5.5.4.** *This theorem was generalized by Hironaka to the case where $X$ is any smooth projective variety over an algebraically closed feld of characteristic* $0$*; in this setting, the morphism $\varepsilon$ is a composition of blow-ups of smooth subvarieties.*

**Corollary 5.5.5.** *Let $\varphi : X \dashrightarrow Y$ be a rational map, where $X$ is a smooth projective surface and $Y$ is projective. If $Y$ contains no rational curves, then $\varphi$ is a morphism.*

**Proof.** Let $\varepsilon : \tilde{X} \to X$ be a minimal composition of blow-ups s.t. $\tilde{\varphi} = \varphi \circ \varepsilon : \tilde{X} \to Y$ is a morphism. If we prove that $\varepsilon$ is an isomorphism, then $\varphi$ is a morphism. Assume by contradiction that $\varepsilon$ is not an isomorphism, and let $E \subseteq \tilde{X}$ be the last exceptional curve. Then $\tilde{\varphi}(E)$ must be a curve, and it must be rational, which contradicts the hypothesis. □

**Theorem 5.5.6.** *Any birational morphism $\varphi : X \to Y$ between smooth projective surfaces is a composition of blow-ups of points and an isomorphism.*

**Proof.** If $\varphi$ is an isomorphism, there is nothing to prove. Otherwise, let $y$ be a point of $Y$ where $\varphi^{-1}$ is not defined and let $\varepsilon : \tilde{Y} \to Y$ be the blow-up of $y$, with exceptional curve $E$. Define $f = \varepsilon^{-1} \circ \varphi : X \dashrightarrow \tilde{Y}$ and $g = f^{-1} : \tilde{Y} \dashrightarrow X$; we want to show that $f$ is a morphism. Assume by contradiction that $f$ is not defined at a point $x \in X$, i.e. that $\varepsilon^{-1}$ is not defined at $\varphi(x)$. Then, there is a curve in $\tilde{Y}$ that $\varepsilon$ maps to $\varphi(x)$, hence this curve must be $E$ and $\varphi(x) = y$ holds; moreover, $g(E) = \{x\}$. Since $\varphi^{-1}$ is not defined at $y$ and $\varphi(x) = y$, there exists a curve $C \subseteq X$ with $x \in C$ s.t. $\varphi(C) = \{y\}$. Now, let $\tilde{y}$ be a point of $E$ where $g$ is defined, and consider the inclusions of local rings $\mathcal{O}_{Y,y} \xhookrightarrow{\varphi^*} \mathcal{O}_{X,x} \xhookrightarrow{g^*} \mathcal{O}_{\tilde{Y},\tilde{y}} \subseteq K(X)$. Choose a system of parameters $(t, v)$ on $\tilde{Y}$ at $\tilde{y}$ (i.e. elements of $\mathfrak{m}_{\tilde{Y},\tilde{y}}$ whose classes in $T_{\tilde{y}}(\tilde{Y}) = \mathfrak{m}_{\tilde{Y},\tilde{y}}/\mathfrak{m}^2_{\tilde{Y},\tilde{y}}$ generate this $K$-v.s.) s.t. $E$ is locally defined by $v$, and a system of parameters $(u, v)$ on $Y$ at $y$ with $u = tv$. Let $w \in \mathfrak{m}_{X,x}$ be a local defining equation for $C$ at $x$. Since $\varphi(C) = y$, we have that $w|u, v$, so we can write $u = aw$ and $v = bw$ for some $a, b \in \mathcal{O}_{X,x}$. Since $v \notin \mathfrak{m}^2_{\tilde{Y},\tilde{y}}$, surely $b \notin \mathfrak{m}_{X,x}$, hence $b$ is invertible and $t = u/v = a/b \in \mathcal{O}_{X,x}$; since $t \in \mathfrak{m}_{\tilde{Y},\tilde{y}}$, it follows that $t \in \mathfrak{m}_{X,x}$. From the fact that $g(E) = \{x\}$ we get that any element of $g^*\mathfrak{m}_{X,x}$ must be divisible in $\mathcal{O}_{\tilde{Y},\tilde{y}}$ by the equation $v$ of $E$: this implies that $v|t$, which is absurd since $(t, v)$ is a system of parameters. This proves that $f$ is a morphism.

Each time $\varphi^{-1}$ is not defined at a point of the image, we can therefore factor $\varphi$ through the blow-up of that point. For each factorization of $\varphi$ as $X \xrightarrow{f'} Y' \to Y$ we have by

Proposition 4.7.3 an injection $(f')^* : N^1(Y')_{\mathbb{R}} \hookrightarrow N^1(X)_{\mathbb{R}}$, hence the Picard numbers of the $Y'$ must remain bounded by $\rho_X$. Since these Picard numbers increase by 1 at each blow-up, the process must stop after finitely many blow-ups of $Y$, in which case we end up with an isomorphism. □

**Corollary 5.5.7.** *Any birational map $\varphi : X \dashrightarrow Y$ between smooth projective surfaces can be factored as the inverse of a composition of blow-ups of points, followed by a composition of blow-ups of points, and an isomorphism.*

**Proof.** By Theorem 5.5.3 there is a composition of blow-ups $\varepsilon : \tilde{X} \to X$ s.t. $\varphi \circ \varepsilon$ is a (birational) morphism, to which the previous Theorem applies. □

**Remark 5.5.8.** *This corollary was generalized in higher dimensions by Abramovich, Karu, Matsuki, Wlodarczyk and Morelli: they proved that any birational map between smooth projective varieties over an algebraically closed field of characteristic* 0 *can be factored as a composition of blow-ups of smooth subvarieties, inverses of such blow-ups and an isomorphism; this is called weak factorization.*

## 5.6 The minimal model program for surfaces

**Remark 5.6.1.** *Let $X$ be a smooth projective surface. From Castelnuovo's Thm. 5.4.4 we know that by contracting exceptional curves on $X$ one arrives eventually at a surface $X_0$ with no exceptional curves: such a surface is called a minimal model for $X$. Notice that this process of contractions must come to an end because the Picard number of the new surface decreases by* 1 *at each step thanks to Proposition 4.7.26. According to the Cone Thm. 5.4.2, we can have two cases:*

- *either $K_{X_0}$ is nef,*
- *or there exists a rational curve $C_i$ as in the theorem. This curve cannot be exceptional, hence $X_0$ is either isomorphic to $\mathbb{P}^2_K$ or a ruled surface, and $X$ has a morphism to a smooth curve whose generic fiber is $\mathbb{P}^1_K$.*

*In particular, if $X$ is not birational to a ruled surface, then it has a minimal model $X_0$ with $K_{X_0}$ nef.*

**Proposition 5.6.2.** *Let $\varphi : X \dashrightarrow Y$ be a birational map between smooth projective surfaces. If $K_Y$ is nef, then $\varphi$ is a morphism. If both $K_X$ and $K_Y$ are nef, then $\varphi$ is an isomorphism.*

**Proof.** Let $f : Z \to Y$ be the blow-up of a point, and let $C \subseteq Z$ be an integral curve other than the exceptional curve $E$. We have that $f^*f(C) \sim C + mE$ for some $m \geqslant 0$, and $K_Z = f^*K_Y + E$. From this, it follows that $(K_Z \cdot C) = ((f^*K_Y + E) \cdot (f^*f(C) - mE)) = = (K_Y \cdot f(C)) - m(E \cdot E) = (K_Y \cdot f(C)) + m \geqslant (K_Y \cdot f(C)) \geqslant 0$. Since any birational morphism $\psi : Z \to Y$ decomposes as a composition of blow-ups by Theorem 5.5.6, by induction on the number of blow-ups we get that $(K_Z \cdot C) \geqslant 0$ for any integral curve $C \subseteq Z$ not contracted by $\psi$.

Now, by Theorem 5.5.3 there exists a (minimal) composition of blow-ups $\varepsilon : \tilde{X} \to X$ s.t. $\tilde{\varphi} = \varphi \circ \varepsilon$ is a morphism. If we can show that $\varepsilon$ is an isomorphism, then $\varphi$ is a morphism. Assume by contradiction that $\varepsilon$ is not an isomorphism; then, its last exceptional curve $E$ is not contracted by $\tilde{\varphi}$, hence it must satisfy $(K_{\tilde{X}} \cdot E) \geqslant 0$, which is absurd since it holds $(K_{\tilde{X}} \cdot E) = -1$.

If also $K_X$ is nef, then also $\varphi^{-1}$ is a morphism; therefore, $\varphi$ is an isomorphism. $\square$

**Remark 5.6.3.** *The previous Remark proves that if $X$ is not birational to a ruled surface, then it has a unique minimal model (up to isomorphism).*

# Chapter 6

# The cone of curves and the minimal model program

In this whole chapter, $K$ will be an algebraically closed field.

## 6.1 Parametrizing morphisms

**Proposition 6.1.1.** *Morphisms $\mathbb{P}^1_K \to \mathbb{P}^n_K$ of degree $d$ are parametrized by a Zariski open subset of $\mathbb{P}((Sym^d(K^2))^{n+1})$; we denote this quasi-projective variety $Mor_d(\mathbb{P}^1_K, \mathbb{P}^n_K)$.*

**Proof.** Let $f : \mathbb{P}^1_K \to \mathbb{P}^n_K$ be a morphism of $deg = d$; then, we can write $f(x, y) =$ $= (F_0(x, y), \dots, F_n(x, y))$, where the $F_i \in K[x, y]_d$ have no nonconstant common factor in $K[x, y]$. We want to show that there exist universal integral polynomials in the coefficients of $F_0, \dots, F_n$ which vanish if and only if they have a nonconstant common factor in $K[x, y]$, i.e. a nontrivial common zero in $\mathbb{P}^1_K$. While ($\Rightarrow$) clearly holds, by the Nullstellensatz the hypothesis of ($\Leftarrow$) tells us that the ideal generated by $F_0, \dots, F_n$ in $K[x, y]$ contains some power of the maximal ideal $(x, y)$. This means that for some $m$, the linear map $(K[x, y]_{m-d})^{n+1} \to K[x, y]_m$ given by $(G_0, \dots, G_n) \mapsto \sum_{i=0}^{n} F_i G_i$ is surjective, hence of rank $m + 1$. Therefore, we conclude that $F_0, \dots, F_n$ have a nonconstant common factor in $K[x, y] \iff$ for any $m$, all $(m + 1)$-minors of some universal matrix whose entries are linear integral combinations of the coefficients of the $F_i$ vanish. This defines a Zariski closed subset of the projective space $\mathbb{P}((Sym^d(K^2))^{n+1})$, defined over $\mathbb{Z}$. $\square$

**Remark 6.1.2.** *With the notations of the previous proof, we have a universal morphism*

$u : \mathbb{P}^1_K \times Mor_d(\mathbb{P}^1_K, \mathbb{P}^n_K) \to \mathbb{P}^n_K$ *given by* $((x, y), f) \mapsto (F_0(x, y), \dots, F_n(x, y))$.
*Moreover, morphisms* $\mathbb{P}^1_K \to \mathbb{P}^n_K$ *are parametrized by the disjoint union of quasi-projective schemes* $Mor(\mathbb{P}^1_K, \mathbb{P}^n_K) = \bigsqcup_{d \geqslant 0} Mor_d(\mathbb{P}^1_K, \mathbb{P}^n_K)$.

**Example 6.1.3.** *In the case of a morphism* $f : \mathbb{P}^1_K \to \mathbb{P}^n_K$ *of degree* $d = 1$*, we can write* $F_i(x, y) = a_i x + b_i y$ *with* $(a_0, \dots, a_n, b_0, \dots, b_n) \in \mathbb{P}^{2n+1}_K$*. The condition that* $F_0, \dots, F_n$ *have no common zeroes is equivalent to* $rk \begin{pmatrix} a_0 & \dots & a_n \\ b_0 & \dots & b_n \end{pmatrix} = 2$*. Its complement* $Z$ *in* $\mathbb{P}^{2n+1}_K$ *is defined by the vanishing of all its* $2 \times 2$*-minors, i.e. by the equations* $a_i b_j - a_j b_i = 0 \ \forall i \neq j$*. The universal morphism* $u : \mathbb{P}^1_K \times (\mathbb{P}^{2n+1}_K \smallsetminus Z) \to \mathbb{P}^n_K$ *is given by*
$((x, y), (a_0, \dots, a_n, b_0, \dots, b_n)) \mapsto (a_0 x + b_0 y, \dots, a_n x + b_n y)$.

**Definition 6.1.4.** *Let* $X$ *be a closed subscheme of* $\mathbb{P}^n_K$ *defined by the homogeneous equations* $G_1, \dots, G_m$*. Morphisms* $\mathbb{P}^1_K \to X$ *of degree* $d$ *are parametrized by the subscheme* $Mor_d(\mathbb{P}^1_K, X)$ *of* $Mor_d(\mathbb{P}^1_K, \mathbb{P}^n_K)$ *defined by the equations* $G_i(F_0, \dots, F_n) = 0 \ \forall i = 1, \dots, m$*. Moreover, morphisms* $\mathbb{P}^1_K \to X$ *are parametrized by the disjoint union of quasi-projective schemes* $Mor(\mathbb{P}^1_K, X) = \bigsqcup_{d \geqslant 0} Mor_d(\mathbb{P}^1_K, X)$.

**Remark 6.1.5.** *We can extend this definition for any quasi-projective variety* $X$*: after embedding* $X$ *into some projective variety* $\overline{X}$*, there is a universal morphism*
$u : \mathbb{P}^1_K \times Mor(\mathbb{P}^1_K, \overline{X}) \to \overline{X}$*, and* $Mor(\mathbb{P}^1_K, X)$ *is the complement in* $Mor(\mathbb{P}^1_K, \overline{X})$ *of the image by the second projection of the closed subscheme* $u^{-1}(\overline{X} \smallsetminus X)$.
*If moreover* $X$ *is defined by the homogeneous equations* $G_1, \dots, G_m$ *with coefficients in a subring* $R$ *of* $K$*, the scheme* $Mor_d(\mathbb{P}^1_K, X)$ *has the same property. If* $\mathfrak{m}$ *is a maximal ideal of* $R$*, let* $X_{\mathfrak{m}}$ *be the reduction of* $X$ *modulo* $\mathfrak{m}$*: this is the subscheme of* $\mathbb{P}^n_{R/\mathfrak{m}}$ *defined by the reductions of the* $G_i$ *modulo* $\mathfrak{m}$*. Since the equations defining the complement of* $Mor_d(\mathbb{P}^1_K, \mathbb{P}^n_K)$ *in* $\mathbb{P}((Sym^d(K^2))^{n+1})$ *are defined over* $\mathbb{Z}$*,* $Mor_d(\mathbb{P}^1_K, X_{\mathfrak{m}})$ *is the reduction of the* $R$*-scheme* $Mor_d(\mathbb{P}^1_K, X)$ *modulo* $\mathfrak{m}$.

**Remark 6.1.6.** *Let* $X, Y$ *be varieties over* $K$*, with* $X$ *projective and* $Y$ *quasi-projective. One can show (see [3] for a reference) that* $K$*-morphisms* $X \to Y$ *are parametrized by a scheme* $Mor(X, Y)$ *which is locally of finite type. Fixed an ample divisor* $H$ *on* $Y$ *and a polynomial* $P$ *with rational coefficients, define the subscheme* $Mor_P(X, Y)$ *of* $Mor(X, Y)$

*as the one parametrizing morphisms $X \to Y$ with fixed Hilbert polynomial $P(m) = \chi(X, mf^*H)$. Notice that $Mor_P(X,Y)$ is quasi-projective over $K$, and that $Mor(X,Y) = \bigsqcup_{P \in \mathbb{Q}[x]} Mor_P(X,Y)$.*
*Moreover, if $X$ is a curve, fixing the Hilbert polynomial is equivalent to fixing the degree of the 1-cycle $f_*X$ for the embedding of $Y$ defined by some multiple of $H$.*

**Proposition 6.1.7.** *Let $X, Y$ be varieties over $K$, with $X$ projective and $Y$ quasi-projective. Then, there is a universal morphism $u : X \times Mor(X,Y) \to Y$ s.t. for any $K$-scheme $Z$, the correspondence $\{$morphisms $Z \to Mor(X,Y)\} \leftrightarrow \{$morphisms $X \times Z \to Y\}$ given by $\varphi \mapsto f(x,z) = u(x, \varphi(z))$ is one-to-one.*

**Example 6.1.8.** *$Mor(SpecK, X) = X$, and the universal morphism $u : SpecK \times X \to X$ is the second projection.*

**Notation.** *Given a morphism $f : X \to Y$, we denote by $[f]$ the corresponding element of $Mor(X,Y)$.*

**Theorem 6.1.9.** *Given $X, Y$ projective varieties over $K$, let $f : X \to Y$ be a $K$-morphism s.t. $Y$ is smooth along $f(X)$. Then, locally around $[f]$ the scheme $Mor(X,Y)$ can be defined by $h^1(X, f^*T_Y)$ equations in a smooth scheme of dimension $h^0(X, f^*T_Y)$. In particular, any irreducible component of $Mor(X,Y)$ through $[f]$ has dimension at least $h^0(X, f^*T_Y) - h^1(X, f^*T_Y)$.*

**Proof.** See [1], Theorem 6.8. □

**Corollary 6.1.10.** *Given $X, Y$ projective varieties over $K$, let $f : X \to Y$ be a $K$-morphism s.t. $Y$ is smooth along $f(X)$. If $H^1(X, f^*T_Y) = \{0\}$, then $Mor(X,Y)$ is smooth at $[f]$.*

**Definition 6.1.11.** *Given $X, Y$ projective varieties over $K$, let $x_1, \ldots, x_r \in X$ and $y_1, \ldots, y_r \in Y$. The morphisms $X \to Y$ which map each $x_i$ to $y_i$ can be parametrized by the fiber over $(y_1, \ldots, y_r)$ of the map $\rho : Mor(X,Y) \to X^r$ given by $[f] \mapsto (f(x_1), \ldots, f(x_r))$; we denote this space by $Mor(X, Y; x_i \mapsto y_i)$.*

**Remark 6.1.12.** *At a point $[f] \in Mor(X,Y;x_i \mapsto y_i)$ s.t. $Y$ is smooth along $f(X)$, the tangent map to $\rho$ is the evaluation $H^0(X, f^*T_Y) \to \bigoplus_{i=1}^{r}(f^*T_Y)_{x_i} \cong \bigoplus_{i=1}^{r} T_{Y,y_i}$, hence the tangent space to $Mor(X,Y;x_i \mapsto y_i)$ is its kernel $H^0(X, f^*T_Y \otimes \mathcal{I}_{x_1,\dots,x_r})$, where $\mathcal{I}_{x_1,\dots,x_r}$ is the ideal sheaf of $x_1,\dots,x_r$ in $X$. By the classical theorems on the dimention of fibers and Theorem 6.1.9, locally around $[f]$ the scheme $Mor(X,Y;x_i \mapsto y_i)$ can be defined by $h^1(X, f^*T_Y) + r\,dim(Y)$ equations in a smooth scheme of dimension $h^0(X, f^*T_Y)$; in particular, its irreducible components at $[f]$ are all of dimension at least*
*$h^0(X, f^*T_Y) - h^1(X, f^*T_Y) - r\,dim(Y)$.*

**Remark 6.1.13.** *Given a curve $C$ and $c_1,\dots,c_r \in C$, let $f : C \to X$ be a morphism. By Riemann-Roch 3.1.7, one has that $dim_{[f]}Mor(C,X) \geqslant \chi(C, f^*T_X) = -K_X \cdot f_*C +$*
*$+ (1-g(C))dim(X)$, and $dim_{[f]}Mor\big(C,X;c_i \mapsto f(c_i)\big) \geqslant \chi(C, f^*T_X) - r\,dim(X) =$*
*$= -K_X \cdot f_*C + (1-g(C)-r)dim(X)$.*

## 6.2 "Bend-and-break" lemmas

We now enter the world of Mori Theory: the whole story began in 1979, with Mori's astonishing proof of a conjecture of Hartshorne characterizing projective spaces as the only smooth projective varieties with ample tangent bundle. The techniques that Mori introduced to solve this conjecture have turned out to have more far reaching applications than Hartshorne's conjecture itself. In particular, Mori's first idea is that if a curve deforms on a projective variety while passing through a fixed point, it must at some point break up with at least one rational component, hence the name "bend-and-break".

**Notation.** *Given a projective variety $X$, let $C$ be a curve on $X$ with irreducible components $C_1,\dots,C_r$, and let $\varphi : C \to X$ be a morphism. Extending Definition 3.4.19, we write $\varphi_*C$ for the effective 1-cycle $\sum_{i=1}^{r} d_i\varphi(C_i)$, where $d_i$ is the degree of $\varphi|_{C_i}$ onto its image. Note that for any Cartier divisor $D$ on $X$, the Projection formula 3.4.20 implies that $(D \cdot \varphi_*C) = deg(\varphi^*D)$.*

**Proposition 6.2.1 (Mori).** *Given a projective variety $X$, let $f : \mathbb{P}^1_K \to X$ be a parametrization of a rational curve. If $dim_{[f]}Mor\big(\mathbb{P}^1_K, X; 0 \mapsto f(0), \infty \mapsto f(\infty)\big) \geqslant 2$, then the 1-cycle*

*$f_*\mathbb{P}^1_K$ is numerically equivalent to a connected nonintegral effective 1-cycle with rational components passing through $f(0)$ and $f(\infty)$.*

**Proof.** Look at [1], Proposition 7.3. $\square$

**Remark 6.2.2.** *Thanks to Remark 6.1.13, if $X$ is smooth along $f(\mathbb{P}^1_K)$, the hypothesis of the previous Proposition are satisfied whenever $(-K_X \cdot f_*\mathbb{P}^1_K) - dim(X) \geqslant 2$.*

**Definition 6.2.3.** *A Fano variety is a smooth projective variety over $K$ with ample anticanonical divisor.*

**Remark 6.2.4.** *A finite product of Fano varieties is a Fano variety.*

**Example 6.2.5.** *The projective space is a Fano variety. Moreover, any smooth complete intersection in $\mathbb{P}^n_K$ defined by equations of degrees $d_1, \ldots, d_s$ with $\sum_{i=1}^{s} d_i \leqslant n$ is a Fano variety.*

**Remark 6.2.6.** *A Del Pezzo surface is a Fano surface.*

**Proposition 6.2.7.** *Given a Fano variety $Y$, let $D_1, \ldots, D_r$ be nef divisors on $Y$. If $\mathcal{E} = \bigoplus_{i=1}^{r} \mathcal{O}_Y(D_i)$, then $X = \mathbb{P}(\mathcal{E})$ is a Fano variety.*

**Proof.** Set $D' = -K_Y - \sum_{i=1}^{r} D_i$; by Proposition 4.2.4, $D'$ is ample. Let $\varphi : X \to Y$ be the canonical map, and let $D$ be a divisor on $X$ associated to the invertible sheaf $\mathcal{O}_X(1)$; notice that $D$ is nef on $X$ since each $D_i$ is nef on $Y$. Then, one can show that it holds $-K_X = rD + \varphi^*D'$. Again from Proposition 4.2.4, we can conclude that $-K_X$ is ample. $\square$

**Theorem 6.2.8 (Miyaoka-Mori Thm.).** *Given a projective variety $X$, let $H$ be an ample divisor on $X$, and let $\varphi : C \to X$ be a morphism from a smooth curve s.t. $X$ is smooth along $\varphi(C)$ and $(K_X \cdot \varphi_*C) < 0$. Then, for any point $x \in \varphi(C)$ there exists a rational curve $\Gamma$ on $X$ through $x$ that satisfies $(H \cdot \Gamma) \leqslant 2dim(X)\frac{(H \cdot \varphi_*C)}{(-K_X \cdot \varphi_*C)}$.*

**Proof.** See [1], Theorem 7.7. $\square$

**Corollary 6.2.9.** *A Fano variety $X$ is covered by rational curves of $(-K_X)$-degree at most $2dim(X)$.*

**Corollary 6.2.10.** *The canonical divisor of a smooth projective complex variety which contains no rational curves is nef.*

**Lemma 6.2.11.** *Given a projective variety $X$ and a positive integer $d$, let $M_d$ be the quasi-projective scheme that parametrizes morphisms $\mathbb{P}^1_K \to X$ of degree at most $d$ with respect to some ample divisor. Then, the image of the evaluation map $\nu_d : \mathbb{P}^1_K \times M_d \to X$, i.e. the set of points of $X$ through which passes a rational curve of degree at most $d$, is closed in $X$.*

**Proof.** One can find it in [1], Lemma 7.8. $\square$

**Theorem 6.2.12 (Bertini's Thm.).** *Let $X$ be a nonsingular closed subvariety of $\mathbb{P}^n_K$. Then, there exists a hyperplane $H \subseteq \mathbb{P}^n_K$ not containing $X$ s.t. the scheme $H \cap X$ is regular at every point. Furthermore, the set of hyperplanes with this property forms an open dense subset of the linear system $|H|$.*

**Proof.** Look at [4], Theorem II.8.18. $\square$

**Theorem 6.2.13.** *If $X$ is a smooth projective variety with $-K_X$ nef, then:*

- *either $K_X$ is numerically trivial,*
- *or there is a rational curve through any point of $X$.*

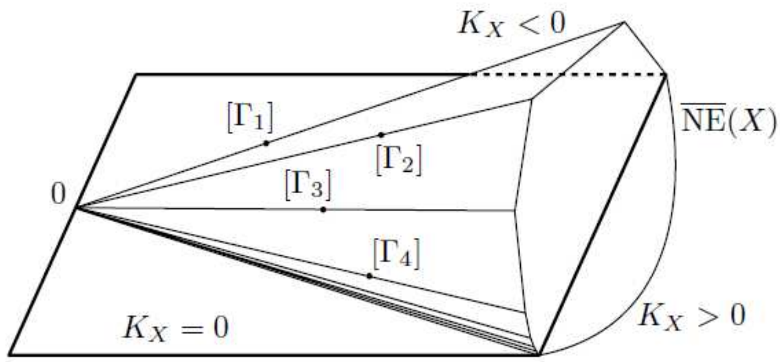
**Proof.** Let $H$ be a very ample divisor on $X$, corresponding to a hyperplane section of an embedding of $X$ in $\mathbb{P}^N_K$. Set $n = dim(X)$.
Assume $(K_X \cdot H^{n-1}) = 0$. For any curve $C \subseteq X$, there exist hypersurfaces $H_1, \dots, H_{n-1}$ in $\mathbb{P}^N_K$ of respective degree $d_1, \dots, d_{n-1}$ s.t. the scheme-theoretic intersection $Z = X \cap \bigcap_{i=1}^{n-1} H_i$ has pure dimension 1 and contains $C$. Since $-K_X$ is nef, we get that $0 \leqslant (-K_X \cdot C) \leqslant$ $\leqslant (-K_X \cdot Z) = \prod_{i=1}^{n-1} d_i(-K_X \cdot H^{n-1}) = 0$. This shows that $(K_X \cdot C) = 0$ for any curve $C$ on $X$, hence $K_X$ is numerically trivial.

Assume now $(K_X \cdot H^{n-1}) < 0$. Taken a point $x$ of $X$, let $C$ be the normalization of the intersection of $n-1$ general hyperplane sections through $x$. By Bertini's Thm. 6.2.12, $C$ is an irreducible curve which satisfies $(K_X \cdot C) = (K_X \cdot H^{n-1}) < 0$. By the Miyaoka-Mori Thm. 6.2.8, there exists a rational curve on $X$ which passes through $x$. $\square$

## 6.3 The cone theorem

**Definition 6.3.1.** *Let $X$ be a smooth projective variety. An extremal ray of $\overline{NE}(X)$ is said to be $K_X$-negative if it meets $N_1(X)_{K_X<0}$.*

**Theorem 6.3.2** (**Mori's Cone Thm.**). *Let $X$ be a smooth projective variety. Then, there exists a family $\{\Gamma_i\}_{i\in\mathbb{N}}$ of rational curves on $X$ s.t. $0 < (K_X \cdot \Gamma_i) \leqslant dim(X) + 1$ and $\overline{NE}(X) = \overline{NE}(X)_{K_X\geqslant 0} + \sum_{i\in\mathbb{N}} \mathbb{R}^+[\Gamma_i]_\equiv$, where the $\mathbb{R}^+[\Gamma_i]_\equiv$ are all the $K_X$-negative extremal rays of $\overline{NE}(X)$; moreover, these rays are locally discrete in $N_1(X)_{K_X<0}$.*




**Proof.** As we saw in §6.1, there are only countably many families of, hence classes of, rational curves on $X$. Pick a representative $\Gamma_i$ for each such class $z_i$ that satisfies $0 < -K_X \cdot z_i \leqslant dim(X) + 1$.

First, we want to show that the rays $\mathbb{R}^+ z_i$ are locally discrete in the half-space $N_1(X)_{K_X<0}$. Let $H$ be an ample divisor on $X$; noticing that $N_1(X)_{K_X<0} = \bigcup_{\varepsilon>0} N_1(X)_{K_X+\varepsilon H<0}$, it is enough to show that for each $\varepsilon > 0$ there are only finitely many classes $z_i$ in $N_1(X)_{K_X+\varepsilon H<0}$. Indeed, if $((K_X + \varepsilon H) \cdot \Gamma_i) < 0$, then $(H \cdot \Gamma_i) < \frac{1}{\varepsilon}(-K_X \cdot \Gamma_i) \leqslant \frac{1}{\varepsilon}(dim(X) + 1)$, and by Kleiman's criterion 4.5.1-*(2)* there are only finitely many such classes of curves on $X$.

Secondly, we want to show that $\overline{NE}(X)$ is equal to the closure of $V = \overline{NE}(X)_{K_X\geqslant 0} +$

$+\sum_{i\in\mathbb{N}}\mathbb{R}^+[\Gamma_i]_\equiv$. If this is not the case, since $\overline{NE}(X)$ contains no lines, there exists by Lemma 4.3.8-*(7)* an $\mathbb{R}$-divisor $D$ on $X$ which is nonnegative on $\overline{NE}(X)$ (so in particular it is nef), positive on $V \smallsetminus \{0\}$ and which vanishes at some nonzero point $z$ of $\overline{NE}(X)$; notice that $z$ cannot be in $V$, hence $K_X \cdot z < 0$. Now, choose a norm on $N_1(X)_\mathbb{R}$ s.t. $\|[C]_\equiv\| \geqslant 1$ for each irreducible curve $C$; notice that this is possible since the set of classes of irreducible curves is discrete. At most by replacing $D$ with a multiple, we can assume (since $D$ is positive on $V \smallsetminus \{0\}$) that $D \cdot v \geqslant 2\|v\| \quad \forall v \in \overline{V}$. Moreover, it holds $2dim(X)(D \cdot z) = 0 < -K_X \cdot z$. Since $[D]_\equiv$ is a limit of classes of ample $\mathbb{Q}$-divisors, and $z$ is a limit of classes of effective rational 1-cycles, there exist an ample $\mathbb{Q}$-divisor $H$ and an effective 1-cycle $Z$ s.t. $2dim(X)(H \cdot Z) < (-K_X \cdot Z)$ and $H \cdot v \geqslant \|v\| \quad \forall v \in \overline{V}$. We may further assume that each component $C$ of $Z$ satisfies $(-K_X \cdot C) > 0$ by taking 0 as the coefficient of the components that do not. Notice that the class of every rational curve $\Gamma$ on $X$ s.t. $(-K_X \cdot \Gamma) \leqslant dim(X) + 1$ is in $\overline{V}$: indeed, it is either in $\overline{NE}(X)_{K_X \geqslant 0}$, or $(-K_X \cdot \Gamma) > 0$ and $[\Gamma]_\equiv$ is one of the $z_i$. This tells us that it holds $(H \cdot \Gamma) \geqslant \|[\Gamma]_\equiv\| \geqslant 1$ by what we have seen above and the choice of the norm. Since $X$ is smooth, the Miyaoka-Mori Theorem 6.2.8 implies that $2dim(X)\frac{(H \cdot C)}{(-K_X \cdot C)} \geqslant 1$ for every component $C$ of $Z$, which contradicts the fact that $2dim(X)(H \cdot Z) < (-K_X \cdot Z)$.

Finally, we would like to show that for any $I \subseteq \mathbb{N}$, the cone $V_I = \overline{NE}(X)_{K_X \geqslant 0} + \sum_{i\in I}\mathbb{R}^+[\Gamma_i]_\equiv$ is closed. By Lemma 4.3.8-*(5)*, it is enough to show that any extremal ray $\mathbb{R}^+ r$ in $\overline{V_I}$ satisfying $K_X \cdot r < 0$ is in $V_I$. Let $H$ be an ample divisor on $X$ and let $\varepsilon > 0$ be s.t. $(K_X + \varepsilon H) \cdot r < 0$. By the first step of this proof, there are only finitely many classes $z_{i_1}, \ldots, z_{i_n}$ with $i_j \in I$ s.t. $(K_X + \varepsilon H) \cdot z_{i_j} < 0$ for all $j = 1, \ldots, n$. Write $r$ as the limit of a sequence $\{r_m + s_m\}_{m \geqslant 0}$, where $r_m \in \overline{NE}(X)_{K_X + \varepsilon H \geqslant 0}$ and $s_m = \sum_{j=1}^{n} \lambda_{m,j} z_{i_j}$. Since $H \cdot r_m$ and $H \cdot z_{i_j}$ are positive and $H \cdot r < -\frac{1}{\varepsilon} K_X \cdot r$ (with the latter that is positive by hypothesis), the sequences $\{H \cdot r_m\}_{m \geqslant 0}$ and $\{\lambda_{m,j}\}_{m \geqslant 0}$ are bounded; by Kleiman's criterion 4.5.1-*(2)* we then have (at most after taking subsequences) that $\{r_m\}_{m \geqslant 0}$ and $\{\lambda_{m,j}\}_{m \geqslant 0}$ have limits. Since $r$ spans an extremal ray in $\overline{V_I}$, $\lim_{m\to\infty} r_m$ and $\lim_{m\to\infty} s_m$ must be nonnegative multiples of $r$; but from $(K_X + \varepsilon H) \cdot r < 0$ we get that $\lim_{m\to\infty} r_m$ must be 0. This tells us that $r$ is a multiple of one of the $z_{i_j}$, hence is in $V_I$. This concludes the proof. $\square$

**Corollary 6.3.3.** *Given a smooth projective variety $X$, let $R$ be a $K_X$-negative extremal ray of $\overline{NE}(X)$. Then, there exists a nef divisor $M_R$ on $X$, called supporting divisor for $R$, s.t. $R = \{z \in \overline{NE}(X) | M_R \cdot z = 0\}$. Moreover, we have that $mM_R - K_X$ is ample for all $m$ large enough.*

**Proof.** With the notation of the previous proof, there exists a unique $j \in \mathbb{N}$ s.t. $R = \mathbb{R}^+ z_j$. By the third step of the same proof, the cone $V_{\mathbb{N} \smallsetminus \{j\}} = \overline{NE}(X)_{K_X \geqslant 0} + \sum_{i \neq j} \mathbb{R}^+[\Gamma_i]_{\equiv}$ is closed, and it is strictly contained in $\overline{NE}(X)$ since it does not contain $R$. By Lemma 4.3.8-*(7)*, there exists a linear form which is nonnegative on $\overline{NE}(X)$, positive on $V_{\mathbb{N} \smallsetminus \{j\}} \smallsetminus \{0\}$ and which vanishes at some nonzero point of $\overline{NE}(X)$, hence on $R$ since $\overline{NE}(X) = V_{\mathbb{N} \smallsetminus \{j\}} + R$. Using item *(3)* of the same Lemma 4.3.8, we have that the intersection of the interior of $V^*_{\mathbb{N} \smallsetminus \{j\}}$ and the rational hyperplane $R^\perp$ is nonempty, hence it contains an integral point, thus there exists a divisor $M_R$ on $X$ which is positive on $V_{\mathbb{N} \smallsetminus \{j\}} \smallsetminus \{0\}$ and vanishes on $R$; in particular, it is nef. This proves the first statement.

Now, choose a norm on $N_1(X)_{\mathbb{R}}$ and set $T = \{z \in V_{\mathbb{N} \smallsetminus \{j\}} | \ \|z\| = 1\}$; notice that $T$ is compact. If we define $a = min\{M_R \cdot z | z \in T\} > 0$ and $b = max\{K_X \cdot z | z \in T\}$, then $mM_R - K_X$ is positive on $V_{\mathbb{N} \smallsetminus \{j\}} \smallsetminus \{0\}$ for any $m > b/a$, and positive on $R \smallsetminus \{0\}$. By Kleiman's criterion 4.5.1-*(1)*, $mM_R - K_X$ is ample for any $m > b/a$. □

## 6.4 Contractions of $K_X$-negative extremal rays

**Theorem 6.4.1 (Kawamata's base-point-free Thm.).** *Given a smooth complex projective variety $X$, let $D$ be a nef divisor on $X$ s.t. $tD - K_X$ is nef and big for some positive $t \in \mathbb{Q}$. Then, $mD$ is generated by its global sections for all $m$ large enough.*

**Corollary 6.4.2.** *Given a smooth complex projective variety $X$, let $R$ be a $K_X$-negative extremal ray of $\overline{NE}(X)$. Then:*

1. *there exists a contraction $c_R : X \twoheadrightarrow Y$ of $R$, with $Y$ being a normal projective variety;*

2. *if $C$ is an integral curve on $X$ with class in $R$, then there is an exact sequence $0 \to Pic(Y) \xrightarrow{c_R^*} Pic(X) \xrightarrow{\delta_C} \mathbb{Z}$, where $\delta_C([D]_\sim) = (D \cdot C)$; moreover, $\rho_Y = \rho_X - 1$.*

**Proof.** *(1)* Let $M_R$ be a supporting divisor for $R$; then, Kawamata's Thm. 6.4.1 tells us that $mM_R$ is generated by its global sections for all $m$ large enough. Therefore, we have an induced morphism $X \to \mathbb{P}^n_K$. By taking its Stein factorization, we get a contraction $c_R$ of $R$ as wanted.

*(2)* First, notice that by definition of $c_R$ there exists a Cartier divisor $D_m$ on $Y$ s.t. $mM_R \sim c_R^* D_m$. Since $(c_R)_*\mathcal{O}_X \cong \mathcal{O}_Y$, by the Projection formula 2.8.4 we have that for any invertible sheaf $\mathcal{L}$ on $Y$ it holds $(c_R)_*(c_R^*\mathcal{L}) \cong \mathcal{L} \otimes (c_R)_*\mathcal{O}_X \cong \mathcal{L}$: this proves that $c_R^*$ is a (split) monomorphism.

Since $Im(c_R^*) \subseteq ker(\delta_C)$ clearly, it remains to show ($\supseteq$): let $D$ be a divisor on $X$ s.t. $(D \cdot C) = 0$. Proceeding as in the proof of Corollary 6.3.3, we see that $mM_R + D$ is nef for all $m$ large enough and vanishes only on $R$. It is therefore a supporting divisor for $R$, hence some multiple $p(mM_R + D)$ also defines its contraction. Since the contraction $c_R$ is unique, there exists a Cartier divisor $E_{m,p}$ on $Y$ s.t. $p(mM_R + D) \sim c_R^* E_{m,p}$. Notice that $(p+1)(mM_R + D) \sim c_R^* E_{m,p+1}$, so by subtracting the former from the latter one gets that $c_R^*(E_{m,p+1} - E_{m,p}) \sim mM_R + D \sim c_R^* D_m + D$, from which we obtain $D \sim c_R^*(E_{m,p+1} - E_{m,p} - D_m)$. This proves that $[D]_\sim \in Im(c_R^*)$; the fact that $\rho_Y = \rho_X - 1$ immediately follows. □

**Remark 6.4.3.** *The same result holds for any $K_X$-negative extremal subcone $V$ of $\overline{NE}(X)$ instead of $R$, in which case the Picard number of $c_V(X)$ is $\rho_X - dim\langle V \rangle$.*

**Remark 6.4.4.** *Item* (2) *together with the relative Kleiman's criterion 4.7.14 imply that $-K_X$ is $c_R$-ample. Moreover, we have the dual exact sequences*
$0 \to N^1(Y)_\mathbb{R} \xrightarrow{c_R^*} N^1(X)_\mathbb{R} \to \langle R \rangle^* \to 0$ *and* $0 \to \langle R \rangle \to N_1(X)_\mathbb{R} \xrightarrow{(c_R)_*} N_1(Y)_\mathbb{R} \to 0.$

## 6.5 Different types of contractions

**Remark 6.5.1.** *Given a smooth complex projective variety $X$, let $R$ be a $K_X$-negative extremal ray of $\overline{NE}(X)$ with contraction $c_R : X \to Y$. Since $c_R$ contracts all curves whose class lies in $R$, we have that $NE(c_R) = R$. Moreover, from the fact that $(c_R)_*\mathcal{O}_X \cong \mathcal{O}_Y$ we have either that $dim(Y) < dim(X)$ or $c_R$ is birational.*

**Definition 6.5.2.** *Let $\varphi : X \to Y$ be a proper birational morphism. We define the exceptional locus $Exc(\varphi)$ of $\varphi$ as the locus of points of $X$ where $\varphi$ is not a local isomorphism.*

**Remark 6.5.3.** *Given a proper birational morphism $\varphi : X \to Y$, set $E = Exc(\varphi)$; notice that $E$ is closed. If $Y$ is normal, Zariski's Main Theorem says that $E = \varphi^{-1}(\varphi(E))$, and the fibers of $\varphi|_E : E \to \varphi(E)$ are connected and everywhere positive-dimensional; in particular, $codim_Y \varphi(E) \geqslant 2$. The largest open set over which $\varphi^{-1} : Y \dashrightarrow X$ is defined is $Y \smallsetminus \varphi(E)$.*

**Definition 6.5.4.** *Given a smooth complex projective variety $X$, let $R$ be a $K_X$-negative extremal ray of $\overline{NE}(X)$ with contraction $c_R : X \to Y$. Then, $locus(R) := Exc(c_R)$ is called the locus of $R$.*

**Remark 6.5.5.** *$locus(R)$ is the union of all curves in $X$ whose classes belong to $R$. We can distinguish 3 cases:*

- *$locus(R) = X$, $dim(Y) < dim(X)$, and $c_R$ is a fiber contraction;*
- *$locus(R)$ is a divisor, and $c_R$ is a divisorial contraction;*
- *$codim_X locus(R) \geqslant 2$, and $c_R$ is a small contraction.*

**Proposition 6.5.6.** *Given a smooth complex projective variety $X$, let $R$ be a $K_X$-negative extremal ray of $\overline{NE}(X)$. If $Z$ is an irreducible component of $locus(R)$, we have that:*

1. *$Z$ is covered by rational curves contracted by $c_R$;*
2. *if $codim_X Z = 1$, then $Z = locus(R)$;*
3. *$dim(Z) \geqslant \frac{1}{2}\big(dim(X) + dim(c_R(Z))\big)$.*

**Proof.** *(1)* Take $x \in Z$; since $x \in locus(R)$, there exists an irreducible curve $C$ through $x$ whose class is in $R$. Let $M_R$ be a supporting divisor for $R$, let $H$ be an ample divisor on $X$, and let $m$ be an integer s.t. $m > 2dim(X)\frac{(H \cdot C)}{(-K_X \cdot C)}$. By Miyaoka-Mori Thm. 6.2.8 applied to $mM_R + H$, there exists a rational curve $\Gamma$ through $x$ s.t.
$((mM_R + H) \cdot \Gamma) \leqslant 2dim(X)\frac{((mM_R+H) \cdot C)}{(-K_X \cdot C)}$. Since for $m$ large enough $mM_R + H$ is ample,

we get that $0 < ((mM_R + H) \cdot \Gamma) \leqslant 2dim(X)\frac{((mM_R+H)\cdot C)}{(-K_X\cdot C)} = 2dim(X)\frac{(H\cdot C)}{(-K_X\cdot C)} < m$, thus $0 < m(M_R \cdot \Gamma) + (H \cdot \Gamma) < m$. This tells us that $(M_R \cdot \Gamma)$ must vanish, and $(H \cdot \Gamma) < m$; therefore, $[\Gamma]_\equiv \in R$, so $\Gamma$ is contained in $locus(R)$, hence in $Z$ since it passes through $x$.

*(2)* If $locus(R) \neq X$, then $c_R$ is birational and $M_R$ is nef and big. Let $H$ be an ample divisor on $X$; as in the proof of Corollary 4.6.4, for $m$ large enough, $mM_R - H$ is linearly equivalent to an effective divisor $D$. Then, a nonzero element in $R$ has negative intersection with $D$, hence with some irreducible component $D'$ of $D$. Any irreducible curve with class in $R$ must then be contained in $D'$, which therefore contains $locus(R)$; since the other inclusion is obvious, we are done.

*(3)* Take $x \in Z$, and pick a rational curve $\Gamma$ in $Z$ through $x$ with class in $R$ and minimal $(-K_X)$-degree. Let $f : \mathbb{P}^1_K \to \Gamma \subseteq X$ be a parametrization of $\Gamma$ that satisfies $f(0) = x$. Let $T$ be a component of $Mor(\mathbb{P}^1_K, X)$ passing through $[f]$, and let $\nu_0 : T \to X$ be the map $t \mapsto f_t(0)$. By Remark 6.1.13, $dim(T) \geqslant dim(X) + 1$. Since each curve $f_t(\mathbb{P}^1_K)$ has the same class as $\Gamma$, it is contained in $Z$, hence $\nu_0(T) \subseteq Z$; moreover, for any component $T_x$ of $\nu_0^{-1}(x)$ we have that $dim(Z) \geqslant dim(T) - dim(T_x) \geqslant dim(X) + 1 - dim(T_x)$.

Now, consider the evaluation $\nu_\infty : T_x \to X$ and let $y \in X$. If $\nu_\infty^{-1}(y)$ has dimension at least 2, Mori's Proposition 6.2.1 implies that $\Gamma$ is numerically equivalent to a connected effective rational nonintegral 1-cycle $\sum_{i=1}^{m} a_i\Gamma_i$ passing through $x$ and $y$. Since $R$ is extremal, each $[\Gamma_i]_\equiv$ must be in $R$, hence $0 < (-K_X \cdot \Gamma_i) < (-K_X \cdot \Gamma)$ for each $i = 1, \ldots, m$. This contradicts the minimality of $\Gamma$ with respect to the $(-K_X)$-degree.

It follows that the fibers of $\nu_\infty$ have dimension at most 1. Since for any $t \in T_x$ the curve $f_t(\mathbb{P}^1_K)$ passes through $x$, it has the same image as $x$ by $c_R$; therefore, $\nu_\infty(T_x) = = \bigcup_{t\in T_x}\{f_t(\infty)\} = \bigcup_{t\in T_x} f_t(\mathbb{P}^1_K)$ is irreducible and contained in $c_R^{-1}(c_R(x))$. This tells us that $dim_x(c_R^{-1}(c_R(x))) \geqslant dim(\overline{\nu_\infty(T_x)}) \geqslant dim(T_x) - 1$, where the last inequality comes from the fact that the fibers of $\nu_\infty$ have dimension at most 1. Since the left-hand side is equal to $dim(Z) - dim(c_R(Z))$, it follows that $dim(Z) \geqslant dim(c_R(Z)) + dim(T_x) - 1$; summing with the previous inequality $dim(Z) \geqslant dim(X) + 1 - dim(T_x)$, one gets that $2dim(Z) \geqslant dim(X) + dim(c_R(Z))$ as wanted. □

**Remark 6.5.7.** *We could have been more precise with some inequalities in the previous* *proof: for any rational curve $\Gamma$ contained in the fiber of $c_R$ through $x$, Remark 6.1.13 tells us that $dim(T) \geqslant dim(X) + (-K_X \cdot \Gamma)$, therefore it holds $dim(Z) \geqslant dim(T) - dim(T_x) \geqslant \geqslant dim(X) + (-K_X \cdot \Gamma) - dim(T_x)$. Moreover, for any positive-dimensional irreducible component $F$ of a fiber of $c_R$ we would get $dim(F) \geqslant dim(T_x) - 1 \geqslant dim(X) + (-K_X \cdot \Gamma) - - dim(locus(R)) - 1 = codim(locus(R)) + (-K_X \cdot \Gamma) - 1$.*

**Definition 6.5.8.** *Given a smooth complex projective variety $X$, let $R$ be a $K_X$-negative extremal ray of $\overline{NE}(X)$. Then, we define the length of $R$ as the integer*
*$\ell(R) = min\{(-K_X \cdot \Gamma) | \Gamma$ rational curve on $X$ with class in $R\}$.*

**Proposition 6.5.9 (Wiśniewski).** *Given a smooth complex projective variety $X$, let $R$ be a $K_X$-negative extremal ray of $\overline{NE}(X)$. Then, any positive-dimensional irreducible component $F$ of a fiber of $c_R$ satisfies $dim(F) \geqslant codim(locus(R)) + \ell(R) - 1$, and $F$ is covered by rational curves of $(-K_X)$-degree at most $dim(F) + 1 - codim(locus(R))$.*

**Proof.** It follows directly from the previous Remark. □

## 6.6 Fiber contractions

**Remark 6.6.1.** *Given a smooth complex projective variety $X$, let $R$ be a $K_X$-negative extremal ray of $\overline{NE}(X)$ with contraction $c_R : X \to Y$ of fiber type. It follows from Proposition 6.5.6-*(1) *that $X$ is covered by rational curves contracted by $c_R$. Moreover, a general fiber $F$ of $c_R$ is smooth, and $-K_F = (-K_X)|_F$ is ample thanks to Remark 6.4.4: this tells us that $F$ is a Fano variety.*

**Proposition 6.6.2.** *Given a smooth complex projective variety $X$, let $R$ be a $K_X$-negative extremal ray of $\overline{NE}(X)$. If the contraction $c_R : X \to Y$ is of fiber type, then $Y$ is locally factorial.*

**Proof.** Recalling Theorem 2.1.10, to conclude is enough to show that all Weil divisors on $Y$ are Cartier divisors. Taken a prime Weil divisor $D$ on $Y$, define $c'_R$ as the restriction of $c_R$ to $c_R^{-1}(Y_{reg})$, and let $D_X$ be the closure in $X$ of $(c'_R)^*(D \cap Y_{reg})$. Since the Cartier divisor $D_X$ is disjoint from a general fiber of $c_R$, it has intersection 0 with any irreducible curve

$C$ of $X$ whose class generates $R$. By Corollary 6.4.2-*(2)*, $[D_X]_\sim \in ker(\delta_C) = Im(c_R^*)$, so there exists a Cartier divisor $D_Y$ on $Y$ s.t. $D_X \sim c_R^* D_Y$. Since $(c_R)_*\mathcal{O}_X \cong \mathcal{O}_Y$, by the Projection formula 2.8.4 we get that $D$ and $D_Y$ are linearly equivalent on $Y_{reg}$, hence on $Y$; this concludes the proof. □

**Example 6.6.3.** *We want to show that a projective bundle is a fiber contraction. Let $\mathcal{E}$ be a locally free sheaf of rank $r$ over a smooth projective variety $Y$. If $X = \mathbb{P}(\mathcal{E})$ is associated to the morphism $\varphi : X \to Y$, let $\ell$ be a line contained in a fiber of $\varphi$. If $\xi$ is the class of the invertible sheaf $\mathcal{O}_X(1)$, it holds $K_X = -r\xi + \varphi^*(K_Y + det(\mathcal{E}))$, hence $(K_X \cdot \ell) = -r$. Noticing that a curve is contracted by $\varphi$ if and only if it is numerically equivalent to a multiple of $\ell$, we conclude that $\mathbb{R}^+[\ell]_\equiv$ is a $K_X$-negative extremal ray of $\overline{NE}(X)$ whose contraction is $\varphi$.*

## 6.7 Divisorial contractions

**Remark 6.7.1.** *Given a smooth complex projective variety $X$, let $R$ be a $K_X$-negative extremal ray of $\overline{NE}(X)$ whose contraction $c_R : X \to Y$ is divisorial. It follows from Proposition 6.5.6-*(2) *and its proof that $locus(R)$ is an irreducible divisor $E$ linearly equivalent to $mM_R - H$ for some ample divisor $H$; therefore, $E \cdot z < 0 \;\; \forall z \in R \smallsetminus \{0\}$.*

**Definition 6.7.2.** *A scheme $X$ is said to be locally $\mathbb{Q}$-factorial if any Weil divisor on $X$ has a nonzero multiple which is a Cartier divisor.*

**Definition 6.7.3.** *Given a locally $\mathbb{Q}$-factorial scheme $X$, let $D$ be a Weil divisor on $X$. For any curve $C$ on $X$, we define $(D \cdot C) = \frac{1}{m} deg\mathcal{O}_C(mD) \in \mathbb{Q}$ for any choice of $m > 0$ s.t. $mD$ is a Cartier divisor.*

**Lemma 6.7.4.** *Let $\varphi : X \to Y$ be a proper birational morphism between varieties, with $Y$ normal. If $F$ is an effective Cartier divisor on $X$ whose support is contained in $Exc(\varphi)$, then it holds $\varphi_*\mathcal{O}_X(F) \cong \mathcal{O}_Y$.*

**Proof.** Since the statement is local on $Y$, it is enough to prove that $H^0(Y, \mathcal{O}_Y) \cong$
$\cong H^0(Y, \varphi_*\mathcal{O}_X(F))$ when $Y$ is affine. By Zariski's Main Theorem we have that

$H^0(Y, \mathcal{O}_Y) \cong H^0(Y, \varphi_*\mathcal{O}_X) \cong H^0(X, \mathcal{O}_X)$, hence if we set $E = Exc(\varphi)$ it holds $H^0(Y, \mathcal{O}_Y) \cong H^0(X, \mathcal{O}_X) \subseteq H^0(X, \mathcal{O}_X(F)) \subseteq H^0(X \smallsetminus E, \mathcal{O}_X(F))$. But $H^0(X \smallsetminus E, \mathcal{O}_X(F)) \cong H^0(X \smallsetminus E, \mathcal{O}_X) \cong H^0(Y \smallsetminus \varphi(E), \mathcal{O}_Y) \cong H^0(Y, \mathcal{O}_Y)$, where the last isomorphism comes from the fact that $Y$ is normal and $codim_Y \varphi(E) \geqslant 2$ (which we know from Remark 6.5.3). Therefore, the former is a chain of isomorphic spaces, hence $H^0(Y, \mathcal{O}_Y) \cong H^0(X, \mathcal{O}_X(F)) \cong H^0(Y, \varphi_*\mathcal{O}_X(F))$. □

**Proposition 6.7.5.** *Given a smooth complex projective variety $X$, let $R$ be a $K_X$-negative extremal ray of $\overline{NE}(X)$. If the contraction $c_R : X \to Y$ is divisorial, then $Y$ is locally $\mathbb{Q}$-factorial.*

**Proof.** Let $C$ be an irreducible curve on $X$ whose class generates $R$, and let $D$ be a prime Weil divisor on $Y$. Define $c'_R$ as the restriction of $c_R$ to $c_R^{-1}(Y_{reg})$, and let $D_X$ be the closure in $X$ of $(c'_R)^*(D \cap Y_{reg})$. Setting $E = locus(R)$, by Remark 6.7.1 we have that $(E \cdot C) < 0$, so there exist integers $a \neq 0$ and $b$ s.t. $aD_X + bE$ has intersection 0 with $C$. By Corollary 6.4.2-*(2)*, there exists a Cartier divisor $D_Y$ on $Y$ s.t. $aD_X + bE \sim c_R^* D_Y$. By the previous Lemma we get that $\mathcal{O}_{Y_{reg}}(D_Y) \cong (c'_R)_* \mathcal{O}_{c_R^{-1}(Y_{reg})}(aD_X + bE) \cong \mathcal{O}_{Y_{reg}}(aD) \otimes (c'_R)_* \mathcal{O}_X(bE) \cong$ $\cong \mathcal{O}_{Y_{reg}}(aD)$, hence $D_Y \sim aD$: this concludes the proof. □

**Example 6.7.6.** *We want to show that a smooth blow-up is a divisorial contraction. Given a smooth projective variety $Y$, let $Z$ be a smooth subvariety of $Y$ of codimension $c$, and let $\varepsilon : X \to Y$ be the blow-up of $Z$, with exceptional divisor $E$. Any fiber $F$ of $E \to Z$ is isomorphic to $\mathbb{P}_K^{c-1}$, and $\mathcal{O}_F(E) \cong \mathcal{O}_F(-1)$. If $\ell$ is a line contained in $F$, from $K_X = \varepsilon^* K_Y + (c-1)E$ we get that $(K_X \cdot \ell) = -(c-1)$. Noticing that a curve is contracted by $\varepsilon$ if and only if it lies in a fiber of $E \to Z$, i.e. if and only if it is numerically equivalent to a multiple of $\ell$, we conclude that $\mathbb{R}^+[\ell]_\equiv$ is a $K_X$-negative extremal ray of $\overline{NE}(X)$ whose contraction is $\varepsilon$.*

**Example 6.7.7.** *We want to construct an example of a divisorial contraction with singular image. Given a smooth projective threefold $Z$, let $C$ be an irreducible curve in $Z$ whose only singularity is a node. This means that in local coordinates $(x, y, z)$, the ideal of $C$ is generated by $xy$ and $z$, hence the blow-up of $Z$ along $C$ is*

*$Y = \{((x, y, z), [u, v]) \in \mathbb{A}^3_K \times \mathbb{P}^1_K | xyv = zu\}$, which is normal and its only singularity is the double point $q = ((0, 0, 0), [0, 1])$. The exceptional divisor is the $\mathbb{P}^1_K$-bundle over $C$ with local equations $xy = 0 = z$. Now, the blow-up $X$ of $Y$ at $q$ is smooth. It contains the proper transform $E$ of the exceptional divisor of $Y$ and an exceptional divisor $Q$, which is a smooth quadric. The intersection $E \cap Q$ is the union of two lines $\ell_1$ and $\ell_2$ belonging to the two different rulings of $Q$. If $\tilde{E} \to E$ and $\tilde{C} \to C$ are the normalizations, each fiber of $\tilde{E} \to \tilde{C}$ is a smooth rational curve, except over the preimages of the node of $C$, where it is the union of two rational curves meeting transversally. One of these curves maps to $\ell_i$, the other one to the same rational curve $\ell$: it follows that $\ell_1$ and $\ell_2$ are algebraically, hence numerically, equivalent on $X$, so they have the same class $L$. Any curve contracted by the blow-up $\varepsilon : X \to Y$ is contained in $Q$, hence its class is a multiple of $L$. A local calculation shows that $\mathcal{O}_Q(K_X)$ is of type $(-1, -1)$, hence $K_X \cdot L = -1$. We conclude that the ray $\mathbb{R}^+ L$ is $K_X$-negative and its (divisorial) contraction is $\varepsilon$ (hence $\mathbb{R}^+ L$ is extremal).*

## 6.8 Small contractions and flips

**Proposition 6.8.1.** *Given a normal and locally $\mathbb{Q}$-factorial variety $Y$, let $\varphi : X \to Y$ be a birational proper morphism. Then, every irreducible component of $Exc(\varphi)$ has codimension 1 in $X$.*

**Proof.** Set $E = Exc(\varphi)$, and let $x \in E$ and $y = \varphi(x)$. If we identify the quotient fields $K(Y)$ and $K(X)$ via the isomorphism $\varphi^*$, then we can see $\mathcal{O}_{Y,y}$ as a proper subring of $\mathcal{O}_{X,x}$. Taken $t \in \mathfrak{m}_{X,x} \setminus \mathcal{O}_{Y,y}$, we can write its divisor as the difference of two effective Weil divisors $D'$ and $D''$ on $Y$ without common components. Since $Y$ is locally $\mathbb{Q}$-factorial, there exists a positive integer $m$ s.t. $mD'$ and $mD''$ are Cartier divisors, hence they define elements $u, v \in \mathcal{O}_{Y,y}$ s.t. $t^m = u/v$. Notice that $v \in \mathfrak{m}_{Y,y}$, otherwise we would have $t^m \in \mathcal{O}_{Y,y} \Rightarrow t \in \mathcal{O}_{Y,y}$ since the latter is integrally closed. This implies that $u = t^m v \in \mathfrak{m}_{X,x} \cap \mathcal{O}_{Y,y} = \mathfrak{m}_{Y,y}$. Therefore, the equations $u = 0 = v$ define a subscheme $Z$ of $Y$ containing $y$ of codimension 2 in some neighborhood of $y$, because it is the intersection of the codimension 1 subschemes $mD'$ and $mD''$. Hence, $\varphi^{-1}(Z)$ is defined by $t^m v = 0 = v$, i.e. by the equation $v = 0$, so it has codimension 1 in $X$, which tells us that it is contained in $E$. This shows that there is a codimension 1 component of $E$ through every point of $E$, which concludes the proof. $\square$

**Remark 6.8.2.** *Given a smooth complex projective variety $X$, let $R$ be a $K_X$-negative extremal ray of $\overline{NE}(X)$ whose contraction $c_R : X \to Y$ is small. Then, the previous Proposition shows that $Y$ cannot even be locally $\mathbb{Q}$-factorial.*

**Remark 6.8.3.** *Fibers of a small contraction $c_R : X \to Y$ contained in locus$(R)$ have dimension at least 2, so by Proposition 6.5.6-(3) we have $\dim(X) \geqslant \dim(c_R(\mathrm{locus}(R)))+4$. In particular, there are no small extremal contractions on smooth threefolds.*

**Definition 6.8.4.** *Let $c : X \to Y$ be a small contraction between normal projective varieties. If $K_X$ is a $\mathbb{Q}$-Cartier divisor s.t. $-K_X$ is $c$-ample, we define a flip of $c$ as a small contraction $c^+ : X^+ \to Y$ s.t. $X^+$ is a normal projective variety, and $K_{X^+}$ is a $\mathbb{Q}$-Cartier divisor which is $c^+$-ample.*

**Proposition 6.8.5.** *Given a locally $\mathbb{Q}$-factorial complex projective variety $X$, let $R$ be a $K_X$-negative extremal ray of $\overline{NE}(X)$ whose contraction $c_R : X \to Y$ is small. If $c_R$ admits a flip $c_R^+ : X^+ \to Y$, then $X^+$ is locally $\mathbb{Q}$-factorial with Picard number $\rho_X$.*

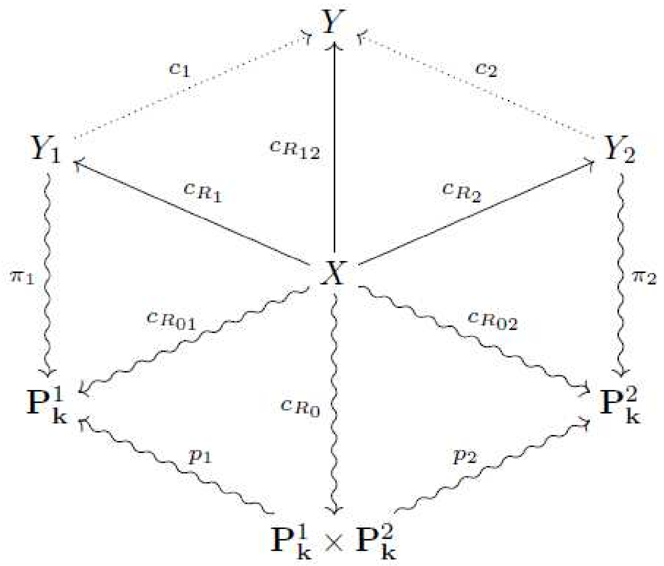
**Proof.** The composition $\varphi_R = c_R^{-1} \circ c_R^+ : X^+ \dashrightarrow X$ is an isomorphism in codimension 1, hence it induces an isomorphism between the Weil divisor class groups of $X$ and $X^+$. Taken a Weil divisor $D^+$ on $X^+$, let $D$ be the corresponding Weil divisor on $X$. If $C$ is an irreducible curve on $X$ whose class generates $R$, let $r \in \mathbb{Q}$ be s.t. $((D + rK_X) \cdot C) = 0$, and let $m \in \mathbb{Z}$ be s.t. $mD$, $mrK_X$ and $mrK_{X^+}$ are Cartier divisors. By Corollary 6.4.2-*(2)*, there exists a Cartier divisor $D_Y$ on $Y$ s.t. $m(D + rK_X) \sim c_R^* D_Y$; therefore, $mD^+ = \varphi_R^*(mD) \sim (c_R^+)^* D_Y - \varphi_R^*(mrK_X) \sim (c_R^+)^* D_Y - mrK_{X^+}$ is a Cartier divisor: this proves that $X^+$ is locally $\mathbb{Q}$-factorial. Moreover, $\varphi_R^*$ induces an isomorphism between $N^1(X)_{\mathbb{R}}$ and $N^1(X^+)_{\mathbb{R}}$, hence the Picard numbers are the same. $\square$

**Example 6.8.6.** *Starting from the Segre embedding $P = \mathbb{P}^1_K \times \mathbb{P}^2_K \subseteq \mathbb{P}^5_K$ and the natural projections $p_i : P \to \mathbb{P}^i_K$, define $Y \subseteq \mathbb{P}^6_K$ as the cone over $P$. Let $\varepsilon : X \to Y$ be the blow-up of the vertex of $Y$, with exceptional divisor $E \subseteq X$. With the notation*

*$\mathcal{O}_P(a,b) = p_1^*\mathcal{O}_{\mathbb{P}^1_K}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2_K}(b)$, we have a projection $\pi : X \to P$ which identifies $X$ with $\mathbb{P}(\mathcal{O}_P \otimes \mathcal{O}_P(1,1))$, and $E$ is a section. Let $\ell_0$ be the class of a fiber of $\pi$, let $\ell_1$ be the class in $X$ of the curve $\{*\}\times\{line\} \subseteq E$, and let $\ell_2$ be the class in $X$ of $\mathbb{P}^1_K \times \{*\} \subseteq E$. Then, one can show that $\rho_X = 3$ and $N_1(X)_{\mathbb{R}} = \bigoplus_{j=0}^{2} \mathbb{R}\ell_j$. If $h_i$ is the nef class of $\pi^* p_i^* \mathcal{O}_{\mathbb{P}^i_K}(1)$, from the fact that $\mathcal{O}_E(E) \cong \mathcal{O}_E(-1,-1)$ we get the multiplication table*

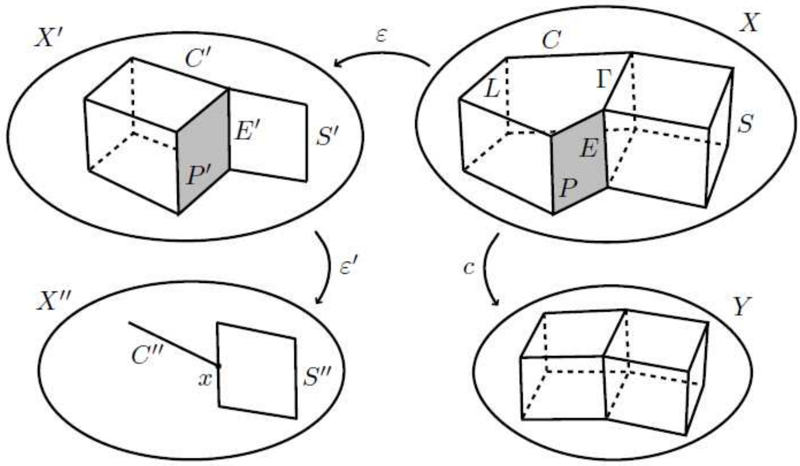
| $\cdot$ | $\ell_0$ | $\ell_1$ | $\ell_2$ |
|---|---|---|---|
| $h_1$ | 0 | 0 | 1 |
| $h_2$ | 0 | 1 | 0 |
| $[E]_\equiv$ | 1 | -1 | -1 |

*Let $C$ be an irreducible curve contained in $X \smallsetminus E$. If $C$ has class $\sum_{j=0}^{2} a_j\ell_j$, then we get that $a_1 = h_2 \cdot C \geqslant 0$, $a_2 = h_1 \cdot C \geqslant 0$, and $a_0 - a_1 - a_2 = (E \cdot C) \geqslant 0$. Therefore, since any curve in $E$ is algebraically equivalent to some nonnegative linear combination of $\ell_1$ and $\ell_2$, we have that $NE(X) = \overline{NE}(X) = \mathbb{R}^+\ell_0 + \mathbb{R}^+\ell_1 + \mathbb{R}^+\ell_2$, and the rays $R_i = \mathbb{R}^+\ell_i$ are extremal. Furthermore, it follows from Example 6.2.5 that $X$ is a Fano variety, hence in characteristic 0 all extremal subcones of $X$ can be contracted. Now, set $R_{ij} = R_i + R_j$. The contraction of $R_0$ is $\pi$, the contraction of $R_{12}$ is $\varphi$, and the contraction of $R_{0i}$ is $p_i \circ \pi : X \to \mathbb{P}^i_K$ and this map must factor through the contraction of $R_i$. Notice that $E$ is contained in locus$(R_i)$. Let us define the fourfolds $Y_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1_K} \oplus \mathcal{O}_{\mathbb{P}^1_K}(1)^{\oplus 3})$ and $Y_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2_K} \oplus \mathcal{O}_{\mathbb{P}^2_K}(1)^{\oplus 2})$, and the natural maps $\pi_i : Y_i \to \mathbb{P}^i_K$. Then, there is a map $X \to Y_i$ which is the contraction $c_{R_i}$; $E$ is therefore the locus of $R_i$, and is mapped onto the image $P_i$ of the section of $\pi_i$ corresponding to the trivial quotient of the defining locally free sheaf on $P_i$. We get the following commutative diagram of contractions:*




*Straight arrows are divisorial contractions, wiggly arrows are contractions of fiber type, and dotted arrows are small contractions (since $c_i$ contracts $P_i$ to the vertex of $Y$). By Example 6.2.5 again, $Y_2$ is a Fano variety, hence $c_2$ is the contraction of a $K_{Y_2}$-negative extremal ray; notice that the latter gives an example where the inequality of Proposition 6.5.6-*(3) *becomes an equality (proving that the inequality is sharp). On the other hand, one can show that the ray contracted by $c_1$ is $K_{Y_1}$-positive; therefore, $c_1$ is a flip of $c_2$.*

**Example 6.8.7.** *We want to construct an example of a small contraction with disconnected exceptional locus. Start from a smooth complex fourfold $X''$ that contains a smooth curve $C''$ and a smooth surface $S''$ meeting transversely at the points $x_1, \ldots, x_r$. Let $\varepsilon' : X' \to X''$ be the blow-up of $C''$; the exceptional divisor $C'$ is a smooth threefold which is a $\mathbb{P}^2_K$-bundle over $C''$. The strict transform $S'$ of $S''$ is the blow-up of $S''$ at the points $x_1, \ldots, x_r$; let $E'_1, \ldots, E'_r$ be the corresponding exceptional curves and let $P'_1, \ldots, P'_r$ be the corresponding projective planes that contain them, i.e. $P'_i = (\varepsilon')^{-1}(x_i)$. Let $\varepsilon : X \to X'$ be the blow-up of $S'$; the exceptional divisor $S$ is a smooth threefold which is a $\mathbb{P}^1_K$-bundle over $S'$. Let $\Gamma_i$ be the fiber over a point of $E'_i$, let $P_i$ be the strict transform of $P'_i$, and let $L$ be a line in one of the $\mathbb{P}^2_K$ in the inverse image $C$ of $C'$. Set $E_i = \varepsilon^{-1}(E'_i)$.*
*For $r = 1$, the picture is something like the following:*




*Since the curves $\Gamma_i$ are fibers of the $\mathbb{P}^1_K$-bundle $S \to S'$, they are all algebraically equivalent in $X$, so they have the same class $[\Gamma]_\equiv$. If we set $\alpha = \varepsilon' \circ \varepsilon$, one can see that the relative effective cone $NE(\alpha)$ is generated by $[\Gamma]_\equiv$, $[L]_\equiv$ and $[E_i]_\equiv$. Since the vector space $N_1(X)_\mathbb{R}/\alpha^* N_1(X'')_\mathbb{R}$ has dimension 2, there must be a relation of the form $E_i \equiv a_i L + b_i \Gamma$ for any $i = 1, \dots, r$. One can check that $(C \cdot E_i) = (C' \cdot E'_i) = -1 = (C' \cdot \varepsilon_*(L)) = (C \cdot L)$, and also that $(S \cdot \Gamma) = -1$. Moreover, $(C \cdot \Gamma) = 0$ because $\Gamma$ is contracted by $\varepsilon'$, $(S \cdot L) = 0$ because $S$ and $L$ are disjoint, and $(S \cdot E_i) = 1$ because $S$ and $P_i$ meets transversally in $E_i$. It follows that $-1 = (C \cdot E_i) = -a_i \Rightarrow a_i = 1$ and $1 = (S \cdot E_i) = -b_i \Rightarrow b_i = -1$, hence the $E_i$ are all numerically equivalent to $L - \Gamma$. The relative cone $NE(\alpha)$ is therefore generated by $[\Gamma]_\equiv$ and $[L-\Gamma]_\equiv$; since it is an extremal subcone of $NE(X)$, the class $[L-\Gamma]_\equiv$ spans a $K_X$-negative extremal ray because it satisfies $(K_X \cdot (L - \Gamma)) = -1$. This tells us that in characteristic $0$ it can be contracted, and the corresponding contraction $X \to Y$ maps each $P_i$ to a point and has exceptional locus $\bigsqcup_{i=1}^{r} P_i$.*

## 6.9 The minimal model program

Let $X$ be a smooth complex projective variety. We saw in §5.6 that when $X$ is a surface, it has a smooth minimal model $X_{min}$ obtained by contracting all exceptional curves on $X$. If $X$ is covered by rational curves, this minimal model is not unique, and is either a ruled surface or $\mathbb{P}^2_K$. Otherwise, the minimal model is unique and has nef canonical divisor.

In higher dimensions, Mori's idea is to try to simplify $X$ by contracting $K_X$-negative extremal rays, hoping to end up with a variety $X_0$ which either has a contraction of fiber type (in which case $X_0$ is covered by rational curves, as we saw in Remark 6.6.1) or has nef canonical divisor (hence no $K_{X_0}$-negative extremal rays). Three main problems arise:

- the end-product of a contraction is usually singular: this means that to continue Mori's program, we must allow singularities. Since most of our methods do not work on singular varieties, different approaches are required.
- The singularities of the target of a small contraction are too severe, and one needs to perform a flip. So we have the problem of existence of flips.
- One needs to know that the process terminates. In case of surfaces, we used that the Picard number decreases when an exceptional curve is contracted; this is still the case for a fiber-type or divisorial contraction, but not for a small one. So we have the additional problem of termination of flips: do there exist infinite sequences of flips?

The first two problems have been overcome, while the third point is still open in full generality.

## 6.10 Minimal models

**Definition 6.10.1.** *Let $\mathcal{C}$ be a birational equivalence class of smooth projective varieties, modulo isomorphisms. If $X, Y \in \mathcal{C}$, we write $Y \preceq X$ if there is a birational morphism $X \to Y$; this defines a partial order on $\mathcal{C}$.*

**Proposition 6.10.2.** *Let $\varphi : Y \to X$ be a birational morphism between varieties, with $X$ smooth. Then, any component of $Exc(\varphi)$ is birational to a product $\mathbb{P}^1_K \times Z$, where $\varphi$ contracts the $\mathbb{P}^1_K$-factor. In particular, if $\varphi$ is projective there exists a rational curve contracted by $\varphi$ through any point of $Exc(\varphi)$.*

**Proof.** Let $E$ be a component of $Exc(\varphi)$; upon replacing $Y$ with its normalization, we may assume that $Y$ is smooth in codimension 1. Upon shrinking $Y$, we may also assume that both $Y$ and $Exc(\varphi)$ are smooth, and that the latter is equal to $E$. Taking the open subset $U_0 = X \smallsetminus Sing(\overline{\varphi(E)})$ of $X$, set $V_1 = \varphi^{-1}(U_0)$; then, $Y \smallsetminus V_1$ has codimension at least 2, $V_1$ and $E \cap V_1$ are smooth, and so is the closure of the image of $E \cap V_1$ in $U_0$. Let $\varepsilon_1 : X_1 \to U_0$ be its blow-up; by the Universal property of blow-ups 3.3.6, since the ideal of $E \cap V_1$ in $\mathcal{O}_{V_1}$ is invertible, there exists a factorization $\varphi|_{V_1} : V_1 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varepsilon_1} U_0$, where $\overline{\varphi_1(E \cap V_1)}$ is contained in the support of $Exc(\varepsilon_1)$. If the codimension of $\overline{\varphi_1(E \cap V_1)}$ in $X_1$ is at least 2, then the divisor $E \cap V_1$ is contained in $Exc(\varphi_1)$. Upon replacing $V_1$ with the complement $V_2$ of a closed subset of codimension at least 2 and $X_1$ by an open subset $U_1$, we may repeat the previous construction. After $i$ steps, we get a factorization $\varphi : V_i \xrightarrow{\varphi_i} X_i \xrightarrow{\varepsilon_i} U_{i-1} \xrightarrow{\varepsilon_{i-1}} \dots \xrightarrow{\varepsilon_1} U_0$ as long as the codimension of $\overline{\varphi_{i-1}(E \cap V_{i-1})}$ in $X_{i-1}$ is at least 2, where $V_i$ is the complement in $Y$ of a closed subset of codimension at least 2. If $E_j \subseteq X_j$ is the exceptional divisor of $\varepsilon_j$ and $E_{i,j}$ is the inverse image of $E_j$ in $X_i$, we get that $K_{X_i} = \varepsilon_i^* K_{U_{i-1}} + c_i E_i = \ldots = (\varepsilon_1 \circ \cdots \circ \varepsilon_i)^* K_X + c_i E_i + \sum_{j=1}^{i-1} c_j E_{i,j}$, where $c_j = codim_{X_{j-1}}(\overline{\varphi_{j-1}(E \cap V_{j-1})}) - 1 > 0$. Since $\varphi_i$ is birational, $\varphi_i^* \mathcal{O}_{X_i}(K_{X_i})$ is a subsheaf of $\mathcal{O}_{V_i}(K_{V_i})$. Moreover, since $\varphi_j(E \cap V_j)$ is contained in the support of $E_j$, the divisor $\varphi_j^* E_j - E|_{V_j}$ is effective, hence so is $E_{i,j} - E|_{V_i}$. It follows that $\mathcal{O}_Y(\varphi^* K_X + E \sum_{j=1}^{i} c_j)|_{V_i}$ is a subsheaf of $\mathcal{O}_{V_i}(K_{V_i}) = \mathcal{O}_Y(K_Y)|_{V_i}$. Since $Y$ is normal and $Y \smallsetminus V_i$ has codimension at least 2, we get that $\mathcal{O}_Y(\varphi^* K_X + E \sum_{j=1}^{i} c_j)$ is a subsheaf of $\mathcal{O}_Y(K_Y)$. Since on a noetherian scheme there are no infinite ascending sequences of subsheaves of a coherent sheaf, the process must come to an end: this tells us that $\overline{\varphi_i(E \cap V_i)}$ is a divisor in $X_i$ for some $i$, hence $E \cap V_i$ is not contained in $Exc(\varphi_i)$. Therefore, $\varphi_i$ induces a dominant map between $E \cap V_i$ and $E_i$, which must be birational since the fibers of $\varphi$ are connected by Zariski's Main Theorem. Since $E_i$ is birationally isomorphic to $\mathbb{P}_K^{c_i - 1} \times (\varphi_{i-1}(E \cap V_{i-1}))$, where $\varepsilon_i$ contracts the $\mathbb{P}_K^{c_i-1}$-factor, the first statement is proved.

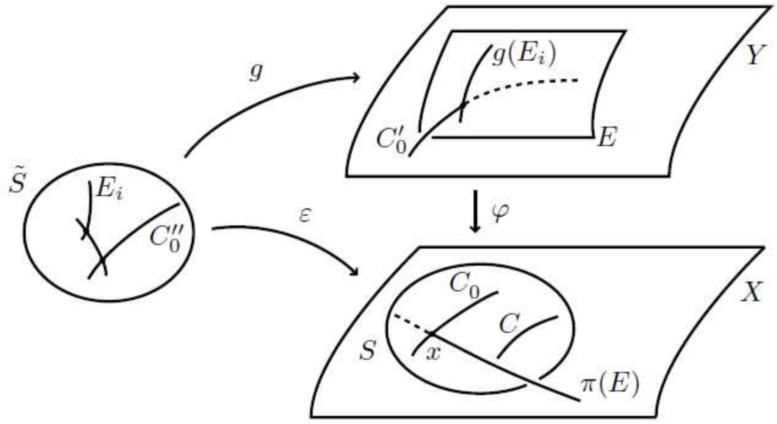
If in particular $\varphi$ is projective, by Lemma 6.2.11 we can conclude. □

**Corollary 6.10.3.** *Let $X, Y$ be projective varieties with $X$ smooth. If $Y$ contains no rational curves, then any rational map $X \dashrightarrow Y$ is defined everywhere.*

**Proof.** Taken a rational map $\varphi : X \dashrightarrow Y$, let $X' \subseteq X \times Y$ be the graph of $\varphi$. The first projection induces a birational morphism $p : X' \to X$. If $Exc(p) \neq \emptyset$, by the previous Proposition there exists a rational curve $C$ on $Exc(p)$ which is contracted by $p$. Since $Y$ contains no rational curves, $C$ must also be contracted by the second projection, which is absurd since $C \subseteq X \times Y$. This shows that $Exc(p) = \emptyset$, so $\varphi$ is defined everywhere. $\square$

**Proposition 6.10.4.** *Let $\varphi : Y \to X$ be a birational morphism between smooth projective varieties. If $\varphi$ is not an isomorphism, there exists a rational curve $C$ on $Y$ contracted by $\varphi$ s.t. $(K_Y \cdot C) < 0$.*

**Proof.** Set $E = Exc(\varphi)$; by Remark 6.5.3, $\varphi(E)$ has codimension at least 2 in $X$, and $E = \varphi^{-1}(\varphi(E))$. Let $x$ be a point of $\varphi(E)$; by Bertini's Thm. 6.2.12, a general hyperplane section of $X$ passing through $x$ is smooth and connected. It follows that by taking $dim(X) - 2$ hyperplane sections we get a smooth surface $S$ in $X$ that meets $\varphi(E)$ in a finite set containing $x$. Moreover, taking one more hyperplane section we get on $S$ a smooth curve $C_0$ that meets $\varphi(E)$ only at $x$ and a smooth curve $C$ that does not meet $\varphi(E)$. By construction, $(K_X \cdot C) = (K_X \cdot C_0)$. One can write $K_Y \sim \varphi^* K_X + R$ for some divisor $R$ whose support is exactly $E$. Since the curve $C' = \varphi^{-1}(C)$ does not meet $E$, we have that $(K_Y \cdot C') = (K_X \cdot C)$. On the other hand, since the strict transform $C_0' = \overline{\varphi^{-1}(C_0 \setminus \varphi(E))}$ of $C_0$ does meet $E = \varphi^{-1}(\varphi(E))$, we have that $(K_Y \cdot C_0') =$ $= ((\varphi^* K_X + R) \cdot C_0') > ((\varphi^* K_X) \cdot C_0') = (K_X \cdot C_0) = (K_X \cdot C) = (K_Y \cdot C')$. By Theorem 5.5.3, the indeterminacies of the rational map $\varphi^{-1} : S \dashrightarrow Y$ can be resolved, i.e. there exists a composition of blow-ups $\varepsilon : \tilde{S} \to S$ of a finite number of points of $S \cap \varphi(E)$ which let us obtain a morphism $g = \varphi^{-1} \circ \varepsilon : \tilde{S} \to Y$, whose image is the strict transform of $S$. The curve $C'' = \varepsilon^* C$ is irreducible, and $g_* C'' = C'$; on the other hand, for $C_0$ we can write $\varepsilon^* C_0 = C_0'' + \sum_{i=1}^{n} m_i E_i$, where the $m_i$ are nonnegative integers, the $E_i$ are exceptional divisors for $\varepsilon$ (hence in particular rational curves), and $g_* C_0'' = C_0'$.




Since $C$ and $C_0$ are linearly equivalent on $S$, we have that $C'' = \varepsilon^* C \sim \varepsilon^* C_0 = = C''_0 + \sum_{i=1}^{n} m_i E_i$ on $\tilde{S}$; hence, by applying $g_*$ we get that $C' \sim C'_0 + \sum_{i=1}^{n} m_i(g_* E_i)$. Taking intersections with $K_Y$, we get $(K_Y \cdot C') = (K_Y \cdot C'_0) + \sum_{i=1}^{n} m_i(K_Y \cdot g_* E_i)$. From the previous inequality $(K_Y \cdot C'_0) > (K_Y \cdot C')$, it follows that $(K_Y \cdot g_* E_i)$ is negative for some $i$. In particular, $g(E_i)$ is not a point, hence it is a rational curve on $Y$. Moreover, $\varphi(g(E_i)) = \varepsilon(E_i) = \{x\}$, hence $g(E_i)$ is contracted by $\varphi$. This concludes the proof. □

**Definition 6.10.5.** *A minimal model is a smooth projective variety with nef canonical divisor.*

**Remark 6.10.6.** *If $(\mathcal{C}, \preceq)$ is as in Definition 6.10.1, Proposition 6.10.4 tells us that any element of $\mathcal{C}$ with nef canonical bundle is minimal: this explains our interest for minimal models (and the reason we call them this way). Moreover, from Corollary 6.10.3 we get that an element of $\mathcal{C}$ which contains no rational curves is the smallest element of $\mathcal{C}$.*

**Remark 6.10.7.** *The main problems regarding the study of minimal models are:*

- *a minimal model can only exist if the variety is not covered by rational curves;*
- *there exist smooth projective varieties which are not covered by rational curves but that are not birational to any smooth projective variety with nef canonical bundle;*
- *in dimension at least 3, minimal models may not be unique, but any two of them are isomorphic in codimension 1.*

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