Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"
Corso di Laurea magistrale in Matematica

Tesi di Laurea Magistrale

On the Siegel Moduli Space with Parahoric Level Structure

Candidato:
Luca Mastella
Matricola 1178390

Relatori:
Dr. Marco Andrea Garuti
Prof. Dr. Ulrich Görtz

17/07/2019
Anno Accademico 2018-2019

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Introduction

In algebraic geometry the notion of moduli space appears very often. Heuristically speaking, a moduli space is a kind of geometric object "parametrizing" isomorphism classes of other geometric objects.

An example of this phenomenon appears naturally in the context of classical algebraic geometry over an algebraically closed field K: in this case one shows that the set of subspaces of K^n of dimension d is in bijection with an algebraic subvariety $G_K(n,d)$ of $\mathbb{P}^N_K$, where $N = \binom{n}{d} - 1$, that is called the Grassmannian variety.

Passing to scheme theory this notion can be reformulated in an abstract and more precise fashion: if S is a base scheme a moduli space is the scheme representing a (representable) functor $F\colon (\mathrm{Sch/S})^{\mathrm{op}} \to (\mathrm{Sets})$, defined by some classification problem, as the one above.

In arithmetic geometry for various reasons we are especially interested in moduli spaces of abelian varieties, that generalize the classic notion of modular curve seen as the moduli space of elliptic curves. The first generalization is the moduli space of principally polarized abelian variety of fixed dimension g with level N structure, for $N \in \mathbb{N}_{\geq 3}$, i.e. the scheme representing the functor $\mathcal{A}_{g,N}$, defined by

$$\mathcal{A}_N(S') = \left\{ (A,\lambda,\eta) : \begin{array}{l} A \text{ is an abelian variety of dimesion } g \text{ over } S', \\ \lambda \text{ a principal polarization of } A, \\ \eta \text{ a level } N \text{ structure} \end{array} \right\} / \cong,$$

S-scheme S'. This scheme is widely study, since Mumford's Geometric Invariant Theory ([12]).

The Siegel moduli space with parahoric level structure goes one more step beyond in generality. By parahoric level structure we will mean a choice of indexes $I = \{ 0 \leq i_0 < \cdots < i_r \leq g \}$ and for such an object we will write $I' = \pm I + 2\mathbb{Z}g$. The Siegel moduli space with parahoric level structure is the scheme representing the functor $\mathcal{A}_{I,N}$, whose S'-valued point are $2g$-periodic chains $(A_i)_{i\in I'}$ of abelian schemes over S' of dimensions g, connected by isogenies, with additional data which corresponds to the principal polarization and a level N structure an A_{i_0}. This is the moduli space in which we are interested.

Unfortunately the geometry of this moduli space is quite difficult to study, but a lot of techniques have been developed in order to solve this problem. In particular the aim of this thesis is to present one of them first introduced by A.J. de Jong in [10]. It consist of the introduction of another scheme $M_N^{\mathrm{loc}}(I)$ that has an "easier nature" as it is defined by some linear algebra data (it is in fact a subscheme of a product of Grassmannian schemes), the so-called local model, and such that there exist a third scheme $\tilde{\mathcal{A}}_{I,N}$ and a diagram of the form

$$\begin{array}{ccc} & \tilde{\mathcal{A}}_{I,N} & \\ {}^{\varphi}\swarrow & & \searrow^{\psi} \\ \mathcal{A}_{I,N} & & M_N^{\mathrm{loc}}(I) \end{array}$$

with φ, ψ smooth of the same relative dimension and φ surjective. The existence of this diagram, called a local model diagram for $\mathcal{A}_{I,N}$, implies that $\mathcal{A}_{I,N}$ is locally isomorphic in the étale topology to $M_N^{\mathrm{loc}}(I)$. Hence it gives a powerful tool in the study of local features of $\mathcal{A}_{I,N}$, as e.g. the nature of the singularities (this is done for instance in [13]).

We will in fact show that it's reasonable (if not easy) to find explicit local equations for $M_N^{\mathrm{loc}}(I)$ and hence étale locally for $\mathcal{A}_{I,N}$.

This thesis will be divided in two chapters:

- In the first chapter we will introduce the categorical formalism needed to talk about moduli spaces, i.e. we will introduce the notion of representability. We will see then an elementary criterion for the representability of a functor $F\colon (\mathrm{Sch/S})^{\mathrm{op}} \to (\mathrm{Sets})$ and that, if $S = \operatorname{Spec} R$ is affine, under some conditions this gives a procedure to introduce schemes giving their functor of T-valued point, for T an R-algebra. We will then use this procedure to introduce the Grassmannian scheme and the Local Model.

  In order to introduce the latter we will deserve a section on a particular chain $(\Lambda_i)_{i\in I'}$ of $\mathbb{Z}_p$-lattices such that the standard alternating pairing on $\mathbb{Q}_p^{2g}$ restrict for any $i \in I'$ to a perfect paring $\langle\cdot,\cdot\rangle_i\colon M_i \times M_{-i} \to \mathbb{Z}_p$. We will show moreover that this construction can be tensored giving a chain of R-lattices $(\Lambda_{i,R})_{i\in I'}$ endowed with "alternating" pairings, for any $\mathbb{Z}_p$-algebra R.

- In the second chapter we will give the definition of the Siegel moduli space with parahoric level stucture and we will describe the local model diagram. In this chapter the main technical tool will be the notion of system of R-modules of type II, i.e. roughly speaking of a chain $(M_i)_{i\in I'}$ if locally free R-module of rank $2g$ connected by a R-linear maps and endowed with "alternating" pairings (e.g. the chain $(\Lambda_{-i,R})_{i\in I'}$) is such an object). We obtain this notion generalizing the corresponding one of [10] from the Iwahori level structure case (i.e. $I = \{0,\dots,g\}$) to the general one. We will prove in particular the following results that generalize the proposition 3.6 in [10] and that imply the surjectivity of φ and the smoothness of φ and ψ:

  Proposition. *Let I be a parahoric level structure, R be a $\mathbb{Z}_p$-algebra and and $M_\bullet$ a system of R-modules of type II for I. If R is a local ring there is an isomorphism $M_\bullet \xrightarrow{\sim} \Lambda_{-\bullet,R}$.*

  Proposition. *Let I be a parahoric level structure, R be a $\mathbb{Z}_p$-algebra and and $M_\bullet$ a system of R-modules of type II for I. Suppose that $\mathfrak{a}$ is a nilpotent ideal of R and that there exist an isomorphism $M_\bullet \otimes_R R/\mathfrak{a} \xrightarrow{\sim} \Lambda_{-\bullet,R/\mathfrak{a}}$, then we can lift it to an isomorphism $M_\bullet \xrightarrow{\sim} \Lambda_{-\bullet,R}$.*

  At the end we will show in some examples the procedure in order to compute local equations for $M_N^{\mathrm{loc}}(I)$. For instance in the case $g = 1$, $I = \{0,1\}$ the R-valued point of the local model are pairs $(\mathcal{F}_0, \mathcal{F}_1)$ of locally free submodules of R^2 of rank 1 such that $\alpha(\mathcal{F}_0) \subseteq \mathcal{F}_1$, where $\alpha\colon R^2 \to R^2$ is the linear map associated with the matrix $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. In local charts they correspond to the vectors $\binom{1}{x}$ and $\binom{y}{1}$, moreover the condition $\alpha(\mathcal{F}_0) \subseteq \mathcal{F}_1$ is equivalent to the existence of $\lambda \in R$ such that

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} p \\ x \end{pmatrix} = \lambda \begin{pmatrix} y \\ 1 \end{pmatrix}.$$

We get therefore the local equation $xy = p$.

Acknowledgments - Ringraziamenti

I am grateful with anyone that during this two years gave me the possibility of complete this new journey inside the world of mathematics. In particular I am very grateful to my advisor Prof. Dr. Ulrich Görtz, without whom this thesis would have never be done, for the (and I hope not lost) time that he spent with me and for all the things that he teached to me during this year.

An other special thanks goes to my fellows ALGANT students and officemates in Essen, in particular to Ju-Feng, the (not only) mathematical discussions with you were always very enlightening.

I am grateful also to the students of the Erasmus group and all the other people that I encountered this year in Essen, they made me feel in holiday, even during the periods of hardest work.

Non può mancare infine un grosso ringraziamento a tutti gli amici di Padova e alla mia famiglia, senza di loro oggi non sarei quello che sono.

Chapter 1

Local Models

The aim of this thesis is to describe the local structure of some Siegel moduli spaces of principally polarized abelian schemes. The moduli problem will play therefore a central role. We can express it in a naive way: a moduli space is a "geometric object" that "parametrizes" a given class of other "geometric objects" modulo isomorphism. This sentence is very vague, let us explain with an example what we have in mind: in the context of Riemann surfaces it is well known that any elliptic curve is isomorphic to a complex torus $\mathbb{C}/\Lambda$, for Λ a lattice in $\mathbb{C}$, and that one can find a particular quotient of the complex upper-halfplane $\mathbb{H}$, called modular curve, in bijection with the set of isomorphism classes of tori (hence of elliptic curves). In this chapter we will give a precise notion of the moduli problem and we will introduce a couple of easy examples of moduli spaces: the Grassmannian scheme and the Local Model. We will follow as main references [5] and [4].

1.1 Representable functors

In this section we will introduce the categorical concept of representability, that generalizes this situation in the context of schemes.

1.1.1 Let $\mathcal{C}$ be a locally small category (i.e. for any X, Y objects in $\mathcal{C}$ the class $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a set) and denote by $\mathcal{C}^\vee$ the category of all contravariant functors $F\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Sets})$. There are some obvious, but very important, objects of $\mathcal{C}^\vee$, namely the contravariant functors $h_X = \mathrm{Hom}_{\mathcal{C}}(-,X)$ for any object X of $\mathcal{C}$. The following classical lemma holds:

Lemma. *(Yoneda) Let $X \in \mathcal{C}$, $F \in \mathcal{C}^\vee$. We have a bijection*

$$\mathrm{Hom}_{\mathcal{C}^\vee}(h_X, F) \xrightarrow{\sim} F(X),$$

functorial in F, given by $\alpha = (\alpha_Z)_{Z\in\mathcal{C}} \mapsto \alpha_X(\mathrm{id}_X)$.

If we specialize the lemma for $F = h_Y$, where Y is another object of $\mathcal{C}$, we get that the functor $h_\bullet\colon \mathcal{C} \to \mathcal{C}^\vee$ defined by $X \mapsto h_X$ is fully faithful, or in other terms that we can embed the category $\mathcal{C}$ into $\mathcal{C}^\vee$. Therefore in the sequel we will not distinguish between $X \in \mathcal{C}$ and the functor $h_X \in \mathcal{C}^\vee$ if it does not cause any confusion; e.g. in the case $\mathcal{C} = (\mathrm{Sch/S})$ we will write $X(T)$ instead of $h_X(T)$ for any $X, T \in (\mathrm{Sch/S})$. The set $X(T)$ is called the set of T-valued points of X. Note that this notation is compatible with the Yoneda lemma in its general form: for any $F \in (\mathrm{Sch/S})^\vee$, $T \in (\mathrm{Sch/S})$ we can view $F(T)$ as the set of morphisms of functors $T \to F$ and if $F = X$ is a scheme they are, as just observed, exactly the morphisms of schemes $T \to X$.

1.1.2 It is interesting to study the functors that, modulo isomorphisms, come from some object X of $\mathcal{C}$.

Definition. A functor $F\colon \mathcal{C} \to (\text{Sets})$ is called representable if it belongs to the essential image of $h_\bullet$, or in other words if there exists an object $X \in \mathcal{C}$ and an isomorphism of functors $\eta\colon h_X \xrightarrow{\sim} F$. If such a pair (X, η) exists, then we say that it represents F or simply that X represents F.

If F is representable the pair (X, η) is unique up to a unique isomorphism: if (Y, ε) is such another pair, then the isomorphism $\varepsilon^{-1} \circ \eta\colon h_X \to h_Y$ is an isomorphism of functors between schemes, hence it comes from an isomorphism of schemes $\vartheta\colon X \to Y$, i.e. $\varepsilon^{-1} \circ \eta = h_\vartheta$. Hence X is isomorphic to Y via ϑ and $\eta = \varepsilon \circ h_\vartheta$.

Example. Let $\mathcal{C} = (\text{Sch/R})$ be the category of R-schemes for R a commutative ring. We define $F\colon (\text{Sch/R})^{\text{op}} \to (\text{Sets})$ by $F(T) = \Gamma(T, \mathcal{O}_T)^n$ for any $T \in (\text{Sch/R})$. It is known that

$$\operatorname{Hom}_R(T, \mathbb{A}^n_R) = \operatorname{Hom}_R\big(R[X_1, \dots, X_n], \Gamma(T, \mathcal{O}_T)\big) = \Gamma(T, \mathcal{O}_T)^n,$$

hence $\mathbb{A}^n_R$ represents F.

This can be generalized to $\mathcal{C} = (\text{Sch/S})$: $\mathbb{A}^n_S$ represent $F\colon (\text{Sch/S})^{\text{op}} \to (\text{Sets})$ defined by $F(T) = \Gamma(T, \mathcal{O}_T)^n$ for any $T \in (\text{Sch/S})$. Note that, by the uniqueness of the representative, we could have defined $\mathbb{A}_S$ to be the representative of F, once we have shown its representability. The rest of this paragraph follows this idea: we will find a criterion of representability that will allow us to define schemes via functors.

Note moreover that the notion of representability can be seen as a formal translation of the naive idea of "parametrizing". In fact, if we collect the objects that we want to parametrize in such a way to define a representable functor F, we have a scheme whose T-valued points parametrize the objects in $F(T)$.

1.1.3 To state the criterion of representability quoted above, we need to study some properties of the functors $F \in (\text{Sch/S})^\vee$. Let's start with the Zariski sheaf property. In the following when it does not lead to any confusion we will identify in the notations a ring with its spectrum, e.g. we will denote the set $F(\operatorname{Spec} A)$ simply by $F(A)$, for any functor $F \in (\text{Sch})^\vee$ and any ring A.

Definition. Let $F\colon (\text{Sch/S})^{\text{op}} \to (\text{Sets})$ be a contravariant functor. We say that F is a Zariski sheaf if for any S-scheme T and any open cover $\{\, U_i \,\}_{i \in I}$ of T

$$F(T) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{i,j \in I} F(U_i \cap U_j),$$

is an exact sequence, where the arrow on the left is induced by the inclusions $U_i \hookrightarrow T$, while the two arrows on the right are induced respectively by the inclusions $(U_i \cup U_j \hookrightarrow U_i)_{i,j}$ and $(U_i \cup U_j \hookrightarrow U_j)_{i,j}$.

Remark 1.1.1. Let's investigate this definition. If we write $s_{|Y}$ for $F(\iota)(s)$, where $s \in F(X)$ and $\iota\colon Y \hookrightarrow X$ is an open inclusion, the map on the left is given by $s \mapsto (s_k)_k$ and the two maps on the right respectively by $(s_k)_k \mapsto (s_{i|U_i \cap U_j})_{i,j}$ and $(s_k)_k \mapsto (s_{j\,|U_i \cap U_j})_{i,j}$. Hence the condition correspond formally to the usual sheaf property for presheaves on topological spaces: these two are in fact two instances of the more general definition of sheaf over a site.

It turns out that the condition of being a Zariski sheaf is a necessary condition for a functor to be representable: the theorem of gluing for morphisms of schemes is equivalent to the fact that for any scheme X, the functor h_X is a Zariski sheaf.

We have moreover a corresponding notion in the affine case.

Definition. Let $F\colon (\text{AffSch/R})^{\text{op}} \to (\text{Sets})$ be a contravariant functor. We say that F is a Zariski sheaf if for any R-algebra T and any $\{\, t_i \,\}_{i\in I} \subseteq T$ generating the unit ideal, or in other word such that $\operatorname{Spec} T = \bigcup_{i\in I} D(t_i)$, the sequence

$$F(T) \longrightarrow \prod_{i\in I} F(T_{t_i}) \rightrightarrows \prod_{i,j\in I} F(T_{t_i t_j})$$

is exact, with arrows induced by the canonical maps.

It's easy to imagine, and not so difficult to prove, that these two definition give rise to the same objects.

Proposition. *Let $F\colon (\text{AffSch/R})^{\text{op}} \to (\text{Sets})$ be a Zariski sheaf. Then we can extend it in a unique way to a Zariski sheaf $\tilde{F}\colon (\text{Sch/R})^{\text{op}} \to (\text{Sets})$.*

Proof. This result is the analog of the fact that to define a sheaf on a topological space it is enough to define it on a basis of the topology. The proof of this latter verbatim extends: the uniqueness is clear by the sheaf property and so it is enough to define for any R-scheme T

$$\tilde{F}(T) = \varprojlim_{\substack{U\subseteq T \\ \text{affine open}}} F(U) = \{\, (s_U)_U : U \subseteq T \text{ affine open, } s_{U|V} = s_V \text{ for any } V \subseteq U \,\}\,.$$

Since any scheme can be covered by affine schemes, it's easy to check that $\tilde{F}$ is a Zariski sheaf. $\square$

1.1.4 In any category $\mathcal{C}$ we can give the following definition:

Definition. Let $F\colon \mathcal{C}^{\text{op}} \to (\text{Sets})$. A subfunctor F' of F is a functor $F'\colon \mathcal{C}^{\text{op}} \to (\text{Sets})$ together with a monomorphism of functors $F' \to F$, i.e. $F'(T) \hookrightarrow F(T)$ for any S-scheme T. In this case we will identify any $F'(T)$ as subset of $F(T)$.

If $\mathcal{C} = (\text{Sch/S})$, for a scheme S, then we can generalize the notion of open subscheme using the embedding into the contravariant functor category, giving a well behaved definition of open subfunctor:

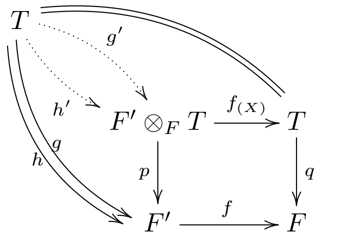
Definition. Let $F, F'\colon (\text{Sch/S})^{\text{op}} \to (\text{Sets})$. We say that a morphism of functors $f\colon F' \to F$ is an open immersion if for any morphism $g\colon X \to F$, where X is an S-scheme, the functor $F' \times_F X$ is representable, say by a scheme Y, and the base change $f_{(X)}\colon Y \to X$ is an open immersion of schemes.

Note that, even if Y is defined only up to isomorphisms, the definition of open immersion of functors make sense: the property of being an open immersion of schemes is indeed invariant under isomorphisms of the domain. The next result says that an open immersion $F' \to F$ gives rise to a subfunctor F' of F; we will say then that F' is an open subfunctor of F.

Proposition. *Let $f : F' \to F$ be an open immersion of functors, then $F'(T) \to F(T)$ is injective for any S-scheme T.*

Proof. We will prove that $\operatorname{Hom}_S(T, F') \to \operatorname{Hom}_S(T, F)$ is injective; this is in fact equivalent to our statement by the Yoneda lemma. We take therefore $g, h\colon T \to F'$ such that $f \circ g = f \circ h = q$.

These morphisms fit in the cartesian diagram:



that defines g', h' using g and h respectively. Moreover $f_{(X)}$ is a monomorphism, as it is an open immersion, and $f_{(X)} \circ g' = \mathrm{id} = f_{(X)} \circ h'$, hence $g' = h'$ and so $g = p \circ g' = p \circ h' = h$. $\square$

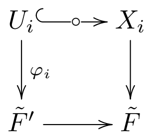
1.1.5 Again we can give an analogous definition in the category of affine schemes over a commutative ring R.

Definition. Let $F, F' \colon (\mathrm{AffSch/R})^{\mathrm{op}} \to (\mathrm{Sets})$. We say that a morphism of functors $f \colon F' \to F$ is an open immersion if for any morphism $g \colon \operatorname{Spec} T \to F$, where T is an R-algebra, the functor $F' \times_F \operatorname{Spec} T$ is representable, say by an R-scheme Y, and the base change $f_{(T)} \colon Y \to \operatorname{Spec} T$ is an open immersion of schemes.

Note that we don't require that Y is affine: indeed not all the open subschemes of an affine scheme are themselves affine. Let's now relate this notion in (AffSch/R) with the previous for (Sch/R).

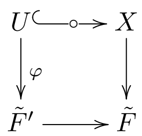
Proposition. *Let $F, F' \colon (\mathrm{AffSch/R}) \to (\mathrm{Sets})$ two are Zariski sheaves and $f \colon F' \to F$ be an open immersion of functors. Then we can extend f to an open immersion $\tilde{f} \colon \tilde{F}' \to \tilde{F}$ of functors from* (Sch/R) $\to$ (Sets)*.*

Proof. The existence of $\tilde{f}$ is guaranteed by the functoriality of the tilde construction. We need to check that it is an open immersion. Let therefore X be an R scheme and $g \colon X \to F$ a morphism of functors. Covering X by open affine subschemes X_i, we have a collection of schemes U_i, open in X_i, and a cartesian diagram (again by functoriality of tilde and by the Zariski sheaf property)



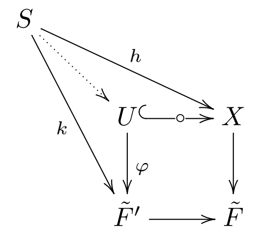
(1)

where the top arrow is an open immersion of schemes. Using Yoneda's lemma we get that $(\varphi_i)_i \in \prod_i \tilde{F}'(U_i)$ agree on the intersections, hence we can glue them to $\varphi \colon U = \bigcup_i U_i \to \tilde{F}'$ since $\tilde{F}'$ is a Zariski sheaf. We get therefore a commutative diagram



(2)

and we claim that this is a cartesian square. Let therefore S be an R-scheme endowed with the two solid arrows h and k as below: we need to show that the dotted arrow exist, i.e. that $h(S) \subseteq U$, as U is an open subset of X.



(3)

Consider therefore $h^{-1}(X_i)$ and let $(S_{i,j})_j$ be an affine open cover of it. As the diagram (1) is cartesian $h(S_{i,j}) \subseteq U_i$ for any i, j. It follows that $h(S) \subset U$. The commutativity of the lower triangle in (3) follows by the definition of φ. $\square$

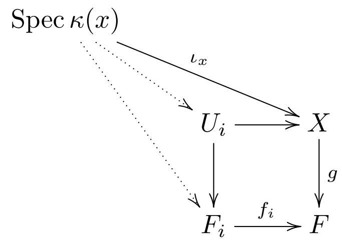
1.1.6 Also the concept of an open covering for functors can be generalized from schemes to functors $F\colon (\mathrm{Sch/S})^{\mathrm{op}} \to (\mathrm{Sets})$:

Definition. Let $F\colon (\mathrm{Sch/S})^{\mathrm{op}} \to (\mathrm{Sets})$, a family $\{ f_i \colon F_i \to F \}_{i \in I}$ of open subfunctors of F is called a Zariski open covering of F if for any S-scheme X and any morphism of functors $g\colon X \to F$, the image of the $(f_i)_{(X)}$'s (as morphisms of schemes) forms an open covering of X.

In this form this definition seems difficult to check as it involves any possible morphism $g\colon X \to F$, with X scheme, but fortunately the following result simplifies a lot this hard task. For convenience we will state it only for $S = \operatorname{Spec} R$ affine, even though the general case can be proved in the same way.

Proposition. *Let $F\colon (\mathrm{Sch/R})^{\mathrm{op}} \to (\mathrm{Sets})$ and $\{ f_i \colon F_i \to F \}_i$ be a collection of open subfunctors of F such that $F(K) = \bigcup_i F_i(K)$ for any R-algebra K that is a field. Then the family $\{ f_i \}_i$ form a Zariski open covering of F.*

Proof. Let $g\colon X \to F$ a morphism of functors, with X a R-scheme. Consider for any i the open immersion $(f_i)_{(X)} \colon F_i \times_F X \to X$ and denote by U_i its image. We have to prove that for any $x \in X$ there is an index i such that $x \in U_i$, or in other terms that the corresponding morphism $\iota_x \colon \operatorname{Spec} \kappa(x) \to X$ factors through U_i. We can now write a commutative diagram



where the existence of the two dotted arrows is equivalent, since $F_i \times_F X \cong U_i$. Hence we are left to prove that the composition $g \circ \iota_x \colon \operatorname{Spec} \kappa(x) \to F$ factors through f_i for some i. Now note that by the Yoneda lemma and by the hypothesis

$$\mathrm{Hom}_R(\operatorname{Spec} \kappa(x), F) = F\big(\kappa(x)\big) = \bigcup_i F_i\big(\kappa(x)\big) = \bigcup_i \mathrm{Hom}_R(\operatorname{Spec} \kappa(x), F_i),$$

hence $g \circ \iota_x$ already lies in $\mathrm{Hom}_R(\operatorname{Spec} \kappa(x), F_i)$ for some index i, i.e. it factors through f_i. $\square$

1.1.7 We have all the tools now to state the criterion of representability that we will use. For the proof we refer to [5], page 209.

Theorem. *Let* $F\colon (\mathrm{Sch/S})^{\mathrm{op}} \to (\mathrm{Sets})$ *a Zariski sheaf and let* $\{ f_i\colon F_i \to F \}_i$ *a Zariski open covering such that any* F_i *is representable. Then* F *is representable.*

We want describe now, as final outcome of this section, a procedure to define a scheme over a commutative ring R that represents a given functor $F\colon (\mathrm{AffSch/R})^{\mathrm{op}} \to (\mathrm{Sets})$:

- show that F is a Zariski sheaf ;
- define a class of subfunctors F_i of F such that $F(K) = \bigcup_i F_i(K)$ for any R-algebra K that is a field;
- show that they are open subfunctors and that they are representable in (AffSch/R).

We can indeed extend F, F_i to Zariski sheaves $\tilde{F}, \tilde{F_i}\colon (\mathrm{Sch/R})^{\mathrm{op}} \to (\mathrm{Sets})$: F is a Zariski sheaf by the hypothesis and the F_i's as they are representable in the category (AffSch/R). We already observed (cf. 1.1.5) that $F_i \to F$ extend to $\tilde{F_i} \to \tilde{F}$ and it is again an open immersion, moreover the condition $F(K) = \bigcup_i F_i(K)$, for any R-algebra K that is a field, ensures that they give rise to a Zariski open cover of $\tilde{F}$ (cf. 1.1.6). Observe moreover any $\tilde{F_i}$ is representable by an affine scheme in the category (Sch/R): let Z_i the affine scheme representing F_i in (AffSch/R), then $\tilde{F_i}$ and h_{Z_i} are two Zariski sheaves that restrict to the same Zariski sheaf on (AffSch/R), hence (by the uniqueness of the extension) they are equal on (Sch/R).

Hence all the conditions of the above theorem are satisfied for $\tilde{F}$ and therefore it is representable by an R-scheme.

1.2 Grassmannian scheme

The previous paragraph introduced the concept of representability and gave us a way to find explicitly (following the proof of the criterion in 1.1.7) the representative of a specific class of functors. In view of the local study of our moduli space we are interested in one such functor in particular: the so-called local model. This is a closed subscheme of a product of Grassmannian schemes, that we are going to introduce in this section.

1.2.1 First, we need to recall some statements about projective modules over commutative rings. Let therefore A be a commutative ring.

Proposition. *([11] App. B, theorem 2.5, theorem 7.12)*

i. Let M be an A-module. Then M be is projective if and only if it is a direct summand of a free A-module;

ii. Let A be a local ring. Then any finite projective module over A is free.

iii. Let M an A-module of finite presentation. Then M is projective if and only if for any $\mathfrak{m} \in \mathrm{MSpec}\, A$ the localization $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module.

By the point *ii.* the following definition makes sense:

Definition. Let M a projective A-module. We define $\mathrm{rk}\colon \operatorname{Spec} A \to \mathbb{N} \cup \{ \infty \}$ to be the function that maps $\mathfrak{p}$ to $\mathrm{rk}_{A_\mathfrak{p}}(M_\mathfrak{p})$, where $\mathrm{rk}_{A_\mathfrak{p}}(M_\mathfrak{p})$ is the rank as free module of the localization of M at $\mathfrak{p}$ over the local ring $A_\mathfrak{p}$. We say that M has rank n if the function rk is constant with value n for any $\mathfrak{p} \in \operatorname{Spec} A$.

Moreover one can show that the rank function is locally constant on $\operatorname{Spec}(A)$:

Proposition. *([3] chapter II, section 5.2, corollary to proposition 2)*
Let M a finitely presented projective A-module. Let $\mathfrak{p} \in \operatorname{Spec} A$ and $M_\mathfrak{p}$ be free of rank d. Then there exists $f \in A \setminus \mathfrak{p}$ such that M_f is free of rank d.

Proposition. *([3] chapter II, section 5.2, theorem 1)*
Let M a finitely generated projective module. Then it is finitely presented.

Combining the previous results we get that if M is a direct summand of A^n, for some positive integer n, the quasi-coherent associated module $\tilde{M}$ is locally free. In fact M is projective and finitely generated (as quotient of the finite free A-module A^n), hence also finitely presented. If we localize at any prime $\mathfrak{p} \in \operatorname{Spec} A$ then $M_\mathfrak{p}$ is a finitely presented module over the local ring $A_\mathfrak{p}$, hence free (of finite rank $\mathrm{rk}_{A_\mathfrak{p}}(M_\mathfrak{p})$). Therefore exists $f \in A \setminus \mathfrak{p}$ such that M_f is free (of the same rank $\mathrm{rk}_{A_\mathfrak{p}}(M_\mathfrak{p})$) i.e. a distinguished open neighborhood $D(f)$ of $\mathfrak{p}$ such that $\tilde{M}_{|D(f)}$ is free (of rank $\mathrm{rk}_{A_\mathfrak{p}}(M_\mathfrak{p})$). Moreover if M has constant rank d, it follows that $\tilde{M}$ has constant rank d.

1.2.2 We pass now to the definition of the grassmannian functor. It generalizes the classical notion of grassmannian variety over an algebraically closed field, i.e. the variety of linear subspaces of K^n of dimension d.

Let R a commutative ring and let's fix n, d two natural numbers such that $0 < d \leq n$. Define for an R-algebra T the set

$$\mathrm{Grass}_{n,d,R}(T) = \{ M \text{ submodule of } T^n : M \text{ direct summand of rank } d \},$$

moreover if $f\colon T_1 \to T_2$ is a morphism of R-algebras define the transition map $M \mapsto M \otimes_{T_1} T_2$, for $M \in \mathrm{Grass}_{n,d,R}(T_1)$. This is well defined as $M \otimes_{T_1} T_2$ is a direct summand of T_2^n (as direct sum and tensor product commute, see [1] prop. 2.14) and it has again rank d (as localization and tensor product commute, see [1] prop. 3.7).

This gives rise to a functor $\mathrm{Grass}_{n,d,R}\colon (\mathrm{AffSch/R}) \to (\mathrm{Sets})$, moreover

Proposition. $\mathrm{Grass}_{n,d,R}$ *is a Zariski sheaf in* (AffSch/R).

Proof. Let T an R-algebra, $\{ t_i \}_i \subseteq T$ generating the unit ideal and, for any i, take M_i a direct summand of $T_{t_i}^n$ of rank d such that $M_i \otimes_{T_{t_i}} T_{t_i t_j} = M_j \otimes_{T_{t_j}} T_{t_i t_j}$ as submodule of $T_{t_i t_j}^n$. We have to show that there exists a unique T-module M, direct summand of T^n of rank d such that $M \otimes_T T_{t_i} = M_i$, as submodules of $T_{t_i}^n$. Note that by [1], prop. 3.5, we will identify the tensor product of a module with its localization: e.g. the above condition becomes $(M_i)_{t_i t_j} = (M_j)_{t_i t_j}$.

First, let's prove the existence of M. As we observed in 1.2.1, we have an $\mathcal{O}_{\operatorname{Spec} T_{t_i}}$-module $\mathscr{F}_i = \tilde{M}_i$, locally free of rank d. Observe that by assumption $X = \operatorname{Spec} T$ is covered by the distinguished open subsets $U_i = D(t_i) \cong \operatorname{Spec}(T_{t_i})$ and moreover we have a collection of isomorphisms

$$\varphi_{i,j} = \mathrm{id}\colon \mathscr{F}_{i|U_i \cap U_j} \xrightarrow{\sim} \mathscr{F}_{j|U_i \cap U_j}$$

hence we can glue them to a sheaf $\mathscr{F}$ on X. This is a quasi-coherent sheaf, since X has an affine open cover of open subsets U_i such that $\mathscr{F}_{|U_i} = \tilde{M}_i$. Hence, as X is affine, $\mathscr{F} = \Gamma(X, \mathscr{F})^{\sim}$. We can therefore take $M = \Gamma(X, \mathscr{F})$. We only need to show that M is a direct summand of T^n of rank d, as by the properties of quasi-coherent modules $M_{t_i} = \Gamma(X, U_i) = M_i$. Writing the sheaf property for $\tilde{M}$ and $\tilde{T}^n$ and the cover $\{ U_i \}_i$ we get the following solid diagram with exact rows:

$$\begin{array}{ccccc} T^n & \longrightarrow & \prod_i T^n_{t_i} & \rightrightarrows & \prod_{i,j} T^n_{t_i t_j} \\ \uparrow & & \uparrow & & \uparrow \\ M & \longrightarrow & \prod_i M_i & \rightrightarrows & \prod_{i,j} (M_i)_{t_i t_j} \end{array}$$

and the existence and injectivity of the dotted arrow follows: let $m \in M$, localizing it gives rise to a collection $(m_i)_i \in \prod_i M_i \subseteq \prod_i T^n_{t_i}$ that agree when localized again in $\prod_{i,j} (M_i)_{t_i t_j} \subseteq \prod_{i,j} T^n_{t_i t_j}$, hence they glue to a uniquely determined $\tilde{m} \in T^n$. Vice versa $\tilde{m}$ can be image of at most one m, as any localization $\tilde{m}_i \in M_i$, then they glue to a unique $m \in M$. Similary one proves the T-linearity of the dotted map, hence we can consider M as a submodule of T^n and more precisely the diagram suggests the explicit description

$$M = \{ m \in T^n : m \in M_i \text{ as an element of the localization } T^n_{t_i} \} .$$

The stalk of M at $\mathfrak{p}$ is isomorphic to the stalk of M_i at $\mathfrak{p}A_{t_i}$, for $t_i \notin \mathfrak{p}$, in particular of the same rank d, hence it is left to prove that M is a direct summand of T^n, or equivalently that the sequence

$$0 \to M \to T^n \to T^n/M \to 0$$

splits. This follows since T^n/M is projective by 1.2.1: in fact $(T^n/M)_{t_i} \cong T^n_{t_i}/M_i$ is finitely presented for any i (it is a direct summand of $T^n_{t_i}$, hence the sequence

$$T^n_{t_i} \to T^n_{t_i} \to T^n_{t_i}/M_i \to 0$$

is exact, where the first map is the composition of the canonical projection $T^n_{t_i} \to M_{t_i}$ followed by the canonical inclusion $M_{t_i} \to T^n_{t_i}$; apply [3] chapter II, section 5.1, corollary to proposition 3); moreover $(T^n/M)_{\mathfrak{m}}$ is free for any $\mathfrak{m} \in \operatorname{MSpec} A$.

Now let's prove the uniqueness: take M, M' such that $M_{t_i} = M'_{t_i}$ for any i. Consider the map $M \cap M' \hookrightarrow M$ and consider the associated map of coherent sheaves: it induces an isomorphism on the stalks, so it is an isomorphism itself. Indeed $(M \cap M')_{t_i} = M_{t_i} \cap M'_{t_i} = M_{t_i}$ for any i, so $(M \cap M')_{\mathfrak{p}} = M_{\mathfrak{p}}$ for any $\mathfrak{p} \in \operatorname{Spec} T$. Since the functor $M \mapsto \tilde{M}$ is fully faithful this implies $M = M'$. ☐

1.2.3 We look now for a Zariski open cover with representable subfunctors. Let therefore $I = \{ i_1 < i_2 < \cdots < i_d \} \subseteq \{ 1, \ldots, n \}$ a choice of d indexes between n. Denote by e_i the standard generators of T^n and by E_I the submodule generated by the e_j's for $j \in \{ 1, \ldots, n \} \setminus I$.

Define the subfunctor $\operatorname{Grass}^I_{n,d,R}$ of $\operatorname{Grass}_{n,d,R}$ by

$$\operatorname{Grass}^I_{n,d,R}(T) = \{ M \subseteq T^n : M \oplus E_I = T^n \} .$$

for any R-algebra T. Observe that if $M \in \operatorname{Grass}^I_{n,d,R}(T)$ then it is free: the projection onto M induces an isomorphism of T-modules $M \cong T^n/E_I = \langle \bar{e}_{i_1}, \ldots, \bar{e}_{i_d} \rangle_{T, i_k \in I}$, where $\bar{e}_j$ denote the image of the j-th standard basis vector in the quotient.

Consider now the affine scheme

$$W = \operatorname{Spec} R[X_{i,j}]_{\substack{i=1,\dots,n \\ j=1,\dots d}}$$

and its closed subset $W_I = V(X_{i_\alpha,\beta} - \delta_{\alpha,\beta})_{\alpha,\beta=1,\dots,d}$, where $\delta_{\alpha,\beta}$ is the Kronecker delta. We have that in particular for any R-algebra T

$$W_I(T) = \{ A \in \mathrm{M}_{n\times d}(T) : A_I = \mathbb{1} \} ,$$

where A_I represents the submatrix of A consisting of the i-th rows, for $i \in I$. We want to prove that $\mathrm{Grass}^I_{n,d,R}$ is representable by the affine scheme W_I, i.e. that $\mathrm{Grass}^I_{n,d,R}(T) \cong W_I(T)$, functorially in T. The map is the correspondence between submodule and matrix of the coordinates of its basis (note that any $M \in \mathrm{Grass}^I_{n,d,R}$ is free) normalized in such a way that the I-th submatrix is the identity.

We explain now why such a normalization is possible. We may indeed without loss of generality assume $I = \{ 1, \dots, d \}$, up to reordering the e_i's. Choose a basis of M: it has a matrix of the form $A_M = \left(\begin{smallmatrix} A_I \\ B \end{smallmatrix}\right)$. Consider moreover the matrix $\left(\begin{smallmatrix} A_I & \mathbb{0} \\ B & \mathbb{1} \end{smallmatrix}\right)$, matrix of the inclusion $M \oplus E_I \hookrightarrow T^n$: the condition $T^n = M \oplus E_I$ is equivalent to asking that this matrix is invertible, or equivalently that A_I invertible. Up to multiplying on the right by A_I^{-1}, operation that corresponds to a change of the basis of M, one can take $A_M \in W_I(T)$.

1.2.4 Note that in the previous argument the assumption $A_I = \mathbb{1}$ has nothing special: we could have taken a different invertible matrix with values in R instead of the identity and the final outcome would have been the same scheme.

There is moreover an elegant way to avoid the choice of such a matrix: one may reformulate what we said above as a a bijection between $\mathrm{Grass}^I_{n,d,R}(T)$ and the set of $n \times d$ matrixes with invertible I-th minor, modulo right multiplication for $d \times d$ invertible matrixes, but we will not use this formulation: for the computations it's important to fix a (non-canonical) choice of A_I.

1.2.5 Note that varying I the covering condition holds for fields:

$$\mathrm{Grass}_{n,d,R}(K) = \bigcup_I \mathrm{Grass}^I_{n,d,R}(K)$$

for any R-algebra K that is a field. In fact for any vector subspace $V \subseteq K^n$ of dimension d any of its associated matrixes has rank d, so we can find a subset of indexes I such that the I-th minor is nonzero, or in other terms $V \oplus E_I = K^n$. It is moreover interesting to remark that $\mathrm{Grass}_{n,d,R}(T) \neq \bigcup_I \mathrm{Grass}^I_{n,d,R}(T)$ for general R-algebras T: in fact $\mathrm{Grass}^I_{n,d,R}$ contains only free modules for any I, but in general there exist non-free projective modules in $\mathrm{Grass}_{n,d,R}(T)$.

1.2.6 Let's now prove that we have truly defined open subfunctors

Lemma. *Let A a ring. The following are equivalent:*

i. $f \in A^\times$;

ii. $f/1 \in A_\mathfrak{p}^\times$ *for any* $\mathfrak{p} \in \operatorname{Spec} A$;

iii. for any set $\{ t_i \}_i$ *generating the unit ideal* $f/1 \in A_{f_i}^\times$ *for any* $i \in I$.

Proof. The implications *i.* $\Rightarrow$ *ii.* and *i.* $\Rightarrow$ *iii.* are clear. For *ii.* $\Rightarrow$ *i.* observe that if $f \notin A^\times$ exist $\mathfrak{m} \in \operatorname{MSpec} A$ such that $f \in \mathfrak{m}$, hence $f/1 \in \mathfrak{m}A_\mathfrak{m} = A_\mathfrak{m} \smallsetminus A_\mathfrak{m}^\times$. By the same argument the implication *iii.* $\Rightarrow$ *ii* follows, as $A_\mathfrak{p} = (A_{t_i})_{\mathfrak{p}A_{t_i}}$ for any $\mathfrak{p} \not\ni t_i$. □

Lemma. *Let T an R-algebra, $M \in \mathrm{Grass}_{n,d,R}(T)$ and I a collection of indexes as above. Then there exists an affine open subset $U_I \subseteq \mathrm{Spec}(T)$ such that, if $\varphi\colon T \to S$ is a morphism of R-algebras, then $f = \varphi^*\colon \mathrm{Spec}(S) \to \mathrm{Spec}(T)$ factors through U_I if and only if $M \otimes_T S \in \mathrm{Grass}^I_{n,d,R}(S)$*

Proof. Let $\{ t_i \}_i$ a (finite) subset of T generating the unit ideal and such that M_{t_i} is free for any i. Let $m_i \in T_{t_i}$ the I-th minor of its matrix. Write $m_i = x_i/t_i^{n_i}$, with $x_i \in T$, and define $U_I = D(\prod_i x_i)$. Note that $\operatorname{Spec} S \to \operatorname{Spec} T$ factors through U_I if and only if $\varphi\colon T \to S$ factors through $T_{\prod x_i}$, i.e. $\varphi(\prod x_i) \in S^\times$. Hence by the lemma above the factorization of f is equivalent to $\varphi(x_i) \in S^\times_{\varphi(t_j)}$ for any i, j.

If the latter holds consider $(M \otimes_T S)_{\varphi(t_i)} = M_{t_i} \otimes_{T_{t_i}} S_{\varphi(t_i)}$. It is free and the I-th minor of its matrix is $\varphi(m_i) = \varphi(x_i)/\varphi(t_i)^{n_i} \in S^\times_{\varphi(t_i)}$, hence $(M \otimes_T S)_{\varphi(t_i)} \in \mathrm{Grass}^I_{n,d,R}(S_{\varphi(t_i)})$. Since $\mathrm{Grass}^I_{n,d,R}$ is representable, therefore a Zariski sheaf, $M \otimes_T S$ lies in $\mathrm{Grass}^I_{n,d,R}(S)$.

Vice versa suppose $M \otimes_T S \in \mathrm{Grass}^I_{n,d,R}(S)$, hence free with I-th minor $u \in S^\times$. Note that $u = \varphi(m_i) = \varphi(x_i)/\varphi(t_i)^{n_i} \in S^\times_{\varphi(t_i)}$ for any i, as we have that $(M \otimes_T S)_{\varphi(t_i)} = M_{t_i} \otimes_{T_{t_i}} S_{\varphi(t_i)}$. Hence $\varphi(x_i) \in S^\times_{\varphi(t_i)}$ for any i. Now fix an index j, then $\varphi(t_i)^{n_i}\varphi(x_k) = \varphi(t_k)^{n_k}\varphi(x_i)$ in $S_{\varphi(t_j)\varphi(t_k)}$ for any k, hence $\varphi(x_i)$ is invertible as $\varphi(x_k)$ is so. Since $\{ \varphi(t_j)\varphi(t_k) \}_k$ generates the unit ideal of $S_{\varphi(t_j)}$ by the lemma $\varphi(x_i) \in S^\times_{\varphi(t_j)}$ for any i, j. This concludes the proof of the equivalence. □

It follows that:

Proposition. *For any choice of I the functor $\mathrm{Grass}^I_{n,d,R}$ is an open subfunctor of $\mathrm{Grass}_{n,d,R}$.*

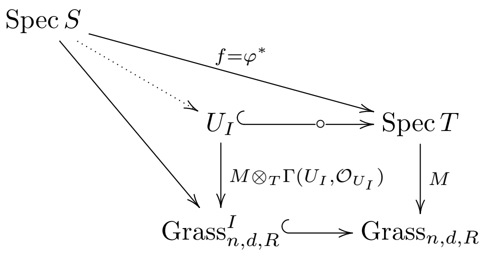
Proof. Consider a morphism of functors $\operatorname{Spec} T \to \mathrm{Grass}_{n,d,R}$. Via Yoneda's lemma it corresponds to $M \in \mathrm{Grass}_{n,d,R}(T)$. Note that for any R-algebra S and $\varphi\colon T \to S$ a morphism of R-algebras the composition

$$\mathrm{Spec}(S) \to \mathrm{Spec}(T) \to \mathrm{Grass}_{n,d,R}$$

corresponds to the element $M \otimes_T S \in \mathrm{Grass}_{n,d,R}(S)$. As U_I defined in the lemma is affine we may apply the lemma to $S = \Gamma(U_I, \mathcal{O}_{U_I})$: $f = \varphi^*$ factors (via the identity) through U_I, hence $M \otimes_T \Gamma(U_I, \mathcal{O}_{U_I}) \in \mathrm{Grass}^I_{n,d,R}\big(\Gamma(U_I, \mathcal{O}_{U_I})\big)$, i.e. the map $U_I \to \mathrm{Grass}_{n,d,R}$ factors through $\mathrm{Grass}^I_{n,d,R}$. We get therefore the commutative square

$$\begin{array}{ccc} U_I & \hookrightarrow & \operatorname{Spec} T \\ {\scriptstyle M\otimes_T\Gamma(U_I,\mathcal{O}_{U_I})}\downarrow & & \downarrow{\scriptstyle M} \\ \mathrm{Grass}^I_{n,d,R} & \hookrightarrow & \mathrm{Grass}_{n,d,R} \end{array}$$

and we need to show that it is cartesian. Let therefore S any R-algebra and $\varphi\colon T \to S$ morphism of R-algebras such that we may complete the solid diagram below



or equivalently (same argument as above) $M \otimes_T S \in \mathrm{Grass}^I_{n,d,R}(S)$. Therefore f factors through U_I, i.e. the dotted arrow exists and it makes the upper triangle commute. The commutativity of the lower triangle follows by the commutativity of the rest of the diagram, since the morphism $\mathrm{Grass}^I_{n,d,R} \to \mathrm{Grass}_{n,d,R}$ is a monomorphism of functors. □

1.2.7 All these properties of the functor $\mathrm{Grass}_{n,d.R}$ where studied in order to apply the procedure in 1.1.7 to find a scheme representing this functor, that we will denote by $G_{n,d,R}$. Following the proof of the criterion of representability in [5] we can describe explicitly how $G_{n,d,R}$ is obtained by gluing affine pieces: as in 1.2.3 we start with

$$W = \mathrm{Spec}\, R[X_{i,j}]_{\substack{i=1,\dots,n \\ j=1,\dots d}} \cong \mathbb{A}^{nd}_R$$

and $W_I = V(X_{i_\alpha,\beta} - \delta_{\alpha,\beta})_{\alpha,\beta=1,\dots,d} \cong \mathbb{A}^{d(n-d)}_R$, where $\delta_{\alpha,\beta}$ is the Kronecker delta. This represent the matrixes with the identity as I-th submatrix. Now define $X = (X_{i,j})_{i,j}$ as the matrix with all the indeterminates and for any J, set of indexes of cardinality d, denote by P_J the J-th minor of X. Now we set

$$W_{I,J} = W_I \cap D(P_J) \cong \mathrm{Spec}\left(\frac{R[X_{i,j}, P_J^{-1}]_{i,j}}{V(X_{i_\alpha,\beta} - \delta_{\alpha,\beta})_{\alpha,\beta}}\right),$$

that represent the matrixes with the identity as I-th submatrix and invertible J-th minor, and define an R. There are moreover isomorphisms

$$\varphi_{I,J} \colon W_{I,J} \xrightarrow{\sim} W_{J,I}$$

given by the map of R-algebras induced by the multiplication on the right of X with X_J^{-1}. We obtain $G_{n,d,R}$ by gluing the W_I's along the $\varphi_{I,J}$'s.

One remark on this construction: as we remarked in 1.2.4 the choice of $A_I = \mathbb{1}$ is completely arbitrary and we may choose any other invertible $d \times d$ matrix. Modifying in a suitable way the definition of W_I's and the bijection described above (that is given in general by $X_J^{-1}X_I$), we get a new construction of the Grassmannian, that *mutata mutandis* is the same described in this paragraph, it gives indeed an isomorphic result.

1.3 Chain of lattices

In this section we will introduce an important chain of $\mathbb{Z}_p$-lattices and we develop some of its features necessary for the definition of local models.

1.3.1 Let g a positive integer and for $i = 0, \dots, 2g$ define the free $\mathbb{Z}_p$-module

$$\Lambda_i = \langle p^{-1}e_1, \dots p^{-1}e_i, e_{i+1}, \dots e_{2g} \rangle_{\mathbb{Z}_p},$$

where $\{ e_1, \dots, e_{2g} \}$ is the standard basis of $\mathbb{Q}_p^{2g}$. We get therefore a sequence of inclusions

$$\mathbb{Z}_p^{2g} = \Lambda_0 \subseteq \Lambda_1 \subseteq \cdots \subseteq \Lambda_{2g} = p^{-1}\Lambda_0$$

that we can extend periodically letting $\Lambda_{i+2gk} = p^{-k}\Lambda_i$, for any $k \in \mathbb{Z}$. Note that in particular $\Lambda_{-i} = \langle e_1, \dots, e_{2g-i}, pe_{2g-i+1}, \dots, pe_{2g} \rangle_{\mathbb{Z}_p}$ for $i = 0, \dots, 2g$.

It is useful to observe that the inclusion $\iota_i \colon \Lambda_i \to \Lambda_{i+1}$ is a $\mathbb{Z}_p$-linear map, for $i = 0, \dots, 2g$, and that its matrix is

$$A_i = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & p & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

(where p is in the $(i+1)$-th place) in the two given bases of Λ_i and Λ_{i+1}. Moreover by the periodic definition in general ι_i has matrix $A_{i'}$, where $i' \in \{ 0, \dots, 2g-1 \}$ is such that $i' \equiv i \mod 2g$.

1.3.2 Endow $\mathbb{Q}_p^{2g}$ with the standard bilinear alternating pairing $\langle -, - \rangle$, i.e. the one given by the matrix $G = \left(\begin{smallmatrix} \mathbb{O} & J \\ -J & \mathbb{O} \end{smallmatrix}\right)$, where $J = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ It is interesting to restrict our pairing to the lattices Λ_i described above.

Lemma. *Let $0 \leq i \leq 2g$. $\Lambda_{-i} = \{ y \in \mathbb{Q}_p^{2g} : \langle x, y \rangle \in \mathbb{Z}_p, \forall x \in \Lambda_i \}$.*

Proof. Observe that $\langle \Lambda_i, y \rangle \in \mathbb{Z}_p$ if and only if $v_p(\langle p^{-1}e_j, y \rangle), v_p(\langle e_{j'}, y \rangle) \geq 0$, where v_p denotes the p-adic valuation, for $0 < j \leq i < j' \leq 2g$. Writing $y = \sum_k a_k e_k$ it follows that $\langle e_j, y \rangle = \pm a_{2g+1-j}$, for any $j = 1, \dots, 2g$, hence $\langle \Lambda_i, y \rangle \in \mathbb{Z}_p$ if and only if $v_p(a_{2g+1-j}) \geq 1$ for $j = 1, \dots, i$ and $v_p(a_{2g+1-j}) \geq 0$ for $j = i+1, \dots, 2g$, i.e. $y \in \Lambda_{-i}$. $\square$

Proposition. *The pairing $\langle -, - \rangle$ on $\mathbb{Q}_p^{2g}$ restricts to a perfect bilinear pairing $\Lambda_i \times \Lambda_{-i} \to \mathbb{Z}_p$, its matrix in the given basis of Λ_i and Λ_{-i} is again G.*

Proof. The validity of the definition follows from the above lemma. Suppose $\langle \Lambda_i, y \rangle = 0$, for $y \in \Lambda_{-i}$; the computation in the proof of the lemma shows that $y = 0$, i.e. the pairing is non degenerate on the right side. Similary for the left side. Hence the two natural morphisms $\Lambda_{-i} \to \Lambda_i^\vee := \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda_i, \mathbb{Z}_p)$ and $\Lambda_i \to \Lambda_{-i}^\vee$ are injective. They are visibly surjective, therefore isomorphisms. $\square$

1.3.3 If R is a $\mathbb{Z}_p$-algebra, then we can tensor all the above constructions with R, and we get similar properties. In the rest of this section all tensor products are taken over $\mathbb{Z}_p$, hence we will omit it from the notations. We define for any $i \in \mathbb{Z}$ the free R-module $\Lambda_{i,R} = \Lambda_i \otimes R$, of rank $2g$. Since the tensor product is only right exact, we don't have in general a chain of inclusions of R module, but we have still a sequence

$$\cdots \to \Lambda_{-2g,R} \to \Lambda_{-2g+1} \to \cdots \to \Lambda_{0,R} \to \Lambda_{1,R} \to \cdots \to \Lambda_{2g,R} \to \cdots$$

of R-linear morphisms ι_i^R with matrix A_i (considered as a matrix with entries in R) and an R-linear isomorphism $\varepsilon_i^R \colon \Lambda_{i-2g,R} \xrightarrow{\sim} \Lambda_{i,R}$ for any $i \in \mathbb{Z}$, with the identity as matrix.

Also the standard pairing extends linearly to $\langle -, - \rangle_R \colon \Lambda_{i,R} \times \Lambda_{-i,R} \to R$, namely we can define $\langle x \otimes r_1, y \otimes r_2 \rangle_R = r_1 r_2 \langle x, y \rangle$. Note that $\langle -, - \rangle_R$ has again G as matrix and it is again perfect:

$$\Lambda_{-i,R} = \Lambda_{-i} \otimes R \cong \Lambda_i^\vee \otimes R = \mathrm{Hom}_{\mathbb{Z}_p}(\Lambda_i, \mathbb{Z}_p) \otimes R \cong \mathrm{Hom}_R(\Lambda_{i,R}, R) = \Lambda_{i,R}^\vee$$

where the last isomorphism follows as Λ_i is free.

1.3.4 By the above lemma, if $\mathcal{F}$ is a submodule of Λ_i for $i = 0, \ldots, 2g$, then its orthogonal $\mathcal{F}^\perp$ is contained in Λ_{-i}. More generally we may define for any $\mathcal{F}$ submodule of $\Lambda_{i,R}$ its orthogonal $\mathcal{F}^\perp := \{\, y \in \Lambda_{-i,R} : \langle x, y \rangle_R = 0 \text{ for all } x \in \Lambda_{i,R} \,\}$. Suppose moreover that we have a submodule $\mathcal{G}$ of $\Lambda_{i+1,R}$ such that $\iota_i^R(\mathcal{F}) \subseteq \mathcal{G}$, then $\iota_{-i-1}^R(\mathcal{G}^\perp) \subseteq \mathcal{F}^\perp$. In fact $GA_{2g-i-1} = A_i G$, therefore if we take $x \in \mathcal{F}$, $y \in \mathcal{G}^\perp$ then

$$\langle x, \iota_{-i-1}^R(y) \rangle_R = \langle \iota_i^R(x), y \rangle_R = 0,$$

as $\iota_i(x) \in \mathcal{G}$.

Hence if we have a chain of submodules $\mathcal{F}_i \subseteq \Lambda_i$ such that

$$\begin{array}{ccccccc} \Lambda_{i_0} & \longrightarrow & \Lambda_{i_1} & \longrightarrow & \ldots & \longrightarrow & \Lambda_{i_r} \\ \uparrow & & \uparrow & & & & \uparrow \\ \mathcal{F}_0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \ldots & \longrightarrow & \mathcal{F}_r \end{array}$$

commutes, then the corresponding diagram

$$\begin{array}{ccccccc} \Lambda_{-i_0} & \longleftarrow & \Lambda_{-i_1} & \longleftarrow & \ldots & \longleftarrow & \Lambda_{-i_r} \\ \uparrow & & \uparrow & & & & \uparrow \\ \mathcal{F}_0^\perp & \longleftarrow & \mathcal{F}_1^\perp & \longleftarrow & \ldots & \longleftarrow & \mathcal{F}_r^\perp \end{array}$$

is again commutative.

1.4 Local models

Finally we are ready to define the (symplectic) local model. Again we will use the procedure described in 1.1.7.

1.4.1 Let $i_\bullet = \{\, 0 \le i_0 < i_1 < \cdots < i_r \le g \,\}$ be a choice of indexes, that we will call parahoric level structure. Let $M^{\mathrm{loc}}(i_\bullet)(R)$ be the set of diagrams

$$\begin{array}{ccccccc} \Lambda_{i_0} & \longrightarrow & \Lambda_{i_1} & \longrightarrow & \dots & \longrightarrow & \Lambda_{i_r} \\ \uparrow & & \uparrow & & & & \uparrow \\ \mathcal{F}_0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \dots & \longrightarrow & \mathcal{F}_r \end{array}$$

where $\mathcal{F}_i$ is a submodule of $\Lambda_{i,R}$ for any $i = i_0, \dots, i_r$, direct summand of rank g, and such that we may complete the above diagram with the dotted arrows (that are part of the datum):

$$\begin{array}{ccccccccccccc} \longrightarrow & \Lambda_{-i_0} & \longrightarrow & \Lambda_{i_0} & \longrightarrow & \dots & \longrightarrow & \Lambda_{i_r} & \longrightarrow & \Lambda_{2g-i_r} & \longrightarrow \\ & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & \\ \longrightarrow & \mathcal{F}_0^\perp & \dashrightarrow & \mathcal{F}_0 & \longrightarrow & \dots & \longrightarrow & \mathcal{F}_r & \dashrightarrow & \varepsilon_{i_r}^R(\mathcal{F}_r^\perp) & \longrightarrow \end{array}$$

where the map $\mathcal{F}_{i_{k+1}}^\perp \to \mathcal{F}_{i_k}^\perp$ is the linear map induced by $\mathcal{F}_{i_k} \to \mathcal{F}_{i_{k+1}}$, as described in the previous section.

Using $(\mathcal{F}_\nu)_\nu \mapsto (\mathcal{F}_\nu \otimes_R R')_\nu$ as connecting morphism, for $R \to R'$ a morphism of $\mathbb{Z}_p$-algebras, we get the functor $M^{\mathrm{loc}}(i_\bullet)\colon (\mathrm{Ring}/\mathbb{Z}_\mathrm{p}) \to (\mathrm{Sets})$.

1.4.2 It is worth for future calculations to observe that in the above definition, in the particular case $i_0 = 0$, the existence condition of the dotted arrow becomes $\mathcal{F}_0^\perp = \mathcal{F}_0$, and similarly for $i_r = g$ it becomes $\varepsilon_g^R(\mathcal{F}_g^\perp) = \mathcal{F}_g$. This follows immediately from the following lemma:

Lemma. *Let M be a finite free R-module and let $N_1 \subseteq N_2$ be direct summands of M of the same (necessarily finite) rank. Then $N_1 = N_2$.*

Proof. If we localize at $\mathfrak{p} \in \mathrm{Spec}(R)$ we get the following commutative diagram with split exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (N_2)_\mathfrak{p} & \longrightarrow & M_\mathfrak{p} & \longrightarrow & (M/N_2)_\mathfrak{p} & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle j} & & \| & & \uparrow{\scriptstyle p} & & \\ 0 & \longrightarrow & (N_1)_\mathfrak{p} & \longrightarrow & M_\mathfrak{p} & \longrightarrow & (M/N_1)_\mathfrak{p} & \longrightarrow & 0 \end{array}$$

and applying the snake lemma $\ker p \cong \operatorname{coker} j$.

Hence $\ker p$ is a finitely generated $R_\mathfrak{p}$-module: in fact $\operatorname{coker} j$ is a quotient of $(N_2)_\mathfrak{p}$, finite free as finitely generated (because quotient of a finite free module) and projective over the local ring $R_\mathfrak{p}$.

Denote by $\mathfrak{m}_\mathfrak{p}$ the maximal ideal of $R_\mathfrak{p}$ and by $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$ the residue field at $\mathfrak{p}$. Note that since the rows are split exact if we tensor with $\kappa(\mathfrak{p})$ they remain exact, hence we get the diagram of $\kappa(\mathfrak{p})$-vector spaces

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (N_2)_{\mathfrak{p}} \otimes \kappa(\mathfrak{p}) & \longrightarrow & M_{\mathfrak{p}} \otimes \kappa(\mathfrak{p}) & \longrightarrow & (M/N_2)_{\mathfrak{p}} \otimes \kappa(\mathfrak{p}) & \longrightarrow & 0 \\
 & & {\scriptstyle j\otimes\kappa(\mathfrak{p})}\uparrow & & \| & & {\scriptstyle p\otimes\kappa(\mathfrak{p})}\uparrow & & \\
0 & \longrightarrow & (N_1)_{\mathfrak{p}} \otimes \kappa(\mathfrak{p}) & \longrightarrow & M_{\mathfrak{p}} \otimes \kappa(\mathfrak{p}) & \longrightarrow & (M/N_1)_{\mathfrak{p}} \otimes \kappa(\mathfrak{p}) & \longrightarrow & 0
\end{array}
$$

with exact rows. The projective modules $(N_1)_{\mathfrak{p}}$ and $(N_2)_{\mathfrak{p}}$ are free and of same rank by assumption and therefore $(N_1)_{\mathfrak{p}} \otimes \kappa(\mathfrak{p})$ and $(N_1)_{\mathfrak{p}} \otimes \kappa(\mathfrak{p})$ have the same dimension. As M was finite free then $M_{\mathfrak{p}} \otimes \kappa(\mathfrak{p})$ is finite dimensional and it follows that also the vector spaces $(M/N_1)_{\mathfrak{p}} \otimes \kappa(\mathfrak{p})$ and $(M/N_2)_{\mathfrak{p}} \otimes \kappa(\mathfrak{p})$ have same (finite) dimension.

Now note that the exact sequence

$$0 \to \ker(p) \to (M/N_1)_{\mathfrak{p}} \to (M/N_2)_{\mathfrak{p}} \to 0$$

is split exact, as $(M/N_2)_{\mathfrak{p}}$ is projective; hence it remains exact after tensoring with $\kappa(\mathfrak{p})$. Therefore

$$0 = \ker\big(p \otimes \kappa(\mathfrak{p})\big) = \ker(p) \otimes \kappa(\mathfrak{p}) = \ker(p)/\mathfrak{m}_{\mathfrak{p}} \ker(p),$$

since $p \otimes \kappa(\mathfrak{p})$ is a surjective linear map between vector spaces of the same dimension. Then it follows by Nakayama's lemma that $\ker(p) = 0$. The equality $(N_1)_{\mathfrak{p}} = (N_2)_{\mathfrak{p}}$ follows for any prime $\mathfrak{p} \in \operatorname{Spec}(R)$ and this implies the equality between N_1 and N_2. □

1.4.3 So far we introduced a subfunctor of a product of Grassmannian functors. We will see now that this functor is represented by a closed subscheme of the corresponding product of Grassmannian schemes.

Proposition. *For any parahoric level structure* $i_\bullet = \{\, 0 \leq i_0 < \cdots < i_r \leq g \,\}$ *the local model* $M^{\mathrm{loc}}(i_\bullet)$ *is a Zariski sheaf*

Proof. Let R a $\mathbb{Z}_p$-algebra and let $\{\, f_k \,\}_k \subseteq R$ generating the unit ideal. Take for any k a collection of submodules $(\mathcal{F}_{i_\nu,k})_\nu \in M^{\mathrm{loc}}(i_\bullet)(R_{f_k})$ that agree on localizing at the element $f_k f_h$ for any pair of indexes (h,k). In the section 1.2.2 we proved that each family $(\mathcal{F}_{i_\nu,k})_k$ glues to a unique direct summand $\mathcal{F}_\nu$ of rank g of $\Lambda_{i_\nu,R}$. Note that in particular $\mathcal{F}_{i_0}^\perp$ coincide with the submodule obtained gluing the $\mathcal{F}_{i_0,k}^\perp$'s. It remains to show the existence of all the connecting morphisms: it follows easily from the explicit description of the $\mathcal{F}_{i_\nu}$'s as in 1.2.2, i.e.

$$\mathcal{F}_{i_\nu} = \{\, f \in R^n : f \in \mathcal{F}_{i_\nu,k} \text{ as element of the localization } R^n_{f_k} \,\}\,.$$ □

Let now $J = (I_0, \ldots, I_r)$, where each I_k is a set of d indexes chosen between 1 and n. For any J we define the subfunctors $M^{\mathrm{loc}}(i_\bullet) \cap \prod_{k=0}^r \mathrm{Grass}^{I_k}_{2g,g,\mathbb{Z}_p}$ of $M^{\mathrm{loc}}(i_\bullet)$. One can show, with the same argument as for the $\mathrm{Grass}^I_{n,d,R}$'s, that they define open subfunctors and an open Zariski cover of $M^{\mathrm{loc}}(i_\bullet)$. Moreover they are representable: write $U_J = W_{I_0} \times \cdots \times W_{I_r} \subseteq \prod_{k=0}^r G_{2g,g,\mathbb{Z}_p}$. Later on (see 2.5) we will show by explicit computations in some particular case (but the computations are analogous in the general case) that for any J there exists a closed subscheme Z_J of U_J such that $M^{\mathrm{loc}}(i_\bullet) \cap \prod_{k=0}^r \mathrm{Grass}^{I_k}_{2g,g,\mathbb{Z}_p} \cong Z_J$ as functors, where the isomorphism is the restriction of $\prod_{k=0}^r \mathrm{Grass}^{I_k}_{2g,g,\mathbb{Z}_p} \cong U_J$. We may therefore apply the criterion of representability in 1.1.7 to show that $M^{\mathrm{loc}}(i_\bullet)$ is representable by a scheme Z. More precisely, we glue finitely many closed subschemes of the U_J's with the same identification that we had in the grassmannian case: hence Z is a closed subscheme of $\prod_{k=0}^r G_{2g,g,\mathbb{Z}_p}$. In the sequel we will denote Z again by $M^{\mathrm{loc}}(i_\bullet)$.

Chapter 2

Siegel Moduli space

In this chapter we will introduce the Siegel moduli space with parahoric level structure, the geometric object we are interested in. We will moreover introduce a powerful technique for the study of its local structure: relating it with the local structure of the local model, introduced in the first chapter. In the whole chapter we will follow mainly the exposition of [10] and sometimes of [8] of the subject.

2.1 Moduli problem

Once again, as we did in the first chapter, we introduce a scheme via the definition of its functor of points. In this section let g be a fixed positive integer, and $I = \{ 0 \leq i_0 < i_1 < \cdots < i_r \leq g \}$ a choice of indexes, that we call a parahoric level structure.

Fixed a parahoric level structure I we need some auxiliary definitions:

- let $I' = \{ \pm i + 2gk : i \in I, k \in \mathbb{Z} \}$;
- let $\mathrm{succ}\colon I' \to I'$ be the "succesive" map, i.e. $\mathrm{succ}(i) = \min \{ j \in I' : j > i \}$;
- given two indexes $i, j \in I'$, let $\mathrm{d}(i,j) = |j - i|$, i.e. the distance between them.

2.1.1 Let p a rational prime and $N \in \mathbb{N}_{\geq 3}$ such that $p \nmid N$. Fix moreover an N-th root of unity ζ_N in $\bar{\mathbb{Q}}_p$ and let $S = \operatorname{Spec} \mathbb{Z}_p[\zeta_N]$. In this setting for any S-scheme S' define $\mathcal{A}_{I,N}(S')$ as the set of equivalence classes of tuples $\big((A_i)_{i\in I'}, (\alpha_i)_{i\in I'}, (\lambda_i)_{i\in I'}, \eta\big)$ where:

S1 $(A_i)_{i\in I'}$ is a family of abelian schemes over S' of (relative) dimension g and such that $A_i = A_{i+2g}$ for any $i \in I'$;

S2 $(\alpha_i)_{i\in I'}$ is a chain of isogenies $\alpha_i\colon A_i \to A_{\mathrm{succ}(i)}$ of degree $p^{\mathrm{d}(i,\mathrm{succ}(i))}$

$$\xrightarrow{\alpha_{-i_1}} A_{-i_0} \xrightarrow{\alpha_{-i_0}} A_{i_0} \xrightarrow{\alpha_{i_0}} \dots \xrightarrow{\alpha_{i_r-1}} A_{i_r} \xrightarrow{\alpha_{i_r}} A_{2g-i_r} \xrightarrow{\alpha_{2g-i_r}}$$

such that for any $i \in I'$ one has that $\alpha_i = \alpha_{i+2g}$ and that the composition

$$A_i \to \cdots \to A_{i+2g} = A_i,$$

is the multiplication by p map;

S3 $(\lambda_i \colon A_i \xrightarrow{\sim} A^\vee_{-i})_{i\in I'}$ is a family of isomorphisms such that $\lambda_{i+2g} = \lambda_i$ for any $i \in I'$ and such that the diagram

$$\begin{array}{ccccccccc} \longrightarrow & A_{-i_1} & \longrightarrow & A_{-i_0} & \longrightarrow & A_{i_0} & \longrightarrow & A_{i_1} & \longrightarrow \\ & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\ \longrightarrow & A^\vee_{i_1} & \longrightarrow & A^\vee_{i_0} & \longrightarrow & A^\vee_{-i_0} & \longrightarrow & A^\vee_{-i_1} & \longrightarrow \end{array}$$

is commutative, where the upper row is given by the chain $(\alpha_i)_{i\in I'}$ (unless in the case $i_0 = 0$ or in the case $i_r = g$, where we use the identity respectively as map $A_0 = A_{-i_0} \to A_{i_0} = A_0$ and $A_g = A_{i_r} \to A_{2g-i_r} = A_g$), with bottom maps the dual isogenies, and such that for any $i \in I'$ the composition $A_i \cong A^\vee_{-i} \to A^\vee_{i-2g} = A^\vee_i$ is a polarization;

S4 $\eta \colon A_{i_0}[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^{2g}$ is a level N structure on A_{i_0}, i.e. an isomorphism making the diagram

$$\begin{array}{ccc} A_{i_0}[N] \times A_{i_0}[N] & \longrightarrow & \mu_N \\ {\scriptstyle \eta\times\eta}\downarrow\cong & & \downarrow\cong \\ (\mathbb{Z}/N\mathbb{Z})^{2g} \times (\mathbb{Z}/N\mathbb{Z})^{2g} & \longrightarrow & \mathbb{Z}/N\mathbb{Z} \end{array}$$

commute, where the upper pairing is the Weil pairing on A_{i_0}, the lower is the standard alternating pairing on $(\mathbb{Z}/N\mathbb{Z})^{2g}$ and the isomorphism $\mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \mu_N$ is the isomorphism given by $a \bmod N \mapsto (\zeta_N)^a$.

2.1.2 Some particular cases of the previous definition are classical. The simplest one is given by the case $I = \{0\}$, this case gives rise to the usual moduli space of principally polarized abelian varieties $\mathcal{A}_{g,N}$, see e.g. [12]. The case $I = \{0, 1, \ldots, g\}$ is called "Iwahori level structure" or "$\Gamma_0(p)$-level structure", is widely studied, see e.g. [10] and [8]; as an exercise we show that in this special case our definition agrees with the one that one can find there. Let therefore $\mathcal{A}'_{I,N}(S')$, for any S-scheme S', be the set of isomorphism classes of tuples $\big((A_i)_{i\in I}, (\alpha_i)_{i\in I}, \lambda_0, \lambda_g, \eta\big)$, where

S1' $(A_i)_{i\in I}$ is a family of abelian schemes over S' of (relative) dimension g;

S2' $(\alpha_i)_{i\in I\smallsetminus\{g\}}$ is a chain of isogenies $\alpha_i \colon A_i \to A_{i+1}$ of degree p;

S3' $\lambda_0 \colon A_0 \xrightarrow{\sim} A^\vee_0$, $\lambda_g \colon A_g \xrightarrow{\sim} A^\vee_g$ are principal polarizations such that in the diagram

$$\begin{array}{ccccccc} A_0 & \longrightarrow & A_1 & \longrightarrow & \ldots & \longrightarrow & A_g \\ {\scriptstyle \lambda_0}\uparrow\cong & & & & & & \cong\downarrow{\scriptstyle \lambda_g} \\ A^\vee_0 & \longleftarrow & A^\vee_1 & \longleftarrow & \ldots & \longleftarrow & A^\vee_g \end{array}$$

the composition of all the maps, starting at any point, gives the multiplication by p map;

S4' $\eta \colon A_0[N] \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^{2g}$ a level N structure on A_{i_0}.

[10] and [8] define the Siegel moduli space with Iwahori level structure via this functor $\mathcal{A}'_{I,N}$, hence we have to show that there is a bijection $\mathcal{A}_{I,N}(S') \cong \mathcal{A}'_{I,N}(S')$ functorial in S'.

From the left to the right: take the same A_i's and isogenies and level N structure. Note that since $i_0 = 0$ and $i_r = g$, by the definition of $\mathcal{A}_{I,N}$, the two maps $A_0 \to A^\vee_0$, $A_g \to A^\vee_g$ are isomorphisms, hence principal polarizations. Using moreover the commutativity of the diagram in S3 the condition on the composition of maps is verified.

From the right to the left: we define $A_{-i} = A_i^\vee$ for any $i = 1, \dots, g-1$ and we extend the family periodically to $\mathbb{Z}$ letting $A_{i+2g} = A_i$; for the chain of isogenies we use the dual isogenies for $A_{i+1}^\vee \to A_i^\vee$, if $i = 1, \dots, g-2$, the composition of the dual isogenies with the principal polarization for $A_1^\vee \to A_0^\vee \cong A_0$, $A_g \cong A_g^\vee \to A_{g-1}^\vee$ and extend periodically the chain to $\mathbb{Z}$. With this definition the condition on the cyclic composition is satisfied and we get the diagram as in S3:

$$\begin{array}{ccccccccccccc}
\longrightarrow & A_{-g} & \longrightarrow & A_{-g+1} & \longrightarrow & \dots & \longrightarrow & A_{-1} & \longrightarrow & A_0 & \longrightarrow & A_1 & \longrightarrow \\
 & \downarrow \cong & & \| & & & & \| & & \downarrow \cong & & \downarrow \cong & \\
\longrightarrow & A_g^\vee & \longrightarrow & A_{g-1}^\vee & \longrightarrow & \dots & \longrightarrow & A_1^\vee & \longrightarrow & A_0^\vee & \longrightarrow & A_1^{\vee\vee} & \longrightarrow
\end{array}$$

where $A_i \to A_i^{\vee\vee}$ is the canonical isomorphism and $A_0 \xrightarrow{\sim} A_0^\vee$, $A_g \xrightarrow{\sim} A_g^\vee$ are the previously defined polarizations. The level N structure remains the same.

2.1.3 The representability of the functor $\mathcal{A}_{I,N}$, follows from the representability of $\mathcal{A}_{g,N}$. In [10] de Jong shows it in the Iwahori case, let's explain the idea of the proof. Denote as $\mathcal{A}_I$ and $\mathcal{A}_g$ the functors defined as $\mathcal{A}_{I,N}$, $\mathcal{A}_{g,N}$ without the level N structure. One proves that he morphism of functors $\mathcal{A}_I \to \mathcal{A}_g$, defined on S'-valued point as the projection onto the first factor (i.e. we forget everythig but A_0 and λ_0) is a representable morphism of functors, i.e. $\mathcal{A}_I \times_{\mathcal{A}_g} X$ is representable by a scheme for any S-scheme X and morphism $X \to \mathcal{A}_g$. As we know already that $\mathcal{A}_{g,N}$ is representable (see [12]), then if we consider the morphism $\mathcal{A}_{g,N} \to \mathcal{A}_g$ that forgets the level N structure, the base change $\mathcal{A}_{I,N} = \mathcal{A}_I \times_{\mathcal{A}_g} \mathcal{A}_{g,N}$ is representable by a scheme.

2.2 Formal power series

We use this section to give some commutative algebra results about formal series that we will use later on.

2.2.1 First we give some result about formal series over a local ring.

Proposition. *Let R a local ring with maximal ideal $\mathfrak{m}$ and residue field k, then $R[[X]]$ is again local with maximal ideal*

$$\tilde{\mathfrak{m}} = (\mathfrak{m}, X) = \left\{ f = \sum_{n=0}^{\infty} a_n X^n \in R[[X]] : a_0 \in \mathfrak{m} \right\}$$

and residue field k.

Proof. Recall that a ring is local if and only if the complement of the group of units is an ideal (and therefore the maximal ideal). It follows that $\mathfrak{m} = R \smallsetminus R^\times$ and that it is enough to prove that $\tilde{m} = R[[X]] \smallsetminus R[[X]]^\times$. It is well known that

$$R[[X]]^\times = \left\{ f = \sum_{n=0}^{\infty} a_n X^n \in R[[X]] : a_0 \in R^\times \right\},$$

hence

$$R[[X]] \smallsetminus R[[X]]^\times = \left\{ f = \sum_{n=0}^{\infty} a_n X^n \in R[[X]] : a_0 \in \mathfrak{m} \right\} = (\mathfrak{m}, X).$$

Now it is an easy observation that $R[[X]]/\tilde{\mathfrak{m}} = R[[X]]/(\mathfrak{m}, X) \cong R/\mathfrak{m} = k$. $\square$

Lemma. *Let R local ring with maximal ideal $\mathfrak{m}$ and let $\tilde{\mathfrak{m}}$ denote the maximal ideal of $R[[X]]$. Then*

$$\tilde{\mathfrak{m}}^k = \left\{ f \in \sum_{n=0}^{\infty} a_n X^n : a_n \in \mathfrak{m}^{k-n} \text{ for any } n = 0, \dots k-1 \right\}.$$

Proof. $\tilde{\mathfrak{m}}^k = (\mathfrak{m}, X)^k = (\mathfrak{m}^k, \mathfrak{m}^{k-1}X, \mathfrak{m}^{k-2}X^2, \dots, X^k)$ hence the inclusion $\supseteq$ is trivial. For the other inclusion let $f = \sum_{n=0}^{\infty} a_n X^n \in \tilde{\mathfrak{m}}$, then $f = \sum_{i=0}^{k} b_i X^{k-i} g_i$, with $b_i \in \mathfrak{m}^i$, $g_i \in R[[X]]$; if we develop the product and we reorder the terms we may take $g_i = 1$ for any $i < k$, up to changing b_i with another coefficient $b_i' \in \mathfrak{m}^{k-i}$. Therefore $a_i = b_i$ for $i = 0, \dots, k-1$ and the thesis follows. $\square$

Lemma. *Let R be a local ring with maximal ideal $\mathfrak{m}$ and let $\tilde{\mathfrak{m}}$ denote the maximal ideal of $R[[X]]$. Then there is an isomorphism of k-vector spaces*

$$\frac{\tilde{\mathfrak{m}}^k}{\tilde{\mathfrak{m}}^{k+1}} \cong \frac{\mathfrak{m}^k}{\mathfrak{m}^{k+1}} \oplus \frac{\mathfrak{m}^{k-1}}{\mathfrak{m}^k} X \oplus \dots \oplus kX^k.$$

Proof. From the explicit description of the powers of $\mathfrak{m}$ above it follows that we have a surjective map

$$\tilde{\mathfrak{m}}^k \twoheadrightarrow \frac{\mathfrak{m}^k}{\mathfrak{m}^{k+1}} \oplus \frac{\mathfrak{m}^{k-1}}{\mathfrak{m}^k} X \oplus \dots \oplus kX^k$$

of abelian groups with kernel $\tilde{\mathfrak{m}}^{k+1}$. Hence we get the isomorphism of the statement, a priori of abelian groups, but it's easy to check that it respects the k-vector space structure. $\square$

2.2.2 The next results are about formal series over complete local rings. In this class of rings we may evaluate any $f = \sum_{n=0}^{\infty} a_n X^n \in R[[X]]$ at any point $x \in \mathfrak{m}$, in fact the series $\sum_{n=0}^{\infty} a_n x^n$ converges in the $\mathfrak{m}$-adic topology, we denote its limit by $f(x)$.

Proposition. *Let R a complete local ring with maximal ideal $\mathfrak{m}$ and $x \in \mathfrak{m}$. If $f \in R[[X]]$ is such that $f(x) = 0$, then there is $g \in R[[X]]$ such that $f = (X - x)g$.*

Proof. Writing $f = \sum_{n=0}^{\infty} a_n X^n$, $g = \sum_{n=0}^{\infty} b_n X^n$ the statement is equivalent to the existence of a sequence $(b_n)_n$ such that $a_0 = -xb_0$ and $a_n = b_{n-1} - xb_n$ for any $n > 0$. As $f(x) = 0$, then $a_0 = -xh_0(x)$, where $h_0 = \sum_{n=0}^{\infty} a_{n+1} X^n$. Define therefore $b_0 = h_0(x)$. Moreover since $h_0 = a_1 + Xh_1$, where $h_1(X) = \sum_{n=0}^{\infty} a_{n+2} X^n$, if we define $b_1 = h_1(x)$, then $b_0 = a_1 + xb_1$, i.e. $a_1 = b_0 - xb_1$, as required. We proceed by induction. $\square$

Note that we may interpret this result in these terms: the kernel of the evaluation at any $x \in \mathfrak{m}$ is the principal ideal generated by $(X - x)$, hence $\mathrm{ev}_x \colon R[[X]]/(X - x) \xrightarrow{\sim} R$.

2.2.3 The main result of this section is the following

Theorem. *Let R_1, R_2 be complete noetherian local rings. Suppose that there exist an isomorphism $R_1[[t]] \cong R_2[[s]]$, then $R_1 \cong R_2$*

We need first a couple of technical lemmas:

Lemma. *Let A, B be noetherian local rings with maximal ideals respectively $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Let $\varphi \colon A \to B$ a local homomorphism and suppose that it induces isomorphisms $\bar{\varphi} \colon A/\mathfrak{m}_1 \xrightarrow{\sim} B/\mathfrak{m}_2$ on the residue fields and $\bar{\varphi} \colon \mathfrak{m}_1/\mathfrak{m}_1^2 \xrightarrow{\sim} \mathfrak{m}_2/\mathfrak{m}_2^2$ on the cotangent spaces. Then φ is surjective.*

Proof. Consider a finite (by noetherianity) set of elements $x_1, \dots, x_n \in \mathfrak{m}_1$ such that their image in $\mathfrak{m}_1/\mathfrak{m}_1^2$ form a basis of it: by Nakayama's lemma (as $\mathfrak{m}_1$ finitely generated) they generate $\mathfrak{m}_1$. Let $y_i = \varphi(x_i)$, then the y_i's generate $\mathfrak{m}_2$: in fact φ induces an isomorphism on cotangent spaces hence if we consider the image of the y_i's in $\mathfrak{m}_2/\mathfrak{m}_2^2$ they form a basis of it and therefore they generate $\mathfrak{m}_2$, again by Nakayama. Fix a set of representatives $\{0, \tau_i\}_i$ of $A/\mathfrak{m}_1$. Then, since φ induces a bijection on residue fields, $\{0, \eta_i\}_i$ gives a set of representative of $B/\mathfrak{m}_2$, with $\eta_i = \varphi(\tau_i)$. By the completeness we may expand any element $y \in B$ as $y = \sum_{\nu \in \mathbb{N}^n} b_\nu y_1^{\nu_1} \dots y_n^{\nu_n}$, with $b_\nu \in \{0, \eta_i\}$. Hence there exists a_ν such that $\varphi(a_\nu) = b_\nu$ and therefore if we define $x = \sum_{\nu \in \mathbb{N}^n} a_\nu x^{\nu_1} \dots x^{\nu_n}$, then $\varphi(x) = y$. □

Lemma. *Let K be a field and $f\colon K^n \to K^n \oplus K$ an injective maps. Then there exist a morphism $g = (\mathbb{1}|a)\colon K^n \oplus K \to K^n$ such that $g \circ f$ is an isomorphism.*

Proof. As f is injective, the images of $e_1, \dots, e_n$ are linearly independent. We may therefore complete them to a bases $\{f(e_1), \dots, f(e_n), v + x\}$ of $K^n \oplus K$, with $v \in K^n$ and $x \in K$. Define $a = -v/x$, hence $\ker g = \langle v + x \rangle$ (since $g = (\mathbb{1}|a)$ has rank n or in other terms the kernel is 1-dimensional) and therefore $\langle f(e_1), \dots, f(e_n) \rangle \cap \ker g = \langle 0 \rangle$. Hence $g\big(f(e_1)\big), \dots, g\big(f(e_1)\big)$ are linearly independent: it follows that $g \circ f$ is injective and hence by dimension considerations an isomorphism. □

Note that one can see this lemma as an instance of the more general property: any short exact sequence of K-vector spaces, for K a field, splits.

Proof. (proposition) First note that R_1 and R_2 have the same residue field, that we denote k, by 2.2.1. Note moreover that by the decomposition of $\tilde{\mathfrak{m}_1}^n/\tilde{\mathfrak{m}_1}^{n+1}$ and $\tilde{\mathfrak{m}_2}^n/\tilde{\mathfrak{m}_2}^{n+1}$described above we have that

$$\dim_k\left(\frac{\mathfrak{m}_1^n}{\mathfrak{m}_1^{n+1}}\right) = \dim_k\left(\frac{\tilde{\mathfrak{m}_1}^n}{\tilde{\mathfrak{m}_1}^{n+1}}\right) - \dim_k\left(\frac{\tilde{\mathfrak{m}_1}^{n-1}}{\tilde{\mathfrak{m}_1}^{n}}\right) =$$

$$= \dim_k\left(\frac{\tilde{\mathfrak{m}_2}^n}{\tilde{\mathfrak{m}_2}^{n+1}}\right) - \dim_k\left(\frac{\tilde{\mathfrak{m}_2}^{n-1}}{\tilde{\mathfrak{m}_1}^{n}}\right) = \dim_k\left(\frac{\mathfrak{m}_2^n}{\mathfrak{m}_2^{n+1}}\right)$$

and by noetherianity these dimensions are finite.

Let's consider now the map

$$\varphi\colon R_1 \to R_1[[t]] \xrightarrow{\sim} R_2[[s]] \to R_2[[s]]/(s - x) \xrightarrow{\sim} R_2,$$

where the last map is the evaluation ev_x for some $x \in \mathfrak{m}$. Note that φ is a local morphism: if $a \in \mathfrak{m}_1$ then correspond to some $b + sg \in \tilde{\mathfrak{m}_2}$, i.e. $b \in \mathfrak{m}_2$, but then $b + xg(x) \in \mathfrak{m}_2$. In particular φ induces a k-linear map on the cotangent spaces

$$\bar{\varphi}\colon \frac{\mathfrak{m}_1}{\mathfrak{m}_1^2} \to \frac{\mathfrak{m}_1}{\mathfrak{m}_1^2} \oplus kt \xrightarrow{\sim} \frac{\mathfrak{m}_2}{\mathfrak{m}_2^2} \oplus ks \to \left(\frac{\mathfrak{m}_2}{\mathfrak{m}_2^2} \oplus ks\right)/(s - \bar{x}) \xrightarrow{\sim} \frac{\mathfrak{m}_2}{\mathfrak{m}_2^2},$$

where $\bar{x}$ is the reduction mod $\mathfrak{m}_2^2$ of x. Suppose now to be able to find an x such that $\bar{\varphi}$ is an isomorphism: the first lemma above implies the surjectivity of φ (as it is clear that it induces an isomorphism on the residue fields). Let's now consider the map induced on the associated graded rings:

$$G(\varphi)\colon G(R_1) = \bigoplus_{n=0}^{\infty} \mathfrak{m}_1^n/\mathfrak{m}_1^{n+1} \to \bigoplus_{n=0}^{\infty} \mathfrak{m}_2^n/\mathfrak{m}_2^{n+1} = G(R_2)$$

this is an isomophism since any $\bar{\varphi}_n \colon \mathfrak{m}_1^n/\mathfrak{m}_1^{n+1} \to \mathfrak{m}_2^n/\mathfrak{m}_2^{n+1}$ is surjective (as φ was surjective) and therefore an isomorphism since the two vector spaces have the same dimensions. By lemma 10.23 of [1] and the completeness of R_1, R_2 we conclude that φ is an isomorphism.

Hence the only thing missing is to find an $x \in \mathfrak{m}_2$ such that $\bar{\varphi}$ is an isomorphism. The existence of this element follows by the second lemma above: in our case f is the map $\frac{\mathfrak{m}_1}{\mathfrak{m}_1^2} \to \frac{\mathfrak{m}_2}{\mathfrak{m}_2^2} \oplus ks$ and if we define $\bar{x} = g(0,1)$, then $g = \mathrm{ev}_{\bar{x}}$, so that $g \circ f = \bar{\varphi}$. Any lift x of $\bar{x}$ therefore does the job. □

2.3 Systems of modules

2.3.1 Let R be a ring. We introduce now the notion of system of R-modules of type II. The "II" in the name is justified since this notion is a generalization of the corresponding one defined in [10], where this arise as a natural name.

Definition. Let I be a parahoric level structure. A system of R-modules of type II for I is a family $(M_i)_{i \in I'}$ of locally free R-modules of rank $2g$ together with a chain $(\alpha_i)_{i \in I'}$ of R-linear maps

$$\xrightarrow{\alpha_{2g-i_r}} M_{i_r} \xrightarrow{\alpha_{i_r}} M_{i_{r-1}} \xrightarrow{\alpha_{i_{r-1}}} \dots \xrightarrow{\alpha_{i_2}} M_{i_1} \xrightarrow{\alpha_{i_1}} M_{i_0} \xrightarrow{\alpha_{i_0}} M_{-i_0} \xrightarrow{\alpha_{-i_0}}$$

and a class $(\langle \cdot,\cdot \rangle_i)_{i \in I'}$ of "alternating" perfect pairings $\langle \cdot,\cdot \rangle_i \colon M_i \times M_{-i} \to R$, in the sense that $\langle x,y \rangle_i = -\langle y,x \rangle_{-i}$ for any $x \in M_i, y \in M_{-i}$, such that

i for any $i \in I'$ there exist an isomorphism $\varepsilon_i \colon M_{i+2g} \xrightarrow{\sim} M_i$, compatible with the chain of maps and the family of pairings;

ii $\langle \alpha_j(x), y \rangle_i = \langle x, \alpha_{-i}(y) \rangle_j$ for any $i \in I'$, $j = \mathrm{succ}(i)$ and $x \in M_j$, $y \in M_{-i}$;

iii $p \operatorname{coker}(M_j \to M_i) = 0$, i.e. the R-module structure on $\operatorname{coker}(M_j \to M_i)$ induce a well defined module structure over R/pR, and $\operatorname{coker}(M_j \to M_i)$ is locally free R/pR-module of rank $\mathrm{d}(i,j)$, for any $i \in I'$, $j = \mathrm{succ}(i)$;

iv for any $i \in I'$ the composition $M_i \xrightarrow{\alpha} \dots \xrightarrow{\alpha} M_{i-2g} \cong M_i$ is the multiplication by p map.

2.3.2 The conditions *ii.* and *iii.* in this definition deserves some more explanations.

Remark. Let $i \in I'$ and $j = \mathrm{succ}(i)$. Let moreover $\mathfrak{p} \in \operatorname{Spec} R$ and consider the localization $\alpha_{j,\mathfrak{p}}$ of $\alpha_j \colon M_j \to M_i$, then there are two possibilities:

- $\operatorname{coker}(\alpha_{j,\mathfrak{p}})$ is free of rank $d(i,j)$ over $R_\mathfrak{p}/pR_\mathfrak{p}$, if $R_\mathfrak{p}$ has residue characteristic p,
- $\alpha_{j,\mathfrak{p}}$ is an isomorphism, if $R_\mathfrak{p}$ has residue characteristic different from p.

In fact if the residue characteristic of $R_\mathfrak{p}$ is different from p, then $p \neq 0$ in $\kappa(\mathfrak{p})$ and therefore $\mathfrak{p} \in R_\mathfrak{p}^\times$. Hence $p \operatorname{coker}(\alpha_{j,\mathfrak{p}}) = 0$ implies $\operatorname{coker}(\alpha_{j,\mathfrak{p}}) = 0$ (note that the rank condition in this case is an empty condition), i.e. $\alpha_\mathfrak{p}$ surjective. But in general a surjective map between finite free modules of same rank is an isomorphism (the proof uses a Nakayama's lemma and a splitting exact sequences argument as in 1.4.2).

We can moreover state *ii.* in a slight different way using the isomorphisms $\lambda_i \colon M_i \xrightarrow{\sim} M_{-i}^\vee$ induced for any $i \in I'$ by the perfect pairings $\langle \cdot,\cdot \rangle_i$, i.e. given by $x \mapsto \langle x, \cdot \rangle$ for any $x \in M_i$.

Lemma 2.3.1. *The the condition ii. in the above definition is equivalent to the commutativity of the following diagram*

$$\begin{array}{ccccccccccccc}
\longrightarrow & M_{2g-i_r} & \longrightarrow & M_{i_r} & \longrightarrow \dots \longrightarrow & M_{i_1} & \longrightarrow & M_{i_0} & \longrightarrow & M_{-i_0} & \longrightarrow \\
& \cong \downarrow \lambda_{2g-i_r} & & \cong \downarrow \lambda_{i_r} & & \cong \downarrow \lambda_{i_1} & & \cong \downarrow \lambda_{i_0} & & \cong \downarrow \lambda_{-i_0} & \\
\longrightarrow & M^\vee_{i_r-2g} & \longrightarrow & M^\vee_{-i_r} & \longrightarrow \dots \longrightarrow & M^\vee_{-i_1} & \longrightarrow & M^\vee_{-i_0} & \longrightarrow & M^\vee_{i_0} & \longrightarrow
\end{array}$$

where the top row uses the α_i's as maps (unless if $i_0 = 0$ or $i_r = g$, in such a case we use the identity respectively for the maps $M_0 \to M_0$ and $M_g \to M_g$) and the bottom row is obtained dualizing the top one.

Proof. Take a pair i, j of consecutive indexes. The commutativity of the square

$$\begin{array}{ccc}
M_j & \xrightarrow{\alpha_j} & M_i \\
\lambda_j \downarrow \cong & & \cong \downarrow \lambda_i \\
M^\vee_{-j} & \xrightarrow{\alpha^\vee_{-i}} & M_{-i}
\end{array}$$

is equivalent to $\lambda_i\big(\alpha_j(x)\big) = \alpha^\vee_{-i}\big(\lambda_j(x)\big) = \lambda_j(x) \circ \alpha_{-i}$ for any x in M_j, or in other terms that $\langle \alpha_j(x), y\rangle_i = \langle x, \alpha_{-i}(y)\rangle_j$ for any $x \in M_j$, $y \in M_{-i}$. □

2.3.3 By induction (or looking at the previous equivalent characterization) is trivial to drop the hypothesis $j = \mathrm{succ}(i)$ in *ii*, as we state in the following lemma. A notational remark for the next lemma and for the sequel: sometimes we will drop the lower indexes and we will denote by α also the compositions of the α_i's, when it is clear from the context which is the source and which the target of them. Some other times, to be more precise, we will use $\alpha^{\mathrm{d}(i,j)}$ for the composition $M_j \to \cdots \to M_i$.

Lemma. *Let $M_\bullet$ be a system of R-modules of type II for I. Let $i, j \in I'$, with $i < j$, and let $x \in M_j$, $y \in M_{-i}$. Therefore $\langle \alpha^{\mathrm{d}(i,j)}(x), y\rangle_i = \langle x, \alpha^{\mathrm{d}(-j,-i)}(y)\rangle_j$.*

A similar discussion holds for *iii.*

Lemma. *Let $M_\bullet$ be a system of R-modules of type II for I, let $i, j \in I'$, with $i < j \leq i + 2g$ and consider the map $\alpha^{\mathrm{d}(i,j)} \colon M_j \to M_i$. Then $p\,\mathrm{coker}(M_j \to M_i) = 0$, i.e. the R-module structure on $\mathrm{coker}(M_j \to M_i)$ induce a well defined module structure over R/pR, and $\mathrm{coker}(M_j \to M_i)$ is locally free R/pR-module of rank $\mathrm{d}(i,j)$.*

Proof. It's enough to check the condition for R local and by the previous remark only in the case of $\mathrm{char}(R) = p$.

Suppose then this condition for any pair of consecutive indexes and take two arbitrary indexes i, j of I' such that $i < j \leq i + 2g$ and enumerate all the middle indexes: $i = k_0, \dots k_n, \dots, k_t = j$; write $\mathrm{d}_n = \mathrm{d}(i, k_n)$.

We prove the claim by induction on n: the base case $n = 1$ is the assumption, suppose then that $\mathrm{coker}(M_{k_n} \to M_{k_0}) = M_{k_0}/\alpha^{d_n}(M_{k_n})$ is free of rank d_n; what we need to prove is that $\mathrm{coker}(M_{k_n+1} \to M_{k_0}) = M_{k_0}/\alpha^{\mathrm{d}_{n+1}}(M_{k_n+1})$ is free of rank d_{n+1}. Consider the exact sequence

$$0 \to \frac{\alpha^{\mathrm{d}_n}(M_{k_n})}{\alpha^{\mathrm{d}_{n+1}}(M_{k_{n+1}})} \to \frac{M_{k_0}}{\alpha^{\mathrm{d}_{n+1}}(M_{k_{n+1}})} \to \frac{M_{k_0}}{\alpha^{\mathrm{d}_n}(M_{k_n})} \to 0,$$

it splits as $M_{k_0}/\alpha^{\mathrm{d}_n}(M_{k_n})$ is free (of rank d_n), hence projective. Note moreover that the module $\alpha^{\mathrm{d}_n}(M_{k_n})/\alpha^{\mathrm{d}_{n+1}}(M_{k_{n+1}})$ is isomorphic to $\bar{\alpha}^{\mathrm{d}_n}\big(M_{k_n}/\alpha(M_{k_{n+1}})\big)$, where $\bar{\alpha}$ is the map induced by α on the cokernels, hence it has d_{n+1} generators (possibly not linearly independent). Hence it is a finite projective modules over a local ring, and therefore free.

So far we have shown that $M_{k_0}/\alpha^{\mathrm{d}_{n+1}}(M_{k_{n+1}})$ is free of rank at most d_{n+1}. Let us see now why the rank is exactly d_{n+1}. Note that by the condition *iv.* of the definition of system of modules $M_{k_0}/\alpha^{2g}(M_{k_0+2g}) = M_{k_0}/pM_{k_0} \cong (R/pR)^{2g}$ and consider the split exact sequence

$$0 \to \frac{\alpha^{\mathrm{d}_{n+1}}(M_{k_{n+1}})}{\alpha^{2g}(M_{k_0+2g})} \to \frac{M_{k_0}}{\alpha^{2g}(M_{k_0+2g})} \to \frac{M_{k_0}}{\alpha^{\mathrm{d}_{n+1}}(M_{k_{n+1}})} \to 0.$$

By a similar argument as above $\alpha^{\mathrm{d}_{n+1}}(M_{k_{n+1}})/\alpha^{2g}(M_{k_0+2g})$ has rank at most $2g-\mathrm{d}_{n+1}$, but since the middle term has rank $2g$ the only possibility is that the two modules on the side have exactly rank $2g-\mathrm{d}_{n+1}$ and d_{n+1} respectively. □

Remark. Looking carefully at the previous proof we can write explicitly a basis for an M_i in the case R local and $\mathrm{char}(R) = p$. With the notation as in the above proof choose a set $\{\,\bar{\eta}_{d_n}, \ldots, \bar{\eta}_{\mathrm{d}_{n+1}-1}\,\}$ of free generators of $\mathrm{coker}(M_{k_{n+1}} \to M_{k_n})$ and for any such an element choose a lift η_k to M_h (with h the only index in I' such that $h \le k < \mathrm{succ}(h)$): the images of the η_k's in M_i form a basis of it as free R-module.

In fact the previous proof shows that the images of the $\bar{\eta}_k$'s in $\mathrm{coker}(\alpha^{2g}) = M_i/pM_i$ form a basis of it over R/pR. Note that M_i finitely generated (direct summand of a finite free module, hence quotient of it) and $p \in \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal, hence we may apply Nakayama's lemma to show that the images of the η_k's in M_i form a basis of it as free R-module.

Example 2.3.2. Let's give an explicit example of what we remarked above.

Let $g = 3$ and $I = \{\,1, 2\,\}$ and assume that $M_i = M_{i+6}$ (but the assumption is not very restrictive). We get a system of modules of the shape

$$\to M_7 = M_1 \to M_5 = M_{-1} \to M_4 = M_{-2} \to M_2 \to M_1 \to M_{-1} \to M_{-2} \to \quad .$$

Let's choose $\eta_1 \in M_1$, $\eta_2, \eta_3 \in M_2$, $\eta_4 \in M_{-2}$, $\eta_5, \eta_0 \in M_{-1}$ such that their reductions give a basis respectively for $\mathrm{coker}(M_2 \to M_1), \mathrm{coker}(M_{-2} \to M_2), \mathrm{coker}(M_{-1} \to M_{-2})$, $\mathrm{coker}(M_1 \to M_{-1})$. Hence M_1 has a basis over R given by

$$\{\,\alpha^4(\eta_0), \eta_1, \alpha(\eta_2), \alpha(\eta_3), \alpha^3(\eta_4), \alpha^4(\eta_5)\,\}\,.$$

By the periodicity conditions in the definition of system of modules the choice of the η_k's that we made is a suitable choice also for all the others M_i's. Hence the sets

$$\begin{gathered}\{\,\alpha^3(\eta_0), \alpha^5(\eta_1), \eta_2, \eta_3, \alpha^2(\eta_4), \alpha^3(\eta_5)\,\}\,,\\ \{\,\eta_0, \alpha^2(\eta_1), \alpha^3(\eta_2), \alpha^3(\eta_3), \alpha^5(\eta_4), \eta_5\,\}\,,\\ \{\,\alpha(\eta_0), \alpha^3(\eta_1), \alpha^4(\eta_2), \alpha^4(\eta_3), \eta_4, \alpha(\eta_5)\,\}\end{gathered}$$

are bases respectively for M_2, M_{-1}, M_{-2}.

2.3.4 In the sequel we will need some further definitions about system of modules:

Definition. Let I a parahoric level structure, $M_\bullet = (M_i, \alpha_i, \langle\cdot,\cdot\rangle_i)_{i\in I'}$ and $N_\bullet = (N_i, \beta_i, [\cdot,\cdot]_i)_{i\in I'}$ two system of R-modules of type II for I.

A morphism $M_\bullet \to N_\bullet$ of system of R-modules of type II is a collection of R-linear maps $(\eta_i \colon M_i \to N_i)_{i\in I'}$ compatible with the two pairings and the two chains of maps (and periodicity isomorphisms).

Remark. Note that in particular the compatibility with the periodic isomorphisms correspond to the condition that for any $i \in I'$ the diagram

$$\begin{array}{ccccccc} M_i & \xrightarrow{\alpha} & \dots & \xrightarrow{\alpha} & M_{i-2g} & \xrightarrow{\cong} & M_i \\ \downarrow{\scriptstyle \eta_i} & & & & \downarrow{\scriptstyle \eta_{i-2g}} & & \downarrow{\scriptstyle \eta_i} \\ N_i & \xrightarrow{\alpha} & \dots & \xrightarrow{\alpha} & N_{i-2g} & \xrightarrow{\cong} & N_i \end{array}$$

commutes.

Definition. Let I be a parahoric level structure, S' a scheme. A system of $\mathcal{O}_{S'}$-modules of type II for I is a family $(\mathcal{M}_i)_{i \in I'}$ of locally free $\mathcal{O}_{S'}$-modules of rank $2g$ endowed with a chain of maps and alternating pairings such that locally on any affine open $\operatorname{Spec} R$ of S the chain give rise to a system of R-modules of type II. A morphism of system of S'-modules is a collection of isomorphism compatible with the chain of maps and the alternating pairings.

2.3.5 Consider the following example of system of R-module of type II.

Example. Let I be a parahoric level structure and R any $\mathbb{Z}_p$-algebra; consider the collection of submodules $(\Lambda_{-i,R})_{i \in I'}$ with connecting linear maps and pairings as described in 1.3. The properties described there ensure that it gives rise to a system of R-modules of type II for I, that we denote as $\Lambda_{-\bullet,R}$.

The following proposition says that if R is a local ring this is the only possible example.

Proposition. *Let I be a parahoric level structure, R be a $\mathbb{Z}_p$-algebra and and $M_\bullet$ a system of R-modules of type II for I. If R is a local ring there is an isomorphism $M_\bullet \xrightarrow{\sim} \Lambda_{-\bullet,R}$.*

Proof. The proof is a tricky but elementary calculation that relies on the fact that, as R is local, any M_i is free. Let's divide the two cases: first suppose that R has residue characteristic different from p. In this case the α_i's are isomorphisms hence the choice of a basis of M_{i_0} induces a choice of a basis on all the others M_i's and hence a family of isomorphisms $\eta_i \colon M_i \to \Lambda_{-i,R}$ compatible with the two chains of maps. Moreover for any $i \in I'$ the diagram

$$\begin{array}{ccccccc} M_i & \underset{\cong}{\xrightarrow{\alpha}} & \dots & \xrightarrow{\alpha} & M_{i-2g} & \xrightarrow{\cong} & M_i \\ \downarrow{\scriptstyle \eta_i} & & & & \downarrow{\scriptstyle \eta_{i-2g}} & & \downarrow{\scriptstyle \eta_i} \\ \Lambda_{-i,R} & \xrightarrow{\alpha} & \dots & \xrightarrow{\alpha} & \Lambda_{2g-i,R} & \xrightarrow{\cong} & \Lambda_{-i,R} \end{array}$$

is commutative by the condition *iv* in the definition of system of modules: take $x \in M_i$, it has image $\eta_i(px)$ following one side of the square and $p\eta_i(x)$ following the other side; they are equal as η_i is R-linear. This proves the compatibility with the periodicity isomorphisms.

To prove that the η_i's are compatible with the pairings it is enough to show (by condition *ii* in the definition) that we may chose a basis of M_{i_0} such that if we endow M_{-i_0} with the corresponding basis via α, than $\langle \cdot, \cdot \rangle_{i_0}$ has matrix $G = \left(\begin{smallmatrix} \mathbb{0} & J \\ -J & \mathbb{0} \end{smallmatrix}\right)$, where $J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$. By some easy matrix computation one can show that for any perfect pairing $R^{2g} \times R^{2g} \to R$ it is possible to find a change of basis $\eta \colon R^{2g} \xrightarrow{\sim} R^{2g}$ such that the pairing has G as matrix. Let's now fix any basis of M_{i_0} and the corresponding on M_{-i_0}, this gives two isomorphisms $\gamma_{i_0} \colon M_{i_0} \xrightarrow{\sim} R^{2g}$ and

$\gamma_{-i_0}\colon M_{-i_0} \xrightarrow{\sim} R^{2g}$. This induces via the isomorphisms a pairing $R^{2g} \times R^{2g} \to R$ and fix η as above. Note that the diagram

$$\begin{array}{ccccc} M_{i_0} & \xrightarrow[\cong]{\gamma_{i_0}} & R^{2g} & \xrightarrow[\cong]{\eta} & R^{2g} \\ \cong \downarrow \alpha & & \| & & \| \\ M_{-i_0} & \xrightarrow[\cong]{\gamma_{-i_0}} & R^{2g} & \xrightarrow[\cong]{\eta} & R^{2g} \end{array}$$

commutes, hence $\eta \circ \gamma_{i_0}$ corresponds to the choice of a basis on M_{i_0} such that the corresponding basis via α corresponds to $\eta \circ \gamma_{-i_0}$, hence the matrix of $\langle \cdot, \cdot \rangle_{i_0}$ in this bases is G.

Let's now discuss the case of R of residue characteristic p. The case for $I = \{ 1, \dots, g \}$ can be found in [10] (proposition 3.6). The general proof is a sight generalization of this latter.

Let's choose η_k, for $k = 0, \dots, 2g-1$ as in the remark in 2.3.3. If we assume (without loss of generality) that the chain is $2g$ periodic (i.e. $M_{i+2g} = M_i$ and the ε_i's are always the identity), it is clear that the example 2.3.3 has nothing special: in general the images of the η_k's via the α_i's in the various M_i give bases of them. Moreover if the numbering of the η_k's follows the pattern of the example it is clear that such a choice of bases gives a sequence of $2g$-periodic isomorphisms $\nu_i\colon M_i \xrightarrow{\sim} \Lambda_{-i,R}$ such that the α's are compatible with the standard chain of maps on $\Lambda_{-\bullet,R}$, e.g. in our special example $\alpha^4(\eta_0), \alpha^4(\eta_5) \in M_1$ are mapped respectively to $\alpha^6(\eta_0) = p\eta_0$, and $\alpha^6(\eta_5) = p\eta_5$, hence the matrix of $\alpha\colon M_1 \to M_{-1}$ in the two chosen bases becomes

$$\begin{pmatrix} p & & & \\ & 1 & & \\ & & \ddots & \\ & & & p \end{pmatrix}.$$

Let's discuss now the compatibility of the pairings. By the condition *ii.* of the definition of system of modules if we know that the matrix of $\langle \cdot, \cdot \rangle_{i_1}$ is $G = \left(\begin{smallmatrix} \mathbb{O} & J \\ -J & \mathbb{O} \end{smallmatrix} \right)$, then the same hold for any other $\langle \cdot, \cdot \rangle_i$, hence the ν_i's define an isomorphism of system of modules $\nu\colon M_\bullet \xrightarrow{\sim} \Lambda_{-\bullet}$.

Hence it is only left to prove that we may change the choice of the η_k's in such a way that the matrix $\langle \cdot, \cdot \rangle_{i_0}$ is G. A notational remark: in the following we will we drop not only the any indexes from α, but also the exponent. This does not lead to any confusion: depending on the different parahoric level structure is possible to recover the correct number of iterations needed. We will do an exception only for α^{2g}, that correspond to the multiplication by p.

Suppose first to have fixed a choice of the η_k's, we get the corresponding bases on M_{i_0} and M_{-i_0}, that we denote as $\{ v_0, \dots, v_{2g-1} \}$ and $\{ w_0, \dots, w_{2g-1} \}$. Write the matrix of $\langle \cdot, \cdot \rangle_{i_0}$ as $\left(\begin{smallmatrix} C & A \\ -{}^tA & D \end{smallmatrix} \right)$, by $B = (b_{i,j})_{i,j=0,\dots,g-1}$ the inverse of $A = (a_{i,j})_{i,j=0,\dots,g-1}$ (that exists as the paring is perfect). We will show the claim in 3 steps.

1. In this step we will change inductively $\eta_0, \dots, \eta_{g-1}$ in such a way that $C = \mathbb{O}$ in this new matrix.

   Let $C = (c_{k,l})_{k,l=0,\dots,g-1}$ and by c_k the k-th row of C. Suppose that $c_{k,l} = \langle v_k, w_l \rangle = 0$ for all $k = 0, \dots, K-1$ and $l = 0, \dots, g$ and define

$$\eta_K^{\text{new}} = \eta_K + \sum_{l,m=0}^{g-1} c_{k,l} b_{m,l}\, \alpha(\eta_{g+m}).$$

   This means that $v_K^{\text{new}} = \left(\begin{smallmatrix} e_k \\ B\,{}^tc_k \end{smallmatrix} \right)$, where e_k denote the column with 1 in the k-th place, 0 else.

Therefore for $l \neq K$

$$c_{K,l}^{\text{new}} = \langle v_K^{\text{new}}, w_l \rangle = \begin{pmatrix} {}^t e_K & c_K\, {}^t B \end{pmatrix} \begin{pmatrix} C & A \\ -\,{}^t A & D \end{pmatrix} \begin{pmatrix} e_l \\ \mathbb{O} \end{pmatrix} = (c_k - c_k\, {}^t B\, {}^t A) e_l = 0$$

hence with this new choice of η_K we have that $c_{k,l} = \langle v_k, w_l \rangle = 0$ for all $k = 0, \ldots, K$ and $l = 0, \ldots, g$.

2. Similarly as above we may change inductively $\eta_g, \ldots, \eta_{2g-1}$ in such a way that $D = \mathbb{O}$. In this case $v_{g+K}^{\text{new}} = \left(\begin{smallmatrix} -\,{}^t B\, {}^t d_K \\ e_k \end{smallmatrix} \right)$, where d_k denotes the k-th row of D.
3. In this third step we change again inductively $\eta_0, \ldots, \eta_g$ in such a way that $A = J$, i.e. the condition $(*)_{i,j}$ is verified, where

$$(*)_{i,j} = \begin{cases} a_{i,j} = 0 & \text{if } i + j \neq g - 1 \\ a_{i,j} = 1 & \text{if } i + j = g - 1 \end{cases} .$$

Suppose then that $C = D = \mathbb{O}$ and $(*)_{i,j}$ hold for any $i = 0, \ldots, g-1$ and $j = 0, \ldots, K-1$ and note that for $L = g - 1 - K$

$$a_{L,j} = \langle v_L, w_{g+j} \rangle = \begin{cases} 0 & \text{if } j < L \text{ (by inductive hypothesis)} \\ \langle \eta_L, \alpha^{2g}\big(\alpha(\eta_{g+j})\big) \rangle = p \langle \eta_L, \alpha(\eta_{g+j}) \rangle \in pR & \text{if } j > L \end{cases} .$$

Hence (since A is invertible and $p = 0$ modulo the maximal ideal) it follows that $a_{L,K} \in R^{\times}$.

We define therefore $u = a_{L,K}$ and

$$\begin{aligned} \eta_i^{\text{new}} &= \eta_i - u^{-1} a_{i,j} \alpha(\eta_L) \quad \text{if } i \neq L; \\ \eta_L &= u_i^{-1} \eta_L. \end{aligned}$$

The corresponding change of coordinates has matrix $\left(\begin{smallmatrix} {}^t E & \mathbb{O} \\ \mathbb{O} & \mathbb{1} \end{smallmatrix} \right)$, where

$$E = \begin{pmatrix} 1 & & & & \\ \vdots & \ddots & & & \\ u^{-1} a_{0,L} & \ldots & 1 & \ldots & u^{-1} a_{g-1,L} \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

It follows that the matrix of $\langle \cdot, \cdot \rangle_{i_0}$ in this new basis is

$$\begin{pmatrix} {}^t E & \mathbb{O} \\ \mathbb{O} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{O} & A \\ -\,{}^t A & \mathbb{O} \end{pmatrix} \begin{pmatrix} E & \mathbb{O} \\ \mathbb{O} & \mathbb{1} \end{pmatrix} = \begin{pmatrix} \mathbb{O} & {}^t E A \\ -\,{}^t A E & \mathbb{O} \end{pmatrix},$$

hence $C^{\text{new}} = D^{\text{new}} = \mathbb{O}$. Moreover the inductive hypothesis is equivalent to ask that

$$A = \begin{pmatrix} 0 & \ldots & 0 & a_{0,K} & \ldots & a_{0,g-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ & & 1 & & & \\ \vdots & \iddots & & \vdots & & \vdots \\ 1 & \ldots & & a_{g-1,K} & \ldots & a_{g-1,g-1} \end{pmatrix}$$

and therefore

$$ {}^{t}EA = \begin{pmatrix} 0 & \cdots & 0 & a'_{0,K+1} & \cdots & a'_{0,g-1} \\ \vdots & & \vdots & \vdots & & \vdots \\ & & 1 & & & \\ \vdots & \iddots & & \vdots & & \vdots \\ 1 & \cdots & & a'_{g-1,K+1} & \cdots & a'_{g-1,g-1} \end{pmatrix}, $$

or equivalentely with the new choice the condition $(*)_{i,j}$ is fulfilled for $i = 0, \dots, g-1$ and $j = 0, \dots, K$.

This 3 steps together show that we may change the η_k still getting lifts of free generators of the cokernels and such that with the induced basis on M_{i_0} and M_{-i_0} the pairing $\langle \cdot, \cdot \rangle_{i_0}$ is G. $\square$

2.3.6 The proof of the following proposition uses the same techniques used in the proof of the proposition above, with some minor changes. We will therefore omit it. The Iwahori case is (briefly) explained in [10].

Proposition. *Let I be a parahoric level structure, R be a $\mathbb{Z}_p$-algebra and and $M_\bullet$ a system of R-modules of type II for I. Suppose that $\mathfrak{a}$ is a nilpotent ideal of R and that there exist an isomorphism $M_\bullet \otimes_R R/\mathfrak{a} \xrightarrow{\sim} \Lambda_{-\bullet, R/\mathfrak{a}}$, then we can lift it to an isomorphism $M_\bullet \xrightarrow{\sim} \Lambda_{-\bullet, R}$.*

2.3.7 We claimed that our notion of system of modules is a generalization of the corresponding one in [10]. Let's conclude this section showing that this is indeed the case.

Let therefore $I = \{ 0, \dots, g \}$ and by simplicity of notations suppose that $M_i = M_{i+2g}$, we will show that the system $M_\bullet$ is always isomorphic to the system given by the chain

$$ M_g \xrightarrow{\alpha_g} \dots \xrightarrow{\alpha_1} M_0 \xrightarrow{\lambda_0} M_0^\vee \xrightarrow{\alpha_1^\vee} M_1^\vee \xrightarrow{\alpha_2^\vee} \dots \xrightarrow{\alpha_g^\vee} M_g^\vee \xrightarrow{\lambda_g^{-1}} M_g $$

extended periodically, together with the pairings $[\cdot, \cdot]_i$ defined by

$$ [x, y]_i = \begin{cases} \langle x, y \rangle_0 & \text{if } i = 0 \\ \langle x, y \rangle_g & \text{if } i = g \\ -y(x) & \text{if } i = 1, \dots, g-1 \\ x(y) & \text{if } i = -1, \dots, -g+1 \end{cases} $$

and again extended periodically.

By the lemma in 2.3.2, the diagram

$$ \begin{array}{ccccccccccccc} \longrightarrow & M_{g+1} & \xrightarrow{\alpha_{g+1}} & M_g & \longrightarrow & \dots & \longrightarrow & M_1 & \xrightarrow{\alpha_1} & M_0 & \xrightarrow{\alpha_0} & M_{-1} & \xrightarrow{\alpha_{-1}} \\ & \lambda_{g+1} \downarrow \cong & & \| & & & & \| & & \| & & \cong \downarrow \lambda_{-1} & \\ \longrightarrow & M_{g-1}^\vee & \xrightarrow{\lambda_g^{-1} \circ \alpha_g^\vee} & M_g & \longrightarrow & \dots & \longrightarrow & M_1 & \xrightarrow{\alpha_1} & M_0 & \xrightarrow{\alpha_1^\vee \circ \lambda_0} & M_1^\vee & \xrightarrow{\alpha_2^\vee} \end{array} $$

is commutative, moreover the compatibility with the pairing follows by the definition if the λ_i's. This shows the equivalence of the two definitions in the Iwahori case.

2.4 Local model diagram

In this section we will finally describe the relation between $\mathcal{A}_{I,N}$ and the local model introduced in the previous chapter.

2.4.1 Let S be a base scheme and $\mathcal{M}$ an S-scheme.

Definition. A local model diagram for $\mathcal{M}$ is a diagram $\mathcal{M} \xleftarrow{\varphi} \tilde{\mathcal{M}} \xrightarrow{\psi} \mathcal{M}^{loc}$, where $\tilde{\mathcal{M}}$ and $\mathcal{M}^{loc}$ are two S-schemes, φ and ψ two smooth morphism of the same relative dimension and φ surjective (as a scheme morphism, i.e. as a map on underlying topological spaces).

The importance of this notion is the fact that it relates étale locally $\mathcal{M}$ with $\mathcal{M}^{loc}$: this is very useful if $\mathcal{M}^{loc}$ is a scheme of "easier nature" than $\mathcal{M}$, as we will see in the case $\mathcal{M} = \mathcal{A}_{I,N}$.

Proposition. *Let $\mathcal{M} \xleftarrow{\varphi} \tilde{\mathcal{M}} \xrightarrow{\psi} \mathcal{M}^{loc}$ be a local model diagram for $\mathcal{M}$. For any point $x \in \mathcal{M}$ choose $z \in \varphi^{-1}(x)$ and set $y = \psi(z)$. Suppose moreover that $\widehat{\mathcal{O}_{\mathcal{M},x}}, \widehat{\mathcal{O}_{\mathcal{M}^{loc},y}}$ are noetherian rings (e.g. if both $\mathcal{M}$ and $\mathcal{M}^{loc}$ are locally noetherian). Then $\widehat{\mathcal{O}_{\mathcal{M},x}} \cong \widehat{\mathcal{O}_{\mathcal{M}^{loc},y}}$.*

Proof. As φ and ψ are smooth of same relative dimension d, hence $\widehat{\mathcal{O}_{\tilde{\mathcal{M}},z}} \cong \widehat{\mathcal{O}_{\mathcal{M},x}}[[X_1, \dots, X_d]]$ and $\widehat{\mathcal{O}_{\tilde{\mathcal{M}},z}} \cong \widehat{\mathcal{O}_{\mathcal{M}^{loc},y}}[[X_1, \dots, X_d]]$ (by [6] vol.4 Prop. 17.5.3.). Hence applying inductively the theorem in 2.2.3 we get the result. □

2.4.2 Our next task is to introduce the candidate local model diagram for $\mathcal{A}_{I,N}$ over the base scheme $S = \operatorname{Spec} \mathbb{Z}_p[\zeta_n]$, chosen a fixed parahoric level structure I. To do that we need to associate to an S'-valued point $\big((A_i)_{i\in I'}, (\alpha_i)_{i\in I'}, (\lambda_i)_{i\in I'}, \eta\big) \in \mathcal{A}_{I,N}(S')$ a system of modules of type II in a canonical way.

Consider for any i the sheaf of $\mathcal{O}_{S'}$-modules $R^1(a_i)_*(\Omega^\bullet_{A_i/S'})$ that we call de Rham cohomology sheaf, where $a_i \colon A_i \to S'$ is the structure morphism of A_i and $R_1(a_i)_*$ is the first right hyperderived functor of $(a_i)_*$. Write moreover $\mathrm{H}^1_{\mathrm{dR}}(A_i/S') = \Gamma\big(S', R^1(a_i)_*(\Omega^\bullet_{A_i/S'})\big)$. If $S = \operatorname{Spec} R$, with R a noetherian ring, then $\mathrm{H}^1_{\mathrm{dR}}(A_i/S')$ coincides with the classical definition of the algebraic de Rham cohomology, as e.g. the one of [9]). In [10] (proposition 3.1) it is shown that if $S' = \operatorname{Spec} R$ is affine $\mathrm{H}^1_{\mathrm{dR}}(A_i/S')$ is a locally free R-module of rank $2g$ and that the Weil pairings on the chain $(A_i)_{i\in I}$ induce alternating pairings on $\big(\mathrm{H}^1_{\mathrm{dR}}(A_i/S')\big)_{i\in I'}$ making it into a system of R-module of type II. Globalizing this property we get that for S' arbitrary the chain $\big(R^1(a_i)_*(\Omega^\bullet_{A_i/S'})\big)_{i\in I'}$ defines a system of $\mathcal{O}_{S'}$-modules of type II.

2.4.3 From now on we denote $M^{\mathrm{loc}}_N(I) = M^{\mathrm{loc}}(I) \otimes \mathbb{Z}_p[\zeta_N]$, where $M^{\mathrm{loc}}(I)$ is the scheme introduced in section 1.4. We want to use it as $\mathcal{M}^{loc}$ in a local model diagram for $\mathcal{A}_{I,N}$ over $\operatorname{Spec} \mathbb{Z}_p[\zeta_N]$.

Moreover as $\tilde{\mathcal{M}}$ we choose the scheme representing the following functor: for any S-scheme S' the set $\tilde{\mathcal{A}}_{I,N}(S')$ is the set of (isomorphism classes of) pairs (x, γ), where $x \in \mathcal{A}_{I,N}(S')$, say $x = \big((A_i)_{i\in I}, \dots\big)$ with $(A_i)_i$ the family of abelian schemes of dimension g, and where $\gamma \colon R^1(a_i)_*(\Omega^\bullet_{A_i/S'}) \xrightarrow{\sim} \Lambda_{-\bullet} \otimes_{\mathbb{Z}_p} \mathcal{O}_{S'}$. In [10] it is proved that:

Proposition. *The functor $\tilde{\mathcal{A}}_{I,N}$ defined above is representable by an S-scheme.*

Let's then define moreover the two morphisms of the diagram: let $\varphi \colon \tilde{\mathcal{A}}_{I,N} \to \mathcal{A}_{I,N}$ to be the morphism "forgetting γ", i.e. such that on S'-valued points $(x, \gamma) \mapsto x$. The definition of ψ is slightly more involved: let $S' = \operatorname{Spec} R$ affine and consider for any $i \in I$ the Hodge submodule $\omega_{A_i/S'} = \Gamma\big(S', (a_i)_*(\Omega_{A_i/S'})\big)$ of $\mathrm{H}^1_{\mathrm{dR}}(A_i/S')$: $\omega_{A_i/S'}$ is a locally free direct summand of rank g functorial in the A_i. Write $\tilde{\gamma}$ for the isomorphism induced by γ on global sections, i.e. $\gamma' \colon \mathrm{H}^1_{\mathrm{dR}}(A_\bullet/S') \xrightarrow{\sim} \Lambda_{-\bullet,R}$. We define then ψ on S'-valued points for S' affine as by the law $(x,\gamma) \mapsto \big(\gamma'(\omega_{A_i/S'})\big)_i$. We have that

Proposition. *$\varphi\colon \tilde{\mathcal{A}}_{I,N} \to \mathcal{A}_{I,N}$ is smooth and surjective.*

Proof. The surjectivity of φ follows from the proposition in 2.3.5: in fact it is enough to check the surjectivity on K-valued points for K field, see e.g. [5], section 4.3. Since any field is a particular local ring given a point $x = \big((A_i)_{i\in I'}, \dots\big) \in \mathcal{A}_{I,N}$ we may apply the proposition for $\mathrm{H}^1_{\mathrm{dR}}(A_\bullet/S')$, for $S' = \operatorname{Spec} K$: therefore there exists an isomorphism of K-vector spaces of type II $\gamma\colon \mathrm{H}^1_{\mathrm{dR}}(A_\bullet/S') \xrightarrow{\sim} \Lambda_{-\bullet,K}$, hence a point of the form $y = (x,\gamma) \in \tilde{\mathcal{A}}_{I,N}$. By definition $\varphi(y) = x$.

The formal smoothness of φ follows from 2.3.6. Consider in fact the solid commutative square

$$\begin{array}{ccc} \operatorname{Spec} R/\mathfrak{a} & \xrightarrow{\ x\ } & \tilde{\mathcal{A}}_{I,N} \\ \big\downarrow & \overset{y}{\nearrow} & \big\downarrow \varphi \\ \operatorname{Spec} R & \longrightarrow & \mathcal{A}_{I,n} \end{array}$$

where $\mathfrak{a}$ be is an ideal of R such that $\mathfrak{a}^2 = 0$. Via Yoneda lemma x correspond to a point $(x_0,\gamma) \in \tilde{\mathcal{A}}_{I,N}(R/\mathfrak{a})$ and consequently $\varphi \circ x$ to $x_0 \in \mathcal{A}_{I,N}(R/\mathfrak{a})$; but by the commutativity of the diagram it the projection of a point $x_1 \in \mathcal{A}_{I,N}(R)$. If we consider now the de Rham cohomology system of modules corresponding to x_0 and x_1 we are exactly in the situation of the proposition: we may lift γ_0 to γ_1 such that the point $(x_1,\gamma_1) \in \tilde{\mathcal{A}}_{I,N}(R)$ projects to (x_0,γ_0). By the Yoneda lemma (x_1,γ_1) correspond to the dotted arrow y and it makes the two triangles commute.

Since moreover one can show that φ is locally of finite presentation the formal smoothness implies the smoothness (see [2] paragraph 8.4, theorem 8). ☐

The following theorem ([10], theorem 2.1) proves the formal smoothness of ψ

Theorem. *(Grothendieck-Messing) Let A a local artin ring, let $\mathfrak{a}$ be an ideal of A such that $\mathfrak{a}^2 = 0$ and X_0 an abelian scheme over $A/\mathfrak{a}$. Then there exist an abelian scheme X over A lifting X_0, i.e. $X_0 = X \times_{\operatorname{Spec} A} \operatorname{Spec} A/\mathfrak{a}$ and an equivalence of categories (note that $\mathrm{H}^1_{\mathrm{dR}}(X/A)$ is independent of the chosen lift X of X_0)*

$$\left(\textit{lifts of } X_0 \textit{ to } A\right) \longrightarrow \left(\mathcal{F} \subseteq \mathrm{H}^1_{\mathrm{dR}}(X/A) \ \textit{direct summand of rank} \dim X\right).$$

$$X' \longmapsto \omega_{X'/A} \subseteq \mathrm{H}^1_{\mathrm{dR}}(X'/A) = \mathrm{H}^1_{\mathrm{dR}}(X/A)$$

It follows that

Proposition. *$\psi\colon \tilde{\mathcal{A}}_{I,N} \to M^{\mathrm{loc}}_N(I)$ is smooth.*

For the proof we need the following refinement of the formally smooth criterion for smoothness (see [7], exposée III, theorem 3.1)

Lemma. *Let $f\colon X \to Y$ a morphism locally of finite type between locally noetherian schemes. Suppose that for any $Y' = \operatorname{Spec} A$, with A local artin ring and for any nilpotent ideal $\mathfrak{a}$ of A it is possible to fill any commutative the square of the shape*

$$\begin{array}{ccc} Y'_0 & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow f \\ Y' & \longrightarrow & X \end{array} \quad ,$$

where $Y'_0 = \operatorname{Spec} A/\mathfrak{a}$, with the dotted diagonal arrow. Then f is a smooth morphism.

Note moreover that by an inductive argument it is enough to check the above condition only for ideals $\mathfrak{a}$ such that $\mathfrak{a}^2 = 0$.

Proof. (proposition) First we show that ψ is locally of finite type: it is a formal consequence of the fact that φ is so. In fact the condition of being locally of finite type is stable under composition and if f, g are morphism of schemes such that $g \circ f$ locally of finite type, then also f is so. We already know that φ is locally of finite type and $\mathcal{A}_{I,N}$ locally of finite type over $\operatorname{Spec} \mathbb{Z}_p[\zeta_N]$, hence $\tilde{\mathcal{A}}_{I,N}$ is locally of finite type over $\operatorname{Spec} \mathbb{Z}_p[\zeta_N]$; therefore $\tilde{\mathcal{A}}_{I,N} \xrightarrow{\psi} M_N^{\mathrm{loc}}(I) \to \operatorname{Spec} \mathbb{Z}_p[\zeta_N]$ is locally of finite type hence ψ is so. Let therefore A be a local artin ring and $\mathfrak{a}$ an ideal such that $\mathfrak{a}^2 = 0$. Consider the solid square

$$\begin{array}{ccc} \operatorname{Spec} A/\mathfrak{a} & \longrightarrow & \tilde{\mathcal{A}}_{I,N} \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} A & \longrightarrow & M_N^{\mathrm{loc}}(I) \end{array} \quad ;$$

by the Yoneda lemma the top arrow correspond to a point of $\tilde{\mathcal{A}}_{I,N}(A/\mathfrak{a})$, i.e. a pair (x, γ), where $x = \big((A_i)_{i \in I'}, \dots\big) \in \mathcal{A}_{I,N}(A/\mathfrak{a})$ and $\gamma\colon \mathrm{H}^1_{\mathrm{dR}}\left(A_i/(A/\mathfrak{a})\right) \xrightarrow{\sim} \Lambda_{-i,A/\mathfrak{a}}$.

The map $\operatorname{Spec} A/\mathfrak{a} \to M_N^{\mathrm{loc}}(I)$ correspond to a point of $M_N^{\mathrm{loc}}(I)(A/\mathfrak{a})$ and since this map factors trough $\tilde{\mathcal{A}}_{I,N}$ then the chain of submodules $(\mathcal{G}_i)$ of the $\Lambda_{i,A/\mathfrak{a}}$'s correspond to the chain of Hodge submodules of the $\mathrm{H}^1_{\mathrm{dR}}\left(A_i/(A/\mathfrak{a})\right)$'s. This maps factors also through $\operatorname{Spec} A$, i.e. the chain $(\mathcal{G}_i)$ can be lifted to a chain of submodules $(\mathcal{F}_i)_i$ of the $\Lambda_{i,A}$.

By the Grothendieck-Messing theorem (and the proposition in 2.3.6) we may lift any A_i to $\tilde{A}_i$ over A, $\tilde{\gamma}$ to $\gamma\colon \mathrm{H}^1_{\mathrm{dR}}(A_i/A) \xrightarrow{\sim} \Lambda_{-i,A}$ in such a way that the Hodge submodule ω_i of $\mathrm{H}^1_{\mathrm{dR}}(A_i/A)$ correspond to $\mathcal{F}_i$ via γ. Moreover one can show that the fact that $(\mathcal{F}_i)_i \in M_N^{\mathrm{loc}}(I)$ gives the polarizations and the conditions required to make $(A_i)_i$ into a point of $\tilde{\mathcal{A}}_{I,N}(A)$: we omit this (although important) detail as goes a little bit further into the constriction of the A_i's. The previous discussion shows the existence of the dotted arrow and to the commutativity of the two triangles. By the above lemma we get therefore the smoothness. ☐

2.4.4 The outcome of all this section is that following

Proposition. *$\mathcal{A}_{I,N} \xleftarrow{\varphi} \tilde{\mathcal{A}}_{I,N} \xrightarrow{\psi} M_N^{\mathrm{loc}}(I)$ is a local model diagram for $\mathcal{A}_{I,N}$. Hence $\mathcal{A}_{I,N}$ and $M_N^{\mathrm{loc}}(I)$ are étale locally isomorphic in the sense of the proposition in 2.4.1.*

Proof. It is left to prove only that φ and ψ have the same relative dimension: see the second part of the proof of 4.6 in [10]. ☐

2.5 Examples

In this paragraph we will do some examples to illustrate the power of the local model diagram: the local model $M^{\mathrm{loc}}(I)$ is defined by some linear algebra computations, we will show that is possible to write explicitly local equations. These local equations hold also for $M_N^{\mathrm{loc}}(I)$, hence the previous discussion says that these are local equations (in the étale sense) for of $\mathcal{A}_{I,N}$. With these computations we will fill moreover the missing point in the proof of the representability of $M^{\mathrm{loc}}(I)$ (see 1.4.3)

2.5.1 We begin with an easiest nontrivial example. Let $g = 1$, $I = \{0, 1\}$ (i.e. we consider a Iwahori level structure). Let's choose the affine open chart U of $\mathrm{Grass}_{2,1,\mathbb{Z}_p} \times \mathrm{Grass}_{2,1,\mathbb{Z}_p} = \mathbb{P}^1_{\mathbb{Z}_p} \times \mathbb{P}^1_{\mathbb{Z}_p}$ such that the set of R-valued point is

$$U(R) = \left\{ \left(\begin{pmatrix} 1 \\ x \end{pmatrix}, \begin{pmatrix} y \\ 1 \end{pmatrix} \right) : x, y \in R \right\},$$

where we identify the matrixes with the free submodule that they define (see the paragraph 1.2 for the details on the charts).

U and $M^{\mathrm{loc}}(I)$ are both subfunctors of $\mathbb{P}^1_{\mathbb{Z}_p} \times \mathbb{P}^1_{\mathbb{Z}_p}$, hence we may see $U(R)$ and $M^{\mathrm{loc}}(I)(R)$ as subset of $\mathbb{P}^1_{\mathbb{Z}_p}(R) \times \mathbb{P}^1_{\mathbb{Z}_p}(R)$, for R any $\mathbb{Z}_p$-algebra. Under this identification $U(R) \cap M^{\mathrm{loc}}(I)(R)$ is the set of diagrams

$$\begin{array}{ccc} R^2 & \xrightarrow{\alpha} & R^2 \\ \uparrow & & \uparrow \\ \mathcal{F}_0 & \longrightarrow & \mathcal{F}_1 \end{array}$$

with $\alpha = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\mathcal{F}_0 = \left(\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right)$ and $\mathcal{F}_1 = \left(\begin{smallmatrix} y \\ 1 \end{smallmatrix}\right)$ (the isotropic condition is trivial for submodules of rank 1, since the pairings are alternating).

Therefore the condition $\alpha(\mathcal{F}_0) \subseteq \mathcal{F}_1$ is equivalent to the existence of $\lambda \in R$ such that

$$\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} p \\ x \end{pmatrix} = \lambda \begin{pmatrix} y \\ 1 \end{pmatrix},$$

i.e. $x = \lambda$ and $p = \lambda y$. In other words the condition gives for the R-valued point an equation $xy = p$.

It follows that $U \cap M^{\mathrm{loc}}(I) = V(xy - p) \subset \mathbb{A}^2_{\mathbb{Z}_p}$ as functors and therefore the discussion of 1.4.3 says that $V(xy - p)$ is an open affine piece of $M^{\mathrm{loc}}(I)$. If we repeat this calculations for any different chart U of $\mathbb{P}^1_{\mathbb{Z}_p} \times \mathbb{P}^1_{\mathbb{Z}_p}$ we find other local equations and we may write explicitely the gluing datum defining $M^{\mathrm{loc}}(I)$. Obviously this is needless for the local study we are interested in.

2.5.2 The following example is studied in [13] to determine the local structure of the Siegel modular threefold around its singularities.

Let $g = 2$, $I = \{1\}$. Hence $M^{\mathrm{loc}}(I)(R) = \{ \mathcal{F}_1 \subseteq \Lambda_{1,R} : \alpha^2(\mathcal{F}_1^{\perp}) \subseteq \mathcal{F}_1 \}$. This time the interesting local chart is the one where $\mathcal{F}_1$ is free with matrix

$$\begin{pmatrix} 1 & 0 \\ x & y \\ z & w \\ 0 & 1 \end{pmatrix}.$$

In fact $\mathcal{F}_1^\perp$ with respect to the standard alternating pairing has matrix

$$\begin{pmatrix} -w & y \\ 1 & 0 \\ 0 & 1 \\ z & -x \end{pmatrix},$$

hence $\alpha^2(\mathcal{F}_1^\perp) \subseteq \mathcal{F}_1$ correspond to

$$\begin{pmatrix} -w & y \\ p & 0 \\ 0 & p \\ z & -x \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & p & & \\ & & p & \\ & & & 1 \end{pmatrix} \begin{pmatrix} -w & y \\ 1 & 0 \\ 0 & 1 \\ z & -x \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ x & y \\ z & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & \eta \\ \mu & \varepsilon \end{pmatrix} = \begin{pmatrix} \lambda & -\eta \\ \lambda x + \mu y & \eta x + \varepsilon y \\ \lambda z + \mu w & \eta z + \varepsilon w \\ \mu & \varepsilon \end{pmatrix}.$$

comparing the two side we get therefore the equation $xw - zy + p = 0$, functorially on R.

2.5.3 We give one more general example. Let $g = 2$, $I = \{0, 2\}$. Then $M^{\mathrm{loc}}(I)(R)$ is the set of diagrams of the form

$$\begin{array}{ccc} R^4 & \xrightarrow{\alpha} & R^4 \\ \uparrow & & \uparrow \\ \mathcal{F}_0 & \longrightarrow & \mathcal{F}_2 \end{array}$$

and such that $\mathcal{F}_0, \mathcal{F}_2$ are totally isotropic. Once again lets choose a local chart, namely the one where

$$\mathcal{F}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ x & y \\ z & w \end{pmatrix}, \quad \mathcal{F}_1 = \begin{pmatrix} a & b \\ c & d \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore $\mathcal{F}_0 = \mathcal{F}_0^\perp$ if and only if

$$\begin{pmatrix} 1 & 0 & x & z \end{pmatrix} \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ y \\ w \end{pmatrix} = 0,$$

i.e. $w = x$. Analogously $\mathcal{F}_2^\perp = \mathcal{F}_2$ if and only if $a = d$.

The condition $\alpha(\mathcal{F}_0) \subseteq \mathcal{F}_2$ becomes

$$\begin{pmatrix} p & 0 \\ 0 & p \\ x & y \\ z & w \end{pmatrix} = \begin{pmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ x & y \\ z & w \end{pmatrix} =$$

$$\begin{pmatrix} a & b \\ c & d \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & \eta \\ \mu & \varepsilon \end{pmatrix} = \begin{pmatrix} \lambda a + \mu b & \eta a + \varepsilon b \\ \lambda c + \mu d & \eta c + \varepsilon d \\ \lambda & \eta \\ \mu & \varepsilon \end{pmatrix};$$

comparing the two side and recalling $x = w$ and $a = d$ we get the system of equations

$$\begin{cases} x = w & a = d \\ xa + zb = p & xc + za = 0 \\ ya + xb = 0 & yc + xa = p \end{cases} .$$

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Erklärung

Hiermit versichere ich, dass die vorliegende Arbeit

On the Siegel Moduli Space with Parahoric Level Structure

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Essen, den 10/07/2019

Luca Mastella