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Higher-Derivative Corrections to Extremal Black Holes and the Weak Gravity Conjecture

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Abstract

Weak Gravity Conjecture (WGC) arguments suggest that higher order derivative corrections to the black hole horizon can make the extremal black hole configurations violate the naive extremality bound of charge-to-mass ratio. These corrections could change black holes mechanics allowing them to decay through a splitting process. We analyze such a possibility considering the effects of higher derivative corrections to the entropy law.
We also show in the framework of 4 derivative Einstein–Maxwell theory that the mild form of the WGC together with the Electric–Magnetic duality implies the same conditions as unitarity, locality and positivity of the scattering amplitudes of the same theory.

Contents

Dedicated to Miriam and my family.

Introduction

Quantum field theory (QFT) is up to now the best framework we have to describe the fundamental interactions. It is capable to provide predictions which are in astonishing agreement with experimental data. For instance, the measure of the electron g-factor[1] matches perfectly the prediction of quantum electrodynamics (QED) obtained considering $O(\alpha^4)$ contributions which involve 891 eight-order Feynman diagrams (see [1]). Exploiting such formalism it has been possible to develop a unified and consistent description of all the known fundamental forces except gravity. The attempts to quantize General relativity, i.e. our classical macroscopic theory of gravity, have failed so far because it turns out to be a non renormalizable theory. Although the theory is one-loop finite, Goroff and Sagnotti proved for the first time that it is two-loop divergent (see [2]). However, this is not enough to conclude that QFT formalism cannot be used to provide a unified and coherent description of gravity. We can still hope that such issue affects only the IR regime and that there exists a consistent UV uplift. Exploiting what we learned with the Standard Model construction, we can guess that the divergent behavior is the consequence of an incorrect identification of the fundamental degrees of freedom (DOFs) of the theory. Another possibility is instead that finiteness arises once we couple gravity to other particles.

So far, string theory is the best example we have of a quantum theory of gravity, though we are not yet able to pinpoint specific phenomenological signatures that would test its validity. This problem arises because of the richness of the theory. For instance, the way we take the theory IR limit is not unique. String theory is indeed a special theory, highly constrained, whose consistency conditions fix the number of spacetime dimensions (see [3]). However, its vacuum structure is not fixed and in order to connect with experiments we need to integrate out extra dimensions, producing as many different IR limits as possible vacuum geometries. The collection of all the effective field theories (EFTs) which arise as an IR limit of string theory is called the *string landscape*. This concept can be extended to any general theory of quantum gravity: we call *landscape* the collection of all the EFTs which admit a consistent UV completion that includes gravity. Despite the string landscape is huge on its own, there are a lot of EFTs that cannot be derived from string theory. The collection of such theories is called the *string swampland*. The *swampland* is then the collection of all the theories which do not admit a generic UV completion.
In this framework takes place the *swampland program* which aims to clarify the criteria that allow to identify which EFT cannot be part of the *landscape*. Due to their heuristic

[1] It is the proportionality constant that relates (in suitable units) the observed magnetic moment of the electron to its spin times the Bohr magneton.

nature, swampland's criteria are often formulated as conjectures, supported by a wide collection of arguments and examples but without a rigorous proof. As one could expect, it turns out that the more rigorous the arguments, the less the conjecture constrains the EFTs. Among all the criteria, the Weak Gravity Conjecture (WGC) has a special role. It is the conjecture which started the swampland program (see [4]) and it is probably the best established one. The thesis work consists then in the study of some constraints produced by such criterion.

The WGC has different formulations (see [5]). However, we will focus on a particular one: the so called Electric Weak Gravity Conjecture (EWGC). The EWGC applies whenever we couple a gauge theory which admits a U(1) gauge charge to gravity. It corresponds to the requirement that there exist a state in the theory spectrum whose charge-to-mass ratio is greater than one (in suitable units). Depending on the type of state we require to satisfy the inequality, we can have a stronger or weaker criterion. If the inequality can be satisfied by an extended state such as a black hole (BH), we do not have necessarily a constraint on the microscopic physics. We talk therefore of the mild EWGC.

The interpretation and the derivation of the EWGC inequality is clear in Einstein–Maxwell theory. It is nothing but the condition which allows Reissner–Nordström (RN) black holes to discharge. Indeed, the charge-to-mass ratio of RN black holes is constrained by the cosmic censorship principle[2] to be smaller than one (in suitable units) and a decay process which respects energy and electric charge conservation can occur only if among the decay products there is a state whose charge-to-mass ratio is greater than the charge-to-mass ratio of the decaying object (see section 1.1.1). The EWGC is therefore strictly related to the structure of extremal BH solutions.

A natural question is then whether an extension of Einstein–Maxwell theory could provide modifications of such structure. It is indeed possible that introducing corrections which depend on the curvature of the black hole horizon, the cosmic censorship bound on charge-to-mass ratio is modified. Such computations have been performed in the 4 derivative extension of the Einstein–Maxwell theory first by Kats, Motl and Padi (see [6]). They obtained that the bound is shifted by a quantity whose modulus decreases with the black hole mass and whose sign depends on the higher order terms coefficients. Later works tried to fix the correction sign with different approaches. In particular, there is a wide production which exploits positivity bounds obtained from crossing symmetries, S-matrix analicity and unitarity (see [7],[8],[9]). They eventually found a positive correction. The direct consequence of such result is that the EWGC holds trivially in its mild form. Indeed, every BH can discharge decaying in a smaller quasi-extremal BH with greater charge-to-mass ratio fulfilling charge and energy conservation constraints.

The first question we address is then whether the discharge of an extremal black hole could happen through a splitting process or not. If the EWGC inequality was not satisfied by extended states then discharge could happen only through the Hawking evaporation and the Schwinger process (see [10]). The former is dominant when the black hole is small and its hawking temperature is big enough to allow the thermal production of charged particles. The latter is dominant when the black hole is cold and large enough to assume the uniformity of the electric field near the horizon and the activation of pair production processes. In both cases the discharge occurs because the external gauge field attracts

[2]The principle in its weak form asserts that can be no naked singularities.

the particles with opposite charge and repels those with the same charge of the black hole. However, being the EWGC inequality satisfied by extremal BHs, there is no reason to exclude a process where a black hole splits in two smaller black holes one of which is extremal with charge-to-mass ratio higher than that of the decaying BH.

In order to evaluate if such process is dynamically allowed we compute the classical black hole entropy using Wald's formalism. Iyer and Wald have been able to derive a formula for the black hole entropy which holds in a general theory of gravity with higher order derivate terms and to show that it satisfies the first law of thermodynamic (see [11],[12],[13]). Although the determination of the entropy in 4 derivative Einstein–Maxwell theory has been already performed in previous works (see [14]), such computations display some problems in the extremal limit. Therefore, as far as we know, the computation of the entropy correction in the extremal case is an original result of this thesis work. Exploiting such result, we have been able to produce an original discussion about the possibility of extremal black hole splitting, concluding that the process can not occur as long as we limit all the ingredients to the regime of validity of our work. The black hole that should be emitted together with the extremal one corresponds indeed to a so called *small black hole*, which violates the perturbative regime.

The second issue we address in the thesis is the understating of the EWGC constraints when duality transformations are taken into account. Duality transformations are symmetries of the equations of motion which do not leave necessarily the Lagrangian invariant. They had an important role in recent development of string theory and they seems to be more fundamental than Lorentz invariance itself (see [15],[16],[17],[18]). Assuming that duality is a symmetry of the UV theory it must hold at all orders in perturbative expansion and it can be completely characterized through leading order terms invariance. Following the idea of the works [19] and [20] it is possible to constrain the generic higher order corrections identified with a bottom-up approach imposing that they do not break the duality group.

The positivity bounds obtained exploiting S-matrix analiticity, crossing symmetries and unitarity imply that the inequality of the EWGC is always satisfied by an extremal BH, but the reverse is not true. We wonder therefore if duality constraints together with the assumption that EWGC is trivialized allow us to derive the former conditions. The reason for why we are interested in having such equivalence is to establish a deeper interpretation of WGC: it would be no more the condition which allows charged black holes to discharge but it would became the condition that must be imposed to guarantee unitarity and microcausality. [3]

The duality group of the Einstein–Maxwell theory is U(1) and can be easily obtained considering a generic $GL(2, \mathbb{R})$ rotation of equations of motion (EOMs) and the Bianchi identity and requiring the invariance of stress energy tensor (see [22],[23]). Imposing that the 4 derivative Einstein–Maxwell theory operators preserve such duality group we have been able to derive constraints on their coefficients. Most of the higher order corrections we considered have been already studied in the literature (see [24]), however such computations have been performed with an original approach which generalizes and systematized the ideas of [23]. Exploiting such constraints we finally verify the claimed equivalence.

The thesis work is organized as follows. In the first chapter we review the swampland

[3] S-matrix analicity and crossing symmetries are both implied by microcausality, see [21].

program. We give a detailed analysis of the WGC and its relation with other swampland conjectures. In the second chapter we discuss the structure of leading order corrections to the Einstein–Maxwell theory, focusing on the ambiguity associated with field redefinitions, and the issue of defining quantities in higher order derivative theories. In particular, we will present Wald's entropy formula and we will show that it satisfies the first law of thermodynamics. In the third chapter we will solve perturbatively the fourth order Einstein–Maxwell theory paying attention to the presence of small black hole solutions which breaks perturbative regime. We conclude discussing the possibility of black holes splitting. In the fourth chapter we analyze positivity bounds and we show how to derive them from S-matrix properties. In the fifth chapter we discuss the duality constraints and we present our generalized approach. In the sixth chapter we conclude with the summary of the thesis work and we present further developments.

CHAPTER 1

The Swampland Program

Due to their heuristic nature, swampland's criteria are often formulated as conjectures, supported by a wide collection of arguments and examples but lacking a rigorous proof. Such arguments are usually of three kinds: conditions derived from microscopic models (such as the constraints due to the presence of monopoles, see section 1.1.2), common characteristics expressed by string theory vacua (such as infinite towers of states, see section 1.2.1) and constraints derived from EFTs (such as black hole based arguments, see section 1.1.1). Moreover, it turns out that the more rigorous the arguments, the less the conjectures constrain EFTs. Looking at the criteria individually it is therefore difficult to believe that the swampland program is a reliable approach to study quantum gravity. However, its relevance becomes evident once the conjectures are considered all together. They are often strictly related and point in a common direction. A single conjecture should be therefore regarded as a node of a web which is constantly expanding and collecting more and more arguments.

In this chapter we start presenting two different formulations of the Weak Gravity Conjecture (WGC): the Electric Weak Gravity Conjecture (EWGC) and the Magnetic Weak Gravity Conjecture (MWGC). We review their black hole based arguments showing that they are dual relations. After that, we present other swampland's conjectures showing that there is an asymmetry between the two formulations. The MWGC can be indeed interpreted as a consequence of the No Global Symmetry Conjecture (NGSC) and can be related with the Species Scale Conjecture (SSC) and the Swampland Distance Conjecture (SDC). EWGC has instead no well established theoretical interpretation. We finally discuss how the the EWGC arguments suggest that higher order derivative corrections to black hole horizon can strengthen or weaken the conjecture constraints.

1.1 The Weak Gravity Conjecture

The Weak Gravity Conjecture (WGC) is the swampland's criterion that was established first. Although there are a lot of refinements (considering for instance central charges associated with scalar fields) and extensions (defining the conjecture for multiple U(1) in higher dimensions) of the WGC (see [5]), the minimal formulation in 4 dimension is enough for the thesis' purposes. It states:

Weak Gravity Conjecture ($D = 4$)
Given a theory coupled to gravity with a $U(1)$ gauge symmetry, let g be the gauge coupling. Then:

- (Electric WGC) There exist a state with mass M and charge q satisfying
$$M \leq \sqrt{2} g q M_P \,. \tag{1.1}$$
- (Magnetic WGC) Exist a cutoff scale Λ such that
$$\Lambda \lesssim g M_p \,. \tag{1.2}$$

It is important to observe that the original formulation (see [4]) does not specify which state should satisfy the relation (1.1) and different cases are considered. It could be:

1. the state of minimal charge;
2. the lightest charged state;
3. the state with highest charge-to-mass ratio.

The various formulations of the WGC conjecture have different strength. (3) is associated with the weakest one (notice that (1) and (2) both imply it), but it is the only statement which is truly supported by black hole based arguments. (1) could be ruled out by string theory based counterexamples (see [4]).

We highlight that extended states (e.g. black holes) are not excluded, but then the WGC would provide a weaker constraint on microscopic physics. In such a case, we talk of the mild version of the WGC.

1.1.1 Electric WGC

The Electric Weak Gravity Conjecture (EWGC) has a natural interpretation within classical $3 + 1$ Einstein–Maxwell theory. In such a framework, it is equivalent to require that Reissner–Nordström black holes are able to discharge themselves.

Given the action
$$S = \int d^4x \;\; e \left[\frac{M_P^2}{2} R - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right] \,, \qquad e = \sqrt{|g|} \,, \tag{1.3}$$
Einstein's equations are
$$\begin{cases} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{M_P^2} T_{\mu\nu} \,, \\ T_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\mu\nu}} \left(\sqrt{|g|} \mathcal{L}_{EM} \right) = F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \,. \end{cases} \tag{1.4}$$
Imposing spherical symmetry, staticity and the presence of a point-like U(1) charge, we obtain the Reissner–Nordström (RN) black hole solution (see [25])
$$ds^2 = -e^{2U(r)} dt^2 + e^{-2U(r)} dr^2 + r^2 d\Omega_{S_2}^2 \,, \tag{1.5}$$

$$F = \frac{Q}{4\pi r^2} dr \wedge dt\,, \tag{1.6}$$

$$e^{2U(r)} = 1 - \frac{2MG}{r} + \frac{Q^2 G}{4\pi r^2}\,. \tag{1.7}$$

Making explicit the dependencies on the gauge coupling constant $Q = gq$ and on the planck mass $M_P^2 = (8\pi G)^{-1}$, the condition to avoid the $r = 0$ singularity is

$$M \geq \sqrt{2} Q M_P = \sqrt{2} q g M_P\,. \tag{1.8}$$

In particular, extremal black holes satisfy $M = \sqrt{2} Q M_P$.

The discharge condition of a generic charged state can be instead derived in a completely general setting just exploiting charge and energy conservation. Let (M_0, Q_0) and $\{(M_i, Q_i)\}$ be the masses and charges of the initial state and the decay products. Then energy and charge conservation imply

$$\begin{cases} M_0 \geq \sum_i M_i\,, \\ Q_0 = \sum_i Q_i\,. \end{cases} \tag{1.9}$$

Exploiting equations (1.9) we get

$$\frac{M_0}{Q_0} \geq \frac{1}{Q_0} \sum_i \frac{M_i}{Q_i} Q_i \geq \left(\frac{M_i}{Q_i}\right)\Big|_{min} \sum_i \frac{Q_i}{Q_0} = \left(\frac{M_i}{Q_i}\right)\Big|_{min}\,. \tag{1.10}$$

Combining equations (1.8) and (1.10) and requiring that extremal black holes are able to discharge themselves, we finally get that should exist a state with parameters M and Q such that

$$\sqrt{2} M_P = \frac{M_{ext}}{Q_{ext}} \geq \frac{M}{Q} \Rightarrow M \leq \sqrt{2} g q M_P\,. \tag{1.11}$$

Relation (1.11) is nothing but the EWGC.

1.1.2 Magnetic WGC

The Magnetic Weak Gravity Conjecture (MWGC) can be derived as the dual of the EWGC, but we will see that there are some subtleties. If we extend Maxwell theory introducing magnetic charge as the dual of the electric charge we get the classical equations

$$\begin{cases} d \star F = \star J_e\,, \\ dF = \star J_m\,. \end{cases} \tag{1.12}$$

Modifying equation (1.6) it is possible to produce dyonic black holes. We have indeed

$$\begin{cases} Q_e = \int_\Sigma \star J_e = \int_{\partial\Sigma} \star F = Q\,, \\ Q_m = \int_\Sigma \star J_m = \int_{\partial\Sigma} F = P\,, \end{cases} \tag{1.13}$$

so that

$$F = \frac{Q}{4\pi r^2} dr \wedge dt + \frac{P}{4\pi} \sin(\theta)\, d\theta \wedge d\phi\,, \tag{1.14}$$

$$e^{2U(r)} = 1 - \frac{2MG}{r} + \frac{G(Q_e^2 + Q_m^2) M_P^2}{4\pi r^2}\,. \tag{1.15}$$

Repeating the computations of the previous section and assuming that the black hole is able to lose its magnetic charge we get the analogous of (1.1):

$$M \leq \sqrt{2}PM_P = \sqrt{2}pg_m M_P \,. \tag{1.16}$$

Recalling that whenever we add magnetic charges the Bianchi identity is no longer vanishing, we get that $F = dA$ holds only locally. It follows that A is not globally defined. In particular, the maximum extension of A over a flat space with a magnetic monopole is $\mathbb{R}^3/\{\text{Dirac String}\}$ (see [26]). Because different strings can be mapped into each other adding a closed form to A, i.e. through a gauge transformation, their position cannot be detected by any physical observable. Considering complex phases associated with Wilson loops, we get by consistency that they have to reduce to 1 whenever loops are shrunk to a point. This induces a quantization of magnetic and electric charge, the so called Dirac quantization. Considering for instance a magnetic monopole with magnetic charge Q_m, a Dirac string oriented towards positive z and a particle with charge Q_e whose trajectory is a circumference $\mathcal{C}$ with axis on positive z direction, we have:

$$A_t = A_r = A_\theta = 0\,, \quad A_\phi = \frac{Q_m}{4\pi}(\cos(\theta)+1)\,, \tag{1.17}$$

$$W[C] = Tr\left[\mathcal{P}\exp\left(iQ_e\oint_{\mathcal{C}} dx^\mu A_\mu\right)\right] = \exp\left[\frac{i}{2}Q_e Q_m(\cos(\theta)+1)\right]\,, \tag{1.18}$$

$$W[C] \xrightarrow{r_{\mathcal{C}}\to 0} 1 \iff Q_e Q_m \in 2\pi\mathbb{Z} \iff g_e = \frac{2\pi}{g_m}\,. \tag{1.19}$$

So far, we have considered symmetric magnetic and electric charges which are mapped into each other through the $\star$ operator. However, monopoles are typically introduced through topological solitons and they are related to a symmetry breaking process. It follows that the duality is no more exact but holds only under a certain scale Λ (the monopoles themselves exist only in the IR regime). In this framework the Λ scale of MWGC can be actually interpreted as the scale associated with the SSB process which produces monopoles. To derive equation (1.2) we consider then a simple example, the Georgi–Glashow model. It is a SU(2) gauge model with gauge field A_μ and scalar field ϕ transforming in the adjoint representation. The Lagrangian is

$$\begin{aligned}
\mathcal{L}_{GG} &= -\frac{1}{4}\text{tr}\left[F^2\right] + \frac{1}{2}\text{tr}\left[D_\mu\phi D^\mu\phi\right] - \frac{\lambda}{4}(\text{tr}\left[\phi^2\right] - v^2)^2\,, &(1.20)\\
D_\mu\phi^a &= \partial_\mu\phi^a + g\epsilon^{abc}A^b_\mu\phi^c\,, &(1.21)\\
F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon^{abc}A^b_\mu A^c_\nu\,. &(1.22)
\end{aligned}$$

The scalar fields moduli space is given by the constraint

$$\text{tr}[\phi^2] = \phi_1^2 + \phi_2^2 + \phi_3^2 = v^2\,. \tag{1.23}$$

Fixing the unitary gauge $\phi_1 = \phi_2 = 0$, $\phi_3 = h + v$ we have that SU(2) breaks to U(1). The system is therefore suitable to produce monopoles, indeed the second homotopy group is non trivial

$$\pi_2\left(\text{SU}(2)/\text{U}(1)\right) = \pi_1(\text{U}(1)) = \mathbb{Z}\,. \tag{1.24}$$

Moving to fields mass basis we get

$$A_\mu = A^3_\mu\,, \qquad m_A = 0\,, \tag{1.25}$$

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(A^1_\mu \pm A^2_\mu\right)\,, \qquad m_W = gv\,, \tag{1.26}$$

$$h = \phi_3 - v\,, \qquad m_h = \sqrt{2\lambda}v\,. \tag{1.27}$$

We are interested now in deriving mass parametric dependence of a static monopole. Such configuration has no kinetic energy, therefore the mass M equals the energy E. It holds (see [27])

$$M = E = \int d^3x\left[\frac{1}{4}\mathrm{tr}[F_{ij}F_{ij}] + \frac{1}{2}\mathrm{tr}[D_i\phi D_i\phi] + \frac{\lambda}{4}(\mathrm{tr}[\phi^2] - v^2)^2\right]\,. \tag{1.28}$$

In order to avoid the computation of the integral we have to guarantee that it is a $O(1)$ factor in the parameters. We introduce therefore the following rescaling of the fields and the spacetime coordinates:

$$\begin{cases} y^i = gvx^i\,, \\ \phi^a(x) = vf^a(y)\,, \\ A^a_i(x) = vBa_i(y)\,. \end{cases} \tag{1.29}$$

The energy takes then the form

$$E = \frac{v}{g}\int d^3y\left[\frac{1}{4}\mathrm{tr}[B_{ij}B_{ij} + \frac{1}{2}\mathrm{tr}[D_ifD_if] + \frac{\lambda}{4g^2}(\mathrm{tr}[f^2] - 1)^2\right]\,. \tag{1.30}$$

Assuming that $m_H^2 \sim m_W^2 \sim \Lambda_{SSB}$ we get that $\frac{\lambda}{4g^2} = \frac{m_H^2}{8m_W^2} \sim O(1)$. So, without other efforts, we can guess the parametric scaling $E \sim \frac{v}{g}$. Recalling that we are in the static case and exploiting equation (1.26) we finally get

$$M \sim E \sim \frac{\Lambda}{g^2}\,. \tag{1.31}$$

Equation (1.31) has been derived for Georgi-Glashow model, however such parametric dependence is typical for all the models with monopoles. Combining equations (1.16), (1.19) and (1.31) together we finally get

$$\frac{\Lambda}{g^2} \sim M \leq \frac{2\pi}{g}pM_p \quad \Longrightarrow \quad \Lambda \lesssim gM_p\,. \tag{1.32}$$

Equation (1.32) is nothing but the MWGC.

Notice that in this framework the MWGC can be intepreted as the condition that at least one monopole in not a black hole. Indeed, relation (1.32) can be derived requiring that the minimally charged monopole satisfy equation (1.8).

1.2 Other Swampland Conjectures

So far, we proved in a simple model the equivalence of (1.1) and the possibility that black holes can discharge. Moreover, we proved (1.2) exploiting the dual relation (1.16) and treating magnetic monopoles as solitons produced after a SSB process. However, we did not provide any argument supporting the necessity of black holes discharge. According to [5] it is still an open question whether the presence of stable remnants would provide inconsistencies or not. However, it is clear that the presence of a fundamental inconsistency associated with the existence of stable black holes would provide a proof of the WGC.

Although the whole conjecture is not proven, there are arguments supporting the MWGC which are independent of the EWGC validity. Such asymmetry between the two conjectures is due to the fact that the core of the MWGC is the existence of an EFT lowered cutoff scale Λ and not the introduction of a bound on magnetic charges. Therefore, despite (1.2) is related to (1.1), it should not be intended simply as its dual relation (unlike (1.16)).

The existence of a cutoff scale lower that Planck scale M_P is central in a lot of swampland conjectures. In order to explicit the web they form we review some of them: the Species Scale Conjecture (SSC), the Swampland Distance Conjecture (SDC) and the No Global Symmetry Conjecture (NGSC). All the arguments we present in this section supports therefore indirectly the MWGC.

1.2.1 Species Scale Conjecture

The Species Scale Conjecture (SSC) states:

Species Scale Conjecture
Consider a theory of gravity in d-dimensions with Planck mass M_P^d which admits N_S particle species under a cutoff scale Λ. Then, in the weakly coupled regime it holds:

$$\Lambda < \Lambda_S = \frac{M_P^d}{N_S^{\frac{1}{d-2}}} \,. \tag{1.33}$$

The main argument which inspired the conjecture exploits a particular class of particles, those associated with the Kaluza–Klein (KK) modes. KK modes arise from compact dimensions integration, thus they are a clear signature that the UV uplift of the theory should be described in a higher dimensional spacetime. Therefore, the presence of KK modes implies the existence of compact extra dimensions and provides a lowering of the Planck mass scale.

Let us consider for instance a flat metric in d+1 dimensions with a S^1 compact dimension normalized in order to have $\tau \sim \tau + 1$:

$$ds^2 = g_{MN} dx^M dx^N = \eta_{\mu\nu} dx^\mu dx^\nu + (2\pi R)^2 (d\tau)^2 \,. \tag{1.34}$$

The action which describes a free massless real $(d+1)$-dimensional scalar field on such background is then

$$S = \left(M_P^{(d+1)}\right)^{d-1} \int d^{\,d+1}x \;\, \frac{e}{2}\, \partial_M \Phi \, \partial^M \Phi \,, \qquad e = \sqrt{|\det\left(g_{MN}\right)|} \,, \tag{1.35}$$

Exploiting τ periodicity, we can express Φ through the periodic expansion

$$\Phi = \sum_{n=-\infty}^{+\infty} \phi_n(x^\mu) e^{2\pi i n \tau} \,, \qquad \phi_n^* = \phi_{-n} \,, \tag{1.36}$$

and the action yields

$$S = \left(M_P^{(d+1)}\right)^{d-1} \int d^d x \, d\tau \, (2\pi R \, e) \sum_{m,n} \left\{ -\frac{1}{2} e^{2\pi i (n+m)\tau} \phi_m \left[\Box + \left(\frac{n}{R}\right)^2 \right] \phi_n \right\} . \quad (1.37)$$

where $e = \sqrt{|\det(g_{\mu\nu})|}$. Equation (1.37) is nothing but the action of an infinite tower of scalar fields in d-dimensions with increasing masses given by $M_n^2 = \left(\frac{n}{R}\right)^2$. We get then

$$S \; = \; \left(M_P^{(d)}\right)^{d-2} \sum_n \int d^d x \; e \left\{ -\frac{1}{2} \phi_n^* \left[\Box + m_n^2 \right] \phi_n \right\} , \qquad e = \sqrt{|\det(g_{\mu\nu})|} , \quad (1.38)$$

where we introduced the d-dimensional Planck mass $M_P^{(d)}$ which satisfies

$$\left(M_P^{(d)}\right)^{d-2} \; = \; 2\pi R \left(M_P^{(d+1)}\right)^{d-1} . \quad (1.39)$$

Equation (1.39) implies that the reduced d-dimensional Planck mass is higher than the $(d+1)$-dimensional Planck mass. It follows that the presence of extra compact dimensions lowers the cutoff scale of the EFT. It is possible then to relate such lowered scale with the maximum number N_S of particle species contained in the theory spectrum. We have just to impose that the theory breaks at M_{N_S}, i.e at the mass scale of the N_S-th mode. Above such scale the theory spectrum should contain indeed a number of KK modes that is greater than the maximum number of species contained, providing an inconsistency. We impose therefore

$$M_{N_S} = M_P^{(d+1)} . \quad (1.40)$$

Exploiting equation (1.39) and the definition of M_n we get

$$\frac{N_S}{R} = M_P^{(d+1)} = \left[\frac{1}{2\pi R} \left(M_P^{(d)}\right)^{d-2} \right]^{\frac{1}{d-1}} = N_S \left(\frac{1}{2\pi}\right)^{\frac{1}{d-2}} N_S^{\frac{1-d}{d-2}} M_P^{(d)} , \quad (1.41)$$

where the last equivalence is obtained replacing R expression computed from the equivalence of the first and third term. The lowered cutoff scale then becomes

$$\frac{M_P^{(d)}}{N_S^{\frac{1}{d-2}}} = M_P^{(d+1)} (2\pi)^{\frac{1}{d-2}} = \Lambda_S . \quad (1.42)$$

Equation (1.42) is nothing but the SSC.

We verify now that considering 3+1 Einstein–Maxwell theory the SSC cutoff scale can be bounded through the gauge coupling constant. Choosing g in order to have minimum charge equal to 1, the number of BHs species with different charge is given by the maximum charge that a black hole can assume. Such charge is nothing but the charge of an extremal black hole whose mass is at the cutoff scale Λ. We have therefore

$$N_S \sim q_{max} = \frac{M_{max}}{\sqrt{2} g M_P^{(4)}} \sim \frac{\Lambda}{g M_P^{(4)}} , \quad (1.43)$$

Applying (1.33) we get

$$\Lambda \lesssim \Lambda_S = \frac{M_P^{(4)}}{\sqrt{\frac{\Lambda}{g M_P^{(4)}}}} \quad \Longrightarrow \quad \Lambda \lesssim g^{\frac{1}{3}} M_P^{(4)} . \quad (1.44)$$

Equation (1.44) presents a suppression factor driven by g coupling constant as (1.2), but the power scaling is different.

In order to understand the correct link between the MWGC and the SSC, it is important to notice that despite the argument proposed to present the SSC exploits the presence of extra dimensions, they are not strictly necessary to derive (1.33). The conjecture can indeed be related to an entropy bound (see [28]) which locally constrains the number of particle species of the theory. Moreover, the bound can be consistently applied to extended states (e.g. black holes) assuming that their internal structure do not produce any not trivial effect. It follows that the SSC can be intended as an extension of the MWGC because it is able to provide bounds on the lowered cutoff scale exploiting a wider class of particle species. However, the bound obtained applying the SSC to black hole species is weaker than the MWGC one.

1.2.2 Swampland Distance Conjecture

In the previous section we introduced KK modes and we used their typical mass scale $M_{KK} \sim \frac{1}{R}$ to evaluate the number N_S of particles species that have mass lower than the reduced Planck mass and verify the SSC. Although we observed that an alternative derivation can be provided through entropy bounds, it is worth taking a closer look to the infinite tower of states structure because it is strictly related to the Swampland Distance Conjecture (SDC).

Let us consider a bosonic free string which lives on a spacetime with only one compact dimension. It is possible to show (see [5]) that integrating over S^1 we will produce a spectrum of particles which contains all the KK modes previously found and some new modes associated with the not trivial topology of the string warping around S^1. The mass of such states can be computed to be:

$$M_{n,w}^2 = \left(\frac{n}{R}\right)^2 + \left(\frac{Rw}{\alpha'}\right)^2 , \qquad m,\, \omega \in \mathbb{Z} . \tag{1.45}$$

Moreover, turning on gravity, the radius R becomes dynamical (it is proportional to the dilaton exponential) and we have

$$R = e^{-\beta\phi} , \qquad M_{n,0} \sim e^{2\beta\phi} , \qquad M_{0,w} \sim e^{-2\beta\phi} , \qquad \beta > 0 . \tag{1.46}$$

Equations (1.46) imply that whenever ϕ takes big values there is at least one infinite tower of states whose mass scale is exponentially suppressed. The SDC is nothing but the assumption that this property holds in every theory of the landscape and for every direction taken in the moduli space.

Swampland Distance Conjecture
Consider a theory of gravity with moduli space $\mathcal{M}$ which is parameterized by the expectation value of some free field ϕ^i. Then

- $\forall\ P \in \mathcal{M},\ s > 0\ \exists\ Q_s \in \mathcal{M}$ such that $d(P, Q_s) > s$, i.e. every configurations admits a boundary at infinite geodesic distance.
- There exist an infinite tower of states with mass scale $M(Q_\infty)$ such that

$$M(Q_\infty) \sim M(P) e^{-\beta d(P,Q_\infty)} , \qquad \beta > 0 . \tag{1.47}$$

We see that the SDC on its own does not imply a bound on the cutoff scale of the EFTs. However, it guarantees that approaching the boundary of the moduli space there is at least a tower of states which becomes exponentially massless. It follows that applying the SSC at such tower we are able to provide a constraint on the cutoff scale that grows exponentially with the parametrization of the moduli space. Therefore, the SDC and the SSC together provide a stronger version of the latter. It follows that the arguments which support the SDC and the SSC can be consider as strengthening the MWGC.

1.2.3 No Global Symmetry Conjecture

We finally consider the swampland conjecture which is at the same time the one with strongest arguments and the one with the most qualitative statement.

No Global Symmetry Conjecture
A theory coupled to gravity with a finite number of states cannot admit global symmetries.

Before reviewing the arguments that support this conjecture, we observe that the No Global Symmetry Conjecture (NGSC) implies that the coupling constants of the gauge interactions $g^{(i)}$ must be bounded from below, i.e. there exist $g^{(i)}_{min}$ such that

$$g^{(i)}_{min} < g^{(i)} \,. \tag{1.48}$$

Indeed, if the coupling constants of the gauge interactions did not have a lower bound, nothing would prevent them to flow smoothly to zero. Thus, a bound on the cutoff scale of the form

$$\Lambda < f(g) M_P \,, \qquad |f(g)| < 1, \; f(g) \xrightarrow{g\to 0} 0 \,, \tag{1.49}$$

would guarantee us that whenever the gauge symmetry approaches the global symmetry regime the cutoff scale Λ drops to zero.

Evidences of the NGSC are derived with different and independent approaches (perturbative string theory, AdS/CFT and black holes physics). The black hole based arguments rely on the possibility of preparing black hole configurations with fixed mass and arbitrarily large global charge. A black hole cannot lose its global charge[1] through Hawking radiation or through the Schwinger process because the global charge does not produce a field outside the black hole. It is therefore not possible to distinguish among the particles produced near the horizon and repel those with the same charge of the black hole. Because a black hole cannot lose its global charge it is possible then to produce a configuration with arbitrary large global charge and fixed mass. We can indeed prepare such system throwing inside the black hole particles charged under the global symmetry of interest and letting it radiate the extra mass. Then

- Because the black hole structure does not depend on global charges (no hair theorem, see [25]), increasing its global charge is possible to violate thermodynamic entropy bounds (see [4]).

[1] At least in semiclassical regime.

- It is possible to approach planck mass regime with arbitrary large charge. In particular such an object can be prepared having a charge to mass ratio greater than any other particle of the theory. The configuration is therefore stable independently of quantum gravity effects. It follows that it is possible to produce an infinite number of remnants.

Thus in both cases inconsistencies arise.

1.3 Extremal Black Holes Instability

An interesting scenario to investigate is the possibility that extremal black holes are able to decay, i.e. that the product of the discharge process which satisfy equation (1.1) is a black hole itself. Proving that black holes are able to decay emitting a smaller black hole one would provide a proof of the mild version of the WGC because equation (1.1) would be trivially satisfied by extended states. Conversely, one proving that black holes cannot decay would rule out the possibility that the EWGC trivializes, but would not provide any inconsistency associated with the existence of a stable remnants.

Exploiting equation (1.10) we get that extremal black holes decay requires a correction on extremality bound of the form:

$$z = \frac{\sqrt{2} M_P Q}{M} = 1 + \epsilon(M)\,, \tag{1.50}$$

$$\epsilon(M) > 0\,, \qquad \epsilon(M_1) < \epsilon(M_2) \quad \text{iff} \quad M_1 > M_2\,. \tag{1.51}$$

The natural guess for corrections that could provide such scale-dependent behavior is a theory with higher order derivative operators. Notice that equations (1.50) and (1.51) suggest that a decay process through a splitting process is possible.

Since the possibility of black hole instability has been suggested by the WGC presentation (see [4]) several models have been studied. In the next chapter we will study the simple example of the Einstein–Maxwell theory.

CHAPTER 2

Einstein–Maxwell Theory

We concluded the previous chapter highlighting that higher order derivative corrections could modify the charge-to-mass ratio of the extremal black hole configurations. Such fact has been verified in several models such as Einstein–Maxwell theory (see [14]), Einstein–Maxwell–dilaton theory (see [20]) and heterotic string theory (see [29]). However, in order to verify whether the WGC trivializes or not it is not enough to determine the charge-to-mass ratio correction. We have indeed to fix its sign. This has been usually accomplished fixing the coefficients of the higher order derivative corrections imposing the matching with IR expansion of string theory (see [20]). In the case of Einstein–Maxwell theory is instead possible to avoid string theory matching and fix the correction sign exploiting positivity bounds due to S-matrix analicity, unitarity and crossing symmetries (see [7]). In order to discuss the possibility of black holes splitting we concentrate then on the simple case of Einstein–Maxwell theory.

In this chapter we identify the lowest order correction of Einstein–Maxwell theory. Exploiting suitable identities and field redefinitions we remove redundant terms reducing the action to a form easy to handle. We discuss then the correctness of the field redefinition itself and the effective equivalence of theories mapped by such transformation. We finally introduce Wald's formalism which allows to compute the Noether charges and the entropy in theories with higher order derivative terms.

2.1 Lowest Order Corrections

Let's consider the action

$$S = \int d^4x \, e \left(\mathcal{L}_2 + \Delta\mathcal{L}\right) , \qquad \mathcal{L}_2 = \frac{M_P^2}{2} R - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} , \qquad e = \sqrt{|g|} \, . \tag{2.1}$$

We want to derive the lowest order correction $\Delta\mathcal{L}$ to Einstein–Maxwell theory. We notice that gauge invariance imposes the use of $F^{\mu\nu}$ instead of A^μ, thus we can built $\Delta\mathcal{L}$ using only tensors with an even number of indices. It follows that a scalar Lagrangian with no free indices requires an even number of covariant derivatives. We have therefore that higher order corrections are dominated by fourth order terms, i.e. $\Delta\mathcal{L} = \mathcal{L}_4 \sim O(D^4)$.

Finally, CP invariance implies that an even number of $F^{\mu\nu}$ operators is required. We have indeed

$$D_\mu \xrightarrow{P} (-)_\mu D_\mu \xrightarrow{C} (-)_\mu D_\mu \,, \tag{2.2}$$

$$A_\mu \xrightarrow{P} (-)_\mu A_\mu \xrightarrow{C} -(-)_\mu A_\mu \,, \tag{2.3}$$

$$F_{\mu\nu} \xrightarrow{P} F_{\mu\nu} \xrightarrow{C} -F_{\mu\nu} \,, \tag{2.4}$$

where we introduced $(-)_\mu = -\delta^0_\mu + \delta^i_\mu$. Thus, the general fourth order term takes the form

$$\mathcal{O}_4 = (R)^p (R_{\mu\nu})^q (R_{\mu\nu\rho\sigma})^r (\nabla_\mu)^{2s} (F_{\mu\nu})^{2t}, \quad p+q+r+s+t = 2\,, \tag{2.5}$$

where the constraint (2.5) fixes the derivative order. Neglecting total derivatives, the maximum extension of Einstein–Maxwell theory involves the terms

$$\begin{aligned}
dim = 4:\quad & R^2,\quad (R_{\mu\nu})^2,\quad (R_{\mu\nu\rho\sigma})^2,\quad RF_{\mu\nu}F^{\mu\nu},\quad R_{\mu\nu}F^{\mu\rho}F^{\nu}{}_{\rho}\,,\\
& R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}.\\
dim = 6:\quad & (D_\mu F^{\mu\nu})^2,\quad (D_\mu F_{\nu\rho})^2,\quad (D_\mu F_{\nu\rho})(D^\nu F^{\mu\rho}).\\
dim = 8:\quad & (F_{\mu\nu}F^{\mu\nu})^2\,,\quad (F_{\mu\nu}\tilde{F}^{\mu\nu})^2,\quad F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}.
\end{aligned}$$

where we have introduced the dual field strength tensor $\tilde{F}_{\mu\nu} = \frac{\sqrt{|g|}}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, with $\epsilon_{\mu\nu\rho\sigma}$ the Levi Civita symbol.[1] However, not all of them are independent. Through the relations

$$(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 \;=\; -\,2\,(F_{\mu\nu}F^{\mu\nu})^2 + 4\,(F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu})\;, \tag{2.6}$$

$$(D_\mu F_{\nu\rho})^2 \;=\; 2(D_\mu F_{\nu\rho})(D^\nu F^{\mu\rho})\,, \tag{2.7}$$

$$(D_\mu F_{\nu\rho})^2 \;=\; -\,2R_{\mu\nu}F^{\mu\rho}F^{\nu}{}_{\rho} + R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + 2(D_\mu F^{\mu\nu})^2 + 2D_\mu\Lambda^\mu\,, \tag{2.8}$$

$$\Lambda^\mu \;=\; F_{\nu\rho}D^\nu F^{\mu\rho} - F^{\mu\rho}D_\nu F^{\nu}{}_{\rho}\;, \tag{2.9}$$

we see that the last 8-dim operator and the last two 6-dim operators are redundant. Moreover, $(R_{\mu\nu\rho\sigma})^2$ cancels thanks to Gauss–Bonnet term G which is a total derivative (see [30]):

$$G = R^2 - 4(R_{\mu\nu})^2 + (R_{\mu\nu\rho\sigma})^2\,, \tag{2.10}$$

$$\star\, G = R^{AB}\wedge R^{CD}\epsilon_{ABCD} = d^D\left[\left(\omega^{AB}\wedge R^{CD} - \frac{1}{3}\omega^{AB}\wedge\omega^{C}{}_{F}\wedge\omega^{FD}\right)\epsilon_{ABCD}\right]. \tag{2.11}$$

Considering variations of $g_{\mu\nu}$ and A^μ of order $O(D^2)$ we can perform a field redefinition which is equivalent to replace the tree level equations of motion

$$\begin{cases} g'^{\mu\nu} = g^{\mu\nu} + \delta g^{\mu\nu}\,,\\ A'_\mu = A_\mu + \delta A_\mu\,,\\ \left(\sqrt{|g|}\mathcal{L}\right)' = \sqrt{|g|}\mathcal{L} + \delta\left(\sqrt{|g|}\mathcal{L}\right)\,, \end{cases} \tag{2.12}$$

[1] Notice that $\epsilon_{\mu\nu\rho\sigma}$ takes a minus sign under CP transformation, therefore in a single term is admitted only an even number of Levi Civita tensors.

where

$$\begin{aligned}\delta\left(\sqrt{|g|}\mathcal{L}\right) &= \delta\left(\sqrt{|g|}\mathcal{L}_2+\sqrt{|g|}\mathcal{L}_4+O(D^6)\right)=\delta\left(\sqrt{|g|}\mathcal{L}_2\right)\;+\;O(D^6)=\\ &= \sqrt{|g|}\left\{\frac{M_P^2}{2}\delta g^{\mu\nu}\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}-\frac{1}{M^2}T_{\mu\nu}\right)\right\}\\ &\quad+\sqrt{|g|}\left\{\delta A_\rho(D_\sigma F^{\sigma\rho})\right\}+O(D^6)\,.\end{aligned}\tag{2.13}$$

Fixing properly field redefinition parameters it is possible to cancel R^2, $(R_{\mu\nu})^2$, RF^2, $R_{\mu\nu}(F^{\mu\rho}F^{\nu}{}_{\rho})$ and $(D_\mu F^{\mu\nu})^2$ terms. We have indeed:

$$\begin{cases}\delta g^{\mu\nu}=\frac{c_1}{M_P^4}g^{\mu\nu}F^2+\frac{c_2}{M_P^4}F^{\mu\rho}F^{\nu}{}_{\rho}+\frac{c_3}{M_P^2}g^{\mu\nu}R+\frac{c_4}{M_P^2}R^{\mu\nu}\,,\\ \delta A_\mu=\frac{b}{M_P^2}D^\nu F_{\nu\mu}\,,\end{cases}\tag{2.14}$$

and therefore

$$\begin{aligned}\delta\left(\sqrt{|g|}\mathcal{L}\right) = &-\frac{(2c_3+c_4)}{4}R^2+\frac{c_4}{2}(R_{\mu\nu})^2-\frac{(4c_1+2c_2-c_4)}{8M_P^2}RF^2\\ &+\frac{(c_2-c_4)}{2M_P^2}R_{\mu\nu}(F^{\mu\rho}F^{\nu}{}_{\rho})+\frac{c_2}{8M_P^4}\,(F_{\mu\nu}F^{\mu\nu})^2\\ &+\frac{c_2}{2M_P^4}(F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu})+\frac{b}{M_P^2}(D_\mu F^{\mu\nu})^2\,.\end{aligned}\tag{2.15}$$

The Einstein–Maxwell Lagrangian at lowest order in higher derivative correction takes the form

$$\begin{aligned}\mathcal{L}' = &\frac{M_P^2}{2}R-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{\alpha_1}{4M_P^4}(F_{\mu\nu}F^{\mu\nu})^2\\ &+\frac{\alpha_2}{4M_P^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2+\frac{\alpha_3}{2M_P^2}(F_{\mu\nu}F_{\rho\sigma}R^{\mu\nu\rho\sigma})\,.\end{aligned}\tag{2.16}$$

We can finally express $R_{\mu\nu\rho\sigma}$ in terms of the Weyl tensor $W_{\mu\nu\rho\sigma}$.[2] In 4 dimensions we have

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}-\left(g_{\mu[\rho}R_{\sigma]\nu}-g_{\nu[\rho}R_{\sigma]\mu}\right)+\frac{1}{6}Rg_{\mu[\rho}g_{\sigma]\nu}\,,\tag{2.17}$$

$$W_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} = R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}-2R_{\mu\nu}F^{\mu\rho}F^{\nu}{}_{\rho}+\frac{1}{3}RF^2\,.\tag{2.18}$$

Applying a proper field redefinition on Lagrangian (2.16) which fixes $R_{\mu\nu}(F^{\mu\rho}F^{\nu}{}_{\rho})$ and RF^2 coefficients we eventually get the fourth-order Lagrangian

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_2+\mathcal{L}_4\\ &= +\frac{M_P^2}{2}R-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{\alpha_1}{4M_P^4}(F_{\mu\nu}F^{\mu\nu})^2\\ &\quad+\frac{\alpha_2}{4M_P^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2+\frac{\alpha_3}{2M_P^2}(F_{\mu\nu}F_{\rho\sigma}W^{\mu\nu\rho\sigma})\,.\end{aligned}\tag{2.19}$$

[2]This choice will be clarified in section 4.3. It is necessary to constrain the α_i coefficients through the positivity bounds found in the literature.

2.2 Field Redefinitions

In the previous section we selected a particular 4 derivative extension of Einstein–Maxwell theory performing a field redefinition. However, equation (2.14) mixes photon and graviton degrees of freedom, therefore the related quantum field theories are not equivalent. A quantum field theory is indeed invariant under field redefinitions only if the degrees of freedom are not mixed (cfr. S-matrix equivalence theorem, see [31]). In general, the theories are not equivalent classically too. The non linear transformations can indeed change the number of DOFs that propagate. However, assuming that higher order derivatives terms are small and considering solutions which do not break the perturbative expansion we should be able to guarantee the theories equivalence. The typical scale of the phenomena we are going to study would be indeed smaller than the cutoff scale at which could appear the new DOFs effects.

The above observations implies that the dynamics of big enough black hole solutions should not be influenced by a field redefinition. However, we highlight that this is not a proven fact but just a claim. It is therefore fundamental to verify it. A possible non trivial test can be obtained through the computation of quantities which influence directly the theory dynamics. Two theories mapped by a field redefinition are equivalent only if their dynamics is the same. Therefore, such quantities must be somehow protected. For instance, being ΔS and Δz fundamental to determine whether a BH can decay or not, a necessary requirement of theories equivalence is that the signs of the corrections are invariant, i.e. that the corrections can not be completely absorbed through a field redefinition[3].

The signs of charge-to-mass ratio and entropy corrections depend on higher order terms coefficients (we will explicit the result in the next chapter). We are interested therefore in understanding whether there exist coefficients combinations which are invariant under fields redefinition. If there were no invariant combinations, there would be no possibility that the field redefinitions leaves the dynamics of big black holes invariant.

Let us consider the general theory

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}^{(2)} + \mathcal{L}_1^{(4)} + \mathcal{L}_2^{(4)}\,, \\
\mathcal{L}^{(2)} &= \frac{M_P^2}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \\
\mathcal{L}_1^{(4)} &= \frac{\alpha_1}{4M_P^4}(F_{\mu\nu}F^{\mu\nu})^2 + \frac{\alpha_2}{4M_P^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + \frac{\alpha_3}{2M_P^2}(F_{\mu\nu}F_{\rho\sigma}W^{\mu\nu\rho\sigma})\,, \\
\mathcal{L}_2^{(4)} &= \frac{\alpha_4}{2M_P^2}R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} + \frac{\alpha_5}{2M_P^2}RF^2 + \alpha_6 R^2 + \alpha_7 (R_{\mu\nu})^2 + \frac{\alpha_8}{2M_P^2}(D_\mu F_{\nu\rho})^2\,.
\end{aligned} \tag{2.20}$$

The operators considered constitute a complete basis for the Einstein–Maxwell theory with 4 derivative corrections[4]. The non linear field redefinition (2.14) introduces then the terms

$$\begin{cases}
\delta g^{\mu\nu} = \frac{c_1}{M_P^4}g^{\mu\nu}F^2 + \frac{c_2}{M_P^4}F^{\mu\rho}F^{\nu}{}_{\rho} + \frac{c_3}{M_P^2}g^{\mu\nu}R + \frac{c_4}{M_P^2}R^{\mu\nu}\,, \\
\delta A_\mu = \frac{b}{M_P^2}D^\nu F_{\nu\mu}\,,
\end{cases} \tag{2.21}$$

[3]Notice that we do not expect that the invariance property is a necessary condition of all physical observable.

[4]Notice that we chose a basis which is slightly different from that of the previous section

$$
\begin{aligned}
\delta\left(\sqrt{|g|}\mathcal{L}\right) = \; & -\frac{(2c_3+c_4)}{4}R^2 + \frac{c_4}{2}(R_{\mu\nu})^2 - \frac{(4c_1+2c_2-c_4)}{8M_P^2}RF^2 \\
& + \frac{(c_2-c_4)}{2M_P^2}R_{\mu\nu}(F^{\mu\rho}F^{\nu}{}_{\rho}) + \frac{c_2}{8M_P^4}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 \\
& + \frac{c_2}{2M_P^4}(F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}) + \frac{b}{M_P^2}(D_\mu F^{\mu\nu})^2 \,.
\end{aligned} \tag{2.22}
$$

Exploiting equations (2.6),(2.8) and (2.18) it is possible to express the operators introduced by (2.22) in the basis of Lagrangian (2.20). We have indeed

$$
\frac{b}{M_P^2}(D_\mu F^{\mu\nu})^2 = \frac{b}{2M_P^2}(D_\mu F_{\nu\rho})^2 + \frac{b}{6M_P^2}RF^2 - \frac{b}{2M_P^2}W_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\,, \tag{2.23}
$$

$$
-\frac{2c_2}{4M_P^2}(F_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu}) = -\frac{c_2}{4M_P^4}(F_{\mu\nu}F^{\mu\nu})^2 - \frac{c_2}{8M_P^4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\,. \tag{2.24}
$$

We obtain therefore the coefficients shifts

$$
\begin{aligned}
\alpha_1' &= \alpha_1 - \frac{c_2}{2}\,, \\
\alpha_2' &= \alpha_2 - \frac{c_2}{2}\,, \\
\alpha_3' &= \alpha_3 - b\,, \\
\alpha_4' &= \alpha_4 + c_2 - c_4\,, \\
\alpha_5' &= \alpha_5 - c_1 - \frac{c_2}{2} + \frac{c_4}{4} + \frac{b}{3}\,, \\
\alpha_6' &= \alpha_6 - \frac{c_3}{2} - \frac{c_4}{4}\,, \\
\alpha_7' &= \alpha_7 + \frac{c_4}{2}\,, \\
\alpha_8' &= \alpha_8 + b\,.
\end{aligned} \tag{2.25}
$$

Having 8 coefficients and 5 independent parameters which parameterize the field redefinitions, we expect that it is possible to built 3 invariant quantities. It is simple to verify that a possible choice for such quantities is:

$$
\begin{aligned}
\beta_1 &= \alpha_1 + \frac{1}{2}\alpha_4 + \alpha_7\,, \\
\beta_2 &= \alpha_2 + \frac{1}{2}\alpha_4 + \alpha_7\,, \\
\beta_3 &= \alpha_3 + \alpha_8\,.
\end{aligned} \tag{2.26}
$$

In particular, we observe that α_5 and α_6 are not involved in β_i definitions. This is a direct consequence of their transformation laws: they are the unique coefficients shifted by c_1 and c_3 respectively.

A convenient choice of $\{c_i\}$ is that which let vanishing α_i coefficients. Exploiting a proper field redefinition we can set:

$$
\begin{aligned}
\alpha_i' &= 0\,, \qquad i = 4,5,6,7,8 \\
\beta_1' &= \alpha_1'\,, \\
\beta_2' &= \alpha_2'\,, \\
\beta_3' &= \alpha_3'\,.
\end{aligned} \tag{2.27}
$$

Therefore, the three invariant quantities equal the coefficients of (2.19). Notice that this is a necessary condition for the equivalence of the particular theory (2.19) and the general theory (2.20). However, equation (2.27) is not enough to conclude that the sign of a generic correction evaluated in (2.19) is invariant under field redefinitions. Although it is driven by a combination of α_1, α_2 and α_3 which can be expressed through the $\{\beta_i\}$, nothing prevents the turning on of other corrections which are proportional to a vanishing α_i. In particular, all the effects due to α_5 or α_6 terms of the general theory are loss. However, we know from the literature that both the charge-to-mass ratio and the entropy[5] are not affected by α_5 and α_6 (see respectively [6] and [14]). Thus, there is no reason to doubt of our claim. In the next chapter we will verify that the carge-to-mass ratio and the extremal black holes entropy are actually invariant.

2.3 Wald Formalism

We present now Wald's formalism. It applies to theories of gravity whose action involves higher order derivative terms. In particular, it provides a generalization of Komar's formalism for general relativity.

We start this section describing how to associate a conserved charge to a Killing vector. After that, we show that black hole masses can be expressed thorough the charge associated with the asymptotic time translation symmetry. In particular, we verify that Wald's definition of mass is a generalization of Komar's definition. Finally, we show that BH entropy can be described as a Noether charge and we verify that it satisfies the first law of thermodynamic.

2.3.1 Noether Charge in Higher Order Derivatives Theories

An important class of coordinate transformations is that of isometries. An isometry is characterized by a section of the spacetime tangent bundle ζ whose Lie derivative[6] satisfies

$$\mathcal{L}_\zeta \, g_{\mu\nu} = 0 \,. \tag{2.28}$$

In general relativity, given a Killing vector ζ it is always possible to define a conserved current. Exploiting the Bianchi identity for Riemann tensor, the Killing equations and $[D_\mu, D_\nu]\zeta_\rho = R_{\mu\nu}{}^\sigma{}_\rho \zeta_\sigma$ it is easy to show that

$$D_\mu D_\nu \zeta^\rho = R_{\mu\sigma}{}^\rho{}_\nu \zeta^\sigma \,. \tag{2.29}$$

Contracting with $g^{\mu\nu}$ we obtain

$$D_\rho D^\rho \zeta_\mu = R_{\mu\nu} \zeta^\nu \,. \tag{2.30}$$

Expressing the relation with differential forms and exploiting Einstein equations we finally get

[5]The entropy has been successfully computed only in the non extremal case. The approximation performed breaks before reaching the extremal limit, therefore it is not obvious whether the result still hold in the extremal case.

[6]We indicate the Lie derivative with the symbol $\mathcal{L}$. It is the same we use for the Lagrangian density, however the right interpretation should be clear given the context.

$$j_\mu = \frac{1}{8\pi} R_{\mu\nu}\zeta^\nu = (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\mu_\mu)\,, \tag{2.31}$$

$$\star\, d \star d\zeta = 8\pi j \quad \Rightarrow \quad d \star j = 0\,. \tag{2.32}$$

The conserved charge related to ζ is then obtained integrating the Komar current over a space-like hypersurface Σ

$$Q_\zeta = \int_\Sigma \star j = \frac{1}{8\pi}\int_{\partial\Sigma} \star d\zeta\,. \tag{2.33}$$

Q_ζ is called the Komar's charge associated with ζ.

In higher order derivative theories it is possible to define a generalization of Komar charges proving their conservation and their definition without using Einstein equations of motion. However, such generalization presents some subtleties. Following [11], let us consider a general theory of gravity $\mathcal{L}$ in n dimensions with at most k derivatives. Exploiting the relation

$$[D_\mu, D_\nu]\psi^\alpha_\beta = R_{\mu\nu}{}^\alpha{}_\gamma \psi^\gamma_\alpha - R_{\mu\nu}{}^\gamma{}_\beta \psi^\alpha_\gamma\,, \tag{2.34}$$

where ψ^α_β is a $(1,1)$ tensor, we can always rewrite $\mathcal{L}$ in order to have only totally symmetric combinations of fields derivatives [7]. Varying the action and defining the n-form $L = \star\mathcal{L}$ we get

$$\delta S = \int \delta L = \int E\delta\phi + d\Theta\,, \tag{2.35}$$

where ϕ indicates a generic dynamical field of the theory and $d\Theta$ is called symplectic prepotential and it is an exact $(n-1)$-form. In particular $\Theta = \Theta(\phi, \delta\phi)$. Equations of motion read then:

$$E = 0\,. \tag{2.36}$$

We consider now a generic section of the tangent bundle ζ (we are not assuming that it is a killing vector) and we define the variation given by the relative Lie derivative:

$$\hat{\delta} = \mathcal{L}_\zeta\,. \tag{2.37}$$

For every ζ can be defined then the $n-1$ form

$$J = \hat{\Theta} - \zeta \cdot L\,, \qquad \hat{\Theta} = \Theta(\phi, \hat{\delta}\phi)\,, \tag{2.38}$$

where the dot is intended as the interior product. Recalling Cartan Identity

$$\mathcal{L}_\zeta \psi = \zeta \cdot d\psi + d(\zeta \cdot \psi)\,, \tag{2.39}$$

we obtain

$$\hat{\delta} L = \mathcal{L}_\zeta = d(\zeta \cdot L)\,. \tag{2.40}$$

Exploiting equation (2.40), (2.35) and imposing equations of motion (2.36) the differential of (2.38) becomes

$$dJ = d\hat{\Theta} - d(\zeta \cdot L) = \hat{\delta} L - E\hat{\delta}\phi - \hat{\delta} L = -E\hat{\delta}\phi = 0\,. \tag{2.41}$$

[7] This step is purely conventional. However, we will show that quantities are affected by ambiguities. It is therefore important to fix the starting of the computations we perform.

Therefore there exists a $(n-2)$-form Q conserved on-shell such that $J = dQ$. Obviously Q is defined up to an exact form dZ, however such ambiguity can be removed integrating over a suitable $(n-2)$-surface

$$q = \int_\Sigma J = \int_\Sigma dQ = \int_{\partial\Sigma} Q \,. \tag{2.42}$$

The $(n-2)$-form Q seems to be a good quantity which generalizes the notion of Komar charge. In particular it seems that exist a conserved charge for every possible tangent bundle section. However, this quantity has some bad features and it is not well defined.

Let us restart from the definition of Θ. Θ is fixed in order to satisfy the Lagrangian variation. However, equation (2.35) tell us that the constraint is imposed on $d\Theta$ and not directly on Θ. It follows that Θ is defined up to an exact form $dY(\phi, \delta\phi)$. This implies that J is defined up to $dY(\phi, \hat{\delta}\phi)$ and that Q is defined up to $Y(\phi, \hat{\delta}\phi)$. Moreover, the definition of Θ has another more important problem: it is sensible to total derivative variations $d\mu$ of the Lagrangian. In particular, it is shifted by a quantity $\delta\mu$. Exploiting (2.39) we obtain that J is instead shifted by:

$$J = \hat{\Theta} - \zeta \cdot L \to J + \hat{\delta}\mu - \zeta \cdot d\mu = J + d(\zeta \cdot \mu) \,. \tag{2.43}$$

Finally, Q gets shifted by $\zeta \cdot \mu$. To sum up, we have that J, Q and Θ are defined up to the transformations

$$\Theta \to \Theta + \delta\mu + dY(\phi, \delta\phi) \,, \tag{2.44}$$

$$J \to J + d(\zeta \cdot \mu) + dY(\phi, \hat{\delta}\phi) \,, \tag{2.45}$$

$$Q \to Q + \zeta \cdot \mu + Y(\phi, \hat{\delta}\phi) + dZ \,. \tag{2.46}$$

Let assume now that ζ is a killing vector and that $\delta\zeta = 0$ (this condition in case of time translations is nothing but the requirement that perturbations are stationary). Being ζ a killing vector it follows that $\hat{\delta}\phi = 0$. The charge (2.42) is therefore defined up to the transformation

$$q = \int_{\partial\Sigma} Q \to q + \int_{\partial\Sigma} \zeta \cdot \mu \,. \tag{2.47}$$

We want to find now a quantity whose variation cancels that of q. We consider then

$$\int_{\partial\Sigma} \zeta \cdot \Theta \,. \tag{2.48}$$

which is defined up to the transformation

$$\int_{\partial\Sigma} \zeta \cdot \Theta \to \int_{\partial\Sigma} \zeta \cdot \Theta + \zeta \cdot \delta\mu + \zeta \cdot dY(\phi, \delta\phi) \,, \tag{2.49}$$

The term (2.48) can not be used to built an invariant definition of charge. However, recalling that $\delta\zeta = 0$ we have that $\zeta \cdot \delta\mu = \delta(\zeta \cdot \mu)$. Moreover, exploiting (2.39) we get

$$\zeta \cdot dY = \mathcal{L}_\zeta Y - d(\zeta \cdot Y) = Y(\phi, \mathcal{L}_\zeta \delta\phi) = \delta Y(\phi, \mathcal{L}_\zeta \phi) \,, \tag{2.50}$$

where in the second step we dropped the second term (we are integrating on a boundary) and in the last two steps we exploited the fact that $\mathcal{L}_\zeta \phi = 0$. It follows that introducing a $(n-1)$ differential form B such that

$$\delta \int_{\partial\Sigma} \zeta \cdot B = \int_{\partial\Sigma} \zeta \cdot \Theta \,, \tag{2.51}$$

we get

$$\int_{\partial\Sigma} \zeta \cdot \Theta \to \int_{\partial\Sigma} \zeta \cdot \Theta + \delta(\zeta \cdot \mu) + \delta Y(\phi, \mathcal{L}_\zeta \phi)\,, \tag{2.52}$$

$$\int_{\partial\Sigma} \zeta \cdot B \to \int_{\partial\Sigma} \zeta \cdot B + \zeta \cdot \mu\,, \tag{2.53}$$

where we exploited $Y(\phi, \mathcal{L}_\zeta \phi) = 0$. Therefore, we can use (2.53) to finally define the conserved charge

$$\tilde{q}_\zeta = \int_{\partial\Sigma} Q - \zeta \cdot B = \int_\Sigma J_\zeta - d(\zeta \cdot B)\,, \tag{2.54}$$

whose definition has no ambiguities. Notice that to build such quantity has been fundamental that ζ was a Killing vector.

2.3.2 Noether Charge Explicit Formula

In section 2.3.1 we proved that it is possible to extend the notion of Komar charge in higher order derivative theories. However, in order to define $\tilde{q}$ we introduced three quantities, Θ, J and Q, whose definition is ambiguous. Therefore, we can not just impose (2.54) to determine an explicit expression for $\tilde{q}$. We have to specify an algorithm which guarantees that Θ, J and Q are computed in a consistent way. We present now the main steps of the algorithm used by Wald. The procedure involves a huge amount of algebra which we don't report. The interested reader can see [11], [12], [32].

Given a theory $\mathcal{L}$, we firstly use equation (2.34) to have only totally symmetric combinations of fields derivatives. Once we get the symmetric lagrangian density $\mathcal{L}_S$ we have to fix its boundary terms. Θ, J and Q are indeed sensible to total a derivative shift. Θ can be computed then through the boundary terms which arise from the action variation (cfr. equation (2.35)). Applying the algorithm provided by [12] to the symmetric Lagrangian density $\mathcal{L}$ it is possible to show that Θ can be written as

$$\Theta = 2E^{bcd} D_d \delta g_{bc} + \Theta'\,. \tag{2.55}$$

Θ' is a $(n-1)$ differential form where the variation δ is always on the left of derivatives operators. It has the structure

$$\begin{aligned} \Theta' \quad &= \quad S^{ab}(\phi)\,\delta g_{ab} + \sum_{l=0}^{k} T_{(l)}(\phi)^{a_1,\dots,a_l} \delta D_{(a_1} \dots D_{a_l)} R_{abcd} \\ &\qquad + \sum_{i\in\{\psi\}} \sum_{l=0}^{k} U_{(i,l)}(\phi)^{a_1,\dots,a_l} \delta D_{(a_1} \dots D_{a_l)} \psi\,, \end{aligned} \tag{2.56}$$

where ϕ indicates all the dynamical fields of the theory and ψ indicates all the dynamical fields except the metric. E^{bcd} is a $(n-1)$ differential form given by

$$E^{bcd} = E^{abcd} \bar{\epsilon}_a\,, \qquad \bar{\epsilon}_a = \frac{\sqrt{-g}}{(n-1)!} \epsilon_{aa_2\dots a_n} dx^{a_2} \wedge \dots \wedge dx^{a_n}\,, \tag{2.57}$$

and E^{abcd} is a tensor which can be interpret as the equations of motion obtained assuming that R_{abcd} was an independent dynamical field, i.e. it holds

$$\begin{aligned}\delta\mathcal{L} = E^{abcd}\delta R_{abcd} = & \Big[\frac{\partial\mathcal{L}_S}{\partial R_{abcd}} - D_{a_1}\frac{\partial\mathcal{L}_S}{\partial D_{a_1}R_{abcd}} + \dots \\ & \dots (-1)^k D_{(a_1}\dots D_{a_k)}\frac{\partial\mathcal{L}_S}{\partial D_{(a_1}\dots D_{a_k)}R_{abcd}}\Big]\delta R_{abcd}\,.\end{aligned} \tag{2.58}$$

Given our choice of Θ, the conserved current J associated with the section of the tangent bundle ζ reads:

$$\begin{aligned} J &= 2E^{bcd}D_d\hat{\delta}g_{bc} + \Theta'(\phi,\hat{\delta}\phi) - \zeta\cdot L \\ &= 2E^{bcd}D_d(D_b\zeta_c + D_c\zeta_b) + \Theta'(\phi,\mathcal{L}_\zeta\phi) - \zeta\cdot L\,.\end{aligned} \tag{2.59}$$

Q is finally defined by $J = dQ$. The algorithm described in [32] guarantees that we can always invert the relation and express Q in terms of the theory fields. However, we are just interested in determining the structure of Q expansion in terms of ζ field (the reason will be clear later), therefore it is not necessary to use it to extract the explicit expression of Q. We start then by looking at the ζ dependency in (2.59). Θ' is linear in ζ and $D\zeta$ (cfr. equation 2.56). Moreover, recalling that E^{bcd} is antisymmetric in the last two indexes and exploiting equation (2.34) we have

$$E^{bcd}\,D_dD_c\zeta_b = \frac{1}{2}E^{bcd}\,[D_d,D_c]\,\zeta_b = -\frac{1}{2}E^{bcd}\,R_{dc}{}^{f}{}_{b}\,\zeta_f\,. \tag{2.60}$$

Therefore, $E^{bcd}D_dD_b\zeta_c$ is the term of highest derivate order of (2.59). Being proportional to $D^2\zeta$, it is the only term which produces contributes in Q proportional to $D\zeta$. It follows that Q takes the general form (for our conventional choices of Θ and J)

$$Q = X^{cd}D_{[c}\zeta_{d]} + \zeta\cdot W(\phi)\,, \tag{2.61}$$

where $W(\phi)$ and X^{cd} are differential forms locally defined through the dynamical fields of the theory with dimensions $(n-1)$ and $(n-2)$ respectively. According to the previous observations we can readily determine X^{cd} relating it with $D^2\zeta$ term in (2.59). It holds

$$X^{cd} = -E^{abcd}\bar{\epsilon}_{ab}\,, \qquad \bar{\epsilon}_{ab} = \frac{\sqrt{-g}}{(n-2)!}\epsilon_{aba_3\dots a_n}dx^{a_3}\wedge\dots\wedge dx^{a_n}\,. \tag{2.62}$$

Equation (2.54) then becomes

$$\tilde{q}_\zeta = \int_{\partial\Sigma} X^{cd}D_{[c}\zeta_{d]} + \zeta\cdot W(\phi) - \zeta\cdot B\,, \tag{2.63}$$

$$\delta\int_{\partial\Sigma}\zeta\cdot B = \int_{\partial\Sigma}\zeta\cdot 2E^{bcd}D_d\delta g_{bc} + \Theta'(\phi,\delta\phi)\,. \tag{2.64}$$

2.3.3 Mass in Higher Order Theories

In Newtonian relativity it is possible to associate a energy density directly to the gravitational field. It holds indeed

$$\mathcal{E} = -\frac{1}{8\pi}|\vec{\nabla}\phi|^2\,, \tag{2.65}$$

where ϕ is the gravitational field. Considering general relativity and taking the Newtonian limit of the theory it turns out that ϕ corresponds to a metric component (see [33]). We can guess then that the energy density could be expressed by an operator quadratic in spacetime metric first derivative. However, it is not possible to built a meaningful operator of that type. Despite there is no general notion of energy density of the gravitational field, it is still possible to define the total energy-momentum 4-vector of an isolated system, i.e. the 4-vector seen by an observer in the asymptotically flat region.

A first definition of the total mass of a asymptotically flat configuration can be given as the bulk mass seen by a test particle moving at infinity. Assuming for simplicity a stationary field configuration with spherical symmetry, Newtonian gravity tell us that a particle which moves in the field ϕ generated by a mass M satisfy:

$$\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\phi\,, \qquad \phi = -G\frac{M}{r}\,, \tag{2.66}$$

where $\vec{x}$ are the space coordinates of the test particle in a particular reference frame and r is understood. Being the particle in the asymptotically flat region, weak field regime applies and general relativity implies

$$\frac{d^2\vec{x}}{dt^2} \approx \frac{1}{2}\vec{\nabla}h_{00}\,, \qquad g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{2.67}$$

Requiring by consistency that general relativity has to reproduce Newtonian gravity in the weak field approximation we get

$$g_{00} \approx (-1 + 2\phi) = -\left(1 - \frac{2GM}{r}\right)\,. \tag{2.68}$$

Thus, in general relativity the bulk mass is proportional to a residual of $1/r$ term of g_{00}.

A second definition of the mass can be formulated in terms of Komar charges. Let us consider a field configuration which is asymptotically flat. It is always possible to define a Killing vector t which correspond to a time translation at infinity. Using such Killing vector it holds

$$Q_t = -\frac{1}{8\pi}\lim_{r\to\infty}\int_{S_r^2} \star\, dt = M\,. \tag{2.69}$$

The total energy can be therefore characterized as the Komar charge associated with time translations and equation (2.32) guarantees its conservation.

We consider now the case of a 4 dimensional Schwarzschild black hole. Such configuration is asymptotically flat, therefore it admits ∂_0 as Killing vector. We will show that the charge $\tilde{q}_t$ with boundary the asymptotic sphere coincides with the Komar mass of the BH. Applying the prescriptions of the previous section to General Relativity we get

$$\begin{aligned}
\Theta &= \frac{1}{16\pi}g^{de}g^{fh}\left(D_f\delta g_{eh} - D_e\delta g_{fh}\right)\bar{\epsilon}_d\,, && (2.70)\\
J &= \frac{1}{8\pi}D_e\left(D^{[e}t^{d]}\right)\bar{\epsilon}_d\,, && (2.71)\\
\bar{\epsilon}_d &= \frac{\sqrt{-g}}{(n-1)!}\epsilon_{da_2\ldots a_n}dx^{a_2}\wedge\cdots\wedge dx^{a_n}\,. && (2.72)
\end{aligned}$$

Comparing equation (2.70) with (2.55) we read that $\Theta' = 0$. Moreover, the structure of the current (2.71) tells us that $W(\phi) = 0$. Therefore, equation (2.63) takes the form

$$\tilde{q}_\zeta = \int_{\partial\Sigma} X^{cd} D_{[c}\zeta_{d]} - \zeta \cdot B \,, \tag{2.73}$$

$$\delta \int_{\partial\Sigma} \zeta \cdot B = \int_{\partial\Sigma} \zeta \cdot \left(2E^{bcd} D_d \delta g_{bc}\right) . \tag{2.74}$$

We compute now the first term of equation (2.73). X^{cd} can be written as

$$X^{ab} = 2s \star E^{ab} \,, \qquad E^{ab} = E^{abcd} g_{cc'} g_{dd'} \frac{dx^{c'} \wedge dx^{d'}}{2} \,, \tag{2.75}$$

therefore we get

$$\mathcal{E}_1 = \int_\infty X^{cd} D_{[c}t_{d]} = 2s \int_\infty \star E^{cd} D_{[c}t_{d]} \,. \tag{2.76}$$

Exploiting equation (A.53) we have then

$$\mathcal{E}_1 = -\int_\infty E^{abcd} \eta_{ab} D_{[c}t_{d]} \, e_\infty = -A\eta_{ab} D_{[c}t_{d]} E^{abcd} \,, \tag{2.77}$$

where all the quantities are intended as evaluated in the limit $r \to \infty$. Recalling that $t^\mu = \delta^\mu_0$, $\eta_{ab} = 2\sqrt{|g_{00}g_{11}|}\delta^{0,1}_{[a,b]}$ and exploiting symmetry properties of E^{abcd} indices we get

$$\mathcal{E}_1 = 2A\partial_c g_{00} \sqrt{|g_{00}g_{11}|} E^{010c} \,. \tag{2.78}$$

Let us assume $M_P = 1$, so that Einstein Lagrangian is just $R/2$. The E^{abcd} tensor then simplifies to

$$E^{abcd} = \frac{1}{4}(g^{ac}g^{bd} - g^{bc}g^{ad}) \,, \tag{2.79}$$

and $g_{00} = -1/g_{11} = -(1 - r_s/r)$. Equation (2.78) becomes

$$\mathcal{E}_1 = -\frac{1}{2} A\partial_1 g_{00} = \lim_{r\to\infty} \left[\frac{1}{2} 4\pi r^2 \frac{2M}{8\pi r^2}\right] = \frac{M}{2} \,, \tag{2.80}$$

where we introduced the Schwarzschild radius $r_s = M/4\pi$.
Variation of the second term of equation (2.63) reads instead

$$\delta\mathcal{E}_2 = -\int_\infty t \cdot \theta = \int_\infty \langle t \cdot E^{bcd}, \star\eta_{ab}\rangle \, 2 \, D_d \delta g_{bc} = 2A\sqrt{|g_{00}g_{11}|} E^{1bcd} (D_d \delta g_{bc}) \,, \tag{2.81}$$

and again all the quantities are intended as evaluated in the limit $r \to \infty$. Exploiting equation (2.79) we get

$$\delta\mathcal{E}_2 = \int_\infty e_\infty \frac{1}{2} D_1 \delta g_{00} = \delta \int_\infty e_\infty \frac{1}{2} \partial_1 \delta g_{00} \,, \tag{2.82}$$

$$\mathcal{E}_2 = \int_\infty e_\infty \frac{1}{2} \partial_1 g_{00} = \frac{M}{2} \,. \tag{2.83}$$

It follows that $\tilde{q}_t = \mathcal{E}_1 + \mathcal{E}_2 = M$. As we claimed $\tilde{q}_t$ reproduces Komar mass.

2.3.4 Entropy Formula

We conclude this section presenting Wald's entropy formula and showing that it satisfies the first law of thermodynamics. Let us consider a black hole configuration and let ζ be the Killing vector associated with its Killing horizon H. Exploiting the fact that ζ vanishes on H and choosing $\partial\Sigma = H$ equation (2.63) becomes

$$\tilde{q}_\zeta = \int_H X^{ab} D_{[a}\zeta_{b]} \,. \tag{2.84}$$

Introducing the surface gravity κ and the binormal to the Killing horizon η_{ab} we get

$$\tilde{q}_\zeta = \kappa \int_H X^{ab}\eta_{ab} \,, \tag{2.85}$$

where we used that $D_{[a}\zeta_{b]} = \kappa\,\eta_{ab}$ and that κ is constant on H.

Let us consider now a non rotating asymptotically flat stationary black hole with spherical symmetry. BH horizon is a Killing horizon H with Killing vector ∂_t. Integrating the associated current on an space-like hypersurface Σ with inner boundary H we get

$$\tilde{q}_t = \int_\Sigma J_t - d(t\cdot B) = -\int_H Q_t + \int_\infty Q_t - t\cdot B \tag{2.86}$$

According with the previous section, we can identify the integral on the asymptotic surface as the mass of the black hole. In the static case it corresponds to the black hole energy. Exploiting equation (2.85) we can write then

$$\tilde{q}_t = -\kappa \int_H X^{ab}\eta_{ab} + \mathcal{E} \tag{2.87}$$

Being $\delta t = 0$, it holds $\delta\kappa = 0$. Therefore, assuming that $\delta\tilde{q}_t = 0$ (see below), varying equation (2.87) we get

$$T\,\delta\left\{2\pi \int_H X^{ab}\eta_{ab}\right\} = \delta\mathcal{E} \,, \tag{2.88}$$

where we introduced the Hawking temperature $T = \frac{\kappa}{2\pi}$. Equation (2.88) is nothing but the first law of thermodynamics. From equation (2.88) we finally read the entropy

$$S = 2\pi \int_{S_H} \eta_{ab} X^{ab} \,. \tag{2.89}$$

We conclude verifying[8] that $\delta\tilde{q}_t = 0$. Varying equation (2.86) we get

$$\delta\tilde{q}_t = \int_\Sigma \delta J_t - \delta\, d(t\cdot B) = \int_\Sigma \delta\,(\hat{\Theta} - t\cdot\mathcal{L}) - d(t\cdot\Theta) \,, \tag{2.90}$$

where in the second step we exploited J and B definition (cfr. equations (2.38), (2.51)). Recalling that we are considering stationary perturbations which satisfy $\delta t = 0$ and exploiting Cartan identity (2.39) we obtain

$$\delta\tilde{q}_t = \int_\Sigma \delta\,\hat{\Theta} - t\cdot\delta\mathcal{L} - d(t\cdot\Theta) = \int_\Sigma \delta\,\hat{\Theta} - \hat{\delta}\,\Theta = \int_\Sigma \omega \,, \tag{2.91}$$

[8] A more general argument which implies that the variation of every Killing charge is null can be found in [32]

where we introduced the so called symplectic potential ω. Notice that although LHS is a well defined quantity, ω presents the same ambiguities of Θ, J and Q. It has the structure

$$\omega(\phi, \delta\phi, \hat{\delta}\phi) = \delta\,\Theta(\phi, \hat{\delta}\phi) - \hat{\delta}\,\Theta(\phi, \delta\phi) \tag{2.92}$$

Considering the second and the fourth term of equation (2.50) we can write ω as

$$\omega = d\,(t \cdot \Theta) \tag{2.93}$$

Replacing equation (2.93) in the RHS of equation (2.91) and applying Stokes theorem we get

$$\delta\tilde{q}_t = \int_\Sigma d\,(t \cdot \Theta) = -\int_H t \cdot \Theta + \int_\infty t \cdot \Theta \tag{2.94}$$

Being t the Killing vector of the Killing horizon H, the first term of (2.94) vanishes. Recalling that asymptotic flatness fixes $\delta\phi(\infty) = 0$, Θ is null at spatial infinity and the second term vanishes too. We have therefore $\delta\tilde{q}_t = 0$.

CHAPTER 3

Black Hole Solutions

In the previous chapter we determined lowest order correction to Einstein–Maxwell theory and Wald's entropy formula. We have therefore the framework and the formalism suitable to evaluate whether black holes splitting processes are allowed or not. The literature guarantees us that the kinematic constraints provided by the EWGC are trivially satisfied once positivity bounds due to S-matrix properties are taken into account (see [7]). However, the dynamics constraints due to the second law of thermodynamics have not been discussed so far. In this chapter we present our original analysis on the existence of configurations which satisfy both the kinematic and thermodynamic constraints. It turns out that the splitting processes are not allowed. They involve indeed small black hole solutions which break the perturbative regime.

We will start by solving perturbatively 4 derivative Einstein–Maxwell equations of motion treating higher order terms as small corrections. Once we have the metric expression, we will determine the corrections to charge-to-mass ratio and to the radius of extremal BH configurations, verifying that higher order corrections do not change the structure of the black hole horizon (it is still a Killing horizon and its surface gravity vanishes in the extremal limit). After that, we will evaluate the correction to the entropy of extremal configurations through Wald's formula. Such computation provides the first relevant original result of the thesis work. It extends indeed the computations of [14] where the entropy diverges in the extremal limit. Exploiting positivity bounds due to S-matrix properties, we will be able to fix the corrections signs and we will use them to prove the mild version of the EWGC and to discuss BH splitting processes. We will finally provide two highly non trivial consistency checks. The first one tests Wald's formalism. We will verify that the thermodynamic temperature computed from Wald's entropy taking the mass derivative vanishes in the extremal limit as well as the Hawking temperature. The second consistency check tests theories equivalence under fields redefinition. We will verify that the entropy and the charge-to-mass ratio of extremal black holes are invariant under such transformations, proving the claim of section 2.2. We conclude this chapter by discussing and analyzing the differences between our result and those of the literature.

3.1 Perturbative Solution of Einstein–Maxwell Theory

Assuming that the variational principle still holds in higher derivative theories, we can compute the exact equations of motion (EOM) of the theory (2.19) varying the action with respect of dynamical fields:

$$
\begin{aligned}
\delta_{A_\nu} S &= \int d^4x \sqrt{|g|}\, D_\mu \left[F^{\mu\nu} - \Delta F^{\mu\nu}\right] \delta A_\nu \,, && (3.1)\\
\delta_{g^{\mu\nu}} S &= \frac{M_P^2}{2} \int d^4x \sqrt{|g|} \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{M_P^2} T_{\mu\nu}\right] \delta g^{\mu\nu} \,, && (3.2)
\end{aligned}
$$

$$
\text{EOM}: \begin{cases} G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{M_P^2} T_{\mu\nu} \,, \\ D_\mu F^{\mu\nu} = J^\nu + D_\mu \Delta F^{\mu\nu} \,, \end{cases} \tag{3.3}
$$

where we introduced the quantities

$$
\begin{aligned}
\Delta F^{\mu\nu} &= \frac{2\alpha_1}{M_P^4} F^2 F^{\mu\nu} + \frac{2\alpha_2}{M_P^4} F\tilde{F}\tilde{F}^{\mu\nu} + \frac{2\alpha_3}{M_P^2} W^{\mu\nu\rho\sigma} F_{\rho\sigma} \,, && (3.4)\\
T_{\mu\nu} &= -\frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\mu\nu}} \left(\sqrt{|g|}\mathcal{L}\right) \,, && (3.5)
\end{aligned}
$$

and Stress-Energy tensor components take the explicit form

$$
T^{(2)}_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\mu\nu}} \left(\sqrt{|g|}\mathcal{L}_2\right) = F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \,, \tag{3.6}
$$

$$
\begin{aligned}
T^{(4)}_{\mu\nu} &= -\frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\mu\nu}} \left(\sqrt{|g|}\mathcal{L}_4\right) \;=\; g_{\mu\nu}\mathcal{L}_4 - 2\delta\mathcal{L}_4 \\
&= \frac{\alpha_1}{4M_P^4} \left\{ g_{\mu\nu} F^4 - 8F^2 F_{\mu\rho} F_\nu{}^\rho \right\} + \frac{\alpha_2}{4M_P^4} \left\{ g_{\mu\nu} (F\tilde{F})^2 - 2 g_{\mu\nu} (F\tilde{F})^2 \right\} \\
&\quad + \frac{\alpha_3}{2M_P^2} \Big\{ -6 F_{\alpha(\nu|} F^{\beta\gamma} R^\alpha{}_{|\mu)\beta\gamma} - 4 D_\beta D_\alpha \left(F^\alpha{}_{(\mu|} F^\beta{}_{|\nu)} \right) \\
&\quad + 8 R_{(\nu|\sigma} F_{|\mu)\rho} F^{\sigma\rho} + 4 R^{\alpha\beta} F_{\alpha\mu} F_{\beta\nu} + 2 g_{\mu\nu} D_\alpha D_\beta (F^\alpha{}_\rho F^{\beta\rho}) \\
&\quad - 4 D_\alpha D_{(\nu|} \left(F_{|\mu)\beta} F^{\alpha\beta} \right) + 2D^2 (F_{\mu\rho} F_\nu{}^\rho) - \frac{4}{3} R F_\mu{}^\sigma F_{\nu\sigma} - \frac{2}{3} F^2 R_{\mu\nu} \\
&\quad + \frac{2}{3} D_{(\mu|} D_{|\nu)} F^2 - \frac{2}{3} g_{\mu\nu} D^2 F^2 + g_{\mu\nu} W_{\alpha\beta\rho\sigma} F^{\alpha\beta} F^{\rho\sigma} \Big\} \,.
\end{aligned} \tag{3.7}
$$

Assuming that the α_i coefficients are small, we can treat $\mathcal{L}_4$ as a perturbation and solve perturbatively the equations of motion. Denoting with (n) the order of approximation in the expansion parameters[1], we have that for general higher order derivative corrections

[1] i.e. $g^{(n)}$ is the solution exact up to α_i^n terms.

hold

$$
\begin{aligned}
M_P^2 G_{\mu\nu}\left[g_{\alpha\beta}^{(n)}\right] &= T_{\mu\nu}\left[g_{\alpha\beta}^{(n)}, A_\gamma^{(n)}\right] \\
&\simeq T_{\mu\nu}^{(2)}\left[g_{\alpha\beta}^{(n)}, A_\gamma^{(n)}\right] + T_{\mu\nu}^{(4)}\left[g_{\alpha\beta}^{(n-1)}, A_\gamma^{(n-1)}\right] + \dots ,
\end{aligned} \tag{3.8}
$$

$$
D_\mu F^{\mu\nu}\left[g_{\alpha\beta}^{(n)}, A_\gamma^{(n)}\right] \simeq J^\nu + D_\mu \Delta F^{\mu\nu}\left[g_{\alpha\beta}^{(n-1)}, A_\gamma^{(n-1)}\right] + \dots , \tag{3.9}
$$

where dots denote corrections with order higher than fourth. Considering only fourth order corrections and using (3.9) to reduce the order of A_γ dependence in (3.8) we get[2]

$$
\begin{aligned}
M_P^2 G_{\mu\nu}\left[g_{\alpha\beta}^{(1)}\right] \simeq\ & T_{\mu\nu}^{(2)}\left[g_{\alpha\beta}^{(1)}, A_\gamma^{(0)}\right] + \Big(2\Delta F_{(\mu|\rho}F_{|\nu)}{}^{\rho} \\
& - \frac{1}{2} g_{\mu\nu}\Delta F_{\rho\sigma}F^{\rho\sigma} + T_{\mu\nu}^{(4)}\Big)\left[g_{\alpha\beta}^{(0)}, A_\gamma^{(0)}\right] .
\end{aligned} \tag{3.10}
$$

Notice that even if we were able to solve the exact equations of motion (3.1) and (3.2), the solution would be meaningless outside the regime of validity of equation (3.10). If the perturbative expansion breaks $\mathcal{L}_4$ is no more a small correction to Einstein–Maxwell theory and we can not neglect terms with order higher than 4. We would have then to consider further corrections.

Now, we look for black hole solutions. Considering stationary, charged and spherical solutions we obtain for the unperturbed case the well known Reissner–Nordström (RN) solution (see [25]):

$$
ds^2 = -f_0(r)^2\, dt^2 + \frac{1}{f_0(r)^2}\, dr^2 + r^2\, d\Omega_{S_2}^2 , \qquad F = \frac{Q}{4\pi r^2}\, dr \wedge dt , \tag{3.11}
$$

$$
f_0(r)^2 = 1 - \frac{M}{4\pi M_P^2 r} + \frac{Q^2}{32\pi^2 M_P^2 r^2} . \tag{3.12}
$$

To derive the perturbed solution we require that in the limit $\alpha_i \to 0$ we must recover the RN solution. The most general ansatz for the metric assuming a stationary, charged and spherical BH is then

$$
ds^2 = -N(r)^2 f_1(r)\, dt^2 + \frac{1}{f_1(r)}\, dr^2 + r^2\, d\Omega_{S_2}^2 , \tag{3.13}
$$

and the asymptotic conditions take the form

$$
g_{tt}^{(1)} = -N(r)^2 f_1(r) = -f_0(r)^2 + \Delta g_{tt} + o(\alpha_i) , \qquad \Delta g_{tt} \xrightarrow{\alpha_i \to 0} 0 , \tag{3.14}
$$

$$
(g_{rr}^{(1)})^{-1} = f_1(r) = f_0(r)^2 + \Delta f + o(\alpha_i) , \qquad \Delta f \xrightarrow{\alpha_i \to 0} 0 . \tag{3.15}
$$

To determine $N(r)$ and $f_1(r)$ we replace the ansatz (3.13) in (3.10) and we take proper combinations of the independent EOM. Let us introduce the more compact notation

$$
\mathcal{E}_\mu^\nu = M_P^2 G_\mu^\nu - T_\mu^\nu , \tag{3.16}
$$

[2] Note that keeping $g^{(1)}$ dependence in $T_{\mu\nu}^{(2)}$ term of equation (3.10) we are considering contributions that are neglected using the techniques presented in [34].

where we have dropped explicit dependence on dynamical fields having in mind the decomposition [3] (3.10). Let us consider the combination:

$$\mathcal{E}_1^1 - \mathcal{E}_0^0 = 0 \qquad \Longleftrightarrow \qquad \frac{\alpha_3\, Q^2\, f_0(r)^2}{12\pi^2 M_P^4\, r^6} + \frac{2 f_1(r) N'(r)}{r N(r)} = 0\,. \tag{3.17}$$

Massaging equation (3.17) we get

$$\begin{aligned} N(r) &= K \exp\left[-\frac{\alpha_3\, Q^2}{24\, M_P^4 \pi^2} \int dr \frac{f_0(r)^2}{r^5 f_1(r)}\right] \simeq K \exp\left[-\frac{\alpha_3\, Q^2}{24\, M_P^4 \pi^2} \int dr \frac{1}{r^5}\right] \\ &\simeq K\left[1 - \frac{\alpha_3\, Q^2}{24\, M_P^4 \pi^2} \int dr \frac{1}{r^5}\right] = \left[1 + \frac{\alpha_3\, Q^2}{96 \pi^2 M_P^4 r^4}\right]\,, \end{aligned} \tag{3.18}$$

where we have removed $f_1(r)$ using eq. (3.15) and neglecting higher order contributions. The integration constant K is fixed imposing the correct limit $g_{tt}^{(1)}$ for vanishing α_i. Let us consider now the relation:

$$\mathcal{E}_0^0 + \mathcal{E}_2^2 = 0\,. \tag{3.19}$$

Replacing equation (3.18) in (3.19) we have:

$$\frac{d}{dr}\left[2 r f_1(r) + r^2 f_1'(r)\right] = 2 + \alpha_1 h_1(r) + \alpha_3 h_3(r)\,, \tag{3.20}$$

$$h_1(r) = -\frac{Q^4}{64\pi^4 M^6 r^6}\,, \tag{3.21}$$

$$\begin{aligned} h_3(r) &= \frac{Q^2}{12\,\pi^2 M_P^4\, r^4}\left[1 - \left(2 f_0(r) - r f_0'(r)\right)^2 - r^2 f_0(r) f_0''(r)\right] \\ &\quad - \frac{Q^2 f_0(r)^2 \left[\alpha_3\, Q^2 f_0(r)^2 - 12\,\pi^2 M_P^4\, r^5 f_1'(r)\right]}{263\,\pi^4\, M_P^8\, r^8\, f_1(r)}\,. \end{aligned} \tag{3.22}$$

Equation (3.20) cannot be solved easily because of the presence of $f_1(r)$ in $h_3(r)$. However, we observe that $h_3(r)$ is multiplied by an α_3 factor. Exploiting (3.15) and neglecting higher order terms in perturbation parameters, equation (3.22) becomes

$$h_3(r) \simeq \frac{Q^2}{12\,\pi^2 M_P^4\, r^4}\left[1 - \left(2 f_0(r) - r f_0'(r)\right)^2 - r^2 f_0(r) f_0''(r)\right] + \frac{24\, Q^2 f_0(r) f_0'(r)}{263\,\pi^2\, M_P^4\, r^3}\,. \tag{3.23}$$

Through a simple double integration[4] we get then:

$$f_1(r) = 1 - \frac{d_2}{r} + \frac{d_1}{r^2} - \frac{\alpha_1\, Q^4}{1280\,\pi^4 M_P^6\, r^6} + \frac{\alpha_3\, Q^2}{M_P^6\, r^6}\left(\frac{5\, M\, r}{384\,\pi^3} - \frac{Q^2}{640\,\pi^4} - \frac{M_P^2\, r^2}{24\,\pi^2}\right)\,, \tag{3.24}$$

[3] In particular we have raised indexes using perturbed and unperturbed metric according to the correct field dependence.

[4] After the first integration we get a standard first order linear differential equation.

where d_1 and d_2 are the integration constants. We can fix them imposing (3.15). We have finally:

$$d_1 = \frac{Q^2}{32\,\pi^2\,M_P^2}\,, \qquad d_2 = \frac{M}{4\pi M_P^2}\,, \tag{3.25}$$

$$g_{tt} = -f_0(r)^2 + \frac{\alpha_1\,Q^4}{1280\,\pi^4 M_P^6\,r^6} - \frac{\alpha_3\,Q^2}{M_P^6\,r^6}\left(\frac{M\,r}{128\,\pi^3} - \frac{7\,Q^2}{7680\,\pi^4} - \frac{M_P^2\,r^2}{48\,\pi^2}\right)\,, \tag{3.26}$$

$$\frac{1}{g_{rr}} = f_0(r)^2 - \frac{\alpha_1\,Q^4}{1280\,\pi^4 M_P^6\,r^6} + \frac{\alpha_3\,Q^2}{M_P^6\,r^6}\left(\frac{5\,M\,r}{384\,\pi^3} - \frac{Q^2}{640\,\pi^4} - \frac{M_P^2\,r^2}{24\,\pi^2}\right)\,, \tag{3.27}$$

$$g_{tt}\,g_{rr} = -1 - \frac{\alpha_3\,Q^2}{48\pi^2 M_P^4 r^4}\,. \tag{3.28}$$

3.2 Radius and Charge-to-Mass Ratio of Extremal Black Holes

In order to study the perturbed BH solution and relate it to WGC we must firstly compute its corrected horizon radius $r_H^{(1)}$ in the extremal case, i.e. for the maximum allowed value of the charge to mass ratio.[5] By definition it is given by:

$$r_{H,\mathrm{ext}}^{(1)} = \max\left\{ r \in \mathbb{R}_{>0} \;:\; \frac{1}{g_{rr}^{(1)}(r)} = 0\,, \quad z = z_{\max}^{(1)} \right\}\,. \tag{3.29}$$

The solution cannot be computed exactly because in our case $r_H^{(1)}$ is the root of a sixth order polynomial. However, a perturbative approach is still possible. The standard techniques used in the literature are those of guessing the polynomial roots dependencies on the perturbative parameters (see [35]) or of considering the expansion $r_H^{(1)} = r_H^{(0)} + \Delta r_H^{(0)}$ and assuming that $\Delta r_H^{(0)}$ is small (see [14]). The method we will use mixes both of them. The unperturbed horizon is given by

$$r_H^{(0)} = \frac{M + \sqrt{M^2 - 2M_P^2 Q^2}}{8\pi M_P^2} = \frac{M}{8\pi M_P^2}\left(1 + \sqrt{1 - z^2}\right)\,, \tag{3.30}$$

$$z = \frac{\sqrt{2} M_P Q}{M}\,, \tag{3.31}$$

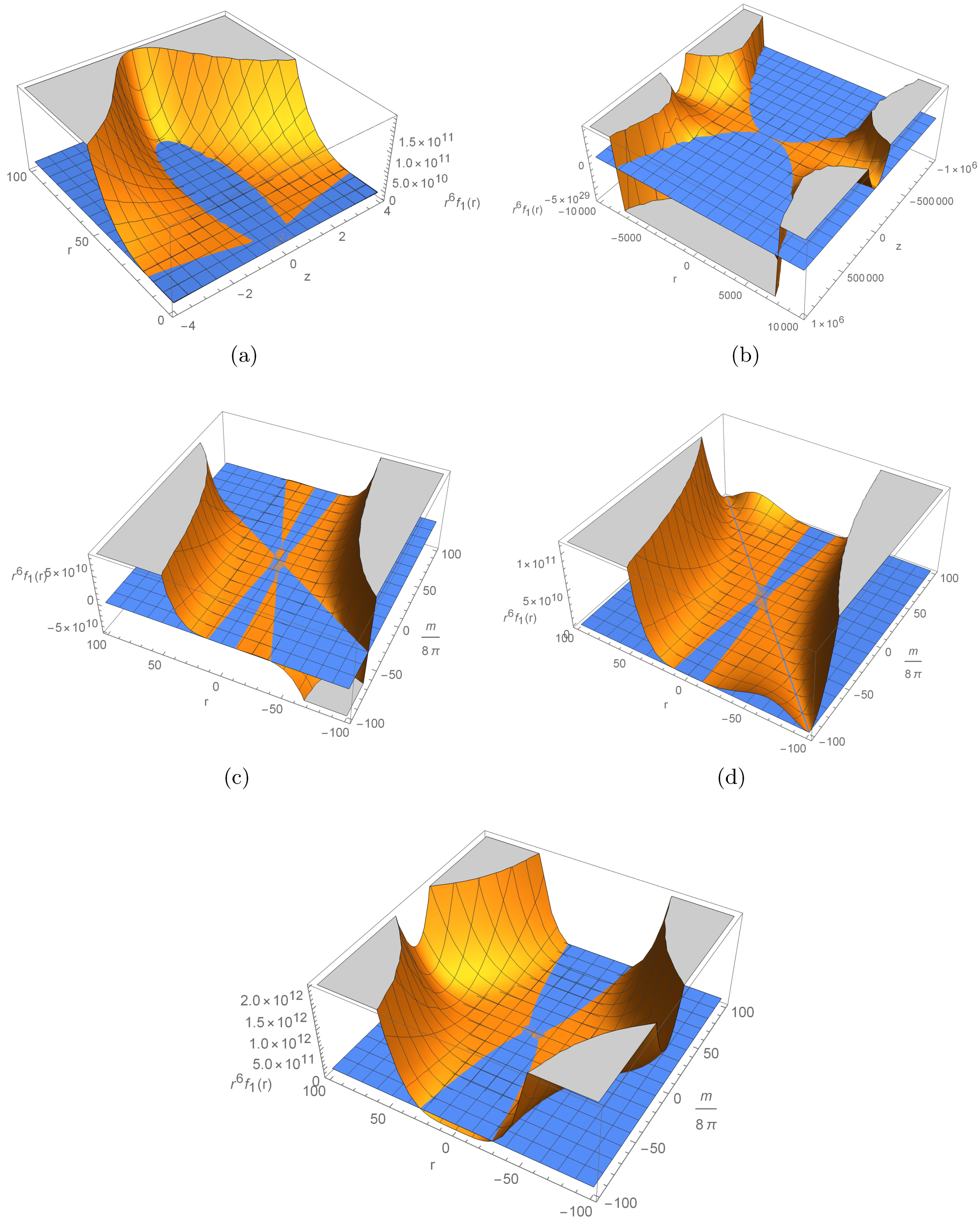
where we introduced the charge-to-mass ratio z. In the unperturbed case z takes values in $[0, 1]$ ($z = 1$ extremal charged BH, $z = 0$ Schwarzschild BH).

We start by evaluating numerically $P(r) := r^6 f_1(r)$, in order to understand which is the best ansatz to apply the perturbative methods. Setting $M_P = 1$, we observe that the polynomial roots present 3 different structures[6] depending on the sign of the z-axis intercept:

[5] The extremal configuration with charge Q can be equivalently defined as that with the minimal mass.

[6] Here we are assuming $M \gg M_P$, $M \gg \alpha_i$. Outside the perturbative regime the solutions structures are pretty different.

- $\alpha_1 + 2\alpha_3 > 0$



(a) (b)

(c) (d)

(e)

Figure 3.1: The orange surface in the plot is the polynomial $P(r) = r^6 f_1(r)$, while the blue surface is the z $= 0$ plane. In every plot we set $\alpha_1 = 1$, $\alpha_3 = 1$, $M_P = 1$. In (a) and (b) we set $m/8\pi = 40$. In (c), (d), (e) we set respectively $z = 0.8$, $z = 1$ and $z = 5$.

- $\alpha_1 + 2\alpha_3 = 0$

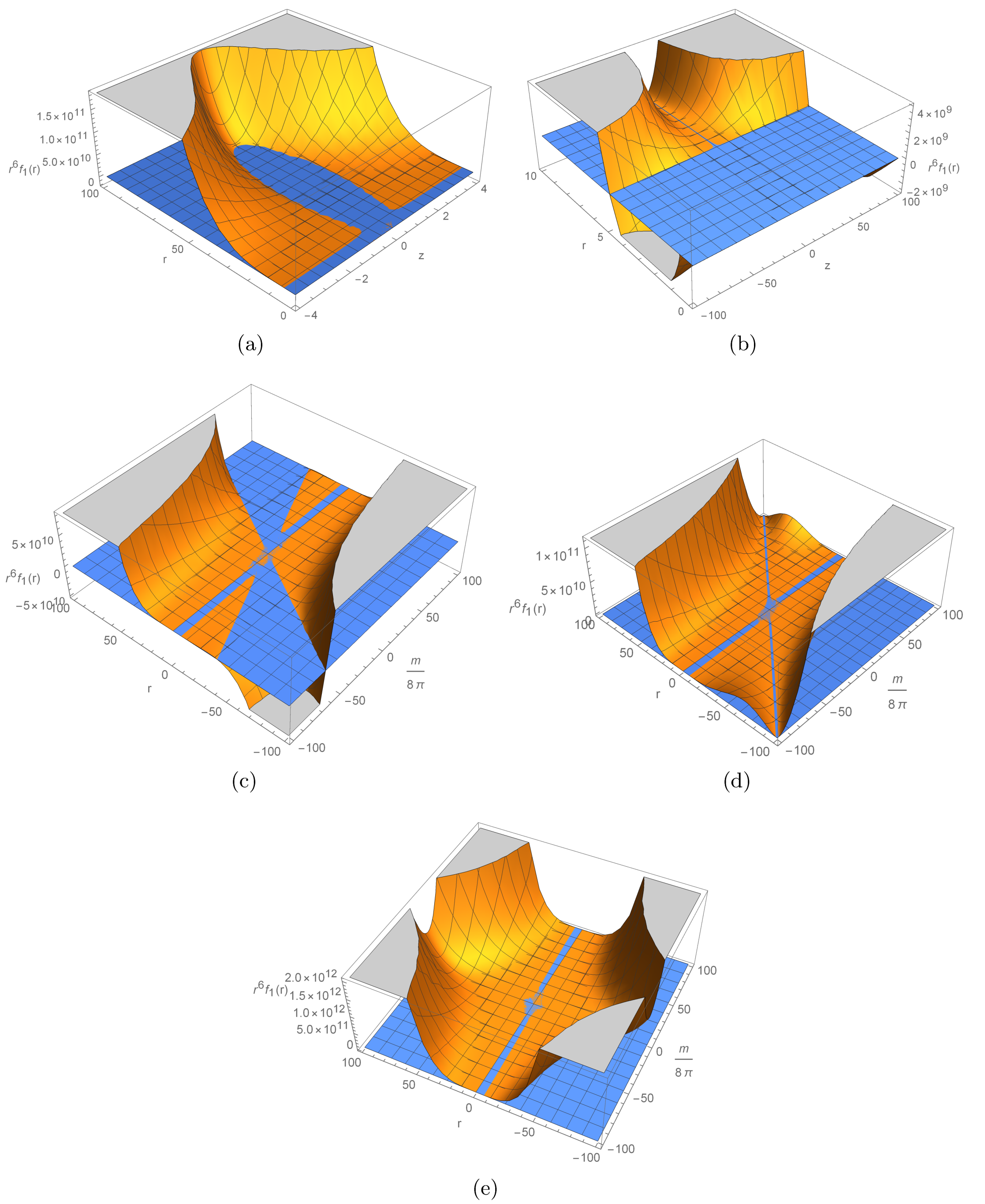


Figure 3.2: The orange surface in the plot is the polynomial $P(r) = r^6 f_1(r)$, while the blue surface is the z = 0 plane. In every plot we set $\alpha_1 = 2$, $\alpha_3 = -1$, $M_P = 1$. In (a) and (b) we set $m/8\pi = 40$. In (c), (d), (e) we set respectively $z = 0.8$, $z = 1$ and $z = 5$.

- $\alpha_1 + 2\alpha_3 < 0$

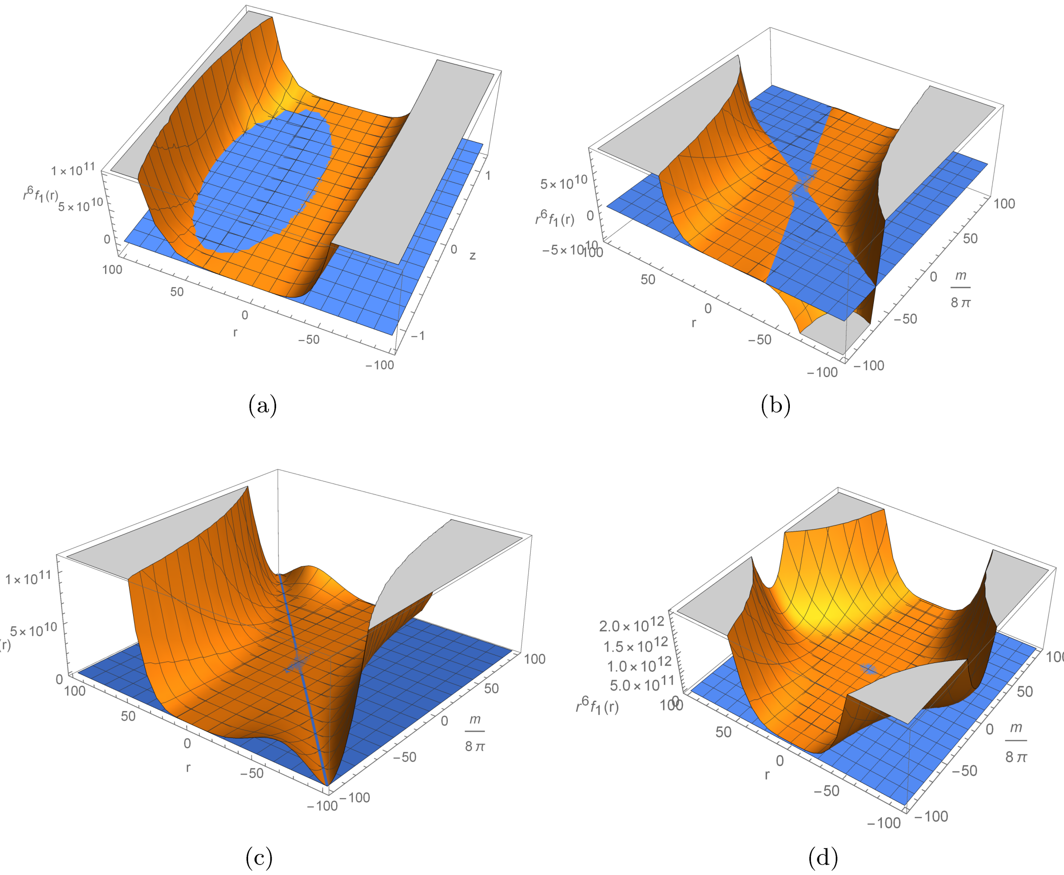


Figure 3.3: The orange surface in the plot is the polynomial $P(r) = r^6 f_1(r)$, while the blue surface is the z = 0 plane. In every plot we set $\alpha_1 = 1$, $\alpha_3 = -1$, $M_P = 1$. In (a) and (b) we set $m/8\pi = 40$. In (c), (d), (e) we set respectively $z = 0.8$, $z = 1$ and $z = 5$.

We notice that only in the case of figure 3.3 the solutions have the same structure of the unperturbed RN solutions (i.e. for any fixed mass M there exists a maximum z at which the BH horizon becomes singular). If $\alpha_1 + 2\alpha_3 \geq 0$ we see instead that there are BH solutions which have an arbitrarily large value of z. Indeed, although there is still a value of z at which the outer horizon vanishes (as in RN case), there is always an inner horizon which prevents from having naked singularity. Finally, the difference between figure 3.1 and figure 3.2 is that in the second case the radius of the black hole is constant for large z values; in the first case it increases.

We proceed now computing the first order corrections to (r, z) for the solutions which correspond to a perturbation of extremal RN black holes. If $\alpha_1 + 2\alpha_3 < 0$ it is clear that such configurations correspond to the extremal ones, i.e. those with maximum charge-to-mass ratio. If $\alpha_1 + 2\alpha_3 \geq 0$ there is instead no bound on z. Although the perturbations of extremal RN black holes are well defined solutions, they do not correspond to the configurations with maximum charge-to-mass ratio. Therefore, the following computations apply for every allowed value of α_1 and α_3, but the configurations they describe seem to be the extremal ones only in the case $\alpha_1 + 2\alpha_3 < 0$. However, we will see that the class of black holes with $z \gg 1$ is composed by *small black holes* which break the perturbative regime (see section 3.5). Therefore, the results that we will derive in this sections assuming to be in the case $\alpha_1 + 2\alpha_3 < 0$ will eventually turn out to describe extremal configuration for every value of α_1 and α_3 (within the perturbative regime). Indeed, in this section we will never exploit the $\alpha_1 + 2\alpha_3 < 0$ bound.

Given the results of the numerical simulation we guess (we drop perturbation order indexes and until the end of this section the quantities are evaluated at the extremal configuration):

$$z = 1 + \Delta z \,, \qquad |\Delta z| \ll 1 \,, \qquad r_H = \frac{M}{8\pi} + \Delta r \,, \qquad \left| \frac{\Delta r}{(M/8\pi)} \right| \ll 1 \,. \tag{3.32}$$

The definition (3.29), tells us that r_H satisfies $P(r_H) = 0$. Assuming the ansatz (3.32) we can consistently expand such condition around $n = M/8\pi$ producing an equation for Δr. Dropping higher order terms we have (we assume $n \gg |\alpha_i| \gtrsim |\Delta r|$):

$$\begin{aligned} 0 \;&=\; n^6 \Big(\frac{\Delta r^6}{n^6} + \frac{4\Delta r^5}{n^5} + \frac{\Delta r^4 z^2}{n^4} + \frac{5\Delta r^4}{n^4} + \frac{4\Delta r^3 z^2}{n^3} - \frac{4\alpha_3 \Delta r^2 z^2}{3n^4} \\ &\quad + \frac{6\Delta r^2 z^2}{n^2} - \frac{5\Delta r^2}{n^2} + \frac{2\alpha_3 \Delta r z^2}{3n^3} + \frac{4hz^2}{n} - \frac{4\Delta r}{n} - \frac{4\alpha_1 z^4}{5n^2} \\ &\quad - \frac{8\alpha_3 z^4}{5n^2} + \frac{2\alpha_3 z^2}{n^2} + z^2 - 1 \Big) \\ &\simeq\; n^6 \Big(\frac{6\Delta r^2 z^2}{n^2} - \frac{5\Delta r^2}{n^2} + \frac{2\alpha_3 \Delta r z^2}{3n^3} + \frac{4\Delta r z^2}{n} - \frac{4\Delta r}{n} - \frac{4\alpha_1 z^4}{5n^2} \\ &\quad - \frac{8\alpha_3 z^4}{5n^2} + \frac{2\alpha_3 z^2}{n^2} + z^2 - 1 \Big) \,. \end{aligned} \tag{3.33}$$

Solving the equation we get

$$\Delta r \;=\; \frac{6n^2(1 - z^2) - \alpha_3 z^2}{18nz^2 - 15n} \pm \frac{\sqrt{\Delta}}{2\,(90nz^2 - 75n)} \,, \tag{3.34}$$

$$\begin{aligned} \Delta \;=\; & -4\left(90nz^2 - 75n\right)\left(15n^3 z^2 - 15n^3 - 12\alpha 1 n z^4 - 24\alpha 3 n z^4 + 30\alpha 3 n z^2\right) \\ & + \left(60n^2 z^2 - 60n^2 + 10\alpha 3 z^2\right)^2 \,. \end{aligned} \tag{3.35}$$

We impose then the extremality condition requiring that Δ vanishes. Dropping higher order terms we can produce a solvable equation for Δz. We have:

$$0 \;=\; \Delta \;=\; n^2 \left[-1800\, n^2 \Delta z + 720\, \alpha_1 - 360\, \alpha_3 + O\left(\frac{\alpha^2}{n^2}\right) \right] \,, \tag{3.36}$$

$$\Delta z \;=\; \frac{64\pi^2}{5M^2} (2\alpha_1 - \alpha_3) \,. \tag{3.37}$$

We observe that Δz parametric dependence is in agreement with the order of the perturbative expansion (3.36). Replacing equation (3.37) in (3.34) and imposing $\Delta = 0$ we get:

$$\Delta r = -\frac{8\pi}{15M} (24\alpha_1 - 7\alpha_3) \,, \tag{3.38}$$

which is in agreement with the order of the perturbative expansion (3.33). Equation (3.37) is in agreement with the results in the literature ([6],[8],[14],[7],[9]).

3.3 Entropy and Positivity Bounds

We proceed now to compute the entropy of the perturbed BH. For stationary BH solutions with stationary perturbations we proved that there exists a quantity, called Wald's entropy, which is defined in generic higher derivative theories and satisfies a relation which has the structure of the first law of thermodynamic (see section 2.3). Although it is still an open problem to understand which is the nature of the microstates associated with BH entropy and there are some unclear issues related to it (such as the information paradox see [36]), Wald's entropy is the most promising formula to compute entropy in the classic regime (see [13]).

We recall the Entropy formula for a D-dimensional theory with derivative order k (cfr. equation 2.89)

$$S = 2\pi \int_{S_H} \eta_{ab} X^{ab} \,, \tag{3.39}$$

$$X^{ab} = 2s \star E^{ab} \,, \qquad E^{ab} = E^{abcd} g_{cc'} g_{dd'} \frac{dx^{c'} \wedge dx^{d'}}{2} \,. \tag{3.40}$$

S_H is given by the $(D-2)$-dimensional intersection between a $(D-1)$-dimensional spacelike hypersurface and the $(D-1)$-dimensional Killing horizon of the BH; s is the sign of the metric determinant, $\star$ is the D-dimensional Hodge star operator, η_{S_H} is the binormal to S_H normalized and oriented according to

$$d^D x \sqrt{|g|} = \eta_{S_H} \wedge e_{S_H} \,, \tag{3.41}$$

where e_{S_H} is the $(D-2)$-dimensional volume element of S_H; E^{abcd} is the tensor obtained varyng the symmetric action[7] as if R_{abcd} was and independent dynamical field, i.e. it holds:

$$\begin{aligned}\delta \mathcal{L}_S = E^{abcd} \delta R_{abcd} = \; & \Big[\frac{\partial \mathcal{L}_S}{\partial R_{abcd}} - D_{a_1} \frac{\partial \mathcal{L}_S}{\partial D_{a_1} R_{abcd}} + \ldots \\ & \ldots (-1)^k D_{(a_1} \ldots D_{a_k)} \frac{\partial \mathcal{L}_S}{\partial D_{(a_1} \ldots D_{a_k)} R_{abcd}} \Big] \delta R_{abcd} \,. \end{aligned} \tag{3.42}$$

Assuming spherical symmetry, equation (3.39) reduces to

$$S = 4\pi s \int_{S_H} \eta_{ab} \star E^{ab} = 2\pi s \int_{S_H} E^{abcd} \eta_{ab} \eta_{cd} \, e_{S_H} = -2\pi A E^{abcd} \eta_{ab} \eta_{cd} \,, \tag{3.43}$$

with

$$A = 4\pi (r_H)^2 \,, \qquad \eta_{ab} = 2\sqrt{|g_{00}|}\sqrt{|g_{11}|}\, \delta^0_{[a} \delta^1_{b]} \,. \tag{3.44}$$

We consider now the 4 derivative Einstein–Maxwell theory (2.19). The Lagrangian density terms are already symmetric in covariant derivative. Applying the definition (3.42) directly to the Lagrangian (2.19) we get:

$$\tilde{E}^{abcd} = \frac{M_P^2}{2} g^{ac} g^{bd} + \frac{\alpha_3}{2M_P^2} \left(F^{ab} F^{cd} - 2 g^{ac} F^{bf} F^{d}{}_{f} + \frac{1}{3} F^2 g^{ac} g^{bd} \right) , \tag{3.45}$$

[7] Given a theory $\mathcal{L}$ we call the symmetric one that is obtained using equation (2.34) to have only totally symmetric combinations of fields derivatives.

where the tilde indicates that the tensor is not correctly symmetrized yet, i.e. does not respect the indexes structure of δR_{abcd}. We have then:

$$E^{abcd} = \frac{1}{8}(\tilde{E}^{abcd} - \tilde{E}^{bacd} - \tilde{E}^{abdc} + \tilde{E}^{badc} + \tilde{E}^{cdab} - \tilde{E}^{dcab} - \tilde{E}^{cdba} + \tilde{E}^{dcba}). \tag{3.46}$$

According with the perturbative expansion performed we can decompose E^{abcd} as

$$E^{(1)}_{abcd} \simeq \frac{\delta \mathcal{L}_2}{\delta R^{abcd}}\left[g^{(1)}_{\mu\nu}\right] + \frac{\delta \mathcal{L}_4}{\delta R^{abcd}}\left[g^{(0)}_{\mu\nu}, A^{(0)}_{\gamma}\right] . \tag{3.47}$$

Exploiting equations (3.28) and (3.47) the perturbed entropy[8] at the first order in α_i takes the form:

$$S_1 = -\left(8\pi^2 M_P^2 r_H^2 + \frac{m^2 z^2}{12 r_H^2}\alpha_3\right)\left(\frac{m^2 z^2}{48\pi^2 r_H^4}\alpha_3 - \left(1 + \frac{m^2 z^2}{192\pi^2 r_H^4}\alpha_3\right)^{-2}\right) . \tag{3.48}$$

Using the explicit results (3.37),(3.38) we get

$$\begin{aligned} z_{\text{ext}} &= 1 + \frac{64\pi^2 M_P^2}{5M^2}(2\alpha_1 - \alpha_3) , && (3.49)\\ r_{H,\text{ext}} &= \frac{M}{8\pi M_P^2} - \frac{8\pi}{15M}(24\alpha_1 - 7\alpha_3) , && (3.50)\\ S_{\text{ext}} &= \frac{M^2}{8M_P^2} - \frac{16}{5}\pi^2 (8\alpha_1 + \alpha_3) . && (3.51) \end{aligned}$$

Recalling that the entropy function for RN solutions is given by (see [37])

$$S_0 = \frac{M^2}{8M_P^2}\left(1 + \sqrt{1 - z^2}\right)^2 , \qquad S_{0,\text{ext}} = \frac{M^2}{8M_P^2} , \tag{3.52}$$

the extremal black holes entropy correction takes the form

$$\Delta S_{\text{ext}} = -\frac{16}{5}\pi^2 (8\alpha_1 + \alpha_3) . \tag{3.53}$$

Notice that the corrections (3.37), (3.38) and (3.53) present the typical parametric scaling (see [19],[35]).

As far as we know, equation (3.51) is an original result of this thesis work. Indeed, the most accurate analysis we found of 4 derivative Einstein–Maxwell theory is that pursued in ([14]). However, the entropy formula they obtained shows a divergent behavior in the extremal limit, therefore the entropy of the extremal configuration have never been evaluated explicitly until now.

We conclude this section by analyzing the implications of the positivity bounds due to S-matrix properties. For the moment, we just assume them without discussing their nature. A more detailed study will be pursued in the next chapter. Exploiting unitarity, crossing symmetries and analiticity of the S-matrix it is possible to constrain the values of α_1 and α_3. According to [7] we have

$$\begin{cases} 2\alpha_1 - \alpha_3 > 0 , \\ 2\alpha_1 + \alpha_3 > 0 , \\ \alpha_2 > 0 . \end{cases} \tag{3.54}$$

[8] Notice that the formula holds for a generic configuration, not only for the extremal ones.

The system (3.54) fixes completely the corrections signs ($\alpha_1 > |\alpha_3|/2$ implies $8\alpha_1+\alpha_3 > 0$). Therefore we get:

$$\Delta z_{\text{ext}} > 0\,, \qquad \Delta r_{\text{ext}} < 0\,, \qquad \Delta S_{\text{ext}} < 0\,. \tag{3.55}$$

The positive correction to the charge-to-mass ratio implies that BHs can spontaneously decay emitting a smaller extremal BH with higher z. EWGC bound is therefore trivially satisfied and the conjecture is proved in its mild form (for our class of theories). The negative sign of extremal entropy correction alone does not allow us instead to determine whether black holes can split or not. We have indeed to evaluate whether the entropy corrections allows us to produce a couple of BH whose entropy is greater than the entropy of the decaying BH. However, before proceeding with such analysis we have to better understand which is the regime of validity of our computations. It is indeed fundamental to test BH splitting processes avoiding BH configurations which are outside the perturbative regime. In order to determine whether a BH breaks the perturbative regime we have just to verify if it satisfy our assumptions. In general, a black hole configuration is reliable if its mass is greater than the theory cutoff scale and if the higher order terms are subdominant once they are evaluated for such solution. We have therefore

$$\begin{cases} M \gg M_P\,, \\ |\mathcal{L}_2| \gg |\Delta\mathcal{L}|\,. \end{cases} \tag{3.56}$$

From the constraints (3.56), it is clear that an heavy black hole can not be outside the perturbative regime. The first condition is indeed trivial. The second one is satisfied because $|\Delta\mathcal{L}|$ increases with the horizon curvature and the bigger is the mass, the bigger is the radius and the smaller is the BH curvature. Therefore, only the light BHs can break perturbative regime. We call such kind of solutions *small black holes*[9]. Finally, an extremal BH together with (3.56) must satisfy

$$\begin{cases} |\Delta z| \ll 1\,, \\ |\Delta r| \ll (M/8\pi)\,. \end{cases} \tag{3.57}$$

Before analyzing the splitting processes we proceed then with a detailed discussion of the computations performed in section 3.2 and of the light black holes which arise for $\alpha_1 + 2\alpha_3 \leq 0$.

3.4 Black Hole Solutions Detailed Analysis

In section 3.2 we obtained the radius and the charge-to-mass ratio for solutions which are the perturbations of extremal RN black holes. However, in order to present as clearly as possible the perturbative computation we did not discussed the details of our assumptions, results and methods.

In this section we start analyzing the last result we obtained, i.e. the charge-to-mass ratio formula for extremal BHs. Higher order corrections introduce a quadratic dependence

[9]Notice that our definition of *small black hole* differs from that commonly used in the literature. We define *small black hole* a BH that is light enough to be outside the perturbative regime. In the literature a *small black hole* is instead a BH that becomes singular in the limit of null perturbations. However, all the BHs we will classify as *small black holes* in the thesis work satisfy both the definitions.

on the BH mass, therefore it is not obvious that for every charge Q is identified only one extremal mass. It is important therefore to understand if there exist more than one configuration and if such extra configurations break or not the perturbative regime. It will turn out that every reliable configuration admits a smaller one which is outside the perturbative regime.

After that, we formalize the perturbative computations performed in the previous section introducing a more general approach. The new algorithm reproduces exactly the same results but works independently of the particular structure of the black hole solution we are considering.

We finally discuss the structure of the black hole horizon in higher order theories. Firstly, we will verify that the sphere with radius defined by (3.29) has actually the structure of a Killing horizon. The condition we used to define the radius is therefore the correct one. Secondly, we will show that whenever the two outermost horizons coincide the BH surface gravity vanishes. Condition (3.36) encodes therefore in the right way the extremality condition.

3.4.1 Charge-to-Mass Ratio in Higher Order Theories

Equation (3.37) tells us that the charge-to-mass ratio it is no more constant but depends on the mass of the BH. However, we notice that now for a fixed charge the bound is not trivially satisfied by a unique mass value. We have indeed

$$\sqrt{2}Q = \left(\frac{M}{M_P}\right) + \frac{64\pi^2}{5\left(\frac{M}{M_P}\right)}(2\alpha_1 - \alpha_3)\,. \tag{3.58}$$

Setting $M_P = 1$ and $\epsilon = 2\alpha_1 - \alpha_3$ the relation can be written as

$$M^2 - \sqrt{2}MQ + \frac{64\pi^2}{5}\epsilon = 0\,. \tag{3.59}$$

Recalling that $Q = |Q|$ in (3.59) and that $\epsilon > 0$ (cfr. equation (3.54)), we have that the two M^+ and M^- should satisfy:

$$\begin{cases} M^+M^- > 0\,, \\ M^+ + M^- > 0\,. \end{cases} \tag{3.60}$$

It follows that whenever the equation (3.59) admits a real solution it admits a couple of positive solutions $M^+ \geq M^- > 0$. It is important then to understand whether M^+ and M^- are reliable solutions. If they were both part of the theory spectrum we could identify two types of extremal black holes: there would be a class of light stable black holes and a class of unstable heavy black holes[10]. Solving equation (3.59) we get:

$$M^\pm = \frac{Q}{\sqrt{2}} \pm \sqrt{\frac{Q^2}{2} - \frac{64\pi^2\epsilon}{5}}\,. \tag{3.61}$$

For large Q values they take the form

[10]Given a fixed value of the charge Q, the light black hole has lower energy, therefore the heavy black hole can not be stable

$$M^+ = \sqrt{2}Q - \frac{64\pi^2\epsilon}{5\sqrt{2}Q}\,, \qquad M^- = \frac{64\pi^2\epsilon}{5\sqrt{2}Q}\,. \tag{3.62}$$

The asymptotic behavior of M^- implies that such configurations will surely exit the perturbative regime. However, it is possible to show that the M^- configurations are never admitted. We recall that we derived equation (3.58) assuming that the mass M is larger than the Planck mass M_P and that Δz is a small perturbation. We have therefore that must hold

$$\begin{cases} M > 1\,, \\ \Delta z = \frac{64\pi^2}{5M}\epsilon < 1\,. \end{cases} \tag{3.63}$$

Moreover, it can be easily verified that $\frac{dM^-}{dQ} \leq 0 \;\; \forall Q$. We have then the bound on M^-

$$M^- \leq M^-\Big|_{Q_{\min}} = \frac{Q_{\min}}{\sqrt{2}} = \sqrt{\frac{64\pi^2\epsilon}{5}} \equiv M_0\,. \tag{3.64}$$

The second relation of (3.63) together with (3.64) implies the mass bound

$$M > \frac{64\pi^2}{5}\,\epsilon = M_0^2\,. \tag{3.65}$$

Condition (3.65) together with condition (3.63) takes then the compact form

$$M > \max\{1, M_0^2\}\,. \tag{3.66}$$

Exploiting equation (3.64) it is easy to show that M^- configurations are *small black holes* completely ruled out by (3.66). We have indeed

$$M_0^2 \geq 1 \quad \rightarrow \quad M > M_0^2 \geq M_0 \geq M^-\,, \tag{3.67}$$

$$M_0^2 \leq 1 \quad \rightarrow \quad M > 1 \geq M_0 \geq M^-\,. \tag{3.68}$$

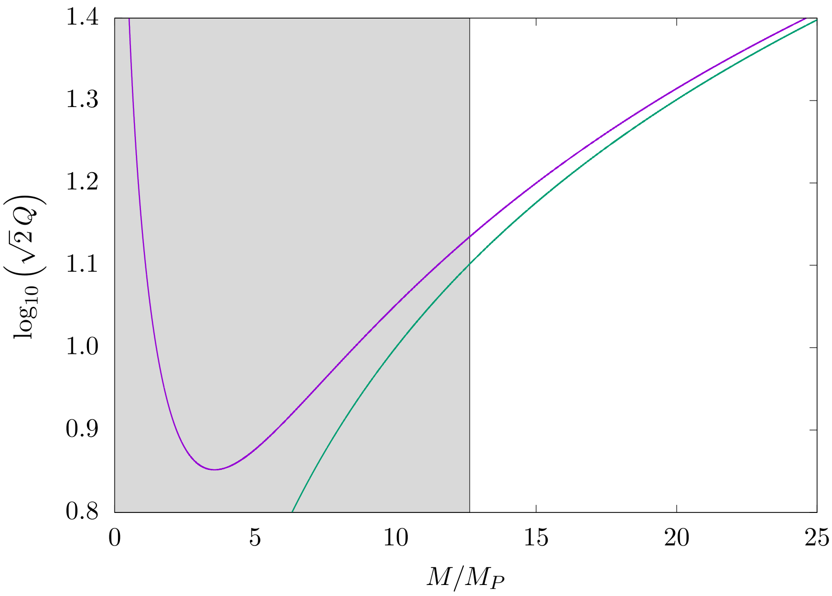


Figure 3.4: The purple curve in the graph is the BH charge as a function of the mass in the extremal configuration with $\epsilon = 0.1$ for the perturbed case. The green curve is the same quantity for the unperturbed case. The intersections with the lines parallel to the x-axis define M^- and M^+ couples. The shadowed region is defined by $M < M_0^2$ and indicates where the perturbative regime breaks.

3.4.2 A More General Algorithm for z and r_H

The computations performed to determine Δz and Δr are highly dependent on the structure of the particular solution we are working with. A possible generalization of the algorithm is suggested then by the parametric dependence of (3.38). Δr could be obtained expanding $f_1(r) \equiv 1/g_{rr}(r)$ around the unperturbed solution and neglecting higher order terms. Indeed, by definition it holds

$$0 \;=\; f_1\left(r_H^{(1)}\right) \;=\; f_1\left(r_H^{(0)}\right) + \partial_r f_1\left(r_H^{(0)}\right)\Delta r + O\left(\Delta r^2\right)\,, \tag{3.69}$$

therefore we get

$$\Delta r = -\frac{f_1\left(r_H^{(0)}\right)}{\partial_r f_1\left(r_H^{(0)}\right)}\,. \tag{3.70}$$

Although equation (3.70) should provide the correct result, to compute the correction Δr_{ext} in the extremal case we have to evaluate Δz_{ext}. This can be achieved easily considering the second order term too in $f_1(r)$ expansion and imposing to have a degenerate solution. In general we have

$$0 \;=\; f_1\left(r_H^{(1)}\right) \;=\; f_1\left(r_H^{(0)}\right) + \partial_r f_1\left(r_H^{(0)}\right)\Delta r + \frac{\partial_r^2 f_1\left(r_H^{(0)}\right)}{2}\Delta r^2 + O(\Delta r^3)\,, \tag{3.71}$$

and the radius takes the form

$$\Delta r \frac{-\partial_r f_1 \pm \sqrt{\Delta}}{\partial_r^2 f_1}\,. \tag{3.72}$$

Thus, in the extremal case the radius and the charge-to-mass ratio are given by

$$\Delta = (\partial_r f_1)^2 - 2 f_1 \partial_r^2 f_1 = 0\,, \tag{3.73}$$

$$\Delta r_{\rm ext} = -\left.\frac{\partial_r f_1}{\partial_r^2 f_1}\right|_{\Delta z_{\rm ext}}. \tag{3.74}$$

It is easy to verify that choosing $r_H^{(0)} = r_{H,\rm ext}^{(0)} = \frac{M}{8\pi}$ and $f_1(r)$ given by

$$f_1 = 1 - \frac{M}{4\pi r} + \frac{Q^2}{32\pi^2 r^2} - \frac{A_2\, Q^2}{24\,\pi^2\, r^4} + \frac{A_1 M\, Q^2}{384\,\pi^3\, r^5} - \frac{A_0\, Q^4}{1280\,\pi^4\, r^6}\,, \tag{3.75}$$

equations (3.74) and (3.73) provides exactly (3.37) and (3.38) (for a suitable choice of A_0, A_1 and A_2). Applying equation (3.73) we get

$$\Delta \propto 64\pi^2(6A_0 - 5A_1 + 10A_2) - 15M^2 \Delta z_{\rm ext} + O\left(\frac{1}{M^2}\right)\,, \tag{3.76}$$

which implies

$$\Delta z_{\rm ext} = \frac{64\pi^2}{15M^2}(6A_0 - 5A_1 + 10A_2)\,. \tag{3.77}$$

Equation (3.74) reads instead

$$\Delta r_{\rm ext} = -\frac{\partial_r f_1}{\partial_r^2 f_1} = -\frac{8\pi}{15M}\left(24A_0 - 15A_1 + 20A_2\right)\,. \tag{3.78}$$

Comparing equations (3.75) and (3.24) we can fix A_i coefficients. We finally get

$$A0 = \alpha_1 + 2\alpha_3\,, \tag{3.79}$$

$$A1 = 5\alpha_3\,, \tag{3.80}$$

$$A2 = \alpha_3\,, \tag{3.81}$$

$$\Delta z_{\rm ext} = \frac{64\pi^2}{5M^2}(2\alpha_1 - \alpha_3)\,, \tag{3.82}$$

$$\Delta r_{\rm ext} = -\frac{8\pi}{15M}(24\alpha_1 - 7\alpha_3)\,. \tag{3.83}$$

It turns out that we found an algorithm which allows us to compute Δz and Δr independently of the particular structure of f_1. However, there are some subtleties. We notice that we can not expand $f_1(r)$ around $r^{(0)} = \frac{m}{8\pi}\sqrt{1-z^2}$ because otherwise the solution would be defined only for $z \leq 1$ and we could not take the extremal limit. It follows that the range of validity of the solution strictly depends on the expansion point and if we use f_0^2 roots it is important to check whether they are well defined or not. [11]

3.4.3 Killing Horizon Structure in Higher Order Theories

During the study of black hole solutions we defined the black hole horizon as the sphere with radius the biggest real root of $1/g_{rr}$ (cfr. equation (3.29)). We will show now that for BH solutions within the perturbative regime this is a good characterization. The surface has indeed the structure of a killing horizon.

[11] These are the reasons why a divergence appears in [14] when quantities approach the extremal limit.

The most general ansatz for the metric of a static, non rotating and spherical black holes is

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2 \tag{3.84}$$

Then, the time derivative $\zeta_t = \partial_0 = \zeta_t^\mu \partial_\mu$ with $\zeta_t^\mu = \delta_0^\mu$ is a Killing vector of the metric (cfr. equation (2.28)). A surface Σ is a Killing horizon iff there is a Killing vector which is null on Σ, i.e.

$$\exists\, \zeta \;:\; 0 = \zeta^2 = \zeta^\mu \zeta_\mu \,, \qquad \forall x \in \Sigma. \tag{3.85}$$

A Killing horizon Σ for ζ_t must satisfy then

$$0 = \delta_0^\mu \delta_0^\nu g_{\mu\nu} = g_{00}(x)\,, \qquad \forall x \in \Sigma. \tag{3.86}$$

It follows that every real positive root of g_{00} is the radius of a sphere which has the structure of a killing horizon with killing vector ∂_0. In particular, the black hole horizon is nothing but the lightlike surface associated to the sphere with the biggest radius.

Let us look at the relations between g_{tt} and $1/g_{rr}$ roots. If our solution is such that $A(r)^{-1} = B(r)$ then the roots of g_{tt} and $1/g_{rr}$ coincide and the biggest positive root of $1/g_{rr}$ is the black hole radius. If we consider instead the general case (3.84), we have no guarantees that the roots coincide and definition (2.19) does not apply. Finally, if we consider the case of a BH obtained perturbing a solution which satisfy $A(r)^{-1} = B(r)$, we expect that the metric takes the form

$$\begin{aligned} ds^2 =& -N(r)^2 A(r) dt^2 + A(r)^{-1} dr^2 + r^2 d\Omega^2\,, && (3.87)\\ A(r) =& \; A_0(r) + \alpha A_1(r) + O(\alpha^2)\,, && (3.88)\\ N(r)^2 =& \; 1 + 2\alpha N_1(r) + O(\alpha^2)\,, && (3.89) \end{aligned}$$

where α is the perturbation parameter and we expanded $N^2(r)$ with $N(r) = 1 + \alpha N_1(r)$.

Let us focus on the third case. Choosing r_H such that $A(r_H) \sim O(\alpha^2)$, then $N(r_H)^2 A(r_H) \sim O(\alpha^2)$. It follows that within the regime of validity of the perturbation theory the vanishing of $1/g_{rr}$ implies the vanishing of g_{tt} up to $O(\alpha^2)$ terms

$$\frac{1}{g_{rr}(r_H)} \sim O(\alpha^2) \quad \Rightarrow \quad g_{tt}(r_H) \sim O(\alpha^2)\,, \tag{3.90}$$

which implies that the roots of $1/g_{rr}$ and g_{tt} correspond and the biggest real positive root of $1/g_{rr}$ defines the black hole horizon.

Explicit check

We conclude verifying explicitly that the largest positive root of g_{tt} coincide with that of $1/g_{rr}$. Equation (3.26) can be written as

$$\begin{aligned} -g_{tt} &= f_0(r)^2 - \frac{A_2\, Q^2}{24\,\pi^2\, r^4} + \frac{A_1 M\, Q^2}{384\,\pi^3\, r^5} - \frac{A_0\, Q^4}{1280\,\pi^4\, r^6}\,, && (3.91)\\ f_0(r)^2 &= 1 - \frac{M}{4\pi M_P^2 r} + \frac{Q^2}{32\pi^2 M_P^2 r^2}\,. && (3.92) \end{aligned}$$

where we set

$$A_0 = \alpha_1 + \frac{7}{6}\alpha_3\,, \tag{3.93}$$

$$A_1 = 3\,\alpha_3\,, \tag{3.94}$$

$$A_2 = \frac{1}{2}\alpha_3\,. \tag{3.95}$$

Equation (3.91) has the structure of (3.75), therefore we can easily compute Δr_{ext} an Δz_{ext} exploiting equations (3.77) and (3.78).

$$\Delta z_{\text{ext}} = \frac{64\pi^2}{15M^2}(6A_0 - 5A_1 + 10A_2) = \frac{64\pi^2}{5M^2}(2\alpha_1 - \alpha_3)\,, \tag{3.96}$$

$$\Delta r_{\text{ext}} = -\frac{8\pi}{15M}(24A_0 - 15A_1 + 20A_2) = -\frac{8\pi}{15M}(24\alpha_1 - 7\alpha_3)\,. \tag{3.97}$$

The corrections obtained are exactly the same computed through $1/g_{rr}$ (cfr. equations (3.37) and (3.38))

3.4.4 Extremality Condition in Higher Order Theories

We will show now that imposing the degeneracy of the biggest positive real root r^+ of $1/g_{rr}$ we get the vanishing of the surface gravity and of the Hawking temperature. It is therefore the right extremality condition.

Let Σ be a Killing horizon and ζ its Killing vector. The surface gravity κ can be consider as a measure of how much the integral curves of ζ fail to be affinely parameterized and is defined by the relation (see [38]):

$$\zeta^\mu D_\mu \zeta^\nu = -\kappa\zeta^\nu\,. \tag{3.98}$$

Using Killing equation $D_{(\mu}\zeta_{\nu)} = 0$ and the fact that $\zeta_{[\mu}D_\nu\zeta_{\sigma]} = 0$ it is straightforward to derive

$$\kappa^2 = -\frac{1}{2}(D_\mu\zeta_\nu)(D^\mu\zeta^\nu)\,. \tag{3.99}$$

Let us assume to have a stationary and non rotating black hole with spherical symmetry. Recalling that ∂_0 is the killing vector of the BH horizon we get

$$\kappa^2 = -\frac{1}{2}g^{\alpha\beta}g_{\mu\nu}(D_\alpha\zeta^\mu)(D_\beta\zeta^\nu) = -\frac{1}{2}g^{\alpha\beta}g_{\mu\nu}(\Gamma^\mu_{\alpha\rho}\delta^\rho_0)(\Gamma^\nu_{\beta\sigma}\delta^\sigma_0)\,. \tag{3.100}$$

The non vanishing Christoffel symbols of the type Γ^a_{b0} are

$$\Gamma^0_{10} = \frac{g'_{tt}}{2g_{tt}}\,, \qquad \Gamma^1_{00} = -\frac{g'_{tt}}{2g_{rr}}\,. \tag{3.101}$$

Replacing in equation (3.100) we finally get

$$\kappa^2 = -\frac{1}{4}\frac{\partial_r g_{tt}}{g_{tt}g_{rr}} \quad \Rightarrow \quad \kappa = \frac{1}{2}g'_{tt}\frac{1}{\sqrt{|g_{tt}g_{rr}|}}\,, \tag{3.102}$$

where we exploit that $\mathrm{sign}(g_{tt}g_{rr}) < 0$ outside the BH horizon. Recalling that the surface gravity has to be evaluated on the the BH horizon we introduce the Hawking temperature

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\left[\frac{1}{\sqrt{-g_{tt}g_{rr}}}\frac{d}{dr}g_{tt}\right]\Bigg|_{r=r^+} . \tag{3.103}$$

Let us consider now the metric structure given by (3.13). The temperature takes the form

$$T = \frac{\kappa}{2\pi} = \frac{1}{4\pi}\left[\frac{1}{N(r)}\frac{d}{dr}\left(N(r)^2 f_1(r)\right)\right]\Bigg|_{r=r^+} , \tag{3.104}$$

Let now $P(r)$ be the monic[12] polynomial associated to f_1 defined as $P(r) = r^n f_1$, where $n = \deg(P)$. Calling r^- the next biggest positive real root the polynomial decomposes as

$$P(r) = (r - r^+)(r - r^-)g(r) , \qquad \deg(g) = n - 2 , \tag{3.105}$$

and replacing in (3.104)

$$T = \frac{1}{4\pi}\frac{N(r^+)}{r^n}(r^+ - r^-)g(r^+) . \tag{3.106}$$

In the limit $r^- \to r^+$ the temperature T as well as the surface gravity κ vanish.

3.5 Small Black Holes

The numerical study of equation (3.27) tells us that if $\alpha_1 + 2\alpha_3 > 0$ there exist configurations with fixed mass $M > M_P$ and arbitrarily large charge $Q \gg M$ (see fig. 3.1 (a): at $z \gg 1$ still exists an horizon which encloses $r = 0$ singularity). However, it is not obvious if these configuration are reliable or not. If they were out of the theory spectrum (as they will turn out to be), extremal configurations would be well defined and described by equations (3.49) and (3.50). If they were instead in the theory spectrum, equations (3.49) and (3.50) would still apply but they would have a different interpretation: they would be associated with the transition configuration from a BH with 3 horizons to a black hole with just 1 horizon (the innermost).

We start by computing the radius of these light BHs. According with the numerical solution (see fig. 3.1), we guess that the greatest real root of (3.27) in the extremal case takes the form $r_H \sim |z|^\beta m^{-\gamma}$, with $0 < \beta < 1$. To identify the dominant terms, we replace the ansatz in $P(r) = r^6 f_1(r)$. Replacing z definition and setting $M_P = 1$ we get

$$\begin{aligned} P(r_H) \quad = \quad & m^{-6(\beta+\gamma)}\left(\sqrt{2}|Q|\right)^{6\beta} - \frac{m^{-5(\beta+\gamma)+1}}{4\,\pi}\left(\sqrt{2}|Q|\right)^{5\beta} \\ & + \frac{m^{-4(\beta+\gamma)}}{64\,\pi^2}\left(\sqrt{2}|Q|\right)^{4\beta+2} - \frac{\alpha_3\, m^{-2(\beta+\gamma)+1}}{48\,\pi^2}\left(\sqrt{2}|Q|\right)^{2\beta+2} \\ & + \frac{5\,\alpha_3\, m^{-\beta-\gamma+1}}{768\,\pi^3}\left(\sqrt{2}|Q|\right)^{\beta+2} - \frac{Q^4(\alpha_1 + 2\alpha_3)}{1280\,\pi^4} . \end{aligned} \tag{3.107}$$

If we take the limit $z \gg 1$ assuming that $M > M_P$ is fixed we get that 2^{nd}, 4^{th} and 5^{th} term of equation (3.107) are surely subdominant. Moreover, we notice that if $0 < \beta < \frac{2}{3}$ the leading order terms in Q of polynomial (3.107) are just

$$P(z^\beta) \simeq \frac{m^{-4(\beta+\gamma)}}{64\,\pi^2}\left(\sqrt{2}|Q|\right)^{4\beta+2} - \frac{Q^4(\alpha_1 + 2\alpha_3)}{1280\pi^4} . \tag{3.108}$$

[12] The coefficient of the highest power $a_n r^n$ satisfy $a_n = 1$.

Considering the same truncation, $P(r)$ reads

$$P(r) \simeq \frac{Q^2 r^4}{32\pi^2} - \frac{Q^4(\alpha_1 + 2\alpha_3)}{1280\pi^4}\,, \tag{3.109}$$

and the root which corresponds to the BH horizon in the extremal case is given by

$$r_{H,\text{ext}} = \sqrt{\frac{|Q|}{2\pi\sqrt{10}}}(\alpha_1 + 2\alpha_3)^{\frac{1}{4}}\,. \tag{3.110}$$

It follows that $\gamma = -\frac{1}{2} = -\beta$, which are in agreement with the range of validity of approximation (3.108). We observe that the relation is well defined because the arguments of square roots are always positive. Moreover, it vanishes smoothly in the limit $\alpha_i \to 0$, which confirm that it is associated with a class of light BH solutions not admitted in the unperturbed case.

In order to evaluate whether light black holes with $z \gg 1$ are reliable or not we check if they violate perturbative regime. Let us firstly fix Q and consider $M \to 0$. The limit requires to take $M < M_P$, therefore this type of solutions are out of perturbative regime. Let us now fix $M > M_P$ and take the limit $Q \gg M$. We recall that we derived equations of motions (3.10) assuming that $\mathcal{L}^{(4)}$ is a small perturbation of $\mathcal{L}^{(2)}$ which implies

$$-\frac{1}{4}F^2 + \frac{\alpha}{4M_P^4}F^4 = -\frac{1}{4}F^2\left(1 - \frac{\alpha_1}{M_P^4}F_{\mu\nu}F^{\mu\nu}\right) = -\frac{1}{4}F^2\left(1+\epsilon\right)\,, \qquad \epsilon \ll 1\,. \tag{3.111}$$

Replacing equation (3.11) we get

$$\epsilon = \frac{\alpha_1 Q^2}{8\pi^2 r^4}\,, \tag{3.112}$$

and evaluating ϵ at the light BH radius scale it reads

$$\epsilon\Big|_{r_{H,\text{ext}}} = \frac{5\alpha_1}{\alpha_1 + 2\alpha_3}\,. \tag{3.113}$$

We have then

$$\epsilon\Big|_{r_{H,\text{ext}}} < 1 \qquad \Longleftrightarrow \qquad 2\alpha_1 - \alpha_3 < 0\,, \tag{3.114}$$

but according to the bound (3.54) $2\alpha_1 - \alpha_3 > 0$. It follows that also this type of solutions is outside the perturbative regime.

Thus, there is no light BH configuration which does not break the perturbative regime: in the former case, M gets smaller than M_P; in the latter, $Q\,\alpha$ becomes big enough that higher order corrections become more relevant than leading order terms. The light BH are therefore *small black holes*.

3.6 Black Holes Splitting

In this section we will finally address the first issue of the thesis work: the study of BH splitting processes. In general, a process is kinetically allowed only if one of the decay products has charge-to-mass ratio higher than the charge-to-mass ratio of the decaying BH. However, exploiting positivity bounds we have been able to prove the mild form of the EWGC (see section 3.3). A splitting process can therefore satisfy the kinematic condition through the emission of an extremal (or quasi extremal) BH. We are interested then in understanding whether the splitting processes are dynamically allowed too, i.e. if there exist for every BH a configuration that respects energy and charge conservation and increases the total entropy.

Let us make explicit the dynamical constraints. A BH with mass and charge (M, Q) can split in a BH with higher charge-to-mass ratio $\tilde{z}$ and mass and charge $(\tilde{M}, \tilde{Q})$ and $n-1$ BHs with masses and charges $\{(M_i, Q_i)\}$, if and only if the following conditions are satisfied:

$$\begin{cases} M \leq \tilde{M} + \sum_i M_i\,, \\ Q = \tilde{Q} + \sum_i Q_i\,, \\ S[M,Q] \leq S[\tilde{M},\tilde{Q}] + \sum_i S[M_i, Q_i]\,. \end{cases} \tag{3.115}$$

In order to evaluate if there exist a configuration which satisfies the constraints (3.115) we start fixing the number n of decay products. Let us consider the simple case with vanishing charges. Assuming $M_P = 1$ the BH entropy is given by (cfr. equation 3.48)

$$S[M, 0] = \frac{M^2}{2}\,, \tag{3.116}$$

which is a convex function of the mass. It holds therefore the identity

$$\frac{M^2}{2} \geq \gamma^2 \frac{M^2}{2} + \rho^2 \frac{M^2}{2}\,, \qquad \gamma, \rho \in [0,1]\,, \quad \gamma + \rho \leq 1\,. \tag{3.117}$$

The relation easily generalizes to the case with an higher number of terms in the RHS. It follows that the entropy of the decaying BH is greater than the entropy of every possible configuration of the decay products. Although equation (3.117) seems useless, it is fundamental to individuate the configuration that could more likely satisfy the constraints (3.115). Indeed, in the general case the entropy is still a convex function of the mass, therefore the products configuration which has the highest entropy is that with only two BHs: an heavy extremal one (whose charge-to-mass ratio satisfy the kinematic constraint) and a light black hole.

Let us consider an extremal BH with mass M. We have to check if it can decay emitting an heavy extremal BH with mass $M_1 = \beta M$ and a light BH with mass $M_2 = (1 - \beta M)$, where $\beta \in (1/2, 1)$. The entropy increasing condition reads

$$S_{\text{ext}}\,[\,M\,,\,Q\,] \leq S_{\text{ext}}\,[\,M_1\,,\,Q_1\,] + S\,[\,M_2\,,\,Q_2\,]\,, \tag{3.118}$$

and the charges of the two BHs can be computed exploiting the extremal charge-to-mass ratio and the charge conservation. They are given by

$$Q_1 = \frac{M_1}{\sqrt{2}} z_{\text{ext}}\,[M_1] = \frac{\beta M}{\sqrt{2}}\left(1 + \frac{\epsilon}{\beta^2 M^2}\right)\,, \tag{3.119}$$

$$\begin{aligned} Q_2 &= Q - Q_1 = \frac{M}{\sqrt{2}} z_{\text{ext}}\,[M] - \frac{\beta M}{\sqrt{2}} z_{\text{ext}}\,[M_1] \\ &= \frac{(1-\beta)M}{\sqrt{2}}\left(1 - \frac{\epsilon}{\beta M^2}\right)\,, \end{aligned} \tag{3.120}$$

$$\epsilon = \frac{64\pi^2}{5}\left(2\alpha_1 - \alpha_3\right)\,. \tag{3.121}$$

The charge-to-mass ratio of the second BH turns out to be smaller then one

$$z\,[M_2] \quad = \quad 1 - \frac{\epsilon}{\beta M^2}\,, \tag{3.122}$$

therefore, the entropy of such configuration can be computed exploiting the formula contained in [14]. Although such formula diverges at $z = 1$, depending on the value assumed by β we could still be in its regime of applicability. It is required indeed

$$\zeta \;=\; \sqrt{1 - z^2\,[M_2]} \;=\; \sqrt{\frac{2\epsilon}{\beta M^2} + O\,(\epsilon)} \;\gg\; O\left(\frac{\sqrt{\alpha_i}}{M_2^2}\right) = O\left(\frac{\sqrt{\alpha_i}}{(1-\beta)^2 M^2}\right)\,. \tag{3.123}$$

The terms of equation (3.118) are then

$$\begin{aligned} S_{\text{ext}}\,[\,M\,,\,Q\,] \quad &= \quad \frac{M^2}{8} + \Delta S_{\text{ext}}\,[M]\;, && (3.124)\\ S_{\text{ext}}\,[\,M_1\,,\,Q_1\,] \quad &= \quad \frac{\beta^2 M^2}{8} + \Delta S_{\text{ext}}\,[M_1]\;, && (3.125)\\ S\,[\,M_2\,,\,Q_2\,] \quad &= \quad S_0\,[\,M_2\,,\,Q_2\,] + \Delta S\,[M_2, Q_2] && \\ &= \quad \frac{(1-\beta)^2 M^2}{8}(1+\zeta)^2 + \Delta S\,[M_2, Q_2]\;, && (3.126) \end{aligned}$$

where $\Delta S_{\text{ext}}\,[M] = \Delta S_{\text{ext}}\,[M_1]$ is given by (3.53) and $\Delta S\,[M_2, Q_2]$ is given by (see [14])

$$\Delta S\,[\,M_2\,,\,Q_2\,] = \frac{\epsilon}{4\zeta} + O\,(\epsilon)\;. \tag{3.127}$$

Therefore, equation (3.118) reduces to

$$\frac{M^2}{8} \le \frac{\beta^2 M^2}{8} + \frac{(1-\beta)^2 M^2}{8}(1 + 2\zeta) + \frac{\epsilon}{4\zeta} + O\,(\epsilon)\;, \tag{3.128}$$

and massaging it we get

$$H(\beta) \equiv \frac{5}{32\pi^2}\,\frac{\beta^3(1-\beta)^2}{(2+\beta-2\beta^2)^2} \le \frac{2\alpha_1 - \alpha_3}{M^2} \equiv \tilde{\epsilon}\,. \tag{3.129}$$

$H(\beta)$ is positive in $(0, 1)$ and vanishes at the extremal points.

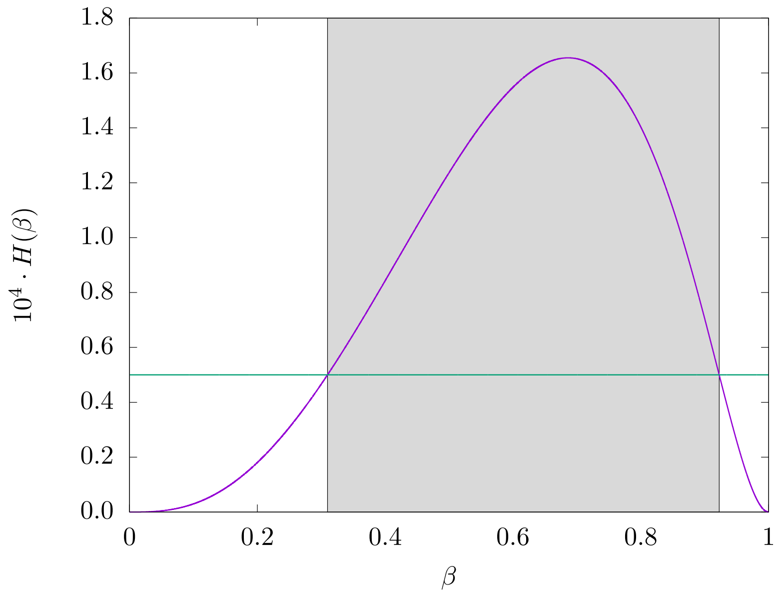


Figure 3.5: The purple curve in the graph is the function $H(\beta)$ and the green curve is the constant function $y = \tilde{\epsilon}$. The shadowed region identifies the values of β which cannot satisfy (3.129) for $y = 5 \cdot 10^{-5}$. Notice that y axis is rescaled by a factor 10^4.

From the graph 3.5 it is clear that we have two different cases depending on the mass of the decaying BH:

- $1 \gg \tilde{\epsilon} > 5 \cdot 10^{-5}$: values of β such that $M_2 \sim M$ are allowed;
- $5 \cdot 10^{-5} \gg \tilde{\epsilon}$: only values of β such that[13] $\beta \sim 1$ and $M_2 \ll M$ are allowed .

Let us consider the second case. We define β_0 as the biggest solution of

$$H(\beta_0) = \tilde{\epsilon}\,, \qquad \beta_0 \in (0,1)\,. \tag{3.130}$$

We have then that $H(\beta)$ is decreasing in $[\beta_0, 1]$. It follows

$$H(\beta) < H(\beta_0)\,, \qquad \forall \beta \in (\beta_0, 1)\,. \tag{3.131}$$

Therefore, the second black hole mass satisfy $M_2 \leq (1 - \beta_0)M$. In order to determine if such BH is a *small black hole* or not we have to compute the maximum mass that it can assume. Given that $\tilde{\epsilon}$ is a small quantity, β_0 can be obtained expanding H around 1. We have therefore

$$H(1-\delta) = \frac{5\,\delta^2}{32\pi^2} + O(\delta^3)\,, \qquad \delta = 1 - \beta \ll 1\,, \tag{3.132}$$

[13] Recall that we assumed $\beta > 1/2$.

which provides us

$$\delta = \sqrt{\frac{32\pi^2}{5M^2}(2\alpha_1 - \alpha_3)} + O\left(\alpha^{\frac{3}{2}}\right), \tag{3.133}$$

$$M_2 = \sqrt{\frac{32\pi^2}{5}(2\alpha_1 - \alpha_3)} + O\left(\alpha^{\frac{3}{2}}\right), \tag{3.134}$$

$$Q_2 = \frac{M_2}{\sqrt{2}}\left(1 - \frac{\epsilon}{\beta M^2}\right) = \frac{M_2}{\sqrt{2}} + O\left(\alpha^{\frac{3}{2}}\right). \tag{3.135}$$

Thus, the second black hole is necessarily a *small black hole.*

Let us consider now the other case, i.e. $1 \gg \tilde{\epsilon} > 5 \cdot 10^{-5}$. The condition on the mass of the decaying BH takes the form

$$\sqrt{2\alpha_1 - \alpha_3} \ll M < \sqrt{2\alpha_1 - \alpha_3}(7 \cdot 10^2). \tag{3.136}$$

It follows that M_2 could have the same order of M only for small M values. Without an estimate of the magnitude order of α_i, we can not explicitly show that a BH with mass $M \sim M_P\sqrt{2\alpha_1 - \alpha_3}\, 10^3$ breaks the perturbative regime. However, we can safely assume that it is small enough to be regarded as a small black hole[14]. It follows that the splitting process can not occur *within our perturbative regime.*

3.7 First Law of Thermodynamic

In the previous section we proved equation (2.88) which has the structure of the first law of thermodynamics and justify Wald's proposal to compute BH entropy. Then, an highly not trivial test of the entropy formula (3.48) can be performed verifying explicitly such relation. We have already showed that Hawking temperature vanishes in the extremal limit (cfr. equation (3.106), therefore theory consistency requires that

$$T_S = \frac{dS}{dM} \xrightarrow{\text{ext}} 0. \tag{3.137}$$

In the following we will verify that (3.137) actually holds.

Let us consider the perturbed entropy formula in the general case (3.48). Neglecting higher order terms in the expansion parameters α_i it reads:

$$S_1 = 8\pi^2 r_H^2 - \frac{\alpha_3 Q^2}{3r_H^2}. \tag{3.138}$$

According to equation (2.88) we define the thermodynamic temperature T_S

$$T_S = \left[\frac{dS_1}{dM}\right]^{-1} = \left[\frac{2Q^2\alpha_3 r'_H}{3r_H^3} + 16\pi^2 r_H r'_H\right]^{-1}, \tag{3.139}$$

which in the extremal limit becomes

[14]Recall that have been observed black holes with mass up to $10^{48} M_P$ (see [39]). A black holes with mass within 2 orders of magnitude form M_P can therefore be regarded as part of the boundary of the perturbative region.

$$T_{S,\text{ext}} = \lim_{z \to z_{\text{ext}}} T_S = \left[\frac{m^2 z_{\text{ext}}^2 \alpha_3 r'_{H\text{ext}}}{3\, r_{H,\text{ext}}^3} + 16\pi^2 r_{H,\text{ext}} r'_{H,\text{ext}} \right]^{-1}, \tag{3.140}$$

where we set

$$r'_{H,\text{ext}} = \lim_{z \to z_{\text{ext}}} \left[\frac{\partial}{\partial m} r_H \right]. \tag{3.141}$$

In general, derivative and limit operations do not commute, therefore in order to evaluate (3.140) we have to compute Δr without assuming to be in the extremal configuration.

We will proceed finding f_1 roots exploiting the same perturbative expansion as before (cfr. equation (3.32). The quantities are no more intended as computed in the extremal case)

$$z = 1 + \Delta z\,, \qquad |\Delta z| \ll 1\,, \qquad r_H = r_0 + \Delta r\,, \qquad \left| \frac{\Delta r}{r_0} \right| \ll 1\,. \tag{3.142}$$

In order to make expansion (3.142) well defined we slightly modify the radius expansion point

$$r_0 = \frac{M}{8\pi} \left(1 + \sqrt{\epsilon} \right), \qquad \epsilon = z_{\text{ext}}^2 - z^2\,, \tag{3.143}$$

which is defined for every admitted BH configuration. Although the choice of r_0 seems to be completely arbitrarily, we notice that r_0 can be interpreted as the unperturbed horizon with the unperturbed extremality charge-to-mass ratio replaced by the perturbed ones,[15] i.e.

$$r_0 \equiv r_{0,H} \left[1 = z_{0,\text{ext}}^2 \to z_{\text{ext}}^2 \right]. \tag{3.144}$$

Moreover, we notice that r_0 choice is in agreement with the computations previously done where we expanded around $m/8\pi$. Its asymptotic behavior is indeed:

$$r_0 \xrightarrow{\alpha_i \to 0}= \frac{m}{8\pi} \left(1 + \sqrt{1 - z^2} \right) \equiv r_{H,0}\,, \qquad r_0 \xrightarrow{z \to z_{\text{ext}}}= \frac{m}{8\pi} \equiv r_{H,0,\text{ext}}\,. \tag{3.145}$$

Exploiting the perturbative expansion (3.142) and equation (3.91) we get

$$\begin{aligned} r'_H &= \frac{d}{dM}(r_0 + \Delta r)\,, && (3.146)\\ r'_0 &= \frac{1}{8\pi} \left(1 + \frac{2\epsilon + M\epsilon'}{\sqrt{\epsilon}} \right), && (3.147)\\ \Delta r' &= \left(\Delta r^+ \right)' = \left(\frac{-\partial_r f_1 + \sqrt{\Delta}}{\partial_r^2 f_1} \Big|_{r=r_0} \right)', && (3.148)\\ \epsilon' &= \frac{4Q^2}{m^3} - \frac{256\pi^2}{5m^3}(2\alpha_1 - \alpha_3) + O\left(\frac{\alpha^2}{m^5} \right), \\ &= \frac{2}{m} \left[1 + 2\,(\Delta z - \Delta z_{\text{ext}}) + O\left(\frac{\alpha^2}{m^4} \right) \right]. && (3.149) \end{aligned}$$

[15] Notice that this definition can be used to define a canonical choice of the expansion point used to compute Δr and Δz through the algorithm described in section 3.4.2.

Taking the extremal limit we obtain

$$\epsilon \to 0\,, \tag{3.150}$$

$$\epsilon' \to \epsilon'_{\text{ext}} = \frac{2}{M}\,, \tag{3.151}$$

$$r_{H,0} \to r_{H,0,\text{ext}} = \frac{M}{8\pi}\,, \tag{3.152}$$

$$r'_{H,0} \to r'_{H,0,\text{ext}} = \frac{1}{8\pi\sqrt{\epsilon}}\,, \tag{3.153}$$

$$\Delta z \to \Delta z_{\text{ext}} = \frac{64\pi^2}{5M^2}(2\alpha_1 - \alpha_3)\,, \tag{3.154}$$

$$\Delta r \to \Delta r_{\text{ext}} = -\frac{8\pi}{15M}(24\alpha_1 - 7\alpha_3)\,, \tag{3.155}$$

$$\Delta r' \to \Delta r'_{\text{ext}} = -\frac{1}{8\pi\sqrt{\epsilon}}\left[1 + \frac{5\alpha_3}{3\sqrt{(46\alpha_1 - 3\alpha_3)\,(2\alpha_1 - \alpha_3)}}\right]\,, \tag{3.156}$$

$$T_S^{-1} \to T_{S,\text{ext}}^{-1} = \frac{M}{\sqrt{\epsilon}}\left[\frac{-5\alpha_3}{12\sqrt{(46\alpha_1 - 3\alpha_3)\,(2\alpha_1 - \alpha_3)}} + O\left(\frac{1}{M^2}\right)\right]\,, \tag{3.157}$$

and we finally get

$$T_S \to T_{S,\text{ext}} = \frac{\sqrt{\epsilon}}{M}\left[\text{const} + O\left(\frac{1}{M^2}\right)\right]^{-1} \equiv 0\,. \tag{3.158}$$

It follows that the thermodynamic relation (2.88) has been verified in the extremal limit.

3.8 Field Redefinitions Invariance

In the previous chapter we identified a particular 4 derivative Einstein–Maxwell theory applying a field redefinitions and we claimed that such class of transformations leaves the dynamics invariant. In this section we will test such a claim verifying that Δz_{ext} and ΔS_{ext} are actually invariant, as we expect from every physical quantity which influences the theory dynamics (such quantities are indeed both related to BHs decay).

In this section we will repeat the computations of sections 3.1 and 3.2 considering the most general 4 derivative Einstein–Maxwell theory which can be obtained without applying a fields redefinition. In particular, we will generalize the previous approach determining a solutions of perturbed EOMs which is independent of the particular perturbation of Einstein–Maxwell theory we consider.

3.8.1 Extended Theory Perturbative Solution

Let us solve the extended theory (2.20). Action variation takes the same form as before (cfr. equation (3.2)):

$$\delta_{A_\nu} S = \int e D_\mu \left[F^{\mu\nu} - \Delta F^{\mu\nu}\right] \delta A_\nu\,, \tag{3.159}$$

$$\delta_{g^{\mu\nu}} S = \frac{M_P^2}{2}\int e\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{M_P^2}T_{\mu\nu}\right]\delta g^{\mu\nu}\,, \tag{3.160}$$

Now stress-energy tensor components take the explicit form:[16]

$$
\begin{aligned}
T_{\mu\nu} &= T^{(2)}_{\mu\nu} + T^{(4)}_{\mu\nu}\,, &(3.161)\\
T^{(2)}_{\mu\nu} &= -\frac{2}{\sqrt{|g|}}\frac{\delta}{\delta g^{\mu\nu}}\left(\sqrt{|g|}\mathcal{L}_2\right) = F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\,, &(3.162)\\
T^{(4)}_{\mu\nu} &= -\frac{2}{\sqrt{|g|}}\frac{\delta}{\delta g^{\mu\nu}}\left(\sqrt{|g|}\mathcal{L}_4\right) \;=\; g_{\mu\nu}\mathcal{L}_4 - 2\delta\mathcal{L}_4 = T^{(4)}_{\mu\nu,1} + T^{(4)}_{\mu\nu,2}\,, &(3.163)
\end{aligned}
$$

$$
\begin{aligned}
T^{(4)}_{\mu\nu,1} = {} & \frac{\alpha_1}{4M_P^4}\Big\{g_{\mu\nu}F^4 - 8F^2F_{\mu\rho}F_\nu{}^\rho\Big\} + \frac{\alpha_2}{4M_P^4}\Big\{g_{\mu\nu}(F\tilde{F})^2 - 2g_{\mu\nu}(F\tilde{F})^2\Big\}\\
& + \frac{\alpha_3}{2M_P^2}\Big\{-6F_{\alpha(\nu|}F^{\beta\gamma}R^\alpha{}_{|\mu)\beta\gamma} - 4D_\beta D_\alpha\left(F^\alpha{}_{(\mu|}F^\beta{}_{|\nu)}\right)\\
& + 8R_{(\nu|\sigma}F_{|\mu)\rho}F^{\sigma\rho} + 4R^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu} + 2g_{\mu\nu}D_\alpha D_\beta(F^\alpha{}_\rho F^{\beta\rho})\\
& - 4D_\alpha D_{(\nu|}\left(F_{|\mu)\beta}F^{\alpha\beta}\right) + 2D^2(F_{\mu\rho}F_\nu{}^\rho) - \frac{4}{3}RF_\mu{}^\sigma F_{\nu\sigma} - \frac{2}{3}F^2R_{\mu\nu}\\
& + \frac{2}{3}D_{(\mu|}D_{|\nu)}F^2 - \frac{2}{3}g_{\mu\nu}D^2F^2 + g_{\mu\nu}W_{\alpha\beta\rho\sigma}F^{\alpha\beta}F^{\rho\sigma}\Big\}\,,
\end{aligned}
$$

$$
\begin{aligned}
T^{(4)}_{\mu\nu,2} = {} & \frac{\alpha_4}{2M_P^2}\Big\{g_{\mu\nu}R^{\alpha\beta}F_{\alpha\gamma}F_\beta{}^\gamma - 4R_{(\nu|\sigma}F_{|\mu)\rho}F^{\sigma\rho} - 2R^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu}\\
& - g_{\mu\nu}D_\alpha D_\beta(F^\alpha{}_\rho F^{\beta\rho}) + 2D_\alpha D_{(\nu|}\left(F_{|\mu)\beta}F^{\alpha\beta}\right) - D^2(F_{\mu\rho}F_\nu{}^\rho)\Big\}\\
& + \frac{\alpha_5}{2M_P^2}\Big\{g_{\mu\nu}RF^2 - 4RF_\mu{}^\sigma F_{\nu\sigma} - 2F^2R_{\mu\nu} + 2D_{(\mu|}D_{|\nu)}F^2 - 2g_{\mu\nu}D^2F^2\Big\}\\
& + \alpha_6\Big\{g_{\mu\nu}R^2 - 4RR_{\mu\nu} + 4D_{(\nu|}D_{|\mu)}R - 4g_{\mu\nu}D^2R\Big\}\\
& + \alpha_7\Big\{g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + 4D_\alpha D_{(\nu|}R_{|\mu)}{}^\alpha - 2D^2R_{\mu\nu} - g_{\mu\nu}D^2R - 4R_{\mu\alpha}R_\nu{}^\alpha\Big\}\\
& + \frac{\alpha_8}{2M_P^2}\Big\{g_{\mu\nu}(D_\alpha F_{\beta\gamma})^2 - 2(D_\mu F_{\alpha\beta})(D_\nu F^{\alpha\beta}) - 4(D_\alpha F_{\beta\mu})(D^\alpha F^\beta{}_\nu)\\
& + 4D_\alpha\left(F_{(\nu|\beta}D^\alpha F_{|\mu)}{}^\beta\right) + 4D_\alpha\left(F_{(\nu|\beta}D_{|\mu)}F^{\alpha\beta}\right) - 4D_\alpha\left(F^\alpha{}_\beta D_{(\nu|}F_{|\mu)}{}^\beta\right)\Big\}\,.
\end{aligned}
$$

Corrections to $\Delta F^{\mu\nu}$ are instead:

$$
\begin{aligned}
\Delta F^{\mu\nu} &= \Delta F_1^{\mu\nu} + \Delta F_2^{\mu\nu}\,, &(3.164)\\
\Delta F_1^{\mu\nu} &= \frac{2\alpha_1}{M_P^4}F^2F^{\mu\nu} + \frac{2\alpha_2}{M_P^4}F\tilde{F}\tilde{F}^{\mu\nu} + \frac{2\alpha_3}{M_P^2}W^{\mu\nu\rho\sigma}F_{\rho\sigma}\,, &(3.165)\\
\Delta F_2^{\mu\nu} &= \frac{2\alpha_4}{M_P^2}R^{[\mu|\alpha}F_\alpha{}^{|\nu]} + \frac{2\alpha_5}{M_P^2}RF^{\mu\nu} - \frac{2\alpha_8}{M_P^2}D^2F^{\mu\nu}\,. &(3.166)
\end{aligned}
$$

[16] We indicate with indexes 1 and 2 respectively the higher order corrections that we already considered and the new ones

Therefore, we get as before the equations of motion:

$$\begin{aligned} M_P^2 G_{\mu\nu}\left[g^{(1)}_{\alpha\beta}\right] \simeq\ & T^{(2)}_{\mu\nu}\left[g^{(1)}_{\alpha\beta}, A^{(0)}_\gamma\right] + \Big(2\Delta F_{(\mu|\rho}F_{|\nu)}{}^{\rho} \\ & - \frac{1}{2}g_{\mu\nu}\Delta F_{\rho\sigma}F^{\rho\sigma} + T^{(4)}_{\mu\nu}\Big)\left[g^{(0)}_{\alpha\beta}, A^{(0)}_\gamma\right] . \end{aligned} \tag{3.167}$$

To solve EOMs we proceed as before considering stationary, charged and spherical solutions and requiring that in the limit $\alpha_i \to 0$ we must recover RN solution. We use therefore the ansatz

$$ds^2 = -N(r)^2 f_1(r)\, dt^2 + \frac{1}{f_1(r)}\, dr^2 + r^2\, d\Omega^2_{S_2}\,, \tag{3.168}$$

and we impose the constraints

$$g^{(1)}_{tt} = -\,N^2 f_1 = -f_0^2 + \Delta g_{tt} + o(\alpha_i)\,, \qquad \Delta g_{tt} \xrightarrow{\alpha_i\to 0} 0\,, \tag{3.169}$$

$$\left(g^{(1)}_{rr}\right)^{-1} = f_1 = f_0^2 + \Delta f + o(\alpha_i)\,, \qquad \Delta f \xrightarrow{\alpha_i\to 0} 0\,, \tag{3.170}$$

$$f_0^2 = 1 - \frac{M}{4\pi M_P^2 r} + \frac{Q^2}{32\pi^2 M_P^2 r^2}\,. \tag{3.171}$$

The computations performed in section 3.1 to determine the perturbative solution are highly dependent on the structure of the particular higher order operators we are considering in our theory. However, such method can be easily generalized. Before computing the actual solution of the EOMs, we will show that it is possible to express $f_1(r)$ and $N(r)$ in terms of $T^{(4)}$ and $f_0^2(r)$ integrals. We recall that EOMs can be written as

$$\mathcal{E}_\mu^\nu = M_P^2 G_\mu^\nu - T_\mu^\nu = 0\,, \tag{3.172}$$

Evaluating Einstein tensor G_μ^ν and ${T^{(2)}}_\mu{}^\nu$ for the ansatz (3.168) we get:

$${G_0}^0 = \frac{1}{r^2}\left[-1 + f_1 + r f_1'\right]\,, \tag{3.173}$$

$${G_1}^1 = \frac{1}{r^2}\left[-1 + f_1 + r f_1'\right] + \frac{2 f_1 N'}{rN}\,, \tag{3.174}$$

$${G_2}^2 = {G_3}^3 = \frac{3 r f_1' N' + N\left(r f_1'' + 2 f_1'\right) + 2 f_1\left(r N'' + N'\right)}{2rN}\,, \tag{3.175}$$

$${T^{(2)}}_0{}^0 = {T^{(2)}}_1{}^1 = -{T^{(2)}}_2{}^2 = -{T^{(2)}}_3{}^3 = -\frac{Q^2}{32\pi^2 r^4 N^2}\,. \tag{3.176}$$

The combination ${\mathcal{E}_1}^1 - {\mathcal{E}_0}^0 = 0$ then reads

$${G_1}^1 - {G_0}^0 = \frac{2 f_1 N'}{rN} = \Delta {T^{(4)}}_1{}^1 - \Delta {T^{(4)}}_0{}^0\,, \tag{3.177}$$

which can be easily integrated providing

$$N = k \exp\left[\frac{1}{2}\int dr\, \frac{r}{f_1}\left({T^{(4)}}_1{}^1 - {T^{(4)}}_0{}^0\right)\right]\,, \tag{3.178}$$

with k constant of integration. Recalling that

$$T^{(4)}{}_{\mu}{}^{\nu} \sim O(\alpha_i)\,, \qquad f_1 = f_0^2 + O(\alpha_i)\,, \tag{3.179}$$

equation (3.168) turns out to be

$$\begin{aligned} N &= k \exp\left[\frac{1}{2}\int dr\,\frac{r}{f_0^2}\left(\Delta T^{(4)}{}_1{}^1 - \Delta T^{(4)}{}_0{}^0\right) + O(\alpha^2)\right] & (3.180)\\ &= \left[1 + \frac{1}{2}\int dr\,\frac{r}{f_0^2}\left(\Delta T^{(4)}{}_1{}^1 - \Delta T^{(4)}{}_0{}^0\right) + O(\alpha^2)\right]\,, & (3.181) \end{aligned}$$

where we fixed k exploiting the constraint on asymptotic behavior.

$$N \xrightarrow{\alpha_i \to 0} 1\,. \tag{3.182}$$

Let us consider now the combination $\mathcal{E}_2{}^2 + \mathcal{E}_0{}^0 = 0$. It reads

$$G_2{}^2 - G_0{}^0 = \Delta T^{(4)}{}_2{}^2 + \Delta T^{(4)}{}_0{}^0\,. \tag{3.183}$$

Setting $N = \exp[Y]$, LHS becomes

$$\frac{1}{2r^2}\left[-2 + r f_1'(4 + 3rY') + r^2 f_1'' + 2f_1(1 + r(Y' + rY'^2 + rY''))\right]\,, \tag{3.184}$$

and neglecting higher order terms in the perturbative expansion parameter it reduces to

$$\frac{1}{2r^2}\left[-2 + 4r f_1' + 3r^2 Y'\left(f_0^2\right)' + r^2 f_1'' + 2f_1 + 2f_0^2 r(Y' + rY'^2 + rY'')\right]\,. \tag{3.185}$$

Massaging equation (3.183) we get the differential equation

$$\frac{d^2}{dr^2}\left[r^2 f_1(r)\right] = 2 + h_1(r) + h_2(r)\,, \tag{3.186}$$

which is solved by

$$f_1 = 1 + \frac{C_1}{r} + \frac{C_2}{r^2} + \frac{H_1}{r^2} + \frac{H_2}{r^2} + O\left(\alpha^2\right)\,, \tag{3.187}$$

$$\begin{aligned} H_1 &= \iint dr\,h_1 = \iint dr\left[-3r^2 Y'\left(f_0^2\right)' - 2f_0^2 rY' - 2r^2 f_0^2 Y''\right]\,, & (3.188)\\ H_2 &= \iint dr\,h_2 = \iint dr\,2r^2\left[\Delta T^{(4)}{}_2{}^2 + \Delta T^{(4)}{}_0{}^0\right]\,, & (3.189) \end{aligned}$$

where C_1 and C_2 are integration constants that can be fixed exploiting the constraint on the asymptotic behavior

$$f_1 \xrightarrow{\alpha_i \to 0} f_0^2\,. \tag{3.190}$$

We have therefore obtained the general solutions we claimed before. Evaluating equations (3.181) and (3.187) for the extended theory (2.20) we finally get

$$C_2 = \frac{Q^2}{32\,\pi^2}\,, \qquad C_1 = -\frac{M}{4\pi}\,, \tag{3.191}$$

$$
\begin{aligned}
N(r) = & \quad 1 + \frac{Q^2}{96\pi^2 r^4}(\alpha_3 + 9\alpha_4 + 30\alpha_5 + 6\alpha_7 - 9\alpha_8)\,, && (3.192)\\
f_1 = & \quad 1 - \frac{M}{4\pi r} + \frac{Q^2}{32\pi^2 r^2} - \frac{Q^2\ (2\alpha_3 + 9\alpha_4 + 24\alpha_5 + 12\alpha_7 - 6\alpha_8)}{48\,\pi^2\, r^4} \\
& + \frac{M\,Q^2\ (5\alpha_3 + 15\alpha_4 + 42\alpha_5 + 18\alpha_7 - 9\alpha_8)}{384\,\pi^3\, r^5} && (3.193)\\
& - \frac{Q^4\ (2\alpha_1 + 4\alpha_3 + 11\alpha_4 + 30\alpha_5 + 12\alpha_7 - 6\alpha_8)}{2560\,\pi^4\, r^6}\,.
\end{aligned}
$$

3.8.2 Extended Theory Observables

Given the expressions (3.192) and (3.193), the metric becomes

$$
\begin{aligned}
\frac{1}{g_{rr}} &= f_1 = 1 - \frac{M}{4\pi r} + \frac{Q^2}{32\pi^2 r^2} - \frac{A_2^{rr}\,Q^2}{24\,\pi^2\, r^4} + \frac{A_1^{rr} M\,Q^2}{384\,\pi^3\, r^5} - \frac{A_0^{rr}\,Q^4}{1280\,\pi^4\, r^6}\,, && (3.194)\\
g_{tt} &= -\,N^2 f_1 = -1 + \frac{M}{4\pi r} - \frac{Q^2}{32\pi^2 r^2} + \frac{A_2^{tt}\,Q^2}{24\,\pi^2\, r^4} - \frac{A_1^{tt} M\,Q^2}{384\,\pi^3\, r^5} + \frac{A_0^{tt}\,Q^4}{1280\,\pi^4\, r^6}\,, && (3.195)\\
g_{rr} g_{tt} &= -\,N^2 = -1 - \frac{Q^2}{48\pi^2 r^4}(\alpha_3 + 9\alpha_4 + 30\alpha_5 + 6\alpha_7 - 9\alpha_8)\,, && (3.196)
\end{aligned}
$$

where we introduced the parameters

$$
\begin{aligned}
A_0^{rr} &= \alpha_1 + 2\alpha_3 + \frac{11}{2}\alpha_4 + 15\alpha_5 + 6\alpha_7 - 3\alpha_8\,, && (3.197)\\
A_1^{rr} &= 5\alpha_3 + 15\alpha_4 + 42\alpha_5 + 18\alpha_7 - 9\alpha_8\,, && (3.198)\\
A_2^{rr} &= \alpha_3 + \frac{9}{2}\alpha_4 + 12\alpha_5 + 6\alpha_7 - 3\alpha_8\,, && (3.199)\\
A_0^{tt} &= \frac{1}{6}\left(6\alpha_1 + 7\alpha_3 - 12\alpha_4 - 60\alpha_5 + 6\alpha_7 + 27\alpha_8\right)\,, && (3.200)\\
A_1^{tt} &= \frac{1}{3}\left(\alpha_3 - \alpha_4 - 6\alpha_5 + 2\alpha_7 + 3\alpha_8\right)\,, && (3.201)\\
A_2^{tt} &= \frac{1}{2}\left(\alpha_3 - 6\alpha_5 + 6\alpha_7 + 3\alpha_8\right)\,. && (3.202)
\end{aligned}
$$

Exploiting equations (3.77) and (3.78), it is easy to check that Δz and Δr coincide for g_{tt} and $1/g_{rr}$ and take the form

$$
\begin{aligned}
\Delta z_{\text{ext}} &= \frac{64\pi^2}{15M^2}(6A_0 - 5A_1 + 10A_2) = \frac{64\pi^2}{5M^2}(2\alpha_1 - \alpha_3 + \alpha_4 + 2\alpha_7 - \alpha_8)\,, && (3.203)\\
\Delta r_{\text{ext}} &= -\frac{8\pi}{15M}\left(24A_0 - 15A_1 + 20A_2\right) \\
&= -\frac{8\pi}{15M}(24\alpha_1 - 7\alpha_3 - 3\alpha_4 + 30\alpha_5 + 6\alpha_7 - 3\alpha_8)\,. && (3.204)
\end{aligned}
$$

We have therefore

$$
\begin{aligned}
z_{\text{ext}} &= 1 + \frac{64\pi^2}{5M^2}(2\alpha_1 - \alpha_3 + \alpha_4 + 2\alpha_7 - \alpha_8)\,, && (3.205)\\
r_{H,\text{ext}} &= \frac{m}{8\pi} - \frac{8\pi}{15M}(24\alpha_1 - 7\alpha_3 - 3\alpha_4 + 30\alpha_5 + 6\alpha_7 - 3\alpha_8)\,. && (3.206)
\end{aligned}
$$

In order to compute entropy corrections we have to evaluate E^{abcd} tensor for the extended theory. Recalling its definition (3.42) we get

$$\begin{aligned}\tilde{E}^{abcd} = \;& \frac{1}{2}g^{ac}g^{bd} + \frac{\alpha_3}{2}\left(F^{ab}F^{cd} - 2g^{ac}F^{bf}F^{d}{}_{f} + \frac{1}{3}F^2 g^{ac}g^{bd}\right) \\ & + \frac{\alpha_4}{2}F^{a\alpha}F^{c}{}_{\alpha}\,g^{bd} + \frac{\alpha_5}{2}F_{\mu\nu}F^{\mu\nu}g^{ac}g^{bd} + 2\alpha_6 R g^{ac}g^{bd} \\ & + 2\alpha_7 R^{ac}g^{bd}\,,\end{aligned} \tag{3.207}$$

where the tilde indicates that the tensor is not correctly symmetrized yet. We have then:

$$E^{abcd} = \frac{1}{8}(\tilde{E}^{abcd} - \tilde{E}^{bacd} - \tilde{E}^{abdc} + \tilde{E}^{badc} + \tilde{E}^{cdab} - \tilde{E}^{dcab} - \tilde{E}^{cdba} + \tilde{E}^{dcba})\,. \tag{3.208}$$

Neglecting higher order correction the perturbed entropy of the extended theory takes the form

$$S = -2\pi A E^{abcd}\eta_{ab}\eta_{cd} = 8\pi^2 r_H^2 - \frac{m^2 z^2}{12 r_H^2}(2\alpha_3 + 3\alpha_4 + 6\alpha_5 + 6\alpha_7)\,, \tag{3.209}$$

and replacing equations (3.205) and (3.206) we obtain the entropy of extremal BH configurations

$$S = \frac{m^2}{8} - \frac{16}{5}\pi^2(8\alpha_1 + \alpha_3 + 4\alpha_4 + 8\alpha_7 + \alpha_8)\,. \tag{3.210}$$

Exploiting equation (2.26), we notice that charge-to-mass ratio and entropy corrections are proportional to invariant combinations of α_i coefficients. We have indeed

$$\Delta z_{\text{ext}} = \frac{64\pi^2}{5M^2}(2\beta_1 - \beta_3)\,, \tag{3.211}$$

$$\Delta S_{\text{ext}} = -\frac{16}{5}\pi^2\,(8\beta_1 + \beta_3)\;. \tag{3.212}$$

In the previous chapter we claimed that the corrections have to be protected in order to not be absorbed by a fields redefinition. Equations (3.211) and (3.212) imply that a much stronger condition holds : they are invariant. It follows that we can determine Δz_{ext} and ΔS_{ext} as well as β_i coefficients in the general case evaluating them in a particular theory. Setting $\alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0$ we move to the theory 2.19 and recalling the positivity bounds of section 3.3 we get

$$\begin{cases} 2\beta_1 - \beta_3 > 0\,, \\ 2\beta_1 + \beta_3 > 0\,, \\ \beta_2 > 0\,. \end{cases} \tag{3.213}$$

It follows that in all the possible 4 derivative extensions of Einstein–Maxwell gravity hold

$$\Delta z_{\text{ext}} > 0\,, \qquad \Delta S_{\text{ext}} < 0\,. \tag{3.214}$$

3.9 Entropy Correction Sign

We conclude this chapter discussing a possible tension between our work and the literature. Cheung et al. [14] claim that for BH configurations in extended Einstein–Maxwell theory $\Delta S > 0$. Moreover, they explicitly check the result in the case of highly charged black holes with $z < 1$. This seems to be in contradiction with our result $\Delta S_{\text{ext}} < 0$, but we will show that this is not the case. The crucial point is that they computed the entropy variation considering configurations with the same mass M and charge Q. However, the values of the charge Q and the mass M which identify an extremal configuration in the unperturbed theory do not identify an extremal configuration in the perturbed theory.

We briefly sketch their argument. They consider extended Einstein–Maxwell theory as an effective field theory which describes a graviton g and a photon A obtained by integrating out the heavy fields ϕ. We have therefore that the corresponding euclidean path integral is

$$\mathcal{Z} = \int \mathcal{D}g\,\mathcal{D}A\,\mathcal{D}\phi\, e^{-I_{\text{UV}}[g,A,\phi]} = \int \mathcal{D}g\,\mathcal{D}A\, e^{-I[g,A]} \,. \tag{3.215}$$

They assume then that higher dimension operators are dominated by heavy fields, therefore

$$I_{\text{UV}}[g, A, 0] = \tilde{I}[g, A] \,, \tag{3.216}$$

where $\tilde{I}$ is the euclidean action of the unperturbed Einstein–Maxwell theory. Choosing boundary conditions properly to have finite temperature β they conclude that

$$-\log \mathcal{Z}(\beta) \approx I_{\text{UV}}[g_{\text{cl}}, A_{\text{cl}}, \phi_{\text{cl}}] < I_{\text{UV}}[\tilde{g}_{\text{cl}}, \tilde{A}_{\text{cl}}, 0] = \tilde{I}[\tilde{g}_{\text{cl}}, \tilde{A}_{\text{cl}}] = -\log \tilde{\mathcal{Z}}(\beta) \,, \tag{3.217}$$

where g_{cl}, A_{cl} and ϕ_{cl} are the classical solution of the UV theory, $\tilde{g}_{\text{cl}}, \tilde{A}_{\text{cl}}$ are the classical solution of the Einstein–Maxwell theory, $\mathcal{Z}$ is the partition function of the extended theory and $\tilde{\mathcal{Z}}$ is the partition function of the Einstein–Maxwell theory. The first equivalence holds because of the saddle point approximation as well as the last one. The inequality is a consequence of the assumption that the action is in a local minimum. The central equality is finally a consequence of (3.216). Using equation (3.217) and the thermodynamics relations

$$\log \mathcal{Z}(\beta) = S - \beta M \,, \tag{3.218}$$
$$\beta = \partial_M S \,, \tag{3.219}$$

it is possible to constrain the entropy variation. In order to compare black holes with the same masses and charges we have to shift β through

$$\beta = \tilde{\beta} + \Delta\beta = \tilde{\beta} + \partial_M \Delta S \,. \tag{3.220}$$

β and $\tilde{\beta}$ are, respectively, the inverse of the temperature of a perturbed and an unperturbed BH, both with mass M and charge Q. It follows

$$\log \tilde{\mathcal{Z}}(\beta) = \log \tilde{\mathcal{Z}}(\tilde{\beta}) - M\partial_M \Delta S \,. \tag{3.221}$$

Combining (3.217), (3.218) and (3.221) we finally get

$$\log \mathcal{Z}(\beta) = S - \beta M > \tilde{S} - \tilde{\beta} M - \Delta\beta M = \log \tilde{\mathcal{Z}}(\beta) \,, \tag{3.222}$$

which yields

$$S[M,Q] - \tilde{S}[M,Q] = \Delta S[M,Q] > 0\,. \tag{3.223}$$

As we anticipated at the beginning of this section, the result of Cheung et al. does not disagree with ours. Once we explicit mass and charge dependencies, we see that equation (3.223) relates configurations with the same M and Q; equation (3.53) relates instead extremal configurations with the same mass but with different charge.

CHAPTER 4

Positivity Bounds

In the previous chapter we assumed the positivity bounds of [7] in order to constrain the coefficients of higher order corrections to the Einstein–Maxwell theory. In this chapter we will discuss the nature of such bounds and we will briefly present the ideas behind the techniques to compute them (see [21], [40], [41]) together with the problems that arise turning on gravity (see [7]). We conclude justifying the choice of the particular 4 derivative Einstein–Maxwell theory we used as our starting point.

4.1 Origin of Positivity Bounds

The reason why such bounds exist is simple (see [40]): the set of conditions one has to impose to obtain a reliable theory[1] cannot be reduced to the request of having a local and Lorentz-invariant Lagrangian (LLI Lagrangian). A LLI Lagrangian guarantees that applying a Lorentz transformation to a solution of the EOMs we obtain another solution of the EOMs. However, it does not guarantee some fundamental features of every physical theory, such as causality. Admitting superluminal signals it is indeed possible to build not reliable theories where there is no Lorentz-invariant notion of time-ordering or such that the time evolution on macroscopic scales can not be described by a local Hamiltonian flow. The bounds on coefficients are therefore the constraints obtained imposing conditions which are not already accounted by the LLI structure.

4.1.1 Example: Scalar Theory

The effects of superluminal signals and the constraints to avoid them can easily understood in the simple case of a massless scalar field π with shift symmetry $\pi \to \pi + \text{const}$. The IR Lagrangian is given by

$$\mathcal{L} = \partial_\mu \pi \partial^\mu \pi + \frac{d}{\Lambda^4} (\partial_\mu \pi \partial^\mu \pi)^2 \,, \tag{4.1}$$

[1] A theory to be reliable must be consistent and does not have to contradict experimental observations.

where d is a dimensionless real constant. Expanding around the background π_0, the perturbation $\phi = \pi - \pi_0$ satisfies the equations of motion

$$\left(\partial_t^2 - v^2\partial_i^2\right)\phi = 0\,, \qquad v \approx 1 - \frac{4d}{\Lambda^4}(\partial_\mu\pi_0)^2\,. \tag{4.2}$$

Being $(\partial_\mu\pi_0)^2$ a kinetic term it holds $(\partial_\mu\pi_0)^2 \geq 0$. Equation (4.2) implies then that the perturbations can be superluminal. If $(\partial_\mu\pi_0)^2$ is not vanishing, we have indeed

$$d < 0 \iff v^2 > 1\,. \tag{4.3}$$

It follows that the existence of superluminal signals is strictly related to the sign of the d coefficient.

We proceed now to analyze the consequences of causality violation. We assume $d < 0$ and we consider a sources configuration which allows to have a ball $\mathfrak{B}$ with not vanishing constant background inside a spacetime with vanishing background. The signals emitted from a point A inside $\mathfrak{B}$ travel faster than light. Outside $\mathfrak{B}$ their speed slows down and they travel at the speed of light. A signal properly oriented can therefore cross the boundary of $\mathfrak{B}$ in a point B in the elsewhere of A. Let us consider a point C outside $\mathfrak{B}$ reached by a signal which passed through B. If there was a Lorentz-invariant notion of time-ordering it would be possible to fix uniquely the orientation of every worldline. But the worldline which connects A, B and C does not have a fixed orientation. C is in the future light cone of B but the position of A, being in B elsewhere, depends on the reference frame. Different observers can see therefore a different orientation of the worldline around B. It follows that the time-ordering is not Lorentz-invariant.

We consider now the boosted EOMs. Applying a Lorentz transformation to (4.2) with β in the $\hat{x}$ direction we get

$$\left[\left(1 - v^2\beta^2\right)\partial_t^2 + 2\beta\left(1 - v^2\right)\partial_t\partial_x - \left(v^2 - \beta^2\right)\partial_x^2 - v^2\partial_\perp^2\right]\phi\,. \tag{4.4}$$

Inside $\mathfrak{B}$ it holds $v > 1$. Moreover, we can choose $\beta > \frac{1}{v}$ such that the coefficient of ∂_t^2 is negative in A. Therefore, the ∂_t^2 coefficient has to vanish along the worldlines of the signals which exit from $\mathfrak{B}$. It follows that there exists a class of reference frames which admits a region of the spacetime where the EOMs become not dynamical constraints: in such regions the time evolution can not be described by a Hamiltonian flow.

4.2 Computation of Positivity Bounds

So far, we have established that to produce a reliable theory we have to constrain properly the LLI Lagrangian. Moreover, we showed through an explicit example the important role of causality constraints. However, the general approach is much more sophisticated than our computations and exploits S-matrix formalism. The conditions to impose are encoded as properties of the S-matrix[2]. In particular, causality is replaced by the stronger condition of microcausality[3].

We proceed now by presenting the general algorithm used to produce positivity bounds and we apply it to the simple case of Euler–Heisenberg theory. After that, we briefly review the problems which arise with gravity.

[2]Notice that the S-matrix formalism can be used to study theories without an off-shell formulation too.
[3]Bosonic (fermionic) operators evaluated at spacelike intervals commute (anticommute)

4.2.1 Positivity Bounds Without Gravity

The standard algorithm is based on three assumptions: microcausality, relativistic invariance and unitarity. Microcausality is implemented assuming that the amplitudes which enter in the S-matrix are real boundary values of analytical functions (in the kinematics invariant quantities) with cuts (see [21], [40]). Relativistic invariance is used to derive crossing symmetries (see below). Unitarity of S-matrix is finally exploited to apply the optical theorem (see [41]).

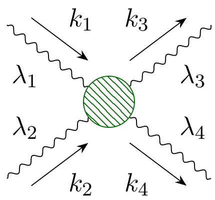
The three ingredients are combined as follows. We consider a forward elastic scattering and we extract its amplitude. Being the amplitude a tensor with indexes which runs over the possible particles polarizations, we use the crossing symmetries to identify the independent entries. We exploit then analicity to express amplitudes derivative in terms of their imaginary part. We conclude applying the optical theorem to produce a positivity bound.

4.2.2 Example: Elastic Scattering of Photons

We apply now the algorithm to a theory which contains 4 photons interactions (see [42]). Let us consider the Euler–Heisenberg theory

$$\mathcal{L} = -\frac{1}{4}F^2 + \frac{a_1}{4m^4}(F^2)^2 + \frac{a_2}{4m^4}(F\tilde{F})^2\,, \tag{4.5}$$

where m is a cutoff scale. The forward elastic elastic scattering of photons with fixed polarization $\gamma\gamma \to \gamma\gamma$ has feymann diagram



and satisfy $t = 0$, $u = -s$. It follows that the amplitude has the structure

$$M_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} = \epsilon^{*\alpha_4}_{\lambda_4}(k_4)\,\epsilon^{*\alpha_3}_{\lambda_3}(k_3)\,\epsilon^{\alpha_2}_{\lambda_2}(k_2)\,\epsilon^{\alpha_1}_{\lambda_1}(k_1)\,\mathcal{M}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}\,, \tag{4.6}$$

$$\mathcal{M}_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} = A(s)\,\eta_{\alpha_1\alpha_3}\,\eta_{\alpha_2\alpha_4} + B(s)\,\eta_{\alpha_1\alpha_4}\,\eta_{\alpha_2\alpha_3} + C(s)\,\eta_{\alpha_1\alpha_2}\,\eta_{\alpha_3\alpha_4}\,, \tag{4.7}$$

where $\eta = \mathrm{diag}(1,-1,-1,-1)$. We assume then that the photons are linearly polarized along $\hat{z}$ axis and we choose the real basis

$$\epsilon_x(k_1) = (0,1,0,0)\,, \tag{4.8}$$
$$\epsilon_y(k_1) = (0,0,1,0)\,, \tag{4.9}$$
$$\epsilon_x(k_2) = (0,-1,0,0)\,, \tag{4.10}$$
$$\epsilon_y(k_2) = (0,0,1,0)\,. \tag{4.11}$$

Eventually there are 8 not vanishing amplitudes

$$M_{xxxx} = M_{yyyy} = A(s) + B(s) + c(s)\,, \tag{4.12}$$
$$M_{xyxy} = M_{yxyx} = A(s)\,, \tag{4.13}$$
$$M_{xyyx} = M_{yxxy} = B(s)\,, \tag{4.14}$$
$$M_{xxyy} = M_{yyxx} = C(s)\,. \tag{4.15}$$

Now we introduce crossing symmetries requiring M to be invariant under legs swapping. Considering for instance the swap of γ_1 and γ_3 we get

$$M_{\lambda_1,\lambda_2,\lambda_3,\lambda_4}(s) \to M_{\lambda_3,\lambda_2,\lambda_1,\lambda_4}(-s)\,. \tag{4.16}$$

Exploiting crossing symmetries we eventually obtain

$$A(-s) = A(s) \qquad C(s) = B(-s)\,. \tag{4.17}$$

It follows that we have only two independent amplitudes

$$M_{xx}(s) \equiv M_{xxxx}(s)\,, \tag{4.18}$$
$$M_{xy}(s) \equiv M_{xyxy}(s)\,, \tag{4.19}$$

and it holds

$$M_{\lambda_1,\lambda_2}(s) = M_{\lambda_1,\lambda_2}(-s)\,. \tag{4.20}$$

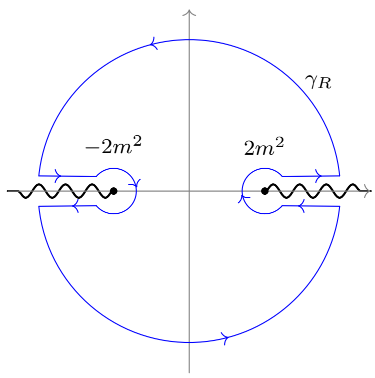
Now we use the analicity of the S-matrix to write amplitudes derivatives trough the Cauchy relation

$$\frac{d^n f}{dz^n} = \frac{n!}{2\pi i}\oint d\zeta \frac{f(\zeta)}{(\zeta - z)^{n-1}}\,, \tag{4.21}$$

where f is a generic analytic function and $\zeta, z \in \mathbb{C}$. In particular, $M_{\lambda_1,\lambda_2}(s)$ is analytical in $s \in \mathbb{C}$ with the two branch cuts (see [40])

$$\Gamma_1 \;=\; \left\{z \in \mathbb{C}\,|\,\mathfrak{Re}(z) \geq 2m^2\,,\ \mathfrak{Im}(z) = 0\right\}\,, \tag{4.22}$$
$$\Gamma_2 \;=\; \left\{z \in \mathbb{C}\,|\,\mathfrak{Re}(z) \leq 2m^2\,,\ \mathfrak{Im}(z) = 0\right\}\,. \tag{4.23}$$

Integrating along the curve γ_R given by



we obtain

$$\frac{d^2 M_{\lambda_1,\lambda_2}}{ds^2}(0) = \frac{1}{\pi i}\oint_{\gamma_R} d\zeta \frac{M_{\lambda_1,\lambda_2}(\zeta)}{\zeta^3}\,. \tag{4.24}$$

Taking the limit $R \to \infty$ the RHS of equation (4.24) reduces to

$$\frac{1}{\pi i}\left\{\int_{2m^2}^{+\infty}\frac{d\sigma}{(\sigma+i\epsilon)^3}M_{\lambda_1,\lambda_2}(\sigma+i\epsilon)-\int_{2m^2}^{+\infty}\frac{d\sigma}{(\sigma-i\epsilon)^3}M_{\lambda_1,\lambda_2}(\sigma-i\epsilon)\right.$$
$$\left.+\int_{-\infty}^{-2m^2}\frac{d\sigma}{(\sigma+i\epsilon)^3}M_{\lambda_1,\lambda_2}(\sigma+i\epsilon)-\int_{-\infty}^{-2m^2}\frac{d\sigma}{(\sigma-i\epsilon)^3}M_{\lambda_1,\lambda_2}(\sigma-i\epsilon)\right\}. \tag{4.25}$$

Exploiting Schwarz reflection principle $M(s^*) = M(s)^*$ and the identity

$$M(\sigma) = \lim_{\epsilon\to 0} M(\sigma + i\epsilon)\,, \tag{4.26}$$

the RHS further simplifies

$$RHS\;:\;\frac{2}{\pi}\left\{\int_{2m^2}^{+\infty}\frac{d\sigma}{\sigma^3}\mathfrak{Im}\left[M_{\lambda_1,\lambda_2}(\sigma)+M_{\lambda_1,\lambda_2}(-\sigma)\right]\right\}. \tag{4.27}$$

Combing equations (4.20) and (4.27), equation (4.24) yieldes the dispersion relation

$$\frac{d^2M_{\lambda_1,\lambda_2}}{ds^2}(0) = \frac{4}{\pi}\int_{2m^2}^{\infty}\frac{d\sigma}{\sigma}\mathfrak{Im}\left[M_{\lambda_1,\lambda_2}(\sigma)\right]\,. \tag{4.28}$$

We exploit now S-matrix unitarity $SS^\dagger = \mathbb{I}$ to constrain the RHS of equation (4.28). Introducing the transfer matrix T given by the expansion $S = \mathbb{I} + iT$ we get

$$\mathbb{I} = SS^\dagger \iff i(T - T^\dagger) + TT^\dagger = 0\,. \tag{4.29}$$

Evaluating (4.29) with elastic the scattering process we obtain then

$$2\mathfrak{Im}\left[M_{\lambda_1,\lambda_2}\right] = \langle\lambda,\lambda_2|\,(TT^\dagger)\,|\lambda,\lambda_2\rangle\,. \tag{4.30}$$

Introducing the identity obtained summing over the projectors on the Hilbert space we get

$$2\mathfrak{Im}\left[M_{\lambda_1,\lambda_2}\right] = \langle\lambda,\lambda_2|\,(TT^\dagger)\,|\lambda,\lambda_2\rangle = \sum_\phi\int d\Pi_\phi\,|M|^2_{\lambda_1\lambda_2\to\phi}\,, \tag{4.31}$$

where ϕ labels a generic state of the Hilbert space. Equation (4.2.2) is nothing but the optical theorem and combined with the dispersion relation (4.28) provides the positivity bounds

$$\frac{d^2M_{\lambda_1,\lambda_2}}{ds^2}(0) > 0\,, \tag{4.32}$$

The amplitudes for the theory (4.5) take the form

$$M_{xxxx}(s) \;=\; A(s) + B(s) + B(-s) \;\propto\; a_1 s^2\,, \tag{4.33}$$
$$M_{xyxy}(s) \;=\; A(s) \;\propto\; a_2 s^2\,, \tag{4.34}$$

and we finally get the constraints

$$a_1 > 0\,,\qquad a_2 > 0\,. \tag{4.35}$$

4.2.3 Positivity Bounds With Gravity

The production of positivity bounds through S-matrix properties in theories of gravity is not completely understood. However, exploiting sophisticated tricks it has been possible to produce some results. An example are the bounds we used to constrain 4 derivative Einstein–Maxwell theory obtained by Bellazzini et al. in [7]. We proceed now reviewing the problems which arises with gravity and the technique used by Bellazzini et al.

Let's start looking at the consequences of turning on gravity. The S-matrix properties we assumed to obtain positivity bounds are consistent with a general theory of gravity and the theorems still hold. However, the algorithm we described previously becomes useless. Implementing gravitons contribution in forward elastic scattering the Cauchy formula turns out to relate divergent quantities. The amplitudes are now dominated by the Coulomb universal singularity

$$\mathcal{M}(s,t\to 0) = -\frac{s^2}{M_P^2 t} + O(s^2)\,, \tag{4.36}$$

which is caused by the soft emission of on-shell massless gravitons. Thus, in order to produce any meaningful results it is necessary to remove the divergence.

The fundamental observation of [7] is that the divergence is due to the infinity flat-space volume. It can be therefore regularized considering a cylindrical space-time[4], i.e. integrating a spatial component on a circle. Applying the dimensional reduction we obtain indeed a theory with a non dynamical 3-dimensional graviton which does not produce the Coulomb singularity. However, the DOFs of the 4-dimensional graviton do not disappear. The dimensional reduction produces a massless dilaton, a massless graviphoton and an infinite tower of Kaluza Klein (KK) modes. Such new fields contributes to the scattering amplitude. In particular, the KK modes introduce a logarithmic divergence and zero modes dominate over the finite IR contributions. Therefore, getting rid of the Coulomb singularity is not enough. In order to produce a positivity bound it is necessary to subtract zero an KK modes contributions. Equation (4.28) is therefore replaced by

$$\frac{d^2 M_{\lambda_1,\lambda_2}}{ds^2}(0) - \frac{d^2 M_{\lambda_1,\lambda_2}}{ds^2}(0)\bigg|_{\text{KK,IR}} = \frac{4}{\pi}\int_{2m^2}^{\infty}\frac{d\sigma}{\sigma}\mathfrak{Im}\left[\tilde{M}_{\lambda_1,\lambda_2}(\sigma)\right]\,. \tag{4.37}$$

Thanks to the optical theorem the LHS is positive. It is possible therefore to use the subtracted amplitude to produce meaningful positivity bounds.

4.3 Positivity Bounds and Field Redefintions

We conclude this section explaining the reasons why we selected the Lagrangian (2.19) with the Weyl tensor instead of the easier-to-handle Lagrangian (2.16).

A fundamental ingredient of the analysis presented in chapter 3 is the set of bounds

$$\begin{cases} 2\alpha_1 - \alpha_3 > 0\,, \\ 2\alpha_1 + \alpha_3 > 0\,, \\ \alpha_2 > 0\,. \end{cases} \tag{4.38}$$

[4]Notice that we can regularize considering just spacetime with finite volume. However in such case we spoil the theory of Lorentz invariance.

Such bounds have been derived by Bellazzini et al. in [7] applying the algorithm we described in the previous section to (2.19). They are therefore implied by the S-matrix structure of (2.19). Thanks to S-matrix equivalence theorem, we know that theories identified by a fields redefinition which does not mixes the DOFs are equivalent. However, we have no guarantees that a general fields redefinition leaves the dynamics invariant. It is evident then that we can not discuss the sings of charge-to-mass ratio and entropy corrections of (2.16) exploiting (4.38) before having at least explicitly verified the corrections invariance. A first reason to consider (2.19) is therefore that a different theory would have needed a more complicated discussion.

Moreover, notice that we do not have a rigorous proof of dynamics invariance. The verification of corrections invariance is indeed only a consistency check. Therefore, the theory (2.19) is the only case where is proved that the dynamics discussion is correct. A second reason is given then by the fact that a different theory would have weakened our analysis.

CHAPTER 5

Duality and the WGC

In the previous chapter we showed that a local and Lorentz-invariant Lagrangian does not necessarily produce a reliable theory. We have indeed to impose causality in order to remove superluminar signals which can break the Lorentz-invariance of time-ordering and the local Hamiltonian description. Moreover, we described a general algorithm which allows to relate the causality and unitarity conditions to a set of bounds on the theory coefficients.

Among the bounds produced for the 4 derivative Einstein–Maxwell theory (2.19) there is one which coincides with the request that the correction to the charge-to-mass ratio of the extremal BHs has a positive sign. It follows that for such class of theories the assumptions of unitarity, locality and positivity of the scattering amplitudes implies the mild form of the EWGC. An interesting question is then whether there are other consistency conditions. If there were further constraints on the theory coefficients it would be possible that EWGC and S-matrix properties provide the same bounds.

In recent works (see [19] and [20]) it has been suggested that a non trivial bound can be obtained by considering duality transformations. Assuming that duality is a symmetry of the UV theory it must hold at all orders in the perturbative expansion and it can be completely characterized through leading order terms invariance. It is possible then to constrain the generic higher order corrections identified with a bottom-up approach imposing that they do not break the duality group.

In this chapter we start representing the action of duality transformations and we derive the duality groups of Maxwell and Einstein–Maxwell theories. We move then to 4 derivative Einstein–Maxwell theory and we study which are the constraints necessary to guarantee the stress-energy tensor invariance. We discuss then our results and the consequences for the EWGC interpretation. After that, we generalize the duality constraints computations exploiting Legendre transformations. Such technique extends that of [23] and constitute the second relevant result of this thesis work.

5.1 Electric–Magnetic Duality

In this section we start defining the Electric–Magnetic (EM) duality transformations following [22]. We derive then the duality group of the free Maxwell theory introducing the notation we will use to present the general techniques based on Legendre transformations. Finally, we compute the duality group of Einstein–Maxwell theory.

5.1.1 Duality Group

Let's consider a Lagrangian of the type

$$d^4x\sqrt{|g|}\,\mathcal{L} = L[A_\mu, g_{\mu\nu}]\,, \tag{5.1}$$

such that $\mathcal{L}$ does not depend on $F^{\mu\nu}$ derivatives. Varying the action with respect to A_μ we get

$$0 = \delta S = \int \delta L = \int d^4x\sqrt{|g|}\delta\mathcal{L} = \int d^4x\sqrt{|g|}\delta F_{\mu\nu}\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}}\,, \tag{5.2}$$

and equations of motion in differential form are given by

$$D_\mu\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} \equiv \frac{1}{2}D_\mu\tilde{G}^{\mu\nu} = 0 \qquad\Longleftrightarrow\qquad dG = 0\,. \tag{5.3}$$

We recall now that $F_{\mu\nu}$ is the field strength of the dynamical field A_μ, thus it satisfy the Bianchi identity

$$D_\mu\tilde{F}^{\mu\nu} = 0 \qquad\Longleftrightarrow\qquad dF = 0\,, \tag{5.4}$$

where we introduced $\tilde{F}_{\mu\nu} = \frac{\sqrt{|g|}}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. It follows that the Bianchi identity and the EOMs have the same structure and can be rotated through a global $\mathrm{GL}(2,\mathbb{R})$ transformation. We have indeed

$$d\mathcal{G} = d\begin{pmatrix}F_{\mu\nu}\\ G_{\mu\nu}\end{pmatrix} = 0 \qquad\Longleftrightarrow\qquad d\mathcal{G}' = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}dF_{\mu\nu}\\ dG_{\mu\nu}\end{pmatrix} = 0\,, \tag{5.5}$$

which induces the transformation

$$\begin{pmatrix}F'\\ G'\end{pmatrix} = \begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}F\\ G\end{pmatrix} \tag{5.6}$$

.

The duality group is then defined as the biggest subgroup of $\mathrm{GL}(2,\mathbb{R})$ which preserves the theory structure. In order to do so we have to impose that F and G definitions are preserved

$$\tilde{G}^{\mu\nu} = 2\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} \qquad\Longleftrightarrow\qquad \tilde{G}'^{\mu\nu} = 2\frac{\partial\mathcal{L}'}{\partial F'_{\mu\nu}}\,. \tag{5.7}$$

Notice that we are requiring that the duality transformations are symmetries of the equations of motions and not of the Lagrangian.[1]

[1] We will see later that in general the Lagrangian is modified.

5.1.2 Free Maxwell Theory

We determine now the duality group of the free Maxwell theory. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^2 \,, \tag{5.8}$$

therefore equation (5.7) takes the form

$$\tilde{G}_{\mu\nu} = -F_{\mu\nu} \,, \tag{5.9}$$

It follows that F and G satisfy the equations

$$\tilde{G} \equiv \star G = -F \qquad \tilde{F} \equiv \star F = G \,. \tag{5.10}$$

which can be written in the more compact form

$$\tilde{\mathcal{G}}^M_{\mu\nu} = \mathbb{C}^M{}_N \mathcal{G}^M_{\mu\nu} \,, \tag{5.11}$$

$$\mathcal{G}^M_{\mu\nu} = \begin{pmatrix} F_{\mu\nu} \\ G_{\mu\nu} \end{pmatrix} , \qquad \tilde{\mathcal{G}}^M_{\mu\nu} = \begin{pmatrix} \tilde{F}_{\mu\nu} \\ \tilde{G}_{\mu\nu} \end{pmatrix} , \qquad \mathbb{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{5.12}$$

We notice then that equation (5.7) invariance is equivalent to equation (5.11) invariance. We proceed therefore imposing the latter. After a rotation $M \in \mathrm{GL}(2, \mathbb{R})$ we get

$$\mathcal{G}'_{\mu\nu} = M\mathcal{G}_{\mu\nu} \,, \qquad \tilde{\mathcal{G}}'_{\mu\nu} = M\tilde{\mathcal{G}}_{\mu\nu} \,, \tag{5.13}$$

which implies

$$M\tilde{\mathcal{G}}_{\mu\nu} = \mathbb{C}M\mathcal{G}_{\mu\nu} \,, \tag{5.14}$$

and comparing with equation (5.11) we obtain the constraint

$$\mathbb{C} = M^{-1}\mathbb{C}M \qquad \Longleftrightarrow \qquad [M, \mathbb{C}] = 0 \,, \tag{5.15}$$

which is solved by

$$A = D \,, \;\; B = -C \,. \tag{5.16}$$

We have therefore that the generic element M of the duality group takes the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \quad A, B \in \mathbb{R} \,. \tag{5.17}$$

Thus, the duality group of the free theory is $\mathrm{GL}(1, \mathbb{C}) \equiv \mathbb{C}$. This is explicit in the complex basis $F^{\pm} = \frac{1}{\sqrt{2}}(F \pm iG)$

$$\mathcal{G}^{\bar{M}}_c = \begin{pmatrix} F^+ \\ F^- \end{pmatrix} = \mathbb{J}^{\bar{M}}{}_N \, \mathcal{G}^N \,, \qquad \mathbb{J} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} , \tag{5.18}$$

$$(M_c)^{\bar{M}}{}_{\bar{N}} = \mathbb{J}^{\bar{M}}{}_N M^N{}_M \mathbb{J}^{\dagger M}{}_{\bar{N}} = \begin{pmatrix} A - iB & 0 \\ 0 & A + iB \end{pmatrix} = \begin{pmatrix} N & 0 \\ 0 & N^* \end{pmatrix} , \qquad N \in \mathbb{C} \,. \tag{5.19}$$

5.1.3 Einstein–Maxwell Theory

Let us turn on gravity. The Lagrangian is now

$$\mathcal{L} = \frac{M_P^2}{2} R - \frac{1}{4} F^2 \,. \tag{5.20}$$

From the previous section we know that Maxwell EOMs are left invariant by GL$(1, \mathbb{C})$. Assuming that the gravitational field is invariant under duality transformations, Einstein equations are invariant if and only if the stress energy tensor is invariant, i.e.

$$T'_{\mu\nu}[F'_{\mu\nu}] = T_{\mu\nu}[F_{\mu\nu}] \,. \tag{5.21}$$

Therefore, the new duality group is the biggest subset of GL$(1, \mathbb{C})$ which satisfy (5.21).

The stress energy tensor is given by

$$T_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \,. \tag{5.22}$$

Therefore, applying a rotation (5.17) and exploiting (5.11)

$$F_{\alpha\beta} = A F_{\alpha\beta} + B G\alpha\beta = A F_{\alpha\beta} + B \tilde{F}\alpha\beta \,, \tag{5.23}$$

$$\begin{aligned} T'_{\mu\nu} = & \left[A^2 F_{\mu\rho} F_\nu{}^\rho + B^2 \tilde{F}_{\mu\rho} \tilde{F}_\nu{}^\rho + 2AB \left(F_{(\mu|\rho} \tilde{F}_{|\nu)}{}^\rho \right) \right] \\ & - \frac{1}{4} g_{\mu\nu} \left[A^2 F^2 + B^2 \tilde{F}^2 + 2AB F \tilde{F} \right] \end{aligned} \,. \tag{5.24}$$

Recalling that

$$\tilde{F}^2 = - F^2 \,, \tag{5.25}$$

$$\tilde{F}_{\mu\rho} \tilde{F}_\nu{}^\rho = F_{\mu\rho} F_\nu{}^\rho - \frac{1}{2} g_{\mu\nu} F^2 \,, \tag{5.26}$$

equation (5.24) reads

$$T'_{\mu\nu} = \left(A^2 + B^2 \right) T_{\mu\nu} + 2AB Q_{\mu\nu} \,, \tag{5.27}$$

$$Q_{\mu\nu} = \left(F_{(\mu|\rho} \tilde{F}_{|\nu)}{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} \tilde{F}^{\alpha\beta} \right) \,. \tag{5.28}$$

However, it is easy to show that $Q_{\mu\nu} \equiv 0$. In 4 dimensions it is not possible to completely anti-symmetrize 5 spacetime indexes. It holds indeed

$$0 = 5\, \delta_\mu^{[\gamma} \epsilon^{\alpha\beta\rho\sigma]} = \delta_\mu^\gamma \epsilon^{\alpha\beta\rho\sigma} - \delta_\mu^\alpha \epsilon^{\gamma\beta\rho\sigma} - \delta_\mu^\beta \epsilon^{\alpha\gamma\rho\sigma} - \delta_\mu^\rho \epsilon^{\alpha\beta\gamma\sigma} - \delta_\mu^\sigma \epsilon^{\alpha\beta\rho\gamma} \,, \tag{5.29}$$

and we have

$$\begin{aligned} 0 &= 5 \left\{ g_{\nu\gamma}\, \delta_\mu^{[\gamma} \epsilon^{\alpha\beta\rho\sigma]} + g_{\mu\gamma}\, \delta_\nu^{[\gamma} \epsilon^{\alpha\beta\rho\sigma]} \right\} F_{\alpha\beta} F_{\rho\sigma} \\ &= -16 \left\{ \frac{1}{2} F_{(\mu|\rho} \tilde{F}_{|\nu)}{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} \tilde{F}^{\alpha\beta} \right\} \,, \end{aligned} \tag{5.30}$$

which implies that $Q_{\mu\nu}$ vanishes as we claimed. It follows that $T_{\mu\nu}$ is invariant if and only if

$$A^2 + B^2 = 1 \,. \tag{5.31}$$

The duality group reduces therefore to $\mathrm{SO}(2,\mathbb{R})$ and the generic element M takes the form

$$M = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} , \quad \alpha \in [0, 2\pi) , \tag{5.32}$$

which reads in the complex basis (5.18)

$$M_c = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} , \quad \alpha \in [0, 2\pi) . \tag{5.33}$$

5.2 Duality Constraints

If the EM duality is an exact symmetry of the UV theory it must hold at every perturbative order. It is possible therefore to use the U(1) duality group of Eintein–Maxwell theory to constrain higher order coefficients.

In this section we start generalizing the computations of section 5.1.3 deriving a simple sufficient condition for stress energy tensor invariance. We specialize then the formulas in the case of theory 2.19. We conclude discussing the implications of the duality constraints.

5.2.1 Stress Energy Tensor Invariance

We want to determine the stress energy tensor transformation under infinitesimal U(1) duality rotations. We recall that in a theory coupled to gravity the stress energy tensor can be defined through the metric variation

$$T_{\mu\nu}\left[\mathcal{L}\right] = -\frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\mu\nu}} \left(\sqrt{|g|}\mathcal{L}\right) , \tag{5.34}$$

where $\mathcal{L}$ is intended without the Hilbert–Einstein Lagrangian density. Exploiting the invariance of the metric under duality rotations, the linearity of (5.34) and considering a general extension of Einstein–Maxwell theory we get

$$[\, \delta_F \, , \, \delta_g \,] = 0 \, , \tag{5.35}$$

$$\delta_F T_{\mu\nu} \, [\, \mathcal{L} \,] = T_{\mu\nu}[\, \delta_F \mathcal{L} \,] = T_{\mu\nu} \, [\, \delta_F \mathcal{L}^{(2)} \,] + T_{\mu\nu} \, [\, \delta_F \Delta\mathcal{L} \,] \, . \tag{5.36}$$

The first term the RHS of (5.36) is nothing but the variation of the free Maxwell theory stress energy tensor and we have already verified that it vanishes if and only if we choose U(1) as duality group (see section 5.1.3).

We introduce now the dual field $G_{\mu\nu}$

$$\tilde{G}_{\mu\nu} = -F_{\mu\nu} + \Delta F_{\mu\nu} \, , \qquad G_{\mu\nu} = \tilde{F}_{\mu\nu} - \Delta\tilde{F}_{\mu\nu} \, , \tag{5.37}$$

where $\Delta F_{\mu\nu}$ and $\Delta\tilde{F}_{\mu\nu}$ are higher order corrections. Choosing the parametrization of the infinitesimal transformation Λ

$$\Lambda = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = -\theta\, \mathbb{C} \, , \qquad \Lambda \in \mathfrak{u}(1) \, , \tag{5.38}$$

$\delta_F \Delta\mathcal{L}$ takes the form

$$\delta_F \Delta\mathcal{L} = \delta F \, \frac{\partial \Delta\mathcal{L}}{\partial F} = -\frac{\theta}{2} G \Delta F = -\frac{\theta}{2} \left(\tilde{F} - \Delta\tilde{F}\right) \Delta F = -\frac{\theta}{2} \tilde{F} \Delta F + O(\alpha^2) \, . \tag{5.39}$$

Exploiting equation (5.39) equation (5.36) reads

$$\delta_F T_{\mu\nu} = -\frac{\theta}{2} g_{\mu\nu} \tilde{F} \Delta F + \theta \delta_g \left[\tilde{F} \Delta F \right] \,, \tag{5.40}$$

and recalling that

$$\delta_g \tilde{F}^{\alpha\beta} = \delta_g \left(\frac{1}{2} \hat{\epsilon}^{\alpha\beta\rho\sigma} F_{\rho\sigma} \right) = \frac{1}{4}\, g_{\mu\nu}\, \hat{\epsilon}^{\alpha\beta\rho\sigma} F_{\rho\sigma} = \frac{1}{2}\, g_{\mu\nu}\, \tilde{F}^{\alpha\beta} \,, \tag{5.41}$$

it further simplify

$$\begin{aligned} \delta_F T_{\mu\nu} &= -\frac{\theta}{2} g_{\mu\nu} \tilde{F} \Delta F + \theta \left[\left(\delta_g \tilde{F}^{\alpha\beta} \right) \Delta F_{\alpha\beta} + \tilde{F}^{\alpha\beta} \delta_g \left(\Delta F_{\alpha\beta} \right) \right] \\ &= \theta\, \tilde{F}^{\alpha\beta}\, \delta_g \left(\Delta F_{\alpha\beta} \right) \,. \end{aligned} \tag{5.42}$$

It follows that the stress energy tensor is invariant under U(1) duality rotations if and only if

$$\tilde{F}^{\alpha\beta}\, \delta_g \left(\Delta F_{\alpha\beta} \right) = 0 \,. \tag{5.43}$$

Notice that massaging (5.43) we can produce a sufficient condition which is easier to verify

$$\begin{aligned} \tilde{F}^{\alpha\beta} \delta_g \Delta F_{\alpha\beta} &= \delta_g \left(\tilde{F}^{\alpha\beta} \Delta F_{\alpha\beta} \right) - \left(\delta_g \tilde{F}^{\alpha\beta} \right) \Delta F_{\alpha\beta} \\ &= \delta_g \left(\tilde{F}^{\alpha\beta} \Delta F_{\alpha\beta} \right) - \frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu} \left(\tilde{F}^{\alpha\beta} \Delta F_{\alpha\beta} \right) \\ &= \left(\delta_g - \frac{1}{2}\, g_{\mu\nu} \delta g^{\mu\nu} \right) \left(\tilde{F}^{\alpha\beta} \Delta F_{\alpha\beta} \right) \,. \end{aligned} \tag{5.44}$$

We finally get

$$\tilde{F}^{\alpha\beta} \Delta F_{\alpha\beta} = 0 \quad \Rightarrow \quad \tilde{F}^{\alpha\beta}\, \delta_g \left(\Delta F_{\alpha\beta} \right) \quad \Longleftrightarrow \quad \delta_F T_{\mu\nu} = 0 \,. \tag{5.45}$$

5.2.2 4 Derivative Einstein–Maxwell Theory

Let us apply the constraints (5.45) to (2.19). In our case it holds

$$\Delta F_{\alpha\beta} = 2\, \alpha_1\, F^2 F_{\alpha\beta} + 2\, \alpha_2\, F \tilde{F} \tilde{F}_{\alpha\beta} + 2\, \alpha_3\, W_{\alpha\beta\rho\sigma}\, F^{\rho\sigma} \,. \tag{5.46}$$

We start with condition (5.43). Neglecting the α_3 term we get

$$\begin{aligned} \tilde{F}^{\alpha\beta}\, \delta_g\, \Delta F^{1,2}_{\alpha\beta} &= 2\, \alpha_1 \left[2 F_{\mu\rho} F_{\nu}{}^{\rho} F_{\alpha\beta} \right] \tilde{F}^{\alpha\beta} \\ &\quad + 2\, \alpha_2 \left[g_{\mu\nu}\, F \tilde{F} \tilde{F}_{\alpha\beta} - 2 F \tilde{F} \tilde{F}_{\alpha(\mu}\, g_{\nu)\beta} \right] \tilde{F}^{\alpha\beta} \\ &= 4\, \alpha_1 \left[F_{\mu\rho} F_{\nu}{}^{\rho} \right] \\ &\quad + 2\, \alpha_2 \left[-2 \tilde{F}_{\mu\rho} \tilde{F}_{\nu}{}^{\rho} F \tilde{F} - g_{\mu\nu} F \tilde{F} F^2 \right] \,, \end{aligned} \tag{5.47}$$

Recalling equation (5.26)

$$\tilde{F}_{\alpha\rho} \tilde{F}_{\beta}{}^{\rho} = F_{\alpha\rho} F_{\beta}{}^{\rho} - \frac{1}{2}\, g_{\alpha\beta}\, F^2 \,, \tag{5.48}$$

we obtain

$$\delta_F T_{\mu\nu} = \theta \left\{ 4\,(\alpha_1 - \alpha_2) \left[F_{\mu\rho} F_\nu{}^\rho F \tilde{F} \right] + 2\,\alpha_3\, \tilde{F}^{\alpha\beta} \delta_g \left[W_{\alpha\beta\rho\sigma} F^{\rho\sigma} \right] \right\}, \tag{5.49}$$

which vanishes if and only if we set

$$\alpha_1 = \alpha_2\,, \qquad \alpha_3 = 0\,. \tag{5.50}$$

We apply now the sufficient condition (5.45). We get then

$$\tilde{F}\Delta F = 2\,\left(\alpha_1 - \alpha_2\right)\, F^2 F\tilde{F} + 2\,\alpha_3\, W_{\mu\nu\rho\sigma}\, \tilde{F}^{\mu\nu} F^{\rho\sigma} = 0\,, \tag{5.51}$$

which provides

$$\alpha_1 = \alpha_2\,, \qquad \alpha_3 = 0\,. \tag{5.52}$$

5.2.3 Weak Gravity Conjecture

We discuss now the effects of duality constraints on the interpretation of the mild EWGC. Without duality constraints the S-matrix properties requires by consistency

$$\begin{cases} 2\alpha_1 - \alpha_3 > 0\,, \\ 2\alpha_1 + \alpha_3 > 0\,, \\ \alpha_2 > 0\,. \end{cases} \tag{5.53}$$

and the charge-to-mass ratio of extremal BHs takes the form

$$z_{\text{ext}} = 1 + \frac{64\pi^2}{5M^2}(2\alpha_1 - \alpha_3)\,. \tag{5.54}$$

It follows that the microcausality imposed through the analicity of the S-matrix and exploiting unitarity, locality and positivity of the scattering amplitudes implies the mild EWGC. It is therefore a condition more general and stronger than the mild EWGC. However, assuming the duality constraints

$$\alpha_1 = \alpha_2\,, \qquad \alpha_3 = 0\,, \tag{5.55}$$

the S-matrix bounds reduce to

$$\alpha_1 > 0\,, \tag{5.56}$$

and the charge-to-mass ratio yields

$$z_{\text{ext}} = 1 + \frac{64\pi^2}{5M^2}(2\alpha_1)\,. \tag{5.57}$$

Now the two conditions coincides.

5.3 Legendre Duality

In section 5.1.2 we learned that to determine the duality group we have to impose together $\tilde{F}$ and $\tilde{G}$ invariance. However, the method used is highly dependent on the structure of the Free Maxwell theory, which is quadratic in $F_{\mu\nu}$. In this section we present a new general technique which relies on Legendre transformations and solve such a problem. Moreover, it can be used used to constrain higher order correction coefficients. The approach we developed can not be considered completely original. Indeed Legendre transformation have been already used to discuss EM duality. However, we managed to systematize and extend the ideas of [22] and [23].

5.3.1 Legendre Formalism

The key idea of our approach is to identify $\tilde{G}$ with the Legendre dual of F and use it to define the dual Lagrangian. $\tilde{G}$ is defined by

$$\tilde{G}_{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial F^{\mu\nu}} \qquad \Longleftrightarrow \qquad G_{\mu\nu} = -\epsilon_{\mu\nu\rho\sigma}\frac{\partial}{\partial F_{\rho\sigma}}\left(\sqrt{-g}\mathcal{L}\right) , \tag{5.58}$$

where $\mathcal{L} = \mathcal{L}[F_{\mu\nu}]$. The dual Lagrangian and the dual fieldstrength obtained applying a Legendre transformation are instead

$$\mathcal{L}_D[F_D^{\mu\nu}] = \mathcal{L}[F^{\mu\nu}] - F_{\mu\nu}F_D^{\mu\nu} , \tag{5.59}$$

$$F_D^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} , \qquad F_{\mu\nu} = -\frac{\partial \mathcal{L}_D}{\partial F_D^{\mu\nu}} . \tag{5.60}$$

It follows

$$\mathcal{L}_D[\tilde{G}^{\mu\nu}] = \mathcal{L}[F^{\mu\nu}] - \frac{1}{2}F_{\mu\nu}\tilde{G}^{\mu\nu} , \tag{5.61}$$

$$\tilde{G}_{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial F^{\mu\nu}} , \qquad F_{\mu\nu} = -2\frac{\partial \mathcal{L}_D}{\partial \tilde{G}^{\mu\nu}} . \tag{5.62}$$

Now, we would like to write a generalized version of equation (5.63). We need then to express $\tilde{F}$ in terms of F and G. To determine $\tilde{F}$ we can use the second equation of (5.62)

$$\tilde{F}_{\mu\nu} = \frac{\sqrt{|g|}}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = -\hat{\epsilon}_{\mu\nu\rho\sigma}\frac{\partial \mathcal{L}_D}{\partial \tilde{G}_{\rho\sigma}} , \tag{5.63}$$

where we introduced $\hat{\epsilon}_{\mu\nu\rho\sigma} = \sqrt{|g|}\,\epsilon_{\mu\nu\rho\sigma}$. Recalling that

$$G_{\alpha\beta} = \frac{s}{2}\hat{\epsilon}_{\alpha\beta\mu\nu}g^{\mu\rho}g^{\nu\sigma}\tilde{G}_{\rho\sigma} , \qquad s = \frac{g}{|g|} , \tag{5.64}$$

$$\frac{\partial G_{\alpha\beta}}{\partial \tilde{G}_{\gamma\delta}} = \frac{s}{2}\hat{\epsilon}_{\alpha\beta\mu\nu}g^{\mu\rho}g^{\nu\sigma}\delta^{\gamma\delta}_{[\rho\sigma]} = \frac{s}{2}\hat{\epsilon}_{\alpha\beta\mu\nu}g^{\mu\gamma}g^{\nu\delta} , \tag{5.65}$$

equation (5.63) reads

$$\tilde{F}_{\mu\nu} = -\hat{\epsilon}_{\mu\nu\rho\sigma}\frac{\partial G_{\alpha\beta}}{\partial \tilde{G}_{\mu\nu}}\frac{\partial \mathcal{L}_D}{\partial G_{\alpha\beta}} = -2\frac{\partial \mathcal{L}_D}{\partial G_{\mu\nu}} . \tag{5.66}$$

We have therefore

$$\begin{aligned} \delta\tilde{F}_{\mu\nu} &= -2\delta\left(\frac{\partial \mathcal{L}_D\left[G_{\alpha\beta}\right]}{\partial G^{\mu\nu}}\right) = -2\left(\frac{\partial^2 \mathcal{L}_D\left[G_{\alpha\beta}\right]}{\partial G^{\mu\nu}\partial G_{\gamma\delta}}\right)\delta G_{\gamma\delta} , && (5.67)\\ \delta\tilde{G}_{\mu\nu} &= 2\delta\left(\frac{\partial \mathcal{L}\left[F_{\alpha\beta}\right]}{\partial F^{\mu\nu}}\right) = 2\left(\frac{\partial^2 \mathcal{L}\left[F_{\alpha\beta}\right]}{\partial F^{\mu\nu}\partial F_{\gamma\delta}}\right)\delta F_{\gamma\delta} , && (5.68) \end{aligned}$$

which can be written in the compact form

$$\delta\tilde{\mathcal{G}}^M_{\mu\nu} = \left(\hat{\mathbb{C}}_{\mu\nu}{}^{\alpha\beta}\right)^M{}_N\,\delta\mathcal{G}^N_{\alpha\beta} , \tag{5.69}$$

$$\hat{\mathbb{C}} = \begin{pmatrix} 0 & -Y \\ X & 0 \end{pmatrix}, \qquad X_{\mu\nu}{}^{\alpha\beta} = 2\frac{\partial^2 \mathcal{L}}{\partial F^{\mu\nu}\partial F_{\alpha\beta}}, \qquad Y_{\mu\nu}{}^{\alpha\beta} = 2\frac{\partial^2 \mathcal{L}_D}{\partial G^{\mu\nu}\partial G_{\alpha\beta}}. \tag{5.70}$$

We proceed now imposing that the structure of (5.69) is preserved by duality transformations. Let us consider the infinitesimal rotation:

$$(\mathcal{G}'_{\mu\nu})^M = (M)^M{}_N (\mathcal{G}_{\mu\nu})^N, \qquad M = \mathbb{I} + \Lambda, \tag{5.71}$$

$$\delta\mathcal{G}^M_{\mu\nu} = \Lambda^M{}_N \mathcal{G}^N_{\mu\nu}, \qquad \delta\tilde{\mathcal{G}}^M_{\mu\nu} = \Lambda^M{}_N \tilde{\mathcal{G}}^N_{\mu\nu}. \tag{5.72}$$

Replacing in equation (5.69) we get

$$\tilde{\mathcal{G}} = \Lambda_1^{-1}\hat{\mathbb{C}}\Lambda_1\mathcal{G}. \tag{5.73}$$

Acting on equation (5.73) with another independent infinitesimal transformation and exploiting (5.69) we obtain

$$\delta_2\tilde{\mathcal{G}} = \delta_2\left[\Lambda_1^{-1}\hat{\mathbb{C}}\Lambda_1\mathcal{G}\right] = \hat{\mathbb{C}}\,\delta_2\mathcal{G}. \tag{5.74}$$

Massaging equation (5.74)

$$\Lambda_1\left[(\delta_2\Lambda_1^{-1})\,\hat{\mathbb{C}}\,\Lambda_1 + \Lambda_1^{-1}\,(\delta_2\hat{\mathbb{C}})\,\Lambda_1 + \Lambda_1^{-1}\,\hat{\mathbb{C}}\,(\delta_2\Lambda_1)\right]\mathcal{G} + \hat{\mathbb{C}}\,\Lambda_1\,\delta_2\mathcal{G} = \Lambda_1\,\hat{\mathbb{C}}\,\delta_2\mathcal{G}, \tag{5.75}$$

$$\left[-(\delta_2\Lambda_1)\Lambda_1^{-1}\hat{\mathbb{C}} + \delta_2\hat{\mathbb{C}} + \hat{\mathbb{C}}(\delta_2\Lambda_1)\Lambda_1^{-1}\right]\delta_1\mathcal{G} = \left[\Lambda_1, \hat{\mathbb{C}}\right]\delta_2\mathcal{G}, \tag{5.76}$$

$$\left(\delta_2\hat{\mathbb{C}}\right)\delta_1\mathcal{G} = \left[\Lambda_1, \hat{\mathbb{C}}\right]\delta_2\mathcal{G} + \left[(\delta_2\Lambda_1)\Lambda_1^{-1}, \hat{\mathbb{C}}\right]\delta_1\mathcal{G}, \tag{5.77}$$

where we exploited $\delta\Lambda^{-1} = -\Lambda^{-1}(\delta\Lambda)\Lambda^{-1}$ and $\mathcal{G} = \Lambda^{-1}\delta\mathcal{G}$. Choosing $\delta_1 = \delta_2$, i.e. $\Lambda_1 = \Lambda_2 = \Lambda$ equation (5.77) reduces to

$$\left\{\delta\hat{\mathbb{C}} - \left[\Lambda + (\delta\Lambda)\Lambda^{-1}, \hat{\mathbb{C}}\right]\right\}\delta\mathcal{G} = 0. \tag{5.78}$$

We notice now that in order to have $\delta\Lambda \neq 0$ the matrix entries have to depend on the field strengths, i.e. $\Lambda = \Lambda(F_{\mu\nu})$. Therefore, considering only linear field transformations[2] we have $\delta\Lambda = 0$ and equation (5.78) takes the form

$$\left\{\delta\hat{\mathbb{C}} - \left[\Lambda, \hat{\mathbb{C}}\right]\right\}\delta\mathcal{G} = 0. \tag{5.79}$$

Recalling equation (5.70) we finally get the constraints

$$\left[\Lambda, \hat{\mathbb{C}}\right] = \begin{pmatrix} Xb + Yc & -Ya + Yd \\ -Xa + Xd & -Xb - Yc \end{pmatrix}, \qquad \delta\hat{\mathbb{C}} = \begin{pmatrix} 0 & -\delta Y \\ \delta X & 0 \end{pmatrix}, \tag{5.80}$$

$$\begin{cases} (-Xb - Yc)\,\delta F \;+\; (-\delta Y + Ya - Yd)\,\delta G = 0, \\ (\delta X + Xa - Xd)\,\delta F \;+\; (Xb + Yc)\,\delta G = 0. \end{cases} \tag{5.81}$$

Notice that we implicitly assumed that Λ is an invertible matrix. It follows that constraints (5.81) do not identify necessarily the full duality group. In general, they identify the subgroup generated by the algebra elements represented by an invertible matrix. However, if there are no scalar fields the duality group of a theory with n vector fields is just $\mathrm{U}(n)$ (see [22]). Therefore, the equations (5.81) identify the full duality group and can be used to constrain the coefficients of Lagrangian higher order terms.

[2] i.e. those compatible with equation (5.5) which preserve equations of motions and the Bianchi identity.

5.3.2 Free Maxwell Theory

We verify now if constraints (5.16) and (5.81) coincide for the free Maxwell theory. Recalling equations (5.8), (5.9), (5.10) and (5.25), $F_D^{\mu\nu}$ and $\mathcal{L}_D$ read

$$F_D^{\mu\nu} = \frac{1}{2}\tilde{G}^{\mu\nu} = -\frac{1}{2}F\,, \qquad G_{\mu\nu} = \tilde{F}_{\mu\nu}\,, \tag{5.82}$$

$$\mathcal{L}_D = -\frac{1}{4}F^2 - \frac{1}{2}F\tilde{G} = -\frac{1}{4}G^2\,. \tag{5.83}$$

Therefore we have

$$X_{\mu\nu}{}^{\alpha\beta} = 2\frac{\partial^2\mathcal{L}}{\partial F^{\mu\nu}\partial F_{\gamma\delta}} = -\delta^{\alpha,\beta}_{[\mu,\nu]}\,, \qquad \delta X = 0\,, \tag{5.84}$$

$$Y_{\mu\nu}{}^{\alpha\beta} = 2\frac{\partial^2\mathcal{L}_D}{\partial G^{\mu\nu}\partial G_{\gamma\delta}} = -\delta^{\alpha,\beta}_{[\mu,\nu]}\,, \qquad \delta Y = 0\,. \tag{5.85}$$

We notice that constraints (5.81) are solved by

$$\begin{cases} Xb + Yc = 0\,, \\ \delta Y = Y(a-d)\,, \\ \delta X = -X(a-d)\,, \end{cases} \tag{5.86}$$

which in our case read

$$\begin{cases} b + c = 0\,, \\ (a-d) = 0\,, \\ (a-d) = 0\,. \end{cases} \tag{5.87}$$

Constraints (5.87) coincide[3] with (5.16).

5.3.3 $U(1)$ Duality Constraints

We showed in the previous section that the constraints (5.81) can be used to determine the duality group of a theory. However, this is not their most important use. The duality group can indeed be determined easily without them. They are fundamental instead once we fix the duality group. They indeed produces highly non trivial constraints on the theory coefficients. We proceed now evaluating them considering proper duality groups.

Imposing that the duality group is $\mathrm{GL}(1,\mathbb{C})$ (i.e. that holds (5.16)) the generic element of algebra $\Lambda \in \mathfrak{gl}(1,\mathbb{C})$ takes the form

$$\Lambda = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\,, \tag{5.88}$$

and we obtain the constraints

$$\begin{cases} -b\,(X-Y)\,\delta F \; - \; (\delta Y)\,\delta G = 0\,, \\ (\delta X)\,\delta F \; + b\,(X-Y)\,\delta G = 0\,. \end{cases} \tag{5.89}$$

[3]Notice that (5.81) are constraints on the algebra and (5.16) are constraints on the group elements. However, $G = GL(1,\mathbb{C}) = \mathbb{C}$, therefore it coincide with its algebra. The constraints define therefore the same group.

Reducing further the duality group to $\mathrm{SO}(2,\mathbb{R})$, the generic element $M \in \mathrm{SO}(2,\mathbb{R})$ takes the form in the complex basis (cfr. equation 5.33)

$$M_c = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \qquad \theta \in [0, 2\pi) \,. \tag{5.90}$$

Thus, the generic element of the algebra Λ_c reads

$$\Lambda_c = \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix} = i\theta\sigma_3 \qquad \Longleftrightarrow \qquad \Lambda = \mathbb{J}^\dagger \, \Lambda_c \, \mathbb{J} = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} = -\theta \, \mathbb{C} \,. \tag{5.91}$$

It follows that constraints (5.89) must hold with $b = -\theta$, $a = 0$. We have then

$$b(X - Y)\,\delta F \;= (-\theta)(X - Y)\,(-\theta G) = (2\theta^2)(\partial^2\mathcal{L} - \partial^2\mathcal{L}_D)\,(G)\,, \tag{5.92}$$

$$b(X - Y)\,\delta G \;= (-\theta)(X - Y)\,(\theta F) = (-2\theta^2)(\partial^2\mathcal{L} - \partial^2\mathcal{L}_D)\,(F)\,, \tag{5.93}$$

$$\delta Y \delta G \;= (\delta Y)\,(\theta F) = (2\partial^3\mathcal{L}_D)(\delta G)(\theta F) = (2\theta^2)(\partial^3\mathcal{L}_D)(FF)\,, \tag{5.94}$$

$$\delta X \delta F \;= (\delta X)\,(-\theta G) = (2\partial^3\mathcal{L})(\delta F)(-\theta G) = (2\theta^2)(\partial^3\mathcal{L})(GG)\,, \tag{5.95}$$

which provide

$$\begin{cases} (2\theta^2)\left[\partial^3\mathcal{L}_D(FF) + \partial^2\mathcal{L}(G) - \partial^2\mathcal{L}_D(G)\right] = 0\,, \\ (2\theta^2)\left[\partial^3\mathcal{L}(GG) - \partial^2\mathcal{L}(F) + \partial^2\mathcal{L}_D(F)\right] = 0\,. \end{cases} \tag{5.96}$$

Imposing that the bounds hold $\forall\, \theta$ we obtain finally

$$\frac{\partial^3\mathcal{L}_D}{\partial G^{\mu\nu}\partial G^{\alpha\beta}\partial G^{\gamma\delta}}(F^{\alpha\beta}F^{\gamma\delta}) + \frac{\partial^2\mathcal{L}}{\partial F^{\mu\nu}\partial F^{\alpha\beta}}(G^{\alpha\beta}) - \frac{\partial^2\mathcal{L}_D}{\partial G^{\mu\nu}\partial G^{\alpha\beta}}(G^{\alpha\beta}) = 0\,, \tag{5.97}$$

$$\frac{\partial^3\mathcal{L}}{\partial F^{\mu\nu}\partial F^{\alpha\beta}\partial F^{\gamma\delta}}(G^{\alpha\beta}G^{\gamma\delta}) - \frac{\partial^2\mathcal{L}}{\partial F^{\mu\nu}\partial F^{\alpha\beta}}(F^{\alpha\beta}) + \frac{\partial^2\mathcal{L}_D}{\partial G^{\mu\nu}\partial G^{\alpha\beta}}(F^{\alpha\beta}) = 0\,. \tag{5.98}$$

5.3.4 4 Derivative Einstein–Maxwell Theory

We apply now our general method to the theory (2.19). The dual field $F_D^{\mu\nu}$ is given by

$$F_D^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} = -\frac{1}{2}\left(F^{\mu\nu} - \Delta F^{\mu\nu}\right) \,. \tag{5.99}$$

Comparing with $\tilde{G}$ definition we obtain

$$\tilde{G}^{\mu\nu} = 2F_D\mu\nu = -F^{\mu\nu} + \Delta F^{\mu\nu} \,. \tag{5.100}$$

Expressing F in terms of G neglecting terms $O(\alpha^2)$ we get

$$\begin{aligned} F_{\mu\nu} &= -\tilde{G}_{\mu\nu} + \Delta F_{\mu\nu} \\ &= -\tilde{G}_{\mu\nu} + \frac{2\alpha_1}{M_P^4}F^2F_{\mu\nu} + \frac{2\alpha_2}{M_P^4}F\tilde{F}\tilde{F}_{\mu\nu} + \frac{2\alpha_3}{M_P^2}W_{\mu\nu\rho\sigma}\,F^{\rho\sigma} \\ &= -\tilde{G}_{\mu\nu} - \frac{2\alpha_1}{M_P^4}\tilde{G}^2\tilde{G}_{\mu\nu} - \frac{2\alpha_2}{M_P^4}\tilde{G}\tilde{\tilde{G}}\tilde{\tilde{G}}_{\mu\nu} - \frac{2\alpha_3}{M_P^2}W_{\mu\nu\rho\sigma}\,\tilde{G}^{\rho\sigma} + O(\alpha^2) \\ &= -\tilde{G}_{\mu\nu} + \frac{2\alpha_1}{M_P^4}G^2\tilde{G}_{\mu\nu} - \frac{2\alpha_2}{M_P^4}\tilde{G}GG_{\mu\nu} - \frac{2\alpha_3}{M_P^2}W_{\mu\nu\rho\sigma}\,\tilde{G}^{\rho\sigma} \\ &= -\tilde{G}_{\mu\nu} + \Delta\tilde{G}_{\mu\nu}\,, \end{aligned} \tag{5.101}$$

where we introduced $\Delta\tilde{G}_{\mu\nu}$ which contains $O(\alpha_i)$ contributes. The dual Lagrangian density reads therefore

$$\begin{aligned}\mathcal{L}_D &= -\frac{1}{4}F^2+\frac{\alpha_1}{4}F^4+\frac{\alpha_2}{4}(F\tilde{F})^2+\frac{\alpha_3}{2}W_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}-\frac{1}{2}F_{\mu\nu}\tilde{G}^{\mu\nu}\\ &= -\frac{1}{4}\tilde{G}^2+\frac{1}{2}\Delta\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu}+\frac{\alpha_1}{4}\tilde{G}^4+\frac{\alpha_2}{4}(\tilde{G}\tilde{\tilde{G}})^2+\frac{\alpha_3}{2}W_{\mu\nu\rho\sigma}\tilde{G}^{\mu\nu}\tilde{G}^{\rho\sigma}\\ &\quad+\frac{1}{2}\tilde{G}^2-\frac{1}{2}\Delta\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu}\\ &= -\frac{1}{4}G^2+\frac{\alpha_1}{4}G^4+\frac{\alpha_2}{4}(\tilde{G}G)^2+\frac{\alpha_3}{2}\tilde{W}_{\mu\nu\rho\sigma}G^{\mu\nu}G^{\rho\sigma}\,,\end{aligned} \tag{5.102}$$

where we defined

$$\tilde{W}_{\mu\nu\rho\sigma}=\frac{1}{4}\hat{\epsilon}_{\mu\nu\alpha\beta}W^{\alpha\beta\gamma\delta}\hat{\epsilon}_{\gamma\delta\rho\sigma}\,. \tag{5.103}$$

The quantities which are involved in constraints (5.97) and (5.98) take then the explicit form

$$\begin{aligned}\frac{\partial^2\mathcal{L}}{\partial F^{\mu\nu}\partial F_{\alpha\beta}} &= -\frac{1}{2}\delta^{\alpha\beta}_{[\mu\nu]}+\alpha_1\left[F^2\delta^{\alpha\beta}_{[\mu\nu]}+2F^{\alpha\beta}F_{\mu\nu}\right]\\ &\quad+\alpha_2\left[\frac{1}{2}\hat{\epsilon}_{\mu\nu}{}^{\alpha\beta}F\tilde{F}+2\tilde{F}^{\alpha\beta}\tilde{F}_{\mu\nu}\right]+\alpha_3 W_{\mu\nu\rho\sigma}g^{\rho\alpha}g^{\sigma\beta}\,,\end{aligned} \tag{5.104}$$

$$\begin{aligned}\frac{\partial^3\mathcal{L}}{\partial F^{AB}\partial F^{\mu\nu}\partial F^{\alpha\beta}} &= \alpha_1\left[2\,g_{\alpha[\mu}\,g_{\nu]\beta}F_{AB}+2\,g_{A[\alpha}\,g_{\beta]B}F_{\mu\nu}+2\,g_{\mu[A}\,g_{B]\nu}F_{\alpha\beta}\right]\\ &\quad+\alpha_2\left[\hat{\epsilon}_{\mu\nu\alpha\beta}\tilde{F}_{AB}+\hat{\epsilon}_{\alpha\beta AB}\tilde{F}_{\mu\nu}+\hat{\epsilon}_{AB\mu\nu}\tilde{F}_{\alpha\beta}\right]\,,\end{aligned} \tag{5.105}$$

$$\begin{aligned}\frac{\partial^2\mathcal{L}_D}{\partial G^{\mu\nu}\partial G_{\alpha\beta}} &= -\frac{1}{2}\delta^{\alpha\beta}_{[\mu\nu]}+\alpha_1\left[G^2\delta^{\alpha\beta}_{[\mu\nu]}+2G^{\alpha\beta}G_{\mu\nu}\right]\\ &\quad+\alpha_2\left[\frac{1}{2}\hat{\epsilon}_{\mu\nu}{}^{\alpha\beta}G\tilde{G}+2\tilde{G}^{\alpha\beta}\tilde{G}_{\mu\nu}\right]+\alpha_3\tilde{W}_{\mu\nu\rho\sigma}g^{\rho\alpha}g^{\sigma\beta}\,,\end{aligned} \tag{5.106}$$

$$\begin{aligned}\frac{\partial^3\mathcal{L}_D}{\partial G^{AB}\partial G^{\mu\nu}\partial G^{\alpha\beta}} &= \alpha_1\left[2\,g_{\alpha[\mu}\,g_{\nu]\beta}G_{AB}+2\,g_{A[\alpha}\,g_{\beta]B}G_{\mu\nu}+2\,g_{\mu[A}\,g_{B]\nu}G_{\alpha\beta}\right]\\ &\quad+\alpha_2\left[\hat{\epsilon}_{\mu\nu\alpha\beta}\tilde{G}_{AB}+\hat{\epsilon}_{\alpha\beta AB}\tilde{G}_{\mu\nu}+\hat{\epsilon}_{AB\mu\nu}\tilde{G}_{\alpha\beta}\right]\,.\end{aligned} \tag{5.107}$$

Let us impose now constraint (5.98). Recalling that $G=\tilde{F}+O(\alpha)$ we have

$$\begin{aligned}\partial^2\mathcal{L}(F)-\partial^2\mathcal{L}_D(F) &= \alpha_1\left[4F^2F_{\mu\nu}-2F\tilde{F}\tilde{F}_{\mu\nu}\right]+\alpha_2\left[4F\tilde{F}\tilde{F}_{\mu\nu}-2F^2F_{\mu\nu}\right]\\ &\quad+\alpha_3 W_{\mu\nu\rho\sigma}F^{\rho\sigma}-\alpha_3\tilde{W}_{\mu\nu\rho\sigma}F^{\rho\sigma}\,,\end{aligned} \tag{5.108}$$

$$\begin{aligned}\partial^3\mathcal{L}(GG) &= \left\{\alpha_1\left[4\,\tilde{F}_{\mu\nu}F\tilde{F}-2\,F^2F_{\mu\nu}\right]\right.\\ &\quad\left.+\alpha_2\left[+4F_{\mu\nu}F^2-2\tilde{F}_{\mu\nu}F\tilde{F}\right]\right\}\,,\end{aligned} \tag{5.109}$$

Which implies

$$\begin{aligned}\partial^3\mathcal{L}(GG)-\partial^2\mathcal{L}(F)+\partial^2\mathcal{L}_D(F) &= 6(\alpha_1-\alpha_2)\left[\tilde{F}_{\mu\nu}F\tilde{F}-F^2F_{\mu\nu}\right]\\ &\quad-\alpha_3\left[W_{\mu\nu\rho\sigma}F^{\rho\sigma}-\tilde{W}_{\mu\nu\rho\sigma}F^{\rho\sigma}\right]\,.\end{aligned} \tag{5.110}$$

In general $\tilde{F}_{\mu\nu} F\tilde{F} - F^2 F_{\mu\nu}$ does not vanish, therefore duality provides us the constraint

$$\alpha_1 = \alpha_2 \,. \tag{5.111}$$

Let us consider now the last term of (5.110).
Evaluating explicitly α_3 coefficient we get that it has some not vanishing components

$$W_{\mu\nu} = \left(W_{\mu\nu\rho\sigma} - \tilde{W}_{\mu\nu\rho\sigma}\right) F^{\rho\sigma} \,, \tag{5.112}$$

$$W_{01} = -W_{10} = -\frac{Q}{16\pi^3 r^6}\left(Q^2 - 4M\pi r\right) \,, \tag{5.113}$$

It follows that we have to require

$$\alpha_3 = 0 \,. \tag{5.114}$$

We have therefore again

$$\alpha_1 = \alpha_2 \,, \qquad \alpha_3 = 0 \,. \tag{5.115}$$

CHAPTER 6

Summary and Outlook

We conclude with a brief summary and a review of the possible further developments. This thesis had two goals. The first one was determining whether higher order derivative corrections allow black holes to decay through a splitting process. The second one was understanding the role of duality in the production of constraints on the effective theories of quantum gravity. Both issues naturally arise as part of a deeper question: investigate the nature of the Weak Gravity Conjecture (WGC).

In chapter 1 we presented the WGC explaining that it has a special role among all the swampland's criteria. It is the first criterion developed and the best established one, thanks to its huge web of different formulations. However, its original and simplest statement, the Electric WGC, is still not fully understood. Indeed, the EWGC can be interpreted as the necessary condition for BHs decay, however it is not clear if there is any inconsistency related to BHs stability.

Such interpretation implies then that the EWGC trivializes if the BHs can decay by themselves. This can occur if the charge-to-mass ratio of the extremal BH solutions decreases with the mass of the BH.

In chapter 2, 3 and 4 we identified the most general 4 derivative extension of Einstein–Maxwell theory and we removed via field redefinitions the redundant terms. We solved the perturbed equations of motion and we computed the radius and the charge-to-mass ratio of extremal BHs. We verified then that the positivity bounds due to S-matrix properties (analiticity, unitarity and crossing symmetries) imply the mild form of the EWGC.

In order to study the possibility of BH splitting, we computed the Wald's entropy in the extremal case (equation (3.51)). This is the first relevant result of the thesis because the entropy formula of [14] diverges in the extremal limit. Exploiting such result we presented our original discussion of splitting processes. In particular, we found that electrically charged, static BHs can not split *within our perturbative regime*.

After that, we tested our result with several consistency checks. The individual tests do not exploit innovative techniques, however as far as we know there is no example in literature of a more comprehensive analysis. Other results we obtained in chapter 3 are the general algorithm to determine the entropy of extremal configurations (section 3.4.2) and the general black hole solution of the perturbed Einstein–Maxwell theory (see equations (3.181) and (3.187)) which generalize the formula for perturbed Einstein theory of [34].

In chapter 5 we presented Electric–Magnetic (EM) duality. Imposing that the 4 derivative Einstein–Maxwell theory has the same duality group of the unperturbed theory we produced constraints on the coefficients of higher order corrections. We showed then that assuming EM duality, the mild form of the EWGC coincide with the positivity bounds obtained from microcausality, unitarity and relativistic invariance.

Another original part of the thesis is the exploitation of the interpretation of duality transformations as the result of a Legendre transformation. Legendre transformations have been already used to characterize EM duality, therefore the technique we described can not be considered a completely original result of this thesis work. However, it is definitely a generalization and systematization of [23], which also allows for a simple application to the cases of theories with higher derivative interactions. In particular, this brought us to the sufficient condition (5.45), which extremely simplifies the computations of duality constraints on higher order terms.

To sum up, we have been able to answer to both our questions in 4 derivative Einstein–Maxwell theory: we can not have splitting processes and the duality constraints provide a new theoretical interpretation of the EWGC. Indeed, for what concerns the first question, despite the fact that we proved the validity of the mild form of the EWGC and that our BH configurations satisfy the kinetics constraints, splitting processes are not thermodynamically allowed within our perturbative regime. For the second question, assuming the duality constraints, we proved that the mild form of the EWGC coincides with the consistency conditions imposed by microcausality, unitarity and relativistic invariance.

A natural question is which of our results are specific to our setup and which can be extended to more general cases, instead. The thesis work has therefore various possible continuations.

For the study of splitting processes, it is important to look at a theory that we control microscopically, i.e. a theory whose UV completion is known to be string theory. A possible candidate is a $\mathcal{N} = 2$ supergravity model. In such class of theories the extremal black holes are obtained from a dimensional reduction of D-branes (in type II string theory) or M-branes (in M-theory) wrapped on a Calabi–Yau manifold (see [37]). Moreover, the 4 derivative corrections are known (see [43]). A starting point would be in particular the STU model which can be interpreted as a low energy limit of type *IIA* string theory compactified on $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. There are 4 magnetic and 4 electric charges obtained by wrapping the various $\mathbb{T}^2$ with the branes $D0$, $D2$, $D4$, $D6$ (see [44]). A partial analysis of the model entropy can be found in [45].

The study of duality constraints and the EWGC interpretation could be done instead considering generic charged black holes coupled with matter. In particular, it would be interesting to analyze the relation with positivity bounds in order to understand if it is necessary to exploit other UV elements to fix WGC.

APPENDIX A

Mathematical Identities

A.1 Definitions

Determinant and determinant sign:

$$g = \det (g_{\mu\nu}) \ , \qquad s = \frac{g}{|g|} \, . \tag{A.1}$$

Christoffel symbols:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu} \right) \, . \tag{A.2}$$

Riemann tensor, Ricci tensor, Ricci scalar:

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} = 2 \left(\partial_{[\mu} \Gamma^{\rho}_{\nu]\sigma} + \Gamma^{\rho}_{[\mu|\tau} \Gamma^{\tau}_{|\nu]\sigma} \right) \, , \tag{A.3}$$

$$R_{\mu\nu} = R_{\rho\mu}{}^{\rho}{}_{\nu} \, , \qquad R = R_{\mu\nu} g^{\mu\nu} \, . \tag{A.4}$$

Weyl tensor:

$$W_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{d-2} \left(g_{\mu[\rho} R_{\sigma]\nu} - g_{\nu[\rho} R_{\sigma]\mu} \right) + \frac{2}{(d-2)(d-1)} R g_{\mu[\rho} g_{\sigma]\nu} \, . \tag{A.5}$$

A.2 Functional Derivatives

A.2.1 $g^{\mu\nu}$ Derivatives

Basic relations:

$$\delta g_{\mu\nu} = -g_{\mu\alpha} \, g_{\nu\beta} \, \delta g^{\alpha\beta} \, , \tag{A.6}$$

$$\delta \sqrt{|g|} = -\frac{\sqrt{|g|}}{2} g_{\mu\nu} \delta g^{\mu\nu} \, , \tag{A.7}$$

$$\delta \frac{1}{\sqrt{|g|}} = \frac{1}{2\sqrt{|g|}} g_{\mu\nu} \delta g^{\mu\nu} \, , \tag{A.8}$$

$$\begin{aligned}
\delta\,\Gamma^{\rho}_{\mu\nu} &= \frac{1}{2} g^{\rho\alpha}\left(D_\mu \delta g_{\nu\alpha} + D_\nu \delta g_{\mu\alpha} - D_\alpha \delta g_{\mu\nu}\right) , && \text{(A.9)}\\
\delta R_{\mu\nu}{}^{\rho}{}_{\sigma} &= 2D_{[\mu}\left(\delta\,\Gamma^{\rho}_{\nu]\sigma}\right) = D_\mu\left(\delta\,\Gamma^{\rho}_{\nu\sigma}\right) - D_\nu\left(\delta\,\Gamma^{\rho}_{\mu\sigma}\right)\\
&= \frac{1}{2} g^{\rho\alpha}\left(D_\mu D_\nu \delta g_{\sigma\alpha} + D_\mu D_\sigma \delta g_{\nu\alpha} - D_\mu D_\alpha \delta g_{\nu\sigma}\right) && \text{(A.10)}\\
&- \frac{1}{2} g^{\rho\beta}\left(D_\nu D_\mu \delta g_{\sigma\beta} + D_\nu D_\sigma \delta g_{\mu\beta} - D_\nu D_\beta \delta g_{\mu\sigma}\right) ,\\
\delta R_{\mu\nu\gamma\sigma} &= \delta R_{\mu\nu}{}^{\rho}{}_{\sigma} g_{\rho\gamma} + R_{\mu\nu}{}^{\rho}{}_{\sigma} \delta g_{\rho\gamma}\\
&= \frac{1}{2}\left(D_\mu D_\nu \delta g_{\sigma\gamma} + D_\mu D_\sigma \delta g_{\nu\gamma} - D_\mu D_\gamma \delta g_{\nu\sigma}\right)\\
&- \frac{1}{2}\left(D_\nu D_\mu \delta g_{\sigma\gamma} + D_\nu D_\sigma \delta g_{\mu\gamma} - D_\nu D_\gamma \delta g_{\mu\sigma}\right) && \text{(A.11)}\\
&+ R_{\mu\nu}{}^{\rho}{}_{\sigma} \delta g_{\rho\gamma} ,\\
\delta R_{\nu\sigma} &= \delta^{\mu}_{\rho}\, \delta R_{\mu\nu}{}^{\rho}{}_{\sigma}\\
&= \frac{1}{2}\left(D^\alpha D_\nu \delta g_{\sigma\alpha} + D^\alpha D_\sigma \delta g_{\nu\alpha} - D^2 \delta g_{\nu\sigma}\right) && \text{(A.12)}\\
&- \frac{1}{2}\left(D_\nu D^\beta \delta g_{\sigma\beta} + D_\nu D_\sigma\left(g^{\mu\beta}\delta g_{\mu\beta}\right) - D_\nu D^\beta \delta g_{\beta\sigma}\right) ,\\
\delta R &= R_{\mu\nu}\delta g^{\mu\nu} + g^{\nu\sigma}\delta R_{\nu\sigma} = \left[-R^{\mu\nu} + D^\mu D^\nu - g^{\mu\nu} D^2\right]\delta g_{\mu\nu} , && \text{(A.13)}\\
\delta F^{\mu\nu} &= g^{\mu\alpha}\delta g^{\nu\beta} F_{\alpha\beta} + \delta g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} , && \text{(A.14)}\\
\delta \tilde{F}^{\mu\nu} &= \frac{1}{2} g_{\alpha\beta}\, \delta g^{\alpha\beta}\, \tilde{F}^{\mu\nu} . && \text{(A.15)}
\end{aligned}$$

Higher derivative terms of Einstein–Maxwell theory (integrated by parts):

$$\begin{aligned}
\delta\left[RF^2\right] &= \left[2RF_\mu{}^{\sigma}F_{\nu\sigma} + F^2 R_{\mu\nu} - D_\mu D_\nu F^2 + g_{\mu\nu} D^2 F^2\right]\delta g^{\mu\nu} , && \text{(A.16)}\\
\delta\left[R_{\mu\nu}F^{\mu\rho}F^{\nu}{}_{\rho}\right] &= \Big[2R_{\nu\sigma}F_{\mu\rho}F^{\sigma\rho} + R^{\alpha\beta}F_{\alpha\mu}F_{\beta\nu} + \frac{1}{2} g_{\mu\nu} D_\alpha D_\beta (F^{\alpha}{}_{\rho}F^{\beta\rho})\\
&\quad - D_\alpha D_\nu (F_{\mu\beta}F^{\alpha\beta}) + \frac{1}{2} D^2 (F_{\mu\rho}F_\nu{}^{\rho})\Big]\delta g^{\mu\nu} , && \text{(A.17)}\\
\delta\left[R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\right] &= \left[3F_{\alpha\nu}F^{\beta\gamma}R^{\alpha}{}_{\mu\beta\gamma} + 2D_\beta D_\alpha (F^{\alpha}{}_{\mu}F^{\beta}{}_{\nu})\right]\delta g^{\mu\nu} , && \text{(A.18)}\\
\delta\left[W_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\right] &= \delta\Big[R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} - \frac{4}{d-2} R_{\mu\nu}F^{\mu\rho}F^{\nu}{}_{\rho}\\
&\quad + \frac{2}{(d-2)(d-1)} R F_{\mu\nu}F^{\mu\nu}\Big] , && \text{(A.19)}\\
\delta\left[(FF)^2\right] &= \left[4F^2 F_{\mu\nu}F_\rho{}^{\nu}\right]\delta g^{\mu\rho} , && \text{(A.20)}\\
\delta\left[(F\tilde{F})^2\right] &= \left[(F\tilde{F})^2 g_{\mu\nu}\right]\delta g^{\mu\nu} . && \text{(A.21)}
\end{aligned}$$

Higher derivative terms of the extended Einstein–Maxwell theory (integrated by parts):

$$\delta\left[R^2\right] = \left[2RR_{\mu\nu} - 2D_{(\nu|}D_{|\mu)}R + 2g_{\mu\nu}D^2R\right]\delta g^{\mu\nu}\,, \tag{A.22}$$

$$\delta\left[(R_{\mu\nu})^2\right] = \left[-2D_\alpha D_{(\nu|}R^\alpha_{|\mu)} + 2D^2R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}D^2R + 2R^\alpha_\mu R_{\alpha\nu}\right]\delta g^{\mu\nu}\,, \tag{A.23}$$

$$\begin{aligned}\delta\left[(D_\mu F_{\nu\rho})^2\right] = &\Big[(D_\mu F_{\alpha\beta})(D_\nu F^{\alpha\beta}) - 4(D_\alpha F_{\beta\mu})(D^\alpha F^\beta{}_\nu) \\ &+ 4D_\alpha\left(F_{(\nu|\beta}D^\alpha F_{|\mu)}{}^\beta\right) + 4D_\alpha\left(F_{(\nu|\beta}D_{|\mu)}F^{\alpha\beta}\right) \\ &- 4D_\alpha\left(F^\alpha{}_\beta\, D_{(\nu|}F_{|\mu)}{}^\beta\right)\Big]\delta g^{\mu\nu}\,.\end{aligned} \tag{A.24}$$

A.2.2 A_μ Derivatives

Higher derivative terms of Einstein–Maxwell theory (integrated by parts):

$$\delta\left[W_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\right] = D_\mu\left[-4\,W^{\mu\nu\rho\sigma}F_{\rho\sigma}\right]\delta A_\nu\,, \tag{A.25}$$

$$\delta\left[(FF)^2\right] = D_\mu\left[-8F^2F^{\mu\nu}\right]\delta A_\nu\,, \tag{A.26}$$

$$\delta\left[(F\tilde{F})^2\right] = D_\mu\left[-8(F\tilde{F})\tilde{F}^{\mu\nu}\right]\delta A_\nu\,. \tag{A.27}$$

Higher derivative terms of the extended Einstein–Maxwell theory (integrated by parts):

$$\delta\left[R_{\mu\nu}F^{\mu\rho}F^\nu{}_\rho\right] = D_\mu\left[-4R^{[\mu|\alpha}F_\alpha{}^{|\nu]}\right]\delta A_\nu\,, \tag{A.28}$$

$$\delta\left[RF^2\right] = D_\mu\left[-4RF^{\mu\nu}\right]\delta A_\nu\,, \tag{A.29}$$

$$\delta\left[(D_\mu F_{\nu\rho})^2\right] = D_\mu\left[4D^2F^{\mu\nu}\right]\delta A_\nu\,. \tag{A.30}$$

A.2.3 R_{abcd} Derivatives

Basic relations:

$$\delta R = g^{ac}g^{bd}\delta R_{abcd}\,, \tag{A.31}$$

$$\delta R_{ab} = g^{cd}\delta R_{acbd}\,. \tag{A.32}$$

Higher derivative terms of Einstein–Maxwell theory:

$$\begin{aligned}\delta\left[F^{ab}F^{cd}W_{abcd}\right] = &\Big[F^{ab}F^{cd} - \frac{4}{D-2}g^{ac}F^{bf}F^d{}_f \\ &+ \frac{4}{2(D-1)(D-2)}F^2g^{ac}g^{bd}\Big]\delta R_{abcd}\,.\end{aligned} \tag{A.33}$$

Higher derivative terms of the extended Einstein–Maxwell theory (integrated by parts):

$$\delta\left[R_{\mu\nu}F^{\mu\rho}F^\nu{}_\rho\right] = \left[F^{a\alpha}F^c{}_\alpha\, g^{bd}\right]\delta R_{abcd}\,, \tag{A.34}$$

$$\delta\left[RF^2\right] = \left[F^2g^{ac}g^{bd}\right]\delta R_{abcd}\,, \tag{A.35}$$

$$\delta\left[R^2\right] = \left[2Rg^{ac}g^{bd}\right]\delta R_{abcd}\,, \tag{A.36}$$

$$\delta\left[(R_{\mu\nu})^2\right] = \left[2R^{ac}g^{bd}\right]\delta R_{abcd}\,. \tag{A.37}$$

A.3 Differential Forms

Levi-Civita tensor definition and products:

$$
\begin{aligned}
\tilde{\epsilon}_{\mu_1\ldots\mu_D} &= \sqrt{sg}\,\epsilon_{\mu_1\ldots\mu_D}\,, && \text{(A.38)}\\
\tilde{\epsilon}^{\mu_1\ldots\mu_D} &= \frac{s}{\sqrt{sg}}\,\epsilon^{\mu_1\ldots\mu_D}\,, && \text{(A.39)}\\
\epsilon^{\mu_1\ldots\mu_p\nu_{p+1}\ldots\nu_D}\epsilon_{\mu_1\ldots\mu_p\rho_{p+1}\ldots\rho_D} &= (D-p)!\,p!\,\delta^{\nu_{p+1}}_{[\rho_{p+1}}\ldots\delta^{\nu_D}_{\rho_D]}\,. && \text{(A.40)}
\end{aligned}
$$

Exterior algebra:

$$
\begin{aligned}
\omega &= \frac{1}{k!}\omega_{\mu_1\ldots\mu_k}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k}\,, && \text{(A.41)}\\
\alpha\wedge\beta &= \frac{(k+q)!}{k!\,q!}\mathcal{A}\left[\alpha\otimes\beta\right] && \\
&= \frac{1}{k!q!}\alpha_{[\mu_1\ldots\mu_p}\beta_{\nu_1\ldots\nu_q]}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k}\wedge dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_q}\,, && \text{(A.42)}\\
d\,\omega &= \frac{1}{k!}D_{[\mu}\omega_{\mu_1\ldots\mu_p}\eta_{\nu_1\ldots\nu_q]}dx^{\mu}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k}\,. && \text{(A.43)}
\end{aligned}
$$

Invariant Hodge operator definition and properties:

$$
\begin{aligned}
g &= \det(g_{\mu\nu})\,, \qquad s = \frac{g}{|g|}\,, && \text{(A.44)}\\
\star\,\omega &= \frac{\sqrt{sg}}{k!(D-k)!}\,\omega_{\mu_1\ldots\mu_k}g^{\mu_1\alpha_1}\ldots g^{\mu_k\alpha_k}\epsilon_{\alpha_1\ldots\alpha_D}dx^{\alpha_{k+1}}\wedge\cdots\wedge dx^{\alpha_D}\,, && \text{(A.45)}\\
\alpha\wedge\star\beta &= \langle\alpha,\beta\rangle\,e = \frac{1}{k!}\alpha_{[\mu_1\ldots\mu_k]}\,\beta_{\nu_1\nu_2\ldots\nu_k}\,g^{\mu_1\nu_1}\ldots g^{\mu_k\nu_k}\,e\,, && \text{(A.46)}\\
\star 1 &= e = \mathrm{Vol}\,, && \text{(A.47)}\\
\star^2\left[\omega\right] &= s(-1)^{(D-k)k}\,\omega \qquad \dim(\omega)=k\,. && \text{(A.48)}
\end{aligned}
$$

Invariant volume element:

$$
e_M = \sqrt{sg}\,dx^0\wedge\cdots\wedge dx^d = \frac{\sqrt{sg}}{D!}\,\epsilon_{\mu_1\ldots\mu_D}\,dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_D}\,. \tag{A.49}
$$

Interior product properties:

$$
\begin{aligned}
i_\omega\circ e_M &= \star\ \omega\,, && \text{(A.50)}\\
i_\omega\circ\eta &= \langle\omega,\eta\rangle\,, \qquad \text{if}\quad \dim(\omega)=\dim(\eta)\,, && \text{(A.51)}\\
i_\omega\circ(\alpha\wedge\beta) &= (i_\omega\circ\alpha)\wedge\beta+(-1)^{\dim(\alpha)}\alpha\wedge(i_\omega\circ\beta)\,, \qquad \text{if}\quad \dim(\omega)=1\,. && \text{(A.52)}
\end{aligned}
$$

Integration of differential forms:

Let M be a D-dimensional manifold, Σ a k-dimensional manifold, $j : \Sigma \hookrightarrow N \subset M$ an embedding map, ω a k-dimensional differential form, $\star$ the D-dimensional invariant Hodge operator and e_M, η_N and e_N the volume element of M, of N and the normal normal volume element such that $e_M = \eta_N \wedge e_N$.

$$\begin{aligned}\int_N \star\omega &= \int_{j(\Sigma)} i_\omega \circ e_M = \int_\Sigma j^* \left[i_\omega \circ (\eta_N \wedge e_N)\right] = \int_\Sigma j^* \left[(i_\omega \circ \eta_N) \wedge e_N)\right] \\ &= \int_N \langle \omega, \eta_N \rangle \; e_N \,,\end{aligned} \tag{A.53}$$

$$\begin{aligned}\int_N \omega &= (-1)^{(D-k)k} s \int_N \star^2 [\omega] = (-1)^{(D-k)k} s \int_N \langle \star\omega, \eta_N \rangle \, e_N \\ &= s \int_N \langle \omega, \star\eta_N \rangle \, e_N \,.\end{aligned} \tag{A.54}$$

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