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Dubrovin's approach to the FPU Problem

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Mathematics is the part of physics where experiments are cheap.
- Vladimir Igorevic Arnol'd

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Contents

1 Introduction **1**

2 The Fermi-Pasta-Ulam Problem **3**
2.1 The system 4
2.2 Connection between the FPU and the Toda lattice 5

3 Dubrovin's theorem and its extension **7**
3.1 Change of coordinates 8
3.2 Extension of the Dubrovin's Theorem 10
3.3 Case of Harmonic Oscillator 14

4 Application for the FPU Problem **17**
4.1 Continuum limit of FPU 17
4.2 Extensions of first integrals 20
4.3 Solutions of the principal PDEs 23

5 Conclusion **29**

A Theory of Nonlinear Wave Equations and Hamiltonian Perturbations **31**
A.1 First integrals of nonlinear wave equation 32
A.2 Perturbation of nonlinear wave equation and deformation of the first integrals 33
A.2.1 Extension of first integrals 35

B Dubrovin's proof **37**
B.1 Proof of the Theorem 38

C Derivation of the coefficients C_n^l and B_n^l **41**

Bibliography **43**

Chapter 1

Introduction

The integrable systems, e.g the Harmonic oscillator, the Toda Lattice or the two body Keplerian problem, are a special kind of models, generally rich of properties and sometimes simple to solve.

Unfortunately, these kind of systems are rare, and most of the times we encounter perturbations of these models, which could have completely different behaviours. Indeed, if in the two body Kleperian problem we add a third mass, smaller then the mass of the two planets of the original system (e.g an asteroid), the system components' motion becomes more complex compared to the original one, giving also rise to chaotic motions. This problem is known as the *three body problem* and it's simpler than the solar system, which have 9 planets plus all other kinds of celestial bodies.

The perturbations of integrable systems, in particular Hamiltonian ones, were originally studied by Poincaré and Birckoff in the end of 19^{th} century and then developed during all the 20^{th} century, thanks to the important contributions of Kolmogorov, Arnold, Moser (KAM Theorem) and Nekhoroshev (Nekhoroshev estimates).

In the study of perturbed Hamiltonian systems, the following theorem (given by Poincaré) is quite relevant [6]:

Theorem 1.0.1. *Given the Hamiltonian:*

$$H(I, \varphi) = H_0(I) + \varepsilon H_1(I, \varphi) + \dots \tag{1.1}$$

with $I \in D \subset \mathbb{R}^n$ and $\varphi \in \mathbb{T}^n$. Suppose that H_0 is non degenerate in D and that the Ponicarè set $\mathbb{B}$[1] *is dense in D, then the Hamiltonian* (1.1) *has no formal integral of this form*

$$F = F_0(I) + \varepsilon F_1(I, \varphi) + \dots \tag{1.2}$$

independent to H, with infinite differentiable function $F_n : D \times \mathbb{T}^n \to \mathbb{R}$.

In other words: non degenerate, integrable Hamiltonian systems under generic perturbations loose all the first integrals in the analytic class.

This means that, if we consider degenerate Hamiltonians or particular perturbations of the systems, it is possible to find extension of first integrals from the unperturbed to the perturbed systems (e.g. see [22]).

Along these lines, methods to extend solutions and first integrals of the unperturbed system have been developed, both for finite Hamiltonian systems and Hamiltonian PDEs.

[1] the set of all $I \in D$ such that exist $n-1$ l.i. vector $k \in \mathbb{Z}^n$ s.t.:

1. $< k_s, \omega(I) >= 0$ with $1 \leq s \leq n-1$
2. $H_{k_s}(I) \neq 0$

where $H_{k_s}(I)$ is the Fourier coefficient of the perturbation $H_1(I, \varphi)$

In recent years, Dubrovin and his co-workers developed new techniques to extend solutions and first integrals of perturbed Hamiltonian of hyperbolic PDEs. The theory, described in [2] and [3], is based on a generic system of first order PDE

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x + B_2(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}) + B_3(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \mathbf{u}_{xxx}) + \dots \tag{1.3}$$

with

$$\mathbf{u} = (u^1(x,t), \dots, u^n(x,t))^T$$

an unknown vector function. We suppose that the entries of the matrix $A(\mathbf{u})$ are smooth on some domain $\mathcal{D} \subset \mathbb{R}^n$ and the characteristic roots $\lambda_1(\mathbf{u}), \dots, \lambda_n(\mathbf{u})$ will be assumed to be pairwise dinstict

$$\det(A(\mathrm{u}) - \lambda \cdot \mathbb{I}) = 0, \quad \lambda_i(\mathbf{u}) \neq \lambda_j(\mathbf{u}) \text{ for } i \neq j, \quad \forall \mathbf{u} \in \mathcal{D}. \tag{1.4}$$

The terms $B_2, \dots, B_k, \dots$ of (1.3) are polynomials of the jet coordinates

$$\mathbf{u}_x = (u^1_x, \dots, u^n_x)^T, \quad \mathbf{u}_{xx} = (u^1_{xx}, \dots, u^n_{xx})^T, \dots$$

graded homogeneus of the degree $2, 3, \dots, k, \dots$

$$\begin{aligned} &\deg\, B_k(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(k)}) = k \\ &\deg\, \frac{\partial^m u^i}{\partial x^m} = m, \quad m > 0, \quad \deg\, u^i = 0, \quad i = 1, \dots, n. \end{aligned} \tag{1.5}$$

The coefficients of these polynomials are smooth functions on the same domain $\mathcal{D}$.

The system (1.3) can be considered as perturbation of the first order quasilinear system

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x \tag{1.6}$$

when considering slowly varying solutions. Let be the natural small parameter

$$h = \frac{1}{L} \tag{1.7}$$

where L is the spatial length where $\mathbf{u}(x)$ change by 1., we estimate the derivatives as

$$\mathbf{u}_x \sim h, \quad \mathbf{u}_{xx} \sim h^2, \dots, \mathbf{u}^{(k)} \sim h^k, \dots. \tag{1.8}$$

It is convenient to introduce *slow variables* by rescaling

$$x \to hx, \quad t \to ht. \tag{1.9}$$

The system (1.3) becomes

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x + hB_2(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}) + h^2 B_3(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \mathbf{u}_{xxx}) + \dots \quad . \tag{1.10}$$

The theory says that there is a way to extend the solutions from the unperturbed system to the perturbed one, and that near the critical point called *gradient catastrophe*, i.e. the point where the derivatives of the solution to the unperturbed equation tend to infinity, the solutions do not depend on the kind of perturbation, but it is given by a certain special solution of Painlevé equations.

After this theory was introduced, Dubrovin, in the article [1] and [4], applied these techniques also to perturbed Hamiltonian PDEs[2]. He used these mathematical tools to extend both solutions and first integrals from the unperturbed system to the perturbed one. By doing so, he developed a perturbative approach to the study of the integrability which can be used for:

1. finding obstructions to the integrability;
2. classification of integrable PDEs.

The aim of the thesis is to apply this new tools to a particular system: the Fermi-Pasta-Ulam problem (of FPU problem). In particular, we want to find out if there are some conditions to extend first integrals to a fixed order, or there is an obstruction to the integrability.

[2]a summary of the main results and theorems of [1] and [4], which we will use in the next chapters, is presented in the appendix A.

Chapter 2

The Fermi-Pasta-Ulam Problem

One of the most important integrable systems is the harmonic oscillator, because it is a model that we encounter in many physical systems. An example of this is the pendulum.

The motion of a pendulum with unitary length is described by ODE

$$\begin{cases} \dot{\theta} = v \\ \dot{v} = -g \sin\theta \end{cases} \implies \ddot{\theta} = -g\sin\theta \quad \theta \in [0, 2\pi]. \tag{2.1}$$

Considering only a small oscillation of the angle ($\theta \ll \pi/18$), we can approximate at the first order the sine of the angle, and the equation becomes

$$\ddot{\theta} = -g\sin\theta \approx -g\theta \tag{2.2}$$

which is exactly the ODE of the harmonic oscillator with frequency $\sqrt{g}$.

Another example is a toy model, which is very important for the solid state physics because it gives us good explanation to some phenomena that we observe in crystals: a 1-D chain of particle that interacts pairwise with a Lennard-Jones potential

$$\phi_{L-J}(x_{n+1} - x_n) = \left[\left(\frac{\sigma}{(x_{n+1} - x_n)} \right)^{12} - \left(\frac{\sigma}{(x_{n+1} - x_n)} \right)^{6} \right]. \tag{2.3}$$

We focus only in the small oscillation around the equilibrium point x^*, so the potential in this approximation becomes

$$\phi_{L-J}(x_{n+1} - x_n) = -\phi_0 + \frac{1}{2}\phi''(x^*)(x_{n+1} - x_n)^2 + \mathcal{O}((x_{n+1} - x_n)^3) \tag{2.4}$$

which is the harmonic potential.

This system and its generalisation in three dimension explain us many important features of the solid state physics (the form in first approximation of the band structure of the phonons, the thermal conductivity, ...).

As said before, however, these are only approximate systems and corrections at further order are important for the physics of the systems (The asynchrony of the period of the pendulum for big θ is explained considering all the ODE (2.1), while the thermal dilatation of solids is explained by taking into consideration also the anharmonic terms of the expansion (2.4) [23]).

It is thus important to study perturbations of the harmonic oscillator.

A particular example of these systems that we want take into account is the Fermi-Pasta-Ulam problem [8].

2.1 The system

In 1954, Enrico Fermi, John Pasta and Stanislav Ulam took advantage of the computer MANIAC I to run some computer simulations, with the aim of studying a particular system, i.e. a chain of N particle that interact pairwise with potential

$$\phi(q_{n+1} - q_n) = \frac{(q_{n+1} - q_n)^2}{2} + \alpha\frac{(q_{n+1} - q_n)^3}{3} \tag{2.5}$$

where α is a constant coefficient in $\mathbb{R}$ and with fixed end points.

The equations for the motion of the system are

$$\ddot{q}_n = (q_{n+1} + q_{n-1} - 2q_n) + \alpha[(q_{n+1} - q_n)^2 - (q_n - q_{n-1})^2] \tag{2.6}$$

for $n = 1, \dots, N$.

The solution of the linear problem ($\alpha = 0$) is a periodic vibration of the string, and this gives us the opportunity to decompose the motion in normal modes

$$Q_k(t) = \sum_{n=1}^{N} q_n(t)\sin(\frac{nk\pi}{N}) \quad \text{with } k = 1, \dots, N$$

and the energy of the k-th mode is

$$E_k = \frac{1}{2}\dot{Q_k}^2 + 2Q_k\sin^2(\frac{k\pi}{2N})$$

Fermi, Pasta and Ulam believed that the presence of nonlinear terms changes the form of the solutions of the linear problem, giving rise to more complicated shapes of the strings. In particular, they expected that, starting with one or few mode excited, the energy would start to redistribute in all the modes, so that the system would reach the equipartition of energy fairly quickly.

Thus, once all the energy had been assigned to the first mode, they started the numerical computation of the q_n for a small number of particle ($N = 64$) and then, after few hundred of cycles, calculated the mode Q_k and the energy E_k. The outputs of the simulation were surprising.

They saw that, during all the computational time (30,000 computation cycles), the system's energy was exchanged only between the first 4/5 modes, while the higher modes weren't excited. Moreover, by observing the energy spectrum of each mode, they found out that the energy of these excited modes presented a periodic behaviour along the time. This suggested that, for not a long time[1], the system had an underlining integrable dynamics.

This result astonished them, because it was completely in contradiction to the expectations on the behaviour of the system.

Ten years later, mathematicians started to study, both analytically and numerically, this problem. In particular, thanks to the new results and theorems from the study of perturbed Hamiltonian systems (one of the most important is the KAM theorem), they try to answer to some of the questions that this system blows up. Why there is such integrable behaviour?, How long take the system to reach equipartition?, How the time of integrability depends on the variable of the system?, ... In particular, we can consider three important results that revolutionized the approach to the problem:

- Zabusky and Kruskal [9] related the periodic behaviour of the modes to the solitonic solutions of the Korteweg-De Vires equation (or KdV)

$$u_t + uu_x + \delta^2 u_{xxx} = 0;$$

[1] In fact, from other numerical simulations performed by computers much more powerful then the MANIAC I, we found that the system reaches equipartition of energy after a very long time, if α is small. In particular, the thermalization times depends on many parameters (the parameter α, the number of particle N, the specific energy $\epsilon, \dots$) [17] [18]

- Izrailev and Chirikov [10] found chaotic motions of the system in the limit of strong non-linearity;
- Ferguson, Flaschka and McLaughlin [11], by numerical simulation, connected the FPU problem to the Toda lattice, which is a nonlinear integrable system.

This problem, called *FPU problem* (formely *FPU paradox*), was the first example of non linear system and these results suggest that these kind of models must be studied more carefully, because the motion and the properties of these systems can be counter-intuitive.

Many other nonlinear chains similar to the FPU problem has become object of study, defining a new family of models called *FPU models.* In particular, we now consider an Hamiltonian

$$H(q,p) = \sum \left(\frac{p_n^2}{2} + \phi(r_n) \right) \quad \text{where } r_n := q_{n+1} - q_n,$$

with potential

$$\phi(r) = \frac{r^2}{2} + \alpha \frac{r^3}{3} + \beta \frac{r^4}{4} + \gamma \frac{r^5}{5} + \dots \tag{2.7}$$

we call:

- FPU α-model if $\alpha \neq 0$;
- FPU β-model if $\alpha = 0$ and $\beta \neq 0$;
- FPU γ-model if $\alpha = \beta = 0$ and $\gamma \neq 0$;

  ⋮
- In general, we call FPU g_d-model if the potential is

  $$\phi(r) = \frac{r^2}{2} + g_d \frac{r^d}{d} + g_{d+1} \frac{r^{d+1}}{d+1} + \dots$$

  where $g_d \neq 0$ and $d \geq 3$.

2.2 Connection between the FPU and the Toda lattice

As previously stated, the FPU system was developed as a perturbation of a chain of particle that interact with a linear potential, but further studies showed other important properties. In particular, in 1982 (almost thirty years after the original FPU article) Ferguson, Flaschka and McLaughlin [11], by numerical simulations, connected the FPU system to another nonlinear system: the Toda Lattice.

This system, presented by Toda [12] in 1967, consisted in a chain of particle that interact pairwise with an exponential potential:

$$\phi_{Toda}(q_{n+1} - q_n) = \frac{e^{\lambda(q_{n+1} - q_n)} - \lambda(q_{n+1} - q_n) - 1}{\lambda^2}. \tag{2.8}$$

We expand ϕ_{Toda} in Taylor series for small oscillation

$$\phi_{Toda}(q_{n+1} - q_n) = \sum_{n=0}^{+\infty} \frac{\lambda^{n-2}}{n!}(q_{n+1} - q_n)^n - \frac{(q_{n+1} - q_n)}{\lambda} - \frac{1}{\lambda^2} = \sum_{n=2}^{+\infty} \frac{\lambda^{n-2}}{n!}(q_{n+1} - q_n)^n =$$
$$= \frac{(q_{n+1} - q_n)^2}{2} + \frac{\lambda}{6}(q_{n+1} - q_n)^3 + \frac{\lambda^2}{24}(q_{n+1} - q_n)^4 + \dots$$

and, defining $\alpha := \lambda/2$, we find the same potential (2.7)

$$\phi_{Toda}(q_{n+1} - q_n) = \frac{(q_{n+1} - q_n)^2}{2} + \frac{\alpha}{3}(q_{n+1} - q_n)^3 + \frac{\beta_{Toda}}{4}(q_{n+1} - q_n)^4 + \frac{\gamma_{Toda}}{5}(q_{n+1} - q_n)^5 + \dots \tag{2.9}$$

where the parameters $\beta_{Toda}, \gamma_{Toda}, \ldots$, depends on the choice of α

$$\beta_{Toda} = \frac{2}{3}\alpha^2, \quad \gamma_{Toda} = \frac{\alpha^3}{3}, \quad \ldots . \tag{2.10}$$

This means that, in the limit of small oscillation, the Toda lattice is tangent at the first order to the FPU system.

However, the most important properties of the Toda lattice are:

1. that the system is integrable (suggested throught to a numerical simulation by Ford, Stoddard and Turner in 1973 [16] and then proved analytically in two different articles of Henon and Flachka in the 1974 [14] [15]);
2. that the equations of motion admits also solitonic solutions [13].

Thus, what Ferguson, Flaschka and McLaughlin showed in their article is that the FPU system is a perturbation of the Toda lattice (a nonlinear integrable system) and not of the harmonic oscillator. In addition to this, in recent years, other simulations pointed out this connection between these two systems using different methods (comparing the spectrum of energies during the time [20], evaluating the Lyapunov's exponent [21], etc...), giving much more credits to this hypothesis.

Chapter 3

Dubrovin's theorem and its extension

After the explanation of the FPU system, we want to focus on a more generic system of the same family: a 1-D chain of particles which interact pairwise and with periodic boundary condition

$$H(q,p) = \sum_{n\in\mathbb{Z}_N} \left(\frac{p_n^2}{2} + \phi(q_{n+1} - q_n)\right) \tag{3.1}$$

where N is the number of particles, $\mathbb{Z}_N := \mathbb{Z}/(N\mathbb{Z})$ and $\phi(q_{n+1} - q_n)$ is a generalized potential, i.e. $\phi'''(q_{n+1} - q_n) \neq 0 \quad \forall\, q_n$.

This Hamiltonian gives us the following equations of motion

$$\begin{aligned} \dot{q}_n &= \frac{\partial H(q_n, p_n)}{\partial p_n} = p_n; \\ \dot{p}_n &= -\frac{\partial H(q_n, p_n)}{\partial q_n} = \phi'(q_{n+1} - q_n) - \phi'(q_n - q_{n-1}). \end{aligned} \tag{3.2}$$

In the article [1], Dubrovin used this system as an example for the application of his theory and, while he was studying the continuum limit of the problem ($N \to \infty$), he found this theorem[1]:
Theorem 3.0.1. *Consider the FPU Hamiltonian* (3.1)*.*

In the continuum limit, the system admits an extension of the first integrals of the unperturbed system at the second order if the potential has this form:

$$\phi(r) = ke^{cr} + ar + b \tag{3.3}$$

where a, b, c, k are real constants.

In particular, when $\phi(r)$ coincides with the Toda potential (2.8)

$$\phi_{Toda}(r) = \frac{e^{2\alpha r} - 2\alpha r - 1}{4\alpha^2} \tag{3.4}$$

the system becomes integrable.

This gives us a strong evidence on the connection between these kind of nonlinear systems and the Toda lattice.

We want to find generalization of this result by entering a general potential in the first order of perturbation.

In particular, we want to see whether the result is valid for all the kind of perturbations or if there are some obstructions to the integrability of the system.

[1]The Dubrovin's proof of this theorem is given in the appendix B.

3.1 Change of coordinates

In his article [1], Dubrovin used a Miura-type transformation to adjust the deformation of the Poisson bracket. Another way to proceed is to start with a different kind of coordinates, already seen in [7], which help us with the discussion of the problem.

We consider the generating function:

$$F(q,p) := \sum_{n\in\mathbb{Z}_N} s_n(q_n - q_{n+1}) \tag{3.5}$$

so that we obtain the following canonical transformation

$$\begin{cases} p_n := \dfrac{\partial F}{\partial q_n} = s_n - s_{n-1} \\ r_n := -\dfrac{\partial F}{\partial s_n} = q_{n+1} - q_n \end{cases}. \tag{3.6}$$

With this new coordinates, the Hamiltonian (3.1) becomes

$$K(r,s) = \sum_{n\in\mathbb{Z}_N} \left[\frac{(s_n - s_{n-1})^2}{2} + \phi(r_n) \right]. \tag{3.7}$$

Here is to notice that the variables r_n defined in (3.6) play the role of the new momenta, with conjugate coordinates s_n.

With the new coordinates, also the equations of motion (3.2) change in relation to the transformation (3.6) but since we apply a canonical transformation, the canonical Poisson tensor $\mathbb{J}_2$ stays invariant

$$\begin{pmatrix} \dot{s}_n \\ \dot{r}_n \end{pmatrix} = \mathbb{J}_2 \nabla K(r,s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla K(r,s)$$

this means that the structure of the equations of motion has not changed:

$$\begin{aligned} \dot{s}_n &= \frac{\partial K(r,s)}{\partial r_n} = \phi'(r_n) \\ \dot{r}_n &= -\frac{\partial K(r,s)}{\partial s_n} = (s_{n+1} - 2s_n + s_{n-1}). \end{aligned} \tag{3.8}$$

Now, we push the system in the continuum limit and see how the solutions of (3.8) can be represented.

We define the parameter $h := 1/N$ and interpolate the coordinates s_n and r_n with two smooth and analytic functions

$$\begin{aligned} r_n(t) &= R(x,\tau) \\ s_n(t) &= \frac{S(x,\tau)}{h} \end{aligned} \tag{3.9}$$

where $x := hn$ and $\tau := ht$.

We study the behaviour of the functions $R(x,\tau)$ and $S(x,\tau)$ for h small ($h \ll 1$).

The equations of motion (3.8) become:

$$\begin{aligned} R_\tau(x,\tau) &= \frac{S(x+h,\tau) - 2S(x,\tau) + S(x-h,\tau)}{h^2} = \Delta_h S(x,\tau) \\ S_\tau(x,\tau) &= \phi'(R). \end{aligned} \tag{3.10}$$

In (3.10), we define the operator Δ_h as

$$\Delta_h f(x) := \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}. \tag{3.11}$$

We are interested in the limit of h small, so we expand in Taylor series both the functions $f(x+h)$ and $f(x-h)$ respect h

$$f(x \pm h) = f(x) \pm hf'(x) + \frac{h^2}{2}f''(x) \pm \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{IV}(x) \pm \dots.$$

Substituting the series in (3.11), the operator becomes:

$$\begin{aligned} \Delta_h f(x) &= \frac{1}{h^2}\left(h^2 f''(x) + \frac{h^4}{12}f^{IV}(x) + \dots\right) \\ &= \left(\partial_x^2 + \frac{h^2}{12}\partial_x^4 + \dots\right) f(x). \end{aligned} \tag{3.12}$$

Therefore, the Hamilton's equations (3.10) in the limit of h small can be rewritten in this way:

$$\begin{aligned} R_\tau &= \Delta_h S = \left(S_{xx} + \frac{h^2}{12}S_{4x} + \dots\right) \\ S_\tau &= \phi'(R). \end{aligned} \tag{3.13}$$

The equations of motion (3.13) are related to the Hamiltonian

$$K[S,R] = \int \left(\phi(R) - \frac{1}{2}S\Delta_h S\right) dx = \int \left(\phi(R) - \frac{S}{2}S_{xx} - \frac{h^2}{24}SS_{4x}\right) dx + \mathcal{O}(h^4). \tag{3.14}$$

We can simplify the system by applying another change in coordinates. Let us define a new function

$$\Xi(R,S) : (S,R) \to (V,R) \tag{3.15}$$

where

$$V(x,\tau) = S_x(x,\tau) \tag{3.16}$$

while the function $R(x,\tau)$ stays the same.

$\Xi(R,S)$ is not a canonical transformation, so the Poisson tensor is transformed by the Jacobian $D\Xi$:

$$\mathbb{J}_2^* = (D\Xi)\mathbb{J}_2(D\Xi)^T = \begin{pmatrix} \partial_x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -\partial_x & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \tag{3.17}$$

and the structure of the Hamilton's equations (3.13) change

$$\begin{aligned} V_\tau &= \partial_x \phi'(R) = \phi''(R)R_x \\ R_\tau &= \left(V_x + \frac{h^2}{12}V_{xxx} + \dots\right) = \partial_x L_h V \end{aligned} \tag{3.18}$$

where we define a new operator L_h

$$L_h := 1 + \frac{h^2}{12}\partial_x^2 + \dots. \tag{3.19}$$

The Hamiltonian related to the new equations of motions is:

$$\mathcal{K}[V,R] = \int \left(\frac{1}{2}VL_hV + \phi(R)\right) dx = \int \left(\frac{V^2}{2} + \phi(R) - \frac{h^2}{24}V_x^2\right) dx + \mathcal{O}(h^4). \tag{3.20}$$

In the limit of $h \to 0$, (3.18) reduce to a nonlinear wave equations

$$\begin{aligned} R_\tau &= V_x \\ V_\tau &= \partial_x \phi'(R). \end{aligned}$$

3.2 Extension of the Dubrovin's Theorem

After introducing the new coordinates, we can easily find a generalization of the result, given by the Theorem 3.0.1, for more general perturbation.

We add a potential $\psi_1(R)$ in the first order of perturbation, and see if there are some limitations on form that must have such that the new system admits an extension of the first integrals.

The Hamiltonian (3.20) can be rewritten as:

$$\begin{aligned} \mathcal{K}[V,R] &= \int \left[\left(\frac{V^2}{2} + \phi(R) \right) + h^2 \left(\psi_1(R) - \frac{V_x^2}{24} \right) \right] dx + \mathcal{O}(h^4) \\ &= \mathcal{K}_0[V,R] + h^2 \mathcal{K}_2[V,R] + \mathcal{O}(h^4). \end{aligned} \tag{3.21}$$

Be $J_0[V,R] = \int j_0(V,R)dx$ a first integral of $\mathcal{K}_0[V,R]$, we are looking for conditions of some functional:

$$J_2[V,R] = \int j_2(R,V,R_x,V_x,R_{xx},V_{xx})dx$$

where $j_2(R,V,R_x,V_x,R_{xx},V_{xx})$ is an homogeneous polynomial of grade 2 in the jets coordinates (R_x,V_x,R_{xx},V_{xx}), such that we can define the extended functional

$$J = J_0 + h^2 J_2 + \mathcal{O}(h^4) \tag{3.22}$$

as the perturbed first integral of $\mathcal{K}$ at the second order

$$\{J,\mathcal{K}\} = \mathcal{O}(h^4). \tag{3.23}$$

The Poisson bracket, in the coordinates (V,R), is defined as:

$$\begin{aligned} \{F,G\} &:= \int (\nabla_{L^2} F \mathbb{J}_2^* \nabla_{L^2} G) dx = \\ &= \int \Big[\frac{\delta F}{\delta R} \partial_x \frac{\delta G}{\delta V} + \frac{\delta F}{\delta V} \partial_x \frac{\delta G}{\delta R} \Big] \end{aligned} \tag{3.24}$$

where ∇_{L^2} is the L^2 gradient, i.e. a vector with components the functional derivatives respect R and V.

In the order to find the conditions on J_2, and eventually the system we have considered, we take advantage of a result of the following lemma

Lemma 3.2.1. *Be $F[u] = \int f(u)dx$ a functional. $F[u] = const \ \forall u$ iff*

$$E_u f = 0$$

where E_u is the Euler-Lagrange operator

$$E_u = \partial_u - \partial_x \partial_{u_x} + \partial_x^2 \partial_{u_{xx}} + \dots$$

Proof.

$$F[u] = const \quad \forall u \iff \delta F[u] = 0 \quad \forall u$$

we know that

$$\delta F[u] = \int E_u f(u) \delta u dx$$

this means that

$$\delta F[u] = 0 \quad \forall u \iff \int E_u f(u) \delta u dx = 0 \quad \forall u, \delta u$$

So we have that:

$$\delta F[u] = 0 \iff E_u f = 0$$

□

From this lemma, we obtain the useful corollary:
Corollary 3.2.2. *Two local functional $F = \int f dx$ and $G = \int g dx$ commute with respect to the Poisson bracket* (3.24) *iff*

$$E_V\left(\frac{\delta F}{\delta R}\partial_x\frac{\delta G}{\delta V}+\frac{\delta F}{\delta V}\partial_x\frac{\delta G}{\delta R}\right)=0$$
$$E_R\left(\frac{\delta F}{\delta R}\partial_x\frac{\delta G}{\delta V}+\frac{\delta F}{\delta V}\partial_x\frac{\delta G}{\delta R}\right)=0.$$

Let us proceed with the calculation of the Poisson bracket (3.23).

Applying the linearity of the Poisson bracket, we obtain

$$\{J;\mathcal{K}\}=\{J_0,\mathcal{K}_0\}+h^2(\{J_2,\mathcal{K}_0\}+\{J_0,\mathcal{K}_2\})=\mathcal{O}(h^4). \tag{3.25}$$

We know that $\{J_0,\mathcal{K}_0\}=0$, so we need to see if the term $\{J_2,\mathcal{K}_0\}+\{J_0,\mathcal{K}_2\}$ is null.

The density of $J_2[V,R]$, up to a total x-derivative, can be written in this form:

$$j_2=\frac{1}{2}(a(V,R)R_x^2+2b(V,R)R_xV_x+c(V,R)V_x^2)+p(V,R)R_x+q(V,R)V_x+d(V,R).$$

Thus, the remaining terms of the Poisson bracket (3.23) give us the following functional:

$$\begin{aligned}\{J,\mathcal{K}\}=&h^2\int dx\Big[\frac{j_{0_R}}{12}V_{xxx}+\frac{1}{2}(c_R-2b_v)V_x^3+\frac{\phi''(R)}{2}(a_V-2b_R)R_x^3+\\&-\frac{1}{2}(a_R+2c_R\phi''(R))R_x^2V_x-\frac{1}{2}(c_V\phi''(R)+2a_V)R_xV_x^2+\\&-aR_{xx}V_x-bV_{xx}V_x-b\phi''(R)R_{xx}R_x-c\phi''(R)V_{xx}R_x+\\&+(q_R-p_V)V_x^2+\phi''(R)(p_V-q_R)R_x^2\\&+d_RV_x+(d_v\phi''(R)+j_{0_V}\psi_1''(R))R_x\Big]+\mathcal{O}(h^4).\end{aligned} \tag{3.26}$$

Denoted I the integrand of (3.26), we apply the Corollary 3.2.2 so that our problem changes in the check of the equations:

$$E_VI=0,\quad E_RI=0. \tag{3.27}$$

In particular, we must check if each coefficients of the jets coordinates vanishes.

The terms of third grade $(R_{xxx},V_{xxx},R_{xx}V_x,\dots)$ give us the conditions on the coefficients a,b and c:

$$\begin{aligned}a&=\left(c+\frac{j_{0_{VV}}}{12}\right)\phi''(R);\quad c=-\frac{j_{0_{VVR}}}{6}\frac{\phi''(R)}{\phi'''(R)}-\frac{j_{0_{VV}}}{12};\\b&=-\frac{j_{0_{3V}}}{6}\frac{\phi''(R)^2}{\phi'''(R)};\quad b_V=c_R+\frac{j_{0_{RVV}}}{12}=\left(\frac{a}{\phi''(R)}\right)_R;\\b_R&=\left(c_V-\frac{j_{0_{3V}}}{12}\right)\phi''(R);\quad c_{VV}\phi''(R)=c_{RR}+\frac{j_{0_{2V2R}}}{6}.\end{aligned} \tag{3.28}$$

Let us consider the fourth equation of (3.28). We obtain explicitly $a(V,R)$ combining the first and the second equations of (3.28)

$$a=-\frac{j_{0_{VVR}}}{6}\frac{\phi''(R)^2}{\phi'''(R)}\Rightarrow\frac{a}{\phi''(R)}=-\frac{j_{0_{VVR}}}{6}\frac{\phi''(R)}{\phi'''(R)}$$

and then we substitute this result in the fourth equation

$$b_V = \left(\frac{a}{\phi''(R)}\right)_R$$
$$\frac{j_{0_{4V}}}{6}\frac{\phi''(R)^2}{\phi'''(R)} = \frac{j_{0_{2V2R}}}{6}\frac{\phi''(R)}{\phi'''(R)} + \frac{j_{0_{VVR}}}{6}\left[\frac{(\phi'''(R))^2 - \phi''(R)\phi^{IV}(R)}{(\phi'''(R))^2}\right].$$

We know that j_0 is the density of the first integral of $\mathcal{K}_0$, so it must satisfy the condition (proved in the Appendix A)

$$j_{0_{RR}} = \phi''(R) j_{0_{VV}}. \tag{3.29}$$

Therefore, we can rewrite the lhs of the previews equation as

$$\frac{j_{0_{4V}}}{6}\frac{\phi''(R)^2}{\phi'''(R)} = \frac{\partial_V^2(j_{0_{VV}}\phi''(R))}{6}\frac{\phi''(R)}{\phi'''(R)} = \frac{j_{0_{2V2R}}}{6}\frac{\phi''(R)}{\phi'''(R)}$$

and the fourth equation of (3.28) reduces to

$$\frac{j_{0_{VVR}}}{6}\left[\frac{(\phi'''(R))^2 - \phi''(R)\phi^{IV}(R)}{(\phi'''(R))^2}\right] = 0.$$

In the end, we obtain a condition on the potential $\phi(R)$

$$\frac{(\phi'''(R))^2 - \phi''(R)\phi^{IV}(R)}{(\phi'''(R))^2} = 0$$

and this is valid only if the numerator is null

$$(\phi'''(R))^2 = \phi''(R)\phi^{IV}(R). \tag{3.30}$$

Now, we need to solve the ODE (3.30): dividing the equation by $\phi'''(R)\phi''(R)$ yields

$$\frac{\phi'''(R)}{\phi''(R)} = \frac{\phi^{IV}(R)}{\phi'''(R)}. \tag{3.31}$$

We can recognise this equation as the condition of equivalence between the logarithmic derivative of $\phi''(R)$ with the logarithmic derivative of $\phi'''(R)$:

$$\partial_R[\ln(\phi'''(R))] = \partial_R[\ln(\phi''(R))]$$
$$\Downarrow$$
$$\phi'''(R) = \tilde{c}\phi''(R)$$

and we obtain an easier ODE to solve. Considering also the condition $\phi'''(R) \neq 0 \ \forall R$, the solution of this equation is:

$$\phi(R) = ke^{\tilde{c}R} + \tilde{a}R + \tilde{b} \tag{3.32}$$

for some constants $\tilde{a}, \tilde{b}, \tilde{c}, k$.

This is exactly the same potential (3.3) that we found in the Theorem 3.0.1, so it is proved also for the new coordinates.

It is easy to show that the last two equations of (3.28) are identities. We start with

$$c_{VV}\phi''(R) = c_{RR} + \frac{j_{0_{2V2R}}}{6}. \tag{3.33}$$

Because we know the form of $c(R,V)$, we calculate the second derivative respect R:

$$
\begin{aligned}
c_{RR} &= \partial_R^2\left(-\frac{j_{0_{VVR}}}{6}\frac{\phi''(R)}{\phi'''(R)}-\frac{j_{0_{VV}}}{12}\right)= \\
&= \partial_R\left[-\frac{j_{0_{2V2R}}}{6}\frac{\phi''(R)}{\phi'''(R)}-\frac{j_{0_{VVR}}}{6}\underbrace{\left(\frac{(\phi'''(R))^2-\phi''(R)\phi^{IV}(R)}{(\phi'''(R))^2}\right)}_{=0}\right]-\frac{j_{0_{2V2R}}}{12}= \\
&= \partial_R\left(-\frac{j_{0_{4V}}}{6}\frac{\phi''(R)^2}{\phi'''(R)}\right)-\frac{j_{0_{4V}}}{12}\phi''(R) = \text{for the condition (3.29)} \\
&= -\frac{j_{0_{4VR}}}{6}\frac{\phi''(R)^2}{\phi'''(R)}-\frac{j_{0_{4V}}}{6}\phi''(R)\left(1+\underbrace{\frac{(\phi'''(R))^2-\phi''(R)\phi^{IV}(R)}{(\phi'''(R))^2}}_{=0}\right)-\frac{j_{0_{4V}}}{12}\phi''(R)= \\
&= -\frac{j_{0_{4VR}}}{6}\frac{\phi''(R)^2}{\phi'''(R)}-\frac{j_{0_{4V}}}{12}\phi''(R)-\frac{j_{0_{2V2R}}}{6}= \\
&= \partial_V^2\left(-\frac{j_{0_{VVR}}}{6}\frac{\phi''(R)}{\phi'''(R)}-\frac{j_{0_{VV}}}{12}\right)\phi''(R)-\frac{j_{0_{2V2R}}}{6}= \\
&= c_{VV}\phi''(R)-\frac{j_{0_{2V2R}}}{6}.
\end{aligned}
$$

Now let us see the last equation

$$
b_R=\left(c_V-\frac{j_{0_{3V}}}{12}\right)\phi''(R). \tag{3.34}
$$

We know the form of $b(R,V)$, so we calculate explicitly b_R:

$$
\begin{aligned}
b_R &= \partial_R\left(-\frac{j_{0_{3V}}}{6}\frac{\phi''(R)^2}{\phi'''(R)}\right)= \\
&= -\frac{j_{0_{3VR}}}{6}\frac{\phi''(R)^2}{\phi'''(R)}-\frac{j_{0_{3V}}}{6}\phi''(R)\left(1+\underbrace{\frac{(\phi'''(R))^2-\phi''(R)\phi^{IV}(R)}{(\phi'''(R))^2}}_{=0}\right)= \\
&= \left(-\frac{j_{0_{3VR}}}{6}\frac{\phi''(R)}{\phi'''(R)}-\frac{j_{0_{3V}}}{6}\right)\phi''(R)=\left(c_V-\frac{j_{0_{3V}}}{12}\right)\phi''(R).
\end{aligned}
$$

Let us continue the check of the coefficients and pass to the quadratic and linear terms related to the jets coordinates.

From the quadratic terms $(R_{xx}, V_x^2, \dots)$ in the equations (3.27), we obtain the only condition

$$
p_V = q_R. \tag{3.35}
$$

This means that $p(V,R)$ and $q(V,R)$ must be the component of the gradient of a function ν

$$
p(V,R)=\nu_R(V,R) \quad q(V,R)=\nu_V(V,R)
$$

and, if we substitute these formulas in j_2, we find that:

$$
j_2=\dots+\nu_R(V,R)R_x+\nu_V(V,R)V_x+d(V,R)=\dots+(\partial_x\nu(V,R))+d(V,R).
$$

The linear part respect V_x and R_x of j_2 is a total derivative of $\nu(V,R)$, so we can ignore it because the integral of this part vanishes.

Now we need to see the linear terms (R_x, V_x). Always from the equations (3.27), we find that $d(R,V)$ must satisfy the following PDE:

$$d_{RR} = \phi''(R) d_{VV} + \psi_1''(R) j_{0_{VV}}. \tag{3.36}$$

This is an equation with two unknown functions, $\psi_1(R)$ and $d(V,R)$. We need to see case by case for which form of $\psi_1(R)$ exist a function $d(V,R)$ such that this equation is satisfied.

In the trivial case $\psi_1''(R) = 0$, i.e $\psi_1(R) = mR + q$ with m, q constants, the eq (3.36) becomes (3.29), thus $d(V,R)$ must be a density of the first integral of the unperturbed system.

3.3 Case of Harmonic Oscillator

We saw that, in the hypothesis $\phi'''(R) \neq 0\ \forall R$, it has a solution a family of exponential potential. In particular, a subfamily of this potentials are the Toda potentials.

If we weaken also this hypothesis (considering also potential with $\phi'''(R) = 0\ \forall R$), we obtain a new trivial solution for the equation (3.30): the harmonic oscillator

$$\phi(R) = \omega \frac{R^2}{2}. \tag{3.37}$$

But, if we substitute (3.37) in the results (3.28), the denominators become null and the functions $a(R,V), b(R,V)$ and $c(R,V)$ blows up to infinity. So, we need to take some steps back and restart from the Poisson brackets (3.26).

$$\begin{aligned}
\{J, \mathcal{K}\} = &\underbrace{\{J_0, \mathcal{K}_0\}}_{=0} + h^2(\{J_2, \mathcal{K}_0\} + \{J_0, \mathcal{K}_2\}) = \\
&= h^2 \int \left[\left(\frac{\delta J_0}{\delta R} \partial_x \frac{\delta \mathcal{K}_2}{\delta V} + \frac{\delta J_0}{\delta V} \partial_x \frac{\delta \mathcal{K}_2}{\delta R} \right) + \left(\frac{\delta J_2}{\delta R} \partial_x \frac{\delta \mathcal{K}_0}{\delta V} + \frac{\delta J_2}{\delta V} \partial_x \frac{\delta \mathcal{K}_0}{\delta R} \right) \right] dx = \\
&\text{substituting at } \phi(R) \text{ with (3.37), we find} \\
=& h^2 \int dx \Big\{ \frac{j_{0_R}}{12} V_{xxx} - b V_{xx} V_x - \omega b R_{xx} R_x - a R_{xx} V_x - \omega c V_{xx} R_x + \\
&- \frac{(\omega c_R - 2a_V)}{2} V_x^2 R_x - \frac{(a_R + 2\omega c_R)}{2} R_x^2 V_x + \\
&+ \frac{(c_R - 2b_V)}{2} V_x^3 + \frac{\omega}{2}(a_V - 2b_R) R_x^3 + (q_R - p_V) V_x^2 + \omega(p_V - q_R) R_x^2 \\
&+ d_R V_x + (d_V \omega + j_{0_V} \psi_1''(R)) R_x \Big\} + \mathcal{O}(h^4)
\end{aligned}$$

and, as we noticed before, we apply the Corollary 3.2.2 and check the equations

$$E_R I = 0, \quad E_V I = 0. \tag{3.38}$$

From the linear component in the jets coordinates of the equations (3.38) we find

$$d_{RR} = \omega d_{VV} + \psi_1''(R) j_{0_{VV}} \tag{3.39}$$

while the quadratic components give us the same result we have found before.

We must now focus on the other components. From the components $R_{xxx}, V_{xx}V_x, R_{xx}V_x, R_{xx}R_x$ and $V_x^2 R_x$ of $E_V I = 0$ we find respectively the following conditions

$$\begin{gathered}
a = \omega c + \frac{j_{0_{RR}}}{12}; \quad b_V = c_R + \frac{j_{0_{RVV}}}{12} = \frac{a_R}{\omega}; \\
b_R = \omega c_V - \frac{j_{0_{RRV}}}{12}; \quad \omega c_{VV} - c_{RR} = \frac{j_{0_{2R2V}}}{6}; \\
j_{0_{3R}} = 0 \text{ (or } j_{0_{RVV}} = 0 \text{ from the (3.29))}
\end{gathered}$$

while from the components $V_{xx}V_x, R_{xx}V_x, R_{xx}R_x$ and $R_x^2V_x$ of $E_RI = 0$ we find respectively

$$a_V = b_R; \quad a_V = \omega c_V \implies j_{0_{RRV}} = 0; \text{ (from the result of the component } R_{xxx} \text{ of } E_V I = 0)$$
$$b_V = c_R - \frac{j_{0_{RVV}}}{12} = c_R \text{ (from the result of the component } R_{xx}R_x);$$
$$c_{RR} - \omega c_{VV} = \frac{j_{0_{2R2V}}}{6} = 0 \text{ (from the result of the component } R_{xx}R_x).$$

All the other components of $E_RI = 0$ and $E_VI = 0$ are identically null.

We found some conditions on the unperturbed first integral and we try to solve them. Let us start from $j_{0_{3R}} = 0$:

$$j_{0_{3R}} = 0 \implies j_0(R, V) = \frac{\varphi(V)}{2}R^2 + \chi(V)R + \varrho(V) \tag{3.40}$$

and we find the form of the functions $\varphi(V), \chi(V)$ and $\varrho(V)$ considering also the other conditions (also (3.29))

$$j_{0_{RRV}} = 0 \implies \varphi'(V) = 0 \implies \varphi(V) = cost. = \gamma;$$
$$j_{0_{RVV}} = 0 \implies \chi''(V) = 0 \implies \chi(V) = \alpha V + \beta :$$
$$j_{0_{RR}} = \omega j_{0_{VV}} \implies \gamma = \omega\varrho''(V) \implies \varrho(V) = \frac{\gamma}{2\omega}V^2 + \delta V + \lambda.$$

Thus, combining all the results, the unperturbed first integral must have this form

$$J_0[R, V] = \int j_0(R, V)dx = \int \left\{ \frac{\gamma}{\omega}\left(\frac{V^2}{2} + \frac{\omega}{2}R^2\right) + \alpha VR + \beta R + \delta V + \lambda \right\} dx \tag{3.41}$$

with $\gamma, \alpha, \beta, \delta$ and λ arbitrary constants.

We recognise some terms in $J_0[R, V]$, in particular:

1. the first term is the unperturbed Hamiltonian, which is by definition a first integral;
2. the second term $\int VRdx = \int S_xRdx$ is connected with the translation symmetry[2] of the unperturbed system, so it is a first integral;
3. the third term $\int Rdx$ is the total "momentum" of the systems, so it's a first integral too;
4. the fourth term $\int Vdx$ is connected to the length of the system, and for the periodic boundary condition this quantity must stay constant.

So $J_0[R, V]$ is a linear combination of first integrals of the unperturbed system. This means that the unperturbed first integrals must depend on the choise of the potential, and the constrains on the $j_0(R, V)$, which we found before, are simply identities.

In the end, we find that it is possible to extend a first integral at the first order of h^2 also in the case of $\phi(R) = \omega R^2/2$, and the functions $a(R, V), b(R, V)$ and $c(R, V)$ are given by this equations:

$$a = \omega c + \frac{j_{0_{RR}}}{12}, \quad b_V = c_R, \quad b_R = \omega c_V \tag{3.42}$$

while the equation (3.39) becomes

$$d_{RR} = \omega d_{VV} + \gamma\psi_1''(R). \tag{3.43}$$

[2]Consider the transformation $f(S, R) = (S(x + s), R(x + s))$, we want to find the Hamiltonian that generates this transformation, so we calculate

$$\left.\frac{\partial f}{\partial s}\right|_{s=0} = (S_x, R_x) = \left(\frac{\delta H_f}{\delta R}\right).$$

Integrating this equations, we find

$$H_f[S, R] = \int S_xRdx$$

Chapter 4

Application for the FPU Problem

In this chapter, we want to apply the techniques and mathematical tools of the previous chapters for the principal object of our thesis: The actual FPU problem.

4.1 Continuum limit of FPU

Let us consider the Hamiltonian (3.1) with potential (2.7). We saw that, if we apply the canonical transformation (3.6), the Hamiltonian becomes (3.7)

$$K_{FPU}(s,r) = \sum_{n\in\mathbb{Z}_N} \left[\frac{(s_n - s_{n-1})^2}{2} + \phi(r_n) \right] \tag{4.1}$$

and the equations of motion (3.2) become the equations (3.8)

$$\begin{aligned} \dot{s}_n &= \frac{\partial K(s,r)}{\partial r_n} = \phi'(r_n) \\ \dot{r}_n &= -\frac{\partial K(s,r)}{\partial s_n} = (s_{n+1} - 2s_n + s_{n-1}). \end{aligned} \tag{4.2}$$

We know that all the FPU systems are tangent at the first order to the Toda lattice ($\beta_{Toda} = \frac{2}{3}\alpha^2$), therefore we can write (3.7) as

$$K_{FPU}(s,r) = K_{Toda}(s,r) + \sum_{n\in\mathbb{Z}_N} \left(\Delta\beta \frac{r_n^4}{4} + \Delta\gamma \frac{r_n^5}{5} + \dots \right) \tag{4.3}$$

with

$$\Delta\beta = (\beta - \beta_{Toda}), \quad \Delta\gamma = (\gamma - \gamma_{Toda}), \dots$$

and

$$K_{Toda}(s,r) = \sum_{n\in\mathbb{Z}_N} \left[\frac{(s_{n+1} - s_n)^2}{2} + \frac{e^{2\alpha r_n} - 2\alpha r_n - 1}{4\alpha^2} \right]. \tag{4.4}$$

We now consider the continuum limit ($N \to \infty$) of the FPU system, and look for smooth and analytic solutions. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the real unit torus. We define

$$h := \frac{1}{N} \text{ (we have already defined this pertibative coefficient)}, \ \varepsilon := \frac{E}{N} \tag{4.5}$$

where E is the energy of the system (ε is called *specific energy*).

We repeat the same strategy we use before, but in this case using another perturbative parameter. So, we suppose that a pair of interpolating smooth and analytic functions

$$(S,R) : \mathbb{R} \times \mathbb{T} \to \mathbb{R}^2 : (\tau, x) \to (S(\tau,x), R(\tau,x))$$

exist, so the solution of (3.8) at time $t \in \mathbb{R}$ is given by

$$\begin{cases} s_n(t) = (\sqrt{\varepsilon}/h) S(\tau,x)|_{\tau=ht;x=hn} \\ r_n(t) = \sqrt{\varepsilon} R(\tau,x)|_{\tau=ht;x=hn} \end{cases} \quad (n \in \mathbb{Z}_N) \ . \tag{4.6}$$

They are similar to the ones (3.9) we used in the previous chapter. The only difference here is that they are re-scaled by a factor $\sqrt{\varepsilon}$.

Inserting the functions (4.6) into (3.8), and removing the restriction on the continuum space variable x, we obtain this system of partial differential equations

$$\begin{cases} S_\tau = \dfrac{1}{\sqrt{\varepsilon}} \phi'(\sqrt{\varepsilon} R) \\ R_\tau = \Delta_h S \end{cases} \tag{4.7}$$

defined on the torus $\mathbb{T}$. The operator Δ_h is the same operator (3.11) that we have defined before.

This equations of motion are related to the following Hamiltonian

$$K_{FPU}[S,R] = \int dx \left[\frac{1}{\varepsilon} \phi(\sqrt{\varepsilon} R) - \frac{1}{2} S \Delta_h S \right] . \tag{4.8}$$

It is more convenient to work with the coordinates (V,R), so we apply the change of coordinates (3.16). In this way the Hamiltonian becomes

$$\mathcal{K}_{FPU}[V,R] = \int dx \left(\frac{1}{2} V L_h V + \tilde{\phi}(R) \right) \tag{4.9}$$

where L_h is the same operator (3.19) defined before and the potential $\tilde{\phi}(R)$ is:

$$\tilde{\phi}(R) := \frac{1}{\varepsilon} \phi(\sqrt{\varepsilon} R) = \frac{R^2}{2} + \sqrt{\varepsilon} \alpha \frac{R^3}{3} + \varepsilon \beta \frac{R^4}{4} + \dots . \tag{4.10}$$

The next step is to see the potential (4.10) as a perturbation of the Toda Potential. So we want to rewrite (4.10) as

$$\begin{aligned} \tilde{\phi}(R) &= \frac{1}{\varepsilon} \phi_{Toda}(\sqrt{\varepsilon} R) + \varepsilon \Delta\beta \frac{R^4}{4} + \varepsilon^{3/2} \Delta\gamma \frac{R^5}{5} + \dots = \\ &= \frac{1}{\varepsilon} \phi_{Toda}(\sqrt{\varepsilon} R) + \psi_1(R) + \psi_2(R) + \dots \end{aligned} \tag{4.11}$$

and the Hamiltonian (4.9) can be rewritten as

$$\begin{aligned} \mathcal{K}_{FPU}[V,R] = \int dx \Big[&\left(\frac{V^2}{2} + \frac{1}{\varepsilon} \phi_{Toda}(\sqrt{\varepsilon} R) \right) + \left(\psi_1(R) - \frac{h^2}{24} V_x^2 \right) \\ &+ \left(\psi_2(R) + \frac{h^4}{720} V_{xx}^2 \right) + \dots \Big]. \end{aligned} \tag{4.12}$$

However, we need to connect the potentials $\psi_1(R), \psi_2(R), \dots$ to the perturbed pieces

$$\varepsilon \Delta\beta \frac{R^4}{4}, \quad \varepsilon^{3/2} \Delta\gamma \frac{R^5}{5}, \dots \tag{4.13}$$

of the FPU potential (4.11). A way to solve this problem is to compare the grade of $\psi_1(R), \psi_2(R)$ and so on, in relation to the perturbed terms of (4.12), and see what part of the FPU potential (4.11) they correspond.

We use a Cauchy estimate of the functions. For fixed τ^*, We define a strip in the complex plane $\Omega_\sigma := \{x \in \mathbb{C} : |Im(x)| \leq \sigma\}$ and

$$v := \max_{\Omega_\sigma}\{|V(x,\tau^*)|, |R(x,\tau^*)|\}. \tag{4.14}$$

Using the Cauchy integral formula, we can estimate the functions V and R, and their derivative, as

$$V \leq v, \quad V_x \leq \frac{v}{\sigma}, \quad V_{xx} \leq \frac{v}{\sigma^2}, \ldots, \quad V^{(n)} \leq \frac{v}{\sigma^n}.$$

To be consistent, each part of the perturbation terms at a fixed order must have the same estimate, and this means that the perturbed potentials must be

$$\psi_1(R) \leq \frac{h^2}{\sigma^2}v^2, \quad \psi_2(R) \leq \frac{h^4}{\sigma^4}v^2, \quad \psi_3(R) \leq \frac{h^6}{\sigma^6}v^2, \ldots$$

and in general:

$$\psi_n(R) \leq \frac{h^{2n}}{\sigma^{2n}}v^2. \tag{4.15}$$

Now, we compare these estimates with the perturbation terms of (4.11). We consider the first hypotesis

$$\begin{cases} \psi_1(R) = \varepsilon \Delta\beta \dfrac{R^4}{4} \\ \psi_2(R) = \varepsilon^{3/2} \Delta\gamma \dfrac{R^5}{5} \\ \quad\vdots \end{cases} \tag{4.16}$$

and, using the estimates obtained from the formula (4.15), we find

$$\begin{aligned} \varepsilon\Delta\beta\frac{R^4}{4} &\leq \frac{h^2}{\sigma^2}v^2 \Rightarrow \sigma \leq \frac{h}{\sqrt{\varepsilon}} \\ \varepsilon^{3/2}\Delta\gamma\frac{R^5}{5} &\leq \frac{h^4}{\sigma^4}v^2 \Rightarrow \sigma \leq \frac{h}{\varepsilon^{3/8}} \\ &\vdots \end{aligned}$$

Basing on these results, we can see how this hypothesis is impossible for two reasons:

1. the estimate of σ at the first order is different from the estimate at the second order;
2. we find that $\sigma \leq \frac{h}{\sqrt{\varepsilon}}$ at the first order, but we know that, from the estimates of the non linear terms of the KdV equation [19], σ must be

$$\sigma \leq \frac{h}{\varepsilon^{1/4}}. \tag{4.17}$$

Thus, we need to consider another hypothesis on the relations between the potentials $\psi_n(R)$ and the perturbed parts of the potental (4.11). The simplest alternative to the hypothesis (4.16) is to start the perturbation at the second order, so that the potentials $\psi_n(R)$ are

$$\begin{cases} \psi_1(R) = 0 \\ \psi_2(R) = \varepsilon \Delta\beta \dfrac{R^4}{4} \\ \psi_3(R) = \varepsilon^{3/2} \Delta\gamma \dfrac{R^5}{5} \\ \quad\vdots \end{cases} \tag{4.18}$$

We repeat the same procedure for this hypothesis and we see that

$$\varepsilon\Delta\beta\frac{R^4}{4} \leq \frac{h^4}{\sigma^4}v^2 \Rightarrow \sigma \leq \frac{h}{\varepsilon^{1/4}}$$
$$\varepsilon^{3/2}\Delta\gamma\frac{R^5}{5} \leq \frac{h^6}{\sigma^6}v^2 \Rightarrow \sigma \leq \frac{h}{\varepsilon^{1/4}}.$$

In this case, the estimate of σ, both for the first and the second order, is equal to the estimate (4.17) that we have see before. But we must verify that the estimate of σ is the same for each order n. Let us consider the general form of the hypothesis (4.18)

$$\psi_n(R) = \begin{cases} 0 & \text{if } n = 1 \\ \varepsilon^{n/2}\Delta g_n \dfrac{R^{n+2}}{n+2} & \text{if } n \geq 2 \end{cases} \tag{4.19}$$

where $\Delta g_n = (g_n - g_{n_{Toda}})$.

From the the formula (4.15), we know the generic estimates of $\psi_n(R)$ and we find

$$\psi_n(R) = \varepsilon^{n/2}\Delta g_n \frac{R^{n+2}}{n+2} \leq \frac{h^{2n}}{\sigma^{2n}}v^2$$
$$\Downarrow$$
$$\sigma \leq \frac{h}{\varepsilon^{1/4}}.$$

Thus, the estimate of σ stay constant for each order of perturbation. This means that the hypothesis (4.18) is correct and we can rewrite the FPU Hamiltonian (4.12) as

$$\begin{aligned}\mathcal{K}_{FPU}[V,R] = \int dx \Big[\Big(\frac{V^2}{2} + \frac{1}{\varepsilon}\phi_{Toda}(\sqrt{\varepsilon}R) \Big) - \frac{h^2}{24}V_x^2 \\ + \Big(\varepsilon\Delta\beta\frac{R^4}{4} + \frac{h^4}{720}V_{xx}^2 \Big) + \dots \Big].\end{aligned} \tag{4.20}$$

4.2 Extensions of first integrals

We now apply the Dubrovin's techniques to find perturbed first integrals of the FPU chain from extensions of the first integrals of the Toda Lattice. We start with the second order. We truncate the Hamiltonian (4.20) up to the second order

$$\mathcal{K}_{FPU}[V,R] = \int dx \left[\left(\frac{V^2}{2} + \frac{1}{\varepsilon}\phi_{Toda}(\sqrt{\varepsilon}R) \right) - \frac{h^2}{24}V_x^2 \right] + \mathcal{O}(h^4) \tag{4.21}$$

which is the same of (3.21) with $\psi_1(R) = 0$, so we already know the solutions for the density $j_2(V,R,V_x,R_x)$

$$\begin{aligned} j_2(V,R,V_x,R_x) &= \frac{a}{2}R_x^2 + bR_xV_x + \frac{c}{2}V_x^2 = \\ &= -\frac{j_{0_{VVR}}}{12}e^R R_x^2 - \frac{j_{0_{3V}}}{6}e^R R_x V_x - \frac{j_{0_{VVR}}}{12}V_x^2 - \frac{j_{0_{VV}}}{24}V_x^2 \end{aligned} \tag{4.22}$$

where we considered $\frac{1}{\varepsilon}\phi_{Toda}(\sqrt{\varepsilon}R) = e^R - R - 1$[1].

[1] In fact, if we calculate $\frac{1}{\varepsilon}\phi_{Toda}(\sqrt{\varepsilon}R)$, we find

$$\frac{1}{\varepsilon}\phi_{Toda}(\sqrt{\varepsilon}R) = \frac{e^{2\alpha\sqrt{\varepsilon}R} - 2\alpha\sqrt{\varepsilon}R - 1}{4\alpha^2\varepsilon}$$

and choosing $2\alpha\sqrt{\varepsilon} = 1$, we obtain $\frac{1}{\varepsilon}\phi_{Toda}(\sqrt{\varepsilon}R) = e^R - R - 1$

We now proceed with the fourth order

$$\begin{aligned}\mathcal{K}_{FPU}[V,R] = \int dx \Big[\Big(\frac{V^2}{2} + \frac{1}{\varepsilon}\phi_{Toda}(\sqrt{\varepsilon}R)\Big) - \frac{h^2}{24}V_x^2 \\ + \Big(\varepsilon\Delta\beta\frac{R^4}{4} + \frac{h^4}{720}V_{xx}^2\Big)\Big] + \mathcal{O}(h^6).\end{aligned} \tag{4.23}$$

We want to find the coefficients of the density $j_4(V,R,V_x,R_x,V_{xx},R_{xx})$, which is a polynomial function on the jet coordinates of order fourth and has the following form:

$$\begin{aligned}j_4(V,R,V_x,R_x,V_{xx},R_{xx}) = {}& \tilde{\alpha}R_{xx}^2 + \tilde{\beta}R_{xx}V_{xx} + \tilde{\gamma}V_x^2 + \tilde{\delta}R_{xx}V_x^2 + \tilde{\epsilon}V_{xx}R_x^2 + \\ &+ \tilde{\mu}R_x^4 + \tilde{\nu}R_x^3V_x + \tilde{\rho}R_x^2V_x^2 + \tilde{\lambda}R_xV_x^3 + \tilde{\omega}V_x^4 + \\ &+ \frac{\eta}{2}R_x^2 + \xi R_xV_x + \frac{\zeta}{2}V_x^2 + \sigma;\end{aligned} \tag{4.24}$$

where $\tilde{\alpha},\tilde{\beta},\tilde{\gamma},\tilde{\delta},\tilde{\epsilon},\tilde{\mu},\tilde{\nu},\tilde{\rho},\tilde{\lambda},\tilde{\omega},\eta,\xi,\zeta$ and σ are functions of R and V, so that the extended first integral

$$\begin{aligned}J[V,R] &= J_0[V,R] + h^2J_2[V,R] + h^4J_4[V,R] = \\ &= \int dx\Big[j_0(V,R) + h^2j_2(V,R,V_x,R_x) + h^4j_4(V,R,V_x,R_x,V_{xx},R_{xx})\Big].\end{aligned} \tag{4.25}$$

commutes with the Hamiltonian (4.20)

$$\{J,\mathcal{K}_{FPU}\} = \mathcal{O}(h^6).$$

We calculate explicitly, thanks to the software *Mathematica*, the Poisson bracket of $J[V,R]$ with (4.20)

$$\begin{aligned}\{J,\mathcal{K}_{FPU}\} = {}& h^4[\{J_0,H_4\} + \{J_2,H_2\} + \{J_4,H_0\}] = \\ = {}& h^4\int dx\Big\{\frac{j_{0R}}{360}V_{5x} + (\tilde{\beta}V_x + 2\tilde{\gamma}R_x\phi''_{Toda})V_{4x} + (2\tilde{\alpha}V_x + \tilde{\beta}\phi''_{Toda}R_x)R_{4x} - \frac{a}{12}V_{xxx}R_{xx} + \\ &- \frac{b}{12}V_{xxx}V_{xx} + \Big(4\tilde{\gamma}_R\phi''_{Toda} - \frac{a_R}{24}\Big)V_{xxx}R_x^2 + \Big(4\tilde{\gamma}_V\phi''_{Toda} + 2\tilde{\beta}_R - 2\tilde{\epsilon} - \frac{a_V}{12}\Big)V_{xxx}R_xV_x + \\ &+ \Big(2\tilde{\delta} + 2\tilde{\beta}_V + \frac{c_R}{24} - \frac{b_V}{12}\Big)V_{xxx}V_x^2 + 4\tilde{\alpha}_VR_{xxx}V_x^2 + (2\tilde{\beta}_V\phi''_{Toda} - 2\tilde{\delta}\phi''_{Toda} + 4\tilde{\alpha}_R)R_{xxx}R_xV_x + \\ &+ (2\tilde{\epsilon}\phi''_{Toda} + 2\tilde{\beta}_R\phi''_{Toda})R_{xxx}R_x^2 + (2\tilde{\alpha}_V + 2\tilde{\beta}_R - 2\tilde{\epsilon})V_{xx}R_{xx}V_x + \\ &+ \phi''_{Toda}(2\tilde{\beta}_V + 2\tilde{\gamma}_R - 2\tilde{\delta})V_{xx}R_{xx}R_x + (5\tilde{\delta}_V + \tilde{\beta}_{VV} - 3\tilde{\nu})V_{xx}V_x^3 + \\ &+ (4\tilde{\gamma}_{RV}\phi''_{Toda} + \tilde{\beta}_{RR} - \tilde{\epsilon}_R - 3\tilde{\lambda} - 6\tilde{\nu}\phi''_{Toda})V_{xx}R_x^2V_x + \\ &+ (2\tilde{\gamma}_{VV}\phi''_{Toda} + 4\tilde{\delta}_R + 2\tilde{\beta}_{RV} - 4\tilde{\rho} - 12\tilde{\omega}\phi''_{Toda} - 2\tilde{\epsilon}_V)V_{xx}R_x^2V_x + \\ &+ 2\phi''_{Toda}(\tilde{\gamma}_{RR} + \tilde{\epsilon}_V - \tilde{\rho})V_{xx}R_x^3 + \phi''_{Toda}(\tilde{\beta}_{RR} + 5\tilde{\epsilon}_R - 3\tilde{\lambda})R_{xx}R_x^3 + \\ &+ (2\tilde{\delta}_R + 2\tilde{\alpha}_{VV} - 2\tilde{\rho})R_{xx}V_x^3 + (2\tilde{\delta} + \tilde{\gamma}_R + \tilde{\beta}_V)V_{xx}^2V_x + 3\tilde{\gamma}_V\phi''_{Toda}V_{xx}^2R_x + \\ &+ \phi''_{Toda}(2\tilde{\epsilon} + \tilde{\alpha}_V + \tilde{\beta}_R)R_{xx}^2R_x + 3\tilde{\alpha}_RR_{xx}^2V_x + \\ &+ (\tilde{\beta}_{VV}\phi''_{Toda} - \tilde{\delta}_V\phi''_{Toda} - 3\tilde{\nu} - 6\tilde{\lambda} + 4\tilde{\alpha}_{RV})R_{xx}R_xV_x^2 + \\ &+ (4\tilde{\epsilon}_V\phi''_{Toda} + 2\tilde{\beta}_{RV}\phi''_{Toda} - 2\tilde{\delta}_R\phi''_{Toda} - 4\tilde{\rho}\phi''_{Toda} + 4\tilde{\alpha}_{RR} - 12\tilde{\mu})R_{xx}R_x^2V_x + \\ &+ (\tilde{\delta}_{VV} - \tilde{\nu}_V + \tilde{\omega}_R)V_x^5 + (2\tilde{\delta}_{RV} - 2\tilde{\rho}_V - 3\tilde{\omega}_V\phi''_{Toda})V_x^4R_x +\end{aligned}$$

$$
\begin{aligned}
&+(\tilde{\delta}_{RR}-3\tilde{\lambda}_V-2\tilde{\nu}_V\phi''_{Toda}-\tilde{\rho}_R-4\tilde{\omega}_R\phi''_{Toda})V_x^3R_x^2+\\
&+(\tilde{\epsilon}_{VV}\phi''_{Toda}-3\tilde{\nu}_R\phi''_{Toda}-\tilde{\rho}_R\phi''_{Toda}-2\tilde{\lambda}_R-4\tilde{\mu}_V)V_x^2R_x^3+\\
&+(2\tilde{\epsilon}_{RV}\phi''_{Toda}-2\tilde{\rho}_R\phi''_{Toda}-3\tilde{\mu}_R)V_xR_x^4+(\tilde{\mu}_V\phi''_{Toda}-\tilde{\lambda}_R\phi''_{Toda}+\tilde{\epsilon}_{RR}\phi''_{Toda})R_x^5+\\
&+\frac{\phi''_{Toda}}{2}(\eta_V-2\xi_R)R_x^3-\frac{1}{2}(\eta_R+2\zeta_R\phi''_{Toda})R_x^2V_x-\frac{1}{2}(\zeta_V\phi''_{Toda}+2\eta_V)R_xV_x^2+\\
&+\frac{1}{2}(\zeta_R-2\xi_V)V_x^3-\xi\phi''_{Toda}R_{xx}R_x-\xi V_{xx}V_x-\zeta\phi''_{Toda}V_{xx}R_x+\\
&\left.-\eta R_{xx}V_x+\sigma_RV_x+(\sigma_V\phi''_{Toda}+3j_{0V}\Delta\beta R^2)R_x\right\}+\mathcal{O}(h^6). \qquad (4.26)
\end{aligned}
$$

To see if (4.26) is null at the fourth order, we apply the Corollary 3.2.2 and see if

$$
\frac{\delta}{\delta R}\{J,H_{FPU}\}=E_RI=0,\quad \frac{\delta}{\delta V}\{J,H_{FPU}\}=E_VI=0 \qquad (4.27)
$$

where I is the integrand of (4.26).

We compute the Euler-Lagrange operators using the software *Mathematica* and we check term by term for which conditions on the coefficients the equation (4.27) are satisfied.

From the terms of fifth grade, we find that both $E_RI=0$ and $E_VI=0$ give us these coefficients:

$$
\begin{aligned}
&\tilde{\alpha}=\frac{j_{0_{4V}}}{120}e^{2R}-\frac{j_{0_{VVR}}}{720}e^R;\quad \tilde{\beta}=\frac{j_{0_{3VR}}}{60}e^R+\frac{j_{0_{3V}}}{120}e^R;\\
&\tilde{\gamma}=\frac{j_{0_{4V}}}{120}e^R+\frac{j_{0_{VVR}}}{180}+\frac{j_{0_{VV}}}{720};\quad \tilde{\epsilon}=\frac{j_{0_{3VR}}}{72}e^R+\frac{j_{0_{3V}}}{120}e^R;\quad \tilde{\delta}=-\frac{j_{0_{4V}}}{180}e^R+\frac{j_{0_{VVR}}}{1440};\\
&\tilde{\mu}=\frac{j_{0_{VVR}}}{2160}e^R-\frac{j_{0_{4V}}}{360}e^{2R}-\frac{17}{4320}j_{0_{4VR}}e^{2R}-\frac{j_{0_{6V}}}{864}e^{3R};\\
&\tilde{\rho}=-\frac{7}{1440}j_{0_{4V}}e^R-\frac{j_{0_{4VR}}}{160}e^R-\frac{j_{0_{6V}}}{144}e^{2R};\quad \tilde{\nu}=-\frac{j_{0_{3VR}}}{1440}-\frac{j_{0_{5V}}}{144}e^R-\frac{j_{0_{5VR}}}{216}e^R;\\
&\tilde{\lambda}=\frac{j_{0_{3VR}}}{540}e^R-\frac{14}{2160}j_{0_{5V}}e^{2R}-\frac{j_{0_{5VR}}}{216}e^{2R};\\
&\tilde{\omega}=-\frac{j_{0_{4V}}}{5760}-\frac{j_{0_{4VR}}}{1080}-\frac{j_{0_{6V}}}{864}e^R.
\end{aligned}
\qquad (4.28)
$$

This result coincides with the Toda hierarchy at the fourth order [24].

We proceed with the computation of lower terms of $E_RI=0$ and $E_VI=0$. Starting from $E_RI=0$, we find:

$V_{xxx}:\quad \zeta e^R-\eta=0\Rightarrow\eta=\zeta e^R$

$V_{xx}V_x:\quad -\xi_R+2\zeta_Ve^R-\eta_V=0\Rightarrow\xi_R=\zeta_Ve^R=\eta_V$

$R_{xx}R_x:\quad \xi_Re^R-\eta_Ve^R-\xi e^R=0\Rightarrow\xi=0$

This means that $\eta_V=0\Rightarrow\eta=f(R)\Rightarrow\zeta=f(R)e^{-R}\Rightarrow\zeta_V=0$

$R_{xx}V_x:\quad 2\zeta_Re^R-\eta_R=0\Rightarrow\zeta_Re^R-\zeta e^R=0$

$\zeta_R=\zeta$

$V_x^3:\quad \zeta_{RR}+\zeta_{VV}e^R=0\Rightarrow\zeta_{RR}=0$

but $\zeta_{RR}=\partial_R(\zeta_R)=\zeta_R=\zeta=0$. This means that $\xi=\eta=\zeta=0$

$V_x:\quad \sigma_{RR}=\sigma_{VV}e^R+3j_{0_{VV}}\Delta\beta R^2$

while all the other equations are identically null. From $E_V I = 0$ we find the same results:

$$\begin{aligned}
&R_{xxx}: \quad \eta - \zeta e^R = 0 \Rightarrow \eta = \zeta e^R \\
&V_{xx}V_x: \quad \xi_V = \zeta_R \\
&V_{xx}R_x: \quad -\zeta_V e^R + 2\eta_V - \xi_R = 0 \Rightarrow \xi_R = \zeta_V e^R = \eta_v \\
&R_{xx}R_x: \quad -\xi_V e^R + 2\eta_R - \zeta_R e^R - 3\zeta e^R = 0 \Rightarrow \zeta e^R = 0 \\
&\text{This means that } \eta = 0 \text{ and } \xi_R = \xi_V = 0 \\
&R_x: \quad \sigma_{RR} = \sigma_{VV} e^R + 3 j_{0_{VV}} \Delta\beta R^2
\end{aligned}$$

while all the other equations are identically null.

Therefore, the density $j_4(V, R, V_x, R_x, V_{xx}, R_{xx})$ becomes

$$\begin{aligned}
j_4(V, R, V_x, R_x, V_{xx}, R_{xx}) = {} & \tilde{\alpha} R_{xx}^2 + \tilde{\beta} R_{xx} V_{xx} + \tilde{\gamma} V_x^2 + \tilde{\delta} R_{xx} V_x^2 + \tilde{\epsilon} V_{xx} R_x^2 + \\
& + \tilde{\mu} R_x^4 + \tilde{\nu} R_x^3 V_x + \tilde{\rho} R_x^2 V_x^2 + \tilde{\lambda} R_x V_x^3 + \tilde{\omega} V_x^4 + \sigma
\end{aligned}$$

where the coefficients $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\epsilon}, \tilde{\mu}, \tilde{\nu}, \tilde{\rho}, \tilde{\lambda}, \tilde{\omega}$ are given by the Toda hierarchy (4.28) and σ must satisfy this PDE

$$\sigma_{RR} = \sigma_{VV} e^R + 3 j_{0_{VV}} \Delta\beta R^2. \tag{4.29}$$

4.3 Solutions of the principal PDEs

We now focus on the most important PDEs that we considered before: the equation (3.29) with $\phi(R) = \phi_{Toda}(R)$ and the equation (4.29) on the extension of this integral

$$\begin{aligned}
j_{0_{RR}} &= e^R j_{0_{VV}} \\
\sigma_{RR} &= \sigma_{VV} e^R + 3 j_{0_{VV}} \Delta\beta R^2.
\end{aligned} \tag{4.30}$$

The first equation describes the first integrals of the continuum dispersionless ($h \to 0$) Toda Lattice.

We know that the density of the Hamiltonian $\mathcal{K}_{Toda}[R, V]$ is a solution of the equation, so the generic j_0 must satisfy the following properties:

- there is no periodicity in V or R;
- j_0 is a polynomial function of V and e^R.

The first integrals that satisfy these two conditions and the equation (3.29) are the continuum limit of the Henon's integrals [14], which are the first integrals of the discrete Toda Lattice.

$$\begin{aligned}
j_0^{(2)} &= \frac{V^2}{2} + e^R; \\
j_0^{(3)} &= \frac{V^3}{6} + V e^R; \\
j_0^{(4)} &= \frac{V^4}{6} + 2V^2 e^R + e^{2R}; \\
j_0^{(5)} &= \frac{V^5}{30} + \frac{2}{3} V^3 e^R + V e^{2R}; \\
&\vdots
\end{aligned} \tag{4.31}$$

For a generic order, we write these first integrals as:

$$\begin{aligned}
j_0^{(2n)} &= \sum_{l=0}^{n} C_n^l V^{2(n-l)} e^{lR} \\
j_0^{(2n+1)} &= \sum_{l=0}^{n} B_n^l V^{2(n-l)+1} e^{lR}
\end{aligned} \qquad \text{for } n \geq 1 \tag{4.32}$$

where the two coefficients C_n^l and B_n^l are given by the formulas:

$$C_n^l = \begin{cases} \dfrac{\prod_{m=l+1}^{n} m^2}{[2(n-l)]!} & \text{if } l = 0;\ldots;n-1 \\ 1 & \text{if } l = n \end{cases} \qquad B_n^l = \begin{cases} \dfrac{\prod_{m=l+1}^{n} m^2}{[2(n-l)+1]!} & \text{if } l = 0;\ldots;n-1 \\ 1 & \text{if } l = n \end{cases} \tag{4.33}$$

Now, we can study the second equation of (4.30). The solution of this equation is composed by two functions

$$\sigma = \sigma_0 + \sigma_p \tag{4.34}$$

where σ_0 is a solution of the homogeneous PDE

$$\sigma_{0_{RR}} = e^R \sigma_{0_{VV}},$$

i.e an Henon's integral, and σ_p is a particular solution of all the PDE

$$\sigma_{p_{RR}} = e^R \sigma_{p_{RR}} + 3\Delta\beta j_{0_{VV}} R^2.$$

Before we try to solve (4.29) for particular solution, we need to choose a $j_0^{(n)}$ to extend. We start with a trivial first integral: $j_0^{(2)}$, i.e. the density of the unperturbed Hamiltonian.

$$j_0^{(2)} = \frac{V^2}{2} + e^R \implies \sigma_{p_{RR}} = e^R \sigma_{p_{RR}} + 3\Delta\beta R^2.$$

Because the extension of the unperturbed Hamiltonian is given by the perturbed Hamiltonian (4.20) itself, σ_p is equal to the perturbed potential $\psi_2(R)$

$$\sigma_p = \psi_2(R) = \frac{\Delta\beta}{4} R^4. \tag{4.35}$$

We analyse now the first nontrivial first integral: $j_0^{(3)}$.

$$j_0^{(3)} = \frac{V^3}{6} + Ve^R \implies \sigma_{p_{RR}} = e^R \sigma_{p_{VV}} + 3\Delta\beta V R^2.$$

In this case, we find another simple solution σ_p:

$$\sigma_p = \frac{\Delta\beta}{4} V R^4. \tag{4.36}$$

The solutions of (4.29) become more complex when we proceed with the other first integrals. In fact, for higher grade n, inside $j_{0_{VV}}^{(n)}$ there are terms in $e^R, Ve^R, \ldots$.

For example, if we take the equation (4.29) for $j_0^{(4)}$ we find

$$\sigma_{RR} = e^R \sigma_{VV} + 3\Delta\beta(2V^2 + 4e^R)R^2. \tag{4.37}$$

Considering the form of the other two solutions (4.35) and (4.36), we suppose that a generic solution σ_p of (4.29) exists and has the following form

$$\sigma_p = \Delta\beta \left[R^4 P_n(V, e^R) + R^3 P_m(V, e^R) + R^2 P_i(V, e^R) + RP_j(V, e^R) + P_k(V, e^R) \right] \tag{4.38}$$

where $P_y(V; e^R)$ is a non homogeneous polynomial of maximum grade y in V and e^R.

We try this ansatz for the PDE (4.37). To simplify the notations, we define $X := e^R$. [2]

Let's calculate the double derivative of (4.38) respect R and V

$$\begin{aligned}\sigma_{p_{RR}} &= \Delta\beta\partial_R^2\left[R^4P_n(V;X)+R^3P_m(V;X)+R^2P_i(V;X)+RP_j(V;X)+P_k(V;X)\right]=\\ &=\Delta\beta\partial_R\Big[R^4P_{n_X}X+R^3(4P_n+P_{m_X}X)+R^2(3P_m+P_{i_X}X)+\\ &+R(2P_i+P_{j_X}X)+P_j+P_{k_X}X\Big]=\\ &=\Delta\beta\Big[R^4(P_{n_{XX}}X^2+P_{n_X}X)+R^3(8P_n+P_{m_{XX}}X^2+P_{m_X}X)+\\ &+R^2(12P_n+6P_{m_x}X+P_{i_{XX}}X^2+P_{i_X}X)+\\ &+R(6P_m+4P_{i_X}X+P_{j_{XX}}X^2+P_{j_X}X)+\\ &+2P_i+2P_{j_X}X+P_{k_{XX}}X^2+P_{k_X}X\Big]\end{aligned} \tag{4.39}$$

$$\sigma_{p_{VV}} = \Delta\beta\left[R^4P_{n_{VV}}+R^3P_{m_{VV}}+R^2P_{i_{VV}}+RP_{j_{VV}}+P_{k_{VV}}\right]. \tag{4.40}$$

We substitute (4.39) and (4.40) in the PDE (4.37) and, comparing the powers of R, we find that the polynomials P_n, P_m, P_i, P_j, P_k must satisfy the following PDEs:

$$\begin{aligned} R^4 &\implies \partial_R^2P_n = e^R\partial_V^2P_n\\ R^3 &\implies 8\partial_RP_n+\partial_R^2P_m = e^R\partial_V^2P_m\\ R^2 &\implies 12P_n+6\partial_RP_m\partial_R^2P_i = e^R\partial_V^2P_i+3(2V^2+4e^R)\\ R &\implies 6P_m+4\partial_RP_i+\partial_R^2P_j = e^R\partial_V^2P_j\\ hom &\implies 2P_i+2\partial_RP_j+\partial_R^2P_k = e^R\partial_V^2P_k.\end{aligned} \tag{4.41}$$

Thus, we move the problem from solving the PDE (4.37) to solving the system of five PDEs (4.41), knowing that P_n, P_m, P_i, P_j, P_k are polynomials.

From the grading of the non homogeneous terms of the PDE (4.37), we understand that the solution (4.38) must be a non homogeneus polynomial of sixth grade. So, we can fix the maximum grade of the polynomials P_y

$$n = 2,\ m = 3,\ i = 4,\ j = 5 \text{ and } k = 6.$$

We now solve, step by step, the system (4.41).

The first equation of (4.41) says that $P_2(V, X)$ must be a first integral of degree 2, so this means it coincides with the density of the Hamiltonian $\mathcal{K}_{Toda}[R, V]$

$$P_2(V,X) = j_0^{(2)}(V,X) = \frac{V^2}{2}+X. \tag{4.42}$$

From this result, we can solve the second equation for $P_3(V, X)$

$$\begin{gathered} 8\partial_RP_2+\partial_R^2P_3 = e^R\partial_V^2P_3\\ \Downarrow\\ 8X+\partial_R^2P_3 = e^R\partial_V^2P_3.\end{gathered} \tag{4.43}$$

[2] this means that the derivatives with respect to R become:

$$\begin{aligned}\partial_Rf(X) &= f_XX\\ \partial_R^2f(X) &= f_{XX}X^2+f_XX\\ &\vdots\end{aligned}$$

The solution of the PDE (4.43) is given, as we saw before, by a solution of the homogeneous PDE, i.e. the Henon's integral $j_0^{(3)}$, and by a particular solution of all the PDE:

$$P_3(V,X) = 8V^2 \text{ or } P_3(V;X) = -8X \tag{4.44}$$

and we can write P_3 as

$$P_3(V,X) = aj_0^{(3)} + 8(\alpha V^2 - \beta X) \tag{4.45}$$

where a, α and β are constants.

To find some restriction on a, α and β, we apply this solution to the PDE (4.43)

$$\begin{gathered} 8X - 8\beta X = 8\alpha X \\ \Downarrow \\ \alpha = 1 - \beta \\ \Downarrow \\ P_3(V,X) = aj_0^{(3)} + 8[(1-\beta)V^2 - \beta X]. \end{gathered}$$

We proceed with the third equation of (4.41)

$$\begin{gathered} 12P_2 + 6\partial_R P_3 + \partial_R^2 P_4 = e^R \partial_V^2 P_4 + 6V^2 + 12e^R \\ \Downarrow \\ 6aVX - 48\beta X + \partial_R^2 P_4 = e^R \partial_V^2 P_4 \end{gathered} \tag{4.46}$$

In this case too, the solution is composed by an Henon's integral and a particular solution of (4.46):

$$P_4(V,X) = bj_0^{(4)} + aV^3 + 48\beta X \tag{4.47}$$

with b as a simple constant.

We continue to solve of the systems of PDEs focusing on the fourth equation

$$\begin{gathered} 6P_3 + 4\partial_R P_4 + \partial_R^2 P_5 = e^R \partial_V^2 P_5 \\ \Downarrow \\ 6aj_0^{(3)} + 48[(1-\beta)V^2 - \beta X] + 4bj_{0_R}^{(4)} + 192\beta X + \partial_R^2 P_5 = e^R \partial_V^2 P_5 \end{gathered} \tag{4.48}$$

since there are not polynomial solutions of this equation that give us pure terms in V, we find that

$$\begin{gathered} a = 0 \quad \beta = 1 \\ \Downarrow \\ P_3(V,X) = -8X \quad P_4(V;X) = bj_0^4 + 48X \end{gathered}$$

and the PDE (4.48) becomes

$$144X + 8b(V^2X + X^2)\partial_R^2 P_5 = e^R \partial_V^2 P_5. \tag{4.49}$$

A solution of the PDE (4.49) is

$$P_5(V,X) = cj_0^{(5)} + b\left(\frac{2}{3}V^4 - 2X^2\right) - 144X \tag{4.50}$$

with c an arbitrary constant.

We now need to solve the last equation

$$2P_4 + 2\partial_R P_5 + \partial_R^2 P_6 = e^R \partial_V^2 P_6$$
$$\Downarrow \tag{4.51}$$
$$2bj_0^{(4)} + 96X + 2cj_{0_R}^{(5)} - 8bX^2 - 288X + \partial_R^2 P_6 = e^R \partial_V^2 P_6.$$

As explained before, there cannot be any power of V in the equation, and this means that also $b = 0$ and the polynomials $P_4(V;X)$ and $P_5(V;X)$ become

$$P_4(V,X) = 48X \quad P_5(V;X) = cj_0^{(5)} - 144X.$$

Having things in mind, we rewrite the PDE (4.51) as

$$-192X + \frac{4}{3}cV^3X + 4cVX^2\partial_R^2 P_6 = e^R \partial_V^2 P_6. \tag{4.52}$$

The $P_6(V,X)$ that satisfies the equation (4.52) is given by:

$$P_6(V,X) = 192X + c\left(\frac{V^5}{15} - VX^2\right) \tag{4.53}$$

(we didn't consider the term j_0^6 because σ_p is define up to a first integral).

We are able to summarize the results: we found out that we can extend at the fourth order for the first two nontrivial first integrals $j_0^{(3)}$ and $j_0^{(4)}$ of the Toda lattice to the FPU system

$$\begin{aligned} j^{(3)}(V,X) =& j_0^{(3)}(V,X) - \frac{h^2}{2}\left(\frac{X}{6}R_xV_x + \frac{V}{12}V_x^2\right) + h^4\bigg(\frac{X}{120}V_{xx}R_{xx} + \frac{V}{720}V_{xx}^2 + \\ &+ \frac{X}{120}V_{xx}R_x^2 + \frac{\Delta\beta}{4}R^4V\bigg) + \mathcal{O}(h^6) \end{aligned} \tag{4.54}$$

$$\begin{aligned} j^{(4)}(V,X) =& j_0^{(4)}(V,X) - \frac{h^2}{3}\left[X^2R_x^2 + VXR_xV_x + \left(3X + \frac{V^2}{4}\right)V_x^2\right] + \\ &+ h^4\bigg[\frac{X^2}{36}R_{xx}^2 + \frac{VX}{30}R_{xx}V_{xx} + \left(\frac{V^2}{360} + \frac{11}{180}X\right) + \\ &- \frac{7}{360}XR_{xx}V_x^2 + \frac{VX}{30}V_{xx}R_x^2 - \frac{X^2}{180}R_x^4 + \\ &- \frac{7}{360}XR_x^2V_x^2 - \frac{V_x^4}{1440} + \tilde{\sigma}(V;X)\bigg] + \mathcal{O}(h^6), \end{aligned} \tag{4.55}$$

where $\tilde{\sigma}(V,X)$ has the form (4.38)

$$\tilde{\sigma}(V,X) = \Delta\beta\left[R^4P_2(V,X) + R^3P_3(V,X) + R^2P_4(V,X) + RP_5(V,X) + P_6(V,X)\right] \tag{4.56}$$

and the polynomials P_2, P_3, P_4, P_5, P_6 are solutions of the system of PDEs (4.41)

$$P_2(V,X) = \frac{V^2}{2} + X; \quad P_3(V,X) = -8X; \quad P_4(V,X) = 48X;$$
$$P_5(V,X) = cj_0^{(5)}(V,X) - 144X; \quad P_6(V,X) = 192X + c\left(\frac{V^5}{15} - VX^2\right).$$

We noticed that no condition was found on the constant c, so it remains a free parameter. This derives by the fact that the homogeneous PDE solutions are defined up to a multiplicative constant, therefore it is carried through all the calculations.

The same idea applied to the extension of $j_0^{(3)}$ and $j_0^{(4)}$ can be applied to all the other first integral (4.31). Indeed, we conjecture that exist an extension at the fourth order for the FPU of the first integral $j_0^{(n)}$, with form (4.38), where the polynomials $P_{n-2}, P_{n-1}, P_n, P_{n+1}, P_{n+2}$ satisfy the following system of PDEs:

$$\begin{cases} \partial_R^2 P_{n-2} = e^R \partial_V^2 P_{n-2} \\ 8\partial_R P_{n-2} + \partial_R^2 P_{n-1} = e^R \partial_V^2 P_{n-1} \\ 12 P_{n-2} + 6\partial_R P_{n-1} + \partial_R^2 P_n = e^R \partial_V^2 P_n + 3 j_{0_{VV}}^{(n)} \\ 6 P_{n-1} + 4\partial_R P_n + \partial_R^2 P_{n+1} = e^R \partial_V^2 P_{n+1} \\ 2 P_n + 2\partial_R P_{n+1} + \partial_R^2 P_{n+2} = e^R \partial_V^2 P_{n+2} \end{cases} \tag{4.57}$$

Chapter 5

Conclusion

At the end of this analysis, we have found that it is possible to extend at the fourth order two no trivial first integrals of the Toda lattice $j_0^{(3)}$ and $j_0^{(4)}$ for the FPU system, in particular we have observed that the solution of this problem follows a particular scheme. We conjecture that this scheme is valid also for the other first integrals of the Toda lattice.

This conjecture will be demonstrated in future works.

The importance of this result is given by two consequences:

1. if we try to find the same result using the method of normal forms, we find out that it is impossible to get the normal form of the FPU system from the Toda lattice, while with this method we find it is possible to construct a perturbative approach for the FPU starting from Toda;

2. from the extension of the first integral, we can obtain also an esteme of the time where the motion of the FPU is similar to the motion of the Toda.

There are other open questions still to study. For example, it should be examine whether the procedure ends here or continue also for the sixth order and further; for which $\psi_1(R)$ the equation (3.39) admits a solution; or what happen in FPU systems with dimension greater then 1.

We have choosen not to proceed for further orders because the amount of calculus considerably increases for each order and the working time for this thesis was limited.

Appendix A

Theory of Nonlinear Wave Equations and Hamiltonian Perturbations

This appendix is a summary of the theory of nonlinear wave equations, focusing on the first integrals of the equations, and the main results of the articles [1] and [4].

We will consider Nonlinear PDEs of this form:

$$u_{tt} - \partial_x^2 \phi'(u) = 0 \tag{A.1}$$

for a given smooth function $\phi(u)$. It's easy to see that (A.1) is linear if $\phi(u)$ is quadratic respect u, but in general we assume that

$$\phi'''(u) \neq 0 \quad \forall\, u.$$

The equation (A.1) is important because it arises in the study of dispersionless limit of various PDEs of higher order. In fact, some example of these kinds of PDEs are:

1. The dispersionless limit of the Boussinesq equation ($\phi(u) = -\frac{u^3}{6}$)

$$u_{tt} + (uu_x)_x = 0$$

2. The long-wave limit of the Toda equations ($\phi(u) = e^u$)

$$u_{tt} = \partial_x^2 e^u$$

We can write the equation (A.1) in Hamiltonian form

$$\begin{aligned} u_t &= v_x = \partial_x \frac{\delta H}{\delta v(x)} \\ v_t &= \partial_x \phi'(u) = \partial_x \frac{\delta H}{\delta u(x)} \end{aligned} \tag{A.2}$$

with Hamiltonian

$$H[u, v] = \int \Big[\frac{v^2}{2} + \phi(u)\Big] dx. \tag{A.3}$$

As we can see from the equation of motions (A.2), we choose the functions $u(x)$ and $v(x)$ so that the Poisson bracket is

$$\mathbb{J}_2^* = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}.$$

Thus, the associated Poisson bracket of two local functionals

$$F = \int f(u, v, u_x, v_x, \dots)dx \quad G = \int g(u, v, u_x, v_x, \dots)dx$$

is given by the following formula:

$$\begin{aligned} \{F, G\} &:= \int (\nabla_{L^2} F \mathbb{J}_2^* \nabla_{L^2} G)dx = \\ &= \int \Big[\frac{\delta F}{\delta u}\partial_x \frac{\delta G}{\delta v} + \frac{\delta F}{\delta v}\partial_x \frac{\delta G}{\delta u}\Big]dx. \end{aligned} \tag{A.4}$$

In particular, the Poisson bracket of the dependent variables $u(x), v(x)$ is

$$\{u(x), v(x)\} = \delta'(x - y). \tag{A.5}$$

A.1 First integrals of nonlinear wave equation

We now study the first integrals of the general nonlinear wave equation (A.1).

Given a functional

$$J[u, v] = \int j(u, v, u_x, v_x, \dots)dx, \tag{A.6}$$

it is a first integral of the Hamiltonian (A.3) if they commute with respect to the Poisson bracket (A.4).

We know that the densities of the local functionals are considered up to a total x-derivative, this means that the Poisson bracket (A.4) vanishes iff the integrand is a total x-derivative.

With this in mind, we want to find some kinds of conditions that the first integrals of the Hamiltonian (A.3) must satisfy.

Lemma A.1.1. *Consider the functional*

$$J[u, v] = \int j(u, v)dx.$$

This functional commutes with the Hamiltonian (A.3) *of the nonlinear wave equation iff the function* $j(u, v)$ *satisfies the PDE*

$$j_{uu} = \phi''(u) j_{vv} \tag{A.7}$$

Proof. The Poisson bracket (A.4) of $J[u, v]$ and $H[u, v]$ reads

$$\{J, H\} = \int \left(\frac{\delta J}{\delta u}\partial_x \frac{\delta H}{\delta v} + \frac{\delta J}{\delta v}\partial_x \frac{\delta H}{\delta u}\right) dx = \int [j_u v_x + j_v \phi''(u) u_x] dx.$$

This Poisson bracket is null if the integrand is a total x-derivative, which means that there must exist a function $g(u, v)$ s.t.:

$$\partial_u g = j_v \phi''(u) \quad \partial_v g = j_u.$$

We can apply the Schwarz's theorem

$$\begin{gathered} \partial_v(\partial_u g) = \partial_u(\partial_v g) \\ \Downarrow \\ j_{vv}\phi''(u) = j_{uu}. \end{gathered}$$

That is the equation (A.7). □

Therefore, all the solutions of the PDE (A.7) are associated to the densities of the first integrals of the nonlinear wave equation (A.1). We can prove that all the functional of the same form of $J[u,v]$ commute pairwise, i.e. the Lie algebra of the symmetries is commutative.

Lemma A.1.2. *Consider two functionals*

$$F[u,v] = \int f(u,v)dx \quad G[u,v] = \int g(u,v)dx.$$

F and G commute with respect to the Poisson bracket iff $f(u,v)$ and $g(u,v)$ are solutions of the linear PDE (A.7).

Proof. Let us calculate the Poisson bracket

$$\{F,G\} = \int (f_u \partial_x g_v + f_v \partial_x g_u)\, dx = \int [(f_u g_{uv} + f_v g_{uu}) u_x + (f_u g_{vv} + f_v g_{uv}) v_x] dx. \tag{A.8}$$

From the previous statement, this Poisson bracket must be null if the integrand is a total x-derivative, so there must exist a density $h(u,v)$ so that:

$$h_u = f_u g_{uv} + f_v g_{uu} \quad h_v = f_u g_{vv} + f_v g_{uv}. \tag{A.9}$$

From the Schwarz's theorem, we impose that the mixed derivative are symmetric

$$\partial_v(h_u) = \partial_u(h_v).$$

Substituting to h_u and h_v the results (A.9), the previous equation become

$$f_{vv} g_{uu} = f_{uu} g_{vv}$$

This equation has a result iff both $f(u,v)$ and $g(u,v)$ are two independent solution of (A.7). In fact:

$$f_{vv} g_{uu} = f_{uu} \phi''(u) g_{uu} = f_{uu} g_{vv}$$

□

A.2 Perturbation of nonlinear wave equation and deformation of the first integrals

We will focus on the perturbation of Hamiltonians. In particular, we will present results and techniques to extend first integrals and solutions of the system. These results, discovered by Dubrovin in [1] and developed in [4], are valid in general, i.e for $\mathbf{u} \in M$ where M is an n-dimensional manifold. Also, the notations like $f(\mathbf{u}(x); \mathbf{u}_x(x); \ldots; \mathbf{u}^{(k)}(x))$ are used for differential polynomials

$$f(\mathbf{u}(x); \mathbf{u}_x(x); \ldots; \mathbf{u}^{(k)}(x)) \in C^\infty[\mathbf{u}_x(x); \ldots; \mathbf{u}^{(k)}(x)] \quad \mathbf{u} \in M,$$

i.e. they are polynomial functions on the jet bundle $J^k(M)$. The degrees of these differential polynomials are given by this rule

$$\deg\, u^i_x = 1, \quad \deg\, u^i_{xx} = 2, \ldots; \quad i = 1, \ldots, n.$$

Given a system of the first order quasi-linear PDEs

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x, \quad \mathbf{u} = (u^1(x,t), \ldots, u^n(x,t)) \tag{A.10}$$

admitting a Hamiltonian description

$$\mathbf{u}_t = \{\mathbf{u}, H_0\}_0, \quad H_0 = \int h_0(\mathbf{u})dx \tag{A.11}$$

with respect to a Poisson bracket of hydrodynamic type, defined by Dubrovin and Novinkov in [16], written as

$$\{u^i(x), u^j(y)\}_0 = \eta^{ij}\delta'(x-y), \quad \eta^{ij} = \eta^{ji} = cost, \quad det(\eta^{ij}) \neq 0. \tag{A.12}$$

Definition 1. We say that a system of this form

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_x + hB_2(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}) + h^2 B_3(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \mathbf{u}_{xxx}) + \dots \tag{A.13}$$

with h as our perturbative parameter; it is an *Hamiltonian deformation* of (A.10) if it can be represented with an Hamiltonian form

$$\mathbf{u}_t = \{\mathbf{u}(x), H\} \tag{A.14}$$

where H is a *perturbed Hamiltonian*

$$\begin{aligned} H &= H_0 + hH_1 + h^2 H_2 + \dots \\ H_k &= \int h_k(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(k)})dx, \quad k \geq 1 \\ \deg\, h_k(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(k)}) &= k \end{aligned} \tag{A.15}$$

and the Poisson bracket becomes

$$\begin{aligned} \{u^i(x), u^j(y)\} &= \{u^i(x), u^j(y)\}_0 + h\{u^i(x), u^j(y)\}_1 + h^2\{u^i(x), u^j(y)\}_2 + \dots \\ \{u^i(x), u^j(y)\}_k &= \sum_{s=0}^{k+1} A^{ij}_{ks}(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(s)})\delta^{(k-s+1)}(x-y), \quad k \geq 1 \\ \deg\, A^{ij}_{ks}(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(s)}) &= s. \end{aligned} \tag{A.16}$$

This kind of Poisson bracket is called *perturbed Poisson bracket.*

We can redefine the Poisson bracket (A.16) such that the delta-function symbol can be spelled out. Be Π^{ij} a matrix of linear differential operator depending on h

$$\begin{aligned} \Pi^{ij} &:= \Pi^{ij}_0 + h\Pi^{ij}_1 + h^2\Pi^{ij}_2 + \dots \\ \Pi^{ij}_0 &:= \eta^{ij}\partial_x, \\ \Pi^{ij}_k &:= \sum_{s=0}^{k+1} A^{ij}_{ks}(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(s)})\partial_x^{(k-s+1)}, \quad k \geq 1. \end{aligned} \tag{A.17}$$

So, the Poisson bracket becomes

$$\{u^i(x), u^j(y)\} = \Pi^{ij}\delta(x-y) \tag{A.18}$$

and the perturbed Hamiltonian system reads

$$u^i_t = \Pi^{ij}\frac{\delta H}{\delta u^j(x)} = \sum_{m\geq 0} h^m \sum_{k+l=m} \Pi^{ij}_k \frac{\delta H_l}{\delta u^j(x)}. \tag{A.19}$$

From this definition, we can find an expression for the perturbative terms of (A.13) in the case of Hamiltonian deformation:

$$B^i_m(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(m+1)}) = \sum_{k+l=m} \Pi^{ij}_k \frac{\delta H_l}{\delta u^j(x)}, \quad m \geq 0, \quad i = 1, \dots, n.$$

An important property of this class of Hamiltonian deformations is that it is invariant with respect to *Miura-type transformation* of the dependent variables

$$\mathbf{u} \mapsto \tilde{\mathbf{u}} = \mathbf{u} + \sum_{k\geq 1} h^k F_k(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(k)}), \quad \deg F_k(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(k)}) = k. \tag{A.20}$$

The transformation of the Hamiltonian is defined by the direct substitution, while the Poisson bracket is transformed by the rule

$$\begin{aligned} \{\tilde{u}^i(x), \tilde{u}^j(x)\} &= \tilde{\Pi}^{ij}\delta(x-y) \\ \tilde{\Pi}^{ij} &= L^i_p \Pi^{pq} L^{\dagger j}_q \end{aligned} \tag{A.21}$$

where L and $L^\dagger$ are respectively the Jacobian of the transformation and its adjoint:

$$L^i_k = \sum_s \frac{\partial \tilde{u}^i}{\partial u^{k,(s)}} \partial^s_x, \quad L^{\dagger i}_k = \sum_s (-\partial_x)^s \frac{\partial \tilde{u}^i}{\partial u^{k,(s)}}. \tag{A.22}$$

We said that two Hamiltonian deformations of the quasi-linear system (A.10) are *equivalent* if they are related by a transformation (A.20). In particular, the Hamitonian deformation is called *trivial* if it is equivalent to the unperturbed system (A.10).

A.2.1 Extension of first integrals

Once defined the Hamiltonian deformation of (A.10), we are interested in knowing what happen to the first integrals of the unperturbed system. In particular, we want see if it is possible to extend them to perturbative system under a fixed order of h.

We consider the Hamiltonian (A.3); so we return in dimension 2, and his perturbation

$$\begin{aligned} H_{pert} &= H_0 + hH_1 + h^2 H_2 + \dots \\ H_k &= \int h_k(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(k)})dx, \quad \deg h_k(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(k)}) = k, \quad k \geq 0 \\ h_0 &= \frac{v^2}{2} + \phi(u). \end{aligned} \tag{A.23}$$

Be $j_0(u, v)$ a solution to the linear PDE (A.7) and J_0 define as

$$J_0 = \int j_0(u, v)dx, \tag{A.24}$$

we know that J_0 commute with the unperturbed Hamiltonian.

The goal is to construct a deformation of J_0

$$\begin{aligned} J &= J_0 + hJ_1 + h^2 J_2 + \dots \\ J_k &= \int j_k(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(k)})dx, \quad \deg j_k(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(k)}) = k, \quad k \geq 0 \end{aligned} \tag{A.25}$$

so that

$$\{J, H_{pert}\} = 0.$$

Definition 2. We say that:

1. The perturbed system H_{pert} is called *N-integrable* if there exist a linear differential operator

$$\begin{aligned} &D_N = D^{[0]} + hD^{[1]} + h^2 D^{[2]} + \dots + h^N D^{[N]} \\ &D^{[0]} := id, \quad D^{[k]} := \sum b^{[k]}_{i_1,\dots,i_{m(k)}}(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(k)}) \frac{\partial^{m(k)}}{\partial u^{i_1} \dots \partial u^{i_{m(k)}}} \\ &\deg b^{[k]}_{i_1,\dots,i_{m(k)}}(\mathbf{u}, \mathbf{u}_x, \dots, \mathbf{u}^{(k)}) = k, \quad k \geq 1 \end{aligned} \tag{A.26}$$

called D-operator, such that for any $f(u,v)$ and $g(u,v)$ solutions to the equation (A.7) we have

$$J_N^f = \int D_N f(u,v)dx + \mathcal{O}(h^{N+1}) \quad J_N^g = \int D_N g(u,v)dx + \mathcal{O}(h^{N+1}) \tag{A.27}$$

satisfy

$$\{J_N^f, J_N^g\} = \mathcal{O}(h^{N+1}). \tag{A.28}$$

Moreover, we require that

$$H_{pert} = \int D_N h_0 dx + \mathcal{O}(h^{N+1}) \tag{A.29}$$

so the Hamiltonian satisfy also

$$\{H_{pert}, J_N^f\} = \mathcal{O}(h^{N+1}) \tag{A.30}$$

for any solution $f(u,v)$ to the equation (A.7).

2. The system H_{pert} is called *integrable* if it's N-integrable for any $N \geq 0$.

In the formula (A.26) $m(k)$ is a positive integer depending on k. It can be noticed that

$$m(k) = \left\lfloor \frac{3k}{2} \right\rfloor \tag{A.31}$$

The summation is taken over all the indices $i_1, \ldots, i_{m(k)}$ from 1 to 2.

Starting from the above definition, we develop a "perturbative" approach to the study of integrability that can be used for:

- finding obstructions to integrability;
- classification of integrable PDEs.

Appendix B

Dubrovin's proof

Here we will show the proof of the Theorem 3.0.1 presented in [1].

Given the Hamiltonian (3.1), the equations of motion are:

$$\begin{aligned} \dot{q_n} &= p_n \\ \dot{p_n} &= \phi'(q_{n+1} - q_n) - \phi'(q_n - q_{n-1}). \end{aligned} \tag{B.1}$$

Defined the parameter $h = 1/N$, we interpolate the distance and the momentum with two smooth analytic functions

$$\begin{aligned} w(x, \tau) &= q_n(t) - q_{n-1}(t) \\ v(x, \tau) &= p_n(t) \end{aligned} \tag{B.2}$$

(where $x = hn$ and $\tau = ht$).

After the interpolation, the Poisson bracket of the system in these coordinates becomes:

$$\begin{aligned} \{w(x), v(y)\} &= \frac{1}{h}[\delta(x-y) - \delta(x-y-h)] = \\ &= \delta'(x-y) - \frac{h}{2}\delta''(x-y) + \frac{h^2}{6}\delta'''(x-y) + \cdots = \\ &= \frac{1}{h}(1 - \Lambda^{-1})\delta(x-y) \end{aligned} \tag{B.3}$$

where $\Lambda^{\pm}$ is the shift operator, defined as:

$$\Lambda^{\pm} f(x) := e^{\pm h\partial_x} f(x) = \sum_{j\geq 0} \frac{(\pm h)^j \partial_x^j}{j!} f(x) = f(x \pm h).$$

Now, to return to a simpler structure of the Poisson bracket, we apply a Miura-type transformation:

$$u = \frac{h\partial_x}{1 - \Lambda^{-1}} w. \tag{B.4}$$

So the Poisson bracket return to the form

$$\{u(x), v(y)\} = \frac{h\partial_x}{1 - \Lambda^{-1}}\{w(x), v(y)\} = \delta'(x-y) \tag{B.5}$$

and the equations of motion become:

$$\begin{aligned} u_t &= v_x \\ v_t &= h^{-1}\left[\phi'(w(x+h)) - \phi'(w(x))\right] = \\ &= \partial_x \phi'(u) + \frac{h^2}{24}\left[2\phi''(u)u_{xxx} + 4\phi'''(u)u_x u_{xx} + \phi^{IV}(u)u_x^3\right] + \mathcal{O}(h^4). \end{aligned} \tag{B.6}$$

The Hamiltonian related to the equations of motion (B.6) is

$$H_{pert}[u,v] = \int \Big[\frac{v^2}{2} + \phi(u) - \frac{h^2}{24}\phi''(u)u_{xx}\Big]dx + \mathcal{O}(h^4) \tag{B.7}$$

and can be considered as a perturbation of the Hamiltonian

$$H_0[u,v] = \int \Big[\frac{v^2}{2} + \phi(u)\Big]dx. \tag{B.8}$$

B.1 Proof of the Theorem

We want now to extend the first integral of the unperturbed system

$$F[u,v] = \int f dx, \quad \{F, H_{pert}\} = \mathcal{O}(h^3) \tag{B.9}$$

where

$$f = f_0 + hf_1(u,v,u_x,v_x) + h^2 f_2(u,v,u_x,v_x,u_{xx},v_{xx})$$

and f_0 is the density of a first integral of H_0.

We calculate the Poisson bracket

$$\{F, H_{pert}\} = \int \Big[\frac{\delta F}{\delta u}\partial_x \frac{\delta H_{pert}}{\delta v} + \frac{\delta F}{\delta v}\partial_x \frac{\delta H_{pert}}{\delta u}\Big]dx$$

and see if the condition (B.9) is valid.

However, in order to proceed correctly, we must start with first order perturbation

$$\{F, H_{pert}\} = \mathcal{O}(h^2).$$

The first correction must be linear in u_x, v_x. Adding a total x-derivative, one can reduce to the study of first perturbation extensions of the form

$$f_1 = p(u,v)v_x$$

We compute the brackets:

$$\{F, H_{pert}\} = h\int p_u[v_x^2 - \phi''(u)u_x^2]dx + \mathcal{O}(h^2).$$

We see that the integrand is never a total derivative unless $p_u = 0$, i.e $p = p(v)$. This means that f_1 is a total x-derivative, so we do not consider this term of the extended integral.

We consider the second order terms. Up to a total x-derivative, they can be written as:

$$f_2 = \frac{1}{2}(a(u,v)u_x^2 + 2b(u,v)u_x v_x + c(u,v)v_x^2).$$

We compute the Poisson brackets and find:

$$\begin{aligned}\{F, H_{pert}\} = h^2 \int \Big\{ &\frac{f_{0v}}{12}\phi'' u_{xxx} + \Big[\Big(\frac{f_{0v}}{6}\phi''' - b\phi''\Big)u_x - av_x\Big]u_{xx} - (c\phi'' u_x + bv_x)v_{xx} + \\ &+\frac{1}{24}(f_{0v}\phi^{IV} + 12\phi'' a_v - 24\phi'' b_u)u_x^3 - \frac{1}{2}(a_u + 2\phi'' c_u)u_x^2 v_x + \\ &-\frac{1}{2}(2a_v + \phi'' c_v)u_x v_x^2 + \frac{1}{2}(c_u - 2b_v)v_x^3\Big\}dx + \mathcal{O}(h^4).\end{aligned} \tag{B.10}$$

Denote with I the integrand of (B.10), we apply the Corollary 3.2.2 and calculate:

$$E_u I = 0, \quad E_v I = 0.$$

From the equation $E_v I = 0$ we find the following conditions:

$$\begin{aligned} a &= (c - \frac{f_{0vv}}{12})\phi''(u); \quad c_u = b_v; \\ c &= -\frac{f_{0uvv}}{6}\frac{\phi''(u)}{\phi'''(u)}; \quad c_v\phi''(u) - b_u - \frac{f_{03v}}{6}\phi''(u) = 0; \end{aligned} \tag{B.11}$$

while, from the equation $E_u I = 0$ we find the same conditions plus a new one

$$b\phi'''(u) - b_u\phi''(u) + c_v\phi''^2(u) = 0 \tag{B.12}$$

Combining (B.12) with the fourth equation of (B.11), we find also

$$b = -\frac{f_{03v}}{6}\frac{(\phi''(u))^2}{\phi'''(u)}.$$

From the second equation of (B.11), we find that the coefficients c and b are the partial derivatives, respectively for $v(x)$ and $u(x)$, of a function $\lambda(u, v)$.

$$c = \lambda_v, \quad b = \lambda_u.$$

Equating the mixed derivatives

$$(\lambda_u)_v = (\lambda_v)_u$$

we obtain a condition on the potential $\phi(u)$

$$\frac{(\phi'''(u))^2 - \phi''(u)\phi^{IV}(u)}{6(\phi'''(u))^2} f_{0uvv} = 0$$

and this is valid only if the numerator is null

$$(\phi'''(u))^2 = \phi''(u)\phi^{IV}(u).$$

This is the same equation (3.30) we have found before, and the solution is

$$\phi(u) = ke^{\tilde{c}u} + \tilde{a}u + \tilde{b}$$

for some constants $\tilde{a}, \tilde{b}, \tilde{c}, k$.

Thus, the Theorem 3.0.1 is proved.

Appendix C

Derivation of the coefficients C_n^l and B_n^l

In this appendix we will explain how we reach the result (4.32). In particular, we will explain why the coefficients must be (4.33).

We start from the Henon's integrals with grade even. The generic form of an homogeneous polynomial in V and e^R of grade $2n$ is

$$j_0^{(2n)}(V;e^R) = y_0 V^{2n} + y_1 V^{2(n-1)} e^R + y_2 V^{2(n-2)} e^{2R} + \cdots + y_n e^{nR}. \tag{C.1}$$

We substitute this polynomial (C.1) in the equation (3.29), with $\phi(R)$ the Toda potential

$$j_{0_{RR}}^{(2n)} = e^R j_{0_{VV}}^{(2n)}. \tag{C.2}$$

Let us calculate explicitly the double derivative with respect to R and V

$$\begin{aligned} j_{0_{RR}}^{(2n)} =& y_1 V^{2(n-1)} e^R + 4y_2 V^{2(n-2)} e^{2R} + \cdots + n^2 y_n e^{nR}; \\ j_{0_{VV}}^{(2n)} =& (2n)(2n-1) y_0 V^{2(n-1)} + (2n-2)(2n-3) y_1 V^{2(n-1)} e^R \\ &+ (2n-4)(2n-5) y_2 V^{2(n-2)} e^{2R} + \cdots + 2y_{n-1} e^{(n-1)R}; \end{aligned} \tag{C.3}$$

and substitute these on (C.2)

$$y_1 V^{2(n-1)} e^R + \cdots + n^2 y_n e^{nR} = (2n)(2n-1) y_0 V^{2(n-1)} e^R + \cdots + 2y_{n-1} e^{nR} \tag{C.4}$$

Comparing term by term both the l.h.s and the r.h.s of (C.4), we find that $y_0, y_1, \ldots, y_n$ must satisfy this system of $n-1$ equation in n unknown variables:

$$\begin{cases} y_1 = 2n(2n-1) y_0 \\ 4y_4 = (2n-2)(2n-3) y_1 \\ \vdots \\ n^2 y_n = 2y_{n-1} \end{cases} \Longrightarrow \begin{cases} y_0 = \dfrac{y_1}{2n(2n-1)} \\ y_1 = \dfrac{4y_2}{(2n-2)(2n-3)} \\ \vdots \\ y_{n-1} = \dfrac{n^2}{2} y_n \end{cases}. \tag{C.5}$$

Therefore, by applying the last equation in the second-last equation and so on until we reach the first one, we rewrite all the coefficients $y_0, \ldots, y_{n-1}$ as a constant multiplied by y_n

$$\begin{cases} y_0 = \dfrac{(\prod_{m=1}^{n} m^2)}{2n!} y_n \\ y_1 = \dfrac{(\prod_{m=2}^{n} m^2)}{[2(n-1)]!} y_n \\ y_2 = \dfrac{(\prod_{m=3}^{n} m^2)}{[2(n-2)]!} y_n \\ \vdots \\ y_{n-1} = \dfrac{n^2}{2} y_n \end{cases} . \tag{C.6}$$

Using these coefficients, the first integral j_0^{2n} becomes

$$j_0^{2n}(V; e^R) = y_n \left(\frac{(\prod_{m=1}^{n} m^2)}{2n!} V^{2n} + \frac{(\prod_{m=2}^{n} m^2)}{[2(n-1)]!} V^{2(n-1)} e^R + \cdots + \frac{n^2}{2} V^2 e^{(n-1)R} + e^{nR} \right) = y_n \tilde{j}_0^{2n}. \tag{C.7}$$

Since each first integral is defined up to a multiplicative constant, $\tilde{j}_0^{(2n)}$ is a first integral as well. Therefore, we have that the first integral becomes

$$\tilde{j}_0^{(2n)}(V; e^R) = \sum_{l=0}^{n} y_l V^{2(n-l)} e^{lR}, \tag{C.8}$$

where the coefficients y_l are define by the formula

$$y_l := \begin{cases} \dfrac{\prod_{m=l+1}^{n} m^2}{[2(n-l)]!} & \text{if } l = 0; \ldots; n-1 \\ 1 & \text{if } l = n \end{cases} = C_n^l. \tag{C.9}$$

We move on and consider the Henon's integrals with grade odd. The generic form of an homogeneous polynomial in V and e^R of grade $2n+1$ is

$$j_0^{(2n+1)}(V; e^R) = z_0 V^{2n+1} + z_1 V^{2n-1} e^R + z_2 V^{2n-3} e^{2R} + \cdots + z_n V e^{nR}. \tag{C.10}$$

We repeat the same procedure and it is easy to see that, in this case, the general Henon's integral of grade $2n+1$ is

$$j_0^{(2n+1)}(V; e^R) = \sum_{l=0}^{n} z_l V^{2(n-l)+1} e^{lR}, \tag{C.11}$$

where the coefficients z_l are given by the formula

$$z_l := \begin{cases} \dfrac{\prod_{m=l+1}^{n} m^2}{[2(n-l)+1]!} & \text{if } l = 0; \ldots; n-1 \\ 1 & \text{if } l = n \end{cases} = B_n^l. \tag{C.12}$$

Bibliography

[1] B. Dubrovin, "On universality of critical behaviour in Hamiltonian PDEs", Amer. Math. Soc. Transl. 224 (2008) 59-109

[2] B. Dubrovin, Si-Qi Liu, Y. Zhang, "On Hamiltonian perturbations of hyperbolic systems of conservation laws I: quasi-triviality of bi-hamiltonian perturbations", Comm. in Pure Appl. Math. 59 (2006) 559 - 615

[3] B. Dubrovin, "On Hamiltonian perturbations of hyperbolic systems of conservation laws II: universality of critical behaviour", Comm. Math. Phys. 267 (2006) 117 - 139

[4] B. Dubrovin, "Hamiltonian Perturbations of Hyperbolic PDEs: from Classification Results to the Properties of Solutions", In: New Trends in Mathematical Physics. Selected contributions of the XVth International Congress on Mathematical Physics , Sidoravicius, Vladas (Ed.), Springer Netherlands, 2009., pp. 231-276

[5] B. Dubrovin, S.P. Nolinkov, "On Poisson brackets of hydrodynamic type", Soviet Math. Dokl. 279:2 (1984), 294-297

[6] V. Kozlov., "Integrability and non-integrability in Hamiltonian mechanics.", Russian Mathematical Surveys, Turpion, 1983, 38 (1), pp.1-76

[7] M. Gallone, A. Ponno, B. Rink, "Hydrodinamics of the FPU problem and its integrable aspects", Unpublished manuscript

[8] E. Fermi, J. Pasta, S. Ulam "Studies of non linear problems", LASL Report LA-1940 (1955); Reprinted in "Collected Papers of E.Fermi",V.II, Univ. of Chicago Press, 1965, p. 978

[9] N.J. Zabusky, M.D. Kruskal, "Interaction of "solitons" in a collisionless plasma and the recurrence of initial states", Phys. Rev. Lett. (1965), 15:240-243

[10] F.M. Izrailev, B.V. Chirikov, "Statistical Properties of a Nonlinear String", Soviet Phys. Dokl. (1966), 11:30

[11] W.E. Ferguson Jr., H. Flaschka, D.W. McLaughlin, "Nonlinear normal modes for the Toda chain", J. of Comp. Phys. 45 (1982), p.157

[12] M. Toda, "Vibration of a chain with nonlinear Interaction", J. Phys. Soc. Japan 22 (1967), p. 431

[13] M. Toda, "Theory of Nonlinear Latticies", Springer Series in Solid-State Sciences (2012), Springer Berlin Heidelberg

[14] M.Henon, "Integrals of the Toda lattice", Phys. Rev. B 9 (1974), p. 1921

[15] H.Flaschka "The Toda lattice. II. Existence of integrals", Phys. Rev. B 9 (1974), p. 1924

[16] J.Ford, D.Stoddard, S.Turner, "On the Integrability of the Toda Lattice", Prog. Theor. Phys. 50 (1973), p. 1547

[17] L. Berchialla, L. Galgani, A. Giorgilli, "Localization of energy in FPU chains", Discr. Cont. Dyn. Syst. A (2004), 11:855-866

[18] G. Benettin, A. Ponno, "Time-Scales to Equipartition in the Fermi-Pasta-Ulam Problem: Finite Size Effects and Thermodynamic Limit", J. Stat. Phys. 144 (2011), 793-812

[19] D. Bambusi, A. Ponno, "On Metastability in FPU", Comm. Math. Phys. 264 (2006), 539-561

[20] A. Ponno, H. Christodoulidi, G. Benettin, "The Fermi-Pasta-Ulam problem and its underlying integrable dynamics", J. Stas. Phys. 152 (2013), p.195

[21] G. Benettin, S. Pasquali, A. Ponno, "The Fermi-Pasta-Ulam problem and its underlying integrable dynamics: an approach through Lyapunov Exponents", J. Stat. Phys. 171 (2018), 521-542

[22] G. Benettin, G. Ferrari, L.Galgani, A.Giorgilli, "An Extension of the Poincaré-Fermi Theorem on the Nonexistence of Invariant Manifolds in Nearly Integrable Hamiltonian Systems", Il Nuovo Cimento 72 (1982), 137-148

[23] N.W.Ashcroft, N.D.Mermin "Solid State Physics", Cenglage Learning Emea, ISBN-13:978-81-315-0052-1, Cap. 25, p.488

[24] G. Carlet, B. Dubrovin, Y. Zhang, "The Extended Toda Hierarchy", Moscow Math. J. 4 (2004), 313-332