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Topological Modular Forms and Quantum Field Theory

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Abstract

Quantum field theories usually depend on a set of parameters that are related to physical observables, such as particle masses or coupling constants measured at some given energy. The parameters take value in a certain parameter space, and one can "deform" the theory by varying the parameters continuously within this space. A more complicated kind of "deformation" is the one induced by the renormalization group (RG) flow, which connects a quantum field theory describing a physical system at very high energies (UV) with the one describing it at very low energies (IR). In general, it can be very difficult to determine whether two QFT's are related via a deformation of parameters or an RG flow - for example, the relevant degrees of freedom in the IR might be completely different from the ones in the UV. A general strategy to attack this problem would be to provide a complete set of invariants, i.e. quantities that can be computed in any QFT (possibly satisfying some conditions), and that do not change under (suitably defined) "continuous deformation". There has been recent progress in implementing this program in certain simple classes of QFT's. In particular, it has been proposed that one- and two-dimensional minimally supersymmetric quantum field theories can be classified, up to deformations, by generalized cohomology theories known as K-theory and topological modular forms, respectively. The goal of this thesis is to describe these proposals and apply them to some simple examples of QFT's.

Contents

Introduction

One of the crucial problems in the study of quantum field theories of the last decades consists in understanding how these theories transform under continuous deformations at fixed energy or along the Renormalization Group (RG) flow. Indeed, quantum field theories can assume different forms at different energy levels, and it can be very difficult to figure out whether two of them can be deformed into one another. Let us consider, for instance, the description of strong interaction. It can be described at high energy, in the so-called UV regime, by QCD, a weakly-coupled theory which has as degrees of freedom quarks and gluons. However, moving down the RG flow and reaching the IR regime, QCD becomes strongly coupled and we need an apparently different theory with degrees of freedom given by mesons and baryons. This example allows us to understand how much the description of the same interaction can change at different energies. A useful approach to this problem consists in finding and computing some quantities, usually called *topological invariants*, that are invariant under continuous deformations.

One of the first and most important of these invariant was introduced by Edward Witten in [Wit82], and is defined as the regularized trace of the fermion number operator for a supersymmetric theory. As argued by Witten, as we will see later on, this quantity is a topological invariant for a particular class of theories, and allows us to figure out if the theory does not break supersymmetry. Other invariant quantities are, for instance, the elliptic genus for supersymmetric field theories or gravitational anomaly for theories in 2 dimensions. The problem that arises for all these invariants is that they are not *complete*, that is, the fact they are equal for two theories is a necessary condition for them to be deformable one on the other, but it is not also a sufficient condition in general. For this reason great efforts were done in order to refine these topological invariants and to obtain a complete one.

In the special case of 2 dimensional theories, Stephan Stolz and Peter Teichner in two works [ST04] [ST11], conjectured the existence of a one-to-one correspondence between two sets, which implies the existence of complete invariants. The sets considered by Stolz and Teichner was the set of 2-dimensional minimally supersymmetric Euclidean field theories SQFT up to deformation on one side, and the set of classes of a generalized cohomology theory known as *topological modular forms* TMF ([Hop95], [Hop02], [Goe09], [Dou+14]) on the other side. This last set was first introduced in order to describe the "universal elliptic cohomology theory", and it owes its name to the fact that the direct sum of all its homotopy groups is rationally isomorphic to the ring of weakly holomorphic integral modular forms. This last property, gives us the possibility to define a more refined version of the Witten index. Indeed, we will see that the index introduced by Witten can be regarded as a map from the set of SQFT to the ring of holomorphic modular forms, and so can be promoted to a *topological version*, describing it as a map from SQFT to TMF. In this way, if the Stolz and Teichner conjecture is assumed to be true, we obtain a complete set of invariants for supersymmetric field theories in 2 dimensions. However, the map that goes from TMF to the ring of weakly holomorphic modular forms, as we will see, is not an isomorphism on the integer numbers. In particular, the kernel of this map is the ideal generated by all the torsion classes in TMF. Hence, besides the Witten genus, which is the first invariant we can obtain from the conjecture, we have also other torsion invariants. Unfortunately, the physical interpretation of all these invariants is not known yet.

From a mathematical point of view, Stolz and Teichner proposal, even if there are a lot of evidence in favor of it, still lacks a precise and satisfactory formulation. The first problem comes from the construction of the set SQFT. Indeed, while the spectrum TMF is mathematically well-defined, we are not able to define properly, from a mathematical point of view, what an element in SQFT is. Also, we cannot completely identify what type of deformations we can consider, or, in other words, we are not able to properly topologize the set SQFT. In order to overcome these problems, we rely on the physical intuition of what a supersymmetric quantum field theory is, and we consider as deformations allowed the composition of small deformations at fixed energy and the motion along the RG flow.

In this framework, following some works by Davide Gaiotto, Theo Johnson-Freyd and Edward Witten ([GJW19], [GJ19], [Joh20]), we study the case of a particular model, the $\mathcal{N} = (0,1)$ supersymmetric sigma model with target S^3 and Wess-Zumino (WZ) coupling k, trying to understand in which cases it spontaneously breaks supersymmetry. This will bring us to study its behavior at low energies, and to describe how the previous authors have introduced an invariant, whose construction is motivated by the consequences of the works by Stolz and Teichner. We will see that this particular model breaks supersymmetry if and only if the WZ coupling has strength $k \equiv 0 \mod 24$, justifying this result in terms of the properties of TMF.

This work is organized as follows. In chapter 1 we introduce some of the basic tools needed to study the theories we are interested in. In particular, we deal with some of the properties of *conformal field theories*, which will be crucial for us since they can be obtained as IR fixed points for non-conformal theories.

In chapter 2, we describe the supersymmetric sigma model with target S^3 and WZ coupling k, which is the main example of the entire thesis. In describing it, we also introduce the Witten index, and in particular we show its crucial properties, some of which are encoded in the so-called *Atiyah-Singer index theorem*, that we state in 2.3.3.

Then, in chapter 3 we study the conjecture due to Stolz and Teichner, describing some aspects of the construction of TMF, and then focusing on the set SQFT, explaining some of the attempts that were done in order to give a proper mathematical definition of its elements, and also describing the properties that it has to inherit from TMF.

In the last two chapters, 4 and 5, we construct the invariant introduced by Gaiotto and Johnson-Freyd (mostly following [GJ19]) and then we compute it for the sigma model we started with. In doing this, we will introduce a way to enlarge the set of supersymmetric quantum field theories that admit elliptic genus, describing what we call *mildly non-compact theories*, and in particular we will focus on studying the way in which the properties of the elliptic genus change for this type of theories. This will allow us to compute the invariant for the sigma model with target S^3 and WZ coupling k, confirming the hypothesis that it spontaneously breaks supersymmetry if and only if $k \equiv 0 \mod 24$.

Chapter 1

Conformal field theories

Our main interest will be the study of field theories in (1+1)-dimensions, and in particular, considering their limit in the IR regime. In this situation, a special type of field theories acquires great importance. These are called *conformal field theories* (*CFT's*). Not all the theories we will encounter will be conformal, however some of the tools we will develop in studying CFT's will be crucial in the construction of the invariant we are looking for, since they appear as IR fixed points under the RG flow.

In this chapter, we will start in section 1.1 giving some information about the general d-dimensional case, but then we will focus on 2-dimensional conformal field theories, in which conformal symmetry will be the crucial constraint in order to obtain the information we need. In this first part we will use the formalism of fields. Then, in section 1.2 we will state some of the properties that arise using the operator formalism. In both case we will present some simple examples, which will reveal as the building blocks of more refined theories.

1.1 Conformal field theories

CFT in d dimensions

Let us start focusing on the general case of a d-dimensional space-time. **Conformal field theories** (**CFT**'s) are theories invariant under the so-called **conformal transformations**, that is, roughly speaking, transformations that preserve the angles between any two lines. More precisely, let us consider two d-dimensional manifolds M and M' with metric tensors g and g' respectively, and a differentiable map

$$\varphi : U \longrightarrow V$$

between two open subsets $U \subset M$ and $V \subset M'$. Then φ is said to be **conformal** if

$$\varphi^* g' = \Lambda g \ . \tag{1.1}$$

Given a coordinate system $x^\mu \in U$, and denoting $x' = \varphi(x)$, we have that the condition (1.1) reads

$$g'_{\rho\sigma}(x') \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = \Lambda(x) g_{\mu\nu}(x) \ , \tag{1.2}$$

with $\Lambda(x)$ a scale factor. For us, the most interesting situation will be the one in which $M = M'$, namely we have the same d-dimensional space-time, and with constant metric tensor $g_{\mu\nu}$. This implies that (1.2) becomes

$$g_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} = \Lambda(x)g_{\mu\nu} \ . \tag{1.3}$$

If we focus on infinitesimal transformations which, up to first order in a small parameter $\epsilon(x) \ll 1$, read

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x) + o(\epsilon^2) \ ,$$

we obtain that the condition for the parameter ϵ in order the transformation to be conformal is (we avoid to write down the computation being an easy expansion of the condition (1.3) in the parameter ϵ)

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}(\partial \cdot \epsilon)g_{\mu\nu} \ . \tag{1.4}$$

CFT in 2 dimensions

Conformal invariance in 2 dimensions require special attention, since, as we will see in a moment, in this case there is an infinite dimensional group of coordinates transformations that are locally conformal. The fact that these transformations are infinite, gives us the possibility to have exact solutions of 2-dimensional CFT's.

Studying the condition on the parameter of infinitesimal transformation in the Euclidean case, that is when the metric is

$$g_{\mu\nu} = \delta_{\mu\nu} \ ,$$

we find, from the relation (1.4)

$$\partial_0\epsilon_0 = \partial_1\epsilon_1 \ , \qquad \partial_0\epsilon_1 = -\partial_1\epsilon_0 \ ,$$

where we can recognize the *Cauchy-Riemann conditions*. This motivates the definition of complex coordinates

$$z := x^0 + ix^1 \ , \qquad \overline{z} := x^0 - ix^1 \ . \tag{1.5}$$

Before going on studying infinitesimal conformal transformation, let us comment on this change of coordinates. We started from a two dimensional space-time parametrized by the coordinates x^0 and x^1, and then, introducing z and $\overline{z}$, we have identified the real plane with the complex one, that is $\mathbb{R}^2 \simeq \mathbb{C}$. This is done using the fact that z and $\overline{z}$ are related one another by complex conjugation. However it will be useful to consider them as independent, hence identifying

$$\mathbb{R}^2 \longrightarrow \mathbb{C}^2 \ .$$

Nevertheless, it should be kept in mind that the physical space is the 2-dimensional submanifold of $\mathbb{C}^2$ identified by $z^* = \overline{z}$, called *real surface*.

Now let us go back to the conformal transformation and notice that, if we define the complex function

$$\epsilon(z) := \epsilon_0 + i\epsilon_1 \ ,$$

we have that $\epsilon(z)$, in order to describe a conformal transformation, has to be holomorphic. So a two dimensional conformal transformation is of the form

$$z \longmapsto f(z) \ ,$$

with $f(z) := z + \epsilon(z)$ an *holomorphic function*. Being this function holomorphic and, of course, the conjugate one *anti-holomorphic*, we can expand them in Laurent series around $z = 0$, getting

$$z' := f(z) = z + \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1} \ , \qquad \overline{z}' := \overline{f}(\overline{z}) = \overline{z} + \sum_{n\in\mathbb{Z}} \overline{\epsilon}_n \overline{z}^{n+1} \ ,$$

with ϵ_n and $\overline{\epsilon}_n$ constant infinitesimal parameters. From here it follows that the generators of a conformal transformation, for a given $n \in \mathbb{Z}$, are

$$l_n := -z^{n+1}\partial_z \ , \qquad \overline{l}_n := -\overline{z}^{n+1}\partial_{\overline{z}} \ , \tag{1.6}$$

and is trivial to verify that they satisfy the *Witt algebra*

$$\begin{aligned} [l_m, l_n] &= (m-n)l_{m+n} \ , \\ [\overline{l}_m, \overline{l}_n] &= (m-n)\overline{l}_{m+n} \ , \\ [l_m, \overline{l}_n] &= 0 \ . \end{aligned} \tag{1.7}$$

Let us notice that the generators that preserve the real surface, i.e. the physical ones, are the linear combinations

$$l_n + \overline{l}_n \ , \qquad i(l_n - \overline{l}_n) \ ,$$

and, in paritcular, $l_0 + \overline{l}_0$ generates dilations, while $i(l_0 - \overline{l}_0)$ generates rotations.

Now let us give an important definition in order to describe the behavior of fields under conformal transformations.

Definition 1.1.1. A field $\varphi(z, \overline{z})$ is said to have **conformal dimension** $(h, \overline{h})$ if, under the scaling $z \longmapsto \lambda z$, it transforms as

$$\varphi(z, \overline{z}) \longmapsto \varphi'(z, \overline{z}) = \lambda^h \overline{\lambda}^{\overline{h}} \varphi(\lambda z, \overline{\lambda}\overline{z}) \ .$$

Definition 1.1.2. A field is said to be a **primary field of conformal dimension** $(h, \overline{h})$ if, under the conformal transformation $z \longmapsto f(z)$, it transforms as

$$\varphi(z, \overline{z}) \longmapsto \varphi'(z, \overline{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}} \varphi\big(f(z), \overline{f}(\overline{z})\big) \ .$$

As we have said, the fact that the algebra of infinitesimal conformal transformations in 2 dimensions is infinite dimensional puts strong constraints on the field theory, and gives us the possibility to study the theory only knowing the behavior under conformal transformation, which are encoded in the energy-momentum tensor. Indeed, given a CFT, from Noether's theorem follows that there exist infinitely many conserved currents related to the conformal symmetry. These currents can be written in terms of the coordinates (x^0, x^1) as

$$j_\mu = T_{\mu\nu}\epsilon^\nu ,$$

where $T_{\mu\nu}$ is the symmetric **energy-momentum tensor** and ϵ^ν is an holomorphic function. From the conservation of the current $\partial^\mu j_\mu = 0$, it follows that the energy-momentum tensor has to be traceless, i.e.

$$T_\mu{}^\mu = 0 .$$

If we apply the change of coordinates $(x^0, x^1) \mapsto (z, \overline{z})$, we easily find that

$$T_{zz} = \frac{1}{2}(T_{00} - iT_{10}) , \quad T_{\overline{z}\overline{z}} = \frac{1}{2}(T_{00} + iT_{10}) , \quad T_{z\overline{z}} = T_{\overline{z}z} = 0 .$$

In particular, again from the conservation of the current in these new coordinates, we have that the non-vanishing components of the energy-momentum tensor split in an holomorphic and an anti-holomorphic part

$$T_{zz}(z, \overline{z}) \equiv T(z) , \qquad T_{\overline{z}\overline{z}}(z, \overline{z}) \equiv \overline{T}(\overline{z}) .$$

Going on in introducing useful tools for the study of CFT's, let us notice that, typically, correlation functions have singularities when the position of some of the fields coincide. In order to figure out how these singularities arise, we use the so-called **operator product expansion** (**OPE**). This means that we represent a product of operators by a sum of single operator terms, well-defined as the positions of the operators approach one to the other, and a function of the difference of the positions, which eventually diverges when they approach. In particular, a primary field φ of conformal dimensions $(h, \overline{h})$ has a peculiar OPE with the energy-momentum tensor, i.e.

$$T(z)\varphi(w, \overline{w}) \sim \frac{h}{(z-w)^2}\varphi(w, \overline{w}) + \frac{1}{z-w}\partial_w\varphi(w, \overline{w}) ,$$
$$\overline{T}(\overline{z})\varphi(w, \overline{w}) \sim \frac{\overline{h}}{(\overline{z}-\overline{w})^2}\varphi(w, \overline{w}) + \frac{1}{\overline{z}-\overline{w}}\partial_{\overline{w}}\varphi(w, \overline{w}) .$$

In these expressions, as we will always do in the OPE's, we use the symbol $\sim$ which indicates that we have neglected the regular terms as $w \to z$.

1.1.1 Free field theories

Now we want to study some theories that will be useful for the next sections. The theories we are going to introduce are actually CFT's, however we will also focus on some properties which are completely general. The aim of studying free field theories relies on the fact that they will be the building blocks for more complicated theories, and also some of the crucial properties that characterize QFT's in 2-dimensions are present in the free case yet.

Free bosonic field theory

Let us give the action of a theory of a scalar field φ in the space-time $\Xi = \mathbb{R}^2$ with Lorentzian metric, given by

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The action for this field theory is given by

$$\begin{aligned} S_\varphi &= \frac{1}{4\pi} \int_\Xi \mathrm{d}x^0\, \mathrm{d}x^1\, \mathcal{L}_\varphi(\varphi, \partial\varphi) = \\ &= \frac{1}{8\pi} \int_\Xi \mathrm{d}x^0\, \mathrm{d}x^1 \left(\big(\partial_0 \varphi(x^0, x^1)\big)^2 - \big(\partial_1 \varphi(x_0, x_1)\big)^2 \right) , \end{aligned}$$

where, as usual

$$\partial_0 := \frac{\partial}{\partial x^0} , \qquad \partial_1 := \frac{\partial}{\partial x^1} .$$

From the action, we can derive the Euler-Lagrange equations which describe the dynamics of the field. They are

$$\big((\partial_0)^2 - (\partial_1)^2\big) \varphi(x_0, x_1) = 0 ,$$

which is solved by

$$\varphi(x^0, x^1) = f(x^0 - x^1) + g(x^0 + x^1) , \tag{1.8}$$

with f and g arbitrary functions. Let us now define the **light-cone coordinates**

$$u := x^0 - x^1 , \qquad v := x^0 + x^1 , \tag{1.9}$$

and let us call **right-moving configurations** the ones depending only on u and **left-moving configurations** the ones depending only on v. From (1.8) it is clear how the general solution is given by the sum of a left- and a right-moving configurations, both at the speed of light and that do not interfere with each other. This is a general property of massless fields in $(1 + 1)$-dimensions, and it means that, in this case, there is a *decoupling of left- and right-moving modes.* Moreover, in these new coordinates, we can rewrite the action as follows. Since

$$\partial_0 = \partial_v + \partial_u , \qquad \partial_1 = \partial_v - \partial_u ,$$

we have that

$$(\partial_0 \varphi)^2 - (\partial_1 \varphi)^2 = 4 \partial_v \varphi \partial_u \varphi ,$$

hence

$$S_\varphi = \frac{1}{4\pi} \int_\Xi \mathrm{d}u\, \mathrm{d}v\, \partial_v \varphi \partial_u \varphi .$$

Up to now we have described this theory as a general QFT. However it is clear that it is invariant under conformal transformations, and so we can study it using the tools we have developed in the case of CFT's. First of all we have to reformulate the theory defining it on an Euclidean space-time. This is achieved thanks to the **Wick rotation**. This transformation is performed substituting the time coordinate as[1]

$$x^0_L \longmapsto -ix^0_E \ .$$

This implies that the action transforms as

$$S \longrightarrow iS_E \ ,$$

with S_E the Euclidean action. In this way the Euclidean action for the scalar field reads

$$S_\varphi = \frac{1}{8\pi}\int_\Xi \mathrm{d}x^0\,\mathrm{d}x^1\Big(\big(\partial_0\varphi\big)^2 + \big(\partial_1\varphi\big)^2\Big) \ .$$

Furthermore, we have seen how the natural framework to study CFT's is $\mathbb{C}^2$, obtained from the change of coordinates (1.5). In this way we have that

$$\big(\partial_0\varphi\big)^2 + \big(\partial_1\varphi\big)^2 = 4\partial_z\varphi\partial_{\overline{z}}\varphi \ ,$$

hence the action becomes

$$S_\varphi = \frac{1}{4\pi}\int \mathrm{d}z\,\mathrm{d}\overline{z}\,\partial_z\varphi\partial_{\overline{z}}\varphi \ .$$

As we can see comparing these with the previous results, we have that, after the Wick rotation, light-cone coordinates are mapped into the complex coordinates z and $\overline{z}$, in particular

$$u \longmapsto -i\overline{z} \ , \qquad v \longmapsto -iz \ .$$

Without performing explicitly the computations, we have that the quantum energy-momentum tensor for this theory is

$$T(z) = -\frac{1}{2}:\!\partial_z\varphi\partial_z\varphi\!: \ ,$$

where we have used the *normal ordering* since it is a composite field. Explicitly we mean

$$T(z) = -\frac{1}{2}\lim_{w\to z}\big(\partial_z\varphi(z)\partial_w\varphi(w) - \langle\partial_z\varphi(z)\partial_w\varphi(w)\rangle\,\big) \ .$$

The relevant OPE's are

$$\partial_z\varphi(z)\partial_w\varphi(w) \sim -\frac{1}{(z-w)^2} \ , \qquad T(z)\partial_w\varphi(w) \sim \frac{\partial_w\varphi(w)}{(z-w)^2} + \frac{\partial^2_w\varphi(w)}{(z-w)} \ ,$$

while

$$T(z)T(w) \sim \frac{1}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \ , \tag{1.10}$$

and analogous results hold for the anti-holomorphic part.

[1] In general we will omit the subscripts L (Lorentzian) and E (Euclidean) since the signature to which the coordinates are referred will always be clear from the context.

Free fermionic field theory

The other free theory we are interested in is the one of free Majorana fermions, which are anti-commuting real spinor fields. Another possibility consists in describing complex spinors, called Dirac fermions. In the $(1+1)$-dimensional space-time with Lorentzian metric, we need first to introduce the Clifford algebra and its generators. The algebra is generated by two elements γ^0 and γ^1 which satisfy

$$\{\gamma^i, \gamma^j\} = 2\eta^{ij} .$$

A possible representation for these generators is given by the 2×2 matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Let us remember how a Clifford algebra is defined in a general framework. Let V be a vector space on a field k and q a symmetric bilinear form on V with values in k

$$q : V \times V \longrightarrow k .$$

The **Clifford algebra** $\mathrm{Cliff}(V, q)$ is the k-algebra generated by V and defined by the relation

$$\{v_1, v_2\} = 2q(v_1, v_2) \cdot 1 \qquad \forall\, v_1, v_2 \in V ,$$

where 1 is the unit in $\mathrm{Cliff}(V, q)$. In particular, we will mostly use to deal with the algebra associated to the usual Euclidean form on $\mathbb{R}$ or $\mathbb{C}$. In this case we will indicate with $\mathrm{Cliff}(\pm n, \mathbb{R})$ the real Clifford algebra generated by n elements $\gamma_1, \ldots, \gamma_n$, such that $\gamma_i^2 = \pm 1$. In the same way we can define the complex Clifford algebra $\mathrm{Cliff}(\pm n, \mathbb{C})$.

Going on studying the free fermion theory, we have that its action in Minkowski $(1+1)$-dimensional space-time is

$$\begin{aligned} S_\psi &= \frac{1}{2\pi} \int_\Xi \mathrm{d}x^0 \, \mathrm{d}x^1 \, \mathcal{L}_\psi(\Psi, \partial\Psi) = \\ &= \frac{1}{4\pi} \int_\Xi \mathrm{d}x^0 \, \mathrm{d}x^1 \, i\Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi . \end{aligned} \tag{1.11}$$

A spinor field like the fermion can be written as a two-components column vector

$$\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} ,$$

where ψ_+ is said to be **chiral** (or with *positive chirality*), while ψ_- is said to be **anti-chiral** (or to have *negative chirality*). Indeed from the Clifford algebra's generators in generic dimension d, we can built a *chirality matrix*[2]

$$\gamma_5 := (-1)^{\frac{d}{4}-1} \gamma^1 \cdots \gamma^d .$$

[2] The notation γ_5 is the one common in physics, where it was built first in the 4-dimensional case. We keep the same notation, since there will be no confusion.

In the $(1+1)$-dimensional case we can choose

$$\gamma_5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

from which it is clear that

$$\gamma_5 \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix} = \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix} , \qquad \gamma_5 \begin{pmatrix} 0 \\ \psi_- \end{pmatrix} = -\begin{pmatrix} 0 \\ \psi_- \end{pmatrix} .$$

In terms of these chirality components, we have that

$$S_\psi = \frac{1}{4\pi}\int_\Xi \mathrm{d}x^0\,\mathrm{d}x^1 \left(i\psi_-(\partial_0 - \partial_1)\psi_- + i\psi_+(\partial_0 + \partial_1)\psi_+ \right) .$$

From the action, the Euler-Lagrange equations follow, and we get

$$\gamma^\mu \partial_\mu \Psi = 0 ,$$

which in terms of the chirality components reads

$$(\partial_0 + \partial_1)\psi_+ = 0 , \qquad (\partial_0 - \partial_1)\psi_- = 0 .$$

These equations are solved by

$$\psi_+(x^0, x^1) = f(x_0 - x_1) \equiv f(u) , \qquad \psi_-(x^0, x^1) = g(x^0 + x^1) \equiv g(v) .$$

Hence the chiral component ψ_+ is a right-moving mode, while the anti-chiral one ψ_- is a left-moving mode.

Rewriting the action in terms of the light-cone coordinates u and v, we get

$$S_\psi = \frac{1}{2\pi}\int_\Xi \mathrm{d}u\,\mathrm{d}v \left(i\psi_+\partial_v\psi_+ + i\psi_-\partial_u\psi_- \right) .$$

When we pass in the Euclidean space-time, we apply Wick rotation, but also we need to change the representation of the γ matrices, using, for instance

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \gamma^1 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ,$$

hence the form of the action (1.11) does not change. In terms of the complex coordinates $(z, \overline{z})$, let us write the spinor as

$$\Psi(z, \overline{z}) = \begin{pmatrix} \psi(z, \overline{z}) \\ \overline{\psi}(z, \overline{z}) \end{pmatrix} ,$$

in such a way that we obtain

$$S_\psi = \frac{1}{2\pi}\int \mathrm{d}z\,\mathrm{d}\overline{z} \left(i\psi\partial_{\overline{z}}\psi + i\overline{\psi}\partial_z\overline{\psi} \right) ,$$

The equations of motion follow and read

$$\partial_z \overline{\psi}(z,\overline{z}) = 0 \ , \qquad \partial_{\overline{z}} \psi(z,\overline{z}) = 0 \ ,$$

which means that ψ is an holomorphic field while $\overline{\psi}$ is anti-holomorphic.

Also, in Euclidean space-time we need to slightly modify the definition of γ_5 which is now given by

$$\gamma_5 := (-1)^{\frac{d}{4}} \gamma^1 \cdots \gamma^d \ .$$

However we have that, with respect to the representation given above, nothing change in the matrix form. Hence we have found the following relations

$$\begin{array}{ccccc} \text{chiral} & \longrightarrow & \text{holomorphic} & \longrightarrow & \text{right-moving} \\ \text{anti-chiral} & \longrightarrow & \text{anti-holomorphic} & \longrightarrow & \text{left-moving} \ . \end{array}$$

In this theory, the holomorphic energy-momentum tensor is

$$T(z) = -\frac{1}{2} :\!\psi(z)\partial_z\psi(z)\!: \ .$$

The relevant OPE's are

$$\psi(z)\psi(w) \sim \frac{1}{z-w} \ , \qquad \partial_z\psi(z)\psi(w) \sim -\frac{1}{(z-w)^2}$$
$$T(z)\psi(w) \sim \frac{1}{2}\frac{\psi(w)}{(z-w)^2} + \frac{\partial_w\psi(w)}{z-w} \ ,$$

where, from the last OPE, we see that ψ is a primary field of conformal dimension $h = \frac{1}{2}$. Finally, the OPE of the energy-momentum tensor with itself reads

$$T(z)T(w) \sim \frac{1}{4}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \ . \qquad (1.12)$$

As usual analogous results hold for the anti-holomorphic part.

Central charge and conformal transformation of the energy-momentum tensor

If we look at the OPE's in (1.10) and (1.12), we can see that they have the same form, except for a constant in front of the first summand. We can generalize this behavior saying that the OPE of the energy-momentum tensor with itself is

$$T(z)T(w) \sim \frac{c}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \ ,$$

where the constant c depends on the specific model and is called **central charge**. In particular, from (1.10) and (1.12) we have that the free bosonic theory has central charge $c = 1$, while the central charge of the free fermionic theory is $c = \frac{1}{2}$.

Now, comparing the OPE of the energy-momentum tensor with itself with the one we gave for a generic primary field, we can see that, for vanishing central charge, the energy-momentum tensor is a primary field of conformal dimension $h = 2$. But this is not true in general. However it can be shown that, under a conformal transformation $f(z)$, the energy momentum tensor transforms as

$$T'(z) = \Big(\frac{\partial f}{\partial z}\Big)^2 T\big(f(z)\big) + \frac{c}{12} S\big(f(z), z\big) \ , \tag{1.13}$$

where $S(w, z)$ is the so-called *Schwarzian derivative*, defined as

$$S(w, z) = \frac{1}{(\partial_z w)^2}\Big((\partial_z w)(\partial_z^3 w) - \frac{3}{2}(\partial_z^2 w)^2\Big) \ .$$

1.2 Operator formalism

Up to now we have studied CFT's with the language of fields, hence all the information where expressed in terms of fields, and we have never mentioned the operator formalism or the Hilbert space. However it will be useful in what follows to deal with them.

Let us start recalling that, from an operator point of view, we need to distinguish between space and time coordinates. This distinction is natural in a Minkowski space-time, thanks to the non-trivial signature. However the same thing cannot be said about Euclidean space-time, in which the selection of space and time is somewhat arbitrary. In order to treat this situation, let us start from a theory defined on an infinite space-time cylinder $S^1_R \times \mathbb{R}$, where time parametrize the $\mathbb{R}$ direction, while the space is compactified identifying the point that differs for $2\pi R$. If we consider this situation in a Minkowski space-time, and then we continue to the Euclidean one, we obtain that it is parametrized by the complex coordinate

$$w := x^0 + i x^1 \quad \text{such that} \quad w \sim w + 2\pi i R \ ,$$

where we have indicated the periodic identification of the spatial direction. Then we can apply a further change of variables, mapping the cylinder in the complex plane. This is achieved introducing the complex coordinate on the plane

$$z := e^{\frac{w}{R}} = e^{\frac{x^0}{R}} e^{\frac{i x^1}{R}} \ ,$$

from which, in particular, we see that the infinite past of the cylinder $x^0 = -\infty$ is mapped to $z = \overline{z} = 0$, whereas the infinite future $x^0 = +\infty$ lies on the point at the infinity.

In these coordinates for the complex plane, we can expand the fields in Laurent series. In particular, given a field of conformal dimension $(h, \overline{h})$, its Laurent expansion around $z = \overline{z} = 0$ reads

$$\varphi(z, \overline{z}) = \sum_{n, \overline{m} \in \mathbb{Z}} x^{-n-h} \overline{z}^{-\overline{m}-\overline{h}} \varphi_{n, \overline{m}} \ .$$

In this formalism we are interested in the operator that generates conformal transformations. What can be seen is that the conformal charge is

$$Q_\epsilon = \frac{1}{2\pi i} \oint \mathrm{d}z \, \epsilon(z) T(z) \ .$$

So let us expand in Laurent series the energy-momentum tensor and the infinitesimal conformal parameter $\epsilon(z)$ as

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \ , \qquad \overline{T}(\overline{z}) = \sum_{n\in\mathbb{Z}} \overline{z}^{-n-2} \overline{L}_n \ ,$$
$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{n+1} \epsilon_n \ .$$

Hence the conformal charge reads

$$Q_\epsilon = \sum_{n\in\mathbb{Z}} \epsilon_n L_n \ .$$

In this way we have found that the modes of the energy-momentum tensor L_n and $\overline{L}_n$ are the generators of the local conformal transformations on the Hilbert space, and are the analogous of the generators of the conformal mapping l_n and $\bar{l}_n$ on the space of functions. In particular, let us notice that the combination $L_0 + \overline{L}_0$ generates dilations on the complex plane, which corresponds to time translations. In the same way we have that $i(L_0 - \overline{L}_0)$ generates rotations, which are spatial translations. This means that these combinations are proportional to the Hamiltonian and the momentum operators respectively, namely

$$H = L_0 + \overline{L}_0 \ , \qquad P = i(L_0 - \overline{L}_0) \ . \tag{1.14}$$

The generators L_n and $\overline{L}_n$ satisfy an algebra analogous to the Witt one (1.7), but with an additional term due to the presence of the central charge. In mathematical terms they satisfy the central extension of the Witt algebra

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \ , \\
[\overline{L}_n, \overline{L}_m] &= (n-m)\overline{L}_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \ , \\
[L_n, \overline{L}_m] &= 0 \ ,
\end{aligned}$$

called the **Virasoro algebra**.

Let us conclude this part discussing very briefly some of the characteristics of the Hilbert spaces of a CFT, which actually can have very intricate structures. First of all we look at the vacuum state $|0\rangle$, which should be invariant under global conformal transformation. In order for this to be true, and also since the quantities $T(z)\,|0\rangle$ and $\overline{T}(\overline{z})\,|0\rangle$ have to be well-defined as $z, \overline{z} \to 0$, we have that

$$L_n\,|0\rangle = 0 \ , \qquad \overline{L}_n\,|0\rangle = 0 \ , \quad n \geq -1 \ .$$

Primary fields acting on the vacuum states create eigenstates of the Hamiltonian, and satisfy the following commutation relations (whose expression can be easily derived from the OPE of the product of primary field and energy-momentum tensor)

$$\begin{aligned}
[L_n, \varphi(z,\overline{z})] &= h(n+1)z^n \varphi(z,\overline{z}) + z^{n+1}\partial_z \varphi(z,\overline{z}) \ , \qquad && n \geq -1 \\
[\overline{L}_n, \varphi(z,\overline{z})] &= \overline{h}(n+1)\overline{z}^n \varphi(z,\overline{z}) + \overline{z}^{n+1}\partial_{\overline{z}} \varphi(z,\overline{z}) \ , \qquad && n \geq -1
\end{aligned}$$

where $\varphi(z,\overline{z})$ is a primary field of conformal dimension $(h,\overline{h})$. If we now define the *asymptotic state*

$$|h,\overline{h}\rangle := \lim_{z,\overline{z}\to 0} \varphi(z,\overline{z})\,|0\rangle \ ,$$

applying the previous commutation relations, we conclude that

$$L_0\,|h,\overline{h}\rangle = h\,|h,\overline{h}\rangle \ , \qquad \overline{L}_0\,|h,\overline{h}\rangle = \overline{h}\,|h,\overline{h}\rangle \ ,$$

and hence $|h,\overline{h}\rangle$ is an eigenstate of the Hamiltonian. Analogously we have

$$L_n\,|h,\overline{h}\rangle = 0 \ , \qquad \overline{L}_n\,|h,\overline{h}\rangle = 0 \ , \quad n > 0 \ .$$

Excited states above the asymptotic states can be obtained by adding ladder operators, which explicitly read, expanding the primary field $\varphi(z,\overline{z})$ in its Laurent modes

$$[L_n, \varphi_m] = \big(n(h-1) - m\big)\varphi_{n+m} \ .$$

In particular

$$[L_0, \varphi_m] = -m\varphi_m \ ,$$

and

$$[L_0, \varphi_0] = 0 \ . \tag{1.15}$$

This last commutation relation tells us that, when it is defined, the zero mode of the primary field creates a degeneracy of the energy eigenstates.

1.2.1 Free fermion theory

The idea now is to study the characteristics of the free fermion theory from an operator perspective, defining it on the cylinder and then using the tools we have introduced in order to figure out how to built its Hilbert space.

We have introduced yet the action and the energy-momentum tensor for this theory, and we have seen the crucial OPE's, thanks to which we have been able to show that the system has central charge $c = \frac{1}{2}$ and the fermion field ψ has conformal dimension $h = \frac{1}{2}$. Now, let us work on the cylinder of radius R. The mode expansion is

$$\psi(w) = \sqrt{\frac{1}{R}} \sum_n \psi_n e^{-n\frac{w}{R}} \ ,$$

where the operators ψ_n obey the anti-commutation relations

$$\{\psi_n, \psi_n\} = \delta_{n+m,0} \ . \tag{1.16}$$

Since the space-time is a cylinder, we have to impose boundary conditions on the fields, and due to the spinorial nature of fermions, we have two possibilities

$$\begin{aligned} &\psi(w + 2\pi i R) = -\psi(w) && \text{Neveu-Schwarz sector (NS)} \\ &\psi(w + 2\pi i R) = +\psi(w) && \text{Ramond sector (R)} \ . \end{aligned}$$

It is clear that, in order for these boundary conditions to be fulfilled, we have some constraints on the index of the sum, indeed

$$
\begin{aligned}
&n \in \mathbb{Z} && \text{Neveu-Schwarz sector (NS)} \\
&n \in \mathbb{Z} + \frac{1}{2} && \text{Ramond sector (R)} .
\end{aligned}
$$

We have yet explained how the cylinder can be mapped on the complex plane, so let us perform this transformation and let us see how the fermionic fields behave. We have to map

$$w \longmapsto z = e^{\frac{w}{R}} ,$$

and, since the fermionic field has conformal dimension $h = \frac{1}{2}$, it transforms as

$$
\begin{aligned}
\psi_{\mathrm{cyl}}(w) \longmapsto \psi_{\mathrm{cyl}}(z) &= \left(\frac{\mathrm{d}z}{\mathrm{d}w}\right)^{\frac{1}{2}} \psi_{\mathrm{pl}}(z) \\
&= \sqrt{\frac{z}{R}} \psi_{\mathrm{pl}}(z)
\end{aligned}
$$

This means that the expansion on the modes on the complex plane is (we do not write anymore the subscript pl since from now on it will be always clear from the context)

$$\psi(z) = \sum_n \psi_n z^{-n-\frac{1}{2}} , \tag{1.17}$$

as we expected. Now we need to translate also the boundary conditions we have imposed on the cylinder. However, due to the presence of the factor $z^{\frac{1}{2}}$, we have to exchange the conditions, obtaining

$$
\begin{aligned}
&\psi(e^{2\pi i} z) = +\psi(z) && \text{Neveu-Schwarz sector (NS)} \\
&\psi(e^{2\pi i} z) = -\psi(z) && \text{Ramond sector (R)} ,
\end{aligned}
$$

which, in terms of the index in the mode expansion read

$$
\begin{aligned}
&n \in \mathbb{Z} + \frac{1}{2} && \text{Neveu-Schwarz sector (NS)} \\
&n \in \mathbb{Z} && \text{Ramond sector (R)} .
\end{aligned}
$$

The next step consists in studying the vacuum energies. In particular we are going to focus on the Ramond sector, in which we will find a peculiar result, due to the presence of the fermionic zero modes. We want to compute the vacuum expectation value of the energy-momentum tensor in the Ramond sector. Hence, remembering the Laurent expansion (1.17) and the relation (1.16), let us start computing the two-point function

$$
\begin{aligned}
\langle \psi(z)\psi(w)\rangle &= \sum_{n,m\in\mathbb{Z}} z^{-n-\frac{1}{2}} w^{-m-\frac{1}{2}} \langle \psi_n \psi_m \rangle = \frac{1}{2\sqrt{zw}} + \sum_{n=1}^{+\infty} z^{-n-\frac{1}{2}} w^{n-\frac{1}{2}} = \\
&= \frac{1}{\sqrt{zw}} \left(\frac{1}{2} + \sum_{n=1}^{+\infty} \left(\frac{w}{z}\right)^n \right) = \frac{1}{2\sqrt{zw}} \frac{z+w}{z-w} = \\
&= \frac{1}{2(z-w)} \left(\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}} \right) .
\end{aligned}
$$

So the vacuum expectation value of the energy-momentum tensor follows

$$
\begin{aligned}
\langle T(z)\rangle &= \frac{1}{2} \lim_{\epsilon\to 0} \Big(- \langle \psi(z+\epsilon)\partial\psi(z)\rangle + \frac{1}{\epsilon^2} \Big) = \\
&= \lim_{w\to z} \left[- \frac{1}{4(z-w)} \left(\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}} \right) + \frac{1}{2(z-w)^2} \right] = \\
&= \frac{1}{16 z^2} .
\end{aligned}
$$

From the definition of the energy-momentum tensor, its expansion in terms of the fermionic modes is

$$
\begin{aligned}
T(z) &= \frac{1}{2} \sum_{n,m\in\mathbb{Z}} \Big(m + \frac{1}{2} \Big) z^{-n-\frac{1}{2}} z^{-m-\frac{3}{2}} :\!\psi_n \psi_m\!:= \\
&= \frac{1}{2} \sum_{n,m\in\mathbb{Z}} \Big(m + \frac{1}{2} \Big) z^{-n-2} :\!\psi_{n-m} \psi_m\!: ,
\end{aligned}
$$

thanks to which we deduce the generators

$$
L_n = \frac{1}{2} \sum_{m\in\mathbb{Z}} \Big(m + \frac{1}{2} \Big) :\!\psi_{n-m} \psi_m\!: \; .
$$

From this general expression we can find the generator L_0, but paying attention to the fact that, since in the Ramond sector the energy-momentum tensor has non-vanishing expectation value, we have to add the contribution of the vacuum, that is

$$
L_0 = \sum_{m\in\mathbb{Z}_{>0}} m \psi_{-m} \psi_m + \frac{1}{16} . \tag{1.18}
$$

Let us end this section with a behavior that is proper of any system with non-vanishing central charge. In what follows will be useful to express the generators of conformal transformations defined on the cylinder. In order to do this, let us apply the usual transformation from the complex plane to the cylinder. Under this change of coordinates, the energy-momentum tensor transforms as (1.13), hence, neglecting the dependence on the radius of the cylinder without loosing generality (since we are in the conformal case), we get

$$T_{\text{cyl}}(w) = z^2 T(z) - \frac{c}{24} = \sum_{n\in\mathbb{Z}} \Big(L_n - \frac{c}{24}\delta_{n,0}\Big) e^{-nw} \ ,$$

from which, in particular

$$(L_{\text{cyl}})_0 = L_0 - \frac{c}{24} \ .$$

Now remembering the definition we gave for the Hamiltonian and the momentum operators in (1.14), we obtain

$$\begin{aligned} H_{\text{cyl}} &= (L_{\text{cyl}})_0 + (\overline{L}_{\text{cyl}})_0 = L_0 + \overline{L}_0 - \frac{c+\overline{c}}{24} \ , \\ P_{\text{cyl}} &= i\big((L_{\text{cyl}})_0 - (\overline{L}_{\text{cyl}})_0\big) = i(L_0 - \overline{L}_0) - i\frac{c+\overline{c}}{24} \ . \end{aligned} \tag{1.19}$$

Chapter 2

$\mathcal{N} = (0, 1)$ supersymmetric models

In this chapter we are going to introduce the example we want to focus on, from a purely physical perspective. We will start in section 2.1, introducing the main features of supersymmetric field theories in $(1 + 1)$-dimensions, with a particular attention in $\mathcal{N} = (0, 1)$ supersymmetric models. Then, in 2.2, we will introduce the sigma model with target S^3 and Wess-Zumino coupling k. In order to do this, we will present the condition under which the model is well-defined, and, following [GJW19], we will describe its behavior in the IR regime. Finally, in 2.3, we will introduce the first topological invariant we encounter, that is the *Witten index*. In doing this, we will explain how to construct the partition function of the supersymmetric field theory, and how to vary this construction in order to properly define the Witten index. We will conclude the chapter with a crucial result on the Witten index, known as *Atiyah-Patodi index theorem*.

2.1 Supersymmetric field theories in $(1 + 1)$ dimensions

In general, supersymmetric theories can be seen as, roughly speaking, theories with an additional fermionic symmetry which maps bosons in fermions and vice versa. In order to study these theories, we need to introduce a new formalism, in which supersymmetric transformations are seen as translations in a specific direction on a properly defined space, called *superspace*.

2.1.1 $\mathcal{N} = (0, 1)$ supersymmetric field theories

Let us start considering a field theory on $\Xi := \mathbb{R}^2$ with space and time coordinates x^0, x^1 and metric given by

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Let us then parametrize the Minkowski space-time through the light-cone bosonic coordinates u and v introduced in (1.9), which, under Lorentz boosts of parameter a, transform as

$$u \longrightarrow e^a u \, , \quad v \longrightarrow e^{-a} v \, . \tag{2.1}$$

Then let us introduce a fermionic (Grassmanian) coordinate θ such that

$$\theta^2 = 0 \; ,$$

and with the properties

$$\int \mathrm{d}\theta \, \theta = 1 \; , \qquad \int \mathrm{d}\theta \, f(u,v) = 0 \; .$$

This coordinate under the transformation (2.1), behaves like

$$\theta \longrightarrow e^{\frac{1}{2}a}\theta \; ,$$

thus it is a chiral coordinate, i.e. right-moving. These three coordinates (u, v, θ) parametrize the so-called **superspace** $\widehat{\Xi}$ with $\mathcal{N} = (0,1)$ supersymmetry, and the $\mathcal{N} = (0,1)$ supersymmetry is generated by the operator

$$Q = \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial u} \; ,$$

called **supercharge**, which satisfies

$$Q^2 = i\frac{\partial}{\partial u} \; .$$

Finally, the *superspace derivative* is defined as

$$D := \frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial u} \; ,$$

and it commutes with the supercharge Q and the ordinary derivatives ∂_u and ∂_v.

In this framework we are interested in the study of **superfields**, which are defined as functions on the superspace. A generic superfield $\mathcal{F}(u, v, \theta)$ can be Taylor expanded as[1]

$$\mathcal{F}(u,v,\theta) = f_0(u,v) + \theta f_1(u,v) \; .$$

Given a set of superfields, we want to introduce a supersymmetric action in order to describe their dynamics. With this new formalism, there is a natural way to do this. Indeed, we have that the integral in superspace of any arbitrary superfield is a supersymmetric invariant quantity. In order to see this property, let us consider a generic superfield $Y(u, v, \theta)$ (which of course can be a generic combination of various superfields), and consider the integral

$$\int \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}\theta \, Y(u,v,\theta) \; . \tag{2.2}$$

The supersymmetry variation take the form

$$\delta_\epsilon = \epsilon Q \; ,$$

[1] Of course this expansion is exact since $\theta^2 = 0$.

hence, varying the integral we get

$$\begin{aligned}\delta_\epsilon \int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\, Y(u,v,\theta) &= \int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\, \delta_\epsilon Y(u,v,\theta) = \\ &= \int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\, \epsilon Q Y(u,v,\theta) = \\ &= \int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\, \epsilon\big(\partial_\theta + i\theta\partial_u\big)\big(y_0(u,v) + \theta y_1(u,v)\big) = \\ &= \int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\, \epsilon\big(y_1(u,v) + i\theta\partial_u y_0(u,v)\big) = \\ &= \int \mathrm{d}u\,\mathrm{d}v\, i\epsilon\partial_u\big(y_0(u,v)\big)\end{aligned}$$

We have found that the variation of the integrand like (2.2) is a total derivative, which vanishes after the integration over $\mathrm{d}u\,\mathrm{d}v$, and hence it is supersymmetric invariant.

Now we want to restrict our attention to some particular classes of fields, and study their actions.

The first type of field is the so-called **scalar superfield** $\Phi(u,v,\theta)$. It can be expanded as

$$\Phi(u,v,\theta) = \varphi(u,v) + i\theta\psi(u,v)\ ,$$

where φ is a scalar field while ψ is a (anti-chiral) right-moving fermion. At the classical level, the supersymmetric free action for this superfield is given by

$$S_\Phi = \frac{i}{2\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\partial_v\Phi D\Phi\ .$$

Expanding we get

$$\begin{aligned}S_\Phi &= \frac{i}{2\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\partial_v\big(\varphi + i\theta\psi\big)(\partial_\theta - i\theta\partial_u)\big(\varphi + i\theta\psi\big) = \\ &= \frac{i}{2\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\big(\partial_v\varphi + i\theta\partial_v\psi\big)\big(i\psi - i\theta\partial_u\varphi\big) = \\ &= \frac{i}{2\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\big(i(\partial_v\varphi)\psi - i\theta\partial_v\varphi\partial_u\varphi + \theta\psi\partial_v\psi\big) = \\ &= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\left(\frac{1}{2}\partial_v\varphi\partial_u\varphi + \frac{i}{2}\psi\partial_v\psi\right)\ .\end{aligned} \tag{2.3}$$

The second superfield we want to consider is the **Fermi superfield** $\Lambda(u,v,\theta)$, i.e. an anti-commuting superfield. In particular it can be expanded as

$$\Lambda(u,v,\theta) = \xi(u,v) + \theta F(u,v)\ ,$$

where ξ is a fermionic (chiral) left-moving field on Ξ, while F is an auxiliary field. The free action for this field is given by

$$S_\Lambda = \frac{1}{2}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\Lambda D\Lambda\ .$$

Expanding it we get

$$
\begin{aligned}
S_\Lambda &= \frac{1}{2\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\big(\xi+\theta F\big)(\partial_\theta - i\theta\partial_u)\big(\xi+\theta F\big) = \\
&= \frac{1}{2\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,(\xi+\theta F)(F - i\theta\partial_u\xi) = \\
&= \frac{1}{2\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,(\xi F + i\theta\xi\partial_u\xi + \theta F^2) = \\
&= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\left(\frac{i}{2}\xi\partial_u\xi + \frac{1}{2}F^2\right)\,, \qquad (2.4)
\end{aligned}
$$

where it becomes clear why F is an auxiliary field.

Up to now we have introduced only kinetic terms for the fields. However we can also introduce interaction terms. For later purpose, let us consider the interaction between n scalar superfields Φ^I and a Fermi superfield Λ. Let W be some real-valued function depending on scalar superfields, called **superpotential**. The coupling term we can add to the action reads

$$
S_W = \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\Lambda W(\Phi^I)\,.
$$

Expanding it we get

$$
\begin{aligned}
S_W &= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\big(\xi+\theta F\big)W(\varphi^I + i\theta\psi^I) = \\
&= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\big(\xi+\theta F\big)\Big(W(\varphi^I) + i\theta\sum_K \psi^K\frac{\partial W(\varphi^I)}{\partial\varphi^K}\Big) = \\
&= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\Big(\xi W(\varphi^I) + i\theta\xi\sum_K \psi_K\frac{\partial W(\varphi^I)}{\partial\varphi^K} + \theta F W(\varphi^I)\Big) = \\
&= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\Big(F W(\varphi^I) + i\xi\sum_K \psi^K\frac{\partial W(\varphi^I)}{\partial\varphi^K}\Big)\,. \qquad (2.5)
\end{aligned}
$$

In the end we have obtained

$$
\begin{aligned}
S_\Lambda + S_W &= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\Big(\frac{i}{2}\xi\partial_u\xi + \frac{1}{2}F^2 + F W(\varphi^I) + i\xi\sum_K \psi^K\frac{\partial W(\varphi^I)}{\partial\varphi^I}\Big) \\
&=: \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathcal{L}_{\Lambda+W}\,,
\end{aligned}
$$

from which we can find the equation of motion of the field F, i.e.

$$
\partial_\mu\frac{\partial\mathcal{L}_{\Lambda+W}}{\partial(\partial_\mu F)} - \frac{\partial\mathcal{L}_{\Lambda+W}}{\partial F} = 0 \implies F = -W(\varphi^I)\,.
$$

Hence, integrating out the auxiliary field F, namely substituting its equation of motion in the action, we obtain

$$
S' = \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\Big(\frac{i}{2}\xi\partial_u\xi - \frac{1}{2}W^2(\varphi^I) - i\xi\sum_K \psi^K\frac{\partial W(\varphi^I)}{\partial\varphi^K}\Big)\,, \qquad (2.6)
$$

from which the *scalar potential* is

$$V(\varphi^I) = \frac{1}{2} W^2(\varphi^I) \; . \tag{2.7}$$

In particular, from (2.6), we see that, if the superpotential W is linear in the fields, or the scalar fields have non vanishing vacuum expectation values (in which case the derivative of W has to be evaluated on the vacuum expectation values), at every point of the field space in which $\mathrm{d}W \neq 0$ (namely its derivatives with respect to φ^I do not vanish), a linear combination of the ψ^I and ξ combine themselves to get a mass.

2.2 Sigma model

Now we want to focus on a particular $\mathcal{N} = (0, 1)$ supersymmetric model in $(1 + 1)$-dimensions, known as **sigma model**. This is defined as the system corresponding to maps from a $(1+1)$-dimensional space Ξ to a target space which is a Riemannian manifold M of dimension n. In this theory we have n scalar superfields $\Phi^I(u, v, \theta)$ $(I = 1, \ldots, n)$ which now describe a map

$$\Phi : \widehat{\Xi} \longrightarrow M \; .$$

In particular, given a set of local coordinates x^I $(I = 1, \ldots, n)$ on M, we have that

$$\Phi^* x^I = x^I \circ \Phi = \Phi^I \; .$$

These superfields can be expanded as

$$\Phi^I(u, v, \theta) = \varphi^I(u, v) + i\theta\psi^I(u, v) \; , \tag{2.8}$$

where the φ^I's are bosonic fields describing the map

$$\varphi : \Xi \longrightarrow M$$

in the same way as before. Instead, the ψ^I's are (chiral) right-moving fermion on Ξ valued on the pull-back $\varphi^*(TM)$, i.e. are sections

$$\psi^I \in \Gamma(\Xi, \varphi^*(TM)) \; .$$

At the classical level, the supersymmetric action is analogous as before, and reads

$$S_\Phi = \frac{i}{2\pi} \int \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}\theta \, g_{IJ} \nabla_v \Phi^I D \Phi^J$$

where g_{IJ} is the metric tensor on M and we have introduced the *covariant derivative* on the target manifold, which acts as

$$\begin{aligned} \nabla_v \varphi^I &= \partial_v \varphi^I \; , \\ \nabla_v \psi^I &= \partial_v \psi^I + i{\Gamma^I}_{JK}(\partial_v \varphi^J)\psi^K \; , \end{aligned} \tag{2.9}$$

where Γ^I_{JK} are the *Christoffel symbols* of the Levi-Civita connection on the target manifold, given by

$$\Gamma^I{}_{JK} = \frac{1}{2} g^{IL}\big(\partial_K g_{LJ} + \partial_J g_{LK} - \partial_L g_{JK}\big) \ .$$

Expanding we get

$$\begin{aligned}
S_\Phi &= \frac{i}{2\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\, g_{IJ}\nabla_v\big(\varphi^I + i\theta\psi^I\big)\big(\partial_\theta - i\theta\partial_u\big)\big(\varphi^J + i\theta\psi^J\big) = \\
&= \frac{i}{2\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\, g_{IJ}\big(\partial_v\varphi^I + i\theta\nabla_v\psi^I\big)\big(i\psi^J - i\theta\partial_u\varphi^J\big) = \\
&= \frac{i}{2\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\, g_{IJ}\big(i(\partial_v\varphi^I)\psi^J - i\theta\partial_v\varphi^I\partial_u\varphi^J + \theta\psi^J\nabla_v\psi^I\big) = \\
&= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\Big(\frac{1}{2}g_{IJ}\partial_v\varphi^I\partial_u\varphi^J + \frac{i}{2}g_{IJ}\psi^J\partial_v\psi^I - \frac{1}{2}\Gamma_{IJK}\psi^I\psi^J\partial_v\varphi^K\Big) \ . \qquad (2.10)
\end{aligned}$$

2.2.1 B-field

The action given above actually is not the most general we can have for a sigma model. Let us consider the case in which the space-time is $\mathbb{R}^2$ with Euclidean signature and the target manifold for the model is a Lie group G, with metric given by the Cartan-Killing form. Then let us compactify the space-time adding a point at the infinity, hence considering the *Riemann sphere*

$$S^2 \simeq \mathbb{R}^2 \cup \{\infty\} \ .$$

The fields of this sigma model are maps

$$g : S^2 \longrightarrow G \ ,$$

whose action, which is equivalent to (2.10), reads

$$S_g = \frac{1}{4a^2}\int_{S^2} \mathrm{d}^2x \ \mathrm{Tr}\left[\partial^\mu g^{-1}\partial_\mu g\right] \ ,$$

with a^2 a positive, dimensionless coupling constant. In the case in which the second homotopy group of G is trivial, $\pi_2(G)=\{0\}$, the image of the 2-sphere through g, $g(S^2)$, can be seen as the boundary of a non-unique open ball $B_3 \subset G$, that is

$$\partial B_3 = g(S^2) \ .$$

Then we can add to the action the so called **Wess-Zumino** (**WZ**) **term** (which is a total derivative up to boundary contributions)

$$\Gamma_{B_3} = -\frac{i}{24\pi}\int_{B_3} \mathrm{d}^3x\ \epsilon_{ijk}\,\mathrm{Tr}\left[g^{-1}(\partial^i g)g^{-1}(\partial^j g)g^{-1}(\partial^k g)\right] \ ,$$

which, in order to be well defined, should depend only on the field g, and not on the chosen ball B_3. So given another ball B_3' such that its boundary coincides with the image through g of the 2-sphere, $\partial B_3' = g(S^2)$, the difference

$$\Gamma_{B_3'} - \Gamma_{B_3} = -\frac{i}{24\pi}\int_{S^3} \mathrm{d}^3x\ \epsilon_{ijk}\,\mathrm{Tr}\left[g^{-1}(\partial^i g)g^{-1}(\partial^j g)g^{-1}(\partial^k g)\right] \ ,$$

should be zero. In the previous integral, the manifold on which we have to integrate is $B'_3 - B_3$, where, with the difference, we mean that we take the union between B'_3 and $\overline{B_3}$, that is B_3 considered with the opposite orientation. It is easy to see that this is equivalent to integrate over the whole compact 3-dimensional space G, which is topologically equivalent to the 3-sphere S^3. The problem is that this integral does not vanish. Actually the term that should be independent from the choice of the B_3 is the factor in the action due to this contribution, that is

$$e^{ik\Gamma_{B_3}} \ ,$$

for a (in principle) generic constant k. It can be shown[2] that the difference is actually an integer multiple of 2π, hence, in order for the term to be well-defined, its coupling, known as **Wess-Zumino coupling**, has to be an integer, $k \in \mathbb{Z}$. In this way we have obtained a more general action given by

$$S_{\sigma+\text{WZ}} = \frac{1}{4a^2} \int_{S_2} \mathrm{d}^2x \ \mathrm{Tr}\left[\partial^\mu g^{-1} \partial_\mu g\right] + k\Gamma_{B_3} \ .$$

If we now flow in the IR limit along the RG trajectories, we have that the coupling k, being an integer, does not get renormalized, but the constant a^2 does. In particular, flowing the RG trajectories, we obtain a fixed point when

$$a^2 = \frac{4\pi}{k} \ .$$

The model obtained in this way is called **Wess-Zumino-Witten mode** (**WZW model**), and is a conformal interacting model (see [Wit84]).

Going on, the WZ term can be rewritten as an integral of a 3-form H on the target manifold $H \in \Omega^3(G)$, that is

$$\Gamma_{B_3} = \int_{B_3} H \ .$$

At the classical level this form results to be the field-strength of a 2-form $B \in \Omega^2(G)$ known as **B-field**

$$H = \mathrm{d}B \ ,$$

from which follows the identity

$$\mathrm{d}H = 0 \ .$$

In this situation, thanks to Stokes' theorem, we can write the WZ term as

$$\Gamma_{B_3} = \int_{B_3} H = \int_{g(S^2)} B = \int_{S^2} g^* B \ .$$

Given a system of coordinates x^I, $I = 1, \ldots, n$ on the target manifold, denoting now the field of the theory as the superfield Φ^I defined as (2.8) (in order to be consistent with the

[2] For details look at [DMS97].

previous results), we get[3]

$$S_{\sigma+WZ}=\frac{k}{16\pi}\int \mathrm{d}u\,\mathrm{d}v\,\Big(\big(g_{IJ}(\varphi)+B_{IJ}(\varphi)\big)\partial_v\varphi^I\partial_u\varphi^J+ \\ +ig_{IJ}\psi^I\partial_v\psi^J-\big(\Gamma_{IJK}+\frac{1}{2}H_{IJK}\big)\psi^I\psi^K\partial\varphi^J\Big)\ ,$$

where of course

$$\begin{aligned} B&=\frac{1}{2}B_{IJ}\,\mathrm{d}x^I\wedge \mathrm{d}x^J\ ,\\ H&=\frac{1}{3!}H_{IJK}\,\mathrm{d}x^I\wedge\mathrm{d}x^J\wedge\mathrm{d}x^K \qquad \text{with } H_{IJK}=\partial_I B_{JK}+\partial_J B_{KI}+\partial_K B_{IJ}\ .\end{aligned}$$

When we try to quantize the theory, we have to pay attention to some aspects. First of all we have yet explained that the form H has to satisfy a Dirac quantization condition. We have imposed yet asking the difference $\Gamma_{B_3'}-\Gamma_{B_3}=\int_{S_3}H$ to be an integer multiple of 2π. This can be restated saying that, in order to define the model, it is necessary to choose a class $x\in \mathrm{H}^3(M,\mathbb{Z})$ which can be represented by the 3-form $H/2\pi$. For instance, in the case in which $M=S^3$, we have that

$$\mathrm{H}^3(S^3,\mathbb{Z})\simeq\mathbb{Z}\ ,$$

so, choosing the class is the same as choosing an integer k, with

$$k=\int_{S^3}\frac{H}{2\pi}\in\mathbb{Z}\ . \tag{2.11}$$

Passing to the quantum level, some anomalies can arise. First of all it is necessary to require the target manifold M to be a spin manifold, i.e. an orientable manifold whose second Stiefel-Whitney class vanishes.

In order to introduce the second Stiefel-Whitney class[4], let us consider the tangent bundle TM to M, where the transition functions are elements of $SO(n)$, $t_{ij}\in SO(n)$. Then let us consider the lift of these functions $\widetilde{t}_{ij}\in \mathrm{Spin}(n)$, such that

$$p(\widetilde{t}_{ij})=t_{ij}\ ,\qquad \widetilde{t}_{ij}^{-1}=\widetilde{t}_{ji}\ ,$$

where

$$p:\mathrm{Spin}(n)\longrightarrow SO(n)$$

is the 2-to-1 homomorphism between the double covering of the group $SO(n)$, that is $\mathrm{Spin}(n)$, and $SO(n)$ itself. Being transition functions, the t_{ij}'s have to satisfy the following properties

$$\begin{aligned} &t_{ii}=\mathrm{id} && \text{in } U_i\\ &t_{ij}=t_{ji}^{-1} && \text{in } U_i\cap U_j\\ &t_{ij}\circ t_{jk}=t_{ik} && \text{in } U_i\cap U_j\cap U_k\end{aligned}$$

[3] Look at [DMS97] and [DPW20] for details.
[4] For details look at [Nak90].

where of course the U_i's are open subsets of M and

$$t_{ij} : U_i \cap U_j \longrightarrow SO(n) \ .$$

From here follows

$$t_{ij} t_{jk} t_{ki} = \mathrm{id} \ ,$$

hence

$$p(\widetilde{t}_{ij} \widetilde{t}_{jk} \widetilde{t}_{ki}) = t_{ij} t_{jk} t_{ki} = \mathrm{id} \ ,$$

which implies

$$\widetilde{t}_{ij} \widetilde{t}_{jk} \widetilde{t}_{ki} \in \ker p = \{\pm \, \mathrm{id}\} \ .$$

If we now define f_2 such that

$$\widetilde{t}_{ij} \widetilde{t}_{jk} \widetilde{t}_{ki} = f_2(i,j,k) \, \mathrm{id} \ ,$$

we have that f_2 defines a cohomology class $[f_2] \in \mathrm{H}^2(M, \mathbb{Z}_2)$. Then the **second Stiefel-Whitney class** is

$$w_2(M) := [f_2] \in \mathrm{H}^2(M, \mathbb{Z}_2) \ .$$

Nevertheless, what we have required is not enough, since we can have also a *sigma-model anomaly.* This anomaly is due to the fact that, since there are chiral fermions, the fermionic measure on the path-integral transforms in a non-trivial way under the $SO(n)$ symmetry group. Since we have added the WZ-term, the B-field has to transform in such a way that it compensates the transformation of the measure. However, introducing these transformations makes the B-field not well-defined as a global 2-form. The way to solve this inconsistency was proposed by Witten in [Wit00]. The idea is that the B-field is defined as a way to associate a phase in the path-integral to each 2-dimensional submanifold $C \subset M$. Now given two cobordant 2-dimensional submanifolds $C, C' \subset M$, namely such that there exists a 3-dimensional manifold U with

$$\partial U = C' - C \ ,$$

then

$$e^{i\left(\int_{C'} B - \int_C B\right)} = e^{i \int_U H} \ ,$$

where now H is a well-defined 3-form also at the quantum level. Once this is done, we have to impose a condition on H in order to cancel anomalies, that is

$$\mathrm{d}H = -\frac{1}{8\pi} \, \mathrm{Tr}(R \wedge R) \ ,$$

which is different from zero. In order to formulate this condition in a different way, let us consider a general manifold M, to which we can associate an integral characteristic class called **first Pontryagin class**

$$p_1 := -\frac{1}{8\pi^2} \, \mathrm{Tr}(R \wedge R) \in \mathrm{H}^4(M, \mathbb{Z}) \ ,$$

with R the curvature 2-form on M. If M is a spin manifold (as required for our situation), then there exists a characteristic class λ such that

$$2\lambda = p_1 \ .$$

In terms of differential forms, of course, it reads

$$\lambda = -\frac{1}{16\pi^2}\,\mathrm{Tr}(R\wedge R)\ . \tag{2.12}$$

In the case where a sigma-model anomaly arises, we can say that H, instead of being a closed 3-form, is a **trivialization** of the class λ, which in terms of differential forms means that

$$\mathrm{d}H = 2\pi\lambda = -\frac{1}{8\pi}\,\mathrm{Tr}(R\wedge R) \neq 0\ . \tag{2.13}$$

A spin manifold with a choice of a trivialization for the characteristic class λ is called **string manifold**, which is the characterization of the manifold we need in order to have a well-defined model.

Looking at the case in which $M = S^3 = SU(2)$, we have seen that the quantization condition on H consists in choosing an integer k given by the relation (2.11). This particular choice is the same as introducing a Wess-Zumino interaction in the supersymmetric sigma model with target S^3. Summarizing, we have that studying the supersymmetric sigma model with target $SU(2)$ and Wess-Zumino coupling k, is the same as studying a supersymmetric sigma model with target the 3-sphere S^3 seen as a string manifold, which we will indicate as S^3_k.

2.2.2 Sigma model on S^3

We want now to study a particular case of a (1+1)-dimensional $\mathcal{N}=(0,1)$ supersymmetric sigma model, that is the one with target space given by $M = S^3$. In order to do that, let us introduce four scalar superfields Φ^I which transform under the vector representation of $O(4)$. Also, they parametrize $M = \mathbb{R}^4$ with a flat metric $g_{IJ} = \delta_{IJ}$. Moreover we introduce a Fermi superfield Λ, which is coupled to the X^I thanks to the superpotential

$$W(\Phi) = \sum_I (\Phi^I)^2 - R^2 \tag{2.14}$$

where $R\in\mathbb{R}$ will be the radius of the sphere S^3. From the general relation (2.6), using the explicit form of the superpotential (2.14), we obtain

$$S' = \int \mathrm{d}u\,\mathrm{d}v\,\Big(\frac{i}{2}\xi\partial_u\xi - \frac{1}{2}\big(\sum_I(\varphi^I)^2 - R^2\big)^2 - 2i\xi\sum_I\varphi^I\psi^I\Big)\ .$$

This superpotential is what ensures us that the fields Φ^I, at low energies, are confined to a sphere S^3 with radius R.

We can generalize this to the case of a sigma model with target S^N requiring that the Φ^I's are $N+1$-component fields with $O(N+1)$ symmetry. In cases like this, for large N, the fields Φ's get mass, and, since these masses are generated from the coupling

$$\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\;\Lambda\Big(\sum_I (\Phi^I)^2 - R^2\Big)\;,$$

the mechanism for the Φ^I's to get mass is that the auxiliary field F in the expansion of Λ gets an expectation value, so the supersymmetry is spontaneously broken. Then in [GJW19] is conjectured that this behavior can be found also for $N=3$, and we are going to present another argument for that in the following part.

Anomalies in the sigma model with generic k

Now we study the anomalies of this model, but, since we have that the 3-sphere is isomorphic to the Lie group $SU(2)$, $S^3 \simeq SU(2)$, we want to focus on the model with a generic Wess-Zumino coupling k. This is done since the anomalies are invariant under the RG-flow in particular, but also under more general deformations. For this reason they constrain the possible behavior of the theory after a deformation.

Let us start noticing that the sigma-model with target S^3 and with coupling $k=0$ introduced previously, has an obvious $O(4)$ symmetry. This symmetry is broken to $SO(4)$ if the coupling k is different from zero. Also, we know that a double covering of $SO(4)$ is the product of two copies of $SU(2)$, which we call

$$SU(2)_l \times SU(2)_r\;,$$

so we will express the next statements in terms of these two $SU(2)$'s.

The model has an anomaly due to the fact that there are only fermions of one chirality. Anomalies for a simple non-abelian Lie group in two dimensions are quantized as integer multiples of a basic invariant. In particular fermions in the vector representation of $SU(2)_l \times SU(2)_r$ have the smallest possible non vanishing anomaly, which we indicate as $(1,1)$.

Now we can add a Wess-Zumino interaction with coupling k. In order to study how this term affects anomalies, we need to focus first on what happen for a purely bosonic sigma model, and then we will generalize to the supersymmetric case.

It is well-known in literature (look at [GJW19] for instance) that Wess-Zumino interaction with coupling k contributes $(-k,k)$ to the $SU(2)_l \times SU(2)_r$ anomalies of a purely-bosonic sigma model with target S^3, since the current algebra of the whole theory satisfies a Kač-Moody algebra at level k. Also, for $|k|$ sufficiently large, there exists a weakly coupled fixed point, that is the WZW model at level k, which has left-moving and right-moving current algebra symmetries[5]

$$SU(2)_L \quad \text{and} \quad SU(2)_R\,.$$

[5] In this case we have used capital letters as subscripts only to distinguish the symmetry groups in the UV from the ones in the IR.

The level means that $SU(2)_R$ has an anomaly $|k|$, while $SU(2)_L$ has an anomaly $-|k|$. This picture is obtained by a perturbative expansion valid for sufficiently large $|k|$, however, as suggested in [GJW19], it is believed that it is actually valid for all k.

Said that, we can focus on the $\mathcal{N}=(0,1)$ supersymmetric case. In this situation it is known that the model flows to a weakly coupled fixed point analogous to the previous one, called $(0,1)$ *supersymmetric WZW model*. In this model there is a left-moving $SU(2)_L$ current algebra at some level κ and a right-moving $\mathcal{N}=1$ supersymmetric $SU(2)_R$ current algebra at the same level κ. From unitarity it is required that $\kappa \geq 0$. As we said, the right-moving current algebra of level κ is supersymmetric. This means that it has actually an $SU(2)_R$ anomaly given by $\kappa+2$. Another way to restate this is that the $\mathcal{N}=(0,1)$ supersymmetric WZW model at level κ is equivalent to the ordinary WZW model at level κ plus the theory of three free right-moving fermions which transform in the adjoint representation of $SU(2)_R$, and which contribute 2 to the $SU(2)_R$ anomaly. Indeed, as found independently in [Di +85] and [AA85] in the $\mathcal{N}=(1,1)$ supersymmetric case (the case we are interested in is obtained simply removing the left-moving fermions from the action), a WZW supersymmetric model can be seen as the bosonic WZW, plus as many free fermions as the number of generators of the Lie algebra associated with the target group, which, in the case of $SU(2)$, is exactly three. This translates in a contribution of 2 to the anomaly, since the fermions satisfy the Kač-Moody algebra at level $h^\vee$, the *dual Coxeter number* of the Lie group, which, in the case of $SU(2)$, is 2.

What we have found is that in the UV regime we have a model with an anomaly given by the one of the sigma model, namely $(1,1)$, plus the one due to the WZ coupling k, i.e. $(-k,k)$, thus

$$SU(2)_l \times SU(2)_r \longrightarrow (-k+1, k+1) \ .$$

In the IR instead we have a bosonic WZW model with anomaly $(-\kappa,\kappa)$ plus three free right-moving fermion with anomaly $(0,2)$, thus

$$SU(2)_L \times SU(2)_R \longrightarrow (-\kappa, \kappa+2) \ .$$

Since anomalies are invariant under RG-flow, the two have to match, which means that

$$\kappa = |k| - 1 \ .$$

In this way we have found, for a generic k, a candidate superconformal fixed point which describes in the infrared the $\mathcal{N}=(0,1)$ supersymmetric sigma model with target S^3. Here it is clear that this argument fails for $k=0$, for which we would have $\kappa<0$. This can be seen as an evidence of the fact that, as we have shown, this model spontaneously breaks supersymmetry.

Summarizing, we have found the following behavior

$$\begin{array}{ll} \text{UV}: & \text{sigma model with target } S^3 + \text{WZ coupling } k \\ & \Big\downarrow \text{RG-flow} \\ \text{IR}: & \text{WZW at bosonic level } |k|-1 + \text{three free right-moving fermion} \end{array}$$

2.2.3 Flowing up and down RG trajectories

We have explained why it has been supposed that for $k = 0$ the model spontaneously breaks supersymmetry. Now the question is what happens for different values of the coupling k. However we want to consider more general deformations then the ones due to the flowing along the RG trajectories. The idea is the following.

Let $\mathcal{F}$ be a generic $\mathcal{N} = (0, 1)$ supersymmetric theory. Let us add to it some arbitrary massive degrees of freedom in a supersymmetric fashion, which, in other words, means that we replace $\mathcal{F}$ with any other theory $\mathcal{F}'$ which is equivalent to $\mathcal{F}$ in the IR. Then we can arbitrarily perturb $\mathcal{F}'$ in a supersymmetric fashion, in such a way that we get some other theory $\mathcal{F}''$. Then the question becomes whether this theory $\mathcal{F}''$ spontaneously breaks supersymmetry. Of course, the procedure can be iterated several times and we will refer to this deformation from $\mathcal{F}$ to $\mathcal{F}''$ as *flowing up and down the RG trajectories.*

Let us focus on the case of the sigma model with target S^3_k. Let Z be any four-dimensional compact spin manifold without boundary, where, as we have explained, the requirement for the manifold to have a spin structure, is needed in order to avoid an anomaly in the $\mathcal{N} = (0, 1)$ supersymmetric sigma model with target Z. However this is not enough and the model is anomalous if

$$\int_Z \lambda \neq 0 \ ,$$

because in this case (2.13) cannot hold for any H. In order to obtain a well-defined sigma model, we can remove a point from the manifold Z, obtaining a new manifold which we call Z'. In this way the 4-form λ becomes topologically trivial, since on Z' there is no compact four-dimensional cycle on which it could be integrated, hence

$$\int_{Z'} \lambda = 0 \ .$$

Now we can define a complete Riemannian metric on Z' so that the missing point is at infinite distance. In this way it has the good properties to be the target space of a well-defined $\mathcal{N} = (0, 1)$ supersymmetric sigma model. It is clear that Z' has a non-compact end, which indeed is topologically equivalent to $\mathbb{R} \times S^3$, in particular

$$\partial Z' = S^3 \ .$$

This allows us to define a metric on Z' which looks asymptotically like the product of a round metric on S^3 times a flat metric on $\mathbb{R}$, which means that, in the asymptotic portion of the field space, the model is described by a sigma model with target S^3 and, decoupled from it, a free chiral superfield which parametrizes the non-compact component $\mathbb{R}$. Moreover, the sigma model can have also a Wess-Zumino coupling. This happens since from the anomaly equation (2.13) follows, using Stoke's theorem,

$$\int_{Z'} \frac{dH}{2\pi} = \int_{S^3} \frac{H}{2\pi} = -\int_{Z'} \frac{\mathrm{Tr}(R \wedge R)}{16\pi^2} \ .$$

However the integral on the right hand side can be easily computed in the following way. We can "close" the non-compact metric on Z' adding an hemisphere at the end, obtaining again the manifold Z. Giving the hemisphere the standard round metric, we have that

$$\mathrm{Tr}(R \wedge R)\big|_{\text{hemisphere}} = 0 \ ,$$

hence, remembering the relation (2.12),

$$-\int_{Z'} \frac{\mathrm{Tr}(R \wedge R)}{16\pi^2} = -\int_{Z} \frac{\mathrm{Tr}(R \wedge R)}{16\pi^2} = \int_Z \lambda \ .$$

In other words we have find that the Wess-Zumino coupling on S^3 is given by the integral of the characteristic class λ over the spin manifold Z

$$\int_{S^3} \frac{H}{2\pi} = \int_Z \lambda \ . \tag{2.15}$$

Since it is known that the integral of $p_1 = 2\lambda$ over a 4-dimensional manifold is always a multiple of 48, a very interesting consequence of (2.15) is that, from here, it can be shown that $\int_Z \lambda$, and hence the Wess-Zumino coupling, is an integer multiple of 24 when S^3 is the boundary of a 4-dimensional manifold with string structure. Now we want to add massive degrees of freedom in a supersymmetric fashion. This is done by adding a Fermi superfield Λ to the sigma model with target Z', thanks to the superpotential coupling

$$S_W = \int \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}\theta \, \Lambda W \ ,$$

for a proper choice of the function W on Z'. In particular we choose the superpotential W as follows. Let us parametrize the non-compact direction by a real variable X, and let us suppose that Z' is, or can be well approximated by, $\mathbb{R} \times S^3$ for $X > 0$, and then this region is glued to Z' in some fashion, in such a way that for $X \ll 0$ the part corresponding to $\mathbb{R} \times S^3$ is missing. Then, for a given constant $x_0 \in R_+$, we can define W on Z' in such a way that

$$W = \begin{cases} m(X - x_0) & X > 0 \\ \text{negative definite} & \text{otherwise} \end{cases} \ . \tag{2.16}$$

Of course, since the complement in Z' of the region $X > 0$ is compact, the function W is bounded below. Moreover we have already seen in (2.7) that a term like this introduces a potential energy given by

$$V = \frac{1}{2} W^2 \ ,$$

which vanishes only at $\{x_0\} \times S^3$. Hence, at low energy, which is equivalent to say that $m \to \infty$, the variable X freezes at the vacuum expectation values x_0, and so we recover the sigma model with target S^3 we have started with.

Finally, we want to perturb the model in such a way that supersymmetry is spontaneously broken. This can be done with the following substitution

$$W \longrightarrow \widetilde{W} := W + c \qquad c \in \mathbb{R}_+ \ .$$

Since, as we have seen, the function W is bounded below, for a constant c sufficiently large, $\widetilde{W}$ is positive definite. This means that the potential energy, which of course is given by

$$V = \frac{1}{2}\widetilde{W}^2 \,,$$

is now everywhere strictly positive, and so supersymmetry is spontaneously broken.

Summarizing, we have found that the $\mathcal{N} = (0,1)$ supersymmetric sigma model with target S^3 and Wess-Zumino coupling k can be connected, in the sense explained above, to a model that spontaneously breaks supersymmetry if k is divisible by 24. In particular the deformation of flowing up and down the RG trajectories was done in the following way:

- we *flowed up* by replacing the target manifold S^3 with the manifold Z' equipped with the superpotential W in equation (2.16);
- we made an ordinary perturbation in a supersymmetric fashion by replacing W with $\widetilde{W}$;
- finally we *flowed down* and found that the model spontaneously breaks supersymmetry.

Generalizing to string boundary

Before going on let us notice that the procedure introduced for the sigma model with target S^3 can be easily generalized. Let M be a generic m-dimensional compact string manifold, namely, a compact orientable manifold with a trivial second Stiefel-Whitney class (i.e. a spin manifold) and with a choice of a trivialization of the class $\lambda := p_1/2$ with p_1 the first Pontryagin class. Then we can say that M is the **string boundary** of a manifold Z, if Z is an $(m+1)$-dimensional string manifold, such that $M = \partial Z$, and the string structure of Z restricts on its boundary to the string structure on M.

Now let us define the manifold Z' as

$$Z' := Z - \partial Z \,,$$

and, on it, we can introduce a complete Riemannian metric that near the infinity looks like $\mathbb{R} \times M$. Then we can follows exactly the same steps as before, replacing, where needed, S^3 with M.

What we have done here shows us that, if M is a string boundary, then the $\mathcal{N} = (0,1)$ supersymmetric sigma model with target M can be deformed, by flowing up and down the RG trajectories, to one that spontaneously breaks supersymmetry.

2.3 Partition function and the Witten index

In this section, we are going to introduce the first invariant we encounter for supersymmetric field theories, that is the *Witten index*. As we will see later on, this invariant has a non trivial kernel and, for this reason, we will need to refine it. However, focusing on its properties will be crucial in order to introduce the invariant we are looking for.

In what follows, we will start considering the construction of the partition function of a supersymmetric field theory and some of its properties. This step will be crucial in understanding how to properly define the Witten index. We will conclude the section presenting an important theorem, the *Atiyah-Singer index theorem*, which gives us a geometric interpretation of the Witten index and also allows us to better understand its behavior.

2.3.1 Partition function

An important quantity we are going to study now is the **partition function**. In order to find it, let us start from an Euclidean space-time $\mathbb{R}^2$, and let us compactify the spatial direction on a circle of radius R. In this way we have that our theory is defined on a cylinder $\mathbb{R} \times S^1_R$, and so it can be regarded as a theory on a 1-dimensional space-time, that is a quantum mechanical model. In this framework it is well known that the partition function is defined as

$$Z(\beta, R) = \mathrm{Tr}\left[e^{-\beta H}\right] ,$$

where the dependence on the radius R is implicit on H, which is obtained as an integral of the Hamiltonian density on the circle of radius R. What we have done is equivalent to evaluate the path integral in the case in which the worldsheet is the Euclidean cylinder of length β and the two boundaries are identified (this identification is due to the presence of the trace). This is clearly equivalent to saying that the worldsheet is a rectangular torus with sides $2\pi R$ and β. However, this is not the most general situation we can describe. Indeed, a possibility is to consider as the worldsheet a torus which is not rectangular, and so in which one end is shifted before the identification by a quantity that we call ν. This is achieved by introducing in the trace the operator

$$e^{-i\nu P} ,$$

where the generator of the translation P is the momentum operator, which, like the Hamiltonian, depends implicitly on the radius of the circle on which we have compactified. In this way we get

$$Z(\beta, \nu, R) = \mathrm{Tr}\left[e^{-\beta H} e^{-i\nu P}\right] .$$

Now we want to factorize the dependence on R of the operators H and P. This is done thanks to the rescaling

$$H \longmapsto \frac{H}{R} , \qquad P \longmapsto \frac{P}{R} ,$$

which means that

$$Z(\beta, \nu, R) = \mathrm{Tr}\left[e^{-\frac{\beta}{R} H} e^{\frac{-\nu i}{R} P}\right] .$$

Now, if we define the operators

$$H_R := \frac{1}{2}(H + P) , \qquad H_L := \frac{1}{2}(H - P) , \tag{2.17}$$

and thanks to the fact that $[H, P] = 0$, we get

$$\begin{aligned} Z(\beta, \nu, R) &= \mathrm{Tr}\left[e^{-\frac{\beta}{R}(H_R+H_L)} e^{\frac{-\nu i}{R}\nu(H_R-H_L)}\right] = \\ &= \mathrm{Tr}\left[e^{i\left(\frac{\nu+i\beta}{R}\right)H_L} e^{-i\left(\frac{\nu-i\beta}{R}\right)H_R}\right] . \end{aligned}$$

Let us introduce the following complex quantities

$$\tau := \frac{\nu + i\beta}{2\pi R} , \qquad q := e^{2\pi i \tau} , \tag{2.18}$$

in such a way that the partition function becomes[6]

$$Z(\tau, \overline{\tau}, R) = \mathrm{Tr}\left[q^{H_L} \overline{q}^{H_R}\right] . \tag{2.19}$$

2.3.2 Witten index and elliptic genus

Now we want to introduce a crucial invariant for supersymmetric field theories known as *Witten index*. All the following construction will be done on *compact theories*, whose name is given by analogy with sigma models, where a model is compact in this sense if the target space of the model is a compact manifold. However, in a general framework, it is actually not so simple to formulate this definition, but we will interpret it as a spectral condition. More precisely, in what follows, we say that a $(d+1)$-dimensional quantum field theory is **compact** if its Wick-rotated partition function *converge absolutely* on all closed space-times. In the Hamiltonian formalism, this occurs when the spectrum of the Hamiltonian H is:

- bounded from below,
- discrete,
- does not grow too slowly.

In a compact supersymmetric theory, we can introduce a topological invariant known as **Witten index** and defined as

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F , \tag{2.20}$$

where $\mathcal{H}$ is the Hilbert space of the theory and $(-1)^F$ is the *fermion number operator*, defined imposing that, acting on fermionic states, they are multiplied by (-1), while the bosonic states are unchanged. Of course the trace in (2.20) is ill-defined, and has to be regularized. In particular it can be done defining the *temperature dependent Witten index*

$$W(\beta) := \mathrm{Tr}_{\mathcal{H}}\left[(-1)^F e^{-\beta H}\right] , \tag{2.21}$$

with β the inverse temperature.

[6]We have written explicitly the dependence on the radius even if we have included it in the definition of τ. This was done since, in the conformal case, the dependence on the radius will disappear, but in terms of τ and $\overline{\tau}$ the expression will be unchanged.

Now, we want to show that, in the case of a compact theory, the Witten index is independent on the inverse temperature β. In doing this, we will also give it a physical meaning. Let us remember some characteristics of a supersymmetric quantum field theory. In general, the Hilbert space $\mathcal{H}$ of any quantum field theory can be decomposed as

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \ , \tag{2.22}$$

where $\mathcal{H}_+$ and $\mathcal{H}_-$ are the spaces of bosonic and fermionic states respectively, which can be identified as the eigenspaces of the operator $(-1)^F$ relative to the eigenvalues ± 1. Furthermore, a supersymmetric theory is, by definition, a theory in which there are N hermitian operators Q_i, $i = 1, \ldots, N$, which map $\mathcal{H}_+$ into $\mathcal{H}_-$ and vice-versa. All these operators have to satisfy the following relations

$$\{(-1)^F, Q_i\} = 0 \ , \qquad [H, Q_i] = 0 \quad \forall i = 1, \ldots, N \ .$$

In addition to this, in the case of a supersymmetric quantum mechanics, we have also to require that

$$Q_i^2 = H \ , \qquad \{Q_i, Q_j\} = 0 \ , \quad i \neq j \ . \tag{2.23}$$

If we want to generalize to the case of a relativistic quantum field theory, we have to notice that the relations (2.23) are not Lorentz invariant. Hence, let us focus on the case (in which we are mainly interested for what follows) of a theory with one time and one space dimensions, in such a way that there is only one momentum operator P. In the case of two supersymmetry operators Q_1 and Q_2 we have that they satisfy

$$(Q_1)^2 = H + P \ , \quad (Q_2)^2 = H - P \ , \quad \{Q_1, Q_2\} = 0 \ , \tag{2.24}$$

from which it follows

$$[Q_i, H] = [Q_i, P] = 0 \ . \tag{2.25}$$

We know that a crucial question in the study of supersymmetric quantum field theories is whether or not they spontaneously break supersymmetry, which corresponds to ask whether there exists a state $|\Omega\rangle$, called *vacuum state*, such that

$$Q_i \, |\Omega\rangle = 0 \quad \forall i \ . \tag{2.26}$$

It is clear form (2.25) that, if a vacuum state exists, then it is also a state of zero energy and zero momentum. Once we have noticed this, we can properly interpret the meaning of the Witten index. First of all, without loss of generality, we can restrict ourselves to the case in which $P = 0$. In this case, again, the Hilbert space of the states annihilated by the momentum, let us say $\mathcal{H}_0$, can be decomposed into bosonic and fermionic components as

$$\mathcal{H}_0 = \mathcal{H}_0^+ \oplus \mathcal{H}_0^- \ .$$

Choosing one of the Q_i's and denoting it simply as Q, we have that $Q^2 = H$ and hence we obtain that non-zero energy states are paired by the action of the supercharge. Indeed, let $|\varphi\rangle$ be a normalized bosonic state of non-zero energy E. Let us define the (normalized) fermionic state

$$|\psi\rangle := \frac{1}{\sqrt{E}} Q\,|\varphi\rangle \ .$$

Hence, we have that Q acts on these states as

$$Q\,|\varphi\rangle = \sqrt{E}\,|\psi\rangle \ , \qquad Q\,|\psi\rangle = \sqrt{E}\,|\varphi\rangle \ ,$$

which means that they are paired in a two-dimensional supermultiplet. Of course this cannot be done for zero-energy states, which are annihilated also by the supercharge Q. For this reason, when we compute the Witten index (2.20), we obtain that it depends only on the sum, with sign, of the zero-energy states

$$\mathrm{Tr}(-1)^F = N_{\mathrm{B}}^{\mathrm{E}=0} - N_{\mathrm{F}}^{\mathrm{E}=0} \ . \tag{2.27}$$

Moreover, interpreting the index in this way, it is quite clear that, varying the parameters of the theory, the energy states can "move" in the spectrum, but always in Bose-Fermi pairs, which means that the Witten index is independent from the parameters in the case of a compact theory. For this reason we are allowed to consider directly the regularized version of the index (2.21).

Thanks to the Witten index, from the relation (2.27), we can obtain important information on the spontaneous breaking of supersymmetry in our theory. Indeed we have the following possibilities:

- $W(\beta) \neq 0$; this means that the difference between bosonic and fermionic supersymmetric ground states is non-vanishing, hence one of them is different from zero. Of course, this implies that supersymmetry is not spontaneously broken;

- $W(\beta) = 0$; in this situation we cannot say anything about the spontaneous breaking. Indeed we cannot distinguish between the case in which $N_{\mathrm{B}}^{\mathrm{E}=0} = N_{\mathrm{F}}^{\mathrm{E}=0} = 0$, where we have a breaking of the supersymmetry, and the case in which they are equal but both different from zero, hence supersymmetry is not broken.

This definition can be generalized in the same way as we have done for the partition function in (2.19), obtaining the expression

$$W(\tau, \overline{\tau}, R) = \mathrm{Tr}\left[(-1)^F q^{H_L} \overline{q}^{H_R}\right] \ . \tag{2.28}$$

First of all let us notice the following fact, which is true also for the partition function of course. We have explained that, in order to define the partition function, we need to compactify the spatial direction on a circle, and so the quantities involved in these definitions are naturally defined on a cylinder. Supposing now for simplicity that our theory is conformal[7], hence loosing the dependence on R, we notice that the definition of

[7]This constraint on the theory is actually not so strong. Indeed we are treating quantities which are invariant under RG flow, so, if the theory is not initially conformal, we can always flow in the infrared and find a conformal model.

H_R and H_L are the same as the ones of $(L_{\text{cyl}})_0$ and $(\overline{L}_{\text{cyl}})_0$ given in (1.19). So now we can restate the definition of the Witten index in terms of the generators defined on the complex plane, obtaining

$$W(\tau, \overline{\tau}) = \text{Tr}\left[(-1)^F q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}}\right] , \tag{2.29}$$

and the same for the partition function

$$Z(\tau, \overline{\tau}) = \text{Tr}\left[q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}}\right] .$$

Furthermore, we have that all these constructions can be reinterpreted in terms of a path integral. Indeed, the Witten index, due to the presence of $(-1)^F$, can be represented as an Euclidean path-integral with *periodic-periodic* (also called *Ramond-Ramond*) *boundary conditions*, i.e. periodic conditions in both spatial and time direction for fermionic fields

$$W(\tau, \overline{\tau}, R) = \int_{\text{PBC}} \mathcal{D}\varphi \mathcal{D}\psi \mathcal{D}\overline{\psi} e^{-S[\varphi, \psi]} .$$

Indeed the path integral is done, as noticed yet in the case of the partition function, on a torus obtained compactifying the space on a cylinder of radius R and length β, and then identifying the boundaries. Since bosonic fields have periodic boundary conditions, in order to preserve supersymmetry, we have to impose these type of conditions also to fermions. As we have explained in section 1.2.1, periodic conditions for fermions means that they are in the Ramond sector, and since we have compactified in both time and space directions, we have to impose the conditions twice. In mathematical terms, the function yet defined, is called **elliptic genus** Z_{RR}, and is equivalent to the definition we gave of the Witten index.

Defined in this way, we have that the elliptic genus depends, in a completely general framework, on the so called **modular parameter** of the torus $(\tau, \overline{\tau})$, which is identified as the τ defined in (2.18), and on the "area" of it. However, we usually work in the IR limit, in which the theory becomes conformal, and so, from now on, we will neglect the dependence on the size. Now let us remember how a torus can be defined, in order to explain the meaning of the modular parameter.

Let us consider the complex plane $\mathbb{C}$ and two non-parallel vectors w_1, $w_2 \in \mathbb{C}$. With these two vectors we can define a torus identifying all the points that differ by an integer combination of them. Of course, thanks to conformal invariance, the only quantity on which the theory depends is the **modular parameter** τ, defined as

$$\tau := \frac{\omega_2}{\omega_1} .$$

It is clear from this definition that different choices of the vectors ω_1 and ω_2 can describe the same torus. In particular this is true if the new complex numbers ω_1' and ω_2' are related from the old ones by the transformation

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} ,$$

with the conditions

$$a,\, b,\, c,\, d \in \mathbb{Z}\ , \qquad ad - bc = 1\ .$$

These relations generate the group $SL(2,\mathbb{Z})$ of invertible 2×2 matrices with integer entries and determinant equals to one. In terms of the modular parameter, this transformation reads

$$\tau \longmapsto \frac{a\tau + b}{c\tau + d} \qquad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\ .$$

It is also clear that nothing changes if all the integers in the matrix change sign. So we conclude the transformation group that acts on the modular parameter and leaves the structure of the torus invariant is given by

$$PSL(2,\mathbb{Z}) := SL(2,\mathbb{Z})/\mathbb{Z}_2\ ,$$

which can be shown to be generated by the transformations

$$T : \tau \longmapsto \tau + 1\ , \qquad S : \tau \longmapsto -\frac{1}{\tau}\ ,$$

called T- and *S-transformation*, respectively[8].

It is clear that, since the elliptic genus depends only on the torus on which we integrate, it is invariant under the transformations that do not change the structure of the torus. In order to better formalize this behavior, let us introduce some useful concepts.

Definition 2.3.1. A **weakly holomorphic modular form of weight** k is an holomorphic function $f : \mathfrak{h} \longrightarrow \mathbb{C}$ such that it transforms as

$$f\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^k f(\tau) \qquad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\ .$$

Recall that $\mathfrak{h}$ is the complex upper half plane

$$\mathfrak{h} = \{\, z \in \mathbb{C} \;\big|\; \mathrm{Im}(z) > 0 \,\}\ .$$

An holomorphic function $f : \mathfrak{h} \longrightarrow \mathbb{C}$ is said to be an **holomorphic modular form of weight** k if it is a weakly holomorphic modular form of weight k and it is "holomorphic at $i\infty$" (or, equivalently, "holomorphic at the cusp").

First of all let us explain what "*holomorphic at* $i\infty$" means. We have that, under the T-transformation, a modular form behaves like

$$f(\tau + 1) = f(\tau)\ ,$$

hence it is invariant under the action of $\mathbb{Z}$ on $\mathbb{C}$ described by

$$\begin{aligned} \mathbb{C} \times \mathbb{Z} &\longrightarrow \mathbb{C} \\ (z, n) &\longmapsto z + n \end{aligned}\ .$$

[8] For details look at [DMS97] and [BP09]

Also, the function

$$\mathbb{C}/\mathbb{Z} \longrightarrow \mathbb{C} \setminus \{0\}$$

$$z \longmapsto q := e^{2\pi i z}$$

is an isomorphism, which implies that every function $f(z)$ invariant under the $\mathbb{Z}$-action given above[9] can be expressed as a function $\widetilde{f}$ of q, which ranges over $\mathbb{C} \setminus \{0\}$. Hence we say that $f(\tau)$ is *holomorphic at* $i\infty$ if the function $\widetilde{f}(q)$ is meromorphic at 0 (note that $\tau \to i\infty$ corresponds to $q \to 0$). In other words, the so-called *q-expansion* of the modular form, namely the Laurent series of $\widetilde{f}(q)$ around 0

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n \ ,$$

is such that

$$a_n = 0 \qquad \text{for} \ \ n < 0 \ .$$

However for later purpose (we will see that this property will characterize the Witten genus in the non-compact case) it is better to relax one of the assumption in this definition. In particular we define a **real-analytic modular form of weight** $(w, \overline{w})$ as a real analytic function $f : \mathfrak{h} \longrightarrow \mathbb{C}$ which transforms as

$$f\Big(\frac{a\tau + b}{c\tau + d}\Big) = (c\tau + d)^w (c\overline{\tau} + d)^{\overline{w}} f(\tau) \qquad \text{for} \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \ .$$

With these new definitions, we can say that the elliptic genus $Z_{RR}(\tau)$ is a weakly holomorphic modular function of weight 0 with no multiplier and an integral q-expansion, that is, the coefficients of its q-expansions are all integer. Let us verify these two properties.

Holomorphicity

In a compact theory, the spectrum of the hamiltonian is discrete and we have that the elliptic genus is an holomorphic function. Indeed, the only states that give a non-vanishing contribution to the elliptic genus are the supersymmetric ground states, which means that, on these states, the action of the supersymmetry is trivial. In $\mathcal{N} = (0,1)$ supersymmetric models, the supersymmetry operator is such that

$$Q^2 = i\partial_u = \frac{i}{2}(\partial_0 - \partial_1) = \frac{1}{2}(H + P) \ ,$$

where, being in a Euclidean spacetime, we have used the definition of the Hamiltonian and momentum operators given by

$$H = i\partial_0 \ , \qquad P = -i\partial_1 \ .$$

Hence, the states that contribute to the elliptic genus are the ones for which

$$H = -P \ .$$

[9] The invariance under the $\mathbb{Z}$ is needed in order to consider the quotient $\mathbb{C}/\mathbb{Z}$ without loosing information.

Now, remembering the definitions in (2.17) we find that

$$H_R = \frac{1}{2}(H + P) \equiv 0 \ ,$$

and the elliptic genus (2.28) reads[10]

$$Z_{RR}(\tau, \overline{\tau}) = \mathrm{Tr}\left[(-1)^F e^{2\pi i \tau H_L}\right] \equiv Z_{RR}(\tau) \ .$$

Integrality

The integrality of the q-expansion of the elliptic genus can be explained in the following way. We have found in (2.28) that the Witten index has the form (we neglect the dependence on the radius of the torus)

$$Z_{RR}(\tau, \overline{\tau}) = \mathrm{Tr}_{\mathcal{H}}\left[(-1)^F q^{H_L} \overline{q}^{H_R}\right] \ .$$

Now, since we have compactified the spatial direction on a circle of radius R, the momentum operator, given by $(H_R - H_L)$, is quantized and it takes values in $\mathbb{Z}$. This allows us to decompose the Hilbert space of the full theory as the sum of the eigenspaces of the momentum operator

$$\mathcal{H} = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_k \ .$$

In this way we have that the elliptic genus can be decomposed as

$$\begin{aligned} Z_{RR}(\tau, \overline{\tau}) &= \mathrm{Tr}_{\mathcal{H}}\left[(-1)^F q^{H_L - H_R} \left(q\overline{q}\right)^{H_R}\right] = \\ &= \sum_k q^k \, \mathrm{Tr}_{\mathcal{H}_k}\left[(-1)^F \left(q\overline{q}\right)^{H_R}\right] = \\ &= \sum_k q^k \, \mathrm{Tr}_{\mathcal{H}_k}\left[(-1)^F e^{-\beta H_R}\right] \ . \end{aligned}$$

We have computed the q-expansion of the elliptic genus, obtaining that, for each k, the coefficient of the expansion is given by the index of a SQM with Hilbert space $\mathcal{H}_k$, Hamiltonian H_R and supercharge Q such that $Q^2 = H_R$. As we have seen, since we are considering the case of a compact theory, we have that the index $\mathrm{Tr}_{\mathcal{H}_k}\left[(-1)^F e^{-\beta H_R}\right]$ is simply given by the difference between bosonic and fermionic ground states, which is of course an integer. Hence we conclude that the q-expansion of the elliptic genus $Z_{RR}(\tau, \overline{\tau})$ is integral.

These properties allow us to see the elliptic genus as a map from the (not yet defined properly, but whose meaning is clear) set of supersymmetric quantum field theories SQFT and the set, which is actually a ring, of weakly holomorphic integral modular forms[11] $\mathrm{wMF}_{\mathbb{Z}}$, namely

$$Z_{RR}(\,\cdot\,) : \mathrm{SQFT} \longrightarrow \mathrm{wMF}_{\mathbb{Z}} \ .$$

[10] In this case we avoid to write the dependence on the radius of the circle on which we compactify the spatial direction, however it is understood.

[11] The ring of weakly holomorphic integral modular forms is a graded ring, given by the direct sum of

2.3.3 Atiyah-Singer index theorem

In this paragraph we are going to focus on supersymmetric quantum mechanics in order to state a crucial theorem known as *Atiyah-Singer index theorem*.

Let us consider a supersymmetric quantum mechanics (SQM in what follows), given by an Hilbert space $\mathcal{H}$, a supercharge Q and an Hamiltonian H. In particular, let us focus on the case of a SQM model describing a supersymmetric particle moving on a compact space M without boundaries and with even dimension. In the situation described above, the supercharge Q can be identified with the Dirac operator of the theory $i\gamma^\mu D_\mu$ and hence the Hamiltonian as the Laplacian. Using a basis in which the chirality operator of the field space, and in turn the fermion number operator, is diagonal

$$(-1)^F = \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we have that

$$Q = i\gamma^\mu D_\mu = \begin{pmatrix} 0 & L^\dagger \\ L & 0 \end{pmatrix} , \qquad H = Q^2 = \begin{pmatrix} L^\dagger L & 0 \\ 0 & LL^\dagger \end{pmatrix} ,$$

where the form of the supercharge is clearly compatible with the fact that it exchanges bosons with fermions and vice-versa. Decomposing again the Hilbert space in the eigenspaces of the chirality operator, in the same way as we have done in (2.22), we see that

$$L : \mathcal{H}_+ \longrightarrow \mathcal{H}_- , \qquad L^\dagger : \mathcal{H}_- \longrightarrow \mathcal{H}_+ .$$

Furthermore, let us notice that

$$(\ker H) \cap \mathcal{H}_+ \equiv \ker(L^\dagger L) = \ker L , \qquad (\ker H) \cap \mathcal{H}_- \equiv \ker(LL^\dagger) = \ker L^\dagger .$$

Proof. Let us focus on the second equivalence. If $|\psi\rangle \in \ker L^\dagger$ we have that

$$LL^\dagger |\psi\rangle = L\big(L^\dagger |\psi\rangle\big) = L(0) = 0 ,$$

which means that

$$\ker L^\dagger \subseteq \ker(LL^\dagger) .$$

Conversely, given $|\psi\rangle \in \ker(LL^\dagger)$ we can write

$$0 = \langle\psi|LL^\dagger\psi\rangle = \langle L^\dagger\psi|L^\dagger\psi\rangle = \left\|L^\dagger\psi\right\|^2 ,$$

all the sets of weakly holomorphic integral modular forms of weight k; in formulae, called $\mathrm{wMF}_k^{\mathbb{Z}}$ the set of weakly holomorphic modular forms of weight k, we have that the graded ring of weakly holomorphic integral modular forms is

$$\mathrm{wMF}_{\mathbb{Z}} = \bigoplus_{k\geq 0} \mathrm{wMF}_k^{\mathbb{Z}} .$$

The same comment is true for holomorphic integral modular forms, in which case their ring is labelled as $\mathrm{MF}_{\mathbb{Z}}$.

from which $L^\dagger |\psi\rangle = 0$ and $|\psi\rangle \in \ker L^\dagger$. In this way we have shown that

$$\ker(LL^\dagger) \subseteq \ker L^\dagger \ ,$$

and so

$$\ker(LL^\dagger) = \ker L^\dagger \ .$$

The first equivalence follows exactly in the same way. □

Remembering that only ground states contribute to the Witten index, we conclude that

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}}\left[(-1)^F e^{-\beta H}\right] &= \dim\left(\ker(L^\dagger L)\right) - \dim\left(\ker(LL^\dagger)\right) = \\ &= \dim\left(\ker L\right) - \dim\left(\ker L^\dagger\right) =: \mathrm{ind}\left(i\gamma^\mu D_\mu\right) \ ,\end{aligned} \tag{2.30}$$

or, in other words, the Witten index equals the so-called *index of the Dirac operator* $\mathrm{ind}(i\gamma^\mu D_\mu)$.

We have shown in section 2.3.2 that, if the theory is compact (i.e. the spectrum of the Hamiltonian is discrete), which in this case means that the manifold M is compact and without boundaries, we have that the Witten index is independent from β. For this reason we can equivalently compute it in the limit $\beta \to 0$, which gives us a different way to interpret the index.

The Witten index can be expressed as an Euclidean path-integral

$$\mathrm{Tr}_{\mathcal{H}}\left[(-1)^F e^{-\beta H}\right] = \int_{\mathrm{PBC}} \mathcal{D}\varphi\mathcal{D}\psi\mathcal{D}\overline{\psi} e^{-S[\varphi,\psi]} \ ,$$

where the fields $\varphi(t)$ and $\psi(t)$ are periodic in the Euclidean time with period β. The action of the SQM is of the form

$$\int \mathrm{d}t \left(\frac{1}{2} g_{IJ}(\varphi)\dot{\varphi}^I\dot{\varphi}^J + \frac{1}{2} g_{IJ}(\varphi)\overline{\psi}^I D_t \psi^J\right) \ ,$$

where the dot on the fields stands for derivative with respect to the time, while g_{IJ} is the metric on the manifold M. The fact that metric depends on the scalar fields φ makes the action very complicated in principle. However, it greatly simplifies in the limit $\beta \to 0$. Indeed, we can expand the fields in terms of the infinitesimal variations

$$\varphi^J(t) = \varphi_0^J + \delta\varphi^J(t) \ , \qquad \psi^J(t) = \psi_0^J + \delta\varphi^J(t) \ ,$$

where the variations are periodic. Hence, we can take the Fourier decompositions

$$\delta\varphi^J(t) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{+\infty} \varphi_n^J e^{2\pi i n \frac{t}{\beta}} \ , \qquad \delta\dot{\varphi}^J(t) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{+\infty} \frac{2\pi i n}{\beta} \varphi_n^J e^{2\pi i n \frac{t}{\beta}} \ ,$$

and the same for $\delta\psi^J(t)$. The non-zero modes in the Fourier expansion vanish due to periodic boundary condition, and so the fields read

$$\varphi^J(t) = \varphi_0^J + \frac{1}{\sqrt{\beta}}\delta\varphi_0^J \ , \qquad \psi^J(t) = \psi_0^J + \frac{1}{\sqrt{\beta}}\delta\psi_0^J \ .$$

Substituting these expressions in the action we can expand it at the lowest order in β, which corresponds to take the second order in the fluctuations, and, rescaling $t \to \beta t$, we arrive to a quadratic action in the fluctuations themselves.

In this way, the functional integral is simply a Gaussian path-integral which can be evaluated, giving us the regularized determinant of the kinetic operators. This determinant depends on the zero modes of the fields, and can be shown[12] that its integration on these modes corresponds to the integration of the so-called *Dirac genus* $\widehat{A}$ on the manifold M. In conclusion, we arrive to the identity

$$\lim_{\beta\to 0} \mathrm{Tr}\left[(-1)^F e^{-\beta H}\right] = \int_M \widehat{A}\,.$$

Since the Witten index on the compact manifold M does not depend on β, we have that the limits for $\beta \to \infty$ and $\beta \to 0$ have to coincide, which gives us the **Atiyah-Singer index theorem**

$$\mathrm{ind}(i\gamma^\mu D_\mu) = \int_M \widehat{A}\,.$$

Let us conclude this section noticing that, from this theorem, a property of characteristic classes we have encountered yet follows. In particular in 2.2.3 we have used the fact that the integral of the first Pontryagin class p_1 on a 4-dimensional manifold with spin structure is a multiple of 48. But, for this type of manifold, a direct computation shows that

$$\widehat{A} = \frac{p_1}{48}\,,$$

hence

$$\int_M \frac{p_1}{48} = \mathrm{ind}(i\gamma^\mu D_\mu) \in \mathbb{Z}\,,$$

which is fulfilled only for p_1 multiple of 48.

[12] For details look at [Win84] and [Eri14].

Chapter 3

The Stolz and Teichner program

In the previous chapter we have seen that if k is divisible by 24, then the sigma model with target S^3 and Wess-Zumino coupling k can be continuously deformed flowing up and down the RG trajectories in a model which spontaneously breaks supersymmetry. Then we can ask if this sufficient condition is also necessary. In order to answer this question we need to find a sufficiently refined invariant of our theory. The first idea is to use the usual Witten index, however can be shown that

$$Z_{RR}(S^3_k) = 0\,,$$

hence we have no information from the Witten index regarding the spontaneous breaking. This means that the Witten index is not a *complete invariant*. The way to overcome this difficulties comes from a conjecture due to Stolz and Teichner ([ST04], [ST11]) which introduce a new *secondary* invariant that can be seen as a topological version of the Witten index. In particular they propose that the Witten index

$$Z_{RR}(\,\cdot\,) : \mathrm{SQFT} \longrightarrow \mathrm{wMF}_{\mathbb{Z}}\ ,$$

can be lifted to a *topological Witten index* Z^{top}_{RR}, from the set of SQFT to a particular set known as Ω*-spectrum of topological modular forms* TMF, namely

$$Z^{\mathrm{top}}_{RR}(\,\cdot\,) : \mathrm{SQFT} \longrightarrow \mathrm{TMF}\ .$$

The crucial assumption of the conjecture is that this new invariant is complete, that is, it gives an identification

$$\mathrm{SQFT} \simeq \mathrm{TMF}\ .$$

Thanks to this last assumption, we have that two theories are homotopic to each other, that is, they can be deformed one into the other, *if and only if* they are in the same TMF-class. The problem consists in the fact that this topological Witten index has, again, a non-vanishing kernel. However, its kernel is composed by all the torsion classes of TMF, which gives us the chance to introduced also invariants useful in studying these elements.

In what next we are going to explain what are the sets involved in this conjecture, what are the problems that this conjecture presents, and then, once assumed as true, to see what are its consequences and how it helps in solving our problem for the sigma-model.

3.1 Topological modular forms

First of all we want to focus on the study of TMF. The idea of this section is to figure out how this construction works, not being too much deep, since its definition is highly non-trivial. In particular we will try to underline the properties that will help us to give a more refined structure to the set SQFT. For more details on the construction of TMF look at [Hop95], [Hop02], [Goe09], [Lur09] and [Dou+14].

3.1.1 Mathematical preliminaries

Let us start giving some basic definitions from algebraic topology, which will be useful in next sections. The first step consists in defining the principal tool of these type of studies, that is the definition of a *category*.

A **category** $\mathcal{C}$ is defined as the datum of a set $\mathrm{Ob}(\mathcal{C})$ of *objects* of $\mathcal{C}$, for each two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ a set $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ of *morphisms* between X and Y, and, for each three objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a map

$$\begin{aligned} \text{“}\circ\text{”} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ (f,g) &\longmapsto g \circ f \end{aligned} ,$$

called *composition* of morphisms. This composition has to satisfy two properties, namely to be *associative*, that is, for all morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, we have

$$(h \circ g) \circ f = h \circ (g \circ f) \ .$$

Also, for every $X \in \mathrm{Ob}(\mathcal{C})$, there exists an *identity map* $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$, such that $\forall\, f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ and $\forall\, g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$, we have

$$f \circ \mathrm{id}_X = f \ , \qquad \mathrm{id}_X \circ g = g \ .$$

Going on, a functor encodes the idea of a “function between categorie”. Formally, given two categories $\mathcal{C}$ and $\mathcal{D}$, a **functor** $F : \mathcal{C} \longrightarrow \mathcal{D}$ consists of a map

$$F : \mathrm{Ob}(\mathcal{C}) \longrightarrow \mathrm{Ob}(\mathcal{D}) \ ,$$

and, for all $X, Y \in \mathrm{Ob}(\mathcal{C})$, of a map

$$F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) \ ,$$

such that, for all $X \in \mathrm{Ob}(\mathcal{C})$

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \ ,$$

and for every functions $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have

$$F(g \circ f) = F(g) \circ F(f) \ .$$

Then, we are ready to introduce the other main object of our discussion, that is cohomology theories. A **cohomology theory** H is a collection of functors H^n and of maps δ^n which satisfies the following axioms, also called **Eilenberg-Steenrod axioms**:

1. *for each $n \in \mathbb{Z}$, H^n is a contravariant functor from the category of pairs of topological spaces $(Y \subseteq X)$ to the one of abelian groups.* This means that it associates to a pair of topological spaces (X, Y) (that is a topological space X and a subspace of it $Y \subseteq X$) an abelian group $\mathrm{H}^n(X, Y)$ known as *n-dimensional cohomology group* of (X, Y); also to a function of topological pairs
$$f : (X, Y) \longrightarrow (X', Y') ,$$
that is a continuous map $f : X \longrightarrow X'$ such that $f(Y) \subseteq Y'$, it associates an homomorphism of abelian groups
$$\mathrm{H}^n(f) : \mathrm{H}^n(X', Y') \longrightarrow \mathrm{H}^n(X, Y) .$$
called *induced map.* In particular this mapping has to be compatible with the composition of morphisms in the two categories;

2. *if $f : X' \longrightarrow X$ is a* **weak homotopy equivalence**, that is it induces an isomorphism of homotopy groups for all basepoints $x \in X$
$$\pi_n(X, x) \xrightarrow{\sim} \pi_n(X', f(x)) \qquad \forall\, n \geq 0 ;$$
then the induced map $\mathrm{H}^n(f)$ is an isomorphism
$$\mathrm{H}^n(f) : \mathrm{H}^n(X) \xrightarrow{\sim} \mathrm{H}^n(X') ,$$
where we have $\mathrm{H}^n(X) := \mathrm{H}^n(X, \emptyset)$;

3. *to every triple of topological spaces $Z \subseteq Y \subseteq X$, there exists an associated* ***long exact sequence***
$$\ldots \longrightarrow \mathrm{H}^n(X, Y) \longrightarrow \mathrm{H}^n(X, Z) \longrightarrow \mathrm{H}^n(Y, Z) \xrightarrow{\delta^n} \mathrm{H}^{n+1}(X, Y) \longrightarrow \ldots .$$
Asking for the sequence to be exact means that, given any two subsequent arrow
$$A \xrightarrow{f} B \xrightarrow{g} C ,$$
we have $g \circ f = 0$ and $\mathrm{Im}\, f \simeq \ker g$. The maps δ^n are called *connecting morphisms*, while the other arrows are induced by inclusions;

4. *let $A, B \subseteq Y$ be two subspaces such that the union of their interiors is equal to Y itself. Then the inclusion $(A, A \cap B) \longrightarrow (Y, B)$ induces an isomorphism*
$$\mathrm{H}^n(Y, B) \xrightarrow{\sim} \mathrm{H}^n(A, A \cap B)$$
for all $n \in \mathbb{Z}$;

5. *given a family of spaces* $\{X_i\}$ *such that*
$$\bigsqcup_i X_i = X \ ,$$
then the map
$$\mathrm{H}^n(X) \longrightarrow \prod_i \mathrm{H}^n(X_i)$$
is an isomorphism for all $n \in \mathbb{Z}$;

6. *if* $X = \{\, point \,\}$, *then*
$$\mathrm{H}^n(X) = \begin{cases} 0 & \text{if } n \neq 0 \\ \mathbb{Z} & \text{if } n = 0 \ . \end{cases}$$

Given a cohomology theory H, we indicate the graded abelian group
$$\mathrm{H}^\bullet(X, A) := \bigoplus_{n=-\infty}^{+\infty} \mathrm{H}^n(X, A) \ ,$$
and define the *coefficient group*
$$\mathrm{H}^\bullet := \mathrm{H}^\bullet(\,\mathrm{point}\,) = \bigoplus_{n=-\infty}^{+\infty} \mathrm{H}^n(\,\mathrm{point}\,) \ .$$

Actually most of the cohomology theories we are interested in do not have the cohomology of the point concentrated in degree zero. This motivates the definition of **generalized cohomology theories**, for which we require the same axioms as before except 6. We can also require other properties for our cohomology theory. In particular we will say that a cohomology theory is **multiplicative** if $\mathrm{H}^\bullet(X)$ is equipped with a structure of graded commutative ring, given by a multiplication map
$$\mathrm{H}^k(X) \times \mathrm{H}^l(X) \longrightarrow \mathrm{H}^{k+l}(X) \ ,$$
with the following grading
$$a \cdot b = (-1)^{|a|\,|b|} b \cdot a \ ,$$
for
$$a \in \mathrm{H}^{|a|}(X) \ , \qquad b \in \mathrm{H}^{|b|}(X) \ .$$
Moreover, a multiplicative cohomology theory H is **even** if
$$\mathrm{H}^{2i+1}(\,\cdot\,) = 0 \qquad \forall\, i \in \mathbb{Z} \ ,$$
while it is said to be **periodic** if there exists an element $\beta \in \mathrm{H}^{-n}(\,\cdot\,)$ for some $n \in \mathbb{Z}$ which is invertible in $\mathrm{H}^\bullet(\,\cdot\,)$, namely β has an inverse $\beta^{-1} \in \mathrm{H}^n(\,\cdot\,)$.

3.1.2 The construction of TMF

In order to figure out how TMF can be defined, we have to start finding some invariants which allow us to study the cohomology $\mathrm{H}^\bullet(X)$ associated to a vector bundle on X. We can simply focus on line bundle and try to generalize the idea of the *Chern class*, which is a complete invariant for line bundles. It is a well-known fact[1] that the infinite dimensional complex projective space $\mathbb{CP}^\infty$ is a *classifying space* for complex line bundles. This means that, for any complex line bundle L on a (well-behaved) space X, there is a classifying map $\varphi : X \to \mathbb{CP}^\infty$ and an isomorphism $L \simeq \varphi^*\mathcal{L}(1)$, where $\mathcal{L}(1)$ is called *universal line bundle*. In the case of ordinary cohomology, for the first Chern class, we know that the cohomology ring of $\mathbb{CP}^\infty$ on $\mathbb{Z}$ is the polynomial ring generated by the first Chern class t of the canonical line bundle, i.e.

$$\mathrm{H}^\bullet(\mathbb{CP}^\infty, \mathbb{Z}) = \mathbb{Z}[t] \ .$$

The computations[2] for this equality relies on the fact that all the cohomology groups of $\mathbb{CP}^\infty$ on $\mathbb{Z}$ of even degree $2i$ are isomorphic to each other and they are infinite-cyclic generated by the i-th power of the first Chern class of the universal line bundle. Also, the cohomology groups of odd degrees are zero. This isomorphism can be extended in the case of a generalized cohomology theory, as[3]

$$\mathrm{H}^\bullet(\mathbb{CP}^\infty) \simeq \mathrm{H}^\bullet(\text{point})[[c_1]] \equiv \mathrm{H}^\bullet[[c_1]] \ ,$$

where c_1 satisfies analogous relations to the ones satisfied by the usual Chern class. In particular this implies that, given two line bundles L_1 and L_2,

$$c_1(L_1 \otimes L_2) = F\big(c_1(L_1), c_1(L_2)\big) \ ,$$

where F is a formal power series known as **formal group law** and satisfies the following properties:

1. $F(x, 0) = F(0, x) = x$;
2. $F(x_1, x_2) = F(x_2, x_1)$;
3. $F\big(F(x_1, x_2),\, x_3\big) = F\big(x_1\,, F(x_2, x_3)\big)$.

A formal group law gives to the commutative ring $\mathrm{H}^\bullet[[c_1]]$ the structure of a so-called *formal group*[4], which is canonically associated to the cohomology theory H. In this way we have found that a choice of a formal group law corresponds to a choice of a cohomology theory. Furthermore, we have that formal groups are related to algebraic groups. Hence, given a one dimensional (connected) group variety G over an algebraically closed field k, we can have one of the following possibilities:

[1] For details look at [Nak90].
[2] For the complete proof look at [Hat02].
[3] Let us recall that, given a ring R and a letter X, we define as $R[X]$ the ring of polynomial $\sum_{i=0}^{n} a_i X^i$ for all $n \in \mathbb{Z}$ with $a_i \in R$; furthermore, we define as $R[[X]]$ the ring of *formal series* in X, that is the ring with elements given by $\sum_{i=0}^{\infty} a_i X^i$ with $a_i \in R$. These elements can be seen as polynomials with infinite terms or, equivalently, as power series without the requirement of convergence.
[4] For details on this construction look at [Dou+14] and the references therein.

1. $G = \mathbb{G}_a$ the additive group,

2. $G = \mathbb{G}_m$ the multiplicative group,

3. G is an elliptic curve.

We want to focus on this last possibility, which gives rise to the so called **elliptic cohomology theory**. Roughly speaking, we have that there is a cohomology theory for every elliptic curve.

Before going on, let us briefly recall what an elliptic curve is. An **elliptic curve** over a field k, is a non-singular curve C in the projective plane $\mathbb{P}^2$ defined by a cubic equation and such that its intersection with the line at the infinity is given by a point $[0 : * : 0]$. In formulae, the line at the infinity is $\mathbb{P}^1_\infty = [0 : * : *]$, hence we need

$$C \cap \mathbb{P}^1_\infty = [0 : * : 0] \ .$$

In affine coordinates[5] (x, y), an elliptic curve can be written in the co-called *Weierstrass form*

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \ .$$

If the characteristic of the field k is neither 2 nor 3, we can rewrite the cubic which describes the curve as, after a change of coordinates

$$y^2 = x^3 - 27c_4x - 54c_6 \ ,$$

where we have defined

$$c_4 = (b_2)^2 - 24b_4 \ , \qquad c_6 = (b_2)^3 + 36b_2b_4 - 216b_6 \ ,$$
$$b_2 = (a_1)^2 + 4a_2 \ , \qquad b_4 = 2a_4 + a_1a_3 \ , \qquad b_6 = (a_3)^2 + 4a_6 \ .$$

A crucial quantity is the *discriminant* of the cubic, defined as

$$\Delta = -(b_2)^2b_8 - 8(b_4)^3 - 27(b_6)^2 + 9b_2b_4b_6 \ ,$$

where

$$b_8 = (a_1)^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2(a_3)^2 - (a_4)^2 \ .$$

In particular it can be shown that the Weierstrass curve is singular if and only if $\Delta = 0$.

A peculiar characteristic of elliptic curves is that they can be described as an algebraic group, defining a group law on their points. This is done as follows. Being defined as a cubic, intersecting an elliptic curve C with any line in $\mathbb{P}^2$ will provide three points (counted with multiplicity) of C. Hence, given two points P and Q on the curve C, we can sum them in the following way:

- let l be the line containing P and Q (or tangent to C at P if $P = Q$);

[5] Let us recall that, given a generic projective space and an hyperplane $H \subseteq \mathbb{P}$, the set $\mathbb{P} \setminus H$ is naturally endowed with a structure of an affine space, and given a set of coordinates on $\mathbb{P}$, $[X_0 : X_1 : \cdots : X_n]$ such that $H = V(X_0)$, the **associated affine coordinates** on $\mathbb{P} \setminus H$ are given by the ratios $\left(\frac{X_1}{X_0}, \ldots, \frac{X_n}{X_0}\right)$.

- let R be the third point in the intersection $l \cap C$;
- let us consider the line l' passing through R and the point $e = [0 : 1 : 0]$;
- let us define the sum $P + Q$ as the third point in the intersection $l' \cap C$.

This construction gives us an abelian group structure on the points of C with identity element $e = [0 : 1 : 0]$. The inverse of this statement is the one we have cited above, according to which, given a one dimensional algebraic group over an algebraically closed field k, it can be the either the additive group, or the multiplicative group or an elliptic curve.

At this point we want to find a sort of "universal" elliptic cohomology, that is a cohomology theory which we can vary in order to obtain all the other elliptic cohomology theories. However, to find this universal elliptic cohomology is not an easy task. What can be done is to find a way to associate an elliptic cohomology to a properly defined set of maps related to elliptic curves

$$\overline{\mathcal{O}} : \{ \text{ maps related to elliptic curves } \} \longrightarrow \{ \text{ elliptic cohomology theories } \} \, .$$

However in order to extract from this a global information, we have to do some work on the target space.

We want now to figured out the behavior of the target space, which gives us information on this universal cohomology theory, which is the set TMF we are interested in. The first step needed consists in replacing the set of cohomology theories by a set of their *representatives*. From the so-called *Brown's representability theorem*, we have that any cohomology theory H has a representing sequence of spaces E_n, such that, for a well-behaved topological space X, we have

$$\mathrm{H}^n(X) \simeq [X, E_n] \, ,$$

with $[X, E_n]$ the set of homotopy classes of maps from X into E_n. By duality, it can be defined a *homology theory* associated to the same sequence of spaces, which we call $\mathrm{H}_n(X)$. They are naturally related by the following relations

$$\pi_\bullet(\mathrm{H}) := \mathrm{H}_\bullet \simeq \mathrm{H}^{-\bullet} \, ,$$

where, of course,

$$H_\bullet := H_\bullet(\,\text{point}\,) = \bigoplus_{n=-\infty}^{+\infty} H_n(\,\text{point}\,) \, ,$$

is the coefficient group of the homology theory. The connecting maps of the cohomology theory, δ^n, endow the sequence of spaces E_n with an additional structure. Indeed, they induce a series of maps[6]

$$s_n : E_n \longrightarrow \Omega E_{n+1} \qquad \forall\, n \in \mathbb{Z} \, ,$$

[6] We will prove this sentence in Appendix A.

such that the s_n's are weak homotopy equivalences. In other words, they induce isomorphisms between all homotopy groups

$$s_n^l : \pi_l(E_n) \xrightarrow{\sim} \pi_l(\Omega E_{n+1}) \simeq \pi_{l+1}(E_{n+1}) \qquad \forall l \in \mathbb{Z} .$$

Here we have that the space ΩE_n is the **loop space** of E_n, that is the space of based loops in E_n or, in other words, the set of the maps from the circle S^1 to E_n, such that a chosen basepoint on S^1 is mapped to a chosen basepoint in E_n. A sequence of spaces with the properties above is called **Ω-spectrum**.

This construction allows us to replace as target space, the space of cohomology theories with the one of Ω-spectra, on which can be introduced global notions. However, the structure is not yet sufficiently well-behaved to give us a way to define it properly. So now we restrict our attention to multiplicative cohomology theories, which are represented by a more rigid structure, known as **E_∞-spectrum**. In order to introduce this notion, we need to define the *E_∞-ring (space)*, which can be defined, roughly speaking, as a topological space A, equipped with the structure of a ring, for which the sum commutes up to homotopy. Any E_∞-ring A determines a cohomology theory H_A, that is, for a well-behaved topological space X, we have that

$$\mathrm{H}_A^{-n}(X) \simeq \pi_n \operatorname{Hom}(X, A) \qquad \text{for } n \geq 0 ,$$

where $\operatorname{Hom}(X, A)$ is endowed with a structure of E_∞-ring, obtained computing all the operations pointwise. This definition gives us a *connective* cohomology theory, that is, it has the property that the n-th cohomology of a generic space X vanishes for $n > 0$, i.e.

$$\mathrm{H}^n(X) = 0 \qquad \text{for } n > 0 .$$

However, in general, we need to describe theories which are not connective. For this reason we want to introduce a slightly more general notion than E_∞-ring. In particular, we define the **E_∞-spectrum**, which is an Ω-spectrum endowed with the structure of an E_∞-ring, in such a way that the related cohomology theory has all the good properties that characterize a cohomology theory associated to an E_∞-ring, but without the requirement to be connective[7]. With this last definition we have the possibility to define the "universal" elliptic cohomology TMF, which, for these reasons, results to be an E_∞-spectrum.

The spectrum TMF we define in this way, owes its name to the fact that the ring of its homotopy groups is rationally isomorphic to the ring of weakly holomorphic modular forms wMF, that is

$$\bigoplus_k \pi_k(\mathrm{TMF}) \xrightarrow[\text{rational}]{\sim} \mathrm{wMF}_\bullet \simeq \mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 - 1728\Delta) . \tag{3.1}$$

Here c_4 and c_6 are the forth and the sixth Eisenstein's series, which are modular forms of weight 4 and 6 respectively, and defined as, in the general case of a k-th series

$$c_k(z) := \frac{1}{2} \sum_{\substack{c,d\in\mathbb{Z} \\ \gcd(c,d)=1}} \frac{1}{(cz+d)^k} .$$

[7]We gave very rough definitions of E_∞-ring and E_∞-spectrum, just to give an intuition of some of their properties. The precise definitions are much more delicate and require some particular attentions on the condition “up to homotopy”. For details some references are [May77], [May09], [Lur17].

The element Δ instead is the discriminant, and is a weight 12 integral modular form defined by

$$\Delta(z) = e^{2\pi i z} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n z}\right)^{24} .$$

It is clear from how this ring is defined, that it is periodic. However, as we have said, the map is an isomorphism only over rational numbers, and in particular can be shown that the discriminant, which corresponds to the periodic element of $\mathrm{wMF}_\bullet$, is not in the homotopy groups of TMF. Nevertheless its twenty-fourth power is

$$\Delta^{24} \in \pi_{24^2}(\mathrm{TMF}) ,$$

and this means that TMF is periodic with period $24^2 = 576$.

The map in (3.1) comes from a well-known map called *elliptic genus map*

$$\pi_n(\mathrm{tmf}) \longrightarrow \mathrm{MF}^{\mathbb{Z}}_{n/2} \qquad \forall\, n \in \mathbb{Z} .$$

Here we have identified with tmf the *non-periodic* and *connective* version of TMF, where connective means that

$$\pi_n(\mathrm{tmf}) = 0 \qquad \forall n \in \mathbb{Z}_{<0} ,$$

while $\mathrm{MF}^{\mathbb{Z}}_\bullet$ is the graded ring of integral modular forms. This map is not an isomorphism in general, since it has both kernel and cokernel. In particular, the kernel is the ideal generated by all the torsion classes, while the cokernel is described by

$$\mathrm{coker}\left(\pi_n(\mathrm{tmf}) \;\rightarrow\; \mathrm{MF}_{\frac{n}{2}}\right) = \begin{cases} \mathbb{Z}/\frac{24}{\gcd(k,24)}\mathbb{Z} & n = 24k , \\ (\mathbb{Z}/2\mathbb{Z})^{\lceil \frac{n-8}{24} \rceil} & n \equiv 4 \mod 8 , \\ 0 & \text{otherwise} . \end{cases}$$

However, the elliptic genus becomes an isomorphism over rational numbers, namely

$$\pi_n(\mathrm{tmf}) \otimes \mathbb{Q} \xrightarrow{\sim} \mathrm{MF}_{n/2} \otimes \mathbb{Q} ,$$

i.e. the ring of homotopy groups of tmf is *rationally isomorphic* to the ring of integral modular forms. This version of the spectrum of topological modular forms is related to the periodic one via

$$\mathrm{TMF} = \mathrm{tmf}[\Delta^{-24}] ,$$

or, at the level of homotopy groups,

$$\pi_\bullet(\mathrm{TMF}) = \pi_\bullet(\mathrm{tmf})[\Delta^{-24}] .$$

3.2 Supersymmetric Quantum Field Theories

Now we want to describe the other construction introduced in the Stolz and Teichner conjecture, namely the set of (1+1)-dimensional supersymmetric QFT. The first difficulties in this definition arise from the fact that, even if from a physical perspective it is clear what a QFT is, we have not yet a precise mathematical definition. The first attempt to formalize this concept was done by Atiyah [Ati89] and Segal [Seg02], who give a geometrical interpretation of a QFT as a functor between two properly chosen categories. An important step forward, that is the one necessary for our purpose, was done in two articles by Stolz and Teichner ([ST04], [ST11]), who gave a more refined, but not yet complete, definition of supersymmetric QFT's.

What we are going to do in the next part, is to give a precise formal definition of a QFT in a functorial perspective in a particular case, namely the *topological quantum field theory (TQFT)*, trying to justify as much as possible our construction from a physical point of view. Then we will see how this definition has to be reformulated in order to describe the so-called *supersymmetric Euclidean field theory*, as it was done by Stolz and Teichner in [ST11], focusing on showing where the difficulties arise.

3.2.1 Topological Quantum Field Theory

Before starting giving the definition of a TQFT, let us recall some trivial facts about the path integral. It is given by

$$Z = \int \mathcal{D}\Phi \, e^{-S[\Phi]} \ ,$$

where $\Phi : M \longrightarrow X$ is a smooth map between two Riemannian manifold M (the *spacetime*) and X (which, once chosen, gives us the nature of Φ as a physical field -scalar, spinor, vector, ...-), $S[\Phi]$ is the so-called *action functional* and is usually of the form

$$S[\Phi] = \int_M \sqrt{\det g} \, \mathrm{d}^n x \, \mathcal{L}\big(\Phi(x), \partial_\mu \Phi(x)\big) \ ,$$

with $\sqrt{\det g}\, \mathrm{d}^n x$ the volume form on M and $\mathcal{L}$ the *lagrangian density*. Then $\int \mathcal{D}\Phi$ is an integral over all Φ, which actually has no meaning in general. Despite it is not well-defined, the path integral approach gives us a lot of information about the theory, and in particular allows us to compute correlations functions. So let $\mathcal{O}_1, \dots, \mathcal{O}_n$ be n observables, i.e. functions from the set of field configurations to complex numbers

$$\mathcal{O}_1, \dots, \mathcal{O}_n : \{\Phi : M \to X\} \longrightarrow \mathbb{C}$$

Then we have that the correlation function of the observables $\mathcal{O}_1, \dots, \mathcal{O}_n$ is

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_g = \frac{1}{Z} \int \mathcal{D}\Phi \, \mathcal{O}_1 \cdots \mathcal{O}_n e^{-S[\Phi]} \ ,$$

where g stands for the metric on M. If this correlation function is independent of g we have a **topological quantum field theory**.

Definition

What we want to do now is to see a quantum field theory as a way of transporting the geometric and dynamical structure of spacetime into the algebraic description of physical states and observables, that is to see it as a sort of map between geometry and algebra which preserves certain structures. In the particular case of a topological quantum field theory, to formalize this idea is relatively easy and was done by Atiyah and Segal introducing the following definition

Definition 3.2.1. An **n-dimensional (oriented closed) TQFT** is a symmetric monoidal functor

$$E : \mathsf{Bord}_n \longrightarrow \mathsf{Vect}_k \ .$$

Now we are going to explain what this definition actually means, while then we will try to make a comparison between the properties of the path integral and the one of this definition.

Let us begin with Vect_k. Its objects are k-vector spaces, with k a field, and morphisms are k-linear maps. As we know, compositions between morphisms are required in the definition of category, and in this case are trivial. However for this category we have an additional structure, indeed it is a **monoidal** category, which means that objects have a composition law too. In particular, given U, $V \in \mathsf{Vect}_k$ we can compose them as $U \otimes_k V \in \mathsf{Vect}_k$, which respects the properties we require for a composition of morphisms, namely it is associative up to isomorphism

$$(U \otimes_k V) \otimes_k W \simeq U \otimes_k (V \otimes_k W) \ , \tag{3.2}$$

and $k \in \mathsf{Vect}_k$ is a unit, since

$$k \otimes_k V \simeq V \simeq V \otimes_k k \ .$$

Moreover, since we can also take tensor products of linear maps, $\otimes_k$ is a functor

$$\otimes_k : \mathsf{Vect}_k \times \mathsf{Vect}_k \longrightarrow \mathsf{Vect}_k \ .$$

We can go on and notice that the monoidal category Vect_k has also a **symmetric** structure: given U, $V \in \mathsf{Vect}_k$, there are natural isomorphisms, known as **braidings**, such that

$$\begin{aligned} \beta_{U,V} :\ & U \otimes_k V \longrightarrow V \otimes_k U \\ & u \otimes v \longmapsto v \otimes u \end{aligned} \ ,$$

compatible with the isomorphism (3.2) and with the symmetry property

$$\beta_{U,V} = \beta^{-1}_{V,U} \ .$$

Then, let us focus on the symmetric monoidal category Bord_n. Its objects are oriented closed $(n-1)$-dimensional real manifolds for some fixed $n \in \mathbb{Z}_{\geq 1}$ (from the physical perspective, we can think of one of these manifolds as a sort of spacial slice of an n-dimensional spacetime).

Now given E, $F \in \mathsf{Bord}_n$, a morphism in the category between these two objects $E \longrightarrow F$ is the equivalence class of a bordism from E to F. A **bordism** $E \longrightarrow F$ is an oriented compact n-dimensional manifold with boundary M, together with smooth maps

$$\iota_{\text{in}} : E \longrightarrow M \ , \qquad \iota_{\text{out}} : F \longrightarrow M \ ,$$

with image in ∂M such that

$$\bar{\iota}_{\text{in}} \sqcup \iota_{\text{out}} : \overline{E} \sqcup F \longrightarrow \partial M$$

is an orientation-preserving diffeomorphism, where $\overline{E}$ denotes E with the opposite orientation. Two bordisms $(M, \iota_{\text{in}}, \iota_{\text{out}})$, $(M', \iota'_{\text{in}}, \iota'_{\text{out}}) : E \longrightarrow F$ are said to be **equivalent** if there exists an orientation-preserving diffeomorphism $\psi : M \longrightarrow M'$ such that the diagram

$$\begin{tikzcd} & M \arrow[dd, "\psi"] & \\ E \arrow[ur, "\iota_{\text{in}}"] \arrow[dr, "\iota'_{\text{in}}"'] & & F \arrow[ul, "\iota_{\text{out}}"'] \arrow[dl, "\iota'_{\text{out}}"] \\ & M' & \end{tikzcd}$$

commutes.
Composition of morphisms $M_1 : E \longrightarrow F$ and $M_2 : F \longrightarrow G$ in Bord_n is given by *gluing M_1 and M_2 along F*. As we have said, the category Bord_n has also the structure of a monoidal category with "multiplication" given by the disjoint union, which is associative by definition, and with unit element given by $\emptyset$ viewed as an $(n-1)$-dimensional manifold, since

$$\emptyset \sqcup E = E = E \sqcup \emptyset$$

for all $E \in \mathsf{Bord}_n$.

Then can be shown[8], via the so-called *cylinder construction*, that a diffeomorphism between two $(n-1)$-dimensional manifolds (which can be seen as objects in Bord_n) induces an isomorphism between the two in Bord_n. This implies that the natural diffeomorphism $E \sqcup F \longrightarrow F \sqcup E$ induces an isomorphism which is the symmetric braiding

$$\beta_{E,F} : E \sqcup F \longrightarrow F \sqcup E$$

in Bord_n.

Motivation

Now we want to compare this functorial definition[9] with the path integral formulation, remembering that we suppose the path integral to be always well-defined, even if it is not true in general.

1. From a path integral point of view, we expect that a TQFT associates to an $(n-1)$-dimensional oriented manifold E a Hilbert spaces $\mathcal{H}_E$ of states on E. We can think of $\mathcal{H}_E$ as the space of functionals on the classical fields on E, that is the space of the maps
$$\{E \to X\} \longrightarrow \mathbb{C}\,.$$

[8] For details look at [CR18].
[9] For all the missing details on this part look at [CR18] and the references therein.

On the other side, looking at the functorial definition, we have that it assigns objects to objects, i.e. to an $(n-1)$-dimensional manifold $E \in \mathsf{Bord}_n$ a $\mathbb{C}$-vector space $Z(E)$, which corresponds exactly to $\mathcal{H}_E$.

Here we have an apparent difference between the two approaches, since the path integral gives us an Hilbert space, while the functor a $\mathbb{C}$-vector space. However, it can be shown that there is a non-degenerate pairing between $Z(E)$ and $Z(\overline{E})$, that is $Z(\overline{E}) \simeq Z(E)^*$, which gives to $Z(E)$ the structure of an Hilbert space.

2. Let φ be a field on E and consider the path integral over the fields Φ's on the n-dimensional manifold M with boundary $\partial M = E$, where the Φ's restrict to φ on the boundary, i.e.

$$Z(M)(\varphi) = \int \mathcal{D}\Phi e^{-S[\Phi]}\Big|_{\Phi \text{ on } M \text{ s.t. } \Phi|_{\partial M}=\varphi} . \tag{3.3}$$

Hence we have that the right-hand side produce a number for each $\varphi : E \longrightarrow X$ we plug in, or, in other words, we have obtained a functional on fields on E associated to the manifold M

$$Z(M) : \; \{E \to X\} \longrightarrow \mathbb{C} .$$

So, from the path integral point of view we expect to obtain a vector $Z(M) \in \mathcal{H}_E$ for each oriented n-manifold M with boundary $\partial M = E$.

On the other hand we know that a functor assigns morphisms to morphisms. This means that Z produces, for every bordism $M : E \longrightarrow E'$, a linear map

$$Z(M) : Z(E) \longrightarrow Z(E') ,$$

which we can think of as describing the evolution along M. In order to relate to the path integral above, we need to think of an n-manifold M with boundary $\partial M = E$ as a bordism $M : \emptyset \longrightarrow E$. Can be shown that $Z(\emptyset) = \mathbb{C}$, in such a way that we obtain

$$Z(M) : \mathbb{C} \longrightarrow Z(E) ,$$

which is the same as giving an element in $Z(E) = \mathcal{H}_E$ by taking the image of $1 \in \mathbb{C}$.

3. Let us consider the space $\mathrm{C}(E_1 \sqcup E_2)$ of maps from $E_1 \sqcup E_2$ to some manifold (i.e. the one needed in order to describe the particular field we want), which can be seen as the set of classical fields. It is quite trivial to see that it equals the Cartesian product $\mathrm{C}(E_1) \times \mathrm{C}(E_2)$, and we have that the linear space $\mathcal{H}_E$ of functionals $C(E) \longrightarrow \mathbb{C}$ satisfies

$$\mathcal{H}_{E_1 \sqcup E_2} = \mathcal{H}_{E_1} \otimes \mathcal{H}_{E_2} .$$

Hence we require that, given an $(n-1)$-manifold E which is a disjoint union $E = E_1 \sqcup E_2$, we have

$$\mathcal{H}_E = \mathcal{H}_{E_1} \otimes \mathcal{H}_{E_2} .$$

Similarly we would like the action to be “local enough” so that, given a field Φ defined on $M_1 \sqcup M_2$, we have

$$S[\Phi] = S[\Phi\big|_{M_1}] + S[\Phi\big|_{M_2}] \ ,$$

and so the path integral in (3.3) splits into a product. We can restate this saying that given two n-dimensional manifolds M_1 and M_2 with $\partial M_1 = E_1$ and $\partial M_2 = E_2$, we expect that

$$Z(M_1 \sqcup M_2) = Z(M_1) \otimes Z(M_2) \ .$$

Indeed we have required the TQFT to be a symmetric monoidal functor, so it has to be compatible with the structure in the target and in the source, which gives us the isomorphisms

$$Z(\emptyset) \simeq \mathbb{C} \ , \qquad Z(E \sqcup F) \simeq Z(E) \otimes_{\mathbb{C}} Z(F) \ ,$$

compatible with the associativity of the products and the braidings; in particular we can assume $Z(\emptyset) = \mathbb{C}$.

It seems we have not described the behavior of the TQFT under the disjoint union of n-dimensional manifolds, however from a general statement about symmetric monoidal functors (the same we are going to use in what follows to solve some apparent incongruences) follows that it is not the case, being this behavior a consequence of the structure of the functor.

4. What we want to require now is that, if we compute the path integral on a thin cylinder, we can distinguish any two states placed on one boundary by selecting an appropriate state to place on the other boundary. This allows us to introduce somehow the idea of *physical states*. Indeed, an example in which this does not happen is gauge theories if we do not identify gauge equivalent field configurations. In this case, the corresponding state would be indistinguishable for thin cylinder and so describe the same physical state.

   In order to translate this requirement in a way which can be compared with the functorial definition, let us consider a cylinder $M = E \times [0,1]$, from which we obtain an element $Z(M) \in \mathcal{H}_E \otimes \mathcal{H}_{\overline{E}}$. We expect this element to be non-degenerate in the following sense. Given two elements u, $v \in \mathcal{H}_{\overline{E}}$, if the contraction of u with the tensor factor relative to $\mathcal{H}_{\overline{E}}$ of $Z(M)$ via the scalar product $\langle \cdot, \cdot \rangle$ is equal, as an element of $\mathcal{H}_E$, to the same contraction but with v, then we have $u = v$. This condition gives us an anti-linear injective map $\mathcal{H}_{\overline{E}} \longrightarrow \mathcal{H}_E$.

   From the functorial perspective this is a bit more difficult to see, and is due to the fact that a functor maps the identity to the identity. So we get

$$Z(E \times [0,1]) = \mathrm{id}_{Z(E)} \ .$$

   Thanks to the statement we cited above, this equality translates in the non-degeneracy condition required.

5. We want to have the possibility to sum over intermediate states. Thus, referring to (3.3), given an $(n-1)$-dimensional manifold U, we want that the integration over all Φ with $\Phi\big|_E = \varphi$ can be split in an integration over all the configuration Φ with $\Phi\big|_E = \varphi$ and $\Phi\big|_U = \psi$ for some $\psi \in \mathrm{C}(U)$ and then an integration over all ψ, in formulae

$$Z(M)(\varphi) = \int\limits_{\psi \text{ on } U} \mathcal{D}\psi \int \mathcal{D}\Phi e^{-S[\Phi]}\Big|_{\Phi \text{ on } M \text{ s.t. } \Phi|_{\partial M}=\varphi\,,\ \Phi|_U=\psi} .$$

In order to compare with the properties of the functorial definition we need to restate this condition in a more handy way. So let us consider an n-dimensional manifold M with boundary $\partial M = E$ and let us embed a closed $(n-1)$-dimensional manifold U into M. Then we cut M open along U. It is quite trivial to see that this operation produces a new n-dimensional manifold N with boundary $\partial N = E \sqcup U \sqcup \overline{U}$. Now let $\{e_i\}$ be an orthonormal basis of $\mathcal{H}_U$ and let $\overline{e}_i$ be the preimage of e_i through the map $\mathcal{H}_{\overline{U}} \longrightarrow \mathcal{H}_U$ defined above. We require that

$$Z(M) = \sum_i \langle e_i \otimes \overline{e}_i\,,\ Z(N)\rangle \in \mathcal{H}_E\ ,$$

where we think of

$$\langle e_i \otimes \overline{e}_i\,,\ \cdot\ \rangle : \mathcal{H}_E \otimes \mathcal{H}_U \otimes \mathcal{H}_{\overline{U}} \longrightarrow \mathcal{H}_E\ .$$

This property is linked to the fact that a functor is compatible with compositions, that is, given two bordisms

$$M : E \longrightarrow F\ , \quad N : F \longrightarrow G\ ,$$

we have

$$Z(M \sqcup_F N) = Z(M) \circ Z(N)\ .$$

Here there is a difference with the path integral motivation. Indeed we have required the gluing of manifolds to compose always *disjoint* n-manifold, which is not what in general happens if we cut a n-bordism along any embedded $(n-1)$-dimensional manifold. However also this difference, as the previous ones, is only apparent and solved by the statement cited above.

3.2.2 Supersymmetric euclidean field theory

Now we want to explain how the definition of a TQFT has to be generalized in order to allow us to state properly the conjecture made by Stolz and Teichner. Unfortunately, the precise definition is very technical, hence we are only going to sketch the main features, underlying the aspects of the previous definition of TQFT we need to refine. For the precise construction look at [ST11].

First of all we need the field theory defined as above to satisfy also the following requirement. Given a $(n-1)$-dimensional closed oriented manifold E and a n-dimensional bordism M of E, we need the vector space $Z(E)$ to depend smoothly on E and the linear map $Z(M)$ to depend smoothly on M. In order to make this requirement precise, roughly speaking, we should replace the categories involved in the definition by a *family version*, in which objects are families of closed $(n-1)$-manifolds or of vector spaces, parametrized by a smooth manifold X, and the same for morphisms.

Then, as we have said, we are interested in *Euclidean* field theories, namely, the $(n-1)$-dimensional closed oriented manifolds in Bord_n and the n-dimensional bordisms have to be endowed with an Euclidean metric, which in this case means a *flat Riemannian* metric (in contrast with a language more common in physics, for which "Euclidean" is used to indicate Riemannian metrics in contrast with Lorentzian ones, hence without the flatness requirement). This particular interest is justified by the fact that, for topological reasons, the only topology for closed manifolds which admits a flat metric is the torus. Hence, only surfaces of genus one can arise in the bordism category. This is what we need in order to relate this construction to TMF, in which only genus one information are used[10]. The requirement for the field theory to be euclidean is solved by endowing our definition of a particular structure known as *rigid geometry* (G, M) with G a Lie group and M a manifold, and properly choosing them.

Another structure we have to endow with our definition of QFT is supersymmetry, defining what we call *supersymmetric field theory of dimension* $n|\delta$, where $\delta \in \mathbb{Z}_{\geq 0}$. Let us make some comments on how to read this definition from a physical perspective. In particular, we have that the non-negative integer δ stands for the number of supersymmetries. Usually in physics we label supersymmetry with respect to minimality, and so the nomenclature makes sense only with respects to the dimension of the space-time, i.e. $\mathcal{N} = 1$ refers to the minimal supersymmetry, $\mathcal{N} = 2$ to twice as much, and so on. In our case we want to consider theories in $(n = 2)$-dimensions, and in particular chiral ones (which can occur only in dimensions $n = 2, 6 \mod 8$). Hence with $\delta = 1$ we are referring to $\mathcal{N} = (0, 1)$ supersymmetry.

Once introduced all these constructions, a good definition of a $2|1$-supersymmetric euclidean field theory can be obtained (in what follows we will refer with no distinction to them as SEFT or SQFT, with the latter as the preferred notation). What we have done up to now is well-defined from a mathematical point of view. However we have not yet given to SQFT a structure compatible with the one of TMF. We are left with two steps, the first one is to define a degree of SQFT which can be seen as a topological degree, and the second one is to "reduce" the set of SQFT. Unfortunately, none of these two steps are well-defined from a mathematical perspective. The problem for the degree is easily solved in physical terms, considering as the cohomological degree of an SQFT its gravitational anomaly. Then, in order to relate properly SQFT to TMF, we need to remove some redundancies in the set of SQFT. This is done first of all taking what is called *concordance classes* of SQFT. The physical interpretation of this equivalence relation is not clear, but the idea we rely on is that theories of the same class are the ones that can be connected by what we have called in section 2.2.3 *flowing up and down the RG trajectories*. In particular, this equivalence gives us the notion of *homotopy* in

[10]Let us remember that there is a relation between elliptic curves and tori.

the set SQFT. For this reason, from now on, when we talk about theories of the same homotopy type, we mean they belong to the same concordance class. Also we need to take only *local* field theories, due to the fact that otherwise there is no hope for this set to fulfill the axiom 3 on page 49 for the definition of a cohomology theory. But there is no formal definition of local euclidean field theory.

Hence, at this stage, we can formulate the conjecture due to Stolz and Teichner as

Conjecture 3.2.1. *There exists an isomorphism*

$$2|1\text{-EFT}^n_{loc}[X] \simeq \mathrm{TMF}^n(X)$$

where $2|1\text{-EFT}^n_{loc}[X]$ *is the set of 2-dimensional* $\mathcal{N} = (0,1)$ *supersymmetric local euclidean field theories of degree* n *over a manifold* X *considered up to concordance, while* TMF *is the generalized cohomology theory of topological modular forms.*

We need to do two more comments on this conjecture. First of all, we have seen how TMF is actually a so-called Ω-spectrum, but, even if we have identified a sort of degree for the SQFT's, we have not said anything about this type of structure. We will show in section 3.2.4 that the set of SQFT has a natural structure of an Ω-spectrum. Then we have also seen that TMF has periodicity 24^2, so we expect that the same property has to be true for SQFT. However this is not proven yet (and so is one of the fact that makes this statement only a conjecture). What Stolz and Teichner proved in [ST11] was that SQFT has a 48 periodicity. In particular, they showed that there exists a field theory $P \in 2|0\text{-EFT}^{-48}$ which is a periodicity element, i.e. it gives an equivalence

$$2|0\text{-EFT}^n(X) \xrightarrow{\sim} 2|0\text{-EFT}^{n-48}(X)\ .$$

Also they showed that theories in 2|0-EFT with spin structure are related to 2|1-EFT, so this periodicity element $P \in 2|0\text{-EFT}$ translates into a periodicity element in 2|1-EFT. The hope, pointed out in [ST11], is that, after properly defining the locality condition for SQFT, the 48 periodicity will turn in a 24^2 periodicity.

3.2.3 Gravitational anomaly

What we want to do now is to give some elementary notions on gravitational anomaly in order to explain why we consider it as the cohomological degree for SQFT.

First of all, let us recall that, roughly speaking, anomalies occur when some symmetries of the classical actions are not preserved at the quantum level. In particular, classical symmetries translate into the so-called *Slavnov-Taylor identities* for the quantum effective action. In general, these identities are crucial in order to prove unitarity and renormalizability of the theory. However, in proving the Slavnov-Taylor identities from the classical symmetries, we need to require the invariance under the symmetries of the integral measure. If this is not true we have an anomaly, i.e. the quantum effective action is not invariant at one-loop level. Anomalies occur also in theories with gravitational couplings only, in which case they are called **(purely) gravitational anomalies** (the term *purely* is due to the fact that also theories with both gauge and gravitational interactions can have anomalies, in which case they are called *mixed gauge-gravitational anomalies*). In particular they can occur only in $(4k+2)$-dimensional theories with $k \in \mathbb{Z}_{\geq 0}$ that contain spin-$\frac{1}{2}$ or spin-$\frac{3}{2}$ fields of definite chirality or an anti-symmetric tensor with $2k+1$ indices which obeys a duality condition ([AW84]). We will not enter in the details of why this happens, but we want to give a very rough justification. First of all we can prove that the only possibly anomalous part of the effective action is the imaginary one. So then, in order to find matter fields in an n dimensional euclidean space-time, we have to look at complex representations of the so-called *holonomy group* of the space-time manifold, which is $O(n)$ or a subgroup of it. But complex representation of $O(n)$ can be found only for $n = 4k+2$. Then it can be seen that the only representations which actually give gravitational anomaly are the ones listed above.

As we have said in section 2.3.2, in a non-anomalous theory, the Witten index is invariant under T-transformation of the modular parameter. If instead $\mathcal{F}$ has a gravitational anomaly $w \in \frac{1}{2}\mathbb{Z}$ we have that $Z_{RR}(\mathcal{F})$ suffers a multiplier under T-transformation, i.e.

$$T\big[Z_{RR}(\mathcal{F})\big] = e^{-2w\frac{2\pi i}{24}} Z_{RR}(\mathcal{F}) \ .$$

In order to explain the reason why this anomalous multiplier appears, let us start recalling from (1.18) that each free fermion gives to the vacuum energy a contribution of $\frac{1}{16}$. So, let us consider a generic theory with n bosons and n anti-chiral fermions, with central charges

$$c = n \ , \qquad \overline{c} = \frac{3}{2}n \ ,$$

from which the gravitational anomaly is

$$w = \overline{c} - c = \frac{n}{2} \ .$$

In this theory we have that the eigenvalue of L_0 on the ground states is 0, while the eigenvalue of $\overline{L}_0$ on the ground states is $\frac{n}{16}$, since there are n anti-chiral fermions. Hence it follows that, for a generic state

$$\begin{aligned}
&\text{the eigenvalues of } L_0 - \frac{c}{24} \text{ are in } \mathbb{Z} - \frac{n}{24} \ , \\
&\text{the eigenvalues of } \overline{L}_0 - \frac{\overline{c}}{24} \text{ are in } \mathbb{Z} + \frac{n}{16} - \frac{3}{2}\cdot\frac{n}{24} = \mathbb{Z} \ .
\end{aligned}$$

This means that, computing the T-transformed elliptic genus

$$Z_{RR}(\tau+1, \overline{\tau}+1) = \mathrm{Tr}\left[(-1)^F q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{\overline{c}}{24}} e^{2\pi i\left(L_0 - \frac{c}{24} - \overline{L}_0 - \frac{\overline{c}}{24}\right)}\right] ,$$

from which we conclude

$$Z_{RR}(\tau+1, \overline{\tau}+1) = e^{-n\frac{2\pi i}{24}} Z_{RR}(\tau, \overline{\tau}) = e^{-2w\frac{2\pi i}{24}} Z_{RR}(\tau, \overline{\tau})$$

However we can "adjust" the Witten index in a clever way, obtaining what is properly called *Witten genus*[11], rescaling it in the following way

$$Z_{RR}(\mathcal{F}) \;\rightarrow\; Z'_{RR}(\mathcal{F}) := Z_{RR}(\mathcal{F})\eta(\tau)^{2w} \; ,$$

where $\eta(\tau)$ is the *Dedekind's eta function*

$$\eta(\tau) = q^{1/24}\prod_{j=1}^{\infty}(1-q^j) \; ,$$

which under T-transformation behaves as

$$\eta(\tau+1) = e^{\frac{2\pi i}{24}}\eta(\tau) \; .$$

This rescaling is useful since, in this way, the Witten genus of a theory $\mathcal{F}$ with gravitational anomaly w is a modular form of weight w and trivial multiplier. In particular the Witten genus maps the theory $\mathcal{F} \in \mathrm{SQFT}$ to a modular form of the same weight as the gravitational anomaly of the theory $Z'_{RR}(\mathcal{F}) \in \mathrm{MF}_w$. But as we have said, there exists the elliptic genus map

$$\pi_n(\mathrm{tmf}) \longrightarrow \mathrm{MF}_{n/2} \qquad \forall\, n \in \mathbb{Z} \; ,$$

thanks to which, for all $n \in \mathbb{Z}$, we can write the following diagram

$$\begin{tikzcd} \mathrm{SQFT} \arrow[rr] \arrow[dr, dotted] & & \mathrm{MF}_{n/2} \\ & \pi_n(\mathrm{tmf}) \arrow[ur] & \end{tikzcd} \; .$$

The map that fills the dotted arrow is the one conjectured by Stolz and Teichner, and, from this diagram, where we can substitute TMF to tmf without any difficulties, we see that, in order to be consistent, the cohomological degree of the theory in SQFT with gravitation anomaly w has to be given by $n = -2w$ (the minus sign is due to the fact that we are talking about *cohomological* degree instead of *homological* one).

On the η-function in the Witten genus

We have defined the Witten genus for a gravitationally anomalous theory rescaling the elliptic genus by $\eta^{2w}(\tau)$, in such a way the result to be a modular form with a trivial multiplier. However, it is useful to restate this procedure in terms of the so-called *spectator fermions*.

Let us start considering a SQFT $\mathcal{F}$ with gravitational anomaly w. In order to "trivialize" the anomaly, we can tensor $\mathcal{F}$ with the holomorphic theory of $n = 2w$ free chiral fermions

$$\mathrm{Fer}(n) := \left(\,\mathrm{Fer}^1\,\right)^{\otimes n} \; .$$

[11] Here we have an ambiguity in the name. As we have seen yet, the Witten index and the elliptic genus are strictly related, while the Witten genus is the same as the elliptic genus, but rescaled. Despite these small differences, we will use the three names without making any difference, since the exact quantity used will be always clear from the context.

The central charge of this theory is $\frac{n}{2}$, from which we can deduce that the eigenvalues of $L_0 - \frac{c}{24}$ are in

$$\mathbb{Z} + \frac{n}{16} - \frac{1}{2} \cdot \frac{n}{24} = \mathbb{Z} + \frac{n}{24} \; .$$

Hence we get that, since the elliptic genus of the product theory is the product of the elliptic genera of the sub-theories, under T-transformation we get an extra factor $e^{n\frac{2\pi i}{24}}$ which cancel the factor due to the anomaly in $\mathcal{F}$. In this way we conclude that the product theory is non-anomalous.

Now let us try to compute the elliptic genus of the non-anomalous theory $\mathcal{F} \otimes \mathrm{Fer}(n)$. The problem now is that the elliptic genus of the theory $\mathrm{Fer}(n)$ is vanishing, due to the presence of the fermionic zero modes in the Ramond-Ramond sector. Indeed, the fermionic zero modes, since they are fermionic operators, exchange bosonic and fermionic states. Also they commute with L_0 and hence with the Hamiltonian, and they do not have neither the kernel nor the cokernel, since $(\psi_0^i)^2 = \frac{1}{2}$. This implies that, for each eigenvalue of L_0, the number of bosonic and fermionic state is the same, hence the elliptic genus vanishes.

In order to solve this problem and obtain a non-vanishing result, let us start rescaling the fermionic zero modes as

$$\gamma^i := 2^{\frac{1}{2}} \psi_0^i \; , \tag{3.4}$$

in such a way that they are the generators of the Clifford algebra $\mathrm{Cliff}(n)$

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij} \; .$$

Then, let us add to the trace the operator

$$\gamma_5 = (-1)^{\frac{n}{4}} \gamma^1 \cdots \gamma^n \; .$$

This is such that

- its square is equal to 1;
- it anti-commutes with all the fermionic zero modes.

This implies that, when it acts on the ground states, it acts as the operator $(-1)^F$ modulo a sign (from now on let us choose the plus sign, however nothing change with the opposite choice). In this way the ground states related by the action of ψ_0^i for certain $i = 1, \ldots, n$ give a contribution with the same sign, and do not cancel each other anymore.

When we consider the action on the other modes ψ_k^i, $k \neq 0$, the operator γ_5 commutes with them, hence things are different. So let us perform the computations. The Hilbert space of the theory $\mathrm{Fer}(n)$ can be factorized dividing the ground states (gs) and the excited states (es). The n-point function computed on the ground states reads

$$\begin{aligned}
\langle (-1)^{\frac{n}{4}} :\psi^1 \cdots \psi^n: \rangle_{\mathrm{gs}} &= \mathrm{Tr}_{\mathrm{gs}} \left[(-1)^F (-1)^{\frac{n}{4}} \psi_0^1 \cdots \psi_0^n q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}} \right] \\
&= 2^{-\frac{n}{2}} \, \mathrm{Tr}_{\mathrm{gs}} \left[(-1)^F (-1)^{\frac{n}{4}} \gamma^1 \cdots \gamma^n \right] = \\
&= 2^{-\frac{n}{2}} \, \mathrm{Tr}_{\mathrm{gs}} \left(\gamma_5^2 \right) = 1
\end{aligned}$$

The complete Hilbert space is obtained by acting on the vacuum state with creation operators

$$\psi^i_k \ , \qquad k > -\frac{1}{2} \ .$$

Since this operators, as we have explained yet, commute with γ_5, we have that the n-point function on the whole Hilbert space is

$$\langle (-1)^{\frac{n}{4}} :\psi^1 \cdots \psi^n: \rangle = \mathrm{Tr}_{\mathrm{gs}} \left[(-1)^F \psi^1_0 \cdots \psi^n_0 \right] \Big(\mathrm{Tr}_{\mathrm{es}} \left[(-1)^F q^{L_0 - \frac{1}{48}} \overline{q}^{\overline{L}_0 - \frac{1}{48}} \right] \Big)^n \ .$$

We are considering only chiral fermions, so the operator $\overline{q}^{\overline{L}_0 - \frac{1}{48}}$ acts trivially. Also let us remember the expression (1.18) and notice that the fermion number operator factorizes on the fermonic modes, that is

$$F = \sum_{m \geq 0} F_m \ , \qquad F_m = \psi_{-k} \psi_k \ , \quad k > 0 \ ,$$

while F_0 is defined such that it is zero when it acts on $|0\rangle$, while it is 1 acting on $\psi_0 \, |0\rangle$. Hence we get

$$\begin{aligned} \mathrm{Tr}_{\mathrm{es}} \left[(-1)^F q^{L_0 - \frac{1}{48}} \right] &= \mathrm{Tr}_{\mathrm{es}} \left[(-1)^{\sum_{m>0} F_m} q^{\sum_{k>0} k \psi_{-k} \psi_k} q^{\frac{1}{24}} \right] = \\ &= q^{\frac{1}{24}} \sum_{m_1=0}^{1} \sum_{m_2=0}^{1} \cdots \langle m_1, m_2, \ldots | \prod_{k>0} (-1)^{F_k} q^{k \psi_{-k} \psi_k} \, |m_1, m_2, \ldots \rangle = \\ &= q^{\frac{1}{24}} \sum_{m_1=0}^{1} \sum_{m_2=0}^{1} \cdots \langle m_1, m_2, \ldots | \prod_{k>0} (-1)^{F_k} q^{k m_k} \, |m_1, m_2, \ldots \rangle = \\ &= q^{\frac{1}{24}} \prod_{k=1}^{\infty} \sum_{m_k=0}^{1} (-1)^{F_k} q^{k m_k} = \\ &= q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - q^k) = \eta(\tau) \ , \end{aligned}$$

where we have used the fact that a generic state is

$$|m_1, m_2 \ldots \rangle = (\psi^1)^{m_1} (\psi^2)^{m_2} \cdots |0\rangle \ ,$$

and the operator $\psi_{-k} \psi_k$ is the fermionic number operator for the k-th mode. Hence, we conclude that

$$\langle (-1)^{\frac{n}{4}} :\psi^1 \cdots \psi^n: \rangle_{\mathrm{Fer}(n)} = \eta(\tau)^n \ . \tag{3.5}$$

When we consider the theory $\mathcal{F} \otimes \mathrm{Fer}(n)$, if we compute the n-point function of the operator $(-1)^{\frac{n}{4}} :\psi^1 \cdots \psi^n:$, we have that the Hilbert space of the whole theory factorizes in the Hilbert spaces of the two sub-theories, and the result is given by the n-point function of the operator in the theory $\mathrm{Fer}(n)$, times the elliptic genus of the theory $\mathcal{F}$, that is

$$\langle (-1)^{\frac{n}{4}} :\psi^1 \cdots \psi^n: \rangle_{\mathcal{F} \otimes \mathrm{Fer}(n)} = \eta(\tau)^n Z_{RR}(\mathcal{F}) \ ,$$

which is exactly the Witten genus we have defined above.

3.2.4 SQFT as an Ω-spectrum

We have seen that the set TMF has a particular structure known as Ω-spectrum. Hence, in order to compare the set SQFT with it, we need the former to have the same structure. What we are going to see now is that SQFT is naturally endowed with this structure, without any further requirement. Let us start from the choice of the basepoint for our spaces. With the definition we have given of a QFT we can define the *zero QFT* $0 \in \mathrm{SQFT}^\bullet$ as the TQFT which associates "0" to every non empty input, i.e. its partition function is zero, its Hilbert space is zero-dimensional, and so on. In order to give a physical interpretation to this theory it is sufficient to notice that asking for a theory $\mathcal{F}$ to spontaneously break supersymmetry is the same as saying that the theory $\mathcal{F}$ flows to 0 under the action of the RG-flow, where the RG-flow is a canonically defined action of the monoid $\mathbb{R}_{\geq 0}$ on $\mathrm{SQFT}^\bullet$. In general we will say that $\mathcal{F}$ flows to $\mathcal{F}_{\mathrm{IR}}$ if $\mathcal{F}_{\mathrm{IR}}$ is the limit under the RG-flow starting from $\mathcal{F}$. We can consider these two notions as equivalent for the following reason. As we know, a theory breaks supersymmetry if there are no supersymmetric ground states, or, in other words, ground states at zero energy. If this is what happens, the energy spectrum has necessarily some strictly positive minima (is a well-known fact that, in a unitary supersymmetric theory, the spectrum of the Hamiltonian is always positive semi-definite). Hence, going down in energy flowing the RG trajectories, we reach at some point below the minima, a theory with no states, that is the zero QFT defined above.

Now we have to build the map

$$\mathrm{SQFT}^n \longrightarrow \Omega\mathrm{SQFT}^{n+1}$$

and then verify it is a weak homotopy equivalence. In particular, let us notice that, from the definition of the loop space, a point in $\Omega\mathrm{SQFT}^n$ is an $\mathbb{R}$-family $x \longmapsto \mathcal{F}(x)$ of theories in SQFT^n that reach the basepoint at the "extrema" of $\mathbb{R}$, hence that breaks supersymmetry for $x \ll 0$ and $x \gg 0$, or, in other words, such that $\mathcal{F}(x)$ goes to the QFT 0 for $x \to \pm\infty$.

In order to build the map we are interested in, let us consider the theory Fer^1, that is a conformal field theory of a single chiral Majorana fermion ξ with trivial supersymmetry. The lagrangian is the same as (2.4), that is, recalling that ξ is the upper component of a Fermi superfield $\Lambda = \xi + \theta F$,

$$\mathcal{L}_\Lambda = \frac{i}{2}\xi\partial_u\xi + \frac{1}{2}F^2 \ .$$

Let us notice that the triviality of the supersymmetry follows once we impose the equation of motion for F given by

$$\frac{\partial\mathcal{L}_\Lambda}{\partial F} = 0 \quad\Longrightarrow\quad F = 0 \ .$$

However, breaking conformal invariance, we can endow our theory with a different supersymmetry. This can be done by adding a superpotential

$$W(\Phi) = -x \qquad x \in \mathbb{R} \ ,$$

from which

$$\begin{aligned} S_W &= \int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\Lambda W(\Phi) = \\ &= \int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,(\xi + \theta F)(-x) = \\ &= \int \mathrm{d}u\,\mathrm{d}v\,(-xF)\ . \end{aligned}$$

From here it follows that the lagrangian of the theory is

$$\mathcal{L}_{\Lambda+W} = \frac{i}{2}\xi\partial_u\xi + \frac{1}{2}F^2 - xF\ , \tag{3.6}$$

and integrating out in a trivial way the auxiliary field F we get

$$\mathcal{L}_{\mathrm{Fer}^1(x)} = \frac{i}{2}\xi\partial_u\xi - \frac{1}{2}x^2\ .$$

It gives us a family of SQFT's depending on a real parameter $x \in \mathbb{R}$ which we call $\mathrm{Fer}^1(x)$. Let us notice that now, once imposed the equation of motion of F, i.e. on-shell, we have that supersymmetry acts on ξ as

$$\xi \ \longrightarrow\ x\ .$$

Also, the potential energy is constant and, for $x \neq 0$, it is strictly positive

$$V = \frac{1}{2}x^2 > 0\ ,$$

hence supersymmetry is spontaneously broken. At this point, given a theory $\mathcal{F} \in \mathrm{SQFT}^n$, we define the corresponding family in $\Omega\mathrm{SQFT}^{n+1}$ as $\mathcal{F} \otimes \mathrm{Fer}^1(x)$, thus defining the map

$$\begin{aligned} \mathrm{SQFT}^n &\longrightarrow \Omega\mathrm{SQFT}^{n+1} \\ \mathcal{F} &\longmapsto \mathcal{F} \otimes \mathrm{Fer}^1(x) \end{aligned}\ .$$

We have seen that, if $x \neq 0$, the theory $\mathrm{Fer}^1(x)$ spontaneously breaks supersymmetry, and so the same is true for $\mathcal{F} \otimes \mathrm{Fer}^1(x)$. This implies that the $\mathbb{R}$-family

$$x \longmapsto \mathcal{F} \otimes \mathrm{Fer}^1(x)$$

is actually a point in $\Omega\mathrm{SQFT}^{n+1}$. What we need to do is to verify that this map just defined is an homotopy equivalence.

Let us start considering the so-called *dynamicalization map*

$$\Omega\mathrm{SQFT}^{n+1} \longrightarrow \mathrm{SQFT}^n$$

which, given a point in $\Omega\mathrm{SQFT}^{n+1}$, i.e. a family $x \longmapsto \mathcal{F}(x)$, promotes the parameter x to a dynamical scalar multiplet. This is done substituting the parameter x with a scalar superfield $\Phi = \varphi + i\theta\psi$, with ψ an anti-chiral fermion field. Let us write the result of this procedure as

$$\mathcal{F}(x) \mapsto \int_\Phi \mathcal{F}(\Phi)\ .$$

Let us assume, without proving it, that, given a family $\big(x \mapsto \mathcal{F}(x)\big) \in \Omega\mathrm{SQFT}^{n+1}$, the theory $\int_\Phi \mathcal{F}(\Phi)$ is compact. This dynamicalization map is our proposed homotopy inverse. Hence, in order to verify that

$$\mathcal{F} \longmapsto \{x \mapsto \mathcal{F} \otimes \mathrm{Fer}^1(x)\}$$

is an homotopy equivalence, it is sufficient to verify that its compositions with $\int_\Phi$ are homotopic to the identity.

Let us start from studying the composition

$$\mathcal{F} \longmapsto \mathcal{F} \otimes \mathrm{Fer}^1(x) \longmapsto \int_\Phi \mathcal{F} \otimes \mathrm{Fer}^1(\Phi)\ .$$

Since the copy of the theory $\mathcal{F}$ comes out of the integral, it is enough to verify the dynamicalization $\int_\Phi \mathrm{Fer}^1(\Phi)$ to be deformable in a continuous way to the trivial theory $1 \in \mathrm{SQFT}^0$. Hence let us come back to the theory $\mathrm{Fer}^1(x)$ and dynamicalize it. The first step is to promote x to a dynamical superfield Φ

$$x \ \longmapsto \ \Phi = \varphi + i\theta\psi\ ,$$

adding of course the kinetic term for it, given by

$$S_\Phi = \frac{i}{2\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\partial_v\Phi D\Phi\ ,$$

whose expansion in terms of ordinary fields is given by (2.3), that is

$$\mathcal{L}_\Phi = \frac{1}{2}\partial_v\varphi\partial_u\varphi + \frac{i}{2}\psi\partial_v\psi\ . \tag{3.7}$$

Moreover, we need to modify also the superpotential interaction

$$W(\Phi) = -x \ \longmapsto \ W(\Phi) = -\Phi\ ,$$

from which the interaction term becomes

$$\begin{aligned}
S_W &= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\Lambda W(\Phi) = \\
&= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,(\xi + \theta F)(-\varphi - i\theta\psi) = \\
&= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,(-\xi\varphi + i\theta\xi\psi - \theta F\varphi) = \\
&= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,(i\xi\psi - F\varphi)\ ,
\end{aligned}$$

in such a way that the full lagrangian is

$$\mathcal{L}_{\Lambda+\Phi+W} = \frac{1}{2}\partial_u\varphi\partial_v\varphi + \frac{i}{2}\xi\partial_u\xi + \frac{i}{2}\psi\partial_v\psi + i\xi\psi - F\varphi + \frac{1}{2}F^2 \ .$$

Integrating out the field F imposing the equation of motion $F = \varphi$, we obtain

$$\mathcal{L}_{\int_\Phi \mathrm{Fer}^1(\Phi)} = \frac{1}{2}\partial_v\varphi\partial_u\varphi - \frac{1}{2}\varphi^2 + \frac{i}{2}\xi\partial_u\xi + \frac{i}{2}\psi\partial_v\psi + i\xi\psi \ ,$$

which we recognize as the theory of a massive Majorana fermion

$$\chi := \begin{pmatrix} \psi \\ i\xi \end{pmatrix} \ , \tag{3.8}$$

and of a massive scalar field φ. This means that the behavior of this theory in the IR is obtained substituting

$$\varphi,\, \xi,\, \psi \ \longmapsto\ 0 \ .$$

In this way we have found that

$$\mathcal{F} \ \mapsto\ \mathcal{F} \otimes \mathrm{Fer}^1(x) \ \mapsto\ \int_\Phi \mathcal{F} \otimes \mathrm{Fer}^1(\Phi) \ \mapsto\ \mathcal{F} \otimes \int_\Phi \mathrm{Fer}^1(\Phi) \ \overset{W}{\longmapsto}\ \mathcal{F} \ ,$$

as we wanted.

The other composition is given by

$$\mathcal{F}(x) \longmapsto \int_\Phi \mathcal{F}(\Phi) \longmapsto \int_\Phi \mathcal{F}(\Phi) \otimes \mathrm{Fer}^1(x) \ .$$

Before going on let us notice one fact we have implicitly assumed yet. When we deal with family of theories, we are considering the case in which the field content of $\mathcal{F}(x)$ or every kinematical information does not depend on the parameter x. What instead depend on x are the lagrangian, the supercharge and every dynamical information. This is quite reasonable. Indeed, if a field exists only for certain values of x, it is sufficient to extend it in such a way that it exists for all x, but it is very massive except for those values it was define earlier. Even if we are not able to topologize the set SQFT, the idea is that it should be done in a way that cares primarily about low-energy effective field theories, hence very massive fields are just small deformations which do not change the homotopy type of the family $\mathcal{F}(\,\cdot\,)$.

Now, in order to study the composition above, we need first of all to dynamicalize the parameter x in the family of theories $\mathcal{F}(x)$. Since the action of the theory $\int \mathrm{d}u\,\mathrm{d}v\ \mathcal{L}_\mathcal{F}$ is supersymmetric, we have that its supersymmetric variation has to be a total derivative, that is

$$Q[\mathcal{L}_\mathcal{F}(x)] = \partial_u Y^u(x) + \partial_v Y^v(x) \ ,$$

for some Y^u and Y^v. Remembering that $Q^2 = i\partial_u$ and that the supercharge commutes with ordinary derivatives, we have

$$Q^2[\mathcal{L}_\mathcal{F}(x)] = i\partial_u\mathcal{L}_\mathcal{F}(x) = \partial_u Q[Y^u(x)] + \partial_v Q[Y^v(x)] \ ,$$

hence

$$\partial_u\Big(i\mathcal{L}_{\mathcal{F}}(x) - Q[Y^u(x)]\Big) = \partial_v Q[Y^v(x)] \ .$$

Now we want to integrate both sides in $\int \mathrm{d}v$. In doing this, we have that the RHS is a boundary term, hence it does not depend on the finite values of v. Looking at the previous identity as an identity of functionals on the fields, we have that also the LHS has to be independent on the finite values of v. In other words, $i\mathcal{L}(x)$ and $Q[Y^u(x)]$, seen as functionals on the fields at finite v, differ by a quantity D which does not depend on u. This allows us to shift the lagrangian as

$$i\mathcal{L}_{\mathcal{F}}(x) \ \longmapsto \ i\mathcal{L}_{\mathcal{F}}(x) - \partial_u(uD) \ ,$$

in such a way that $i\mathcal{L}(x)$ and $Q[Y^u(x)]$ coincide, and replacing $\mathcal{L}(x)$ with $-iQ[Y^u(x)]$ we get the same action. We can write then

$$Q[Y^u(x)] = i\mathcal{L}_{\mathcal{F}}(x) \ ,$$

and applying the supercharge Q to both sides we obtain

$$Q[\mathcal{L}_{\mathcal{F}}(x)] = \partial_u Y^u(x) \equiv \partial_u Y(x) \ ,$$

eliminating in this way the $Y^v(x)$ term. Now let us define the quantity

$$\widetilde{\mathcal{L}}(x) := -iY(x) + \theta\mathcal{L}_{\mathcal{F}}(x) \ .$$

This is a superfield, indeed its supersymmetric variation reads

$$Q[\widetilde{\mathcal{L}}(x)] = -iQ[Y(x)] + \theta Q[\mathcal{L}_{\mathcal{F}}(x)] = \mathcal{L}_{\mathcal{F}}(x) + \theta\partial_u Y(x) = \big(\partial_\theta + i\theta\partial_u\big)\widetilde{\mathcal{L}}(x) \ .$$

It is now clear that

$$\mathcal{L}_{\mathcal{F}}(x) = \int \mathrm{d}\theta \ \widetilde{\mathcal{L}}(x) \ .$$

Finally we promote x to the scalar superfield

$$\Phi(u,v,\theta) = \varphi(u,v) + i\theta\psi(u,v) \ ,$$

obtaining, expanding in θ,

$$\begin{aligned}\int \mathrm{d}\theta \ \widetilde{\mathcal{L}}(\Phi) &= \int \mathrm{d}\theta \ \Big(-iY^u(\Phi) + \theta Y'(\varphi)\psi + \theta\mathcal{L}_{\mathcal{F}}(\varphi)\Big) = \\ &= Y'(\varphi)\psi + \mathcal{L}_{\mathcal{F}}(\varphi) \ .\end{aligned}$$

with clearly $Y'(x) = \partial_x Y(x)$. Also, we need to add the kinetic term for Φ given by equation (3.7). The next step consists in tensoring this theory with $\mathrm{Fer}^1(x)$, which results in adding to the lagrangian the terms in (3.6), obtaining

$$\mathcal{L}_{\mathcal{F}(\Phi)\otimes\mathrm{Fer}^1(x)} = \mathcal{L}_{\mathcal{F}}(\varphi) + Y'(\varphi)\psi + \frac{1}{2}\partial_u\varphi\partial_v\varphi + \frac{i}{2}\psi\partial_v\psi + \frac{i}{2}\xi\partial_u\xi + \frac{1}{2}F^2 - Fx \ .$$

Now let us deform this SQFT introducing the superpotential

$$W(\Phi) = f(\Phi)\,,$$

for some polynomial $f \in \mathbb{R}[x]$. The action term for this potential is given by

$$\begin{aligned}
S_W &= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\Lambda W(\Phi) = \\
&= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,(\xi + \theta F) f(\Phi) = \\
&= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,(\xi + \theta F)\Big(f(\varphi) + i\theta\frac{\partial f(\varphi)}{\partial\varphi}\psi\Big) = \\
&= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\big(\xi f(\varphi) - i\theta\xi f'(\varphi)\psi + \theta F f(\varphi)\big) = \\
&= \frac{1}{\pi}\int \mathrm{d}u\,\mathrm{d}v\,\big(-i\xi f'(\varphi)\psi + F f(\varphi)\big)\,,
\end{aligned}$$

where of course

$$f'(\varphi) := \frac{\partial f(\varphi)}{\partial\varphi}\,.$$

Thanks to this new term the total lagrangian becomes

$$\mathcal{L}_{\mathrm{Tot}} = \mathcal{L}_{\mathcal{F}}(\varphi) + Y'(\varphi)\psi + \frac{1}{2}\partial_u\varphi\partial_v\varphi + \frac{i}{2}\psi\partial_v\psi + \frac{i}{2}\xi\partial_u\xi + \frac{1}{2}F^2 - Fx - i\xi f'(\varphi)\psi + F f(\varphi)\,.$$

Integrating out the field F imposing the equation of motion

$$F = x - f(\varphi)\,,$$

we get

$$\begin{aligned}
\mathcal{L}_{\mathrm{Tot}} &= \cdots + \frac{1}{2}\big(x - f(\varphi)\big)^2 - x\big(x - f(\varphi)\big) + f(\varphi)\big(x - f(\varphi)\big) = \\
&= \mathcal{L}_{\mathcal{F}}(\varphi) + Y'(\varphi)\psi + \frac{1}{2}\partial_u\varphi\partial_v\varphi + \frac{i}{2}\psi\partial_v\psi + \frac{i}{2}\xi\partial_u\xi - i\xi f'(\varphi)\psi - \frac{1}{2}\big(x - f(\varphi)\big)^2\,,
\end{aligned}$$

where the ellipsis stand for those terms that do not change. If we now choose $f(\varphi) = \varphi$, the lagrangian reads

$$\mathcal{L}_{\mathrm{Tot}} = \big[\mathcal{L}_{\mathcal{F}}(\varphi) + Y'(\varphi)\psi\big] + \Big[\frac{1}{2}\partial_u\varphi\partial_v\varphi + \frac{i}{2}\psi\partial_v\psi + \frac{i}{2}\xi\partial_u\xi - i\xi\psi - \frac{1}{2}(x - \varphi)^2\Big]\,.$$

Here, looking at the second bracketed expression, we see that the field φ gets a mass with vacuum expectation value given by x, and also the fermion χ, defined as (3.8), becomes massive. This means that the behavior of the theory in the IR can be obtained substituting

$$\varphi \longmapsto x\,, \qquad \xi, \psi \longmapsto 0\,,$$

hence recovering the original theory $\mathcal{F}(x)$. In this way we have found that

$$\mathcal{F}(x) \mapsto \int_\Phi \mathcal{F}(\Phi) \mapsto \int_\Phi \mathcal{F}(\Phi) \otimes \mathrm{Fer}^1(x) \xmapsto{W} \mathcal{F}(x) \ .$$

In conclusion, even if we are not able to properly define the set SQFT, we have shown that, relying on the physical intuition of what a QFT is, the set SQFT has naturally maps

$$\mathrm{SQFT}^n \longrightarrow \Omega\mathrm{SQFT}^{n+1} \ ,$$

for all $n \in \mathbb{Z}$, which induce isomorphisms at the level of the homotopy groups, and so that give to the set the structure of an Ω-spectrum, as we expected.

Chapter 4

A secondary invariant

We have found in chapter 2 a limit under RG flow of the sigma model with target S^3_k, and we have seen that, if $k = 0 \mod 24$, the theory spontaneously breaks supersymmetry. We have also noticed that the Witten genus of this model is zero, so it gives us no information on the spontaneous breaking of the supersymmetry. In order to see whether the condition $k = 0 \mod 24$ is also a necessary condition for the spontaneous breaking, we need to find a new and more refined invariant for our theories. This will be achieved describing the Witten genus for non-compact theories, and see how its properties change in this situation.

Let us start with some nomenclature. We will say that a $(1+1)$-dimensional theory with $\mathcal{N} = (0,1)$ supersymmetry is **null** if supersymmetry is spontaneously broken, while it is said **nullhomotopic** if it can be connected to a null theory via the deformations that in section 2.2.3 on page 33, we have indicated as flowing up and down the RG flow.

Now we want to focus our attention on a possible way to enlarge the set SQFT, which will bring us to formalize a sort of "mild" non-compactness, that is theories which violate compactness in a controllable manner.

In what follows we will change our previous notation, identifying the gravitational anomaly as

$$n = 2(\bar{c} - c) \ .$$

This will be done in order to have an integer quantity.

4.1 Mildly non-compact theories

The first step consists in enlarging the set SQFT, allowing the theories to have more general properties. In particular, up to now we have only considered compact theories. As we have explained in 2.3.2 on page 37 , this definition is inspired from the case of sigma models, where compact models are the ones with a compact manifold as a target. In defining non-compact theories we want, again, to mimic sigma models whose target manifolds have an asymptotic boundary region which approaches a configuration $\mathbb{R}^+ \times N$, with N a compact manifold, as seen in section 2.2.3.

In the case of a general SQFT, a theory $\mathcal{F}$ of this type is described as follows. The theory depends on a local scalar superfield Φ which parametrizes the non-compact direction. Then, let us add to $\mathcal{F}$ the theory of a single chiral Fermi superfield $\Lambda = \xi + \theta F$, in such a way that the lagrangian of the theory reads (we neglect the computations since they are the same as the ones we have done several times before)

$$\mathcal{L}_{\mathcal{F}(\Phi)\otimes \mathrm{Fer}^1} = \mathcal{L}_{\mathcal{F}(\Phi)} + \frac{i}{2}\xi\partial_u\xi + \frac{1}{2}F^2 \ .$$

Now we add the superpotential

$$W(\Phi) = \Phi - p \qquad \text{with } p \in \mathbb{R} \ ,$$

whose contribution to the lagrangian, expanding as usual the superfield Φ as $\Phi = \varphi + i\theta\psi$, is

$$\begin{aligned} S_W &= \int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,\Lambda W(\Phi) = \\ &= \int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,(\xi + \theta F)(\varphi + i\theta\psi - p) = \\ &= \int \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\theta\,(\xi\varphi - i\theta\xi\psi - \xi p + \theta\varphi F - \theta p F) = \\ &= \int \mathrm{d}u\,\mathrm{d}v\,\big(- i\xi\psi + F(\varphi - p)\big) \ , \end{aligned}$$

hence

$$\mathcal{L}_{\mathcal{B}_p} = \mathcal{L}_{\mathcal{F}(\Phi)} + \frac{i}{2}\xi\partial_u\xi - i\xi\psi + \frac{1}{2}F^2 + F(\varphi - p) \ .$$

Integrating out F substituting it with its equations of motion $F = -(\varphi - p)$, we obtain

$$\mathcal{L}_{\mathcal{B}_p} = \mathcal{L}_{\mathcal{F}(\Phi)} + \frac{i}{2}\xi\partial_u\xi - i\xi\psi - \frac{1}{2}(\varphi - p)^2 \ ,$$

thanks to which we have obtained a family of SQFT's $\mathcal{B}_p$ parametrized by $p \in \mathbb{R}$. We require the family $\mathcal{B}_p$ to be made of compact theories which approach a compact theory $\mathcal{B}$ for $p \gg 0$, and spontaneously break supersymmetry, that is approach the zero QFT, for $p \ll 0$.

In other words, we require that the family $\mathcal{B}_p$, built from $\mathcal{F}$ as we have explained, is a *nullhomotopy* of the compact theory $\mathcal{B}$, that is a deformation of the theory $\mathcal{B}$ which connects it to the zero QFT. Also, we can alway associate to any nullhomotopy of $\mathcal{B}$ a mildly non-compact theory $\mathcal{F}$ dynamicalizing the parameter of the deformation to a chiral superfield, as we have explained in section 3.2.4, which corresponds to reverse the steps above. Summarizing we have find that

$$\begin{array}{c}\text{mildly non-compact SQFT } \mathcal{F} \\ \text{with 1 cylindrical end}\end{array} \xleftrightarrow{\ \sim\ } \text{nullhomotopy of a compact SQFT } \mathcal{B}$$

The condition we have described is identified saying that the theory $\mathcal{F}$ has *cylindrical end*. This definition can be easily generalized producing an homotopy between two theories $\mathcal{B}$ and $\mathcal{B}'$, simply requiring that, for $p \ll 0$ the family of theories approaches $\mathcal{B}'$. However, despite this requirement, the elliptic genus, defined as usual as the path integral on the torus with Ramond-Ramond boundary conditions, has no reasons to converge. In order to make sure that this definition makes sense, we need to ask that the Z_{RR} vanishes on the boundary theory, that is

$$Z_{RR}(\mathcal{B}) = 0 \ . \tag{4.1}$$

This condition is not restrictive at all, since, if the elliptic genus of $\mathcal{F}$ is well-defined, we expect the elliptic genus of $\mathcal{B}$ to vanish, being $\mathcal{F}$ a nullhomotopy of $\mathcal{B}$. Once imposed (4.1), we expect the theory to converge conditionally, indeed the end will be of the form $\mathbb{R}_{\gg 0} \times \mathcal{B}$, hence the contribution to the elliptic genus will be

$$Z_{RR}(\mathcal{B}) \times \mathrm{vol}(\mathbb{R}_{\gg 0}) \simeq 0 \times \mathrm{vol}(\mathbb{R}_{\gg 0}) \ ,$$

which we take to vanish. This ensure us to have a well-defined elliptic genus in this non-compact case. However, some of the properties of the elliptic genus are different than in the compact case, but looking at how they change, we will obtain crucial information on the non-compact theory itself, which will allow us to defined the invariant we need.

4.1.1 Holomorphic anomaly equation

The holomorphicity argument shown in section 2.3.2 is valid only since we are dealing with a theory with a discrete spectrum, in which case we can focus only on ground states. The same thing cannot be done in the case of a non-compact theory as the one defined earlier, and indeed the elliptic genus is not holomorphic anymore. However the anomaly in the holomorphicity of a mildly non-compact theory $\mathcal{F}$ with gravitational anomaly n can be computed, and in [GJ19] was proposed the following **holomorphic anomaly equation**

$$\sqrt{-8\tau_2}\frac{\partial}{\partial\overline{\tau}}\left[\eta^n(\tau)Z_{RR}(\mathcal{F})(\tau,\overline{\tau})\right] = g_{\mathcal{B}}(\tau,\overline{\tau}) \ , \tag{4.2}$$

where $\tau_2 = \frac{1}{2i}(\tau - \overline{\tau})$ is the imaginary part of τ, while

$$g_{\mathcal{B}}(\tau,\overline{\tau}) = (\text{ torus one-point function of } (-1)^{\frac{n}{4}} :\psi_1 \cdots \psi_{n-1}\overline{G}: \text{ in } \mathrm{Fer}(n-1) \otimes \mathcal{B} \ ) \ .$$

Here $\mathcal{B}$ is the compact theory related to $\mathcal{F}$ in the way we have explained, while $\overline{G}$ is the anti-holomorphic component of the supercurrent, that is the superpartner of the energy-momentum tensor.

This equation can be easily generalized in the following way. Let us consider a non-compact theory $\mathcal{F}$ with gravitational anomaly n, to which we can associate a family of theories $\mathcal{B}_p$. Now let us suppose that $\mathcal{B}_p$ stabilizes to a compact SQFT $\mathcal{B}_+$ for $p \gg 0$, and to another compact SQFT $\mathcal{B}_-$ for $p \ll 0$. In this case the holomorphic anomaly equation can be written as

$$\sqrt{-8\tau_2}\frac{\partial}{\partial\overline{\tau}}\left[\eta^n(\tau)Z_{RR}(\mathcal{F})(\tau,\overline{\tau})\right] = g_{\mathcal{B}_+}(\tau,\overline{\tau}) - g_{\mathcal{B}_-}(\tau,\overline{\tau}) \ .$$

4.1.2 Integrality of the q-expansion: compact case

We have seen in 2.3.2 that the elliptic genus, in the compact case, has an integral q-expansion. This was shown in the following way. Let $\mathcal{F}$ be a compact SQFT and let us compactify the spatial direction on a circle S^1 of radius R. In this way we have obtained a new theory $\mathcal{F}[S^1]$ and we have that the action on the circle, i.e. the momentum operator P, is quantized. Hence, for each eigenvalue k of the S^1-action, parametrized by the variable q, we have a supersymmetric quantum mechanics model. Therefore, the Witten genus of the original theory, $\eta^n Z_{RR}(\mathcal{F})$, for each k, restricts to the Witten index of a SQM, which is of course an integer, simply counting with sign the number of the supersymmetric ground states.

Then we want to refine the definition of the index in order to extend it in a useful way to gravitational anomalous theories. The first step is to define what a degree n SQM model is, interpreting this notion in term of the spectator fermions as we have done in the definition of the Witten genus.

A **degree n SQM model** is a SQM model with Hilbert space $\mathcal{H}$, endowed with the action of the n-th Clifford algebra $\mathrm{Cliff}(n)$. In terms of the spectator fermions, this algebra is the one generated by (after a simple rescaling shown in equation (3.4)) the fermions ψ^i's themselves. The SQM models we are interested in are obtained from the compactification of $(1+1)$-dimensional SQFT's, and from this follows that they have also a time-reversal symmetry (see [Guk+18]). The presence of the time reversal symmetry equips the Hilbert space $\mathcal{H}$ with a real structure, hence the Clifford algebra that acts on it is the real one $\mathrm{Cliff}(n, \mathbb{R})$.

In this framework, the supersymmetric ground states are a finite-dimensional $\mathrm{Cliff}(n, \mathbb{R})$-module V, given by the action of the zero modes of the spectator fermions on the vacuum state (indeed, let us recall from equation (1.15) that $[L_0, \psi_0^i] = 0$, hence the vacuum state is degenerate, and its degeneracy is given exactly by the action of the ψ_0^i's). However, the usual supersymmetric Witten index is defined depending on[1] $V \otimes \mathbb{C}$ as a $\mathrm{Cliff}(n, \mathbb{C})$-module, ignoring the time-reversal symmetry. In particular, if n is even, the complex Clifford algebra $\mathrm{Cliff}(n, \mathbb{C})$ has two irreducible modules, which differ by parity. If we choose one of them as the one with positive parity I, we have that

$$V \otimes \mathbb{C} \simeq I^{a|b} = I \otimes_{\mathbb{C}} \mathbb{C}^{a|b} \ ,$$

where $\mathbb{C}^{a|b}$ is the complex super-vector space with graded dimension (a, b). The Witten index is defined as

$$\mathrm{index} := a - b \ .$$

Indeed, from a more physical perspective, we have that a generic $\mathrm{Cliff}(n, \mathbb{C})$-module M is $\mathbb{Z}_2$-graded, hence can be decomposed as

$$M = M_0 + M_1 \ ,$$

[1]The fact that the complex module in $\mathrm{Cliff}(n, \mathbb{C})$ corresponding to a real module V of the real algebra $\mathrm{Cliff}(n, \mathbb{R})$ is given by the complexification of V, is a standard result in the theory of modules of Clifford algebras ([ABS64]). It is related to the fact that complexifing the real Clifford algebra $\mathrm{Cliff}(n, \mathbb{R})$, we get the complex Clifford algebra corresponding to the complexified quadratic form, that is $\mathrm{Cliff}(n, \mathbb{C})$.

where these components will be connected by the grading morphisms given by the action of the zero modes

$$\psi_0^i : M_0 \rightleftarrows M_1 \; .$$

Since ψ_0^i is a fermionic operator, we have that it exchange bosonic and fermionic states. From here follows that choosing the parity of the modules is the same as choosing bosonic and fermionic ground states, and then the definition of the Witten index is the usual one. Analogously, the two irreducible modules M_0 and M_1 can be seen as the eigenspaces of a fermionic number operator $(-1)^F$. In particular, always for even n, the operator on which eigenspaces we are interested in is

$$(-1)^{\frac{n}{4}}\gamma_0^1 \cdots \gamma_0^n \; , \tag{4.3}$$

Here, the γ_0^i's are the generators of the Clifford algebra

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij} \; .$$

They are obtained by the zero modes of the fermions up to a multiplicative factor, $\psi_0^i = 2^{-\frac{1}{2}}\gamma^i$, which generate the algebra

$$\{\psi_0^i, \psi_0^j\} = \delta^{ij} \; .$$

The operator (4.3) satisfies the following properties (whose proofs are trivial)

$$\left((-1)^{\frac{n}{4}}\gamma_0^1 \cdots \gamma_0^n\right)^2 = 1 \; , \qquad \left\{(-1)^{\frac{n}{4}}\gamma_0^1 \cdots \gamma_0^n \, , \gamma_0^i\right\} = 0 \; ,$$
$$\left[(-1)^{\frac{n}{4}}\gamma_0^1 \cdots \gamma_0^n \, , (-1)^F\right] = 0 \; .$$

These imply that $(-1)^{\frac{n}{4}}\gamma_0^1 \cdots \gamma_0^n$ has eigenvalues ± 1 and can be diagonalized simultaneously to $(-1)^F$. In this case the two irreducible modules can be identified by the two different choices of the relation between $(-1)^{\frac{n}{4}}\gamma_0^1 \cdots \gamma_0^n$ and $(-1)^F$, indeed they can be either

$$(-1)^{\frac{n}{4}}\gamma_0^1 \cdots \gamma_0^n = (-1)^F \qquad \text{or} \qquad (-1)^{\frac{n}{4}}\gamma_0^1 \cdots \gamma_0^n = -(-1)^F \; .$$

However we are interested in the situation in which V is a $\mathrm{Cliff}(n, \mathbb{R})$-module, so let us see what happens in this case, remembering that the modules of real Clifford algebras are classified with periodicity 8[2].

- If $n = 0$ we have the usual situation, in which we known the index to be an integer, hence index $\in \mathbb{Z}$.

- If $n = 2$, the generators of the module are

$$\gamma^1 := \frac{1}{\sqrt{2}}\psi_0^1 \; , \qquad \gamma^2 := \frac{1}{\sqrt{2}}\psi_0^2 \; .$$

[2]We are going to study only the cases in which $n \in 2\mathbb{Z}$. Indeed, we are no interested in odd n since there is no modular form of weight not divisible by 2.

Hence the $\mathrm{Cliff}(n,\mathbb{R})$-module J has dimension 4 and it has as a basis

$$|0\rangle \ , \quad \gamma^1 |0\rangle \ , \quad \gamma^2 |0\rangle \ , \quad \gamma^1\gamma^2 |0\rangle \ ,$$

where the first and the last states are bosonic, while the others are fermionic. Once complexified, we have that the module $J \otimes \mathbb{C}$ decomposes into two irreducible modules of complex dimension 2, generated by the basis

$$|\alpha_1\rangle := (\gamma^1 - i\gamma^2)|0\rangle \ , \qquad |\alpha_2\rangle := (1 - i\gamma^1\gamma^2)|0\rangle \ ,$$

and

$$|\beta_1\rangle := (\gamma^1 + i\gamma^2)|0\rangle \ , \qquad |\beta_2\rangle := (1 + i\gamma^1\gamma^2)|0\rangle \ .$$

In both cases, the first state is fermionic, while the second is bosonic. However, looking at the eigenvalues of $i\gamma^1\gamma^2$ relative to these states, we get

$$\begin{aligned} &(i\gamma^1\gamma^2)|\alpha_1\rangle = |\alpha_1\rangle \ , && (i\gamma^1\gamma^2)|\alpha_2\rangle = -|\alpha_2\rangle \\ &(i\gamma^1\gamma^2)|\beta_1\rangle = -|\beta_1\rangle \ , && (i\gamma^1\gamma^2)|\beta_2\rangle = |\beta_2\rangle \ , \end{aligned}$$

from which the difference in parity is clear. Summarizing we have that

$$J \otimes \mathbb{C} \simeq I^{1|1} \ ,$$

so

$$\text{index} = 0 \quad \text{for } n = 2 \mod 8 \ .$$

- If $n = 6$ we get the same behavior as the $n = 2$ case, hence we can summarize the two results saying that

$$\text{index} = 0 \quad \text{for } n = 2 \mod 4 \ .$$

- If $n = 4$ we have 4 generators of the module, that is

$$\gamma^1 := \frac{1}{\sqrt{2}}\psi_0^1 \ , \quad \gamma^2 := \frac{1}{\sqrt{2}}\psi_0^2 \ , \quad \gamma^3 := \frac{1}{\sqrt{2}}\psi_0^3 \ , \quad \gamma^4 := \frac{1}{\sqrt{2}}\psi_0^4 \ .$$

In this case there are two irreducible $\mathrm{Cliff}(n,\mathbb{R})$-modules which differs for the parity, $J^{1|0}$ and $J^{0|1}$, which have as basis respectively

$$\begin{aligned} &(\gamma^1\gamma^3 - \gamma^2\gamma^4)|0\rangle \ , \quad (\gamma^1\gamma^4 + \gamma^2\gamma^3)|0\rangle \ , \quad (\gamma^1\gamma^2 + \gamma^3\gamma^4)|0\rangle \ , (1 - \gamma^1\gamma^2\gamma^3\gamma^4)|0\rangle \ ; \\ &(\gamma^1 - \gamma^2\gamma^3\gamma^4)|0\rangle \ , \quad (\gamma^2 + \gamma^1\gamma^3\gamma^4)|0\rangle \ , \quad (\gamma^3 - \gamma^1\gamma^2\gamma^4)|0\rangle \ , \quad (\gamma^4 + \gamma^1\gamma^2\gamma^3)|0\rangle \ , \end{aligned}$$

and

$$\begin{aligned} &(\gamma^1\gamma^3 + \gamma^2\gamma^4)|0\rangle \ , \quad (\gamma^1\gamma^4 - \gamma^2\gamma^3)|0\rangle \ , \quad (\gamma^1\gamma^2 - \gamma^3\gamma^4)|0\rangle \ , (1 + \gamma^1\gamma^2\gamma^3\gamma^4)|0\rangle \ ; \\ &(\gamma^1 + \gamma^2\gamma^3\gamma^4)|0\rangle \ , \quad (\gamma^2 - \gamma^1\gamma^3\gamma^4)|0\rangle \ , \quad (\gamma^3 + \gamma^1\gamma^2\gamma^4)|0\rangle \ , \quad (\gamma^4 - \gamma^1\gamma^2\gamma^3)|0\rangle \ . \end{aligned}$$

Here, the eigenvalues of $(-1)\gamma^1\gamma^2\gamma^3\gamma^4$ for the states of $J^{1|0}$ are such that they are $+1$ for the bosons and -1 for the fermions, and the opposite for the states in $J^{0|1}$. If we look at the complexified module $J^{1|0}\otimes\mathbb{C}$, we see that it has as basis

$$\begin{aligned}\Big((\gamma^1\gamma^3-\gamma^2\gamma^4)-i(\gamma^1\gamma^4+\gamma^2\gamma^3)\Big)\,|0\rangle &= (\gamma^1-i\gamma^2)(\gamma^3-i\gamma^4)\,|0\rangle\ ,\\ \Big((1-\gamma^1\gamma^2\gamma^3\gamma^4)-i(\gamma^1\gamma^2+\gamma^3\gamma^4)\Big)\,|0\rangle &= (1-i\gamma^1\gamma^2)(1-i\gamma^3\gamma^4)\,|0\rangle\ ,\\ \Big((\gamma^3-\gamma^1\gamma^2\gamma^4)-i(\gamma^4+\gamma^1\gamma^2\gamma^3)\Big)\,|0\rangle &= (1-i\gamma^1\gamma^2)(\gamma^3-i\gamma^4)\,|0\rangle\ ,\\ \Big((1-\gamma^1\gamma^2\gamma^3\gamma^4)-i(\gamma^1\gamma^2+\gamma^3\gamma^4)\Big)\,|0\rangle &= (1-i\gamma^1\gamma^2)(1-i\gamma^3\gamma^4)\,|0\rangle\ ,\end{aligned}$$

and the complex conjugated ones. Hence, it is clear that this module splits in two copies of the irreducible $\mathrm{Cliff}(n,\mathbb{C})$-module I,

$$J^{1|0}\otimes\mathbb{C}\simeq I^{2|0}\ ,\qquad \text{while}\qquad J^{0|1}\otimes\mathbb{C}\simeq I^{0|2}\ ,$$

so the index is automatically even, that is

$$\text{index}\in 2\mathbb{Z}\qquad\text{for } n=4 \mod 8\ .$$

Summarizing all these results we have that the index of the SQM model of degree n is in $r\mathbb{Z}$ for

$$r=\begin{cases}1 & n=0 \mod 8\ ,\\ 2 & n=4 \mod 8\ ,\\ 0 & \text{otherwise}\ .\end{cases}\tag{4.4}$$

This result, in the SQFT case we are interested in, tells us that the Witten genus has a q-expansion in[3] $r\mathbb{Z}((q))$, where r can assume the values in (4.4).

4.1.3 Integrality of the q-expansion: non-compact case

The next step consists in looking at what happens if instead the theory is not compact. In order to do this, we have to slightly modify the conclusions stated by the Atiyah-Singer index theorem explained in 2.3.3 on page 44. In particular, it is not true anymore that the index is independent of all the parameters, due to the fact that the spectrum is not discrete. We are going to show what is the relation between the two limits considered before, that is $\beta\to\infty$ and $\beta\to 0$, showing that they are equal up to a new term.

Let us focus on a 4-dimensional non-compact target manifold M, as the one in Figure 4.1, described as follows: it contains a compact submanifold $\widehat{M}\subset M$ with boundary

$$\partial\widehat{M}=N_+-N_-\ ,$$

[3]Here we mean with $\mathbb{Z}((q))$ the ring of *formal Laurent series* with coefficients in $\mathbb{Z}$. In general, given a ring R and a letter X, we have that a **formal Laurent series**, that is an element in $R((X))$, is a sum like $\sum_{i=n}^{\infty}a_iX^i$ for some $n\in\mathbb{Z}$ and with $a_i\in R$.

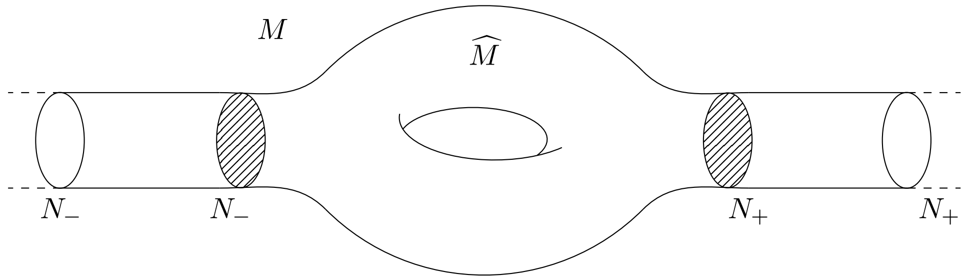

Figure 4.1: The topological structure of the manifold M.

with $N_\pm$ two 3-dimensional manifolds. Moreover, the complement $M \setminus \widehat{M}$ is given by two half-cylinder, that is two asymptotic components described by $\mathbb{R}^+ \times N_+$ and $\mathbb{R}^- \times N_-$. These components are endowed with the metrics

$$ds^2 = du^2 + g^{\pm}_{ij}(y^i)\,\mathrm{d}y^i\,\mathrm{d}y^j \ ,$$

where $u \in \mathbb{R}^\pm$ is the coordinate that parametrize the non-compact directions in both the asymptotic regions, while y^i, $i = 1, 2, 3$ are the coordinates on $N_\pm$ and $g^\pm_{ij}$ are the u-independent metrics on $N_\pm$. Let us study a particular case of this situation, for which the generalization is then trivial. That is, let us consider the case in which the manifold M is topologically of the form

$$M = \mathbb{R} \times N \ ,$$

with N a compact 3-dimensional manifold. This manifold M is then endowed with the metric

$$ds^2 = du^2 + g_{ij}(u, y^k)\,\mathrm{d}y^i\,\mathrm{d}y^j \ , \quad i = 1, 2, 3 \ ,$$

where, again, $u \in \mathbb{R}$ parametrizes the non-compact direction, y^i, $i = 1, 2, 3$ are the coordinates on N and g_{ij} is the metric on N. Finally, we have to require that the metric $g_{ij}(u, y^k)$ becomes constant in u for $u \gg 0$ and $u \ll 0$.

Then, let us consider the 4-dimensional gamma matrices γ^μ, $\mu = 1, \ldots, 4$, in the representation

$$\gamma^i = \begin{pmatrix} 0 & i\widetilde{\gamma}^i \\ -i\widetilde{\gamma}^i & 0 \end{pmatrix} \ , \qquad \gamma^4 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \ ,$$

where we have used the gamma matrices in 3 dimensions $\widetilde{\gamma}^i$, $i = 1, 2, 3$. In this particular representation, the chirality matrix is diagonal, of the form

$$\gamma_5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \ .$$

With this choice of the gamma matrices, the Dirac operator for the manifold $M = \mathbb{R} \times N$ is clearly given by

$$i\gamma^\mu D_\mu = \begin{pmatrix} 0 & i\partial_u - \widetilde{\gamma}^k D_k \\ i\partial_u + \widetilde{\gamma}^k D_k & 0 \end{pmatrix} = \begin{pmatrix} 0 & i(\partial_u + B) \\ i(\partial_u - B) & 0 \end{pmatrix} \ ,$$

where we have introduced the Hermitian, u-dependent, 3-dimensional boundary Dirac operator

$$B(u) := i\widetilde{\gamma}^k D_k \ .$$

Of course, in the general situation described above, this construction holds in the cylindrical ends $\mathbb{R}^\pm \times N_\pm$, with an analogous definition of the boundary Dirac operators $B_\pm$.

Then, we have to look at what happens to the Witten index. Indeed, in the non-compact case, besides the usual discrete components of the spectrum of the Hamiltonian, we have a continuous part of it, due to scattering states. For this reason, it is not clear if the integrality properties we have found previously are still valid, so we have to focus on this situation.

The results found in section 2.3.3 for the limits of the Witten index are still valid, which means that

$$\lim_{\beta\to\infty} \mathrm{Tr}_{\mathcal{H}} \left[(-1)^F e^{-\beta H}\right] = \mathrm{ind}\left(i\gamma^\mu D_\mu\right) \ , \qquad \lim_{\beta\to 0} \mathrm{Tr}_{\mathcal{H}} \left[(-1)^F e^{-\beta H}\right] = \int_M \widehat{A} = \int_{\widehat{M}} \widehat{A} \ .$$

For consistency, we have introduced particular boundary conditions on $\widehat{M}$, known as *APS* (*Atiyah-Patodi-Singer*) *boundary conditions*[4], which ensure us that the Dirac genus $\widehat{A}$ vanishes on the cylindrical ends, from which the last equality above follows. Contrarily to the compact case, now we have that the two limits are not equal, and so the Atiyah-Singer index theorem is not valid anymore. However, in the non-compact case we are dealing with, a new theorem, the **Atiyah-Patodi-Singer index theorem**, holds, and it states that

$$\mathrm{ind}\left(i\gamma^\mu D_\mu\right) = \int_M \widehat{A} - \frac{1}{2}\Big(\eta(B_+) + \dim\big(\ker B_+\big) - \eta(B_-) - \dim\big(\ker B_-\big)\Big) \ .$$

Let us now justify these terms.

First of all we have introduced the **APS η-invariant**, defined as

$$\eta(B) := \sum_{\lambda\neq 0} \mathrm{sign}(\lambda)\big|_{\mathrm{reg}} \ , \tag{4.5}$$

where the sum is over all the non-vanishing eigenvalues of the boundary Dirac operator. Of course, this sum, which is a measure of the asymmetry of the spectrum, has to be regularized. A possibility consists in defining it as

$$\eta(B) = \lim_{s\to 0} \sum_{\lambda\neq 0} \frac{\mathrm{sign}(\lambda)}{|\lambda|^s} =: \lim_{s\to 0} \eta_{\mathrm{APS}}(s) \ ,$$

where we have used the fact that $\eta_{\mathrm{APS}}(s)$ is an analytic function near $s = 0$. The presence of this term can be verified computing the integral of a local density over the cylindrical ends (see [DJR19] for details).

The presence of the dimension of the kernel of the boundary Dirac operator is instead clear, since we have excluded the vanishing eigenvalues in the definition (4.5).

[4] For the explicit expression of these boundary conditions look at [DJR19] and the references therein.

Then, we have that a crucial result holds, namely that the 4-dimensional index counts, with sign, the number of the eigenvalues of B that cross 0 moving from an asymptotic region to the other, that is going from $u = -\infty$ to $u = +\infty$. Let

$$\Psi(u, y) = \begin{pmatrix} \psi_+(u, y) \\ \psi_-(u, y) \end{pmatrix}$$

be an element of the kernel of the Dirac operator $i\gamma^\mu D_\mu$ on M. In particular, let us focus on the restriction of Ψ to one of the cylindrical ends, let us say $\mathbb{R}^+ \times N_+$, in the region in which the boundary Dirac operator B_+ is stable, i.e. constant in u. Hence we have

$$i\gamma^\mu D_\mu \Psi = \begin{pmatrix} 0 & i(\partial_u + B_+) \\ i(\partial_u - B_+) & 0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0$$

from which

$$i(\partial_u + B_+)\psi_- = 0 \ , \qquad i(\partial_u - B_+)\psi_+ = 0 \ . \tag{4.6}$$

It is useful to decompose $\psi_\pm(u, y)$ in terms of the eigenfunctions $\psi_\lambda(y)$ of B_+, i.e.

$$\psi_\pm(u, y) = \sum_\lambda a_\pm^\lambda(u)\psi_\lambda(y) \ , \tag{4.7}$$

in such a way that (4.6), for each eigenvalue λ of B_+, read

$$i(\partial_u + \lambda)a_-^\lambda = 0 \ , \qquad i(\partial_u - \lambda)a_+^\lambda = 0 \ .$$

These equations are solved by

$$a_\pm^\lambda(u) = a^\lambda(0)e^{\pm\lambda u} \qquad u > 0 \ ,$$

where we have $u > 0$ since we have chosen to focus on the asymptotic region $\mathbb{R}^+ \times N_+$. In this way we have that, for $u > 0$

$$\begin{aligned} &a_+^\lambda(u) \text{ is a normalizable wave function only if } \lambda < 0 \ , \\ &a_-^\lambda(u) \text{ is a normalizable wave function only if } \lambda > 0 \ . \end{aligned}$$

Therefore, depending on the sign of λ, we have either a solution of positive chirality or a solution of negative chirality. Of course, the same argument holds for the other asymptotic region, $\mathbb{R}^- \times N_-$, but with the signs reversed due to the different orientation of the boundary manifold. This means that, if we consider a globally defined element of the kernel of the Dirac operator, ψ_+, it has to correspond to a negative eigenvalue of B in the region $\mathbb{R}^+ \times N_+$, and to a positive eigenvalue on the region $\mathbb{R}^- \times N_-$, that is, the corresponding eigenvalue, in passing from $u = -\infty$ to $u = +\infty$, has to change sign.

In conclusion, the ground states, i.e. the elements of the kernel of $i\gamma^\mu D_\mu$, are in one-to-one correspondence with those eigenvalues of the boundary Dirac operator that changes sign.

In order to figure out what kind of contribution the η-invariant gives, let us notice the following fact. We know that the Hamiltonian of the theory is the square of the Dirac operator, hence the eigenvalue equation for the Hamiltonian, using the decomposition (4.7), reads

$$\big(-\partial_u^2+\lambda^2\big)a_\pm^\lambda = E a_\pm^\lambda \ ,$$

which can have the following solutions:

- an exponentially suppressed solution if $E<\lambda^2$, corresponding to the bound states, which means that E is in the discrete spectrum;
- an oscillating solution if $E>\lambda^2$, which means that E is in the continuous spectrum.

Hence, if B has no kernel in any of the two asymptotic regions, there is an energy gap between the continuous and discrete spectrum. In this particular case, the continuous spectrum is bounded from below by the square of the smallest eigenvalue of B, and so, in the low energy limit, the one we are interested in, only the discrete spectrum contributes. For this reason, in this situation, the Witten index has the same integrability condition as in the compact case.

If, instead, B admits zero eigenvalues in the asymptotic regions, the energy gap does not exist anymore, and the continuous spectrum extends to zero energy. This gives to the Witten index the fractional contributions

$$\pm\frac{1}{2}\dim\big(\ker B_\pm\big)\ .$$

In order to explain the presence of the factor $\frac{1}{2}$, let us give the following argument. We can consider two non-compact 4-dimensional manifolds M and M' with asymptotic boundaries N_1-N_2 and N_2-N_3 respectively. Then, we can glue these two manifolds along N_2, obtaining a new 4-dimensional non-compact manifold $M\cup_{N_2}M'$ with boundaries N_1-N_3. Let us suppose that on $M\cup_{N_2}M'$ only an eigenvalue λ changes sign in the following way: it moves from being positive to zero on M, and then from zero to negative on M'. Of course, for what we have said above, the contribution of this eigenvalue to the index of $M\cup_{N_2}M'$ is the sum of the indices of M and M', since they are given by integrals of local densities. Hence, if we call α the contribution to the index of M, and α' the one of M', we have that

$$\alpha+\alpha'=1\ .$$

Also, if we change the manifold M' with a new one M'' in which still λ moves from zero to a negative value, and we call α'' the contribution of M'' to the index, we have again

$$\alpha+\alpha''=1\ ,$$

and so $\alpha'=\alpha''$, or, in other words, the fractional contribution to the index of an eigenvalue λ is the same independently on the particular manifold, whenever λ passes from zero to a negative value. Moreover, by symmetry reasons, if we change the orientation of the boundary 3-manifold, all the eigenvalues of B flip signs and, in turn, the orientation of the 4-manifold and therefore the sign of the index change. This means that we get the opposite contribution to the index, $-\alpha'$, when the eigenvalue moves from zero to a positive value. Finally, changing the signs of all the four coordinates of the 4-manifold, the orientation in preserved, and so the index of the Dirac operator, which has as contribution exactly α'. But the situation described is the same as M, hence $\alpha' = \alpha''$. Since they have to sum to 1, we conclude

$$\alpha = \alpha' = \frac{1}{2} .$$

Of course, what we have said up to now, can be expressed in terms of a u-dependent supercharge $Q(u)$. This is exactly the case we are interested in.

In order to see this, we have to define a *mildly non-compact* SQM model, in such a way that it asymptotically behaves exactly as we have described up to now. This definition is completely analogous to the one of the SQFT case. Hence given a SQM model $\mathcal{M}$ depending on a scalar superfield Φ which parametrizes the non-compact direction, we add a Fermi superfield Λ which couple to the scalar superfield through the superpotential $W(\Phi) = \Phi - u$, with $u \in \mathbb{R}$. In this way we have defined a family of SQM models $\mathcal{N}_u$ all compact and parametrized by the real parameter u. In order for the theory $\mathcal{M}$ to be mildly non-compact, we require that in the limits $u \to \pm\infty$, the family stabilizes to compact SQM models $\mathcal{N}_\pm$ with vanishing indices. As before, we can assume without loss of generality, that the only quantities that vary with u for the family of theories $\mathcal{N}_u$, are the lagrangian and the supersymmetry operator, while the Hilbert space $\mathcal{H}$ of $\mathcal{N}_u$ is independent of u.

Now let us start considering the case in which the degree of the SQM model $\mathcal{M}$ is $n = 4$, from which it follows that $\mathcal{N}_u$ is a SQM model of degree 3. This happens since, when we dynamicalize the parameter u in order to recover the theory $\mathcal{M}$, we promote u to a regular scalar field φ, but also we need to add its fermionic superpartner ψ, which gives a contribution $+1$ to the degree of the SQM model. In this situation we have that the Hilbert space $\mathcal{H}$ of $\mathcal{N}_u$ is a $\mathrm{Cliff}(3,\mathbb{R})$-module. For this algebra we have two canonical isomorphisms

$$\mathrm{Cliff}(3,\mathbb{R}) \simeq \mathbb{H} \otimes \mathrm{Cliff}(-1,\mathbb{R}) .$$

These two isomorphisms are described as follows. Let us call γ_i with $i = 1,2,3$ the generators of $\mathrm{Cliff}(3,\mathbb{R})$, which fulfill the relation $\{\gamma_i,\gamma_j\} = 2\delta_{ij}$. Also, let us identify with e_i, $i = 1,2,3$ the generators of $\mathbb{H}$, which satisfy $\{e_i, e_j\} = -2\delta_{ij}$ and with γ, such that $\gamma^2 = -1$ the generator of $\mathrm{Cliff}(-1,\mathbb{R})$. Then, an element of $\mathbb{H} \otimes \mathrm{Cliff}(-1,\mathbb{R})$ is generated by the products γe_i, so the isomorphisms are described identifying

$$\gamma_i \longleftrightarrow \pm\gamma e_i .$$

It is straightforward to verify that $\pm\gamma e_i$ satisfy the conditions required to the generators of $\mathrm{Cliff}(3,\mathbb{R})$.

Let us choose one of them, thanks to which we have that the generator of the supersymmetry is

$$Q(u) = g(u)\gamma ,$$

with $g(u)$ a quaternionic matrix. The presence of the time-reversal symmetry ensures us that the eigenvalues of $g(u)$ are in $\mathbb{R} \subset \mathbb{H}$. Through a u-dependent change of basis, we can arrive to the situation in which only the spectrum of $g(u)$, which is, by compactness, a discrete subset of $\mathbb{R}$, depends on u. The case we are interested in is the IR limit of our construction, that is, given the index of the theory $\mathcal{M}$, $Z_R(\mathcal{M})$, we will compute it in the limit $\overline{\tau} \to -i\infty$. Also, we have to notice that we have defined the index as the partition function with periodic boundary conditions, which means that, following the notation used above

$$Z_R(\mathcal{M}) = \lim_{\beta \to 0} \mathrm{Tr}_{\mathcal{H}} \left[(-1)^F e^{-\beta H} \right] = \int_M \widehat{A} \, .$$

Hence, the APS index theorem reads

$$Z_R(\mathcal{M}) = \mathrm{ind}\left(i\gamma^\mu D_\mu\right) + \frac{1}{2}\Big(\eta\big(Q(u \to +\infty)\big) + \dim\big(\ker Q(u \to +\infty)\big)\Big) + \\ - \frac{1}{2}\Big(\eta\big(Q(u \to -\infty)\big) + \dim\big(\ker Q(u \to -\infty)\big)\Big) \, .$$

The summand $\mathrm{ind}(i\gamma^\mu D_\mu)$ fulfills the same integrality condition of the compact case, that is, in the degree we are considering,

$$\mathrm{ind}\left(i\gamma^\mu D_\mu\right) \in 2\mathbb{Z} \, .$$

The other two terms, instead, as explained above, give a fractional contribution for each eigenvalue of Q that lands on 0 in the limits $u \to \pm\infty$.

The same argument, that we have explained in the case of a theory of degree $n = 4$, is of course true for all the cases in which $n = 4 \mod 8$, but also, with a slight modification due to the different integrality condition of the index in the compact case, in the case $n = 0 \mod 8$. In order to give a unified description of the integrality condition of the index, let us use the spectator fermions on the boundary theory, complexifying them. Hence, we are interested in the irreducible modules of the algebra $\mathrm{Cliff}(n_\pm, \mathbb{C})$. These modules, since the algebra is the complex one, are classified modulo 2 and in the case in which $n_\pm$ is odd, $\mathrm{Cliff}(n_\pm, \mathbb{C})$ has only one irreducible module, that is the algebra itself. We are actually interested only in this case, since we know that the degree of the full theory has to be even, and so the degree of the boundary theories is odd. Going on, we have that the even subalgebra of $\mathrm{Cliff}(n_\pm, \mathbb{C})$ is isomorphic to $\mathbb{C}$, which allows us to write the irreducible module as $\mathbb{C}^{1|1}$. The supersymmetric ground states of the boundary theories, the ones we are interested in, are again modules for the Clifford algebra and, in particular, they have a decomposition in irreducible modules, i.e.

$$V \otimes \mathbb{C} \simeq \mathbb{C}^{b|b} \, , \qquad b \in \mathbb{Z} \, .$$

However, the theories we are considering have a time-reversal symmetry, which gives to the Clifford algebra a real structure. Hence, in order to count the supersymmetric ground states that land on zero on the boundary theories, we need to consider V as a module of the real algebra. This can described simply rescaling b by the same factor r we have found in (4.4), obtaining

$$V \simeq \mathbb{C}^{a|a} \, , \qquad a \in r\mathbb{Z} \, .$$

In conclusion, calling the factor a the **bosonic index** of the boundary theories $a(\mathcal{N}_\pm)$, we conclude that

$$Z_R(\mathcal{M}) \in r\mathbb{Z} + \frac{1}{2}a(\mathcal{N}_+) - \frac{1}{2}a(\mathcal{N}_-) \; .$$

In the SQFT case we care about, it is straightforward to figure out that this relation becomes

$$Z_{RR}(\mathcal{F}) \in r\mathbb{Z}(\!(q)\!) + \frac{1}{2}a(\mathcal{B}_+) - \frac{1}{2}a(\mathcal{B}_-) \; , \tag{4.8}$$

with clear meaning of the symbols.

4.2 The invariant

Now that we have found the behavior of the Witten index in the non-compact case, we are ready to show how the full invariant can be defined.

Let $\mathcal{B}$ be a compact SCFT with gravitational anomaly

$$2(c_R - c_L) = n - 1 \; , \qquad n \equiv 0 \mod 4 \; .$$

Let us suppose $\mathcal{B}$ to be nullhomotopic. Thanks to this property, we can build an SQFT $\mathcal{F}$ with gravitational anomaly n and a non-compact direction, which has as boundary $\mathcal{B}$. Let $\widehat{f}(\tau,\overline{\tau})$ be the Witten genus of $\mathcal{F}$. Then, from what we have seen in the previous sections

1. $\widehat{f}$ is a real-analytic modular form of weight $(\frac{n}{2}, 0)$. In particular it solves the holomorphic anomaly equation

$$\sqrt{-8\tau_2}\frac{\partial}{\partial\overline{\tau}}\widehat{f}(\tau,\overline{\tau}) = g_{\mathcal{B}}(\tau,\overline{\tau}) \; , \tag{4.9}$$

where we have defined

$$g_{\mathcal{B}}(\tau,\overline{\tau}) = \text{ torus one-point function of } (-1)^{\frac{n}{4}} :\!\psi_1 \cdots \psi_{n-1}\overline{G}\!: \text{ in } \mathrm{Fer}(n-1) \otimes \mathcal{B} \; . \tag{4.10}$$

2. The holomorphic part of the Witten genus

$$f(\tau) = \lim_{\overline{\tau}\to -i\infty} \widehat{f}(\tau,\overline{\tau}) \; ,$$

has a q-expansion

$$f \in f_2(q) + r\mathbb{Z}(\!(q)\!) \; , \tag{4.11}$$

where r is the one introduced in (4.4), while

$$f_2(q) = \frac{1}{2}a\big(\mathcal{B}[S^1]\big) \; . \tag{4.12}$$

Let us notice that we have considered the holomorphic part since what we have said about the q-expansion of the index in the non-compact case is true in the IR limit, which corresponds, exactly in the same way as extracting the holomorphic part, to the limit $\overline{\tau} \to -i\infty$.

Now, if $\mathcal{B}$ is nullhomotopic, then there exists a function $\widehat{f}$ which solves the holomorphic anomaly equation (4.9) and has q-expansion as (4.11).

On the contrary, if we do not know whether $\mathcal{B}$ is nullhomotopic, we actually have to notice that $g_{\mathcal{B}}(\tau, \overline{\tau})$ in (4.10) and $f_2(q)$ in (4.12) depend only on $\mathcal{B}$ itself. We can, for this reason, use the knowledge of g in order to find a real-analytic modular form $\widehat{f_1}$ which is a solution for (4.9). However, it is clear that we have an ambiguity in the choice of the solution. Let $\widetilde{f}$ be an holomorphic modular form of weight $\frac{n}{2}$ and let $\widehat{f_1}$ be a solution for the holomorphic anomaly equation

$$\sqrt{-8\tau_2}\partial_{\overline{\tau}}\widehat{f_1}(\tau, \overline{\tau}) = g \ .$$

Hence we have that also $\widehat{f_1} + \widetilde{f}$ is a real-analytic modular form and a solution for the same equation, indeed

$$\sqrt{-8\tau_2}\partial_{\overline{\tau}}(\widehat{f_1}(\tau, \overline{\tau}) + \widetilde{f}(\tau)) = \sqrt{-8\tau_2}\partial_{\overline{\tau}}\widehat{f_1}(\tau, \overline{\tau}) = g \ .$$

So, let us call f_1 the holomorphic part of $\widehat{f_1}$, then g, and therefore $\mathcal{B}$, determines the class of f_1 in

$$[f_1] \in \frac{\mathbb{C}(\!(q)\!)}{\mathrm{MF}_{\frac{n}{2}}} \ .$$

On the other side, we have that the knowledge of $\mathcal{B}$ can actually determines $f_2 \in \frac{r}{2}\mathbb{Z}(\!(q)\!)$, but we are interested only on it as a class in

$$[f_2] \in \frac{\mathbb{C}(\!(q)\!)}{r\mathbb{Z}(\!(q)\!)} \ .$$

None of these classes are individually deformation invariants, but their difference is so. Hence we can define the deformation invariant

$$[f_1] - [f_2] \in \frac{\mathbb{C}(\!(q)\!)}{\mathrm{MF}_{\frac{n}{2}} + r\mathbb{Z}(\!(q)\!)} := A_n \ . \tag{4.13}$$

Let us show the invariance of this quantity under deformations. In order to do this, let us deform the original SCFT $\mathcal{B}$ in some other theory $\mathcal{B}'$. So we can build a non-compact SQFT $\mathcal{F}$ with boundaries

$$\mathcal{B}_- = \mathcal{B} \ , \qquad \mathcal{B}_+ = \mathcal{B}' \ .$$

Let us call f_1 and f_2 the classes introduced above for $\mathcal{B}$ and f_1' and f_2' the same classes for $\mathcal{B}'$.

Claim 4.2.1. *The quantities f_1' and $f_1 + Z_{RR}^{IR}(\mathcal{F})$ solve the same holomorphic anomaly equation. Here $Z_{RR}^{IR}(\mathcal{F})$ is the Witten genus of the non-compact theory $\mathcal{F}$, computed in the IR limit.*

Proof. We want to show that

$$\sqrt{-8\tau_2}\partial_{\overline{\tau}} f_1' = \sqrt{-8\tau_2}\partial_{\overline{\tau}}\big(f_1 + Z_{RR}^{\mathrm{IR}}(\mathcal{F})\big) \ .$$

In order to do this, we will see that

$$\sqrt{-8\tau_2}\partial_{\overline{\tau}} Z^{\mathrm{IR}}_{RR}(\mathcal{F}) = \sqrt{-8\tau_2}\partial_{\overline{\tau}} f_1' - \sqrt{-8\tau_2}\partial_{\overline{\tau}} f_1 \ .$$

By definition we have that

$$\begin{aligned}\sqrt{-8\tau_2}\partial_{\overline{\tau}} f_1' &= \sqrt{-8\tau_2}\partial_{\overline{\tau}}\big(\lim_{\overline{\tau}\to -i\infty} \widehat{f_1'}\big) = \\ &= \lim_{\overline{\tau}\to -i\infty} \sqrt{-8\tau_2}\partial_{\overline{\tau}}\widehat{f_1'} = \\ &= \lim_{\overline{\tau}\to -i\infty} \langle(-1)^{\frac{n}{4}} :\psi_1\cdots\psi_{n-1}\overline{G}:\rangle_{\mathrm{Fer}(n-1)\otimes\mathcal{B}'} \ ,\end{aligned}$$

and, in the same way

$$\sqrt{-8\tau_2}\partial_{\overline{\tau}} f_1 = \lim_{\overline{\tau}\to -i\infty} \langle(-1)^{\frac{n}{4}} :\psi_1\cdots\psi_{n-1}\overline{G}:\rangle_{\mathrm{Fer}(n-1)\otimes\mathcal{B}} \ .$$

Moreover, by construction, we know that

$$\begin{aligned}&\sqrt{-8\tau_2}\partial_{\overline{\tau}} Z^{\mathrm{IR}}_{RR}(\mathcal{F}) = \lim_{\overline{\tau}\to -i\infty} \sqrt{-8\tau_2}\partial_{\overline{\tau}} Z_{RR}(\mathcal{F}) = \\ &= \lim_{\overline{\tau}\to -i\infty} \Big(\langle(-1)^{\frac{n}{4}} :\psi_1\cdots\psi_{n-1}\overline{G}:\rangle_{\mathrm{Fer}(n-1)\otimes\mathcal{B}'} - \langle(-1)^{\frac{n}{4}} :\psi_1\cdots\psi_{n-1}\overline{G}:\rangle_{\mathrm{Fer}(n-1)\otimes\mathcal{B}} \Big) \ ,\end{aligned}$$

exactly as we wanted. □

What we have shown now, implies that

$$f_1' - f_1 - Z^{\mathrm{IR}}_{RR}(\mathcal{F}) \in \mathrm{MF}_{\frac{n}{2}} \ .$$

Claim 4.2.2. *The quantities f_2' and $f_2 + Z^{IR}_{RR}(\mathcal{F})$ differ by an element in $r\mathbb{Z}(\!(q)\!)$, with the usual meaning of the symbols.*

Proof. We have shown in section 4.1.3 on page 81 that, if $\mathcal{F}$ is a non-compact theory, as it is in our case, we have

$$Z^{\mathrm{IR}}_{RR}(\mathcal{F}) \in r\mathbb{Z}(\!(q)\!) + f_2' - f_2 \ .$$

Hence the claim trivially follows and

$$f_2' - f_2 - Z^{\mathrm{IR}}_{RR}(\mathcal{F}) \in r\mathbb{Z}(\!(q)\!) \ .$$

□

This two claims imply that

$$f_1' - f_1 - Z^{\mathrm{IR}}_{RR}(\mathcal{F}) - f_2' + f_2 + Z^{\mathrm{IR}}_{RR}(\mathcal{F}) \in \mathrm{MF}_{\frac{n}{2}} + r\mathbb{Z}(\!(q)\!) \ ,$$

that is

$$f_1' - f_2' = f_1 - f_2 \mod \mathrm{MF}_{\frac{n}{2}} + r\mathbb{Z}(\!(q)\!) \ ,$$

hence the equivalence class defined in (4.13) is actually a deformation invariant.

Chapter 5

The invariant for the sigma model S^3_k

Let us study now the invariant for the model we started with, that is the sigma model with target the 3-sphere S^3 and Wess-Zumino coupling k we have described in section 2.2. As we have said, it is crucial to first compute the gravitational anomaly of our model, that is the degree of it. This model consists of 4 scalar superfields Φ^I's and 1 Fermi superfield Λ. In terms of the ordinary fields we have 4 bosonic fields and 4 fermionic fields from the expansion of Φ^I, and a chiral fermion field given by the upper component of Λ. Putting all together we have that

$$w = \bar{c} - c = \left(4 + 4 \cdot \frac{1}{2}\right) - \left(4 + \frac{1}{2}\right) = \frac{3}{2} \,, \tag{5.1}$$

which means that the degree of the theory is

$$n = 2w = 3 \,. \tag{5.2}$$

In order to properly compute the invariant, we need to work in the IR regime. We have explained the conjecture on the behavior of the model in this limit in 2.2.2, which proposes that it behaves as a bosonic Wess-Zumino-Witten model at level $\kappa = |k| - 1$, together with the theory of 3 anti-chiral free fermions. Let us underline, just for consistency, how the gravitational anomaly matches in the two regimes, since the bosonic WZW model is not anomalous.

One of the two component of the invariant introduced in section 4.2 is given by the index of the SQM model obtained compactifying the original SQFT on a circle S^1 of radius R. So let us start studying this compactification. In particular let us focus on the limit of small and big radius R. We are interested in the IR regime, but, in the case of small R, there is a sort of "intermediate" regime, in which the energy is much smaller than $1/R$, but still bigger than the threshold energy, hence we treat the model with the same language as we would do in the UV regime. The case with big R, instead, is described as the usual IR limit.

small R**.** In this case we can study the problem using the language of sigma model. In particular, since we are interested in ground states, and so in the low energy situation, we can rely on the fact that, for energy much smaller than $1/R$, those modes that carry momentum around the circle can be neglected. In this approximation, the model becomes a usual SQM model, for which the supercharge Q can be interpreted as the Dirac operator on S^3 acting on section of the spin bundle of S^3, regardless of k. It is well-known that this operator does not have any zero eigenvalues, hence supersymmetry is spontaneously broken regardless of k.

big R**.** Now we have to work in the IR regime, i.e. in the case in which the model is described by the bosonic WZW model and by the three free anti-chiral fermions.

If $k = 0$, the conjecture in [GJW19] says that supersymmetry is spontaneously broken, since our IR fixed point loses its meaning (let us remember that, for the unitarity requirement, the level κ of the bosonic WZW has to be greater or equal to 0).

If $k \neq 0$, the question becomes whether the supersymmetric current algebra of $G \simeq SU(2)$ at level $\kappa = |k| - 1$ has supersymmetric ground states in the Ramond sector, namely whether there exist states with vanishing energy. Actually this does not happen. Indeed, the energy of the ground states is given by

$$E_{\text{g.s.}} = -\frac{c_{\text{WZW}}}{24} - 3 \cdot \left(\frac{c_{\text{Fer}}}{24} - \frac{1}{16} \right) , \tag{5.3}$$

with

$$c_{\text{WZW}} = 3\frac{\kappa}{\kappa + 2} , \qquad c_{\text{Fer}} = \frac{1}{2} . \tag{5.4}$$

Hence the energy of the ground states is

$$E_{\text{g.s.}} = \frac{1}{8}\frac{2}{\kappa + 2} , \tag{5.5}$$

which is greater than zero for all $\kappa \geq 0$.

What we have obtained is that the sigma model with target S^3 and WZ coupling k, once compactified on the circle S^1, spontaneously breaks supersymmetry, regardless of k, and hence its index vanishes.

For this reason, the only non-vanishing contribution to the invariant is the solution to the holomorphic anomaly equation. In order to compute it let us start focusing on a particular case, namely the one in which $k = 1$, and then we will extend it to the general case.

5.1 $k = 1$ case

Now we want to solve the holomorphic anomaly equation for the CFT of 3 free anti-chiral fermions $\overline{\text{Fer}^3}$, described by the energy-momentum tensor

$$\overline{T} = -\frac{1}{2}\overline{\psi}_a \overline{\partial}\, \overline{\psi}_a , \tag{5.6}$$

and supercurrent
$$\overline{G} = \sqrt{-1}\,\overline{\psi}_1\overline{\psi}_2\overline{\psi}_3\ . \tag{5.7}$$
The holomorphic anomaly equation in this case reads
$$\sqrt{-8\tau_2}\partial_{\overline{\tau}}\widehat{f} = \langle(-1)\ {:}\psi_1\psi_2\psi_3\overline{G}{:}\rangle_{\mathrm{Fer}(3)\otimes\overline{\mathrm{Fer}}(3)}\ . \tag{5.8}$$
In order to compute the torus one-point function, let us remember that the Hilbert space of the product theory is given by the product of the Hilbert spaces of the sub-theories, and hence the one-point function factorizes. In particular, rememebering the result in (3.5),
$$\begin{aligned}\langle(-1)\ {:}\psi_1\psi_2\psi_3\overline{G}{:}\rangle_{\mathrm{Fer}(3)\otimes\overline{\mathrm{Fer}}(3)} &= -(\sqrt{-1})\ \langle{:}\psi_1\psi_2\psi_3\overline{\psi}_1\overline{\psi}_2\overline{\psi}_3{:}\rangle_{\mathrm{Fer}(3)\otimes\overline{\mathrm{Fer}}(3)} = \\ &= -\sqrt{-1}\ \langle{:}\psi_1\psi_2\psi_3{:}\rangle_{\mathrm{Fer}(3)}\cdot\langle{:}\overline{\psi}_1\overline{\psi}_2\overline{\psi}_3{:}\rangle_{\overline{\mathrm{Fer}}(3)} = \\ &= -\sqrt{-1}(-1)^{-\frac{3}{4}}\big(\eta(\tau)\big)^3\cdot(-1)^{-\frac{3}{4}}\big(\overline{\eta(\tau)}\big)^3 = \big|\eta(\tau)\big|^6\ .\end{aligned} \tag{5.9}$$

Then we want to verify that the function
$$F_1(\tau) = -\frac{1}{24} + \sum_{n=1}^{\infty} n\frac{q^n}{1-q^n} + \sum_{n=1}^{\infty}(-1)^{n-1}n\frac{q^{\frac{n(n+1)}{2}}}{1-q^n}\ , \tag{5.10}$$
has a modular non-holomorphic completion $\widehat{F}_1$ of weight 2 such that $2\widehat{F}_1$ satisfies the anomaly equation, that is
$$\sqrt{-8\tau_2}\partial_{\overline{\tau}}\big(2\widehat{F}_1\big) = \big|\eta(\tau)\big|^6\ . \tag{5.11}$$
In order to verify this statement, let us use the integral form
$$\widehat{F}_1 = \frac{1}{8\pi^2}\int_0^1\int_0^1 \wp(u_1+\tau u_2,\tau)H_1(u_1,u_2;\tau,\overline{\tau})\ \mathrm{d}u_1\,\mathrm{d}u_2\ , \tag{5.12}$$
where $\wp(u_1+\tau u_2,\tau)$ is the Weierstrass elliptic function, while
$$H_1(u_1,u_2\ \tau,\overline{\tau}) := \sum_{n,m\in\mathbb{Z}} e^{2\pi i(nu_2-mu_1)}(-1)^{n+m+nm}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\ . \tag{5.13}$$
This function can be Poisson resummed to
$$H_1(u_1,u_2;\tau_1,\tau_2) = \sqrt{2\tau_2}\big|\theta(u_1+\tau u_2,\tau)\big|^2 e^{-2\pi\tau_2(u_2)^2}\ , \tag{5.14}$$
where
$$\begin{aligned}\theta(u,\tau)\equiv\theta_1(u,\tau) &:= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2}e^{2\pi i(n+\frac{1}{2})(u+\frac{1}{2})} = \\ &= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n+\frac{1}{2})^2}e^{2\pi iu(n+\frac{1}{2})}e^{\pi i(n+\frac{1}{2})}\end{aligned} \tag{5.15}$$

Indeed, let us Poisson resum the expression (5.13) with respect to n, hence focusing on

$$\sum_{n\in\mathbb{Z}} e^{2\pi i n u_2}(-1)^{n(1+m)}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} = \sum_{n\in\mathbb{Z}} e^{2\pi i n u_2}e^{-\pi i n(1+m)}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} \ . \tag{5.16}$$

The Poisson resummation formula reads

$$\sum_{n\in\mathbb{Z}}\varphi(x+n) = \sum_{r\in\mathbb{Z}}\Big(\int_{\mathbb{R}}\varphi(t)e^{-2\pi i r t}\ \mathrm{d}t\Big)e^{2\pi i r x} \equiv \sum_{r\in\mathbb{Z}}\widetilde{\varphi}(r)e^{2\pi i r x} \ , \tag{5.17}$$

which, in our case, means that we need to find the Fourier inverse transform of

$$\widetilde{\varphi}(n) = e^{\pi i n(1+m)}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} \ , \tag{5.18}$$

that is

$$\varphi(t) = \int_{\mathbb{R}} e^{\pi i n(1+m)}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}e^{2\pi i n t}\ \mathrm{d}n \ . \tag{5.19}$$

Focusing only on the exponents we obtain

$$\begin{aligned}\pi i n(1+m+2t) - \frac{\pi}{2\tau_2}(m\tau+n)(m\overline{\tau}+n) &= \\ &= -\frac{\pi}{2\tau_2}\chi^2 + \pi i\Big(-\big(m^2+m(1-2t)\big)\overline{\tau} + \frac{i\tau_2}{2}(1+2t)^2\Big) \ .\end{aligned} \tag{5.20}$$

where we have introduced

$$\chi := n + m\overline{\tau} + i\tau_2(1+2t) \ , \tag{5.21}$$

which is only a shift of the original variable n (and so the measure of the integral for φ change trivially). We have obtained that

$$\begin{aligned}\varphi(t) &= \int_{\mathbb{R}} e^{-\frac{\pi}{2\tau_2}\chi^2}e^{-\pi i\left(m^2+m(1+2t)\right)\overline{\tau}}e^{-\frac{\pi\tau_2}{2}(1+2t)^2}\ \mathrm{d}\chi = \\ &= \Big(\int_{\mathbb{R}} e^{-\frac{\pi}{2\tau_2}\chi^2}\ \mathrm{d}\chi\Big)e^{-\pi i\left(m^2+m(1+2t)\right)\overline{\tau}}e^{-\frac{\pi\tau_2}{2}(1+2t)^2} = \\ &= \sqrt{2\tau_2}e^{-\pi i\left(m^2+m(1+2t)\right)\overline{\tau}}e^{-\frac{\pi\tau_2}{2}(1+2t)^2}\end{aligned} \tag{5.22}$$

Hence, the Poisson resummation formula tells us that

$$\sum_{n\in\mathbb{Z}} e^{2\pi i n u_2}(-1)^{n(1+m)}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} = \sqrt{2\tau_2}\sum_{n\in\mathbb{Z}} e^{-\pi i\left(m^2+m(1+2n+2u_2)\right)\overline{\tau}}e^{-\frac{\pi\tau_2}{2}(1+2n+2u_2)^2} \ . \tag{5.23}$$

Substituting this expression in (5.13) we easily find

$$\begin{aligned}H_1(u_1,u_2;\tau,\overline{\tau}) = \sqrt{2\tau_2}e^{-2\pi\tau_2(u_2)^2} &\sum_{m,n\in\mathbb{Z}}(-1)^m e^{-2\pi i\overline{u}m}\times \\ &\times\ e^{-\pi i\overline{\tau}\left(m^2+m(1+2n)\right)}e^{-\frac{\pi}{2}(1+2n)^2}e^{-2\pi\tau_2 u_2(1+2n)} \ .\end{aligned} \tag{5.24}$$

Let us perform the following change in the index of the sum

$$m \longmapsto k - n\,, \tag{5.25}$$

with of course $k \in \mathbb{Z}$, in such a way that

$$\begin{aligned} H_1 = \sqrt{2\tau_2} e^{-2\pi\tau_2 (u_2)^2} \sum_{k,n\in\mathbb{Z}} (-1)^{k-n} e^{-2\pi i\overline{u}(k-n)} \times \\ \times\; e^{-\pi i\overline{\tau}\big((k-n)^2+(k-n)(1+2n)\big)} e^{-\frac{\pi}{2}\tau_2(1+2n)^2} e^{-2\pi\tau_2 u_2(1+2n)}\,. \end{aligned} \tag{5.26}$$

Let us focus only on the exponents, except the one due to the factor $(-1)^{k-n}$, which become

$$\begin{aligned} -\,2\pi i\overline{u}(k-n) - \pi i\overline{\tau}\big((k-n)^2 + (k-n)(1+2n)\big) - \frac{\pi}{2}\tau_2(1+2n)^2 - 2\pi\tau_2 u_2(1+2n) = \\ = -2\pi i\overline{u}\big(k+\frac{1}{2}\big) - \pi i\overline{\tau}\big(k+\frac{1}{2}\big)^2 + \pi i\tau\big(n+\frac{1}{2}\big)^2 + 2\pi i u\big(n+\frac{1}{2}\big) \end{aligned} \tag{5.27}$$

Inserting these exponents back in (5.26) and noticing that

$$(-1)^{k-n} = (-1)^{k+n}\,, \tag{5.28}$$

we get

$$\begin{aligned} H_1 &= \sqrt{2\tau_2} e^{-2\pi\tau_2 (u_2)^2} \sum_{m,n\in\mathbb{Z}} (-1)^{n+m} e^{2\pi i\overline{u}(m+\frac{1}{2})} e^{-2\pi i\overline{\tau}\frac{1}{2}(m+\frac{1}{2})^2} e^{2\pi i u(n+\frac{1}{2})} e^{2\pi i\tau\frac{1}{2}(n+\frac{1}{2})^2} = \\ &= \sqrt{2\tau_2} e^{-2\pi\tau_2 (u_2)^2} \sum_{m,n\in\mathbb{Z}} (-1)^{n+m} \overline{x}^{n+\frac{1}{2}} \overline{q}^{\frac{1}{2}(k+\frac{1}{2})^2} x^{n+\frac{1}{2}} q^{\frac{1}{2}(n+\frac{1}{2})^2} = \\ &= \sqrt{2\tau_2}\big|\theta(u_1+\tau u_2,\tau)\big|^2 e^{-2\pi\tau_2 (u_2)^2} \end{aligned} \tag{5.29}$$

Let us specialize the expression of H_1 in equation (5.13) to the IR regime, which means taking the limit $\overline{\tau} \to -i\infty$. In this limit the function $\wp$ does not change being an holomorphic function, and the only factor in H_1 that is affected by the limit is the exponent

$$\begin{aligned} -\frac{\pi}{2\tau_2}|m\tau+n|^2 &= -\frac{\pi}{2\tau_2}(m\tau+n)(m\overline{\tau}+n) = \\ &= -\frac{\pi}{\frac{2}{2i}(\tau-\overline{\tau})}\big(m^2\tau\overline{\tau} + mn(\tau+\overline{\tau}) + n^2\big) = \\ &= \frac{i\pi}{\overline{\tau}-\tau}\big(\overline{\tau}(m^2\tau+mn) + mn\tau + n^2\big)\,. \end{aligned} \tag{5.30}$$

Hence

$$e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} \longrightarrow e^{i\pi(m^2\tau+mn)}\,, \tag{5.31}$$

and the function H_1 becomes

$$H_1(\tau;u_1,u_2) = \sum_{n,m\in\mathbb{Z}} e^{2\pi i(nu_2-mu_1)}(-1)^{n+m} e^{\pi i m^2\tau}\,. \tag{5.32}$$

This means that we get

$$F_1 = \frac{1}{8\pi^2}\int_0^1\int_0^1 \wp(u_1+\tau u_2,\tau)\sum_{n,m\in\mathbb{Z}} e^{2\pi i(nu_2-mu_1)}(-1)^{n+m}e^{\pi i m^2\tau}\,\mathrm{d}u_1\,\mathrm{d}u_2\ , \tag{5.33}$$

which is the Poisson resummation of

$$F_1 = \frac{1}{8\pi^2}\int_0^1 \wp(u_1+\frac{\tau}{2},\tau)\sum_{m\in\mathbb{Z}} e^{-2\pi i m u_1}(-1)^m e^{i\pi m^2\tau}\,\mathrm{d}u_1\ . \tag{5.34}$$

Indeed, focusing only on the sum on n, we have

$$\sum_{n\in\mathbb{Z}} e^{2\pi i n u_2}e^{-\pi i n} = \sum_{n\in\mathbb{Z}} e^{2\pi i n(u_2-\frac{1}{2})} = \sum_{r\in\mathbb{Z}}\delta\Big(r-\big(u_2-\frac{1}{2}\big)\Big)\ . \tag{5.35}$$

Since $u_2\in[0,1]$, there exists only one term in the sum that gives a non-vanishing contribution, that is

$$\sum_{n\in\mathbb{Z}} e^{2\pi i n u_2}(-1)^n = \delta\big(u_2-\frac{1}{2}\big)\ . \tag{5.36}$$

Substituting this expression in (5.33) we get

$$\begin{aligned} F_1 &= \frac{1}{8\pi^2}\int_0^1\int_0^1 \wp(u_1+\tau u_2,\tau)\sum_{m\in\mathbb{Z}} e^{-2\pi i m u_1}(-1)^m e^{i\pi m^2\tau}\delta\big(u_2-\frac{1}{2}\big)\,\mathrm{d}u_1\,\mathrm{d}u_2 = \\ &= \frac{1}{8\pi^2}\int_0^1 \wp(u_1+\frac{\tau}{2},\tau)\sum_{m\in\mathbb{Z}} e^{-2\pi i m u_1}(-1)^m e^{i\pi m^2\tau}\,\mathrm{d}u_1\ , \end{aligned} \tag{5.37}$$

as we wanted.

Now let us use the Fourier expansion of the Weierstrass elliptic function on the circle, given by

$$\frac{1}{(2\pi i)^2}\wp(\xi,\tau) = -2G_2(\tau) + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{n}{1-q^n}e^{2\pi i\xi n}\ , \tag{5.38}$$

where $G_2(\tau)$ is the Eisenstein series of weight two

$$G_2 = -\frac{1}{24} + \sum_{n=1}^{\infty} n\frac{q^n}{1-q^n}\ . \tag{5.39}$$

In this way we get

$$F_1 = \frac{(2\pi i)^2}{8\pi^2} \int_0^1 \Big(-2G_2(\tau) + \sum_{n\in\mathbb{Z}^*} \frac{n}{1-q^n} e^{2\pi i(u_1+\frac{\tau}{2})n} \Big) \Big(\sum_{m\in\mathbb{Z}} e^{-2\pi i m u_1} (-1)^m e^{i\pi m^2 \tau} \Big)\, \mathrm{d}u_1 =$$

$$= -\frac{1}{2} \int_0^1 \big(-2G_2(\tau) \big) \Big(\sum_{m\in\mathbb{Z}} e^{-2\pi i m u_1} (-1)^m e^{i\pi m^2\tau} \Big)\, \mathrm{d}u_1 +$$

$$- \frac{1}{2} \int_0^1 \sum_{m,n\in\mathbb{Z}|n\neq 0} \frac{n}{1-q^n} e^{2\pi i n(u_1+\frac{\tau}{2})} e^{-2\pi i m u_1} (-1)^m e^{i\pi m^2\tau}\, \mathrm{d}u_1 =$$

$$= G_2(\tau) \sum_{m\in\mathbb{Z}} \int_0^1 e^{-2\pi i m u_1} e^{\pi i m} e^{\pi i m^2 \tau}\, \mathrm{d}u_1 +$$

$$- \frac{1}{2} \sum_{m,n\in\mathbb{Z}|n\neq 0} e^{\pi i m} e^{\pi i m^2\tau} e^{\pi i n\tau} \frac{n}{1-q^n} \int_0^1 e^{2\pi i(n-m)u_1}\, \mathrm{d}u_1 \,. \tag{5.40}$$

Let us focus separately on the two summands. Regarding the first one we have

$$\sum_{m\in\mathbb{Z}} \int_0^1 e^{-2\pi i m u_1} e^{\pi i m} e^{\pi i m^2\tau}\, \mathrm{d}u_1 = 1 + \sum_{m\in\mathbb{Z}^*} \frac{1}{-2\pi i m} \big(e^{-2\pi i m} - 1\big) e^{\pi i m} e^{\pi i m^2\tau} \,, \tag{5.41}$$

but being $m \in \mathbb{Z}$

$$e^{-2\pi i m} - 1 \equiv 0 \,. \tag{5.42}$$

For the second summand, since the expansion of $\wp$ is on the circle, we have

$$\int_0^1 e^{2\pi i(n-m)u_1}\, \mathrm{d}u_1 = \delta_{m,n} \,, \tag{5.43}$$

hence

$$- \frac{1}{2} \sum_{m,n\in\mathbb{Z}|n\neq 0} e^{\pi i m} e^{\pi i(m^2+n)\tau} \frac{n}{1-q^n} \delta_{m,n} = \sum_{m>0} (-1)^m q^{\frac{m(m+1)}{2}} \frac{m}{1-q^m} \,. \tag{5.44}$$

Putting all together we arrive at

$$F_1 = G_2(\tau) - \sum_{n>0} \frac{n}{1-q^n} (-1)^n q^{\frac{n(n+1)}{2}} \,, \tag{5.45}$$

which, with the expression (5.39), becomes

$$F_1 = -\frac{1}{24} + \sum_{n=1}^{\infty} n \frac{q^n}{1-q^n} - \sum_{n>0} n \frac{(-1)^n}{1-q^n} q^{\frac{n(n+1)}{2}} =$$

$$= -\frac{1}{24} + \sum_{n=1}^{\infty} n \frac{q^n}{1-q^n} + \sum_{n>0} n \frac{(-1)^{n-1}}{1-q^n} q^{\frac{n(n+1)}{2}} \,, \tag{5.46}$$

that is what we had in (5.10).

Now let us verify that the function $2\widehat{F}_1$ satisfies the holomorphic anomaly equation. In order to do that, let us notice that the expansion in Fourier modes around the circle of $\wp$ is still valid also if we do not consider the limit $\overline{\tau} \to -i\infty$. Hence

$$\begin{aligned}\widehat{F}_1 = \frac{-4\pi^2}{8\pi^2}\int_0^1\int_0^1 \Big(-2G_2(\tau) + \sum_{n\in\mathbb{Z}^*}\frac{n}{1-q^n}e^{2\pi i n(u_1+\tau u_2)}\Big)\times \\ \times \sum_{n,m\in\mathbb{Z}} e^{2\pi i(nu_2-mu_1)}(-1)^{n+m+nm}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\ . \qquad (5.47)\end{aligned}$$

Let us focus on the two summands separately. The first one reads

$$\begin{aligned}G_2(\tau)\Big[\int_0^1\int_0^1 \mathrm{d}u_1\,\mathrm{d}u_2 + \sum_{n,m\in\mathbb{Z}^*}(-1)^{n+m+nm}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\int_0^1 e^{2\pi i n u_2}\,\mathrm{d}u_2\int_0^1 e^{-2\pi i n u_1}\,\mathrm{d}u_1\Big] = \\ = G_2(\tau)\Big[1+\sum_{n,m\in\mathbb{Z}^*}(-1)^{n+m+nm}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\delta_{n,0}\delta_{m,0}\Big] = G_2(\tau)\ . \qquad (5.48)\end{aligned}$$

The second one, instead, reads

$$\begin{aligned}-\frac{1}{2}\sum_{n,m,l\in\mathbb{Z}|l\neq0}\int_0^1\int_0^1\frac{l}{1-q^l}e^{2\pi i(u_1+\tau u_2)l}e^{2\pi i n u_2}e^{-2\pi i m u_1}(-1)^{n+m+nm}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\,\mathrm{d}u_1\,\mathrm{d}u_2 = \\ = -\frac{1}{2}\sum_{n,m,l}\int_0^1\frac{l}{1-q^l}e^{2\pi i(l\tau+n)u_2}(-1)^{n+m+nm}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\delta_{l,m}\,\mathrm{d}u_2 = \\ = -\frac{1}{2}\sum_{n,m\in\mathbb{Z}|m\neq0}\int_0^1\frac{m}{1-q^m}e^{2\pi i(m\tau+n)u_2}(-1)^{m+n+nm}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\,\mathrm{d}u_2 = \\ = -\frac{1}{2}\sum_{n,m\in\mathbb{Z}|m\neq0}\frac{1}{2\pi i}\Big(e^{2\pi i(m\tau+n)}-1\Big)\frac{(-1)^{m+n+mn}}{m\tau+n}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\frac{m}{1-q^m} = \\ = -\frac{i}{4\pi}\sum_{n,m\in\mathbb{Z}|m\neq0}\frac{m}{m\tau+n}(-1)^{m+n+mn}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\ . \qquad (5.49)\end{aligned}$$

Putting all together we arrive at

$$\begin{aligned}\widehat{F}_1 &= G_2(\tau) - \frac{i}{4\pi}\sum_{n,m\in\mathbb{Z}|m\neq0}\frac{m}{m\tau+n}(-1)^{m+n+mn}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} \\ &= -\frac{1}{24}+\sum_{n=1}^{\infty}n\frac{q^n}{1-q^n} - \frac{i}{4\pi}\sum_{n,m\in\mathbb{Z}|m\neq0}\frac{m}{m\tau+n}(-1)^{m+n+mn}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} \qquad (5.50)\end{aligned}$$

Then let us derive with respect to $\overline{\tau}$. The only term affected by this derivation is

$$e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\ . \qquad (5.51)$$

Focusing on the derivation of the exponent, we have

$$\partial_{\overline{\tau}}\Big(-\frac{\pi}{2\tau_2}|m\tau+n|^2\Big) = \frac{\pi i}{4(\tau_2)^2}(m\tau+n)^2 \ , \tag{5.52}$$

from which

$$\begin{aligned}\partial_{\overline{\tau}}\widehat{F}_1 &= -\frac{i}{4\pi}\sum_{n,m\in\mathbb{Z}|m\neq 0}(-1)^{n+m+nm}\frac{m}{m\tau+n}\frac{\pi i}{4(\tau_2)^2}(m\tau+n)^2 e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} = \\ &= \frac{1}{16(\tau_2)^2}\sum_{n,m\in\mathbb{Z}|m\neq 0}(-1)^{n+m+nm}m(m\tau+n)e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} \ .\end{aligned} \tag{5.53}$$

This expression is the Poisson resummed of the holomorphic anomaly equation. Indeed let us notice that

$$\begin{aligned}\partial_{\overline{u}}\partial_{u_1}H_1 &= \partial_{\overline{u}}\partial_{u_1}\Big(\sum_{m,n\in\mathbb{Z}} e^{2\pi i(nu_2-mu_1)}(-1)^{n+m+nm}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\Big) = \\ &= \frac{1}{\tau-\overline{\tau}}(\tau\partial_{u_1}-\partial_{u_2})\Big(-2\pi i\sum_{n,m\in\mathbb{Z}} e^{2\pi i(nu_2-mu_1)}m(-1)^{n+m+nm}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\Big) = \\ &= \frac{2\pi^2 i}{\tau_2}\sum_{n,m\in\mathbb{Z}} m(m\tau+n)(-1)^{n+m+nm}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}e^{2\pi i(nu_2-mu_1)} \ ,\end{aligned} \tag{5.54}$$

hence, in particular, we have that

$$\partial_{\overline{u}}\partial_{u_1}H_1\big|_{u_1=u_2=0} = \frac{2i\pi^2}{\tau_2}\sum_{m,n\in\mathbb{Z}} m(m\tau+n)(-1)^{n+m+nm}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} \ . \tag{5.55}$$

Comparing this equation with (5.53), and using (5.14) we obtain

$$\begin{aligned}\partial_{\overline{\tau}}\widehat{F}_1 &= \frac{1}{16(\tau_2)^2}\sum_{n,m\in\mathbb{Z}} m(m\tau+n)(-1)^{n+m+nm}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} = \\ &= -\frac{i}{32\pi^2\tau_2}\partial_{\overline{u}}\partial_{u_1}H_1\big|_{u_1=u_2=0} = \\ &= \frac{1}{2}\frac{1}{\sqrt{-8\tau_2}}\big|\eta(\tau)\big|^6 \ ,\end{aligned} \tag{5.56}$$

where we have used the fact that

$$\begin{aligned}\partial_{\overline{u}}\partial_{u_1}H_1\big|_{u_1=u_2=0} &= (\partial_{\overline{u}}\partial_u + \partial_{\overline{u}}^2)\big(\sqrt{2\tau_2}\big|\theta(u,\tau)\big|^2 e^{-2\pi\tau_2(u_2)^2}\big)\big|_{u_1=u_2=0} = \\ &= \big(\sqrt{2\tau_2}\partial_u\theta(u,\tau)\partial_{\overline{u}}\overline{\theta(u,\tau)} + \dots\big)\big|_{u_1=u_2=0} = \\ &= \sqrt{2\tau_2}4\pi^2\big|\eta(\tau)\big|^6 \ ,\end{aligned} \tag{5.57}$$

where the ellipsis stand for those terms that go to zero, since

$$\theta(u,\tau)\big|_{u=0} = 0 \ , \tag{5.58}$$

and also

$$\partial_u\theta(u,\tau)\big|_{u=0} = 2\pi\eta^3(\tau)\ . \tag{5.59}$$

The same result can be obtained in a different way, noticing that the function $H_1(u_1,u_2;\tau,\overline{\tau})$ satisfy the heat equation

$$\partial_{\overline{\tau}} H_1(u_1,u_2;\tau,\overline{\tau}) = \frac{i}{4\pi}\partial^2_{\overline{u}} H_1(u_1,u_2;\tau,\overline{\tau})\ , \tag{5.60}$$

with

$$\partial_{\overline{u}} = \frac{1}{\tau-\overline{\tau}}\big(\tau\partial_{u_1} - \partial_{u_2}\big)\ . \tag{5.61}$$

Indeed

$$\begin{aligned}\partial_{\overline{\tau}} H_1(u_1,u_2;\tau,\overline{\tau}) &= \sum_{m,n\in\mathbb{Z}} e^{2\pi i(nu_2-mu_1)}(-1)^{n+m+nm}\partial_{\overline{\tau}}\Big(e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\Big) = \\ &= \frac{\pi i}{4(\tau_2)^2}\sum_{m,n\in\mathbb{Z}} e^{2\pi i(nu_2-mu_1)}(-1)^{n+m+nm}(m\tau+n)^2 e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\ ,\end{aligned} \tag{5.62}$$

where we have used the derivative found in (5.52). Then

$$\begin{aligned}\partial_{\overline{u}} H_1(u_1,u_2;\tau,\overline{\tau}) &= \frac{1}{\tau-\overline{\tau}}\big(\tau\partial_{u_1}H_1 - \partial_{u_2}H_1\big) = \\ &= -\frac{\pi}{\tau_2}\sum_{m,n\in\mathbb{Z}}(m\tau+n)e^{2\pi i(nu_2-mu_1)}(-1)^{m+n+mn}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\ .\end{aligned} \tag{5.63}$$

It is clear from here that the double derivative acts trivially as

$$\partial^2_{\overline{u}} H_1(u_1,u_2;\tau,\overline{\tau}) = \frac{\pi^2}{(\tau_2)^2}\sum_{m,n\in\mathbb{Z}}(m\tau+n)^2 e^{2\pi i(nu_2-mu_1)}(-1)^{m+n+mn}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2}\ , \tag{5.64}$$

which, compared with (5.62), verifies the heat equation (5.60).

Now let us use the heat equation in order to compute the holomorphic anomaly equation, hence

$$\begin{aligned}\partial_{\overline{\tau}}\widehat{F}_1 &= \frac{1}{8\pi^2}\int_0^1\int_0^1 \wp(u_1+\tau u_2,\tau)\partial_{\overline{\tau}}H_1(u_1,u_2;\tau,\overline{\tau})\ \mathrm{d}u_1\,\mathrm{d}u_2 = \\ &= \frac{1}{8\pi^2}\int_0^1\int_0^1 \wp(u_1+\tau u_2,\tau)\Big(\frac{i}{4\pi}\partial^2_{\overline{u}}H_1(u_1,u_2;\tau,\overline{\tau})\Big)\ \mathrm{d}u_1\,\mathrm{d}u_2 = \\ &= -\frac{i}{32\pi^3}\int_0^1\int_0^1 \big(\partial_{\overline{u}}\wp(u_1+\tau u_2,\tau)\big)\big(\partial_{\overline{u}}H_1(u_1,u_2;\tau,\overline{\tau})\big)\ \mathrm{d}u_1\,\mathrm{d}u_2\ .\end{aligned} \tag{5.65}$$

We have

$$\partial_{\overline{u}}\big(u_1+\tau u_2\big) = 0\ , \qquad \partial_{\overline{u}}\Big(\frac{1}{(u_1+\tau u_2)^2}\Big) = -\frac{\pi}{\tau_2}\partial_u\big(\delta(u_1)\delta(u_2)\big)\ , \tag{5.66}$$

where

$$\partial_u = \frac{1}{\overline{\tau} - \tau}\left(\overline{\tau}\partial_{u_1} - \partial_{u_2}\right) . \tag{5.67}$$

Since the Weierstrass elliptic function has a double pole in the origin with coefficient 1 and no residue, we have that, when we compute the derivative with respect to $\overline{u}$, the only term that contributes is the double pole. Hence it localizes H_1 on $u_1 = u_2 = 0$ and, after an integration by parts, we get

$$\partial_{\overline{\tau}}\widehat{F}_1 = -\frac{i}{32\pi^2\tau_2}\Big(\partial_u\partial_{\overline{u}}H_1(u_1, u_2; \tau, \overline{\tau})\Big)\big|_{u_1=u_2=0} . \tag{5.68}$$

From here we can directly derive the holomorphic anomaly equation, indeed

$$\begin{aligned}
\partial_u\partial_{\overline{u}}H_1(u_1, u_2; \tau, \overline{\tau}) &= \frac{1}{\overline{\tau} - \tau}\left(\overline{\tau}\partial_{u_1} - \partial_{u_2}\right)\left(\partial_{\overline{u}}H_1(u_1, u_2; \tau, \overline{\tau})\right) = \\
&= \frac{\pi}{2i(\tau_2)^2}\sum_{m,n\in\mathbb{Z}}(m\tau + n)\Big(\overline{\tau}\left(\partial_{u_1}e^{-2\pi imu_1}\right)e^{2\pi inu_2} + \\
&\qquad - \left(\partial_{u_2}e^{2\pi imu_2}\right)e^{-2\pi inu_1}\Big)(-1)^{n+m+nm}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} = \\
&= -\frac{\pi^2}{(\tau_2)^2}\sum_{m,n\in\mathbb{Z}}|m\tau + n|^2 e^{2\pi i(nu_2 - mu_1)}(-1)^{n+m+nm}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} .
\end{aligned} \tag{5.69}$$

Imposing the condition $u_1 = u_2 = 0$ we get

$$\partial_u\partial_{\overline{u}}H_1(u_1, u_2; \tau, \overline{\tau})\big|_{u_1=u_2=0} = -\frac{\pi^2}{(\tau_2)^2}\sum_{m,n\in\mathbb{Z}}|m\tau+n|^2(-1)^{n+m+mn}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} , \tag{5.70}$$

hence

$$\begin{aligned}
\partial_{\overline{\tau}}\widehat{F}_1 &= -\frac{i}{32\pi^2\tau_2}\frac{-\pi^2}{(\tau_2)^2}\sum_{m,n\in\mathbb{Z}}|m\tau + n|^2(-1)^{m+n+mn}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} = \\
&= \frac{i}{32(\tau_2)^3}\sum_{m,n\in\mathbb{Z}}|m\tau + n|^2(-1)^{m+n+mn}e^{-\frac{\pi}{2\tau_2}|m\tau+n|^2} ,
\end{aligned} \tag{5.71}$$

which gives directly the holomorphic anomaly equation. Indeed, thanks to the expression (5.14) for H_1, we have that

$$\begin{aligned}
\partial_u\partial_{\overline{u}}H_1\big|_{u_1=u_2=0} &= \left(\sqrt{2\tau_2}\partial_u\theta(u, \tau)\partial_{\overline{u}}\overline{\theta(u, \tau)}e^{-2\pi\tau_2(u_2)^2} + \dots\right)\big|_{u_1=u_2=0} = \\
&= \sqrt{2\tau_2}4\pi^2\left|\eta(\tau)\right|^6 ,
\end{aligned} \tag{5.72}$$

where the ellipsis stand for those terms that goes to zero when we impose $u_1 = u_2 = 0$. From here we trivially conclude.

5.2 The case of general k

Now we are going to focus on the general case, where we have a sigma model with target S^3 and Wess-Zumino coupling k, which in the IR goes to a $\mathcal{N} = (0, 1)$ supersymmetric WZW model at bosonic level $\kappa = |k| - 1$ that, as we have yet explained, is equivalent to a bosonic WZW model at level κ plus the theory of three free anti-chiral fermions. The supercurrent in this theory is

$$\overline{G} = \sqrt{-1}\sqrt{\frac{2}{\kappa+2}}\,\overline{\psi}_1\overline{\psi}_2\overline{\psi}_3 + \dots\ , \tag{5.73}$$

where the ellipsis stand for terms of the type

$$\sum_a \psi_a J^a_b\ , \tag{5.74}$$

with J^a_b the currents of the bosonic WZW model. Since they are bosonic currents, the one-point function still vanishes due to the presence of the fermionic zero modes. This in particular means that the one-point function can be factorized in the sub-models which form the whole theory, and it will be

$$g_\kappa(\tau,\overline{\tau}) = \sqrt{\frac{2}{\kappa+2}}\big|\eta(\tau)^6\big|Z^{\text{WZW}}_\kappa(\tau,\overline{\tau})\ . \tag{5.75}$$

The partition function on the torus of the WZW model can be expanded in terms of the characters of the current algebra since the theory is a rational CFT, that is

$$Z^{\text{WZW}}_\kappa(\tau,\overline{\tau}) = \sum_{2j+1=1}^{\kappa+1}\big|\chi^{(\kappa)}_j(\tau)\big|^2\ . \tag{5.76}$$

The characters of the current algebra are defined by

$$\chi^{(\kappa)}_j(\tau) = \frac{q^{-\frac{1}{8}}}{\prod_{n>0}(1-q^n)^3}\sum_{m\in\mathbb{Z}+\frac{j+\frac{1}{2}}{\kappa+2}} q^{(\kappa+2)m^2}\big(2m(\kappa+2)\big)\ . \tag{5.77}$$

If we define the theta functions of weight 3/2 as

$$\Theta_{k,l}(\tau) = \sum_{m\in\mathbb{Z}+\frac{l}{2k}} m q^{km^2}\ , \tag{5.78}$$

we can rewrite

$$\chi^{(\kappa)}_j(\tau) = \frac{2(\kappa+2)\Theta_{\kappa+2,2j+1}(\tau)}{\eta(\tau)^3}\ . \tag{5.79}$$

With this definition we get

$$
\begin{aligned}
g_\kappa(\tau,\overline{\tau}) &= \frac{\sqrt{2}}{\sqrt{\kappa+2}}\big|\eta(\tau)\big|^6 \sum_{2j+1=1}^{\kappa+1} \big(2(\kappa+2)\big)^2 \frac{\Theta_{\kappa+2,2j+1}(\tau)\overline{\Theta}_{\kappa+2,2j+1}(\overline{\tau})}{\eta(\tau)^3\overline{\eta}(\overline{\tau})^3} = \\
&= 4\sqrt{2}(\kappa+2)^{\frac{3}{2}}\big|\eta(\tau)\big|^6 \sum_{2j+1=1}^{\kappa+1} \frac{\big|\Theta_{\kappa+2,2j+1}(\tau)\big|^2}{\big|\eta(\tau)\big|^6} = \\
&= 4(\kappa+2)\sqrt{2(\kappa+2)} \sum_{2j+1=1}^{\kappa+1} \big|\Theta_{\kappa+2,2j+1}(\tau)\big|^2 \ .
\end{aligned} \tag{5.80}
$$

In what follows it will be useful to define the *flavoured WZW characters*

$$
\chi_j^{(\kappa)}(\xi;\tau) = \frac{\vartheta_{\kappa+2,2j+1}(\xi;\tau) - \vartheta_{\kappa+2,2j+1}(-\xi;\tau)}{\theta(\xi;\tau)} \ , \tag{5.81}
$$

where

$$
\vartheta_{k,l}(\xi;\tau) = \sum_{m\in\mathbb{Z}+\frac{l}{2k}} x^{km} q^{km^2} \ , \qquad x = e^{2\pi i\xi} \ q = e^{2\pi i\tau} \ , \tag{5.82}
$$

Let us find now the generic solution of the holomorphic anomaly equation that, in analogy with what we have done in the case $k=1$, we call $\widehat{F}_k$. Let us define

$$
\widehat{F}_k = \frac{1}{8\pi^2}\int_0^1\int_0^1 \wp(u_1+\tau u_2,\tau) H_k(u_1,u_2;\tau,\overline{\tau}) \ \mathrm{d}u_1\,\mathrm{d}u_2 \ , \tag{5.83}
$$

where

$$
H_k(u_1,u_2;\tau,\overline{\tau}) = \sqrt{(\kappa+2)\tau_2}e^{-(\kappa+2)\pi\tau_2(u_2)^2} \sum_{2j+1=1}^{\kappa+1} \big|\chi_j^{(\kappa)}(u_1+\tau u_2;\tau)\big|^2 \big|\theta(u_1+\tau u_2;\tau)\big|^2 \ . \tag{5.84}
$$

This last function satisfies the heat equation

$$
\partial_{\overline{\tau}} H_k(u_1,u_2;\tau,\overline{\tau}) = \frac{i}{2\pi(\kappa+2)}\partial_{\overline{u}}^2 H_k(u_1,u_2;\tau,\overline{\tau}) \ . \tag{5.85}
$$

Thanks to the heat equation, for the same reasons as in the case of $k=1$, we have that

$$
\partial_{\overline{\tau}}\widehat{F}_k = -\frac{i}{16\pi^2(\kappa+2)\tau_2}\Big(\partial_u\partial_{\overline{u}} H_k(u_1,u_2;\tau,\overline{\tau})\Big)\Big|_{u_1=u_2=0} \ . \tag{5.86}
$$

Let us compute the derivatives in the RHS, using the relation

$$
\partial_u\big(\chi_j^{(\kappa)}(u,\tau)\theta(u,\tau)\big)\big|_{u=0} = 2\pi i\eta(\tau)^3\chi_j^{(\kappa)}(\tau) \ . \tag{5.87}
$$

In this way

$$\partial_u \partial_{\overline{u}} H_k \big|_{u_1=u_2=0} = \sqrt{(\kappa+2)\tau_2}\, e^{-(\kappa+2)\pi\tau_2 (u_2)^2} \times \\ \times \sum_{2j+1=1}^{\kappa+1} \partial_u \big(\chi_j^{(\kappa)}(u;\tau)\theta(u;\tau)\big) \partial_{\overline{u}} \big(\overline{\chi_j^{(\kappa)}(u;\tau)}\overline{\theta(u;\tau)}\big)\big|_{u_1=u_2=0} = \\ = 4\pi^2 \sqrt{(\kappa+2)} \big|\eta(\tau)\big|^6 \sum_{2j+1=1}^{\kappa+1} \big|\chi_j^{(\kappa)}(\tau)\big|^2 \,, \tag{5.88}$$

from which we conclude that

$$\partial_{\overline{\tau}} \widehat{F}_k = \frac{1}{2\sqrt{-8\tau_2}} \sqrt{\frac{2}{\kappa+2}} \big|\eta(\tau)\big|^6 Z_k^{\mathrm{WZW}}(\tau,\overline{\tau}) \,, \tag{5.89}$$

that is

$$\sqrt{-8\tau_2}\, \partial_{\overline{\tau}} (2\widehat{F}_k) = \sqrt{\frac{2}{\kappa+2}} \big|\eta(\tau)\big|^6 Z_k^{\mathrm{WZW}}(\tau,\overline{\tau}) = g_k(\tau,\overline{\tau}) \,. \tag{5.90}$$

The last step consists in extracting the holomorphic part F_k of $\widehat{F}_k$. In order to do this, let us notice that the function $H_k(u_1,u_2;\tau,\overline{\tau})$ can be split in two parts

$$H_k^{(1)}(u_1,u_2;\tau,\overline{\tau}) = \sqrt{(\kappa+2)\tau_2}\, e^{-(\kappa+2)\pi\tau_2(u_2)^2} \sum_{2j+1=0}^{2\kappa+3} \big|\vartheta_{\kappa+2,2j+1}(u;\tau)\big|^2 \\ H_k^{(2)}(u_1,u_2;\tau,\overline{\tau}) = -\sqrt{(\kappa+2)\tau_2}\, e^{-(\kappa+2)\pi\tau_2(u_2)^2} \sum_{2j+1=0}^{2\kappa+3} \vartheta_{\kappa+2,2j+1}(u;\tau) \overline{\vartheta_{\kappa+2,2\kappa+4-2j-1}(u;\tau)} \,. \tag{5.91}$$

Indeed, we have

$$\big|\vartheta_{\kappa+2,2j+1}(u;\tau) - \vartheta_{\kappa+2,2j+1}(-u;\tau)\big| = \big|\vartheta_{\kappa+2,2j+1}(u;\tau)\big|^2 + \big|\vartheta_{\kappa+2,2j+1}(-u;\tau)\big|^2 + \\ - \vartheta_{\kappa+2,2j+1}(-u;\tau)\overline{\vartheta_{\kappa+2,2j+1}(u;\tau)} - \vartheta_{\kappa+2,2j+1}(u;\tau)\overline{\vartheta_{\kappa+2,2j+1}(-u;\tau)} \,. \tag{5.92}$$

A trivial property ensures us that

$$\vartheta_{k,l}(-u;\tau) = \vartheta_{k,2k-l} \,, \tag{5.93}$$

thanks to which we can see that

$$\sum_{2j+1=1}^{\kappa+1} \big|\vartheta_{\kappa+2,2j+1}(-u;\tau)\big|^2 = \sum_{2j+1=1}^{\kappa+1} \big|\vartheta_{\kappa+2,2\kappa+4-2j-1}(u;\tau)\big|^2 = \\ = \sum_{2l+1=\kappa+3}^{2(\kappa+2)} \big|\vartheta_{\kappa+2,2l+1}(u;\tau)\big|^2 \,, \tag{5.94}$$

where we have first extended the sum to $2j+1=0$, since for this value the ϑ-function vanishes, and then introduced the new variable l such that

$$2\kappa+4-2j-1=2l+1\ . \tag{5.95}$$

Then, since $\chi^{(\kappa)}_{\frac{\kappa+1}{2}}=0$, we can extend the sum to $2l+1=\kappa+2$, in such a way that

$$\sum_{2j+1=1}^{\kappa+1}\left(\left|\vartheta_{\kappa+2,2j+1}(u,\tau)\right|^2+\left|\vartheta_{\kappa+2,2j+1}(-u,\tau)\right|^2\right)=\sum_{2j+1=0}^{2\kappa+3}\left|\vartheta_{\kappa+2,2j+1}(u;\tau)\right|^2\ . \tag{5.96}$$

In the same way we can see that

$$\sum_{2j+1=1}^{\kappa+1}\vartheta_{\kappa+2,2j+1}(-u;\tau)\overline{\vartheta_{\kappa+2,2j+1}(\chi,\tau)}=\sum_{2l+1=\kappa+3}^{2(\kappa+2)}\vartheta_{\kappa+2,2l+1}(u;\tau)\overline{\vartheta_{\kappa+2,2\kappa+4-2l-1}(u,\tau)}\ , \tag{5.97}$$

and so we can justify also the second contribution in (5.91). These two terms Poisson resum to

$$\begin{aligned} H_k^{(1)}(u_1,u_2;\tau,\overline{\tau}) &= \sum_{m,n\in\mathbb{Z}}(\kappa+2)e^{2\pi i(\kappa+2)(nu_2-mu_1)}e^{-\frac{(\kappa+2)\pi}{\tau_2}|m\tau+n|^2}\ , \\ H_k^{(2)}(u_1,u_2;\tau,\overline{\tau}) &= -\sum_{n,m\in\mathbb{Z}}e^{2\pi i(nu_2-mu_1)}e^{-\frac{\pi}{(\kappa+2)\tau_2}|m\tau+n|^2}\ . \end{aligned} \tag{5.98}$$

Let us verify these results, starting from studying $H_k^{(2)}$. We have to find the Fourier transformation of

$$e^{2\pi inu_2}e^{-\frac{\pi}{(\kappa+2)\tau_2}|m\tau+n|^2}\ , \tag{5.99}$$

that allows us to write

$$H_k^{(2)}=-\sqrt{(\kappa+2)\tau_2}\sum_{m,n\in\mathbb{Z}}e^{-2\pi imu_1}e^{-\frac{\pi}{(\kappa+2)\tau_2}\left(2i\tau_1\tau_2m(\kappa+2)(u_2-n)+(\tau_2)^2(\kappa+2)^2(u_2-n)^2+m^2(\tau_2)^2\right)}\ . \tag{5.100}$$

Defining the new variables

$$a:=(\kappa+2)n+m\ ,\qquad b:=(\kappa+2)n-m\ , \tag{5.101}$$

we arrive at

$$\begin{aligned} H_k^{(2)} &= -\sqrt{(\kappa+2)\tau_2}\sum_{a,b\in\mathbb{Z}}e^{2\pi i\tau\frac{a^2}{4(\kappa+2)}}e^{-2\pi i\overline{\tau}\frac{b^2}{4(\kappa+2)}}e^{2\pi iu\frac{a}{2}}e^{-2\pi i\overline{u}\frac{b}{2}}e^{-(\kappa+2)\pi\tau_2(u_2)^2}= \\ &= -\sqrt{(\kappa+2)\tau_2}e^{-(\kappa+2)\pi\tau_2(u_2)^2}\sum_{2j+1=0}^{2\kappa+3}\vartheta_{\kappa+2,2j+1}(u;\tau)\overline{\vartheta_{\kappa+2,2\kappa+4-2j-1}(u;\tau)}\ . \end{aligned} \tag{5.102}$$

Here, we have used the fact that, remembering the definition (5.82),

$$\sum_{2j+1=0}^{2\kappa+3} \vartheta_{\kappa+2,2j+1}(u,\tau)\overline{\vartheta_{\kappa+2,2\kappa+2-2j-1}(u,\tau)} =$$
$$= \sum_{2j+1=0}^{2\kappa+2} \Bigg(\sum_{m\in\mathbb{Z}+\frac{2j+1}{2(\kappa+2)}} x^{(\kappa+2)m} q^{(\kappa+2)m^2} \Bigg) \Bigg(\sum_{n\in\mathbb{Z}+\frac{2(\kappa+2)-(2j+1)}{2(\kappa+2)}} \overline{x}^{(\kappa+2)m} \overline{q}^{(\kappa+2)m^2} \Bigg) =$$
$$= \sum_{2j+1=0}^{2\kappa+3} \sum_{m\in\mathbb{Z}+\frac{2j+1}{2(\kappa+2)}} \sum_{n\in\mathbb{Z}-\frac{2j+1}{2(\kappa+2)}} x^{(\kappa+2)m} q^{(\kappa+2)m^2} \overline{x}^{(\kappa+2)n} \overline{q}^{(\kappa+2)n^2} \ . \tag{5.103}$$

It is clear that the following relation holds

$$\sum_{2j+1=0}^{2\kappa+3} \sum_{m\in\mathbb{Z}\pm\frac{2j+1}{2(\kappa+2)}} = \sum_{m\in\mathbb{Z}} \Big|_{m\mapsto\frac{m}{2(\kappa+2)}} \ , \tag{5.104}$$

from which we conclude that

$$\sum_{2j+1=0}^{2\kappa+3} \vartheta_{\kappa+2,2j+1}(u,\tau)\overline{\vartheta_{\kappa+2,2\kappa+2-2j-1}(u,\tau)} = \sum_{m,n\in\mathbb{Z}} x^{\frac{m}{2}} q^{\frac{m^2}{4(\kappa+2)}} \overline{x}^{\frac{n}{2}} \overline{q}^{\frac{n^2}{4(\kappa+2)}} \ . \tag{5.105}$$

Let us now focus on $H_k^{(1)}$. In this case we have to find the Fourier transform of

$$e^{2\pi i(\kappa+2)nu_2} e^{-\frac{(\kappa+2)\pi}{\tau_2}|m\tau+n|^2} \ , \tag{5.106}$$

thanks to which we can write

$$H_k^{(1)} = \sum_{m,n\in\mathbb{Z}} \sqrt{(\kappa+2)\tau_2} \sum_{m,n\in\mathbb{Z}} e^{-2\pi i(\kappa+2)mu_1} e^{-(\kappa+2)\pi\left(2i\tau_1 m(u_2-\frac{n}{\kappa+2})+\tau_2(u_2-\frac{n}{\kappa+2})^2+m^2\tau_2\right)} \ . \tag{5.107}$$

Defining the new variables

$$a := (\kappa+2)m+n \ , \qquad b := -(\kappa+2)m+n \ , \tag{5.108}$$

we get

$$H_k^{(1)} = \sqrt{(\kappa+2)\tau_2} \sum_{a,b\in\mathbb{Z}} e^{2\pi i\tau\frac{a^2}{4(\kappa+2)}} e^{-2\pi i\overline{\tau}\frac{b^2}{4(\kappa+2)}} e^{2\pi iu\frac{a}{2}} e^{-2\pi iu\frac{b}{2}} e^{-(\kappa+2)\pi\tau_2(u_2)^2} =$$
$$= \sqrt{(\kappa+2)\tau_2} e^{-(\kappa+2)\pi\tau_2(u_2)^2} \sum_{2j+1=0}^{2\kappa+3} \left|\vartheta_{\kappa+2,2j+1}(u;\tau)\right|^2 , \tag{5.109}$$

as we wanted. In this case we have used the same argument as before, noticing that

$$\begin{aligned}
&\sum_{2j+1=0}^{2\kappa+3} \left|\vartheta_{\kappa+2,2j+1}(u;\tau)\right|^2 = \\
&= \sum_{2j+1=0}^{2\kappa+3} \Bigg(\sum_{m\in\mathbb{Z}+\frac{2j+1}{2(\kappa+2)}} x^{(\kappa+2)m} q^{(\kappa+2)m^2} \Bigg) \Bigg(\sum_{n\in\mathbb{Z}+\frac{2j+1}{2(\kappa+2)}} \overline{x}^{(\kappa+2)n} \overline{q}^{(\kappa+2)n^2} \Bigg) = \\
&= \sum_{2j+1=0}^{2\kappa+3} \sum_{m\in\mathbb{Z}+\frac{2j+1}{2(\kappa+2)}} \sum_{n\in\mathbb{Z}+\frac{2j+1}{2(\kappa+2)}} x^{(\kappa+2)m} q^{(\kappa+2)m^2} \overline{x}^{(\kappa+2)n} \overline{q}^{(\kappa+2)n^2} = \\
&= \sum_{m,n\in\mathbb{Z}} x^{\frac{m}{2}} q^{\frac{m^2}{4(\kappa+2)}} \overline{x}^{\frac{n}{2}} \overline{q}^{\frac{n^2}{4(\kappa+2)}} \ .
\end{aligned} \tag{5.110}$$

The two expressions found in (5.98), can be unified as

$$H_k(u_1,u_2;\tau,\overline{\tau}) = \sum_{n,m\in\mathbb{Z}} \epsilon^{\kappa+2}_{n,m} e^{2\pi i(nu_2 - mu_1)} e^{\frac{2\pi i}{\kappa+2}m(m\tau+n)} \ , \tag{5.111}$$

with

$$\epsilon^{\kappa+2}_{n,m} = \begin{cases} \kappa+1 & n,m \equiv 0 \mod \kappa+2 \ , \\ -1 & \text{otherwise} \ . \end{cases} \tag{5.112}$$

Then we have to take the limit $\overline{\tau} \to -i\infty$. It is clear how in the expression of H_k, the only term affected by the limit procedure is

$$e^{-\frac{\pi}{(\kappa+2)\tau_2}|m\tau+n|^2} \ , \tag{5.113}$$

which, focusing on the exponent, becomes

$$-\frac{\pi}{(\kappa+2)} \frac{1}{\frac{\tau-\overline{\tau}}{2i}} (m\tau+n)(m\overline{\tau}+n) \xrightarrow{\overline{\tau}\to -i\infty} \frac{2\pi i}{\kappa+2} m(m\tau+n) \ . \tag{5.114}$$

Hence, in this limit, we get

$$H_k(\overline{\tau}\to -i\infty) = \sum_{n,m\in\mathbb{Z}} \epsilon^{\kappa+2}_{n,m} e^{2\pi i(nu_2 - mu_1)} e^{\frac{2\pi i}{\kappa+2}m(m\tau+n)} \ , \tag{5.115}$$

from which

$$F_k = \frac{1}{8\pi^2} \int_0^1 \int_0^1 \wp(u_1+\tau u_2,\tau) \sum_{n,m\in\mathbb{Z}} \epsilon^{\kappa+2}_{n,m} e^{2\pi i(nu_2 - mu_1)} e^{\frac{2\pi i}{\kappa+2}m(m\tau+n)} \, du_1 \, du_2 \ . \tag{5.116}$$

This expression can be split in two parts

$$F_k^{(1)} = \frac{\kappa+2}{8\pi^2}\int_0^1\int_0^1 \wp(u_1+\tau u_2,\tau)\sum_{n,m\in\mathbb{Z}} e^{2\pi i(\kappa+2)(nu_2-mu_1)}e^{2\pi i(\kappa+2)m^2\tau}\,\mathrm{d}u_1\,\mathrm{d}u_2\ , \tag{5.117a}$$

$$F_k^{(2)} = -\frac{1}{8\pi^2}\int_0^1\int_0^1 \wp(u_1+\tau u_2,\tau)\sum_{n,m\in\mathbb{Z}} e^{2\pi i(nu_2-mu_1)}e^{\frac{2\pi i}{\kappa+2}m^2\tau}e^{\frac{2\pi i}{\kappa+2}mn}\,\mathrm{d}u_1\,\mathrm{d}u_2\ . \tag{5.117b}$$

This splitting is justified as follows. Let us start from focusing only on the case in which the indices of the sum $\widetilde{m}$ and $\widetilde{n}$ are both divisible by $\kappa+2$, that is

$$\widetilde{m}\equiv\widetilde{n}\equiv 0 \mod \kappa+2\ , \tag{5.118}$$

from which we have that there exist two integers $m,n\in\mathbb{Z}$ such that

$$\widetilde{m}=(\kappa+2)m\ ,\qquad \widetilde{n}=(\kappa+2)n\ . \tag{5.119}$$

Hence, in this case

$$\begin{aligned}F_k &= \frac{\kappa+1}{8\pi^2}\int_0^1\int_0^1 \wp(u_1+\tau u_2,\tau)\sum_{\widetilde{m},\widetilde{n}\in\mathbb{Z}} e^{2\pi i(\widetilde{n}u_2-\widetilde{m}u_1)}e^{\frac{2\pi i}{\kappa+2}\widetilde{m}(\widetilde{m}\tau+\widetilde{n})}\,\mathrm{d}u_1\,\mathrm{d}u_2 = \\ &= \frac{\kappa+1}{8\pi^2}\int_0^1\int_0^1 \wp(u_1+\tau u_2,\tau)\sum_{m,n\in\mathbb{Z}} e^{2\pi i(\kappa+2)(nu_1-mu_2)}e^{2\pi i(\kappa+2)m^2\tau}\,\mathrm{d}u_1\,\mathrm{d}u_2\ ,\end{aligned} \tag{5.120}$$

where we have used the fact that

$$(\kappa+2)m\,n = \widetilde{m}\,n\in\mathbb{Z}\ , \tag{5.121}$$

and so $e^{2\pi i(\kappa+2)m\,n}=1$. If instead $m,n\not\equiv 0 \mod \kappa+2$, then

$$F_k = -\frac{1}{8\pi^2}\int_0^1\int_0^1 \wp(u_1+\tau u_2,\tau)\sum_{\substack{m,n\in\mathbb{Z}\\ m,n\not\equiv 0 \mod \kappa+2}} e^{2\pi i(nu_2-mu_1)}e^{\frac{2\pi i}{\kappa+2}m^2\tau}e^{\frac{2\pi i}{\kappa+2}mn}\,\mathrm{d}u_1\,\mathrm{d}u_2\ . \tag{5.122}$$

If in this last relation we deleted the condition on the indices m and n, we would obtain an expression whose error would be given by the extra counting of all the summands with $m\equiv n\equiv 0 \mod \kappa+2$. This error, with an obvious meaning of the following notation, reads

$$\begin{aligned}&-\frac{1}{8\pi^2}\int_0^1\int_0^1 \wp(u_1+\tau u_2)\sum_{\substack{\widetilde{m},\widetilde{n}\in\mathbb{Z}\\ \widetilde{m},\widetilde{n}\equiv 0 \mod \kappa+2}} e^{2\pi i(\widetilde{n}u_1-\widetilde{m}u_2)}e^{\frac{2\pi i}{\kappa+2}\widetilde{m}^2\tau}e^{\frac{2\pi i}{\kappa+2}\widetilde{m}\widetilde{n}}\,\mathrm{d}u_1\,\mathrm{d}u_2 = \\ &= -\frac{1}{8\pi^2}\int_0^1\int_0^1 \wp(u_1+\tau u_2,\tau)\sum_{m,n\in\mathbb{Z}} e^{2\pi i(\kappa+2)(nu_2-mu_2)}e^{2\pi i(\kappa+2)m^2\tau}\,\mathrm{d}u_1\,\mathrm{d}u_2\ ,\end{aligned} \tag{5.123}$$

which can be reabsorbed changing the constant in front of (5.120)

$$\kappa+1 \longmapsto \kappa+2\,. \tag{5.124}$$

This justifies the splitting in (5.117).

Looking at the expression of $F_k^{(1)}$ in (5.117a), we have that the sum over n is only given by

$$\sum_{n\in\mathbb{Z}} e^{2\pi i(\kappa+2)nu_2}\,, \tag{5.125}$$

from which follows

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} e^{2\pi i(\kappa+2)u_2 n} &= \sum_{l=0}^{\kappa+1} \delta\big(u_2(\kappa+2)-l\big) = \\
&= \sum_{l=0}^{\kappa+1} \delta\Big(\frac{1}{\kappa+2}\big(u_2-\frac{l}{\kappa+2}\big)\Big) = \\
&= \sum_{l=0}^{\kappa+1} \frac{1}{\kappa+2}\delta\Big(u_2-\frac{l}{\kappa+2}\Big)\,.
\end{aligned} \tag{5.126}$$

This allows us to rewrite

$$\begin{aligned}
F_k^{(1)} &= \frac{\kappa+2}{8\pi^2}\int_0^1\int_0^1 \wp(u_1+\tau u_2,\tau)\sum_{m\in\mathbb{Z}} e^{-2\pi i(\kappa+2)mu_1}e^{2\pi i(\kappa+2)m^2\tau}\times \\
&\qquad\qquad \times\sum_{l=0}^{\kappa+1}\frac{1}{\kappa+2}\delta\Big(u_2-\frac{l}{\kappa+2}\Big)\,\mathrm{d}u_1\,\mathrm{d}u_2 = \\
&= \frac{1}{8\pi^2}\sum_{l=0}^{\kappa+1}\sum_{m\in\mathbb{Z}}\int_0^1 \wp\Big(u_1+\tau\frac{l}{\kappa+2},\tau\Big)q^{(\kappa+2)m^2}e^{-2\pi i(\kappa+2)mu_1}\,\mathrm{d}u_1 = \\
\big(u_1\mapsto u_1-\tau\tfrac{l}{\kappa+2}\big)\qquad &= \frac{1}{8\pi^2}\sum_{l=0}^{\kappa+1}\sum_{m\in\mathbb{Z}}\int_{\tau\frac{l}{\kappa+2}}^{1+\tau\frac{l}{\kappa+2}} \wp(u_1,\tau)q^{(\kappa+2)m^2}e^{-2\pi i(\kappa+2)m(u_1-\tau\frac{l}{\kappa+2})}\,\mathrm{d}u_1 = \\
&= \frac{1}{8\pi^2}\sum_{l=0}^{\kappa+1}\sum_{m\in\mathbb{Z}} q^{(\kappa+2)m^2+ml}\int_{\tau\frac{l}{\kappa+2}}^{1+\tau\frac{l}{\kappa+2}} \wp(u_1,\tau)e^{-2\pi i(\kappa+2)mu_1}\,\mathrm{d}u_1\,,
\end{aligned} \tag{5.127}$$

Let us focus for one second to the region of the complex plane on which the elliptic function $\wp$ is defined, in order to change in a convenient way the extremes of the integration. Let us refer to Figure 5.1 where we have identified with the dashed line one period (of period τ of course) of the lattice on which $\wp$ is defined, and then we have represented a path of integration, where the point B corresponds to the complex number $1+\tau\frac{l}{\kappa+2}$, while C corresponds to the number $\tau\frac{l}{\kappa+2}$ (let us underline that it is crucial that $\frac{l}{\kappa+2}<1$). In particular, the integral we have in (5.127) is done on the line between C and B. However,

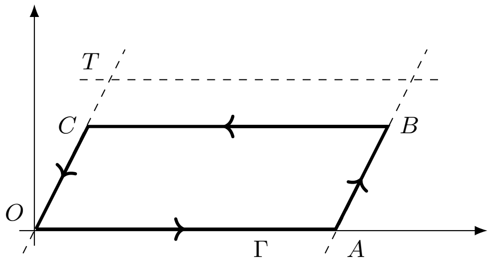

Figure 5.1: Domain of $\wp$ and path of integration.

the lines $\overline{AB}$ and $\overline{OC}$ are on the sides of the lattice, hence they are identified but, since they have the opposite directions, they cancel each other. For this reason, we can conclude that the integral on the path from C to B is the same if computed from O to A. Hence

$$F_k^{(1)} = \sum_{l=0}^{\kappa+1} \sum_{m\in\mathbb{Z}} q^{(\kappa+2)m^2+ml} \frac{1}{8\pi^2} \int_0^1 e^{-2\pi i(\kappa+2)mu_1} \wp(u_1,\tau)\ \mathrm{d}u_1\ . \tag{5.128}$$

Let us study now the expression of $F_k^{(2)}$ in (5.117b). If we focus only on the sum on n, we can Poisson resum it as

$$\sum_{n\in\mathbb{Z}} e^{2\pi i n u_2} e^{\frac{2\pi i}{\kappa+2}mn} = \sum_{n\in\mathbb{Z}} \delta\Big(n - u_2 - \frac{m}{\kappa+2}\Big)\ , \tag{5.129}$$

however, in the RHS, the only term that contributes is the one for which $u_2 \in [0,1)$, that is

$$\delta\Big(u_2 - \big[-\frac{m}{\kappa+2}\big]\Big)\ , \tag{5.130}$$

where we have used the following notation

$$\big[-\frac{m}{\kappa+2}\big] := -\frac{m}{\kappa+2} \mod 1\ . \tag{5.131}$$

Hence

$$\begin{aligned}
F_k^{(2)} &= -\frac{1}{8\pi^2}\int_0^1\int_0^1 \wp(u_1+\tau u_2,\tau)\sum_{m\in\mathbb{Z}} e^{-2\pi i m u_1} e^{\frac{2\pi i}{\kappa+2}m^2\tau}\delta\Big(u_2+\frac{m}{\kappa+2}\Big)\ \mathrm{d}u_1\,\mathrm{d}u_2 = \\
&= -\frac{1}{8\pi^2}\sum_{m\in\mathbb{Z}}\int_0^1 \wp\big(u_1+\tau\big[-\frac{m}{\kappa+2}\big],\tau\big)e^{-2\pi i m u_1}e^{\frac{2\pi i}{\kappa+2}m^2\tau}\ \mathrm{d}u_1 = \\
\overset{u_1\mapsto u_1-\tau[-\frac{m}{\kappa+2}]}{} &= -\frac{1}{8\pi^2}\sum_{m\in\mathbb{Z}}\int_0^1 \wp(u_1,\tau)e^{-2\pi i m(u_1-\tau[\frac{m}{\kappa+2}])}e^{\frac{2\pi i}{\kappa+2}m^2\tau}\ \mathrm{d}u_1 = \\
&= -\frac{1}{8\pi^2}\sum_{m\in\mathbb{Z}} q^{\frac{m^2}{\kappa+2}+m[-\frac{m}{\kappa+2}]}\int_0^1 e^{-2\pi i m u_1}\wp(u_1,\tau)\ \mathrm{d}u_1\ .
\end{aligned} \tag{5.132}$$

Since we are interested on the function F_k up to $2\mathbb{Z}((q)) + \mathrm{MF}_2$, let us consider only the terms of order q^0. In the case of $F_k^{(1)}$, we have to look at the contribution for $m = 0$ in (5.128), that is, using the Fourier expansion of $\wp$ on the circle,

$$
\begin{aligned}
F_k^{(1)}\Big|_{m=0} &= \sum_{l=0}^{\kappa+1} \frac{1}{8\pi^2} \int_0^1 \wp(u_1,\tau)\,\mathrm{d}u_1 = \\
&= \sum_{l=0}^{\kappa+1} \frac{1}{8\pi^2} \int_0^1 \Big(8\pi^2 G_2(\tau) - 4\pi^2 \sum_{n\in\mathbb{Z}^*} \frac{n}{1-q^n} e^{2\pi i u_1 n}\Big)\,\mathrm{d}u_1 = \\
&= (\kappa+2)G_2(\tau) + \frac{i}{4\pi}(\kappa+2) \sum_{n\in\mathbb{Z}^*} \frac{(e^{2\pi i n}-1)}{1-q^n} = \\
&= (\kappa+2)G_2(\tau)\ ,
\end{aligned}
\tag{5.133}
$$

where, since $n \in \mathbb{Z}$ we have of course that

$$
e^{2\pi i n} - 1 \equiv 0\ . \tag{5.134}
$$

We have now to focus on $F_k^{(2)}$. Here, the contribution of order q^0 are the ones of the kind

$$
q^{\frac{m^2}{\kappa+2} + m[-\frac{m}{\kappa+2}]} = q^0\ , \tag{5.135}
$$

that is the ones with $m = -(\kappa+1), -\kappa, -(\kappa-1), \ldots, -1, 0$. Using again the Fourier expansion of the elliptic function $\wp$, we get

$$
\begin{aligned}
F_k^{(2)}\big|_{q^0} &= -\frac{1}{8\pi^2} \sum_{m=-\kappa-1}^{0} \int_0^1 e^{2\pi i m u_1} \wp(u_1,\tau)\ \mathrm{d}u_1\Big|_{q^0} = \\
&= -\frac{1}{8\pi^2} \sum_{m=-\kappa-1}^{\kappa+1} \int_0^1 e^{-2\pi i m u_1} \Big(8\pi^2 G_2(\tau) - 4\pi^2 \sum_{n\in\mathbb{Z}^*} \frac{n}{1-q^n} e^{2\pi i u_1 n}\Big)\ \mathrm{d}u_1\Big|_{q^0} = \\
&= -G_2(\tau) \sum_{m=-\kappa-1}^{0} \int_0^1 e^{-2\pi i m u_1}\ \mathrm{d}u_1 + \frac{1}{2} \sum_{m=-\kappa-1}^{0} \sum_{n\in\mathbb{Z}^*} \frac{n}{1-q^n} \int_0^1 e^{2\pi i(n-m)u_1}\ \mathrm{d}u_1\Big|_{q^0} = \\
&= -G_2(\tau) + \frac{1}{2} \sum_{m=-\kappa-1}^{-1} \frac{m}{1-q^m}\Big|_{q^0}
\end{aligned}
\tag{5.136}
$$

In order to expand the last sum, and then take only the terms of order q^0, we have to notice that $|q| < 1$ and $m < 0$, hence

$$
\frac{m}{1-q^m} = m q^{-m} \frac{1}{q^{-m}-1} \simeq -m \sum_{l=0}^{\infty} q^{-(l+1)m}\ , \tag{5.137}
$$

from which it is clear that the term we are looking for vanishes. This means that

$$
F_k^{(2)}\big|_{q^0} = -G_2(\tau)\ . \tag{5.138}
$$

In conclusion, the contribution to the invariant of F_k, that is its value modulo $2\mathbb{Z}(\!(q)\!) + \mathrm{MF}_2$, is given by the sum of (5.133) and (5.138), which gives

$$F_k\big|_{q^0} = (\kappa+1)G_2(\tau)\big|_{q^0} = -\frac{k}{24}\ . \tag{5.139}$$

Let us comment on this result. We have computed only the terms in the expansion of F_k of order q^0, since the solution of the holomorphic anomaly equation is $2\widehat{F}_k$. So, since our invariant is defined modulo $2\mathbb{Z}(\!(q)\!)$, and F_k has integral q expansion (except, eventually, for the constant term), we have that the only possible obstruction for the equivalence class to be zero is given exactly by the constant term. In this way, the invariant we were looking for results to be

$$[f_1]_{S^3_k} = -\frac{k}{12} \mod 2\ . \tag{5.140}$$

Hence, the sigma model with target S^3 and Wess-Zumino coupling k is nullhomotopic, i.e. breaks supersymmetry, if and only if $-k/12 \in 2\mathbb{Z}$, that is

$$k \equiv 0 \mod 24\ . \tag{5.141}$$

Conclusion

In this thesis we have described a new deformation invariant for $(1+1)$-dimensional supersymmetric field theories. The motivation for the construction of the invariant comes from the Stolz and Teichner conjecture, which state that the spectrum TMF defined in section 3.1 is equivalent to the spectrum SQFT of supersymmetric field theories defined in section 3.2. The first invariant expected from this conjecture is the well-known *Witten genus*. However, the Witten genus, seen as a map from TMF to modular forms, has a non-vanishing kernel and cokernel. The kernel in particular contains all the torsion classes of TMF. For this reason it is necessary to find some "secondary" invariant able to classify the elements of the kernel. Following [GJ19] we have defined one of these invariants. We have first looked at a deformation of a theory $\mathcal{B}$ of degree n as a non-compact theory of degree $n+1$. Then we have studied how the properties of the Witten genus change in the non-compact case. This, in particular, brought us to find an equation for the holomorphic anomaly of the Witten genus (4.2), and then studying the integrality condition of its q-expansion, finding the result in (4.8). From these conditions we have defined the invariant (4.13).

Then, in chapter 5, we applied this construction to a particular case, that is the sigma model with target S^3 and Wess-Zumino coupling k. In this case, we have found that the model spontaneously breaks supersymmetry only whether $k \equiv 0 \mod 24$, as expected. Indeed, in chapter 2, we have performed a sequence of transformations we have identified as *flowing up and down the RG trajectories*, finding that a sufficient condition for the model to break supersymmetry was to have a coupling multiple of 24.

This invariant, though, captures only some of the torsion classes in the space of supersymmetric quantum field theories. As underlined by [GJ19], indeed, if we consider the TMF classes represented by the group manifolds $\mathrm{Sp}(2)$, G_2 and $\mathrm{G}_2 \times \mathrm{U}(1)$, they are non-zero, in fact they have orders 3, 2 and 2 respectively. Both the elliptic genus and the invariant defined in chapter 4 vanish in these cases, but, due to the conjectured relation between TMF and SQFT, we expect that those theories are not nullhomotopic. From here the failure of the secondary invariant introduced above to capture all the torsion classes follows. For this reason the construction we have described in this thesis has to be refined in order to obtain a complete set of invariants, which can distinguish between the elements in the different elements in the kernel of the elliptic genus map.

Despite these limits, there are some possible ways to enlarge the range of application of the conjecture:

1. the first one consist in actually restricting the set of possible deformation allowed, but in a useful way for applications. Indeed, we have considered only supersymmetry-preserving deformations. But, at this point, a natural question arises, that is what happens if we consider deformation that preserve supersymmetry and a flavor symmetry with symmetry group given by G. In order to answer to this question, we need to focus on a different space of supersymmetric field theories, invariant also under the symmetry group G, let us say SQFT_G. Moreover, we should consider a G-equivariant version of TMF, let us say TMF_G. Unfortunately, this set has not been properly studied yet in the mathematical literature (as explained in [Guk+18]), and it would be interesting to do so;

2. the other possible generalization consists in increasing the dimension of the theory, or enhancing supersymmetry. The problem in this case would rely on the fact that, even if a ring structure on the set of theories can always be defined, we are not sure whether this ring has also the structure of a spectrum. Moreover, requiring the manifold on which we define our model to have a string structure is not enough. So far, a conjecture has been proposed only in the 6-dimensional case. The idea, underlined in [Guk+18], comes from string theory, in which, in order to cancel the anomalies (analogously to what we have done in 2.2.1) a new structure has to be imposed, that is the *fivebrane structure* (for details look at [SSS09] and [Guk+18]). So, the idea is that, in 6-dimensional theories, imposing the fivebrane structure on the 6-manifold on which the theory is defined, we can perform an analogous construction to the one described in this thesis. However, neither this idea has been properly studied yet.

Appendix A

On the Brown's representability theorem

We have used as a crucial result the so-called *Brown's representability theorem*, which ensures us that any cohomology theory H can be represented by a sequence of spaces $\{E_n\}$ which forms an Ω-spectrum. We will not prove this theorem[1], however we will explain how the structure of Ω-spectrum arises. Before giving this justification, we need to give a basic construction in algebraic topology known as *suspension*.

Definition A.0.1. Given a topological space X, the **suspension** SX is the quotient $X \times [0,1]$ obtained collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point.

An easy example that helps to understand the motivation for this construction is given by the n-sphere $X = S^n$. In this case it is clear that $SX = S^{n+1}$, where the two *suspension points* are identified with the poles $(0, \dots, 0, \pm 1)$ of S^{n+1}.

In the case we need to work with basepointed spaces, the suspension gives us an ambiguity in the choice of its basepoint, which can be chosen arbitrarily in an entire segment. For this reason it is convenient to introduce the following definition

Definition A.0.2. Given a topological space X with basepoint x_0, we define the **reduced suspension** ΣX as the basepointed space

$$\Sigma X := \frac{SX}{\{x_0\} \times [0,1]} \, ,$$

with basepoint given by the image of $\{x_0\} \times [0,1]$ in ΣX.

For the reduced suspension holds the following adjoint relation, that is, given two topological spaces X, Y with basepoints, we have

$$[\Sigma X, Y] = [X, \Omega Y] \, .$$

[1] The complete proof of Brown's representability theorem can be found in [Hat02].

This adjoint relation holds since a basepoint preserving map $\Sigma X \to Y$ and a basepoint preserving map $X \to \Omega Y$ are identified associating to the map $f : \Sigma X \to Y$ the family of the loops obtained by restricting f to the images of the segments $\{x\} \times [0,1]$ in ΣX.

Let us apply what we have said to the case of Brown's representability theorem. Let H be a generalized cohomology theory and $\{E_n\}$ the associated sequence of spaces such that

$$\mathrm{H}^n(X) = [X, E_n] \ .$$

We want to show that there exist a weak homotopy equivalence

$$E_n \longrightarrow \Omega E_{n+1} \ .$$

From the exactness of the sequence in the axiom 3 on page 49, it follows that there exists the *suspension isomorphism*

$$\mathrm{H}^n(X) \simeq \mathrm{H}^{n+1}(\Sigma X) \ , \tag{A.1}$$

In order to prove the validity of this isomorphism, let us consider the case of the unreduced suspension, for which it is easier to deduce some topological properties. It is clear, however, that we can replace SX with ΣS without any difficulties. The unreduced suspension SX can be seen as the union of two cones CX and $C'X$ such that $CX \cap C'X = X$. Indeed, a cone is generally defined as

$$CX := \frac{X \times [0,1]}{X \times \{0\}} \ .$$

So, let us apply the axiom 3 to $\{\mathrm{pt}\} \subseteq X \subseteq CX$, that is

$$\ldots \longrightarrow 0 \longrightarrow \mathrm{H}^n(X, \{\mathrm{pt}\}) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(CX, X) \longrightarrow 0 \longrightarrow \ldots \ , \tag{A.2}$$

where we have used the fact that the cohomology groups $\mathrm{H}^n(CX, \{\mathrm{pt}\})$ are trivial, since CX is *contractible*, that is it has the same homotopy type of the point. Now the isomorphism (A.1) follows from the diagram

$$\begin{tikzcd}
\mathrm{H}^n(X,\{\mathrm{pt}\}) \arrow[r, "\delta_n", "\sim"'] \arrow[ddr] & \mathrm{H}^{n+1}(CX,X) \\
 & \mathrm{H}^{n+1}(SX,C'X) \arrow[u, "f", "\wr"'] \arrow[d, "g"', "\wr"] \\
 & \mathrm{H}^{n+1}(SX,\{\mathrm{pt}\}) \ .
\end{tikzcd}$$

where δ_n is an isomorphism due to the exactness of the sequence in (A.2). Let us explain where the isomorphisms f and g come from. Recalling the axiom 4 on page 49, choosing $Y = SX$ and $A = CX$, $B = C'X$, which of course respect the properties required in the axiom, we obtain directly

$$\mathrm{H}^n(SX, C'X) \xrightarrow[\sim]{f} \mathrm{H}^n(CX, CX \cap C'X) = \mathrm{H}^n(CX, X) \ .$$

The isomorphism g, instead, exists due to the fact that $C'X$ is contractible. Hence we conclude.

The isomorphism (A.1) corresponds to the natural bijection

$$[X, E_n] \simeq [\Sigma X, E_{n+1}] = [X, \Omega E_{n+1}] \ ,$$

which we call Γ. Thanks to this bijection, for all the $f : X \to E_n$, we have the following commutative diagram

$$\begin{tikzcd}
{[E_n, E_n]} \arrow[r, "\mathrm{H}^n(f)"] \arrow[d, "\Gamma"] & {[X, E_n]} \arrow[d, "\Gamma"] \\
{[E_n, \Omega E_{n+1}]} \arrow[r, "\mathrm{H}^n(f)"'] & {[X, \Omega E_{n+1}]}
\end{tikzcd} \ . \tag{A.3}$$

Let us define now the map

$$s_n := \Gamma(\mathrm{id}) : E_n \longrightarrow \Omega E_{n+1} \ ,$$

and, thanks to the commutativity of (A.3), we obtain, for all $f : X \to E_n$

$$\Gamma(f) = \Gamma f^*(\mathrm{id}) = f^*\Gamma(\mathrm{id}) = f^*(s_n) = s_n f \ ,$$

or, in other words, we have that the morphism

$$\Gamma : [X, E_n] \longrightarrow [X, \Omega E_{n+1}]$$

acts by composition with s_n. Since Γ is a bijection, if we choose $X = S^i$, we find that s_n induces an isomorphism on $\pi_i(E_n)$ for all i, that is

$$\Gamma : [S^i, E_n] \equiv \pi_i(E_n) \xrightarrow{\sim} [S^i, \Omega E_{n+1}] \equiv \pi_i(\Omega E_{n+1}) \ .$$

We conclude that s_n is a weak homotopy equivalence.

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