Università degli studi di Padova

DIPARTIMENTO DI FISICA E ASTRONOMIA

Master Degree in Physics

Arising of the moduli space of Riemann surfaces from the study of topological strings

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Contents

1 Introduction

Throughout the history of science, the interplay between mathematics and physics has yielded profound discoveries, reshaping our understanding of the natural world. At its core there is the recognition that the description of natural phenomena often demands the elegant language of mathematics and the creation of novel mathematical constructs in order to be able to predict the behaviour of nature.

From Isaac Newton's revolutionary equations, which described the laws of motion, to Albert Einstein's formulation of general relativity, mathematics has stood as the indispensable tool for physicists seeking to grasp the essence of reality. Yet, the connection between these disciplines is not one-way. In fact there are plenty of historical examples of the converse: the pursuit of deeper insights into physics has also sparked the development of new mathematical concepts, enriching the mathematical landscape in unexpected ways.

This thesis tries to go deeper on one such example. The quest to describe quantum gravity has posed one of the most formidable challenges for physicists, spanning from the formalization of the standard model and of general relativity to the present day. One of the approaches in order to achieve this has been string theory a theory that offers the potential to unite gravity and the standard model within a quantum field theory framework. However, the consistency of string theory remains unproven, and its most significant achievements have been within the realm of mathematics.

In the realm of mathematics, a fundamental objective is to describe topological spaces and answer a critical question: can we determine if two topological spaces are fundamentally the same? In mathematical terms this is formalized by the existence of a homeomorphism between them. While this is a challenging task for general spaces, mathematics provides a potent tool known as topological "invariants", which can immediately determine whether two spaces are distinct. For instance, in the case of two-dimensional topological surfaces without boundaries, there exists just one invariant, which allows us to distinguish all possible topological surfaces: the genus g.

However, deeper structures than topology, such as symplectic or Kähler structures, come into play. In such cases, we seek spaces to be equivalent only if there exist homeomorphisms preserving these additional structures. This is where quantum field theory plays a crucial role, particularly in the study of a non-linear σ model between a two-dimensional surface and symplectic (though, we will primarily focus on Kähler) target spaces. This leads to the discovery that correlation functions of this theory serve as invariants of the target spaces, and give us a recipe of plenty of new invariants.

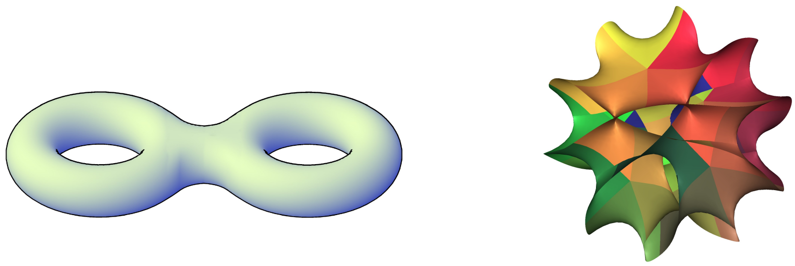

Figure 1: An example of a two dimensional Riemann surface $g = 2$, and a 2D slice of a Calabi Yau manifold, which in particular is a Kähler manifold

This exploration is not only fascinating from a mathematical point of view but also holds profound significance in the world of physics. At the heart of our discussions there are in fact topological string theories, a simplified setting of string theories. These theories offer a valuable framework to model and understand properties of ordinary string theory and derive results in related fields.

This work will follow a path that primarily focuses on the physical journey leading to the discovery of correlation functions as invariants. Chapter 1 will introduce the physical framework, which includes supersymmetric theories, notation, and the relevant symmetries. In Chapter 2, the concept of topological twisting will be presented, allowing the consideration of supersymmetric theories on curved worldsheets and demonstrating the topological nature of the theory, independent of the worldsheet's metric.

Chapters 4-5 will be dedicated to the formalization of topological field theories and their properties, enabling the computation of correlation functions using recursion relations based on three-point functions. The A-twist will be discussed as an example of these theories. In this context, we will find that the supersymmetric charge aligns with the De Rham's operator of the target space, and the computation of correlation functions becomes an integration over the fibers of the map from the moduli space of stable maps, $\mathcal{M}_{g,n}(X)$, to the moduli space of Riemann surfaces, $\mathcal{M}_{g,n}$. This will showcase the relevance of the moduli space in the com-

putation of correlation functions.
Up to now the worldsheet metric was not considered as a physical degree of freedom, however, there is the possibility to include it entering the realm of topological strings. Since the moduli space of stable maps is very complicated to deal with, we will invert the setting we were working on: the only degree of freedom will be the metric and physical observables will be cohomology classes on the moduli space of Rieman surfaces, $\mathcal{M}_{g,n}$.
Thus correlation functions will be intergrals of cohomology classes over this moduli space.
Hence in Chapter 6 we will introduce $\mathcal{M}_{g,n}$ more formally and will give a some examples of cohomology classes. In Chapter 7 we will discuss the importance of the model from physical point of view and finally Chapter 8 will offer examples for computing correlation functions and hints at the existence of recursion relations for generic g, demonstrating the far-reaching potential of this exploration.

This thesis aims to provide a comprehensive understanding of the quite recent discoveries between mathematics, physics, and topological string theories, paving the way for further inquiries at that are still open problems of this topics.

2 $\mathcal{N} = (2, 2)$ supersymmetry

In the first sections of this work we give the physical setting and the motivations to reach the study of the moduli space of stable maps.
At the heart of this discussion there is the study of σ-models displaying supersymmetry with $\mathcal{N} = (2, 2)$ supercharges.
There are several reasons to deal with supersymmetric theories

- localization/deformation invariance which will play a crucial role;
- path integral measure behaves better under fields re-definitions;
- physical operators should be invariant under supersymmetric transformations. We will see that supercharges, that are the generators supersymmetric transformations, will correspond to De Rham operator.
  Thus physical operators will correspond to elements of De Rham cohomology ring.

In particular the last statement is what we are interested into, in fact the possibility to recover De-Rham's cohomology means that we can learn topological information by studying this type of supersymmetric theories.

2.1 The model

We use this section mainly to introduce our notation and the basic notions in order to reach topological field theories which will be fundamental for our discussion.
The physical model is the one related to non-linear σ-models, in particular 2-dimensional ones. This means that we will choose a target manifold M, of dimension d (we will only consider complex manifolds of complex dimension d).
The bosonic fields that we will consider will be the continuous maps

$$\phi : \Sigma \to M$$

where Σ is our $2d$-worldsheet.
We will be mostly interested in theories of this type adding also supersymmetry. This means that we will have to introduce odd fields and supercharges to our discussion, which are better described in the context of superspace, which is an enlargement of ordinary space-time, by adding additional odd coordinates.

2.2 Superspace

We will start our discussion from the case where $\Sigma = \mathbb{R}^2 \cong \mathbb{C}$, thus since we need 2 supercharges we have to add to our complex plane coordinates $(z, \bar{z})$, four fermionic coordinates $\theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-$, so that we have superspace coordinates $\tilde{\Sigma}$ given by

$$(z, \bar{z}, \theta^{\pm}, \bar{\theta}^{\pm})$$

We have to specify how these coordinates behave under $SO(2) \cong U(1)$

$$\begin{aligned} z &\mapsto e^{i\alpha} z \\ \bar{z} &\mapsto e^{-i\alpha} z \\ \theta^{\pm} &\mapsto e^{\pm i\alpha/2} \theta^{\pm} \\ \bar{\theta}^{\pm} &\mapsto e^{\pm i\alpha/2} \bar{\theta}^{\pm} \end{aligned} \tag{1}$$

Note: We are dealing here with euclidean space, but looking back at the physics we know that our space time should be Lorentzian. We will use (x_0, x_1) as the physical coordinates while $z = x_1 + ix_0$ as the coordinates on $\mathbb{C}$ considering ix_0 as the euclidean time.

Let us make some definitions

- Superfields: these are functions defined on the superspace, they can be Taylor expanded in terms of $\theta^{\pm}$, $\bar{\theta}^{\pm}$, in this case we have 16 terms in this expansion;
- Supercharges: these are the 4 odd differential operators defined on the superspace characterizing the $\mathcal{N} = (2,2)$ supersymmetry

$$Q_{\pm} = \frac{\partial}{\partial\theta^{\pm}} + i\bar{\theta}^{\pm}\partial_{\pm} \qquad \bar{Q}_{\pm} = -\frac{\partial}{\partial\bar{\theta}^{\pm}} - i\theta^{\pm}\partial_{\pm}$$

  Satisfying

$$\{Q_{\pm}, \bar{Q}_{\pm}\} = -2i\partial_{\pm}$$

  Where

$$\partial_{+} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^0}\right) = \partial_z \qquad \partial_{-} = \frac{1}{2}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^0}\right) = \partial_{\bar{z}}$$

- We also define other differential operators, $D_{\pm}, \bar{D}_{\pm}$, which anticommute with the supercharges defined by

$$D_{\pm} = \frac{\partial}{\partial\theta^{\pm}} - i\bar{\theta}^{\pm}\partial_{\pm} \qquad \bar{D}_{\pm} = -\frac{\partial}{\partial\bar{\theta}^{\pm}} + i\theta^{\pm}\partial_{\pm}$$

  such that

$$\{D_{\pm}, \bar{D}_{\pm}\} = 2i\partial_{\pm}$$

Definition 2.1. A superfield Φ is chiral ($\bar{\Phi}$ is antichiral) if it satisfies the two conditions $\bar{D}_\pm\Phi = 0$ ($D_\pm\bar{\Phi} = 0$)

From now we will discuss about chiral fields. We may note that the constraint $\bar{D}_\pm\Phi$ heavily constrains Φ in particular, let us define (using the notation $x^+ = z$, $x^- = \bar{z}$)

$$y^\pm = x^\pm - i\theta^\pm\bar{\theta}^\pm$$

then

$$\bar{D}(y^\pm) := \Big(-\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\partial_\pm\Big)(x^\pm - i\theta^\pm\bar{\theta}^\pm) = i(\theta^\pm - \theta^\pm) = 0$$

So Φ must be of the form

$$\Phi = \phi(y^\pm) + \theta^\alpha\psi_\alpha(y^\pm) + \theta^+\theta^- F(y^\pm)$$

that we can further expand in terms of the θs, this means that there are a total of 4 degrees of freedom.

These fields are components of the supermultiplet, ϕ is the bosonic one and is a scalar field, ψ_α are spinor fields, while F are scalar fields with no kinetic terms which can be easily integrated out once specified the action S.
Chiral fields also have the property that their variation under supercharges is still a chiral field, this is due to the fact that $\{Q_\pm, \bar{D}_\pm\} = 0$ and $\{\bar{Q}_\pm, \bar{D}_\pm\} = 0$. This implies that we can give the explicit variation of the fields under supersymmetric transformations

$$\delta\Phi = i[\underbrace{\epsilon_+ Q_- - \epsilon_- Q_+ - \bar{\epsilon}_+\bar{Q}_- + \bar{\epsilon}_-\bar{Q}_+}_{\epsilon Q}, \Phi]$$

So that we obtain the variations

$$\begin{cases} \delta\phi = i\epsilon_+\psi_- - i\epsilon_-\psi_+ \\ \delta\psi_\pm = \mp 2\bar{\epsilon}_\mp\partial_\pm\phi + i\epsilon_\pm F \\ \delta F = 2\epsilon\partial_-\psi_+ + 2\bar{\epsilon}_-\partial_+\psi_- \end{cases}$$

We are now ready to build supersymmetric actions invariant under the action of the supercharges.
It is quite easy to see that under the action of the supercharges any term of the following type are invariant,

$$\int d^2x d^4\theta K(F_i) = \int d^2x d\theta^+ d\theta^- d\bar{\theta}^- d\bar{\theta}^+ K(F_i)$$

where K is an arbitrary differentiable function of some generic superfields F_i. This can be shown for every of the four supercharges, for example

$$\int d^2xd^4\theta Q_\pm\big(K(F_i)\big) = \int d^2xd^4\theta\Big(\frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm\partial_\pm\Big)\big(K(F_i)\big)$$

The first term vanishes since $d^4\theta$ selects the terms with top θs while the derivative never allow it to exists, while the second term is a total derivative after integrating $d^4\theta$. We will call this type of action D-term, and we will use them to build kinetic terms.

Another type is the F-term, which is a functional of some chiral superfields Φ_i

$$\int d^4xd^2\theta W(\Phi_i) = \int d^2xd\theta^+d\theta^- W(\Phi_i)|_{\bar{\theta}^\pm=0}$$

where $W(\Phi_i)$ is a holomorphic function of the Φ_is.

We can show that this is invariant under supersymmetric transformations, in fact under $Q_\pm$

$$\int d^2xd\theta^+d\theta^- Q_\pm\big(W(\Phi_i)\big)|_{\bar{\theta}^\pm=0} = \int d^2xd\theta^+d\theta^-\Big(\frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm\partial_\pm\Big)\big(W(\Phi_i)\big)|_{\bar{\theta}^\pm=0}$$

the first term as before vanishes, while second term vanishes since computed at $\bar{\theta}^\pm = 0$.
Instead under $\bar{Q}_\pm$ we should note that $\bar{Q}_\pm = \bar{D}_\pm - 2i\theta^\pm\partial_\pm$, then

$$\int d^2xd\theta^+d\theta^-\bar{Q}_\pm\big(W(\Phi_i)\big)|_{\bar{\theta}^\pm=0} = \int d^2xd\theta^+d\theta^-(\bar{D}_\pm - 2i\theta^\pm\partial_\pm)\big(W(\Phi_i)\big)|_{\bar{\theta}^\pm=0}$$

the first term vanishes since Φ^i is chiral and W is a holomorphic function of them, while the second term vanishes since is a total derivative, these type of therms will be potential terms for our theory.

Thus we can build our first superysmmetric action for a chial field

$$S = S_{kin} + S_W = \int d^2xd^4\theta\bar{\Phi}\Phi + \int d^2xd^2\theta W(\Phi) + c.c.$$

There are further global symmetries to this action, called R-symmetries acting on generic superfields F (q_V and q_A are the corresponding conserved quantum numbers)

$$\begin{aligned} \text{vector } e^{i\alpha F_V} &: F \mapsto e^{i\alpha q_V}F(x^\mu, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar{\theta}^\pm) \\ \text{axial } e^{i\beta F_A} &: F \mapsto e^{i\beta q_A}F(x^\mu, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\bar{\theta}^\pm) \end{aligned} \tag{2}$$

in our specific case we have that both $d^4\theta$ and $d^2\theta$ are invariant under axial rotations so that by assigning $q_A = 0$ charge to Φ give us axial symmetry.
Instead for the vector rotation $d^4\theta$ is invariant while $d^2\theta \mapsto e^{-2i\alpha}d^2\theta$ so that S_W is invariant only if W is monomial

$$W(\Phi) = c\Phi^k \mapsto ce^{ik\alpha q_V}\Phi^k$$

so that the symmetry is preserved when $k = 2/q_V$.

After expanding our action S we obtain that

$$\begin{aligned} S = \int d^2z 2\partial_z\phi\partial_{\bar{z}}\bar{\phi} + 2\partial_{\bar{z}}\phi\partial_z\bar{\phi} + 2i\bar{\psi}_-\partial_z\psi_- + 2i\bar{\psi}_+\partial_{\bar{z}}\psi_+ + \\ + |F|^2 + W'(\phi)F - W''(\phi)\psi_+\psi_- + c.c. \end{aligned}$$

That we can refine by integrating out the auxiliary fields $F, \bar{F}$.

We are mainly interested in theories involving only kinetic terms, where only the first line of this action appears, these type of theories are called *σ-models*.
Instead, if we add potential terms the theory gets the name of *Landau-Ginzburg* model or LG-model.

2.3 The algebra

We should also give an operator point of view, regarding them as acting on the superspace and giving the commutation relations between them.
Obviously to our list of operators we should not forget the generators of the Poincaré algebra, that in our euclidean setting is given by

- The generator of time translations H

$$H = -i\frac{\partial}{\partial(ix^0)} = -i(\partial_+ - \partial_-)$$

- The generator of space translations

$$P = -i\frac{\partial}{\partial x^1} = -i(\partial_+ + \partial_-)$$

- The generator of $U(1)$ rotations

$$M = 2z\partial_z - 2\bar{z}\partial_{\bar{z}} + \theta^+\partial_{\theta^+} - \theta^-\partial_{\theta_-} + \bar{\theta}^+\partial_{\bar{\theta}^+} - \bar{\theta}_-\partial_{\theta_-}$$

These operators already satisfy the natural algebra

$$[H, P] = 0 \qquad [H, M] = 2P \qquad [P, M] = 2H$$

If we now include our supercharge operators it is quite obvious that they commute with H, P. However with M we get

$$[M, Q_\pm] = \mp Q_\pm \qquad [M, \bar{Q}_\pm] = \mp \bar{Q}_\pm$$

At last we can compute

$$\{Q_\pm, \bar{Q}_\pm\} = P \pm H \tag{3}$$

This equation will be useful later and characterizes how worldsheet derivatives are linked to operators.
We can also give an operator point of view for R-charges. Looking at how they act on superspace coordinates we have

$$F_V = -\theta^+ \frac{\partial}{\partial\theta^+} - \theta^- \frac{\partial}{\partial\theta^-} + \bar{\theta}^+ \frac{\partial}{\partial\bar{\theta}^+} + \bar{\theta}^- \frac{\partial}{\partial\bar{\theta}^-}$$
$$F_A = -\theta^+ \frac{\partial}{\partial\theta^+} + \theta^- \frac{\partial}{\partial\theta^-} + \bar{\theta}^+ \frac{\partial}{\partial\bar{\theta}^+} - \bar{\theta}^- \frac{\partial}{\partial\bar{\theta}^-}$$

Thus we get the other relations that will be fundamental later

$$\begin{aligned} [F_V, Q_\pm] &= Q_\pm & [F_A, Q_\pm] &= \pm Q_\pm \\ [F_V, \bar{Q}_\pm] &= -\bar{Q}_\pm & [F_A, \bar{Q}_\pm] &= \mp \bar{Q}_\pm \end{aligned} \tag{4}$$

2.4 Non-linear case: Kähler target space

We would like to generalize the above discussion, to the non-linear case and by doing this we would like, at least locally, to preserve the $\mathcal{N} = (2, 2)$ supersymmetry for this generalized model. This imposes great constraints for our target space.

Thus we move to the case where the worldsheet manifold is not necessarily flat, and we study a theory with $\Phi^1, ..., \Phi^n$ chiral fields representing the local coordinates of out target manifold. We note that the kinetic term we discussed must be expressed in terms of a function of the superfields $K(\Phi, \bar{\Phi})$. We also want to recover a formulation for the standard bosonic σ model where we recover the kinetic term as $g_{IJ}\phi^I\phi^J$.
The only way of obtaining this is by requiring that the target manifold is Kähler and in particular the kinetic term is given by the Kähler potential

$$\mathcal{L}_{kin} = K(\Phi, \bar{\Phi})$$

By expanding this we obtain, in fact, the term

$$\frac{1}{2}g_{i\bar{j}}(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}}+\partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}})$$

To have correct signs we assume that $g_{i\bar{j}}$ to be positive definite.

In practice, in order to construct this theory we determine a Kähler metric on $\mathbb{C}^n$ which also endows it with a Levi-Civita connection on $T\mathbb{C}^n$. The kinetic lagrangian then becomes (after integrating out auxiliary field degrees of freedom)

$$\begin{aligned}\mathcal{L}_{kin} &= \int d^4\theta K(\Phi^i,\bar{\Phi}^{\bar{i}}) = \\ &= g_{i\bar{j}}(2\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}}+2\partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}}+2i\psi_-^{\bar{j}}D_z\psi_-^i+2i\psi_+^{\bar{j}}D_{\bar{z}}\psi_+^i)+R_{i\bar{j}k\bar{l}}\psi_+^i\psi_-^k\psi_+^{\bar{j}}\psi_-^{\bar{l}}\end{aligned} \tag{5}$$

where

$$D_z\psi_\pm^i=\partial\psi_\pm^i+\partial_z\phi^j\Gamma^i_{jk}\psi_\pm^k \tag{6}$$

We may note that the above expression is covariant in terms of the holomorphic coordinates of the target space (we may identify ϕ^i with z^i) so that we may identify this action as the action for the patch U_i covering a Kähler manifold.
In particular the action is invariant under Kähler transformations

$$K(\Phi^i,\bar{\Phi}^{\bar{i}})\mapsto K(\Phi^i,\bar{\Phi}^{\bar{i}})+f(\Phi^i)+\bar{f}(\bar{\Phi}^{\bar{i}}) \tag{7}$$

since the presence of mixed derivatives in the action cancels these terms.

We can apply this construction on different patches covering the target manifold M and then glue them together through coordinate and Kähler transformations. The model will be that one of an action for maps from the worldsheet to a Kähler target space manifold.

$$\phi:\Sigma\to X$$

where the spinorial fields $\psi_\pm,\bar{\psi}_\pm$ are sections of

$$\begin{aligned}\psi_\pm &\in \Gamma(\Sigma,\phi^*TM^{(1,0)}\otimes S_\pm)\\ \bar{\psi}_\pm &\in \Gamma(\Sigma,\phi^*TM^{(0,1)}\otimes S_\pm)\end{aligned} \tag{8}$$

So that the covariant derivative acting on them is given by the pull-back on the wordlsheet of the covariant derivative on M, exactly as in (6).

Remark: This is not a global formulation of the model, but just a formulation given on each patch of an open cover of M, thus we need to check supersymmetry on each open however attaching patches may give rise to problems on curved spaces.

Regarding R-symmetries we have that the above discussion generalizes to the case with n-chiral fields.

As before looking at the kinetic term we have that $d^4\theta$ is invariant under both vector and axial rotations and if $K(\Phi^i, \bar{\Phi}^{\bar{i}})$ depends only on $\Phi^i \bar{\Phi}^{\bar{i}}$ then the D-term is invariant whatever is the q_A, q_V charge assignment.

The potential term is invariant instead if $q_A^i = 0$ and if $W(\Phi^i)$ is a quasi-homogeneous function in the Φ's such that

$$W(e^{i\alpha q_V^i}\Phi^i) = e^{2i\alpha}W(\Phi^i)$$

We know that the classical level discussion is not enough to ensure the existence of symmetries also at the quantum level.
In order to discuss the symmetries at the quantum level we need to refer to some results given for non-linear σ models, with respect to R-currents.

- The vector symmetry is not anomalous, thus we only need to refer to the classical discussion
- Axial symmetry is anomalous thus in general q_A is not a good quantum number, it can be show that anomaly lies in the path integral measure [4]

  $$\mathcal{D}\psi\mathcal{D}\bar{\psi} \to e^{2ik\beta}\mathcal{D}\psi\mathcal{D}\bar{\psi}$$

  where k is given by

  $$k = \int_\Sigma c_1(\phi^* TM^{(1,0)}) = \int_\Sigma \phi^* c_1(TM^{(1,0)})$$

  thus we can see that when $c_1(M) = 0$, which means that M is Calabi-Yau, the axial symmetry is always preserved.

This implies that is much more easier to deal with CY non-linear sigma models.

Actually one can show that axial anomaly and RG-flow of the Kähler class (and hence the metric) of the target space are linked to each other. On CY target space we also have that the Kähler class do not flow, thus is believed that in this case the metric flows to a unique metric compatible with conformal invariance.

Remark: We only have discussed theories for chiral and anti-chiral fields, however is can generalize them to theories using twisted-chiral fields $(U, \bar{U})$. The kinetic term will still have the same properties, and R-symmetries conservation holds always for CY target spaces.

Remark: Up to now we have by no means discussed the manifold structure/topology of the world-sheet. This will be relevant later since it may modify the axial anomaly. Moreover this is the main problem to define supersymmetry on a curved woroldsheet.

2.4.1 Why is supersymmetry in general not preserved?

We have stated before that for this model supersymmetry is not, in general, preserved.
In fact in our definition of supersymmetric variations we have (for example)

$$\delta_{\epsilon_+} \mathcal{F}^i = i\epsilon_+ \{Q_-, \mathcal{F}\}$$

If we now have a non-flat metric on our worldsheet (in fact this can only be done locally, only at the level of the coordinate patch that we are discussing, not globally), we need to replace our worldsheet derivatives with covariant ones.
The problem now is: is ϵ_- always commuting with covariant derivatives?
The answer is in general no, in fact this happens only if ϵ_- is a covariantly constant spinor.
Since we do not want to impose a particular metric we would need a covariantly constant spinor with respect to all of them. This is impossible to require. Thus in general supersymmetry is not a symmetry of the action.
This is where twisting comes to help. In fact ϵ is not a spinor, but a scalar, of course we can choose ϵ to be constant.

3 Topological twisting

In this section we will explore the tool of topological twisting. This is useful in order to deal with one of the main problem we encountered in the previous section: the fact that supersymmetric non-linear sigma models were formulated just on open subsets of the worldsheet manifold.
Since we want to preserve at least some of the underlying supersymmetry we will consider some linear combinations of the supercharges, and consider Q to be one of them

$$Q_A = \bar{Q}_+ + Q_- \qquad Q_B = \bar{Q}_+ + \bar{Q}_-$$

note that $Q^2 = 0$.

We then consider the set of operators $\mathcal{O}$ of our theory. They can be seen as acting on the fields of our theory that are sections as in (8), thus operators have explicit dependence on the worldsheet coordinates.
In general we can define them as pairing between between forms (with values on the cotangent bundle of the target space) paired with cycles of the worldsheet, however we will be mainly interested in local ones, paired with points $p \in \Sigma$: $\mathcal{O}(p)$.
In particular we are interested in physical operators, this means Q closed ones: $[Q_\pm, \mathcal{O}] = 0$, $[\bar{Q}_\pm, \mathcal{O}] = 0$. Physically this idea comes from BRST quantization where physical operators are in one to one correspondence with cohomology classes of the BRST operators.

However we have seen that is not possible to preserve all the supercharges so let us consider just one supercharge (and its complex conjugate), so that physical operators, depending on theory, just refer to one of the supercharges $Q = Q_{A,B}$. Then physical operators correspond to elements of Q-cohomology classes.

Definition 3.1. An operator $\mathcal{O}$ is is called a chiral operator if $[Q_B, \mathcal{O}] = 0$, and twisted chiral if $[Q_A, \mathcal{O}] = 0$.

We can now work with them to show some important relations. By looking at the commutation relations in section (2.3)

$$\begin{aligned} \{Q_A, Q_+ - \bar{Q}_-\} &= \{\bar{Q}_+ + Q_-, Q_+ - \bar{Q}_-\} = 2H \\ \{Q_A, Q_+ - \bar{Q}_-\} &= \{\bar{Q}_+ + Q_-, Q_+ + \bar{Q}_-\} = 2P \end{aligned} \tag{9}$$

$$\begin{aligned} \{Q_B, Q_+ - \bar{Q}_-\} &= \{\bar{Q}_+ + \bar{Q}_-, Q_+ - Q_-\} = 2H \\ \{Q_B, Q_+ - \bar{Q}_-\} &= \{\bar{Q}_+ + \bar{Q}_-, Q_+ + Q_-\} = 2P \end{aligned} \tag{10}$$

Since we have seen that the worldsheet derivatives may be written as linear combinations of H and P, this implies that ∂_z and $\partial_{\bar{z}}$ are Q-exact.

Thus for any physical operator $\mathcal{O}$, we have that $\partial_z\mathcal{O}$ is Q-trivial and we should not consider it in our theory.

In physics correlation functions always play an important role, thus our main goal should be the computation of them. They are closely connected to the following definition

Definition 3.2. Given two operators $\mathcal{O}_1, \mathcal{O}_2$ that are Q-closed, then their product $\mathcal{O}_1\mathcal{O}_2$ is still Q-closed. Thus we can define the Q-cohomology ring as the chiral ring if $Q = Q_B$ and twisted chiral ring if $Q = Q_A$.

We can take a basis for the Q-cohomology group of operators given by $\{\mathcal{O}_i\}_{i=0}^{M}$ where M is the dimension of the ring. Then since we are looking at the Q-cohomology ring we can build the structure constants of it (up to Q-exact terms)

$$\mathcal{O}_i\mathcal{O}_j = C_{ij}^k\mathcal{O}_k + [Q, \Lambda] \tag{11}$$

Where the symmetry in the indexes i, j depends on the bosonic/fermionic nature of the operators $\mathcal{O}_i, \mathcal{O}_j$.
By defining $\mathcal{O}_0 = \mathbf{1}$ we can give a definition for the so called *topological metric*

$$C_{i0}^k = C_{0i}^k = \delta_j^k$$

Chiral rings will be fundamental in the computation of correlation functions for topological field theories, as we will show later.

3.1 Definition of twisting

The basic idea of twisting is to change the nature of the Q operator, making it a scalar, which is well defined on the whole worldsheet. This means that we need to change the nature of the fields on which Q acts.
This is done by changing the representation under which a certain field transforms under the group of rotations, by combining the generator of rotations with an R-symmetry generator.

Definition 3.3. Let M be the generator of rotations and let R be the generator of an R-symmetry.
Twisting consists in replacing the $U(1)_E$ group of euclidean rotations with the diagonal subgroup of $U(1)_E \times U(1)_R$ generated by $M' = M - R$, we will define

- A-twist: $R = F_V$
- B-twist: $R = F_A$

Clearly in order to make an R-twist we need to check that the R-symmetry is not broken, both at classical and at the quantum level, thus A-twist can be performed on a generic target space Kähler manifold, while B needs a CY target.

By changing the representation under which a field transforms we are also changing the bundle on which a field takes value as we are changing the principal bundle and thus the transition functions.
Consider a theory of a scalar superfield Φ, with R-charges $q_A = q_V = 0$, then

- ϕ is scalar under both $U(1)_E$ and $U(1)_R$ so that after the twisting is still scalar
- ψ_- is a section of S_+ thus it has M charge 1, moreover since Φ is invariant, looking at 2 it has $q_V = 1$, $q_A = -1$ thus after A-twist $q_M = 0$ and after B-twist $q_M = 2$. In term of representations this means that after A-twist it becomes a scalar, while after B-twist it becomes a section of $T\Sigma$.

Similarly we can generalize this to the case of a Kähler target space using $\psi^i\pm$ and $\bar{\psi}^{\bar{i}}_{\pm}$, where originally spinors are sections as in (8)

				A-twist		B-twist	
	$U(1)_E$	$U(1)_V$	$U(1)_A$	$U(1)'_E$	bundle	$U(1)'_E$	bundle
ψ_-	1	1	-1	0	$\phi^*(TM)$	2	$T\Sigma \otimes \phi^*(TM)$
$\bar{\psi}_-$	1	-1	1	2	$T\Sigma \otimes \phi^*(\overline{TM})$	0	$\phi^*(\overline{TM})$
ψ_+	-1	1	1	-2	$\overline{T\Sigma} \otimes \phi^*(TM)$	-2	$\overline{T\Sigma} \otimes \phi^*(TM)$
$\bar{\psi}_+$	-1	-1	-1	0	$\phi^*(\overline{TM})$	0	$\phi^*(\overline{TM})$

Table 1: Result of twisting

Note: Twisting does not change the Grassmann odd nature of the ψ fields, but now we have a global definition of some of them, since they are now scalars.

This has heavy consequences on the supercharges, in fact by looking at the commutation relations in (4) we get the following table

				A-twist		B-twist	
	$U(1)_E$	$U(1)_V$	$U(1)_A$	$U(1)'_E$	bundle	$U(1)'_E$	bundle
Q_-	1	1	-1	0	$\phi^*(TM)$	2	$T\Sigma \otimes \phi^*(TM)$
$\bar{Q}_-$	1	-1	1	2	$T\Sigma \otimes \phi^*(\overline{TM})$	0	$\phi^*(\overline{TM})$
Q_+	-1	1	1	-2	$\overline{T\Sigma} \otimes \phi^*(TM)$	-2	$\overline{T\Sigma} \otimes \phi^*(TM)$
$\bar{Q}_+$	-1	-1	-1	0	$\phi^*(\overline{TM})$	0	$\phi^*(\overline{TM})$

Table 2: Result of twisting for supercharges

This means that after twisting we change the “spin” of the supercharges. We can understand this as where the supercurrents take value (before twisting they were sections as in 8), or how the new supercharges act on the supersymmetric multiplet.
For example we know that Q_- sends ϕ to ψ_-, but after A-twist, ψ_- is a scalar, thus also Q_- is a scalar.

Remark: Note that $Q_A = \bar{Q}_+ + Q_-$ becomes a scalar after A-twist, same for $Q_B = \bar{Q}_+ + \bar{Q}_-$ after B-twist, thus a supersymmetric theory having them as supercharges is globally defined.

Another consequence of twisting (and in fact it can be formulated exactly in these terms) is the modification of the energy momentum tensor $T_{\mu\nu}$, this, given an action S, is defined at the classical level by the variation of the action by the worldsheet metric h

$$\delta S = \frac{1}{4\pi}\int \sqrt{h} d^2 z \delta h^{\mu\nu} T_{\mu\nu}$$

this relation can be generalized at the quantum level using correlation functions

$$\delta_h \langle \mathcal{O} \rangle = \left\langle \mathcal{O} \frac{1}{4\pi} \int \sqrt{h} d^2 z \delta h^{\mu\nu} T_{\mu\nu} \right\rangle$$

It happens in most supersymmetric theories, and this is the case for twisting, that the energy momentum tensor is Q-exact

$$T^{twisted}_{\mu\nu} = \{Q, G_{\mu\nu}\}$$

for some symmetric fermionic tensor $G_{\mu\nu}$.
It is quite hard to show what G really is, however the easiest way to think about

it is that G is the Q-superpartner of T. The twisted energy momentum tensor is defined to be

$$T_{\mu\nu}^{twisted} = T_{\mu\nu} + \frac{1}{4}(\epsilon_{\mu}^{\lambda}\partial_{\lambda}J_{\nu}^{R} + \epsilon_{\nu}^{\lambda}\partial_{\lambda}J_{\mu}^{R}) \tag{12}$$

Where $\epsilon_{\mu}^{\lambda} = \eta^{\mu\alpha}\epsilon_{\alpha\lambda}$ and J^R is the conserved R-current used for the twisting.
This implies that, given $\mathcal{O}_1, ..., \mathcal{O}_s$ physical operators, we have that the correlation function does not vary by variations of the worldsheet metric

$$\begin{aligned}\delta_h\langle\mathcal{O}_1...\mathcal{O}_s\rangle &= \left\langle\mathcal{O}_1...\mathcal{O}_s\frac{1}{4\pi}\int\sqrt{h}d^2z\delta h^{\mu\nu}\{Q, G_{\mu\nu}\}\right\rangle = \\ &= \left\langle\{Q, \mathcal{O}_1...\mathcal{O}_s\frac{1}{4\pi}\int\sqrt{h}d^2z\delta h^{\mu\nu}G_{\mu\nu}\}\right\rangle = 0\end{aligned}$$

since the last term is Q-exact.
This is the final justification for the term topological. We just have shown that any correlation function of physical operators is now independent on the metric of the worldsheet and thus only depends on the topology of the worldsheet, namely the genus if we restrict to the study of compact orientable surfaces without boundary.

Remark: The name “topological” may be misleading, in fact the theory is topological only with respect to the worldsheet metric, however the target space may still have great influence on it.
In particular one can be interested on the set of possible theories given the topology of the target space. In the case of CY target spaces this corresponds to the moduli space of the CY manifold, that divides into the *Kähler moduli* and the *complex structure moduli*, we will not go any further into this, but this means that the theory strongly depends on deeper structures than the topological one looking at the target space.

4 Topological field theories

We arrived before to the term topological field theory. Thus in this section we will provide a definition of them and how can we work out correlation functions in this setting.
We are mainly interested in 2-dimensional field theories where our base manifold is the wordlsheet Σ endowed with a metric choice h and where we do not consider the metric h dynamical (in the path integral we do not integrate over all possible metric configurations). Thus we consider our theory topological if correlation functions do not depend on h.

Note: this imposition together with coordinate invariance, which always is satisfied in physical context, heavily simply the theory. In fact we can always perform a coordinate change (for example we can act on our manifold by translations) and transform the metric accordingly. However in this case we can transform back the metric to its original value. This will result in just a change in position of the observables of the type

$$\langle \mathcal{O}_1(x_1)...\mathcal{O}_s(x_s)\rangle$$

So that it doesn't really matter where we insert them. This implies that when we compute correlation functions we should not bother about time ordering since we can always change the time insertion point.

There are several types of topological field theories arising from physical contexts. For example Chern-Simons ones, that explicitly avoid the introduction of the metric. However our case above is in the context of cohomological field theories in fact it satisfies

- The existence of a global fermionic symmetry such that $Q^2 = 0$, which is not spontaneously broken, so that the vacuum of the theory satisfies

$$Q|0\rangle = 0$$

- Physical operators are elements of the Q-cohomology ring.
- The energy-momentum tensor is Q-exact. In fact this is exactly how we have shown the independence of correlation functions from the metric before.

All of these requests are satisfied in our twisted theories so that we can apply some of the results of these theories to our examples.

To see how much this simplify correlation functions let us consider the following. A straight way of respecting the above requests is to take the lagrangian to be Q-exact

$$\mathcal{L} = \{Q, V\} \tag{13}$$

For some fermionic current V.
We can now compute correlation functions by reintroducing back the Planck constant $\hbar$

$$\frac{\partial}{\partial\hbar}\langle\mathcal{O}_1...\mathcal{O}_s\rangle = \frac{\partial}{\partial\hbar}\int\mathcal{D}\phi\mathcal{O}_1...\mathcal{O}_s e^{-\frac{1}{\hbar}S[\phi]} = \int\mathcal{D}\phi\mathcal{O}_1...\mathcal{O}_s\frac{1}{\hbar^3}\Big\{Q,\int_\Sigma V\Big\}^2 e^{-\frac{1}{\hbar}S[\phi]} = 0$$

since the computation of correlation functions containing Q-exact operators is always vanishing.
In practice we have seen that the correlation function do not depend on $\hbar$ so that we can take the classical limit $\hbar \to 0$ to compute them.

4.1 Correlation functions and chiral rings

In the case of two dimensional cohomological field theories we have another result that allows us to consider only Riemann surfaces without boundaries and to reduce the computation of correlation functions to the computation of the chiral ring.

Recall that there is a correspondence between state in the operator formalism and boundary conditions in the path integral formalism given by

$$\langle BC1|\mathcal{O}_1...\mathcal{O}_s|BC2\rangle = \int_{BC1}^{BC2}\mathcal{D}\phi\mathcal{O}_1...\mathcal{O}_s e^{-S[\phi]}$$

Where we omitted time ordering since we are dealing with a topological field theory. Explicitly this means that when we assign at a time t_1 a field configuration $f(t_1)$, the path integral lower extremum is that field configuration. In terms of field operators the boundary conditions are defined by

$$\phi(t_1)|BC1\rangle = f(t_1)|BC1\rangle$$

Since we may have linear combinations of boundary conditions a state will be specified by the coefficients of the linear combination specifying the state (in general we will have a distribution not a discrete sequence). This means that in the path integral formalism a boundary condition is an operator which specifies a number to each possible boundary condition on the fields.

Let us look at what in practice this means in the case of our two dimensional theory. In this case boundaries are circles we would like to know how to build an operator out of a boundary condition. Let us assume that the Hilbert space of field configurations is separable. Thus a boundary condition is given by a state

$$|BC1\rangle = \sum_n a_n |\psi_n\rangle$$

Then in the path integral formalism we can take a surface $\mathcal{S}$ with the shape of an hemisphere and define the operator $\mathcal{O}_a$ such that

$$\int_{\mathcal{S}}^{BC:\psi_n} \mathcal{D}\phi \mathcal{O}_a = a_n$$

Since we are dealing with a topological field theory we can take the hemisphere to be infinitely long, this is exactly what we define usually as an asymptotic state in computing S-matrix elements.
In this way we can take a boundary condition to be nothing but an insertion of a infinitely long cylinder ending with a hemisphere together with an operator inserted on it.
However, since our theory is topological, we are allowed to take a finite sized cylinder.
Using this formalism expectation values of operators will read.

$$\langle \mathcal{O}_a | \mathcal{O}_1 ... \mathcal{O}_s | \mathcal{O}_b \rangle$$

This can be pictured schematically as in figure (2)

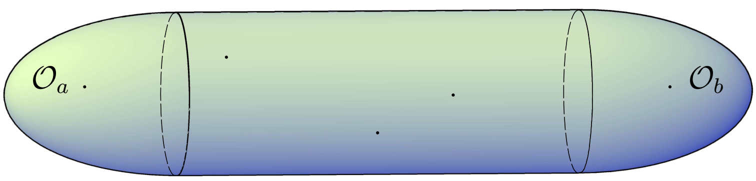

Figure 2: Expectation value by using operator-asymptotic state correspondence

In this way we can restrict the study of correlation functions to compact and orientable Riemann surfaces without boundaries by taking out boundaries and boundary conditions, and substituting them by attaching hemispheres and inserting operators on them.

Let us consider now correlation functions

$$\langle \mathcal{O}_1 ... \mathcal{O}_s \rangle$$

Then, since we are dealing with a topological field theory, we can insert an infinite cylinder between two operators, allowing only asymptotic states to propagate in the throat of the cylinder. This is equivalent to cutting the throat at the two bases of the cylinder and adding the same boundary conditions to the two cuts on the left ($\mathcal{O}_a$) and to the two cuts on the right ($\mathcal{O}_b$). This must be done with all possible boundary conditions each of them weighted by a factor η^{ab} that we will compute later.
Schematically this can be represented as in figure (3).
Thus we get the formula

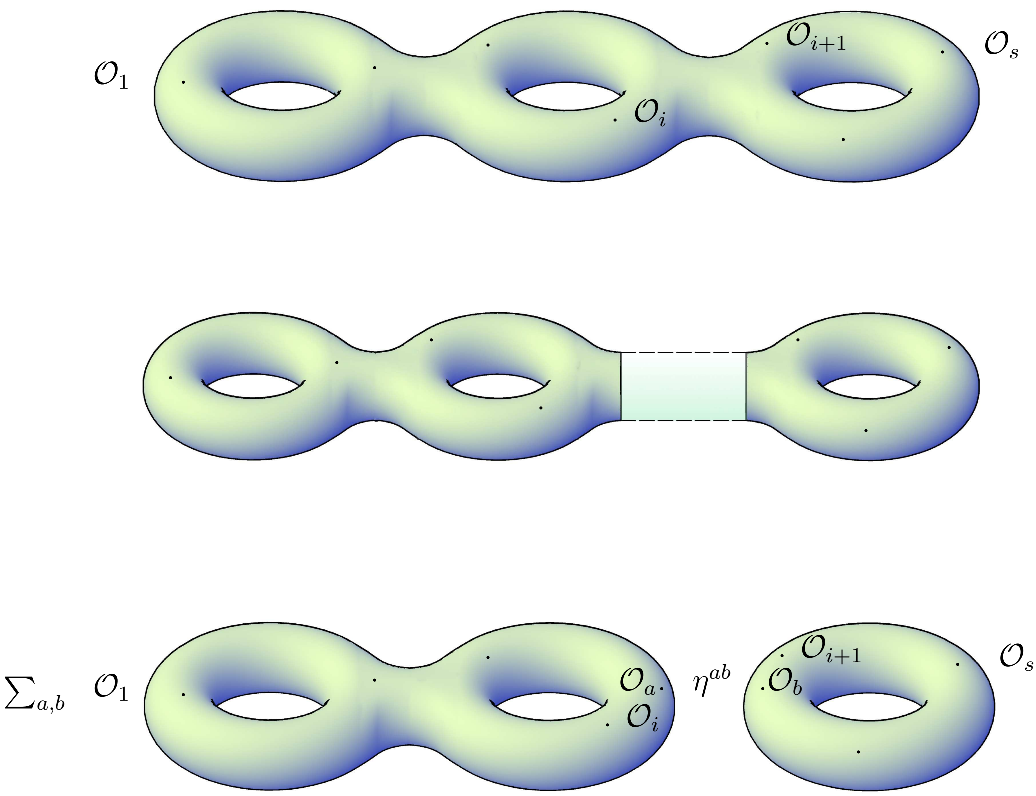

Figure 3: Surgery operation splitting genus

$$\langle \mathcal{O}_1 ... \mathcal{O}_s \rangle_\Sigma = \sum_{ab} \langle \mathcal{O}_1 ... \mathcal{O}_i | \mathcal{O}_a \rangle_{\Sigma_1} \eta^{ab} \langle \mathcal{O}_b | \mathcal{O}_{i+1} ... \mathcal{O}_s \rangle_{\Sigma_2} \tag{14}$$

where $g(\Sigma) = g(\Sigma_1) + g(\Sigma_2)$.
This property can also give us an answer for η^{ab}. In fact consider the two point function defined on the sphere $C_{ab} = \langle \mathcal{O}_a \mathcal{O}_b \rangle_0$, then by previous property we get that

$$C_{cd} = C_{ca} \eta^{ab} C_{bd}$$

Thus η^{ab} is nothing but the inverse of C_{ab}, that we will define as η_{ab} the *topological metric*.
An important thing we should ask ourselves is whether or not the operators we are summing on $(\mathcal{O}_a, \mathcal{O}_b)$ are physical. Of course the answer is not affirmative if we consider all asymptotic states, but unphysical ones do not give any contribution to the correlation function.
In fact let us take a complete basis of asymptotic states $|\mathcal{O}_A\rangle$, such that $\langle \mathcal{O}_A | \mathcal{O}_B \rangle = \eta_{AB}$, so that the completeness relation can be written as $\mathbf{1} = \sum_{AB} \eta^{AB} |\mathcal{O}_A\rangle\langle \mathcal{O}_B|$. Now we can divide the Hilbert space of asymptotic states into a direct sum

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$$

where $|\psi\rangle \in \mathcal{H}_0$ if $Q|\psi\rangle = 0$ and $\mathcal{H}_1 = \mathcal{H}_0^\perp$.
Then, since any state of the type $\mathcal{O}_1 ... \mathcal{O}_s |0\rangle$, satisfies $Q\mathcal{O}_1 ... \mathcal{O}_s |0\rangle = 0$, we have that $\mathcal{O}_1 ... \mathcal{O}_s |0\rangle \in \mathcal{H}_0$. Thus in the sum (14) all unphysical states do not contribute giving us no trouble in consider the correspondence between asymptotic physical states and operators.

Another simplifying formula can be deduced looking at picture (4). Note that we can always change the position of the operators so that they lay on the first of the two handles.
Take a handle on a genus g Riemann surface and extend it to infinite length, we can cut the throat obtaining two boundaries on which we impose boundary conditions $\mathcal{O}_a$ and $\mathcal{O}_b$ and summing over all the possible boundary conditions gives us the following

$$\langle \mathcal{O}_1 ... \mathcal{O}_s \rangle_g = \sum_{a,b} (-1)^F \eta^{ab} \langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_1 ... \mathcal{O}_s \rangle_{g-1}$$

Where F is the fermion number of $\mathcal{O}_a$ (and of $\mathcal{O}_b$).

Using iteratively this formula together with formula (14) give us the possibility to compute all correlation functions in terms of three point functions on the sphere. Define them as

$$\langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle_0 = C_{ijk}$$

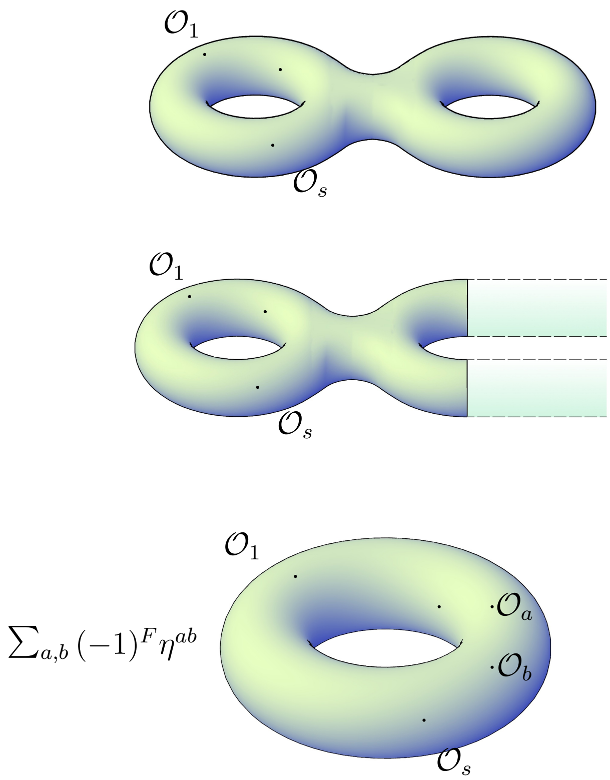

Figure 4: Surgery operation reducing the genus

Then by recalling the definition of chiral ring structure constants we get

$$C_{ijk} = \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle_0 = \langle \big(C^l_{ij} \mathcal{O}_l \mathcal{O}_k + \{Q, \Lambda\} \big) \mathcal{O}_k \rangle = C^l_{ij} \eta_{lk}$$

Which means that the chiral ring structure constants are related to three-point functions on the sphere simply by the use of the topological metric.

Finally we can conclude that in two-dimensional topological field theories all the information of topological observables (which are correlation functions) is enclosed in the chiral ring structure constants and in the computation of two point functions on the sphere.

5 A-twist example

In this section we actually perform the A-twist of the supersymmetric non-linear σ model, without introducing any potential term, $W = 0$, in our discussion.
We will use the tool of supersymmetric localization, which is a huge subject per se, we will just borrow some results.

Consider a supersymmetric theory and define $\mathcal{F}_Q$ to be the set of field configurations such that $\{Q, \phi\} = 0$. Then the path integral for the computation of correlation functions of physical operators receives contributions just from $\mathcal{F}_Q$. In particular we will see that in our case the path integral will localize to $\mathcal{M}_\Sigma$ the space of holomorphic maps.
There are some subtleties about this result, we actually need to add a factor, however in the case we will discuss below, this factor is just 1, so that we do not need to go deeper discussing localization.

Let us start with an action like in (5) and let us perform the A-twit using the results of Table 1. We see that ψ^i_- and $\bar{\psi}^{\bar{i}}_+$ are now scalars, while $\bar{\psi}^{\bar{i}}_-$ and ψ^i_+ are 1 forms. Thus we introduce the following notation to underline this

$$\begin{aligned} \chi^i &:= \psi^i_- \qquad & \chi^{\bar{i}} &:= \bar{\psi}^{\bar{i}}_+ \\ \rho^i_{\bar{z}} &:= \psi^i_+ \qquad & \rho^{\bar{i}}_z &= \bar{\psi}^{\bar{i}}_- \end{aligned}$$

We can rewrite the action and the supersymmetric transformations acting on the fields

$$S = 2\int d^2 z g_{i\bar{j}}\left(\partial_z\phi^i\partial_{\bar{z}}\phi^{\bar{j}} + \partial_{\bar{z}}\phi^i\partial_z\phi^{\bar{j}} + i\rho^{\bar{j}}_z D_{\bar{z}}\chi^i + i\rho^i_{\bar{z}} D_z\chi^{\bar{j}}\right) + \frac{1}{2}R_{i\bar{j}k\bar{l}}\rho^i_{\bar{z}}\rho^{\bar{j}}_z\chi^k\chi^{\bar{l}}) \tag{15}$$

The supersymmetric transformations can be easily found by imposing $\bar{\epsilon}_+ = \epsilon_- = 0$ (do not consider the i factor which just complicates the discussion)

$$\begin{aligned} \delta\phi^i &= \epsilon_+\chi^i \qquad & \delta\bar{\phi}^{\bar{i}} &= \bar{\epsilon}_-\chi^{\bar{i}} \\ \delta\chi^i &= 0 \qquad & \delta\chi^{\bar{i}} &= 0 \\ \delta\rho^i_{\bar{z}} &= 2i\bar{\epsilon}_-\partial_{\bar{z}}\phi^i + \epsilon_+\Gamma^i_{jk}\rho^j_{\bar{z}}\chi^k \qquad & \delta\rho^{\bar{i}}_z &= -2i\epsilon_+\partial_z\bar{\phi}^{\bar{i}} + \bar{\epsilon}_-\Gamma^{\bar{i}}_{\bar{j}\bar{k}}\rho^{\bar{j}}_z\chi^{\bar{k}} \end{aligned} \tag{16}$$

and by imposing $\epsilon_+ = \bar{\epsilon}_- = \epsilon$ one can also find their variation under $Q = Q_A$. In particular we can make the following identifications

$$\begin{aligned} \phi^i &\leftrightarrow z^i \qquad & \chi^i &\leftrightarrow dz^i \\ \bar{\phi}^{\bar{i}} &\leftrightarrow \bar{z}^{\bar{i}} \qquad & \chi^{\bar{i}} &\leftrightarrow d\bar{z}^{\bar{i}} \end{aligned}$$

which give us a hint to identify

$$Q_- \leftrightarrow \partial \qquad \bar{Q}_+ \leftrightarrow \bar{\partial} \qquad Q_A = \bar{Q}_+ + Q_- \leftrightarrow \partial + \bar{\partial} = d$$

This can be made rigorous by considering only physical operators (Q_A closed). Consider just operators associated to points $\mathcal{O}^{(0)}$ (and not to one-form $\mathcal{O}^{(1)}$ or two-form $\mathcal{O}^{(2)}$ ones).
These are 0-form operators on the worldsheet, thus we may take ϕ, χ, but we cannot take their derivatives, since they are Q-exact by (10). We could try to use ρ's, but since they are already one-forms we need to contract them with the metric, in fact let $\mathcal{O}_z$ and $\mathcal{O}'_{\bar{z}}$ be one-form physical operators, then

$$\mathcal{O}^{(0)} = h^{z\bar{z}}\mathcal{O}_z\mathcal{O}'_{\bar{z}}$$

This cannot be done if we want to extract only topological quantities, since we are introducing back the metric.
This implies that we can consider physical operators of the type

$$\omega_{i_1...i_p\bar{j}_1...\bar{j}_q}(\phi)\chi^{i_1}...\chi^{i_p}\chi^{\bar{j}_1}...\chi^{\bar{j}_q} \leftrightarrow \omega_{i_1...i_p\bar{j}_1...\bar{j}_q}(z)dz^{i_1}...dz^{i_p}d\bar{z}^{\bar{j}_1}...d\bar{z}^{\bar{j}_q} \tag{17}$$

thus Q_A cohomology classes of operators are identified with d-cohomology classes of differential forms.

$$\{\text{physical operators}\} \cong H^*_{DR}(X)$$

A useful representation of physical operators is given by dual homology cycles. Let N be a homology cycle, represented by a submanifold of codimension m, $N \in H_{n-m}(X)$, then its Poincaré dual $[N] \in H^n(X)$ and may be represented as a δ-function m-form supported on N. Let this operator be $\mathcal{O}_N$, let z be a point on Σ, then the 0-form operator, with respect to the worldsheet, is given by

$$\mathcal{O}^{(0)}(z) = \mathcal{O}_N(\phi(z))$$

so that the operator vanishes if $\phi(z) \notin N$.
We can also assign R-charges to physical operators, looking at table 1, and recalling the substitutions made in A-twist case an operator of the type (17), $\mathcal{O} \in H^{p,q}$ has R-charges

$$\begin{aligned} q_V &= q - p \\ q_A &= q + p \end{aligned} \tag{18}$$

these will be useful defining some selection rules for correlation functions.

5.1 Correlation functions

A generic correlation function of physical operators is given by the path integral

$$\langle \mathcal{O}_1 ... \mathcal{O}_s \rangle = \int \mathcal{D}\phi \mathcal{D}\chi \mathcal{D}\rho \, \mathcal{O}_1 ... \mathcal{O}_s e^{-S[\phi,\chi,\rho]}$$

Since we have to sum over all possible field configurations, we may classify the ϕ maps based on $\beta = [\phi(\Sigma)] = \phi_*(\Sigma) \in H_2(X)$, then

$$\langle \mathcal{O}_1 ... \mathcal{O}_s \rangle = \sum_{\beta \in H_2(X)} \langle \mathcal{O}_1 ... \mathcal{O}_s \rangle_\beta$$

$$\text{where } \langle \mathcal{O}_1 ... \mathcal{O}_s \rangle_\beta = \int_{\phi_*(\Sigma)=\beta} \mathcal{D}\phi \mathcal{D}\chi \mathcal{D}\rho \, \mathcal{O}_1 ... \mathcal{O}_s e^{-S[\phi,\chi,\rho]}$$

In order to compute them we will use the tool of supersymmetric localization discussed discussed briefly previously.

First of all looking at the the variations in (16) we have the fixed points of supersymmetry $\mathcal{F}_Q$ contain for sure the field configurations

$$\partial_{\bar{z}} \phi^i = 0 \qquad \partial_z \bar{\phi}^{\bar{i}} = 0 \qquad \chi^i = 0 \qquad \chi^{\bar{i}} = 0 \tag{19}$$

Thus it would be natural for the path integral to localize to field configurations where $\chi^i = \chi^{\bar{i}} = 0$ and such that the map ϕ is holomorphic, this maps can be defined univoquely once we have fixed a complex structure on Σ.

Since our theory should be topological from our previous discussion, we may try to look if it is possible to write the lagrangian in the same way as in (13).

$$\mathcal{L} = \{Q, V\}$$

For some fermionic current V, so that the theory is not only topological, but is also equivalent to the classical limit.
It turns out that we are almost able to do this, in fact one can write

$$\mathcal{L}' = \{Q, V\}$$

Where

$$V = g_{i\bar{j}}(\rho^i_{\bar{z}} \partial_z \phi^{\bar{j}} + \partial_{\bar{z}} \phi^i \rho^{\bar{j}}_z)$$

And

$$\mathcal{L} = \mathcal{L}' + \phi^*(K) \qquad K \text{ is the Kähler class on } X$$

Thus we have that correlation functions depend only on the Kähler class ω and get contributions only from homolorphic maps $\partial_z \phi = 0$. Since we have to integrate over all possible ϕ holomorphic, it is useful to consider the set

$$\mathcal{M}_\Sigma(X, \beta) = \{\phi : \Sigma \to X \text{ s.t. } \phi \text{ holomorphic } \phi_*(\Sigma) = \beta\} \tag{20}$$

We will assume that this set is smooth manifold, then our path integral localize to a finite dimensional integral over $\mathcal{M}_\Sigma(X, \beta)$, the moduli space of holomorphic maps od degree β for fixed Σ.

It remains now to give a prescription to evaluate correlation functions in this setting.
To know which is the dimension of $\mathcal{M}_\Sigma(X, \beta)$, we need to look at infinitesimal deformations of a holomorphic map

$$\phi^i \to \phi^i + \delta\phi^i \qquad \text{s.t.} \quad \partial_{\bar{z}}\delta\phi^i = 0$$

this implies that deformations of a holomorphic map ϕ corresponds to zero-modes of the operator $\partial_{\bar{z}}$. However,given an infinitesimal deformation, there may be an obstruction to integrating it in order to obtain a finite deformation. Obstructions lie in the co-kernel of $\partial_{\bar{z}}$ (which is the kernel of $\partial_{\bar{z}}^\dagger$), this means that the dimension of the moduli space is given by the index of $\partial_{\bar{z}}$.

Let us formalize this discussion. We know that ϕ^i fields are sections of $\phi^*(TX^{(1,0)}) \cong \Omega^{(0,0)}(\Sigma) \otimes \phi^*(TX)$.
Thus we can treat $\partial_{\bar{z}}$ as an elliptic operator in the complex

$$0 \to \Omega^{(0,0)}(\Sigma) \otimes \phi^*(TX) \to \Omega^{(0,1)}(\Sigma) \otimes \phi^*(TX) \to 0$$

Thus we can use Hirzebruch-Riemann-Roch theorem for the Dolbeaut complex to compute the index

$$\begin{aligned} k :&= \mathrm{ind}\partial_{\bar{z}} = \int_\Sigma \mathrm{ch}(\phi^* TX)\mathrm{td}(\Sigma) = \int_\Sigma (\mathrm{dim}X + \phi^* c_1(X))\big(1 + \frac{c_1(\Sigma)}{2}\big) = \\ &= c_1(X) \cdot \beta + \dim X(1-g) \end{aligned} \tag{21}$$

This corresponds to the complex dimension of $\mathcal{M}_\Sigma(X, \beta)$ and if we are integrating over it, we should integrate volume forms; this provides some selection rules that we need to respect in order to have non-zero correlation functions.
In fact, if we expand our action around ϕ holomorphic we see that

$$D_{\bar{z}}\chi^i = \partial_{\bar{z}}\chi^i + \underbrace{\partial_{\bar{z}}\phi^j}_{=0}\Gamma^i_{jk}\chi^k = \partial_{\bar{z}}\chi^i$$

Thus around these configurations, the zero modes contributions to the kinetic terms in the lagrangian correspond exactly to the $\partial_{\bar{z}}\chi^i = \partial_z\chi^{\bar{i}} = 0$ zero modes.
We would like to know, also, which is the number of ρ zero modes. This can be done by looking at our action. And since by considering (for example)

$$\begin{aligned} &g_{i\bar{j}}\rho_z^{\bar{j}}\partial_{\bar{z}}\chi^i = g_{i\bar{j}}(h^{z\bar{z}}\rho_z^{\bar{j}}\partial_{\bar{z}}\chi^i) = \\ &= g_{i\bar{j}}(\rho^{\bar{j}\bar{z}}\partial_{\bar{z}}\chi^i) \end{aligned}$$

we have that the ρ zero modes lie in the co-kernel of $\partial_{\bar{z}}$. And since

$$\mathrm{coker}(\bar{\partial}) \cong \ker(\bar{\partial}^\dagger)$$

we obtain that, for the considered term

$$k = (\#\chi \text{ zero modes}) - (\#\rho \text{ zero modes})$$

This number corresponds to the axial anomaly hidden in the path integral integration measure.
Note that ($\#\chi^i$ zero modes) = ($\#\chi^{\bar{i}}$ zero modes) and that ($\#\rho_z^{\bar{j}}$ zero modes) = ($\#\rho_{\bar{z}}^j$ zero modes), so that the integration measure gets rotated by (looking at table (1)

$$\mathcal{D}\chi^i\mathcal{D}\chi^{\bar{i}}\mathcal{D}\rho_z^{\bar{j}}\mathcal{D}\rho_{\bar{z}}^j \to e^{2ik\alpha}\mathcal{D}\chi^i\mathcal{D}\chi^{\bar{i}}\mathcal{D}\rho_z^{\bar{j}}\mathcal{D}\rho_{\bar{z}}^j$$

This together with the non-anomalous vector R-charge, gives us some selection rules for correlation functions, in fact these must be invariant under R-symmetry action.
Thus let $\mathcal{O}^i \in H^{p_i,q_i}(X)$, then

$$q_V(\mathcal{O}_1 \ldots \mathcal{O}_s) = \sum_i q_i - \sum_i p_i = 0 \implies \sum_i q_i = \sum_i p_i$$

Axial symmetry instead since is anomalous, thus we get non-vanishing correlation functions only if

$$2k = q_A(\mathcal{O}_1 \ldots \mathcal{O}_s) = \sum_i p_i + \sum_i q_i$$

These two rules together give us

$$\sum_i p_i = \sum_i q_i = k$$

Let us consider now the case where there are no ρ zero modes, this is equivalent to ask that there are no obstructions to the integration of infinitesimal variations. In this case the path integral reduces to an integral over the moduli space, and

the localization procedure gives no extra factor, because of supersymmetry.
We need now to make sense of observables on $\mathcal{M}_\Sigma(X,\beta)$.
Let $\mathcal{O}^i \in H^{p_i,q_i}$ be inserted at $x_i \in \Sigma$, we can define the evaluation map

$$\begin{aligned} ev_i : \mathcal{M}_\Sigma(X,\beta) &\to X \\ \phi &\mapsto \phi(x_i) \end{aligned}$$

thus we can pullback observables $ev_i^*\mathcal{O}^i \in H^{p_i,q_i}(\mathcal{M}_\Sigma(X,\beta))$, thus our selection rules tell us nothing but correlation functions of operators are non vanishing only if the wedge product of operators are volume forms on the moduli space.

$$\langle \mathcal{O}_1 ... \mathcal{O}_s \rangle_\beta = e^{-\omega\cdot\beta} \int_{\mathcal{M}_\Sigma(X,\beta)} ev_1^*\mathcal{O}_1 ... ev_s^*\mathcal{O}_s$$

The main objective is now to compute the above integral. Let D_i be the Poincaré dual cycles to to $\mathcal{O}_i$, then we may choose as a representative of the cohomology class of $\mathcal{O}_i$ the differential form $\omega_i \in H^{p_i+q_i}(X)$ that is a δ function supported on D_i. Then, as noted before, the integral is non-vanishing only if $\phi(x_i) \in D_i \ \forall i$. We define

$$n_{\beta,D_1...D_s} = \#\{\phi : \Sigma \to X \text{ s.t. } \phi \text{ holo. } , \phi(x_i) \in D_i \ \forall i, \ \phi_*(\Sigma) = \beta\}$$

Thus finally

$$\langle \mathcal{O}_1 ... \mathcal{O}_s \rangle = \sum_{\beta \in H_2(X)} e^{-\omega\cdot\beta} n_{\beta,D_1...D_s}$$

We can give an important interpretation to this.
Note that $\omega \cdot \phi \geq 0$ since $\phi(\Sigma)$ is the holomorphic image of a Riemann surface, thus the pullback of the Kähler form is positive definite on Σ. Thus in the large volume limit (when ω becomes large, which is the IR limit) we have that the main contribution is given when $\beta = 0$, that is when $\phi_*(\Sigma)$ is contractible to a point.
The moduli space reduces to the space of constant maps

$$\mathcal{M}_\Sigma(X,0) \cong X$$

all evaluation maps reduce to the identity. Thus taking as representative of $\mathcal{O}_i$ the δ functions supported on D_i we get

$$\langle \mathcal{O}_1 ... \mathcal{O}_s \rangle_0 = \int_X \omega_1 \wedge ... \wedge \omega_s = \#(D_1 \cap .... \cap D_s)$$

Quantum correlation functions are just a quantum correction of classical intersection numbers of submanifolds.

We do not consider the case where there are ρ-zero modes for simplicity.

5.1.1 Example: the chiral ring of $X \cong \mathbb{CP}^1$

We now make an example where we actually compute correlation functions for an A-twisted theory.
We take $X \cong \mathbb{CP}^1 \cong S^2$ and also $\Sigma \cong \mathbb{CP}^1$, we now that in this case

$$H^0(\mathbb{CP}^1) \cong \mathbb{Z} \qquad H^1(\mathbb{CP}^1) \cong \{1\} \qquad H^2(\mathbb{CP}^1) \cong \mathbb{Z}$$

Where $H^0(\mathbb{CP}^1)$ is generated by the constant function 1, and $H^2(\mathbb{CP}^1)$ is generated by the hyperplane class, the Poincaré dual to a point. Then

$$\int_{\mathbb{CP}^1} H \cdot 1 = \int_{\mathrm{pt}} 1 = 1$$

Let P, Q be operators respectively represented by 1 and H, then

$$q_V(P) = q_A(P) = 0 \qquad q_V(Q) = q_A(Q) = 1$$

Note that P, Q are of even degree so that they commute inside correlation functions In this case

$$k = \phi_*(\Sigma) \cdot e(\mathbb{CP}^1) + 1(1-0) = 2n+1 \qquad (k \text{ is odd})$$

Where n is the winding number of our map ϕ.
Thus selection rules tell us that that

$$\langle PPP \rangle = \langle PQQ \rangle = 0$$

For $k = 1$ we have $n = 0$, thus ϕ is a constant function, thus we must count all constant functions mapping to a point $p \in X$, p=p.d.(H). Since there is just one of this maps

$$\langle PPQ \rangle = 1$$

Thus we obtain our topological metric

$$\eta_{PP} = \eta_{QQ} = 0 \qquad \eta_{PQ} = \eta_{QP} = 1$$

Finally for $k = 3$, $n = 1$ we may compute $\langle QQQ \rangle$.
The representative of the three hyperplane classes are three point $y_1, y_2, y_3 \in X$, thus we have to count all degree one maps mapping to these three fixed points, we know that fixing three points in $\mathbb{CP}^1$ fixes just one automorphism ϕ, thus

$$n_{1,y_1,y_2,y_3} = 1 \qquad \langle QQQ \rangle = \langle QQQ \rangle_1 = e^{-\omega \cdot 1} 1$$

Thus we get computed the twisted chiral ring, that is

$$\begin{aligned} PP &= P \\ PQ &= QP = Q \\ QQ &= e^{-\omega \cdot 1} P \end{aligned}$$

Classically we would have expected that $QQ = 0$ (since the intersection of three distinct points is 0). We recover exactly the same by taking the IR-limit in fact this corresponds to the large volume case where we take the limit $\omega \to \infty$.

5.2 Some natural questions

We have seen how the moduli space arises from the discussion of topological twisting and how it is related to the computation of correlation functions.
There are some questions, now, that are natural, arising from our whole discussion.

- We have described locally the moduli space of holomorphic maps from our genus g worldsheet to a general Kähler target space. Is this space a smooth complex manifold? Which are its topological properties? Is it closed? If closed, is it compact?
- We have seen the importance of the De Rham cohomology ring, since in our A-twist example there was a 1-1 correspondence between it and physical observables. Is there any cohomology class of the De Rham cohomology ring of the moduli space which is physically relevant?
- The twisted theories in this case are topological in the sense that we are treating the worldsheet metric as a background field. Can we try to include it as a dynamical degree of freedom?

All this questions are deeply connected to each other and will be at the heart of the discussion in the next chapters. The main goal will be to try to generalize the above discussion and to include gravity in it.
The great problem, while trying to do this, is that in general the moduli space of stable maps $\mathcal{M}_{g,n}(X, \beta)$ may be deeply singular and in this case is not even easy to compute its dimension so the study of Gromov-Witten invariant is even harder.

Remark: we should make here a quick note in order to avoid confusion that may arise from the path that we have followed.
We have defined a cohomological field theory in section (4), however we have seen that the theory we have worked out is a *topological* one. This creates confusion

since mathematicians refer to *cohomological* field theories to theories where physical operators are cohomology classes defined on the moduli space $\mathcal{M}_{g,n}$. If we include gravity the moduli space that we have to work on is $\mathcal{M}_{g,n}(X,\beta)$, which is the moduli space where each point represents a holomorphic map from a Riemann surface $\phi : \Sigma \to X$ and, in this case, the complex structure of Σ is not fixed!

We will meet later $\mathcal{M}_{g,n}$ the moduli space of Riemann surfaces, for our discussion it is enough to know that this is the set of complex structures of Riemann surfaces with genus g and n fixed points and has dimension $3g-3+n$. Consider the natural map

$$\begin{aligned} p : \mathcal{M}_{g,n}(X,\beta) &\to \mathcal{M}_{g,n} \\ (\Sigma,\phi) &\mapsto \Sigma \end{aligned}$$

Thus fixing $\Sigma = \bar{\Sigma}$ and looking at the holomorphic maps $\phi : \bar{\Sigma} \to X$ corresponds to look at the fibers of p so that

$$\mathcal{M}_{\bar{\Sigma}}(X,\beta) = p^{-1}(\bar{\Sigma})$$

This is the space where the action we have studied above localize.
Thanks to the computation of $\mathcal{M}_{\Sigma}(X,\beta)$ we can say that the virtual dimension (which is not necessarily the dimension) of $\mathcal{M}_{g,n}(X,\beta)$ is

$$\text{virdim}\mathcal{M}_{g,n}(X,\beta) = \text{virdim}\mathcal{M}_{\Sigma}(X,\beta) + \text{virdim}\mathcal{M}_{g,n} = c_1(X)\cdot\beta + (\text{dim}X - 3)(1-g)$$

In this setting we can understand that adding gravity as a degree of freedom drastically changes the situation, even if we are not including gravitational observables!

Let $\omega_1, ..., \omega_n \in H^*(X)$ be physical observables, $\omega_i \in H^{p_i,q_i}$ inserted at the points $p_1, ..., p_n$ of our worldsheet. As we have done before we can use the evalutation maps, (note that in this case evaluation maps are slightly different from the previous ones, since they are defined on different spaces)

$$\begin{aligned} ev_i : \mathcal{M}_{g,n}(X,\beta) &\to X \\ (\Sigma,\phi) &\mapsto \phi(p_i) \end{aligned}$$

And define correlation functions to be

$$\langle \mathcal{O}_i ... \mathcal{O}_n \rangle = \int_{\mathcal{M}_{g,n}(X,\beta)} ev_1^*\omega_1 \wedge ... \wedge ev_n^*\omega_n$$

- First of all, even the degree of the correlation functions is different, in fact if we want a correlation function to be non vanishing we need, that

$$\sum_i p_i = \sum_i q_i = \text{virdim}\mathcal{M}_{g,n}(X,\beta)$$

While in our previously considered case

$$\sum_i p_i = \sum_i q_i = \text{virdim}\mathcal{M}_\Sigma(X, \beta)$$

Which means that in this case the degree of operators in the correlation function must be equal to the dimension of the fiber $\mathcal{M}_\Sigma(X, \beta) = p^{-1}(\Sigma)$.

- This means that what we were actually doing in section (5.1) is

$$\langle ev_1^*\omega_1 ... ev_n^*\omega_n, p^{-1}(\Sigma)\rangle = \langle p_*(ev_1^*\omega_1 ... ev_n^*\omega_n), \Sigma\rangle$$

  Where Σ is a point in $\mathcal{M}_{g,n}$, this shows that in fact correlation functions of this type are topological, in fact since $\mathcal{M}_{g,n}$ is connected, then any points Σ and Σ' are equivalent in homology.

- This is true only in this particular case, since $p_*(ev_1^*\omega_1 ... ev_n^*\omega_n) \in H^0(\mathcal{M}_{g,n})$, the set of cohomology classes of constant functions on $\mathcal{M}_{g,n}$, however by taking higher degree operators we get cohomology classes with explicit dependence on Σ, and this happens even if we are not considering gravitational observables.

To conclude, the theory we have previously studied, the A-twist example, where we have fixed Σ is just a special type of cohomological field theory, in fact it studies degree 0 cohomology classes on $\mathcal{M}_{g,n}$ reducing to a topological field theory.

As we said including gravity implies working with $\mathcal{M}_{g,n}(X, \beta)$ however this space is very complicated, so that we need to simplify our treatment. We could try to do this in two different ways.

- The first is to consider the genus of worldsheet as the order of the amplitude, so that the relevant amplitudes are the ones with lowest genus $g = 0$. In this case and under some assumptions of the target space X (that $h^1(\mathcal{O}, \phi^*TX) = 0$)) we have that $\mathcal{M}_{(0,n)}(X, \beta)$ is a non-singular orbifold and has the expected dimension

$$\int_\beta c_1(X) + \dim(X) + n - 3$$

  Thus the treatment in this case simplifies giving us more chances.

- Another approach, and this is the one that we will be focusing on, is to simplify our target space to a point. This of course makes only sense in the case we are including gravity in our discussion: we do not include any matter

field and try to study gravitational observables.
Note that if we take $\beta = 0$ then $\phi : \Sigma \to X$ is homologically equivalent to the constant function, thus

$$\mathcal{M}_{g,n}(X, \beta = 0) \cong \mathcal{M}_{g,n} \times X$$

Where $\mathcal{M}_{g,n}$ is the moduli space of Riemann surfaces of genus g with n marked points, and if we take X to be a point we have that

$$\mathcal{M}_{g,n}(\mathrm{pt}, \beta = 0) \cong \mathcal{M}_{g,n}$$

In this case our discussion heavily simplifies. In fact $\mathcal{M}_{g,n}$ is almost non-singular as a complex manifold (it has some singularities which makes it an orbifold). Moreover we will see that gravitational observables are very natural characteristic classes defined on $\mathcal{M}_{g,n}$ and that we can also find (as in the case of topological field theories) some recursion relations that will allow us to compute correlation functions.

6 Moduli space

In the previous sections we have seen the importance of compact orientable surfaces without boundaries. It is natural now, if we want the metric to be dynamical, to study the moduli space of Riemann surfaces, since, as in the context of bosonic string theory, we will have to integrate over it, if computing correlation functions. Before diving directly into this topic let us recall some results regarding complex curves.

6.1 Complex curves

A topological surface is a topological space of two real dimensions which is endowed with a differentiable structure. If we restrict to the case of orientable, compact, without boundaries surfaces there is just one topological invariant: the genus g which corresponds to the number of handles of the surface.
Clearly this implies that only with this structure there is not much variety of cases. This is why, from the mathematical point of view, we add finer structures on it, in particular

- A *Riemannian metric*, given by a positive definite tensor

$$g = g_{ij} dx^i \otimes dx^j$$

  which gives a notion of distance on Σ_g (the topological surface).

- Once given a Riemannian metric it is possible to introduce a *conformal structure*, which is given by the data of the Riemannian metric, up to scaling by a positive function on Σ_g. This means an equivalence relation between Riemannian metrics

$$g \simeq \eta \iff \eta = gf$$

  This corresponds to fixing the angles between tangent vectors on Σ_g, but not their length.

- An *almost complex structure* which is an automorphism

$$J : T\Sigma_g \to T\Sigma_g \text{ s.t. } J^2 = -2$$

  This corresponds to rotating tangent vectors by $\pi/2$, and since a conformal structure fixes angles, one can always fix an almost complex structure from a conformal one. The converse is also true: given an almost complex structure at any point $p \in \Sigma_g$ one can take a tangent vector $v \in T_p\Sigma_g$ and the vector $u = Jv$ and specify the metric with respect to this basis to be

$$g_{\alpha\beta} = \delta_{\alpha\beta}$$

This is in agreement with the fact that locally on any surface we can always perform a coordinate change such that

$$g_{\alpha\beta} = f(z)\delta_{\alpha\beta}$$

- If the almost complex structure is integrable (which is always true for surfaces) then we have a *complex structure*. This is equivalent to state that transition functions in the differentiable structure may be chosen to be holomorphic.
  A surface Σ_g with a given complex structure is called *Riemann surface*.
- Every Riemann surface carries also an *algebraic structure*, which means that can be described as the vanishing set of homogeneous polynomials defined on $\mathbb{CP}^n$. This is a useful fact since it allows to study Riemann surfaces using algebraic results.

Conformal structure is fundamental in the study of strings, and we have seen here that for surfaces

$$\text{conformal structure} \iff \text{complex structure} \iff \text{algebraic structure}$$

We will see that if we just consider smooth curves our moduli space will be open and that singular curves will play a fundamental role in their compactification.
In particular the simplest singularity (and the key one) is the node. It is described locally by the analytical equation defined in $\mathbb{C}^2$

$$xy = 0$$

A nodal curve may be represented as in figure (5)

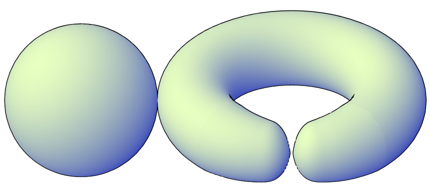

Figure 5: An example of nodal curve displaying two nodes, note that it appears that the two components at the node are tangent, this is not true in general, but is the only way of representing a node in 3 dimensions

We also define its *normalization* $\tilde{\Sigma}$ to be the disconnected Riemann surface obtained by ungluing its nodes. Consider the map, called the normalization map

$$\nu : \tilde{\Sigma} \to \Sigma$$

We will define the preimages of the nodes in $\tilde{\Sigma}$ to be the *node branches* and the connected components of $\tilde{\Sigma}$ to be the *irreducible components* of Σ.
Finally we define the *arithmetic genus* of Σ to be the genus of the curve obtained by smoothing the nodes in such a way to obtain a smooth connected curve.
We will just refer to it as the genus of the singular curve. Note that the genus of a curve is not the genus of its normalization, in general, for example in figure (5) genus of the curve is 1, while the genus of its normalization is 0.

6.2 The moduli space of Riemann surfaces

The definition of the moduli space useful in the physical context is in terms of smooth orbifolds in the analytic setting.

The idea is to collect the set of all possible complex structures that a fixed topological surface may display, each element of this set represent an equivalence class of complex structures, where the equivalence relation is given by complex isomorphisms.
This set admits a differentiable structure, since there exist infinitesimal variations of the complex structure, however it is not smooth as a differentiable manifold and the proper mathematical setting is the one of orbifolds. Roughly speaking an orbifold is locally isomorphic to an open ball in $\mathbb{C}^n$ factorized by a finite group action, this group exactly represents the complex isomorphisms.

Definition 6.1. Let X be a topological space and let $V \in X$ be an open subset. Then ϕ is called an *orbifold chart* if it is a homeomorphism

$$\phi : U/G \to V$$

Where $U \in \mathbb{C}^n$ contractible and G is a finite group acting on U in a bi-holomorphic way.

Definition 6.2. A subchart ϕ' of ϕ is a homeomorphism

$$\phi' : U'/G' \to V'$$

where $V' \subset V$ and there exists a group homomorphism $G' \to G$ and a holomorphic embedding $U' \hookrightarrow U$ such that:

- The embedding and the group homomorphism are equivariant w.r.t. the group action
- The stabilizers in G and G' at any point are always isomorphic
- The embedding commutes with the homeomorphisms ϕ and ϕ'

Definition 6.3. Two orbifold chart on V_1 and V_2 are compatible if every point of $V_1 \cap V_2$ is contained in an orbifold chart V_3 which is a subchart of both V_1 and V_2.

Finally

Definition 6.4. An orbifold is a topological space X together with an open cover $\{V_\alpha\}_{\alpha \in I}$ of compatible charts.

Thanks to this definition we can generalize any tool that have a local definition (like diffferential forms and vector bundles) by requiring the the local definition to be invariant under the group action of the chart that we are considering.
Thus cohomology and homology may still be defined and moreover Poincaré duality holds, note however that it is safer to consider homology with coefficients in $\mathbb{Q}$: $H_*(X, \mathbb{Q})$, because of the quotient by a finite group. In the following we will always assume that this is the case.

For example for the case of a differential form we require it to be invariant under the G action, or for vector bundles we require local trivializations to be G invariant.

At last we can give the definition of map between orbifolds, such that each fiber over a point is a manifold. This will be enough for the study of moduli space.

Definition 6.5. A *map of orbifolds with manifold fibers* is a map $f : X \to Y$ such that it admits a continuous map of the underlying topological spaces $\tilde{f} : \tilde{X} \to \tilde{Y}$ together with:

- the choice $\forall y \in Y$ of a chart $\phi_y : U_y/G \to V_y$ where $y \in V_y$;
- a holomorphic map $F : U_x \to U_y$;
- a lifting of the group action on U_y to U_x commuting with F
- an isomorphism $\phi_x : U_x/G \to V_x$ with an open suborbifold $V_x \in X$ such that

$$\phi_y \circ F = \tilde{f} \circ \phi_x$$

From now on we will turn our attention to the moduli of Riemann surfaces and in particular we will start considering the moduli space $\mathcal{M}_{g,n}$ where n are fixed points.

The main reason for considering fixed points is reducing the isomorphisms available in G, so that G is finite and we get a smooth orbifold.
For example the $\mathcal{M}_{0,0}$ has a infinite group of isomorphisms: $SL(2,\mathbb{C})$. Thus we have a group action which is infinite which means a singular orbifold.
This can be fixed by fixing three points, for example $0, 1, \infty$. This completely fixes the isomorphisms just to the identity, so that the space $\mathcal{M}_{0,3} = \text{pt}$.

Remark: We have a criterion thanks to which we can immediately say if the moduli space is a smooth orbifold. It turns out that the isomorphism group of a Riemann surface satisfying

$$2 - 2g - n < 0 \tag{22}$$

is always finite (this condition is equivalent to ask negative Euler characteristic of the Riemann surface).
In this case the moduli space $\mathcal{M}_{g,n}$ has dimension $\dim\mathcal{M}_{g,n} = 3g - 3 + n$. Thus the only ill-behaved moduli spaces are $\mathcal{M}_{0,0}$, $\mathcal{M}_{0,1}$, $\mathcal{M}_{0,2}$, $\mathcal{M}_{1,0}$.

All of this definitions allow us to define another important object: the *universal curve*. The following result comes to help

Theorem 6.1. *Let Σ be a Riemann surface satisfying (22), and let G be its finite isomorphism group. Then there exists*

- *An open ball $U \subset \mathbb{C}^{3g-3+n}$.*
- *A map $\pi : \mathcal{C} \to U$ where each fiber on a point is a Riemann surface with n marked points.*
- *A G action on $\mathcal{C}$ descending to a G action on U given by $\pi \circ G$.*

satisfying

- *The fiber $C_0 = \pi^{-1}(0)$ is isomorphic to C.*
- *The action $G|_{C_0}$ preserves C_0 and acts on it as the isomorphism group.*
- *For any family $\mathcal{C}_B \to B$ of smooth curves with n marked points homeomorphic to C such that C_b is isomorphic to C for some $b \in B$. Then there exists $b \in B' \subset B$ and a map $\phi : B' \to U$ such that the family fibration $\mathcal{C}'_B \to B'$ is the pullback of the family fibration $\mathcal{C} \to U$ under ϕ.*

The above construction implies that the map π, called the universal curve, is map between orbifolds with manifold fibers. The two orbifolds we are considering are the moduli space $\mathcal{M}_{g,n}$ covered by local charts U/G and $\mathcal{C}_{g,n}$ covered by local charts $\pi^{-1}(U)/G$.

6.3 The Deligne-Mumford compactification

Up to now we have discussed about the local description of the moduli space in terms of open neighbourhoods. It turns out that in general $\mathcal{M}_{g,n}$ is not compact. For example we can take $\mathcal{M}_{0,4}$. This corresponds to fix three points on the sphere $0, 1, \infty$ and then the four point can be added anywhere but the former three points, this means that

$$\mathcal{M}_{0,4} \cong \mathbb{CP}^1 \setminus \{0, 1, \infty\}$$

which clearly is not compact. what happens if we take the limit the 4-th point t to approaches one of the other three points?
In practice let us parametrize the fourth point to be $[t : 1]$ it seems now that the fourth points in the $\lim t \to 0$

$$([0 : 1], [1 : 1], [1 : 0], [t : 1]) \to ([0 : 1], [1 : 1], [1 : 0], [0 : 1])$$

However if we change coordinates by performing a projective transformation we can fix the fourth point x_4 to be fixed at $[1 : 1]$ and give the mobility freedom to the second one. In practice we can multiply our coordinates by the non singular matrix

$$\begin{pmatrix} 1/t & 0 \\ 0 & 1 \end{pmatrix}$$

obtaining for $t \lim 0$

$$([0 : 1], [1/t : 1], [1/t : 0], [1 : 1]) \to ([0 : 1], [1 : 0], [1 : 0], [1 : 1])$$

And now the second and the third point are approaching each other. So that just by changing the representative of our fiber on $\mathcal{M}_{g,n}$ we have that there is another pair of points approaching each other. This is solved in terms of bubbling, which means by introducing a nodal curve where the node separates the two pair of points, as in Figure (6).

We have talked about nodal curves earlier, however we can only consider them when they carry a finite isomorphism subgroup. This lead us to the notion of *stable curves*, which is a nodal curve such that

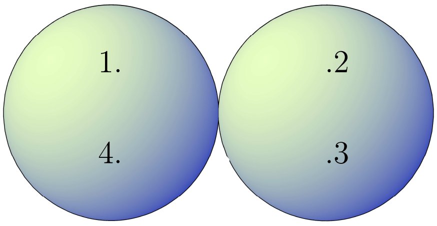

Figure 6: The bubbling related to the compactification of $\mathcal{M}_{0,4}$

- The only singularities of the curve are nodal ones.
- The marked points do not coincide with the nodes (in fact we will allow the limit of marked point to a node, but this will result in the formation of a bubble containing the marked point between the two components of the node)
- All components of the normalization of the curve satisfies (22), where the nodes are treated as marked points.

An example of stable curve is given in Figure (7).
The following theorem gives a complete justification for our construction

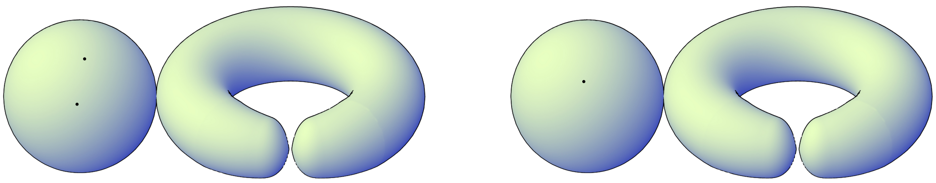

Figure 7: An example of a stable and of an unstable curves

Theorem 6.2. *When (22) is satisfied there exist a smooth compact orbifold* $\overline{\mathcal{M}}_{g,n}$ *of dimension* $3g-3+n$*, a smooth compact orbifold* $\overline{\mathcal{C}}_{g,n}$ *of dimension* $3g-2+n$ *and a map* $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$*, such that*

- $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ *is an open dense suborbifold and the same holds for its preimage* $\pi^{-1}(\mathcal{M}_{g,n}) = \mathcal{C}_{g,n}$ *inside* $\overline{\mathcal{C}}_{g,n}$.
- *the fibers of* π *are stable curves of genus* g *and* n *marked points.*

- *The stabilizer at point* $p \in \overline{\mathcal{M}}_{g,n}$ *is the isomorphism group of the curve* $C_t = \pi^{-1}(p)$.

The space $\overline{\mathcal{M}}_{g,n}$ is called the *Deligne-Mumford* compactification. And the set $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{gn}$ is the *boundary* of the compactification and we will see that is has great mathematical and physical importance.
In particular is a codimension 1 orbifold which means that is a divisor inside $\overline{\mathcal{M}}_{g,n}$. An important result is that it consists of co-dimension 1 divisors which always intersect transversally.
This is a key fact: we have seen that the compactification involves the formation of nodes, thus intersection points of the boundary divisor may be seen as the boundary of the boundary divisor itself. This means that intersection of two components form a co-dimension 2 submanifold and further intersections form codimension k subvarieties and the fibers over these intersections in $\overline{\mathcal{C}}_{g,n}$ have k nodes in them. A visual example of this will be clearer later and can be visualized in the half dimensional picture (8).

Disclaimer: The Deligne-Mumford compactification is not the only possible compatification for the moduli space. In the mathematical setting there are a lot of non-equivalent ones. From the physical point of view it is not that clear why this should be the most natural one. For sure it has been the most relevant since the '90s due to the intense progress made by Witten in that period.

6.4 Forgetful maps

There is a key relationship between moduli spaces sharing the same genus. In fact we can define the the map called the *forgetful map* acting on a curve $(C, x_1, ..., x_n, x_{n+1}, ..., x_m)$ where $x_1, ..., x_m$ are the fixed points, such that

$$(C, x_1, ..., x_n, x_{n+1}, ..., x_m) \mapsto (C, x_1, ..., x_n)$$

However we must pay attention to stability of the resulting curve, in fact if we delete one or two points on a spherical component with only one node (or on a spherical component with two nodes), the curve will not be stable. This is why we compose this operation with the *stabilization map*. Whenever a spherical component will result unstable we contract it to a point.

$$(C, x_1, ..., x_n) \mapsto (\tilde{C}, \tilde{x}_1, ..., \tilde{x}_n)$$

Thus we may define the forgetful map

$$p : (C, x_1, ..., x_n, x_{n+1}, ..., x_m) \to (\tilde{C}, \tilde{x}_1, ..., \tilde{x}_n)$$

Note that the problematic only regards the boundary divisor and only particular strata of it.

It is useful to consider the forgetful map of just a point, in fact the following result holds

Theorem 6.3. *Consider $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ and the forgetful map $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, then π and p are isomorphic as orbifolds maps with manifold fibers.*

There is a nice way to picture this isomorphism of maps, in fact one can think as the $n+1$ fixed point as the coordinate running on the fiber in $\overline{\mathcal{C}}_{g,n}$. We just have to keep track of bubbling/stability, let us consider the following cases

- If we have that $(C, x_1, ..., x_n)$ is stable then

$$(C, x_1, ..., x_n) = (\tilde{C}, \tilde{x}_1, ..., \tilde{x}_n)$$

  Thus x_{n+1} will just be the coordinate running on the fiber inside $\overline{\mathcal{C}}_{g,n}$.

- If x_{n+1} lies on a genus 0 component that has one other marked point and only one node, then its image in $\overline{\mathcal{C}}_{g,n}$ will be the node of the stable curve with the spherical component contracted.

- If x_{n+1} lies on spherical component with just two nodes then its image in $\overline{\mathcal{C}}_{g,n}$ will be the node in the surface with that spherical component contracted.

We can also construct the inverse map for this and in this way we can prove that it is in fact an isomorphism. Note that the construction of the inverse map, implies that the image of a marked point represents a nodal curve in $\overline{\mathcal{M}}_{g,n}$, thus the ith section represents an element of the boundary divisor: the one with a node at x_i attached with a sphere containing x_i, x_{n+1}.

We can look Figure (8), to give an example of this isomorphism. This example considers the half dimensional representation of $\mathcal{M}_{1,1}$ and its universal curve, where the self intersected curve represents the nodal curve, then we look at $\overline{\mathcal{C}}_{1,1}$ to be $\overline{\mathcal{M}}_{1,2}$.

The key thing to note here, is that when playing with the forgetful map of the point x_{n+1} (which is actually equivalent to the universal curve) one is losing some information on the boundary divisor.
Basically, let Δ_n and Δ_{n+1} be the boundary divisors in $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n+1}$ respectively, we can try to compare the pull-back of the boundary divisor Δ_n with Δ_{n+1}. Thus we need to consider the divisors $\delta_{i,n+1}$ and the co-dimension two subvariety

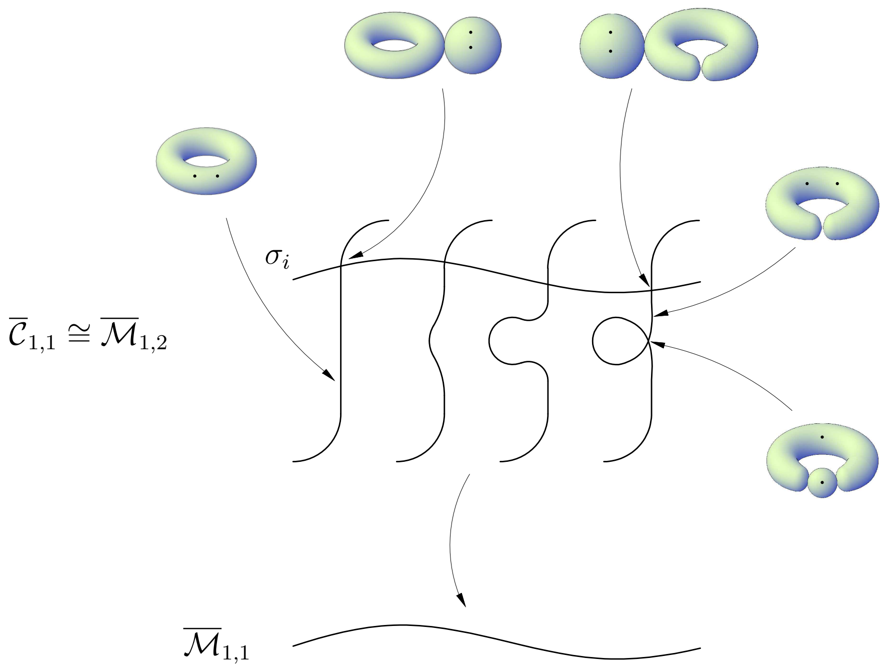

Figure 8: Through this example we may also notice that in the in the intersection of the boundary divisor of $\overline{\mathcal{M}_{1,2}}$, there is are curves with 2 nodes

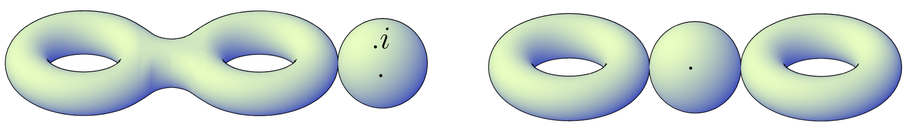

Figure 9: The boundary subvarieties that are are forgotten in the forgetful morphism, the $n+1$ marked point is the unnamed one

δ_{n+1} which respectively encodes the curves as in figure 9.

This is exactly the information that one loses by the forgetful map. We will see later how this plays in terms of cohomology classes.

Another type of maps that we can define in a similar way, and these are the best suited for the description of of boundary divisors are attaching maps.

Let S', S'' be a disjoint partition of $S = \{1, ..., n\}$, let $g = g' + g''$, such that

$$\overline{\mathcal{M}}_{g',S'\cup x'} \qquad \overline{\mathcal{M}}_{g'',S''\cup x''}$$

are stable.

We define the *attaching map of separating kind* to be the map

$$q : \overline{\mathcal{M}}_{g',S'\cup x'} \times \overline{\mathcal{M}}_{g'',S''\cup x''} \to \overline{\mathcal{M}}_{g,n}$$

The map that assigns to two stable curves, the curve where we identify the points x', x'' creating a separating node between the two components.

Similarly we can define the *attaching map of non-separating kind* to be the map

$$q : \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}$$

that assigns to a stable curve the curve obtained by identifying the marked points $n+1$ and $n+2$.

6.5 Some cohomology classes

As we have seen in the context of topological twisting, De Rham cohomology (in that case of the target space) plays an important important role since it is identified with physical operators. When coupling to gravity, then, it is natural to extend this to cohomology classes on the moduli space. In this case not all the cohomology ring may be relevant. We will make here some examples, however we will need some definitions and to state some facts about differential forms for nodal curves.

Recall that a differential on a Riemann surface is given by fdg (where f, g are meromorphic functions) and, in particular, on a genus $g \geq 1$ curve there are g linearly independent differentials (this is reflected in $h^{(1,0)} = g$).

Definition 6.6. A *meromorphic differential* α on a stable curve C with marked points is a meromorphic one form on C such that

- The only poles of α are at the marked points or at nodes of C.
- The poles are at most simple.
- The residues of the poles on to two branches meeting at a node must be opposite to each other.

By means of this definition one can show that on a genus g stable curve there are g meromorphic differentials. Thus we will define cohomology classes on $\overline{\mathcal{C}}_{g,n}$ in terms of these forms.

It is worth saying that the whole De Rham's cohomology ring is very complicated to study, and there are some results in literature, that it is very difficult to try to study generic cohomology classes of $\overline{\mathcal{M}}_{g,n}$ starting from low values of g, n. This is why we restrict ourselves to the study of the *tautological rings*.
A tautological ring is a minimal subring $R^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$, that is closed under the push-forward by all forgetting and attaching maps.

We will use two ways for building cohomology classes, which will turn out to be equivalent

- The first is to consider submanifolds of the moduli space and to consider them as homology cycles in $H_*(\mathcal{M}_{g,n})$. We know that Poincaré duality holds for orbifolds, so this is a good way of producing cohomology classes. Which submanifolds should we choose?
  It comes out that the most interesting ones are the boundary ones. For example looking at the previous example we have that (we will denote with square brackets the Poincaré dual of a cycle)
  $$\delta_{i,n+1} \in H_{6g-4+2n}(\overline{\mathcal{M}}_{g,n+1}) \implies [\delta_{i,n+1}] \in H^2(\overline{\mathcal{M}}_{g,n+1})$$
- Another way is by considering characteristic classes of vector bundles defined on $\mathcal{M}_{g,n}$. We know that they encode more topological information than seen simply as cohomology classes but in this way we can build cohomology classes of particular interest. We will mostly deal with line bundles, and for them we have that the first chern class has great relevance.

In order to build a bridge between these two points of view let us recall some facts regarding divisors and line bundles.

Recall that a line bundle is a fibration $\pi : L \to X$ whose fibers are lines ($\mathbb{C}$), and such that there exist an open cover of $\{U_i\}_{i\in I}$ over which π is trivial
$$\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}$$
Moreover we need a compatibility relation between these local trivialization on non-empty intersections $U_i \cap U_j \neq \emptyset$, which can be done through transition functions
$$\begin{aligned}&\phi_{ij} : U_i \cap U_j \times \mathbb{C} \to U_i \cap U_j \times \mathbb{C}\\ &\psi_i = \phi_{ij}\psi_j\end{aligned}$$

Which are holomorphic functions defined on $U_i \cap U_j$. These must satisfy

$$\phi_{ij} \circ \phi_{ji} = 1 \qquad \phi_{ij} \circ \phi_{jk} \circ \phi_{ki} = 1$$

A *section* of a line bundle is given by a map $\sigma : X \to L$ such that the composition $\pi \circ \sigma = I_X$. Sections are global objects so they also need to satisfy the compatibility relation

$$s_i = \phi_{ij} s_j$$

If one is able to find a non-vanishing section then the line bundle is trivial, namely $L = X \times \mathbb{C}$. In this case if X is compact the only holomorphic sections are constant ones. Instead if L is not trivial there are no non-vanishing sections, and the zeroes of them define a divisor.
A *divisor* is a formal sum of hypersurfaces V_i with integer coefficients

$$D = \sum_i n_\alpha V_\alpha \qquad n_\alpha \in \mathbb{Z}$$

These hypersurfaces are defined locally on an open cover U_i as zeroes of holomorphic functions $f_i^\alpha = 0$, up to rescaling by a non-vanishing holomorphic function. Formally we say that the divisor D has

$$\begin{aligned} &\text{a zero of order } n_i \text{ along } V \cap U_i \text{ (if } n_i > 0) \\ &\text{a pole of order } n_i \text{ along } V \cap U_i \text{ (if } n_i < 0) \end{aligned}$$

The bridge is now quite clear

- Let V be an irreducible hypersurface, then we have that on $U_i \cap U_j \neq 0$, the defining local functions f_i, f_j must agree (up to a rescaling of a non-vanishing function given by $\phi_{ij} = f_i/f_j$). Since these are non-vanishing and since they respect the compatibility conditions we can use them as transition functions for a line bundle!

- One can also show that any holomorphic section of a non-trivial L may vanish on different divisors, however they do not differ as homology cycles. So that if a section s_0 has zeroes on D and another section s_1 has zeroes on D', then cycle

  $$D - D' = 0 \in H_{n-2}(X)$$

  This implies that any line bundle defined on a compact space X defines a cohomology class that in particular is exactly the *first chern class* $c_1(X) = [D] \in H^2(X)$.

In conclusion over a compact space X a divisor D determines a line bundle L, and sections of L determine a divisor D' and they are equivalent as homology cycles

$$D \sim D' \in H_{n-2}(X)$$

All the above discussion can be made more rigorous by using sheaf cohomology, however we will need only the stated results.

6.5.1 ψ classes

Given a stable Riemann surface c we can define on it the cotangent line bundle on the non-singular points of it, given by the dual of the tangent bundle. We can also extend it to the nodes of a stable curve. Thus in this way we are able to build a line bundle over $\overline{\mathcal{C}}_{g,n}$ given by the cotangent lines of the fibers. We denote this line bundle by $\mathcal{L}$, and its sections will be exactly meromorphic sections as described above.

Definition 6.7. Let us consider the moduli space $\overline{\mathcal{M}}_{g,n}$ and its universal curve $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, let $s_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{C}}_{g,n}$ be the sections corresponding to the ith marked point. We define the line bundles

$$\mathcal{L}_i := s_i^*(\mathcal{L})$$

Then we define the ψ_i classes $\in H^2(X)$, to be

$$\psi_i := c_1(\mathcal{L}_i)$$

Example: We can explicitly describe the ψ classes in the case $g = 0$. We will do this exactly by building sections α_i on $\mathcal{L}_i$ and by looking at the zeroes of the section, the Poincaré dual of this divisor will give us $c_1(\mathcal{L}_i)$.

Take pairwise distinct $i, j, k \in \{1, ..., n\}$ and denote by $\delta_{i|jk}$ the divisor of stable curves where a node separates x_i from x_k, x_j marked points. Then we can construct a meromorphic section α of $\mathcal{L}$, in the following way.
Consider a stable curve $C \in \overline{\mathcal{C}}_{g,n}$ and take a meromorphic one form $\alpha|_C$

- $\alpha|_C$ has a simple pole a with residue 1 at x_j and a simple pole with residue -1 at x_k.
- If the curve C is nodal then it will be a "tree" of spheres. Consider the subtree connecting x_j to x_k, then we ask $\alpha|_C$ to have a simple pole with residue 1 on the nodes connecting to x_j and a simple pole with residue -1 on the nodes connecting to x_k

- $\alpha|_C$ vanishes on all the components outside the connecting sub-tree.

We can take this to be a section on $\mathcal{L}$ by defining it on any fiber of $\overline{\mathcal{C}}_{g,n}$. Now let us look at the values that it takes on s_i, $\alpha|_{s_i}$:

- If x_i is in the sub-tree connecting x_j to x_k then α is non-vanishing at x_i.
- If x_i is not on the connecting sub-tree then α vanishes at x_i

This implies that since $\alpha|_{s_i}$ is a section of $\mathcal{L}_i$ (which is built just by taking the cotangent fibers at x_i), that $\alpha|_{s_i}$ vanishes on $\mathcal{L}_i$ exactly on the divisor $\delta_{i|jk}$. Thus

$$\psi_i = \delta_{i|jk}$$

As we may understand from this example ψ classes are closely related to particular boundary divisors.

7 Topological strings

We will now return our attention back to the study of topological strings, starting from the twisted non-linear σ models.
We have seen how they are described and how they turn out to be cohomological field theories.
If we want the metric to be dynamical degree of freedom we have to go through the following steps

- We have to rewrite the lagrangian of the theory in a covariant way by replacing the flat metric by dynamical one h, introducing covariant derivatives (instead of $z, \bar{z}$ ones) and multiplying the measure by $\sqrt{h}$
- Then one needs to introduce the kinetic term for the metric, given by the Einstein-Hilbert lagragian, and since we want to preserve some supersymmetry we are likely to insert ghost fields.
- Then we need to perform the integration over the moduli space of stable maps $\mathcal{M}_{g,n}(X, \beta)$

Note that in order to define strings in a proper way we will need a CY target space, that supports conformal invariance.
We will not discuss the very first point since $\mathcal{M}_{g,n}(X, \beta)$ is away from our discussion. Thus since our target space will be the one of a point the twisted action will reduce to 0 (obviously a point is CY).
We will still perform twisting at the level of the conformal algebra, since we know how twisting acts on the energy momentum tensor and we will see that this has strong implications in terms of central charges and conformal anomaly.

7.1 Twisting the conformal algebra

Conformal invariance plays a huge role in string theory since it allows to reduce the path integral to an ordinary integral over the moduli space of conformal structures (which is also the moduli space of complex structures). Usually this has to be done paying attention to conformal anomalies and this is why bosonic string theory has the critical dimension of target space $D = 26$.
The conformal algebra is usually given in terms of the modes of the Fourier expansion of the energy momentum tensor, this is where open and closed strings differ, we will discuss only the closed case.

The energy momentum tensor corresponds to the conserved charges w.r.t. world-sheet change of coordinates. It is a rank two tensor with four components

$$T(z,\bar{z}) = \begin{pmatrix} T_{zz} & T_{z\bar{z}} \\ T_{\bar{z}z} & T_{\bar{z}\bar{z}} \end{pmatrix}$$

Note: The metric expressed in terms of $z, \bar{z}$ is antidiagonal so that

$$\partial^z = \partial_{\bar{z}} \qquad \partial^{\bar{z}} = \partial_z$$

Conformal invariance implies that $T_{z\bar{z}} = T_{\bar{z}z} = 0$, while Nöther current conservation $(\partial_\alpha T^\alpha_\beta)$ implies $\partial_{\bar{z}} T_{zz} = 0$, while $\partial_z T_{\bar{z}\bar{z}} = 0$. Thus we may define $T(z) = T_{zz}$ and $\bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}$. In the closed string case these are independent degrees of freedom. We can expand them in Laurent modes, giving

$$T(z) = \sum_m L_m z^{-m-2} \qquad \bar{T}(\bar{z}) = \sum_m \bar{L}_m \bar{z}^{-m-2}$$

Satsfying the commutation relations of the "quantum" Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n}$$

Where c is the central charge of the theory and is the term creating conformal anomaly, that, for example, in the bosonic case may be canceled only in 26 dimensions.
The complex conjugate operators satisfy similar commutation relations. Thus we get two sets of Virasoro algebras.

We now introduce the R-symmetry currents.
In particular let us consider a conserved $U(1)$ current $J_\alpha(z,\bar{z})$. Then in our coordinates we have two possibilities (since it has to be conserved $\partial_\alpha J^\alpha = 0$)

$$\partial_{\bar{z}} J_z + \partial_z J_{\bar{z}} = 0 \quad \Longrightarrow \quad J(z) := J_z \qquad \bar{J}(\bar{z}) := J_{\bar{z}}$$

And these two components may be seen as the conserved currents acting separately on the left and right-moving modes of the string. (The name can be explained by returning back to the Lorentzian signature)
Thus we jave two generators that act only on the holomorphic/anti-holomorphic modes of the string. By looking at the algebra (4) this means that

$$F_V = \frac{1}{2}(F_L + F_R) \qquad F_A = \frac{1}{2}(F_L - F_R)$$

We can now expand these currents into Laurent modes

$$J(z) = \sum_m J_m z^{-m-1}$$

Now from the commutation relations that the fields satisfy, and writing each mode in terms of the fields one can find that the symmetry algebra is

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n} \\
[L_m, J_n] &= -nJ_{m+n} \\
[J_m, J_n] &= \frac{c}{3}m\delta_{m+n}
\end{aligned}$$

Now we have all the ingredients to make the topological twisting.
In fact we know that the generator of a symmetry is just the integration over a space-like hypersurface of the conserved current. Since we have done Wick's rotation in order to go to Euclidean setting, time is now the radial direction. Thus a spacelike curve is, for example a circle C_r around the origin, then

$$F_L = \int_{C_r} J(z) = \int_{C_r} \sum_m J_m z^{-m-1} = \int_{C_r} J_0 z^{-1} = 2\pi i J_0$$

Where we have used Cauchy's theorem. Analogously $F_R = 2\pi i \bar{J}_0$
Recalling the twisting operation in Table (1), we have that

$$\begin{aligned}
M_A = M - F_V = M - \frac{1}{2}(F_L + F_R) = M - \frac{1}{2}(J_0 + \bar{J}_0) \\
M_B = M - F_A = M - \frac{1}{2}(F_L - F_R) = M - \frac{1}{2}(J_0 - \bar{J}_0)
\end{aligned}$$

And recalling that it holds that $M = 2\pi i(L_0 - \bar{L}_0)$ we have that

$$\begin{aligned}
L_0 \to L_{0,A} = L_0 - \frac{1}{2}J_0 \qquad \bar{L}_0 \to \bar{L}_{0,A} = L_0 + \frac{1}{2}\bar{J}_0 \\
L_0 \to L_{0,B} = L_0 - \frac{1}{2}J_0 \qquad \bar{L}_0 \to \bar{L}_{0,B} = L_0 - \frac{1}{2}\bar{J}_0
\end{aligned}$$

An this can be achieved by modifying the energy momentum tensor such that we get another rank two conserved tensor which components are

$$\tilde{T}(z) = T(z) + \frac{1}{2}\partial J(z)$$

Note the discrepancy between the two Laurent expansion, this make sense now since

$$\partial J(z) = -\sum_m J_m(m+1)z^{-m-2}$$

So that

$$\tilde{T}(z) = \sum_m \big(L_m - \frac{1}{2}(m+1)\big) z^{-m-2} \qquad \Longrightarrow \qquad L_{m,A} = L_m - \frac{1}{2}(m+1)J_m$$

And clearly is a conserved current since $\partial_{\bar{z}}\tilde{T}(z) = 0$.
We can perform similar calculations for the right-moving sector (paying attention to a different sign in the A case). In both cases if we compute again the algebra for the twisted Virasoro generators we get (for example in the A-case)

$$\begin{aligned}
[L_{m,A}, L_{n,A}] = [L_m, L_n] - \frac{1}{2}(m+1)[J_m, L_n] - \frac{1}{2}(n+1)[L_m, J_n] + \\
+ \frac{1}{4}(m+1)(n+1)[J_m, J_n] = \\
= (m-n)L_{m+n} + \cancel{\frac{c}{12}m(m^2-m)\delta_{m+n}} + \\
- \frac{1}{2}(m+1)mJ_{m+n} - \frac{1}{2}(n+1)(-nJ_{m+n}) + \\
+ \cancel{\frac{1}{4}(m+1)(-m+1)\frac{c}{3}m\delta_{m+n}} = \\
= (m-n)\big(L_{m+n} - \frac{1}{2}(m+n+1)J_{m+n}\big) = (m-n)L_{m+n,A}
\end{aligned}$$

In this way we have seen how twisting modifies the superconformal algebra and one can show [22]-[21] that this is the only part that changes.
Note that the central charge in the commutation relations is gone! This implies that for twisted strings we do not have a critical dimension for our target space, in particular topological string theories are well defined target spaces of any dimension.
This also means that we may compute correlation function by doing integrals over the moduli space $\mathcal{M}_{g,n}(X, \beta)$.
In particular this shows also that it is possible to define a string theory with target space the point.

7.2 2D quantum gravity and physical operators

In this case there is an interest point to discuss. In fact we are dealing with a string theory without any matter field, the only degree of freedom is gravity. Thus this model could be seen as an attempt to study a quantum field theory of a two dimensional universe, in some sense a toy model for $2D$ gravity.
Our goal is to describe this theory as a cohomological one and this is where we need a supercharge (which is nilpotent) acting on our only degree of freedom left: the metric.

It is not actually easy to give a complete formulation of the lagragian describing this toy model. This is usually done by physicist by trying to gauge all symmetries of the lagrangian, which in particular are reparametrization and conformal invariance. This has been discussed for example in [3]-[16]-[17].

Briefly the way to do this is to introduce a super-partner of the metric $\psi_{\alpha\beta}$, and then proceeding with gauge fixing by adding ghost fields. This defines the supercharge Q, together with supersymmetric transformations of the ghost fields.
Thus, we know that physical operators are exactly equivalence classes of the Q-cohomology. However in this case the Q supersymmetric operator is not acting on our target space, but on the metric itself, which after gauge fixing, is a degree of freedom varying on $\mathcal{M}_{g,n}$, so Q acts on $\mathcal{M}_{g,n}$.

It turns out by studying supersymmetric transformations acting on the metric and on ghost fields, that physical operators are exactly elements of the tautological rings that we have discussed above. This discussion is quite complicated and out of the scope of this thesis, however this is the reason for which physicists are interested computing correlation functions of tautological classes.

In particular, thanks to the non-anomalous conformal invariance, we are able in the case topological gravity to localize our action to the moduli space $\mathcal{M}_{g,n}$, this means, as in the example of the A-twist, that correlation functions are integrals over the moduli space we have localized on.
Clearly the most natural objects that we can integrate over $\mathcal{M}_{g,n}$ are cohomology classes of $\mathcal{M}_{g,n}$, and this is why they are so special in this context.
From mathematics we also know that the generic cohomology ring of $\mathcal{M}_{g,n}$ is very difficult to get hold of, thus we restrict to the study of tautological classes, so that intersection numbers of them are both interesting from both the mathematical and the physical point of view.
In the next section we will show that we are able to compute correlation functions of one particular type of classes, the ψ ones.

8 Correlation functions for topological strings

As promised we will give here an overview concerning correlation functions for topological strings, and give some examples that clarify their computation.
We have seen how in the case of topological correlation functions for twisted theories all the physical information was contained in the chiral ring and that through some recursive relations we were able to obtain any correlation function for any genus g of the worldsheet, just by knowing the topological metric and the correlation functions of three physical operators on the $g = 0$ case.

We would like to obtain an analogous for this new case. This can be done by giving some results concerning the boundary divisor and a better comprehension of the forgetful map forgetting x_{n+1} which is isomorphic to the universal curve.
However it happens that we can almost obtain the same thing, and that for generic genus there is much more to discuss on this topic, concerning Witten's conjecture and the relation to integrable systems of PDEs (though this will be out of the scope of the thesis).

8.1 The boundary

The boundary of $\overline{\mathcal{M}}_{g,n}$, as stated above, is nice as one would wish: all intersections of the boundary with itself are transverse.
Moreover one can give an interpretation of the irreducible components composing it, thanks to the attaching map of separating kind that we have discussed previously.
Let $S = \{x_1, ..., x_n\}$ be the marked points, then for each partition of S, $S' \cup S'' = S$ and of the genus $g' + g'' = g$, such that $2 - 2g' - \#S' < 0$, $2 - 2g'' - \#S'' < 0$. Then there is the irreducible component of the boundary divisor

$$D = D(S', g'|S'', g'')$$

that can be described by moduli spaces of lower dimension

$$\overline{\mathcal{M}}' := \overline{\mathcal{M}}_{g', S' \cup \{x'\}} \qquad \overline{\mathcal{M}}'' := \overline{\mathcal{M}}_{g'', S'' \cup \{x''\}}$$

Such that D is isomorphic to the image of the attaching morphism ρ_D that identifies the points x' and x'' (on their respective universal curve). Let τ', τ'' be the projection on the first and on the second element in the product $\overline{\mathcal{M}}' \times \overline{\mathcal{M}}''$, this situation can be summarised in the following

$$\begin{tikzcd} \overline{\mathcal{M}}' & \overline{\mathcal{M}}' \times \overline{\mathcal{M}}'' \arrow[l, "\tau'"'] \arrow[r, "\tau''"'] \arrow[d, "\rho_D"'] & \overline{\mathcal{M}}'' \\ & D \subset \overline{\mathcal{M}}_{g,n} & \end{tikzcd}$$

So that any point of D represents a nodal curve with node on $x' \sim x''$ and the two components at the node are encoded by the elements of $\overline{\mathcal{M}}', \overline{\mathcal{M}}''$.

Example: consider $\delta_{i,n+1} \subset \overline{\mathcal{M}}_{g,n+1}$ the divisor with one node dividing the x_i, x_{n+1} contained on a genus $g = 0$ component.
In this case $S' = \{x_i, x_{n+1}, x'\}$, $g' = 0$ and $S'' = \{x_1, ..., \hat{x}_i, ..., x_n, x''\}$, $g'' = g$.
So that

$$\overline{\mathcal{M}}' = \overline{\mathcal{M}}_{0,3} \cong \{pt\} \qquad \overline{\mathcal{M}}'' = \overline{\mathcal{M}}_{g,n}$$

Thus this case, then $\overline{\mathcal{M}}' \times \overline{\mathcal{M}}'' \cong \overline{\mathcal{M}}'' \cong \overline{\mathcal{M}}_{g,n}$ and the attaching map is trivial to the identity. Thus

$$\delta_{i,n+1} \simeq \overline{\mathcal{M}}_{g,n}$$

And by looking at the isomorphism (6.3), it is clear that $\rho_D = \sigma_i$ as seen as a map

$$\sigma_i : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{C}}_{g,n} \cong \overline{\mathcal{M}}_{g,n+1}$$

in fact the points $p \in \sigma_i$ represent exactly curves where the added point coincide with x_i, but since they cannot coincide they describe the same curve with a node at x_i and a genus $g = 0$ bubble containing x_i, x_{n+1}.
Note that

- The forgetful map $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ restricted to $\delta_{i,n+1}$ becomes an isomorphism

- By our definition of $\delta_{i,n+1}$ there is no intersection

$$\delta_{i,n+1} \cap \delta_{j,n+1} = [\delta_{i,n+1}] \cdot [\delta_{j,n+1}] = 0 \qquad \forall i \neq j \tag{23}$$

The following proposition will give us an important result

Proposition 8.1. *By using the above notation we have that*

$$\rho_D^* \mathcal{L}_i = \begin{cases} \tau'^* \mathcal{L}_i & \text{if } x_i \in S' \\ \tau''^* \mathcal{L}_i & \text{if } x_i \in S'' \end{cases}$$

Proof. As we have heuristically stated above, points in $\rho_D(\overline{\mathcal{M}}' \times \overline{\mathcal{M}}'')$ represent stable curves with one separating node at $x' \sim x''$. This can be formalized by looking at the two universal curves

$$\pi' : \overline{\mathcal{C}}' \to \overline{\mathcal{M}}' \qquad \pi'' : \overline{\mathcal{C}}'' \to \overline{\mathcal{M}}''$$

By using them we can work out the following commutative diagram

$$\begin{tikzcd}
 & & \tau'^*\overline{\mathcal{C}}' \cup_{x'\sim x''} \tau''^*\overline{\mathcal{C}}'' \arrow[dd] & & \\
 & \tau'^*\overline{\mathcal{C}}' \arrow[dl] \arrow[dr, "\bar{\pi}'"] & & \tau''^*\overline{\mathcal{C}}'' \arrow[dl, "\bar{\pi}''"'] \arrow[dr] & \\
\overline{\mathcal{C}}' \arrow[dr, "\pi'"] & & \overline{\mathcal{M}}' \times \overline{\mathcal{M}}'' \arrow[dl, "\tau'"] \arrow[dr, "\tau''"'] & & \overline{\mathcal{C}}'' \arrow[dl, "\pi''"'] \\
 & \overline{\mathcal{M}}' & & \overline{\mathcal{M}}'' &
\end{tikzcd}$$

Where $\tau'^*\overline{\mathcal{C}}' \cup_{x'\sim x''} \tau''^*\overline{\mathcal{C}}''$ is the union of the two universal curves where we identify $\sigma(x')$ with $\sigma(x'')$ forming a node at the identification.
In this way we may see how we can think $\tau'^*\overline{\mathcal{C}}' \cup_{x'\sim x''} \tau''^*\overline{\mathcal{C}}''$ to e a suborbifold of the universal curve $\overline{\mathcal{C}}_{g,n}$

$$\begin{tikzcd}
\tau'^*\overline{\mathcal{C}}' \cup_{x'\sim x''} \tau''^*\overline{\mathcal{C}}'' \arrow[r] \arrow[d] & \overline{\mathcal{C}}_{g,n} \arrow[d, "\pi"] \\
\overline{\mathcal{M}}' \times \overline{\mathcal{M}}'' \arrow[r, "\rho_D"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

Thus $\rho_D^*\mathcal{L}_i$ is given by the pull-back of the cotangent bundle inside $\tau'^*\overline{\mathcal{C}}' \cup_{x'\sim x''} \tau''^*\overline{\mathcal{C}}''$

$$\rho_D^*\mathcal{L}_i = \tilde{\sigma}_i^*\mathcal{L} \qquad \text{where} \quad \tilde{\sigma}_i : \overline{\mathcal{M}}' \times \overline{\mathcal{M}}'' \to \tau'^*\overline{\mathcal{C}}' \cup_{x'\sim x''} \tau''^*\overline{\mathcal{C}}''$$

However the i section restricted to this set is always in one of the two components of the union, and we can construct the line bundle, just by looking at the relevant component. So that

$$\tilde{\sigma}_i^*\mathcal{L} = \begin{cases} \tau'^*\sigma_i'^*\mathcal{L} & \text{if } i \in S' \\ \tau''^*\sigma_i''^*\mathcal{L} & \text{if } i \in S'' \end{cases}$$

$\square$

Remark: if we take the divisor $\delta_{i,n+1}$ we have that (recall that $\rho_D = \sigma_i$, $\tau'' = $id)

- If $j \neq i, n+1$ then $\sigma_i^* \mathcal{L}_j = \mathcal{L}_j$ on $\overline{\mathcal{M}}_{g,n}$, so that
$$\sigma_i^* \psi_j = \psi_j$$
- $\sigma_i^* \psi_i = 0$ since we need to look at the line bundle over a point, which of course is trivial.
Since pulling back through σ_i corresponds to intersecting ψ_i, with $\sigma_i = \delta_{i,n+1}$ and then pushing down the intersection, then
$$0 = \sigma_i^*(\psi_i) = \pi_*(\delta_{i,n+1} \cap \psi_i)$$
But since the restriction of π to the $\delta_{i,n+1}$ is an isomorphism, then, on $\overline{\mathcal{M}}_{g,n+1}$, it holds
$$\psi_i \cap \delta_{i,n+1} = 0 \tag{24}$$

We will work out now an important comparison result relating the ψ_i classes of $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n+1}$.
As we have noted before, all the information for $\mathcal{L}_i$ is encoded by the cotangent bundle at the $i-th$ marked point, so that the two line bundles will differ exactly at the points where the line bundles over the two respective universal curves will change at the $i-th$ marked point. This happens exactly at the divisor $\delta_{i,n+1}$. The following proposition formalize the discussion

Proposition 8.2. *Let $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}$ be the forgetful morphism. Then (by using some abuse of notation), it holds*
$$\psi_i = p^*\psi_i + \delta_{i,n+1} \qquad \in H^2(\overline{\mathcal{M}}_{g,n+1}) \tag{25}$$
Where the ψ-class on the RHS refers to $\overline{\mathcal{M}}_{g,n}$.

Proof. As noted above the only divisor that could change ψ is $\delta_{i,n+1}$, So that
$$\psi_i = p^*\psi_i + r\delta_{i,n+1} \qquad r \in \mathbb{Z}$$
The line bundle should be then
$$\mathcal{L}_i = p^*\mathcal{L}_i \otimes \mathcal{O}(r\delta_{i,n+1}) = p^*\mathcal{L}_i \otimes \mathcal{O}(\delta_{i,n+1})^{\otimes r}$$
Recall that $\sigma_i^*\psi_i = 0$, so that
$$\sigma_i^*\mathcal{L}_i$$
should be trivial. This means $\sigma_i^*\mathcal{L}_i = \overline{\mathcal{M}}_{g,n} \times \mathbb{C}$. This allows us to compute $r \in \mathbb{Z}$. In fact
$$\sigma_i^*\mathcal{L}_i = \sigma_i^* p^*\mathcal{L}_i \otimes \sigma_i^*\mathcal{O}(\delta_{i,n+1})^{\otimes r} = \mathcal{L}_i \otimes \mathcal{L}_i^{*\otimes r}$$
Which is trivial in general only if $r = 1$. □

Remark: By using (24) and (25), we get an important identity

$$0 = \psi_i \cdot \delta_{i,n+1} = (p^*\psi_i + \delta_{i,n+1}) \cdot \delta_{i,n+1}$$

This implies that

$$\delta_{i,n+1}^2 = -p^*\psi_i \cdot \delta_{i,n+1} \tag{26}$$

Proposition 8.3. *In our setting it holds*

$$\psi_i^a = p^*\psi_i^a + p^*\psi_i^{a-1}\delta_{i,n+1} \tag{27}$$

Proof. This can be easily be proven by induction on a.

- $a = 1$ is trivial from formula (25)
- Assuming the inductive hypothesis, then

$$\begin{aligned}
\psi_i^{a+1} &= \psi_i^a\psi_i = (p^*\psi_i^a + p^*\psi_i^{a-1}\delta_{i,n+1})(p^*\psi_i + \delta_{i,n+1}) = \\
&= p^*\psi_i^{a+1} + p^*\psi_i^a\delta_{i,n+1} + p^*\psi_i^a\delta_{i,n+1} + p^*\psi_i^{a-1}\underbrace{\delta_{i,n+1}^2}_{-p^*\psi_i\cdot\delta_{i,n+1}} = \\
&= p^*\psi_i^{a+1} + p^*\psi_i^a\delta_{i,n+1}
\end{aligned}$$

□

8.2 The string and dilaton equation

The previous results will be used here to determine two recursion relations that can simplify the computation of physical correlation functions.
First we can prove the string equation

Proposition 8.4. *This relation allows to lower the number of marked points of the moduli space we are treating, obviously we need the arriving moduli space to be well defined:* $2 - 2g - n < 0$.

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{a_1} \cdot \ldots \cdot \psi_n^{a_n} = \int_{\overline{\mathcal{M}}_{g,n}} \sum_{i=1}^n \psi_i^{a_i-1} \cdot \prod_{j\neq i} \psi_j^{a_j}$$

Proof. By using (25)

$$\prod_{i=1}^n \psi_i^{a_i} = \prod_{i=1}^n (p^*\psi_i^{a_i} + p^*\psi_i^{a_i-1}\delta_{i,n+1})$$

recalling by (23), that for $i \neq j$: $\delta_{i,n+1} \cdot \delta_{j,n+1} = 0$.
This implies that in the product all terms involving two or more δ factors vanish. So that

$$\prod_{i=1}^{n} \psi_i^{a_i} = \prod_{i=1}^{n} p^*\psi_i^{a_i} + \sum_{j=1}^{n} p^*\psi_j^{a_j-1}\delta_{j,n+1} \prod_{i \neq j} p^*\psi_i^{a_i}$$

The first term in the sum pulls-back to 0, since is of higher order than the dimension of $\mathcal{M}_{g,n}$. The second term can be integrated

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \sum_{j=1}^{n} p^*(\psi_j^{a_j-1}\psi_1^{a_1}...\hat{\psi}_j^{a_j}...\psi_n^{a_n})\delta_{j,n+1} =$$

$$= \int_{\overline{\mathcal{M}}_{g,n+1}} \sum_{j=1}^{n} p^*(\psi_j^{a_j-1}\psi_1^{a_1}...\hat{\psi}_j^{a_j}...\psi_n^{a_n})\sigma_j(\overline{\mathcal{M}}_{g,n}) =$$

$$= \sum_{j=1}^{n} \int_{\sigma_j(\overline{\mathcal{M}}_{g,n})} p^*(\psi_j^{a_j-1}\psi_1^{a_1}...\hat{\psi}_j^{a_j}...\psi_n^{a_n}) =$$

$$= \sum_{j=1}^{n} \int_{\overline{\mathcal{M}}_{g,n}} \sigma_j^* p^*(\psi_j^{a_j-1}\psi_1^{a_1}...\hat{\psi}_j^{a_j}...\psi_n^{a_n}) = \sum_{j=1}^{n} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1}... \psi_j^{a_j-1}...\psi_n^{a_n}$$

□

The second relation is the *dilaton* equation. In order to prove it let us make some remarks

- Recall that the forgetful morphism $p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ restricted to $\delta_{i,n+1}$ is an isomorpshism, thus

  $$p_*(\delta_{i,n+1}) = \overline{\mathcal{M}}_{g,n}$$

  Which in terms of cohomology classes, means that

  $$p_*(\delta_{i,n+1}) = 1 \qquad \in H^0(\overline{\mathcal{M}}_{g,n})$$

- Note that as for (24), it also holds that

  $$\psi_{n+1} \cdot \delta_{i,n+1} = 0 \tag{28}$$

  This easily follows from the fact that restricted to the divisor $\delta_{i,n+1}$ the line bundle over the universal curve pulls back to a trivial line bundle over $\delta_{i,n+1}$, since it is the line bundle over a point.

Proposition 8.5. *By means of the same notation and when $2-2g-n<0$, it holds*

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{a_1}...\psi_n^{a_n}\psi_{n+1} = (2g-2+n)\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1}...\psi_n^{a_n}$$

Where the factors on the LHS are ψ-classes defined on $\overline{\mathcal{M}}_{g,n+1}$, while the ones on the RHS are defned on $\overline{\mathcal{M}}_{g,n}$

Proof. As for the string equation we can expand the integrand by using (25), and then applying (28)

$$\psi_1^{a_1}...\psi_n^{a_n}\psi_{n+1} = \psi_{n+1}\prod_{i=1}^{n}(p^*\psi_i+\delta_{i,n+1}) = \psi_{n+1}\prod_{i=1}^{n}p^*\psi_i^{a_i}$$

Thus, recalling the push-forward formalism for cohomology classes, then our integral can be seen as a pairing between the integrand and $\overline{\mathcal{M}}_{g,n+1}$:

$$\begin{aligned}
&\langle\psi_{n+1}\cdot p^*\big(\prod_{i=1}^{n}\psi_i^{a_i}\big),\overline{\mathcal{M}}_{g,n+1}\rangle = \\
&= \langle\psi_{n+1}\cdot p^*\big(\prod_{i=1}^{n}\psi_i^{a_i}\big),p^{-1}(\overline{\mathcal{M}}_{g,n})\rangle = \\
&= \langle p_*\Big(\psi_{n+1}\cdot p^*\big(\prod_{i=1}^{n}\psi_i^{a_i}\big)\Big),\overline{\mathcal{M}}_{g,n}\rangle = \\
&= \langle p_*(\psi_{n+1})\cdot\big(\prod_{i=1}^{n}\psi_i^{a_i}\big),\overline{\mathcal{M}}_{g,n}\rangle
\end{aligned}$$

and claiming that

$$p_*(\psi_{n+1}) = (2g-2+n)\cdot 1 \qquad \in H^0(\overline{\mathcal{M}}_{g,n})$$

we get the statemnt.
The claim can be proven by noting that $p_*(\psi_{n+1})$ is a zero form that can be paired with a generic point $x\in\overline{\mathcal{M}}_{g,n}$, so that

$$\langle p_*\psi_{n+1},x\rangle = \langle\psi_{n+1},p^{-1}(x)\rangle$$

so that the result is nothing but the integration of ψ_{n+1} restricted to a curve, $C=p^{-1}(x)$ inside the universal curve.
But now thinking about the isomorphism between $\overline{\mathcal{M}}_{g,n}\cong\overline{\mathcal{C}}_{g,n}$, we have that, except on the other marked points $i=1,...,n$, $\psi_{n+1}|_C$ is nothing else but the pullback

$$\psi_{n+1}|_C = c_1(TC^*) = -c_1(TC) = -e(C)$$

the Euler class, so that

$$\langle \psi_{n+1}, p^{-1}(x) \rangle = -\langle e(C), C \rangle = -\chi_C = 2g - 2 + n$$

This proves the claim

$$p_*(\psi_{n+1}) = (2g - 2 + n) \cdot 1 \qquad \in H^0(\overline{\mathcal{M}}_{g,n})$$

$\square$

This two relations are very powerful, in particular they allow us to compute all intersection numbers, that correspond to correlation functions, on $\mathcal{M}_{g,n}$, for $g = 0, 1$.

In order to show this it is best to introduce Witten's notation to define intersection numbers.

$$\langle \tau_{a_1} ... \tau_{a_n} \rangle_g$$

To be the intersection number

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} ... \psi_n^{a_n}$$

Sometimes it is also used another type of notation introduced since the ordering of the marked points is not relevant when defining intersection numbers. This corresponds to associate, to the above intersection number

$$\langle \tau_0^{k_0} \tau_1^{k_1} \tau_2^{k_2} ... \rangle_g$$

Where:

- n, the number of marked points, corresponds

  $$n = \sum_i k_i$$

  This clearly implies that there are finite $k_i \neq 0$.

- each k_i corresponds to the number of ψ classes that appear as ψ^i in the integral.

In particular we have that

$$3g - 3 + n = \sum_{i=1} i \cdot k_i$$

Using this notation it is explicit that the order of the marked points is not relevant.

Example: let us fix $g = 4$, $n = 5$, then

$$\langle \tau_0^1 \tau_2^1 \tau_4^3 \rangle_3 = \int_{\overline{\mathcal{M}}_{3,5}} \psi_1^4 \psi_2^4 \psi_3^4 \psi_4^2 \psi_5^0$$

8.2.1 Example: $g = 0$ case

For the $g = 0$ case we should note that for $n \geq 3$

$$\dim \overline{\mathcal{M}}_{g,n} = n - 3$$

So that we have non-vanishing intersection number only for $\sum_{i=1}^{n} a_i = n-3$. Since $a_i \geq 0$, we have that at least one of the $a_i = 0$, this give us the possibility to use the string equation.

Proposition 8.6. *Consider $\overline{\mathcal{M}}_{g,n}$, then*

$$\langle \tau_{a_1} ... \tau_{a_n} \rangle_0 = \frac{(n-3)!}{a_1! ... a_n!}$$

Proof. The proof entirely relies on the multinomial identity

$$\frac{p!}{q_1!...q_n!} = \sum_{j=1}^{n} \frac{(p-1)!}{(q_j - 1)! \prod_{i \neq j} q_i!}$$

where $\sum_{i=1}^{n} q_i = p$.
We can use this identity in our inductive step of the proof.

- For $n = 3$, then $\overline{\mathcal{M}}_{0,3}$ is a point, and $a_i = 0$, so that

$$\langle \tau_0 \tau_0 \tau_0 \rangle_0 = 1 = \frac{0!}{0!0!0!}$$

- In the inductive step we can use the string equation, consider on $\overline{\mathcal{M}}_{g,n+1}$, and $a_1, ..., a_{n+1}$ such that $\sum_{i=1}^{n+1} = n - 2$. Then at least one of the $a_i = 0$, so we can use the string equation for the forgetful map forgetting the $i - th$ point

$$\langle \tau_{a_1} ... \tau_{a_n} \tau_0 \rangle_0 = \sum_{j=1}^{n} \langle \tau_{a_1} ... \tau_{a_j - 1} ... \tau_{a_n} \rangle_0$$

Then by using the inductive hypothesis and the multinomial identity

$$\langle \tau_{a_1} ... \tau_{a_n} \tau_0 \rangle_0 = \sum_{j=1}^{n} \frac{(n-3)!}{(a_j - 1)! \prod_{i \neq j} a_i!} = \frac{(n-2)!}{a_1!...a_n!} = \frac{(n+1-3)!}{a_1!...a_n! a_{n+1}!}$$

since $a_{n+1} = 0$.

□

8.2.2 Example: $g = 1$ case

This case is already not so trivial. First we should know a starting point, in particular, we assume (and is not trivial to show), that

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}$$

this could work as our starting point for the computation of intersection numbers for $g = 1$.
In fact we are actually able to find them.

Consider $\langle \tau_{a_1} ... \tau_{a_n} \rangle_1$, then the correlation function is non-vanishing for

$$\sum_{i=1}^{n} a_i = n \qquad (\dim \overline{\mathcal{M}}_{1,n} = n)$$

Since $a_i \geq 0$, then at least one of the $a_i \leq 1$ so we can use the dilaton or the string equation, to reduce the number of marked points. This can be done inductively up to $n = 1$, where we know the result.
Note that the coefficient for this case of the dilaton equation is n.

Let us compute for example

$$\begin{aligned}
\langle \tau_2 \tau_2 \tau_1 \tau_0 \tau_0 \rangle_1 &= 5 \cdot \langle \tau_2 \tau_2 \tau_0 \tau_0 \rangle_1 = \\
&= 5 \cdot (\langle \tau_1 \tau_2 \tau_0 \rangle_1 + \langle \tau_2 \tau_1 \tau_0 \rangle_1) = \\
&= 5 \cdot (3 \langle \tau_2 \tau_0 \rangle_1 + 3 \langle \tau_2 \tau_0 \rangle_1) = \\
&= 30 \cdot \langle \tau_1 \rangle_1 = 30 \frac{1}{24} = \frac{5}{4}
\end{aligned}$$

We can clearly see that this treatment is not enough for the generic g case. In fact the string and the dilaton equations cannot be applied anymore for generic intersection numbers.

However, thanks to the work of Witten and Kontsevic, a more powerful method has been found, that allows to compute them for any genus, here we will not give a complete description of this, but just state some facts.

Basically the generating functional of the intersection numbers obliges a partial differential equation called *Korteweg-de Vries* or *KdV* equation.
This together with the string equation completely determines all intersection numbers, and it is quite an astonishing result, that have come from the physical discussion of 2 dimensional gravity from two different point of view: the one of topological

strings that we have discussed and a different approach called *matrix models*.
In particular there is a huge physical consequence: correlation functions of physical operators in the setting of topological strings can be computed at any order analogously to the topological observables in the context of cohomological field theories.

9 Conclusions

The aim of this work was illustrating how topological invariants in the mathematical context, in particuar Gromov-Witten invariants for Kähler manifolds, can arise from the study of two dimensional quantum field theories where the target space is the Kähler manifold we want to study, so from a physical theory.

In particular by following the construction of these invariants from non-linear σ models, with fixed worldsheet, we have seen how we entered the realm of topological field theories, or equivalently of cohomological field theories of 0-degree.
For this type of theories we have seen that there are some recursion relations that allow us to compute any correlation functions just by knowing three point correlation functions where the worldsheet is forced to be the sphere.
In this context we have seen that in the A-twist example, the physical observables where De Rham's cohomology classes of the target space. And the computation of them was an integration over a subset of the moduli space of stable maps $\mathcal{M}_{g,n}(X)$, this subset corresponding to the fiber, over the fixed Σ, of the map $p : \mathcal{M}_{g,n}(X) \to \mathcal{M}_{g,n}$.

$$\mathcal{M}_{\Sigma} \cong p^{-1}(\Sigma)$$

In this sense the theory we where looking at was a degree 0 cohomological field theory, since the push forward of the cohomology classes we where considering was a constant function on $\mathcal{M}_{g,n}$.

The next step was asking ourselves how we could include the worldsheet's metric as a physical degree of freedom, and since the theory we where studying where twisted ones we entered the realm of topological strings.
In this setting it is natural to define physical operators to be cohomology classes on $\mathcal{M}_{g,n}$ and to study higher degree cohomological field theories. The space we have to integrate on in this case then is the whole $\mathcal{M}_{g,n}(X)$. However since this space is difficult to tackle, we decided to simplify the situation, by using the metric as the only degree of freedom, so that $X = \{\text{pt}\}$, so that $\mathcal{M}_{g,n}(X) \cong \mathcal{M}_{g,n}$.

This can be seen as model for a two-dimensional quantum gravity, and this is why it is important to try to study it from a physical point of view.
The last chapters where, thus, dedicated to the study of the moduli space and to the computation of correlation functions of this toy model. We where indeed able to show that, in this simplified case, we can give some recursion relations that allow to compute generic correlation function for $g = 0, 1$. Moreover, it has been shown by Witten and Kontsevich that this can be done for generic g. So that the case where the target space is a point has been solved.

There are still plenty of questions that may be interesting for further development. Some of them are

- Some work has been done to try to show how ψ classes are related to gravitational observables, however it is not clear this passage and the study of this aspect can give further physical meaning to other classes on $\mathcal{M}_{g,n}$.
- As we have seen the generating function of intersection ψ correlation functions obey a an integrable system of PDEs, it has been shown that this is not the only case where these happens. Thus it is very interesting to study cohomological field theories in order to enrich our current knowledge of integrable systems of PDEs.
- We have defined the tautological ring and we have studied just a the subset of ψ classes. One can be interested on trying to find a set of generators of it and relations, in order to complete its study, and finally end the study of tautological cohomolgy classes for $\mathcal{M}_{g,n}$.

10 Appendix

10.1 Poincaré duality and intersections

There is a precise reason why computation of correlation functions of these type of theories is called *intersection theory* on the moduli space. We recall here some results on this topic that we have used throughout the previous sections.

Consider X to be a compact differentiable manifold of dimension n without boundary $\partial X = 0$, and let H^* be its cohomology ring.
We now that the wedge product descends to a map on cohomology

$$\wedge : H^k \otimes H^l \to H^{k+l}$$

in fact it does not depend representative of the cohomology class

$$\omega \wedge (\eta + d\theta) = \omega \wedge \theta + (-1)^k d(\theta \wedge \eta)$$

since θ is closed.
This together with stokes theorem ensures the existence of a bilinear map

$$\begin{aligned} \langle\ ,\ \rangle : H^k \otimes H^{n-k} &\to \mathbb{R} \\ [\omega] \otimes [\eta] \quad &\mapsto \int_X \omega \wedge \eta \end{aligned}$$

This map is non-degenerate, thus

$$H^k(X) \cong H^{n-k}(X)$$

We can work out a similar construction by defining the bilinear map (C_r is the set of r-chains)

$$\begin{aligned} (\ ,\ ) : C_r \otimes \Omega^r &\to \mathbb{R} \\ c \otimes \omega \ &\mapsto \int_c \omega \end{aligned}$$

In this setting Stokes' theorem reads

$$(c, d\omega) = (\partial c, \omega)$$

Thanks to stokes theorem, this map descends to homology cycles and cohomology classes

$$\begin{aligned} (\ ,\ ) : H_r \otimes H^r &\to \mathbb{R} \\ [c] \otimes [\omega] &\mapsto \int_c \omega \end{aligned}$$

One can show, and this is called De Rham's theorem, that if X is compact this map is non-degenerate. This implies that

$$H_r(X) \cong H^r(X)$$

Poincaré duality puts together these two results, as we have seen

$$H^k(X) \cong H^{n-k}(X) \cong H_{n-k}(X)$$

So that to any element $[\omega] \in H^k$ we can assign, by means of the two dualities explained, a co-dimension k homology cycle and viceversa.
Explicitly, given an homology cycle $C \in H^k$, this defines a linear map

$$\int_C : H^k \to \mathbb{R}$$

and Poincaré duality says that we can represent this map by $\eta_C \in H^{n-k}$ such that

$$\int_C \omega = \int_X \omega \wedge \eta_C$$

where η_C is called Poincaré dual class.

It is not easy to work out what η_C really is, however we can think about it as a differential form of degree $n-k$ that has support only on along C (or within an arbitrary small neighbourhood of C).
Thus, given two homology cycles C and D whose co-dimensions add up to n we can define

$$\begin{aligned} \cdot \quad &: H_k \otimes H_{n-k} \to \mathbb{R} \\ C \otimes D \quad &\mapsto C \cdot D := \int_X \eta_C \wedge \eta_D \end{aligned}$$

And this, given the fact that η_C and η_D can be chosen to have support along C and D, $C \cdot D$ picks up contributions only from the intersection points $x \in C \cap D$, this is exactly the reason why the computation of physical observables in theories considered is also called intersection theory on the moduli space.
It turns out that intersections and wedge product are actually Poincaré dual to each other

$$\eta_{C \cap D} = \eta_C \wedge \eta_D$$

This is exactly what we have done when expressing physical observables in terms of their dual Poincaré classes.

10.2 Push-forward of cohomology classes

We know that given a morphism of two manifolds $f : Y \to X$ we can always define the pullback map

$$f^* : \Omega^k(X) \to \Omega^k(Y)$$

This morphism commutes with the exterior derivative, namely

$$d \circ f^* = f^* \circ d$$

Thus the pullback of a closed (exact) form is a closed (exact) form. Moreover one can show that the pullback of differential forms respects

$$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$$

These two results imply that the pullback descends to ring morphism between De Rham's cohomology rings of the two manifolds.

$$\begin{aligned} f^* : H^*(X) &\to H^*(Y) \\ [\omega] \mapsto f^*[\omega] &= [f^*\omega] \end{aligned}$$

Satisfying

$$f^*[\omega] \wedge f^*[\eta] = f^*[\omega \wedge \eta]$$

We can ask ourselves if given $f : Y \to X$ we can build a similar construction, but going on the direction of the morphism $Y \to X$. The answer is yes, and we can do it by means of Poincaré duality.
In fact we can define the push-forward map between r-chains

$$f_* : C_r(Y) \to C_r(X)$$

Acting linearly on singular r-simplexes. Explicitly, let $\sigma_Y : \Delta^r \to Y$ be a singular r-complex on Y, then we can define a singular r-complex on X by

$$f_*\sigma_Y = f \circ \sigma_X : \Delta^r \to X$$

Note that f_* do not preserve the dimension of the image of σ_Y.
This map has similar properties to the pull-back map that we have defined above, one can show that

$$\partial f_* = f_* \partial$$

So that f_* descends to a map between homology groups

$$f_* : H_*(Y) \to H_*(X)$$

Now by means of Poincaré duality we can define the push-forward map $f_* : H^*(Y) \to H_*(X)$ by pre-composing and post-composing the map $f_* H_*(Y) \to$

$H_*(Y)$ with Poincaré duality isomorphisms.
This actually coincides with the following construction, consider the pairing we have defined above

$$(\, , \,) : H^r(Y) \otimes H_r(Y) \to \mathbb{R}$$
$$[c] \otimes [\omega] \mapsto \int_c \omega$$

Let $f : Y \to X$ be a proper morphism of manifolds (which means that inverse images of compact sets are compact), then we can define

$$f_* : H^*(Y) \to H^*(X)$$

defined as the adjoint of the inverse image, which means by integration of fibers

$$(f_*\omega, C) = (\omega, p^{-1}(C))$$

where $\omega \in H^*(Y)$, $C \in H_*(X)$.

This clearly does not preserve the cohomology degree. For example let us assume that f is a surjective morphism with fibers of constant dimension and consider $\omega \in H^r(Y)$, then the pairing is non vanishing only if $p^{-1}(C) \in H_n(Y)$. The dimension of fibers over points is constant and equal to $\mathrm{dim}Y - \mathrm{dim}X$, so that

$$\mathrm{dim}C = \mathrm{dim}p^{-1}(C) + \mathrm{dim}X - \mathrm{dim}Y = r + \mathrm{dim}X - \mathrm{dim}Y$$

Thus $f_*\omega \in H^{r+\mathrm{dim}X-\mathrm{dim}Y}(X)$ and has a lower degree than $\omega \in H^r(Y)$.

This implies that f_* and f^* are not inverse to each other. However there is a useful formula connecting them, in particular

$$f_*(\alpha f^*\beta) = f_*(\alpha)\beta$$

where one can subsitute $\alpha = 1 \in H^0(Y)$.

References

[1] E. Witten, Two-dimensional gravity and intersection theory on moduli space. In Surveys in differential geometry (Cambridge, Mass., 1990), Vol. I, Lehigh University, Bethlehem, PA, 1991, 243-310.

[2] D.Zvonkine, An introduction to moduli spaces of curves and their intersection theory. 2014.

[3] J. M. F. Labastida, M. Pernici and E. Witten, Topological Gravity in Two-Dimensions, Nucl. Phys. B **310** (1988), 611-624 doi:10.1016/0550-3213(88)90094-6

[4] S. Chiantese, On the moduli space of generalized holomorphic maps, [arXiv:hep-th/0609174 [hep-th]].

[5] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa et al., Mirror symmetry, vol. 1 of Clay mathematics monographs, AMS, Providence, USA (2003).

[6] V. Pestun, M. Zabzine, F. Benini, T. Dimofte, T. T. Dumitrescu, K. Hosomichi, S. Kim, K. Lee, B. Le Floch and M. Marino, *et al.* Localization techniques in quantum field theories, J. Phys. A **50** (2017) no.44, 440301 doi:10.1088/1751-8121/aa63c1 [arXiv:1608.02952 [hep-th]].

[7] S. Cremonesi, An Introduction to Localisation and Supersymmetry in Curved Space, PoS **Modave2013** (2013), 002 doi:10.22323/1.201.0002

[8] E. Witten, Mirror manifolds and topological field theory, AMS/IP Stud. Adv. Math. **9** (1998), 121-160 [arXiv:hep-th/9112056 [hep-th]].

[9] E. Witten, Topological Sigma Models, Commun. Math. Phys. **118** (1988), 411 doi:10.1007/BF01466725

[10] N. Garner and N. M. Paquette, Mathematics of String Dualities, PoS **TASI2021** (2023), 007 doi:10.22323/1.403.0007 [arXiv:2204.01914 [hep-th]].

[11] C. Closset and S. Cremonesi, Comments on $\mathcal{N} = (2, 2)$ supersymmetry on two-manifolds, JHEP **07** (2014), 075 doi:10.1007/JHEP07(2014)075 [arXiv:1404.2636 [hep-th]].

[12] U. Lindström, $2d$ Sigma Models and Geometry, [arXiv:2203.02974 [hep-th]].

[13] K. Hori and C. Vafa, Mirror symmetry, [arXiv:hep-th/0002222 [hep-th]].

[14] B. Zumino, Supersymmetry and Kahler Manifolds, Phys. Lett. B **87** (1979), 203 doi:10.1016/0370-2693(79)90964-X

[15] M. Rocek, Supersymmetry and Nonlinear σ Models, Physica D **15** (1985), 75-82 ITP-SB-84-30.

[16] R. Dijkgraaf, H. L. Verlinde and E. P. Verlinde, Notes on topological string theory and 2-D quantum gravity, PUPT-1217.

[17] R. C. Myers and V. Periwal, Topological Gravity and Moduli Space, Nucl. Phys. B **333** (1990), 536-550 doi:10.1016/0550-3213(90)90050-N

[18] E. Witten, On the Structure of the Topological Phase of Two-dimensional Gravity, Nucl. Phys. B **340** (1990), 281-332 doi:10.1016/0550-3213(90)90449-N

[19] J.Kock, Notes on psi classes, 2001

[20] M. Vonk, A Mini-course on topological strings, [arXiv:hep-th/0504147 [hep-th]].

[21] W. Lerche, C. Vafa and N. P. Warner, Chiral Rings In N=2 Superconformal Theories, Nucl. Phys. B 324, 427 (1989).

[22] S. Cecotti and C. Vafa, Topological - antitopological fusion, Nucl. Phys. B 367, 359 (1991).

[23] E. Verlinde, H. Verlinde, A solution of two-dimensional topological quantum gravity, Nucl. Phys. B348 457-489 (1991).