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Dynamic formulations of L^1 optimal transport problems

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Abstract

The purpose of this work is to analyse the relationships between different formulations of Monge-Kantorovich transport problem, in the case of L^1 cost. In particular, we focus on the famous Benamou and Brenier's dynamical formulation, finding a way to switch their model from the quadratic cost to our cost function. Helped by the solutions of the other formulations, we find a way to solve the dynamical problem, with L^1 cost, and we compare the solution to the one found by Benamou and Brenier, with L^2 cost. For this aim, we also analyse the intermediate cases of L^p costs, with $1 < p < 2$, recurring to a numerical scheme based on a smart adaptation of Newton method.

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Introduction

The history of optimal transport began in 1781, when the French mathematician Gaspard Monge proposed to the *Académie des Sciences* a problem written in a report called "Mémoire sur la Theorie des Déblais et des Remblais" [37]. This problem consisted, briefly, in finding the best way to move a pile of soil ("déblais") from the ground to a hole ("remblais"), in order to fill it (see Figure 1).

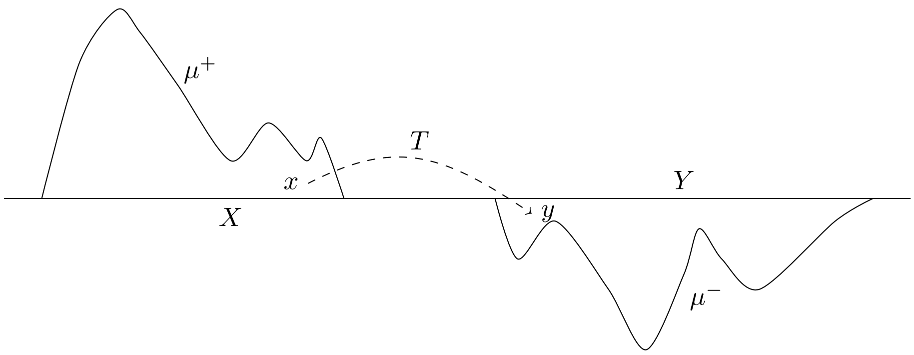


FIGURE 1. Graphic representation of the mass transportation problem.

Obviously, the pile and the hole must have the same volume. Thus, both the pile and the hole are modeled by finite measures μ^+ and μ^-, with the same mass, defined on some subspaces X and Y of $\mathbb{R}^n$. The movement is described by a map T, which has to "push" the initial configuration onto the final arrangement. Moving the sand needs clearly some effort which is modelled by a cost function $c(x, y)$ defined on $X \times Y$. In the description of Monge, the cost function was just the Euclidean distance $c(x, y) = \|x - y\|$. Apparently, it seems a simple problem, but if one tries to deal with it, he immediately discovers that it hides a highly nonlinear structure. This was the reason why the *Académie des Sciences* offered a price to anyone who would be able to solve this problem. The French mathematician Paul Émile Appell won this price, even if he was just able to provide only marginal results about its solution. Thus, the problem remained without a solution, until progress was made by Kantorovich [32], [33]. He inserted the problem into a more general framework, which gave the possibility to use standard mathematical tools and, later, to provide a solution. To do that, he started to look at the problem from a different point of view, which allowed him to see a link between this problem and linear programming. Moreover, he widely generalized the problem, taking into account general cost functions $c(x, y)$, different from the one considered by Monge, defined on more general measure spaces. The case of quadratic cost function $c(x, y) = \|x - y\|^2$, physically related to the kinetic energy, was of particular interest in literature, as, more generaly, strictly convex cost functions on $x - y$. In these cases, as first shown by Brenier [14] in the L^2 cost case, the optimal solution of Kantorovich's reformulation provides a solution to the problem formulated by Monge. Thus, while in these cases things worked quite well, with the original L^1 cost function adopted by Monge, a solution had not been found yet. Sudakov [43] was the first to propose a solution with this cost function. He decomposed $\mathbb{R}^n$ in the disjoint union of segments with special features, reduced the problem into a family of 1D problems and finally glued together the 1D maps constructed on these segments. The strategy seemed to work properly, but its complete definition was proposed only later by Ambrosio [1]. Starting from a deformation argument due to Dacorogna and Moser [20], Evans and Gangbo [27] proposed an alternative method based on differential equations. Finally, a solution was found.

This problem, however, can not be merely reduced to the study about the existence and the properties of a solution. Indeed, the beauty and the greatness of the problem introduced by Monge are due to its generality. It has a number of diverse applications, coming from different branches of mathematics. It is linked to flow minimization problems describing traffic issues, to defining distances in the space of probabilities, image processing, economics, meteorology, etcetera. After this brief historical background, a normal question which arises could be: “why this thesis?” The literature is full of books fully describing almost everything about optimal transport, starting from the “bible” of the field medalist Villani [46] or the more recent work of Santambrogio [41] or, even, the lecture notes of Ambrosio [1]. But the “problem” that we found is exactly this: the literature is full of books or articles about optimal transport and, especially in the case of L^1, the results are scattered on different works and the situation is quite chaotic. Indeed, if the problem is connected with several applications coming from different fields, it is normal that there will be many different reformulations of the same problem. Thus, this work born from this need to try to put some order in the situation, in the case of L^1 cost, and find the relations between different formulations. Our study starts from one of these formulations, taken from fluid dynamic, firstly introduced in Benamou and Brenier [5]. In this famous work, they introduced a time variable and looked at the way in which the system evolved in time. The problem we have to face is that they treated the L^2 cost case and their procedure is not adaptable to L^1. Therefore, we need an alternative formulation of Benamou and Brenier's dynamic problem and we find an original way to adapt this problem to the L^1 cost case, taking inspiration from Chen, Georgiou, and Pavon [18]. Thus, keeping this new dynamic formulation at the center of the attention, using the information obtained from the other formulations, we try to recover the optimal solution of this formulation with L^1 cost, in the same way Benamou and Brenier did, but with a different method. The result we achieve, compared with the solution with a quadratic cost function, is somewhat unexpected. Thus, we focus on the result obtained, trying to understand the main differences between the two cases, both from geometrical and analytical point of views. To focus once more these differences, we want to graphically see how the dynamical behavior of the optimal solutions differ in time, when we change the cost function and when we change the initial and the final data. First, we see some simulations in the already known cases of L^1 and L^2 costs. Later, helped by a numerical scheme based on Newton method, we see how optimal densities change in the intermediate cases of L^p costs, with $1 < p < 2$. The structure of the thesis is the following.

In Chapter 1 we provide some preliminary concepts, grouped in different mathematical fields, in order to improve the flow of the reading.

Chapter 2 summarizes the chronological progresses that contributed to this work, starting from the rigorous mathematical description of Monge and Kantorovich formulations of the optimal transport problem and passing through other versions linked to these classical ones, in order to obtain, at the end, a way to recover an optimal transport map in the case of L^1 cost function.

In Chapter 3, after we presented two more formulations of the optimal transport in the L^1 cost case, we try to find the relationships underlying these different formulations, attempting to untangle the complex situation.

In Chapter 4 we focus on some of these formulations, trying to go deeper into the underlying reasons and motivations. In order to do that, we also present some numerical and graphical examples which allow the reader to better understand these particularly important results.

In this work, we divided the original proofs from those taken from other books or articles, specifying every time the source.

I would like to thank my advisors, Professor Mario Putti and Michele Pavon, and my co-advisor, Enrico Facca, for the invaluable support and the precious advice they have offered me throughout this work together.

CHAPTER 1

Notation and some preliminary results

In the course of this work, we shall use some notations and well-known results, which will be reported briefly in this chapter in order to clarify the meaning of the different symbols and to improve the readability. First, the main notation will be listed and then the preliminary results used throughout the work will be reported, grouped into homogeneous areas.

1. Notation

- $\mathbb{R}_+$: non-negative real numbers, i.e. $[0, +\infty[$.
- $\overline{\mathbb{R}}$: the extended real number system, i.e. $\mathbb{R} \cup \{-\infty, +\infty\}$.
- $\mathbb{R}^n$: the standard n-dimensional Euclidean space.
- π_x, π_y, π_t: the projection of a product space onto one of its components.
- ν_Ω: the outward normal vector to a given domain Ω.
- δ_a: the Dirac mass concentrated at the point a.
- $\mathcal{L}^n$: the Lebesgue measure on $\mathbb{R}^n$.
- $M \times N$: the product space between M and N.
- $\mu \otimes \nu$: the product measure of μ and ν, i.e. $\mu \otimes \nu(A \times B) := \mu(A)\nu(B)$.
- $\sim$: $X \sim Y$ this symbol stands for two random variables X and Y with the same distributions.
- a.e.: given a measure space $(\Omega, \mathcal{A}, \mu)$, we say that a property P holds almost everywhere (a.e.) if $\mu(\{x \in \Omega \ : \ \overline{P}(x)) = 0$.

Consider a matrix A:

- A^T: is the transpose matrix of A.
- $\det(A)$: is the determinant of the matrix A.
- $\mathrm{tr}(A)$: is the trace of A.
- A_{ij}: is the minor of the entry (i, j) of the square matrix A, i.e. the determinant of the submatrix obtained deleting the i-th row and the j-th column.
- $\mathrm{cof}(A)$: is the cofactor matrix of A, i.e. the matrix whose (i, j) entry is obtained by multypling the minor A_{ij} by $(-1)^{i+j}$.
- e_j: is the unitary vector in the standard basis, with 1 in the j-th coordinate and 0's elsewhere, i.e. $e_j := (0, \dots, 0, 1, 0, \dots, 0)^T$.
- $\mathbb{1}$: is the vector of ones, i.e. $\mathbb{1} := (1, \dots, 1)^T$.
- $\mathbb{I}$: is the identity matrix, i.e. $\mathbb{I} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$.

Consider a scalar function $f : \mathbb{R}^n \to \mathbb{R}$:

- $\mathrm{supp}(f) := \{x \in \mathbb{R}^n \ : \ f(x) \neq 0\}$ is the support of f.
- $\mathrm{dom}(f)$: is the natural domain of a function, i.e. the maximal set of values for which the function is defined. In this case it is $\mathrm{dom}(f) = \{x \in \mathbb{R}^n \ : \ f(x) \in \mathbb{R}\}$.
- $f^+(x) := \max\{f(x), 0\}$, $f^-(x) := \max\{-f(x), 0\}$.
- ∇: the gradient, which is the vector $\nabla f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)^T$.
- $\nabla\cdot$: the divergence, which is defined as $\nabla \cdot f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}$.

- Δ: the Laplacian, which is $\Delta f := \nabla \cdot (\nabla u) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$.
- D^2: the Hessian matrix, i.e. $D^2 f := \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$.
- D: if we consider a function $T : \mathbb{R}^n \to \mathbb{R}^m$, the Jacobian matrix DT is $DT(x) = \begin{pmatrix} \frac{\partial T_1(x)}{\partial x_1} & \cdots & \frac{\partial T_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial T_n(x)}{\partial x_1} & \cdots & \frac{\partial T_n(x)}{\partial x_n} \end{pmatrix}$.
- $dJ(u, \phi)$: if we consider a functional $J : V \to W$, with V, W Banach spaces, the Gâteaux differential of J at $u \in U \subset X$ (U open set) in the direction $\phi \in X$ is $\frac{d}{d\varepsilon} J(u+\varepsilon\phi)\big|_{\varepsilon=0}$. Note that it will also be called the first variation of J in direction ϕ.

Consider a set Ω:

- Ω^{C}: is the complement of Ω.
- $\overline{\Omega}$: is the closure of Ω, i.e. the smallest closed set containing Ω.
- Ω°: is the interior of Ω, i.e. the largest open set contained in Ω.
- $\mathcal{P}(\Omega)$: is the power set of Ω, i.e. the collection of subsets of Ω.
- χ_Ω: is the characteristic function of Ω, i.e. $\chi_\Omega(x) := \begin{cases} 1, & \text{if } x \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$
- $\mathcal{C}(\Omega)$, $\mathcal{C}_b(\Omega)$, $\mathcal{C}_c(\Omega)$, $\mathcal{C}_0(\Omega)$: the sets of functions from Ω to $\mathbb{R}$ which are continuous, bounded continuous, compactly supported continuous and continuous vanishing at infinity functions, respectively. If the codomain is different, it will be specified: for example, $\mathcal{C}(\Omega; \mathbb{R}^n)$.
- $\mathrm{Lip}_C(\Omega)$: is the set of all Lipschitz continuous functions with Lipschitz constant $\mathrm{Lip}(f) = C$, i.e. all the functions f such that $|f(x) - f(y)| \leq C|x-y|$, for all x, $y \in \Omega$.
- $\mathrm{AC}(I; \Omega)$: is the set of all the absolutely continuous functions defined on the interval $I \subset \mathbb{R}$, i.e. $\gamma \in \mathrm{AC}(I; \Omega)$ if and only if γ is differentiable a.e. and $\gamma' \in L^1(I)$.

Consider a general topological space (X, τ):

- $\mathcal{B}(X)$: is the Borel σ-algebra, i.e. the smallest σ-algebra containing the topology τ.
- $\mathcal{M}_+(X)$: is the space of the positive finite Borel measures on X.

In this last part, we report some notations we will define throughout this work. Anyway, in order to make the reading easier, we report them here. Given a measurable space $(\Omega, \mathcal{A})$, we denote by:

- $L^p(\Omega)$, $L^1_{\mathrm{loc}}(\Omega)$, $W^{1,p}(\Omega)$, $W^{1,p}_0(\Omega)$: the L^p space, locally integrable space, Sobolev space and compactly supported Sobolev space of functions, respectively.
- $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$: the scalar product defined, for $f, g \in L^2(\Omega)$, as:
$$\langle f, g \rangle_{L^2(\Omega)} := \int_\Omega f(x) g(x) d\mu, \quad f, g \in L^2(\Omega).$$
- $\langle \cdot, \cdot \rangle_{L^p(\Omega)}$: the dual pairing defined, for $\phi \in (L^p(\Omega)^*$, as:
$$\langle \phi, u \rangle_{L^p(\Omega)} = \int_\Omega u \cdot g \; d\mu \quad \forall u \in L^p(\Omega).$$
- $\langle \cdot, \cdot \rangle_{\mathcal{C}_0(\Omega; \mathbb{R}^n)}$: the dual pairing, defined for $\Phi \in (\mathcal{C}_0(\Omega; \mathbb{R}^n))^*$, as:
$$\langle \Phi, f \rangle_{\mathcal{C}_0(\Omega; \mathbb{R}^n)} = \int_\Omega f \cdot d\mu = \sum_{i=1}^{n} \int_\Omega f_i d\mu_i, \quad \forall f \in \mathcal{C}_0(\Omega; \mathbb{R}^n).$$

- $\mathcal{M}^1(\Omega) = \mathcal{M}(\Omega)$: the space of the finite signed Borel measures.
- $\mathcal{M}^n(\Omega) := \{\mu : \mathcal{B}(\Omega) \to \mathbb{R}^n \ : \ \mu \text{ is a finite vector measure}\}$, the set of all the finite vector measures defined on the Borel set $\mathcal{B}(\Omega)$ with values in $\mathbb{R}^n$.
- $\mathcal{M}^n_{\text{div}}(\Omega) := \left\{\mu \in \mathcal{M}^n(\overline{\Omega}) \ : \ \nabla \cdot \mu \in \mathcal{M}(\overline{\Omega})\right\}$;
- $\mathcal{M}^n_{\text{div},0}(\Omega) := \{\mu \in \mathcal{M}^n_{\text{div}}(\Omega) \ : \ \mu \cdot \nu_\Omega = 0\}$.
- $\mathbb{P}(\Omega)$: the set of all the probabilitiy measures defined on the measurable space $(\Omega, \mathcal{B}(\Omega))$.
- $\mathbb{P}_p(\Omega) = \left\{\mu \in \mathbb{P}(\Omega) \ : \ \int_\Omega \|x\|^p \, d\mu(x) < +\infty\right\}$.

Given two finite positive measures μ^+ and μ^-:

- $\mathcal{T}(\mu^+, \mu^-) := \{T : X \to Y | \text{ T is Borel measurable, one-to-one and } T_\#(\mu^+) = \mu^-\}$.
- $\Pi(\mu^+, \mu^-) := \left\{\mu \in \mathcal{M}_+(\mathbb{R}^n \times \mathbb{R}^n) \mid \pi_x(\mu) = \mu^+, \ \pi_y(\mu) = \mu^-\right\}$.
- $\mathcal{X} := \mathcal{C}^1([0,1]; \mathbb{R}^n)$.
- $\mathcal{Q}^1(\mu^+, \mu^-) := \left\{P \in \mathbb{P}(\mathcal{X}) \mid (e_0)_\# P = \mu^+ \text{ and } (e_1)_\# P = \mu^-\right\}$.
- $$\mathcal{Q}^p(\mu^+, \mu^-) := \{Q \in \mathbb{P}(\mathcal{X}) \mid (e_0)_\# Q = \mu^+, \ (e_1)_\# Q = \mu^- \text{ and } \\ di_Q(x) = f_Q(x)dx \text{ with } f_Q(x) \in L^p(\Omega)\}.$$
- $d_{KR_p}(\mu^+, \mu^-)$: the p-Kantorovich-Rubinstein distance between μ^+ and μ^-.

Given a metric space (V, d):

- $\text{CSG}([0,1]; V) := \{\gamma \in \text{AC}([0,1]; V) \ : \ \gamma \text{ is a constant speed geodesic in } V\}$.

Given two functions f^+ and f^-:

- $\mathbb{P}(f^+, f^-) := \left\{\rho : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n \mid \rho(0,x) = f^+(x) \text{ and } \rho(1,y) = f^-(y)\right\}$.

2. Elements of Real and Functional Analysis

Before starting with some preliminary results taken from real and functional analysis, let us make a small clarification. Throughout the whole work, any integral will be thought in the sense of Lebesgue and the integral of any (Lebesgue) measurable scalar function $f : \Omega \to \mathbb{R}$ will be denoted by:

$$\int_\Omega f(x)dx.$$

For a reader who wants to go deeper into the details about Lebesgue integration, we suggest a couple of standard books, such as Folland [29] and Rudin [39].

We want to focus, now, on some important spaces of functions and on their main properties. For a deeper reading about this argument we suggest, again, Rudin [39] and Folland [29]. Consider a measure space $(\Omega, \mathcal{A}, \mu)$.

Definition 1.1 (L^p spaces). For $1 \le p < +\infty$, the L^p space is defined as:

$$L^p(\Omega, \mathcal{A}, \mu) = L^p(\Omega) := \left\{f \text{ is } \mu\textit{-measurable} \ : \ \|f\|_p < +\infty\right\},$$

where the p-norm is defined as:

$$\|f\|_p := \left(\int_\Omega |f(x)|^p \, d\mu\right)^{\frac{1}{p}}.$$

For $p = +\infty$, the L^∞ space is defined as:

$$L^\infty(\Omega, \mathcal{A}, \mu) = L^\infty(\Omega) := \{f \text{ is } \mu\textit{-measurable} \ : \ \|f\|_\infty < +\infty\},$$

where the *essential supremum* is defined as:

$$\|f\|_\infty := \inf\{M \ge 0 \ : \ |f(x)| \le \text{ for } \mu\text{-a.e. } x \in \Omega\}.$$

It can be checked that the couple $(L^p(\Omega), \|\cdot\|_p)$ is a Banach space and, in particular, for $p = 2$, the couple $(L^2(\Omega), \langle\cdot,\cdot\rangle_{L^2(\Omega)})$ is a Hilbert space equipped with the inner product:

$$\langle f, g\rangle_{L^2(\Omega)} := \int_\Omega f(x)g(x)d\mu, \quad f, g \in L^2(\Omega). \tag{1.1}$$

Note that, to be precise, in order to satisfy the property of the norm:

$$\|f\|_p = 0 \implies f = 0,$$

we should identify functions which are equal almost everywhere. Moreover, we should consider the L^p spaces as quotient spaces. Anyway, the L^p spaces and their quotients are usually identified, for simplicity. Throughout this work, another particular space will also appear:

Definition 1.2 (Locally integrable). A Lebesgue measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *locally integrable*, and it is denoted by $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, if for every compact set $K \subset \mathbb{R}^n$ we get $f \in L^1(K)$.

This kind of functions are used, for example, inside the following classical theorem, which is a well-known result, taken from real analysis.

Theorem 1.1 (Lebesgue differentiation). *Consider $f \in L^1_{loc}(\mathbb{R}^n)$. Then for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$ we get:*

$$\lim_{r\to 0^+} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} f(y)dy = f(x).$$

After recalling that a measure μ, defined on a measurable space $(\Omega, \mathcal{A})$ is *σ-finite* if Ω is a countable union of measurable sets with finite measure, we state another famous result which will be useful in the course of this work.

Theorem 1.2 (Fubini-Tonelli). *Suppose $(M, \mathcal{A}, \mu)$ and $(N, \mathcal{B}, \nu)$ two σ-finite measure spaces.*

(i) (Fubini) If $f \in L^1(\mu \otimes \nu)$, then, if we denote by:

$$f_x(y) := f^y(x) := f(x, y), \quad \forall (x, y) \in M \times N,$$

then $f_x \in L^1(\nu)$ for a.e. $x \in M$, $f^y \in L^1(\mu)$ for a.e. $y \in N$. Moreover, the a.e.-defined functions $g(x) := \int_N f_x(y)d\nu(y)$ and $h(y) := \int_M f^y(x)d\mu(x)$ are in $L^1(\mu)$ and $L^1(\nu)$, respectively. Lastly, the following equality holds:

$$\begin{aligned}\int_{M\times N} f(x,y)d(\mu\otimes\nu)(x,y) &= \int_M \left(\int_N f(x,y)d\nu(y)\right) d\mu(x) \\ &= \int_N \left(\int_M f(x,y)d\mu(x)\right) d\nu(y).\end{aligned} \tag{1.2}$$

(ii) (Tonelli) If $f : M \times N \to [0, +\infty]$ is a non-negative $\mu \otimes \nu$-measurable function, then $g(x) = \int_N f_x(y)d\nu(y)$ and $h(y) = \int_M f^y(x)d\mu(x)$ are μ-measurable and ν-measurable, respectively, and (1.2) *holds.*

Inside these spaces we can define different types of convergence which are possible for a sequence $(f_n)_{n\in\mathbb{N}}$. We recall, here, just those modes of convergence which will be used more inside this thesis.

Definition 1.3 (Different modes of convergence).

- **Pointwise convergence**: we say that $f_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, converges *pointwise* to $f : \Omega \to \mathbb{R}$ if:

$$\lim_{n\to+\infty} f_n(x) = f(x), \quad \forall x \in \Omega.$$

- **Almost everywhere convergence**: we say that $f_n : \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, converges μ-*almost everywhere* (μ-a.e.) to $f : \Omega \to \mathbb{R}$ if there exists a set $N \subset \Omega$ such that $\mu(N) = 0$ and for all $x \in \Omega \setminus N$, we have:
$$\lim_{n\to+\infty} f_n(x) = f(x).$$
- **Strong convergence**: let $1 \leq p \leq +\infty$. We say that $f_n \to f$ *strongly* in L^p if $f_n, f \in L^p(\Omega)$ and
$$\lim_{n\to+\infty} \|f_n - f\|_p = 0.$$
- **Weak convergence**: let $1 \leq p < +\infty$. We say that f_n converges *weakly* to f in $L^p(\Omega)$, and we denote it by $f_n \rightharpoonup f$ in L^p, if $f, f_n \in L^p(\Omega)$ and for all $g \in L^q(\Omega)$, with $1 < q \leq +\infty$ the Hölder conjugate of p, i.e.
$$\frac{1}{p} + \frac{1}{q} = 1, \tag{1.3}$$
we have:
$$\lim_{n\to+\infty} \int_\Omega f_n(x)g(x)d\mu = \int_\Omega f(x)g(x)d\mu.$$
- **Convergence in measure**: given f_n, f measurable functions, $n \in \mathbb{N}$, we say that $f_n \to f$ in measure if for every $\varepsilon > 0$ we have:
$$\lim_{n\to+\infty} \mu\left(\{x \in \Omega \ : \ |f_n(x) - f(x)| \geq \varepsilon\}\right) = 0.$$

Let us remind the reader about another group of spaces: the Sobolev spaces. For a complete description and characterization of Sobolev spaces see, for example, Burenkov [16].

Definition 1.4 (Sobolev spaces $W^{1,p}$). Consider $\Omega \subset \mathbb{R}^n$ open set and $1 \leq p \leq \infty$. The *Sobolev space* $W^{1,p}(\Omega)$ is defined by:

$$W^{1,p}(\Omega) := \Big\{ f \in L^p(\Omega) \ : \ \exists g_1, \dots, g_n \in L^p(\Omega) \text{ such that:} \\ -\int_\Omega f(x)\frac{\partial \phi(x)}{\partial x_i}dx = \int_\Omega g_i(x)\phi(x)dx \quad \forall i = 1, \dots, n, \ \forall \phi \in \mathcal{C}_c^\infty(\Omega) \Big\}.$$

Note that, calling the functions g_i *distributional derivatives*, we get the following link between these spaces and L^p spaces:

- if $f \in L^p(\Omega)$, then $f \in W^{1,p}(\Omega)$ if and only if all of its distributional derivatives are in $L^p(\Omega)$;
- if $f \in \mathcal{C}^1(\Omega) \cap L^p(\Omega)$, then $f \in W^{1,p}(\Omega)$ if and only if $\frac{\partial f}{\partial x_i} \in L^p(\Omega)$ for all $i = 1, \dots, n$.

The Sobolev space $W^{1,p}(\Omega)$ is equipped with the following norm:

$$\|f\|_{W^{1,p}} := \|f\|_p + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_p, \text{ or, equivalently,}$$
$$\|f\|_{W^{1,p}} := \left(\|f\|_p^p + \sum_{i=1}^n \left\| \frac{\partial u}{\partial x_i} \right\|_p^p \right)^{\frac{1}{p}}.$$

We, now, want to introduce some tools taken from functional analysis, starting from preliminary definitions, just to have everything we will need properly defined, and proceeding with some non-trivial results. For the details about this part of functional analysis we refer mainly to Brezis [15]. Consider two normed spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|)_W$.

Definition 1.5 (Operator norm). Given a linear operator $T : V \to W$, the *operator norm* is:
$$\|T\|_{\text{op}} := \sup_{x\neq 0} \frac{\|Tx\|_W}{\|x\|_V}.$$

In this context, a linear operator $T : V \to W$ is continuous if and only if $\|T\|_{\text{op}} < +\infty$, i.e. bounded operators and continuous operators coincide. We will denote the space of linear and continuous operators by:

$$\mathcal{L}(V, W) := \{T : V \to W \ : \ T \text{ is linear and continuous}\} .$$

Moreover:

DEFINITION 1.6 (Dual space). Given a normed vector space $(E, \|\cdot\|)$, the *dual space* is:

$$E^* := \mathcal{L}(E, \mathbb{R}).$$

This is a normed space, equipped with the operator norm:

$$\|T\|_{\text{op}} := \sup_{x \neq 0} \frac{|\langle T, x\rangle_*|_F}{\|x\|_E},$$

where $\langle \cdot, \cdot \rangle_* \to \mathbb{R}$, is the dual pairing of these spaces, i.e. the evaluation of T against x.

Note that the $*$ below the scalar product is used to distinguish this dual product from a normal scalar product. However, in the following Riesz Theorems, we will use a specific notation for any scalar product, depending on the particular space in which is defined, always having in mind that they are dual pairings.

THEOREM 1.3 (Riesz representation for L^p spaces). *Let $1 \leq p < +\infty$ and $(\Omega, \mathcal{A}, \mu)$ is σ-finite. Then for all $\phi \in (L^p(\Omega))^*$, there exists one and only one $g \in L^q(\Omega)$, with q Hölder conjugate of p as defined in* (1.3)*, such that:*

$$\langle \phi, u\rangle_{L^p(\Omega)} = \int_\Omega u \cdot g \, d\mu \quad \forall u \in L^p(\Omega). \tag{1.4}$$

Moreover, $\|\phi\|_{op} = \|g\|_q$.

Note that this theorem says that for $1 \leq p < +\infty$ the map:

$$\begin{aligned} (L^p(\Omega))^* &\to L^q(\Omega) \\ \phi &\mapsto g \end{aligned}$$

is a linear surjective isometry, which allow us to identify the two spaces. Note, also, that the case $p = +\infty$ is excluded, because the dual space of $L^\infty(\Omega)$ is strictly bigger then $L^1(\Omega)$, i.e. there are functionals on $(L^\infty(\Omega))^*$ that cannot be represented as an integral against a function $g \in L^1(\Omega)$. We are going to recall, now, some famous and important definitions in this field:

DEFINITION 1.7 (Compactness, sequential compactness and relative compactness). Given a topological space (X, τ), a subspace $K \subset X$ is:

(i) *compact* if from every open cover of K, we can extract a finite subcover, i.e. if $K \subset \bigcup_{i \in I} A_i$, with $A_i \in \tau$ for all $i \in I$, there exist $A_1, \ldots, A_n$ such that $K \subset \bigcup_{i=1}^n A_i$.
(ii) *sequentially compact* if for every sequence contained in K, $\{x_n\} \subset K$, there exist a convergent subsequence, $x_{n_k} \overset{k\to\infty}{\longrightarrow} x \in K$.
(iii) *relatively compact* if $\overline{K}$ is compact.

In the case of $\mathbb{R}^n$, for example, the conditions of being closed and bounded fully characterize compact sets, but, in general, in infinite dimensional spaces it is not as easy to understand. We have the following:

THEOREM 1.4. *Given a metric space (X, d), then:*

$$K \subset X \textit{ is compact} \iff K \subset X \textit{ is sequentially compact.}$$

In this part we will just focus our attention on the conditions characterizing sequentially compact sets in the case of L^p spaces, which will be the only case considered in this work. Note that, the convergence of this subsequence will be “accepted” in any kind of way. Thus, we need to introduce the weak and weak-* convergence, which arise from the notion of weak and weak-* topologies, but, in the way in which we need these tools, we can give self-sufficient and equivalent definitions.

Definition 1.8 (Weak and weak-* convergence). Given a normed space $(E, \|\cdot\|_E)$, we say that:

(i) the sequence $\{x_n\}_{n\in\mathbb{N}} \subset E$ is *weakly convergent* to $x \in E$, and we write $x_n \rightharpoonup x$, if for every $f \in E^*$, we have $\langle f, x_n\rangle_* \to \langle f, x\rangle_*$.

(ii) the sequence $\{f_n\}_{n\in\mathbb{N}} \subset E^*$ is *weakly-* convergent* to $f \in E^*$, and we write $f_n \overset{*}{\rightharpoonup} f$, if for every $x \in E$, we have $\langle f_n, x\rangle_* \to \langle f, x\rangle_*$.

This definition elucidates the definition of the weak convergence given above in the case of L^p spaces (Definition 1.3), using Riesz representation's Theorem 1.3. We need, now, another definition, in order to capture the difference between the case of L^p spaces, with $1 < p < +\infty$, and L^1. This distinction between the two cases will be used later.

Definition 1.9 (Reflexive space). Consider a normed space $(E, \|\cdot\|_E)$, with the canonical injection map $J : E \to E^{**}$ given by:

$$\langle J(x), f\rangle_* = \langle f, x\rangle_*, \quad \forall x \in E,\ \forall f \in E^*.$$

The space is said *reflexive* if J is surjective.

An example of reflexive spaces are, indeed, the L^p spaces for $1 < p < +\infty$, while the L^1 space is *not* reflexive. We have the following:

Theorem 1.5. *If the normed space $(E, \|\cdot\|_E)$ is reflexive, every bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ admits a weakly converging subsequence (i.e. the bounded sets are relatively sequentially weakly compact).*

In the case of L^1 spaces, we cannot use this result. In this case, there is another characterization:

Theorem 1.6 (Dunford-Pettis). *Given $\Omega \subset \mathbb{R}^n$ measurable set, $F \subset L^1(\Omega)$ bounded set. Then F is relatively weakly compact $\iff$ F is* equintegrable*, i.e.*

(i) for all $\varepsilon > 0$ it exists a $\delta > 0$ such that:

$$\int_A |f|\, dx < \varepsilon, \quad \forall A \subset \Omega \text{ measurable with } \mathcal{L}^n(A) < \delta,\ \forall f \in F;$$

(ii) for all $\varepsilon > 0$, it exists $\omega \subset \mathbb{R}^n$ with finite measure such that:

$$\int_{\Omega\setminus\omega} |f|\, dx < \varepsilon, \quad \forall f \in F.$$

Obtaining a weakly converging subsequence in applications is very important. For example:

Proposition 1.1 (Direct method in the Calculus of Variations). *Consider a Banach space V and a functional $J : V \to (-\infty, +\infty]$ bounded from below. Then the* Direct method in the Calculus of Variations *is divided into three steps:*

(i) Take a minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \subset V$, i.e. $\lim_{n\to+\infty} u_n = \min\{J(u) \ :\ u \in V\}$.

(ii) Show that u_n admits some subsequence u_{n_k} that converges weakly to some $u_0 \in V$, i.e. $u_{n_k} \rightharpoonup u_0$.

(iii) Show that J is weakly sequentially lower-semicontinuous, i.e. for any weakly convergent sequence $u_n \rightharpoonup u_0$, it follows that $\liminf_{n\to+\infty} J(u_n) \geq J(u_0)$.

Once these conditions are proved, we have that u_0 is a minimizer for J.

To conclude this section, we want to present an example that mixes together integration, measure theory and duality properties, which will be used later in this thesis. For more details about this part we refer to Rudin [39].

Definition 1.10 (Vector measure). A *vector measure* on a measurable space $(\Omega, \mathcal{A})$ is a map with values in $\mathbb{R}^n$, i.e. $\mu : \mathcal{A} \to \mathbb{R}^n$ such that:

- $\mu(\emptyset) = 0$;
- If $A_k \in \mathcal{A}$, $k \in \mathbb{N}$ are mutually disjoint sets, i.e. $A_i \cap A_j = \emptyset$, $\forall i \neq j$, then:

$$\mu\left(\bigcup_{k=1}^{+\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k), \quad \text{with the absolute convergence.}$$

In the case of $n = 1$ we say that μ is a *signed measure*; in the case of $n = 2$ it is called *complex measure*. A *finite vector measure* is a vector measure μ such that $\mu(A) < +\infty$, for all $A \in \mathcal{A}$.

Given a measurable space $(\Omega, \mathcal{A})$, we denote by:

- $\mathcal{M}^1(\Omega) = \mathcal{M}(\Omega)$: the space of the finite signed Borel measures.
- $\mathcal{M}^n(\Omega) := \{\mu : \mathcal{B}(\Omega) \to \mathbb{R}^n \ : \ \mu \text{ is a finite vector measure}\}$, the set of all the finite vector measures defined on the Borel set $\mathcal{B}(\Omega)$ with values in $\mathbb{R}^n$.
- $\mathcal{M}^n_{\text{div}}(\Omega) := \left\{\mu \in \mathcal{M}^n(\overline{\Omega}) \ : \ \nabla \cdot \mu \in \mathcal{M}(\overline{\Omega})\right\}$;
- $\mathcal{M}^n_{\text{div},0}(\Omega) := \{\mu \in \mathcal{M}^n_{\text{div}}(\Omega) \ : \ \mu \cdot \nu_\Omega = 0\}$.

For every such vector measure, we can associate a positive scalar measure, called *total variation measure*, defined as:

Definition 1.11 (Total variation measure). Consider a finite vector measure $\mu \in \mathcal{M}^n(\Omega)$. The *total variation measure* $\|\mu\| \in \mathcal{M}_+(\Omega)$ is a map $\|\mu\| : \mathcal{B}(\Omega) \to [0, +\infty]$, which associates to every $A \in \mathcal{B}(\Omega)$:

$$\|\mu\|(A) := \sup\left\{\sum_{k=1}^{+\infty} \|\mu(A_k)\| \ : \ A = \bigcup_{k=1}^{\infty} A_k \text{ with } A_k \in \mathcal{B}(\Omega) \text{ and } A_i \cap A_j = \emptyset,\ \forall i \neq j\right\}$$

Note that $\sum_{k=1}^{+\infty} \|\mu(A_k)\| < +\infty$, because the series is absolutely convergent, thanks to the Definition 1.10. Moreover, note that the quantity $\|\mu\| := \|\mu\|(\Omega)$ is a norm on the space $\mathcal{M}^n(\Omega)$. We, now, need to recall a modified version of the Riesz representation theorem:

Theorem 1.7 (Riesz representation for continuous function). *Given $\Omega \subset \mathbb{R}^n$ open bounded set, then every bounded linear functional Φ on $(\mathcal{C}_0(\Omega; \mathbb{R}^n))^*$ (with $\mathcal{C}_0(\Omega; \mathbb{R}^n)$) considered with the supremum norm) is represented by a unique finite vector measure $\mu \in \mathcal{M}^n(\Omega)$, in the sense that:*

$$\langle \Phi, f \rangle_{\mathcal{C}_0(\Omega;\mathbb{R}^n)} = \int_\Omega f \cdot d\mu = \sum_{i=1}^{n} \int_\Omega f_i d\mu_i, \quad \forall f \in \mathcal{C}_0(\Omega; \mathbb{R}^n). \tag{1.5}$$

Moreover, the map:

$$\begin{gathered}(\mathcal{C}_0(\Omega; \mathbb{R}^n))^* \to \mathcal{M}^n(\Omega)\\ \Phi \mapsto \mu\end{gathered}$$

is a linear surjective isometry, i.e. $\|\Phi\| = \|\mu\|(\Omega)$.

In practice, we construct the integral in (1.5) as the sum of the integrals of scalar functions according to scalar measures. For the proof of this theorem, in Rudin [39] there is a different version, but it is enough to apply that version piecewise. The integral of a function $f \in \mathcal{C}_0(\Omega; \mathbb{R}^n)$ with respect to $\mu \in \mathcal{M}^n(\Omega)$, as defined in (1.5), is well defined, if $\|f\| \in L^1(\Omega, \|\mu\|)$. Another important fact about vector measures, consequence of the Radon-Nikodym theorem, is the following:

PROPOSITION 1.2. *For every $\mu \in \mathcal{M}^n(\Omega)$, there exists a Borel function $u : \Omega \to \mathbb{R}^n$, called the density of μ with respect to $\|\mu\|$, such that $\mu = u \cdot \|\mu\|$ and $\|u\| = 1$ a.e. for the measure $\|\mu\|$. In particular,*

$$\int_\Omega f \cdot d\mu = \int_\Omega (f \cdot u) d\,\|\mu\|\,, \quad \forall f \in \mathcal{C}_0(\Omega; \mathbb{R}^n).$$

3. Elements of Probability Theory

In this section, we just want to refresh the mind of the reader about well-known results and properties of the integration in a probability setting, mainly in the case of absolutely continuous random variables. For the details about this argument, we refer to Jacod and Protter [31]. Let us start consider a measurable space $(\Omega, \mathcal{A})$ and for future reasons, denote:

- $\mathbb{P}(\Omega)$: the set of all the probabilitiy measures defined on the measurable space $(\Omega, \mathcal{B}(\Omega)$.

Note, also, that a probability measure P is just a measure that has the property $P(\Omega) = 1$. Moreover, let us recall two key definitions inside probability theory.

DEFINITION 1.12 (Random variable). A measurable function $X : E \to F$, with $(E, \mathcal{E})$ and $(F, \mathcal{F})$ two measurable spaces, is a *random variable* if $(E, \mathcal{E}) = (\Omega, \mathcal{A})$. Moreover, we say that X is a random variable with values in $\mathbb{R}$, if $(F, \mathcal{F}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

DEFINITION 1.13 (Distribution). Given a probability space $(\Omega, \mathcal{A}, P)$ and a random variable $X : \Omega \to E$, then the *distribution* of X is defined as:

$$P^X(B) := P(X^{-1}(B)) = P(\{w \in \Omega \ : \ X(\omega) \in B\}).$$

Note that P^X is a probability measure on $(E, \mathcal{E})$. It is possible to define the integral of a random variable in the same way as the Lebesgue integral of a measurable function. This particular integral is called *expectation* of a random variable and it is denoted by $E(X)$, if $X : \Omega \to \mathbb{R}$ is a random variable. Moreover, in the same way as before, we can define the $\mathcal{L}^p$ spaces of random variables as:

$$\mathcal{L}^p(\Omega, \mathcal{A}, P) = \mathcal{L}^p(\Omega) := \left\{X : \Omega \to \mathbb{R} \ : \ X \text{ is a random variable and } |X|^p \in \mathcal{L}^1(\Omega, \mathcal{A}, P)\right\}.$$

We want to report, here, a result which links the expectation of a random variable X to the integral with respect to the distribution P^X.

THEOREM 1.8 (Expectation rule). *Let $X : \Omega \to E$ be a random variable with values in $(E, \mathcal{E})$ and $g : E \to \mathbb{R}$ be a measurable function. Then:*

(i) $g(X) \in \mathcal{L}^1(\Omega, \mathcal{A}, P)$ if and only if $g \in \mathcal{L}^1(E, \mathcal{E}, P^X)$;
(ii) If $g \geq 0$ or if g satisfies one of the equivalent conditions in (i), we have:

$$E(g(X)) = \int g(x) dP^X(x). \tag{1.6}$$

As for the expectation, we can define different types of convergence of random variables, with properties similar to the case of the convergence of measurable functions. Thus, it is possible to define the convergence almost everywhere, the convergence in $\mathcal{L}^p$, the convergence in probability, in the same way as we did in Definition 1.3. The only case that differs is the following:

DEFINITION 1.14 (Convergence in Distribution). Consider a sequence of probabilities $(\mu_n)_{n\in\mathbb{N}} \in \mathbb{P}(\mathbb{R}^d)$ and $\mu \in \mathbb{P}(\mathbb{R}^d)$, then we say that μ_n *converges in distribution* or *converges weakly* to μ, and we write $\mu_n \rightharpoonup \mu$, if:

$$\lim_{n\to+\infty} \int f(x) d\mu_n(x) = \int f(x) d\mu(x), \quad \forall f : \mathbb{R}^d \to \mathbb{R} \text{ bounded and continuous.}$$

We say that the sequence of random variables $(X_n)_{n\in\mathbb{N}}$ with values in $\mathbb{R}^d$ *converges in distribution* or *weakly* to the random variable X, if their distributions $P^{X_n} \rightharpoonup P^X$, i.e. for all $f : \mathbb{R}^d \to \mathbb{R}$ bounded and continuous, using (1.6), we have:

$$\lim_{n\to+\infty} E(f(X_n)) = E(f(X)).$$

An important theorem in this context, used later in this thesis, is the following:

THEOREM 1.9 (Prokhorov). *Consider a sequence of probability measures in $\mathbb{R}^d$, $(\mu_n)_{n\in\mathbb{N}} \subset \mathbb{P}(\mathbb{R}^d)$, which satisfies the* tightness *property, i.e. for every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset \mathbb{R}^d$ such that:*

$$\mu_n(\mathbb{R}^d \setminus K_\varepsilon) < \varepsilon \quad \forall n \in \mathbb{N}. \tag{1.7}$$

Then there exists a subsequence $(\mu_{n_k})_{k\in\mathbb{N}}$ and $\mu \in \mathbb{P}(\mathbb{R}^d)$, such that $\mu_{n_k} \rightharpoonup \mu$.

For the proof of this theorem, Jacod and Protter [31] there is only the case $d = 1$. For the proof in general case see, for example, Billingsley [7]. We need, now, to introduce an important concept of the probability theory, which will be used often in this study.

DEFINITION 1.15 (Absolutely continuous measure and random variable). Consider a Lebesgue measurable function $f : \mathbb{R}^n \to [0, +\infty)$ such that:

$$\int_{\mathbb{R}^n} f(x)dx = 1.$$

Then, function:

$$\mu(A) = \int_A f(x)dx, \quad A \in \mathcal{B}(\mathbb{R}^n),$$

is a probability measure on $\mathcal{B}(\mathbb{R}^n)$, and we say that μ is an *absolutely continuous measure with density* f. We say that the random variable X with values in $\mathbb{R}^n$ is *absolutely continuous*, if its distribution P^X is an absolutely continuous measure. The density of X is the density of P^X

Note that, by the definition of absolutely continuous measure μ with density f, we obtain the rule:

$$\int g(x)d\mu(x) = \int g(x)f(x)dx, \quad \forall g \in L^1(\Omega, \mathcal{A}, \mu).$$

As a consequence, we obtain:

PROPOSITION 1.3. *If X is an absolutely continuous random variable with values in $\mathbb{R}^n$ and density f_X, for all $g : \mathbb{R}^n \to \mathbb{R}$ such that $g(X) \in \mathcal{L}^1$, then:*

$$E(g(X)) = \int g(x)f_X(x)dx. \tag{1.8}$$

Note that comparing Equations (1.6) and (1.8), we can denote the fact that a random variable X is absolutely continuous with density f_X by:

$$dP^X(x) = f_X(x)dx.$$

A particular case in which the densities of the random variables are used is the following:

THEOREM 1.10. *Assume that (X, Y) is a random variable with values in $\mathbb{R}^2$, absolutely continuous with density $f_{X,Y}$, then both X and Y are absolutely continuous random variables with densities given by, respectively:*

$$f_X(x) = \int f_{X,Y}(x, y)dy, \quad f_Y(y) = \int f_{X,Y}(x, y)dx.$$

These densities f_X and f_Y are called the *marginal densities of* f. We finally turn our attention to a change of variable technique taken from standard calculus, in order to answer the question: if $X = (X_1, \ldots, X_n)$ has the density f, what is the density of $Y = g(X)$? We start recalling the change of variable theorem, taken for example from Stroock [42].

THEOREM 1.11 (Change of variable or Jacobi's transformation). *Suppose that G is an open set in $\mathbb{R}^n$ and $g : G \to \mathbb{R}^n$ is continuously differentiable, injective and its Jacobian Dg never vanishes. If f is Lebesgue measurable on $g(G)$, then $f \circ g$ is Lebesgue measurable on G. Moreover, if $f \in L^1(g(G), \mathcal{A}, \mathcal{L}^n)$, then:*

$$\int_{g(G)} f(y)dy = \int_G f(g(x)) \left|det(Dg(x))\right| dx. \tag{1.9}$$

As an application of this theorem, we can answer the previous question with the following:

THEOREM 1.12. *Consider $X = (X_1, \dots, X_n)$ a random variable with values in $\mathbb{R}^n$, absolutely continuous with density f_X and consider $g : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable, injective and with Jacobian Dg never vanishing. Then $Y = g(X)$ is an absolutely continuous random variable with density:*

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left|det(Dg^{-1}(y))\right| & \text{if } y \in Im(g), \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

4. Elements of Linear Programming

For the first introductory part about linear programming we refer to Vanderbei [44], while for a deeper reading about duality we suggest the book of Boyd and Vandenberghe [10]. We start with some general definitions:

DEFINITION 1.16 (Linear program). A *linear program* is a constrained optimization problem, where the *objective function* f is linear and the *constraints* are described by linear equalities or inequalities. The general case is:

$$\begin{array}{rl} \min \ (\text{or } \max) & c_1x_1 + \cdots + c_nx_n \\ \text{subject to} & a_{11}x_1 + \cdots + a_{1n}x_n \sim b_1, \\ & \vdots \qquad \vdots \\ & a_{m1}x_1 + \cdots + a_{mn}x_n \sim b_m, \end{array}$$

where "$\sim$" stands for one of the following symbols: "$\geq$", "$\leq$" or "$=$". In the problem above, $x_1, \dots, x_n$ are the *variables*; a_{ij}, b_i and c_j are the *parameters*.

In general, every linear program can be written in the *standard form* (possibly after an addition of some variables, called "slack" or "surplus" variables, or after some changes of the sign) as:

$$\begin{aligned} &\min c^T x \\ &\text{s.t. } Ax = b \\ &\quad x \geq 0, \end{aligned} \tag{1.11}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$. Here we report some standard notations and a basic result characterizing the different types of linear programs.

We say that $x^* \in \mathbb{R}^n$ is:

- *feasible* if it satisfies all the constraints. In the case of the standard form, for example, x^* has to be non-negative and $Ax^* = b$.
- an *optimal solution* if it is feasible and, in the case of minimization problems,
$$c^T x^* \leq c^T x, \quad \text{for all the feasible } x \in \mathbb{R}^n.$$
If x^* is an optimal solution, the corresponding value $c^T x^*$ is said the *optimal value.*

THEOREM 1.13. *Given a linear program in the standard form, one and only one of the following alternatives holds:*

(i) the problem has at least one optimal solution;
(ii) the problem is unfeasible, *i.e. there are no admissible solutions;*

(iii) the problem is boundless*, i.e. for every $K \in \mathbb{R}$ it exists an admissible solution $x \in \mathbb{R}^n$ such that $c^T x < K$.*

We have just introduced the main features and definitions of the linear programming. Actually, for the aim of this work, we just need one property of these linear programs: the existence of a *dual program.* If we call a general linear problem, as those we described before, *primal problem*, then we can construct a *dual problem*, arising from the primal problem, which can give us some useful information about it. However, before going on with the full list of the rules for the construction of a dual problem, let us briefly see some motivations and interpretations of this duality property. Often, in this linear programming framework, the dual program has an *economic interpretation.* For example, in the cases of problems linked to allocation of resources, production or transportation, which have clear economic features, the dual problem describes the issue of an operator who wants to penetrate the market and propose an alternative to our business. It is called *shadow market.* The constraints of the dual represent the fact that the proposal of the operator must be competitive, hence it must be economically interesting for us. Conversely, the objective function represents the will of the operator to maximize the profit. Later in this thesis, we will focus on the case of the assignment problem and the economic interpretation of the dual will be clearer through this specific example. Furthermore, it exists another interpretation of the dual problem, which is the most natural one and it holds in more general conditions. It is called *Lagrangian duality.* Consider a general optimization problem written in a standard way as:

$$\min_{x\in S} f(x), \tag{MP}$$

where the set of feasible solutions S can be written as intersections of equalities and inequalities, i.e.

$$S := \{x \in \mathbb{R}^n \ : \ g_i(x) \le 0,\ h_j(x) = 0,\ \forall i = 1,\dots,m,\ \forall j = 1,\dots,p\}. \tag{1.12}$$

We assume S non-empty and we denote by x^* the optimal solution. It is often useful to find a lower bound for a minimization problem (or an upper bound for a maximization problem). The basic idea in the Lagrangian duality is to take the constraints inside the definition of S into account by augmenting the objective function with a weighted sum of the constraint functions. Indeed we define the Lagrangian $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ as:

$$\mathcal{L}(x, \lambda, \nu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \nu_j h_j(x). \tag{1.13}$$

These λ_i and ν_j are called the *Lagrange multipliers* associated with the i-th inequality constraint $g_i(x) \le 0$ and with the j-th equality constraint $h_j(x)$, respectively. We, then, consider the *Lagrange dual function* $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ defined as:

$$g(\lambda, \nu) := \inf_{x\in\mathbb{R}^n} \mathcal{L}(x, \lambda, \nu) = \inf_{x\in\mathbb{R}^n} \left(f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \nu_j h_j(x) \right). \tag{1.14}$$

Note that it is the pointwise infimum of a family of affine functions of (λ, ν), thus it is concave even without any particular assumptions on the principal problem (MP). The basic property of this dual function is that it yields lower bounds on the optimal value $f(x^*)$ of (MP). Indeed, for any $\lambda \ge 0$ and any ν, supposing $\tilde{x} \in S$, we get:

$$\sum_{i=1}^{m} \lambda_i g_i(\tilde{x}) + \sum_{j=1}^{p} \nu_j h_j(\tilde{x}) \le 0.$$

Therefore,

$$\mathcal{L}(\tilde{x}, \lambda, \nu) = f(\tilde{x}) + \sum_{i=1}^{m} \lambda_i g_i(\tilde{x}) + \sum_{j=1}^{p} \nu_j h_j(\tilde{x}) \le f(\tilde{x}).$$

Hence, simply noting that $g(\lambda, \nu)$ of (1.14) is less than or equal to the Lagrange function $\mathcal{L}(\tilde{x}, \lambda, \nu)$ in the previous inequality, we get that $g(\lambda, \nu) \leq f(\tilde{x})$ holds for any $\tilde{x} \in S$. Furthermore, we get that, for any $\lambda \geq 0$ and any ν, the following inequality holds:

$$g(\lambda, \nu) \leq f(x^*). \tag{1.15}$$

Note that the previous inequality is non-trivial only when $\lambda \geq 0$ and $(\lambda, \nu) \in \mathrm{dom}(g)$, i.e. $g(\lambda, \nu) > -\infty$. We refer to the couple (λ, ν), with the features just described, as *dual feasible*, for reasons which will be clear later. Now, once we discovered that each of the dual feasible couple gives us a lower bound on the optimal value x^*, a natural question is: what is the best lower bound that can be obtained from the Lagrange dual function?

The answer to this question leads to the following optimization problem:

$$\begin{aligned} &\max\ g(\lambda, \nu) \\ &\text{s.t. } \lambda \geq 0, \end{aligned} \tag{DP}$$

where the constraint $(\lambda, \nu) \in \mathrm{dom}(g)$ is implicit. The problem (DP) is usually called *Lagrange dual problem*. Moreover, the term *dual feasible* to describe the pair (λ, ν) with $\lambda \geq 0$ and $(\lambda, \nu) \in \mathrm{dom}(g)$ now becomes clearer, because it stands for a couple which is feasible for the dual problem. We look for the optimal couple (λ^*, ν^*) that solves (DP), called *dual optimal* or *optimal Lagrange multipliers*. Note that Lagrange dual problem (DP) involves the maximization of a concave function under convex constraints, thus it is a convex problem. Thus, in general, a dual problem can be easier than the primal problem for its convexity properties and, sometimes, a primal problem with a lot of difficult constraints can be reduced into a dual problem with less constraints, easier to handle. Let us see in details, now, the case in which the primal problem is a linear program in the standard form as in (1.11). In this case, in order to transform feasible set into an intersections of equalities and inequalities like the set S, we consider the functions $g_i(x) = -x_i$, $i = 1, \dots, n$. Thus, taking λ_i for the n inequality constraints and ν_j for the m equality constraints, we get:

$$\mathcal{L}(x, \lambda, \nu) = c^T x - \sum_{i=1}^{n} \lambda_i x_i + \nu^T (Ax - b) = -b^T \nu + (c + A^T \nu - \lambda)^T x.$$

Therefore, the dual function in this case is:

$$g(\lambda, \nu) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda, \nu) = -b^T \nu + \inf_{x \in \mathbb{R}^n} (c + A^T \nu - \lambda)^T x = \begin{cases} -b^T \nu & \text{if } A^T \nu - \lambda + c = 0 \\ -\infty & \text{otherwise,} \end{cases}$$

since a linear function is bounded from below only when it is identically zero. Hence, g is finite only when $A^T \nu - \lambda + c = 0$. Thus, the Lagrange dual problem, making these equality constraints explicit and considering $u = -\nu$, becomes:

$$\begin{aligned} &\min\ b^T u \\ &\text{s.t. } -A^T u - \lambda + c = 0 \\ &\qquad \lambda \geq 0, \end{aligned} \tag{1.16}$$

This problem, removing the constraints $\lambda \geq 0$, can be expressed as:

$$\begin{aligned} &\min\ b^T u \\ &\text{s.t. } A^T u \leq c. \end{aligned} \tag{1.17}$$

Hence, this example gives us the following duality rule:

$$\begin{array}{lcl} \min c^T x, & & \max b^T u, \\ \text{s.t. } Ax = b, & \overset{\text{Dual}}{\Longleftrightarrow} & \text{s.t. } A^T u \leq c, \\ x \geq 0 & & u \in \mathbb{R}^m. \end{array} \tag{1.18}$$

In general, it is easy to see, proceeding like above, that the following rules for the construction of the dual problem, in this case of linear programming, hold:

- the dual of a minimum problem is a maximum problem and vice versa.
- the type of the primal constraint determines the sign of the dual variable, taking the same one. The sign of a primal variable determines the type of the dual constraint, taking the opposite one.
- the vector c defining the objective function of the primal problem becomes the right-hand side of the constraints in the dual; vice versa, the vector b of the primal appears in the objective function of the dual.
- if the constraints of the primal problem are defined by a matrix $A \in \mathbb{R}^{m\times n}$, then the constraints of the dual problem are defined by the transpose matrix $A^T \in \mathbb{R}^{n\times m}$.

Table 1 summarizes all the rules written above and fully describes the construction of a dual linear problem. Note that, given a matrix $A \in \mathbb{R}^{m\times n}$, the i-th row of A is denoted by a_i^T and the j-th column of A is denoted by A_j.

min $c^T x$	max $b^T u$	
$a_i^T x \leq b_i$	$u_i \leq 0$	$i = 1, \dots, m_1$
$a_i^T x \geq b_i$	$u_i \geq 0$	$i = m_1 + 1, \dots, m_2$
$a_i^T x = b_i$	u_i free	$i = m_2 + 1, \dots, m$
$x_j \geq 0$	$A_j^T u \leq c_j$	$j = 1, \dots, n_1$
$x_j \leq 0$	$A_j^T u \geq c_j$	$j = n_1 + 1, \dots, n_2$
x_j free	$A_j^T u = c_j$	$j = n_2 + 1, \dots, n$

TABLE 1. The rules for the construction of the dual problem of a linear program.

We state here some important theorems in this context, where in every theorem we suppose to consider a couple of primal-dual linear problems.

THEOREM 1.14 (Weak duality). *If x is feasible for the primal problem and u is feasible for the dual problem, then:*

$$b^T u \leq c^T x.$$

This theorem follows directly from the definition of the Lagrange dual function, which is a lower bound for the optimal value of the primal problem. In particular, is a lower bound for any feasible point. Indeed, Theorem 1.14 is just (1.15) in the case of a linear program. A direct consequence is the following:

COROLLARY 1.1. *If x^* is feasible for the primal problem, u^* is feasible for the dual problem and $c^T x^* = b^T u^*$, then x^* and u^* are optimal solutions for the primal and the dual problems, respectively.*

Another important result, also following from the general framework of the Lagrange duality, is the following:

THEOREM 1.15 (Strong duality). *The primal problem has an optimal solution x^* if and only if the dual problem has an optimal solution u^*. In that case, the optimal values coincide, i.e.*

$$c^T x^* = b^T u^*.$$

Starting from the same problem written in (MP), in the case in which functions $g_1, \dots, g_m$ and $h_1, \dots, h_p$ are differentiable, we can obtain some necessary conditions for optimality, which will be useful later in this thesis. The idea is, again, to use the Lagrangian function $\mathcal{L}(x, \lambda, \nu)$ defined in Equation (1.13) and to impose the fact that its gradient must vanish at optimal solution x^*. We get that:

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) + \sum_{j=1}^{p} \nu_j^* \nabla h_j(x^*) = 0,$$

if (λ^*, ν^*) is optimal for the dual problem written in (DP). Thus, we obtain:

PROPOSITION 1.4 (Karush–Kuhn–Tucker (KKT) conditions). *If x^* is a local solution of constrained minimization problem* (MP) *and (λ^*, ν^*) is a solution of constrained maximization problem* (DP)*, the following conditions hold:*

$$\begin{cases} \nabla_X \mathcal{L}(x^*, \lambda^*, \nu^*) = 0, & \\ g_i(x^*) \leq 0 & i = 1, \dots, m \\ h_j(x^*) = 0 & j = 1, \dots, p \\ \lambda_i^* \geq 0 & i = 1, \dots, m \\ \lambda_i^* g_i(x^*) = 0 & i = 1, \dots, m \end{cases} \tag{KKT}$$

We refer again to Boyd and Vandenberghe [10] for a deeper analysis about this result.

5. Elements of Convex Analysis

In this section, we want to start by describing some tools of convex analysis. For a deeper reading about the introductory concepts and the preliminary results set out below, see both Ekeland and Temam [23] and Brezis [15].

DEFINITION 1.17 (Convex set). Given a vector space V over $\mathbb{R}$, we say that $C \subset V$ is a *convex set* if:

$$\lambda u + (1 - \lambda) v \in C, \quad \forall u, v \in C, \ \forall \lambda \in [0, 1].$$

DEFINITION 1.18 (Convex function). Given a vector space V and a convex subset $C \subset V$, a function $F : C \to \overline{\mathbb{R}}$ is said to be *convex* if

$$F(tx + (1 - t)y) \leq tF(x) + (1 - t)F(y) \quad \forall x, y \in C, \quad \forall t \in [0, 1].$$

It is called *strictly convex* if the inequality above is strict for all $x \neq y$ and for all $t \in (0, 1)$.

DEFINITION 1.19 (Concave function). Given a vector space V and a convex subset $C \subset V$, a function $F : C \to \overline{\mathbb{R}}$ is said to be *concave* if

$$F(tx + (1 - t)y) \geq tF(x) + (1 - t)F(y) \quad \forall x, y \in C, \quad \forall t \in [0, 1].$$

It is called *strictly concave* if the inequality above is strict for all $x \neq y$ and for all $t \in (0, 1)$.

In this work, we will need also a suitable generalization of this definition, which adapts the notion of concavity to the geometry of a function $c(x, y)$.

DEFINITION 1.20 (c-concave function). Given two vector spaces V and W and a function $c : V \times W \to [0, +\infty]$, a function $F : V \to \overline{\mathbb{R}}$ is said to be *c-concave* if there exists $G : W \to \overline{\mathbb{R}}$, $G \not\equiv -\infty$, such that:

$$F(x) = \inf_{y \in W} [c(x, y) - G(y)], \quad \forall x \in V.$$

Indeed, the previous generalization is motivated by the fact that, in general, the pointwise infimum of a family of affine functions is a concave function.

The following proposition collects together some properties of the convex functions:

PROPOSITION 1.5. *Given a vector space V and a convex subset $C \subset V$:*

(i) If $F : C \to \overline{\mathbb{R}}$ is convex and if $\lambda \geq 0$, then λF is convex.
(ii) If F and G are convex functions from C into $\overline{\mathbb{R}}$, then $F + G$ is convex.
(iii) If $(F_i)_{i \in I}$ is any family of convex functions of C into $\overline{\mathbb{R}}$, their pointwise supremum $F(x) = \sup_{i \in I} F_i(x)$ is convex.

Definition 1.21 (Epigraph). Given a vector space V and a function $F : V \to \overline{\mathbb{R}}$, the *epigraph* of F is the set:

$$\mathrm{epi}(F) = \{(x, a) \in V \times \mathbb{R} \ : \ F(x) \le a\}\,.$$

The following result links together this final concept with the definition of convex function.

Proposition 1.6. *Given a convex subset $C \subset V$ of a vector space V, a function $F : C \to \mathbb{R}$ is convex if and only if its epigraph is a convex set.*

Another important preliminary definition, taken from this framework, is:

Definition 1.22 (Lower semi-continuous). Given a metric space V, $F : V \to \overline{\mathbb{R}}$ is l.s.c. if and only if for every $x \in V$ and every sequence $\{x_n\}$ such that $x_n \to x$, we have:

$$F(x) \le \liminf_{n\to\infty} F(x_n).$$

Note that, using the definition of a closed set as a set which contains all its limit points, we get the following characterization:

Proposition 1.7. *Given a normed vector space V, it follows that a function F is l.s.c if and only if epi(F) is closed.*

Furthermore, as in Proposition 1.5 about convex functions, also in this case the property of being lower semi-continuous is linear and it is preserved for the pointwise supremum of a family of lower semi-continuous functions. Another important tool is the following:

Definition 1.23 (Legendre-Fenchel transform). Given a normed vector space V and a function $F : V \to \overline{\mathbb{R}}$, the *Legendre-Fenchel transformation* is the function $F^* : V^* \to \overline{\mathbb{R}}$ defined as:

$$F^*(f) := \sup_{v\in V} \left[\langle f, v\rangle_* - F(v)\right] \quad \forall f \in V^*,$$

where V^* is the usual dual vector space.

Note that the Legendre-Fenchel transformation F^* is always convex and l.s.c. on V^*. Indeed, for any $v \in V$, the function $f \mapsto \langle f, v\rangle_* - F(v)$ is convex and l.s.c. on V^*. It follows that the pointwise supremum for $v \in V$ is a convex and l.s.c function, thanks to the properties of convex and l.s.c functions stated before. The last important result from convex analysis is the following:

Definition 1.24 (Subgradient and subdifferential). Given a normed space V and a function $F : V \to \overline{\mathbb{R}}$. We say that $u^* \in V^*$ is a *subgradient* of F at $x_0 \in V$ if the following *subgradient inequality* holds:

$$\langle u^*, x - x_0\rangle_* + F(x_0) \le F(x), \quad \forall x \in V. \tag{1.19}$$

The set $\partial F(x_0) := \{u^* \in V \ : \ u^* \text{ is a subgradient of } F \text{ at } x_0\}$ is called the *subdifferential* of F at x_0.

For the details about this subtle argument we refer to the convex analysis chapter in the book of Clarke [19]. Another characterization of the subderivative, using the Legendre-Fenchel transform, is given by the following:

Proposition 1.8. *Consider a normed space V, a function $F : V \to \overline{\mathbb{R}}$ and its Legendre-Fenchel transform. Then $u^* \in \partial F(x_0)$ if and only if:*

$$F(x_0) + F^*(u^*) = \langle u^*, x_0\rangle_*. \tag{1.20}$$

Just to have an intuition about this delicate and non-trivial tool, if we consider the affine function $h(x) := F(x_0) + \langle u^*, x - x_0\rangle_*$, the graph of h is a non-vertical supporting hyperplane of epi(F) at the point $(x_0, F(x_0))$, as in Figure 2. It is easy to understand, also helped by this figure, that the subderivative and the subdifferential are tools introduced

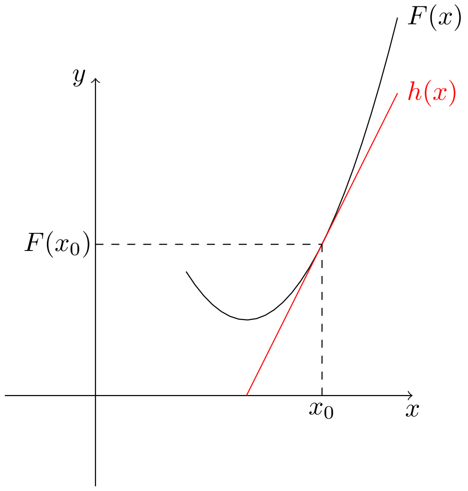


FIGURE 2. Graphic representation of the meaning of a subgradient of F at x_0.

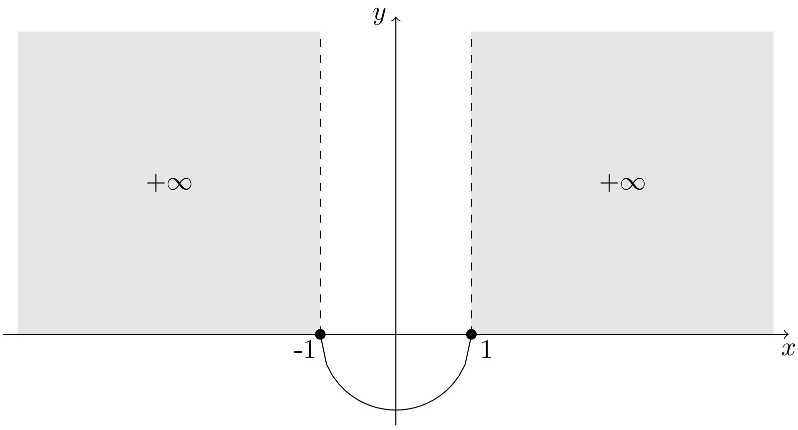


FIGURE 3. Graphic representation of function $F(x)$.

to generalize the concept of derivative of functions which are not differentiable. Indeed, if we look, for example, at the case of the absolute value on $\mathbb{R}$, i.e. $F : \mathbb{R} \to \mathbb{R}$ defined as $F(x) = |x|$. We know that F is not differentiable at 0, but, in this case, the subdifferential of F at 0 is $\partial F(0) = [-1, 1]$. However, note there is no guarantee that the subdifferential is a priori non-empty, as we can see in the following example.

EXAMPLE 1.1. Consider $F : \mathbb{R} \to \mathbb{R}$, defined as:

$$F(x) = \begin{cases} -\sqrt{1 - |x|^2} & \text{if } |x| \leq 1, \\ +\infty & \text{if } |x| > 1. \end{cases}$$

The function is represented in Figure 3. In this case the $\partial F(-1) = \partial F(1) = \emptyset$, because it is not possible to find an non-vertical hyperplane passing through $(1, F(1))$ or $(-1, F(-1))$ remaining below epi(F).

Thus, the following characterization of the subdifferential is non-trivial and very useful.

PROPOSITION 1.9. *Let V be a normed space and $F : V \to \overline{\mathbb{R}}$. Then, for every $x_0 \in V$, the subdifferential $\partial F(x_0)$ is a convex subset of V^* which is also closed for the weak-* topology. Moreover, in the case in which the function F is convex and continuous at $x_0 \in dom(F)$, then $\partial F(x_0) \neq \emptyset$.*

Furthermore, another characterization of the subdifferential holds in a more particular case, which, once more, stresses the fact that this concept generalize the definition of derivative.

Proposition 1.10. *Let V be a normed space and let $F : V \to \overline{\mathbb{R}}$ be a convex function, with $x_0 \in dom(F)$. If F is Gâteaux differentiable at x_0, then:*

$$\partial F(x_0) = \{DF(x_0)\}.$$

We introduced this concept, because it is linked with optimization problems. Before going on with the result, we need to do a latter consideration, which will be useful to remove the constraints of an optimization problem. If we consider a function $f : \mathbb{R}^n \to \mathbb{R}$ and a convex subset $C \subset \mathbb{R}^n$, the following are equivalent:

$$\min_{x\in C} f(x) \iff \min_{x\in\mathbb{R}^n} f(x) + I_C(x), \text{ where } I_C(x) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C. \end{cases} \tag{1.21}$$

The function I_C is called the *indicator function* of C. Moreover, its subdifferential $N_C(x) := \partial I_C(x)$ is another particular structure, called *normal cone*.

Proposition 1.11. *Suppose that $C \subset V$ is convex subset of a normed space and that $F : V \to \overline{\mathbb{R}}$ is a continuous convex function, with $x_0 \in C$. Then:*

$$F(x_0) = \min_{x\in C} F(x) \iff 0 \in \partial(F(x_0) + I_C(x_0)) = \partial F(x_0) + N_C(x_0). \tag{1.22}$$

Note that this result simply follows from the subgradient inequality 1.19. Let us present, now, two important theorems taken from convex analysis, whose applications will be of great importance in this work.

Theorem 1.16 (Fenchel-Moreau). *Assume that V is a normed vector space and that $F : V \to \overline{\mathbb{R}}$ is a convex and l.s.c function. Then $F^{**} = F$.*

For the proof of this theorem, we suggest Brezis [15]. Now, let us consider V and W two vector spaces, with the analogous standard dual vector spaces V^* and W^*, and consider a linear continuous operator from V into W, $\Lambda \in \mathcal{L}(V, W)$, whose *adjoint* $\Lambda^* \in \mathcal{L}(W^*, V^*)$, is defined by:

$$\langle \Lambda^* p^*, u\rangle_* := \langle p^*, \Lambda u\rangle_* \quad \forall p^* \in W^*, \quad \forall u \in V. \tag{1.23}$$

Let us assume, also, that $F : V \to \overline{\mathbb{R}}$ and $G : W \to \overline{\mathbb{R}}$ are two functions. Define, then, a principal problem as:

$$\inf_{u\in V} [F(u) + G(\Lambda u)]. \tag{P$_1$}$$

Recalling the concept of dual problem taken from linear programming and using a technique similar to the Lagrangian duality, properly extended to a convex analysis framework, we want to find a correspondence between the solutions of the primal and the dual problems, as in the Strong Duality Theorem 1.15. All the details are explained in Ekeland and Témam [23]. We have that the dual formulation of the problem (P_1) is:

$$\sup_{p^*\in W^*} \left[-F^*(\Lambda^* p^*) - G^*(-p^*)\right]. \tag{P$_1^*$}$$

The functions $F^* : V^* \to \overline{\mathbb{R}}$ and $G^* : W^* \to \overline{\mathbb{R}}$ are the Legendre-Fenchel transformations of F and G, respectively. With all these preliminary definitions, the following result holds:

Theorem 1.17 (Fenchel–Rockafellar). *Let us assume that F and G are convex functions, that* inf (P_1) *is finite and that there exists $u_0 \in V$ such that $F(u_0) < +\infty$, $G(\Lambda u_0) < +\infty$, with G continuous in Λu_0. Then, it follows that:*

$$\inf(\mathrm{P}_1) = \sup(\mathrm{P}_1^*).$$

The proof of this theorem can also be found inside Ekeland and Témam [23].

CHAPTER 2

The Monge-Kantorovich Mass Transfer

In this chapter, we start to deal with Monge-Kantorovich optimal transport problem. For the sake of the exposition, we need to spend some words on the structure of this chapter and on the way in which it will be developed. Starting from the original formulation due to Monge [37], we describe the mathematical background underlying this apparently simple problem and we hit against the high non linearity of the problem set like that. Taking inspiration from Kantorovich [32] and [33], we change the point of view. Indeed, looking at the initial problem from a different perspective, we start considering a "relaxation" of it, passing through the discrete case. Noting that this reformulation is connected with the linear programming, we can apply standard mathematical tools coming from this field and we can consider the dual problem. These new formulations of the original problem are solvable under reasonable assumptions, as they are linear problem, just depending on the cost function c and on linear constraints. Coming back to the original issue, with these new information taken from Kantorovich formulations, we are also able to solve Monge problem. In the case of a cost function $c(x, y) = h(x-y)$ with h strictly convex, we use a result due to Gangbo-McCann [30]. Thus, we concentrate on the original L^1 cost function $c(x, y) = \|x-y\|$, for which this result does not hold. In this case, we follow the idea used in Evans and Gangbo [27], who thought of using a differential equation technique involving a deformation argument which dates back to Dacorogna and Moser [20]. Following this idea, we give a "recipe" for the construction of an optimal solution of Monge problem.

1. Formulation of the problem

The original formulation of the optimal mass transfer problem was proposed by Gaspard Monge in 1781, with a report called *"Mémoire sur la Theorie des Déblais et des Remblais"* (see the front page in Figure 4). He asked which was the best way to move a pile of soil or rubble ("déblais") from the ground to an excavation or fill ("remblais"), with the least amount of work. In mathematical words, consider two positive finite Borel measures μ^+ (the mass distribution of the "déblais") and μ^- (the mass distribution of the "remblais") on $\mathbb{R}^n$, i.e. μ^+, $\mu^- \in \mathcal{M}_+(\mathbb{R}^n)$, satisfying the *mass balance* condition:

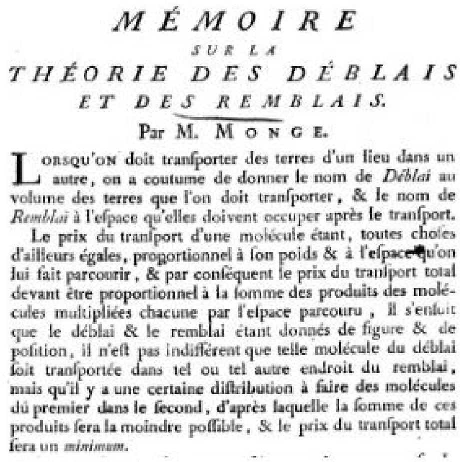
$$\mu^+(\mathbb{R}^n) = \mu^-(\mathbb{R}^n) < \infty. \tag{2.1}$$

Note that we could consider measures μ^+ and μ^- defined on general complete and separable spaces, but, for the aim of this thesis, it is enough to consider $\mathbb{R}^n$ or some proper subspaces of it. Moreover, if we divide the equality in (2.1) by a constant, it is the same to consider $\mu^+(\mathbb{R}^n) = \mu^-(\mathbb{R}^n) = 1$, and, thus, μ^+, $\mu^- \in \mathbb{P}(\mathbb{R}^n)$. Denoting by $X = \operatorname{supp}(\mu^+)$ and by $Y = \operatorname{supp}(\mu^-)$, we will call *transport map* (or, alternatively *mass rearrangement map* or *mass transfer*) $T : X \to Y$ a one-to-one and Borel measurable map, which tells us how a certain point x moves to $y = T(x)$. This map T, in order to be feasible for the Monge optimal mass problem, must also satisfy the property of rearranging μ^+ into μ^-. This property is expressed through:

$$T_{\#}(\mu^+) = \mu^-, \tag{2.2}$$

where $\#$ denotes the *push-forward* of a measure, defined by:

$$T_{\#}(\mu^+)(B) = \mu^+(T^{-1}(B)), \quad B \in \mathcal{B}(Y). \tag{2.3}$$



MÉMOIRE
SUR LA
THÉORIE DES DÉBLAIS
ET DES REMBLAIS.
Par M. MONGE.

LORSQU'ON doit tranſporter des terres d'un lieu dans un autre, on a coutume de donner le nom de *Déblai* au volume des terres que l'on doit tranſporter, & le nom de *Remblai* à l'eſpace qu'elles doivent occuper après le tranſport.

Le prix du tranſport d'une molécule étant, toutes choſes d'ailleurs égales, proportionnel à ſon poids & à l'eſpace qu'on lui fait parcourir, & par conſéquent le prix du tranſport total devant être proportionnel à la ſomme des produits des molécules multipliées chacune par l'eſpace parcouru, il s'enſuit que le déblai & le remblai étant donnés de figure & de poſition, il n'eſt pas indifférent que telle molécule du déblai ſoit tranſportée dans tel ou tel autre endroit du remblai, mais qu'il y a une certaine diſtribution à faire des molécules du premier dans le ſecond, d'après laquelle la ſomme de ces produits ſera la moindre poſſible, & le prix du tranſport total ſera un *minimum*.

FIGURE 4. Front page of Monge's *"Mémoire sur la Theorie des Déblais et des Remblais"*.

This last constraint can also be written, using a change of variable, in the following way:

$$\int_X h(T(x))d\mu^+(x) = \int_Y h(y)d(T_\#(\mu^+))(y), \ \forall h \in L^1(Y, T_\#(\mu^+)). \tag{2.4}$$

Thus, we denote the set of all possible transport maps by:

$$\mathcal{T}(\mu^+, \mu^-) := \{T : X \to Y | \text{ T is Borel measurable, one-to-one and } T_\#(\mu^+) = \mu^-\}.$$

Moreover, consider the *work* or *cost density* function:

$$c : \mathbb{R}^n \times \mathbb{R}^n \to [0, \ \infty) .$$

This function tells us the work required to move a unit of mass from the position $x \in X$ to the position $y \in Y$. Hence, if $d\mu^+(x)$ denotes the mass associated with a small neighborhood near x, the infinitesimal cost for to transport $d\mu^+(x)$ from x to $T(x)$ is $c(x, T(x))d\mu^+(x)$. With this notation in our mind, we can define the *total cost* associated to a mass rearrangement map T by:

$$I(T) := \int_X c(x, T(x))d\mu^+(x). \tag{2.5}$$

Then, the original Monge problem reads as:

Problem ($\mathcal{M}$).

Given two positive finite measures $\mu^+, \mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ satisfying (2.1)*, find an* optimal mass transfer $T^* \in \mathcal{T}(\mu^+, \mu^-)$ *such that:*

$$I(T^*) = \inf_{T \in \mathcal{T}(\mu^+, \mu^-)} I(T). \tag{2.6}$$

We will see from the following two examples, which can be found for example in Ambrosio [1], that the existence or the uniqueness of an optimal transport map are not always guaranteed.

EXAMPLE 2.1 (Non existence). Consider the case in which $X = Y = [-1, +1]$, $\mu^+ = \delta_0$ and $\mu^- = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$, then Problem ($\mathcal{M}$) has no solution simply because mass cannot split. The problem is *unfeasible*, i.e. there is no map such that $T_\#(\mu^+) = \mu^-$.

EXAMPLE 2.2 (Non uniqueness). Consider the case in which $X = Y = \mathbb{R}$, $\mu^+ = \chi_{[0,n]}dx$,$\mu^- = \chi_{[1,n+1]}dx$ and there is an L^1 cost function, i.e. $c(x,y) = \|x-y\|$. Then, consider the two transport maps:

$$T_1(x) = x+1; \tag{2.7}$$

$$T_2(x) = \begin{cases} x+n, & \text{on } [0,1) \\ x, & \text{on } [1,n]. \end{cases} \tag{2.8}$$

Now, note that if we consider a function $h : \mathbb{R} \to \mathbb{R}$ with $\mathrm{Lip}(h) = 1$, then

$$\int_Y h(y)d\mu^-(y) - \int_X h(x)d\mu^+(x) \overbrace{=}^{(2.4)} \int_X [h(T(x)) - h(x)]\, d\mu^+(x) \overbrace{\leq}^{\mathrm{Lip}(h)=1} \int_X \|x - T(x)\|\, d\mu^+(x).$$

If we consider $h(x) = x$, for which $\mathrm{Lip}(h) = 1$, then

$$\int_X h(x)d\left(\mu^- - \mu^+\right) = \int_1^{n+1} xdx - \int_0^n xdx = \frac{x^2}{2}\Big|_{x=1}^{x=n+1} - \frac{x^2}{2}\Big|_{x=0}^{x=n} = n.$$

Thus, any transport map is characterized by the fact that its cost has to be greater than or equal to n and both T_1 and T_2 satisfy the equality:

$$\int_X \|x - T_1(x)\|\, d\mu^+(x) = \int_X \|x - T_2(x)\|\, d\mu^+(x) = n.$$

Hence, we can conclude that they are both optimal and we loose the uniqueness of the minimizer.

The difficulty of Problem $(\mathcal{M})$ is not only due to the fact that the existence and the uniqueness of the minimizer are not a priori ensured, but also that, still now, after more than two hundred years, it is really difficult to solve a problem set like that, because of its high nonlinearity. Indeed, consider, for example, absolutely continuous measures with smooth densities f^+ and f^-, as defined in Definition 1.15 inside Section 3 of Chapter 1. This can briefly be written as:

$$d\mu^+ = f^+dx, \quad d\mu^- = f^-dy. \tag{2.9}$$

Note that, throughout the whole work, we will consider in most of the cases this type of measures. One reason of this particular choice is that in this case, it is possible to write the constraint expressed in (2.2) in a different way, starting from (2.4) and using the change of variable Theorem 1.11. For any $h \in L^1(Y, \mu^-)$, we get:

$$\int_X h(T(x))f^+(x)dx \overbrace{=}^{(2.4)} \int_Y h(y)f^-(y)dy \overbrace{=}^{(1.9)} \int_X h(T(x))|\det\left(DT(x)\right)|f^-(T(x))dx.$$

Looking at the right-hand side and at the left-hand side of the previous equality, by the arbitrariness of function h, it follows that the constraint in (2.2) reads as:

$$f^+(x) = f^-(T(x))|\det\left(DT(x)\right)|. \tag{2.10}$$

Even in this case, the structure of the constraint is *highly nonlinear* and it is not obvious that there exists any mapping satisfying the constraint (as for Example 2.1). Indeed, in order to briefly understand this nonlinearity, we will try to apply the Direct method in the Calculus of Variations, written in Proposition 1.1. Thus, supposing that it exists, consider a minimizing sequence $\{T_k\}_{k=1}^{+\infty} \subseteq \mathcal{T}(\mu^+, \mu^-)$, i.e.

$$I(T_k) \overset{k\to+\infty}{\longrightarrow} \inf_{T\in\mathcal{T}(\mu^+,\mu^-)} I(T).$$

The second step would be to find a subsequence $\{T_{k_j}\}_{j=1}^{+\infty}$ converging to the optimal mass allocation map T^*, i.e.

$$T_{k_j} \overset{j\to+\infty}{\longrightarrow} T^*,$$

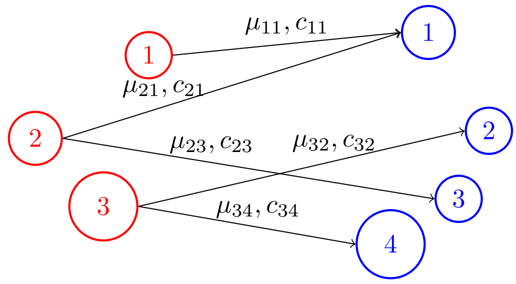


FIGURE 5. Graphic example of the optimal transport problem in the discrete case.

in some type of metric. The problem is that there is no clear way to extract such a subsequence, converging in any reasonable sense to the minimizer. Moreover, even if we want to show the lower semi-continuity of this functional, we will have problems. In general, in any Banach space V, the goal is reached if we show that the functional $I(\cdot)$ satisfies the following property:

$$I(T) \geq \alpha \|T\|^q - \beta, \quad \text{for some } \alpha > 0, q \geq 1 \text{ and } \beta \geq 0,$$

for whatever metric. This property is linked to the concept of *coercivity*, intended, in a general way, as the characteristic of a function that tends to infinity when $\|T\|$ goes to infinity. The problem, also in this case, is that we are not able to show coerciveness in any type of Sobolev space, hence the Direct method of the Calculus of Variations fails both for the step of the sequential compactness and for the step of the lower semi-continuity.

2. Kantorovich formulation: Relaxation and Duality

Kantorovich in his works [32] and [33], in the 1940's, tried to solve this highly nonlinear Problem $(\mathcal{M})$ using the following two steps:

(i) "relax" the original Monge's mass transfer problem;
(ii) use a dual variational principle.

The main idea of (i) is to transform Problem $(\mathcal{M})$ into a linear problem. Just to have some intuitions about the problem, let us pass through the discrete case, which is, as announced in Section 4 of Chapter 1, an example of linear programming. The discrete case of the optimal transport problem is also known in literature as *assignment problem*. The details about this formulation can be found, for example, in a classical book about linear programming such as Sakarovitch [40]. Suppose that we want to move an initial configuration of resources (soil in Monge's formulation) positioned in n different places, everyone with a non-negative fixed amount of resources μ_i^+, $i = 1, \dots, n$. Note that the initial configuration, in general, could also be seen as n different warehouses, each with a different amount of stock. The aim is to move these resources into m different places, representing the final fixed configuration, for example the excavation, each with a requested non-negative quantity of material μ_j^- . The unitary cost for the transport of a single unit of resource from the position $i \in \{1, \dots n\}$ to the place $j \in \{1, \dots, m\}$ is c_{ij}. Suppose, finally, that the volume expected in the final configuration is the same as the volume present in the initial configuration, i.e. the *volume balance* condition is satisfied:

$$\sum_{i=1}^{n} \mu_i^+ = \sum_{j=1}^{m} \mu_j^- . \tag{2.11}$$

We have to decide how to better move the resources, in such a way that every resource in the initial configuration is moved into the final and the final configuration is fulfilled, minimizing the transportation cost. To do that, we have to define the variables μ_{ij}, which stands for the quantity of resources moved from the initial place i to the final place j. The situation is graphically represented in Figure 5. Now, we have all the parameters and the variables in order to describe the linear program that has to accomplish the aim described before.

Problem (P).

Find an optimal solution $\mu = \{\mu_{ij}\}_{i,j}$ *which satisfies:*

$$\begin{aligned} \min \quad & \sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}\mu_{ij} \\ \textit{subject to} \quad & \sum_{j=1}^{m}\mu_{ij} = \mu_i^+ \quad i = 1,\dots,n; \\ & \sum_{i=1}^{n}\mu_{ij} = \mu_j^- \quad j = 1,\dots,m; \\ & \mu_{ij} \geq 0 \quad \forall i,j. \end{aligned}$$

The continuous version of this discrete problem means that the mass of the initial and of the final configurations are no more divided into different places, but they are arranged with initial and final "distributions". Hence, let us define the set of all the admissible maps, taking into account the three constraints used in the discrete case, this time in a continuous way. This set is defined by:

$$\Pi(\mu^+,\mu^-) := \left\{\mu \in \mathcal{M}_+(\mathbb{R}^n \times \mathbb{R}^n) \mid \pi_x(\mu) = \mu^+,\ \pi_y(\mu) = \mu^-\right\}. \tag{2.12}$$

This is the affine space of the measures defined on the product space $\mathbb{R}^n \times \mathbb{R}^n$, whose elements are characterized by the fact that the standard projections on the first n coordinates and on the last n coordinates are μ^+ and μ^-. Such measures are called *transport plans.* The projections of such measures are also called marginal distributions, as defined in the probability Section 3 of Chapter 1. Note, also, that the space $\Pi(\mu^+,\mu^-)$ is always not empty because it contains, for example, the product measure of μ^+ and μ^-, i.e. $\mu^+ \otimes \mu^-$, and it is convex. Now, given $\mu \in \Pi(\mu^+,\mu^-)$, we define the *relaxed cost functional*

$$J(\mu) := \int_{\mathbb{R}^n\times\mathbb{R}^n} c(x,y)d\mu(x,y), \tag{2.13}$$

a continuous version of the objective function in Problem (P). Using these definitions, Problem $(\mathcal{M})$ is "relaxed" into:

Problem $(\mathcal{K})$.

Given two positive finite measures $\mu^+,\mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ *satisfying* (2.1), *find an* optimal transport plan $\mu^* \in \Pi(\mu^+,\mu^-)$ *such that:*

$$J(\mu^*) = \inf_{\mu\in\Pi(\mu^+,\mu^-)} J(\mu). \tag{2.14}$$

The correspondence between Kantorovich and Monge formulations is that if $T : X \to Y$ belongs to $\mathcal{T}(\mu^+,\mu^-)$, then the transport plan defined by:

$$\mu_T(E) := \mu^+\left(\{x \in X \mid (x,T(x)) \in E\}\right), \quad E \subset \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^n)$$

belongs to $\Pi(\mu^+,\mu^-)$, using (2.2). Moreover, if it exists, the induced transport plan has the same cost of the transport map, i.e.

$$I(T) = \int_{\mathbb{R}^n} c(x,T(x))d\mu^+(x) = \int_{\mathbb{R}^n\times\mathbb{R}^n} c(x,y)d\mu_T(x,y) = J(\mu_T). \tag{2.15}$$

Furthermore, the link between the two formulations is even stronger. As the title of the first step suggested, we are going to motivate the term “relax”, used to describe Kantorovich way of proceeding. Let us define:

DEFINITION 2.1 (Relaxation). Let V be a metric space and let $F : V \to \overline{\mathbb{R}}$ be a functional, bounded from below. We define the *relaxation* of F as:

$$\overline{F} := \max\left\{G : V \to \overline{\mathbb{R}} \ : \ G \text{ is lower semi-continuous and } G \leq F\right\}.$$

Note that, as the supremum of an arbitrary family of l.s.c. functionals, it is also l.s.c, as we noted in the preliminary Chapter 1 in Section 5. Moreover, we also have the representation formula:

$$\overline{F}(x) = \inf\left\{\liminf_n F(x_n) \ : \ x_n \to x\right\}. \tag{2.16}$$

A consequence of this representation is that:

$$\inf_{x\in V} F(x) = \inf_{x\in V} \overline{F}(x). \tag{2.17}$$

In order to define both the problems on the same domain, we re-write Monge problem $(\mathcal{M})$, using (2.15), as:

$$\inf\left\{M(\mu) \ : \ \mu \in \Pi(\mu^+, \mu^-)\right\},$$

where

$$M(\mu) = \begin{cases} J(\mu) = I(T) & \text{if } \mu = \mu_T \\ +\infty & \text{otherwise.} \end{cases} \tag{2.18}$$

It is the same to consider this problem or Monge problem, because we are forced to consider only those plans induced by a transport map. Now, M and J are defined in the same set and, clearly, $J(\mu) \leq M(\mu)$, $\mu \in \Pi(\mu^+, \mu^-)$, by the definition of M. Thus:

THEOREM 2.1. *Suppose that both $X, Y \subset \mathbb{R}^n$ the supports of μ^+ and μ^-, respectively, are compact and convex subsets. Assume, also, c continuous in $X \times Y$ and μ^+* atomless, *i.e. $\mu^+(\{x\}) = 0$ for all $x \in X$. Then J is the* relaxation *of M and, in particular, the optimal values of Monge Problem* (P) *and Kantorovich Problem* $(\mathcal{K})$ *coincide, i.e.*

$$\inf_{\mu\in\Pi(\mu^+,\mu^-)} J(\mu) = \inf_{\mu\in\Pi(\mu^+,\mu^-)} M(\mu). \tag{2.19}$$

OUTLINE OF THE “PROOF” (SANTAMBROGIO [41]). $(\leq)$ Note that the functional J is continuous, thus l.s.c., and we have $J \leq M$. Then, necessarily, K is smaller than the relaxation $\overline{M}$ of M, because J belongs to the set along which is taken the infimum in Definition 2.1.

$(\geq)$ For the vice versa, we want to use the representation formula of the relaxation given by (2.16). Thus, we need to prove that for each $\mu \in \Pi(\mu^+, \mu^-)$, we are able to find a sequence of transport maps $\{T_n\}_{n\in\mathbb{N}}$, such that $\mu_{T_n} \rightharpoonup \mu$ and $M(\mu_{T_n}) \to J(\mu)$, in order to conclude that the infimum in the sequential characterization in (2.16) will be smaller than J. Actually, since $M(\mu) = J(\mu)$ for $\mu = \mu_{T_n}$, by the definition of M in (2.18), and since J is continuous, we know that if such a sequence of maps $\{T_n\}_{n\in\mathbb{N}}$ exists, the limit $M(\mu_{T_n}) \to J(\mu)$ will be automatically true. We only need to produce the sequence $\{T_n\}_{n\in\mathbb{N}}$ such that $\mu_{T_n} \rightharpoonup \mu$. This is possible thanks to the density of the set of plans induced by a transport map, i.e. of the form μ_T, along the set of plans $\Pi(\mu^+, \mu^-)$. This result is shown in Santambrogio [41, Theorem 1.32], in the case of μ^+ atomless, which is an hypothesis of the theorem we are proving. Thus we have shown that J is the relaxation of M. To conclude that the optimal values of the two problems coincide, we just need to apply the property of the relaxation, given by (2.17). □

For the complete proof of this theorem see Ambrosio [1] or Santambrogio [41].

This change of formulation of the problem is due to the fact that, unlike Monge's formulation, in this case the existence is ensured, under reasonable conditions. Indeed, we want briefly to discuss the existence in a couple of cases. For the first case, we need the following:

LEMMA 2.1. *Let V be a metric space and consider $f : V \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous function, bounded from below. Then the functional $K : \mathcal{M}_+(V) \to \mathbb{R} \cup \{+\infty\}$ defined by:*

$$K(\mu) := \int_V f(x) d\mu(x),$$

is lower semi-continuous for the weak convergence of measures.

THEOREM 2.2 (Existence case (I)). *Suppose that both X, $Y \subset \mathbb{R}^n$, the supports of μ^+, $\mu^- \in \mathcal{M}_+(\mathbb{R}^n)$, are compact and convex subsets. Moreover, suppose $c : X \times Y \to [0, +\infty]$ lower semi-continuous and bounded from below. Then $(\mathcal{K})$ admits a solution.*

The idea of this proof is to use the Direct method of the Calculus of Variations written in Proposition 1.1 to show the lower semi-continuity existence. The strategy is, first of all, to show the sequential compactness of $\Pi(\mu^+, \mu^-)$ and, then, to use Lemma 2.1, with $f = c$ defined on $X \times Y$, to prove that function J defined in (2.13) is lower semi-continuous for the weak convergence. The second existence case described below is even more general and it is true for every X and Y Polish spaces, i.e. complete and separable metric spaces. In this case, let us assume $X = Y = \mathbb{R}^n$.

THEOREM 2.3 (Existence case (II)). *Consider μ^+, $\mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ and $c : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ lower semi-continuous. Then $(\mathcal{K})$ admits a solution.*

The idea, in this case, is to show that any sequence in $\Pi(\mu^+, \mu^-)$ satisfies the *tightness*, a property of a sequence of probabilities previously introduced in Equation (1.7). Once shown this property, we are able to deduce, using Prokhorov's Theorem 1.9, the weakly sequential compactness of the set $\Pi(\mu^+, \mu^-)$. For the complete proofs of Lemma 2.1 and both Theorem 2.2 and 2.3, see Section 1.1 of Santambrogio [41]. To provide some examples of the existence, let us consider again Example 2.1 and Example 2.2. For the first one, we saw that the optimal transport map does not exist, but the optimal transport plan is given by

$$\mu(x, y) = \frac{1}{2}\delta_0 \times \delta_{-1}(x, y) + \frac{1}{2}\delta_0 \times \delta_1(x, y).$$

In general, however, also in this case uniqueness fails because of the linearity of J and because of the convexity of $\Pi(\mu^+, \mu^-)$. Example 2.2 highlights this lack of uniqueness. We showed before that there are at least two optimal maps T_1 and T_2, defined in (2.7) and (2.8). Is possible to show that, for any $t \in [0, 1]$, the measure

$$t\,(Id \times T_1)_\#\,(\mu^+) + (1 - t)\,(Id \times T_2)_\#\,(\mu^-),$$

is optimal.

Thus, under some assumptions, the existence of Kantorovich relaxed Problem $(\mathcal{K})$, which is called also "weak solution" of the Monge's original problem, is ensured. We also showed that, under other assumptions, the optimal values of the two problems coincide. The problem is that the information about the form of an optimal solution of Problem $(\mathcal{M})$ are still missing. Perhaps the next idea, also provided by Kantorovich, will help us finding the answer to this problem.

The second intuition (ii) provided by Kantorovich, was to take into account a *dual formulation* of Problem $(\mathcal{K})$. The more intuitive way to understand this formulation is to consider the finite dimensional case explained, for example, in Evans [26], but also in other classical book of linear programming, such as Sakarovitch [40]. We have already shown, at the beginning, of this section, that the Kantorovich relaxed reformulation of Monge original problem comes from a linear program, in the discrete case. Thus, we want to start considering the dual problem of a linear program, with the techniques and the motivations we introduced in Section 4 of Chapter 1. Note that, primal Problem (P), can be written in a compact way. Indeed, consider $c \in \mathbb{R}^N$, $b \in \mathbb{R}^M$ defined by:

$$\begin{cases} N = nm,\ M = n+m,\ x = (\mu_{11}, \dots, \mu_{1m}, \mu_{21}, \dots, \mu_{2m}, \dots, \mu_{nm})^T, \\ c = (c_{11}, \dots, c_{1m}, c_{21}, \dots, c_{2m}, \dots, c_{nm})^T,\ b = \left(\mu_1^+, \dots, \mu_n^+, \mu_1^-, \dots, \mu_m^-\right)^T \end{cases}$$

Moreover, consider the matrix $A \in \mathbb{R}^{M \times N}$:

$$\begin{array}{r} n \text{ rows}\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ m \text{ rows}\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \end{array} \begin{pmatrix} \mathbb{1} & 0 & \dots & 0 \\ 0 & \mathbb{1} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \mathbb{1} \\ e_1 & e_1 & \dots & e_1 \\ e_2 & e_2 & \dots & e_2 \\ \vdots & \vdots & \vdots & \vdots \\ e_m & e_m & \dots & e_m \end{pmatrix}.$$

Here the vectors $\mathbb{1}$, 0 and e_j are row vectors in $\mathbb{R}^m$, respectively, $\mathbb{1} = (1, \dots, 1)$, $0 = (0, \dots, 0)$ and $e_j = (0, \dots, 1, 0, \dots, 0)$, with the one in the j-th slot, for $j = 1, \dots, m$. Then, Problem (P) becomes:

Problem (P).

Find an optimal solution $x^* \in \mathbb{R}^N$ satisfying:

$$\begin{aligned} \min\ & c^T x \\ s.t.\ & Ax = b \\ & x \geq 0, \end{aligned}$$

Once we write the primal Problem (P) in this way, we can derive the dual formulation simply using the rules summarized in Table 1.

Problem (D).

Find an optimal solution $y^* \in \mathbb{R}^M$ satisfying:

$$\begin{aligned} \max\ & b^T y, \\ s.t.\ & A^T y \leq c, \\ & y \in \mathbb{R}^M \end{aligned}$$

Finally, calling $y = (u_1, \dots, u_n, v_1, \dots, v_m) \in \mathbb{R}^{n+m}$ and employing the explicit form of the matrix A^T, we obtain that Problem (D) is equivalent to:

Problem (D).

Find and optimal couple $y^* = (u^*, v^*)$ such that:

$$\begin{aligned} \max\ & \sum_{i=1}^n u_i \mu_i^+ + \sum_{j=1}^m v_j \mu_j^- \\ s.t.\ & u_i + v_j \leq c_{ij} \quad \forall i = 1, \dots, n,\ \forall j = 1, \dots, m. \end{aligned}$$

This dual program has an interesting economic interpretation that can be deduced from the preliminary chapter, in Section 4 about linear programming. In this case the *shadow market* is represented by an operator who offers to buy the resources that are in the initial configuration and to sell the same material at the final configuration, assuming for himself the cost of transportation. The operator offers to pay the price u_i for a unit of material placed at point $i \in \{1, \ldots, n\}$ and to sell it at $j \in \{1, \ldots, m\}$ for the unit price v_j. The constraints ensure the competitiveness of the offer: for every initial and final location i and j, the balance of the operator cost of a unit of resource in i and the operator remuneration for the sale at point j cannot be more than the amount of money a buyer could spend in buying the good in i and transporting it to j. The objective function represents the fact that the operator wants to maximize his incomes. Furthermore, we have, by Strong Duality Theorem 1.15, that any competitive optimal offer made by the operator must coincide in value with the cost of an optimal solution of the discrete transport Problem (P), i.e.

$$\sum_{i=1}^{n}\sum_{j=1}^{m} c_{ij}\mu_{ij}^* = \sum_{i=1}^{n} u_i^*\mu_i^+ + \sum_{j=1}^{m} v_j^*\mu_j^- . \tag{2.20}$$

We can now guess the dual variational formulation of Problem ($\mathcal{K}$), by using a continuous variant of the discrete Problem (D). First of all, we define the space:

$$\mathcal{L} := \{(u, v) \mid u, v : \mathbb{R}^n \to \mathbb{R},\ u(x) + v(y) \leq c(x, y) \text{ s.t. } x, y \in \mathbb{R}^n\} . \tag{2.21}$$

Here, u and v are intended to live in the space of continuous functions, i.e. $\mathcal{C}(\mathbb{R}^n)$. Moreover, we define the functional:

$$K(u, v) := \int_X u(x)d\mu^+(x) + \int_Y v(y)d\mu^-(y). \tag{2.22}$$

The continuous *dual problem* of ($\mathcal{K}$) becomes:

Problem ($\mathcal{D}$).

Given two positive finite measures $\mu^+, \mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ satisfying (2.1)*, find an optimal couple $(u^*, v^*) \in \mathcal{L}$ such that*

$$K(u^*, v^*) = \max_{(u,v)\in\mathcal{L}} K(u, v). \tag{2.23}$$

As an extension of the Strong Duality Theorem 1.15 in the discrete case, reported in (2.20), we have that in the case the optimal couple exists, the dual and the primal problem must achieve the same value:

$$K(u^*, v^*) = \max_{(u,v)\in\mathcal{L}} K(u, v) = \min_{\mu\in\Pi(\mu^+,\mu^-)} J(\mu) = J(\mu^*). \tag{2.24}$$

We introduced this discrete procedure in order to understand better this transition to the dual problem, which is motivated and justified by the Lagrange duality introduced in the linear programming section. We can obtain this dual reformulation of Kantorovich problem, using results of convex analysis in infinite dimensional spaces, which we briefly introduced in Section 5 of Chapter 1. For this duality result we have taken inspiration from Santambrogio [41]. Suppose that the supports X and Y of our measures μ^+ and μ^- are compact and that $c : X \times Y \to \mathbb{R}$ is continuous. We want to introduce a function $H : \mathcal{C}(X \times Y) \to \mathbb{R}$, that is defined for every continuous function $p \in \mathcal{C}(X \times Y)$ as:

$$H(p) := -\max_{(u,v)} \left\{ \int_X u(x)d\mu^+(x) + \int_Y v(y)d\mu^-(y) \ : \ u(x) + v(y) \leq c(x, y) - p(x, y) \right\}.$$

This function coincides with the opposite of the value of Problem ($\mathcal{D}$) in the case of a cost function $c - p$. The main tool we plan to use to prove the duality correspondence between Problems ($\mathcal{K}$) and ($\mathcal{D}$) is Fenchel-Moreau Theroem 1.16. In order to apply it, we use a result that can also be found in [41]:

Lemma 2.2. *The function $H : \mathcal{C}(X \times Y) \to \mathbb{R}$ is convex and l.s.c with respect to the uniform convergence on the compact space $X \times Y$.*

This result tells us that the conditions of the Fenchel-Moreau theorem are satisfied and we can apply it to the function H. Before proving the duality result, let us compute the Legendre-Fenchel transformation $H^* : \mathcal{M}(X \times Y) \to \mathbb{R} \cup \{+\infty\}$ of the function H. For every $\mu \in \mathcal{M}(X \times Y)$, which is, recalling Riesz representation Theorem 1.7, the dual space of $\mathcal{C}(X \times Y)$, the Legendre-Fenchel transformation is given by:

$$H^*(\mu) = \sup_p \left[\int_{X\times Y} p \; d\mu + \sup_{(u,v)} \left\{ \int_X u d\mu^+ + \int_Y v d\mu^- \; : \; u(x) + v(y) \le c(x,y) - p(x,y) \right\} \right].$$

Note that we used the scalar product $\langle \mu, p\rangle_{\mathcal{C}_0(X\times Y)} = \int_{X\times Y} p \; d\mu$ defined inside the Riesz representation theorem recalled above. Now:

- if $\mu \notin \mathcal{M}_+(X\times Y)$, then there exists a function $p_0 \le 0$ s.t. $\int p_0 d\mu > 0$ and, taking $u = v = 0$, $p = c + np_0$ and letting $n \to \infty$, we obtain $H^*(\mu) = +\infty$;
- if $\mu \in \mathcal{M}_+(X \times Y)$, we choose the largest possible p, i.e. $p(x, y) = c(x, y) - u(x) - v(y)$.

Hence, for all $\mu \in \mathcal{M}_+(X \times Y)$, we obtain, using the largest p:

$$H^*(\mu) = \int_{X\times Y} c(x,y) d\mu(x,y) + \sup_{(u,v)} \left\{ \int_X u d\mu^+ + \int_Y v d\mu^- - \int_{X\times Y} (u(x) + v(y)) d\mu(x,y) \right\}.$$

Note, also, that:

$$\sup_{u,v} \left(\int_X u d\mu^+ + \int_Y v d\mu^- - \int_{X\times Y} (u(x) + v(y))\, d\mu(x,y) \right) = \begin{cases} 0 & \text{if } \mu \in \Pi(\mu^+, \mu^-), \\ +\infty & \text{otherwise.} \end{cases}$$

We get, from the definition of J in (2.13):

$$H^*(\mu) = \begin{cases} J(\mu) & \text{if } \mu \in \Pi(\mu^+, \mu^-), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.25}$$

Finally, we are ready to prove the duality theorem:

THEOREM 2.4. *If X, Y are compact spaces and the cost function $c : X \times Y \to \mathbb{R}$ is continuous, then the optimal values of Problems $(\mathcal{K})$ and $(\mathcal{D})$ coincide.*

PROOF (SANTAMBROGIO [41]). By Lemma 2.2, H is convex and l.s.c. Thus, we can use Fenchel-Moreau Theorem 1.16 and we get:

$$\max(\mathcal{D}) = -H(0) = -H^{**}(0).$$

Moreover, by the definition of Legendre-Fenchel conjugate and (2.25), we have:

$$H^{**}(0) = \sup_\mu [\langle 0, \mu\rangle_* - H^*(\mu)] = -\inf_\mu H^*(\mu) = -\min(\mathcal{K}).$$

Putting together the last two equalities, we obtain:

$$\max(\mathcal{D}) = \min(\mathcal{K}).$$

□

If we assume the same conditions of Theorem 2.1, it follows:

$$\min_{T\in\mathcal{T}(\mu^+,\mu^-)} I(T) = \min_{\mu\in\Pi(\mu^+,\mu^-)} J(\mu) = \max_{(u,v)\in\mathcal{L}} K(u,v). \tag{2.26}$$

Summarizing, instead of solving a highly nonlinear Problem $(\mathcal{M})$, we can consider a relaxation and, then, a dualization which puts us in a setting where we can use standard mathematical tools. This allows us to construct a minimizer as a limit of a minimizing sequence, in some metric, as we saw before. However, even if we can find an optimal pair $(u^*, v^*) \in \mathcal{L}$ or an optimal plan $\mu^* \in \Pi(\mu^+, \mu^-)$, we are still far away from the original issue. Thus, we can not be pleased with this "weak" solution and the question remains: *how can we recover the optimal transport map T^* and what is its form?* We want to start answering this question in a general case, where the cost function c is a strictly convex function of $x - y$. Recalling Definition 1.20 of c-concave function, the following theorem holds:

THEOREM 2.5 (Gangbo-McCann). *Suppose that* μ^+, $\mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ *are absolutely continuous measure with densities* f^+ *and* f^-, *respectively. Moreover, suppose that* $c(x,y) = h(x-y)$ *with* $h : \mathbb{R}^n \to \mathbb{R}$ *strictly convex and that there exists* $\tilde{\mu}$ *such that:*

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} h(x-y) d\tilde{\mu}(x,y) < +\infty.$$

Then, there exists a unique optimizer μ^* *for the relaxed Kantorovich Problem* $(\mathcal{K})$. *Furthermore, the unique optimal* μ^* *is induced by a transport map* T^*, *i.e.*

$$\mu^* = (Id \times T^*)_{\#}\mu^+. \tag{2.27}$$

This T^* *is uniquely determined* μ^+*-a.e by the conditions* $(T^*)_{\#}\mu^+ = \mu^-$ *and*

$$T^*(x) = x - \nabla h^* \left(\nabla \phi(x)\right), \tag{2.28}$$

where h^* *is the usual Legendre-Fenchel transform of* h *and* ϕ *is a c-concave function.*

The proof of this theorem is due to Gangbo and McCann and can be found in [30]. Note, that, typical examples of strictly convex functions are $c(x,y) = \|x-y\|^p$, with $p > 1$. In these cases, we are able to recover existence and uniqueness of Monge's original Problem $(\mathcal{M})$ and we know the form of the optimal transport map T^*. For the case $p = 1$, instead, Theorem 2.5 does not apply. Thus, the previous question about the existence and the form of the optimal T^* remains without an answer.

3. Introduction to the L^1 case and some first results

From now on, we will just focus on the non-strictly convex cost $c(x,y) = \|x-y\|$, which is Monge's original cost. Thus, Problem $(\mathcal{M})$, in this case of L^1 cost function, becomes:

Problem $(\mathcal{M}_1)$.

Given two positive finite measures $\mu^+, \mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ *satisfying* (2.1), *find an* optimal mass transfer $T^* \in \mathcal{T}(\mu^+,\mu^-)$ *such that:*

$$I_1(T^*) = \inf_{T \in \mathcal{T}(\mu^+,\mu^-)} I_1(T), \tag{2.29}$$

with:

$$I_1(T) = \int_X \|x - T(x)\| \, d\mu^+(x). \tag{2.30}$$

We start by building some intuitions about the form of $T^* \in \mathcal{T}(\mu^+,\mu^-)$. We begin with some informal insights, without proofs, for the moment, trying to have some information about the optimal transport map and about its geometric properties. This reasoning is taken from Evans [26].

Let us assume that $T^* \in \mathcal{T}(\mu^+,\mu^-)$ minimizes Problem $(\mathcal{M}_1)$. Fix a positive integer $m \geq 2$, select some distinct points $\{x_k\}_{k=1}^m \subset X = \mathrm{supp}(\mu^+)$ and assume we can find m small disjoint balls around x_k (Figure 6)

$$E_k := B(x_k, r_k) \quad (k = 1, \ldots, m). \tag{2.31}$$

The radii $\{r_k\}_{k=1}^m$ are chosen in such a way that

$$\mu^+(E_1) = \cdots = \mu^+(E_m) = \varepsilon. \tag{2.32}$$

Next, set $y_k := T^*(x_k)$, $F_k := T^*(E_k)$. Since T^* satisfies (2.2), we have that:

$$\mu^-(F_1) = \cdots = \mu^-(F_m) = \varepsilon. \tag{2.33}$$

We, then, construct another mapping $T \in \mathcal{T}(\mu^+,\mu^-)$ by cyclically permuting the images of $\{E_k\}_{k=1}^m$, i.e.

$$\begin{cases} T(x_k) = y_{k+1}, \quad T(E_k) = F_{k+1}, \quad k = 1, \ldots, m \\ T \equiv T^* \text{ on } X \setminus \bigcup_{k=1}^m E_k, \end{cases} \tag{2.34}$$

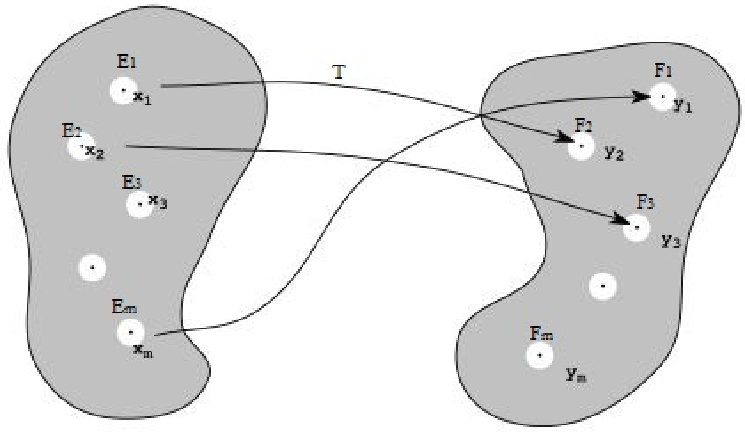


FIGURE 6. Representation of $T \in \mathcal{T}(\mu^+, \mu^-)$, defined in (2.34)

as it is represented in Figure 6, where $y_{m+1} := y_1$, $F_{m+1} := F_1$ (this picture is taken from Evans [26]). Now, since, by the minimality of T^*, we have $I_1(T^*) \leq I_1(T)$, then, we get:

$$\sum_{k=1}^{m} \int_{E_k} \|x - T^*(x)\| \, d\mu^+(x) \leq \sum_{k=1}^{m} \int_{E_k} \|x - T(x)\| \, d\mu^+(x). \tag{2.35}$$

Dividing by ε and sending $\varepsilon \to 0$, we obtain, using Lebesgue Differentiation Theorem 1.1:

$$\sum_{k=1}^{m} \|x_k - y_k\| \leq \sum_{k=1}^{m} \|x_k - y_{k+1}\|. \tag{2.36}$$

Let us try to gain some geometrical intuition about inequality (2.36). Consider a closed curve, contained in X, $C := \{r(t) \mid 0 \leq t \leq 1\}$ $(r(0) = r(1))$, with constant speed parametrization. Take m equally spaced points along C, i.e. $x_k := r(\frac{k}{m})$. Choosing $\tau > 0$, s.t. $\tau = \frac{1}{m}$ and letting $m \to \infty$, we get from (2.36):

$$i(\tau) := \int_0^1 \|r(t) - T^*(r(t))\| \, dt \leq \int_0^1 \|r(t) - T^*(r(t+\tau))\| \, dt$$
$$\overbrace{=}^{t+\tau=s} \int_0^1 \|r(s-\tau) - T^*(r(s))\| \, ds.$$

Hence, $i(\tau)$ has a minimum in $\tau = 0$, which implies:

$$0 = i'(0) = \int_0^1 \left(\frac{T^*(r(t)) - r(t)}{\|T^*(r(t)) - r(t)\|} \right) \cdot r'(t) dt = \int_C \xi \cdot ds,$$

where

$$\xi(x) := \frac{T^*(x) - x}{\|T^*(x) - x\|}.$$

As this holds for all closed curves C, it follows, thanks to a property of the holomorphic functions, see, for example, Rudin [39], that ξ is a gradient of some function $u^* : \mathbb{R}^n \to \mathbb{R}$, i.e.:

$$\frac{T^*(x) - x}{\|T^*(x) - x\|} = -\nabla u^*(x). \tag{2.37}$$

This function is called the *Kantorovich potential* and satisfies the *eikonal equation*:

$$\|\nabla u^*\| = 1 \quad \text{on } X = \operatorname{supp}(\mu^+). \tag{2.38}$$

In other words, at point x, the optimal transport map T^* moves in the direction of the gradient of a potential u^*, by the distance $d^*(x) := \|T^*(x) - x\|$, i.e.

$$T^*(x) = x - d^*(x)\nabla u^*(x). \tag{2.39}$$

Remark 2.1. Note that the same result can be achieved using a *Lagrange multiplier* argument for the constraint $f^+ = f^-(T)|\det(DT)|$, as in equation (2.10), in the case of absolutely continuous measures with densities f^+ and f^-. In this case, the *augmented work functional* is:

$$\widetilde{I}(T) := \int_{\mathbb{R}^n} \|x - T(x)\|\, f^+(x) + \lambda(x)\left[f^-(T(x))|\det(DT(x))| - f^+(x)\right] dx.$$

Annihilating the first variation $\frac{d}{d\varepsilon}\widetilde{I}(T + \varepsilon\phi)\big|_{\varepsilon=0}$ of $\widetilde{I}$ in the direction ϕ, we obtain the equality:

$$\frac{\partial}{\partial x_i}\left(\lambda f^-(T^*)\mathrm{cof}(DT^*)_{ki}\right) = \frac{T_k^* - x_k}{\|T^* - x\|} f^+(x) + \lambda \frac{\partial}{\partial y_k}\left(f^-(T^*)\right)\det(DT^*), \tag{2.40}$$

with $(\mathrm{cof}(DT^*))_{ki}$ the $(k,i)-$th entry of the cofactor matrix of DT^*. The cofactor matrix appears here, because for any matrix $A \in \mathbb{R}^{n\times m}$, chosen a column j, the determinant can be written as:

$$\det(A) = \sum_{i=1}^{n} a_{ij}\mathrm{cof}(A)_{ij}. \tag{2.41}$$

Now, standard matrix identities assert:

(i) $\dfrac{\partial}{\partial x_i}(\mathrm{cof}(DT^*))_{ki} = 0$;

(ii) $\dfrac{\partial T_j^*}{\partial x_i}(\mathrm{cof}(DT^*))_{ki} = \delta_{kj}(\det(DT^*))$;

(iii) $\dfrac{\partial T_k^*}{\partial x_j}(\mathrm{cof}(DT^*))_{ki} = \delta_{ij}(\det(DT^*))$.

Then, we get:

$$\begin{aligned}
&\frac{\partial}{\partial x_i}\left(\lambda f^-(T^*)\mathrm{cof}(DT^*)_{ki}\right) = \frac{\partial \lambda}{\partial x_i} f^-(T^*)\mathrm{cof}(DT^*)_{ki} + \\
&+ \lambda \sum_{j=1}^{n} \frac{\partial}{\partial y_j}(f^-(T^*))\frac{\partial T_j^*}{\partial x_i}\mathrm{cof}(DT^*)_{ki} + \lambda f^-(T^*)\frac{\partial}{\partial x_i}(\mathrm{cof}(DT^*))_{ki} \quad \text{using (i) and (ii)} \\
&= \frac{\partial \lambda}{\partial x_i} f^-(T^*)\mathrm{cof}(DT^*)_{ki} + \lambda \frac{\partial}{\partial y_k}\left(f^-(T^*)\right)\det(DT^*).
\end{aligned}$$

Thus, using this equality inside (2.40), we obtain:

$$\frac{\partial \lambda}{\partial x_i} f^-(T^*)\mathrm{cof}(DT^*)_{ki} = \frac{T_k^* - x_k}{\|T^* - x\|} f^+(x).$$

Now, using the identity written in (2.41), multiplying by $\frac{\partial T_k^*}{\partial x_i}$ the previous equality and summing on k, we deduce:

$$\frac{\partial \lambda}{\partial x_i} f^-(T^*)\det(DT^*) = \sum_{k=1}^{n} \frac{T_k^* - x_k}{\|T^* - x\|}\frac{\partial T_k^*}{\partial x_i} f^+(x).$$

Recalling that f^+ and f^- satisfy the constraint written in (2.10), we obtain the final representation:

$$\frac{\partial \lambda}{\partial x_i} = \sum_{k=1}^{n} \frac{T_k^* - x_k}{\|T^* - x\|}\frac{\partial T_k^*}{\partial x_i}.$$

Next, setting $u^* : \mathbb{R}^n \to \mathbb{R}$ by:

$$\lambda(x) = -u^*(T^*(x)).$$

Then:

$$\sum_{k=1}^{n} \frac{\partial u^*}{\partial y_k}(T^*)\frac{\partial T_k^*}{\partial x_i} = -\frac{\partial \lambda}{\partial x_i} = -\sum_{k=1}^{n} \frac{T_k^* - x_k}{\|T^* - x\|}\frac{\partial T_k^*}{\partial x_i}.$$

Hence, finally we get:

$$\nabla u^*(T^*(x)) = -\frac{T^*(x) - x}{\|T^*(x) - x\|},$$

which is exactly the same equality as in (2.37). □

Now, the idea is to transform the previous heuristic calculations into a proof of the existence of an optimal transport map T^*, using its explicit form written in (2.39). Thus, we should start from the existence of the Kantorovich potential u^*. Note that, looking at Remark 2.1 and at the way in which we introduced the Lagrange duality in Section 4 of Chapter 1, it is clear that this potential u^* is linked to dual Problem $(\mathcal{D})$, in the case of L^1 cost function. Thus, first of all, let us precisely write both Problems $(\mathcal{K})$ and $(\mathcal{D})$, in the specific case of L^1 cost function. For the first one, just considering the explicit formulation of c, we get:

Problem $(\mathcal{K}_1)$.

Given two positive finite measures $\mu^+, \mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ satisfying (2.1)*, find an* optimal transport plan $\mu^* \in \Pi(\mu^+, \mu^-)$ *such that:*

$$J_1(\mu^*) = \inf_{\mu \in \Pi(\mu^+, \mu^-)} J_1(\mu), \tag{2.42}$$

with:

$$J_1(\mu) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - y\| \, d\mu(x, y). \tag{2.43}$$

For the dual case, we want to consider simple case of absolutely continuous measures with densities f^+ and f^-, which, as in (2.9), can be briefly written as $d\mu^+ = f^+dx$, $d\mu^- = f^-dy$. As in the general case, f^+ and f^- have to satisfy the *mass balance* condition:

$$\int_X f^+(x)dx = \int_Y f^-(y)dy, \tag{2.44}$$

where X and Y are the usual supports of f^+ and f^-. Problem $(\mathcal{D})$ defined before becomes, with an L^1 cost:

Problem $(\mathcal{D}_1)$.

Given two positive finite absolutely continuous measures $\mu^+, \mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ with densities f^+ and f^- satisfying (2.44)*, find an optimal couple (u^*, v^*) which* maximizes*:*

$$K_1(u, v) = \int_X u(x)f^+(x)dx + \int_Y v(y)f^-(y)dy \tag{2.45}$$

under the constraint:

$$u(x) + v(y) \leq \|x - y\| \quad x \in X, \ y \in Y. \tag{2.46}$$

The first important result, here exposed, is the proof of the existence of Kantorovich potential u^*, which, as we already said, is linked to the optimal solution of dual Problem $(\mathcal{D}_1)$. Also in this case, this result is taken from Evans [26].

Lemma 2.3. *Given two positive finite absolutely continuous measures $\mu^+, \mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ with densities f^+ and f^- satisfying* (2.44)*, there exists an optimal couple (u^*, v^*) solving Problem $(\mathcal{D}_1)$. Moreover, we can take*

$$v^* = -u^*, \tag{2.47}$$

with $u^* : \mathbb{R}^n \to \mathbb{R}$ Lipschitz continuous with Lipschitz constant 1, i.e. $u^* \in Lip_1(\mathbb{R}^n)$.

PROOF (EVANS [26]). Take the pair (u, v) which satisfies (2.46). Then:

$$u(x) \leq \inf_{y \in Y}(\|x-y\| - v(y)) =: \widehat{u}(x), \tag{2.48}$$

with $(\widehat{u}, v)$ satisfying the constraint

$$\widehat{u}(x) + v(y) \leq \|x-y\| \quad x \in X,\ y \in Y.$$

This implies

$$v(y) \leq \min_{x \in X}(\|x-y\| - \widehat{u}(x)) =: \widehat{v}(y) \tag{2.49}$$

Also in this case

$$\widehat{u}(x) + \widehat{v}(y) \leq \|x-y\| \quad x \in X,\ y \in Y. \tag{2.50}$$

Moreover, since $v \leq \widehat{v}$ and (2.48) holds, we have:

$$\widehat{u}(x) \geq \inf_{y \in Y}(\|x-y\| - \widehat{v}(y)),$$

which, added to (2.50), gives us:

$$\widehat{u}(x) = \min_{y \in Y}(\|x-y\| - \widehat{v}(y)). \tag{2.51}$$

Now, since $f^{\pm} \geq 0$ and $\widehat{u} \geq u$, $\widehat{v} \geq v$, we see that $K(u, v) \leq K(\widehat{u}, \widehat{v})$. Next, focusing on this pair $(\widehat{u}, \widehat{v})$, take $z \in X \cap Y$ (if $X \cap Y \neq \emptyset$) and suppose

$$\widehat{u}(z) + \widehat{v}(z) < 0. \tag{2.52}$$

Take, then, $x \in X$, $y \in Y$ so that, using definitions in (2.49) and (2.51), satisfy

$$\begin{cases} \widehat{v}(z) = \|x-z\| - \widehat{u}(x) \\ \widehat{u}(z) = \|z-y\| - \widehat{v}(y) \end{cases}$$

Adding the two rows, we get:

$$\|x-z\| + \|z-y\| = \widehat{u}(x) + \widehat{v}(y) + \widehat{u}(z) + \widehat{v}(z) \overbrace{<}^{(2.52)} \widehat{u}(x) + \widehat{v}(y) \overbrace{\leq}^{(2.50)} |x-y|,$$

which contradicts the triangle inequality. It follows, then, that $\widehat{u} + \widehat{v} \geq 0$ on $X \cap Y$, but, clearly, (2.50) implies $\widehat{u} + \widehat{v} \leq 0$ on $X \cap Y$ and so the equality holds:

$$\widehat{u} + \widehat{v} = 0 \text{ on } X \cap Y. \tag{2.53}$$

Using this property, we can extend the definition of $\widehat{u}$, by setting:

$$\widehat{u} := -\widehat{v} \quad \text{ on } Y,$$

and the inequality in (2.50) reads as:

$$\widehat{u}(x) - \widehat{u}(y) \leq \|x-y\| \quad (x \in X,\ y \in Y).$$

Finally, once we have this intuition on the form of $\widehat{u}$, we can extend it to $\mathbb{R}^n$, in such a way that the inequality above is satisfied in the whole space. For example, one way to extend a general function $f : S \to \mathbb{R}$, with S subset of $(X, \|\cdot\|)$ arbitrary normed space, to $g : X \to \mathbb{R}$ in such a way that $\mathrm{Lip}(g) = \mathrm{Lip}(f)$ is given by Kirszbraun's Theorem, which can be found for example in [28, Theorem 2.10.43]. The explicit form of this function g given by:

$$g(x) := \inf_{s \in S}[f(s) + \mathrm{Lip}(f) \cdot \|x-s\|], \quad \forall x \in X.$$

Thus, we can extend $\widehat{u}$ to all $\mathbb{R}^n$, calling this extension u^*, in such a way that it is a Lipschitz function with Lipschitz constant 1. $\square$

In conclusion, we saw that there exists an optimal solution (u^*, v^*) of Problem $(\mathcal{D}_1)$, which is equivalent to the following:

Problem $(\mathcal{D}_1')$.

Given two positive finite absolutely continuous measures $\mu^+, \mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ with densities f^+ and f^- satisfying (2.44)*, find an optimal u^** maximizing*:*

$$K_1(u) := \int_{\mathbb{R}^n} u(x)(f^+(x) - f^-(x))dx \tag{2.54}$$

under the constraint:

$$\begin{aligned} \|\nabla u\| &\leq 1 \qquad a.e.\ in\ \mathbb{R}^n, \\ \|\nabla u\| &= 1 \qquad a.e\ in\ X \cup Y. \end{aligned} \tag{2.55}$$

We want to focus, just for a while, on this maximal u^*, called, as before, *Kantorovich potential* and for doing that we need first some definitions. Note that the ball $B(0,R)$ will be intended as a ball large enough to contain both the supports of the measures μ^+ and μ^-.

Definition 2.2 (Transport set). Given the Kantorovich potential u^*, we call *transport set* the following:

$$\begin{aligned} T_s := \{z \in B(0,R)\ &: u^*(z) = u^*(x) - \|x - z\| \text{ for some point } x \in X \\ &\text{and } u^*(z) = u^*(y) + \|z - y\| \text{ for some } y \in Y\}. \end{aligned}$$

Observe that if $z \in T_s$, then the corresponding x, y, z are collinear. Indeed, using the identities $u^*(x) - u^*(z) = \|x - z\|$, $u^*(z) - u^*(y) = \|z - y\|$, we deduce:

$$\|x - z\| + \|z - y\| = u^*(x) - u^*(y) \leq \|x - y\|.$$

Thus, applying the triangle inequality, we get the colinearity. Moreover, it is possible to show, that $X \cup Y \subset T_s$. We need to introduce another definition.

Definition 2.3 (Transport ray). Given $z_0 \in T_s$, we define the set:

$$R_{z_0} := \{z \in B(0,R)\ :\ \|u(z_0) - u(z)\| = \|z_0 - z\|\}.$$

Note that if u^* is differentiable at z_0, then R_{z_0} is a line segment. In that case, it is called *transport ray* passing through z_0.

Observe that this is the set containing z_0 along which u^* changes at the maximum rate 1. It is also possible to show that one end of R_{z_0}, call it a_0, lies in X and the other end, call it b_0, lies in Y. Thus, we think to R_{z_0} as pointing "downhill" from a_0 to b_0, because u^* decreases at rate 1 as we move along R_{z_0}, from a_0 to b_0. We call $R_{z_0} \setminus \{a_0, b_0\}$ the *interior* of a transport ray. These segments are constructed in such a way that u^* behaves in a particular way along these structures and it satisfies a lot of peculiar properties. We do not want to go deeper into the details of these transport rays, because it is not the aim of this thesis and it is a delicate and complex issue. However, for all the details we refer to [27] or to [41]. Here we are interested just in a particular property, which will be useful later on and it is taken from [27]:

Lemma 2.4. *If z belongs to the interior of a transport ray $]a_0, b_0[$, i.e. exists $t \in (0,1)$ such that $z = (1-t)a_0 + tb_0$, then u^* is differentiable at z and $\nabla u^*(z) = \frac{a_0 - b_0}{|a_0 - b_0|}$.*

4. Existence of an optimal mass transport map

In this section, we want to use the Kantorovich transport potential u^* in order to build an optimal mass transfer map T^*. Looking at Equation (2.39), where it appears the link between optimal T^* and Kantorovich potential u^*, we only miss the information about distance $d^*(x)$. This problem was originally solved in [43] by decomposing almost all X in *transport rays*. He, then, built the optimal transport map by gluing together the 1-dimensional transport maps constructed in each ray. This technique worked under restrictive assumptions later relaxed (see [1]). Here, we consider a different approach based differential equations. We are going to present only an empirical discussion of this procedure, referring to [27] for a complete proof. The idea is to extract the missing information of $d^*(x)$ from the variational Problem $(\mathcal{D}_1')$, approximated by the solution of a partial differential equation called *p-Laplacian PDE*, as $p \to +\infty$. We start with the following:

Definition 2.4 (p-Laplacian). Let $1 < p < \infty$, $\Omega \subset \mathbb{R}^n$ a convex and open set and $u : \Omega \to \mathbb{R}$ sufficiently regular, usually in $W^{1,p}(\Omega)$. We define the **p-Laplacian**, a quasi-linear elliptic partial differential operator of the second order, as:

$$\Delta_p u := \nabla \cdot \left(\|\nabla u\|^{p-2} \nabla u \right). \tag{2.56}$$

When $p = 2$, it is the normal Laplacian operator.

Let us consider, now, the p-Laplacian PDE:

$$-\Delta_p u_p = -\nabla \cdot \left(\|\nabla u_p\|^{p-2} \nabla u_p \right) = f^+ - f^- = f \quad (n+1 \leq p \leq \infty) \tag{2.57}$$

and note that this equation is the Euler-Lagrange equation for the problem of maximizing:

$$K_p(u) := \int_\Omega u(x)(f^+(x) - f^-(x)) - \frac{1}{p} \|\nabla u(x)\|^p \, dx, \tag{2.58}$$

over all $u \in W_0^{1,p}(\Omega)$. Note also that, if u_p is the solution of (2.57), then $\|\nabla u_p\|^{p-2}$ is very large in any region $\{\|\nabla u_p\| > 1 + \varepsilon\}$ and very small in any region $\{\|\nabla u_p\| < 1 - \varepsilon\}$, if $\varepsilon > 0$. The limit as $p \to \infty$ can be interpeted as an "infinitely fast / infinitely slow" diffusion limit (see the work of Evans, Feldman, and Gariepy [25]). Also helped by the intuition just stated, it is not very hard to prove that if $u_p \to u^*$, as $p \to \infty$, then u^* is Lipschitz continuous with Lipschitz constant 1 and it maximizes the functional K_1 in $(\mathcal{D}_1')$. Thus, u^* is the Kantorovich potential. But there is still the issue of the missing information about the distance $d^*(x)$. The most important discovery due to Evans-Gangbo is that PDE (2.57), in fact, contains, in the limit as $p \to \infty$, a "transport density", which allow us to compute this distance. The idea is to start using a maximum principle argument, taken from Bhattacharya-DiBenedetto-Manfredi [6], which tells us that:

$$\sup_p \|u_p\|, \|\nabla u_p\|, \|\nabla u_p\|^p \leq C < \infty$$

for some constant C. Next, we proceed using this information to see that it exists a subsequence $p_k \to \infty$ such that:

$$\begin{cases} u_{p_k} \to u^* & \text{locally uniformly} \\ Du_{p_k} \to \nabla u^* & \text{boundedly, a.e.} \end{cases}$$

for some function u^*, which is differentiable almost everywhere. To prove the differentiability, it can be used a not so famous result called Rademacher's Theorem ([28, Theorem 3.1.6]). The most important result is:

$$\|\nabla u_{p_k}\|^{p-2} \rightharpoonup a^* \quad \text{weakly-* in } L^\infty$$

for some non-negative bounded function a^*, called the *transport density*. Thus, from all these intuitions, it follows:

Lemma 2.5 (PDE Version). *Assume that the mass densities f^+ and f^- are non-negative, Lipschitz continuous functions on $\mathbb{R}^n$ with disjoint compact supports X and Y having smooth boundaries. Moreover, assume that they satisfy the mass balance condition* (2.44). *Then, it follows that:*

(i) There exists a non-negative L^∞ function a^* such that

$$-\nabla \cdot (a^*(x) \nabla u^*(x)) = f^+(x) - f^-(x), \quad x \in \mathbb{R}^n, \tag{2.59}$$

in a weak sense.

(ii) Furthermore

$$\begin{cases} \|\nabla u^*\| = 1 & \textit{a.e. on the set } \{a^* > 0\} \\ \|\nabla u^*\| \leq 1 & \textit{a.e. on the set } \{a^* \geq 0\} \end{cases} \tag{2.60}$$

Note that, with "weak sense", we mean that:

$$-\int_{\mathbb{R}^n} \nabla \cdot (a^*(x)\nabla u^*(x))\phi(x)dx = \int_{\mathbb{R}^n} (f^+(x) - f^-(x))\phi(x)dx, \quad \forall \phi \in \mathcal{C}_c^1(\mathbb{R}^n).$$

This condition, if we use the integration by parts and the fact that $\phi \in \mathcal{C}_c^1(\mathbb{R}^n)$, becomes:

$$\int_{\mathbb{R}^n} a^*(x)\nabla u^*(x))\nabla\phi(x)dx = \int_{\mathbb{R}^n} (f^+(x) - f^-(x))\phi(x)dx, \quad \forall \phi \in \mathcal{C}_c^1(\mathbb{R}^n). \tag{2.61}$$

Summarizing, we have the following:

Problem ($\mathcal{PDE}$).

Consider two positive finite measures $\mu^+, \mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ with mass densities f^+ and f^- non-negative, Lipschitz continuous on $\mathbb{R}^n$, with disjoint compact supports X and Y and with smooth boundaries, respectively. Moreover, assume that they satisfy the mass balance condition (2.44)*. Find the optimal pair $(a^*, u^*) \in (L^\infty(\mathbb{R}^n), Lip_1(\mathbb{R}^n))$ that satisfies:*

$$\begin{cases} -\nabla \cdot (a^*(x)\nabla u^*(x)) = f^+(x) - f^-(x), & x \in \mathbb{R}^n \textit{ (in a weak sense)}, \\ \|\nabla u^*\| = 1 & \textit{a.e. on the set } \{a^* > 0\}, \\ \|\nabla u^*\| \leq 1 & \textit{a.e. on the set } \{a^* \geq 0\}. \end{cases} \tag{2.62}$$

We want to spend some words about the origin of this transport density a^* and about Problem ($\mathcal{PDE}$), due to its importance in this work. Indeed, even if we saw that this version came from a limit in weak-* topology of the p-Laplacian solutions and it provided us the missing information about distance d^*, it has also other characteristics and meanings. Here, we report a couple of other interpretations of the origin of this PDE, taken from different fields. Moreover, in the next chapter, we will see that this transport density will appear also in other contexts, providing us a link between this PDE formulation and other formulation we will introduce later.

(i) **Optimization**: Consider the KKT conditions (Proposition 1.4). We consider Problem ($\mathcal{D}_1'$), written in the same form as the constrained minimization problem expressed in (MP). In our case, function f in (MP) is $-K_1$ (eq. (2.54)) and the function representing the constraints of S (eq. (1.12)) is $g(u) := \|\nabla u\| - 1 \leq 0$. Note that we changed sign of K_1, because Problem (MP) is a minimization problem, while Problem ($\mathcal{D}_1'$) is a maximization problem. The Lagrangian $\mathcal{L}$ can, then be defined as:

$$\mathcal{L}(u, \lambda) := \int_{\mathbb{R}^n} u(x)(f^-(x) - f^+(x)) + \lambda(x) \left(\|\nabla u(x) - 1\|\right) dx.$$

Now, if we define $\widetilde{g}(u)$ as:

$$\widetilde{g}(u) := \frac{1}{2}\left(\|\nabla u\|^2 - 1\right),$$

we obtain that:

$$\widetilde{g}(u) \leq 0 \;\Rightarrow\; (\|\nabla u\| - 1) \cdot (\|\nabla u\| + 1) \leq 0 \;\Rightarrow\; g(u) \leq 0.$$

Vice versa,

$$g(u) \leq 0 \;\Rightarrow\; \|\nabla u\|^2 \leq 1 \;\Rightarrow\; \widetilde{g}(u) \leq 0.$$

Hence, it is equivalent to consider the constraint given by $\widetilde{g}(u)$ and, thus, the Lagrangian function:

$$\widetilde{\mathcal{L}}(u, \lambda) := \int_{\mathbb{R}^n} u(x)(f^-(x) - f^+(x)) + \frac{\lambda(x)}{2}\left(\|\nabla u(x)\|^2 - 1\right) dx.$$

We want to annihilate, now, the derivative with respect to the first variable, in order to satisfy the first condition of the KKT Proposition 1.4. Note that, since the Lagrangian $\widetilde{\mathcal{L}}$ is defined, in the first variable, on a space of functions, it is not possible to consider a standard derivative in that variable. Thus, we consider the Gâteaux derivative in the variable u, i.e. the first variation of $\tilde{\mathcal{L}}$. For doing that, consider $\varepsilon > 0$, $\phi \in \mathcal{C}_c^1(\mathbb{R}^n)$ and calculate the following derivative, as it was recalled in the notation in the preliminary chapter:

$$\begin{aligned}\frac{d}{d\varepsilon}\tilde{\mathcal{L}}(u+\varepsilon\phi,\lambda)\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\left(\int_{\mathbb{R}^n}(u+\varepsilon\phi)(f^- - f^+) + \frac{\lambda}{2}\left(\|\nabla u + \varepsilon\nabla\phi\|^2 - 1\right)dx\right)\Big|_{\varepsilon=0} = \\ &= \left(\int_{\mathbb{R}^n}\phi(f^- - f^+) + \frac{\lambda}{2}\cdot 2\left(\nabla u + \varepsilon\nabla\phi\right)\nabla\phi\; dx\right)\Big|_{\varepsilon=0} = \\ &= \left(\int_{\mathbb{R}^n}\phi(f^- - f^+) + \lambda\nabla u\cdot\nabla\phi\; dx\right).\end{aligned}$$

Annihilating the last equality, we obtain that (2.61) is verified for an optimal couple (u^*, λ^*). Note, in addition, that condition (2.60) is exactly the last KKT condition. Thus, this result can be summarized by saying that if u^* is the optimal solution of $(\mathcal{D}_1')$, **$\mathbf{a}^*$ is the optimal Lagrange multiplier** corresponding to the constraint $\|\nabla u^*\| \leq 1$, satisfying, paired with u^*, the KKT conditions for the constrained maximization Problem $(\mathcal{D}_1')$.

(ii) **Convex analysis**: Let us define:

$$\mathbb{K} := \{u \in L^2(\mathbb{R}^n) \mid \|\nabla u\| \leq 1 \text{ a.e.}\}$$

and the indicator function of this set

$$I_{\mathbb{K}}(u) := \begin{cases} 0 & \text{if } u \in \mathbb{K} \\ +\infty & \text{otherwise.}\end{cases}$$

Now, note that $\mathbb{K}$ is convex and $K_1(u) = \langle f^+ - f^-, u\rangle_{L^2(\mathbb{R}^n)}$, using the scalar product defined in (1.1), is continuous and convex, thus we are in the hypothesis of Proposition 1.11. Note also that, reasoning as we did in equation (1.21), maximizing the function K_1, in $(\mathcal{D}_1')$, on the set $\mathbb{K}$, is the same as maximizing the function $K_1 - I_{\mathbb{K}}$ on all $L^2(\mathbb{R}^n)$. Indeed, if $u \notin \mathbb{K}$, then $K_1(u) - I_{\mathbb{K}}(u) = -\infty$. The idea is that doing a constrained maximization is the same as maximizing in the whole space with a *penalization* for those elements not in the right space. Moreover, note that the following two problems are equivalent:

$$\max_{u\in L^2(\mathbb{R}^n)} K_1(u) - I_{\mathbb{K}}(u) \iff -\left(\min_{u\in L^2(\mathbb{R}^n)} -\left(K_1(u) - I_{\mathbb{K}}(u)\right)\right).$$

Applying, now, Proposition 1.11, together with the previous equality, Kantorovich potential u^* is the maximizer of Problem $(\mathcal{D}_1')$ if and only if $0 \in \partial\left(-K_1(u^*) + I_{\mathbb{K}}(u^*)\right)$. By definition of subdifferential (eq. 1.24), we obtain:

$$\langle 0, v - u^*\rangle_* - K_1(u^*) + I_{\mathbb{K}}(u^*) \leq -K_1(v) + I_{\mathbb{K}}(v), \quad \forall v \in L^2(\mathbb{R}^n).$$

Using the definition of K_1 and the scalar product in L^2, which we recalled above, we obtain:

$$I_{\mathbb{K}}(v) \geq I_{\mathbb{K}}(u^*) + \langle f^+ - f^-, v - u^*\rangle_{L^2(\mathbb{R}^n)}, \quad \forall v \in L^2(\mathbb{R}^n),$$

i.e.

$$f^+ - f^- \in \partial I_{\mathbb{K}}(u^*),$$

This final representation is the same result as in Lemma 2.5, but in a convex analysis language.

Now, let us go on with the core problem of this chapter: we want to find the optimal transport map. Before we directly address this issue, we need a deformation argument due to Dacorogna and Moser [20]. We need this result, because the idea of Evans and Gangbo was to use this deformation argument to build an optimal transport map, using a "recipe" we will see later. We will show also an outline of the original proof of this result, because the idea used inside this proof will often be used in order to prove that two functions f^+ and f^- satisfy the constraint expressed in (2.10).

Theorem 2.6 (Dacorogna-Moser). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\mathcal{C}^{k+3}$ regular boundary $\partial\Omega$ and let $f \in \mathcal{C}^k(\overline{\Omega})$, positive and such that*

$$\int_\Omega f(x)dx = meas(\Omega).$$

Then there exists a $\mathcal{C}^k$-diffeomorphism $u : \bar{\Omega} \to \bar{\Omega}$ satisfying:

$$\begin{cases} det(Du(x)) = f(x), & x \in \Omega \\ u(x) = x, & x \in \partial\Omega \end{cases} \tag{2.63}$$

Outline of the "proof" (Dacorogna-Moser [20]). Let us define for $t \in [0,1]$, $z \in \Omega$:

$$v_t(z) := \frac{v(z)}{t + (1-t)f(z)}, \tag{2.64}$$

where v satisfies

$$\begin{cases} \nabla \cdot v = f - 1 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.65}$$

Such v exists thanks to Dacorogna and Moser [20, Theorem 2]. We then define $\Phi : [0,1] \times \Omega \to \mathbb{R}^n$ as the solution of the following ODE:

$$\begin{cases} \frac{\partial}{\partial t}\left(\Phi(t,x)\right) = v_t(\Phi(t,x)) & \text{for } t > 0 \\ \Phi(0,x) = x \end{cases} \tag{2.66}$$

Now, we change notation in a standard way calling $\Phi_t(x) := \Phi(t,x)$. Note that Φ_t is uniquely defined for every $t \in [0,1]$. We also have that for every $t \in [0,1]$:

$$\Phi_t(x) \equiv x \quad \text{if } x \in \partial\Omega.$$

This follows from the uniqueness of the solution of (2.66), from the fact that, on $\partial\Omega$, $v = 0$ and $\Phi_0(x) = x$. Our claim is now that $u(x) := \Phi_1(x)$ is the solution of (2.63). We have just verified the boundary conditions. Hence, we only need to check that $\det\left(D\Phi_1(x)\right) = f(x)$. To prove this, let us define:

$$h(t,x) := \det(D\Phi_t(x))\left(t + (1-t)f(\Phi_t(x))\right). \tag{2.67}$$

Note that if we show that:

$$\frac{\partial}{\partial t}h(t,x) = 0 \quad \forall t \in [0,1], \tag{*}$$

it follows that:

$$h(1,x) = \det\left(D\Phi_1(x)\right) = f(x) = h(0,x),$$

which is exactly what we want to prove. We therefore only need to prove (*). To do that, we need to use the property that if A is an $n \times n$ matrix and ψ is a matrix valued function satisfying $\psi'(t) = A(t)\psi(t)$, then:

$$(\det\psi)' = \operatorname{tr}(A) \cdot \det\psi. \tag{2.68}$$

We prove this identity for 2×2 matrices and it is easy to extend it for general $n\times n$ matrices. Thus, consider the equality:

$$\begin{pmatrix} \psi'_{11} & \psi'_{12} \\ \psi'_{21} & \psi'_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}.$$

Looking at the matrix multiplication at each entry we obtain:

$$\begin{cases} \psi'_{11} = a_{11}\psi_{11} + a_{12}\psi_{21}; \\ \psi'_{12} = a_{11}\psi_{12} + a_{12}\psi_{22}; \\ \psi'_{21} = a_{21}\psi_{11} + a_{22}\psi_{21}; \\ \psi'_{22} = a_{21}\psi_{12} + a_{22}\psi_{22}; \end{cases}$$

Hence,

$$\begin{aligned}(\det(\psi))' &= (\psi_{11}\psi_{22} - \psi_{12}\psi_{21})' = \psi'_{11}\psi_{22} + \psi_{11}\psi'_{22} - \psi'_{12}\psi_{21} - \psi_{12}\psi'_{21} \\ &= (a_{11}\psi_{11} + a_{12}\psi_{21})\psi_{22} + \psi_{11}(a_{21}\psi_{12} + a_{22}\psi_{22}) - \\ &- (a_{11}\psi_{12} + a_{12}\psi_{22})\psi_{21} - \psi_{12}(a_{21}\psi_{11} + a_{22}\psi_{21}) \\ &= \operatorname{tr}(A) \cdot \det(\psi).\end{aligned}$$

Now, in order to use Equation (2.68), note that:

$$\frac{\partial}{\partial t}(D\Phi_t(x)) = D\frac{\partial \Phi_t(x)}{\partial t} \overbrace{=}^{(2.66)} Dv_t(\Phi_t(x)) \cdot D\Phi_t(x).$$

Thus, in our case, we have $A(t) = Dv_t(\Phi_t(x))$, that implies, using (2.68):

$$\frac{\partial}{\partial t}(\det D\Phi_t(x)) = \det(D\Phi_t(x)) \cdot \operatorname{tr}(Dv_t(\Phi_t(x))) = \det(D\Phi_t(x)) \cdot (\nabla \cdot (v_t(\Phi_t(x)))) . \quad (2.69)$$

Differentiating, now, (2.67), we get:

$$\begin{aligned}\frac{\partial}{\partial t}h(t,x) &= \frac{\partial}{\partial t}(\det(D\Phi_t(x))) \cdot [t + (1-t)f(\Phi_t(x))] + \\ &+ \det(D\Phi_t(x)) \cdot \left[1 - f(\Phi_t(x)) + (1-t)\nabla f(\Phi_t(x)) \cdot \left(\frac{\partial}{\partial t}\Phi_t(x)\right)\right].\end{aligned}$$

Using (2.66) and (2.69), we get:

$$\begin{aligned}\frac{\partial}{\partial t}h(t,x) &= \det(D\Phi_t(x)) \cdot [\nabla \cdot (v_t(\Phi_t(x))) \cdot (t + (1-t)f(\Phi_t(x))) .+ \\ &+ (1 - f(\Phi_t(x))) + (1-t)\nabla f(\Phi_t(x)) \cdot v_t(\Phi_t(x))].\end{aligned}$$

Finally, taking the divergence in the definition of v_t written in (2.64), it follows that:

$$\nabla \cdot (v(x)) = (t + (1-t)f(x)) \cdot (\nabla \cdot v_t(\Phi_t(x))) + (1-t)\nabla f(x) \cdot v_t(x).$$

Combining this relation with the previous identity, we get:

$$\frac{\partial}{\partial t}h(t,x) = \det(D\Phi_t(x)) \cdot [\nabla \cdot (v(\Phi_t(x))) + (1 - f(\Phi_t(x)))] .$$

The definition of v in (2.65) gives immediately (*) and the proof is complete. □

Note that this proof is just an outline, because we did not prove any regularity of the function u. For the complete proof we refer to [20]. Now, we are ready to show the existence of an optimal mass transportation map that is given by the following **recipe**:

(1) Solve Problem $(\mathcal{PDE})$, which gives us the couple (a^*, u^*);
(2) Fix $x_0 \in X = \operatorname{supp}(\mu^+)$ and use (u^*, a^*) to describe the *flow* of a Cauchy problem involving ∇u^* and a^*, given by:

$$\begin{cases} \dot{z}(t) = b(z(t), t) \quad (0 \le t \le 1) \\ z(0) = x_0 \end{cases} \tag{2.70}$$

for the time dependent *vector field*, depending on time:

$$b(z,t) := \frac{-a^*(z)\nabla u^*(z)}{tf^-(z) + (1-t)f^+(z)} \tag{2.71}$$

(3) Calling $z(t, x_0)$ the solution of (2.70), note that we are in the same situation of Dacorogna-Moser's Theorem 2.6 with:

$$\begin{cases} v(x) = -a^*(x)\nabla u^*(x) \qquad (velocity) \\ \nabla \cdot (v(x)) = f^+(x) - f^-(x) \\ v_t(x) = \dfrac{v(x)}{t\underbrace{f^-(x)}_{\text{Added}} + (1-t)f^+(x)}. \end{cases}$$

Using the same idea as in that theorem, we define the *optimal mass transfer* as the time-one map of the flow z, i.e.:

$$\boxed{T^*(x) := z(1, x).} \tag{2.72}$$

Once we have all this intuitive background, we would like to verify if this is actually the map we were looking for. We have the following theorem, taken from [26] or [27].

Theorem 2.7. *Assume that densities f^+ and f^- of the absolutely continuous measures μ^+ and μ^-, respectively, are non-negative Lipschitz continuous functions on $\mathbb{R}^n$ with disjoint compact supports X and Y having smooth boundaries. Moreover, assume that they satisfy the mass balance condition* (2.44). *Consider T^* defined in* (2.72). *Then:*

(i) $T^* : X \to Y$ is bijective.

(ii) T^* satisfies equation (2.10) *and, thus:*

$$\int_X h(T^*(x))f^+(x)dx = \int_Y h(y)f^-(y)dy, \ \forall h \in L^1(Y, \mu^-). \tag{2.73}$$

(iii) For all $T : X \to Y$ satisfying (2.73), *we get:*

$$\int_X \|x - T^*(x)\| f^+(x)dx \leq \int_X \|x - T(x)\| f^+(x)dx.$$

Thus, T^* defined in (2.72) *is an optimal solution of Monge's original Problem* $(\mathcal{M}_1)$.

Outline of the "proof" (Evans-Gangbo [27]). (i) T^* is bijective thanks to Dacorogna-Moser's Theorem 2.6, using the fact that the solution of Equation (2.63) is a diffeomorphism.

(ii) The idea is the same of the proof of Dacorogna-Moser's Theroem 2.6. Thus, calling $z(t, x)$ the solution of (2.70) starting from x and calling:

$$J(t, x) := \det(Dz(t, x)) \cdot \left(tf^-(z(t, x)) + (1 - t)f^+(z(t, x))\right),$$

we follow step-by-step their proof. Hence, we want to show, also in this case, that $\frac{\partial}{\partial t}J(t, x) = 0$, in order to equate the value of J at times 0 and 1. Using (2.69), we obtain:

$$\begin{aligned} \frac{\partial}{\partial t}J(t, x) = \det(Dz) \cdot [\nabla \cdot (b(z, t)) \left(tf^-(z) + (1 - t)f^+(z)\right) + \\ + \left(f^-(z) - f^+(z)\right) + \left(t\nabla f^-(z) \cdot b(z, t) + (1 - t)\nabla f^+(z) \cdot b(z, t)\right)], \end{aligned} \tag{2.74}$$

where $z = z(t, x)$. Now, following again the idea of Dacorogna-Moser, using the definition of b in (2.71), we get:

$$-\nabla\cdot(a^*(z)\nabla u^*(z)) = (\nabla\cdot b(z, t))\cdot(tf^-(z)+(1-t)f^+(z))+b(z, t)\cdot(t\nabla f^-(z)+(1-t)\nabla f^+(z)).$$

Finally, recalling the PDE in (2.59), we get that the summands inside the parenthesis (2.74) are equal to zero. Furthermore, using the definition of T^* in (2.72), we get:

$$J(0, x) = f^+(x) = \det(DT^*(x))f^-(T^*(x)) = J(1, x),$$

that confirms assertion (ii), just using a change of variable, as we did to obtain (2.10). Note that, in that case there was an absolute value, but both f^+ and f^- are positive and T^* is invertible, thus $\det(DT^*)$ remains strictly positive.

(iii) Finally, we need to use a property of the transport rays. Indeed, it follows from the construction of these particular segments, that for a.e. $x_0 \in X$ the point $y_0 = T^*(x_0)$ lies on the transport ray R_{x_0}. By Definition 2.3 of transport ray, we obtain:

$$u^*(x) - u^*(T^*(x)) = \|x - T^*(x)\| \, . \tag{2.75}$$

For the details about transport rays and this particular result we refer again to Evans and Gangbo [27]. Thus, we obtain:

$$\begin{aligned}
\int_X \|x - T^*(x)\| \, f^+(x)dx &= \int_X \left[u^*(x) - u^*(T^*(x))\right] f^+(x)dx \quad \text{by (2.75)} \\
&= \int_X u^*(x)f^+(x)dx - \int_Y u^*(y)f^-(y)dy \quad \text{by (ii)} \\
&= \max_{u\in \mathrm{Lip}_1(\mathbb{R}^n)} \int_{\mathbb{R}^n} u(x)(f^+(x) - f^-(x))dx, \quad \text{by def. of } u^* \\
&= \min_{S\in \mathcal{T}(\mu^+,\mu^-)} \int_X \|x - S(x)\| \, f^+(x)dx \quad \text{by (2.26)} \\
&\leq \int_X \|x - T(x)\| \, f^+(x)dx. \qquad \square
\end{aligned}$$

Remark 2.2. This proof is in "quotations", because, actually, neither a^* nor ∇u^* nor $f^{\pm}$ are smooth enough to justify these computations. However, it is still useful to outline the general idea in order to figure out the main intuitions and tools used for the proof, even if it is not rigorous. For a careful proof, with all the analytic details, see again Evans-Gangbo [27]. They used the same idea of employing Dacorogna-Moser's theorem, but "mollifying" (which means regularizing) $-(a^*(z)\nabla u^*(z))_\varepsilon$, f^+_ε and f^-_ε, in order to get the "permission" to do the computations. They regularized, also, the analogous vector field $b_{\varepsilon,\delta}$ and the solution $z_{\varepsilon,\delta}$ of a smooth variant of (2.70). Finally they took the solution $z_{\varepsilon,\delta}(1, x_0)$ as an approximation of $T_{\varepsilon,\delta}$ and passing to limits $\varepsilon \to 0$ and $\delta \to 0$ they obtained our T^*. $\square$

CHAPTER 3

Other formulations of the mass transfer problem

In this chapter, we want to take a different look at the Monge-Kantorovich transport problem. We start introducing a dynamic version of the problem, taking inspiration from the celebrated Benamou and Brenier [5]. After the description of their original idea, developed with a quadratic cost function, we would like to adapt it to the case of L^1 cost, remaining consistent with the rest of the thesis. The problem is that their procedure is not adaptable to L^1. Thus, we need another way to move Monge-Kantorovich problem with L^1 cost function to a dynamical framework. To do that, we want to adapt to this case a technique provided by Chen, Georgiou and Pavon [18], who also worked with L^2 cost functions. However, their way of proceeding can be adjusted to the L^1 cost case. In their work, they were also able to find a solution to this dynamical reformulation doing a variational analysis. In our case, instead, after the failure of this mathematical technique, we try to solve this problem in another way. Our idea is to get some useful information from another version of the mass transportation problem, this time a static one, introduced firstly by Beckmann [3]. Adding these other two, the different formulations count to six: we feel an urgent need of order and clarity in the situation. This need to sort things out and to find the relationships between the different formulations is also evident in literature: we can cite the work of Ambrosio [1] or a more recent paper written by Brasco and Petrache [11] or, even, the book of Santambrogio [41], just to have some examples. This need is due to the fact that the different formulations come from different settings and backgrounds, but they all describe the same problem. Thus a normal question to ask is: what are the connections between them? The final section tries to answer this question.

1. A fluid-dynamic version of the optimal transport problem

In this section, we will introduce an alternative formulation for the Monge-Kantorovich optimal transport problem. The idea we are going to use was firstly introduced in the famous work Benamou and Brenier [5]. They reset the whole problem into a dynamical framework, originally for numerical purposes. In fact, a continuum mechanics formulation was implicitly contained in the original problem addressed by Monge, who decided to eliminate the time variable just to reduce the dimension of the problem. Benamou and Brenier, instead, decided to reintroduce the time variable, because, from a computational point of view, it is more convenient to solve a convex space-time minimization problem. In their work they used the quadratic cost function and, unfortunately, their way of proceeding is not suited for the L^1 cost. Let us briefly see their original problem and why it does not fit well for our purposes. First of all, they fixed a time interval, which we suppose $[0, 1]$, and they considered all the possible time-dependent density and velocity fields $\rho(t, x) \geq 0$, $v(t, x) \in \mathbb{R}^n$, smooth enough, subject to some constraints described in the following:

Problem ($\mathcal{DYN}_2$).

Given two positive finite absolutely continuous measures $\mu^+, \mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ with densities

f^+ and f^- satisfying (2.44)*, find an* optimal couple (ρ^*, v^*) *satisfying:*

$$\begin{cases} \inf_{(\rho,v)} \int_{\mathbb{R}^n} \int_0^1 \|v(t,x)\|^2 \rho(t,x) dt dx, \\ \dfrac{\partial \rho}{\partial t}(t,x) + \nabla \cdot (v(t,x)\rho(t,x)) = 0 \quad \textit{(continuity eq.)}, \\ \rho(x,0) = f^+(x), \quad \rho(y,1) = f^-(y) \quad \textit{(initial and final conditions)}. \end{cases} \tag{3.1}$$

In order to show why this dynamic problem is related with Monge-Kantorovich optimal transport problem, let us see the precise connection between this problem and Problem $(\mathcal{M})$ with cost function $c(x,y) = \|x-y\|^2$, taken directly from [5].

THEOREM 3.1. *Consider $\mu^+, \mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ two absolutely continuous measures with smooth and compactly supported densities f^+ and f^-. Then:*

$$\min(\mathcal{M}) = \min(\mathcal{DYN}_2). \tag{3.2}$$

PROOF (BENAMOU AND BRENIER [5]). ($\leq$) Consider an admissible pair (ρ, v) for Problem $(\mathcal{DYN}_2)$ and consider the flow:

$$\begin{cases} X'(t,x) &= v_t(X(t,x)) \\ X(0,x) &= x, \end{cases}$$

with the usual notation $v_t(x) = v(t,x)$. It can be proved (we will precisely state this result later) that curve ρ satisfying the continuity equation with initial datum f^+, can be represented by $\rho_t = (X(t,\cdot))_\# f^+$. Using this characterization and using Jensen's inequality (see for example Rudin [39, Theorem 3.3]) with convex function $\phi(x) = \|x\|^2$, we get:

$$\begin{aligned} \int_0^1 \int_{\mathbb{R}^n} \|v(t,x)\|^2 \rho(t,x) dx dt &= \int_0^1 \int_{\mathbb{R}^n} \|v_t(X(t,x))\|^2 f^+(x) dx dt \quad \text{by def. of } X(t,x) \\ &= \int_0^1 \int_{\mathbb{R}^n} \|X'(t,x))\|^2 f^+(x) dx dt \quad \text{by Fubini-Tonelli Theorem 1.2} \\ &= \int_{\mathbb{R}^n} \int_0^1 \|X'(t,x))\|^2 f^+(x) dt dx \quad \text{by Jensen's inequality} \\ &\geq \int_{\mathbb{R}^n} \|X(1,x) - x\|^2 f^+(x) dx \geq \min(\mathcal{M}). \end{aligned}$$

Note that, in the last part, used the fact that $f^- = X(1,\cdot)_\# f^+$, which can be proved following exactly what we did in the proof of Dacorogna and Moser Theorem 2.6. Thus $T(x) = X(1,x)$ is an admissible transport map for $(\mathcal{M})$, with cost function $c(x,y) = \|x-y\|^2$ and the last inequality of the previous calculation is justified. This inequality tells us, by the arbitrariness of the couple (ρ, v), that:

$$\min(\mathcal{M}) \leq \min(\mathcal{DYN}_2). \tag{3.3}$$

($\geq$) For the other inequality, we consider the optimal transport map T^*, uniquely given by Gangbo-McCann Theorem 2.5, in the case of strictly convex cost $c(x,y) = \|x-y\|^2$. Setting $X_t(x) = (1-t)x + tT^*(x)$ and considering v of the form:

$$v_t(x) = T^*(X_t^{-1}(x)) - X_t^{-1}(x),$$

(the proof of the fact that such X_t is invertible can be found in [41, Lemma 5.29]) we get:

$$\begin{aligned} &X'(t,x) = T^*(x) - x = T^*(X_t^{-1}(X(t,x))) - X_t^{-1}(X(t,x)) = v_t(X(t,x)), \\ &X(0,x) = x. \end{aligned}$$

Thus, X_t is the flow map of v and, as before, $\rho_t = (X_t)_\# f^+$ is a solution of the continuity equation with velocity field v and initial and final data equal to f^+ and f^-, respectively.

With this choice of (ρ, v) we get:

$$\int_{\mathbb{R}^n}\int_0^1 \|X'(t,x))\|^2 f^+(x)dtdx = \int_{\mathbb{R}^n} \|T^*(x)-x\|^2 f^+(x)dx = \min(\mathcal{M})$$

Thus, noting that the left-hand side of the equality above is exactly equal to

$$\int_0^1 \int_{\mathbb{R}^n} \|v(t,x)\|^2 \rho(t,x)dxdt,$$

we recover (3.2) and we conclude the proof. □

Note that we can even say more about the last proof. In the last part, we actually proved, using (3.3), that:

$$\int_0^1 \int_{\mathbb{R}^n} \|v(t,x)\|^2 \rho(t,x)dxdt = \min(\mathcal{M}) \le \min(\mathcal{DYN}_2).$$

Hence, we obtained that the optimal couple solving $(\mathcal{DYN}_2)$ is given by:

$$\begin{cases} \boxed{\rho_t(x) = ((1-t)x + tT^*(x))_{\#} f^+(x)}; \\ v_t(x) = T^*(X_t^{-1}(x)) - X_t^{-1}(x). \end{cases} \tag{3.4}$$

This boxed equality represents density ρ_t with a nonlinear interpolation formula, which we will describe, later, in any details. Note that, if we try to adapt the previous statement to the L^1 cost case we fail, because, as already said, Gangbo-McCann Theorem 2.5 can not be used in the case of non strictly convex cost function $c(x,y) = \|x-y\|$. Thus, we want to recover a correspondence between classical formulations and this dynamic formulation similar to the one given by Theorem 3.1, but in our L^1 cost case. To do that, we will take inspiration from the recent Chen, Georgiou and Pavon [18]. Even in this work they used an L^2 cost function, but their technique is adaptable to the L^1 cost case. Let us start defining the space:

$$\mathcal{X} := \mathcal{C}^1([0,1];\mathbb{R}^n). \tag{3.5}$$

Observe, then, the elementary fact:

$$\|x-y\| = \inf_{\gamma\in\mathcal{X}_{xy}} \int_0^1 \|\dot\gamma(t)\|\, dt, \tag{3.6}$$

where $\mathcal{X}_{xy} = \{\gamma \in \mathcal{X} \mid \gamma(0) = x,\ \gamma(1) = y\}$. This simple observation comes from the fact that the shortest path connecting x and y is the straight line $\gamma_{x,y}(t) = (1-t)x + ty$. Now, doing the same observation from a stochastic point of view, we get that the probability measure on $\mathbb{P}(\mathcal{X})$ concentrated on the path $\{\gamma_{x,y}(t) \mid 0 \le t \le 1\}$ solves the problem:

$$\inf_{P_{xy}\in\mathcal{Q}^1(\delta_x,\delta_y)} \mathbb{E}_{P_{xy}} \left\{\int_0^1 \|\dot\gamma(t)\|\, dt\right\}, \tag{3.7}$$

where $\mathcal{Q}^1(\delta_x,\delta_y)$ is the set of all the probability measures on $\mathcal{X}$ whose marginal at time 0 and 1 are, respectively, the Dirac measures concentrated at x and y, i.e.

$$\mathcal{Q}^1(\delta_x,\delta_y) := \{P_{xy} \in \mathbb{P}(\mathcal{X}) \mid (e_0)_{\#}P_{xy} = \delta_x \text{ and } (e_1)_{\#}P_{xy} = \delta_y\}. \tag{3.8}$$

Function e_t used above is the *evaluation map at time* t, i.e.

$$\begin{aligned} e_t : \mathcal{X} &\to \mathbb{R}^n \\ \gamma &\mapsto \gamma(t). \end{aligned} \tag{3.9}$$

Thus, using the fact that (3.7) is another representation of $\|x-y\|$, recalling that Problem $(\mathcal{K}_1)$ with L^1 cost is:

$$\inf_{\mu\in\Pi(\mu^+,\mu^-)} \int_{\mathbb{R}^n\times\mathbb{R}^n} \|x-y\|\, d\mu(x,y)$$

and plugging (3.7) in the relaxed cost functional $J(\mu)$ (eq. (2.13)), we obtain:

$$\inf_{\mu\in\Pi(\mu^+,\mu^-)} \int_{\mathbb{R}^n\times\mathbb{R}^n} \left[\inf_{P_{xy}\in\mathcal{Q}^1(\delta_x,\delta_y)} \mathbb{E}_{P_{xy}} \left\{ \int_0^1 \|\dot\gamma(t)\|\, dt \right\} \right] d\mu(x,y).$$

Now observe that, if we have $P_{xy} \in \mathcal{Q}^1(\delta_x, \delta_y)$ and $\mu \in \Pi(\mu^+, \mu^-)$, we obtain that:

$$P = \int_{\mathbb{R}^n\times\mathbb{R}^n} P_{xy} d\mu(x,y) \tag{3.10}$$

is a probability measure in

$$\mathcal{Q}^1(\mu^+,\mu^-) := \left\{ P \in \mathbb{P}(\mathcal{X}) \mid (e_0)_\# P = \mu^+ \text{ and } (e_1)_\# P = \mu^- \right\}. \tag{3.11}$$

Note that, also the converse is true, i.e. every measure $P \in \mathcal{Q}^1(\mu^+, \mu^-)$ can be "disintegrated" with respect to the initial and final positions in $P_{xy} \in \mathcal{Q}^1(\delta_x, \delta_y)$ and $\mu \in \Pi(\mu^+, \mu^-)$ such that (3.10) is verified. Hence, we get that Problem $(\mathcal{K}_1)$ is equivalent to:

$$\inf_{P\in\mathcal{Q}^1(\mu^+,\mu^-)} \mathbb{E}_P \left\{ \int_0^1 \|\dot\gamma(t)\|\, dt \right\}. \tag{3.12}$$

Now, if instead of this static version we take an *hydrodynamic version* of the equality in (3.6), we can write:

$$\begin{cases} \|x-y\| = \inf_{v\in\mathcal{V}_y} \int_0^1 \|v(t,\gamma^v(t))\|\, dt, \\ \dot\gamma^v(t) = v(t,\gamma^v(t)), \\ \gamma^v(0) = x, \text{ a.e.} \end{cases}$$

where $\mathcal{V}_y = \{v : [0,1]\times\mathbb{R}^n \to \mathbb{R}^n \mid v \text{ is continuous and } \gamma^v(1) = y\}$. Following the same idea as before, we get that optimal transport Problem $(\mathcal{K}_1)$ is equivalent to the following stochastic control problem with atypical boundary conditions:

$$\begin{cases} \inf_{v\in\mathcal{V}} \mathbb{E} \left\{ \int_0^1 \|v(t,\gamma^v(t))\|\, dt \right\}, \\ \dot\gamma^v(t) = v(t,\gamma^v(t)), \\ \gamma^v(0) \sim \mu^+, \quad \gamma^v(1) \sim \mu^-, \end{cases} \tag{3.13}$$

where $\mathcal{V}$ is simply the set of all continuous $v(\cdot,\cdot)$. Finally, let us suppose, as usual, that we have absolutely continuous measures with smooth densities $d\mu^+(x) = f^+(x)dx$, $d\mu^-(y) = f^-(y)dy$ and that there exists a density $\rho(t,x)$ such that $\gamma^v(t) \sim \rho(t,x)dx$. Moreover, suppose that such density satisfies (weakly) the continuity equation expressing the conservation of mass, i.e.

$$\frac{\partial\rho}{\partial t}(t,x) + \nabla\cdot(v(t,x)\rho(t,x)) = 0. \tag{3.14}$$

Using this new definition of the variable $\gamma^v(t)$ and using the rule in Proposition 1.3, we get that the probabilistic average becomes:

$$\mathbb{E}\left\{ \int_0^1 \|v(t,\gamma^v(t))\|\, dt \right\} \overbrace{=}^{\gamma^v(t)\sim\rho(t,x)dx} \int_{\mathbb{R}^n} \int_0^1 \|v(t,x)\|\, \rho(t,x) dt dx.$$

Furthermore, putting this last equality inside (3.13), we obtain the L^1 cost case of the "fluid dynamic" version of the optimal transport:

Problem $(\mathcal{DYN}_1)$.

Given two positive finite absolutely continuous measures $\mu^+, \mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ with densities f^+ and f^- satisfying (2.44)*, find an* optimal couple (ρ^*, v^*) *satisfying:*

$$\begin{cases} \inf\limits_{(\rho,v)} \displaystyle\int_{\mathbb{R}^n}\int_0^1 \|v(t,x)\|\,\rho(t,x)dtdx, \\ \dfrac{\partial \rho}{\partial t}(t,x) + \nabla\cdot(v(t,x)\rho(t,x)) = 0, \\ \rho(0,x) = f^+(x), \quad \rho(1,y) = f^-(y). \end{cases} \tag{3.15}$$

Note that the last line is obtained using the definition of $\gamma^v(t)$ and (3.13). Following, again, the idea of the work of Chen, Georgiou and Pavon [18], we want to do the variational analysis of dynamic Problem $(\mathcal{DYN}_1)$. It can be done in many different ways: let us write here an example of the way in which this can be carried out. First of all, let us start defining the family of the probability densities:

$$\mathbb{P}(f^+, f^-) := \left\{\rho : [0,1]\times\mathbb{R}^n \to \mathbb{R}^n \mid \rho(0,x) = f^+(x) \text{ and } \rho(1,y) = f^-(y)\right\}. \tag{3.16}$$

Consider the unconstrained minimization problem of the Lagrangian over the space $\mathbb{P}(f^+, f^-)\times\mathcal{V}$:

$$\mathcal{L}(\rho, v) := \int_{\mathbb{R}^n}\int_0^1 \left[\|v(t,x)\|\,\rho(t,x) + \lambda(t,x)\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(v\rho)\right)\right] dtdx, \tag{3.17}$$

where $\lambda : [0,1]\times\mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{C}^1$ Lagrange multiplier. Now, integrating by parts this Lagrangian and supposing that the density satisfies $\rho(t,x) \overset{\|x\|\to+\infty}{\longrightarrow} 0$, $\mathcal{L}(\rho,v)$ can be written as:

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_0^1 \left(\|v(t,x)\| - \frac{\partial\lambda}{\partial t}(t,x) - v(t,x)\cdot\nabla\lambda(t,x)\right)\rho(t,x)dtdx + \\ &+ \int_{\mathbb{R}^n}\left(\lambda(x,1)f^+(x) - \lambda(x,0)f^-(x)\right)dx. \end{aligned}$$

Thus, noting that the last part is simply a constant not depending on (ρ, v), Problem $(\mathcal{DYN}_1)$ is the same as solving:

$$\inf_{(\rho,v)} \int_{\mathbb{R}^n}\int_0^1 \left(\|v(t,x)\| - \frac{\partial\lambda}{\partial t}(t,x) - v(t,x)\cdot\nabla\lambda(t,x)\right)\rho(t,x)dtdx, \tag{3.18}$$

over $\mathbb{P}(f^+, f^-)\times\mathcal{V}$. To solve it, we follow a two-step optimization procedure. Thus, we first fix $\rho \in \mathbb{P}(f^+, f^-)$ and minimize (3.18) along all $v \in \mathcal{V}$. The integrand in (3.18), for a fixed ρ, is convex and differentiable for $v \neq 0$. Thus we obtain the *optimality condition*:

$$\frac{v^*_\rho(t,x)}{\|v^*_\rho(t,x)\|} = \nabla\lambda(t,x), \qquad \text{whenever } \{\rho(t,x) > 0\}. \tag{3.19}$$

Note that this optimality condition implies that:

- $\|\nabla\lambda\| = 1$ on the support of $\rho(t,x)$;
- v^*_ρ is the gradient of a potential.

Plugging this optimal v^*_ρ in (3.18) yields the second minimization problem:

$$\inf_{\rho\in\mathbb{P}(f^+,f^-)} H(\rho) := \int_{\mathbb{R}^n}\int_0^1 \left(-\frac{\partial\lambda}{\partial t}(t,x) + \|v(t,x)\|\left(1 - \|\nabla\lambda(t,x)\|^2\right)\right)\rho(t,x)dtdx.$$

Note that if $\lambda(t,x)$ does not depend on time and if v^*_ρ is of the form (3.19), we get that the functional H just defined, is automatically zero over $\mathbb{P}(f^+, f^-)$. Hence, every $\rho \in \mathbb{P}(f^+, f^-)$ is a minimizer of the Lagrangian together with the optimal v^*_ρ defined on (3.19). All this considerations are summarized into the following:

PROPOSITION 3.1. *Consider a function $\lambda(t,x) = \lambda(x)$ not depending on time and a couple $(\rho^*, v^*) \in \mathbb{P}(f^+, f^-) \times \mathcal{V}$ satisfying:*

$$\begin{cases} \dfrac{\partial \rho^*}{\partial t}(t,x) + \nabla \cdot (\|v^*(t,x)\| \, \nabla\lambda(x)\rho^*(t,x)) = 0, & \\ \|\nabla\lambda\| = 1 & \text{on } \{\rho(t,x) > 0\}, \\ \frac{v^*}{\|v^*\|} = \nabla\lambda & \text{on } \{\rho(t,x) > 0\}. \end{cases} \tag{3.20}$$

Then (ρ^*, v^*) is a solution of $(\mathcal{DYN}_1)$, provided that $\rho(t,x) \overset{\|x\|\to+\infty}{\longrightarrow} 0$.

If we compare this result with the variational analysis for the L^2 dynamic Problem $(\mathcal{DYN}_2)$, which can be found always in [18], or also in [5], it easy to see that there are a lot of differences. In the case of L^2 cost, they obtained a result which completely characterized all the optimal couples solving the dynamical problem. Indeed, doing the same analysis we did above, in that case they obtained that, if the Lagrange multiplier λ solves the Hamilton-Jacobi equation:

$$\frac{\partial \lambda}{\partial t} + \frac{1}{2}\|\nabla\lambda\|^2 = 0,$$

with some boundary condition, then the couple (ρ^*, v^*), with $v^* = \nabla\lambda$ and ρ^* solving the continuity equation with this vector field, will be a solution for the dynamic problem with the L^2 cost function. Here, instead, we obtain a result which does not give us an explicit solution without knowing the quantity $\|v^*\|$. This problem reminded us, in some way, of problem we addressed in the previous chapter, of finding T^* given by (2.39), not having information about the distance d^*. Indeed, we will see that, also in this case, the quantity a^* and u^* will be involved to solve this problem. However, the variational analysis, in this case, is useless and this problem in the case of L^1 cost function must be solved in a different way. Our strategy will be to use the other formulations of the Monge-Kantorovich problem in order to find a relation similar to the one given by Theorem 3.1 and extract useful information from this correspondence, in such a way that we will be able to write an explicit solution, as we did for the L^2 cost case (eq. (3.4)).

2. Beckmann's problem

In this section, we want to discuss a model for the optimal transport problem proposed by Beckmann in [3] and discussed more recently in [41]. Here we present just a particular case of a wider class of convex optimization problems, which are of the form $\min\{\int H(w)\, dx \ : \ \nabla \cdot w = f^+ - f^-\}$ and we will treat just the case of the non-strictly convex function $H(z) = \|z\|$, being coherent with the rest of the work. Beckmann called this problem *continuous transportation model.* However, later in this work, we will get back to the general case of the convex minimization problems written above and we will see the similarities and the differences with this L^1 case. Note that, even if Beckmann worked in the same years of Kantorovich, he was not aware of Kantorovich's papers on related matters. Just to introduce some features of this model, instead of looking at dynamic model, as in the previous section, here we are interested in a "static" model. Since it may be difficult to understand what "static" means when we want to describe the movement, let us look at one example first. We want to consider the traffic intensity in each point of the space, but, instead of considering the traffic intensity hour by hour, here we want to consider the average traffic intensity at such a point during the whole day. In this case, we call w the mass flow rate and, consequently, $\nabla \cdot w$ stands for the excess of mass which is injected into the motion at every point. Being more precise, if particles are injected into the motion according to the distribution μ^+ and exit with the distribution μ^-, the vector field w must satisfy:

$$\nabla \cdot w = \mu^+ - \mu^-. \tag{3.21}$$

Formally, we will consider the following:

Problem ($\mathcal{B}$).

Consider an open convex subset $\Omega \subset \mathbb{R}^n$ which contains $X = supp(f^+)$ and $Y = supp(f^-)$, where f^+ and f^- are the densities of μ^+ and μ^-, respectively, satisfying (2.44). *Then, Beckmann minimization problem consists in finding an* optimal vector field $w \in L^1(\Omega)$ *satisfying:*

$$\inf_w \left\{ \int_\Omega \|w(x)\| \, dx \; : \; w : \Omega \to \mathbb{R}^n, \; \nabla \cdot w = f^+ - f^-, \; w \cdot \nu_\Omega = 0 \textit{ on } \partial\Omega \right\}. \tag{3.22}$$

Here the divergence condition is to be interpreted in a weak sense and, with the Neumann boundary condition, i.e.:

$$-\int_\Omega \nabla\phi(x) w(x) dx = \int_\Omega \phi(x)(f^+(x) - f^-(x)) dx, \quad \forall \phi \in \mathcal{C}^1_c(\Omega). \tag{3.23}$$

The first observation on this problem is that it is not a priori well-posed, in the sense that there may not exist an L^1 vector field minimizing Problem ($\mathcal{B}$). This can be understood using a standard method of the Calculus of Variations to prove the existence. Following the idea stated in Proposition 1.1, we consider a minimizing sequence and we would like to extract a subsequence converging to the minimum, in order to find a minimizer using some lower semi-continuity property. Thus, starting with $w_n \to w$, it is easy to prove that such w satisfies the divergence constraint $\nabla \cdot w = f^+ - f^-$, using the fact that:

$$-\int_\Omega \nabla\phi \; w_n dx = \int_\Omega \phi(f^+ - f^-) dx$$

and letting $n \to \infty$. The fact is that, even if we have that the sequence is bounded in L^1, i.e. $\int_\Omega \|w_n\| \, dx \leq C$, it is not enough to extract a converging subsequence, even weakly. Indeed, the space L^1 is *not reflexive*, as we showed in the Section 2 of the Chapter 1. Thus bounded sequences are not guaranteed to have weakly converging subsequences, because we cannot use Theorem 1.5. In order to automatically deduce the weakly sequential compactness of a subspace in L^1, we need also the *equintegrability* condition, a property described inside Dunford-Pettis Theorem 1.6. If we are able to show this property, we can use that theorem and conclude. To avoid this difficulty of well-posedness, we will choose a different setting for ($\mathcal{B}$): the framework of *vector measures*. We already described vector measures (see Section 2 of Chapter 1), looking at the definition and at some properties of these objects. In particular, we saw that if $\mu \in \mathcal{M}^n(\Omega)$, we can define the total variation measure $\|\mu\|(\Omega)$ (Definition 1.11). Hence, we can reformulate Problem ($\mathcal{B}$), in this context, by:

Problem ($\mathcal{B}'$).

Consider an open convex subset $\Omega \subset \mathbb{R}^n$ which contains $X = supp(f^+)$ and $Y = supp(f^-)$, where f^+ and f^- are the densities of μ^+ and μ^-, respectively, satisfying (2.44). *Then, Beckmann minimization problem in a vector measures form consists in finding an* optimal vector measure $w^* \in \mathcal{M}^n_{div,0}(\Omega)$ *satisfying:*

$$\inf_w \left\{ \|w\|(\Omega) \; : \; w \in \mathcal{M}^n_{div,0}(\Omega), \; \nabla \cdot w = f^+ - f^- \right\}. \tag{3.24}$$

Here the two constraints have a meaning similar to (3.23), but in the context of the space of measures $\mathcal{M}^n_{\mathrm{div},0}(\Omega)$ (see notation), they become, also recalling Riesz representation Theorem 1.7:

$$\int_\Omega \nabla\phi \cdot dw = \langle \nabla\phi, w \rangle_{\mathcal{C}_0(\Omega;\mathbb{R}^n)} = \int_\Omega \phi \left(f^- - f^+\right) dx, \qquad \forall \phi \in \mathcal{C}^1_c(\Omega). \tag{3.25}$$

3. Relationship between different formulations

With all these formulations of the same problem coming from different fields, it becomes important to find a correspondence between the problems, as stated in the introduction of this chapter. Moreover, if possible, it is also important to find a correlation between the minimizers/maximizers of different formulations. If a solution of a problem can be built from the minimizer/maximizer of another problem we say that the two problems are *equivalent*. The situation is complex and a bit confused, thus let us proceed step by step. Let us start recalling the fact that Problem $(\mathcal{K}_1)$ is the relaxation of Problem $(\mathcal{M}_1)$, as shown in Theorem 2.1 of Chapter 2 and that Problems $(\mathcal{K}_1)$ and $(\mathcal{D}_1)$ have the same optimal value thanks to Duality Theorem 2.4. The first step of this section is the relation between Problem (3.24) just introduced, and the dual Problem $(\mathcal{D}_1')$. To do that, we want to use a duality result, recalled in Convex Analysis Section 5 of Chapter 1. The idea of using Fenchel-Rockafellar Theorem 1.17 in this particular case, is taken from Barrett and Prigozhin [2]. To be precise and to give the formal background, in the whole section we will consider a compact convex subset $\Omega \subset \mathbb{R}^n$ and, as always, two absolutely continuous measures with densities f^+ and f^-, respectively. Moreover, suppose that the supports of these densities f^+ and f^- are contained in Ω. We are ready, then, for the following:

Theorem 3.2. *The minimal value of Beckmann Problem $(\mathcal{B}')$ is the same as the maximal value of the dual Problem $(\mathcal{D}_1')$, i.e.*

$$\min(\mathcal{B}') = \max(\mathcal{D}_1'). \tag{3.26}$$

Proof (Barrett-Prigozhin [2]). First of all, note that if $\mu = \mu^+ - \mu^-$, then $\mu \in \mathcal{M}(\Omega)$. Moreover, if $f = f^+ - f^-$, we write the duality pairing (1.5) of Riesz representation Theorem 1.7 using, with some abuse of notation, the absolutely continuous measure $d\mu = f dx$:

$$\langle f, u\rangle_{\mathcal{C}(\Omega)} = \int_\Omega u \cdot f dx \quad \forall u \in C(\Omega).$$

Thus, Problem $(\mathcal{D}_1')$ can be written in the form:

$$\max\left\{\langle f, u\rangle_{\mathcal{C}(\Omega)} \ : \ u \in \mathrm{Lip}_1(\Omega)\right\}.$$

With Fenchel-Rockafellar Theorem 1.17 in mind, we want to find proper functions F and G and proper spaces V and W, in order to write $(\mathcal{D}_1')$, written in the form just stated, as the principal problem (P_1). Then, consider spaces $V := \mathcal{C}^1(\Omega)$, $W := \mathcal{C}(\Omega;\mathbb{R}^n)$ and linear operator $\Lambda : V \to W$, defined as

$$\Lambda u := \nabla u.$$

Moreover, consider the functions $F : V \to \overline{\mathbb{R}}$ and $G : W \to \overline{\mathbb{R}}$, defined as:

$$F(u) := -\langle f, u\rangle_{\mathcal{C}(\Omega)} \quad \text{and} \quad G(v) := \begin{cases} 0 & \text{if } \|v(x)\| \le 1 \ \forall x \in \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

Then it is easy to see that the principal problem:

$$\inf_{u\in V}[F(u) + G(\Lambda u)], \tag{P$_3$}$$

with the functions and the spaces defined above, is the same as Problem $(\mathcal{D}_1')$. Note that F and G are convex functions and if we take $u_0 = 0$, we obtain $F(u_0) = 0$. Moreover, the function $G(v)$ is continuous at $\Lambda u_0 = 0$. Thus, the hypothesis of Fencel-Rockafellar Theorem 1.17 are verified. Before using that theorem, we have to write down explicitly the dual formulation in this particular case. First of all, recall that the dual space of W is, using Riesz representation Theorem 1.7, $W^* = \mathcal{M}^n(\Omega)$ and that $\Lambda^* : W^* \to V^*$ is the adjoint of Λ, as defined in (1.23). Using the definition of Legendre-Fenchel conjugate function, let us try to understand how are defined functions F^* and G^*. Considering $u^* \in V^*$, we have that:

$$F^*(u^*) = \sup_{u\in V}\left[\langle u^*, u\rangle_* + \langle f, u\rangle_{\mathcal{C}(\Omega)}\right] = \begin{cases} 0 & \text{if } u^* + f = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore, looking at the dual formulation (P_1^*), it follows that:

$$-F^*(\Lambda^* p^*) > -\infty \iff \Lambda^* p^* + f = 0.$$

Thus, the supremum in (P_1^*) should be considered only on the following convex set:

$$\begin{aligned} Z &:= \left\{p^* \in W^* \ : \ \langle \Lambda^* p^*, u\rangle_* + \langle f, u\rangle_{\mathcal{C}(\Omega)} = 0 \quad \forall u \in \mathcal{C}^1(\Omega)\right\} = \text{ by def. of } \Lambda^* \\ &= \left\{p^* \in W^* \ : \ \langle p^*, \nabla u\rangle_{\mathcal{C}(\Omega)} + \langle f, u\rangle_{\mathcal{C}(\Omega)} = 0 \quad \forall u \in \mathcal{C}^1(\Omega)\right\} = \text{ by int. by parts} \\ &= \left\{w \in \mathcal{M}^n_{\mathrm{div},0}(\Omega) \ : \ \langle \nabla w - f, \eta\rangle_{\mathcal{C}(\Omega)} = 0 \quad \forall \eta \in \mathcal{C}(\Omega)\right\}. \end{aligned}$$

Note that this set is exactly the set described by the two constraints in the minimization Problem ($\mathcal{B}'$). Moreover, looking at G^*, we have that if $w \in W^*$, then:

$$G^*(w) = \sup\left\{\langle w, v\rangle_{\mathcal{C}_0(\Omega;\mathbb{R}^n)} \ : \ v \in W \text{ and } \|v(x)\| \le 1 \quad \forall x \in \Omega\right\} = \|w\|(\Omega).$$

The last equality is due to the definition of the scalar product in Riesz representation Theorem 1.7 plus Proposition 1.2. Hence, the dual problem can be written as:

$$\sup\{-\|w\|(\Omega) \ : \ w \in Z\}. \tag{P_3^*}$$

Therefore, using now the result of Fenchel-Rockafellar Theorem 1.17, we obtain:

$$\max(\mathcal{D}_1') = \max\{\langle f, u\rangle_{\mathcal{C}(\Omega)} \ : \ u \in \mathrm{Lip}_1(\Omega)\} = \inf\{\|w\|(\Omega) \ : \ w \in Z\} = \min(\mathcal{B}'). \tag{3.27}$$

□

The second new relation is between the relaxed Kantorovich Problem ($\mathcal{K}_1$) and, again, the Beckmann problem ($\mathcal{B}'$). The interesting fact is that we can even recover equivalence of the two problems, in the sense stated at the beginning of this section, as we can see in the following theorem, taken from Section 4.2.2 of Santambrogio [41].

Theorem 3.3. *The minimal value of Problem ($\mathcal{B}'$) equals the minimal value of Problem ($\mathcal{K}_1$), i.e.:*

$$\min(\mathcal{B}') = \min(\mathcal{K}_1). \tag{3.28}$$

Moreover, a solution of Beckmann Problem ($\mathcal{B}'$) can be built from a solution of Kantorovich Problem ($\mathcal{K}_1$).

Proof (Santambrogio [41]). ($\ge$) We start taking an arbitrary function $\phi \in \mathcal{C}^1(\Omega)$ with $\phi \in \mathrm{Lip}_1(\Omega)$, so that $\|\nabla\phi\| \le 1$. Consider any $w \in \mathcal{M}^n_{\mathrm{div},0}(\Omega)$ with $\nabla \cdot w = f^+ - f^-$. We get, then:

$$\|w\|(\Omega) = \int_\Omega 1 d\|w\| \ge \int_\Omega (-\nabla\phi)\cdot dw \overbrace{=}^{(3.25)} \int_\Omega \phi(f^+ - f^-)dx.$$

Now, if we consider a sequence in $\mathrm{Lip}_1(\Omega) \cap \mathcal{C}^1(\Omega)$ uniformly converging to the Kantorovich potential u^* (for instance the convolution $\phi_\varepsilon = \chi_\varepsilon * u^*$, with χ_ε the standard mollifier, as described, for example, in Brezis [15]), we obtain:

$$\int_\Omega u^*(f^+ - f^-)dx = \max(\mathcal{D}_1') = \min(\mathcal{K}_1),$$

where we used the Strong Duality Theorem 1.15 for the last equality. Moreover, we have that:

$$\|w\|(\Omega) \ge \min(\mathcal{K}_1),$$

for any w with the property as above, i.e. for any vector measure satisfying the constraints of the minimization Problem ($\mathcal{B}'$). This means:

$$\min(\mathcal{B}') \ge \min(\mathcal{K}_1). \tag{3.29}$$

($\le$) For the converse, we shall show that it is possible to construct an optimal w starting from an optimal μ for ($\mathcal{K}_1$). This idea is taken from Bouchitté and Buttazzo [8]. We start by defining, given an arbitrary transport plan $\mu \in \Pi(\mu^+, \mu^-)$, a *scalar measure* $\sigma_\mu \in \mathcal{M}(\Omega)$ as:

$$\langle \sigma_\mu, \phi\rangle_{\mathcal{C}_0(\Omega)} := \int_{\Omega\times\Omega}\int_0^1 \left\|\gamma'_{x,y}(t)\right\| \phi(\gamma_{x,y}(t))dt \, d\mu(x,y), \qquad \forall \phi \in \mathcal{C}_0(\Omega). \tag{3.30}$$

Here $\gamma_{x,y} \in \mathcal{X}_{xy}$ (space defined in the first section of this chapter) is a parametrization of the segment $[x, y]$, taken for simplicity with constant speed, e.g. $\gamma_{x,y}(t) = (1 - t)x + ty$ (again already defined in the first section of the this chapter). Note that, using the duality isometry given by Riesz representation Theorem 1.7 that correlates $(\mathcal{C}_0(\Omega))^*$ and $\mathcal{M}(\Omega)$, this σ_μ is effectively a scalar measure. Define, also, a *vector measure* $w_\mu \in \mathcal{M}^n(\Omega)$, depending on μ, as:

$$\langle w_\mu, \xi\rangle_{\mathcal{C}_0(\Omega;\mathbb{R}^n)} := \int_{\Omega\times\Omega}\int_0^1 \gamma'_{x,y}(t)\cdot\xi(\gamma_{x,y}(t))dt\; d\mu(x, y), \qquad \forall\xi \in \mathcal{C}_0(\Omega; \mathbb{R}^n). \tag{3.31}$$

Also in this case we use Riesz representation Theorem 1.7 to confirm that w_μ is actually a vector measure. Note that if $\xi = \nabla\phi \in \mathcal{C}_0(\Omega; \mathbb{R}^n)$, then:

$$\int_0^1 \gamma_{x,y}(t)\cdot\nabla\phi(\gamma_{x,y}(t))dt = \int_0^1 \frac{d}{dt}\left(\phi(\gamma_{x,y}(t))\right)dt = \phi(y) - \phi(x).$$

Hence,

$$\langle w_\mu, \nabla\phi\rangle_{\mathcal{C}_0(\Omega;\mathbb{R}^n)} = \int_{\Omega\times\Omega}\phi(y)d\mu(x, y) - \int_{\Omega\times\Omega}\phi(x)d\mu(x, y) \overbrace{=}^{\mu\in\Pi(\mu^+,\mu^-)} \int_\Omega \phi(x)\left(f^-(x) - f^+(x)\right)dx.$$

This equality, using the definition of the scalar product $\langle\cdot,\cdot\rangle_{\mathcal{C}_0(\Omega;\mathbb{R}^n)}$ in (1.5), tells us that w_μ satisfies the divergence constraint in the weak sense and with the Neumann boundary condition, i.e.

$$-\int_\Omega \nabla\phi\cdot dw_\mu = \int_\Omega \phi\left(f^+ - f^-\right)dx, \qquad \forall\phi \in \mathcal{C}^1_c(\Omega). \tag{3.32}$$

Recalling that this constraint is the same as $w_\mu \in \mathcal{M}^n_{\mathrm{div},0}(\Omega)$ and $\nabla\cdot w_\mu = f^+ - f^-$ in a weak sense, we can easy say that w_μ is in the search set for Problem $(\mathcal{B}')$. Simply using Cauchy-Schwarz inequality, it is easy to see that the mass of w_μ can be estimated by:

$$\|w_\mu\| \le \sigma_\mu \quad \forall\mu \in \Pi(\mu^+, \mu^-). \tag{3.33}$$

Considering, now, an optimal μ which solves Problem $(\mathcal{K}_1)$ and looking at the mass of σ_μ, we obtain:

$$\int_\Omega d\sigma_\mu = \int_{\Omega\times\Omega}\int_0^1 \left\|\gamma'_{x,y}(t)\right\| dt\; d\mu(x, y) = \int_{\Omega\times\Omega}\|x - y\|\, d\mu(x, y) = \min(\mathcal{K}_1) \overbrace{\le}^{(3.29)} \min(\mathcal{B}').$$

Finally, this inequality plus inequality (3.33), tells us that w_μ is a solution of Beckmann's Problem $(\mathcal{B}')$. □

Remark 3.1. Actually, we can even say more about the relationship between the vector measure w_μ and the scalar measure σ_μ. Using the Kantorovich potential u^*, we have:

$$\gamma'_{x,y}(t) = -\|x - y\|\frac{x - y}{\|x - y\|} = -\|x - y\|\nabla u^*(\gamma_{x,y}(t)), \quad \forall t \in (0, 1), \quad \forall x, y \in \mathrm{supp}(\mu),$$

where we used the fact that $\gamma_{x,y}(t)$ is in the interior of a transport ray $]x, y[$ and we used Lemma 2.4 in Chapter 2. This allows us to write, for every $\xi \in \mathcal{C}_0(\Omega; \mathbb{R}^n)$:

$$\begin{aligned}\langle w_\mu, \xi\rangle_{\mathcal{C}_0(\Omega;\mathbb{R}^n)} &= \int_{\Omega\times\Omega}\int_0^1 -\|x - y\|\nabla u^*(\gamma_{x,y}(t))\cdot\xi(\gamma_{x,y}(t))dt\; d\mu(x, y) = \\ &= -\int_0^1 dt\int_{\Omega\times\Omega}\nabla u^*(\gamma_{x,y}(t))\cdot\xi(\gamma_{x,y}(t))\,\|x - y\|\, d\mu(x, y).\end{aligned}$$

Introducing the function $\pi_t : \Omega\times\Omega \to \Omega$ given by $\pi_t(x, y) := \gamma_{x,y}(t) = (1 - t)x + ty$ and defining the measure $c\cdot\mu \in \mathcal{M}_+(\Omega\times\Omega)$ as the absolutely continuous measure with density $c(x, y) = \|x - y\|$, we can write:

$$\langle w_\mu, \xi\rangle_{\mathcal{C}_0(\Omega;\mathbb{R}^n)} = -\int_0^1 dt\int_\Omega \nabla u^*(z)\cdot\xi(z)d((\pi_t)_\#(c\cdot\mu)). \tag{3.34}$$

The same kind of computations gives:

$$\langle \sigma_\mu, \phi \rangle_{\mathcal{C}_0(\Omega)} = \int_0^1 dt \int_\Omega \phi(z) d((\pi_t)_\#(c \cdot \mu)). \tag{3.35}$$

Hence, comparing these last two equalities, we get:

$$\langle w_\mu, \xi \rangle_{\mathcal{C}_0(\Omega;\mathbb{R}^n)} = \langle \sigma_\mu, -\xi \cdot \nabla u^* \rangle_{\mathcal{C}_0(\Omega)}. \tag{3.36}$$

By the arbitrariness of ξ and using the dual pairing in (1.5), it follows that:

$$w_\mu = -\nabla u^* \cdot \sigma_\mu. \tag{3.37}$$

This confirms (3.33) and tells us that $-\nabla u^*$ is the density of w_μ w.r.t σ_μ, in the sense of Proposition 1.2, i.e.

$$\int_\Omega \xi \cdot dw_\mu = \langle w_\mu, \xi \rangle_{\mathcal{C}_0(\Omega;\mathbb{R}^n)} = \langle \sigma_\mu, -\xi \cdot \nabla u^* \rangle_{\mathcal{C}_0(\Omega)} = -\int_\Omega (\xi \cdot \nabla u^*) d\sigma_\mu.$$

Note also that Proposition 1.2 tells us that:

$$\|\nabla u^*\| = 1 \quad \sigma_\mu\text{-a.e.}. \tag{3.38}$$

Moreover, using the equality in (3.35) and using (1.5), we obtain an explicit representation of σ_μ given by:

$$\sigma_\mu = \int_0^1 ((\pi_t)_\#(c \cdot \mu)) dt. \tag{3.39}$$

□

Scalar measure σ_μ described in the proof above and in the previous remark is strictly related to a^* of Lemma 2.5 introduced in [27]. In this case, this object is connected to some shape-optimization problems treated in Bouchitté and Buttazzo [8] and [9]. This relation between σ_μ and a^*, using some considerations which can be found in both Ambrosio [1] and Santambrogio [41], is described in the following:

Proposition 3.2. *Starting from an optimal $\mu \in \Pi(\mu^+, \mu^-)$ solving Problem $(\mathcal{K}_1)$, the scalar measure σ_μ defined in* (3.30) *solves, together with the Kantorovich potential u^*, Problem $(\mathcal{PDE})$.*

Proof. For all $\phi \in \mathcal{C}_c^1(\Omega)$, we get:

$$-\int_\Omega \nabla\phi \cdot \nabla u^* d\sigma_\mu = -\langle \sigma_\mu, \nabla\phi \cdot \nabla u^* \rangle_{\mathcal{C}_0(\Omega)} \overbrace{=}^{(3.36)} \langle \nabla\phi, w_\mu \rangle_{\mathcal{C}(\Omega,\mathbb{R}^n)} \overbrace{=}^{(3.32)} \int_\Omega \phi(f^- - f^+) dx.$$

This means that, looking at the weak formulation of Equation (2.59) of $(\mathcal{PDE})$ as written in (2.61), the scalar measure σ_μ is absolutely continuous with density exactly equal to a^*, i.e.

$$d\sigma_\mu(x) = a^*(x) dx. \tag{3.40}$$

Thus, in this sense, the couple (σ_μ, u^*) solves equation (2.59), where with σ_μ in this case we mean its density. Note, also, that we have already shown the other condition (2.60) of Problem $(\mathcal{PDE})$ in the previous remark, precisely in (3.38). □

With an abuse of language, both scalar measure σ_μ and density a^* are usually called with the same name *scalar density*. This is due to the fact that the density of an absolutely continuous measure and the measure itself are, often, improperly identified.

Thus, we have seen the correspondences between Beckmann Problem $(\mathcal{B}')$ and the "classic versions" of the Monge-Kantorovich Problems $(\mathcal{K}_1)$ and $(\mathcal{D}_1')$, that also provides a way to obtain a solution of $(\mathcal{PDE})$.

We want to focus, now, on the relation between Problem $(\mathcal{B}')$ and the dynamic Problem $(\mathcal{DYN}_1)$. Indeed, in order to fill the gap left at the end of Section 1 of this chapter, by the failure of the variational analysis, we want to see $(\mathcal{DYN}_1)$ from a different perspective. The idea is to take information from the Beckamann Problem $(\mathcal{B}')$ in order to obtain an optimal couple that minimizes the dynamical problem. Before doing that, note that if we consider the case of a proper, compact and convex subset $\Omega \subset \mathbb{R}^n$, such that the supports of f^+ and f^- are contained inside this set, we can substitute $\mathbb{R}^n$ with Ω inside the integral of $(\mathcal{DYN}_1)$, provided that the admissible velocity vector fields v satisfy Neumann boundary condition $v \cdot \nu_\Omega = 0$ on the boundary $\partial\Omega$. In this case, the continuity equation has to be intended in a weak sense, i.e. for all $\phi \in \mathcal{C}^1([0,1] \times \Omega)$:

$$\begin{aligned}\int_\Omega \phi(1,x)f^-(x)dx - \int_\Omega \phi(0,x)f^+(x)dx = &\int_0^1 \int_\Omega \frac{\partial \phi}{\partial t}(t,x)\rho(t,x)dxdt \\ &+ \int_0^1 \int_\Omega \nabla\phi(t,x) \cdot (v(t,x)\rho(t,x))dxdt.\end{aligned} \tag{3.41}$$

Note that we obtained the equation above just integrating the continuity equation (3.14) multiplied by a test function $\phi \in \mathcal{C}^1([0,1]\times\Omega)$ and using zero Neumann boundary conditions.

After this clarification, we are ready to prove the following relationship, which takes inspiration from an idea used in Brasco [12]:

THEOREM 3.4. *The minimal value of Problem $(\mathcal{B}')$ equals the minimal value of Problem $(\mathcal{DYN}_1)$, i.e.:*

$$\min(\mathcal{B}') = \min(\mathcal{DYN}_1). \tag{3.42}$$

Moreover, starting from $w \in \mathcal{M}^n_{div,0}(\Omega)$ solution of $(\mathcal{B}')$, assumed absolutely continuous, it is possible to construct an optimal pair (ρ_w, v_w) solving $(\mathcal{DYN}_1)$.

PROOF. $(\leq)$ Given a feasible couple (ρ, v) for Problem $(\mathcal{DYN}_1)$, i.e. a couple satisfying:

$$\begin{cases} \dfrac{\partial\rho}{\partial t}(t,x) + \nabla \cdot (v(t,x)\rho(t,x)) = 0 \\ \rho(x,0) = f^+(x), \quad \rho(y,1) = f^-(y), \end{cases}$$

we consider the function:

$$f_{\tilde{w}}(x) := \int_0^1 v(t,x)\rho(t,x)dt. \tag{3.43}$$

Assuming enough regularity for the couple (ρ, v) so that we can interchange ∇ with the integral sign, we see that $f_{\tilde{w}}$ solves, for all $\phi \in \mathcal{C}^1_c(\Omega)$:

$$\begin{aligned}\int_\Omega \nabla\phi(x)\ f_{\tilde{w}}(x)dx &= -\int_\Omega \phi(x) \int_0^1 \nabla \cdot (v(t,x)\rho(t,x))\,dtdx \\ &= \int_\Omega \phi(x) \int_0^1 \frac{\partial\rho}{\partial t}(t,x)dtdx = \int_\Omega \phi(x)\left(f^-(x) - f^+(x)\right)dx.\end{aligned}$$

This means, comparing this equality with the one written in (3.25), that the absolutely continuous measure with density $f_{\tilde{w}}$, i.e. $d\tilde{w} = f_{\tilde{w}}dx$, satisfies Beckmann's Problem $(\mathcal{B}')$ weak constraint. Now, using Minkowski's integral inequality with $p = 1$, we get:

$$\int_\Omega \int_0^1 \|v(t,x)\|\,\rho(t,x)dt\ dx \geq \int_\Omega \left\|\int_0^1 v(t,x)\rho(t,x)dt\right\| dx = \int_\Omega \|f_{\tilde{w}}(x)\|\,dx = \int_\Omega d\,\|\tilde{w}\|$$

By the arbitrariness of the couple (ρ, v), the inequality above implies:

$$\|\tilde{w}\|(\Omega) \leq \inf(\mathcal{DYN}_1).$$

Thus, using also the arbitrariness of $\tilde{w}$, we obtain:

$$\min(\mathcal{B}') \leq \inf(\mathcal{DYN}_1). \tag{3.44}$$

$(\geq)$ For the converse, we consider an absolutely continuous measure w, with density f_w, that solves of $(\mathcal{B}')$, satisfying the divergence constraint (3.25) in a weak sense. Consider the couple, depending on w, given by:

$$\rho_w(t,x) := (1-t)f^+(x) + tf^-(x); \qquad v_w(t,x) := \frac{f_w(x)}{\rho_w(t,x)}. \tag{3.45}$$

For all $\phi \in \mathcal{C}_c^1([0,1]\times\Omega)$ we can write:

$$\int_0^1 \int_\Omega \frac{\partial \phi}{\partial t}(t,x)\rho_w(t,x)dxdt + \int_0^1 \int_\Omega \nabla\phi(t,x)\cdot(v_w(t,x)\rho_w(t,x))dx\ dt =$$
$$= \int_\Omega (\phi(1,x)-\phi(0,x))f^+(x)dx + \int_\Omega (f^-(x)-f^+(x))\left(\int_0^1 t\cdot\frac{\partial\phi}{\partial t}(t,x)\ dt\right)dx$$
$$+ \int_0^1\int_\Omega \nabla\phi(t,x)\cdot f_w(x)dx\ dt = \quad \text{(integrating by parts the second term)}$$
$$= \int_\Omega \phi(1,x)f^-(x)dx - \int_\Omega \phi(0,x)f^+(x)dx - \int_\Omega (f^-(x)-f^+(x))\left(\int_0^1 \phi(t,x)dt\right)dx$$
$$+ \int_\Omega \nabla\left(\int_0^1 \phi(t,x)dt\right)\cdot f_w(x)dxdt = \int_\Omega \phi(1,x)f^-(x)dx - \int_\Omega \phi(0,x)f^+(x)dx.$$

Note that we obtain the last equality using the fact that f_w satisfies the divergence constraint in a weak sense written in (3.25), with the test function $\Phi(x) = \int_0^1 \phi(t,x)dt \in \mathcal{C}_c^1(\Omega)$. Clearly, we assumed enough regularity in order to take the gradient out of the integral sign and in order to use Fubini-Tonelli Theorem 1.2. The last calculation tells us that the couple (ρ_w, v_w) satisfies the continuity equation in a weak sense, i.e.

$$\begin{cases} \dfrac{\partial\rho_w}{\partial t}(t,x) + \nabla\cdot(v_w(t,x)\rho_w(t,x)) = 0, & \text{in a weak sense;} \\ \rho_w(x,0) = f^+(x), \quad \rho_w(y,1) = f^-(y), & \text{by the definition of } \rho_w \text{ in (3.45).} \end{cases}$$

This means that the couple (ρ_w, v_w) belongs to the feasible set of Problem $(\mathcal{DYN}_1)$. Finally, note that:

$$\int_\Omega\int_0^1 \|v_w(t,x)\|\,\rho_w(t,x)dt\ dx = \int_\Omega \|f_w(x)\|\,dx = \|w\|(\Omega) = \min(\mathcal{B}') \overbrace{\le}^{(3.44)} \min(\mathcal{DYN}_1).$$

This inequality tells us that this pair (ρ_w, v_w) is actually the minimizer we were looking for. □

Note that density $\rho(t,x)$ solution of Problem $(\mathcal{DYN}_1)$ is simply represented by the affine interpolation between f^+ and f^-. As we already said, in general, the affine interpolation differs greatly from the optimal density of Problem $(\mathcal{DYN}_2)$, represented in (3.4). We will come back to this relation between L^1 and L^2 optimal density and we will discuss it in detail, from both an analytical and a geometrical point of view. Instead, now, we want to highlight another parallelism between the dynamic formulation $(\mathcal{DYN}_1)$ and the Kantorovich formulation $(\mathcal{K}_1)$, just putting together Theorem 3.3 and Theorem 3.4. Moreover, using also the deformation argument explained in Theorem 2.6, we will connect almost all the ideas used in the theorems above, in order to find a correlation with the optimal transport map of Evans defined in Theorem 2.7.

Theorem 3.5. *The minimal value of Problem $(\mathcal{K}_1)$ is equal to the minimal value of Problem $(\mathcal{DYN}_1)$, i.e.:*

$$\min(\mathcal{K}_1) = \min(\mathcal{DYN}_1). \tag{3.46}$$

Moreover, starting from an optimal transport plan $\mu \in \Pi(\mu^+,\mu^-)$ solution of $(\mathcal{K}_1)$, it is possible to construct an optimal couple (ρ_μ, v_μ) solution of $(\mathcal{DYN}_1)$.

Proof. The idea of the proof is, as already stated, simply to connect Theorem 3.3 and Theorem 3.4. Indeed, for the proof of the first part, we just need to combine the equalities (3.28) and (3.42) to get:

$$\min(\mathcal{K}_1) = \min(\mathcal{B}') = \min(\mathcal{DYN}_1).$$

For the second part, we start from an optimal $\mu \in \Pi(\mu^+,\mu^-)$ and we want to use both proofs of Theorems (3.5) and (3.6), combined together as before. Thus, we start defining the scalar measure $\sigma_\mu \in \mathcal{M}(\Omega)$ and the vector measure $w_\mu \in \mathcal{M}^n(\Omega)$, respectively as in (3.30) and (3.31). Recall that w_μ and σ_μ are related by (3.37), i.e.

$$w_\mu = -\nabla u^* \cdot \sigma_\mu,$$

where u^* is the Kantorovich potential. Moreover, w_μ satisfies the divergence constraint in a weak sense, as in (3.32), and is a solution of the Beckmann Problem $(\mathcal{B}')$. In order to use Theorem 3.4, starting from this optimal w_μ, we want to find the density of this measure. To do this, we combine (3.37) and (3.40) to yield:

$$dw_\mu = f_{w_\mu} dx \overbrace{=}^{(3.37)} = -\nabla u^* \cdot d\sigma_\mu \overbrace{=}^{(3.40)} -\nabla u^* \cdot a^* dx.$$

This equality tells us that the density of the optimal measure w_μ is:

$$f_{w_\mu}(x) = -a^*(x) \cdot \nabla u^*(x). \tag{3.47}$$

We are now ready to apply Theorem 3.4. We start defining the couple:

$$\rho_{w_\mu}(t,x) = (1-t)f^+(x) + tf^-(x); \quad v_{w_\mu}(t,x) = \frac{f_{w_\mu}(x)}{\rho_{w_\mu}(t,x)} = \frac{-a^*(x) \cdot \nabla u^*(x)}{\rho_{w_\mu}(t,x)}, \tag{3.48}$$

as in (3.45). Hence, we obtain that the pair $(\rho_{w_\mu}, v_{w_\mu})$ is optimal for Problem $(\mathcal{DYN}_1)$, following step-by-step the final part of the proof of previous Theorem 3.4. □

Remark 3.2. As stated before, we can use the proof of the previous theorem to correlate the optimal couple $(\rho^*, v^*) = (\rho_{w_\mu}, v_{w_\mu})$, defined in (3.48), solution of $(\mathcal{DYN}_1)$, to the optimal transportation map T^* given by Evans in [26]. Let us consider the velocity vector field v^* and denote by $v_t^*(x) = v^*(t,x)$, in a standard compact way. Consider, then, its flow $Y(t,x) = Y_t(x) = y_x(t)$, solution of the Cauchy problem

$$\begin{cases} y_x'(t) = v_t^*(y_x(t)), \\ y_x(0) = x. \end{cases}$$

Thus, defining:

$$T^*(x) := y_x(1) = Y_1(x), \quad \forall x \in \Omega, \tag{3.49}$$

and, doing the same calculations we did inside the proof of Dacorogna and Moser Theorem 2.6, we get that:

$$\frac{d}{dt}h(t,x) = \frac{d}{dt}\left(\det\nabla Y(t,x)\left((1-t)f^+(Y(t,x)) + tf^-(Y(t,x))\right)\right) = 0.$$

Hence, $h(0,x)$ and $h(1,x)$ must be equal and we obtain:

$$f^+(x) = f^-(T^*(x))\det(DT^*(x)).$$

As already noted in Chapter 2, this determinant is always strictly positive. Now, looking at the constraint in (2.10), obtained by (2.2) just using a change of variable, we obtain, proceeding backward, the same push-forward constraint $T^*_\#(f^+) = f^-$. Thus, T^* is a transport map, i.e. it belongs to $\mathcal{T}(\mu^+, \mu^-)$. Furthermore, noting that T^* is defined exactly as in Theorem 2.7 as the time-one map of the flow of an ODE with the same velocity $-a^* \cdot \nabla u^*$, T^* is exactly the same optimal transport map that solves Monge's original problem $(\mathcal{M}_1)$ and that was found by [26].

Note that, with this last remark, we finally achieved a result similar to the one obtained with the L^2 cost case (Theorem 3.1), recovering a correspondence between the classical Monge's formulation $(\mathcal{M}_1)$ and the dynamic formulation $(\mathcal{DYN}_1)$, also in the case of L^1 cost function. Doing that, we also found many other correspondences between different formulations. In search for a quick summary of the complex relationships between the different formulations, we show in Figure 7 a visual representation of the different formulations and of the theorems used for their analysis.

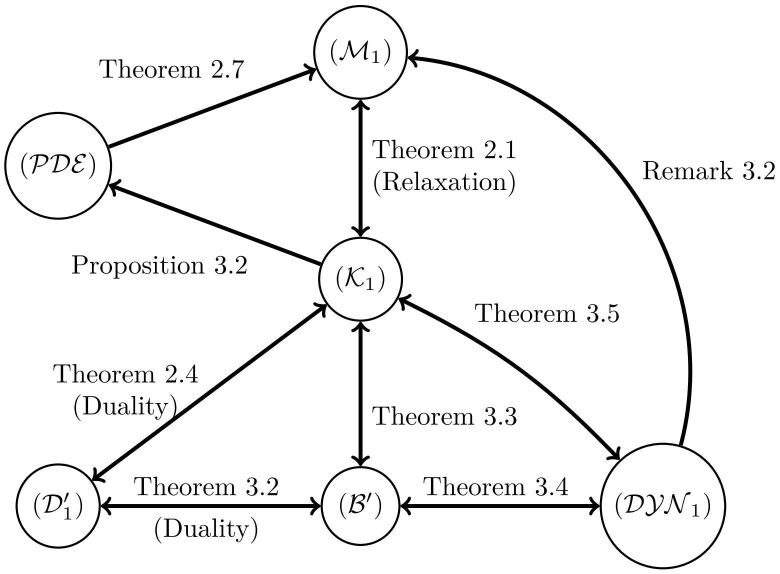


FIGURE 7. Summary of the relations between different formulations of Monge-Kantorovich problem.

CHAPTER 4

Supplementary analysis of the results obtained

This chapter born from the will to examine in depth some aspects we treated quickly before. We start from the study of the density that solves Problem $(\mathcal{DYN}_1)$. Trying to interpret the reasons underlying its form, we compare this solution with the solution of the dynamical problem in the case of a cost function $c(x, y) = \|x - y\|^p$, with $p > 1$. We find a lot of differences between them, both from analytical and geometrical point of views. The geometric perspective is particularly important, because allow us to introduce an important distance, defined in some particular spaces of probabilities, which, actually, is the real reason of the differences between the two cases considered. In the second part, we want to see the correspondence between Beckmann Problem $(\mathcal{B})$ and dynamical Problem $(\mathcal{DYN}_1)$ from a wider point of view. For this aim, we look at some extensions of the two problems and we prove an equivalence between these wider class of problems, analogous to the one given by Theorem 3.4. The interesting fact is that this relationship is achieved using many procedures we have already introduced. In the final part, we want to present some simulations and numerical solutions, graphically representing the time evolutions of the optimal densities of $(\mathcal{DYN}_1)$ and $(\mathcal{DYN}_2)$. Our initial will is to underline the differences between these two problems. However, once graphically confirmed the already known theoretical differences between these two cases, we pass to examine the intermediate cases of L^p costs, with$1 < p < 2$. To do that, we consider two different examples of initial and final data. In the first case, we are able to recover an exact solution, while in the second case we need to recur to a numerical scheme, taken from Delzanno and Finn [22], based on Newton method.

1. Optimal densities solving dynamical problems

In this part, we want to analyse the solution of dynamical formulation $(\mathcal{DYN}_1)$, in order to understand why the optimal density $\rho \in \mathbb{P}(f^+, f^-)$ (eq. (3.16)), is just the affine interpolation between f^+ and f^- written in (3.45). We want to dwell on this result and study the analytic and the geometric details underlying it, because it was somewhat unexpected. The reason why it is surprising is that the dynamical formulation may be used, in applications, for example, to morph two different images. This technique consists, briefly, on transforming one image into another, taking into account also the intermediate images. In the case of affine interpolation, it physically corresponds to a *teleport* phenomenon: at every time, the mass disappears from the initial configuration and appears in the final configuration, as we will see later with an example. This is the reason why it is interesting to go deeper into the details of this result and to understand the reasons behind this behavior. To do that, we will do a comparison between the optimal solutions of the dynamical problems in the case of costs $c(x, y) = \|x - y\|^p$, with $p = 1$ and $p > 1$. This will be useful in order to outline the geometrical differences between the two cases and to understand the different ways in which the optimal density changes in time.

We consider first the case $p = 1$. We will start reporting a uniqueness result for continuity equation (3.14). The following materials is summarized from Santambrogio [41], where more details and proofs can be found.

THEOREM 4.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is either a convex, open subset or $\mathbb{R}^n$ itself. Suppose that $v_t : \Omega \to \mathbb{R}^n$, defined as $v_t(x) := v(t, x)$ is Lipschitz continuous in x, uniformly* *continuous in t and uniformly bounded, and consider its flow $X(t,x)$, such that:*

$$\begin{cases} X'(t,x) = v_t(X(t,x)), \\ X(0,x) = x. \end{cases} \tag{4.1}$$

Suppose also that, for every $x \in \Omega$ and every $t \in [0,1]$, we have $X(t,x) \in \Omega$ (which is obvious for $\Omega = \mathbb{R}^n$, and requires suitable Neumann conditions on v_t otherwise). Then, for every positive finite Borel measure $\mu^+ \in \mathcal{M}_+(\Omega)$, the measures $\rho_t := (X(t,\cdot))_\# \mu^+$ solve the continuity equation with initial datum μ^+, i.e.

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(t,x) + \nabla \cdot (v(t,x)\rho(t,x)) = 0, & (t,x) \in \Omega \times [0,1] \\ \rho(x,0) = \mu^+(x) & x \in \Omega, \end{cases} \tag{4.2}$$

where $\rho(t,x) := \rho_t(x)$. Moreover, every solution of the same equation, if ρ_t are absolutely continuous for every t, is necessarily obtained as $\rho_t := (X(t,\cdot))_\# \mu^+$. In particular, the continuity equation admits a unique solution.

Using this uniqueness result, we want to motivate the fact that the optimal solution of dynamical Problem $(\mathcal{DYN}_1)$ is exactly the affine interpolation between μ^+ and μ^-. Just to be precise, we will consider the case of absolutely continuous function with densities f^+ and f^-. We will apply the theorem above considering the density f^+, as initial condition. As done in the proof of Theorem 3.4, consider an optimal measure $w \in \mathcal{M}^n_{\mathrm{div},0}(\Omega)$ satisfying weakly the constraint $\nabla \cdot w = f^+ - f^-$ and solving $(\mathcal{B}')$. Let us suppose that this optimal measure w is absolutely continuous with density f_w. Then the optimal solution of $(\mathcal{DYN}_1)$ is given by the couple:

$$f_t(x) := (1-t)f^+(x) + tf^-(x); \qquad v_t(x) := v(t,x) := \frac{f_w(x)}{f_t(x)}, \qquad (t,x) \in [0,1] \times \Omega.$$

Consider, then, the flow $X(t,x)$ of the following Cauchy problem:

$$\begin{cases} X'(t,x) = v_t(X(t,x)), \\ X(0,x) = x. \end{cases}$$

Thanks to Theorem 4.1, $\rho_t = X(t,\cdot)_\# f^+$ solves the continuity equation $\frac{\partial \rho_t}{\partial t} + \nabla \cdot (v_t \rho_t) = 0$ with initial datum f^+. Moreover, if we consider (f_t, v_t) defined above, we get:

$$\frac{\partial f_t}{\partial t} = f^- - f^+; \qquad \nabla \cdot (f_t v_t) = \nabla \cdot w = f^+ - f^-.$$

By the uniqueness result of Theorem 4.1 and by the fact that $\rho_0 = f^+$, we obtain the equality:

$$f_t(x) = (1-t)f^+(x) + tf^-(x) = X(t,\cdot)_\# f^+, \qquad \forall x \in \Omega, \quad \forall t \in [0,1]. \tag{4.3}$$

This result confirms once more that the affine interpolation is the optimal solution for $(\mathcal{DYN}_1)$. We want to focus, now, on the dynamical formulation of the optimal transportation problem, in the case of a cost function $c(x,y) = \|x-y\|^p$, with $p > 1$, in order to get some intuitions about the differences between this case and the L^1 case. Note that we already described the dynamic formulation $(\mathcal{DYN}_2)$ with a quadratic cost function, taken from [5]. However, in order to extend this dynamic formulation to a general $p > 1$, as we did for the case of $p = 1$, we use the procedure provided by [18] (Section 1 Chapter 3). Looking at the way in which we proceeded in the case of L^1 cost and noting that equality (3.6) is still true in the case of any $p \geq 1$, it is easy to obtain the general version of the fluid dynamic formulation, given by:

Problem $(\mathcal{DYN}_p)$.

Given two positive finite absolutely continuous measures $\mu^+, \mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ *with densities* f^+ *and* f^- *satisfying* (2.44)*, find an* optimal couple (ρ^*, v^*) *satisfying:*

$$\begin{cases} \inf_{(\rho,v)} \int_{\mathbb{R}^n} \int_0^1 \|v(t,x)\|^p \rho(t,x) dt dx, \\ \frac{\partial \rho}{\partial t}(t,x) + \nabla_x \cdot (v(t,x)\rho(t,x)) = 0, \\ \rho(x,0) = f^+(x), \quad \rho(y,1) = f^-(y). \end{cases} \tag{4.4}$$

We want to get some geometric intuitions about the differences underlying the cases of $p = 1$ and $p > 1$. To do that, we need to introduce a geometrical formulation of the optimal transport problem, which can be viewed as a distance on the space of the probability measures. Note, that we do not want to go deep into the geometrical details of the problem, which will be really challenging and not required by the aim of this thesis. However, we want to give an intuition about the geometry underlying this problem and, to do that, we have to introduce some tools just to be able to answer this issue. Hence, first of all, let us recall some definitions about metric spaces.

Definition 4.1 (Length of a curve). Given a metric space (V, d) and a curve $\gamma \in \mathrm{AC}([0,1]; V)$, we define the *length* ℓ of γ as:

$$\ell(\gamma) := \int_0^1 \|\gamma'(t)\|_d \, dt.$$

Definition 4.2 (Geodesic and constant speed geodesic). Given a metric space (V, d), a curve $\gamma \in \mathrm{AC}([0,1]; V)$ is said to be:

- a *geodesic* between x and y in V if it minimizes the length among all curves such that $\theta(0) = x$ and $\theta(1) = y$, i.e.
$$\ell(\gamma) = \min \{\ell(\theta) \ : \ \theta \in \mathrm{AC}([0,1]; V) \text{ and } \theta(0) = x, \ \theta(1) = y\}.$$
- a *constant speed geodesic* between x and y in V is a geodesic for which
$$d(\gamma(t), \gamma(s)) = |t - s| \, d(x, y), \quad \forall t, s \in [0,1].$$

We denote the set of the constant speed geodesics between $\gamma(0)$ and $\gamma(1)$ in V as:

$$\mathrm{CSG}([0,1]; V) := \{\gamma \in \mathrm{AC}([0,1]; V) \ : \ \gamma \text{ is a constant speed geodesic in } V\}. \tag{4.5}$$

From now on, consider $\Omega \subset \mathbb{R}^n$ convex and unbounded, which, clearly, includes the case of $\Omega = \mathbb{R}^n$. In order to define the previously announced distance on the space of probabilities, we introduce, for $p \geq 1$ the following space:

$$\mathbb{P}_p(\Omega) = \left\{ \mu \in \mathbb{P}(\Omega) \ : \ \int_\Omega \|x\|^p \, d\mu(x) < +\infty \right\}. \tag{4.6}$$

This is the set of probability measures with finite p-momentum. Note that, whenever Ω is bounded, this space clearly coincides with the whole $\mathbb{P}(\Omega)$. We are ready, now, to define the *Kantorovich-Rubinstein-Wasserstein distance*. Note that this distance, in the literature, is known just as Wasserstein distance, even if Kantorovich and Rubinstein discovered its existence before Wasserstein and he just did a brief article about its relation with Markov fields. Thus, we decided to call it like that for historical reasons and we included the name of Wasserstein so that it will be more recognisable. For a further clarification about the name, we cite a brief article [45] about the history and the work of Kantorovich.

Definition 4.3 (Kantorovich-Rubinstein-Wasserstein distance). Consider $p \geq 1$ and μ^+, $\mu^- \in \mathbb{P}_p(\Omega)$. Then the *$p$-Kantorovich-Rubinstein-Wasserstein distance* between μ^+ and μ^- is given by:

$$d_{KRW_p}(\mu^+, \mu^-) := \inf \left\{ \int_{\Omega \times \Omega} \|x - y\|^p \, d\mu(x,y) \ : \ \mu \in \Pi(\mu^+, \mu^-) \right\}^{\frac{1}{p}}. \tag{4.7}$$

Note the obvious relationship between this distance and the Kantorovich formulation of the optimal transport problem $(\mathcal{K})$, with $c(x,y) = \|x-y\|^p$. The assumption μ^+, $\mu^- \in \mathbb{P}_p(\Omega)$ guarantees finiteness of this value, thanks to the inequality $\|x-y\|^p \le C(\|x\|^p + \|y\|^p)$. Moreover, it can be shown that d_{KRW_p} is actually a distance on the space $\mathbb{P}_p(\Omega)$. Thus:

Definition 4.4 (Kantorovich-Rubinstein-Wasserstein space). For any $p \ge 1$, we define the *Kantorovich-Rubinstein-Wasserstein space* of order p as the metric space $(\mathbb{P}_p(\Omega), d_{KRW_p})$.

There are a number of studies that characterize the geometrical and topological properties of this distance. For example, the fact that it "metrizes" weak convergence, i.e.

$$\mu_k \rightharpoonup \mu \text{ in } \mathbb{P}_p(\Omega) \iff d_{KRW_p}(\mu_k, \mu) \to 0, \text{ as } k \to \infty,$$

But, also, the fact that it is a Polish space, i.e. a separable and complete metric space, if Ω is a Polish space. As said at the beginning, we do not want to spend a lot of time on this geometrical details. For the proofs of what we just stated and for more details about this topic, we suggest the book of Villani [46]. Chapter 6 of this book is entirely dedicated to the Kantorovich-Rubinstein-Wasserstein distances and we refer to it for any deepening about the geometrical and topological details. The definitions above involve also the case of $p = 1$, but, we want to focus for a while on the case of $p > 1$ strictly greater than 1. We want to present a precise connection between this minimization Problem $(\mathcal{DYN}_p)$ and the distance $d_{KRW_p}(\mu^+, \mu^-)^p$, in order to obtain some information about the form of the optimal density ρ, in the case of $p > 1$. Recalling the link between the distance d_{KRW_p} and Problem $(\mathcal{K})$ and proceeding like we did in the proof of Theorem 3.1, it would be easy to prove that the correspondence between classical versions of Monge-Kantorovich problem and the dynamical Problem $(\mathcal{DYN}_p)$ with $p = 2$, given by the abovementioned theorem, holds also in the case of a general $p > 1$.

Theorem 4.2. *Consider $\mu^+, \mu^- \in \mathcal{M}_+(\mathbb{R}^n)$ two absolutely continuous measures with smooth and compactly supported densities f^+ and f^-. Then, for every $p > 1$, the p-Kantorovich-Rubinstein distance between μ^+ and μ^- is characterized as follows:*

$$d_{KRW_p}(\mu^+, \mu^-)^p = \min\,(\mathcal{DYN}_p). \tag{4.8}$$

This theorem gives, recalling again the link between the distance d_{KRW_p} and Problem $(\mathcal{K})$ with a cost function $c(x,y) = \|x-y\|^p$, a generalization of the equivalence given in Theorem 3.5. Moreover, note that we can even say more. Indeed, doing the same observation we did after the proof of Theorem 3.1, we can prove, exactly in the same way, that the optimal couple solving $(\mathcal{DYN}_p)$ is given by:

$$\begin{cases} \boxed{\rho_t(x) = ((1-t)x + tT^*(x))_\# f^+(x)}; \\ v_t(x) = T^*(X_t^{-1}(x)) - X_t^{-1}(x). \end{cases} \tag{4.9}$$

Note that this couple is exactly the same as in the case of $p = 2$ (eq. (4.35)). Thus, note that the optimal curve ρ, in the case of $p > 1$, is not anymore given by the affine interpolation between f^+ and f^-, but is given by a linear relation involving the optimal transport map. As previously announced, to understand the real difference between this case and the case of $p = 1$, we have to understand the geometry behind dynamical Problem $(\mathcal{DYN}_p)$. Specifically, we want to look at the geodesics in the Kantorovich-Rubinstein-Wasserstein spaces and, in particular, we want to define and characterize the constant speed geodesics in these spaces. Just to have an intuition about their form, we want to start with a basic example, taken from[12].

Example 4.1. Fix two distinct points $x_0, x_1 \in \Omega$. Clearly, the constant speed geodesic between these two points in Ω is the segment $\overline{x_0x_1}$, parametrized by $\overline{x_0x_1}(t) = (1-t)x_0 + tx_1$. Then, it is straightforward to see that curve $\mu : [0,1] \to \mathbb{P}_p(\Omega)$ concentrated on this segment, i.e. given by:

$$\rho_t := \delta_{\overline{x_0x_1}(t)}, \quad t \in [0,1],$$

verifies the following equality:

$$d_{KRW_p}(\rho_t, \rho_s) = |t-s| \, \|x_0 - x_1\| = |t-s| \, d_{KRW_p}(\rho_0, \rho_1).$$

This means that ρ_t is a constant speed geodesic in $\mathbb{P}_p(\Omega)$, connecting the two Dirac measures $\rho_0 = \delta_{x_0}$ and $\rho_1 = \delta_{x_1}$. Note that this ρ can be written in the form $\rho_t = (e_t)_\# Q$, where e_t is the usual evaluation map at time t, defined in (3.9) and Q, belonging to the space $\mathbb{P}(\mathcal{X})$, is defined as:

$$Q = \delta_{\overline{x_0 x_1}}.$$

Here $\mathcal{X}$ is the usual space of $\mathcal{C}^1$ curves, defined in (3.5).

This example gives an insight about the characterization of the constant speed geodesics in these Kantorovich-Rubinstein-Wassersstein spaces. Indeed, we will present here a result which fully describes the space $\mathrm{CSG}([0,1]; \mathbb{P}_p(\Omega))$ in terms of the constant speed geodesics in Ω, exactly as in the case of the previous example. For the proof, we refer the reader to Lisini [35, Theorem 4.2].

THEOREM 4.3. *For $p > 1$, a curve $\mu \in AC([0,1]; \mathbb{P}_p(\Omega))$ is a constant speed geodesic if and only if there exists $Q \in \mathbb{P}(\mathcal{X})$ such that:*

(i) $\rho_t = (e_t)_\# Q$, *for every* $t \in [0,1]$*;*
(ii) Q *is concentrated on the set of constant speed geodesics* $CSG([0,1]; \Omega)$*;*
(iii) $d_{KRW_p}(\mu_0, \mu_1)^p = \int_{\mathcal{X}} \|\gamma(0) - \gamma(1)\|^p \, dQ(\gamma)$.

Note that theorem above is written in general terms, because is taken from a work in which instead of $\Omega \subset \mathbb{R}^n$ it was considered a general Polish space (V, d). In the case of $\Omega \subset \mathbb{R}^n$, we can be more precise about the form of the constant speed geodesics and we can derive an explicit formulation for the probability Q. Indeed every element in space $\mathrm{CSG}([0,1]; \Omega)$ is the segment $\overline{xy}$, parametrized, as in the previous example, by $\overline{xy}(t) = (1-t)x + ty$. Now, consider $\mu^* \in \Pi(\mu^+, \mu^-)$ the optimal transport plan minimizing (4.7). Using (iii) we get that:

$$d_{KRW_p}(\mu^+, \mu^-)^p = \int_{\Omega \times \Omega} \|x - y\|^p \, d\mu^*(x,y) = \int_{\mathcal{X}} \|\gamma(0) - \gamma(1)\| \, dQ(\gamma).$$

Furthermore, if we add the fact that the measure Q is concentrated on the set of constant speed geodesics $\mathrm{CSG}([0,1]; \Omega)$, which are simply the straight line connecting two points x and y we obtain the following form for the probability measure Q:

$$Q = \int_{\Omega \times \Omega} \delta_{\overline{xy}} d\mu^*(x,y). \tag{4.10}$$

In other words, Q is the probability measure whose disintegration with respect to the optimal μ^* is given by the family of Dirac masses $\{\delta_{\overline{xy}}\}_{(x,y) \in \Omega \times \Omega}$ concentrated on the segments $\overline{xy}$. Then, using (i) of the Theorem 4.3, we get that $\mu \in \mathrm{AC}([0,1]; \mathbb{P}_p(\Omega))$ is a constant speed geodesic if and only if it has the form:

$$\rho_t = (e_t)_\# Q = ((1-t)\pi_x + t\pi_y)_\# \mu^*, \quad t \in [0,1]. \tag{4.11}$$

This is explicitly represented, using the integral representation of a measure in terms of Riesz representation Theorem 1.7, by the following:

$$\int_\Omega \phi(x) d\rho_t(x) = \int_{\Omega \times \Omega} \phi((1-t)x + ty) d\mu^*(x,y), \quad \forall \phi \in \mathcal{C}_0(\Omega)$$

Formula (4.11) is usually referred to as *displacement interpolation*. This terminology has been introduced for the first time in the work of McCann [36]. He was working on a model for interacting gases and he was looking for an interpolant ρ_t, different from the linear interpolation. His aim was to obtain convexity on the total energy $E(\rho)$ with respect to the interpolation parameter t, i.e. $E(\rho_t) \leq (1-t)E(\rho_0) + tE(\rho_1)$. He discovered that a curve ρ_t in the form of displacement interpolation satisfied his requests. We refer to the work cited above for all the details. Here we are interested in understanding the differences between this displacement interpolation and the linear interpolation obtained for $p = 1$. Note that Theorem 2.5 states that in the case of strictly convex cost, which is the case of $c(x, y) = \|x - y\|^p$, with $p > 1$, the unique optimal transport plan μ^* is induced by a transport map T^*. This relationship, already represented in Equation (2.27), is given by:

$$\mu^* = (\mathrm{Id} \times T^*)_\# \mu^+.$$

Observe that, in this case, displacement interpolation formula (4.11) can be rewritten as:

$$\rho_t = ((1-t)\mathrm{Id} + tT^*)_\# \mu^+, \quad t \in [0, 1], \tag{4.12}$$

which means that Q is given by $Q = \int_\Omega \delta_{\overline{xT^*(x)}} d\mu^+(x)$. Note that the equality written above, in the case of μ^+ absolutely continuous with density f^+, is exactly equal to the optimal ρ_t written in (4.9). This means that in the case of $p > 1$, the optimal solutions of the dynamical Problems $(\mathcal{DYN}_p)$ are **constant speed geodesics** on p-Kantorovich-Rubinstein-Wasserstein spaces $\mathbb{P}_p(\Omega)$. Moreover, these geodesics can all be written as **displacement interpolations**. On the contrary, in the case of $p = 1$, we showed that the optimal solution of dynamical Problem $(\mathcal{DYN}_1)$ is the affine interpolation written in (4.3). We want to show, now, that in this case the optimal solution is not obtained via displacement interpolation. Indeed, consider for example two Dirac measures $\rho_0 = \delta_{x_0}$ and $\rho_1 = \delta_{x_1}$ with $x_0, x_1 \in \mathbb{R}^n$, as in Example 4.1. In this case, the optimal solution of dynamical Problem $(\mathcal{DYN}_1)$ with initial and final data ρ_0 and ρ_1, respectively, is given by the linear interpolation $\rho_t = (1-t)\delta_{x_0} + t\delta_{x_1}$ such that:

$$d_{KRW_1}(\rho_t, \rho_s) = |t - s| \, \|x_0 - x_1\| = |t - s| \, d_{KRW_1}(\rho_0, \rho_1), \quad \text{for every } t, s \in [0, 1].$$

However, in this case, it is not true that ρ_t can be written as a displacement interpolation. Indeed, noting that the optimal plan in this case is $\mu^*(x, y) = \delta_{x_0} \times \delta_{x_1}(x, y)$, we obtain, using (4.11), that the curves verifying the displacement interpolation equality are given by:

$$\int_\Omega \phi(x) d\eta_t(x) = \int_{\Omega \times \Omega} \phi((1-t)x + ty) d\mu^*(x, y) = \phi((1-t)x_0 + tx_1), \quad \forall \phi \in \mathcal{C}_0(\Omega).$$

On the other hand, the linear interpolation ρ_t satisfies:

$$\int_\Omega \phi(x) d\rho_t(x) = \int_\Omega \phi(x) d((1-t)\delta_{x_0} + t\delta_{x_1})(x) = (1-t)\phi(x_0) + t\phi(x_1), \quad \forall \phi \in \mathcal{C}_0(\Omega).$$

We obtain that such ρ_t can not be written in the form of displacement interpolation, because the two equalities above are not the same for all $\phi \in \mathcal{C}_0(\Omega)$ (they are not all linear). Thus, we showed that in the case of $p > 1$, the optimal solution of the dynamical problem ρ_t is linked to the displacement interpolation, while in the case of $p = 1$ it is represented by linear. The simulations in the final section of this chapter will underline once more the difference between these two cases and will also help us to visualize it.

2. More on the relation between $(\mathcal{B}')$ and $(\mathcal{DYN}_1)$

In this section we take a different point of view in the analysis of the relations between Problems $(\mathcal{B}')$ and $(\mathcal{DYN}_1)$. As announced in Section 2 of Chapter 3, Beckmann minimization Problem $(\mathcal{B})$ can be extended to a wider class of minimization problems. Before describing all the formal details about this class of problems, let us spend some words about the reasons behind this extension of the Beckmann continuous transportation model. This generalization of $(\mathcal{B}')$ comes from the idea of taking into account a non-uniform cost for the movement, due to the presence of some obstacles or of particular configurations. To do that, we have to introduce a positive function $k : \Omega \to \mathbb{R}_+$, where $k(x)$ stands for the local cost at x per unit length of a path passing through x. Indeed, if we know a priori the congestion function k, this problem can be modeled by the following minimization problem:

$$\min \left\{ \int k(x) \, |w(x)| \, dx \; : \; \nabla \cdot w = \mu^+ - \mu^-, \; w \cdot \nu_\Omega = 0 \text{ on } \partial\Omega \right\}. \tag{4.13}$$

Yet, as it happens in many situations, for example in urban traffic, this function k is not known a priori, but it depends on the traffic vector field w itself. The easiest model for this problem, chosen also by Beckmann, is to consider k as a function of the modulus of w, i.e.

$$k(x) = g(\|w(x)\|). \tag{4.14}$$

We consider the following problem (for more details see [13]):

Problem $(\mathcal{B}_{\mathcal{H}})$.

Given $\Omega \subset \mathbb{R}^n$ a convex and compact set with a smooth boundary and two measures μ^+, $\mu^- \in \mathcal{M}_+(\Omega)$ satisfying the mass balance condition restricted in Ω,

$$\mu^+(\Omega) = \mu^-(\Omega) < +\infty.$$

Given the function $\mathcal{H} : \mathbb{R}^n \to \mathbb{R}$ satisfying the following properties:

(i) $\mathcal{H}$ is a convex radially symmetric function with $\mathcal{H}(0) = 0$;

(ii) there exist some positive real constants a, b and $p \in [1, +\infty)$ such that:

$$a\,\|x\|^p \leq \mathcal{H}(x) \leq b(\|x\|^p + 1), \quad x \in \mathbb{R}^n;$$

(iii) $\mathcal{H}$ is differentiable in $\mathbb{R}^n \setminus \{0\}$ and there exists a positive constant c such that:

$$\|\nabla\mathcal{H}(x\| \leq c(|x|^{p-1} + 1), \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Find an optimal vector field $w \in L^p(\Omega; \mathbb{R}^n)$ *that minimizes the following class of problems:*

$$\inf_{w \in L^p(\Omega;\mathbb{R}^n)} \left\{ \int_\Omega \mathcal{H}(w(x))\, dx \;:\; \nabla \cdot w = \mu^+ - \mu^-,\; w \cdot \nu_\Omega = 0 \text{ on } \partial\Omega \right\}, \tag{4.15}$$

where, as usual, the divergence constraint has to be interpreted in a weak sense.

We will consider two special cases $\mathcal{H}(\cdot) = H_1(\|\cdot\|)$ and $\mathcal{H}(\cdot) = H_2(\|\cdot\|)$ with $H_1, H_2 : \mathbb{R}_+ \to \mathbb{R}_+$ defined as $H_1(t) = g(t)t$ and $H_2'(t) = g(t)$, $H_2(0) = 0$. The reasons why we consider these two functions H_1 and H_2 will be clearer later. Note that Problem $(\mathcal{B}_{\mathcal{H}})$, in the case of H_1, is exactly the initial intuitive minimization problem (4.13), with a metric k depending on w. The functions H_1 and H_2 satisfy the same properties as $\mathcal{H}$, without the modulus in the inequalities. Moreover, $g : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function that models the congestion effect and has to satisfy similar conditions, namely $at^{p-1} \leq g(t) \leq b(t^{p-1}+1)$ for some positive real constants a, b. Note that the case of $\mathcal{H}(x) = \|x\|$ is exactly Beckmann minimization problem $(\mathcal{B})$, as we have already said in Section 2 of Chapter 3. Note, also, that in the case of $\mathcal{H}$ strictly convex the problem is well posed. Indeed, in the case $w \in L^p$, with $p > 1$, L^p reflexive space, enable the application of Theorem 1.5 in order to use the Direct method in the Calculus of Variations. Thus, we will consider in this section just the case of $p > 1$. However, also in this case, follow the previous approach and cast the problem within the setting of vector measures, reasoning as for $(\mathcal{B}')$ and taking as domain of minimization a proper subspace of $\mathcal{M}^n_{\mathrm{div},0}(\Omega)$.

Before going on with the analogous generalization of the dynamical Problem $(\mathcal{DYN}_1)$, we want to consider first a discrete problem which can give us some good insights about its form, passing to the limits. We refer to the original idea of Wardrop in [47] or to the more recent work [41] of Santambrogio for all the details. The main ingredients of the model are:

- a finite oriented connected graph $\mathcal{G} = (V, E)$ modeling the network;
- some travel time functions $g_e : \mathbb{R}_+ \to \mathbb{R}_+$, continuous and increasing, giving, for each edge $e \in E$, the travel time on e when the flow is $\theta \in \mathbb{R}_+$. Note that this functions g_e are meant to capture the congestion effects, which can be different according to different edges, since some roads may be longer or wider;
- for each pair of nodes $(x, y) \in N^2$ interpreted as origins/destinations, we consider a transport plan $\mu_{x,y}$ representing the "mass" that has to be sent from x to y;

- we will denote by $\mathbb{X}_{x,y}$ the set of all simple paths connecting x and y. With simple path we mean a path on the graph $\mathcal{G} = (V, E)$ that contains no repeated vertices. Moreover, we will denote by $\mathbb{X} := \cup_{(x,y)\in N^2}\mathbb{X}_{x,y}$ the set of all the simple paths.

The unknowns of the problem are the flow configurations, divided into the following variables:

- the *edge flows* $i = (i_e)_{e\in E}$, where i_e is the total flux on the edge e;
- the *path flows* $q = (q_\gamma)_{\gamma\in\mathbb{X}}$, where q_γ is the mass traveling on each generic path γ.

The values i_e and q_γ are all non-negative and have to satisfy the *mass conservation* constraints:

$$\mu_{x,y} = \sum_{\gamma\in\mathbb{X}_{x,y}} q_\gamma, \quad \forall (x,y)\in N^2 \tag{4.16}$$

and

$$i_e = \sum_{\gamma\in\mathbb{X}\,:\,e\in\gamma} q_\gamma, \quad \forall e\in E. \tag{4.17}$$

The last equation tells us that i is a function of q. Finally, we want to define, given the edge flows $i = (i_e)_{e\in E}$, the *total cost* or *weighted length* of the path $\gamma\in\mathbb{X}$ by:

$$L_i(\gamma) = \sum_{e\in\gamma} g_e(i_e), \tag{4.18}$$

depending on the already mentioned travel time functions g_e. In this case, we do not want to solve a standard maximization or minimization linear program, as we did for the discrete version of Kantorovich problem, but we want to find a particular flow configuration called *Wardrop equilibrium*. Intuitively speaking, we want to find a flow configuration such that every actually used path is an optimal path for the cost L_i. Formally, let us give the following:

Definition 4.5 (Wardrop equilibrium). A *Wardrop equilibrium* is a flow configuration $q = (q_\gamma)_{\gamma\in\mathbb{X}}$ satisfying $q_\gamma \geq 0$ and constraint (4.16). Moreover, when we compute the values i_e, for all $e\in E$, with equation (4.17), for every $(x,y)\in N^2$ and every $\gamma\in\mathbb{X}_{x,y}$ with $q_\gamma > 0$, we have:

$$L_i(\gamma) = \min_{\eta\in\mathbb{X}_{x,y}} L_i(\eta). \tag{4.19}$$

Wardrop equilibria can be characterized by a variational principle, as it is stated in the following theorem, taken from [4]:

Theorem 4.4. *The flow configuration $q = (q_\gamma)_{\gamma\in\mathbb{X}}$ is a Wardrop equilibrium if and only if it solves the convex minimization problem:*

$$\min\left\{\sum_{e\in E} H_e(i_e) \;:\; q\geq 0 \textit{ satisfies } (4.16)\right\}, \tag{4.20}$$

where, for each e, we define H_e as: $H_e' = g_e$.

The proof of this characterization theorem dates back to [4], but it can be found also in Santambrogio [41]. Note that the previous result gives us for free an existence result for i and, also, its uniqueness, in the case of g_e strictly increasing, i.e. H_e strictly convex.

Remark 4.1. Wardrop equilibria, which solves (4.20), does not, in general, represent the natural total social cost measured by the total time lost which would rather be

$$\sum_{e\in E} i_e g_e(i_e). \tag{4.21}$$

We refer to the transport configuration that minimizes (4.21) as *efficient* transport patterns. It would be tempting to deduce, from Theorem 4.4, that equilibria are efficient, but, in general, $H_1(\theta) = \theta g(\theta)$ and $H_2'(\theta) = g(\theta)$ lead to very different states. For this difference between efficiency and equilibrium, we considered two different types of function H for

Problem $(\mathcal{B}_{\mathcal{H}})$. However, for example, efficient and equilibria configurations coincide in the special case of $H_e(\theta) = a_e\theta^p$. The difference between efficiency and equilibrium is a topic especially treated in the case of finite-dimensional network, where it is usually known as *price of anarchy* (see [38]).

REMARK 4.2. In the problem just presented, the transport plan μ is fixed. This may be interpreted as a *short-term problem*. Alternatively, we can consider the *long-term problem*, where we fix only the initial and final distributions μ^+, μ^-. In this case, one also obtains an optimality condition for μ, in the sense that it minimizes among all transport plans in $\Pi(\mu^+, \mu^-)$ (thought in the discrete sense), the total cost:

$$\sum \mu_{x,y} d_i(x, y) \text{ with } d_i(x, y) := \min_{\gamma \in \mathbb{X}_{x,y}} L_i(\gamma). \tag{4.22}$$

In the case of long-term problem, we have the same convex minimization problem as in (4.20), but minimizing also over μ.

We want to generalize the previous analysis to a continuous framework, trying to identify each instrument and parameter with their continuous counterpart. Consider $\Omega \subset \mathbb{R}^n$ and two measures μ^+, $\mu^- \in \mathcal{M}_+(\Omega)$ as in the beginning of this section. We proceed in steps:

- The framework is, as always, $\Pi(\mu^+, \mu^-)$. This means that we consider a long-term problem, in the sense of Remark 4.2.
- As a continuous version of the set of all the simple paths $\mathbb{X}$, we will consider the same space $\mathcal{X} = \mathcal{C}^1([0, 1]; \Omega)$ considered in Section 1 of Chapter 3.
- We will consider continuous and increasing travel times functions $g : \mathbb{R}_+ \to \mathbb{R}_+$.
- We choose as continuous path flows, extensions of the discrete flows q, the probability measures defined on the space $\mathcal{X}$, i.e. $Q \in \mathbb{P}(\mathcal{X})$.
- For the continuous counterpart of i, we look at the relation written in (4.17), which tells us that i is a function depending on q. Thus, recalling the definition of the scalar measure σ_μ written in (3.30), we define the measure $i_Q \in \mathcal{M}_+(\Omega)$, called the *traffic intensity*, as:

$$\langle i_Q, \phi\rangle_{\mathcal{C}_0(\Omega;\mathbb{R}_+)} := \int_{\mathcal{X}} \left(\int_0^1 \phi(\gamma(t)) \, \|\gamma'(t)\| \, dt \right) dQ(\gamma), \qquad \forall \phi \in \mathcal{C}_0(\Omega; \mathbb{R}_+). \tag{4.23}$$

Note that, reasoning as for σ_μ and using Riesz representation Theorem 1.7, we know that i_Q is actually a scalar measure. It is a generalization of the discrete i, but the interpretation, in this case, is that for every subregion A, $i_Q(A)$ represents the total cumulated traffic in A induced by Q. It means that for every path we compute "how long" it stays in A and, then, we average on paths. The link with σ_μ, starting from its definition in (3.30), is that if we define the measure which concentrates on the transport rays as:

$$Q_\mu := \int \delta_{\overline{xy}} d\mu(x, y), \quad \text{where } \overline{xy} \text{ stands for the segment connecting } x \text{ and } y,$$

with $\mu \in \Pi(\mu^+, \mu^-)$, then:

$$\langle i_{Q_\mu}, \phi\rangle_{\mathcal{C}_0(\Omega;\mathbb{R}_+)} = \langle \sigma_\mu, \phi\rangle_{\mathcal{C}_0(\Omega)}.$$

Furthermore, to any measure $Q \in \mathbb{P}(\mathcal{X})$ we need to associate another measure which is not linked to any object used in the discrete case, but which will be useful later. Thus, we define the measure $w_Q \in \mathcal{M}^n(\Omega)$, called *traffic flow*, as:

$$\langle w_Q, \xi\rangle_{\mathcal{C}_0(\Omega;\mathbb{R}^n)} := \int_{\mathcal{X}} \left(\int_0^1 \xi(\gamma(t)) \cdot \gamma'(t) dt \right) dQ(\gamma), \qquad \forall \xi \in \mathcal{C}_0(\Omega; \mathbb{R}^n). \tag{4.24}$$

Note that, also in this case, w_Q is a measure linked via Riesz representation Theorem 1.7 to the vector measure w_μ defined in (3.31). Moreover, taking a gradient field $\xi = \nabla\phi \in$

$\mathcal{C}_0(\Omega;\mathbb{R}^n)$, we get:

$$\langle w_Q, \nabla\phi\rangle_{\mathcal{C}_0(\Omega;\mathbb{R}^n)} = \int_{\mathcal{X}} [\phi(\gamma(1)) - \phi(\gamma(0))]\, dQ(\gamma) = \int_{\Omega} \phi \; d((e_1)_{\#}Q - (e_0)_{\#}Q).$$

From now on, we will restrict our attention just to the *admissible plans* which, recalling the discrete constraint in (4.16), will be given by the following set:

$$\begin{aligned} \mathcal{Q}^p(\mu^+, \mu^-) := \{Q \in \mathbb{P}(\mathcal{X}) \mid (e_0)_{\#}Q = \mu^+,\ (e_1)_{\#}Q = \mu^- \text{ and} \\ di_Q(x) = f_Q(x)dx \text{ with } f_Q(x) \in L^p(\Omega)\}. \end{aligned} \tag{4.25}$$

With $di_Q(x) = f_Q(x)dx$ we mean, using a compact notation, that i_Q is an absolutely continuous measure with density f_Q (Section 3 of Chapter 1). Note that this resembles the set defined in (3.11) in Section 1 of Chapter 3, but with an L^p constraint (to be compared with the set of the admissible vector field $w \in L^p(\Omega;\mathbb{R}^n)$ of Problem $(\mathcal{B}_{\mathcal{H}})$). At first glance, it may seem hard to check if $\mathcal{Q}^p(\mu^+, \mu^-) \neq 0$. However, if, for example, we assume that the density of the measures μ^+ and μ^- are in L^p, we can conclude that there is at least one $Q \in \mathbb{P}(\mathcal{X})$ such that its density $f_Q \in L^p$. This follows from a regularity result shown in [21]. Coming back, for a while, to the vector measure w_Q defined above, using an admissible plan $Q \in \mathcal{Q}^p(\mu^+, \mu^-)$, we get that it satisfies the constraint:

$$\nabla \cdot w_Q = \mu^+ - \mu^-. \tag{4.26}$$

Obviously this constraint is always meant in the weak sense, automatically endowed with Neumann boundary conditions, as in (3.23). We see immediately that the total variation of the vector measure w_Q is less than or equal to the scalar measure i_Q, i.e.

$$\|w_Q\| \leq i_Q, \quad \forall Q \in \mathcal{Q}^p(\mu^+, \mu^-). \tag{4.27}$$

This result can be easily shown using, again, Cauchy-Schwarz inequality. In this context, we define the congested function k_Q associated to $Q \in \mathcal{Q}^p(\mu^+, \mu^-)$, considering an absolutely continuous measure $i_Q \in \mathcal{M}_+(\Omega)$ with density f_Q, as:

$$k_Q(x) := g(f_Q(x)). \tag{4.28}$$

The last ingredients to complete the description of the continuous case are the continuous analogues of the total cost defined on (4.19) and the function d_{k_Q}, continuous version of the d_i defined in (4.22). They are defined by:

$$L_{k_Q}(\gamma) = \int_0^1 g(f_Q(\gamma(t)))\, \|\gamma'(t)\|\, dt$$

and

$$d_{k_Q}(x, y) := \inf_{\gamma \in \mathcal{X}_{xy}} L_{k_Q}(\gamma), \tag{4.29}$$

where $\mathcal{X}_{xy} = \{\gamma \in \mathcal{X} \mid \gamma(0) = x,\ \gamma(1) = y\}$, as in Section 1 of Chapter 3. Note that, using Definition 4.2, paths in $\mathcal{X}$ such that $d_{k_Q}(\gamma(0), \gamma(1)) = L_{k_Q}(\gamma)$ are *geodesics* between $\gamma(0)$ and $\gamma(1)$ in $\mathcal{X}$. Now, we are ready to define the notion of Wardrop equilibrium in the continuous case, adapting the idea used in Definition 4.5 in a continuous framework. This means that a Wardrop equilibrium is just an admissible plan Q, such that almost every path in $\mathcal{X}$ should be optimal for the cost L_{k_Q}.

Definition 4.6 (Wardrop equilibrium in continuous case). A Wardrop equilibrium is an admissible plan $Q \in \mathcal{Q}(\mu^+, \mu^-)$ such that:

$$Q(\{\gamma \ : \ L_{k_Q}(\gamma) = d_{k_Q}(\gamma(0), \gamma(1))\}) = 1.$$

Existence or, even, well-posedness of these equilibria are not always straightforward. However, it is possible to link these Wardrop equilibria to solutions of minimal traffic problem, extending Theorem 4.4 to a continuous framework. The variational problem, continuous version of the congested traffic dynamic issue, represented in minimization problem (4.20), is the following:

Problem ($\mathcal{DYN}_\mathcal{H}$).

Given $\Omega \subset \mathbb{R}^n$ a convex and compact set with a smooth boundary and two positive measures μ^+, $\mu^- \in \mathcal{M}_+(\Omega)$ that satisfy the mass balance condition, as in (2.1)*, but restricted in this domain, i.e.*

$$\mu^+(\Omega) = \mu^-(\Omega) < +\infty.$$

Find an optimal probability measure $Q \in \mathcal{Q}^p(\mu^+, \mu^-)$ *satisfying the following minimization problem:*

$$\inf_{Q \in \mathcal{Q}^p(\mu^+,\mu^-)} \int_\Omega H_2(f_Q(x))dx. \tag{4.30}$$

This minimization problem is called ($\mathcal{DYN}_\mathcal{H}$), because it is strictly linked with Problem ($\mathcal{DYN}_1$). Indeed, as in the case of ($\mathcal{B}_\mathcal{H}$), this ($\mathcal{DYN}_\mathcal{H}$) is a wider problem containing, as a particular case, the dynamical Problem ($\mathcal{DYN}_1$). Thus, considering $H_2(t) = t$ and, accordingly, $p = 1$, the problem just introduced becomes:

$$\inf_{Q \in \mathcal{Q}^1(\mu^+,\mu^-)} \int_\Omega f_Q(x)dx = \inf_{Q \in \mathcal{Q}^1(\mu^+,\mu^-)} \int_\Omega di_Q(x) = \inf_{Q \in \mathcal{Q}^1(\mu^+,\mu^-)} \int_\mathcal{X} \left(\int_0^1 \|\gamma' t\| \, dt \right) dQ(w).$$

Note that the right-hand side of the last equality, using Theorem 1.8, is equivalent to (3.12), i.e. it is another way of writing problem ($\mathcal{DYN}_1$). Moreover, ($\mathcal{DYN}_\mathcal{H}$) is also linked to dynamical Problem ($\mathcal{DYN}_p$), as can be seen by repeating the same argument but with $H_2(t) = t^p$, for $p > 1$. Once motivated the name ($\mathcal{DYN}_\mathcal{H}$), we are ready to state the previously announced characterization of Wardrop equilibria using the dynamical variational problem ($\mathcal{DYN}_\mathcal{H}$), generalization in a continuous framework of Theorem 4.4. The full proof requires to take care of some regularity issues in details. However, the following result summarizes the work done by Carlier, Jimenez and Santambrogio in [17]. Thus, we refer to this article for the details and its complete proof.

THEOREM 4.5. *Under the assumptions on $\mathcal{H}$ and, thus, on H_2 and provided that the set $\mathcal{Q}^p(\mu^+, \mu^-)$ is non-empty, the problem ($\mathcal{DYN}_\mathcal{H}$) admits at least one minimizer. Moreover, $Q \in \mathcal{Q}^p(\mu^+, \mu^-)$ solves ($\mathcal{DYN}_\mathcal{H}$) if and only if Q is a Wardrop equilibrium and $\mu_Q := (e_0, e_1)_\# Q$ solves the optimization problem:*

$$\min_{\mu \in \Pi(\mu^+,\mu^-)} \int_{\Omega\times\Omega} d_{k_Q}(x,y) d\mu(x,y) \tag{4.31}$$

REMARK 4.3. Note that the second condition means that μ_Q solves a Monge-Kantorovich problem for a distance cost depending on Q itself, which is a new equilibrium condition. This second equilibrium condition depends on the fact that we choose a long-term problem and, thus, we have also to consider an optimality condition for μ, as explained for the discrete case inside Remark 4.2. Indeed, the minimization problem in (4.31) is exactly the continuous version of the the discrete minimization problem written in (4.22). It is also interesting to note that the last part of the previous theorem gives us a link between the extension of dynamical Problem ($\mathcal{DYN}_\mathcal{H}$) and Problem ($\mathcal{K}$), with a non-uniform cost given by $d_k(x, y)$. Thus, this result not only links Wardrop equilibria with solutions of variational Problem ($\mathcal{DYN}_\mathcal{H}$), but also, in some way, extends Theorem 3.5 for a wider classes of problems.

We are ready, now, to show the core theorem of this section which, as announced, gives us the correspondence between optimal values of minimization Problems ($\mathcal{B}_\mathcal{H}$) and ($\mathcal{DYN}_\mathcal{H}$), extending, in some way, Theorem 3.4. As in that theorem, we will start with inequality $\min(\mathcal{B}_\mathcal{H}) \le \min(\mathcal{DYN}_\mathcal{H})$ and, then, we will show that starting from an optimal w solution of ($\mathcal{B}_\mathcal{H}$), we are able to recover an optimal $Q_w \in \mathcal{Q}^p(\mu^+, \mu^-)$. Thus, we are able to show the equivalence of the two problems. We will consider, inside this theorem, just the case of H_2. However, we remind that for $H(t) = at^p$, which is the case of Problems ($\mathcal{DYN}_1$) and ($\mathcal{DYN}_p$), the conditions defining H_1 and H_2 are the same.

THEOREM 4.6. *Suppose that μ^+, μ^- are absolutely continuous measure with densities f^+, f^- and that the set $\mathcal{Q}^p(\mu^+,\mu^-)$ is non-empty. Then the minimal value of Problem $(\mathcal{B}_{\mathcal{H}})$ equals the minimal value of Problem $(\mathcal{DYN}_{\mathcal{H}})$, i.e.:*

$$\min(\mathcal{B}_{\mathcal{H}}) = \min(\mathcal{DYN}_{\mathcal{H}}). \tag{4.32}$$

Moreover, starting from an optimal $w \in L^p(\Omega;\mathbb{R}^n)$ solution of $(\mathcal{B}_{\mathcal{H}})$ it is possible to construct an optimal probability measure Q_w solving $(\mathcal{DYN}_{\mathcal{H}})$.

OUTLINE OF THE "PROOF". ($\leq$) Let us start considering an arbitrary $Q \in \mathcal{Q}^p(\mu^+,\mu^-)$. Note that, by the fact that Q belongs to this set, i_Q is an absolutely continuous measure and, thus, it follows that also w_Q is absolutely continuous. Just to clarify the notation, we get:

$$di_Q(x) = f_Q(x)dx, \quad dw_Q(x) = g_Q(x)dx,$$

for some $g_Q : \Omega \to \mathbb{R}^n$. Note, also, that using (4.26), we get that g_Q satisfies the divergence constraint in the weak sense and with Neumann boundary condition, i.e.

$$-\int_\Omega \nabla\phi \cdot g_Q(x)dx = \int_\Omega \phi\left(f^+ - f^-\right)dx, \qquad \forall \phi \in \mathcal{C}^1_c(\Omega).$$

Furthermore, recalling the inequality in (4.27), we get that:

$$\|g_Q(x)\| \leq f_Q(x), \quad \forall x \in \Omega. \tag{4.33}$$

Thus, we obtain also that $g_Q \in L^p(\Omega;\mathbb{R}^n)$, i.e. it satisfies all the conditions in order to be inside the feasible set of minimization Problem $(\mathcal{B}_{\mathcal{H}})$. We are ready, now, to show the first inequality, using (4.33) and the fact that H_2 is increasing, since its derivative g is always positive. Indeed, we get:

$$\int_\Omega \mathcal{H}(g_Q(x))dx = \int_\Omega H_2(\|g_Q(x)\|)dx \leq \int_\Omega H_2(f_Q(x))dx.$$

By the arbitrariness of $Q \in \mathcal{Q}^p(\mu^+,\mu^-)$, we get:

$$\min(\mathcal{B}_{\mathcal{H}}) \leq \min(\mathcal{DYN}_{\mathcal{H}}).$$

($\geq$) Conversely, given an optimal $w \in L^p(\Omega;\mathbb{R}^n)$ solution of $(\mathcal{B}_{\mathcal{H}})$, we want to construct a probability measure $Q_w \in \mathcal{Q}^p(\mu^+,\mu^-)$ that will be optimal for $(\mathcal{DYN}_{\mathcal{H}})$. To show the optimality of Q_w, the strategy is equivalent to show that:

$$f_{Q_w}(x) = \|w(x)\|, \quad \forall x \in \Omega, \tag{4.34}$$

with f_{Q_w}, as usual, the density of the absolutely continuous measure i_{Q_w}. Since

$$\int_\Omega H_2(f_{Q_w}(x))dx = \int_\Omega H_2(\|w(x)\|)dx = \int_\Omega \mathcal{H}(w(x))dx = \min(\mathcal{B}_{\mathcal{H}}) \leq \min(\mathcal{DYN}_{\mathcal{H}}),$$

we can automatically conclude the optimality of Q_w. The idea for the construction of this Q_w is to start defining a couple (ρ, v) as in (3.45), inside the proof of Theorem (3.4). We proceed, again, applying the deformation argument of Dacorogna-Moser, introduced in Theorem 2.6. Then, we define the probability measure Q_w using something similar to the displacement interpolation formula written in (4.12). In that case we obtained that the Q, involved in the displacement interpolation, was given by $Q = \int_\Omega \delta_{\overline{xT^*(x)}} d\mu^+(x)$. Now, instead, we do not consider the optimal T^*, but we define Q_w as the probability measure concentrated in the flow X of the ODE used inside Dacorogna-Moser deformation argument. Define:

$$\rho_t(x) := (1-t)\mu^+(x) + t\mu^-(x); \qquad \widehat{v_t}(x) := \widehat{v}(t,x) := \frac{w(x)}{\rho_t(x)}. \tag{4.35}$$

and consider the following Cauchy problem:

$$\begin{cases} X'(t,x) = \widehat{v_t}(X(t,x)), \\ X(0,x) = x. \end{cases} \tag{4.36}$$

We define the candidate probability measure as the disintegration $Q_w = \int_\Omega Q^x d\mu^+(x)$, where, for μ^+-a.e. x, Q^x is the Dirac delta concentrated on this curve X, that is:

$$Q^x = \delta_{X(\cdot,x)}. \tag{4.37}$$

Just to be more concrete, this means that Q_w is represented as:

$$\int_{\mathcal{X}} F(\gamma) dQ_w(\gamma) = \int_\Omega F(X(\cdot,x)) d\mu^+(x), \quad \forall F \in \mathcal{C}(\mathcal{X};\mathbb{R}).$$

Note that, so far, the proof has been rigorous and should not be in "quotations". The fact that requires the proof to be just an outline is that, here, we have to assume some regularity on the function $\widehat{v_t}$, in order to have uniqueness for Cauchy problem (4.36) and for the continuity equation. We assume, looking at the hypothesis of Theorem 4.1, that the function $\widehat{v_t}$ is at least Lipschitz. There are two possible ways to proceed further, both showing that Q_w is feasible for Problem $(\mathcal{DYN}_\mathcal{H})$, but from two different perspectives. In the first case, we can follow exactly the idea of the proof of Dacorogna-Moser Theorem 2.6, seeing the problem from a *Lagrangian* point of view. In the second case, we can use the uniqueness of the continuity equation and, thus, Theorem 4.1, following an *Eulerian* point of view.

(i) **Lagrangian**: As we did inside the proof of Dacorogna-Moser Theorem 2.6, we define the function h as in (2.67), i.e.:

$$h(t,x) = \det\nabla_x X(t,x) \left((1-t)f^+(X(t,x)) + tf^-(X(t,x))\right).$$

Proceeding exactly as in that proof, it is possible to show that $\frac{d}{dt}h(t,x) = 0$. This implies, in particular, that $h(1,x) = h(0,x)$, yielding:

$$f^+(x) = f^-(X(1,x))\det\nabla_x X(1,x).$$

Looking at the way in which we found the constraint in (2.10) and reasoning backwards, we obtain $\mu^- = (X(1,\cdot))_\# \mu^+$. Note that, this is exactly what we also did inside Remark 3.2. Moreover, we get:

$$\begin{aligned} &(e_0)_\# Q_w = \mu^+, \\ &(e_1)_\# Q_w = \int \delta_{X(1,x)} d\mu^+(x) = \int d\left((X(1,\cdot))_\# \mu^+\right) = \mu^-. \end{aligned} \tag{4.38}$$

This means that $Q_w \in \mathcal{Q}^1(\mu^+,\mu^-)$.

(ii) **Eulerian**: As we already stated, we suppose enough regularity on $\widehat{v_t}$ in order to apply uniqueness Theorem 4.1 for the continuity equation with initial datum μ^+, which, we recall, is:

$$\begin{cases} \partial_t \rho_t + \nabla_x \cdot (\widehat{v_t}\rho_t) = 0, \\ \rho_0 = \mu^+, \end{cases}$$

where $\rho_t(x) = \rho(t,x)$. We saw, in the uniqueness theorem recalled above, that the unique solution of this problem is given by $\rho_t = (X(t,\cdot))_\# \mu^+$. However, also the linear interpolation μ_t defined above satisfies the initial condition μ^+ and:

$$\partial_t \mu_t + \nabla_x \cdot (\widehat{v_t}\mu_t) = \mu^- - \mu^+ + \nabla_x \cdot \left(\frac{w}{\mu_t}\mu_t\right) = \mu^- - \mu^+ + \nabla_x \cdot w = 0.$$

Thus by the uniqueness of the Cauchy problem for the continuity equation, with initial datum μ^+, we have that:

$$\mu_t = (X(t,\cdot))_\# \mu^+ \tag{4.39}$$

Note, also, that this confirms the fact that $\mu^- = (X(1,\cdot))_\# \mu^+$ and, thus, (4.38).

We are ready, now, for the final part of the proof. We want to show the equality in (4.34), in order to conclude. To this aim, invoke the disintegration of Q_w and Fubini-Tonelli Theorem 1.2. Using these tools, we get:

$$\begin{aligned}\int_\Omega \phi(x) di_{Q_w}(x) &= \int_{\mathcal{X}}\int_0^1 \phi(\gamma(t)) \|\gamma'(t)\| \, dt dQ_w(\gamma) \quad \text{by Fubini-Tonelli}\\ &= \int_0^1 \int_{\mathcal{X}} \phi(\gamma(t)) \|\gamma'(t)\| \, dQ_w(\gamma) dt \quad \text{by def. of } Q_w \text{ and by (4.36)}\\ &= \int_0^1 \int_\Omega \int_{\mathcal{X}} \phi(\gamma(t)) \|\widehat{v}_t(\gamma(t))\| \, dQ_w^x(\gamma) d\mu^+(x) dt \quad \text{by (4.39)}\\ &= \int_0^1 \int_\Omega \phi(x) \|\widehat{v}_t(x)\| \, d\mu_t(x) dt \quad \text{by (4.35)}\\ &= \int_0^1 \int_\Omega \phi(x) \|w(x)\| \, dx dt.\end{aligned}$$

From the last calculation, noting that in the last term of the equality there is no dependence on time, we get:

$$\int_\Omega \phi(x) di_{Q_w}(x) = \int_\Omega \phi(x) \|w(x)\| \, dx, \forall \phi \in \mathcal{C}_0(\Omega; \mathbb{R}_+).$$

This clearly implies that i_{Q_w} is an absolutely continuous measure with density $f_{Q_w}(x) = \|w(x)\|$, and, thus, that $f_{Q_w} \in L^p(\Omega)$. Hence, this tells us that $Q_w \in \mathcal{Q}^p(\mu^+, \mu^-)$ and that (4.34) is verified, thus proving the theorem □

Remark 4.4. This theorem takes inspiration from a recent paper due to Brasco, Carlier and Santambrogio [13]. Note that it is possible to remove the quotation marks from the proof. Indeed, without supposing the Lipschitz regularity for the velocity vector field $\widehat{v}$, it is still possible to prove this theorem, following the same calculations and almost the same idea. The only thing we are not allowed to do without Lipschitz vector field $\widehat{v}$ is to conclude that $\mu_t = (X(t, \cdot)_{\#}\mu^+$. In the general case, the idea is not anymore to use uniqueness of the continuity equation, but the notion of *superposition principle*. This tool can be interpreted as a probabilistic version of the method of the characteristics and allows us to avoid using the uniqueness of the solution of the continuity equation. All the details about this general case can be found always in the just cited paper [13].

Just to say some final considerations, we showed this extension of the correspondence between the Beckmann Problem $(\mathcal{B}')$ and the dynamical Problem $(\mathcal{DYN}_1)$, in order to see that the techniques, used in the previous chapter to show the various relationships, are taken from a wider context. Indeed, the ideas of defining a scalar and a vector measures (Theorem 3.3) or the idea of invoking Dacorogna and Moser's Theorem 2.6 clearly took inspiration from the above proof.

3. Some simulations and numerical results

We want to present, below, some examples which will help us to understand, once more, the main differences between the optimal solutions of dynamical Problems $(\mathcal{DYN}_p)$ in the cases of $p > 1$ and $p = 1$. To do that, we present here some simulations describing the way in which these optimal densities ρ_t change in time. We want to present two different examples of initial and final data. In the first one, actually, we do not need any numerical solution, because we are able to recover the exact solutions, using the theoretical representations of the optimal densities in the two cases analyzed. This is due to the particular choice of the initial and final conditions. In the second example, instead, we need to use a numerical scheme, because we are not able to recover an exact solution, proceeding like in the first example. We want to include, also, some simulations and numerical results in order to show not only a list of theoretical results, but also the concrete way in which these results work.

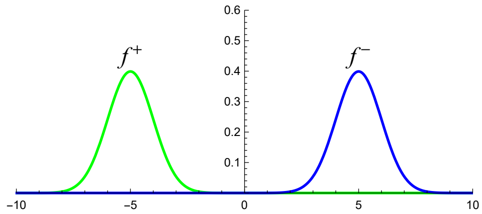
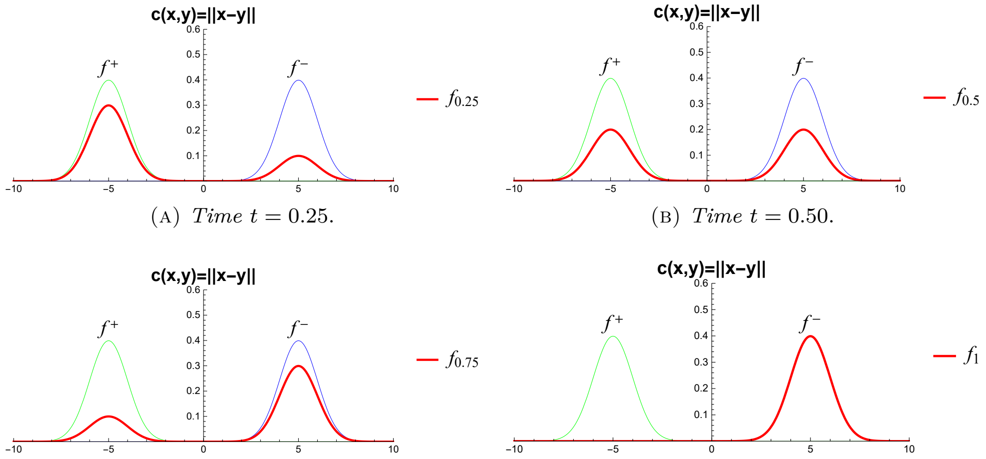


FIGURE 8. Initial and final data f^+ and f^-.



(A) *Time* $t = 0.25$.

(B) *Time* $t = 0.50$.

(C) *Time* $t = 0.75$.

(D) *Time* $t = 1$.

FIGURE 9. Time evolution of density (4.41) in the L^1 cost case.

3.1. Two Gaussians obtained one from the other through a translation. Consider the case of μ^+ and μ^- two measures with Gaussian densities:

$$f^+(x) = \frac{1}{\sqrt{2\pi}} \exp^{-\frac{(x+5)^2}{2}}, \quad f^-(x) = \frac{1}{\sqrt{2\pi}} \exp^{-\frac{(x-5)^2}{2}}, \tag{4.40}$$

i.e. with the densities of a standard normal distributed random variable, translated by -5 and $+5$, as shown in Figure 8.

In the L^1 cost case, we have shown that the optimal solution of the dynamical Problem $(\mathcal{DYN}_1)$ is simply the affine interpolation given by the density:

$$f_t(x) = (1-t)f^+(x) + tf^-(x), \quad \forall t \in [0,1]. \tag{4.41}$$

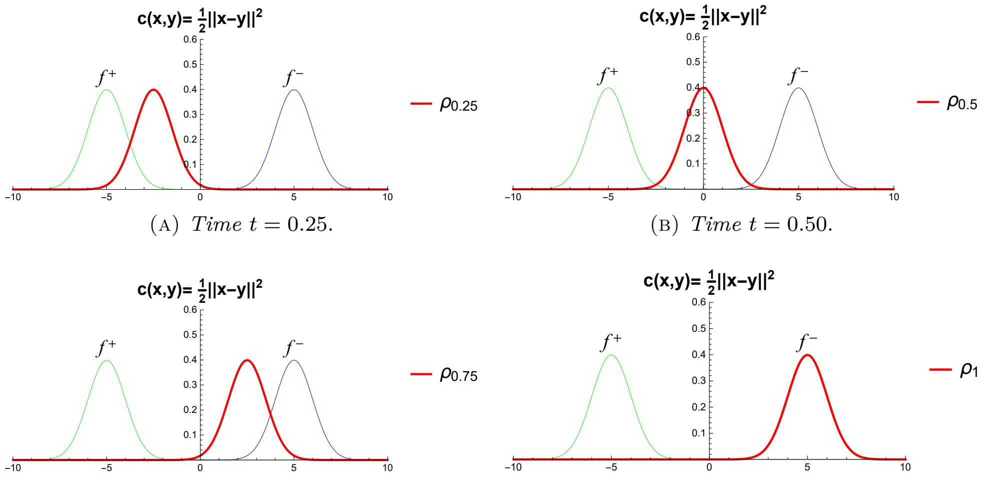
In Figure 9 we show the shape of f_t, as time increases. Note that, as already stated, the process can be thought as a *teleport* phenomenon, in the sense that the mass in initial datum f^+ disappears and reappears in f^-.

In the quadratic cost case, instead, the optimal solution of $(\mathcal{DYN}_p)$ with $p = 2$, is not anymore the affine interpolation, but it is given by the displacement interpolation formula (4.12). In this particular case, $f^-(x) = f^+(x-10)$, thus the final datum is obtained from the initial one through a translation. Here we want to show an example taken from [5], that considers the particular case $p = 2$ and a final density f^- obtained from an initial f^+ through a dilatation and a translation, i.e.:

$$f^-(x) = rf^+(r(x-c)), \tag{4.42}$$

for some $r > 0$ and some $c \in \mathbb{R}$. It can easily be verified that, in this particular situation, the optimal transport map T^* is given by:

$$T^*(x) = r^{-1}x + c. \tag{4.43}$$



(A) *Time* $t = 0.25$.

(B) *Time* $t = 0.50$.

(C) *Time* $t = 0.75$.

(D) *Time* $t = 1$.

FIGURE 10. Time evolution of density (4.44) in the L^2 cost case.

In our case, $r = 1$ and $c = 10$ yields $T^*(x) = x+10$. Thus, the density ρ solving displacement interpolation formula (4.12) is given by:

$$\rho_t(y) = f^+((1-t)\mathrm{Id} + tT^*)^{-1}(y)) = f^+(X_t^{-1}(y)).$$

Now, using the definition of T^*, it is easy to see that the inverse of the function above is given by $X_t^{-1}(y) = y - 10t$. Hence, the optimal density solving the dynamical formulation of optimal transport Problem $(\mathcal{DYN}_p)$ with $p = 2$ and these particular initial and final data is given by:

$$\rho_t(x) = f^+(x - 10t). \tag{4.44}$$

Figure 10 represents the evolution of this solution ρ_t. We can see that, contrary to the case $p = 1$, here there is a real transport, in which the initial state f^+ moves towards the final configuration f^-.

Now, a natural question is what happens in the intermediate case of $c(x, y) = \|x - y\|^p$ with $1 < p < 2$. One could think that, for p close enough to 1, the optimal density ρ_t solving the dynamical problem $(\mathcal{DYN}_p)$, could be something in between the cases of $p = 1$ and $p = 2$. Thus, we hypothesized that the optimal velocity v^*, solving Problem $(\mathcal{DYN}_p)$ together with ρ_t, could decrease a bit for small times and, then, could increase in order to have $\rho_1 = f^-$. We tried to solve this problem doing, again, a variational analysis. Thus, recalling what we did in Section 1 of Chapter 3, we start by defining the Lagrangian function of Problem $(\mathcal{DYN}_p)$ as:

$$\mathcal{L}_p(\rho, v) := \int_{\mathbb{R}^n}\int_0^1 \left[\frac{\|v(t,x)\|^p}{p}\rho(t,x) + \lambda(t,x)\left(\frac{\partial \rho}{\partial t} + \nabla_x \cdot (v\rho)\right)\right] dtdx,$$

defined over the space $\mathbb{P}(f^+, f^-) \times \mathcal{V}$. Here, $\lambda \in \mathcal{C}^1([0,1] \times \mathbb{R}^n)$ is a smooth Lagrange multiplier, $\mathbb{P}(f^+, f^-)$ is the space defined in (3.16) and $\mathcal{V}$ is simply the space of continuous velocity field v. Note that we put a $\frac{1}{p}$ before the norm of velocity v to simplify the computations, since the problem remains the same. Following step-by-step the idea used for the variational analysis of Problem $(\mathcal{DYN}_1)$, integrating by parts, we obtain the equivalent problem of finding the minimum of:

$$\int_{\mathbb{R}^n}\int_0^1 \left(\frac{\|v(t,x)\|^p}{p} - \frac{\partial \lambda}{\partial t}(t,x) - v(t,x)\cdot\nabla_x\lambda(t,x)\right)\rho(t,x)dtdx, \tag{4.45}$$

again over $\mathbb{P}(f^+, f^-) \times \mathcal{V}$. Going on as in that case, we fix $\rho \in \mathbb{P}(f^+, f^-)$ and we minimize along all $v \in \mathcal{V}$, using a two-step minimization procedure. Then, differentiating with respect to v the integrand of (4.45), we obtain a characterization of the optimal v, i.e.

$$v = \|v\|^{2-p} \nabla\lambda. \tag{4.46}$$

Defining q as the Hölder conjugate of p (eq. (1.3)) and taking the norm in both the left-hand side and the right-hand side in the previous equality, we obtain:

$$\|v\| = \|\nabla\lambda\|^{\frac{1}{p-1}} = \|\nabla\lambda\|^{q-1}.$$

Thus, putting it inside (4.46), we get:

$$v = \|\nabla\lambda\|^{\frac{2-p}{p-1}} \nabla\lambda = \|\nabla\lambda\|^{q-2} \nabla\lambda. \tag{4.47}$$

Looking at the integrand in (4.45), using this v just defined, we obtain a differential equation for λ, which is:

$$\frac{\partial\lambda}{\partial t} = -v \cdot \nabla\lambda + \frac{\|v\|^p}{p} = -\|\nabla\lambda\|^q + \frac{1}{p}\|\nabla\lambda\|^{(q-1)p} = -\frac{1}{q}\|\nabla\lambda\|^q. \tag{4.48}$$

This is the Hamilton-Jacobi equation with Hamiltonian $H(v) = \frac{\|v\|^q}{q}$ and it is a result analogous to what Benamou and Brenier obtained in [5], but with a general $p > 1$. Once we have that λ satisfies this equation and v is of the form of (4.47), we obtain that the integral in (4.45) is automatically zero for any $\rho \in \mathbb{P}(f^+, f^-)$. Hence, summarizing, we need to solve the following system of equations:

$$\begin{cases} \frac{\partial\lambda}{\partial t} + \frac{1}{q}\|\nabla\lambda\|^q = 0; \\ v = \|\nabla\lambda\|^{q-2}\nabla\lambda; \\ \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho v) = 0, \end{cases} \tag{4.49}$$

with some initial condition $\lambda(0, x) = \lambda_0(x)$ for the Hamilton-Jacobi equation and the standard initial condition $\rho(0, x) = f^+(x)$ for the continuity equation. Note that, as we already said in Section 1 of Chapter 3, a similar result for L^2 cost was shown in [18]. However, we took the idea of doing the variational analysis in order to solve Problem $(\mathcal{DYN}_p)$ also for $1 < p < 2$ from Delzanno and Finn [22]. In this work, they obtained a numerical scheme looking at system (4.49). In our case, instead, where we have that our initial data are just two Gaussians obtained one from the other through a translation, we are even luckier, because we are able to find an exact solution for optimal ρ given by system (4.49). Indeed, it is easy to understand that the only missing ingredient in order to solve the above system of equations is the initial datum $\lambda_0(x)$. Our idea is to find this λ_0 taking inspiration from the case of $p = 2$. In that case, we have that $\nabla\lambda(0, x) = v(0, x) = T^*(x) - x$, just using the definition of v and Equation 4.35. Using, now, the explicit solution of the optimal transport map, given in this case by $T^* = x + 10$, we obtain:

$$\lambda(0, x) = \lambda_0(x) = 10x. \tag{4.50}$$

Hence, in the case of general $1 < p < 2$, we do not have an explicit solution for T^* and, thus, for λ_0, but we can try to solve a similar problem, i.e.

$$\begin{cases} \frac{\partial\lambda}{\partial t} + \frac{1}{q}|\nabla\lambda|^q = 0, \\ \lambda(0, x) = cx, \end{cases} \tag{4.51}$$

with $c > 0$ that has to be found. This nonlinear PDE can be solved using the method of characteristics for nonlinear equations, which can be found in Evans [24], for example. If we consider a parametrization $x(t) \in \mathbb{R}$, $t \in [0, 1]$ and we call $y(t) = \frac{\partial\lambda}{\partial x}(t, x)$ and $z(t) = \lambda(t, x(t))$, the characteristics, in the case of Hamilton-Jacobi equations are defined by:

$$\begin{cases} \dot{y}(t) = -\frac{\partial}{\partial x}H(x(t), y(t)); \\ \dot{x}(t) = \frac{\partial}{\partial y}H(x(t), y(t)); \\ \dot{z}(t) = \frac{\partial}{\partial y}H(x(t), y(t)) \cdot y(t) - H(x(t), y(t)). \end{cases} \tag{4.52}$$

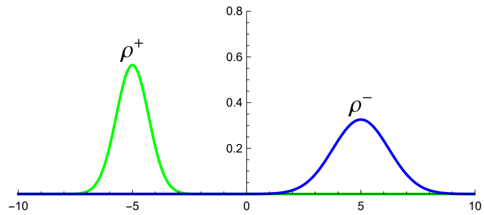


FIGURE 11. Initial and final data ρ^+ and ρ^-.

The last equation is simply obtained deriving $z(t) = \lambda(t, x(t))$ and using the other two characteristics equation and the definition of Hamilton-Jacobi equation. If we focus, now, on our specific case, in which the Hamiltonian is given by $H(x, y) = \frac{|y|^q}{q}$, we can notice that it only depends on y. Thus, we obtain:

$$\dot{y}(t) = 0 \implies y(t) = \frac{\partial \lambda}{\partial x}(0, x) = c.$$

Then, looking at the evolution in time of the variable x, we get:

$$\dot{x}(t) = \frac{\partial}{\partial y} H(y(t)) = |y|^{q-2}\, y = c^{q-1} \implies x(t) = x_0 + tc^{q-1}.$$

Thus, by inverting the last equality, we obtain $x_0 = x(t) - tc^{q-1}$. Noting, also, that the third characteristic equation is:

$$\dot{z}(t) = \frac{\partial}{\partial y} H(y(t)) \cdot y(t) - H(y(t)) = (1 - \frac{1}{q})c^q = \frac{1}{p}c^q,$$

we obtain an explicit representation for $\lambda(t, x)$, solving (4.51), given by:

$$\lambda(t, x) = z(0) + t\dot{z}(t) = c(x - tc^{q-1}) + t(\frac{1}{p}c^q) = cx + tc^q(-1 + \frac{1}{p}) = cx - \frac{t}{q}c^q. \tag{4.53}$$

Going on with the second equation of (4.49), we obtain the constant velocity $v(t, x) = c^{q-1}$. Inserting this v inside the third equation of (4.49), we obtain that we have to solve the transport equation:

$$\begin{cases} \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} = 0, \\ \rho(0, x) = f^+(x). \end{cases} \tag{4.54}$$

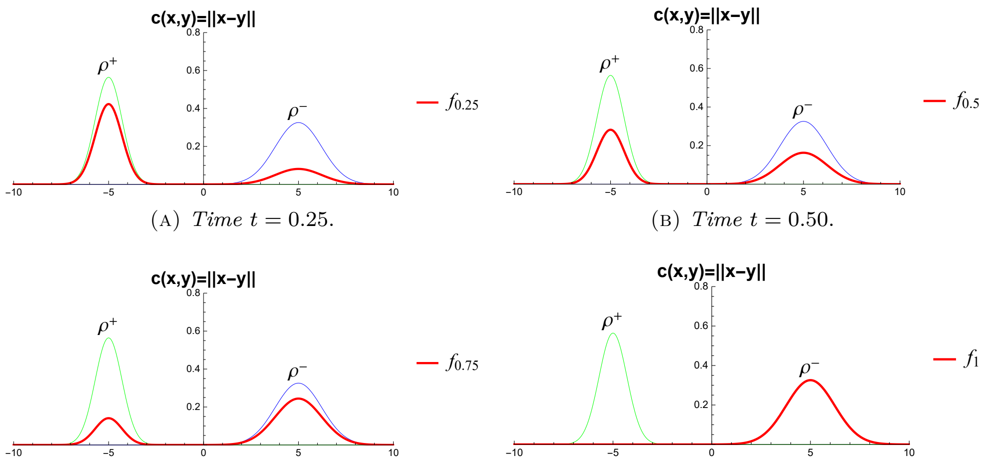
Using again the method of characteristics, this equation is solved by:

$$\rho(t, x) = f^+(x - c^{q-1}t), \tag{4.55}$$

as it can be checked, again, in Evans [24]. In order to find the right constant c, putting together the fact that ρ at time 1 has to be equal to f^- and that $f^-(x) = f^+(x - 10)$, we obtain:

$$\rho(1, x) = f^+(x - c^{q-1}) = f^+(x - 10) \implies \boxed{c = 10^{\frac{1}{q-1}}}.$$

Note that our initial guess was wrong, because optimal velocity v is constant and optimal density $\rho(t, x)$ given by (4.55), with the right c, is exactly equal to the optimal density solving (4.4) for $p = 2$, given by (4.44). Thus, for any $1 < p < 2$, even with p very close to 1, the time evolution of the optimal density ρ_t is equal to the one represented in Figure 10. Note, also, that this case was really peculiar because the initial and the final data where just a translation one from the other and the optimal velocity remained constant in time.



(A) *Time* $t = 0.25$.

(B) *Time* $t = 0.50$.

(C) *Time* $t = 0.75$.

(D) *Time* $t = 1$.

FIGURE 12. Time evolution of density (4.57) in the L^1 cost case.

3.2. Two Gaussians obtained one from the other through a translation and a dilatation. If we consider as initial and final data the densities of two normal distributions with different variance, the velocity is no more constant in time and we have some troubles to find an exact solution. Indeed, consider the following functions:

$$\rho^+(x) = \frac{1}{\sqrt{\pi}} \exp^{-(x+5)^2}, \quad \rho^-(x) = \frac{1}{\sqrt{3\pi}} \exp^{-\frac{(x-5)^2}{3}}, \tag{4.56}$$

which are the densities of two normal distributed random variables in $N(-5, \frac{1}{2})$ and in $N(5, \frac{3}{2})$, respectively, as can be seen in Figure 11.

In the case of L^1 cost function, it follows from what we have seen in Section 1 of Chapter 4 that the optimal density is again the affine interpolation between ρ^+ and ρ^-, given by:

$$f_t(x) = (1-t)\rho^+(x) + t\rho^-(x). \tag{4.57}$$

The dynamical evolution is presented in Figure 12 and is again a "teleport" phenomenon, similar to the one shown in Figure 9.

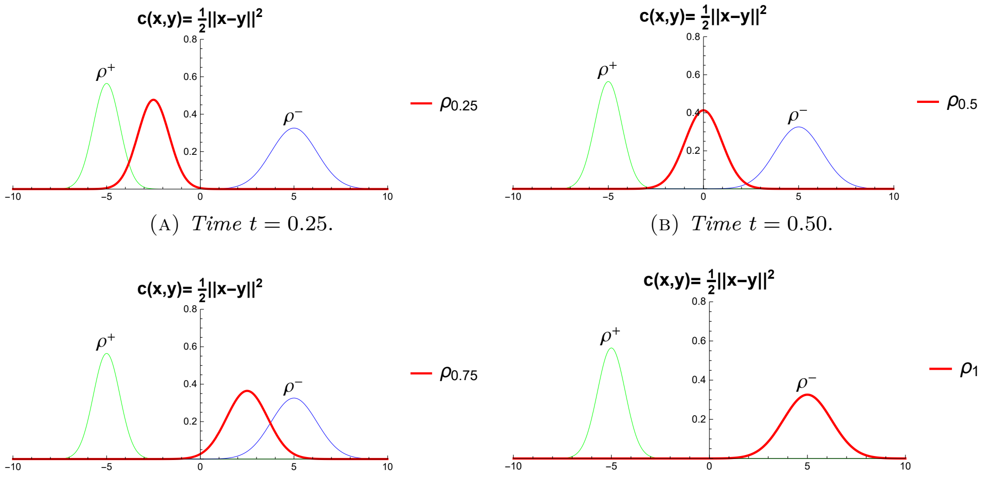
In the case of $(\mathcal{DYN}_2)$, on the other hand, the optimal density is represented by the displacement interpolation formula (4.12) and we are able to recover its explicit formulation, using, again, the same idea as in the previous example (Subsection 3.1). Indeed, note that, also in this case, the final density ρ^- is a dilatation and a translation of the initial density, as in Equation (4.42) with $r = \frac{1}{\sqrt{3}}$ and $c = 5(1+\sqrt{3})$. Thus, using again the representation of the optimal transport map given by (4.43), we have that:

$$T^*(x) = \sqrt{3}x + 5(1+\sqrt{3}).$$

With this optimal transport map, it can be verified that $X_t^{-1}(y) = ((1-t)\mathrm{Id} + tT^*)^{-1}(y) = \frac{y - t5(1+\sqrt{3})}{1-t(1-\sqrt{3})}$. Now, comparing the evaluation at time 1 of ρ_t, given by displacement interpolation formula (4.12), with this X_t^{-1}, and f^-, we notice that we have also to rescale everything by $\frac{1}{1+t(\sqrt{3}-1)}$. Hence, we obtain that the optimal density solving dynamical Problem $(\mathcal{DYN}_2)$ and initial and final data given by (4.56), is:

$$\rho_t(x) = \frac{1}{1+t(\sqrt{3}-1)} f^+\left(\frac{x - t5(1+\sqrt{3})}{1-t(1-\sqrt{3})}\right). \tag{4.58}$$

Note that, at time 1, we have $\rho(1,x) = \frac{1}{\sqrt{3}} f^+(\frac{x-5(1+\sqrt{3})}{\sqrt{3}}) = f^-(x)$, confirming that it is the right density. Its dynamical evolution in time is pictured in Figure 13.



(A) *Time* $t = 0.25$.

(B) *Time* $t = 0.50$.

(C) *Time* $t = 0.75$.

(D) *Time* $t = 1$.

FIGURE 13. Time evolution of density (4.58) in the L^2 cost case.

Using, again, the same strategy of the previous Subsection 3.1, in order to obtain optimal density ρ_t solving Problem $(\mathcal{DYN}_p)$, with $1 < p < 2$, we need to solve system of equations (4.49). The problem is, again, that we do not know initial datum $\lambda_0(x)$ for the Hamilton-Jacobi equation, thus we have to get some intuitions about its form from the case of $p = 2$. Following the same idea of the previous example, in the case of $p = 2$, we know that $\nabla\lambda(0, x) = v(0, x) = T^*(x) - x$. However, this time T^* is quite different and we obtain $\nabla\lambda(0, x) = (\sqrt{3}-1)x + 5(1+\sqrt{3})$. Taking the integral in the both sides of the last equality, we obtain that the initial datum of the Hamilton-Jacobi equation with $p = 2$ is:

$$\lambda(0, x) = \lambda_0(x) = (\sqrt{3} - 1)\frac{x^2}{2} + 5(1 + \sqrt{3})x. \tag{4.59}$$

Thus, if we want to proceed like in the previous example, in order to find an explicit solution of the Hamilton-Jacobi equation for a general $1 < p < 2$, we could consider, as initial datum, a general quadratic polynomial and solve the following PDE:

$$\begin{cases} \frac{\partial\lambda}{\partial t} + \frac{1}{q}\,|\nabla\lambda|^q = 0, \\ \lambda(0, x) = ax^2 + bx + c. \end{cases}$$

Using, again, the method of characteristics, we should solve the equations in (4.52), but, this time, we are not able to find an explicit solution. Indeed, $p(t)$ is again constant, because the Hamiltonian is still not depending on x, but, now, it has the form:

$$p(t) = p(0) = 2ax_0 + b.$$

This dependence on the initial datum x_0 is the real problem, because, this time, if we try to proceed as before, we must be able to invert $x(t, x_0)$ given by:

$$x(t, x_0) = x_0 + t\,|2ax_0 + b|^{q-2}\,(2ax_0 + b).$$

It is clear that this is not an easy task. Thus, we need to proceed numerically. We are going to use the numerical scheme described in [22]. The problem, clearly, is to find the right initial datum of Hamilton-Jacobi equation. It has to be chosen in such a way that the density solving (4.49), with the velocity inside continuity equation given by the solution of Hamilton-Jacobi equation starting from that initial datum, must be equal to the final density ρ^- defined in (4.56). In other words, we would like to find an initial condition of Hamilton-Jacobi equation λ_0 such that, if $\rho(t, x, \lambda_0)$ is the solution of (4.49), "starting" from λ_0, then function G, defined by:

$$G(x, \lambda_0) = \rho(1, \lambda_0, x) - \rho^-(x), \tag{4.60}$$

will be constantly equal to 0. The idea recast the original problem into the problem of finding a zero of G, which can be approximated by means of Newton method. Before explaining how to apply Newton method in this particular case, we need to find a robust discretization of the three equations in (4.49). First of all, note that we are in 1D and we can transform Hamilton-Jacobi equation in conservative form:

$$\begin{cases} \phi_t + H(\phi_x) = 0, \\ \phi(0, x) = \phi^0(x), \end{cases} \quad \overbrace{\Longrightarrow}^{u=\phi_x} \quad \begin{cases} u_t + H(u)_x, \\ u(0, x) = \frac{\partial \phi^0}{\partial x}(x). \end{cases} \tag{4.61}$$

Thus, we have that both Hamilton-Jacobi equation and continuity equation inside (4.49), if considered in 1D, are of the form:

$$\begin{cases} u_t + F(u)_x = 0, & \text{for } x \in [a, b],\ t \in [0, T], \\ u(0, x) = u_0(x), & \text{for } x \in [a, b], \end{cases} \tag{4.62}$$

for a right function F. The conservative form is fundamental to obtain a stable and accurate Finite-Volume based scheme. For all the details about this part we refer to Leveque [34]. The idea is to discretize our time and space intervals, in order to use a Finite Volume Method and solve in any subinterval equation (4.62). The basis of finite volume schemes is the Divergence Theorem, a well-known result which can be found in any standard analysis book. The scheme is the following:

(1) Take a partition of $[a, b]$ and $[0, T]$. For example, we will consider a uniform subdivison of space interval $[a, b]$ into N_x subintervals of dimension Δx and a uniform subdivision of time interval $[0, T]$ into N_t subintervals of dimension Δt. We denote by x_i, for $i = 1, \ldots, N_x$, the center of the i-th cell $[x_{i-1/2}, x_{i+1/2}]$, with right and left boundary points $x_{i+1/2}$ and $x_{i-1/2}$, respectively. Moreover, we denote by $[t^k, t^{k+1}]$, with $k = 1, \ldots, N_t$, the k-th time subinterval. Note that $\Delta x = \frac{(b-a)}{N_x}$ and $\Delta t = \frac{1}{N_t}$.

(2) Define a piecewise smooth function $\phi(x)$ given by the characteristic function of the i-th space subinterval and the k-th time subinterval:

$$\phi_{i,k}(x, t) = \chi_i(x)\chi_k(t), \text{ where } \chi_i(x) = \begin{cases} 1 & \text{if } x \in [x_{i-1/2}, x_{i+1/2}], \\ 0 & \text{otherwise;} \end{cases}$$
$$\chi_k(t) = \begin{cases} 1 & \text{if } t \in [t^k, t^{k+1}], \\ 0 & \text{otherwise.} \end{cases}$$

(3) In order to find a weak solution of Conservation Equation (4.62), annihilate the integral of the PDE inside this conservative equation multiplied by the previous test function, i.e.

$$\int_{t^k}^{t^{k+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} (u_t + F(u)_x) dx\ dt = 0, \quad \forall i = 1, \ldots, N_x,\ \forall k = 1, \ldots, N_t. \tag{4.63}$$

(4) Apply the Divergence Theorem and use Fubini-Tonelli Theorem 1.2 in order to exchange space and time integrals:

$$\int_{t^k}^{t^{k+1}} \left(\int_{x_{i-1/2}}^{x_{i+1/2}} u_t dx - \left[F(u)\big|_{x_{i+1/2}} - F(u)\big|_{x_{i-1/2}} \right] \right) dt$$
$$= \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{t^k}^{t^{k+1}} u_t dx\ dt - \int_{t^k}^{t^{k+1}} \left[F(u)\big|_{x_{i+1/2}} - F(u)\big|_{x_{i-1/2}} \right] dt.$$

(5) Define the cell average at time t^k, our unknown, as:

$$u^k_{h,i} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} u(t^k, x)dx. \tag{4.64}$$

Using a forward Euler scheme, i.e. using a simple "left-evaluation" (time t^k) quadrature rule for the time-integrals of the flux functions, we obtain the scheme:

$$\boxed{u^{k+1}_{h,i} = u^k_{h,i} - \frac{\Delta t}{\Delta x}\left[G^k_{h,i+1/2} - G^k_{h,i-1/2}\right]}, \tag{4.65}$$

where flux operator $G^k_{h,i}$ is defined as:

$$G^k_{h,i} = \frac{1}{\Delta t}F(u(t^k, x_i)). \tag{4.66}$$

Now, we just need to evaluate the fluxes $G^k_{h,i+1/2}$ and $G^k_{h,i-1/2}$. These are obtained solving a *Riemann problem* at each cell interface $\ell = i - 1/2$ and $\ell = i + 1/2$. This problem, whose theoretical details can be found again in Leveque [34], consists in solving the same problem (4.62), where the initial conditions are given by constant states u^L and u^R just on the left and on the right of x_ℓ. Since we are using an explicit time-stepping scheme, u^L and u^R are the numerical approximations of the cell averages at the previous time step t^k. Thus, the Riemann problems read as:

Problem ($\mathcal{R}$).

Find a solution u^* of the equation:

$$u_t + F(u)_x = 0, \quad x = x_\ell,$$

subject to the following initial condition:

$$u(x, 0) = \begin{cases} u^L = u^k_{h,\ell-1/2}, & \text{if } x < x_\ell \\ u^R = u^k_{h,\ell+1/2}, & \text{if } x > x_\ell \end{cases}$$

It can be shown that a weak solution of Riemann Problem ($\mathcal{R}$) is given by:

$$u^*(u^L, u^R) = \begin{cases} u^L, & \text{if } \theta > 0, \\ u^R, & \text{if } \theta < 0, \end{cases} \tag{4.67}$$

where $\theta = \frac{F(u^R) - F(u^L)}{u^R - u^L}$ is given by the **Rankine-Hugoniot** jump condition. The solution of the Riemann Problem ($\mathcal{R}$) can be used to evaluate the numerical flux. Note that the numerical flux can be written as a function of u^L and u^R, i.e., the left and right states of the local Riemann Problem. For example,

$$G^k_{h,i+1/2}(u^k_{h,L}, u^k_{h,R}) = G^k_{h,i+1/2}(u^k_{h,i}, u^k_{h,i+1}).$$

One way to obtain a numerical flux is with the standard Godunov Scheme, which is the one we will use in our work, where the flux is simply given by:

$$G^k_{h,i+1/2}(u^k_{h,i}, u^k_{h,i+1}) = F(u^*(u^k_{h,i}, u^k_{h,i+1})).$$

The resulting scheme displays first order convergence in both space and time, i.e., the norm of the error tends to zero linearly as Δt and Δx tend to zero:

$$\left\|u^k_{h,i} - u(x_i, t^k)\right\| \leq C_1\Delta t + C_2\Delta x, \tag{4.68}$$

with u exact solution of (4.62). Thus, before going on with Newton method, we want to show this result of convergence in the cases of $F(u) = \frac{u^2}{2}$, which is Hamilton-Jacobi equation with $q = 2$, and $F(u) = v$, which is the continuity equation with constant velocity. We will do this, because in these particular cases we have explicit solutions and because we want to test if we can trust our schemes.

- **Hamilton-Jacobi's equation with** $q = 2$ **or Burger's equation**: we want to test the thoretical convergence of our implementation of Godunov's scheme in the case in which initial and final data are given by (4.56), with $p = 2$. In this case, we know that the initial solution of Hamilton-Jacobi equation λ_0 is given by (4.59). Thus, using the method of characteristics, in the same way we tried to do in the case of a general $1 < p < 2$, we obtain three equations:

$$\begin{cases} y(t, x_0) = y(0) = (\sqrt{3} - 1)x_0 + 5(1 + \sqrt{3}), \\ x(t, x_0) = x_0 + t((\sqrt{3} - 1)x_0 + 5(1 + \sqrt{3})), \\ z(t, x_0) = \lambda_0(x_0) + \frac{t}{2}((\sqrt{3} - 1)x_0 + 5(1 + \sqrt{3})). \end{cases}$$

In this case, we are able to invert the second equation and, if we call $a = (\sqrt{3} - 1)$ and $c = 5(1 + \sqrt{3})$, we get:

$$x_0 = \frac{x - ct}{1 + at}.$$

Using this x_0 inside $z(t, x_0)$ and deriving with respect to x, in order to obtain a solution of the "conservative" form of the Hamilton-Jacobi equation, using what we observed in (4.61), we obtain the explicit solution:

$$\frac{\partial \lambda}{\partial x}(t, x) = \frac{c + 2ax}{1 + 2at}. \tag{4.69}$$

This is our exact solution which will be used inside (4.68) in order to test the convergence of our scheme, in this particular case. The numerical solution $u^k_{h,i}$, instead, is exactly given by (4.65), following the same scheme we stated before. Note that, in this case, θ given by the Rankine-Hugoniot condition is equal to $(u^L + u^R)/2$ and $F(x) = x^2/2$, as we already recalled. Moreover, note that we choose an L^2 norm for the error, which is calculated by evaluating in every cell the difference between $u^k_{h,i}$ and $(u(t^k, x_i) + u(t^k, x_{i+1}))/2$, because we know the approximation of the solution at $x_\ell = x_{i+1/2}$. Thus, the error is defined by:

$$e^k_s(N_x, N_t) = \left(\sum_{i=1}^{N_x} \left(u^k_{h,i} - \frac{u(t^k, x_i) + u(t^k, x_{i+1})}{2} \right)^2 \Delta x \right)^{\frac{1}{2}}. \tag{4.70}$$

We will evaluate this error always at the final time step $k = N_t$. Thus, we have all the ingredients in order to check the convergence of the solution of Hamilton-Jacobi equation with $p = 2$ with a Godunov scheme. The idea is to double, at every step, the time and space discretizations and to look at the new error: if halve the error, then we will conclude that our scheme satisfies (4.68) and that there is a first order convergence in both time and space. Table 2 shows that the errors decrease at the correct rate, thus providing the necessary confidence on the correctness of our implementation.

Step	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$
N_x	100	200	400	800	1600
N_t	150	300	600	1200	2400
$\mathbf{e_{s+1}/e_s}$	**0.5010**	**0.5005**	**0.5002**	**0.5001**	/

TABLE 2. Convergence rate of Godunov scheme solving Hamilton-Jacobi equation with $p = 2$.

- **Continuity equation with constant velocity or Transport equation**: we want to test the convergence for continuity equation with constant velocity $v = 10$ with initial data given by (4.40). Indeed, in this case it is easy to see that the solution of (4.54) is simply given, using the standard method of characteristics, by:

$$\rho(t,x) = f^+(x-10t).$$

Using this exact solution and arguing in the same way we did before, with θ given by Rankine-Hugoniot condition which, this time, is constantly equal to v and with the same L^2 error given by formula (4.70), we obtain the following Table 3.

Step	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$
N_x	100	200	400	800	1600
N_t	150	300	600	1200	2400
$\mathbf{e_{s+1}/e_s}$	**0.6346**	**0.5812**	**0.5455**	**0.5242**	/

TABLE 3. Convergence rate of Godunov scheme solving continuity equation with constant velocity v.

Now, we are confident enough in order to apply these schemes for solving the system of equations in (4.49) and use a Newton method to find the right initial condition of Hamilton-Jacobi equations. The precise algorithm to obtain the optimal density solving $(\mathcal{DYN}_p)$ for a general $1<p<2$, with initial and final data given by (4.56), is given by:

(1) Start with an initial guess $S_0 = \frac{\partial \lambda_0}{\partial x}(x)$, as we are solving Hamilton-Jacobi for the derivative. Note that, in the case of $p=2$, we have $\frac{\partial \lambda_0}{\partial x}(x) = (\sqrt{3}-1)x+5(1+\sqrt{3})$. Thus, with a general $1<p<2$, we can start with something similar.
(2) Call $r_0 = \frac{\|G(S_0)\|_2}{\|S_0\|_2}$, with G as in (4.60), but depending on S_0 (the derivative of the initial condition=. Fix the number of steps at $n=0$.
(3) If $\frac{\|G(S_n)\|_2}{\|S_n\|_2} < \delta$, with δ parameter fixed by the user in order to check the convergence, then STOP. Otherwise:
(4) Calculate the "Newton update" dS by:

$$J_n dS = -G(S_n),$$

where $J_n = \frac{\partial G}{\partial S}\big|_{S_n}$. In this case, it is not possible to find an explicit derivative of G. Thus, we will use the Gâteaux derivative:

$$\frac{\partial G}{\partial S}\Big|_{S_n} v = \lim_{\varepsilon\to 0} \frac{G(S_n+\varepsilon v) - G(S_n)}{\varepsilon},$$

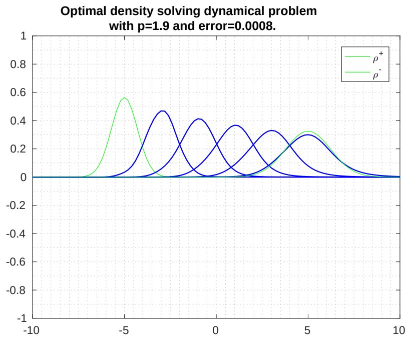
with ε small enough. Thus, the Jacobian matrix becomes:

$$(J_n)_{ij} = \frac{G_i(S_n+\varepsilon e_j) - G_i(S_n)}{\varepsilon}, \tag{4.71}$$

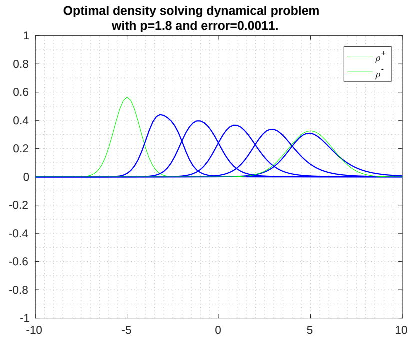
where ε is very small (in our case $\varepsilon = 10^{-15}$) and e_j is the standard vector with 1 in the j-th coordinate and 0's elsewhere.
(5) Once we find dS, update $S_{n+1} = S_n + \alpha dS$, with a proper $0 \le \alpha \le 1$. Update also $n = n+1$ and go back to (3).

With this Newton method we are able to find an optimal density solving the dynamical formulation $(\mathcal{DYN}_p)$, with $1<p<2$. The solutions starting from the initial datum of the Hamilton-Jacobi equation given by this Newton's method are represented in Figure 14, in the cases of $p = 1.9, 1.8, 1.7, 1.6$. Let us spend some words on the results obtained. In the first two cases of $p=1.9$ and $p=1.8$ we choose $N_x = 100$ and $N_t = 150$, while for the other two cases $N_x = 100$ and $N_t = 200$. We have to choose the number of time discretizations higher then the number of space discretizations, because of the *Courant–Friedrichs–Lewy* (CFL) condition. This is a necessary condition ensuring the stability of the numerical approximation. In general, it is given by:

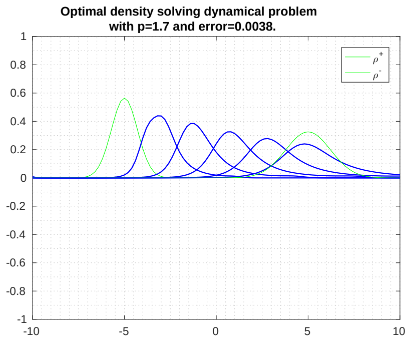
$$\frac{v\Delta t}{\Delta x} \le C_{\max}, \tag{4.72}$$



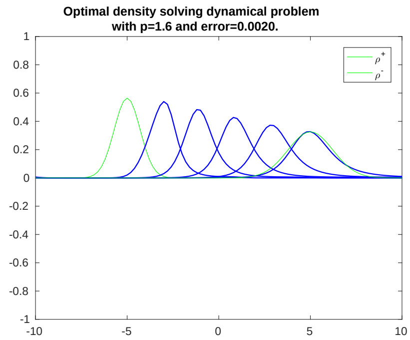
(A) $p = 1.9$



(B) $p = 1.8$



(C) $p = 1.7$



(D) $p = 1.6$

FIGURE 14. Time evolution the optimal densities found with the Newton's method, in the cases of $p = 1.9, 1.8, 1.7, 1.6$.

where $C_{\max}$, in the case of explicit time schemes, as our case, is usually equal to 1. This v represents the magnitude of the velocity, which in our case is given by $v = \|\nabla\lambda\|^{q-2}\nabla\lambda$. Thus, when p tends to 1, q increases and we have to provide the stability of the method increasing the number of time discretizations N_t. Note that there are some errors due to the method, but it seems that the optimal densities, with these p, behave like in the case of $p = 2$ (Figure 13). Unfortunately, we are not able to find the optimal density solving $(\mathcal{DYN}_p)$ with this method, for a p smaller. Indeed, there are two main reasons why we can not extend this method for that p:

(i) the already defined CFL number (eq. (4.72)), which forces us to excessively increase the number of time discretizations N_t, creating numerical diffusion;
(ii) Godunov scheme is a very diffusive method. Thus, to increase the precision of the method and to reduce the diffusion effect, it is possible to consider a second order Godunov scheme, in both time and space. For a higher order explicit time integration we can employ Runge-Kutta method. Obtaining second order in space needs, instead, higher order interpolation of the cell values, in order to obtain second order accurate left and right states for the Riemann solution. It is clearly out of the competences and the aims of this thesis.

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