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Abstract

The *Benchmark Approach* (BA) represents an alternative framework to quantitative finance that relies on the existence of a *Growth Optimal Portfolio* (GOP) to be used as *Numeraire* for financial modeling. When employed as a benchmark, the GOP makes any other non-negative portfolio either trendless or mean-decreasing. This property is known as *supermartingale property*, and it allows to exclude ex-ante some basic forms of arbitrage in the financial markets. Moreover, the GOP is constructed to maximize the long-term growth rate of the investment, and it delivers the best long-term performance when compared to any other strictly positive portfolio. These results are particularly interesting to apply the BA framework in the context of portfolio optimization and valuation of contingent claims. By introducing the *Diversification Theorem*, Platen and Heath (2006) show that *Diversified Portfolios* (DPs) converge to the Numeraire when composed by a sufficiently large number of constituents. In the present research, we build on this result and exploit naïve-diversification as a tool to construct valid proxies of the GOP. More specifically, we follow the methodology proposed by Platen and Rendek (2020) to construct a *Hierarchically Weighted Index* (HWI), a particular class of equally-weighted strategies that proved to be very robust to approximate the GOP. We evaluate the performance of different specifications of the HWI against the traditional *Equally Weighted Index* (EWI) and the *MSCI-World Index.* As a final result, we prove that the HWI approximates well the GOP, by showing robust statistical evidence that the supermartingale property of benchmarked returns cannot be easily rejected when our preferred HWI specification is used as a benchmark.

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1 Introduction

Modern mathematical finance relies extensively on the existence of an *Equivalent (Local) Martingale Measure* (ELMM) to exclude the possibility of arbitrage opportunities in the market. In particular, the existence of an ELMM is a sufficient condition to rule out all forms of arbitrage, as this is defined under the *No Free Lunch with Vanishing Risk* (NFLVR) theory[1]. This result has been crucial in the context of portfolio optimization and asset pricing theory.

However, several authors have shown that during periods of acute financial turbulence, some financial market anomalies can be observed, which are totally consistent with the classical no-arbitrage theory. This is for example the case of stock price bubbles. Indeed, in the presence of a bubble, discounted stock prices are not true martingales under the risk-neutral probability, with the consequence that classical results, like the put-call parity relationship, may break down. Another interesting example is provided by the dynamics observed in the interest rate market after the 2008 financial crisis, when markets started to face negative interest rates for the first time. In most recent years, we observed an increase in the frequency of unexpected and unusual macroeconomic and financial shocks. These developments raised questions on whether non-traditional frameworks that do not require the existence of an ELMM might be more appropriate for financial modeling.

Given the impossibility of relying on a risk-neutral probability measure when an ELMM fails to exist, some alternatives have been developed which allow to model financial quantities and exclude strong forms of arbitrage. A general class of diffusion-based models that does not rely on an ELMM has been studied by Fontana and Runggaldier (2013). In this work, the authors have proposed some weaker definitions of arbitrage, under which financial markets are still viable and the existence of a risk-neutral probability measure is not required for a fair valuation of contingent claims. This approach leaves room for some limited forms of arbitrage, partially explaining the existence of market anomalies. Financial markets are viable if strong forms of arbitrage can be excluded. This is the

1. See Delbaen and Schachermayer (1994), Delbaen and Schachermayer (2006).

case, even if an ELMM fails to exist, when financial markets are complete, that is, if any payoff can be replicated starting from some initial investment.

In this research we propose a modeling framework based on the *Benchmark Approach* (BA), which has proven to work even when a risk-neutral probability measure does not exist. The BA enlarges the possibilities for modeling financial quantities also when the classical approaches seem to break down. The main pillar of this framework is the Growth Optimal Portfolio (GOP), that takes the role of Numeraire and represents the portfolio that, when used as benchmark, forces any other portfolio to evolve as local-martingale. The GOP is closely linked to the concept of *Martingale Deflator*, the weaker counterpart of traditional martingale measures (Fontana and Runggaldier 2013). This relationship establishes a direct link between the concept of Numeraire and the no-arbitrage criteria.

The BA proved to be suitable and very flexible for a range of applications in the quantitative finance field. For instance, this is the case of portfolio optimization problems, which can be meaningfully solved if financial markets are viable. In addition, the GOP provides a convenient way to identify systemic risk, becoming relevant for regulatory purposes. Last but not least, another interesting area of application is the valuation of contingent claims. Indeed, the BA allows to model prices under the real world probability, opening the road to a range of applications that could not be explored otherwise.

The Benchmark Approach has been studied in a consistent stream of literature. The monograph of Platen and Heath (2006) collects a wide range of notions on the BA and provides some guidance on how the main concepts can be linked together. However, the work lacks of some of the latest (and possibly interesting) developments in the field. In order to close this gap, the first objective of this research is to complement the material of the authors, and reorganize some concepts into a unified, concise and updated framework.

Platen and Heath (2006) have discussed some interesting theoretical results that allow to link *Diversified Portfolios* (DPs) to the GOP. Platen and Rendek (2012a) and Platen and Rendek (2020) represent two successful attempts of approximating empirically the GOP via well diversified strategies. The second objective of this research is inspired by their results. To prove that the Benchmark Approach can be robust even in the presence of market anomalies, we challenge their methodology to verify if their findings hold when using updated data. We confirm the results of the authors by showing that the GOP can be efficiently approximated by mean of well diversified portfolios. In particular, a Hierarchically Weighted Index (HWI) proves to be suitable for this purpose. Indeed, the economic structure of the market provides sufficient information to disentangle systematic and idiosyncratic risk, and to efficiently get rid of the latter via diversification. This result is particularly interesting because it is model independent. Avoiding the need of estimating model parameters (or other complex factors) is not only beneficial in terms of mathematical modeling but also allows to achieve a better portfolio performance. The approximation methodology proposed by Platen and Rendek (2012a) and Platen and Rendek (2020) to construct proxies of the GOP is robust and allows to preserve GOP properties even when data reflects periods of unexpected and unusual financial turmoil and macroeconomic turbulence, like the Covid-19 crisis of 2020. The local martingale property is verified by mean of robust statistical tests, which show that is not possible to easily reject the hypothesis of zero expected instantaneous returns, when these are benchmarked by the GOP.

The present research is structured as follows. Chapter 2 provides a detailed overview of the Benchmark Approach. We introduce here the concept of Growth Optimal Portfolio and its role as Numeraire, following what has been proposed by Long (1990). We provide the theoretical framework to approximate the GOP by mean of Diversified Portfolios and we analyze some interesting applications highlighting some of the benefits achievable with this approach, compared to classical finance theories. Chapter 3 illustrates some portfolios that are suitable to approximate the GOP and provides some methodological guidance to implement these strategies using real market data. We present the dataset employed for the analyses and we discuss some preliminary results by looking at some descriptive statistics. Finally, in Chapter 4 we assess comparatively the performance of the portfolios constructed, by calculating some common metrics in the field of fund management. Furthermore, we provide evidence that the local martingale property cannot be easily rejected for benchmarked returns when our preferred GOP proxy is used as a benchmark.

2 Financial modeling under the Benchmark Approach

In this chapter the *Benchmark Approach* (BA) is presented, along with the set of mathematical structures and technical assumptions that constitute its building blocks. It represents a unified modeling framework for continuous financial markets. The material illustrated is mainly based on Platen and Heath (2006) and augmented with a consistent stream of literature, in an attempt of collecting and re-organizing a range of relatively unexplored concepts that do not fit under the currently prevailing approaches in the quantitative finance field.

In Section 2.1 the general market settings to model financial markets under the BA is presented. A broad range of notation, relationships and definitions of central importance have been included to support the discussion of subsequent concepts. The Growth Optimal Portfolio (GOP) is introduced and explicitly derived in a continuous time framework in Section 2.2. The GOP is a portfolio constructed to maximize the long-term growth rate of the investment, and delivers the best performance when compared to any other strictly positive portfolio. Thanks to these and other properties that are presented later, the GOP constitutes the main pillar of the BA, taking the role of Numeraire. Section 2.3 collects some of the most interesting applications of the Benchmark Approach in quantitative finance, highlighting the advantages offered by this framework in terms of mathematical modeling in the context of portfolio selection, risk measurement and asset pricing. Finally, in Section 2.4 the *Diversification Theorem* is derived. Under some regularity conditions, it allows to bridge between abstract mathematical modeling and real world implementation of the BA, and represents a simple and convenient tool to construct proxies the GOP.

2.1 Market setting and general framework

To illustrate the main elements of the market setting under the Benchmark Approach, we follow Platen and Heath (2006). The financial market is modeled in continuous time, by relying on a filtered probability space $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$, where $\underline{\mathcal{A}} = (\mathcal{A}_t)_{t\in[0.\infty)}$ represents the set of information available at market participants, and $\mathcal{A}_t$ the information available at a specific point in time $t \in [0, \infty)$. Trading uncertainty is modeled as a set of independent standard $(\underline{\mathcal{A}}, P)$-Wiener processes $W^k = \{W_t^k,\ t \in [0, \infty)\}$, for $k \in \{1, 2, \ldots, d\}$ and $d \in \mathbb{N}$.

Primary Security Accounts

We consider a market composed by $d + 1$ primary security accounts. These include d non-negative risky primary security account processes $S^j = \{S_t^j,\ t \in [0, \infty)\}$ – with $j \in \{1, 2, \ldots, d\}$ being a single type of security (*i.e.* stocks, bonds, foreign savings accounts, derivatives, indexes, etc.) – where all proceeds are compounded, and a locally riskless primary security account $S^0 = \{S_t^0,\ t \in [0, \infty)\}$, the savings account.
At time $t \in [0, \infty)$, the value of the jth risky asset satisfies the SDE:

$$dS_t^j = S_t^j \left(a_t^j \ dt + \sum_{k=1}^{d} b_t^{j,k} \ dW_t^k \right) \tag{2.1.1}$$

where $b^{j,k} = \{b_t^{j,k},\ t \in [0, \infty)\}$ is the *volatility* of the jth primary security account with respect to the kth Wiener process W^k, and $a^j = \{a_t^j,\ t \in [0, \infty)\}$ is the *appreciation rate*.
At time t the value of the risk-free account is given by:

$$S_t^0 = \exp\left\{ \int_0^t r_s \ ds \right\} < \infty \tag{2.1.2}$$

where $r = \{r_t,\ t \in [0, \infty)\}$ is the short-term risk-free rate.

Market Price of Risk

The number of Wiener processes considered for the modeling will be exactly the same as the number of risky primary security accounts. In this way we are implicitly assuming that markets are complete and avoiding the possibility of including redundant primary security accounts. To simplify the exposition, we express the previous processes in vector notation. $\boldsymbol{S} = \{\boldsymbol{S}_t = (S_t^0, S_t^1, \ldots, S_t^d)^\top,\ t \in [0, \infty)\}$ is the vector of primary security account processes, $\boldsymbol{a} = \{\boldsymbol{a}_t = (a_t^1, \ldots, a_t^d)^\top,\ t \in [0, \infty)\}$ the vector of appreciation rate processes, $\boldsymbol{b} = \{\boldsymbol{b}_t,\ t \in [0, \infty)\}$ the volatility matrix process, and $\boldsymbol{r} = \{\boldsymbol{r}_t = (r_t, r_t^1, \ldots, r_t^d)^\top,\ t \in [0, \infty)\}$ a vector that includes the short-term rate process r_t and d dividend rates processes r_t^j, corresponding to each primary risky account.

Definition 2.1.1. *The market $\mathcal{S}^C_{(d)} = (\boldsymbol{S}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{r}, \underline{\mathcal{A}}, P)$ is defined as a continuous financial market (CFM), if the volatility matrix $\boldsymbol{b}_t = [b_t^{j,k}]_{j,k=1}^d$ is invertible for Lebesgue-almost every $t \in [0,\infty)$, with inverse matrix $\boldsymbol{b}_t^{-1} = [b_t^{-1 j,k}]_{j,k=1}^d$.*

The assumption on the invertibility of the volatility matrix $\boldsymbol{b}_t$ is necessary to guarantee that markets are complete.

We denote the kth *market price of risk* with respect to the kth Wiener process as

$$\theta_t^k = \sum_{j=1}^d b_t^{-1\ j,k} \left(a_t^j - r_t \right) \tag{2.1.3}$$

and using Equation (2.1.3) to isolate the market price of risk, we can re-write SDE (2.1.1) to obtain

$$dS_t^j = S_t^j \left(r_t \ dt + \sum_{k=1}^d b_t^{j,k} \ (\theta_t^k \ dt + dW_t^k) \right) \tag{2.1.4}$$

Self-financing Portfolios

A *strategy* is defined as a predictable stochastic process $\boldsymbol{\delta} = \{\boldsymbol{\delta}_t = (\delta_t^0, \delta_t^1, \dots, \delta_t^d)^\top, \ t \in [0,\infty)\}$, where $\delta_t^j, \ j \in \{0,1,\dots,d\}$ is the number of units of the jth primary security account that are held a time $t \in [0,\infty)$ in the corresponding portfolio $S^\delta = \{S_t^\delta, \ t \in [0,\infty)\}$. At each point in time, the value of the entire investment portfolio is represented by

$$S_t^\delta = \sum_{j=0}^d \delta_t^j \ S_t^j \tag{2.1.5}$$

For the rest of the analysis we will assume that all strategies and portfolios are *self-financing*. A portfolio is said to be self-financing if all the cash flows (capital gains and dividends) generated by holding the portfolio from time t to time $t+h$, where $t \in [0,\infty)$ and $h > 0$, are reinvested (or absorbed if negative). More formally, we will consider the strategy δ and the corresponding portfolio S^δ to be self-financing if

$$dS_t^\delta = \sum_{j=0}^d \delta_t^j \ dS_t^j \tag{2.1.6}$$

for $t \in [0,\infty)$.

From (2.1.6) and SDE (2.1.4) we derive the following SDE of a self-financing portfolio

$$dS_t^\delta = S_t^\delta \; r_t \; dt + \sum_{k=1}^{d} \sum_{j=0}^{d} \delta_t^j \; S_t^j \; b_t^{j,k} (\theta_t^k \; dt + dW_t^k) \tag{2.1.7}$$

for $t \in [0, \infty)$.

It is important to note that the value of the portfolio can become zero or negative, as no restrictions are technically imposed. In what follows, we will consider only a subset of strictly positive portfolios that we denote by $\mathcal{V}^+$.

Fractions

For convenience it is useful to express the proportion of wealth invested in each primary security account in relative terms, instead of units. We define the *fraction* $\pi_{\delta,t}^j$ invested in the the jth risky account S_t^j, $j \in \{0, 1, \ldots, d\}$ as

$$\pi_{\delta,t}^j = \delta_t^j \; \frac{S_t^j}{S_t^\delta} \tag{2.1.8}$$

Fractions can be negative, but by definition will always sum up to 1.

By Equation (2.1.8) and the SDE specified in (2.1.7) we obtain

$$dS_t^\delta = S_t^\delta \left(r_t \; dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi_{\delta,t}^j \; b_t^{j,k} \; (\theta_t^k \; dt + dW_t^k) \right) \tag{2.1.9}$$

for $t \in [0, \infty)$.

2.2 Growth Optimal Portfolio (GOP)

The Growth Optimal Portfolio (GOP) has been discovered by Kelly (1956) in a research motivated by questions arising in the field of information theory, where the author derives an optimal gambling strategy that allows to accumulate more wealth than any other strategy. More precisely, a gambler can use the knowledge received from a communication channel – where the states of the channel are binary and determine the probability of winning – to cause its money to grow exponentially. Assuming that the gambler has the possibility of reinvesting the profits generated, and that he can freely vary the amount of money bet at every round, then his optimal strategy would be determining the value of the next bet such that the expected value of the logarithm of his capital is maximized.

This results is drawn from a real life situation, but has been applied widely to analyze other economic phenomena. The information channel can be interpreted as the quantity of inside information available to investors at time t, and the result of the gamble as the unknown future performance of an investment in a risky financial asset.

Latane (1959) uses a similar set up to investigate which is the optimal criteria to make a choice among multiple risky ventures. He concluded that the optimal investment allocation is always the combination of assets that allows to maximize the expected growth rate of the initial capital. This result is completely independent from any consideration on investor's utility function. Since the logarithm is additive in repeated choices (bets), by the law of large numbers, investor's (or gambler's) capital will surpass the capital of any other investor adopting a different strategy, with probability one.

Thorp (1969) in a study related to optimal gambling systems highlights some deficiencies of the portfolio selection theory introduced earlier by Markowitz (1952). The author supports with empirical evidence the idea that the Kelly criterion based on the expected logarithmic utility is superior to the one of Markowitz which is based on first and second moments only. Only later, the author formalized his own findings in a detailed investigation on the benefits of using the $E(\log X)$ as a criterion to guide investment decisions (Thorp 1975).

The aforementioned contributions are just a smaller part of a consistent stream of literature[2] that leveraged on the results of Kelly (1956), to advance in the fields of gambling, portfolio optimization and derivative pricing, between the '60s and the '70s.

Of outstanding importance is the work of Long (1990), that represents the first attempt to formalize some important properties of the Kelly criterion by looking at the GOP as a *Numeraire*. In the discussion that follows, we provide a formal definition of the GOP and we analyze its role as Numeraire. We also model the GOP in a continuous financial market (CFM) setting, to show that it can be considered as the best performing portfolio under multiple perspectives.

2.2.1 Definition and properties

Long (1990) identifies the following as the three main characteristics of a Numeraire:

(1) It is self-financing. At any time, the portfolio can be rebalanced at zero-cost, with no need to invest extra capital, nor to withdraw money.

2. See as additional examples Breiman et al. (1961), Hakansson (1971), Markowitz (1976).

(2) *Its value is always positive.* This assumption guarantees that there is always at least one portfolio that can be used as denominator to derive benchmarked returns.

(3) *Zero is always the best conditional forecast of numeraire-denominated rate of return of every asset in the market.* This feature is closely linked to the absence of profit (arbitrage) opportunities in the market. As shown by Long (1990), the existence of the Numeraire portfolio is inconsistent with the presence of such opportunities[3,4]. Absence of arbitrage is a crucial condition for financial modeling in any context. This property of the Numeraire represents the main pillar on which the Benchmark Approach is based, and represents the reason why this framework is suitable for several applications (*e.g.*, for pricing derivatives without relying on risk-neutral probability measure, see Section 2.3.3).

For a given list of assets, the Numeraire portfolio coincides with the portfolio that would be chosen by a log utility investor if his trading were restricted to that list (Long 1990). Following Kelly (1956), this portfolio guarantees the maximum capital growth, outperforming any other self-financing strategy constructed from that list. When the list includes all assets traded in the market, the Numeraire portfolio correspond to the Growth Optimal Portfolio. For the rest of the discussion we will use the concept of Numeraire and GOP interchangeably.

The notion of Numeraire portfolio can be related to several concepts commonly investigated in the quantitative finance field. For instance, in the derivative pricing context, if a claim's payoff can be replicated by a combination of securities and a risk-free asset, then it exists an equilibrium relation between the price of the claim, the price of the the securities and the interest rate, that holds if all investors are assumed to be risk neutral. When pricing the claim, one has to substitute the actual probability with a so-called risk neutral probability measure, to artificially recreate the aforementioned equilibrium and ensure absence of arbitrage opportunities. The power of the GOP is that, when it replaces the savings account as a discount factor, profits opportunities are already eliminated by definition. Discounted returns behave as local-martingales, and derivatives can be priced under a real world probability measure. More details are provided in Section 2.3.3.

3. Following Ingersoll (1987), a profit opportunity arises if an investor can get something for nothing, either by buying a claim to a non-negative future payoff that will be positive with a positive probability, for a zero or negative current price (first type opportunities), or by selling a claim to a certain zero payoff for a positive current price (second type opportunities).

4. Fontana and Runggaldier (2013) study some definitions of arbitrage which are particularly interesting in the context of the Benchmark Approach.

Another implication of the properties discussed at the beginning of this Section, with further details provided in Section 2.3.2, is that returns on the GOP coincide with returns on the market portfolio. As a consequence, Numeraire denominated returns can be intended as a measure of asset-specific returns (or abnormal returns), in the same sense as market-model residuals. However, the GOP is a superior benchmark when compared to market-models, since its construction (see Section 2.4) does not require the estimation of parameters that can potentially introduce noise in the calculations. Some interesting examples related to the role of the Numeraire portfolio under the Benchmark Approach are collected in Section 2.3.

2.2.2 Continuous time setting

For the generalization of the GOP properties in a continuous time setting, Platen and Heath (2006) consider a continuous financial market $\mathcal{S}^C_{(d)}$ and a strictly positive portfolio $S^\delta \in \mathcal{V}^+$. By application of the Itô formula to (2.1.9) we obtain the SDE for $\ln(S^\delta_t)$ in the form:

$$d\ln(S^\delta_t) = g^\delta_t \; dt + \sum_{k=1}^{d} \sum_{j=1}^{d} \pi^j_{\delta,t} \; b^{j,k}_t \; dW^k_t \tag{2.2.1}$$

with *growth rate* g^δ_t defined as:

$$g^\delta_t = r_t + \sum_{k=1}^{d} \left(\sum_{j=1}^{d} \pi^j_{\delta,t} \; b^{j,k}_t \; \theta^k_t - \frac{1}{2} \left(\sum_{j=1}^{d} \pi^j_{\delta,t} \; b^{j,k}_t \right)^2 \right) \tag{2.2.2}$$

Definition 2.2.1. *In a CFM $\mathcal{S}^C_{(d)}$ a strictly positive portfolio process $S^{\delta*} = \{S^{\delta*},\ t \in [0,\infty)\} \in \mathcal{V}^+$ is called a GOP if for all $t \in [0,\infty)$ and all strictly positive portfolios $S^\delta \in \mathcal{V}^+$, its growth rate satisfies*

$$g^{\delta*}_t \geq g^\delta_t \tag{2.2.3}$$

almost surely.

To derive the GOP analytically, one has to compute the optimal fraction of wealth $\pi^j_{\delta_*,t}$ to be invested in each risky primary security account in the investment universe. The first order condition, derived from equation (2.2.2), with respect to $\pi^j_{\delta,t}$, is given by:

$$0 = \sum_{k=1}^{d} b^{j,k}_t \left(\theta^k_t - \sum_{\ell=1}^{d} \pi^\ell_{\delta,t} \; b^{\ell,k}_t \right) \tag{2.2.4}$$

for all $t \in [0,\infty)$ and $j \in \{1, 2, \ldots, d\}$.

By solving equation (2.2.4), the jth optimal fraction satisfy the equation:

$$\pi^j_{\delta_*,t} = \sum_{k=1}^{d} \theta^k_t \; b_t^{-1\;j,k} \tag{2.2.5}$$

for $j \in \{1, 2, \ldots, d\}$.

Alternatively, by mean of Equation (2.1.3), and using vector notation:

$$\boldsymbol{\pi}_{\delta_*,t} = (\boldsymbol{b}_t^{-1})^\top \boldsymbol{\theta}_t = (\boldsymbol{b}_t^{-1})^\top \boldsymbol{b}_t^{-1}(\boldsymbol{a}_t - r_t \mathbf{1}) \tag{2.2.6}$$

By substituting the optimal fractions into the SDE (2.1.9) it is possible to show that the GOP $S^{\delta_*}_t$ satisfies the SDE:

$$dS^{\delta_*}_t = S^{\delta_*}_t \left(r_t \; dt + \sum_{k=1}^{d} \theta^k_t \left(\theta^k_t \; dt + dW^k_t \right) \right) \tag{2.2.7}$$

for $t \in [0, \infty)$.

Filipovic and Platen (2009) report some details on this optimization problem in a generalized framework. They identify some necessary conditions such that a solution to SDE (2.2.7) exists. When these conditions hold, the GOP exists and its value process is uniquely determined for some fixed initial value $S^{\delta_*}_0 > 0$, albeit the GOP strategy δ_* may not be unique.

However, the market setting proposed in the present research relies on the notion of CFM (see Definition 2.1.1), that implies that financial markets are complete by imposing a condition on the invertibility of the volatility matrix. Fontana and Runggaldier (2013) show that when markets are complete and strong forms of arbitrage are excluded, the GOP and its fractions are both uniquely determined. Furthermore, for the approximation of the GOP (see Section 2.4), we rely on a specific and well defined set of risky assets. This guarantees ex-ante the uniqueness of the optimal fractions.

2.2.3 Best performing portfolio

A detailed discussion on the GOP is key for a deep understanding of the Benchmark Approach framework. One of the most relevant properties of the GOP, that makes it versatile to pursue several objectives, is that, by construction, it is the best performing portfolio. To prove this result, we will leverage on the notion of benchmarked non-negative portfolios and on the so-called supermartingale property of benchmarked returns.

We denote the value of a portfolio S_t^δ when benchmarked by the GOP as

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}} \tag{2.2.8}$$

for $t \in [0, \infty)$.

By application of the Itô formula to the SDE (2.1.9) and (2.2.7), one can obtain the SDE of the benchmarked portfolio

$$d\hat{S}_t^\delta = \sum_{j=0}^{d} \delta_t^j \ \hat{S}_t^j \sum_{k=1}^{d} \left(b_t^{j,k} - \theta_t^k\right) \, dW_t^k \tag{2.2.9}$$

for $t \in [0, \infty)$.

By observing that the SDE (2.2.9) is driftless, and following some properties of the Itô integral with respect to Wiener processes[5], one can conclude that any benchmarked portfolio evolve as an $(\underline{\mathcal{A}}, P)$-local martingale. In addition, any non-negative $(\underline{\mathcal{A}}, P)$-local martingale is also an $(\underline{\mathcal{A}}, P)$-supermartingale (see Appendix A.2 for more technical details). We can summarize these notions and derive the *supermartingale property of benchmarked portfolio.*

Proposition 2.2.2. *In a CFM $\mathcal{S}_{(d)}^C$ the benchmarked value $\hat{S}^\delta$ of any non-negative portfolio $S^\delta \in \mathcal{V}$ is an $(\underline{\mathcal{A}}, P)$-supermartingale.*

Arbitrage in a CFM

For a financial market model to be valid, some basic form of arbitrage must be excluded. Platen (2002) provides the following definition of arbitrage opportunities.

Definition 2.2.3. *A non-negative portfolio $S^\delta \in \mathcal{V}$ is considered an arbitrage if its starting value S_0^δ is zero, and its value at a later bounded stopping time $\tau \in (0, \infty)$ is strictly positive with a strictly positive probability. In details,*

$$P(S_\tau^\delta > 0) > 0 \tag{2.2.10}$$

One could argue that the definition above is weak with respect to other definitions, and covers only non-negative portfolios. This leaves room for some arbitrage opportunities to arise if agents can construct negative portfolios. However, an important feature of real markets is that, by law, investors are subject to a limited liability constraint. If

5. More details and technicalities are discussed in Protter (2004) and reported in Platen and Heath (2006).

their portfolio becomes negative, they are forced to declare bankruptcy and they are not permitted to trade anymore. Negative portfolios can be constructed only if agents hold a credit line. This condition restricts substantially the possibility to exploit negative strategies, and limits arbitrage opportunities to an extent that is negligible in the context of financial modeling. More details on arbitrage conditions under the Benchmark Approach will be discussed in the context of valuation of contingent claims, see Section 2.3.3.

By combining Definition 2.2.3 and Proposition 2.2.2 one can derive the following statement.

Proposition 2.2.4. *A CFM $\mathcal{S}^C_{(d)}$ does not allow arbitrage opportunities with any of its non-negative portfolios.*

Expected return of the GOP

In a first place, the GOP can be characterized as best performing portfolio in terms of growth rate and expected return. From inequality (2.2.3) it follows that at any point in time t, the growth rate $g_t^{\delta *}$ of the GOP is always greater than the growth rate g_t^{δ} of any strictly positive portfolio $S^{\delta} \in \mathcal{V}^+$. Furthermore, from Proposition 2.2.2 we know that any benchmarked portfolio $\hat{S}_t^{\delta}$ behave as an $(\underline{\mathcal{A}}, P)$-supermartingale when the GOP is selected as the benchmark. This implies that over any time period $[t, t+h] \subseteq [0, \infty)$ where $h > 0$ represent the length of the period, the value $\hat{S}_t^{\delta}$ of the portfolio is

$$\hat{S}_t^{\delta} \geq E\left(\hat{S}_{t+h}^{\delta} \middle| \mathcal{A}_t\right) \tag{2.2.11}$$

or, equivalently, the expected returns associated to $\hat{S}_t^{\delta}$ are

$$E\left(\frac{\hat{S}_{t+h}^{\delta} - \hat{S}_t^{\delta}}{\hat{S}_t^{\delta}} \middle| \mathcal{A}_t\right) \leq 0 \tag{2.2.12}$$

Outperforming long term growth rate

We will show in Section 3.3 that if we consider a sufficiently long time horizon, the trajectory of the GOP lays above the trajectory of any other strictly positive portfolio constructed using the same universe of assets. We define the *long term growth rate* $\bar{g}^{\delta}$ as the almost-sure upper limit

$$\bar{g}^{\delta} \overset{\text{a.s.}}{=} \limsup_{T \to \infty} \frac{1}{T} \ln\left(\frac{S_T^{\delta}}{S_0^{\delta}}\right) \tag{2.2.13}$$

assuming that this exists.

Theorem 2.2.5. *In a CFM $\mathcal{S}^{\mathcal{C}}_{(d)}$ the GOP S^{δ_*} achieves almost surely the greatest long-term growth rate when compared to the long-term growth rate of all other strictly positive portfolios $S^\delta \in \mathcal{V}^+$. In mathematical terms, the inequality*

$$\bar{g}^{\delta_*} \geq \bar{g}^{\delta} \tag{2.2.14}$$

holds almost surely.

A proof of Theorem 2.2.5 is provided in Appendix B.3.

Systematic outperformance

Equation (2.2.14) provides the second characterization of best performance, which represents a desirable pathwise feature of the GOP for long-term investors. Since the GOP is unique in the entire investment universe, it will deliver almost-surely the best possible outcome when the investment horizon is sufficiently long.

From a different perspective, it can be relevant to study what is the minimum amount of time that is necessary to benefit of the best long-term performance property with a reasonably high probability. More precisely, for an investor it is of interest to know if the GOP can systematically outperform any other portfolio over a short time period. This depends on the dynamics of the underlying market. Platen (2004a) provides the following definition, that that represents the basis of our third characterization of best performance.

Definition 2.2.6. *A strictly positive portfolio $S^\delta \in \mathcal{V}^+$ will systematically outperform another strictly positive portfolio, lets say $\bar{S}^\delta \in \mathcal{V}^+$, if for some bounded stopping times $\tau \in [0, \infty)$ and $\sigma \in [\tau, \infty)$ with $S^\delta_\tau = S^{\bar{\delta}}_\tau$ and $S^\delta_\sigma \geq S^{\bar{\delta}}_\sigma$ it holds almost surely that*

$$P\big(S^\delta_\sigma > S^{\bar{\delta}}_\sigma \big| \mathcal{A}_\tau\big) > 0 \tag{2.2.15}$$

From a different point of view, Definition 2.2.6 implies the absence of relative arbitrage opportunities. If a non-negative portfolio systematically outperforms the GOP, then this portfolio can generate with a strictly positive probability better outcomes than those achieved by the GOP. If this is the case, one of the necessary conditions for a portfolio to be defined as the GOP is not verified. Proof of this result is provided in Appendix B.4.

2.3 Applications of the Benchmark Approach

After presenting the framework of the Benchmark Approach, in the present Section we discuss some of the most important applications of the Growth Optimal Portfolio in quantitative finance. In details, Platen (2005b) and Platen (2006) show how the GOP can be interpreted as the fundamental building block of a unified framework for financial modeling, portfolio optimization, risk management and derivative pricing.

Markowitz (1959) introduced for the first time the concept of efficient frontier with his mean-variance theory, setting up the ground for the development of modern portfolio theories. Later, Merton (1973) modeled portfolio selection in a generalized continuous time framework, with the *Intertemporal Capital Asset Pricing Model* (ICAPM). Most of the traditional portfolio selection methodologies assume that investors are able to choose their preferred portfolio according to their utility function. A utility functions is usually suited for mathematical modeling, but do not necessarily provide an adequate description of investor's attitude toward risk. In addition, these models are sometimes difficult to implement due to the uncertainty underlying the estimation of their parameters. The Benchmark Approach and the properties of the GOP allow to obtain significant advantages when modeling optimal investment strategies.

The *Arbitrage Pricing Theory* (APT), firstly proposed by Ross (1976) and further developed in an extensive literature, has always played a central role in the context of derivative pricing. For the modeling of asset price dynamics under the APT, several authors have focused on different quantities (*e.g.*, pricing kernel, stochastic discount factor, state price density, to mention some) and proposed different frameworks. However, all these approaches are based on the existence of a risk-neutral probability measure, to exclude arbitrage opportunities. With the fair pricing methodology (Platen 2002), which exploits the GOP as a Numeraire, asset pricing can be implemented under the real world probability.

In this Section we show that the Benchmark Approach provides a flexible modeling framework, and that the properties of the GOP can be exploited to relax some traditional assumptions, thus extending the range of possible applications to new problems.

2.3.1 Portfolio optimization

The Benchmark Approach and the notion of CFM introduced in Section 2.1 allow us to derive and extend a range of classical results in the field of portfolio optimization. In particular, some considerations can be made by leveraging on the properties of the GOP to answer the question of how an investor should optimally allocate his wealth among different assets, according to his preferences.

We already defined the GOP as the investment strategy that achieves the highest possible long-term growth rate. However, Platen and Heath (2006) demonstrated that when the investment horizon is short, the investment must be separated into two funds: the GOP and the savings account. The resulting allocation is known under the name of *fractional Kelly strategy*.

If the investor maximizes an expected utility from discounted terminal wealth, the optimal portfolio obtained by investing money in the the GOP and the savings account is an efficient portfolio in the sense of Markowitz (1959). Furthermore, a fractional Kelly strategy is always positioned on the efficient investment frontier, that is, it always achieves the maximum Sharpe ratio. This result can be used to generalize the Intertemporal Capital Asset Pricing Model (ICAPM) proposed my Merton (1973) under very weak assumptions (Platen and Heath 2006).

In line with the purposes of this research, we provide a general overview of these concepts as they appear in the literature, where a more detailed technical discussion can be found.

Locally Optimal Portfolio

To show the potential of the Benchmark Approach in terms of portfolio selection, we consider a portfolio $S^\delta \in \mathcal{V}^+$ discounted by a savings account which is locally riskless (*i.e.*, does not face short-term fluctuations):

$$\bar{S}_t^\delta = \frac{S_t^\delta}{S_t^0} \tag{2.3.1}$$

By application of the Itô formula to (2.1.9) and (2.1.2) we obtain the SDE

$$d\bar{S}_t^\delta = \sum_{k=1}^{d} \psi_{\delta,t}^k \; (\theta_t^k \; dt + dW_t^k) \tag{2.3.2}$$

with kth *diffusion coefficient*

$$\psi_{\delta,t}^k = \sum_{j=1}^{d} \delta_t^j \; \bar{S}_t^j \; b_t^{j,k} \tag{2.3.3}$$

and *discounted drift*

$$\alpha_t^{\delta} = \sum_{k=1}^{d} \psi_{\delta,t}^{k} \, \theta_t^{k} \tag{2.3.4}$$

at time $t \in [o, \infty)$.

The *aggregate diffusion coefficient*, obtained as

$$\gamma_t^{\delta} = \sqrt{\sum_{k=1}^{d} \left(\psi_{\delta,t}^{k}\right)^2} \tag{2.3.5}$$

measures the magnitude of trading uncertainty (*i.e.*, variance per unit of time) of the discounted portfolio $\bar{S}_t^{\delta}$, at time $t \in [0, \infty)$.

Platen (2005b) generalizes the mean-variance optimality derived in Markowitz (1952) and Markowitz (1959) in a continuous time setting, by introducing the following notion of optimal portfolio.

Definition 2.3.1. *In a CFM $\mathcal{S}_{(d)}^{C}$ the strictly positive portfolio $S^{\bar{\delta}} \in \mathcal{V}^{+}$ is defined as locally optimal, if for all $t \in [0, \infty)$ and all strictly positive portfolios $S^{\delta} \in \mathcal{V}^{+}$ with aggregate diffusion coefficient*

$$\gamma_t^{\delta} = \gamma_t^{\bar{\delta}} \tag{2.3.6}$$

it has the largest discounted drift:

$$\alpha_t^{\delta} \leq \alpha_t^{\bar{\delta}} \tag{2.3.7}$$

What is shown in (2.3.6) and (2.3.7) is that investors prefer "more rather than less" (Platen 2006), that is, higher returns for the same variance. To be noted is that this optimality criterion neither relies on the notion of utility function, nor on any specific time horizon.

Portfolio selection and Two Funds Theorem

Denote the *Sharpe ratio* as

$$s_t^{\delta} = \frac{\alpha_t^{\delta}}{\gamma_t^{\delta}} \tag{2.3.8}$$

and the *total market price of risk* as

$$|\boldsymbol{\theta}_t| = \sqrt{\sum_{k=1}^{d} (\theta_t^k)^2} \tag{2.3.9}$$

for $t \in [0, \infty)$.

The following portfolio selection theorem is derived by Platen and Heath (2006) under the assumptions that *(i)* the market price of risk is strictly greater than zero and finite, and *(ii)* the fraction of wealth invested in the savings account is not equal to one.

Theorem 2.3.2. *Consider a CFM $\mathcal{S}^C_{(d)}$. For any strictly positive portfolio $S^\delta \in \mathcal{V}^+$ with non-zero aggregate diffusion coefficient and aggregate volatility $b^\delta_t = \frac{\gamma^\delta_t}{\bar{S}^\delta_t}$, the Sharpe ratio s^δ_t satisfies the inequality:*

$$s^\delta_t \leq |\boldsymbol{\theta}_t| \tag{2.3.10}$$

for all $t \in [0, \infty)$, where the equality arises when S^δ is locally optimal. The value $\bar{S}^\delta_t$ of a discounted locally optimal portfolio satisfies the SDE

$$d\bar{S}^\delta_t = \bar{S}^\delta_t \, \frac{b^\delta_t}{|\boldsymbol{\theta}_t|} \sum_{k=1}^{d} \theta^k_t \, (\theta^k_t + dW^k_t), \tag{2.3.11}$$

with fractions

$$\pi^j_{\delta,t} = \frac{b^\delta_t}{|\boldsymbol{\theta}_t|} \pi^j_{\delta_*,t} \tag{2.3.12}$$

for all $j = \{1, 2, , \dots, d\}$ and $t \in [0.\infty)$.

Platen and Heath (2006) refer to Theorem 2.3.2 as *Two Funds Separation Theorem*, emphasizing that any locally optimal portfolio can be formed using only these two funds (*i.e.*, by mean of a fractional Kelly strategy). As a consequence, any discounted portfolio that satisfies SDE (2.3.12) is locally optimal. This result can be achieved in several ways, including by minimizing the aggregate volatility for a given risk premium or by maximizing expected utility, with further details discussed in Platen and Heath (2006)[6] and Platen (2002). We can exploit the fractions in Equation (2.3.12) to derive the fractions of wealth held in the savings account and the GOP.

Corollary 2.3.3. *Under the same assumptions valid for Theorem 2.3.2, any locally optimal portfolio $S^\delta \in \mathcal{V}^+$ can be decomposed into a fraction of wealth invested in the GOP*

$$\frac{b^\delta_t}{|\boldsymbol{\theta}_t|} = \frac{1 - \pi^0_{\delta,t}}{1 - \pi^0_{\delta_*,t}} \tag{2.3.13}$$

and the remaining fraction that is held in the savings account

$$\pi^0_{\delta,t} = 1 - \frac{b^\delta_t}{|\boldsymbol{\theta}_t|}(1 - \pi^0_{\delta_*,t}) \tag{2.3.14}$$

6. See Chapter 11 for reference.

Risk aversion coefficient and utility maximization

One can interpret the quantity in Equation (2.3.13) as a *risk aversion coefficient* that equals 1 when investor's wealth is fully invested in the GOP, and explodes to infinity when all the wealth is invested in the savings account.

$$J_t^\delta = \frac{b_t^\delta}{|\boldsymbol{\theta}_t|} = \frac{1-\pi^0_{\delta,t}}{1-\pi^0_{\delta_*,t}} \tag{2.3.15}$$

The risk-aversion coefficient is a useful flexible parameter process for modeling the evolution of risk-aversion over time (Platen 2005b), and allows to relate what we have discussed so far to the concept of expected utility maximization. An advantage of this framework is that it allows to implicitly determine the risk aversion coefficient of an investor from the actual value of the discounted portfolio at a given point in time, without any need of computing the corresponding value for a given specific utility function. As a result, optimal portfolios can be modeled for a wide range of risk aversion coefficients, to fit the preferences of all the investors in the context of investment planning.

Efficient Frontier

To conclude the discussion on portfolio selection, it is useful to reconcile the concepts discussed with the notion of *efficient investment frontier* in the sense of Markowitz (1952). From SDE (2.3.11) we can define the *risk premium* of the locally optimal portfolio process S^δ as

$$p_{S^\delta}(t) = b_t^\delta \; |\boldsymbol{\theta}_t| \tag{2.3.16}$$

with *aggregate volatility*

$$b_t^\delta = \frac{1-\pi^0_{\delta,t}}{1-\pi^0_{\delta_*,t}} \; |\boldsymbol{\theta}_t| \tag{2.3.17}$$

In a CFM we can identify a family of *efficient portfolios*, parameterized by the squared aggregate volatility, which correspond to the continuous time generalization of the one period mean-variance methodology of Markowitz (1959). We define the *appreciation rate* of the portfolio process S^δ as

$$a_t^\delta = r_t + p_{S^\delta}(t) \tag{2.3.18}$$

Definition 2.3.4. *In a CFM $S^C_{(d)}$, we define S^δ an efficient portfolio if the appreciation rate a^δ_t, as a function of its squared volatility $(b^\delta_t)^2$, lies on the efficient frontier defined as*

$$a^\delta_t = r_t + \sqrt{(b^\delta_t)^2}\ |\boldsymbol{\theta}_t| \tag{2.3.19}$$

for all $t \in [0, \infty)$.

According to Platen and Heath (2006) we can also formulate the following result.

Corollary 2.3.5. *Any optimal portfolio $S^\delta \in \mathcal{V}^+$ as defined by Theorem 2.3.2 is also an efficient portfolio.*

By combining the definition of Sharpe ratio provided in (2.3.8) and Equation (2.3.10) it follows that it is not possible to construct a strictly positive portfolio that is located above the efficient frontier in terms of rate of return and volatility. Due to the continuous time nature of this framework, the efficient frontier randomly changes intercept and slope over time. However, locally optimal portfolios constructed as a combination of the savings account and the GOP are always located on top of it, and record the highest Sharpe ratio that is achievable at any time t.

2.3.2 GOP as market portfolio and risk measurement

By linking the concepts of GOP and market portfolio (MP) Platen and Heath (2006) show that the modeling framework proposed under the Benchmark Approach can be exploited to isolate and measure different components of the *market risk*, which are relevant in terms of regulatory requirements and risk management practices, see Platen and Stahl (2003) and Basle (1995). Furthermore, interpreting the GOP as the MP provide the basis for the selection of the Morgan Stanley Capital International Developed Markets Total Return Index (MSCI-World) as a candidate proxy of the GOP that is analyzed in the following chapters.

To explain the concept of market portfolio, Platen and Heath (2006) assume that the market is composed by $n \in \mathbb{N}$ investors, who hold the sum of all the units of primary security accounts that are traded in the market (total tradable wealth). More precisely, each ℓth investor with $\ell \in \{1, 2, \ldots, n\}$, hold an optimal portfolio $S^{\delta_\ell}_t$ which is non-negative, as per the limited liability constraint already discussed, and denominated in the domestic currency.

At time $t \in [0, \infty)$, the market portfolio corresponds to the sum of all the investors' portfolios, and is defined as

$$S_t^{\delta_{MP}} = \sum_{\ell=1}^{n} S_t^{\delta_\ell} \tag{2.3.20}$$

Platen and Heath (2006) demonstrate that he MP is itself a locally optimal portfolio, since it is entirely composed by optimal portfolios[7].

The market portfolio can be reasonably interpreted as a broadly diversified index, even if its composition is not precisely defined and largely depends on the composition of investors' portfolios. In Section 2.4 we demonstrated through the Diversification Theorem that diversified portfolios approximate well the GOP for an increasing number of primary risky assets. As a result, the GOP can be considered a good proxy of the MP, and, following Platen (2005b), it can be used as a reference unit that allows to disentangle *systematic risk* (or *general market risk*) from *specific market risk.*

The *total market price of risk* is represented by the volatility of the GOP and describes in a natural way the general market risk.

$$|\boldsymbol{\theta}_t| = \sqrt{\sum_{k=1}^{d} (\theta_t^k)^2} \tag{2.3.21}$$

By application of the Itô formula on (2.1.4) and (2.2.7), we obtain the (driftless) SDE of the benchmarked primary security account

$$d\hat{S}_t^j = -\hat{S}_t^j \sum_{k=1}^{d} \sigma_t^{j,k} dW_t^k \tag{2.3.22}$$

for $t \in [0, \infty)$.

The parameter $\sigma_t^{j,k}$ in Equation (2.3.22) represents the (j,k)th *specific volatility* and measures the jth specific market risk with respect to the process W^k, of the benchmarked primary security account $\hat{S}_t^j$ (Platen 2005b). This quantity is calculated as

$$\sigma_t^{j,k} = \theta_t^k - b_t^{j,k} \tag{2.3.23}$$

where θ_t^k is the general market volatility of the GOP with respect to the kth Wiener process, and $b_t^{j,k}$ is the volatility of the jth primary security account with respect to the same W^k process.

7. For a proof see Section 11.2 of Platen and Heath (2006).

2.3.3 Pricing of contingent claims

The main advantage of the Benchmark Approach in the field of contingent claims valuation is that it allows to model price dynamics in a generalized framework and under weak assumptions, compared to traditional risk-neutral and actuarial pricing methodologies. Thanks to the properties of the GOP we can significantly extend the range of asset pricing models that can be implemented, also to those situations where other approaches are not available[8].

In the literature, several pricing frameworks have been developed using a variety of approaches. Some of them are designed to explain the relationship between the financial and the intrinsic economic value of the underlying assets, answering the question of how price reacts to changes in fundamental values. However, the price of certain asset classes is sometimes difficult to link to underlying economic values. Derivatives are a typical example in this context, since they securitize multiple uncertainties to which the market is subject. Most of the pricing approaches are based on the existence of a Numeraire. In general terms, a Numeraire is a financial quantity on which the valuation is anchored to simplify the mathematical framework necessary to deal with expectations and the uncertainty embedded in future outcomes of price dynamics.

Under the Benchmark Approach, the Numeraire that allows to achieve a particular interesting set of mathematical results is the GOP. Using the GOP as Numeraire allows to price contingent claims under the *real world probability measure* P. We show in what follows the *real world pricing framework*, and compare it against actuarial and risk-neutral pricing, to highlight specific advantages of the Benchmark Approach in this context.

Real World Pricing

We showed in the previous sections that the value process of a benchmarked portfolio $\hat{S}_t^\delta$ forms an $(\mathcal{A}, P)$-local martingale when the GOP is used as benchmark. The local martingale property is associated to the absence of arbitrage in CFMs[9] and it represents the building block of the following notion of *fair value process*[10].

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8. For instance, the Benchmark Approach provides several advantages when modeling the so-called non-replicable payoffs (*i.e.*, payoffs that cannot be replicated by a fair portfolio of primary security accounts). See Section 11.4 of Platen and Heath (2006) for further details.

9. See Appendix B.2 for an analytical proof.

10. This notion of value process is intended to be general, and does not necessarily represent a portfolio process, thus the use of a slightly different notation with respect to other sections.

Definition 2.3.6. *A security price process* $V = \{V_t, t \in [0,\infty)\}$ *is called fair if its benchmarked value* $\hat{V}_t = \frac{V_t}{S_t^{\delta_*}}$ *forms an* $(\mathcal{A}, P)$*-martingale.*

Pricing models derived under the Benchmark Approach are more general than those derived under traditional continuous market models. In particular, the BA does not rely on the existence of an *Equivalent Local Martingale Measure* (ELMM) to exclude arbitrage opportunities as defined under the *No Free Lunch with Vanishing Risk* (NFLVR) theory[11]. When an ELMM fails to exist, and the NFLVR is not verified, financial market may still be viable in the sense that strong forms of arbitrage are still banned.

Fontana and Runggaldier (2013) analyzed the notions of *increasing profits*, and *arbitrage of first type*, and highlighted that what is crucial to meaningfully solve portfolio optimization and asset pricing problems is the notion of *completeness of the market.* Markets are complete when any contingent claim payoff can be replicated starting from some initial investment. This concept is linked to the so-called *squared integrability* of the market price of risk, but does not depend on the existence of an ELMM. If markets are complete, mild forms of arbitrage may still exist. This is not only more realistic, but also not problematic in terms of mathematical modeling. Under the Benchmark Approach, the local martingale property of benchmarked returns, is sufficient to exclude strong forms of arbitrage, and to model price dynamics without relying on a risk-neutral probability measure.

From Definition 2.3.6 and by application of the martingale property of $\hat{V}$, one can derive the following pricing formula:

Corollary 2.3.7. *For any security price process* $\hat{V}_t = \frac{V_t}{S_t^{\delta_*}}$ *and for any time* $t \in [0,\infty)$ *and* $T \in [t,\infty)$*, one can define the following real world pricing formula:*

$$V_t = S_t^{\delta_*} E\left(\frac{V_T}{S_T^{\delta_*}}\middle|\mathcal{A}_t\right) \tag{2.3.24}$$

The result achieved by using the GOP as Numeraire, is that the expectation in Equation (2.3.24) is taken from the real world probability measure P. This implies that the only necessary condition to apply the fair pricing framework under the Benchmark Approach is the existence of the GOP.

11. See Delbaen and Schachermayer (1994), Delbaen and Schachermayer (2006).

Risk-Neutral and Actuarial Pricing

To emphasize even more the potential of the Benchmark Approach in derivative pricing we report some of the results discussed by Platen (2005b), and show that risk-neutral pricing methodologies appear as a particular case of fair pricing under the real world probability. To prove this result, we assume the existence of a presumed risk-neutral probability measure P_θ, associated with the Radon-Nikodym derivative

$$\Lambda_\theta(t) = \frac{dP_\theta}{dP}\bigg|_{\mathcal{A}_t} = \frac{\hat{S}_t^0}{\hat{S}_0^0} = \frac{S_t^0 S_0^{\delta_*}}{S_t^{\delta_*} S_0^0} \tag{2.3.25}$$

with $t \in [0, \infty)$ and being S_t^0 the savings account.

By mean of Equation (2.3.25) one can re-write Equation (2.3.24) as

$$\begin{aligned} U_H(t) &= E\left(\frac{S_T^0}{S_T^{\delta_*}} \frac{S_t^{\delta_*}}{S_t^0} \frac{S_t^0}{S_T^0} \, H \,\bigg|\, \mathcal{A}_t \right) \\ &= E\left(\frac{\Lambda_\theta(T)}{\Lambda_\theta(t)} \frac{S_t^0}{S_T^0} \, H \,\bigg|\, \mathcal{A}_t \right) \end{aligned} \tag{2.3.26}$$

By the Girsanov Theorem and the Bayes Rule[12], if the Radon-Nikodym derivative process Λ is an $(\mathcal{A}, P)$-martingale, then relation (2.3.26) correspond to the risk neutral pricing formula

$$U_H(t) = E_\theta\left(\frac{S_t^0}{S_T^0} \, H \,\bigg|\, \mathcal{A}_t \right) \tag{2.3.27}$$

where E_θ denotes the expectation under the risk neutral probability P_θ, and the savings account S_t^0 is employed as numeraire.

It is common in traditional literature to specify a model under a risk-neutral probability measure. A limitation of this approach is that, when dealing with realistic models, the candidate Radon-Nikodym derivative process may not evolve as an $(\mathcal{A}, P)$-martingale, with the consequence that the probability P_θ may not exist. Under the Benchmark Approach, fair derivative prices can always be directly computed as conditional expectations under the real world probability P, using the GOP $S_t^{\delta_*}$ as Numeraire. By relaxing the assumption on the existence of the risk-neutral probability measure, one can significantly extend the range of models available for contingent claim pricing.

The fair pricing concept does not only generalize the risk-neutral pricing framework,

12. See Section 9.5 of Platen and Heath (2006) for additional details

but covers also the traditional approach of actuarial pricing, widely used in many areas including accounting and insurance. We denote by $P(t,T)$ the payoff of a zero coupon bond, which pays one unit of the domestic currency at maturity $T \in [0,\infty)$. By Equation (2.3.24) we have that:

$$P(t,T) = S_t^{\delta_*} E\left(\frac{1}{S_T^{\delta_*}} \;\middle|\; \mathcal{A}_t\right) \tag{2.3.28}$$

Platen (2002) and Platen (2006) show that, when a contingent claim H_T, with maturity T, is independent from the GOP $S_T^{\delta_*}$, the fair pricing formula (2.3.24) yields the actuarial pricing formula (alternatively known as present value pricing formula):

$$U_H(t) = S_t^{\delta_*} E\left(\frac{1}{S_T^{\delta_*}} \;\middle|\; \mathcal{A}_t\right) E(H_T|\mathcal{A}_t) = P(t,T)\ E\left(H_T \;\middle|\; \mathcal{A}_t\right) \tag{2.3.29}$$

for $t \in [0,T]$ and $T \in [0,\infty)$.

2.4 Methodology for the approximation of the GOP

In the previous sections we have presented the mathematical properties and the features that characterize the GOP. For the application of the Benchmark Approach it is necessary to identify a methodology to obtain suitable proxies for which all the properties of the GOP are verified. We need a tradable portfolio strategy that can be efficiently implemented in the real world, even when considering all the constraints and frictions arising from trading activity in the financial markets (*i.e.*, transaction costs, portfolio rebalancing frequency, etc.). In this section we follow Platen (2004b) and Platen (2005a) to demonstrate that *Diversified Portfolios* (DPs) approximate well the GOP under some regularity condition, and without relying on any particular modeling assumption, which could introduce additional levels of uncertainty in the implementation of an investment strategy.

2.4.1 Diversified portfolios and sequences of approximate GOP

The convergence of DPs toward the GOP is probably the most important result for a practical application of the Benchmark Approach, and the main pillar on which the approximation methodology proposed in this research is built upon. To prove this outcome, we will introduce in the following order the notion of *(i)* sequence of CFMs, *(ii)* sequence of diversified portfolios, *(iii)* sequence of regular CFMs and *(iv)* sequence of approximate

GOPs. These notions constitute all the technical framework needed to show the role that diversification plays as our tool to efficiently obtain proxies of the GOP.

Recall from Definition (2.1.1) that a market $\mathcal{S}^C_{(d)}$ can be defined as a continuous financial market (CFM) if the volatility matrix of the risky security accounts $\boldsymbol{b}_t = [b^{j,k}_t]^d_{j,k=1}$ is invertible, meaning that markets are complete and assets are not redundant.

In a sequence of CFMs $(\mathcal{S}^C_{(d)})_{d\in\mathbb{N}}$, indexed by the number d of primary security accounts in $\mathcal{S}^C_{(d)}$, the dth CFM includes the savings account $S^0_{(d)} = \{S^0_{(d)}(t), t \in [0,\infty)\}$ which is locally riskless, and d non-negative risky primary security account processes $S^j_{(d)} = \{S^j_{(d)}(t), t \in [0,\infty)\}$.

For exploring the concept of diversification, it is useful to define the *volatility* as

$$b^{j,k}_{(d)}(t) = \sigma^{0,k}_{(d)} - \sigma^{j,k}_{(d)} \tag{2.4.1}$$

and *market price of risk* as

$$\theta^k_{(d)}(t) = \sigma^{0,k}_{(d)} \tag{2.4.2}$$

for $j, k \in \{1, 2, \ldots, d\}$.

Using these two definitions, SDE (2.1.4) can be written as

$$dS^j_{(d)}(t) = S^j_{(d)}(t)\left(r_t\ dt + \sum_{k=1}^{d}\left(\sigma^{0,k}_{(d)} - \sigma^{j,k}_{(d)}\right)\left(\sigma^{0,k}_{(d)}(t)\ dt + dW^k_t\right)\right) \tag{2.4.3}$$

Representing volatilities and the market price of risk as per the definitions (2.4.1) and (2.4.2) is convenient to disentangle diversifiable and non-diversifiable risk. On one hand, the kth volatility of the GOP represented by θ^k_t can be seen as the *general market volatility* with respect to the kth Wiener process, that is, the fluctuation of the market as a whole caused by W^k. On the other hand, the predictable process $\sigma^{j,k} = \{\sigma^{j,k}_{(d)}(t),\ t \in [0,\infty)\}$ represents the (j,k)th *specific volatility*, that is, according to SDE (2.2.9), the negative volatility of the jth primary security account, caused by W^k, but not captured by the GOP. In other words, the kth *specific market risk* with respect to W^k (Platen and Stahl 2003).

Before continuing the discussion, it is useful to adapt the notation presented in the previous sections to deal with the settings of a sequence of continuous financial markets

(CFMs). For $d \in \mathbb{N}$ we can rewrite Equation (2.1.5) as

$$S^{\delta}_{(d)}(t) = \sum_{j=0}^{d} \delta^j_t \; S^j_{(d)}(t) \tag{2.4.4}$$

to represent the portfolio associated to the strategy $\boldsymbol{\delta} = \{\boldsymbol{\delta}_t = (\delta^0_t, \delta^1_t, \ldots, \delta^d_t)^\top, t \in [0, \infty)\}$ under the dth CFM $\mathcal{S}^C_{(d)}$.

Analogously, we can rewrite Equation (2.1.8) as

$$\pi^j_{\delta,t} = \delta^j_t \; \frac{S^j_{(d)}(t)}{S^{\delta}_{(d)}(t)} \tag{2.4.5}$$

for $t \in [0, \infty)$ and $j \in \{0, 1, \ldots, d\}$, and with a small abuse of notation, given that both the strategy δ and the fractions π depend on d.

SDE (2.2.7) can be rewritten using the alternative definition of market price of risk in Equation (2.4.2), to represent the dth GOP as

$$dS^{\delta_*}_{(d)}(t) = S^{\delta_*}_{(d)}(t) \left(r_t \; dt + \sum_{k=1}^{d} \sigma^{0,k}_{(d)}(t) \left(\sigma^{0,k}_{(d)}(t) \; dt + dW^k_t \right) \right) \tag{2.4.6}$$

for $t \in [0, \infty)$.

From (2.2.8), the benchmarked portfolio value $\hat{S}^{\delta}_{(d)}(t)$, corresponding to the dth GOP $S^{\delta_*}_{(d)}(t)$, is given by

$$\hat{S}^{\delta}_{(d)}(t) = \frac{S^{\delta}_{(d)}(t)}{S^{\delta_*}_{(d)}(t)} \tag{2.4.7}$$

Making use of (2.4.5) we can adapt the driftless SDE (2.2.9) as

$$d\hat{S}^{\delta}_{(d)}(t) = -\hat{S}^{\delta}_{(d)}(t) \sum_{j=0}^{d} \pi^j_{\delta,t} \sum_{k=1}^{d} \sigma^{j,k}_{(d)}(t) \; dt + dW^k_t \tag{2.4.8}$$

for $t \in [0, \infty)$, and provided that $S^{\delta_*}_{(d)}(0) > 0$.

Sequence of continuous financial markets (CFMs)

With all the aforementioned elements, and following Platen and Heath (2006), it is possible to provide a formal definition of a sequence of CFMs.

Definition 2.4.1. *A sequence* $(\mathcal{S}^C_{(d)})_{d\in\mathbb{N}}$ *is defined as a sequence of CFMs if in* $\mathcal{S}^C_{(d)}$ *(i) all the primary security accounts satisfy the SDEs of the type (2.4.3), (ii) the volatility matrix*

$$\boldsymbol{b}_{(d)}(t) = [b^{j,k}_{(d)}(t)]^d_{j,k=1} = [\sigma^{0,k}_{(d)}(t) - \sigma^{j,k}_{(d)}(t)]^d_{j,k=1} \tag{2.4.9}$$

is invertible for Lebesgue-almost every $t \in [0,\infty)$*, and (iii) for all* $j \in \{0,1,\dots\}$*,* $k \in \mathbb{N}$ *and* $t \in [0,\infty)$ *the* (j,k)*th specific volatility* $\sigma^{j,k}_{(d)}(t)$ *converges almost surely to a finite limit* $\sigma^{j,k}(t)$ *as* $d \to \infty$*, that is*

$$\lim_{d\to\infty} \sigma^{j,k}_{(d)}(t) \stackrel{a.s.}{=} \sigma^{j,k}(t) < \infty \tag{2.4.10}$$

Sequence of Diversified Portfolios (DPs)

Sequences of diversified portfolios (DPs) are a particular class of sequences of CFMs that are suitable to approximate well the corresponding sequences of GOPs. The notion of DPs is of central importance to prove that diversification allows to obtain a mathematical object that approximates asymptotically the GOP. For these portfolios, the share of wealth invested in each security is vanishing as the number d of securities increases.

Definition 2.4.2. *For a sequence of CFMs* $(\mathcal{S}^C_{(d)})_{d\in\mathbb{N}}$ *we call the corresponding sequence* $(\mathcal{S}^{\delta_d}_{(d)})_{d\in\mathbb{N}}$ *of strictly positive portfolio processes* $\mathcal{S}^{\delta_d}_{(d)}$ *with* $\mathcal{S}^{\delta_d}_{(d)}(0) = 1$ *a sequence of diversified portfolios (DPs) if some constants* $K_1, K_2 \in [0,\infty)$ *and* $K_3 \in \mathbb{N}$ *exist, independent of* d*, and such that for* $d \in \{K_3, K_3+1,\dots\}$ *the inequality*

$$\left|\pi^j_{\delta_d,t}\right| \le \frac{K_2}{d^{\frac{1}{2}+K_1}} \tag{2.4.11}$$

holds almost surely for all $j \in \{0,1,\dots,d\}$ *and* $t \in [0,\infty)$.

From a practical point of view, condition (2.4.11) requires that the fraction $\pi^j_{\delta_d,t}$ vanishes sufficiently fast as $d \to \infty$, and that the (absolute) fractions in the sequence are not of large magnitude when compared to the value $\frac{1}{d+1}$. For example, the inequality is satisfied by a sequence of equally weighted portfolios with $\pi^i_{\delta_d,t} = \pi^j_{\delta_d,t}$ for all $d \in \mathbb{N}$, $t \in [0,\infty)$ and $i,j \in \{0,1,\dots,d\}$.

Regular sequence of CFMs

We defined the GOP as the investment strategy the maximizes the growth rate of investor's wealth in the long term, outperforming any other allocation that can be constructed with the same universe of assets. In practical terms, to exploit diversification and to maximize the returns achievable with that particular strategy, one should remove all the sources of idiosyncratic risk, given that the market remunerates only non-diversifiable risks. Formally, we need to specify a condition such that each of the independent source of trading uncertainty influences only a restricted group of benchmarked primary security accounts, in order to be able to capture and leverage on that uncertainty to maximize returns.

The jth benchmarked primary security account process $\hat{S}^j_{(d)} = \{\hat{S}^j_{(d)}(t),\ t \in [0,\infty)\}$, associated with

$$\hat{S}^j_{(d)}(t) = \frac{S^j_{(d)}(t)}{S^{\delta_*}_{(d)}(t)}, \tag{2.4.12}$$

satisfies by (2.4.5) and (2.4.8) the SDE

$$d\hat{S}^j_{(d)}(t) = -\hat{S}^j_{(d)}(t) \sum_{k=1}^{d} \sigma^{j,k}_{(d)}(t)\ dW^k_t \tag{2.4.13}$$

By leveraging on SDE (2.4.13) one can define the kth *total specific volatility* for the dth CFM $S^C_{(d)}$ as

$$\hat{\sigma}^k_{(d)}(t) = \sum_{j=0}^{d} |\sigma^{j,k}_{(d)}(t)| \tag{2.4.14}$$

This quantity sums the values of all the specific volatilities $\sigma^{j,k}_{(d)}(t)$ with respect to the kth Wiener process in $\mathcal{S}^C_{(d)}$, and can be interpreted as the measure of the total kth specific market risk. If the total specific volatility is small for some k, it means that only a restricted number of primary security accounts have a larger specific volatility with respect to the kth Wiener process, and one can be reasonably sure that there is a portfolio strategy capable to capture non-diversifiable risk (Platen and Stahl 2003). The following definition of regular sequence of CFMs summarizes these considerations under a simple mathematical condition.

Definition 2.4.3. *A sequence of CFMs $(\mathcal{S}^C_{(d)})_{d\in\mathbb{N}}$ is called regular if it exists a constant $K_5 \in [0,\infty)$, independent of d, such that*

$$E\left(\left(\hat{\sigma}^k_{(d)}(t)\right)^2\right) \leq K_5 \tag{2.4.15}$$

for all $t \in [0,\infty)$, $d \in \mathbb{N}$ and $k \in \{1,2,\ldots,d\}$.

Sequence of approximate GOPs

To identify in practical terms the GOP, one would need a complex and accurate model to estimate volatilities and the market price of risk, and determine optimal investment fractions. As mentioned before, given the limited amount of historical data available, it is not possible to estimate model parameters in a sufficiently precise way. DeMiguel, Garlappi, and Uppal (2009) employed parameters calibrated on US stock market data to derive analytically the length of the estimation period that would be necessary to obtain an optimizing model capable of outperforming a $1/N$ naive diversified strategy. To achieve such result for an investment universe of 50 assets, one would need an estimation window of about 6000 months, hundred times longer than the 60/120 months timeseries usually employed for parameters estimation.

For the following discussion, it is useful to consider a portfolio process $S^{\delta_d}_{(d)}$, defined in the dth CFM $\mathcal{S}^C_{(d)}$ and associated with the strategy $\boldsymbol{\delta}_d = \{\boldsymbol{\delta}_d(t) = \left(\delta^0_d(t), \delta^1_d(t), \ldots, \delta^d_d(t)\right)^\top,\ t \in [0,\infty)\}$. To understand if a portfolio $S^{\delta_d}_{(d)}$ is close enough to the GOP, Platen (2005a) introduce the *tracking rate* $R^{\delta_d}_{(d)}(t)$, defined as

$$R^{\delta_d}_{(d)}(t) = \sum_{k=1}^{d}\left(\sum_{j=0}^{d} \pi^j_{\delta_d,t}\ \sigma^{j,k}_{(d)}(t)\right)^2 \tag{2.4.16}$$

at time $t \in [0,\infty)$.

From Equations (2.4.1), (2.4.2) and (2.2.6) we have that

$$\sum_{j=0}^{d} \pi^j_{\delta_*,t}\ \sigma^{j,k}_{(d)}(t) = 0 \tag{2.4.17}$$

for all $k \in \{1,2,\ldots,d\}$.

As a consequence, for the GOP $S^{\delta_*}_{(d)}$, the tracking rate $R^{\delta_*}_{(d)}(t)$ is zero for all $t \in [0, \infty)$. Moreover, it follows logically that a portfolio $S^{\delta_d}_{(d)}$ equals the dth GOP almost surely, if and only if

$$R^{\delta_d}_{(d)}(t) = 0 \tag{2.4.18}$$

for all $t \in [0, \infty)$.

Combining all these results, Platen and Heath (2006) provide the following formal definition of sequence of approximate GOP.

Definition 2.4.4. *For a sequence of CFMs $(\mathcal{S}^C_{(d)})_{d\in\mathbb{N}}$ we call a sequence $(S^{\delta_d}_{(d)})_{d\in\mathbb{N}}$ of strictly positive portfolio processes which starts at the value of one, a sequence of approximate GOPs, if for all $t \in [0, \infty)$ the corresponding sequence of tracking rates vanishes in probability. In particular, we have that*

$$\lim_{d\to\infty} R^{\delta_d}_{(d)}(t) \stackrel{p}{=} 0 \tag{2.4.19}$$

Equation (2.4.19) implies that, for a sequence of approximate GOPs, the tracking rate becomes smaller as the number of securities considered to implement the portfolio strategy (*i.e.*, the level of diversification) increases.

2.4.2 Diversification Theorem

Definition 2.4.4 and some of the previous results are exploited in this Section to present the *Diversification Theorem*, firstly proposed by Platen (2005a)[13]. This theorem reveals a fundamental robustness property of DPs, and provides a theoretical basis for the empirical analysis that is conducted in the following chapters. In particular, it allows to exploit diversified portfolios to approximate the GOP, avoiding the need to determine the exact GOP fractions with a model that implies accurate estimates of the volatilities and market prices of risk.

Theorem 2.4.5. *For a regular sequence of CFMs $(\mathcal{S}^C_{(d)})_{d\in\mathbb{N}}$ any sequence of $(\mathcal{S}^{\delta_d}_{(d)})_{d\in\mathbb{N}}$ of DPs converges toward a sequence of GOPs.*

It is important to note that this result is achieved without relying on any strict assumption on the dynamics of the market, but only on a regularity condition such that each of the independent sources of trading uncertainty influences only a restricted group of primary

13. A detailed proof of Diversification Theorem can be found in Section 10.6 of the monograph of Platen and Heath (2006).

security accounts[14]. In mathematical terms, this is true if inequality (2.4.15) holds for all $t \in [0, \infty)$.

To prove Theorem 2.4.5 we can slightly modify equation (2.4.16) to hold with a weak inequality

$$R^{\delta_d}_{(d)}(t) \leq \sum_{k=1}^{d} \left(\sum_{j=0}^{d} |\pi^j_{\delta_d,t}| \; |\sigma^{j,k}_{(d)}(t)| \right)^2 \tag{2.4.20}$$

which lead by Equations (2.4.11) and (2.4.14) to

$$E\big(R^{\delta_d}_{(d)}(t)\big) \leq \frac{(K_2)^2}{d^{1+2K_1}} \sum_{k=1}^{d} E\left(\left(\hat{\sigma}^k_{(d)}(t) \right)^2 \right) \leq \frac{(K_2)^2}{d^{2K_1}} K_5 \tag{2.4.21}$$

for all $t \in [0, \infty)$.
Given that by definition 2.4.2 the constant $K_1 > 0$, it follows from (2.4.21) that the expected tracking rate $E\big(R^{\delta_d}_{(d)}(t)\big)$ vanishes as the number d of risky assets included in the investment strategy grows.

The fact that DPs converge towards the GOP is consistent with the idea that, in general, broad market indices fluctuate in a very similar manner. Indeed, broad market indices are well diversified by definition, as they are constructed with a large number of securities to represent the entire stock market. In Section 2.3.2 this result provides an additional argument why the concept of GOP is also close to the concept of *market portfolio*.

By relying on a Black-Scholes settings Platen and Rendek (2012b) provides some interesting examples of sequences of DPs which qualify as sequences of approximate GOPs. By mean of simulations, the authors show that the *Equally Weighted Index* (EWI) and the *Total Return Index* converge to the GOP for relatively small values of d. These results are exploited later for some empirical analysis, in particular to select suitable GOP proxy candidates to be evaluated.

14. Platen and Rendek (2012a) proves that this regularity condition holds for the existing global financial markets, by showing that the information that is intrinsically embedded in the economic structure of the market can be exploited to design an investment strategy that is capable of efficiently extracting diversifiable risk, see also Section 3.1.1.

3 Approximation of the GOP with real market data

In the previous chapter we have presented the theoretical framework underlying the Benchmark Approach. The Growth Optimal Portfolio (GOP) is a "hidden object", an optimal allocation strategy defined in terms of a set of mathematical features and financial concepts. Filipovic and Platen (2009) identify the necessary conditions for the GOP to exists. If these conditions hold, the GOP is identified by its unique value process, for some initial value $S_0^{\delta^*}$. The optimal fractions π_t^* such that the portfolio growth rate (*i.e.*, the drift of its logarithm) is maximized, can be calculated according to Equation (2.2.5). However, this would require the estimation of unknown parameters such as the market price of risk and the volatility matrix. Achieving good results for this notably difficult statistical task, with any estimation methodology, is conditional on disposing of a large amount of historical data that does not exist.

In Section 2.4 we have discussed how well-diversified portfolios approximate asymptotically the GOP, for an increasing number of stocks. The *Diversification Theorem* (see Section 2.4.2) provides a robust framework that allows us to exploit diversification as a tool to construct valid proxies of the GOP, (*i.e.* investment strategies for which the features discussed in Section 2.2 are verified).

DeMiguel, Garlappi, and Uppal (2009) provided empirical evidence that naive diversified portfolios outperform the strategies constructed with sample-based mean-variance models, including all the extensions to these models designed to reduce the estimation error. Given that the GOP must be the global best performing strategy, we leverage on their results (and further validate their claim that equally-weighted indexes generate the highest growth in the investors' wealth in the long term) by constructing different naive-diversified portfolios and show empirically that they are suitable candidates to approximate the GOP.

This chapter is organized in three main sections. In the first section we discuss the methodology to construct hierarchically weighted indexes – a more complex and structured class of equally-weighted strategies that proved to be particularly suitable to approximate the GOP – with a focus on the main specification adopted by Platen and Rendek (2020). Furthermore, we propose a set of alternative strategies, which are themselves potentially valid proxies of the GOP, and that are used to validate empirical results. In the second section we describe the data collected to construct the portfolios. Considering the complexity and the size of the dataset, we provide details about the selection of the relevant sample of stocks, and a summary of the cleaning procedures implemented. In the last section some preliminary results and descriptive statistics are presented.

3.1 Methodology

According to finance theory, only systematic risk should attract a risk premium. Moreover, an investment strategy that best captures non-diversifiable risk is most likely to outperform any other allocation. In the present work, finding an efficient diversification strategy that is capable of maximizing the portfolio performance is crucial to bring the portfolio closer to the true GOP.

In this section we present the methodology to construct a *Hierarchically Weighted Index* (HWI), an investment strategy designed to exploit the economic structure of the market to efficiently remove systematic risk components of the investment. We also introduce in the analysis the *Equally Weighted Index* (EWI) and the MSCI-World as two natural benchmarks to evaluate if the HWI, in three different specifications, can be a suitable GOP proxy. We found preliminary evidence that, in the long-run, the HWI outperforms the EWI and the MSCI-World, reporting a significantly higher long-term growth rate.

3.1.1 Hierarchically Weighted Index (HWI)

Most of the literature on the Benchmark Approach converges toward the choice of a hierarchically-weighting methodology to approximate the GOP. Many well-known optimal portfolio strategies aim at the minimum variance portfolio as in Clarke, De Silva, and Thorley (2011); the risk parity portfolio, as in Maillard, Roncalli, and Teiletche (2010); the maximum diversification portfolio, as in Choueifaty and Coignard (2008); or the hierarchical risk parity portfolio, as in Lopez De Prado (2016), to mention some. Estimation errors that arise in the process of building these strategies can easily offset the theoretical benefits of such optimal portfolio structures.

Despite the construction of a *Hierarchically Weighted Index* (HWI) being more structured when compared to the plain equally-weighting approach, it still allows to avoid the explicit estimation of covariances of returns and expected returns which would require much longer observation windows than those usually available. In addition, naive diversification does not rely on the modelling of complex factors to improve the performance of the allocation strategy (*e.g.*, momentum, fundamental values, size, correlations with market returns, etc.).

The economic structure of the market represents the source of information that is exploited to systematically remove diversifiable risk and optimize the performance of the portfolio. For example, stocks of companies operating in the same industry are likely to be highly correlated, and in a similar fashion, industries in a given country are subject to similar macroeconomic uncertainties. The underlying idea, as suggested by Platen and Rendek (2012a), is that forming a hierarchical industrial and geographical grouping of stocks is a natural solution to capture within groups similarities. Assuming that factors that capture the exposure to similar uncertainties are intrinsically embedded in the groups formed, one can avoid the need to model and estimate those factors by recursively forming equally weighted indexes of the elements within the previous group (*i.e.*, stocks in the lowest group and intermediate indexes formed in the previous steps for the rest of the hierarchies).

When building the HWI we use the ICB industry classification, and information on the country where the headquarter of the company is located, to form our groups (technical details of the ICB classification are reported in Section 3.2).

According to Platen and Rendek (2020), a four-level hierarchy is adequate to obtain statistically significant evidence of the local martingale property for numeraire-denominated returns, and the best portfolio performance. We decided to model the HWI following the same set up. At a certain time $t \geq 0$ stocks are classified into geographical macro-region, country, and industry. The industrial breakdown (*i.e.* supersector, sector or subsector) selected for each country depends on the total number of domestic stocks that are listed in the national stock exchange. This choice has been made to ensure that an appropriate number of constituents is included in each industry bucket at any point in time (again, further details will be provided in Section 3.2).

As previously mentioned, the dataset is hierarchical and self-contained, thus the total number of stocks in the investment universe is given by:

$$N_t = \sum_{j_1=1}^{M_t} \sum_{j_2=1}^{M_t^{j_1}} \sum_{j_3=1}^{M_t^{j_1,j_2}} M_t^{j_1,j_2,j_3} \tag{3.1.1}$$

where M_t is the number of geographical regions, $M_t^{j_1}$ is the number of countries in the j_1th region, $M_t^{j_1,j_2}$ is the number of industrial grouping in the j_2th country of the j_1th region and $M_t^{j_1,j_2,j_3}$ is the number of stocks in the j_3th industrial grouping, of the j_2th country, of the j_1th region.

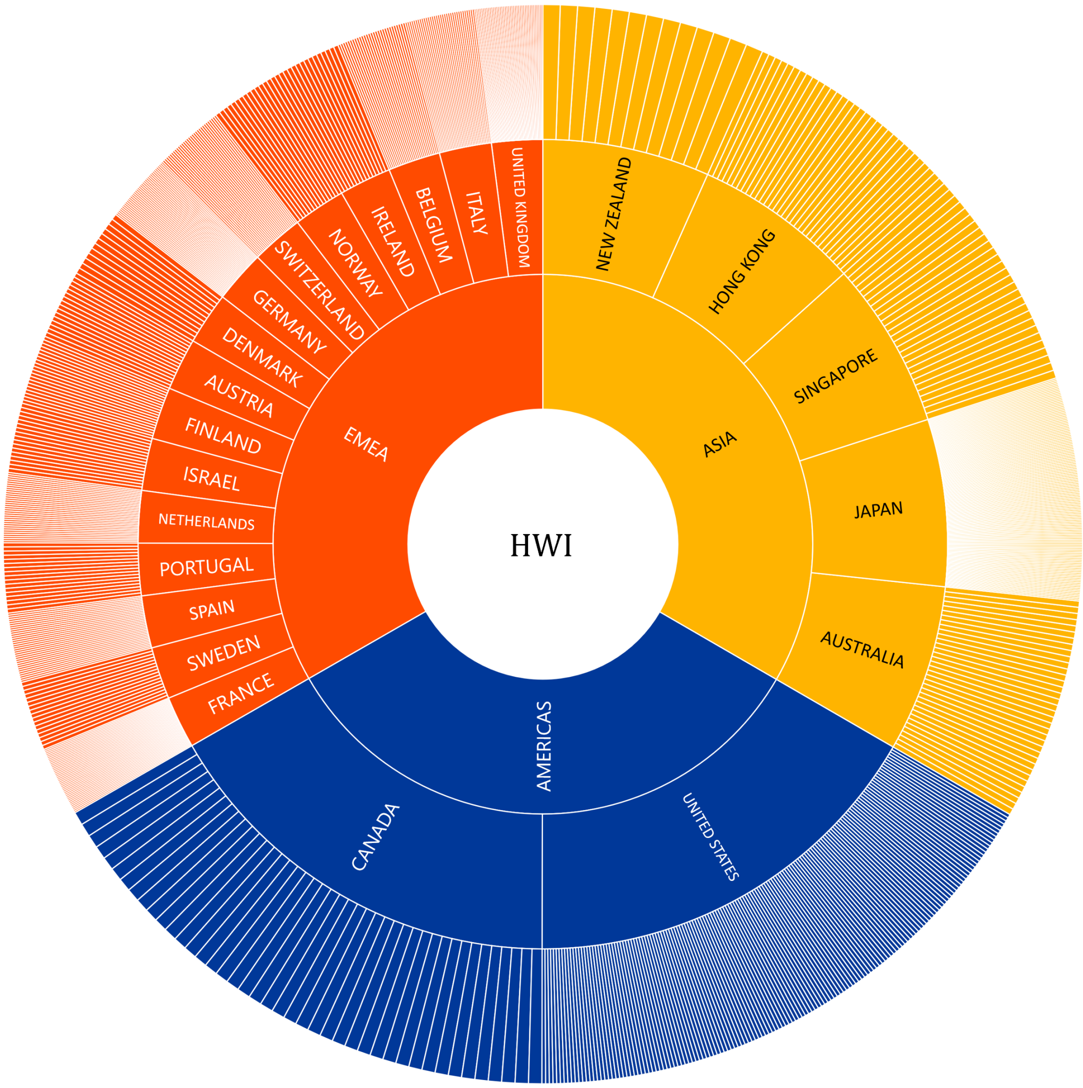
A naive diversified strategy usually invests equal fractions of wealth in each risky security. To obtain the HWI one has to follow a conceptually similar approach. Given the hierarchical nature of this portfolio strategy, the constituents of each group are themselves equally-weighted indexes of the indexes formed at the previous levels, except for the lowest, where constituents are stocks. At time t, the weight of the jth stock with $j = (j_1, j_2, j_3, j_4)$ is of the form:

$$\pi_t^{HWI,j} = \frac{1}{M_t} \frac{1}{M_t^{j_1}} \frac{1}{M_t^{j_1,j_2,}} \frac{1}{M_t^{j_1,j_2,j_3}} \tag{3.1.2}$$

When constructing equally weighted indexes it is necessary to regularly rebalance capital among the securities in the investment universe. In this context, it is worth mentioning that what is mathematically desirable is not always economically feasible: the real world implementation of a strategy implies bearing some transaction costs incurred when performing reallocations, which can ultimately wipe-out part of the returns. Platen and Rendek (2012a) provide an analytical assessment of the sensitivity of the HWI performance to the reallocation frequency and the transaction cost structure, and found that more frequent reallocations diminish, in general, the Sharpe ratio. We constructed the HWI portfolios using daily, monthly and quarterly rebalancing, concluding in favor of the findings mentioned. We took into account these results when choosing the optimal rebalancing frequency, designing all the portfolios to be rebalanced at the end of every quarter. Furthermore, an additional rebalance of the portfolio is triggered every time that a company is delisted.

Figure 1 provides a graphical representation of the HWI structure using a sunburst chart. This visualization makes clear that each group comprises a family of equally-weighted indexes. Outside the circle (not displayed) we have all the primary security accounts included in the dataset, which are equally weighted to form the sector indexes (within each country) in the most external layer. In the central layer of the circle, sector indexes are equally weighted to form country indexes. Lastly, country indexes are equally weighted to form region indexes (internal circle) and region indexes are combined to obtain the HWI.

Figure 1: Structure of HWI



Source: Refinitiv Datastream and author's calculations.
Notes: Primary security accounts used to form sector indexes in the external layer are not displayed.

3.1.2 Equally Weighted Index (EWI)

To evaluate the performance of the HWI we construct other allocation strategies that are themselves suitable to approximate the GOP. The first natural candidate is the *Equally Weighted Index* (EWI). By definition, to obtain the EWI one has to invest the same proportion of wealth in every security $j \in \{1, 2, \ldots, N_t\}$. The weights $\pi_t^{EWI,j}$ at time $t \in [0, \infty)$ are given by:

$$\pi_t^{EWI,j} = \frac{1}{N_t} \tag{3.1.3}$$

The main differences in the EWI and HWI allocations are reported in Table 1. By construction, country-level exposures are stable for the HWI, since countries are equally weighted within the same region and thus represent a fixed proportion of the portfolio. The weight of each country depends on the number of countries classified in each region[15]. Each region represents exactly one third of the portfolio, thus the 16 countries of the EMEA represent about 2% of the total portfolio, Asia-pacific countries accounts for a 6.6% while Canada and United States have the most relevant exposure of 16.6%.

On the other hand, the EWI methodology does not imply any specific rule to constraint the allocation. Consequently, the share of each country depends on the stocks that compose the asset universe at each specific point in time. United States and Japan are predominant as they both account for more than 20% of the total portfolio. The biggest differences when compared to the HWI are Canada, New Zealand, Singapore, and the United Kingdom.

With a hierarchically weighted allocation we privilege some smaller economies if they are classified into Americas or Asia-Pacific, and suppress substantially the share of other relevant economies in the EMEA. This mechanism could be counterintuitive at first sight, but allows to protect the portfolio from macroeconomic shocks that are historically very different in each of the three macro-regions.

In terms of exposure to different sectors, the HWI and EWI are very similar. The main differences are recorded for Financial Services, Real Estate, Industrial Goods and Basic Resources (more than 2% absolute deviation) and are mainly due to the concentration of certain sectors in specific countries.

15. Further details of regional classification are reported in Section 3.2

Table 1: Supersector and country weights for HWI and EWI

Country	EWI	HWI	ICB Supersector	EWI	HWI
Australia	3.513	6.667	Technology	8.971	6.934
Austria	0.973	2.083	Telecommunications	2.203	2.854
Belgium	1.667	2.083	Health Care	8.277	8.246
Canada	6.729	16.667	Banks	4.505	4.525
Denmark	0.973	2.083	Financial Services	10.202	8.016
Finland	0.953	2.083	Insurance	2.500	2.580
France	5.260	2.083	Real Estate	8.932	12.203
Germany	4.982	2.083	Automobiles & Parts	1.865	1.266
Hong Kong	2.422	6.667	Consumer Products	4.466	3.661
Ireland	0.635	2.083	Media	1.865	1.421
Israel	1.092	2.083	Retail	3.374	2.823
Italy	3.057	2.083	Travel and Leisure	3.811	4.259
Japan	20.445	6.667	Food & Beverage	4.069	5.028
Netherlands	2.025	2.083	Personal Care & Drugs	2.024	1.752
New Zealand	0.992	6.667	Construction & Materials	3.989	3.648
Norway	0.933	2.083	Industrial Goods	14.946	12.591
Portugal	0.913	2.083	Basic Resources	4.545	7.205
Singapore	1.945	6.667	Chemicals	2.540	1.745
Spain	2.342	2.083	Energy	3.374	4.434
Sweden	1.449	2.083	Utilities	3.533	4.798
Switzerland	2.759	2.083			
United Kingdom	10.421	2.083			
United States	23.521	16.667			

Source: Refinitiv Datastream and author's calculations.

Notes: Weights reported refer the end of the year 2020.

Figure 2 complements what is shown in Table 1 by visualizing how sectors' weights change over time for the HWI and EWI. Although the structure of the market is relatively stable and the adjustments recorded are minor, it is possible to distinguish some changes in the overall composition of the portfolio. Weights tend to fluctuate in a similar way for both strategies.

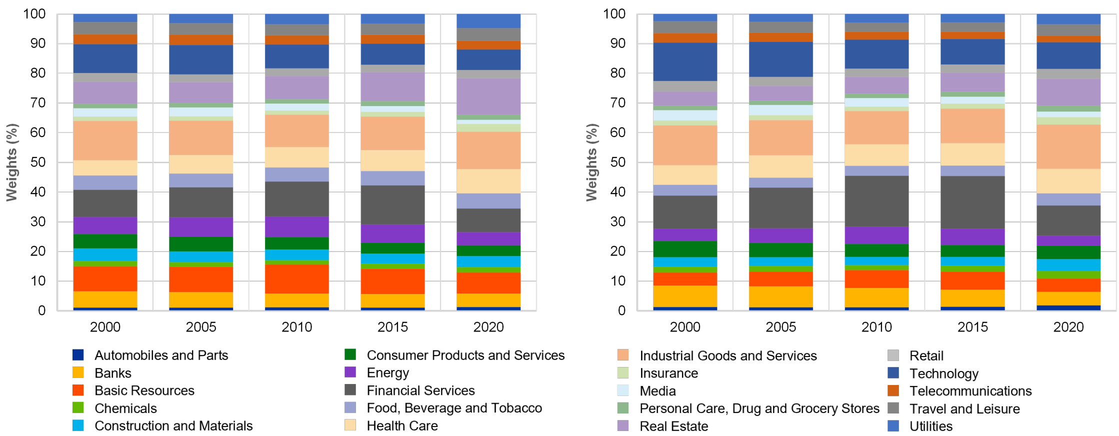
3.1.3 Market Capitalization Weighted Index (MSCI-World)

Following what has been already proposed in the literature[16], another candidate proxy of the GOP (and natural benchmark to run a comparison with the HWI) is the *Market Capitalization Weighted Index* (MCI). To construct this investment strategy, the amount of wealth to be invested in the jth primary security account at time $t \in [0, \infty)$ is proportional to its market value MV_t^j relative to the whole market capitalization.

16. See for instance Platen and Heath (2006), Platen and Rendek (2012a), Platen and Rendek (2020)

Figure 2: Evolution of supersector weights over time

(a) HWI (b) EWI



Source: Refinitiv Datastream and author's calculations.
Notes: The figure displays companies traded worldwide without country distinction.

In mathematical terms, the weights $\pi_t^{MCI,j}$ of the MCI are determined as

$$\pi_t^{MCI,j} = \frac{MV_t^j}{\sum_{k=1}^{N_t} MV_t^k} \tag{3.1.4}$$

for $j \in \{1, 2, \ldots, N_t\}$ and $i \in \{0, 1, \ldots\}$.

Although a MCI constructed using the same universe of assets of the HWI and EWI would ensure a more appropriate comparison of all the GOP candidates, we found little value added in computing this strategy in-house. In light of the fact that the construction of each portfolio is a computationally-intense process, we decided to replace the MCI with the *Morgan Stanley Capital International Developed Countries Total-Return Index* (MSCI-World), downloaded from Refinitiv Datastream.

The MSCI-World represents a traditional benchmark for fund management. Furthermore, it is constructed following the same methodology of the MCI and comprises assets from the same set of 23 developed economies that we consider for the other portfolios. Even if it is composed by a significantly lower number of constituents, the coverage in terms of free float market capitalization is more than 85% in each country[17]. As a consequence, any additional constituent on top of those included in the MSCI-World would have a weight that is close to zero and an almost negligible effect on the overall performance of the MCI portfolio. Additional support to our choice is provided by Platen and Rendek (2020), who found that the MSCI-World deviates only marginally from the MCI

17. See MSCI-World index fact sheet available at: https://www.msci.com/documents/10199/cad25553-6265-4a1b-9942-cb5be891015d (accessed: October 2022)

in terms of long-term performance and evolution over time.

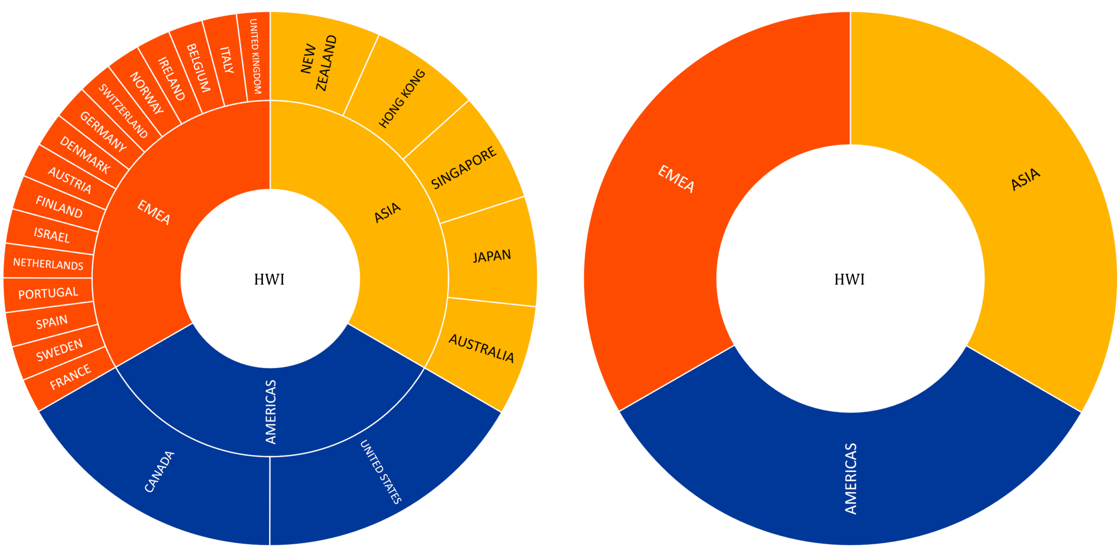
3.1.4 Other portfolio specifications

The first issue to tackle when constructing the HWI portfolio is the identification of the optimal number of diversification layers allowing to achieve the zero-mean property for instantaneous benchmarked returns with sufficient accuracy. The ultimate objective of the hierarchical diversification strategy is grouping companies according to their exposure to similar uncertainties to extract systematically that portion of non-diversifiable risk that is remunerated by the market. Tweaking the groups at lower or higher level of the hierarchy still provides well-diversified indexes, but their performance can significantly differ.

To further validate our results we designed the HWI under two additional alternative settings, one considering only one level of diversification – the hierarchically weighted index diversified by region (HWI.r) – and one considering two levels of diversification – the hierarchically weighted index diversified by country and by region (HWI.r.c). The structures of these two strategies are visualized in Figure 3. Additional combinations of region, country and industry can be considered. However, different attempts to build the portfolios with different hierarchies led to overlapping results. The three HWI specifications selected are those for which some major differences can be appreciated.

Figure 3: Structure of HWI alternative specifications

(a) HWI.r.c (b) HWI.r



Source: Refinitiv Datastream and author's calculations.
Notes: Primary security accounts used to form indexes in the external layer are not displayed.

3.2 Data infrastructure

The Hierarchically Weighted Index (HWI) is constructed for stocks in developed markets, following closely the analytical design proposed by Platen and Rendek (2020). Stock prices have been retrieved from *Refinitiv Datastream* (RD) that provides a comprehensive set of indexes calculated at country level, along with the price timeseries of their constituents, distinguishing between companies which are currently active on the market and securities that have been delisted.

In this study we have considered the stock market of 23 developed economies following the MSCI annual market classification[18] updated on June 2021. It is worth mentioning that Israel is included in the dataset, following its incorporation among the developed markets in May 2010, while Greece has not been considered due to its downgrading to emerging market on November 2013. Countries have been classified into three macro-regions, in line with the methodology adopted for the calculation of the MSCI-World: Americas, Asia-pacific and EMEA (that comprises Europe, Africa and Middle East).

The base year for each country index included in our investment universe is reported in Table 2, together with the corresponding RD identifier. Stock prices have been retrieved up to the end of the first quarter of 2021.

For the industry classification of the companies we adopted the *Industry Classification Benchmark* (ICB), which provides a comprehensive structure for classifying companies into specific sectors (Reuters 2008). This transparent classification methodology ensures the coverage of all the major industries in the world stock market, and guarantees that companies are assigned to a specific group according to their business nature as determined by their largest source of revenues. In details, the ICB allows to classify a company among 11 industries, 20 supersectors, 45 sectors and 173 subsectors[19]. ICB codes are available in RD and have been downloaded for all the companies in our dataset. A small set of companies for which the industry label was missing has been removed.

Given the large difference between countries in the number of constituents available, and mirroring the methodology adopted by Platen and Rendek (2020), we selected a more granular industrial breakdown based on the number of currently active companies, to ensure that a sufficient number of constituents is assigned to each industry within each country. The rule for the selection of the *supersector* breakdown imposes that a country

18. More information are available at: https://www.msci.com/our-solutions/indexes/market-classification (accessed: December 2022)

19. More information are available at: https://www.ftserussell.com/data/industry-classification-benchmark-icb (accessed: November 2022)

Table 2: Refinitiv Datastream stock lists and corresponding base dates

Country	Active	Dead	Base year
Australia	LTOTMKAU	DEADAU	Jan-1973
Austria	LTOTMKOE	DEADOE	Jan-1973
Belgium	LTOTMKBG	DEADBG	Jan-1973
Canada	LTOTMKCN	DEADCN	Jan-1973
Denmark	LTOTMKDK	DEADDK	Jan-1973
Finland	LTOTMKFN	DEADFN	Jan-1988
France	LTOTMKFR	DEADFR	Jan-1973
Germany	LTOTMKBD	DEADBD	Jan-1973
Hong Kong	LTOTMKHK	DEADHK	Jan-1973
Ireland	LTOTMKIR	DEADIR	Jan-1973
Israel	LTOTMKIS	DEADIS	Jan-1992
Italy	LTOTMKIT	DEADIT	Jan-1973
Japan	LTOTMKJP	DEADJP	Jan-1973
Netherlands	LTOTMKNL	DEADNL	Jan-1973
New Zealand	LTOTMKNZ	DEADNZ	Mar-1988
Norway	LTOTMKNW	DEADNW	Jan-1980
Portugal	LTOTMKPT	DEADPT	Jan-1990
Singapore	LTOTMKSG	DEADSG	Jan-1973
Spain	LTOTMKES	DEADES	Jan-1986
Sweden	LTOTMKSD	DEADSD	Jan-1982
Switzerland	LTOTMKSW	DEADSW	Jan-1973
United Kingdom	LTOTMKUK	DEADUK	Jan-1973
United States	LTOTMKUS	DEADUS	Jan-1973

Source: Refinitiv Datastream.

Notes: Identifiers displayed were relevant when data has been originally downloaded.

has less than 80 active stocks. We assigned the *sector* to those countries with a number of stocks between 80 and 900 and the *subsector* to the countries with more than 900 constituents. Table 3 reports the region and the level of sectoral granularity assigned to each country, with the corresponding number of constituents downloaded, distinguishing between active and delisted (dead) securities.

The investment universe considered for this research is composed by 45,646 companies whose prices have been downloaded at daily frequency for a time horizon of more than 48 years. Given that the data infrastructure plays a crucial role for the implementation of the methodologies to approximate the GOP (recall that the Diversification Theorem presented in Section 2.4.2 works only if the investment is spread over a sufficiently large number of primary security accounts), it is worth providing some details about data cleaning procedures and relevant data pre-processing.

Table 3: Number of constituents downloaded, region and industrial grouping assigned to each country

Country	Region	Industrial grouping	No. Active	No. Dead	Total
Australia	Asia-Pacific	Sector	155	1656	1811
Austria	EMEA	Supersector	49	185	234
Belgium	EMEA	Sector	84	250	334
Canada	Americas	Sector	244	4885	5129
Denmark	EMEA	Supersector	46	269	315
Finland	EMEA	Supersector	47	125	172
France	EMEA	Sector	246	1416	1662
Germany	EMEA	Sector	237	1814	2051
Hong Kong	Asia-Pacific	Sector	122	296	418
Ireland	EMEA	Supersector	30	89	119
Israel	EMEA	Supersector	49	515	564
Italy	EMEA	Sector	152	389	541
Japan	Asia-Pacific	Subsector	969	1849	2818
Netherlands	EMEA	Sector	99	324	423
New Zealand	Asia-Pacific	Supersector	50	241	291
Norway	EMEA	Supersector	45	434	479
Portugal	EMEA	Supersector	46	128	174
Singapore	Asia-Pacific	Sector	99	385	484
Spain	EMEA	Sector	116	229	345
Sweden	EMEA	Supersector	61	408	469
Switzerland	EMEA	Sector	138	278	416
United Kingdom	EMEA	Sector	511	4504	5015
United States	Americas	Subsector	943	20439	21382

Source: Refinitiv Datastream.

Notes: Countries are sorted in alphabetical order.

For each constituent of a country index RD provides several information. Following the approach of Platen and Rendek (2020), for companies that hold more than one equity security we kept only the one associated to the highest market capitalization, by filtering only the rows whose variable "MAJOR" = "Y". To avoid duplicate entries and ensure consistency, we kept only the securities for which the country in which the company is headquartered ("GEOGN") is reported to be the same as the country in which the primary security is listed ("GEOLN"). Each company is labeled into an industry according to the ICB classification. Datastream provides this information under the labels "FTAG3" for supersectors, "FTAG4" for sectors and "FTAG5" for subsectors.

For the downloaded stocks we have obtained the timeseries of (total return) prices and the market capitalization, taking care of converting the values in USD when they were expressed in other currencies (this was the case for most of the delisted stocks). Prices equal to zero have been removed from the timeseries to avoid problems when calculating stock returns. When a company is delisted, the security is inserted into the index of dead companies for the country. After delisting, RD repeats the last price recorded for all the subsequent days. To avoid the generation of a sequence of zero returns (from the day of delisting until the day of the download), we removed all the observations reported after the delisting with an algorithm, given that this information was not directly available in RD. Rows associated with more missing values than the 95^{th} percentile (calculated for each country separately) are removed from the dataset.

We denote with S_t^j the price (in USD) of the jth stock in the dataset at time $t \in [0, \infty)$. Daily returns have been calculated according to:

$$R_t^j = \frac{S_t^j - S_{t-1}^j}{S_t^j} \tag{3.2.1}$$

We found necessary to winsorize daily returns respectively smaller than the 1^{st} percentile or larger than the 99^{th} percentile (calculated on the vector of returns recorded at each time t). With this procedure we removed data points associated with an unreasonable extreme behavior, most likely due to reporting mistakes.

3.3 Descriptive statistics and preliminary results

Focusing on a discrete time setting, we introduce by $0 = t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots$ the rebalancing times for the portfolio S^δ. We also denote with $S_{t_i}^j$ the cum-dividend price of the jth stock denominated in US dollars. At time t_i the value of the portfolio $S_{t_i}^\delta$ can be calculated recursively as

$$S_{t_i}^\delta = S_{t_{i-1}}^\delta \left(1 + \sum_{j=1}^{N_{t_{i-1}}} \pi_{t_{i-1}}^j \frac{S_{t_i}^j - S_{t_{i-1}}^j}{S_{t_{i-1}}^j} \right) \tag{3.3.1}$$

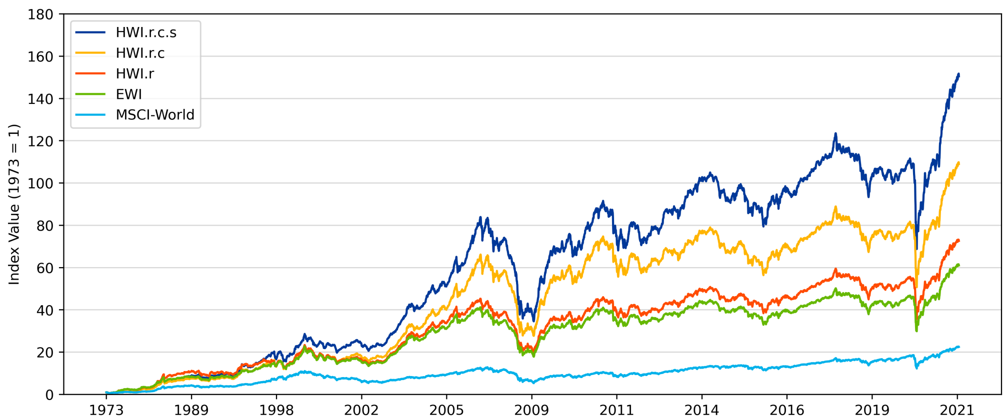
for $i \in \{1, 2, \dots\}$ and being $S_{t_0}^\delta = 1$.

We use Equation (3.3.1) to calculate the value over time of the HWI and EWI by plugging-in the respective weights as specified in Equations (3.1.2) and (3.1.3). From now on we refer to the three different specifications of the HWI as HWI.r.c.s (region-country-sector), HWI.r.c (region-country) and HWI.r (region) according to the hierarchies considered.

For the MSCI-World, we adjusted the timeseries downloaded by forcing its value at inception $S_{t_0}^{MSCI} = 1$. We also cut out all the daily observations for which we do not have a corresponding value for the other portfolios, since those are in general rebalanced with non-fixed frequency due to the rebalancing rule already presented.

In Figure 4 we display the evolution over time of the five investment strategies under analysis. All the trajectories follow a similar pattern but the HWI.r.c.s appears to have outperformed the others from the beginning, increasing its value up to 149 units over the whole history of 48 years. It follows the HWI.r.c, with a total increase with respect to the starting value of 109 units. HWI.r and EWI evolve in a very similar way, overlapping frequently until the year 2009, when the HWI.r started to record a more consistent increase that led to a stable, even if small, gap between the two strategies. The MSCI-World index is clearly the strategy which records the worst performance, achieving a value of 22 units in the same time horizon. These preliminary results already give an idea of the robustness of the approach chosen, showing that naive-diversified investments tend to outperform substantially the MSCI-World in the long-term.

Figure 4: Trajectories of GOP candidates

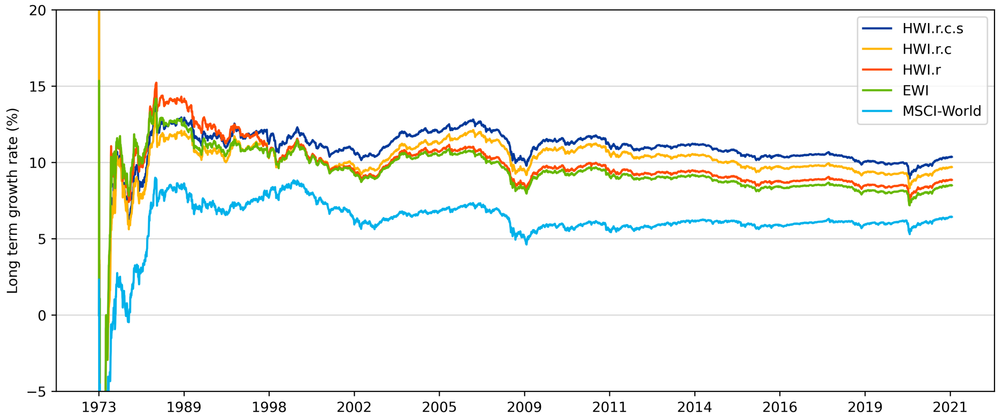


Source: Refinitiv Datastream and author's calculations.

Using again $S_{t_i}^{\delta}$ for the value at time t_i of a strictly positive portfolio corresponding to the strategy δ, which is associated to the weights process $\pi_{t_i}^{\delta}$, we write the respective growth rate (GR) at time $t_i > 0$ in the form:

$$G_{t_i}^{\delta} = \frac{1}{t_i} \log \left(\frac{S_{t_i}^{\delta}}{S_{t_0}^{\delta}} \right) \tag{3.3.2}$$

Figure 5: Long-term growth rates of GOP candidates



Source: Refinitiv Datastream and author's calculations.
Notes: Results at the beginning of the period are partially hidden to improve chart's readability.

Figure 5 shows the long term annualized growth rate for the five GOP proxy candidates, calculated as in Equation (3.3.2). Despite the first period, in which growth rates have been very volatile, due to the observation window still being short, they start to stabilize around 1985. From 1990 onward it is possible to appreciate how the strategies performed, recording similar fluctuations in the GRs and constant deviations relative to each other.

We observe that the HWI.r.c.s outperforms the HWI.r.c, which in-turn outperforms the HWI.r, with the EWI coming last. This ranking already indicate that the addition of extra hierarchical levels delivers an improvement in the GR of the portfolio. Consequently, we can confirm that the information intrinsically embedded in the geographical and industrial classification of the companies is representative of the macroeconomic uncertainties that companies face. By looking at the results, and recalling that the GOP is the portfolio that maximizes the long term GR of investor's wealth, we can already understand that the HWI.r.c.s is the best GOP proxy among those that are analyzed.

In the next chapter all the GOP proxy candidates presented are evaluated by mean of different performance measures. Moreover, the local martingale property is tested by showing that expected instantaneous returns are never strictly greater than zero when benchmarked by the GOP.

4 Empirical results and GOP proxies validation

This chapter provides empirical evidence that the portfolio specifications previously proposed and described are suitable candidates to approximate the Growth Optimal Portfolio. In the first section of this chapter we compare and rank the GOP proxy candidates by mean of the most common metrics of performance in the field of equity fund management, with the objective of identifying the optimal one. The measures reported are similar to those proposed by Platen and Rendek (2020), to make the results of the two researches comparable.

In the second section the local martingale property is tested for all the given strategies, using robust statistical tests to assess whether the stochastic processes associated to the benchmarked primary security accounts behave as supermartingales. Results are consistent with the assertion that a hierarchical weighting methodology based on the industrial and geographical classification of the companies is an effective solution to implement diversified strategies and efficiently disentangle systemic and systematic risks. We obtain evidence that the HWIs are in general outperforming both the EWI and MSCI-World. Furthermore, when additional hierarchies (*i.e.*, layers of diversification) are considered, the long term performance of the portfolio improves significantly, and the defining mathematical features of the GOP can be demonstrated with stronger confidence.

4.1 Performance measures

To validate the theoretical framework derived in the previous chapters, and to evaluate comparatively the performance of GOP proxy candidates, we report in this section the most common metrics and benchmarks employed to back-test trading strategies.

As already discussed, the numeraire portfolio should theoretically represent the optimal strategy among all the different allocations that one could potentially construct with a specific universe of assets, say Φ. Important to note is that it exists an entire subset $\phi \in \Phi$ of strategies that are suitable to approximate the GOP. However, the preferred candidate must be capable of outperforming all the other portfolios $\in \Phi$.

Due to the rebalancing rule chosen to construct the portfolios, the data points of the resulting timeseries are not equally distant from each other, depending on how many stocks have been delisted during a given quarter. To ensure readability of the results, and comparability with the main findings of the literature, some of the statistics that will be presented are calculated on annual basis, by taking into account only the value that each index reaches on the last trading day of the year.

We found evidence that the Morgan Stanley Capital International Developed Markets Total Return Index (MSCI-World) – that stands as our benchmark for the market-capitalization weighted indexes – substantially and systematically underperforms naive-diversified strategies. This is the case when looking at long time horizons but also when considering shorter time intervals. The MSCI-World draws stocks from the same sample of 23 developed countries employed to construct the other portfolios, but allocates wealth approximately among 1700 securities, being less effective in capturing different sources of uncertainty. This result reveals that naive-diversified strategies, are at the same time less complex to construct, less risky, and more profitable than other strategies which are traditionally standard benchmarks in the industry.

First column of Table 4 reports the annualized percentage growth rate of portfolios' returns, computed as in Equation (2.2.2), over the longest observation window that is available (*i.e.*, the growth rate achieved in 48 years, from 1973 to 2021). Evidence is that naive diversified portfolios outperformed historically the MSCI-World, from a minimum of 207 basis points for the EWI, to a maximum of 393 basis points for the HWI.r.c.s.

From the same table it is possible to appreciate how the introduction of additional hierarchies has been effective in pushing the HWIs closer to the GOP, in support of the idea that making use of the ICB industry classification has been effective to capture the exposure of stocks to similar sources of uncertainty. This effect is visible when looking at the growth rate, the average return and the Sharpe ratio, which are gradually deteriorating toward the values of the EWI when eliminating the sector and the country grouping sequentially.

It should also be noted that introducing additional hierarchies comes at the cost of a slightly higher volatility. This effect is reasonably expected, since introducing more granular groupings means weighting fewer securities in each family positioned in the first layer of the portfolio. Furthermore, the increase in the standard deviation is more than compensated by a higher average return, making our choices justified. HWI.r.c.s shows the highest Sharpe-ratio, 25% larger than the one achieved by the MSCI-World. This means that we have been able to increase substantially the performance for each unit of uncertainty just by refining the diversification strategy.

As an additional robustness check we compare our results with the findings of Platen and Rendek (2020)[20], who conducted a similar exercise but using 2014 as cutoff year for the data. The two analysis are similar, but the investment strategies to proxy the GOP are constructed with a slightly different methodology. In particular, the authors put a cap on the maximum number of securities that can be included for a country at a specific point in time. When more than 1000 companies were active, they only kept the ones with the highest market value. In the case of our research, we decided to include all the securities downloaded without any restriction.

As a logical result of including data recorded during the Covid-19 pandemic, that hit around March 2020 with an almost immediate shock on the financial markets, the average returns and growth rates of the portfolios we considered are about 220 basis point lower than what has been calculated by Platen and Rendek (2020) for the same strategies. In terms of volatility, the MSCI-World shows a substantially lower standard deviation when compared to the EWI and HWI. The contrary was shown by Platen and Rendek (2020), where the MSCI-World proved to be the most volatile strategy. To investigate this difference we tried to calculate our performance measures cutting off our data to the end of 2014. We observed that volatilities of HWI, EWI and MSCI-World converge to similar values. Thus, we can reasonably attribute the differences observed to the time horizon considered. Finally when looking at the Sharpe-ratios, the ranking of the strategies under the two analysis is the same. Although performance measures slightly diverge, the main result is consistent across our work and the one of Platen and Rendek (2020), with HWI.r.c.s being the best performing strategy and the MSCI-World the worst.

To complement with additional considerations the results shown in Table 4, in Figure 6 average annualized daily returns are displayed against average annualized volatilities. Theory suggests that the GOP is also an efficient portfolio in the sense of Markowitz (1959), thus we expect our preferred candidate to exhibit the best mean-variance combination, and the highest Sharpe-ratio, when compared to the other candidates (see Section 2.3.1). In addition, the efficient investment frontier is characterized by portfolios that share the same Sharpe-ratio. The strategy HWI.r.c.s is positioned on the highest boundary, and is associated to the optimal returns-volatility structure.

20. "Approximating the Growth Optimal Portfolio and stock price bubbles", see Table 3 on Page 13.

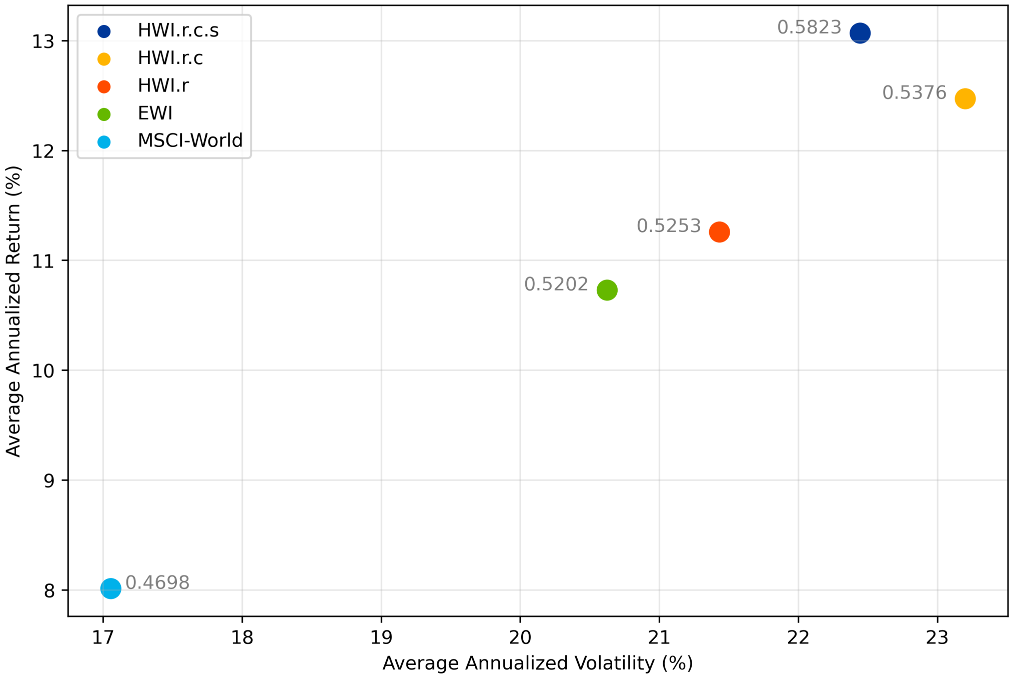
Table 4: Growth rate, average return, volatility and Sharpe-ratio of GOP candidates

Index	Growth rate	Average return	Volatility	Sharpe ratio
HWI.r.c.s	10.36	13.06	22.44	0.582
HWI.r.c	9.70	12.47	23.20	0.537
HWI.r	8.86	11.25	21.43	0.525
EWI	8.50	10.72	20.62	0.520
MSCI-World	6.43	8.01	17.05	0.469

Source: Refinitiv Datastream and author's calculations.

Notes: Sharpe ratio has been calculated assuming a risk-free interest rate of 0.00%.

Figure 6: Average return vs. average volatility of GOP candidates



Source: Refinitiv Datastream and author's calculations.
Notes: Observations are labelled with their corresponding Sharpe-ratio.

We emphasize that the long term growth rate is a key measure in the context of the benchmark approach, as per the definition of the GOP already provided. Table 5 reports the difference on average annualized percentage GR for all the GOP candidate strategies compared to the HWI.r.c.s (*i.e.*, the best performing strategy according to the metrics already analyzed) and the relative 95% confidence intervals for the estimates. The difference in the growth rates is calculated and reported over five different (rolling) observation windows, to highlight that when considering longer time periods the divergence in the growth rates increases gradually (this is consistent with the GOP being defined as optimal portfolio allocation that maximizes the GR of the investor's wealth in the long-run).

Confidence intervals are calculated as:

$$\left[\mu_w \pm 1.96 \cdot \frac{\sigma_w}{\sqrt{n_w}}\right] \tag{4.1.1}$$

where μ_w and σ_w are respectively the average and the standard deviation of the growth rates recorded on all the rolling observation windows of length $w \in \{1, 2, 3, 5, 10\}$, and n_w is the corresponding number of observations.

The deviation in the annualized GRs of all the given strategies when compared to the GR of the HWI.r.c.s are persistently positive and significantly different from 0, providing further evidence that the latter performs better and is asymptotically closer to the GOP. The largest divergence in the GR is achieved when adding the country grouping to the portfolio, with a 123 basis points increase between the HWI.r and the HWI.r.c over the 10 years observation window.

Over the 10 years window, switching from a market capitalization-weighted index (MSCI-World) to an equally-weighted index (EWI) allows to achieve an increase in the GR of 197 basis points, at further support of naive-diversification better performing in the long-run. Confidence intervals provided in Table 5 become wider for shorter time windows. Indeed, if the GR is computed over a longer time period, estimates are less affected by short-term price fluctuations. This implies that the standard deviation that goes into Equation (4.1.1) is smaller, driving the corresponding CI to narrow down.

Differences in the growth rates are again in line with the findings of Platen and Rendek (2020). For some of the strategies we calculated smaller differences, again as a result of the fact that we capture an additional shock in the financial markets by including data for the year 2020. Furthermore, we observe that confidence intervals are wider for our analysis. This is probably due to different ways of annualizing growth rates. In our case, growth rates are calculated based on the last value of each year, which ultimately lead to a large reduction of the sample. Taking as reference Equation (4.1.1), and provided that portfolios' volatilities are comparable, the only explanation can be a different number of observations. Even if the two studies diverge in some aspects, the main result is preserved. The hierarchically weighting allocation strategy is not only the best performing one, but seems also to be the most resilient to huge shocks in the financial markets.

Table 5: Difference on average annualized percentage growth rate over different observation windows (95% confidence intervals in parenthesis) for HWI.r.c., HWI.r, EWI and MSCI-World compared to HWI.r.c.s

Years	HWI.r.c	HWI.r	EWI	MSCI-World
1	0.65 (0.12,1.18)	1.47 (0.14,3.10)	1.84 (0.35,3.32)	3.88 (1.09,6.68)
2	0.65 (0.21,1.09)	1.45 (0.24,2.66)	1.82 (0.74,2.89)	3.91 (1.62,6.20)
3	0.67 (0.30,1.05)	1.51 (0.54,2.47)	1.85 (0.95,2.74)	4.04 (2.04,6.05)
5	0.71 (0.42,0.99)	1.60 (0.80,2.41)	1.97 (1.20,2.73)	4.21 (2.50,5.92)
10	0.72 (0.54,0.89)	1.95 (1.25,2.64)	2.31 (1.70,2.92)	4.28 (3.19,5.37)

Source: Refinitiv Datastream and author's calculations.

Table 6 reports observed frequencies for the event of outperforming the MSCI-World over daily, quarterly and yearly observation windows. The largest frequencies are observed for the HWI.r.c.s that outperformed the benchmark 61% of the times when looking at yearly returns. In addition, all the strategies tend to outperform the MSCI-World more frequently when returns are referred to longer periods. This outcome validates the systematic outperformance property of the GOP, see Section 2.2.3.

Our findings are again aligned with Platen and Rendek (2020). The ranking of the strategies is stable, and, as already mentioned, some minor differences can be attributed to the data and to the methodological approach.

Table 6: Relative frequency of outperforming the MSCI-World Index for different period lengths for HWI.r.c.s, HWI.r.c, HWI.r and EWI

Period length	HWI.r.c.s	HWI.r.c	HWI.r	EWI
Daily	0.5195	0.5140	0.5107	0.5073
Quarterly	0.5927	0.5618	0.5000	0.5257
Yearly	0.6122	0.5918	0.5918	0.5306

Source: Refinitiv Datastream and author's calculations.

Table 7 reports the average relative drawdown, the maximum drawdown, the Calmar ratio and the average recovery time for all the GOP strategies under analysis. In the first column it is again possible to appreciate how the performance of the portfolio improves when increasing the layers of diversification. The average drawdown for the HWI.r.c.s, assessed at 10.76% is almost 400 basis points lower than the average drawdown recorded for the MSCI-World index.

When looking at the maximum drawdown, results could appear counterintuitive at first sight. It seems that HWI.r and EWI portfolios are the best performing ones, while for the remaining the differences are negligible and all the maximum drawdowns recorded are close to 58%. However, when observing the Calmar ratio, that corresponds to the ratio between the average return recorded over the entire time horizon and the maximum drawdown, the ranking of the strategies that have been achieved and discussed before is re-established. This result confirms again how the GOP is effective in maximizing the long-run growth of the wealth, even if other strategies can perform better when considering shorter time intervals (*e.g.*, those periods around a financial crisis). Moreover, a worse performance in terms of maximum drawdown, which is recorded in a very specific point in time, is more than compensated by the long-term returns achieved with that strategy.

This effect is clearly visible in Figure 7 that shows the drawdown sequences for all the strategies from the base date to the end of 2021. The peak in the portfolio losses are concentrated around the three major crisis in the last 20 years (*i.e.*, the Internet bubble of 2002, the Financial crisis of 2008 and the Covid-19 pandemic of 2020).

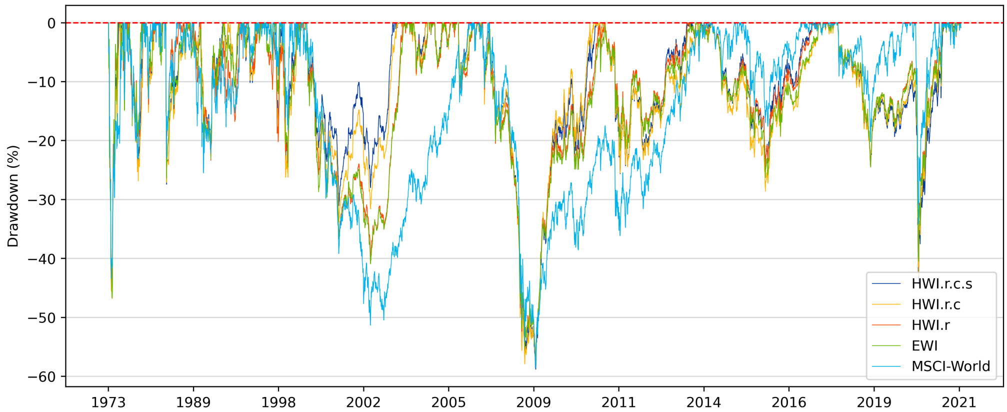
The last column of Table 7 shows that the average recovery time of each portfolio is increasing when decreasing the layers of diversification, going from 74 days for the most diversified portfolio (HWI.r.c.s), up to 112 days for the less diversified one (EWI). The recovery time for the MSCI-World is much smaller because the volatility of this strategy is about 1.4 times higher than the volatility of the others, which causes returns to fluctuate more in both directions and ultimately to recover completely in less time.

Table 7: Average relative drawdown, Calmar ratio and average recovery time (in days) for HWI.r.c.s, HWI.r.c, HWI.r, EWI and MSCI-World

Index	Avg. Drawdown	Max. Drawdown	Calmar Ratio	Avg. Recovery
HWI.r.c.s	10.76	58.80	0.0019	74
HWI.r.c.	11.67	58.11	0.0018	87
HWI.r	11.82	56.49	0.0017	96
EWI	12.18	56.70	0.0016	112
MSCI-World	14.47	58.54	0.0013	76

Source: Refinitiv Datastream and author's calculations.

Figure 7: Drawdown Sequence for the GOP candidates



Source: Refinitiv Datastream and author's calculations.

From the perspective of an equity fund manager it is important to take into account transaction costs associated to the construction and maintenance of the portfolio. These fees could substantially reduce the returns that the investment delivers. Many articles and publications, including Platen and Heath (2006), Platen and Rendek (2012a), Platen and Rendek (2020), already provided evidence that transaction costs calibrated according to common practices in the financial markets do not significantly impact the performances of HWI strategies, nor causes the EWI and MSCI-World to overcome their performance. Given the robustness of the results achieved in the literature, we do not account for transaction costs in our analysis.

4.2 Local martingale property

In this section we provide empirical evidence that the *local martingale property* cannot be easily rejected when the HWI is used as benchmark. This is a defining property of the GOP, that, when used as a benchmark, causes instantaneous returns of benchmarked portfolios to be zero (Platen and Rendek 2020). Moreover, as shown by Protter (2004), any non-negative local martingale process is a also a supermartingale[21]. This result is mentioned in the literature as the *supermartingale property* of non-negative benchmarked securities, and justifies the fact that benchmarked returns can be negative over strictly positive time periods. By showing that the mean of returns for stocks benchmarked by that portfolio is lower or equal to zero, but never positive, one can assess if a portfolio is a good proxy of the GOP.

21. The supermartingale property of non-negative local martingales is also discussed in the Appendix A.2, while in Appendix B.1 an analytical proof is provided.

To test these properties we use a one-tailed Z-test (Mode 1966) to reject the hypothesis that the mean returns for stocks benchmarked by each candidate proxy of the GOP are strictly positive. The test is performed over a sample of more than fifty million benchmarked returns, obtained by combining all the data points associated to the 45,646 securities that took part to the analysis. We rely on a standard statistical test – which requires observations to be independent and identically distributed – under the assumption that benchmarked returns are reasonably independent when observed on different days. Furthermore, discounting returns for the GOP allows to eliminate systemic risk stemming from the broad macroeconomic environment and conditions (Platen and Rendek 2020). Benchmarked returns of different stocks are driven only by their idiosyncratic uncertainties, and can be treated as independent when observed in the same day[22].

The system of hypothesis that will be tested is:

$$H_0 : \mu \leq 0 \qquad vs. \qquad H_1 : \mu > 0$$

where μ denotes the "true" expected average return of all benchmarked stocks when all the periods are considered.

In a discrete time setting, we denote $0 = t_0 < t_1 < \cdots < t_i < t_{i+1} < \cdots$ the rebalancing times for the portfolio. We also define the jth benchmarked primary security account as:

$$\hat{S}^j_{t_i} = \frac{S^j_{t_i}}{S^{\delta_*}_{t_i}} \tag{4.2.1}$$

where $S^{\delta_*}_{t_i}$ is the value of the benchmark portfolio at a specific rebalancing date t_i.

We will refer to the returns $\hat{R}^j_{t_i}$ associated to $\hat{S}^j_{t_i}$ as benchmarked returns, where the benchmarks will be all the GOP proxy candidates under analysis:

$$\hat{R}^j_{t_i} = \frac{\hat{S}^j_{t_i} - \hat{S}^j_{t_{i-1}}}{\hat{S}^j_{t_{i-1}}} \tag{4.2.2}$$

As mentioned before, the intervals between rebalancing dates for our portfolios are not necessarily constant. In Section 4.1 returns have been annualized by taking into account

22. Empirically testing the local martingale property is a complex task, due to the fact that the property is expressed mathematically through an expected value, while we can only observe a single realization of the stochastic processes involved. However, the test proposed are robust and the assumptions made plausible. Platen and Rendek (2020) verify the local martingale property by mean of a bootstrap test, in order to relax the assumption of independent and identically distributed returns. Their results are consistent with those obtained with our methodology.

only the values registered in the last trading day of the year. However, in the context of testing the local martingale property, we found important to preserve a large number of observations, in order not to break the minimum assumption of *IID* returns already presented. As a solution, we decided to annualize benchmarked returns (assuming that they are not compounded over time, to ensure more stable outputs of the transformation) according to:

$$\left(\hat{R}^j_{t_i}\right)^{\frac{365}{t_i - t_{i-1}}} \tag{4.2.3}$$

where $t_i - t_{i-1}$ is the distance in days between rebalancing dates.

Table 8 summarizes some statistics calculated on benchmarked returns to assess the validity of the local martingale property. It reports the sample mean, the standard error, the 99% confidence interval for the "true" expected daily return of benchmarked stocks, the score of the Z-Test and the corresponding P-Value.

The HWI, in the three specifications proposed, appears to be the best strategy to proxy the GOP, leading the daily annualized returns of the benchmarked stocks to be strictly negative. In support of what has been presented in the previous section, it is possible to note that the level of diversification achieved through an EWI is not adequate to verify with sufficient robustness the local martingale property. This output underlines the importance of systematically extracting non-diversifiable risk and reinforces the choice of the ICB classification as the tool to achieve this objective

When used as a benchmark the HWI.r.c.s is the diversification strategy that appears to be the most suitable to approximate the Growth Optimal Portfolio, leading daily annualized benchmarked stock returns to be -3.04% on average. Decreasing the granularity and the hierarchies of diversification in the HWIs pushes the portfolios toward the results obtained for the EWI. When discounted by the HWI.r.c, stock returns are on average -2.76%. When the benchmarks are the HWI.r and EWI the 99% confidence intervals include some positive values, meaning that statistical significance is decreasing, even if P-values are high enough not to reject the null hypothesis.

Finally, discounting with the MSCI-World Index does not allow to achieve the local martingale property, making this strategy not suitable to approximate the GOP and reinforcing the idea that traditional market-capitalization portfolio allocation can underperform substantially naive diversification strategies, if the time horizon is sufficiently long.

Table 8: One-sided Z-test for the mean of daily annualized percentage returns of benchmarked stocks

Benchmark	Sample Mean	Std Error	Z-Score	99% L-CI	99% U-CI	P-Value
HWI.r.c.s	-3.04297	0.13186	-23.08	-3.38318	-2.70276	1.00
HWI.r.c	-2.76498	0.13188	-20.97	-3.10523	-2.42473	1.00
HWI.r	-0.05016	0.13176	-0.38	-0.39009	0.28978	0.65
EWI	0.00000	0.13164	0.00	-0.33962	0.33962	0.49
MSCI-World	2.92669	0.13344	21.93	2.58242	3.27096	0.00

Source: Refinitiv Datastream and author's calculations.

We demonstrated that using the GOP as a discounting factor leads the returns of the underlying primary securities to be negative. Furthermore, from the theoretical framework presented, we know that the GOP is the only allocation that forces all the other non-negative portfolios constructed with the same universe of assets to behave as supermartingales, when benchmarked. As a robustness check, we want to assess if the local martingale property holds when benchmarking other GOP proxies. Following the same logic and framework applied previously for the primary security accounts, and the same set of hypothesis already presented, we perform a one-sided Z-test on the average annualized returns of all the GOP candidates when benchmarked by the HWI.r.c.s (*i.e.*, the best performing portfolio according to performance metrics evaluated).

Table 9 displays the results of the Z-test. In the last column, P-values close to 1 indicate that the average annualized returns of all the GOP proxies are negative with a high probability when discounted by the HWI.r.c.s. This result is consistent with the idea that capturing in a more precise and systematic way non-diversifiable risk allows to obtain a GOP proxy with the correct characteristics. HWI.r.c.s is the specification which makes all the other portfolios to behave as strict-supermartingales. This result implicitly implies that HWI.r.c.s is the only strategy that can be considered as proxy of the GOP, given that it overcomes all the others.

Table 9: One-sided Z-test for the mean of daily annualized percentage returns of all the portfolio candidates benchmarked by the HWI.r.c.s

Benchmark	Sample Mean	Std Error	Z-Score	99% L-CI	99% U-CI	P-Value
HWI.r.c	-0.31743	0.33806	-0.94	-1.18962	0.55476	0.83
HWI.r	-2.45772	0.83353	-2.95	-4.60822	-0.30722	0.99
EWI	-2.65707	0.81945	-3.24	-4.77124	-0.54290	0.99
MSCI-World	-3.57860	2.39491	-1.49	-9.75746	2.60026	0.93

Source: Refinitiv Datastream and author's calculations.

From Table 9 we can also note how the Standard Error recorded for benchmarked MSCI-World returns is almost 3 times larger than the one compute for the other portfolios. This results in a wide confidence interval. The effect is driven by the significantly larger volatility of the benchmarked MSCI-World. While the other indices are constructed using the same universe of assets, the MSCI-World relies on a different set of securities which is much smaller in size. By design, the evolution of the portfolio values is not necessarily (or at least is less) synchronized with the evolution of the benchmark. This results in a (annualized) standard deviation of 165% compared to the 56% of the rest of the benchmarked returns.

In addition, compared to what has been shown in Table 8, the number of benchmarked returns on which we performed the Z-test is significantly lower, as it corresponds only to the returns calculated on the data points associated to a specific index (4670 when considering all the rebalancing performed, vs 50 Mln.). The wider confidence interval results in a slightly lower P-Value, which is anyways large enough to verify the local martingale property.

5 Conclusions

In the present research we illustrated the Benchmark Approach (BA) to quantitative finance, a framework for financial modeling that proved to be suitable when classical theories, based on strong notions of absence of arbitrage, turn out to impose unnecessary restrictions. In particular, it has been showed that market anomalies may arise when financial markets face periods of dramatic and unusual turbulence. As a consequence, some assumptions that hold in normal times could not be verified anymore, highlighting the need for more flexible mathematical frameworks to solve portfolio optimization and asset pricing problems.

In the first chapter we introduced the concept of Growth Optimal Portfolio (GOP), an investment strategy obtained by maximizing the growth rate of the wealth process, and an interesting object of study thanks to its peculiar properties, see Section 2.2.3. It has been shown that the GOP achieves the highest long-term growth rate when compared to any other non-negative portfolio, and its existence implies the impossibility of relative arbitrages (or, systematic outperformance). Furthermore, it maximises the expected logarithmic utility function, and can be connected to mean-variance investment theory through the two-fund separation theorem (see Section 2.3).

In the context of the BA, the GOP plays the role of *Numeraire*. Long (1990) provided a first definition of Numeraire as a non-negative self-financing portfolio, that, when used as a benchmark (*i.e.*, discounting factor), makes any other non-negative portfolio process either trendless or mean-decreasing. This property is equivalent to the absence of arbitrage of first kind as defined by Fontana and Runggaldier (2013), and makes the GOP and the BA suitable for several applications which have been analyzed in Section 2.3.

In Section 2.4 we showed that *Diversified Portfolios* (DPs) converge to the GOP for an increasing number of securities. This is a major result derived from the *Diversification Theorem* first proposed by Platen and Heath (2006). The convergence property of DPs allows to create good proxies of the Numeraire, without any need of modelling complex and unknown quantities, which is usually the main limitation of other approaches, see

DeMiguel, Garlappi, and Uppal (2009). This result provides the basis for the second part of this research, in which we focused on the approximation of the GOP, using real market data.

Our dataset includes stocks prices collected at daily frequency for more than 45 thousands companies, listed in 23 developed countries. In total, our observations are collected over a period of 48 years, from 1973 to 2021 (*i.e.*, the longest timeseries that was available in Refinitiv Datastream for the constituents of country indexes that we selected). For the data collection we refined the methodology of Platen and Rendek (2020), by performing additional data quality checks. Working with an extremely large number of assets is crucial to create diversified portfolios that converge asymptotically to the GOP.

Platen and Rendek (2012a) and Platen and Rendek (2020) already run similar exercises to the one we are proposing, with promising results achieved. However, their datasets do not include the recent past, and most importantly, do not include the outbreak of the Covid-19, that triggered a huge (and unusual) market reaction that could potentially affect the robustness of their results.

To this effect, we analyzed different proxies of the GOP, with the final objective of evaluating their performance and suitability to take the role of Numeraire. In particular we evaluated three different specifications of a Hierarchically Weighted Index (HWI), a more structured methodology for constructing equally-weighted portfolios, that relies on the information embedded in the economic structure of the market to optimize the performance of the portfolio. We also constructed an Equally Weighted Index (EWI) and downloaded data for the MSCI-World Index. It can be proved that these two investment strategies present the characteristics of Diversified Portfolios, and thus, both could be meaningful proxies of the GOP. Furthermore, they are commonly used as benchmarks in the field of equity fund management.

We confirmed the findings of Platen and Rendek (2012a) and Platen and Rendek (2020), who showed that a hierarchical diversification is the most efficient methodology to systematically extract from the investment the portion of idiosyncratic risk that is not remunerated by the markets. We evaluated the performance of each GOP proxy candidate by mean of traditional performance metrics, and performed a statistical test of the local martingale property over a large set of benchmarked risky assets. We concluded that the HWI, in its specification that includes 4 layers of diversification (also mentioned as HWI.r.c.s in the main text), is the best proxy of the GOP. It recorded an average annualized return of 13.06%, compared to 10.72% and 8.01% achieved by the EWI and the

MSCI-World. Furthermore, the HWI outperformed the EWI and MSCI-World, recording a gap in the long-term annualized growth rate of 231 and 428 basis points respectively. On a yearly basis, the returns of the HWI were higher than those of the MSCI-World 61% of the times. Finally, the HWI proved an outstanding ability to recover from periods of downturn, achieving the lowest average drawdown and recovery time.

Besides these specific results, two main general conclusions can be drawn from the analysis presented, which in our view, represent a valuable contribution to the existing literature on the Benchmark Approach.

(i) Diversified Portfolios, and in particular the HWI, proved to be suitable to approximate the GOP. According to what we have presented in Section 2.2.3, the GOP, in the quality of Numeraire, is considered the best performing portfolio under three main dimensions. The HWI, as good proxy of the GOP, achieved the best performance in terms of expected returns, long-term growth rate and proved to systematically outperform all other strategies. In addition, the HWI proved to be extremely resilient to acute financial shocks, recording the smallest deterioration of the performance. This result can be seen as a further validation of the results achieved by Platen and Rendek (2020), which have never been validated with an updated dataset.

(ii) When the HWI is used as benchmark, the local martingale property of benchmarked returns cannot be easily rejected. The local martingale property implies that benchmarked return processes are either mean decreasing or trendless. We tested this property by obtaining statistical evidence that benchmarked returns of primary risky assets and of other portfolios are on average negative. This is a relevant empirical result in support of our theoretical framework, which stands as an additional proof that the HWI approximates well the GOP. This is again in line with what has been reported in Platen and Rendek (2020).

To conclude, we collected some potential ideas that could be considered for future development of this project. Firstly, although several authors applied the benchmark approach for the valuation of contingent claims, all their works remain focused on theoretical modeling. To our knowledge, there is no evidence of a study that involves both the empirical approximation of the GOP and its application in the context of asset pricing. An interesting exercise could be a comparison between the valuations produced under the

BA and more traditional methodologies, using real data. Secondly, we studied that the benchmark approach can be applied in the context of portfolio optimization. Similarly to what we did using the EWI and MSCI-World, it could be interesting to compare classical mean-variance, or more generally, model-based allocation strategies, against the HWI. Finally, in the context of this research, we dedicated a significant effort in setting up a data infrastructure to construct a good proxy of the GOP, and evaluate it. The procedure we put in place could be automatized and transformed into a data processing pipeline to ingest and clean daily market data. This would allow to compute frequently the value of the GOP, and obtain a market index that could be published and easily employed for several applications.

A Notions of probability and stochastic processes

The stochastic nature of the financial phenomena require the use of flexible mathematical structures and tools that provide the potential for modeling abstract objects. More precisely, asset prices and other financial quantities evolve randomly over time, thus, the best representation of their dynamics is a probabilistic one. For the purposes of this research, we found relevant to report some results of general probability theory and present some useful notation to support the theoretical discussion and the empirical analysis. Given their central role in the quantitative finance field, we collected some notions about stochastic processes, with a focus on those that are more meaningful for the benchmark approach framework. In addition, some effort has been put on re-organizing some traditional concepts that have already been discussed in the literature, to showcase some key properties of stochastic processes that are frequently applied in the main text.

A.1 Probability space and filtrations

The triplet $(\Omega, \mathcal{A}, P)$ describes the essential probabilistic information that characterizes a random experiment, and is called a *probability space* when its elements (*i.e.*, the sample space Ω, the collection of events $\mathcal{A}$ and the probability measure P) satisfy specific relationships that allow to form a consistent set of rules for modeling probabilities. More precisely, if A and B are exclusive events (*i.e.*, $A \cap B = \emptyset$) taken from $\mathcal{A}$, the following set of relationships holds:

$$0 \leq P(A) \leq 1 \tag{A.1.1}$$

$$P(A^C) = 1 - P(A) \tag{A.1.2}$$

$$P(\emptyset) = 0, \qquad P(\Omega) = 1 \tag{A.1.3}$$

$$P(A \cup B) = P(A) + P(B) \tag{A.1.4}$$

where $P(\cdot)$ is the probability of occurrence of an event.

If we consider infinite collections of events, $\mathcal{A}$ is defined as a *sigma-algebra*, which means that

$$\Omega \in \mathcal{A} \tag{A.1.5}$$

$$\text{if } A \in \mathcal{A} \text{ then } A^C \in \mathcal{A} \tag{A.1.6}$$

$$\text{if } A \in \mathcal{A} \text{ and } B \in \mathcal{A} \text{ then } A \cup B \in \mathcal{A} \tag{A.1.7}$$

$$\text{if } A_i \in \mathcal{A} \text{ for any } i \in \mathbb{N} = \{1, 2, \dots\} \text{ then } \left(\cup_{i=1}^{\infty} A_i\right) \in \mathcal{A} \tag{A.1.8}$$

If for any $i \neq j$ it holds that $A_i \cap A_j = \emptyset$, that is, the events are pairwise disjoint, Equation A.1.4 can be replaced with:

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \tag{A.1.9}$$

Market participants are in general interested in producing their best estimate of the future evolution of financial quantities. In this context, different kind of information are relevant and crucial to determine the accuracy of those estimates. We denote by $\mathcal{A}_t$ the *information set* available to the market participants at time $t \in [0, \infty)$. More precisely, $\mathcal{A}_t$ is the sigma-algebra

$$\mathcal{A}_t = \sigma\{\boldsymbol{X}_{\boldsymbol{s}} : s \in [0, t]\} \tag{A.1.10}$$

generated from the observation of the evolution of the vector process $\boldsymbol{X}$ up to time t, where the latter can describe quantities or represent information which is sensitive to the financial markets (*e.g.*, security prices, interest rates, balance sheet of companies, macroeconomic variables, indicators of the occurrence of political events, etc.).

In this setting, it is reasonable to assume that market participants are able to collect and retain information from the past. Thus, if all the information sets $\mathcal{A}_t$ are sub-sigma-algebras of $\mathcal{A}_\infty$, for any $0 \leq t_1 \leq t_2 \leq \cdots \leq \infty$ it holds the relation $\mathcal{A}_{t_1} \subseteq \mathcal{A}_{t_2} \subseteq \cdots \subseteq \mathcal{A}_\infty = \cup_{t \in [0,\infty)} \mathcal{A}_t$. We define the increasing right-continuous family of all information sets

$$\underline{\mathcal{A}} = \{\mathcal{A}_t, t \in [0, \infty)\} \tag{A.1.11}$$

a *filtration*.

Every right-continuous stochastic process $X = \{X_t,\ t \in [0, \infty)\}$ generates a filtration $\underline{\mathcal{A}}^X = \{\mathcal{A}_t^X,\ t \in [0, \infty)$ that can be interpreted as the complete record of all movements of the process X up to a specific point in time t. We define a process $Z = \{Z_t,\ t \in [0, \infty)\}$

to be $\mathcal{A}_t^{\boldsymbol{X}}$-measurable when the history of the process until time t is covered by the information set $\mathcal{A}_t^{\boldsymbol{X}}$, where $\boldsymbol{X}$ is the vector process that describes the total evolution of the model and the $\mathcal{A}_t^{\boldsymbol{X}}$ the corresponding increasing families of information sets.

Given that the propagation of the information is the main driver of the market dynamics, the concept of filtration and the technical assumptions presented above becomes of extreme importance for modeling financial markets. A *filtered probability space* $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$ equipped with the filtration $\underline{\mathcal{A}}$ satisfies the conditions (A.1.1) – (A.1.3) and (A.1.9) to define a probability space.

A.2 Stochastic processes

Stochastic processes are the mathematical tool that allows to model the evolution of random variables over time. This is particularly useful when modeling financial quantities, given the uncertain nature of their dynamic. In line with the purposes of the present research, three main classes of continuous time stochastic processes are presented, along with some related technical concepts which provides the mathematical basis to support a complete and clear illustration of the benchmark approach.

The evolution of asset prices can be described under a common underlying probability space $(\Omega, \mathcal{A}, P)$ by a collection of random variables $X_{t_0}, X_{t_1}, \dots, X_{\mathcal{T}}$ indexed over a set of observation times $t_0 < t_1 < \cdots < \mathcal{T}$. The family $X = \{X_t, t \in \mathcal{T}\}$ of random variables $X_t \in \Re$ is called a stochastic process, where $\Re$ is a the set on non-negative real numbers (or a subset of it), and $\mathcal{T}$ is the time set. Since asset prices are observable at any instant, the benchmark approach is developed on a *continuous time* setting. As a results, $\mathcal{T}$ is defined on the interval $[0, T]$ where $T \in [0, \infty)$.

Wiener Process (Brownian Motion)

An important class of stochastic processes for financial modeling, with suitable mathematical properties are the so called *processes with stationary independent increments*, where the random increments $X_{t_{j+1}} - X_{t_j}$ with $j \in \{0, 1, \dots, n-1\}$ are independent for any combination of time instants $t_0 < t_1 < \cdots < t_n$. The fundamental class of stochastic process used to model the motion of stock prices (or, more in general, to model processes that are characterized by continuous and strongly fluctuating random dynamics) is known under the name of *Wiener process* or *Brownian motion*.

Definition A.2.1. *We define a standard Wiener process* $W = \{W_t, t \in [0, \infty)\}$ *as a process with Gaussian stationary independent increments and continuous sample paths for which*

$$W_0 = 0, \qquad \mu(t) = E(W_t) = 0, \qquad Var(W_t - W_s) = t - s \tag{A.2.1}$$

for all $t \in [0, \infty)$ *and* $s \in [0, t)$.

Continuous Time Markov Process

Another type of process that is suitable for modeling financial markets is the *Markov process.* Markov processes satisfy the so called *Markov property* that allows to characterize the future evolution of a random variable without need to rely on the past history. More precisely, the predictions that one could make by looking at the full history of the process are just as precise as the one made by looking solely at the present value. Financial quantities like stock prices or exchange rates are characterized by continuous random movements. If the Markov property holds, modelling, statistical inference, or any other numerical analysis is simplified considerably, since probability distribution of the stock price at a particular point in time in the future depends only on the current stock price.

Martingale Process

The ultimate objective of investors is to produce the best estimate of the actual value of future payoffs. To do so, they rely on three main elements: *(i)* their individual information set, up to that specific point in time, which under the assumption of efficient markets, is assumed to be the same for all the investors, *(ii)* a probability measure for forming some expectations, and *(iii)* a benchmark (or numeraire) that provides the units in which the estimates are formulated (*e.g.*, the rate of return of a savings account or of the market portfolio).

Of particular importance for the modelling of financial markets is the notion of Martingale, a specific class of stochastic processes that does not show systemic trends in its dynamic and whose last recorded value provides the best forecast for its future evolution. A martingale is defined with respect to a probability measure P that denotes the likelihood of an event, and a filtration $\underline{\mathcal{A}}$ that represent the relevant family of information sets. We follow Platen and Heath (2006) to provide the following definition of martingale process.

Definition A.2.2. *We define a continuous time stochastic process* $X = \{X_t, t \in [0, \infty)\}$ *a* $(\underline{\mathcal{A}}, P)$*-martingale if for all* $t \in [0, \infty)$ *and all* $s \in [0, t]$ *it satisfies the property*

$$X_s = E(X_t | \mathcal{A}_s) < \infty \tag{A.2.2}$$

and the integrability condition

$$E(|X_t|) < \infty \tag{A.2.3}$$

If a price process is a Martingale, the best estimate produced at time $s \in [0, t]$ for its future value at time $t \in [s, \infty)$ is formed on the basis of all the information available at the present moment and contained in $\mathcal{A}_s$. Under the benchmark approach, derivative prices when expressed in units of the benchmark, are modelled to form martingales to exploit exactly this property.

Super- and Submartingales

To best represent the dynamic of asset prices, which in general are not completely trendless, it is useful to introduce the definition of Super- and Submartingales. These processes share the same technical features of the Martingales, but on average they decreases (increases) their value over time.

Definition A.2.3. *The* $\underline{\mathcal{A}}$*-adapted process* $X = \{X_t, t \in [0, \infty)\}$ *is defined as an* $(\underline{\mathcal{A}}, P)$*-supermartingale (-submartingale) if*

$$X_s \underset{\leq}{\overset{(\geq)}{}} E(X_t | \mathcal{A}_s) \tag{A.2.4}$$

and

$$E(|X_t|) < \infty \tag{A.2.5}$$

for $s \in [0, \infty)$ *and* $t \in [s, \infty)$.

Supermartingales are of central interest for this research, and for the benchmark approach to quantitative finance. In fact, when the price of a security is expressed in units of a particular benchmark (*i.e.* the Growth Optimal Portfolio) its time evolution can be modelled as a *strict supermartingale*, where for a strict supermartingale Equation (A.2.4) holds with strict inequality.

Stopping Times

In the real world, it is common that stochastic processes do not appear with the characteristics of martingales. More frequently, they become martingales when properly stopped. Before introducing the concept of *local martingales*, which is particularly relevant in the framework of the benchmark approach, it is worth presenting the notion of *stopping time*.

Definition A.2.4. *We define a random variable $\tau : \Omega \to [0, \infty)$ a stopping time with respect to the filtration $\underline{\mathcal{A}}$ if the relation*

$$\{\tau \leq t\} \in \mathcal{A}_t \tag{A.2.6}$$

hold for all $t \in [0, \infty)$.

The sigma-algebra associated with the stopping time τ is defined as:

$$\mathcal{A}_\tau = \sigma\{A \in \mathcal{A} : A \cap \{\tau \leq t\} \in \mathcal{A}_t \quad for \quad t \in [0, \infty)\} \tag{A.2.7}$$

and following the usual interpretation, it represents the information available before and until the stopping time τ.

Local Martingales

As anticipated, stopping times are useful to define local martingales, stochastic processes that show the features of martingales only if properly stopped.

Definition A.2.5. *The stochastic process $X = \{X_t, t \in [0, \infty)\}$ is defined as an $(\underline{\mathcal{A}}, P)$-local martingale if it is possible to identify an increasing sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times such that $\lim_{n\to\infty} \tau_n \overset{a.s.}{=} \infty$ and each stopped process*

$$X^{\tau_n} = \{X_t^{\tau_n} = X_{\min{(t, \tau_n)}}, \;\; t \in [0, \infty)\} \tag{A.2.8}$$

is an $(\underline{\mathcal{A}}, P)$-martingale.

It useful to highlight that an $(\underline{\mathcal{A}}, P)$-local martingale is not necessarily an $(\underline{\mathcal{A}}, P)$-martingale, while an $(\underline{\mathcal{A}}, P)$-martingale is also an $(\underline{\mathcal{A}}, P)$-local martingale. In the former case, the $(\underline{\mathcal{A}}, P)$-local martingale is defined *strict local martingale*.

Local martingale processes are extremely relevant in the context of this research. The following statement is formulated in Protter (2004). For completeness, a proof based on Platen and Heath (2006)[23] is reported in Appendix B.1.

Lemma A.2.6. *A non-negative (negative)* $(\underline{\mathcal{A}}, P)$*-local martingale* $X = \{X_t,\ t \in [0, \infty)\}$ *with* $E(X_t|\mathcal{A}_s) < \infty$ *for all* $0 \leq s \leq t \leq \infty$ *is an* $(\underline{\mathcal{A}}, P)$*-supermartingale (submartingale).*

Connecting the notion of local martingale to the notion of supermartingale is key to explain the behaviour of stock returns when they are benchmarked by the GOP, and allows to establish absence of arbitrage opportunities when modeling financial quantities under the benchmark approach.

23. An alternative proof based on Fatou's Lemma is provided by Rogers and Williams (2000).

B Proofs

B.1 Supermartingale property

Lemma (A.2.6). *A non-negative* $(\underline{\mathcal{A}}, P)$*-local martingale* $X = \{X_t,\ t \in [0,\infty)\}$ *with* $E(X_t|\mathcal{A}_s) < \infty$ *for all* $0 \leq s \leq t \leq \infty$ *is an* $(\underline{\mathcal{A}}, P)$*-supermartingale.*

Proof. If the process $X = \{X_t,\ t \in [0,\infty)\}$ is an $(\underline{\mathcal{A}}, P)$-local martingale, then one can also identify an increasing sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times, with respect to the filtration $\underline{\mathcal{A}}$, such that each stopped process $X^{\tau_n} = \{X_t^{\tau_n} = X_{\min t, \tau_n},\ t \in [0,\infty)\}$ is an $(\underline{\mathcal{A}}, P)$-martingale and we have $\tau_n \to \infty$ almost surely. Consequently, for each $n \in \mathbb{N}$ and $0 \leq s \leq t \leq \infty$ we have:

$$\begin{aligned} E(X_t|\mathcal{A}_s) &= E\left(\mathbf{1}_{\{\tau_n \geq t\}} X_t \middle| \mathcal{A}_s\right) + E\left(\mathbf{1}_{\{\tau_n < t\}} X_t \middle| \mathcal{A}_s\right) \\ &= E\left(\mathbf{1}_{\{\tau_n \geq t\}} X_t^{\tau_n} \middle| \mathcal{A}_s\right) + E\left(\mathbf{1}_{\{\tau_n < t\}} X_t \middle| \mathcal{A}_s\right) \\ &\leq E\left(X_t^{\tau_n} \middle| \mathcal{A}_s\right) + E\left(\mathbf{1}_{\{\tau_n < t\}} X_t \middle| \mathcal{A}_s\right) \\ &= X_s^{\tau_n} + E\left(\mathbf{1}_{\{\tau_n < t\}} X_t \middle| \mathcal{A}_s\right) \end{aligned} \tag{B.1.1}$$

By definition, $\mathbf{1}_{\{\tau_n \geq t\}} X_t$ approaches X_t from below almost surely as $n \to \infty$, for each $t \in [0,\infty)$.

The following monotone convergence theorem provides some relationships that allow to conclude the proof.

Theorem B.1.1 (Monotone Convergence). *Let* $Y, X, X_1, X_2, \ldots$ *be random variables.*

(i) If $X_n \geq Y$ *for all* $n \in \mathbb{N}$, $E(Y) > -\infty$ *and the sequence* $(X_n)_{n\in\mathbb{N}}$ *is monotone increasing, where* $\lim_{n\to\infty} X_n \overset{a.s.}{=} X$, *then*

$$\lim_{n\to\infty} E(X_n) = E(X) \tag{B.1.2}$$

(ii) If $X_n \leq Y$ for all $n \in \mathbb{N}$, $E(Y) < \infty$ and the sequence $(X_n)_{n\in\mathbb{N}}$ is monotone decreasing, where $\lim_{n\to\infty} X_n \overset{a.s.}{=} X$, then

$$\lim_{n\to\infty} E(X_n) = E(X) \tag{B.1.3}$$

By (B.1.2) and (B.1.1) the difference

$$E\left(X_t \middle| \mathcal{A}_s\right) - E\left(\mathbf{1}_{\{\tau_n<t\}} X_t \middle| \mathcal{A}_s\right) = E\left(\mathbf{1}_{\{\tau_n\geq t\}} X_t \middle| \mathcal{A}_s\right) \tag{B.1.4}$$

approaches almost surely the conditional expectation $E(X_t \big| \mathcal{A}_s)$ from below as $n \to \infty$. As a result, the quantity $E\left(\mathbf{1}_{\{\tau_n<t\}} X_t \middle| \mathcal{A}_s\right)$ converges to zero as $n \to \infty$. Furthermore, Theorem B.1.1 implies that $\lim_{n\to\infty} X_s^{\tau_n} \overset{a.s.}{=} X_s$. For $n \to \infty$, it holds $E(X_t \big| \mathcal{A}_s) \leq X_s$. Following the definition provided in Equation (A.2.4), the process $X = \{X_t,\ t \in [0,\infty)\}$ is a supermartingale. □

B.2 Absence of arbitrage in a CFM

Proposition (2.2.4)**.** *A CFM $\mathcal{S}^C_{(d)}$ does not allow arbitrage opportunities with any of its non-negative portfolios.*

Proof. To prove the absence of arbitrage in a CFM, we need to introduce the optional sampling theorem, firstly proposed by Doob (1953):

Theorem B.2.1 (Optional Sampling)**.** *If $X = \{X_t, t \in [0,\infty)\}$ is a right continuous $(\underline{\mathcal{A}}, P)$-supermartingale on $(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$, then for two bounded stopping times τ and τ', with $\tau \leq \tau'$ it holds almost surely that*

$$E(X_{\tau'} \mid \mathcal{A}_\tau) \leq X_\tau \tag{B.2.1}$$

If X is also a an $(\underline{\mathcal{A}}, P)$-martingale, then Equation (B.2.1) holds with equality.

Given the supermartingale property of benchmarked portfolio $\hat{S}^\delta$, for any non-negative portfolio $S^\delta \in \mathcal{V}$ with $S^\delta_0 = 0$, and for any unbounded stopping time $\tau \in [0,\infty)$, we have that:

$$0 = \hat{S}^\delta_0 \geq E\left(\hat{S}^\delta_\tau \mid \mathcal{A}_0\right) = E\left(\hat{S}^\delta_\tau\right) \geq 0 \tag{B.2.2}$$

Given that by definition $\hat{S}^{\delta}$ is non-negative, and the GOP S^{δ_*} used as benchmark in Equation (2.2.8) is strictly positive, it must hold that

$$P(S^{\delta}_{\tau} > 0) = P(\hat{S}^{\delta}_{\tau} > 0) = 0 \tag{B.2.3}$$

that implies that all the trajectories of investor's portfolio are absorbed at zero, and ultimately it proves the absence of arbitrage opportunities in a CFM. □

B.3 Long term growth rate

Theorem (2.2.5)**.** *In a CFM $\mathcal{S}^{\mathcal{C}}_{(d)}$ the GOP S^{δ_*} record almost surely the greatest long-term growth rate when compared to the long-term growth rate of all other strictly positive portfolios $S^{\delta} \in \mathcal{V}^{+}$. In mathematical terms, the inequality*

$$\bar{g}^{\delta *} \geq \bar{g}^{\delta} \tag{B.3.1}$$

holds almost surely.

Proof. Follow Karatzas et al. (1998) and Platen and Heath (2006), we consider a strictly positive portfolio $S^{\delta} \in \mathcal{V}^{+}$ with

$$S^{\delta}_0 = S^{\delta_*}_0 \tag{B.3.2}$$

Doob (1953) shows that if $X = \{X_t, t \in [0, \infty)\}$ is a right continuous supermartingale, for any $\lambda > 0$, it holds that

$$\lambda P\Bigg(\sup_{t \in [0,\infty)} X_t \geq \lambda \mid \mathcal{A}_0 \Bigg) \leq E(X_0 \mid \mathcal{A}_0) + E\Bigg(\max(0, -X_0) \mid \mathcal{A}_0 \Bigg) \tag{B.3.3}$$

By Definition 2.2.2 the strictly positive benchmarked portfolio $\hat{S}^{\delta}$ is an $(\underline{\mathcal{A}}, P)$-supermartingale, which combined to Equation (B.3.3), yields:

$$\exp\{\varepsilon k\} P\Bigg(\sup_{k \leq t < \infty} \hat{S}^{\delta}_t > \exp\{\varepsilon k\} \mid \mathcal{A}_0 \Bigg) \leq E\Big(\hat{S}^{\delta}_k \mid \mathcal{A}_0 \Big) \leq \hat{S}^{\delta}_0 = 1 \tag{B.3.4}$$

for $k \in \mathbb{N}$ and $\varepsilon \in (0, 1)$. For fixed $\varepsilon \in (0, 1)$:

$$\sum_{k=1}^{\infty} P\Bigg(\sup_{k \leq t < \infty} \ln\Big(\hat{S}^{\delta}_t\Big) > \varepsilon k \mid \mathcal{A}_0 \Bigg) \leq \sum_{k=1}^{\infty} \exp\{-\varepsilon k\} < \infty \tag{B.3.5}$$

The *Borel-Cantelli Lemma*[24] states that, for a sequence of events $A_1, A_2, \ldots$ in $\mathcal{A}$, if $\sum_{k=1}^{\infty} P(A_k) = \infty$, then the event that consists of the realization of infinitely many of the events $A_1, A_2, \ldots$ has probability zero. This implies the existence of a random variable K_ε such that

$$\ln\left(\hat{S}_t^\delta\right) \leq \varepsilon k \leq \varepsilon t \tag{B.3.6}$$

for $k \geq K_\varepsilon$ and $t \geq k$.

In addition, for $k \geq K_\varepsilon$, one has almost surely

$$\sup_{T \geq k} \frac{1}{T} \ln\left(\hat{S}_T^\delta\right) \leq \varepsilon \tag{B.3.7}$$

and therefore

$$\limsup_{T\to\infty} \frac{1}{T} \ln\left(\frac{S_t^\delta}{S_0^\delta}\right) \leq \limsup_{T\to\infty} \frac{1}{T} \ln\left(\frac{S_T^{\delta_*}}{S_0^{\delta_*}}\right) + \varepsilon \tag{B.3.8}$$

Since (B.3.8) holds for all $\varepsilon \in (0,1)$, then inequality (B.3.1) follows straight from the definition of the long-term growth rate $\bar{g}^{\delta*}$. □

B.4 Systematic outperformance

Following Platen and Heath (2006) one can derive the following statement from Definition 2.2.6:

Corollary B.4.1. *In a CFM $\mathcal{S}_{(d)}^C$ no strictly positive portfolio systematically outperforms the GOP.*

Proof. We consider a benchmarked non-negative portfolio $\hat{S}_\tau^\delta = \{\hat{S}_\tau^\delta, t \in [0,\infty)\}$. We assume that at a stopping time $\tau \in [0,\infty)$ the benchmarked value is

$$\hat{S}_\tau^\delta = 1, \tag{B.4.1}$$

and for a bounded stopping time $\sigma \in [\tau,\infty)$ the inequality

$$\hat{S}_\sigma^\delta \geq 1 \tag{B.4.2}$$

holds almost surely.

24. See Section 2.7 of Platen and Heath (2006).

By combining Definition 2.2.6, Equation (B.4.1), the supermartingale property derived in Lemma A.2.6, and the optional sampling theorem already introduced in Appendix B.2, one can verify that

$$0 \geq E\Big(\hat{S}^{\delta}_{\sigma} - \hat{S}^{\delta}_{\tau} \mid \mathcal{A}_{\tau}\Big) = \big(\hat{S}^{\delta}_{\sigma} - 1 \mid \mathcal{A}_{\tau}\big) \geq 0. \tag{B.4.3}$$

By (B.4.2) it follows that the benchmarked value $\hat{S}^{\delta}_{\sigma}$ cannot be strictly grater than $\hat{S}^{\delta}_{\tau} = 1$ with any strictly positive probability. In other words $\hat{S}^{\delta}_{\sigma} = \hat{S}^{\delta_*}_{\sigma}$, since $\hat{S}^{\delta}_{\sigma} = 1$ almost surely. □

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