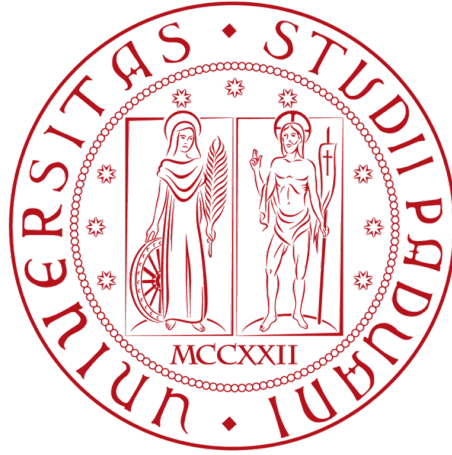# Università degli Studi di Padova

Dipartimento di Fisica ed Astronomia "Galileo Galilei"

Laurea in Fisica

Correlation functions and perturbative expansion

Funzioni di correlazione e serie perturbativa

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Contents

1   Introduction

The very first insight which led to the path integral formulation of quantum mechanics is due to Dirac in the work [1] where he was looking for the meaning of the action functional in quantum mechanics, later Feynman gave it the well known physical interpretation of "sum over all paths".
More recently the path integral formalism has proven itself to be a key tool in quantum field theories. One of the reasons of its success is that since for most theories it makes direct use of the Lagrangian, all the symmetries of the theory are preserved.
Another strong point of the formalism is the ease with which correlation functions can be computed, in perturbation theory, using Schwinger's identity. Computing the correlation function can be considered the goal for every quantum field theory: they can be related to transition amplitudes which ultimately determine the predictions of the theory itself.
In this work we illustrate the method exposed in the paper [2] and we make use of it to compute the 4-point connected Green's function to all perturbation orders for a normal ordered potential $V(\phi) = \frac{\lambda}{n!} : \phi^n :$ without using Feynman's rules.
The central point of the method is to consider the exponential interaction as a master potential: in this way functional derivatives are replaced by ordinary derivatives when computing the correlation functions, thus making possible to control the combinatorics in the explicit calculations. Furthermore the exponential allows to absorb at once all the singularities due to the normal ordering.
The work is organized as follows: we first introduce the path integral formalism in quantum mechanics and scalar quantum field theories, then we extend the path integral formalism to these theories in section 4. In section 5 we introduce exponential interactions which will be the starting point for the method. The method is fully illustrated in section 6 where is also given an expression for the 4-point Green's function, explicit calculations used in computing it are attached in the appendix.
Everywere in this work we use the convention $\hbar = c = 1$.

2   Path integral in quantum mechanics

In this section we will calculate the transition amplitude $\langle Q, T | q, t \rangle$ using the path integral formalism as in [3].
We start by splitting up the interval $[t, T]$ into N infinitesimal time intervals $t_k = t + k\epsilon$ with $N\epsilon = T - t$. Using the completeness relation $\mathbf{1} = \int dq \, |q, t\rangle \, \langle q, t|$ for

each t_k we obtain

$$\langle Q,T|q,t\rangle = \int dq_1 \ldots dq_{N-1} \langle Q,T|q_{N-1},t_{N-1}\rangle \langle q_{N-1},t_{N-1}|q_{N-2},t_{N-2}\rangle$$
$$\cdots \langle q_1,t_1|q,t\rangle \tag{2.1}$$

Thus we only need to know how to calculate the transition amplitude for infinitesimal time intervals. Inserting a complete set of momentum eigenstate at time t' between t_k and $t_k + \epsilon$ we can write

$$\langle q_{k+1},t_k + \epsilon|q_k,t_k\rangle = \int dp \langle q_{k+1},t_k + \epsilon|p,t'\rangle \langle p,t'|q_k,t_k\rangle \tag{2.2}$$

Using the canonical commutation relations between $\hat{q}$ and $\hat{p}$ we can write $\hat{H}(\hat{q},\hat{p})$ so that all the $\hat{q}$ operators appear on the left of the $\hat{p}$ operators. Let $\hat{H}_+$ be the operator written in such a way. Recalling that

$$\langle q_{k+1},t_k + \epsilon|p,t'\rangle = \langle q_{k+1},0|\, e^{-i\hat{H}(\hat{q},\hat{p})(t_k+\epsilon-t')}\,|p,0\rangle$$

to the first order in $t_k + \epsilon - t'$ we have

$$\langle q_{k+1},t_k + \epsilon|p,t'\rangle \approx \langle q_{k+1},0|\,(1 - i\hat{H}_+(\hat{q}_{k+1},\hat{p})(t_k + \epsilon - t'))\,|p,0\rangle$$
$$= \langle q_{k+1},0|p,0\rangle\,(1 - iH_+(q_{k+1},p)(t_k + \epsilon - t'))$$
$$\approx \frac{1}{\sqrt{2\pi}}e^{i(pq_{k+1}-H_+(q_{k+1},p)(t_k+\epsilon-t'))} \tag{2.3}$$

where the function $H_+(q_{k+1},p)$ is obtained from the operator $\hat{H}_+(\hat{q}_{k+1},\hat{p})$ substituting $\hat{p}$, $\hat{q}$ with their eigenvalues.
With similar calculations:

$$\langle p,t'|q_k,t_k\rangle \approx \frac{1}{\sqrt{2\pi}}e^{i(-pq_k-H_-(q_k,p)(t'-t_k))} \tag{2.4}$$

With H_- obtained putting all the position operators to the right of the momentum operators by using the canonical commutation relations. Choosing $t' = \frac{t_k+t_{k+1}}{2}$ and using equations (2.3) and (2.4), equation (2.2) becomes

$$\langle q_{k+1},t_k + \epsilon|q_k,t_k\rangle \approx \int \frac{dp}{2\pi}e^{i\epsilon[\frac{p(q_{k+1}-q_k)}{\epsilon}-H_c(q_{k+1},q_k,p)]} \tag{2.5}$$

We have set $H_c(q_{k+1},q_k,p) = \frac{H_+(q_{k+1},p)+H_-(q_k,p)}{2}$.
Defining $\dot{q}_k \equiv \frac{q_{k+1}-q_k}{\epsilon}$ and $H(q_k,p) \equiv H_c(q_{k+1},q_k,p)$ we obtain

$$\langle q_{k+1},t_k + \epsilon|q_k,t_k\rangle \approx \int \frac{dp}{2\pi}e^{i\epsilon L(q_k,p)} \tag{2.6}$$

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With $L(q_k, p) \equiv p\dot{q}_k - H(q_k, p)$. Inserting (2.6) in (2.1) and taking the limit $N \to \infty$, with $N\epsilon$ fixed, we find

$$\langle Q, T | q, t \rangle = \lim_{N \to \infty} \int \left(\prod_{k=1}^{N-1} dq_k \right) \left(\prod_{k=0}^{N-1} \frac{dp_k}{2\pi} \right) e^{i \sum_{k=0}^{N-1} \epsilon L(q_k, p_k)}$$

$$= \lim_{N \to \infty} \int \left(\prod_{k=1}^{N-1} dq_k \right) \left(\prod_{k=0}^{N-1} \frac{dp_k}{2\pi} \right) e^{i \int_t^T dt L(q, p)} \tag{2.7}$$

In the following we will write (2.7) using the notation

$$\langle Q, T | q, t \rangle = \int \mathcal{D}p \mathcal{D}q \, e^{iS(t, T, [q], [Q])} \tag{2.8}$$

where the square brackets indicate that S is a function over all trajectories such that $q(t) = q$ and $q(T) = Q$. When the Hamiltonian has the form $H(q, p) = \frac{p^2}{2m} + V(q)$ we can explicitly integrate over the momenta obtaining

$$\langle Q, T | q, t \rangle = \lim_{N \to \infty} \int \left(\sqrt{\frac{m}{2i\pi\epsilon}} \right)^N \left(\prod_{k=1}^{N-1} dq_k \right) e^{i \int_t^T dt L(q, p)} \tag{2.9}$$

We will naively write (2.9) as

$$\langle Q, T | q, t \rangle = N \int \mathcal{D}q \, e^{iS(t, T, [q], [Q])} \tag{2.10}$$

It is useful to consider the following representation of the propagator:

$$\langle Q, T | q, t \rangle = e^{-iH(T-t)} \delta(Q - q) \tag{2.11}$$

In fact $\delta(Q - q)$ is the probability amplitude of finding the particle in the position Q knowing it is in position q. If we know that at time t the particle was in position q, the wave function is $\psi_q(q', t) = \delta(q' - q)$ which satisfies $\hat{q}' \psi_q(q', T) = q' \psi_q(q', T)$. At time T we have $\psi(q', T) = e^{-iH(T-t)} \delta(q' - q)$: this is the probability amplitude of finding the particle in q' at time T knowing that it was in q at time t. Thus $\langle Q, T | q, t \rangle$ corresponds to $\psi(Q, T) = e^{-iH(T-t)} \delta(Q - q)$.

3   Scalar quantum field theories

A quantum field assigns an operator $\hat{\phi}(\mathbf{x})$ to every point in the space. In Heisenberg picture operators are time dependent $\hat{\phi}(x) = e^{i\hat{H}t} \hat{\phi}(\mathbf{x}) e^{-i\hat{H}t}$ thus both position and time are now just labels on operators. We will focus on real scalar quantum fields discussing at first the free theory and then the interacting one following the approach of [4].

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3.1 The free theory

We start from a classical field ϕ with Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$. Such a field is subjected to the equation of motion $(\partial_\mu\partial^\mu + m^2)\phi(\mathbf{x}, t) = 0$, called the Klein-Gordon equation. In order to quantize the classical theory we promote the classical field ϕ and his conjugate momentum $\pi \equiv \frac{\partial\mathcal{L}}{\partial\partial_0\phi}$ to operators [1] satisfying the following commutation rules (at $t = 0$)

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$$
$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0$$

$$(3.1)$$

The Hamiltonian $\mathcal{H} = \phi\pi - \mathcal{L}$, being a function of ϕ and π, will also become an operator. In order to find the spectrum of the Hamiltonian we start again from the classical field expanding it in his Fourier series

$$\phi(t, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(t, \mathbf{p})$$

$$(3.2)$$

with $\phi(\mathbf{p}) = \phi^*(-\mathbf{p})$ being ϕ real. The Klein-Gordon equation becomes:

$$\left[\frac{\partial^2}{\partial t^2} + (|\mathbf{p}|^2 + m^2)\right]\phi(t, \mathbf{p}) = 0$$

which is the same equation of motion of a simple harmonic oscillator with frequency $\omega_\mathbf{p} = \sqrt{|\mathbf{p}|^2 + m^2}$.

We can find the spectrum of the Hamiltonian for the Klein-Gordon field treating each Fourier mode as an independent oscillator. In analogy with the simple harmonic oscillator we introduce the creation and destruction operators $a_\mathbf{p}$ and $a_\mathbf{p}^\dagger$ such that

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}}(a_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} + a_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}}(a_\mathbf{p} + a_{-\mathbf{p}}^\dagger)e^{i\mathbf{p}\cdot\mathbf{x}}$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_\mathbf{p}}{2}}(a_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} - a_\mathbf{p}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}) = \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_\mathbf{p}}{2}}(a_\mathbf{p} - a_{-\mathbf{p}}^\dagger)e^{i\mathbf{p}\cdot\mathbf{x}}$$

$$(3.3)$$

Creation and destruction operators satisfy

$$[a_\mathbf{p}, a_{\mathbf{p}'}^\dagger] = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{p}')$$

$$(3.4)$$

it can be verified that such a definition gives the correct commutation relation between ϕ and π: $[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$.

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[1] For brevity, in this section we will omit the hat over operators.

We can express the Hamiltonian in terms of $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$:

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2}[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]) \tag{3.5}$$

From (3.4) we find that the second term is infinite, it is the sum over all (infinite) modes of the zero point energy. Experimentally we can measure only energy differences so we can set this term to zero.

We can find the spectrum of the Hamiltonian using the same techniques used for the harmonic oscillator.

The state $|0\rangle$ such that $a_{\mathbf{p}} |0\rangle = 0$ for every $\mathbf{p}$, called the vacuum state, is an eigenstate of the Hamiltonian with energy $E = 0$. All other energy eigenstates are obtained acting on $|0\rangle$ with creation operators, for example the state $a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger + \ldots |0\rangle$ is an eigenstate of the Hamiltonian with energy $E = \omega_{\mathbf{p}} + \omega_{\mathbf{q}} + \ldots$.

3.2 The interacting theory

We will now consider a potential $V(\phi)$ which is a smooth function of the field ϕ. There are no known ways to obtain an explicit expression of the eigenvalues and eigenstates for a general $V(\phi)$ thus we will consider the potential as a small perturbation and we will use perturbation theory to obtain an approximated expression for the spectrum. We will write $H = H_0 + \lambda H_{int}$ where λ is a coupling constant, H_0 is the Hamiltonian of the free theory and H_{int} is the perturbing potential.

The perturbation λH_{int} acts in two ways: it changes the fields in the definition of the Heisenberg field $\phi(x) = e^{iHt}\phi(\mathbf{x})e^{-iHt}$ and it changes the vacuum state (and all others eigenstates) which we will indicate with the symbol $|\Omega\rangle$ for the interacting theory.

For every fixed t_0 we can expand ϕ in terms of the creation and destruction operators

$$\phi(t_0, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}) \tag{3.6}$$

For an arbitrary t we have $\phi(t, \mathbf{x}) = e^{iH(t-t_0)}\phi(t_0, \mathbf{x})e^{-iH(t-t_0)}$.

When $\lambda = 0$ H becomes H_0 and we have

$$\phi(t, \mathbf{x})\big|_{\lambda=0} = e^{iH_0(t-t_0)}\phi(t_0, \mathbf{x})e^{-iH_0(t-t_0)} \equiv \phi_I(t, \mathbf{x}) \tag{3.7}$$

We will call ϕ_I the interaction picture field.

Since we can diagonalize H_0, we can have an explicit expression for ϕ_I:

$$\phi_I(t, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} (a_{\mathbf{p}} e^{ip\cdot x} + a_{\mathbf{p}}^\dagger e^{-ip\cdot x})\bigg|_{x^0=t-t_0} \tag{3.8}$$

We will now express the full Heisenberg field in terms of ϕ_I:

$$\phi(t, \mathbf{x}) = e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\phi_I(t, \mathbf{x})e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$$
$$\equiv U^\dagger(t, t_0)\phi_I(t, \mathbf{x})U(t, t_0)$$
(3.9)

where we have defined the (unitary) operator $U(t, t_0) \equiv e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$.
We note that $U(t, t_0)$ is the unique solution of the differential equation

$$i\frac{\partial}{\partial t}U(t, t_0) = H_I(t)U(t - t_0)$$
(3.10)

with initial condition $U(t_0, t_0) = 1$, where $H_I(t) = e^{iH_0(t-t_0)}(\lambda H_{int})e^{-iH_0(t-t_0)}$. The solution of this equation is

$$U(t, t_0) = T\left\{e^{-i\int_{t_0}^t dt' H_I(t')}\right\}$$
(3.11)

where T is the time ordering symbol defined by

$$T\{A_1(t_1)A_2(t_2)\cdots A_n(t_n)\} \equiv T\{A_1(t_{i_1})A_2(t_{i_2})\cdots A_n(t_{i_n})\}$$
(3.12)

where $A_1(t_1)A_2(t_2)\cdots A_n(t_n)$ is an arbitrary product of operators and $t_{i_1}, t_{i_2}, \ldots, t_{i_n}$ are the times $t_1, \ldots, t_n$ relabelled such that $t_{i_n} > t_{i_{n-1}} > \ldots > t_1$. It is simple to generalize (3.11) to arbitrary values of the second argument, obtaining

$$U(t, t') = T\left\{e^{-i\int_{t'}^t dt' H_I(t')}\right\}$$
(3.13)

We will now give an expression of $|\Omega\rangle$. We start with $|0\rangle$, the ground state of H_0, and we let it evolve through time with H. Using a complete set of H eigenstates we can write

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_n T}|n\rangle\langle n|0\rangle$$
(3.14)

We assume that $|\Omega\rangle$ has some overlap with $|0\rangle$, i.e $\langle\Omega|0\rangle \neq 0$. Then the above series contains $|\Omega\rangle$ leading to:

$$e^{-iHT}|0\rangle = e^{-iE_0 T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_n T}|n\rangle\langle n|0\rangle$$
(3.15)

with $E_0 = \langle\Omega|H|\Omega\rangle$.
We can get rid of all the $n \neq 0$ terms in the series by sending $T \rightarrow \infty(1 - i\epsilon)$ Since $E_n > E_0$ the first term dies slowest, so we have

$$\lim_{T\rightarrow\infty(1-i\epsilon)} e^{-i\hat{H}T}|0\rangle = \lim_{T\rightarrow\infty(1-i\epsilon)} e^{-iE_0 T}|\Omega\rangle\langle\Omega|0\rangle$$
(3.16)

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Since T is very large we can shift it by a small constant, obtaining

$$
\begin{aligned}
|\Omega\rangle &= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-iH(T+t_0)}|0\rangle \\
&= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-iH(t_0-(-T))}e^{-iH_0(-T-t_0)}|0\rangle \qquad (3.17)\\
&= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} U(t_0,-T)|0\rangle
\end{aligned}
$$

4 Path integral in scalar quantum field theories

We will now apply the path integral formalism to scalar quantum field theories. Our goal is to calculate the n-point correlation functions, denoted with the symbol $\langle\Omega|\phi(x_1)\dots\phi(x_n)|\Omega\rangle$, which can be related to measurable quantities through the Lehmann–Symanzik–Zimmermann reduction formula.
In analogy with formula (2.8) we define the transition amplitude

$$
\langle\phi_b(x_2)|\phi_a(x_1)\rangle = \langle\phi_b(\mathbf{x})|e^{-i\hat{H}(t_2-t_1)}|\phi_a(\mathbf{x})\rangle = N\int\mathcal{D}\phi\mathcal{D}\pi e^{i\int_{t_1}^{t_2}d^4x\pi\dot{\phi}-\mathcal{H}} \qquad (4.1)
$$

Where $\mathcal{D}\phi$ ($\mathcal{D}\pi$) stands for $\prod_k d\phi(x_k)\left(\prod_k d\pi(x_k)\right)$ and $\phi(x)$ is constrained to the configurations $\phi_a(\mathbf{x})$ at $x^0 = t_1$ and $\phi_b(\mathbf{x})$ at $x^0 = t_2$.
In order to calculate correlation functions it is useful to consider the theory in the presence of an arbitrary source term $J(x)$, i.e. adding to the Lagrangian the term $J(x)\phi(x)$. We then define the generating functional

$$
W[J] \equiv \langle\Omega|\Omega\rangle_J = N'\int\mathcal{D}\phi\mathcal{D}\pi e^{i\int d^4x\pi\dot{\phi}-\mathcal{H}+J\phi} \qquad (4.2)
$$

For a scalar field we consider the Hamiltonian $H = \int d^3x[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)]$. Integrating over π as in (1.10) equation (2.2) becomes

$$
W[J] = N\int\mathcal{D}\phi e^{i\int d^4x\mathcal{L}+J\phi} \qquad (4.3)
$$

with $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - V(\phi)$.
Integral (2.3) is oscillatory, there are two ways to fix this problem:
a) we can put in a convergence factor $e^{-\frac{1}{2}\epsilon\phi^2}$ with $\epsilon > 0$;
b) we can define the generating functional in the euclidean space setting $x_0 = -i\bar{x}_0$, $x_i = \bar{x}_i$ and obtaining

$$
W_E[J] = N_E\int\mathcal{D}\phi e^{-\int d^4\bar{x}\mathcal{L}_E-J\phi} \qquad (4.4)
$$

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With $\mathcal{L}_E = \frac{1}{2}\bar{\partial}_\mu\phi\bar{\partial}^\mu\phi + \frac{1}{2}m^2\phi^2 + V(\phi)$.

The free theory

We will calculate $W[J]$ for a free field using the $-i\epsilon$ procedure.
The Lagrangian for a free field is $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$. Equation (4.3) with the $-i\epsilon$ procedure becomes

$$W_0[J] = \int \mathcal{D}\phi e^{i\int d^4x \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J\phi}. \tag{4.5}$$

Expressing the exponent in the integral in terms of the Fourier transforms of ϕ and J we obtain

$$W_0[J] = N \int \mathcal{D}\phi e^{\frac{i}{2}\int d^4p[\tilde{\phi}'(p)[p^2-m^2+i\epsilon]\tilde{\phi}'(-p)-\tilde{J}(p)[p^2-m^2+i\epsilon]^{-1}\tilde{J}(-p)]} \tag{4.6}$$

where $\tilde{\phi}'(p) = \tilde{\phi}(p) + [p^2 - m^2 + i\epsilon]^{-1}\tilde{J}(p)]$.
The new variable ϕ' differs from ϕ by a constant in function space so $\mathcal{D}\phi' = \mathcal{D}\phi$, thus

$$W_0[J] = N e^{-\frac{i}{2}\int d^4p \frac{|\tilde{J}(p)|^2}{p^2-m^2+i\epsilon}} \int \mathcal{D}\phi e^{\frac{i}{2}\int d^4x \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J\phi}$$

$$= W_0[0] e^{-\frac{i}{2}\int d^4p \frac{|\tilde{J}(p)|^2}{p^2-m^2+i\epsilon}} \tag{4.7}$$

N can be chosen such that $W_0[0] = 1$, with such a choice we have

$$W_0[J] = e^{-\frac{i}{2}\langle J_1 \Delta_{12} J_2\rangle_{12}} \tag{4.8}$$

where $J_1 = J(x_1)$, $J_2 = J(x_2)$ and $\Delta_{12} \equiv \Delta(x_1 - x_2) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(x_1-x_2)}}{p^2-m^2+i\epsilon}$. We have also used the notation

$$\langle f(x_1, \ldots, x_n)\rangle_{j\ldots k} = \int d^4x_j \ldots \int d^4x_k f(x_1, \ldots, x_n)$$

4.1 Correlation functions

We now show, as can be found in [4], that correlation functions can be calculated taking the functional derivatives of $W[J]$ with respect to J at $J = 0$, i.e.

$$\frac{\langle\Omega| T\{\phi(x_1)\cdots\phi(x_n)\}|\Omega\rangle}{\langle\Omega|\Omega\rangle} = \frac{1}{i^N}\frac{1}{W[0]}\frac{\delta^n W[J]}{\delta J(x_1)\ldots\delta J(x_n)}\bigg|_{J=0} \tag{4.1.9}$$

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The right-hand side of equation (4.1.9) is simply

$$\frac{\delta^n W[J]}{\delta J(x_1) \dots \delta J(x_n)}\bigg|_{J=0} = i^N N \int \mathcal{D}\phi \, \phi(x_1) \dots \phi(x_n) e^{i \int d^4 x \mathcal{L}} \qquad (4.1.10)$$

For simplicity we consider $\int \mathcal{D}\phi \, \phi(x_1)\phi(x_2) e^{i \int_{-T}^{T} d^4 x \mathcal{L}}$, the generalization to the n-point correlation functions is straightforward. First we break up the functional measure

$$\int \mathcal{D}\phi(x) = \int \mathcal{D}\phi_1(\mathbf{x}) \int \mathcal{D}\phi_2(\mathbf{x}) \int_* \mathcal{D}\phi(x)$$

Where the * means that the integral is constrained by the conditions

$$\begin{cases} \phi(x_1^0, \mathbf{x}) = \phi_1(\mathbf{x}) \\ \phi(x_2^0, \mathbf{x}) = \phi_2(\mathbf{x}) \\ \phi(T, \mathbf{x}) = \phi_b(\mathbf{x}) \\ \phi(-T, \mathbf{x}) = \phi_a(\mathbf{x}) \end{cases} \qquad (4.1.11)$$

Thus, assuming $x_2^0 > x_1^0$, we have

$$\int \mathcal{D}\phi(x)\phi(x_1)\phi(x_2) e^{i \int_{-T}^{T} d^4 x \mathcal{L}}$$

$$= \int \mathcal{D}\phi_1(\mathbf{x}) \int \mathcal{D}\phi_2(\mathbf{x})\phi_1(\mathbf{x_1})\phi_2(\mathbf{x_2}) \int_* \mathcal{D}\phi(x) e^{i \int_{-T}^{T} d^4 x \mathcal{L}}$$

$$= \int \mathcal{D}\phi_1(\mathbf{x}) \int \mathcal{D}\phi_2(\mathbf{x})\phi_1(\mathbf{x_1})\phi_2(\mathbf{x_2}) \int_* \mathcal{D}\phi(x) e^{i\left(\int_{x_2^0}^{T} d^4 x \mathcal{L} + \int_{x_1^0}^{x_2^0} d^4 x \mathcal{L} + \int_{-T}^{x_1^0} d^4 x \mathcal{L}\right)}$$

$$= \int \mathcal{D}\phi_1(\mathbf{x}) \int \mathcal{D}\phi_2(\mathbf{x})$$

$$\phi_1(\mathbf{x_1})\phi_2(\mathbf{x_2}) \langle \phi_b | e^{-i\hat{H}(T-x_2^0)} |\phi_2\rangle \langle \phi_2| e^{-i\hat{H}(x_1^0-x_2^0)} |\phi_1\rangle \langle \phi_1| e^{-i\hat{H}(x_1^0+T)} |\phi_a\rangle$$

$$(4.1.12)$$

In the last line we have used (4.1).

Using the relation[2] $\phi_S(\mathbf{x_1}) |\phi_1\rangle = \phi_1(\mathbf{x_1}) |\phi_1\rangle$ and the completeness relation

$$\int D\phi \, |\phi\rangle \langle \phi| = \mathbf{1}$$

we obtain

$$\int \mathcal{D}\phi(x)\phi(x_1)\phi(x_2) e^{i \int_{-T}^{T} d^4 x \mathcal{L}} =$$

$$\int \mathcal{D}\phi_1(\mathbf{x}) \int \mathcal{D}\phi_2(\mathbf{x}) \langle \phi_b | e^{-i\hat{H}(T-x_2^0)}\phi_S(\mathbf{x_2}) e^{-i\hat{H}(x_1^0-x_2^0)}\phi_S(\mathbf{x_1}) e^{-i\hat{H}(x_1^0+T)} |\phi_a\rangle$$

$$(4.1.13)$$

_____

[2]In the following we will use the subscript S to denote Schrödinger operators and the subscript H to denote Heisenberg operators.

We can turn a Schrödinger operator into an Heisenberg one using $\phi_H(T, \mathbf{x}) = e^{-i\hat{H}T}\phi_S(\mathbf{x})e^{i\hat{H}T}$ so (4.1.13) becomes

$$\int \mathcal{D}\phi\,\phi(x_1)\phi(x_2)e^{i\int_{-T}^{T} d^4x \mathcal{L}} = \langle\phi_b|\, e^{-i\hat{H}T}\phi_H(x_2)\phi_H(x_1)e^{-i\hat{H}T}\, |\phi_a\rangle \qquad (4.1.14)$$

If $x_1^0 > x_2^0$ the order in which they appear in (4.1.14) would be inverted, so we can write (4.1.14) for arbitrary x_1^0, x_2^0 using the time ordering operator:

$$\int \mathcal{D}\phi\,\phi(x_1)\phi(x_2)e^{i\int_{-T}^{T} d^4x \mathcal{L}} = \langle\phi_b|\, e^{-i\hat{H}T}T\{\phi_H(x_2)\phi_H(x_1)\}e^{-i\hat{H}T}\, |\phi_a\rangle \qquad (4.1.15)$$

Let's consider the term $e^{-i\hat{H}T}|\phi_a\rangle$ in the limit $T \to \infty(1 - i\epsilon)$. Using a complete set of $\hat{H}$ eigenstates we can write $e^{-i\hat{H}T}|\phi_a\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|\phi_a\rangle$. Assuming $\langle\Omega|\phi_a\rangle \neq 0$ we have

$$e^{-i\hat{H}T}|\phi_a\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|\phi_a\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|\phi_a\rangle$$

with $E_0 = \langle\Omega|\hat{H}|\Omega\rangle$.
Since $E_n > E_0$ the first term dies slowest, so we have

$$\lim_{T\to\infty(1-i\epsilon)} e^{-i\hat{H}T}|\phi_a\rangle = \lim_{T\to\infty(1-i\epsilon)} e^{-iE_0T}|\Omega\rangle\langle\Omega|\phi_a\rangle \qquad (4.1.16)$$

Analogously

$$\lim_{T\to\infty(1-i\epsilon)} \langle\phi_b|\, e^{-i\hat{H}T} = \lim_{T\to\infty(1-i\epsilon)} e^{-iE_0T}\langle\phi_b|\Omega\rangle\langle\Omega| \qquad (4.1.17)$$

Inserting (4.1) and (4.1.17) in (4.1.15) we obtain

$$\lim_{T\to\infty(1-i\epsilon)} \int \mathcal{D}\phi\,\phi(x_1)\phi(x_2)e^{i\int_{-T}^{T} d^4x \mathcal{L}}$$
$$= \langle\phi_b|\Omega\rangle\langle\Omega|\phi_a\rangle\, e^{-iE_0 2\cdot\infty(1-i\epsilon)}\langle\Omega|\, T(\phi_H(x_1)\phi_H(x_2))\, |\Omega\rangle \qquad (4.1.18)$$

Repeating the procedure with $\int \mathcal{D}\phi\, e^{i\int_{-T}^{T} d^4x \mathcal{L}}$ we find

$$\int \mathcal{D}\phi\, e^{i\int_{-T}^{T} d^4x \mathcal{L}} = \langle\phi_b|\, e^{-2i\hat{H}T}\, |\phi_a\rangle = e^{-2iE_0T}\langle\Omega|\phi_a\rangle\langle\phi_b|\Omega\rangle \qquad (4.1.19)$$

Finally we have:

$$\langle\Omega|\, T\{\phi_H(x_1)\phi_H(x_2)\}\, |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi\,\phi(x_1)\phi(x_2)e^{i\int_{-T}^{T} d^4x \mathcal{L}}}{\int \mathcal{D}\phi\, e^{i\int_{-T}^{T} d^4x \mathcal{L}}} \qquad (4.1.20)$$

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So, if we pick N' in (4.2) such that

$$W[J] = \frac{\int \mathcal{D}\phi\, e^{i \int_{-T}^{T} d^4x \mathcal{L} + J\phi}}{\int \mathcal{D}\phi\, e^{i \int_{-T}^{T} d^4x \mathcal{L}}} \tag{4.1.21}$$

taking care that the temporal limits of integration are $\pm\infty(1 - i\epsilon)$ and remembering (4.1.10) we obtain what desired:

$$\langle \Omega | T\{\phi(x_1)\phi(x_2)\} | \Omega \rangle = \frac{1}{i^2} \frac{\delta^2 W[J]}{\delta J(x_1)\delta J(x_2)} \bigg|_{J=0} \tag{4.1.22}$$

4.2 Connected Green's Function

We define the n-point Green's function by

$$G^{(n)}(x_1, \ldots, x_n) = \frac{\langle \Omega | T\{\phi(x_1)\cdots\phi(x_n)\} | \Omega \rangle}{\langle \Omega | \Omega \rangle} \tag{4.2.1}$$

With the choice of the generating functional as in (4.1.21) we have $\langle \Omega | \Omega \rangle = 1$. We now define the connected Green's functions recursively

$$G^{(1)}(x_1) = G_c^{(1)}(x_1)$$
$$G^{(2)}(x_1, x_2) = G_c^{(2)}(x_1, x_2) + G_c^{(1)}(x_1)G_c^{(1)}(x_2)$$
$$G^{(3)}(x_1, x_2, x_3) = G_c^{(3)}(x_1, x_2, x_3) + G_c^{(1)}(x_1)G_c^{(2)}(x_2, x_3)$$
$$+ G_c^{(1)}(x_2)G^{(2)}(x_1, x_3) + G_c^{(1)}(x_3)G_c^{(2)}(x_1, x_2)$$
$$+ G_c^{(1)}(x_1)G_c^{(1)}(x_2)G_c^{(1)}(x_3)$$

And for an arbitrary n

$$G^{(n)}(x_1, \ldots, x_n) = \sum G_c^{(n_1)}(x_{p_1}, \ldots, x_{p_{n_1}})G_c^{(n_2)}(x_{q_1}, \ldots, x_{q_{n_2}}) \\ \cdots G_c^{(n_k)}(x_{r_1}, \ldots, x_{r_{n_k}}) \tag{4.2.2}$$

where we sum over all indices n_i such that $n_1 + \ldots + n_k = n$ and over all the permutations from 1 to n of the indices in the set $p_1, \ldots, p_{n_1}$, etc.

We can write (4.1.9) in the form

$$W[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \left\langle J(x_1)\cdots J(x_n)G^{(n)}(x_1, \ldots, x_n) \right\rangle_{x_1, \ldots, x_n} \tag{4.2.3}$$

where $\langle \cdots \rangle_{x_1, \ldots, x_n}$ stands for $\int dx_1 \ldots dx_n$.
Defining $Z[J]$ such that

$$W[J] = e^{iZ[J]} \tag{4.2.4}$$

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we now show,following [5], that $Z[J]$ generates the connected Green's functions, i.e.

$$iZ[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \left\langle J(x_1) \cdots J(x_n) G_c^{(n)}(x_1, \ldots, x_n) \right\rangle_{x_1, \ldots, x_n} \qquad (4.2.5)$$

We start by rewriting (4.2.2) as

$$G_c^{(n)}(x_1, \ldots, x_n) =$$
$$\sum_{\{\sigma_k\}} \sum_{\mathcal{P}} \mathcal{P}([\underbrace{G_c^{(1)}(\cdots)) \cdots G_c^{(1)}(\cdots))}_{\sigma_1 \text{ factors}}][\underbrace{G_c^{(2)}(\cdots)) \cdots G_c^{(2)}(\cdots))}_{\sigma_2 \text{ factors}}] \cdots) \qquad (4.2.6)$$

where $\{\sigma_1, \ldots, \sigma_n\}$ is a partition of n such that $n = \sigma_1 + \ldots + n\sigma_n$ with σ_k denoting the number of copies of $G_c^{(k)}(\cdots)$, the dots denote the coordinates $x_1, \ldots, x_n$ in some fixed order and $\mathcal{P}$ denotes the permutations of these coordinates leading to inequivalent products of connected Green's functions. The number of such permutations is

$$\frac{n!}{\sigma_1! \cdots \sigma_n!(1!)^{\sigma_1}(2!)^{\sigma_2} \cdots (n!)^{\sigma_n}}$$

In fact there are $n!$ permutations of the n coordinates to which we have to subtract $\sigma_k!$ permutations, for every k, corresponding to the exchange of the factors $G^{(k)}(x_1, \ldots, x_k)$ and another $k!$ permutations for every $G^{(k)}(x_1, \ldots, x_k)$ because Green's functions are invariant under permutation of their arguments.

Inserting (4.2.6) in (4.2.3) all the terms in the second summation give the same contribution being integrated over all the coordinates, so we have

$$W[J] = \sum_{N=0}^{\infty} \sum_{\{\sigma_k\}} i^N \frac{\left(\int d^4x\, G_c^{(1)}(x)J(x) \right)^{\sigma_1}}{\sigma_1!(1!)^{\sigma_1}} \frac{\left(\int d^4x d^4y\, G_c^{(2)}(x,y)J(x)J(y) \right)^{\sigma_2}}{\sigma_2!(2!)^{\sigma_2}} \cdots$$
$$(4.2.7)$$

with the second summation subjected to the condition $\sum_{k=1}^{N} k\sigma_k = N$.

We note that $\sum_{N=0}^{\infty} \sum_{\{\sigma_k\}} = \sum_{\{\sigma_k\}}$ where the right-hand side summation has no restriction. Thus each σ_k is independently summed from 0 to ∞ yielding

$$W[J] = \sum_{\sigma_1=0}^{\infty} \frac{i^{\sigma_1}}{\sigma_1!} \left[\langle G_c^{(1)}(x)J(x) \rangle \right]^{\sigma_1} \sum_{\sigma_2=0}^{\infty} \frac{i^{2\sigma_2}}{\sigma_2!} \left[\langle G_c^{(2)}(x,y)J(x,y) \rangle \right]^{\sigma_2} \cdots \qquad (4.2.8)$$

Noting that each factor in (4.2.8) is the expansion of an exponential we have

$$W[J] = e^{i\left\langle G_c^{(1)}(x)J(x) \right\rangle + \frac{i^2}{2!}\left\langle G_c^{(2)}(x,y)J(x,y) \right\rangle + \cdots} = e^{\sum_{N=0}^{\infty} \frac{i^N}{N!}\left\langle G_c^{(N)}(x_1,\ldots,x_n)J(x_1)\cdots J(x_n) \right\rangle} \qquad (4.2.9)$$

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Finally using definition (4.2.4) we obtain

$$iZ[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \langle J(x_1) \cdots J(x_n) G_c^n(x_1, \ldots, x_n) \rangle \tag{4.2.10}$$

(4.2.5) implies

$$G_c^{(n)}(x_1, \ldots, x_n) = i^{n-1} \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} \tag{4.2.11}$$

In the euclidean space we define

$$W_E[J] = e^{-Z_E[J]} \tag{4.2.12}$$

Thus obtaining

$$G_{c,E}^{(n)}(x_1, \ldots, x_n) = (-1) \frac{\delta^n Z_E[J]}{\delta J(x_1) \ldots \delta J(x_n)} \tag{4.2.13}$$

5   Exponential interaction

In the following we will consider a D dimensional space. We will also make use of the relation due to Schwinger

$$W[J] = N e^{-\left\langle V\left(\frac{\delta}{\delta J}\right)\right\rangle} e^{-Z_0[J]} \tag{5.1}$$

Where $Z_0[J]$ is the connected Green's functions generating functional of the free theory.
Let us prove (5.1):

$$W[J] = \int \mathcal{D}\phi\, e^{\langle \mathcal{L}_0 - V(\phi) + J\phi \rangle} = e^{-\langle V(\phi) \rangle} e^{-Z_0[J]} \tag{5.2}$$

Formally for a quite regular function F

$$F\left(\frac{\delta}{\delta J}\right) e^{\langle J\phi \rangle} = \sum_{n=0}^{\infty} c_n \left(\frac{\delta}{\delta J}\right)^n e^{\langle J\phi \rangle} = \sum_{n=0}^{\infty} c_n \phi^n e^{\langle J\phi \rangle} = F(\phi) e^{\langle J\phi \rangle} \tag{5.3}$$

In particular

$$W[J] = N e^{-\langle V(\phi) \rangle} e^{Z_0[J]} = N e^{-\left\langle V\left(\frac{\delta}{\delta J}\right)\right\rangle} e^{-Z_0[J]} \tag{5.4}$$

We will now consider the interaction dictated by an exponential potential: $V(\phi) = \mu^D e^{\alpha\phi}$.
Using (5.1), the generating functional for this potential is

$$W[J] = e^{-\left\langle \mu^D e^{\alpha \frac{\delta}{\delta J}}\right\rangle} e^{-Z_0[J]} = \sum_{k=0}^{\infty} \frac{(-\mu)^D}{k!} \left\langle e^{\alpha \frac{\delta}{\delta J}} \right\rangle^k e^{-Z_0[J]} \tag{5.5}$$

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Setting $a_x(y) \equiv \alpha \delta(x - y)$, we have

$$e^{\alpha \frac{\delta}{\delta J(x)}} e^{-Z_0[J]} = e^{-Z_0[J+a_x]} \tag{5.6}$$

which is the functional analogous of the translation operator, where

$$
\begin{aligned}
Z_0[J + \alpha_x] &= \\
&- \frac{1}{2} \int d^D y \, d^D z (J(y) + \alpha \delta(x - y)) \Delta(y - z)(J(z) + \alpha \delta(x - z)) \\
&= Z_0[J] - \frac{\alpha^2}{2} \Delta(0) - \alpha \int d^D y J(y) \Delta(y - x)
\end{aligned}
\tag{5.7}
$$

Using (5.6) the generating functional (5.5) becomes

$$W[J] = \sum_{k=0}^{\infty} \frac{(-\mu^D)^k}{k!} \left\langle e^{-Z_0[J+\alpha_{x_1}+\ldots+\alpha_{x_n}]} \right\rangle = e^{Z_0[J]} \sum_{k=0}^{\infty} \frac{(-\mu^D)^k}{k!} e^{\frac{k\alpha^2}{2}\Delta(0)} G_k[J] \tag{5.8}$$

Where the $G_k[J]$ are defined as follows:

$$G_0[J] = 1$$

$$G_k[J] = \int d^D z_1 \ldots \int d^D z_k e^{\alpha \int d^D z J(z) \sum_{j=1}^{k} \Delta(z-z_j)+\alpha^2 \sum_{l>j}^{k} \Delta(z_j - z_l)} \tag{5.9}$$

Noting that functional derivatives commute to the right of the first exponential in (5.5), Green's functions in the euclidean space are given by

$$G_E^{(N)}(x_1, \ldots, x_N) = \sum_{k=0}^{\infty} \frac{(-\mu^D)^k}{k!} \left\langle (-1) \frac{\delta^N e^{-Z_0[J+\alpha_{z_1}+\ldots+\alpha_{z_k}]}}{\delta J(x_1) \ldots J(x_N)} \right\rangle_{z_1, \ldots, z_k} \tag{5.10}$$

6 Exponential interaction as master potential

In this section we will calculate the generating functional for polynomial interactions considering the exponential interaction as a master potential.
The starting point is to note that

$$\phi^n = \partial_\alpha^n e^{\alpha \phi} \big|_{\alpha=0} \tag{6.1}$$

Thus if we consider the potential $V(\phi) = \frac{\lambda}{n!} \phi^n$ we can write the generating functional, using (5.1), as

$$
\begin{aligned}
W^{(n)}[J] &= e^{-\frac{\lambda}{n!} \partial_\alpha^n \langle e^{\alpha \frac{\delta}{\delta J}} \rangle} e^{-Z_0[J]} \Big|_{\alpha=0} \\
&= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!n!} \partial_{\alpha_1}^n \ldots \partial_{\alpha_k}^n \langle e^{\alpha_1 \frac{\delta}{\delta J}} \rangle \cdots \langle e^{\alpha_k \frac{\delta}{\delta J}} \rangle e^{-Z_0[J]} \Big|_{\alpha^{(k)}=0}
\end{aligned}
\tag{6.2}
$$

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Whit $\alpha^{(k)} = (\alpha_1, \ldots, \alpha_k)$.

We will now consider normal-ordered potentials $: \phi^n(x) :$ i.e. where all creation operators are to the left of the destruction operators.

For an arbitrary regular function of ϕ Wick's theorem states

$$T\{F[\phi]\} = e^{\frac{1}{2}\langle \frac{\delta}{\delta\phi(x)} \Delta(x-y) \frac{\delta}{\delta\phi(y)}\rangle_{x,y}} : F[\phi] : \qquad (6.3)$$

Obviously if there is only one operator $T\{F[\phi]\} = F[\phi]$, thus isolating $: F[\phi] :$ in (6.3) we have

$$: F[\phi] := e^{-\frac{1}{2}\langle \frac{\delta}{\delta\phi(x)} \Delta(x-y) \frac{\delta}{\delta\phi(y)}\rangle_{x,y}} F[\phi] \qquad (6.4)$$

We will now compute $: e^{\alpha\phi(x)} :$ using (6.4):

$$: e^{\alpha\phi(x)} := e^{-\frac{1}{2}\langle \frac{\delta}{\delta\phi(x_1)} \Delta(x_1-x_2) \frac{\delta}{\delta\phi(x_2)}\rangle} e^{\alpha\phi(x)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2} \left\langle \frac{\delta}{\delta\phi(x_1)} \Delta(x_1 - x_2) \frac{\delta}{\delta\phi(x_2)} \right\rangle \right)^n e^{\alpha\phi(x)} \qquad (6.5)$$

We note that

$$\left\langle \frac{\delta}{\delta\phi(x_1)} \Delta(x_1 - x_2) \frac{\delta}{\delta\phi(x_2)} \right\rangle_{x_1,x_2} e^{\alpha\phi(x)}$$

$$= \left\langle \frac{\delta}{\delta\phi(x_1)} \Delta(x_1 - x_2) \alpha\delta(x - x_2) e^{\alpha\phi(x)} \right\rangle_{x_1,x_2}$$

$$= \langle \Delta(x_1 - x)\alpha\delta(x - x_1)e^{\alpha\phi(x)}\rangle_{x_1} = \alpha^2 \Delta(0) e^{\alpha\phi(x)}$$

Thus (6.5) becomes:

$$: e^{\alpha\phi(x)} := \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2}\alpha^2 \Delta(0) \right)^n e^{\alpha\phi(x)} = e^{-\frac{1}{2}\alpha^2 \Delta(0)} e^{\alpha\phi(x)} \qquad (6.6)$$

Remembering that $\partial_x^n f(x)g(x) = \sum_{k=0}^{n} \binom{n}{k} \partial_x^k f(x) \partial_x^{n-k} g(x)$, from (6.6) and (6.1) we find:

$$: \phi(x)^n := \partial_\alpha^n : e^{\alpha\phi(x)} :|_{\alpha=0} = \sum_{k=0}^{n} \binom{n}{k} \partial_\alpha^k e^{-\frac{\alpha^2}{2}\Delta(0)}\Big|_{\alpha=0} \phi^{n-k} \qquad (6.7)$$

Thus using (5.1) we can write the generating functional as in (6.2), with the difference that each $\langle e^{\alpha_k \frac{\delta}{\delta J}}\rangle$ is multiplied by $e^{-\frac{\alpha_k^2}{2}\Delta(0)}$, i.e.

$$W^{(n)}[J] =$$

$$\sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!n!} \partial_{\alpha_1}^n \ldots \partial_{\alpha_k}^n \langle e^{\alpha_1 \frac{\delta}{\delta J}}\rangle e^{-\frac{\alpha_1^2}{2}\Delta(0)} \cdots \langle e^{\alpha_k \frac{\delta}{\delta J}}\rangle e^{-\frac{\alpha_k^2}{2}\Delta(0)} e^{Z_0[J]}\Big|_{\alpha^{(k)}=0} \qquad (6.8)$$

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So when we apply the translation operator $\langle e^{\alpha_1 \frac{\delta}{\delta J}} \rangle$ we can replace $Z_0[J + \alpha_x]$ in (5.7) with

$$\tilde{Z}_0[J + \alpha_x] = Z_0[J] - \alpha \int d^D y J(y) \Delta(x - y) \tag{6.9}$$

We then obtain

$$W^{(n)}[J] = e^{-Z_0[J]} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{(n!)^k k!} \partial_{\alpha_1}^n \ldots \partial_{\alpha_k}^n \langle e^{-\tilde{Z}_0[J + \alpha_{x_1} + \ldots + \alpha_{x_n}]} \rangle$$

$$= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{(n!)^k k!} \partial_{\alpha_1}^n \ldots \partial_{\alpha_k}^n \tag{6.10}$$

$$\int d^D z_1 \ldots \int d^D z_k e^{\int d^D y J(y) \sum\limits_{j=1}^{k} \alpha_j \Delta(y - z_j) + \sum\limits_{l>j}^{k} \alpha_j \alpha_l \Delta(z_j - z_l)} \Big|_{\alpha^{(k)} = 0}$$

6.1 An explicit expression for $W^{(n)}[J]$

We will now derive a more explicit expression for (6.10).
We start by considering the case $J = 0$

$$W^{(n)}[0] = \sum_{k=0, k \neq 1}^{\infty} \frac{(-\lambda)^k}{n!^k k!} \partial_{\alpha_1}^n \ldots \partial_{\alpha_k}^n \langle e^{\sum\limits_{l>j}^{k} \alpha_j \alpha_l \Delta(z_j - z_l)} \rangle \Big|_{\alpha^{(k)} = 0} \tag{6.1.1}$$

Expanding the exponential the only terms in the summation giving contributions are, for each k, the ones containing $\alpha_1^n \ldots \alpha_k^n$. Thus when kn is odd there are no contributions to the kth term of the summation. Instead when kn is even the derivatives select, after setting $\alpha^{(k)} = 0$, $n!^k$ times the coefficient of $\alpha_1^n \ldots \alpha_k^n$.
In what follows it will be useful to consider the multinomial identity

$$\frac{1}{\left(\frac{kn}{2}\right)!} \left(\sum_{l>j}^{k} \alpha_j \alpha_l \Delta(z_j - z_l) \right)^{\frac{kn}{2}} = \sum_{\sum\limits_{l>j}^{k} m_{jl} = \frac{kn}{2}} \frac{1}{\prod\limits_{l>j}^{k} m_{jl}!} \prod_{l>j}^{k} (\alpha_j \alpha_l \Delta(z_j - z_l))^{m_{jl}} \tag{6.1.2}$$

where $0 \leq m_{jl} \leq \frac{kn}{2}$.
For each $l = 1, \ldots, k$ the total exponent of α_l in (6.1.1) is

$$p_l \equiv \sum_{i=1}^{l-1} m_{il} + \sum_{j=l+1}^{k} m_{lj} \tag{6.1.3}$$

There are contributions to the kth term of the summation (6.1.1) only when

$$p_1 = \ldots = p_k = n \tag{6.1.4}$$

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which gives $\sum_{l=1}^{k} p_l = kn$. Noting that equation (6.1.3) implies $\sum_{l=1}^{k} p_l = 2\sum_{l>j}^{k} m_{jl}$, we have that condition (6.1.4) includes the condition $\sum_{l>j}^{k} m_{jl} = \frac{kn}{2}$ in the summation in the right hand side of (6.1.2).

Thus

$$W^{(n)}[0] = \sum_{k=0, k\neq 1}^{\infty} \frac{(-\lambda)^k}{n!^k k!} \partial_{\alpha_1}^n \ldots \partial_{\alpha_k}^n \left\langle \sum_{s=0}^{\infty} \frac{1}{s!} \left(\sum_{l>j}^{k} \alpha_j \alpha_l \Delta(z_j - z_l) \right)^s \right\rangle \Bigg|_{\alpha^{(k)}=0}$$

$$= \sum_{k=0, k\neq 1}^{\infty} \frac{(-\lambda)^k}{k!} \sum_{p_1 = \ldots = p_k = n} [k|m] \prod_{l>j}^{k} \langle \Delta(z_j - z_l)^{m_{jl}} \rangle$$

(6.1.5)

with

$$[k|m] \equiv \begin{cases} \dfrac{1}{\prod\limits_{l>j}^{k} m_{jl}!} & \text{for } kn \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

(6.1.6)

We will now extend the analysis for an arbitrary J.

For each k in the (6.10) the terms in the expansion of the exponential containing $\alpha_1^n \ldots \alpha_k^n$, unless in the case $J = 0$ with kn odd, satisfy

$$\sum_{p=0}^{\left[\frac{kn}{2}\right]} \frac{1}{(kn-2p)!} \frac{1}{p!} \left\langle J(y) \sum_{i=1}^{k} \alpha_i \Delta(y - z_i) \right\rangle_y^{kn-2p} \left(\sum_{l>j}^{k} \alpha_j \alpha_l \Delta(z_j - z_l) \right)^p$$

$$= \sum_{p=0}^{\left[\frac{kn}{2}\right]} \sum_{\sum_{i=1}^{k} q_i = kn-2p} \sum_{\sum_{l>j}^{k} m_{jl} = p} \frac{1}{\prod\limits_{i=1}^{k} q_i! \prod\limits_{l>j}^{k} m_{jl}!}$$

(6.1.7)

$$\prod_{i=1}^{k} \langle J(y)\alpha_i \Delta(y - z_i)\rangle_y^{q_i} \prod_{l>j}^{k} (\alpha_j \alpha_k \Delta(z_j - z_l))^{m_{jl}}$$

where $[a]$ denotes the integer part of a, $0 \leq q_i \leq kn - 2p$, $0 \leq m_{jl} \leq p$.

Let us first show that (6.1.7) reproduces (6.1.2) for $J = 0$. When kn is even, for $J = 0$, all the terms $\langle J(y)\alpha_i \Delta(y - z_i)\rangle_y^{q_i}$, $i = 0, \ldots, k$ in the right-hand side of (6.1.7) are zero unless $q_i = 0$.

Similarly, when $J = 0$, the term $\left\langle J(y) \sum_{i=1}^{k} \alpha_i \Delta(y - z_i) \right\rangle_y^{kn-2p}$ contributes only for $kn - 2p = 0$. Therefore, for $J = 0$, the summation over p in (6.1.7) reduces to

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the term with $p = \frac{kn}{2}$. Since $q_1, \ldots, q_k = 0$, then $\prod_{i=1}^{k} q_i! = 1$ and (6.1.7) reduces to (6.1.2).

When kn is odd and $J = 0$ the kth term in (6.10) is zero. Let us show that (6.1.7) reproduces this result. If kn is odd and $J = 0$ then again the right-hand side of (6.1.7) is zero unless $q_1 = \ldots = q_k = 0$, but this would imply $\sum_{i=1}^{k} q_i = 0$ that cannot be equal to the odd number $kn - 2p$. So the only configuration that would give contribution is not included in the summation, thus (6.1.7) is zero for $J = 0$ and kn odd.

For each $l = 1, \ldots, k$ and for each choice of the m_{jl}'s the total exponent of α_l in (6.1.7) is

$$p_l \equiv \sum_{i=1}^{l-1} m_{il} + \sum_{j=l+1}^{k} m_{lj} + q_l \tag{6.1.8}$$

The conditions in the summations indices in the right-hand side of (6.1.7) imply

$$\sum_{l=1}^{k} p_l = 2 \sum_{l>j}^{k} m_{jl} + \sum_{l=1}^{k} q_l = 2p + kn - 2p = kn \tag{6.1.9}$$

The only terms that give contribution in (6.10) are the one containing $\alpha_1^n \ldots \alpha_k^n$ i.e. with $p_1 = \ldots = p_k = n$, but this condition implies (6.1.9), thus we have

$$W^{(n)}[J] = e^{-Z_0[J]} \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \sum_{p=0}^{\left[\frac{kn}{2}\right]} \sum_{\substack{k \\ \sum_{i=1}^{k} q_i = kn-2p}} \sum_{p_1 = \ldots = p_k = n}$$

$$[k|m, q] \prod_{i=1}^{k} \langle\langle J(y)\Delta(y - z_i)\rangle\rangle_y^{q_i} \prod_{l>j}^{k} \Delta(z_j - z_l)^{m_{jl}}\rangle \tag{6.1.10}$$

whit $[k|m, q] \equiv \dfrac{1}{\prod_{i=1}^{k} q_i! \prod_{l>j}^{k} m_{jl}!}$.

6.2 Four-point connected Green's function

We will now calculate the four-point connected Green's function for the potential $: \phi^n :$ in the euclidean space using the method exposed in this section.

We will give an expression in function of the one, two and three point functions. An expression for the one and two point functions can be found in [2], the three point function can be calculated easily repeating the procedure we will now follow

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for the four point function.

It is easier to start again from expression (6.10). In the following we will use the notation $\langle \phi_1 \cdots \phi_n \rangle \equiv G_{E,c}(x_1, \ldots, x_n)$.

We have

$$
\begin{aligned}
\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle &= \left. \frac{\delta^4 \log(W^{(n)}[J])}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \right|_{J=0} \\
&= - \langle \phi_1 \phi_3 \phi_4 \rangle \langle \phi_2 \rangle - \langle \phi_1 \phi_3 \rangle \langle \phi_2 \phi_4 \rangle - \langle \phi_1 \phi_4 \rangle \langle \phi_2 \phi_3 \rangle - \langle \phi_1 \rangle \langle \phi_2 \phi_3 \phi_4 \rangle \\
&\quad - \langle \phi_1 \phi_4 \rangle \langle \phi_2 \rangle \langle \phi_3 \rangle - \langle \phi_1 \rangle \langle \phi_2 \phi_4 \rangle \langle \phi_3 \rangle - \langle \phi_1 \rangle \langle \phi_2 \rangle \langle \phi_3 \phi_4 \rangle - \langle \phi_1 \phi_2 \phi_4 \rangle \langle \phi_3 \rangle \\
&\quad - \langle \phi_1 \phi_2 \rangle \langle \phi_3 \phi_4 \rangle - \langle \phi_1 \phi_3 \rangle \langle \phi_2 \rangle \langle \phi_4 \rangle - \langle \phi_1 \rangle \langle \phi_2 \phi_3 \rangle \langle \phi_4 \rangle - \langle \phi_1 \rangle \langle \phi_2 \rangle \langle \phi_3 \rangle \langle \phi_4 \rangle \\
&\quad - \langle \phi_1 \phi_2 \rangle \langle \phi_3 \rangle \langle \phi_4 \rangle - \langle \phi_1 \phi_2 \phi_3 \rangle \langle \phi_4 \rangle + \frac{1}{W^{(n)}[0]} \left. \frac{\delta^4 W^{(n)}[J]}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \right|_{J=0}
\end{aligned}
\tag{6.2.1}
$$

Setting

$$
\begin{aligned}
F[J] &\equiv \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{(n!)^k k!} \partial_{\alpha_1}^n \ldots \partial_{\alpha_k}^n \\
&\quad \left. \int d^D z_1 \ldots \int d^D z_k e^{\int d^D y J(y) \sum_{j=1}^{k} \alpha_j \Delta(y - z_j) + \sum_{l>j}^{k} \alpha_j \alpha_l \Delta(z_j - z_l)} \right|_{\alpha^{(k)}=0}
\end{aligned}
\tag{6.2.2}
$$

we have, from expression (6.10),

$$
W^{(n)}[J] = e^{\frac{1}{2} \langle J \Delta J \rangle} F[J]
\tag{6.2.3}
$$

Thus

$$
\begin{aligned}
&\frac{1}{W^{(n)}[0]} \left. \frac{\delta^4 W^{(n)}[J]}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \right|_{J=0} \\
&= \Delta_{12}(\langle \phi_3 \rangle \langle \phi_4 \rangle - \langle \phi_3 \phi_4 \rangle - \Delta_{34}) + \Delta_{13}(\langle \phi_2 \rangle \langle \phi_4 \rangle - \langle \phi_2 \phi_4 \rangle - \Delta_{24}) \\
&\quad + \Delta_{14}(\langle \phi_2 \rangle \langle \phi_3 \rangle - \langle \phi_2 \phi_3 \rangle - \Delta_{23}) + \Delta_{23}(\langle \phi_1 \rangle \langle \phi_4 \rangle - \langle \phi_1 \phi_4 \rangle) \\
&\quad + \Delta_{24}(\langle \phi_1 \rangle \langle \phi_3 \rangle - \langle \phi_1 \phi_3 \rangle) + \Delta_{34}(\langle \phi_1 \rangle \langle \phi_2 \rangle - \langle \phi_1 \phi_2 \rangle) \\
&\quad + \frac{1}{W^{(n)}[0]} \left. \frac{\delta^4 F[J]}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \right|_{J=0}
\end{aligned}
\tag{6.2.4}
$$

where we have used the notation $\Delta_{ij} = \Delta(x_i - x_j)$.

Let us calculate the last addend in the summation:

$$\left.\frac{\delta^4 F[J]}{\delta J_1 \delta J_2 \delta J_3 \delta J_4}\right|_{J=0} = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{(n!)^k k!} \partial_{\alpha_1}^n \ldots \partial_{\alpha_k}^n$$

$$\int d^D z_1 \ldots \int d^D z_k \sum_{r,s,t,u}^{k} \alpha_r \alpha_s \alpha_t \alpha_u \Delta(x_1 - z_r) \Delta(x_2 - z_s)$$

$$\Delta(x_3 - z_t) \Delta(x_4 - z_u) e^{\sum_{l>j}^{k} \alpha_j \alpha_l \Delta(z_j - z_l)}\Bigg|_{\alpha^{(k)}=0} \qquad (6.2.5)$$

$$= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{(n!)^k k!} \partial_{\alpha_1}^n \ldots \partial_{\alpha_k}^n \int dZ \sum_{r,s,t,u}^{k} \alpha_r \alpha_s \alpha_t \alpha_u \lhd_{r,s,t,u}^{1,2,3,4} e^{\sum_{l>j}^{k} \alpha_j \alpha_l \Delta(z_j - z_l)}\Bigg|_{\alpha^{(k)}=0}$$

Where we have set $\int dZ \equiv \int d^D z_1 \ldots \int d^D z_k$ and $\lhd_{r,s,t,u}^{1,2,3,4} \equiv \Delta(x_1 - z_r)\Delta(x_2 - z_s)\Delta(x_3 - z_t)\Delta(x_4 - z_u)$. Again, as in (6.1.1), there are contributions only when kn is even and expanding the exponential the terms containing $\alpha_1^n \ldots \alpha_k^n$ are

$$\frac{1}{\left(\frac{kn-4}{2}\right)!} \sum_{r,s,t,u}^{k} \alpha_r \alpha_s \alpha_t \alpha_u \lhd_{r,s,t,u}^{1,2,3,4} \left(\sum_{l>j}^{k} \alpha_j \alpha_l \Delta(z_j - z_l)\right)^{\frac{kn-4}{2}} \qquad (6.2.6)$$

For simplicity we will restrict the analysis to $n \geq 4$.

Let us consider separately the cases $k = 1$ and $k = 2$.

When $k = 1$, because the second summation in (6.2.6) starts from $k = 2$, there will be contributions only when the exponent is zero i.e. when $n = 4$. Thus the contribution to the series for $k = 1$ is

$$\delta_4^n \left(-\frac{\lambda}{4!} \partial_{\alpha_1}^4 \int d^D z_1 \alpha_1^4 \lhd_{1,1,1,1}^{1,2,3,4}\Bigg|_{\alpha_1=0}\right) = -\lambda \left\langle \lhd_{1,1,1,1}^{1,2,3,4}\right\rangle_{z_1} \delta_4^n$$

$$= -\lambda \left\langle \Delta(x_1 - z_1)\Delta(x_2 - z_1)\Delta(x_3 - z_1)\Delta(x_4 - z_1)\right\rangle_{z_1} \delta_4^n \qquad (6.2.7)$$

When $k = 2$ we have

$$\frac{\lambda^2}{(n!)^2 (n-2)!} \partial_{\alpha_1}^n \partial_{\alpha_2}^n \int d^D z_1 \int d^D z_2$$

$$\sum_{r,s,t,u}^{2} \alpha_r \alpha_s \alpha_t \alpha_u \lhd_{r,s,t,u}^{1,2,3,4} \left(\sum_{l>j}^{k} \alpha_j \alpha_l \Delta(z_j - z_l)\right)^{n-2}\Bigg|_{\alpha_1,\alpha_2=0} \qquad (6.2.8)$$

Noting that $\left(\sum_{l>j}^{2} \alpha_j \alpha_l \Delta(z_j - z_l)\right)^{n-2} = (\alpha_1 \alpha_2 \Delta(z_1 - z_2))^{n-2}$, the only term that gives contributions is the one containing $\alpha_1^2 \alpha_2^2$ in the first summation. Thus (6.2.8)

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becomes

$$\frac{\lambda^2}{(n!)^2(n-2)!}\partial^n_{\alpha_1}\partial^n_{\alpha_2}\int d^D z_1 d^D z_2$$

$$\alpha_1^2\alpha_2^2(\lhd^{1,2,3,4}_{1,1,2,2} + \lhd^{1,2,3,4}_{1,2,1,2} + \lhd^{1,2,3,4}_{1,2,2,1} + \lhd^{1,2,3,4}_{2,1,2,1} + \lhd^{1,2,3,4}_{2,2,1,1}$$

$$+ \lhd^{1,2,3,4}_{2,1,1,2})(\alpha_1\alpha_2\Delta(z_1 - z_2))^{n-2}\big|_{\alpha_1,\alpha_2=0} \tag{6.2.9}$$

$$= \frac{\lambda^2}{(n-2)!}\int d^D z_1 \int d^D z_2(\lhd^{1,2,3,4}_{1,1,2,2} + \lhd^{1,2,3,4}_{1,2,1,2} + \lhd^{1,2,3,4}_{1,2,2,1} + \lhd^{1,2,3,4}_{2,1,2,1}$$

$$+ \lhd^{1,2,3,4}_{2,2,1,1} + \lhd^{1,2,3,4}_{2,1,1,2})\Delta(z_1 - z_2)^{n-2}$$

We will now extend the analysis to $k \geq 3$. The term containing $\alpha_1^n \ldots \alpha_k^n$ is (6.2.6), so the only contributions are given by

$$\sum_{k=3}^{\infty} \frac{(-\lambda)^k}{(n!)^k k!}\partial^n_1 \ldots \partial^n_k \int d^D z_1 \ldots \int d^D z_k$$

$$\frac{1}{\left(\frac{kn-4}{2}\right)!}\sum_{r,s,t,u}^{1,2,3,4}\alpha_r\alpha_s\alpha_t\alpha_u \lhd^{1,2,3,4}_{r,s,t,u}\left(\sum_{l>j}^{k}\alpha_j\alpha_l\Delta(z_j - z_l)\right)^{\frac{kn-4}{2}}\Bigg|_{\alpha^{(k)}=0} \tag{6.2.10}$$

Using (6.1.2), replacing $\frac{kn}{2}$ with $\frac{kn-4}{2}$, the total exponent of α_l in the second summation is

$$p_l \equiv \sum_{i=1}^{l-1} m_{jl} + \sum_{j=l+1}^{k} m_{lj} \tag{6.2.11}$$

with the m_{jl}'s constrained by the condition

$$\sum_{j>l}^{k} m_{jl} = \frac{kn-4}{2} \tag{6.2.12}$$

The terms in the second summation that give contributions are the one of the kind

$$\alpha_1^n \ldots \alpha_{i_1}^{n-1} \ldots \alpha_{i_2}^{n-1} \ldots \alpha_{i_3}^{n-1} \ldots \alpha_{i_4}^{n-1} \ldots \alpha_k^n$$
$$\alpha_1^n \ldots \alpha_{i_1}^{n-2} \ldots \alpha_{i_2}^{n-1} \ldots \alpha_{i_3}^{n-1} \ldots \alpha_k^n$$
$$\alpha_1^n \ldots \alpha_{i_1}^{n-2} \ldots \alpha_{i_2}^{n-2} \ldots \alpha_k^n \tag{6.2.13}$$
$$\alpha_1^n \ldots \alpha_{i_1}^{n-3} \ldots \alpha_{i_2}^{n-1} \ldots \alpha_k^n$$
$$\alpha_1^n \ldots \alpha_{i_1}^{n-4} \ldots \alpha_k^n$$

where clearly the i_js are not necessarily in order and can be equal to 1 or k as well. It is also understood that when $k = 3$ at least two of the indices i_1, i_2, i_3, i_4 are equal thus reducing the first case to one of the cases below. It is crucial that

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$n \geq 4$ to have the last case.

We have $\sum_{l=1}^{k} p_l = kn - 4$ that implies $\sum_{l>j}^{k} m_{jl} = \frac{kn-4}{2}$ which is exactly condition (6.2.12). Thus, using (6.1.2) and taking the derivatives, (6.2.10) becomes

$$\sum_{k=3}^{\infty} \frac{(-\lambda)^k}{k!} \sum_{i_1,i_2,i_3,i_4=1}^{k} \sum_{\{p\}_{n,i_1,i_2,i_3,i_4}}$$

$$\left\langle \Delta(x_1 - z_{i_1})\Delta(x_2 - z_{i_2})\Delta(x_3 - z_{i_3})\Delta(x_4 - z_{i_4}) \prod_{l>j}^{k} \Delta(z_j - z_l)^{m_{jl}} \right\rangle_{z_1,...,z_k} \qquad (6.2.14)$$

$$= \sum_{k=3}^{\infty} \frac{(-\lambda)^k}{k!} \sum_{i_1,i_2,i_3,i_4} \sum_{\{p\}_{n,i_1,i_2,i_3,i_4}} \left\langle \triangleleft_{i_1,i_2,i_3,i_4}^{1,2,3,4} \prod_{l>j}^{k} \Delta(z_j - z_l)^{m_{jl}} \right\rangle_{z_1,...,z_k}$$

Where $\sum_{\{p\}_{n,i_1,i_2,i_3,i_4}}$ means that the summation is constrained to the m_{jl}s which give

$$\begin{cases} p_{i_1} = n - 1 - \delta_{i_1}^{i_2} - \delta_{i_1}^{i_3} - \delta_{i_1}^{i_4} \\ p_{i_2} = n - 1 - \delta_{i_2}^{i_1} - \delta_{i_2}^{i_3} - \delta_{i_2}^{i_4} \\ p_{i_3} = n - 1 - \delta_{i_3}^{i_1} - \delta_{i_3}^{i_2} - \delta_{i_3}^{i_4} \\ p_{i_4} = n - 1 - \delta_{i_4}^{i_1} - \delta_{i_4}^{i_2} - \delta_{i_4}^{i_3} \\ p_j = n \text{ if } j \neq i_1, i_2, i_3, i_4 \end{cases} \qquad (6.2.15)$$

Putting the various contributions together we obtain, for $n \geq 4$

$$\frac{\delta^4 F[J]}{\delta J_1 \delta J_2 \delta J_3 \delta J_4}\bigg|_{J=0} = -\lambda \left\langle \triangleleft_{1,1,1,1}^{1,2,3,4} \right\rangle_{z_1} \delta_4^n$$

$$+ \frac{\lambda^2}{(n-2)!} \int d^D z_1 \int d^D z_2 (\triangleleft_{1,1,2,2}^{1,2,3,4} + \triangleleft_{1,2,1,2}^{1,2,3,4} + \triangleleft_{1,2,2,1}^{1,2,3,4} + \triangleleft_{2,1,2,1}^{1,2,3,4}$$

$$+ \triangleleft_{2,2,1,1}^{1,2,3,4} + \triangleleft_{2,1,1,2}^{1,2,3,4})\Delta(z_1 - z_2)^{n-2} \qquad (6.2.16)$$

$$+ \sum_{k=3}^{\infty} \frac{(-\lambda)^k}{k!} \sum_{i_1,i_2,i_3,i_4} \sum_{\{p\}_{n,i_1,i_2,i_3,i_4}} \left\langle \triangleleft_{i_1,i_2,i_3,i_4}^{1,2,3,4} \prod_{l>j}^{k} \Delta(z_j - z_l)^{m_{jl}} \right\rangle_{z_1,...,z_k}$$

Inserting (6.2.16) and (6.2.4) in (6.2.1) we finally obtain

$$\langle\phi_1\phi_2\phi_3\phi_4\rangle$$
$$= -\langle\phi_1\phi_3\phi_4\rangle\langle\phi_2\rangle - \langle\phi_1\phi_3\rangle\langle\phi_2\phi_4\rangle - \langle\phi_1\phi_4\rangle\langle\phi_2\phi_3\rangle - \langle\phi_1\rangle\langle\phi_2\phi_3\phi_4\rangle$$
$$- \langle\phi_1\phi_4\rangle\langle\phi_2\rangle\langle\phi_3\rangle - \langle\phi_1\rangle\langle\phi_2\phi_4\rangle\langle\phi_3\rangle - \langle\phi_1\rangle\langle\phi_2\rangle\langle\phi_3\phi_4\rangle - \langle\phi_1\phi_2\phi_4\rangle\langle\phi_3\rangle$$
$$- \langle\phi_1\phi_2\rangle\langle\phi_3\phi_4\rangle - \langle\phi_1\phi_3\rangle\langle\phi_2\rangle\langle\phi_4\rangle - \langle\phi_1\rangle\langle\phi_2\phi_3\rangle\langle\phi_4\rangle - \langle\phi_1\rangle\langle\phi_2\rangle\langle\phi_3\rangle\langle\phi_4\rangle$$
$$- \langle\phi_1\phi_2\rangle\langle\phi_3\rangle\langle\phi_4\rangle - \langle\phi_1\phi_2\phi_3\rangle\langle\phi_4\rangle + \Delta_{12}(\langle\phi_3\rangle\langle\phi_4\rangle - \langle\phi_3\phi_4\rangle - \Delta_{34})$$
$$+ \Delta_{13}(\langle\phi_2\rangle\langle\phi_4\rangle - \langle\phi_2\phi_4\rangle - \Delta_{24}) + \Delta_{14}(\langle\phi_2\rangle\langle\phi_3\rangle - \langle\phi_2\phi_3\rangle + \Delta_{23})$$
$$+ \Delta_{23}(\langle\phi_1\rangle\langle\phi_4\rangle - \langle\phi_1\phi_4\rangle) + \Delta_{24}(\langle\phi_1\rangle\langle\phi_3\rangle - \langle\phi_1\phi_3\rangle)$$
$$+ \Delta_{34}(\langle\phi_1\rangle\langle\phi_2\rangle - \langle\phi_1\phi_2\rangle) + \frac{1}{W^{(n)}[0]}\left[-\lambda\langle\lhd^{1,2,3,4}_{1,1,1,1}\rangle_{z_1}\delta^n_4\right.$$
$$+ \frac{\lambda^2}{(n-2)!}\int d^D z_1 \int d^D z_2(\lhd^{1,2,3,4}_{1,1,2,2} + \lhd^{1,2,3,4}_{1,2,1,2} + \lhd^{1,2,3,4}_{1,2,2,1} + \lhd^{1,2,3,4}_{2,1,2,1}$$
$$+ \lhd^{1,2,3,4}_{2,2,1,1} + \lhd^{1,2,3,4}_{2,1,1,2})\Delta(z_1 - z_2)^{n-2}$$
$$\left.+ \sum_{k=3}^{\infty}\frac{(-\lambda)^k}{k!}\sum_{i_1,i_2,i_3,i_4}^{k}\sum_{\{p\}_{n,i_1,i_2,i_3,i_4}}\langle\lhd^{1,2,3,4}_{i_1,i_2,i_3,i_4}\prod_{l>j}^{k}\Delta(z_j - z_l)^{m_{jl}}\rangle\right]_{z_1,...,z_k}$$

$$(6.2.17)$$

7 Appendix: Explicit calculations

In the following we will sometimes use the compact notation $W_{1,...,k}[J] \equiv \frac{\delta^n W[J]}{\delta J_1 ... \delta J_n}$

7.1 $\langle\phi_1\phi_2\phi_3\phi_4\rangle$

We will calculate in chain the functional derivatives of $\log W[J]$.

$$\frac{\delta \log W[J]}{\delta J_1} = \frac{1}{W[J]}\frac{\delta W[J]}{\delta J_1}$$

$$\frac{\delta^2 \log W[J]}{\delta J_1 \delta J_2} = -\frac{1}{W^2[J]}\frac{\delta W[J]}{\delta J_1}\frac{\delta W[J]}{\delta J_2} + \frac{1}{W[J]}\frac{\delta^2 W[J]}{\delta J_1 \delta J_2}$$
$$= -\frac{\delta \log W[J]}{\delta J_1}\frac{\delta \log W[J]}{\delta J_2} + \frac{1}{W[J]}\frac{\delta^2 W[J]}{\delta J_1 \delta J_2}$$

$$\frac{\delta^3 \log W[J]}{\delta J_1 \delta J_2 \delta J_3} = -\frac{\delta^2 \log W[J]}{\delta J_1 \delta J_3}\frac{\delta \log W[J]}{\delta J_2} + \frac{\delta \log W[J]}{\delta J_1}\frac{\delta^2 \log W[J]}{\delta J_2 \delta J_3}$$

$$\underbrace{-\frac{1}{W^2[J]}\frac{\delta^2 W[J]}{\delta J_1 \delta J_2}\frac{\delta W[J]}{\delta J_3}}_{\left(-\frac{\delta \log W[J]}{\delta J_1}\frac{\delta \log W[J]}{\delta J_2} - \frac{\delta^2 \log W[J]}{\delta J_1 \delta J_2}\right)\frac{\delta \log W[J]}{\delta J_3}} + \frac{1}{W[J]}\frac{\delta^3 W[J]}{\delta J_1 \delta J_2 \delta J_3}$$

$$= -\frac{\delta^2 \log W[J]}{\delta J_1 \delta J_3}\frac{\delta \log W[J]}{\delta J_2} - \frac{\delta \log W[J]}{\delta J_1}\frac{\delta^2 \log W[J]}{\delta J_2 \delta J_3}$$

$$- \frac{\delta \log W[J]}{\delta J_1}\frac{\delta \log W[J]}{\delta J_2}\frac{\delta \log W[J]}{\delta J_3} - \frac{\delta \log W[J]}{\delta J_3}\frac{\delta^2 \log W[J]}{\delta J_1 \delta J_2}$$

$$+ \frac{1}{W[J]}\frac{\delta^3 W[J]}{\delta J_1 \delta J_2 \delta J_3}$$

$$(7.1.1)$$

$$\frac{\delta^4 \log W[J]}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} = -\frac{\delta^3 \log W[J]}{\delta J_1 \delta J_3 \delta J_4}\frac{\delta \log W[J]}{\delta J_2} - \frac{\delta^2 \log W[J]}{\delta J_1 \delta J_3}\frac{\delta^2 \log W[J]}{\delta J_2 \delta J_4}$$

$$- \frac{\delta^2 \log W[J]}{\delta J_1 \delta J_4}\frac{\delta^2 \log W[J]}{\delta J_2 \delta J_3} - \frac{\delta^3 \log W[J]}{\delta J_2 \delta J_3 \delta J_4}\frac{\delta \log W[J]}{\delta J_1}$$

$$- \frac{\delta^2 \log W[J]}{\delta J_1 \delta J_4}\frac{\delta \log W[J]}{\delta J_2}\frac{\delta \log W[J]}{\delta J_3} - \frac{\delta^2 \log W[J]}{\delta J_2 \delta J_4}\frac{\delta \log W[J]}{\delta J_1}\frac{\delta \log W[J]}{\delta J_3}$$

$$- \frac{\delta^2 \log W[J]}{\delta J_3 \delta J_4}\frac{\delta \log W[J]}{\delta J_1}\frac{\delta \log W[J]}{\delta J_2} - \frac{\delta^3 \log W[J]}{\delta J_1 \delta J_2 \delta J_4}\frac{\delta \log W[J]}{\delta J_3}$$

$$- \frac{\delta^2 \log W[J]}{\delta J_1 \delta J_2}\frac{\delta^2 \log W[J]}{\delta J_3 \delta J_4} - \underbrace{\frac{1}{W[J]}\frac{\delta^3 W[J]}{\delta J_1 \delta J_2 \delta J_3}}_{*}\underbrace{\frac{1}{W[J]}\frac{\delta W[J]}{\delta J_4}}_{\frac{\delta \log W[J]}{\delta J_4}}$$

$$+ \frac{1}{W[J]}\frac{\delta^4 W[J]}{\delta J_1 \delta J_2 \delta J_3 \delta J_4}$$

Where $*$ stands for

$$- \frac{\delta^2 \log W[J]}{\delta J_1 \delta J_3}\frac{\delta \log W[J]}{\delta J_2} - \frac{\delta \log W[J]}{\delta J_1}\frac{\delta^2 \log W[J]}{\delta J_2 \delta J_3}$$

$$- \frac{\delta \log W[J]}{\delta J_1}\frac{\delta \log W[J]}{\delta J_2}\frac{\delta \log W[J]}{\delta J_3} - \frac{\delta \log W[J]}{\delta J_3}\frac{\delta^2 \log W[J]}{\delta J_1 \delta J_2}$$

$$- \frac{\delta^3 \log W[J]}{\delta J_1 \delta J_2 \delta J_3}$$

Remembering that $\langle \phi_1 \cdots \phi_N \rangle = \frac{\delta^N \log W[J]}{\delta J_1 \ldots \delta J_N}$ we finally have

$$
\begin{aligned}
&\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle \\
&= - \langle \phi_1 \phi_3 \phi_4 \rangle \langle \phi_2 \rangle - \langle \phi_1 \phi_3 \rangle \langle \phi_2 \phi_4 \rangle - \langle \phi_1 \phi_4 \rangle \langle \phi_2 \phi_3 \rangle - \langle \phi_1 \rangle \langle \phi_2 \phi_3 \phi_4 \rangle \\
&\quad - \langle \phi_1 \phi_4 \rangle \langle \phi_2 \rangle \langle \phi_3 \rangle - \langle \phi_1 \rangle \langle \phi_2 \phi_4 \rangle \langle \phi_3 \rangle - \langle \phi_1 \rangle \langle \phi_2 \rangle \langle \phi_3 \phi_4 \rangle - \langle \phi_1 \phi_2 \phi_4 \rangle \langle \phi_3 \rangle \\
&\quad - \langle \phi_1 \phi_2 \rangle \langle \phi_3 \phi_4 \rangle - \langle \phi_1 \phi_3 \rangle \langle \phi_2 \rangle \langle \phi_4 \rangle - \langle \phi_1 \rangle \langle \phi_2 \phi_3 \rangle \langle \phi_4 \rangle - \langle \phi_1 \rangle \langle \phi_2 \rangle \langle \phi_3 \rangle \langle \phi_4 \rangle \\
&\quad - \langle \phi_1 \phi_2 \rangle \langle \phi_3 \rangle \langle \phi_4 \rangle - \langle \phi_1 \phi_2 \phi_3 \rangle \langle \phi_4 \rangle + \frac{1}{W^{(n)}[0]} \frac{\delta^4 W^{(n)}[J]}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \bigg|_{J=0}
\end{aligned}
\tag{7.1.2}
$$

7.2 $\quad \frac{1}{W[0]} W_{1,\ldots,k}[0]$

Calculating in chain the derivatives using the form (6.2.3) for the generating functional we have

$$
\begin{aligned}
\frac{\delta W}{\delta J_1} &= \langle J(y) \Delta(y - x_1) \rangle_y \, e^{\frac{1}{2}\langle J\Delta J \rangle} F[J] + e^{\frac{1}{2}\langle J\Delta J \rangle} \frac{\delta F[J]}{\delta J_1} \\
&= \left(\langle J(y) \Delta(y - x_1) \rangle_y F[J] + \frac{\delta F[J]}{\delta J_1} \right) e^{\frac{1}{2}\langle J\Delta J \rangle}
\end{aligned}
\tag{7.2.1}
$$

$$
\begin{aligned}
\frac{\delta^2 W}{\delta J_1 \delta J_2} &= \left(\Delta_{12} F[J] + \langle J(y) \Delta_{y1} \rangle_y \frac{\delta F[J]}{\delta J_2} + \frac{\delta^2 F[J]}{\delta J_1 \delta J_2} \right) e^{\frac{1}{2}\langle J\Delta J \rangle} \\
&\quad + \langle J(y) \Delta_{y2} \rangle_y \frac{\delta W}{\delta J_1}
\end{aligned}
\tag{7.2.2}
$$

where $\Delta_{y1} \equiv \Delta(y - x_1)$.

$$
\begin{aligned}
\frac{\delta^3 W}{\delta J_1 \delta J_2 \delta J_3} &= \left(\Delta_{12} \frac{\delta F[J]}{\delta J_3} + \Delta_{13} \frac{\delta F[J]}{\delta J_2} + \langle J(y) \Delta_{y1} \rangle_y \frac{\delta^2 F[J]}{\delta J_2 \delta J_3} \right. \\
&\quad \left. + \frac{\delta^3 F[J]}{\delta J_1 \delta J_2 \delta J_3} \right) e^{\frac{1}{2}\langle J\Delta J \rangle} + \langle J(y) \Delta_{y3} \rangle_y \frac{\delta^2 W}{\delta J_1 \delta J_2} + \Delta_{23} \frac{\delta W}{\delta J_1} \\
&\quad + \langle J(y) \Delta_{y2} \rangle_y \frac{\delta^2 W}{\delta J_1 \delta J_3}
\end{aligned}
\tag{7.2.3}
$$

$$\frac{\delta^4 W}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} = \left(\Delta_{12} \frac{\delta^2 F[J]}{\delta J_3 \delta J_4} + \Delta_{13} \frac{\delta^2 F[J]}{\delta J_2 \delta J_4} + \Delta_{14} \frac{\delta^2 F[J]}{\delta J_2 \delta J_3} \right.$$
$$\left. + \langle J(y) \Delta_{y1} \rangle_y \frac{\delta^3 F[J]}{\delta J_2 \delta J_3 \delta J_4} + \frac{\delta^4 F[J]}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \right) e^{\frac{1}{2} \langle J \Delta J \rangle}$$
$$+ \Delta_{34} \frac{\delta^2 W}{\delta J_1 \delta J_2} + \langle J(y) \Delta_{y3} \rangle_y \frac{\delta^3 W}{\delta J_1 \delta J_2 \delta J_4} + \Delta_{23} \frac{\delta^2 W}{\delta J_1 \delta J_4} \qquad (7.2.4)$$
$$+ \Delta_{24} \frac{\delta^2 W}{\delta J_1 \delta J_3} + \langle J(y) \Delta_{y2} \rangle_y \frac{\delta^3 W}{\delta J_1 \delta J_3 \delta J_4}$$
$$+ \langle J(y) \Delta_{y4} \rangle_y \frac{\delta^3 W}{\delta J_1 \delta J_3 \delta J_4}$$

Setting $J = 0$

$$\frac{\delta^4 W}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \bigg|_{J=0} = \Delta_{12} \frac{\delta^2 F[J]}{\delta J_3 \delta J_4} \bigg|_{J=0} + \Delta_{13} \frac{\delta^2 F[J]}{\delta J_2 \delta J_4} \bigg|_{J=0} + \Delta_{14} \frac{\delta^2 F[J]}{\delta J_2 \delta J_3} \bigg|_{J=0}$$
$$+ \frac{\delta^4 F[J]}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \bigg|_{J=0} + \Delta_{34} \frac{\delta^2 W}{\delta J_1 \delta J_2} \bigg|_{J=0} \qquad (7.2.5)$$
$$+ \Delta_{23} \frac{\delta^2 W}{\delta J_1 \delta J_4} \bigg|_{J=0} + \Delta_{24} \frac{\delta^2 W}{\delta J_1 \delta J_3} \bigg|_{J=0}$$

From (7.2.2) we have

$$\frac{\delta^2 F[J]}{\delta J_a \delta J_b} \bigg|_{J=0} = \frac{\delta^2 W[J]}{\delta J_a \delta J_b} \bigg|_{J=0} - W[0] \Delta_{ab} \qquad (7.2.6)$$

Furthermore from (7.1.1)

$$\frac{1}{W[0]} \frac{\delta^2 W[J]}{\delta J_1 \delta J_2} \bigg|_{J=0} = \langle \phi_a \rangle \langle \phi_b \rangle - \langle \phi_a \phi_b \rangle \qquad (7.2.7)$$

Thus

$$\frac{1}{W^{(n)}[0]} \frac{\delta^4 W^{(n)}[J]}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \bigg|_{J=0}$$
$$= \Delta_{12}(\langle \phi_3 \rangle \langle \phi_4 \rangle - \langle \phi_3 \phi_4 \rangle - \Delta_{34}) + \Delta_{13}(\langle \phi_2 \rangle \langle \phi_4 \rangle - \langle \phi_2 \phi_4 \rangle - \Delta_{24})$$
$$+ \Delta_{14}(\langle \phi_2 \rangle \langle \phi_3 \rangle - \langle \phi_2 \phi_3 \rangle - \Delta_{23}) + \Delta_{23}(\langle \phi_1 \rangle \langle \phi_4 \rangle - \langle \phi_1 \phi_4 \rangle) \qquad (7.2.8)$$
$$+ \Delta_{24}(\langle \phi_1 \rangle \langle \phi_3 \rangle - \langle \phi_1 \phi_3 \rangle) + \Delta_{34}(\langle \phi_1 \rangle \langle \phi_2 \rangle - \langle \phi_1 \phi_2 \rangle)$$
$$+ \frac{1}{W^{(n)}[0]} \frac{\delta^4 F[J]}{\delta J_1 \delta J_2 \delta J_3 \delta J_4} \bigg|_{J=0}$$

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