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The Adelic point of view on Abelian L-functions

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Introduction

In this thesis, the Iwasawa-Tate theory will be expounded, following an interpretation much closer to the modern automorphic point of view of the Langlands program. The subject matter, usually known as Tate's thesis, aims to prove the functional equation of a large family of L-functions through the harmonic analysis of the ring of adèles associated with a number field. To appreciate the relevance of the result, it is helpful to review a hint of the deep connection between arithmetic and L-functions starting with what can be considered the prototype of an L-function, the Riemann's ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\,.$$

This series was considered by Riemann as a function in the complex variable s, initially defined for $\Re(s) > 1$, where the sum converges absolutely. He was able to extend it analytically on the whole complex plane except for a simple pole at $s = 1$. The importance of the ζ for arithmetic comes from the link between its zeros on the strip $0 \leq \Re(s) \leq 1$ and the prime numbers. To get such a connection, one relies on some special properties of the function ζ:

(i) *Euler Product.* On the right half-plane $\Re(s) > 1$ the function ζ is equal to a product over all primes p

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}\,.$$

(ii) *Gamma Factor and Functional Equation.* After one multiplies ζ by a factor involving the Gamma function, getting the completed zeta function

$$\hat{\zeta}(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)\,,$$

one has a functional equation for $\hat{\zeta}$

$$\hat{\zeta}(s) = \hat{\zeta}(1 - s)\,.$$

The functional equation and the Euler product cause the zeros of $\hat{\zeta}$ to be confined to the region $0 \leq \Re(s) \leq 1$ and contour integrals on this region lead to the so-called explicit formulas relating critical zeros of ζ and prime numbers. Knowing more about the zeros gives better information about the distribution of prime numbers. For example, the Prime Number Theorem, which is about the asymptotic behaviour of the number of primes between 0 and N as $N \to \infty$, can be deduced from the non-vanishing of ζ in the region $\Re(s) \geq 1$. This culminates with the famous Riemann Hypothesis, which predicts that there are no zeros in the region $\Re(s) > \frac{1}{2}$ and is equivalent to a precise estimate of the difference between the prime-counting function mentioned above and its asymptotic approximation. Dirichlet studied a variant of ζ, the Dirichlet L-function

$$L(s,\chi) = \sum_{n\in\mathbb{N},\,(n,N)=1} \frac{\chi(n)}{n^s}$$

attached to a character χ of the group of units of $\mathbb{Z}/N\mathbb{Z}$. He obtained a proof of the homonymous theorem on primes in arithmetic progression using the non-vanishing of $L(s,\chi)$ at $s=1$. All this generalizes to number fields, where Riemann's ζ becomes a special case of the Dedekind ζ-function, and Hecke L-functions generalize all the previous examples. Those mentioned are part of a larger family of functions, called L-functions. For every L-function, one is interested in the properties corresponding to those satisfied by Riemann's ζ:

(i) Euler product over primes,

(ii) analytic continuation,

(iii) Gamma factors,

(iv) functional equation,

(v) Riemann hypothesis.

The Iwasawa-Tate theory provides a framework where one obtains all the properties from (i) to (iv) for Hecke L-functions. This framework is based on the ring of adèles, a locally compact topological ring associated with a number field and constructed using all its completions. The advantage of working with the adèles is that we can use the techniques coming from the duality and measure theory of locally compact abelian groups. Furthermore, above all, the adèles convey local-to-global techniques: working in the *local*, namely on a completion of the number field, allows, through the adèles, to obtain *global* data, i.e. information on the number field. As an example, consider the field of rational numbers $\mathbb{Q}$. Its ring of adèles is the restricted direct product

$$\mathbb{A} = \mathbb{R} \times \prod_{p \text{ prime}}{}' \mathbb{Q}_p$$

which is a construction that produces a locally compact ring from the completions of $\mathbb{Q}$. The field $\mathbb{Q}$ is embedded diagonally in $\mathbb{A}$ as a lattice, in the sense that $\mathbb{A}/\mathbb{Q}$ is compact and $\mathbb{Q}$ is discrete in $\mathbb{A}$. As a locally compact abelian group, $\mathbb{A}$ is isomorphic to its Potryagin dual, the group $\widehat{\mathbb{A}}$ of unitary characters of $\mathbb{A}$. It has a translation-invariant measure, the Haar measure, which is determined uniquely by the choice of an isomorphism $\mathbb{A} \cong \widehat{\mathbb{A}}$. The latter is induced by a unitary character ψ of the quotient $\mathbb{A}/\mathbb{Q}$. This way, $\mathbb{Q}$ becomes a self-dual lattice in $\mathbb{A}$. The relation between $\mathbb{Q}$ and $\mathbb{A}$ is analogous to the one between the integers and the real number line. The argument used by Riemann to obtain the functional equation of the completed ζ-function was based on the self-duality of $\mathbb{Z}$ inside $\mathbb{R}$: for $\Re(s) > 1$, the function $\hat{\zeta}$ can be expressed as an integral

$$\hat{\zeta}(s) = \int_0^\infty x^{\frac{s}{2}-1}\frac{1}{2}(\theta(x)-1)\,dx\,,$$

where θ is the function

$$\theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} \,.$$

The Poisson summation formula for the self-dual lattice $\mathbb{Z}$ of the real line implies a functional equation for θ,

$$\theta(x) = \frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right) \,,$$

which in turn allows us to eliminate the problematic integral over the interval $(0,1)$. Finally, one obtains an expression

$$\hat{\zeta}(s) = -\frac{1}{s} - \frac{1}{1-s} + \int_1^\infty \left(x^{\frac{s}{2}-1} + x^{\frac{1-s}{2}-1}\right)\frac{1}{2}(\theta(x)-1)\,dx$$

for $\hat{\zeta}$ which make sense for all $s \in \mathbb{C} \setminus \{0,1\}$ and is invariant by the symmetry $s \mapsto 1-s$. The framework developed by Tate allowed him to transport Riemann's argument into the adèlic context. He considered a functional, called *zeta integral*, operating on a space of nice enough functions $f : \mathbb{A} \to \mathbb{C}$, where the Fourier transform induces a linear automorphism. To define the zeta integral, one needs to introduce the group of *idèles* $\mathbb{A}^\times$. This is the group of units of the ring $\mathbb{A}$ and it is again a restricted direct product involving the groups of invertible elements of every completion of $\mathbb{Q}$:

$$\mathbb{A}^\times = \mathbb{R}^\times \times {\prod_p}' \mathbb{Q}_p^\times \,.$$

The multiplicative group of invertible rational numbers $\mathbb{Q}^\times$ is a discrete sub-group of the idèles. The characters

$$\omega : \mathbb{A}^\times \longrightarrow \mathbb{C}^\times$$

which are trivial on $\mathbb{Q}^\times$ are called *idèle class characters* and they are part of the definition of the zeta integral. There is a special idèle class character

$$|\cdot| : \mathbb{A}^\times \longrightarrow \mathbb{R}_+^\times$$

with target the multiplicative group of positive real numbers, which is called *idèlic norm*. It gives an inclusion of the complex plane inside the space of idèle class characters via the map $s \mapsto |\cdot|^s$. The zeta integral associated with a couple (s,ω) is then defined on a function $f : \mathbb{A} \to \mathbb{C}$ by

$$\zeta(s,\omega;f) = \int_{\mathbb{A}^\times} f(x)\omega(x)|x|^s \; d^\times x \,,$$

if it does exist. The Poisson summation formula

$$\sum_{y\in\mathbb{Q}} f(xy) = |x|^{-1} \sum_{y\in\mathbb{Q}} \widehat{f}(x^{-1}y)$$

is available thanks to the self-duality of $\mathbb{Q}$ in $\mathbb{A}$. It enables us to use the same ideas of Riemann to get the analytic continuation of $\zeta(s,\omega;f)$ and the functional equation

$$\zeta(1-s,\omega^{-1};\widehat{f}) = \zeta(s,\omega;f) \,.$$

What does the zeta integral have to do with L-functions? Dirichlet characters χ are examples of idèle class characters and evaluating $\zeta(s,\chi)$ on a particular function f produces the Dirichlet L-function completed with its Gamma factors. Moreover, the restricted direct product structure of the adèles and idèles makes it possible to analyze the zeta integral and characters locally on the completions $\mathbb{Q}_p$, including the real numbers as the case of the "infinite prime" $p = \infty$. Explicitly, idèle class characters ω are determined by families of local characters $\omega_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, and the global zeta integral is determined by its local analogue

$$\zeta_p(s, \omega_p; f_p) = \int_{\mathbb{Q}_p^\times} f_p(x_p)\omega_p(x_p)|x|_p^s \, d^\times x_p \, .$$

Thanks to this, the Euler product decomposition of the L-function is already built into the global zeta integral and the Euler factors are obtained by the local zeta integrals.

This thesis aims to present the above-discussed theory in the context of representation theory, following the guideline given by S. Kudla in [Kud04]. This perspective focuses on a representation $\mathscr{S}'(\mathbb{A})$ of the idèles whose vectors are tempered distributions defined on the additive adèlic group. From this point of view, the zeta integral is a meromorphic section of

$$\coprod_{\omega} \mathscr{S}'(\omega) \, ,$$

where ω ranges over idèle class characters and $\mathscr{S}'(\omega)$ is the 1-dimensional sub-representation of $\mathscr{S}'(\mathbb{A})$ with character ω. In this approach, the greatest effort is put into proving that $\mathscr{S}'(\omega)$ has dimension one, a result that relies heavily on working locally at each completion of the number field. Indeed, there is local-global parallelism in the whole setup: taking the field $\mathbb{Q}$ as an example, there are representations $\mathscr{S}'(\mathbb{Q}_p)$ of the multiplicative groups $\mathbb{Q}_p^\times$ for every completion of $\mathbb{Q}$. Local characters ω_p identify 1-dimensional sub-representations $\mathscr{S}'(\omega_p)$ and the adèlic construction are recovered by these. The Fourier transform of tempered distributions induces isomorphisms

$$\mathscr{S}'(\omega^{-1}|\cdot|^{1-s}) \cong \mathscr{S}'(\omega|\cdot|^s) \, ,$$

whose appearance resembles the functional equation. This holds both locally and globally. Analysing the local zeta integrals, one can construct non-zero distributions $\zeta_p^o(\omega_p)$ generating the space $\mathscr{S}'(\omega_p)$ for all local characters. Then, the Euler factors $L_p(s,\omega_p)$ are recovered by proportionality between the local zeta integral and $\zeta_p^o(\omega_p|\cdot|_p^s)$. The product of the local data produces a non-zero vector $\zeta^o(\omega)$ of $\mathscr{S}'(\omega)$ for each idèle class character ω and the global L-function is recovered as a factor of proportionality between the zeta integral and $\zeta^o(\omega|\cdot|^s)$. The functional equation of the completed global L-function is then derived from the functional equation of the zeta integral and the isomorphism $\mathscr{S}'(\omega^{-1}|\cdot|^{1-s}) \cong \mathscr{S}'(\omega|\cdot|^s)$ induced by the Fourier transform.

Content of the chapters. In Chapter 1, we study absolute values and completions of *global fields*. These are the fields to which the adèlic methods developed by Tate apply. We explain the notion of *place* of a field, a class of equivalent absolute values corresponding to a completion of the field. For number fields, places generalize primes by including some "infinite primes" whose corresponding complete fields are $\mathbb{R}$ or $\mathbb{C}$, along with usual (finite) primes and p-adic complete fields. The completions of global fields are called *local fields* and they are the working ground for the local analysis present at each step of the Iwasawa-Tate theory. Local fields are locally compact topological fields and the adèles form a locally compact topological ring. For this reason, we include in Chapter 2 a synthesis of the theory of locally compact abelian groups on which we recall all the tools needed in the rest of the thesis. They include the theory of characters and Pontryagin duality, the integration theory with Haar measures, the Fourier transform and Poisson summation formula, the *module* of locally compact rings and fields which links to the absolute values of Chapter 1, and finally, a discussion about restricted direct products in general. In Chapter 3 we introduce the adèles $\mathbb{A}$ of a global field and we describe its additive and multiplicative structure from the point of view of the previous chapter's notions. Regarding the additive structure, the goal is to show that the ring $\mathbb{A}$ is its own Potryagin dual and that the global field becomes a self-dual lattice by this identification. This is achieved through the additive self-duality of local fields and the construction of a suitable additive character of the adèles. Then we study the group of idèles. This group encodes arithmetic information of the global field, for example, it has quotients isomorphic to the group of fractional ideals (or the divisor group) and the ideal class group (or the divisor class group). Another example is the Product Formula (Proposition 3.4.5) for global fields, a property resembling the fact that principal divisors have zero degree. We define the idèlic norm and non-canonical decompositions of the group of idèles which are useful to describe idèle class characters (Section 3.5). Chapter 4 is the last one and contains the representation-theoretic approach discussed above. It ends with the proof of the functional equation of the zeta integral for all global fields in Section 4.4.

Chapter 1

Global Fields

The notion of *global field* combines number fields and their geometrical analogue, function fields, which are the fields of algebraic functions on the projective curves defined over a finite field. The reason for us to introduce this notion is the possibility of developing the same adèlic formalism for both types of fields, but the analogy between the two has a long history and is known as *function field analogy*. The chapter presents the theory of valuations/absolute values and completion in full generality, although we are interested in valuations of global fields. An extension of fields $k \hookrightarrow K$ induces a map between the set of valuations in the other direction just like a morphism of rings induces a map between the spectra. Then, the study of places (equivalent classes of valuations) of global fields is reduced into two parts: the classification of places of $\mathbb{Q}$ and the field $\mathbb{F}_q(T)$ of rational functions on the projective line over the finite field $\mathbb{F}_q$ with q elements (Theorem 1.1.16), and the study of the fibres over the set of valuations of $\mathbb{Q}$ and $\mathbb{F}_q(T)$. In the classification theorem, we see that the places of $\mathbb{Q}$ correspond to the primes p and the complete field $\mathbb{Q}_p$ of p-adic numbers, plus one more place, the infinite prime ∞, whose corresponding complete field is $\mathbb{R}$. For the field $\mathbb{F}_q(T)$, places correspond to the points of the projective line over $\mathbb{F}_q$. Knowing the places of a global field K is then the same as knowing the ways each valuation of $\mathbb{Q}$ or $\mathbb{F}_q(T)$ extend to K, and this is determined by the action of the Galois group (Proposition 1.3.1). A reference for this chapter is Chapter II of [CF67].

1.1 Valuations and places

The most familiar case of an absolute value is perhaps the usual one defined on $\mathbb{Q}$, the one used to define the field of real numbers via completion. The next more familiar example may be the norm of a complex number. Continuing along these lines, one might have met the p-adic absolute values used to define the p-adic numbers. The definition 1.1.1 of valuation encompasses all the above examples with possibly an exponent. The need to consider exponents is motivated by measure theory. For example, the square of the usual norm of a complex number is a more natural thing to consider, from the point of view of measure theory, because it measures how multiplication affects the size of subsets of the field (see Section 2.3).

Notation. The field of real numbers is denoted by $\mathbb{R}$. The convention for intervals is the following:

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\},$$

for $a < b$. We denote by $\mathbb{R}_+$ the set $[0, +\infty)$ of non-negative real numbers and $\mathbb{R}_+^\times$ is the group of invertible positive real numbers. The symbol $^\times$ is used to indicate the group of invertible elements of monoids, rings or fields.

Definition 1.1.1. Let K be any field. A *valuation* or *absolute value* is a function $|\cdot| : K \to \mathbb{R}_+$ satisfying the three properties written below:

A1. $|x| = 0$ if and only if $x = 0$ in K;

A2. for all $x, y \in K$ the equality $|xy| = |x||y|$ holds;

A3. there is some real constant $C > 0$ such that $|x + y| \leq C \cdot \sup \left\{|x|, |y|\right\}$ for all $x, y \in K$.

If axiom A3 is satisfied with constant $C = 1$ then $|\cdot|$ is said *ultrametric* or *non-archimedean*, if not then is said *archimedean*. A field with a fixed valuation is called *valued field* for short. The term absolute value will be used mostly to indicate the image $|x|$ of an element $x \in K$ as the *absolute value of* x via the valuation $|\cdot|$.

Given the first two axioms, it must be $|u| = 1$ for any roots of unity $u \in K$ and axiom A3 become equivalent to the finiteness of the supremum

$$\sup_{|x| \leq 1} |1 + x| \,.$$

The constant C in the axiom corresponds to any real number C that realizes the bound

$$|1 + x| \leq C \,, \quad |x| \leq 1 \,.$$

Example 1.1.2. Let $K = \mathbb{R}$, the field of real numbers. The function $|\cdot|_2 : \mathbb{R} \to \mathbb{R}_+$ defined by $|x|_2 = x^2$ for all $x \in \mathbb{R}$ is a valuation in the sense of Definition 1.1.1: it is multiplicative, the zero is sent to itself and

$$\sup_{|x| \leq 1} |1 + x|_2 = 4.$$

Remark 1.1.3. In the literature, the term *discrete valuation* usually refers to a map $v : K \to \mathbb{Z} \cup \{\infty\}$ satisfying

(i) $v(x) = \infty$ if and only if $x = 0$;

(ii) $v(xy) = v(x) + v(y)$ for all $x, y \in K$;

(iii) $v(x + y) \geq \min\big(v(x), v(y)\big)$ for all $x, y \in K$,

where ∞ is assumed to be larger than any integer. Given a real number $t \in (0, 1)$, we recover an ultrametric valuation $|\cdot|_{v,t}$ in the sense of Definition 1.1.1 by defining $|x|_{v,t} := t^{v(x)}$ for all $x \in K$, where t^∞ is set equal to 0.

Example 1.1.4. Let R be a Dedekind domain with fraction field K, and $\mathfrak{p}$ a maximal ideal of R (for example R can be the ring of integers). If $x \in K$ is non-zero, there is a unique integer n such that $x \in \mathfrak{p}^n$ but $x \notin \mathfrak{p}^{n+1}$. Call it $\mathrm{ord}_\mathfrak{p}(x)$. For a real number $t \in (0, 1)$, define $|x|_{\mathfrak{p},t} = t^{\mathrm{ord}_\mathfrak{p}(x)}$ for all non-zero $x \in K$ and set $|0|_{\mathfrak{p},t} := 0$. Then $|\cdot|_{\mathfrak{p},t}$ is an ultrametric valuation of K.

Every valuation $|\cdot|$ induces a topology on K where a pre-basis of open neighbourhoods of an element $x_0 \in K$ consists of the subsets $\{x \in K : |x - x_0| < \varepsilon\}$ for all $\varepsilon > 0$. The next goal is to understand this topology, starting with a couple of lemmas on valuations.

Lemma 1.1.5. *For any valuation $|\cdot|$ on K and any $\lambda > 0$ the function $|\cdot|^\lambda$ is a valuation.*

Proof. The map $\mathbb{R}_+ \to \mathbb{R}_+,\ x \mapsto x^\lambda$ is a continuous, multiplication-preserving bijection that fixes 0 and 1, then axioms A1 and A2 are obviously preserved. The map is also increasing and then, for all $x \in K$, the conditions $|x| \leq 1$ and $|x|^\lambda \leq 1$ are equivalent, and the same is true for the conditions $|1+x| \leq C$ and $|1+x|^\lambda \leq C^\lambda$. This is enough to verify that axiom A3 holds for $|\cdot|^\lambda$. □

Lemma 1.1.6. *If a valuation $|\cdot|$ on K satisfies*

$$\sup_{|x|\leq 1} |1+x| \leq 2,$$

then it satisfies the triangle inequality:

$$|x+y| \leq |x| + |y|$$

for any $x, y \in K$

Proof. The bound $\sup_{|x|\leq 1}|1+x| \leq 2$ implies the inequality $|x+y| \leq 2 \cdot \sup\{|x|, |y|\}$ for all $x, y \in K$ and we can iterate this to obtain inequalities for the sum of two, four, eight and so on elements of K. For example $|(x_1+x_2)+(y_1+y_2)| \leq 2 \cdot \sup\{|x_1+x_2|, |y_1+y_2|\}$ and with an iteration of the inequality one gets $|x_1+x_2+y_1+y_2| \leq 2 \cdot 2 \cdot \sup\{|x_1|, |x_2|, |y_1|, |y_2|\}$, for all $x_1, x_2, y_1, y_2 \in K$. By induction one has the formula

$$\left|\sum_{i=1}^{2^n} x_i\right| \leq 2^n \cdot \sup_{1\leq i\leq 2^n} |x_i|$$

for all $x_1, \ldots, x_{2^n} \in K$, n positive integer. Noticing that any positive integer n lies between two consecutive powers of 2 and any sum of n elements of K can be completed to a sum of a power of 2 elements by adding zeroes, one has the relation

$$\left|\sum_{i=1}^{n} x_i\right| \leq 2n \cdot \sup_{1\leq i\leq n} |x_i|\ ,$$

for any n elements $x_1 \ldots, x_n$ of K. In particular, letting x_i be the identity of K for all i, produces the estimate $|n \cdot 1_K| \leq 2n$. Given that, we can estimate the absolute value of the sum of two elements $x, y \in K$ as follows:

$$\begin{aligned}
|x+y| &= \sqrt[n]{|(x+y)^n|} \\
&= \sqrt[n]{\left|\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}\right|} \\
&\leq \sqrt[n]{2(n+1) \cdot \sup_{0 \leq i \leq n} \left|\binom{n}{i} x^i y^{n-i}\right|} \\
&\leq \sqrt[n]{4(n+1) \cdot \sup_{0 \leq i \leq n} \binom{n}{i} |x|^i |y|^{n-i}} \\
&\leq \sqrt[n]{4(n+1) \cdot \sum_{i=0}^{n} \binom{n}{i} |x|^i |y|^{n-i}} \\
&= \sqrt[n]{4(n+1)} \cdot \big(|x| + |y|\big) \,,
\end{aligned}$$

with n an arbitrarily large positive integer. Since $\sqrt[n]{4(n+1)}$ approaches 1 as n tends to infinity, the triangle inequality is established. □

Proposition 1.1.7. *The topology induced by a valuation $|\cdot|$ on K is equivalent to a topology induced by a metric. Moreover, the addition, multiplication and inversion of K are continuous, making it a topological field.*

Proof. Let C be the constant of axiom A3 for the absolute value. Since $|1| = 1$, the constant is forced to be greater or equal to 1. If $C = 1$ then the triangle inequality is automatically satisfied for $|\cdot|$ and the function $d(x, y) = |x - y|$ is a metric on K. If $C > 1$ we have that $|\cdot|^\lambda$, for $\lambda = \frac{\log 2}{\log C}$, is a valuation satisfying triangle inequality by the previous two lemmas. Therefore, the function $d(x, y) = |x - y|^\lambda$ is a metric and the equality

$$\big\{x \in K : |x - x_0| < \varepsilon\big\} = \Big\{x \in K : d(x, x_0) < \varepsilon^\lambda\Big\}$$

guarantees that the two topologies induced by $|\cdot|^\lambda$ and $|\cdot|$ respectively are equivalent, hence K is metrizable and d is a metric inducing the topology of K. From general, basic theory of metric spaces we know that the sets $B(x_0, \varepsilon) := \big\{x \in K : d(x, x_0) < \varepsilon\big\}$, for all $x_0 \in K$ and $\varepsilon > 0$, form a basis of the topology and that d is a continuous function from $K \times K$ to the real numbers. In the rest of the proof, we can assume that $|\cdot|$ already satisfies the triangle inequality, since the continuity of the operations on K does not depend on the parameter λ.

To prove continuity of the addition, take $(x_0, y_0) \in K \times K$ and a ball of radius $\varepsilon > 0$ around $x_0 + y_0$. If $(x, y) \in B(x_0, \frac{\varepsilon}{2}) \times B(y_0, \frac{\varepsilon}{2})$ then

$$|x + y - x_0 - y_0| \leq |x - x_0| + |y - y_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \,,$$

so that $x + y$ belongs to the open set $B(x_0 + y_0, \varepsilon)$.

Continuity of multiplication: consider again a ball of radius $\varepsilon > 0$ around the product $x_0 y_0$. Observing that

$$d(xy, x_0y_0) = |xy - xy_0 + xy_0 - x_0y_0| \leq |x| \cdot |y - y_0| + |y_0| \cdot |x - x_0|$$

and that the absolute value $|x|$ is less or equal than $|x - x_0| + |x_0|$, we can consider the conditions $d(x, x_0) < \delta$ and $d(y, y_0) < \delta$ for δ positive real number, and it follows the estimate

$$d(xy, x_0y_0) < (\delta + |x_0|) \cdot \delta \ + |y_0| \cdot \delta \, .$$

If δ is sufficiently small, than $(\delta + |x_0|) \cdot \delta \ + |y_0| \cdot \delta < \varepsilon$ and $xy \in B(x_0y_0, \varepsilon)$.

Continuity of inversion: let $x \in K$ be non-zero and let $B(x^{-1}, \varepsilon)$ be a ball contained in $K \setminus \{0\}$, so $0 < \varepsilon < |x|^{-1}$. For any $y \in K \setminus \{0\}$, the distance between x^{-1} and y^{-1} is

$$d(x^{-1}, y^{-1}) = \frac{|x - y|}{|xy|} \, .$$

If $d(x, y) < \delta < |x|$, then $|y| > |x| - \delta$ and

$$\frac{|x - y|}{|xy|} < \frac{\delta}{|x| \cdot \big(|x| - \delta\big)} \, ,$$

ensuring that $y^{-1} \in B(x^{-1}, \varepsilon)$ when δ is small enough. □

For a valuation $|\cdot|$ of K we saw that any real number $\lambda > 0$ produces another valuation $|\cdot|^{\lambda}$ that induces the same topology as the original one. The next proposition shows that all valuations that induce the same topology as $|\cdot|$ are obtained in this way.

Proposition 1.1.8. *Let $|\cdot|_1$ and $|\cdot|_2$ be two valuation on K. If they induce the same topology, then there is a real number $\lambda > 0$ such that $|\cdot|_2 = |\cdot|_1^{\lambda}$*

Proof. Observe that for any $x \in K$ and any valuation $|\cdot|$ inducing the topology of K, the sequence $(x^n)_{n \in \mathbb{N}}$ of powers of x approaches zero if and only the sequence of absolute values $|x^n|$ goes to zero. Knowing that $|x^n| = |x|^n$, it follows that the topological property

$$\lim_{n \to \infty} x^n = 0$$

is equivalent to $|x|_i < 1$ regardless of i being 1 or 2, because they induce the same topology. Then the two conditions $|x|_1 < 1$ and $|x|_2 < 1$ are equivalent and the same argument for x^{-1} implies that $|x|_1 > 1$ and $|x|_2 > 1$ are equivalent conditions too. If one of the two valuations is trivial, i.e. $|x|_i = 1$ for any non-zero $x \in K$, then also the other is trivial, because the condition $|x|_i = 1$ is the negation of

$$|x|_i < 1 \quad \text{or} \quad |x|_i > 1$$

which is independent of i. Excluding the trivial case, we can fix an element $z \in K$ with absolute value strictly greater than 1. Define $a = |z|_1$ and $b = |z|_2$, they are both strictly greater than 1 and $b = a^{\lambda}$ for a unique $\lambda > 0$. The purpose of doing so is to measure the absolute values $|x|_1$ and $|x|_2$ of any $x \in K$ with the absolute values of z and to deduce that $|x|_2 = |x|_1^{\lambda}$. The function

$$\mathbb{R} \to \mathbb{R}_+^{\times} \, , \quad t \mapsto |z|_i^t$$

parametrizes the positive part of the real line, then, for any $x \in K^\times$, there exists a unique $t \in \mathbb{R}$ such that $|x|_i = |z|_i^t$. By completeness of the real numbers, t is the infimum of the set of rational numbers strictly larger than t or the supremum of the rationals strictly smaller than t and this fact is transposed to exponentials because of the monotonicity of this operation so that we can assert that

$$|x|_i = \inf_{\substack{q\in\mathbb{Q}\\ q>t}} |z|_i^q \; .$$

Let $\frac{m}{n}$ be a rational number and suppose that $|x|_1 < a^{\frac{m}{n}}$. Then the observation made at the beginning of the proof together with the multiplicativity of the valuation implies what follows:

$$|x|_1 < a^{\frac{m}{n}} \quad \text{is equivalent to} \quad \frac{|x|_i^n}{a^m} < 1\,,$$

by multiplicativity $\frac{|x|_1^n}{a^m} = \left|\frac{x^n}{z^m}\right|_1$ and

$$\left|\frac{x^n}{z^m}\right|_1 < 1 \quad \text{is equivalent to} \quad \left|\frac{x^n}{z^m}\right|_2 < 1\,.$$

Reversing the previous algebraic manipulations one obtains that

$$\left|\frac{x^n}{z^m}\right|_2 < 1 \quad \text{if and only if} \quad |x|_2 < b^{\frac{m}{n}}$$

and after an exponentiation by λ one has that the two inequalities

$$|x|_1^\lambda < b^{\frac{m}{n}} \quad \text{and} \quad |x|_2 < b^{\frac{m}{n}}$$

are equivalent. Taking the infimum for $\frac{m}{n} > t$ it yields the equalities

$$\begin{aligned} |x|_1^\lambda &= b^t \\ &= |x|_2 \end{aligned}$$

and the proof is complete. □

Remark 1.1.9. As the proof shows, the topology is completely determined by the set of *topologically nilpotent elements* i.e. the elements of the field whose increasing powers approach zero. The operation of raising a valuation to a positive constant preserves the property of being or not being archimedean, so it makes sense to say that the topological field K is or is not archimedean. In the second case the closed set $\mathcal{O}$ of elements $x \in K$ with $|x| \leq 1$ is a valuation sub-ring of K, in the sense that $\mathcal{O}$ is a sub-ring of K with the property that for all non-zero $x \in K$, at least one of x or its inverse x^{-1} belongs to $\mathcal{O}$. The set of topologically nilpotent elements is the unique maximal ideal $\mathcal{O}$, making it a local ring, and the set of invertible elements is $\mathcal{O}^\times = \left\{x \in K : |x| = 1\right\}$, which coincide with the set of non-zero elements $x \in K$ such that neither x nor x^{-1} is topologically nilpotent.

Definition 1.1.10. Two valuations $|\cdot|_1$ and $|\cdot|_2$ on a field K are said *equivalent* if they induce the same topology. The equivalence class of a valuation is called a *place*. The set of valuations of K is denoted by $\mathscr{V}_K$ and the set of places by $\mathscr{P}_K$, with the convention that the place corresponding to the trivial valuation is excluded from the set $\mathscr{P}_K$.

By the proposition 1.1.8, the place corresponding to a valuation $|\cdot|$ is identified with the set

$$\left\{|\cdot|^{\lambda} : \lambda > 0\right\}$$

and, as highlighted in the remark 1.1.9, it makes sense to speak about the archimedeanity of places. Indeed this property is characterized by the absolute value of the elements $n \cdot 1_K$ for n a positive integer, as it will be described soon. The fact that any natural number has a basis expansion with respect to all integer b larger or equal to 2 produces the following estimate inside a valued field:

Lemma 1.1.11. *Let $|\cdot|$ be a valuation on a field K. Suppose n, b are positive integers with $b > 1$. Then*

$$|n \cdot 1_K| \leq \max(1, |b \cdot 1_K|)^{\frac{\log n}{\log b}}$$

Proof. Let $n = a_0 + a_1 b + \cdots + a_r b^r$ the base-b expansion of n. Rising the valuation to some positive real number does not change the validity of the inequalities stated, so we can suppose that $|\cdot|$ satisfies the triangle inequality. In this case $|m \cdot 1_K| \leq m$ for any positive integer m, in particular $|a_i \cdot 1_K| \leq b$ for all $i = 0, \ldots, r$ and we get estimates

$$\begin{aligned}
|n \cdot 1_K| &\leq \sum_{i=0}^{r} \left|a_i b^i \cdot 1_K\right| \\
&= \sum_{i=0}^{r} |a_i \cdot 1_K| |b \cdot 1_K|^i \\
&\leq b \sum_{i=0}^{r} |b \cdot 1_K|^i \\
&\leq (r+1) \cdot b \cdot \max(1, |b \cdot 1_K|)^r \\
&\leq (\frac{\log n}{\log b} + 1) \cdot b \cdot \max(1, |b \cdot 1_K|)^{\frac{\log n}{\log b}} .
\end{aligned}$$

If one use the inequality with n^s, for $s \in \mathbb{N}$, in the place of n, they get

$$|n \cdot 1_K|^s \leq (s \frac{\log n}{\log b} + 1) \cdot b \cdot \max(1, |b \cdot 1_K|)^{s \frac{\log n}{\log b}}$$

and

$$|n \cdot 1_K| \leq \max(1, |b \cdot 1_K|)^{\frac{\log n}{\log b}}$$

after an exponentiation to $\frac{1}{s}$, with $s \to \infty$. □

Corollary 1.1.12. *Let $|\cdot|$ be a valuation on a field K. The following are equivalent:*

(i) $|\cdot|$ *is non-archimedean,*

(ii) $|1_K + 1_K| \leq 1$,

(iii) *the set* $\left\{|n \cdot 1_K| : n \in \mathbb{Z}\right\}$ *is bounded.*

Proof. "(i) $\Longrightarrow$ (ii)" is obvious.

"(ii) $\Longrightarrow$ (iii)": recall that $|-n \cdot 1_K| = |n \cdot 1_K|$ for all $n \in \mathbb{N}$. By the lemma 1.1.11

$$|n \cdot 1_K| \leq \max(1, |2 \cdot 1_K|)^{\frac{\log n}{\log 2}},$$

but $|2 \cdot 1_K| = |1_K + 1_K| \leq 1$ and then $|n \cdot 1_K| \leq 1$ for all integers n.

"(iii) $\Longrightarrow$ (i)": Suppose that $C > 0$ is a constant that dominates the absolute value of any integer multiple of 1_K. For any real number $\lambda > 0$, the bound

$$\sup_{n \in \mathbb{N}} |n \cdot 1_K| \leq C$$

is equivalent to

$$\sup_{n \in \mathbb{N}} |n \cdot 1_K|^\lambda \leq C^\lambda ,$$

and the valuation $|\cdot|$ is non-archimedean if and only if $|\cdot|^\lambda$ is non-archimedean, therefore the implication "(iii) $\Longrightarrow$ (i)" holds for $|\cdot|$ if and only if it holds for $|\cdot|^\lambda$. This means that we can replace $|\cdot|$ with an equivalent valuation that satisfies the triangle inequality and prove the implication "(iii) $\Longrightarrow$ (i)" for that valuation. We can assume without loss of generality that $|\cdot|$ satisfies the triangle inequality. Let n be a non-zero positive integer and $x, y \in K$.

$$\begin{aligned} |x+y| &= \sqrt[n]{|(x+y)^n|} \\ &= \sqrt[n]{\left|\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}\right|} \\ &\leq \sqrt[n]{\sum_{i=0}^{n} \left|\binom{n}{i} \cdot 1_K\right| |x|^i |y|^{n-i}} \\ &\leq \sqrt[n]{\sum_{i=0}^{n} C \cdot \max(|x|, |y|)^n} \\ &= \sqrt[n]{C \cdot (n+1)} \max(|x|, |y|) . \end{aligned}$$

If $n \to \infty$ then $\sqrt[n]{C \cdot (n+1)} \to 1$ and the ultrametric equality is established. □

Corollary 1.1.13. *If K has positive characteristic then any valuation on it is non-archimedean.*

Proof. The set $\{|n \cdot 1_K| : n \in \mathbb{Z}\}$ is finite, hence bounded. □

Corollary 1.1.14. *If $|\cdot|$ is an archimedean valuation on a field K then there is a real number $\lambda > 0$ such that for all positive integers n*

$$|n \cdot 1_K| = n^\lambda.$$

Proof. The set $\{n \in \mathbb{N} : |n \cdot 1_K| > 1\}$ must be non-empty, otherwise, the absolute value would be non-archimedean. Let n be an integer in this set and let $b \in \mathbb{N}$ be strictly larger than 1. By lemma 1.1.11,

$$|n \cdot 1_K| \leq \max\left(1, |b \cdot 1_K|\right)^{\frac{\log n}{\log b}} .$$

If $|b \cdot 1_K| \leq 1$ then $|n \cdot 1_K| \leq 1$, which contradicts the assumption on n, therefore $|b \cdot 1_K| > 1$ and

$$|n \cdot 1_K| \leq |b \cdot 1_K|^{\frac{\log n}{\log b}}$$

for all $b > 1$. In particular, any integer larger or equal to 2 has absolute value strictly larger than 1 and given n, m like that

$$|n \cdot 1_K| \leq |m \cdot 1_K|^{\frac{\log n}{\log m}}$$

hold, or equivalently

$$|n \cdot 1_K|^{\frac{1}{\log n}} \leq |m \cdot 1_K|^{\frac{1}{\log m}} .$$

Since the above inequality is symmetric in n and m, we have the equality

$$|n \cdot 1_K|^{\frac{1}{\log n}} = |m \cdot 1_K|^{\frac{1}{\log m}}$$

for all $n, m > 1$. Define $\lambda = \frac{\log|2 \cdot 1_K|}{\log 2}$, then

$$\begin{aligned} |n \cdot 1_K| &= |2 \cdot 1_K|^{\frac{\log n}{\log 2}} \\ &= n^{\lambda} . \end{aligned}$$

□

We can summarize all the corollaries after lemma 1.1.11 by saying that the archimedeanity of a place ν of a field K depends only on the absolute value of multiples of the multiplicative identity of the field. Moreover, if ν is archimedean then K has characteristic zero and there is a unique absolute value $|\cdot|_\nu$ in the class ν such that

$$|n \cdot 1_K|_\nu = n$$

for all natural numbers n.

Any field embedding $\sigma : k \to K$ induces a map between valuation sets

$$\mathscr{V}_K \to \mathscr{V}_k , \quad |\cdot| \mapsto |\cdot|_\sigma$$

defined by $|x|_\sigma = \left|\sigma(x)\right|$ for all $x \in k$ and all valuation $|\cdot|$ of K. In light of the characterization of places given by Proposition 1.1.8 it's clear that the map of valuations also restricts to a well-defined map between the sets of places.

Definition 1.1.15. Let $\mathfrak{p}, \mathfrak{P}$ be places of k and K respectively. We say that $\mathfrak{P}$ *lies over* $\mathfrak{p}$ and we write $\mathfrak{P}|\mathfrak{p}$ if $\mathfrak{p}$ is the image of $\mathfrak{P}$ under the map $\mathscr{P}_K \to \mathscr{P}_k$ just mentioned.

Any field is an algebra over the rational numbers or some finite field, then the first step to analyze the places is their description in the case of prime fields. Note that any field that is algebraic over a finite one consists, excluding the zero, of roots of unity, thus any valuation on it is trivial. To get some non-trivial valuation on a field of positive characteristic there must be at least one transcendental element over its prime field. The next theorem, due to A. Ostrowski, describes the places in the two non-trivial simplest cases:

Theorem 1.1.16. *Denote as usual the rational numbers by $\mathbb{Q}$ and let $\mathbb{F}_q(T)$ be the field of rational functions in one variable T over the finite field $\mathbb{F}_q$ with q elements. Every non-trivial valuation on $\mathbb{Q}$ is equivalent to one and only one of the following valuations:*

- *for $p \in \mathbb{N}$ prime, the non-archimedean p-adic valuation $|\cdot|_p$, defined on each prime l by*

$$|l|_p = \begin{cases} \frac{1}{p} & \text{if } l = p \\ 1 & \text{if } l \neq p \end{cases}$$

  and then extended multiplicatively;

- *the archimedean valuation $|\cdot|_\infty$ defined, as usual, by*

$$|x|_\infty = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise}. \end{cases}$$

Analogously, every non-trivial valuation on $\mathbb{F}_q(T)$ is equivalent to one and only one of the following valuations:

- *for a maximal ideal $\mathfrak{p}$ of $\mathbb{F}_q[T]$ generated by an irreducible polinomial of degree d, the non-archimedean $\mathfrak{p}$-adic valuation $|\cdot|_\mathfrak{p}$, defined on each irreducible polynomial $f \in \mathbb{F}_q[T]$ by*

$$|f|_\mathfrak{p} = \begin{cases} \frac{1}{q^d} & \text{if } f \in \mathfrak{p} \\ 1 & \text{if } f \notin \mathfrak{p} \end{cases}$$

  and then extended multiplicatively;

- *the non-archimedean valuation $|\cdot|_\infty$ defined on every $\frac{f}{g} \in \mathbb{F}_q(T)$, with $f, g \in \mathbb{F}_q[T]$, by*

$$\left|\frac{f}{g}\right|_\infty = \begin{cases} q^{\deg f - \deg g} & \text{if } \frac{f}{g} \neq 0 \\ 0 & \text{otherwise}. \end{cases}$$

Proof. By corollary 1.1.14, an archimedean valuation on the field of rational numbers is equivalent to a valuation $|\cdot|$ such that $|n| = n$ for all natural number n. Since the valuation is multiplicative and $|-1| = 1$ it must be $|x| = |x|_\infty$. Let $|\cdot|$ be a non-trivial, non-archimedean valuation on $\mathbb{Q}$, then any integer has absolute value less or equal to 1 and there is a non-zero integer n, which can be assumed to be positive, such that $|n| < 1$. Let p be the minimal, positive, non-zero integer with $|p| < 1$. Then it must be irreducible: if $p = p_1 p_2$ for two positive integers p_1, p_2, then $|p_1| \cdot |p_2| < 1$ and it must be that $|p_i| < 1$ for at least one $i = 1, 2$. Since p is the minimum of such natural numbers, $p \leq p_i$, but p_i is a factor of p and therefore $p = p_i$. If l is a prime different from p then $1 = ap + bl$ for some integers a, b, hence $1 \leq \max(|ap|, |bl|)$. Knowing that $|a| \leq 1$ and $|b| \leq 1$, we have

$$\begin{aligned} 1 &\leq \max(|ap|, |bl|) \\ &\leq \max(|p|, |l|) \\ &\leq 1, \end{aligned}$$

so $\max(\,|p|\,,|l|\,) = 1$, but $|p| < 1$ forces $|l| = 1$. The equality between valuation $|l| = |l|_p^{\lambda}$, for $\lambda = -\frac{\log|p|}{\log p}$, is true for every prime l and thus holds for all the rational numbers.

Now suppose that $|\cdot|$ is a valuation of $\mathbb{F}_q(T)$. There are two possibilities: $|T| \leq 1$ or $|T| > 1$. In the first case $\mathbb{F}_q[T]$ is contained in the subset of elements with absolute value less or equal to 1. Note that $\mathbb{F}_q[T]$ is an euclidean domain with fraction field $\mathbb{F}_q(T)$, like $\mathbb{Z}$ for $\mathbb{Q}$, hence the valuation is determined by its behaviour on that sub-ring and we can repeat the same argument just presented for the integers working instead with the degree of polynomials. The case $|T| > 1$ is traced back to the previous case: define $U = T^{-1}$, then $\mathbb{F}_q(T) = \mathbb{F}_q(U)$ and $|U| < 1$. □

The place defined by the valuation $|\cdot|_\infty$, described in Theorem 1.1.16 for both $\mathbb{Q}$ and $\mathbb{F}_q(T)$, is called the *place at infinity* or the *infinite place* and denoted by ∞, while the other places are said *finite places*.

Definition 1.1.17. A *global field* is a field K that is of one of the following two types:

1. a finite extension of the field $\mathbb{Q}$ of rational numbers, or
2. a finite extension of the field $\mathbb{F}_q(T)$ of rational functions in one variable over a finite field $\mathbb{F}_q$ with q elements.

In the first case, we call K a *number field*, as opposed to the second case in which K is called a *function field*.

Remark 1.1.18. For a finite extension $K/\mathbb{F}_q(T)$ there is always an element $U \in K$, transcendental over $\mathbb{F}_q$, such that K is finite and separable over $\mathbb{F}_q(U)$, so any function field is a finite, separable extension of a field isomorphic to $\mathbb{F}_q(T)$. Recalling that $\mathbb{Q}$ is a perfect field we can say that any global field is a finite and *separable* extension of $\mathbb{Q}$ or $\mathbb{F}_q(T)$.

The description of all the places of fields in the aforementioned family is the main goal of these sections. Each place has its own topology which is metrizable and, just like the real numbers are obtained as the completion of the rational numbers with respect to the archimedean absolute value, we can take completions of fields with respect to any place.

1.2 Completions

Although we are interested in global fields, completions make sense for any valuation on a general field, so we let K and $|\cdot|$ be general.

Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in the metric space K is Cauchy if, given an arbitrary $\varepsilon > 0$, there is some positive integer N such that the distance between x_n and x_m is smaller then ε for all $n, m \geq N$. The distance function on K is constructed by choosing an appropriate valuation equivalent to $|\cdot|$ that satisfies the triangle inequality, but it's easy to see that for any real parameter $\lambda > 0$, the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy if and only if for all $\varepsilon > 0$ there is an integer $N \geq 0$ such that $|x_n - x_m|^{\lambda} < \varepsilon$ for all $n, m \geq N$. Therefore, it's reasonable to speak about Cauchy sequences and completions with respect to places. Mimicking the construction of the completion of a metric space with equivalence classes of Cauchy sequences one obtains analogous results in the case of fields and places. They are summarized in the following theorem:

Theorem 1.2.1. *Let K be a field, ν a place of it and $|\cdot|$ a valuation in the class of ν. There exists a field K_ν extending K, with the following properties:*

i. K_ν *is complete with respect to a valuation* $|\cdot|_\nu$ *that extends the valuation of* K*, in the sense that* $|x|_\nu = |x|$ *for all* $x \in K$*;*

ii. K *is dense in* K_ν*;*

iii. K_ν *satisfies the subsequent universal property: for any field* F *complete with respect to a valuation* $|\cdot|_F$ *and any field embedding* $\sigma : K \to F$ *satisfying* $\left|\sigma(x)\right|_F = |x|$ *for all* $x \in K$*, there is a unique field embedding* $\sigma' : K_\nu \to F$ *compatible with the valuations such that* σ' *restricted on* K *is equal to* σ*.*

Moreover, K_ν *is unique: suppose* K' *is a field over* K*, which is complete with respect to a valuation* $|\cdot|'$ *that extends the valuation of* K *and satisfies the universal property in iii. Then there is a unique isomorphism of fields* $\sigma : K_\nu \to K'$ *that is the identity on* K *and such that* $|x|_\nu = \left|\sigma(x)\right|'$ *holds for all* $x \in K_\nu$*.*

Remark 1.2.2. If K is a complete field with respect to a valuation $|\cdot|$ and F is a sub-field of K, then the restriction of $|\cdot|$ to F is a valuation and the closure of F in K is a completion of F with respect to that valuation. For example, if K has characteristic zero it must contain the rational numbers. The valuation on them is trivial or equivalent to the p-adic valuation or the usual absolute value, therefore K is an algebra over $\mathbb{Q}$, $\mathbb{Q}_p$ or $\mathbb{R}$ respectively.

One of the many benefits of working with complete fields is that finite dimensional vector space over them carries a topology in a unique way, exactly as it happens in the case of real vector spaces.

Definition 1.2.3. Let K be a field with a valuation $|\cdot|$, normalized to have the triangle inequality property, and let V be a vector space over K. A *norm* is a function $\|\cdot\| : V \to \mathbb{R}_+$ such that for any $v, u \in V$ and for any $a \in K$

- $\|v\| = 0$ if and only if $v = 0$;
- $\|a \cdot v\| = |a| \cdot \|v\|$
- $\|v + u\| \leq \|v\| + \|u\|$

V is equipped with the metric topology induced by the distance function defined as $d(u, v) = \|u - v\|$ for $u, v \in V$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are called *equivalent* if there are positive real constants A, B such that for all $v \in V$

$$\|v\|_1 \leq A \cdot \|v\|_2 \quad \text{and} \quad \|v\|_2 \leq B \cdot \|v\|_1 \ .$$

If two norms are equivalent then define the same topology and the same Cauchy sequences, thus completions depend only on the equivalence class of norms.

Example 1.2.4. For n positive integer, the canonical n-dimensional vector space K^n has the norm defined by taking the maximum absolute value of the coordinates of a vector:

$$\left\|(x_1, \ldots, x_n)\right\| = \max_{1 \leq i \leq n} |x_i|$$

This norm induces the product topology on K^n. For any vector space V of dimension n, each linear isomorphism $V \to K^n$ defines a norm by pulling back the one mentioned above.

Lemma 1.2.5. *Let K and V be as in definition 1.2.3. Suppose that V has finite dimension n. Let $e_1, \dots e_n$ be a basis with induced norm*

$$\|x_1 e_1 + \cdots + x_n e_n\|_e = \max_{1 \le i \le n} |x_i| \quad \textit{for all} \quad (x_1, \dots, x_n) \in K^n \,.$$

Then there is a positive real constant C that depends on the base chosen such that

$$\|v\| \le C \cdot \|v\|_e \quad \textit{for all} \quad v \in V.$$

Moreover, if K is complete then V is complete with respect to $\|\cdot\|_e$.

Proof. Take $(x_1, \dots, x_n) \in K^n$, then

$$\begin{aligned} \|x_1 e_1 + \cdots + x_n e_n\| &\le \|x_1 e1\| + \cdots + \|x_n e_n\| \\ &= |x_1| \cdot \|e_1\| + \cdots + |x_n| \cdot \|e_n\| \\ &\le (\|e_1\| + \cdots + \|e_n\|) \cdot \max_{1 \le i \le n} |x_i| \end{aligned}$$

and so the value $C = \|e_1\| + \cdots + \|e_n\|$ works. Let $v_1, v_2, v_3, \dots$ be a Cauchy sequence of vectors for the norm $\|\cdot\|_e$. Let $\xi_1, \dots, \xi_n$ be the dual basis. For all $i = 1, \dots, n$ and all vectors v

$$\left|\xi_i(v)\right| \le \|v\|_e$$

by definition. Then, for all i, the sequence $\xi_i(v_1), \xi_i(v_2), \xi_i(v_3), \dots$ is Cauchy in K, which is complete, thus it converges to some $x_i \in K$. Define $v = x_1 e_1 + \dots x_n e_n$, then

$$\left\|v - v_j\right\|_e = \max_{1 \le i \le n} \left|x_i - \xi_i(v_j)\right|$$

goes to zero as $j \to \infty$ and v is the limit of the sequence of vectors. □

Theorem 1.2.6. *Let K be a field complete with respect to a valuation $|\cdot|$ and V a vector space of dimension n over K. Then any two norms on V are equivalent and V is complete for any of them.*

Proof. Let $e_1, \dots, e_n$ be a basis of V, with $\xi_1, \dots, \xi_n$ the dual basis and define $\|\cdot\|_e$ as in the lemma. The proof goes by induction on the dimension n. The theorem holds for $n = 1$ because any norm is of the form $\|e_1\| |\cdot|$ when it's pulled back by an isomorphism $K \to V, a \mapsto a e_1$. Suppose $n > 1$ and that the theorem holds for $n - 1$. It's enough to prove that all norms are equivalent to the max-norm $\|\cdot\|_e$ and by the lemma 1.2.5 it's sufficient to show that for any norm $\|\cdot\|$ there is a constant $C > 0$ such that to $\|\cdot\|_e \le C \|\cdot\|$. By contradiction, suppose that for any $C > 0$ there is some $v \in V$ such that $\|v\|_e > C \|v\|$. If $\left|\xi_i(v)\right| = \|v\|_e$ then we can substitute v with $\xi_i(v)^{-1} v$ in the inequality without change it. So there is a sequence $(v_k)_{k \ge 1}$ of vectors such that $\|v_k\| < \frac{1}{k}$, their components $\xi_1(v_k), \dots, \xi_n(v_k)$ have absolute value less or equal to 1 and $\xi_i(v_k) = 1$ for at least one $i = 1, \dots, n$. Up to passing to a subsequence and up to reordering the basis, we can suppose that for all $k \ge 1$ the vector v_k has the n-th component equal to 1. Define $u_k = v_k - e_n$ and note that

$$\begin{aligned} \|u_k - u_l\| &= \left\|v_k - e_n - (v_l - e_n)\right\| \\ &= \|v_k - v_l\| \,, \end{aligned}$$

thus the sequence $(u_k)_{k \ge 1}$ is Cauchy for $\|\cdot\|$ because $\|v_k\| \to 0$ as $k \to \infty$. The induction hypothesis implies that $\|\cdot\|$ restricted to the subspace U generated by the vectors $e_1, \dots, e_{n-1}$ is equivalent to the max-norm, for which U is complete (lemma 1.2.5). Therefore $(u_k)_{k \ge 1}$ converges to some $u \in U$ for the norm $\|\cdot\|$ and then $(v_k)_{k \ge 1}$ converges to $u + e_n$ for the same norm. But $\|v_k\| \to 0$ as $k \to \infty$, which forces $u = -e_n$, this is a contradiction: $u \in U$ but $-e_n \notin U$. □

If K/k is a finite extension of fields, with k complete for a valuation $|\cdot|_k$, then K, as a vector space over k, is topologized in a unique way. Any norm $\|\cdot\|$ of K viewed as a k-vector space induces the topology of K, but it isn't necessarily a valuation on K as a field. If there is a norm $\|\cdot\|$ that commutes with the multiplication of the field K, then $\|\cdot\|$ is also a valuation of the field K that extends the valuation $|\cdot|_k$ of k. Indeed the norm of a k-vector is compatible with the valuation $|\cdot|_k$ by definition of norm, hence for all $x \in k$

$$\begin{aligned} \|x\| &= \|x \cdot 1\| \\ &= |x|_k \cdot \|1\| \end{aligned}$$

and the equality $\|1\| = 1$ can be easily deduced by the assumption that $\|\cdot\|$ is multiplicative. Conversely, a valuation $|\cdot|$ of K that extends $|\cdot|_k$ satisfies the definition of the norm of a k-vector space. This fact combined with Theorem 1.2.6 has the effect that any two valuations $|\cdot|, |\cdot|'$ of K extending $|\cdot|_k$ are equivalent as norms, therefore they define the same topology. Proposition 1.1.8 implies that there is a real $\lambda > 0$ such that $|x|^\lambda = |x|'$ for all $x \in K$ when we view $|\cdot|$ and $|\cdot|'$ as valuations of the field K. If we view them as norms of the same k-vector space K instead, we know by Theorem 1.2.6 and Lemma 1.2.5 that there is a positive, real constant C such that

$$\frac{1}{C}|x| \leq |x|' \leq C|x| \ , \quad \text{for all} \quad x \in K \, .$$

These inequalities are compatible with the identity $|x|^\lambda = |x|'$ if and only if $\lambda = 1$. What we observed so far accounts for the proof of the uniqueness part stated in the following result:

Proposition 1.2.7. *Let k be a field complete with respect to a valuation $|\cdot|_k$ and K a finite extension of k. Then there is a unique valuation $|\cdot|_K$ on K such that for all $x \in k$*

$$|x|_K = |x|_k \, .$$

Moreover, K is complete with respect to it and

$$|x|_K = \left| N_{K/k}(x) \right|_k^{\frac{1}{[K:k]}} \quad \text{for } x \in K,$$

where $N_{K/k}(x)$ is the determinant of the k-linear map on K corresponding to the multiplication by x.

To complete the proof of Proposition 1.2.7 we should check that a valuation $|\cdot|_K$ that extends $|\cdot|$ exists. The problem of the existence can be addressed by separating the archimedean case from the non-archimedean. For the archimedean case, there is a result of Ostrowski:

Theorem 1.2.8. *Any complete, archimedean, valued field is isomorphic, both algebraically and topologically, to $\mathbb{R}$ or $\mathbb{C}$.*

Proof. See Theorem 1.1 in Chapter 3 of [Cas86] or Theorem 4.2 in Chapter II of [NS99]. □

By Theorem 1.2.8 the only non-trivial extension of complete, valued, archimedean fields is $\mathbb{C}/\mathbb{R}$ and in this case, we know well how to extend the absolute value of $\mathbb{R}$ to $\mathbb{C}$. In the case of finite extensions of non-archimedean fields, the proof that the valuation of the smaller field can be extended to the bigger one is given in [NS99], Chapter II, Theorem 4.9.

Remark 1.2.9. Proposition 1.2.7 can be extended to the case in which K is a possibly infinite algebraic extension of k because any element $x \in K$ is contained in a finite extension of k. In that case, the absolute value of x would be

$$|x|_K = \left| N_{k(x)/k}(x) \right|^{\frac{1}{[k(x):k]}} ,$$

where $k(x)$ is the smallest subfield of K containing k and x. Furthermore, if σ is an automorphism of K/k, then $|x|_K = \left|\sigma(x)\right|_K$ for all $x \in K$ because also $\left|\sigma(\cdot)\right|_K$ is a valuation on K that extends the valuation of k. In particular, for K/k Galois of finite degree n, the formula

$$\begin{aligned} |x|_K^n &= \prod_\sigma \left|\sigma(x)\right|_K \\ &= \left| \prod_\sigma \sigma(x) \right|_K \end{aligned}$$

holds for all $x \in K$ and with σ running through all the automorphism of K/k. One can read again the correct formula for $|\cdot|_K$ by recalling that $N_{K/k}(x)$ is the product of all conjugates of x in K in the Galois case.

1.3 Classification of Places for Global Fields

This section is devoted to the classification of places in a global field. We fix some notation for this goal:

- k denotes a field and K an algebraic extension of it;
- $\mathfrak{p}, \mathfrak{P}$ denotes places of k and K respectively;
- the completion of a field is indicated with the sub-script: for example $k_\mathfrak{p}$ stands for the completion of k with respect to the place $\mathfrak{p}$;
- for any place $\mathfrak{p}$ of k fix an algebraic closure $\overline{k}_\mathfrak{p}$ of the complete field $k_\mathfrak{p}$;
- $G_\mathfrak{p} = \mathrm{Aut}(\overline{k}_\mathfrak{p}/k_\mathfrak{p})$ is the absolute Galois group of $k_\mathfrak{p}$

Recall that this Galois group has a left action on the set $\mathrm{Hom}_k(K, \overline{k}_\mathfrak{p})$ by left-composition.

Proposition 1.3.1. *Let K/k be a finite and separable extension of fields. Then, for any place $\mathfrak{p}$ of k and any valuation $|\cdot|$ in the class $\mathfrak{p}$, there is a bijection*

$$G_\mathfrak{p} \cdot \sigma \mapsto |\cdot|_\sigma ,$$

from the set $G_\mathfrak{p} \backslash \mathrm{Hom}_k(K, \overline{k}_\mathfrak{p})$ of left cosets to the set of valuations of K that extends $|\cdot|$.

Proof. Let n be the degree of the extension K/k. By separability, there are exactly n morphisms of k-algebras from K to any algebraically closed field that contains k as a sub-field. Fix a place $\mathfrak{p}$ of k and choose a valuation $|\cdot|$ in the equivalence class $\mathfrak{p}$. We can extend uniquely $|\cdot|$ to $\overline{k}_\mathfrak{p}$ by Theorem 1.2.1 and Remark 1.2.9. Now define the map

$$\mathrm{Hom}_k(K, \overline{k}_\mathfrak{p}) \to \mathscr{V}_K, \quad \sigma \mapsto |\cdot|_\sigma \tag{1.1}$$

that sends a k-algebra morphism σ to the valuation $|\cdot|_\sigma$ defined by $|x|_\sigma = |\sigma(x)|$ for all $x \in K$. Since σ is the identity on k, the valuation $|\cdot|_\sigma$ is equal to $|\cdot|$ on k, thus the map (1.1) has image contained in the set of valuations that extends $|\cdot|$ from k to K. If $\tau \in G_\mathfrak{p}$, then $|z| = |\tau(z)|$ for all $z \in k_\mathfrak{p}$ by Remark 1.2.9, so the map (1.1) is well-defined on the coset space $G_\mathfrak{p} \backslash \mathrm{Hom}_k(K, \overline{k}_\mathfrak{p})$. Now we prove that the image of the map (1.1) is the set declared in the statement. If $|\cdot|_\mathfrak{P}$ is a valuation of K with associated place $\mathfrak{P}$ that extends the one of k, we can extend it furthermore to the completion $K_\mathfrak{P}$. Since this field is complete, contains k and the absolute value agrees with $|\cdot|$ on k, there is a completion isomorphic to $k_\mathfrak{p}$ inside $K_\mathfrak{P}$ and we can assume without loss of generality that $k_\mathfrak{p}$ is indeed the topological closure of k in $K_\mathfrak{P}$. Choose a basis $x_1, \ldots, x_n$ of K over k as a vector space, which is the same as a choice of a k-linear isomorphism

$$\varphi : k^n \to K$$

that sends the canonical basis vector e_i to x_i. The space k^n is dense in $k_\mathfrak{p}^n$ and we can extend φ to a $k_\mathfrak{p}$-linear map $\tilde{\varphi} : k_\mathfrak{p}^n \to K_\mathfrak{P}$ by setting $\tilde{\varphi}(e_i) = x_i$ for all $i = 1, \ldots, n$. The square

$$\begin{array}{ccc} k^n & \xrightarrow{\varphi} & K \\ \downarrow & & \downarrow \\ k_\mathfrak{p}^n & \xrightarrow{\tilde{\varphi}} & K_\mathfrak{P} \end{array}$$

is commutative, thus $\tilde{\varphi}(k^n)$ is dense in $K_\mathfrak{P}$. The image of $\tilde{\varphi}$ is closed because it's a $k_\mathfrak{p}$-vector sub-space and it contains the dense set $\tilde{\varphi}(k^n)$, hence the image of $\tilde{\varphi}$ is all $K_\mathfrak{P}$. This means that $K_\mathfrak{P}$ is a finite extension of $k_\mathfrak{p}$, so there exists an embedding $\sigma : K_\mathfrak{P} \to \overline{k}_\mathfrak{p}$. The uniqueness part of Proposition 1.2.7 implies that the valuation $|\cdot|_\mathfrak{P}$ is equal to $|\cdot|_\sigma$ on $K_\mathfrak{P}$ and even more so it is on K. Now we prove that the map is injective on the coset space. Take an embedding $\sigma \in \mathrm{Hom}_k(K, \overline{k}_\mathfrak{p})$ and an element $x \in K$. Denote by $\lambda(X)$ the minimal polynomial of x over k and consider its factorization

$$\lambda(X) = \prod_{i=1}^{r} \lambda_i(X)$$

into irreducibles polynomials in the ring $k_\mathfrak{p}[X]$. The factors are irreducible and all distinct because the separability of the extension K/k implies that $\lambda(X)$ is a separable polynomial. The element $z := \sigma(x)$ is a zero of $\lambda_i(X)$ for a unique i and all roots of $\lambda_i(X)$ are the (not necessarily distinct) conjugates of z under the action of $G_\mathfrak{p}$. We have that, for all $\tau \in G_\mathfrak{p}$,

$$\left|\lambda_j(\tau(z))\right| = 0 \text{ if and only if } j = i$$

and, for all $w \in \overline{k}_\mathfrak{p}$,

$$\left|\lambda_i(w)\right| = 0 \text{ if and only if } w \in \left\{\tau(z) : \tau \in G_\mathfrak{p}\right\} .$$

Separable, finite field extensions admit primitive elements, so we can suppose x is one of them for the extension K/k. If $\sigma' : K \to \overline{k}_\mathfrak{p}$ is an embedding that does not belongs to the coset $G_\mathfrak{p} \cdot \sigma$, it must be $\tau(\sigma(x)) \neq \sigma'(x)$ for all $\tau \in G_\mathfrak{p}$ and this means that $z' := \sigma'(x)$ is a root $\lambda(X)$ but not a root of $\lambda_i(X)$. There must be a factor $\lambda_j(X)$ different from $\lambda_i(X)$ such that $\lambda_j(z) \neq 0$ and $\lambda_j(z') = 0$. For all real numbers $\varepsilon, \varepsilon' > 0$ we can find a polynomial $f(X)$ with coefficients in k such that

$$\left|f(z) - \lambda_i(z)\right| < \varepsilon \quad \text{and} \quad \left|f(z') - \lambda_i(z')\right| < \varepsilon' .$$

One can convince himself of this fact by observing that for any integer $d > 0$ and any $w, w' \in \overline{k}_{\mathfrak{p}}$, the map

$$\begin{aligned} k_{\mathfrak{p}}^{d+1} &\longrightarrow k_{\mathfrak{p}}(w) \oplus k_{\mathfrak{p}}(w') \\ (a_0, \dots, a_d) &\longmapsto (a_0 + a_1 w + \dots + a_d w^d, a_0 + a_1 w' + \dots + a_d w'^d) \end{aligned}$$

is a linear homomorphism between finite-dimensional $k_{\mathfrak{p}}$-vector spaces, hence continuous for the unique topology induced by any norm, and k^{d+1} is a dense subset of the domain. Note that $\lambda_i(z) = 0$ and $\lambda_i(z') \neq 0$, so that we can choose $\varepsilon, \varepsilon'$ small enough to have $\left|f(z)\right| < \frac{1}{2}\left|\lambda_i(z')\right|$ and $\left|f(z')\right| > \frac{1}{2}\left|\lambda_i(z')\right|$. Finally, recall that z, z' are the image of x via the k-algebra morphisms σ, σ' respectively, therefore

$$\begin{aligned} \left|f(x)\right|_{\sigma} &= \left|\sigma(f(x))\right| \\ &= \left|f(\sigma(x))\right| \\ &< \frac{1}{2}\left|\lambda_i(z')\right| \end{aligned}$$

and

$$\begin{aligned} \left|f(x)\right|_{\sigma'} &= \left|\sigma'(f(x))\right| \\ &= \left|f(\sigma'(x))\right| \\ &> \frac{1}{2}\left|\lambda_i(z')\right| . \end{aligned}$$

This implies that $\left|f(x)\right|_{\sigma} \neq \left|f(x)\right|_{\sigma'}$, which is enough to conclude the proof. □

The bijection in Proposition 1.3.1 induces a bijection between the set of left cosets $G_{\mathfrak{p}} \backslash \operatorname{Hom}_k(K, \overline{k}_{\mathfrak{p}})$ and the set of places $\mathfrak{P}$ of K that lie over $\mathfrak{p}$, therefore the map of places

$$\mathscr{P}_K \longrightarrow \mathscr{P}_k$$

is surjective with finite fibres of cardinality less or equal to the degree of the extension K/k. If K is a global field we can always find a subfield $k \subseteq K$ isomorphic to $\mathbb{Q}$ or to $\mathbb{F}_q(T)$ such that K/k is finite and separable. Theorem 1.1.16 describes the set $\mathscr{P}_k$ and Proposition 1.3.1 characterizes the fibre of $\mathscr{P}_K \to \mathscr{P}_k$, together they provide a classification of places of any global field:

Corollary 1.3.2 (Classification of places of a global field)**.** *Let K be a global field, finite and separable over the field k, with k isomorphic to the field of rational numbers or rational functions in one variable over a finite field. For each place $\mathfrak{p}$ of k let $|\cdot|_{\mathfrak{p}}$ be the choice of valuation defined in Theorem 1.1.16 extended to $\overline{k}_{\mathfrak{p}}$. Then, for all places $\mathfrak{P}$ of K there is a unique place $\mathfrak{p}$ of k and an embedding $\sigma : K \to \overline{k}_{\mathfrak{p}}$ of k-algebras such that $\mathfrak{P}|\mathfrak{p}$ and the valuation*

$$K \to \mathbb{R}_+, \quad x \mapsto \left|\sigma(x)\right|_{\mathfrak{p}}$$

represents the class $\mathfrak{P}$. The embedding σ is unique up to conjugation by elements of the absolute Galois group of $k_{\mathfrak{p}}$.

Definition 1.3.3. Let k be a global field. The places of k that lie over the place ∞ are called the *infinite places* of k. Let $S \subset \mathscr{P}_k$ be a finite set of places that contains the set of infinite places of k. The sub-ring of k

$$\mathcal{O}_{k,S} := \left\{ x \in k \ : \ |x|_{\mathfrak{p}} \leq 1 \quad \text{for all place } \mathfrak{p} \notin S \right\}$$

is called the *ring of S-integers* of k.

The ring $\mathcal{O}_{k,S}$ is a Dedekind domain with fraction field k. If K is a finite, separable extension of k, then the integral closure of $\mathcal{O}_{k,S}$ is the ring of S'-integers of K, where S' is the set of places $\mathfrak{P}$ of K such that there is some $\mathfrak{p} \in S$ with $\mathfrak{p}|\mathfrak{P}$. Observe that, with this notation, the ring of ∞-integers of the rational numbers is $\mathbb{Z}$, thus we get back the classical ring of integers of a number field K by considering its ring of S-integers, where S is the set of infinite places. The notion of "place" can be seen as a generalization of the primes, as the set of finite places of a global field k is in bijection with the set of non-zero prime ideals of $\mathcal{O}_k$. Precisely:

Theorem 1.3.4. *Let k be a global field and S be a finite set of places containing the infinite ones. Then:*

(i) *the rule*

$$\mathfrak{p} \longmapsto \left\{ x \in \mathcal{O}_{k,S} \ : \ |x|_{\mathfrak{p}} < 1 \right\}$$

defines a bijection between the complementary of S in $\mathscr{P}_k$ and the set of non-zero prime ideals of $\mathcal{O}_{k,S}$;

(ii) *the inverse of the bijection of point* (i) *is given by assigning a prime ideal $\mathcal{P}$ of $\mathcal{O}_{k,S}$ to the place $\mathfrak{p}$ represented by the valuation $|\cdot|_{\mathfrak{p}}$ defined by*

$$|x|_{\mathfrak{p}} = t^{v(x|\mathcal{P})} \quad \textit{for all non-zero } x \in k\,,$$

where t is any fixed real number satisfying $0 < t < 1$ and $v(x|\mathcal{P})$ is the maximal integer n for which $x \in \mathcal{P}^n$;

(iii) *if K is a finite, separable extension of k and $\mathfrak{p}$ is a place of the smaller field with associated prime ideal $\mathcal{P}$, then, the map in point* (i) *induces a bijection between the set of places $\mathfrak{P}$ of K such that $\mathfrak{P}|\mathfrak{p}$ and the set of prime ideals of $\mathcal{O}_{K,S'}$ containing $\mathcal{P}$, where S' is the set of places of K lying over places of S.*

Proof. It's easy to check that the maps of (i) and (ii) are mutual inverses, and (iii) follows from that. □

Chapter 2

Harmonic Analysis on Locally Compact Abelian Groups

This chapter is a summary of what is needed in the theory of locally compact abelian groups and their harmonic analysis, collecting the results from the first three chapters of [RV99]. We start with some basic properties of the category of locally compact abelian groups, then we introduce unitary characters and the dual of a group, namely the group of unitary characters on it. The first section ends with the Pontryagin duality Theorem, stating that a locally compact abelian group is naturally isomorphic to its double dual. In Section 2.2 we introduce the Haar measure of a locally compact abelian group. This is a Radon measure invariant by the group's operation, and, through it, the Fourier transform can be defined. It is an integral transform sending functions of a group to functions on its dual, and it combines well with Pontryagin duality, defining isomorphisms between functional spaces attached to the group and its dual. The Fourier transform is used in Chapter 4 for the additive groups of local fields (see Definition 2.3.5) and adèle rings. These are special in the sense that they are identified with their dual group, a case examined in Example 2.2.6. For these special groups, one can choose uniquely a Haar measure and make the Fourier transform act internally on functional spaces of the group. The last result of the section is a slightly more general version of the Poisson summation formula (Theorem 2.2.8). Sections 2.3 introduce the *module* of an automorphism of a locally compact abelian group, which is a real number that measures how the automorphism changes the Haar measure. In the case of a locally compact field, the module is a valuation of the field. Indeed locally compact fields that are not topologically discrete are classified and they are precisely the completions of global fields. The module gives then to local fields a canonical choice for the valuation that defines their topology. Finally, in Section 2.4 we treat the restricted direct product of families of locally compact abelian groups. In some sense, this construction lies in between the coproduct and the direct product of groups. We describe how to construct a Haar measure on it, and how the integration of functions works. Regarding characters, the dual of a restricted direct product is the restricted direct product of the duals (see Theorem 2.4.7). The content of Section 2.4 is useful to study the adèles and idèles in Chapter 3, as they are both restricted direct products.

2.1 Basic properties of locally compact abelian groups

Definition 2.1.1. A *locally compact abelian group* is, by definition, a topological group $(A, +, 0)$ that is Hausdorff and locally compact as a topological space and abelian as a group. We will write that A is a **LCA** group for short. If A, B are **LCA** groups, a morphism between them is a continuous group homomorphism $A \to B$. The term **LCA** is also used to indicate the category of locally compact abelian groups and morphism between them.

Remark 2.1.2. Since an **LCA** group is Hausdorff as a topological space, there is no confusion in the definition of local compactness: it means equivalently that any point of the group has a compact neighbourhood or that it has a basis of compact neighbourhoods.

The following proposition summarises the properties of sub-objects and quotients.

Proposition 2.1.3 (subgroups and quotients)**.** *Let A be a* **LCA** *group, $B \subseteq A$ a subgroup with the subspace topology and $C = A/B$ the quotient group with the topology induced by the projection $p : A \to C$. Then the following holds for A, B and C:*

(i) *B is Hausdorff;*

(ii) *B is locally compact if and only if it is closed in A;*

(iii) *the projection $p : A \to C$ is an open map;*

(iv) *C is locally compact;*

(v) *C is Hausdorff if and only if B is closed in A;*

(vi) *if B is open then it is automatically closed;*

(vii) *C is discrete if and only if B is open.*

Proof. See [RV99], propositions 1-4 and 1-6. □

Remark 2.1.4. The category **LCA** is pre-abelian, in the sense that it is $\mathbb{Z}$-linear and it has finite limits and colimits, in particular, kernels are computed in the usual way and the cokernel of a morphism $\varphi : A \to B$ is given by the quotient of B by the closure of the algebraic image of f. It is not an abelian category because there are morphisms with trivial kernel and cokernel that aren't isomorphisms, like the morphism $\mathbb{R}^{\text{disc}} \to \mathbb{R}$, where the domain is the set of real numbers with the discrete topology and the map is the identity on elements. In general, every morphism $\varphi : A \to B$ of **LCA** lies in a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \big\downarrow{\scriptstyle \text{proj}} & & \big\uparrow \\ A/\ker f & \xrightarrow{\tilde{\varphi}} & \overline{f(A)} \end{array}$$

but the induced morphism $\tilde{\varphi}$ may not be an isomorphism, contrary to the case of abelian categories. Still, one may consider *strict morphisms*, which are morphisms $\varphi : A \to B$ in **LCA** such that $\tilde{\varphi}$ is an isomorphism, and use the formalism of abelian categories for them. For example, one can work with exact sequences in the usual way provided that each morphism of the sequence is strict. For what we need, is not necessary to introduce this notion of strict morphism, but it is nevertheless convenient for us to use short exact sequences

$$0 \longrightarrow B \xrightarrow{\varphi} A \xrightarrow{\psi} C \longrightarrow 0 \tag{2.1}$$

as a visual aid. Therefore, we adopt the convention that a sequence like (2.1) is *exact* if ψ is a cokernel of φ and φ is a kernel of ψ. This is equivalent to asking that φ and ψ are strict.

The category **LCA** can be enriched over topological Hausdorff abelian groups in the sense that the abelian group $\mathrm{Hom}_{\mathbf{LCA}}(A, B)$ of morphisms between two locally compact abelian groups A, B has a topology that makes it a topological Hausdorff abelian group and the composition of morphisms is continuous for this topology. This is the compact-open topology, determined by the following pre-basis of open subsets: for all compact subsets K of A and all open subsets U of B, a basic open subset of $\mathrm{Hom}_{\mathbf{LCA}}(A, B)$ is the subset $W(K, U)$ of morphisms φ that satisfy $\varphi(K) \subseteq U$.

Lemma 2.1.5. *For all* **LCA** *groups A, B, the topological group $\mathrm{Hom}_{\mathbf{LCA}}(A, B)$ is Hausdorff. Moreover, for any C in the same category of A and B, the composition*

$$\circ : \mathrm{Hom}_{\mathbf{LCA}}(B, C) \times \mathrm{Hom}_{\mathbf{LCA}}(A, B) \to \mathrm{Hom}_{\mathbf{LCA}}(A, C)\,, \quad (\psi, \varphi) \mapsto \psi \circ \varphi$$

is continuous, where the domain has the product topology of the compact-open topologies of each factor.

Proof. If $\varphi \neq \psi$ in $\mathrm{Hom}_{\mathbf{LCA}}(A, B)$, then there is an element $a \in A$ such that $\varphi(a) \neq \psi(a)$ and we can choose disjoint open neighbourhood U_1, U_2 in B of $\varphi(a)$ and $\psi(a)$ because B is Hausdorff. If K is any compact neighbourhood of a in A, it follows that $W(K, U_1)$ and $W(K, U_2)$ are disjoint open neighbourhood separating φ and ψ, so that the topology of $\mathrm{Hom}_{\mathbf{LCA}}(A, B)$ is Hausdorff.

To prove the continuity of the composition, suppose that K is compact in A and U is open in C and observe that the pre-image of the basic open set $W(K, U)$ of the target is the basic open set $W(\varphi(K), U) \times W(K, \psi^{-1}U)$ of the domain. □

Our case of interest is when B is the unit circle.

Definition 2.1.6. Let A be an **LCA** group. Define the topological group $\widehat{A}$ as the group $\mathrm{Hom}(A, \mathbb{S}^1)$ with the compact-open topology, where $\mathbb{S}^1$ is the unit circle in the complex plane. The group $\widehat{A}$ is called *Pontryagin dual* or the group of *unitary characters* of A.

Remark 2.1.7. Here, the term *character* refers to continuous group-homomorphism to the multiplicative group of complex numbers $\mathbb{C}^\times$. The adjective *unitary* is used to indicate a character with image contained in the unit circle. In this chapter, we deal almost only with unitary characters, the non-unitary case will be important in the last chapter.

Lemma 2.1.8. *Let A be an* **LCA** *group, then $\widehat{A}$ is also an* **LCA** *group. In particular, the rule $A \mapsto \widehat{A}$ defines a functor from* $\mathbf{LCA}^{\mathrm{op}}$ *to* **LCA** *such that any morphism $\varphi : A \to B$ is sent to the morphism $\varphi^* : \widehat{B} \to \widehat{A}$ defined by $\varphi^*(\chi) = \chi \circ \varphi$ for any unitary character $\chi \in \widehat{B}$.*

Proof. See Proposition 3-2 of [RV99] for the local compactness of $\widehat{A}$. The functoriality of $A \mapsto \widehat{A}$ is provided by the equality $\varphi^* \circ \psi^* = (\psi \circ \varphi)^*$ valid for all composable couples of morphisms φ, ψ of **LCA**. □

The above functor is a duality of the category **LCA** in the sense expressed by the following result and it is known as Pontryagin duality.

Theorem 2.1.9 (Pontryagin duality)**.** *The functor $A \mapsto \widehat{A}$ defines a contravariant equivalence on the category* **LCA** *and there is a functorial isomorphism $A \cong \widehat{\widehat{A}}$ given by the map*

$$\tau : A \to \widehat{\widehat{A}}, \quad a \mapsto \tau_a ,$$

with $\tau_a(\chi) = \chi(a)$ for all unitary character χ of A.

Proof. See Chapter 3 of [RV99] for the proof that the map $a \to \tau_a$ is an isomorphism. Functoriality is obvious by definition: for all morphisms $\varphi : A \to B$ in **LCA**, the image of $a \in A$ by the map $\varphi^{**} \circ \tau$ is exactly the unitary character $\tau_{\varphi(a)}$ of $\widehat{B}$, since they both produce $\chi(\varphi(a))$ when evaluated at any $\chi \in \widehat{B}$. □

The following is an immediate consequence:

Corollary 2.1.10. *The Potryagin dual commutes with limits and colimits in* **LCA**. *In particular, if B is a closed subgroup of an* **LCA** *group A and $C = A/B$, then $\widehat{C}$ is isomorphic to $B^{\perp}$, the subgroup of unitary characters of A that are trivial on B, and $\widehat{B}$ is isomorphic to $\widehat{A}/B^{\perp}$.*

Proof. The Pontryagin duality is an equivalence by Theorem 2.1.9 and equivalences of categories always respect limits and colimits when they exist. The sequence

$$0 \longrightarrow B \xrightarrow{\varphi} A \xrightarrow{\psi} C \longrightarrow 0$$

is exact (recall the convention declared in Remark 2.1.4). Then, also

$$0 \longrightarrow \widehat{C} \xrightarrow{\psi^*} \widehat{A} \xrightarrow{\varphi^*} \widehat{B} \longrightarrow 0$$

is exact. In particular, $\widehat{C}$ is isomorphic to $\ker(\varphi^*)$ that is equal to $B^{\perp}$, and $\widehat{B}$ is the cokernel of the inclusion of $B^{\perp} \hookrightarrow \widehat{A}$. □

Corollary 2.1 shows how Pontryagin duality sub-groups with quotients and the next proposition shows that it permutes compact groups with discrete groups.

Proposition 2.1.11. *Let A be an* **LCA** *group and $\widehat{A}$ its Pontryagin dual. If A is compact, then $\widehat{A}$ is discrete, if A is discrete then $\widehat{A}$ is compact.*

Proof. Suppose that A is discrete. The underlying set of $\widehat{A}$ is the set of algebraic homomorphisms from A to the circle group and it is also a closed subset of the product $\prod_A \mathbb{S}^1$ of copies of the circle indexed by A. Moreover, the topology of $\widehat{A}$ is the subspace topology: compact subsets K of A are finite subsets and the basic open set $W(K, U)$ of $\widehat{A}$, for U open subset of the circle, is equal to the intersection inside $\prod_A \mathbb{S}^1$ of $\widehat{A}$ with the open subset $\{f \in \prod_A \mathbb{S}^1 : f(a) \in U \text{ for all } a \in K\}$. Since $\mathbb{S}^1$ is compact and Hausdorff, then the same is true for the product $\prod_A \mathbb{S}^1$ and all its closed subset, so $\widehat{A}$ is compact.

Suppose that A is compact. Let U be an open neighbourhood of the identity in the circle group and consider an open subset of $\widehat{A}$ of the form $W(A, U)$. If $\chi \in W(A, U)$, we have that $\chi(A)$ is a compact subgroup of the circle contained in U. The lemma 2.1.12 below ensures that for U small enough $W(A, U)$ is the subset containing only the trivial character and this implies that points of the topological space $\widehat{A}$ are open. □

Lemma 2.1.12. *Let $\mathbb{C}^\times$ be the group of invertible complex numbers. Then there is a neighbourhood of the identity that does not contain non-trivial subgroups.*

Proof. Consider the exponential map $\exp : \mathbb{C} \to \mathbb{C}^\times$, $z \mapsto e^z$. For a radius $r > 0$ small enough, the map exp is a homeomorphism from the ball $B(0, r)$ to an open neighbourhood $U \subset \mathbb{C}^\times$ of 1. Let G be a subgroup of $\mathbb{C}^\times$ contained in U. Then $\exp^{-1}(G)$ is a subgroup of $(\mathbb{C}, +)$ isomorphic to G contained in $B(0, r)$. Then, for all $z \in \exp^{-1}(G)$ we have $|n \cdot z| < r$ for all integer n, but this is possible if and only if $z = 0$ because $\mathbb{C}$ is an archimedean field, thus $\exp^{-1}(G) = \{0\}$ and G is the trivial group. □

2.2 The Haar measure and the Fourier transform

The motivation behind the choice of **LCA** as the category to work with is the nice interplay between topology, algebraic structure and measure theory. A locally compact abelian group always has a special measure on it that is invariant by translations and it is unique up to a positive constant. This is the notion of a Haar measure.

Definition 2.2.1. Let A be an **LCA** group. A *Haar measure* is a Radon measure μ on A such that

$$\int_A f(x - a)\, d\mu(x) = \int_A f(x)\, d\mu(x)$$

for all compactly supported, real-valued, continuous function f on A and every $a \in A$.

Remark 2.2.2. By the Riesz–Markov–Kakutani representation theorem, the definitions of Radon measure given in terms of well-behaved measures on the σ-algebra of Borel sets or continuous linear functionals on the space of compactly supported, continuous functions are equivalent. The translation invariant property of a Haar measure μ given in the definition corresponds to the equality $\mu(E) = \mu(a + E)$ for a measurable set E and an element a of the group. For reference, see theorem 7.2 of [Fol99].

Recall that for an isomorphism $\varphi : B \to A$ in the category **LCA** there is an associated linear isomorphism between the spaces of Radon measures, that sends a measure μ on A to its pullback $\varphi^\star \mu$ defined on B by the integral

$$\int_B f\, d(\varphi^\star \mu) = \int_A f \circ \varphi^{-1}\, d\mu$$

for compactly supported, continuous functions $f : B \to \mathbb{R}$. The pullback of a Haar measure is again a Haar measure since φ respects the group law.

Theorem 2.2.3. *Let A be an* **LCA** *group. Then there exists a non-zero Haar measure μ on A and for all Haar measure λ there is a unique positive, real constant c such that $\lambda = c\mu$.*

Proof. See Theorem 1 of Chapter 1 in the book [Bou04]. □

As usual, $L^p(A,\mu)$ denotes the space of μ-measurable functions $f : A \to \mathbb{C}$ such that

$$\int_A \|f(x)\|_{\mathbb{C}}^p \, d\mu(x)$$

is finite. Since Haar measures are proportional to each other, the finiteness of the integral of a positive measurable function is independent of the choice of the Haar measure, so we can drop the μ in the notation of the L^p-space and write $L^p(A)$ for the space of power-p summable functions of A. For an L^1-function $f : A \to \mathbb{C}$, the *Fourier transform* $\widehat{f}$ of f is defined by the formula

$$\widehat{f}(\chi) = \int_A f(x)\overline{\chi(x)}\, d\mu(x)$$

for all unitary characters $\chi \in \widehat{A}$. If ν is a Haar measure on $\widehat{A}$ and $\widehat{f} \in L^1(\widehat{A})$, by Pontryagin duality we may regard the Fourier tranform of $\widehat{f}$ as a function on A via the formula

$$\widehat{\widehat{f}}(x) = \int_{\widehat{A}} \widehat{f}(\chi)\overline{\chi(x)}\, d\nu(\chi)$$

for all $x \in A$. As in the case of functions of real variables, suitable functions on the group A can be represented as a "weighted sum" of unitary characters, where the Fourier transform $\widehat{f}(\chi)$ is the weight of the character χ in the representation. To be explicit, the formula

$$f(x) = c\int_{\widehat{A}} \widehat{f}(\chi)\chi(x)\, d\nu(\chi)$$

holds for sufficiently nice functions f, where c is a constant independent of f but dependent on the choice of the Haar measure ν. The unique measure ν on $\widehat{A}$ that makes the constant c equal to 1 is called the *dual measure* of μ. All this is formalized in the next theorem.

Theorem 2.2.4. *Let A be an* **LCA** *group with a Haar measure μ. There is a unique Haar measure $\widehat{\mu}$ on $\widehat{A}$ such that for all $f \in L^1(A)$ the formula*

$$\widehat{\widehat{f}}(x) = f(-x) \tag{2.2}$$

holds for all $x \in A$ whenever $\widehat{f}$ is also in $L^1(\widehat{A})$. Moreover, the Fourier transform induces an isometry between the Hilbert spaces $L^2(A)$ and $L^2(\widehat{A})$ equipped with the standard L^2 inner products.

Proof. See the theorem 4.21 and 4.25 of [Fol95]. □

Example 2.2.5. Suppose that A is compact and let μ be a Haar measure. Let f be the function that is constant and equal to 1 on all A. The function f is continuous, it's L^p for all $p \in (1,\infty]$ and its Fourier transform is

$$\widehat{f}(\chi) = \int_A \overline{\chi(x)}\, d\mu(x)\,, \quad \chi \in \widehat{A}$$

The problem of computing the above Fourier transform is equivalent to the problem of computing the integral of a unitary character on A. Since the Haar measure is invariant by translation of any element $a \in A$, we have

$$
\begin{aligned}
\widehat{f}(\chi) &= \int_A \overline{\chi(x)}\, d\mu(x) \\
&= \int_A \overline{\chi(x-a)}\, d\mu(x) \\
&= \overline{\chi(-a)} \int_A \overline{\chi(x)}\, d\mu(x) \\
&= \chi(a)\widehat{f}(\chi)
\end{aligned}
$$

for all $a \in A$. So, $\widehat{f}$ is zero when calculated on any non-trivial unitary character and it is equal to $\mu(A)$ on the trivial character. Since A is compact, the group $\widehat{A}$ is discrete and the counting measure is a Haar measure on any discrete abelian group. So, the dual measure $\widehat{\mu}$ is equal to the counting measure multiplied by a constant c. The formula of Theorem 2.2.4 determines c:

$$
\begin{aligned}
f(x) &= \int_{\widehat{A}} \widehat{f}(\chi)\chi(x)\, d\widehat{\mu}(\chi) \\
&= c \sum_{\chi \in \widehat{A}} \widehat{f}(\chi)\chi(x) \\
&= c\mu(A)\,,
\end{aligned}
$$

where the last equality comes from the fact that $\widehat{f}$ is the characteristic function of the trivial character multiplied by the measure of A. The function $f(x)$ is constant and equal to 1, so $c = \mu(A)^{-1}$, in particular, if the Haar measure of A is normalized to give measure 1 to the whole group, then the dual measure is simply the counting measure.

Example 2.2.6. Suppose that A is an **LCA** group with an isomorphism $\psi : A \to \widehat{A}$. It corresponds to a bilinear continuous map

$$
\Psi : A \times A \to \mathbb{S}^1\,, \quad (\xi, x) \mapsto \psi(\xi)(x)
$$

and we assume that Ψ is symmetric: $\Psi(\xi, x) = \Psi(x, \xi)$, which is the same of asking that the diagram

$$
\begin{array}{ccc}
\widehat{\widehat{A}} & \xrightarrow{\psi^*} & \widehat{A} \\
\wr\uparrow & \overset{\psi}{\nearrow} & \\
A & &
\end{array}
$$

commutes. Let μ be a Haar measure on A and $\widehat{\mu}$ its dual. Then there is another measure on A, namely the pullback of the measure $\widehat{\mu}$ by ψ. Up to rescaling μ, we can suppose that μ is itself the pullback of $\widehat{\mu}$. In this case, μ is said to be *self-dual* with respect to the isomorphism ψ. The identification of the group A with its dual permits to define the Fourier transform as a linear automorphism $\mathscr{F}$ of $L^2(A)$ that sends an L^2 function f to $\widehat{f} \circ \psi$. To be explicit, the operator $\mathscr{F}$ is defined on the functions f that belong to the dense subspace $L^1(A) \cap L^2(A)$ by

$$
\left[\mathscr{F}(f)\right](\xi) = \int_A f(x)\overline{\Psi(\xi, x)}\, d\mu(x)
$$

for all $\xi \in A$. The newly defined Fourier transform satisfies the same properties of the abstract transform stated in Theorem 2.2.4, namely, it is an isometry of $L^2(A)$ and the double transform $\mathscr{F} \circ \mathscr{F}$ is the operator that sends a function $f(x)$ in the variable x to the function $f(-x)$. To understand why, note that the condition $\psi^\star \widehat{\mu} = \mu$ implies that

$$\int_A f(x)\overline{\Psi(\xi,x)}\, d\mu(x) = \int_{\widehat{A}} f\big(\psi^{-1}(\chi)\big)\overline{\Psi\big(\xi,\psi^{-1}(\chi)\big)}\, d\widehat{\mu}(\chi)$$

for all functions $f \in L^1(A)$ and all $\xi \in A$, and

$$\begin{aligned}\Psi\big(\xi,\psi^{-1}(\chi)\big) &= \Psi\big(\psi^{-1}(\chi),\xi\big)\\ &= \chi(\xi)\end{aligned}$$

thanks to the symmetry of Ψ. So, the equality of integrals is interpreted by the equality $\widehat{f} \circ \psi = \widehat{f \circ \psi^{-1}}$ of functions, which is sufficient to transport the properties of the abstract Fourier transform to $\mathscr{F}$. For a short exact sequence

$$0 \longrightarrow B \xrightarrow{\ \iota\ } A \xrightarrow{\ \pi\ } C \longrightarrow 0$$

there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^\perp & \longrightarrow & A & \longrightarrow & A/B^\perp & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow & & \\ 0 & \longrightarrow & \widehat{C} & \xrightarrow{\ \pi^*\ } & \widehat{A} & \xrightarrow{\ \iota^*\ } & \widehat{B} & \longrightarrow & 0 \end{array}$$

induced by the the identification $A \cong \widehat{A}$, where the vertical arrows are all isomorphisms and the group $B^\perp$ consists of the elements $\xi \in A$ satisfying $\Psi(\xi,\iota(y)) = 1$ for all $y \in B$.

Now we explore how Haar measures and Fourier transforms interact with subgroups and quotients. Let

$$0 \longrightarrow B \xrightarrow{\ \iota\ } A \xrightarrow{\ \pi\ } C \longrightarrow 0$$

be an exact sequence in **LCA**. Suppose that α, β, γ are Haar measures of A, B, C respectively. If f is a continuous function with compact support on A, we can define a continuous function $f^\flat$ with compact support on C by the average values of f on the B-cosets of A:

$$f^\flat(\pi(x)) = \int_B f(x+\iota(y))\, d\beta(y)$$

for all $x \in A$. The rule $f \mapsto f^\flat$ defines a continuous linear map from the space $\mathscr{C}_c(A)$ of continuous functions with compact support on A to the corresponding functional space for C. For any element $a \in A$ and all functions $f \in \mathscr{C}_c(A)$, this map sends the shifted functions $x \mapsto f(x-a)$ to the shifted function $z \mapsto f^\flat\big(z-\pi(a)\big)$, so that the continuous linear functional

$$\mathscr{C}_c(A) \to \mathbb{R}\,, \quad f \mapsto \int_C f^\flat\, d\gamma$$

defines a Haar measure on A. Thus, up to rescaling one of the three Haar measures, we can integrate on fibres:

$$\int_A f(x)\, d\alpha(x) = \int_C \int_B f(z+y)\, d\beta(y)\, d\gamma(z) \tag{2.3}$$

for all functions $f \in \mathscr{C}_c(A)$, where $f(z+y)$ is a short notation to indicate the value $f(x+\iota(y))$ for any lift $x \in A$ of $z \in C$ obtained via the surjective map π.

Example 2.2.7. If the sequence is split, i.e. $A = B \oplus C$, the measure α satisfies the normalization of equation (2.3) also if B and C are permutated. Then the integral of a $\mathscr{C}_c$-function $f(y,z)$ in the variables $(y,z) \in B \times C$ is computed equivalently by

$$\int_C \int_B f(y,z)\, d\beta(y)\, d\gamma(z) \qquad \text{or} \qquad \int_B \int_C f(y,z)\, d\gamma(z)\, d\beta(y)\,.$$

In this case, the measure α is called the *product measure* of β and γ.

The exact sequence involving the groups A, B, C is coupled with the exact mirror-sequence

$$0 \longrightarrow \widehat{C} \xrightarrow{\pi^*} \widehat{A} \xrightarrow{\iota^*} \widehat{B} \longrightarrow 0$$

and one can wonder what relationship there is between the Fourier transform and the averaging transform $f \mapsto f^\flat$. Since the Fourier transform of a continuous function with compact support is not necessarily compactly supported on the Pontryagin dual, it is useful to note that the L^1-norm of f dominates the L^1-norm of $f^\flat$, and this allows us to extend the averaging transform to the space of L^1-functions.

Theorem 2.2.8 (Poisson summation formula)**.** *With the same notation as above, assume that f is a continuous, integrable function on A with integrable Fourier transform. Then*

$$\int_B f\big(\iota(y)\big)\, d\beta(y) = \int_{\widehat{C}} \widehat{f}\big(\pi^*(\hat{z})\big)\, d\widehat{\gamma}(\hat{z})$$

Proof. Let g be the Fourier transform of f. The equality to prove is equivalent to the equality of $f^\flat$ and $g^\flat$ when they are both calculated on the neutral element of C and $\widehat{B}$ respectively. The key is the Fourier inversion formula (2.2) for $f^\flat$ combined with the relation $\widehat{f^\flat} = g \circ \pi^*$ between the Fourier transforms of $f^\flat$ and f:

$$f^\flat(z) = \int_{\widehat{C}} \widehat{f^\flat}(\hat{z}) \cdot \hat{z}(z)\, d\widehat{\gamma}(\hat{z}) \quad \text{for any } z \in C$$

and, in particular, for $z = 0$

$$f^\flat(0) = \int_{\widehat{C}} \widehat{f^\flat}(\hat{z})\, d\gamma(\hat{z})\,. \tag{2.4}$$

Let's prove that the Fourier transform of $f^\flat$ is $g \circ \pi^*$.

$$\widehat{f^\flat}(\hat{z}) = \int_C f^\flat(z) \cdot \overline{\hat{z}(z)}\, d\gamma(z)$$

and the integrand expands as

$$f^\flat(z) \cdot \overline{\hat{z}(z)} = \int_B f\big(x+\iota(y)\big) \cdot \overline{\hat{z}(\pi(x))}\, d\beta(y)\,, \quad \text{with} \quad z = \pi(x) \quad \text{for some } x \in A.$$

Observe that for the character $\hat{z} \circ \pi$ is trivial on $\iota(B)$, hence the expression $f\big(x + \iota(y)\big) \cdot \overline{\hat{z}\big(\pi(x)\big)}$ can be replaced with $h\big(x + \iota(y)\big)$, where h is the function defined by

$$h(a) = f(a) \cdot \overline{\hat{z}\big(\pi(a)\big)} \quad \text{for all } a \in A.$$

So, we have that $f^\flat \cdot \hat{z}$ is equal to $h^\flat$ and the value $\widehat{f^\flat}(\hat{z})$ is computed by the integral of $h^\flat$ on C. Recall that the measures α, β, γ satisfy the identity 2.3, which means that

$$\begin{aligned} \widehat{f^\flat}(\hat{z}) &= \int_C h^\flat \, d\gamma \\ &= \int_A h \, d\alpha \\ &= \int_A f(x) \cdot \overline{\hat{z}\big(\pi(x)\big)} \, d\alpha(x) \,. \end{aligned}$$

The bottom expression is the Fourier transform of f calculated on the character $\pi^*(\hat{z})$, therefore $\widehat{f^\flat}$ is equal to g composed with the morphism π^*. To end the proof simply put together equation (2.4) with the identity $\widehat{f^\flat} = g \circ \pi^*$ and note that

$$\int_{\widehat{C}} g\big(\pi^*(\hat{z})\big) \, d\gamma(\hat{z}) = g^\flat(0) \,.$$

□

Remark 2.2.9. The argument used in the proof of Theorem 2.2.4 implies the stronger identity of functions in the variable $x \in A$:

$$\int_B f\big(x + \iota(y)\big) \, d\beta(y) = \int_{\widehat{C}} \widehat{f}\big(\pi^*(\hat{z})\big) \cdot \hat{z}\big(\pi(x)\big) \, d\widehat{\gamma}(\hat{z}) \,.$$

2.3 Locally compact fields

Before we move on to the main examples it will be useful to note that the automorphisms of a locally compact abelian group act on its set of Haar measures: if an **LCA** group A is equipped with a Haar measure μ and φ is an automorphism of A, then, by pullback, we get another Haar measure $\varphi^\star\mu$ on A. Since any Haar measure on A is a positive multiple of μ, there must be a unique positive real number $|\varphi|$, called the *module* of φ, such that $\varphi^\star\mu = |\varphi| \, \mu$. Note that the module of φ is the same if we exchange the measure μ with $c\mu$ for some $c > 0$. This defines a homomorphism $|\cdot|$ from the group of automorphism of A to the multiplicative group of the positive real line. Thus, for any group G that acts continuously and linearly on A there is a homomorphism $|\cdot| : G \to \mathbb{R}_+^\times$ that computes the module of an element $g \in G$ viewed as an automorphism of A.

Definition 2.3.1. A *locally compact ring* is defined to be a topological, commutative ring $(A, +, \cdot)$ such that the additive group $(A, +)$ belongs to the category **LCA**. A Haar measure of A is simply a Haar measure of the additive group of A. A locally compact ring K is said to be a *locally compact field* if the underlying ring of K is a field and the inversion map $K^\times \to K \,, \; x \mapsto x^{-1}$ is continuous for the sub-space topology on the domain.

Remark 2.3.2 (Notation for measure and integration). In the previous sections, we used the notation

$$\int_A f(x)\, d\mu(x)$$

for the integral of a function f on a locally compact group A. From now on we can drop the μ in the notation if the Haar measure is clear from the context and we write

$$\int_A f(x)\, dx$$

instead. We use the symbol dx for the measure μ and if φ is an automorphism of A, we use the symbol $d\varphi(x)$ for the measure $\varphi^\star \mu$.

Let A be a locally compact ring with Haar measure dx, then the group of units $A^\times$ acts on the additive group of A by automorphisms in the category **LCA**. So it is well defined the module homomorphism $|\cdot| : A^\times \to \mathbb{R}_+^\times$ that satisfies the formal identity $d(ax) = |a|\, dx$ for all $a \in A^\times$.

Lemma 2.3.3. *Let K be a topologically non-discrete, locally compact field and $|\cdot|$ the module of $K^\times$. Then $|\cdot|$ is an open continuous map whose image is a closed sub-group of $\mathbb{R}_+^\times$. Moreover, if $|\cdot|$ is extended to K by imposing $|0| = 0$, then $|\cdot|$ is a valuation of K and the topology of K is the same as the induced valuation-topology and K.*

This lemma is a synthesis of many results that are proved in Section 4.1 of [RV99] and in Chapter VII, Section 10-11 of [Bou04]. These results lead to the classification of (non-discrete) locally compact fields.

Theorem 2.3.4 ([RV99], Theorem 4-12)**.** *Let K be a topologically non-discrete, locally compact field. Then, K is isomorphic, as a topological field, to one of the following:*

(i) *the field of real or complex numbers;*

(ii) *a finite extension of the field of p-adic numbers $\mathbb{Q}_p$ for some prime integer p.*

(iii) *the field $k((T))$ of Laurent series with coefficients in a finite field k.*

Proof. The field K with the module $|\cdot|$ is a valued field by Lemma 2.3.3. Locally compactness implies that K is complete, as any Cauchy sequence with a convergent sub-sequence is convergent. If K is archimedean then it is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ by Theorem 1.2.8. If K is non-archimedean then let $\mathcal{O}$ be the ring of elements x with $|x| \leq 1$ and $\mathfrak{p}$ the unique maximal ideal of $\mathcal{O}$, which is equal to the set of elements x with $|x| < 1$. The ring $\mathcal{O}$ is different from K and $\mathfrak{p} \neq 0$ because K is not topologically discrete. The ring $\mathcal{O}$ is the closed ball of radius 1 of K and the ideal $\mathfrak{p}$ is the open ball of the same radius. But $\mathcal{O}$ is also open because it is a union of cosets of $\mathfrak{p}$. The ideal $\mathfrak{p}$ is also closed because it is an open sub-group of the additive groups of K and $\mathcal{O}$ (see point (vi) of Proposition 2.1.3). The group of units $\mathcal{O}^\times$ is the set of elements of absolute value equal to 1, so it is the complement of $\mathfrak{p}$ in $\mathcal{O}$. Since the last two groups are both open and closed in K, then $\mathcal{O}^\times$ is also both open and closed in K. This means that the quotient $K^\times/\mathcal{O}^\times$ is discrete. The valuation $|\cdot|$ is open, hence its image is topologically isomorphic to $K^\times/\mathcal{O}^\times$, which is discrete. The discrete sub-groups of $\mathbb{R}_+^\times$ are free of rank 1. Let $\pi \in K$ be such that $|\pi|$ generates the image of $|\cdot|$. Then $\{\pi^n \mathcal{O} : n \in \mathbb{Z}\}$ is the set of all balls of K with centre at zero. By local compactness, there must be a compact ball centred at zero, so $\pi^n \mathcal{O}$ is compact for some $n \in \mathbb{Z}$. The operation of multiplication by π^n is a homeomorphism of K, then $\mathcal{O} \cong \pi^n\mathcal{O}$ also topologically and $\mathcal{O}$ must be compact. The residue field $\kappa := \mathcal{O}/\mathfrak{p}$ is compact and discrete, hence finite. So far we have that K is a complete, valued field with a finite residue field and $K^\times/\mathcal{O}^\times \cong \mathbb{Z}$. It is known that such fields are finite extensions of $\mathbb{Q}_p$ or fields the form $\kappa((T))$ (see Proposition 5.2 of [NS99]). $\square$

Given the classification of locally compact fields of Theorem 2.3.4:

Definition 2.3.5. A *local filed* is defined to be a locally compact, topologically non-discrete field or, equivalently, one of the fields of Theorem 2.3.4.

2.4 Restricted direct products

In this section, we introduce a general construction, with all the relevant properties, that will be useful for the analysis of the ring of adèles and its multiplicative group of units. All the results come from Tate's exposition of his thesis in [CF67], Chapter XV, Section 3. Let $\mathscr{P}$ be a non-empty, infinite set and $\{A_\nu\}_{\nu\in\mathscr{P}}$ a family of **LCA** groups with a specified compact, open sub-group $K_\nu \subset A_\nu$ for all $\nu \in \mathscr{P}$ exept for a finite subset S_0. For all finite subset $S \subset \mathscr{P}$ that contains S_0, define A_S as the topological sub-group of the product of all A_ν

$$A_S = \left\{(x_\nu)_{\nu\in\mathscr{P}} : x_\nu \in K_\nu \text{ for all } \nu \notin S\right\} .$$

The group A_S is topologically isomorphic to

$$\prod_{\nu\in S} A_\nu \times \prod_{\nu\notin S} K_\nu$$

which belongs to the category **LCA**, since it is the product of the compact, Hausdorf group $\prod_{\nu\notin S} K_\nu$ with a finite product of **LCA** groups.

Definition 2.4.1. The restricted direct product of the groups A_ν with respect to the compact, open sub-groups K_ν is the group

$$\prod_{\nu\in\mathscr{P}}{}' A_\nu := \left\{(x_\nu)_{\nu\in\mathscr{P}} : x_\nu \in K_\nu \text{ for almost all } \nu \in \mathscr{P}\right\}$$

with the maximal topology that makes the inclusion maps

$$A_S \hookrightarrow \prod_{\nu\in\mathscr{P}}{}' A_\nu$$

continuous for all finite subsets $S \subset \mathscr{P}$ that contains S_0.

For the rest of the section, A will denote the restricted product and the symbols $S, S', S'', \ldots$ are used for finite subsets of $\mathscr{P}$ that contain S_0. The subsets

$$\prod_{\nu\in\mathscr{P}} U_\nu$$

of A, with U_ν open subset of A_ν for all ν and $U_\nu = K_\nu$ for almost all ν, form a pre-basis of open subset for the topology of A. It is easy to see that A is Hausdorff and locally compact because it is the union of the open sub-groups A_S for $S \subset \mathscr{P}$ finite, indeed any two distinct elements a, b of A, are contained in some A_S, hence there are two disjoint open neighbourhoods of a and b in A_S respectively because A_S is Hausdorff, and a basis of compact neighbourhoods of the neutral element of A is given by taking products of compact neighbourhoods $T_\nu \subseteq A_\nu$ of the neutral element of A_ν with $T_\nu = K_\nu$ for almost all ν. In abstract terms, A is the colimit in the category **LCA** of the filtrant diagram given by the natural inclusion-maps

$$A_S \hookrightarrow A_{S'}$$

for $S \subset S'$. From this point of view, it's clear that the restricted direct product is independent of the choice of the finite set S_0.

Construction. Since A is locally compact, it has Haar measures, but one can choose one canonically. Let dx_ν be a Haar measure on A_ν given for any ν in such a way that K_ν has measure 1 for almost all ν. We can enlarge the finite set S_0, where the compact, open sub-group is not specified, to include the finite set of indexes ν where the condition on the measure of K_ν is not imposed. Enlarging S_0 does not affect the restricted direct product, so, without loss of generality, we can suppose that, for all $\nu \notin S_0$, the measure of K_ν is 1. Let K_S be the product of all the compact groups K_ν for $\nu \notin S$. It can be naturally seen as a compact subgroup of A contained in any open subgroup $A_{S'}$ such that $S \subseteq S'$. Moreover, the sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_S & \hookrightarrow & A_S & \longrightarrow & \prod_{\nu \in S} A_\nu & \longrightarrow & 0 \\ & & & & (x_\nu)_{\nu \in \mathscr{P}} & \longmapsto & (x_\nu)_{\nu \in S} & & \end{array}$$

is split-exact. Equip K_S with the unique Haar measure of total measure 1. This choice is compatible with the decomposition

$$K_S = K_{S'} \times \prod_{\nu \in S' \setminus S} K_\nu$$

for $S \subset S'$, because K_ν has measure 1 by assumption. Equip A_S with the product measure induced by the decomposition

$$A_S = K_S \times \prod_{\nu \in S} A_\nu \,.$$

Note that any compact subset of A is contained in some A_S because A is the union of the increasing sequence of open sub-groups A_S, so any compactly supported, continuous function of A is the extension by zero of a unique function belonging to the space $\mathscr{C}_c(A_S)$ for some S large enough. To be more precise, the topological vector space $\mathscr{C}_c(A)$ is the colimit of the filtrant diagram obtained by the obvious inclusion maps

$$\mathscr{C}_c(A_S) \to \mathscr{C}_c(A_{S'})$$

that extend a function giving it the value zero on $A_{S'} \setminus A_S$, for $S \subset S'$. This observation and the conditions imposed on the measures defined so far ensure that there is a unique Haar measure dx on A whose restriction to the open subgroup A_S is the measure chosen before in a way that is compatible for inclusions $S \subset S'$. If not specified, it will be assumed that the restricted direct product has the measure constructed like dx, called the *restricted product* of the measures dx_ν.

The consequences of the construction can be summarized in the next proposition. Since the inclusion $A \subset \prod_\nu A_\nu$ is still continuous, also the projections $A \to A_\nu\,,\ x \mapsto x_\nu$ are continuous. If we have a collection of functions f_ν defined on A_ν, for each $\nu \in \mathscr{P}$, such that $f_\nu(K_\nu) = 1$ for almost all ν, then the product

$$f(x) = \prod_\nu f_\nu(x_\nu)$$

defines a function of A. In this case, f is called *factorizable* with factors f_ν and is denoted by $\otimes_\nu f_\nu$.

Proposition 2.4.2. *Let f be a measurable function on A. Then f is integrable if and only if*

$$\sup_S \int_{A_S} \left|f(x)\right| \, dx$$

is bounded, and in that case the integral of f is computed by

$$\int_A f(x)\, dx = \lim_S \int_{A_S} f(x)\, dx\,.$$

Moreover, if f is factorizable with factors f_ν for each ν, then

$$\int_A f(x)\, dx = \prod_\nu \int_{A_\nu} f_\nu(x_\nu)\, dx_\nu\,.$$

We conclude the section by describing the Pontryagin dual of the restricted direct product A. It is again a restricted direct product, precisely the one corresponding to the data $(\widehat{A_\nu}, K_\nu^\perp)$.

Lemma 2.4.3. *Let B be an* **LCA** *group with a compact, open sub-group C. Then $C^\perp$ is also a compact, open sub-group.*

Proof. Recall that $C^\perp$ is isomorphic to the dual of B/C. Since C is open, the quotient group B/C is discrete, therefore its dual must be compact. To see that $C^\perp$ is also open consider the quotient $\widehat{B}/C^\perp$. It is isomorphic to the dual of C, but the dual of a compact group is discrete and then $\widehat{B}/C^\perp$ is discrete, implying that $C^\perp$ is open in $\widehat{B}$. □

Lemma 2.4.4. *Let B be an* **LCA** *group with a compact, open sub-group C. Suppose that dx is a Haar measure on B and $d\chi$ is a Haar measure on $\widehat{B}$. Then, dx and $d\chi$ are dual to each other if and only if*

$$\left(\int_C dx\right) \cdot \left(\int_{C^\perp} d\chi\right) = 1$$

Proof. Let f be the locally constant function equal to 1 on C and 0 on the complement. Its Fourier transform calculated on the character $\chi \in \widehat{B}$ is

$$\widehat{f}(\chi) = \int_C \overline{\chi(x)}\, dx\,.$$

By the observations made in the example 2.2.5, that integral is equal to the measure of C when χ is trivial on C, i.e. when $\chi \in C^\perp$, and it is zero otherwise. Let $c > 0$ be the constant such that $d\chi$ is the dual measure multiplied by c. By Fourier inversion formula

$$f(x) = c \cdot \int_{\widehat{B}} \widehat{f}(\chi)\chi(x)\, d\chi$$

and for $x = 0$

$$\begin{aligned} 1 &= c \cdot \int_{C^\perp} \left(\int_C dx \right) d\chi \\ &= c \cdot \left(\int_C dx \right) \cdot \left(\int_{C^\perp} d\chi \right) . \end{aligned}$$

To conclude, observe that the identity in the statement of the lemma is equivalent to the equality $c = 1$ which in turn is equivalent to $d\chi$ being the dual measure of dx. □

By the above lemmas, the of groups $\widehat{A}_\nu$ are equipped with an open, compact sub-group $K_\nu^\perp$ and the measure $d\chi_\nu$ dual to dx_ν gives measure 1 to the set $K_\nu^\perp$ for almost all ν, so that the restricted product of $\widehat{A}_\nu$ is equipped with the restricted product measure $d\chi$ of the measures $d\chi_\nu$. Note that an element of $\prod'_\nu \widehat{A}_\nu$ is a tuple $(\chi_\nu)_\nu$ of unitary characters $\chi_\nu \in \widehat{A}_\nu$ such that χ_ν belongs to the sub-group $K_\nu^\perp$ for almost all ν, i.e. the character χ_ν is trivial on K_ν for almost all ν. If a is an element of A, the product

$$\prod_\nu \chi_\nu(a_\nu)$$

makes sense because almost all factors are equal to 1. The rule $a \mapsto \prod_\nu \chi_\nu(a_\nu)$ defines a homomorphism $A \to \mathbb{S}^1$ that we denote by $\otimes_\nu \chi_\nu$. Conversely, if χ is a unitary character of A, we can consider the homomorphism $\chi_\nu : A_\nu \to \mathbb{S}^1$ obtained by the restriction of χ to the copy of A_ν inside A and ask if the tuple $(\chi_\nu)_\nu$ is an element of the restricted product $\prod'_\nu \widehat{A}_\nu$. The verification that these constructions define morphisms between $\widehat{A}$ and $\prod'_\nu \widehat{A}_\nu$ is accomplished in the next lemmas. Since it will be useful in the future, they are stated for more general characters.

Lemma 2.4.5. *Let $(\omega_\nu)_\nu$ be an element of the product $\prod_\nu \mathrm{Hom}_{\mathbf{LCA}}(A_\nu, \mathbb{C}^\times)$ such that ω_ν is trivial on the sub-group K_ν for almost all ν. Then the rule*

$$\omega : A \longrightarrow \mathbb{C}^\times , \quad x \longmapsto \prod_\nu \omega_\nu(x_\nu)$$

defines a (not necessarily unitary) character of A.

Proof. Let S_0 be a finite set of indexes large enough to contain all index ν for which ω_ν is non-trivial on K_ν. Let S range over finite subsets of $\mathscr{P}$ that contain S_0. The restricted direct product A is the union of the open-subgroups A_S, so, to give an element of $\omega \in \mathrm{Hom}_{\mathbf{LCA}}(A, \mathbb{C}^\times)$ is the same as to give a compatible system of elements of $\omega_S \in \mathrm{Hom}_{\mathbf{LCA}}(A_S, \mathbb{C}^\times)$, where $\omega_S = \omega|_{A_S}$. □

Lemma 2.4.6. *Let $\omega : A \to \mathbb{C}^\times$ be a (not necessarily unitary) character of A. Then, for any index ν, the map $\omega_\nu : A_\nu \to \mathbb{C}^\times$ defined by pre-composing ω with the morphism*

$$\iota_\nu : A_\nu \longrightarrow A , \quad x_\nu \longmapsto (\ldots, 0, x_\nu, 0, \ldots)$$

is a morphism of locally compact groups, and for all $x \in A$ the equality

$$\omega(x) = \prod_\nu \omega_\nu(x_\nu)$$

holds.

Proof. The map ω_ν is obtained by composing morphisms in **LCA**: for any finite set S of indexes containing ν, the group-homomorphism ι_ν factors through the inclusion $A_S \hookrightarrow A$. Since A_ν has the product topology, the map $A_\nu \to A_S$ is continuous and so is ι_ν, therefore it is a morphism in **LCA**. To obtain the product formula for ω, choose an open neighbourhood $U \subset \mathbb{C}^\times$ that is small enough to not contain sub-groups of $\mathbb{C}^\times$ except for the trivial one. The pre-image $\omega^{-1}(U)$ is an open neighbourhood of the neutral element of A, thus it contains a basic open subset $V = \prod_\nu V_\nu$ where V_ν is an open neighbourhood of $0 \in A_\nu$ for all ν and V_ν is equal to the sub-group group K_ν for almost all index ν. If S is the set of indexes ν for which $V_\nu \neq K_\nu$, then the compact group K_S is contained in V and in the bigger open set $\omega^{-1}(U)$, so the group $\omega(K_S)$ is contained in U, forcing it to be trivial. Any element $x \in A$ is contained in some sub-set of the form $A_{S'}$ and we may suppose that S' is a finite set of indexes containing S. The compact group $K_{S'}$ is contained in K_S, so $\omega(K_{S'})$ is still the trivial group, moreover x is equal to

$$\sum_{\nu \in S'} \iota_\nu(x_\nu)$$

modulo the sub-group $K_{S'}$. The two facts together imply that

$$\omega(x) = \prod_{\nu \in S'} \omega(\iota_\nu(x_\nu))\,,$$

an equality that can be written as

$$\omega(x) = \prod_\nu \omega_\nu(x_\nu)$$

by observing that the definition of ω_ν is precisely the composition $\omega \circ \iota_\nu$ and $\omega_\nu(x_\nu) = 1$ for $\nu \notin S'$. □

Theorem 2.4.7. *With notations as above, the two maps*

$$\Phi : \widehat{A} \longrightarrow {\prod_\nu}' \widehat{A}_\nu\,, \quad \chi \longmapsto (\chi_\nu)_\nu$$

and

$$\Psi : {\prod_\nu}' \widehat{A}_\nu \longrightarrow \widehat{A}\,, \quad (\chi_\nu)_\nu \longmapsto \otimes_\nu \chi_\nu$$

are morphisms in the category **LCA**, *one the inverse of the other. With this identification, the dual measure of A corresponds to the restricted product of the dual measures $d\chi_\nu$.*

Proof. Thanks to the lemmas 2.4.6 and 2.4.5 it's easy to see that the two maps in the statement are well-defined homomorphisms of groups, one the inverse of the other. It remains to check that they are continuous for the restricted-product topologies. Recall that the restricted direct product of the groups $\widehat{A}_\nu$, which in this proof is denoted by A', is the filtered colimit of the groups

$$A'_S := \left(\bigoplus_{\nu \in S} \widehat{A}_\nu \right) \oplus \prod_{\nu \notin S} K_\nu^\perp$$

while, by duality, we have a description of $\widehat{A}$ in terms of a cofiltered projective limit:

$$\begin{aligned}\widehat{A} &= \mathrm{Hom}_{\mathbf{LCA}}(A, \mathbb{S}^1)\\ &\cong \varprojlim_S \mathrm{Hom}_{\mathbf{LCA}}(A_S, \mathbb{S}^1)\\ &\cong \varprojlim_S \left[\left(\bigoplus_{\nu\in S} \mathrm{Hom}_{\mathbf{LCA}}(A_\nu, \mathbb{S}^1)\right) \oplus \mathrm{Hom}_{\mathbf{LCA}}\left(\prod_{\nu\notin S} K_\nu\ ,\ \mathbb{S}^1\right)\right].\end{aligned}$$

So, Ψ is continuous if and only if its restrictions Ψ_S to the open sub-groups A'_S are continuous as S gets larger and larger. In turn, Ψ_S is continuous if and only if the morphisms $\Psi_{S,S'}$ obtained by composition with the projections of the limit are continuous for S' containing S. More explicitly, $\Psi_{S,S'}$ is the morphism

$$\Psi_{S,S'} : A'_S \to \mathrm{Hom}_{\mathbf{LCA}}(A_{S'}, \mathbb{S}^1)\,, \quad (\chi_\nu)_\nu \mapsto (\otimes_\nu \chi_\nu)\big|_{A_{S'}}\,.$$

We can consider $\Psi_{S,S'}$ as the restriction of $\Psi_{S',S'}$ and prove that the latter map is continuous. Recall that, in a linear category with biproducts, one can identify the group of morphisms

$$\mathrm{Hom}(X_1 \oplus X_2, Y_1 \oplus Y_2)$$

with two-by-two matrixes of morphisms whose entrances belong to $\mathrm{Hom}(X_i, Y_j)$ for all the possible combination of i, j varying between 1 and 2. In what follows we prove that $\Psi_{S',S'}$ is a morphism corresponding to a matrix with three null components and one component that is an isomorphism in the category **LCA**, so in particular, continuous. Note that, regarding the decomposition

$$\mathrm{Hom}_{\mathbf{LCA}}(A_{S'}, \mathbb{S}^1) \cong \mathrm{Hom}_{\mathbf{LCA}}\left(\bigoplus_{\nu\in S'} A_\nu\ ,\ \mathbb{S}^1\right) \oplus \mathrm{Hom}_{\mathbf{LCA}}\left(\prod_{\nu\notin S'} K_\nu\ ,\ \mathbb{S}^1\right),$$

the image of $\Psi_{S',S'}$ is contained in the left component because a tuple $(\chi_\nu)_\nu$ belonging to $A'_{S'}$ has the property $\chi_\nu(x_\nu) = 1$ for all $x_\nu \in K_\nu$ and all $\nu \notin S'$, so its image $\chi := \otimes_\nu \chi_\nu$, that is defined by the product $\chi(x) = \prod_\nu \chi_\nu(x_\nu)$ for all $x \in A$, is trivial on all elements x for which $x_\nu \in K_\nu$ when $\nu \notin S'$ and $x_\nu = 0$ when $\nu \in S'$, that is an element of $K_{S'}$. Note also that $\Psi_{S',S'}$ is trivial on the subgroup of $A'_{S'}$ corresponding to the factor $\prod_{\nu\notin S'} K_\nu^\perp$ in the decomposition

$$A'_{S'} = \left(\bigoplus_{\nu\in S'} \widehat{A}_\nu\right) \oplus \prod_{\nu\notin S'} K_\nu^\perp$$

because $\prod_\nu \chi_\nu(x_\nu) = 1$ for all $x \in A'_{S'}$ if χ_ν is the trivial character for all $\nu \in S'$ and $\chi_\nu \in K_\nu^\perp$ for the rest of the indexes ν. Thus, ignoring the three null components of $\Psi_{S',S'}$, we are left with the topological isomorphism that gives the identification

$$\bigoplus_{\nu\in S'} \widehat{A}_\nu \cong \mathrm{Hom}_{\mathbf{LCA}}\left(\bigoplus_{\nu\in S'} A_\nu\ ,\ \mathbb{S}^1\right)$$

in the category **LCA**, precisily because of the definition of $\otimes_\nu \chi_\nu$ when ν ranges in a finite set.

Now we prove that Φ is continuous. Let Φ_ν be the composition of Φ with the projection from A' to $\widehat{A}_\nu$.

Claim. The map Φ_ν is continuous.

To see this consider the inverse image $\Phi_\nu^{-1}(W(C,U))$ of a (pre-)basic open set, where $C \subseteq A_\nu$ is compact and U is an open set of the circle. Define the compact subset C_ν of $\widehat{A}$ as the set of elements $x \in A$ for which $x_\nu \in C$ and $x_\upsilon = 0$ for $\upsilon \neq \nu$, then

$$\Phi_\nu^{-1}(W(C,U)) = \left\{\chi \in \widehat{A} : \chi(C_\nu) \subseteq U\right\},$$

which the open set $W(C_\nu, U)$ in $\widehat{A}$. Now, for the continuity of Φ, consider an open neighbourhood V of the neutral element of A' which is of the form $V = \prod_\nu V_\nu$, with V_ν open neighbourhood of 1 in $\widehat{A}_\nu$ for all ν and $V_\nu = K_\nu^\perp$ for all $\nu \notin S$, given a finite S. Consider the inverse-image $\Phi^{-1}(V)$ and an open set U of the circle containing only the trivial sub-group of $\mathbb{S}^1$ and no other sub-group.

Claim.

$$\Phi^{-1}(V) = \left(\bigcap_{\nu \in S} \Phi_\nu^{-1}(V_\nu)\right) \cap W(K_S, U).$$

If this is true then $\Phi^{-1}(V)$ would be open because $W(K_S, U)$ is a basic open set of $\widehat{A}$ and Φ_ν is continuous. To prove the claim observe that χ belongs to $\Phi^{-1}(V)$ if and only if satisfies the conditions

$$\begin{cases} \chi_\nu(K_\nu) = 1 & \text{for all } \nu \notin S; \\ \chi_\nu \in V_\nu & \text{for all } \nu \in S. \end{cases}$$

The first condition is equivalent to $\chi(K_S) = 1$ and the second condition is equivalent to χ belonging to the intersection of the sets $\Phi_\nu^{-1}(V_\nu)$ for ν varying in S. Finally, observe that $\chi \in W(K_S, U)$ if and only if $\chi(K_s) = 1$, because $\chi(K_S)$ is a sub-group of U and U contains only the trivial group.

The statement about the measure is a consequence of Proposition 2.4.2.

□

All the notions about restricted direct products in this section are used in the next chapters to study the ring of *adèles* and the group of the *idèles* of a global field. Thus it is useful to observe here that the restricted direct product of locally compact rings with given open and compact sub-rings is again a locally compact ring if the multiplication on the product is defined pointwise.

Chapter 3

Harmonic Analysis on the Ring of Adèles

In this chapter, we study the ring of adèles using all the techniques developed in the previous ones. Before starting, we summarize all the fundamental properties of the ring of adèles used in the final chapter. The adèle ring $\mathbb{A}$ of a global field K is the restricted direct product of the local fields K_ν for all places ν and it sits in the middle of a short exact sequence

$$0 \longrightarrow K \longrightarrow \mathbb{A} \longrightarrow \frac{\mathbb{A}}{K} \longrightarrow 0\,,$$

where K is discrete and $\mathbb{A}/K$ is compact. The Pontryagin dual of the quotient $\mathbb{A}/K$ is naturally identified with the group of unitary characters of $\mathbb{A}$ that are trivial on K. Every non-trivial character $\psi \in \widehat{\mathbb{A}/K}$ induces an isomorphism of locally compact groups $\mathbb{A} \cong \widehat{\mathbb{A}}$ which identifies K with $\widehat{\mathbb{A}/K}$. The isomorphism is given by the map

$$\mathbb{A} \longrightarrow \widehat{\mathbb{A}}\,, \quad a \longmapsto \psi_a\,,$$

where $\psi_a(x) = \psi(ax)$ for all $x \in \mathbb{A}$. The character ψ is of the form $\otimes_\nu \psi_\nu$ for local characters ψ_ν of K_ν inducing isomorphisms $K_\nu \cong \widehat{K}_\nu$ analogous to that of $\mathbb{A}$ induced by ψ. Attached to ψ and ψ_ν there are self-dual measures dx and dx_ν on $\mathbb{A}$ and K_ν respectively. The global measure dx is the restricted product of the local ones dx_ν. The group of units $\mathbb{A}^\times$ turn out to be the restricted direct product of the groups $K_\nu^\times$ with respect to the sub-groups $\mathcal{O}_\nu^\times$. One defines a multiplicative Haar measure $d^\times x_\nu := m_\nu |x_\nu|_\nu^{-1}\, dx_\nu$ for each place ν, where the constant m_ν ensures that $\mathcal{O}_\nu^\times$ gets measure 1 for almost all places. The group $\mathbb{A}^\times$ is called the group of idèles and it acts as an automorphism group of the additive structure of $\mathbb{A}$ via multiplication. The module of this action

$$|\cdot| : \mathbb{A}^\times \longrightarrow \mathbb{R}_+^\times$$

is the idèlic norm. It is computed by the product of all local modules $|\cdot|_\nu$ of $K_\nu^\times$. The kernel of the idèlic norm, denoted by $\mathbb{A}^{\times,1}$, contains the multiplicative group $K^\times$. A consequence of this is that the self-dual measure dx of $\mathbb{A}$ is independent of the choice of the additive character ψ. The image $|\mathbb{A}^\times|$ of the idèlic norm is $\mathbb{R}_+^\times$ for K a number field and it is a free rank 1 sub-group of $\mathbb{R}_+^\times$ if K is a function field. In both cases $|\cdot|$ admits a continuous splitting (or section) inducing non-canonical isomorphisms

$$\mathbb{A}^\times \cong \mathbb{A}^{\times,1} \oplus \left|\mathbb{A}^\times\right| \qquad \text{and} \qquad \mathbb{A}^\times/K^\times \cong \mathbb{A}^{\times,1}/K^\times \oplus \left|\mathbb{A}^\times\right| .$$

The multiplicative group of the global field is discrete in $\mathbb{A}^{\times,1}$ and the quotient $\mathbb{A}^{\times,1}/K^\times$ is compact. This and the non-canonical isomorphisms simplify integrals over the idèles and the description Ω_K, the space of idèle class characters. These are of main interest in Chapter 4 and they are the (not necessarily unitary) characters of the quotient $\mathbb{A}^\times/K^\times$, the idèle class group. A choice of a section of the idèlic norm induces an isomorphism

$$\Omega_K \cong \begin{cases} \widehat{\mathbb{A}^{\times,1}/K^\times} \oplus \mathbb{C} & \text{if } K \text{ is a number field,} \\ \widehat{\mathbb{A}^{\times,1}/K^\times} \oplus \mathbb{C}/\frac{2\pi i}{\log q}\mathbb{Z} & \text{if } K \text{ is a function field and } q^{-1} \text{ generates } \left|\mathbb{A}^\times\right|. \end{cases}$$

Since the Pontryagin dual of a compact group is discrete, we get the space Ω_K parametrized by a discrete group and a complex variable s. The restricted direct product structure of $\mathbb{A}^\times$ gives a decomposition of an idèle class character ω into local characters ω_ν. Theorem 3.5.8 completes the description of idèle class characters with the classification of local characters.

The results on the adèles are derived first from the analysis of local fields, so we start by studying them together with their additive unitary characters. The main result of the next Section is the self-duality of local fields (Theorem 3.1.3).

3.1 Pontryagin duality for local fields

As summarized in Section 2.3, there is a canonical choice for the valuation of a local field, that is the *module*. The following results show how to calculate the module for finite extensions of local fields.

Lemma 3.1.1. *Let K be a local field and V an n-dimensional vector space over it, with the topology given by any of its equivalent norms. Then V is a* **LCA** *group and the module of a linear automorphism φ of V is equal to $\left|\det(\varphi)\right|$, where $|\cdot|$ is the module of K.*

Proof. The vector space V is isomorphic to K^n topologically and linearly, so it is locally compact and Hausdorff. The module is a homomorphism, hence the formula can be checked on Gauss operations, which is easy using the product measure of K^n. □

Corollary 3.1.2. *Let K/k be a finite extension of local fields, $|\cdot|_k$ the module of k and $|\cdot|_K$ the module of K. Then*

$$|x|_K = \left|N_{K/k}(x)\right|_k \quad \text{for all } x \in K .$$

By Corollary 3.1.2 it is sufficient to know the module of the various completions of $\mathbb{Q}$ and $\mathbb{F}_q(T)$ to know the module of an arbitrary local field. Indeed the valuations described in Theorem 1.1.16 are already normalized to compute the module of the corresponding local field. The verification for the field $\mathbb{R}$ is done in Example 3.1.5 and for non-archimedean local fields K we can make the following observation which also gives another way to compute the module: let $\mathcal{O}$ be the local ring of K and μ a Haar measure of K satisfying $\mu(\mathcal{O}) = 1$. Let π be a generator of the maximal ideal of $\mathcal{O}$. Recall that any element x of K is of the form $x = u\pi^n$ for a unique unit $u \in \mathcal{O}^\times$ and a unique integer n. The group $\mathcal{O}^\times$ is also the set of $x \in K$ such that $|x| = 1$, hence the module $|\cdot|$ is determined by the value of $|\pi|$. The module of π can be calculated as the ratio $\mu(\pi E)/\mu(E)$ for any subset E with a non-zero measure. By the normalization of the measure, $|\pi| = \mu(\pi\mathcal{O})$. If q is the cardinality of the residue field $\mathcal{O}/\pi\mathcal{O}$ and $x_1, \ldots, x_q$ form a complete set of representatives of the cosets of $\mathcal{O}/\pi\mathcal{O}$ in $\mathcal{O}$, we can decompose the local ring as the disjoint union

$$\mathcal{O} = \bigcup_{i=1}^{q} x_i + \pi\mathcal{O}$$

of the q cosets. They have the same measure of $\pi\mathcal{O}$ because μ is invariant by translations, therefore $q \cdot \mu(\pi\mathcal{O}) = \mu(\mathcal{O})$, which implies that $|\pi| = q^{-1}$. You can check that the valuations chosen in Theorem 1.1.16 to represent non-archimedean places are of this form.

From the point of view of harmonic analysis, the advantage of working with local fields is that they are Pontryagin-self-dual, as explained in detail in the next Proposition, leading to a Fourier calculus very close to the classical theory on the real vector spaces.

Theorem 3.1.3 (Lemma 2.2.1 of [CF67], Chapter XV)**.** *Let K be a local field and consider its additive group. Let χ be a non-trivial, unitary character of K. Then the morphism*

$$\psi : K \to \widehat{K}$$

induced by the bilinear map

$$\Psi : K \times K \to \mathbb{S}^1\,, \quad (\xi, x) \mapsto \psi(\xi)(x) := \chi(\xi x)$$

is an isomorphism of the locally compact additive group of K with its own Pontryagin dual.

Proof. The map Ψ is the composition of χ with the multiplication of K. The latter is $\mathbb{Z}$-bilinear and continuous, and the former is $\mathbb{Z}$-linear and continuous, hence Ψ is $\mathbb{Z}$-bilinear and continuous. This implies that ψ is a morphism of locally compact abelian groups. The morphism ψ is a monomorphism: if $\xi \in K$ is non-zero then $\chi(\xi x) \neq 1$ for $x = \xi^{-1}y$, where y is an element of $K \setminus \ker\chi$, and therefore $\psi(\xi)$ is non-trivial. Let τ be the canonical isomorphism that identifies K with the Pontryagin dual of $\widehat{K}$. We have that $\psi = \psi^* \circ \tau$ because

$$\begin{aligned}
\psi^*(\tau(\xi))(x) &= \tau(\xi)(\psi(x)) \\
&= \psi(x)(\xi) \\
&= \chi(x\xi) \\
&= \chi(\xi x) \\
&= \psi(\xi)(x)\,.
\end{aligned}$$

The fact that ψ is a monomorphism implies that the dual morphism ψ^* is an epimorphism, but the identity $\psi = \psi^* \circ \tau$ ensures that ψ has the same property of ψ^*. This means in particular that the image $\psi(K)$ is dense in $\widehat{K}$. Now we prove that ψ is a homeomorphism from K to the image $\psi(K)$ considered as a topological sub-group of $\widehat{K}$. It is enough to check that $\psi^{-1} : \psi(K) \to K$ is continuous in the trivial character. The family of open balls $B(0, \varepsilon)$, for $\varepsilon > 0$, form a basis for the filter of neighbourhoods of $0 \in K$. We have to show that $V_\varepsilon := \psi\big(B(0, \varepsilon)\big)$ is a neighbourhood of the trivial character inside $\psi(K)$, namely that for all positive real ε there is a neighbourhood $W \subseteq \widehat{K}$ of the trivial character such that $W \cap \psi(K) \subseteq V_\varepsilon$. We search for a W of the form $W(C_m, U)$, where C_m is the compact set of elements $x \in K$ such that $|x| \leq m$, for $m > 0$, and U is an open neighbourhood of $1 \in \mathbb{S}^1$. The character χ is non-trivial by assumption, hence there is a $x_0 \in K$ such that $\chi(x_0) \neq 1$. Let U be small enough to satisfy $\chi(x_0) \notin U$ and m large enough to have $|x_0| m^{-1} \leq \varepsilon$. Take $\eta \in W(C_m, U) \cap \psi(K)$, so $\eta = \psi(\xi)$ for a unique $\xi \in K$ and $\eta(C_m) \subseteq U$. If η is the trivial character then the relation $\eta \in V_\varepsilon$ is obvious, so assume that η is not trivial, which means that $\xi \neq 0$. By definition $\eta(x) = \chi(\xi x)$ for all $x \in K$, hence the fact that $\chi(x_0) \notin U$ is equivalent to $\eta(x) \notin U$ if we define $x := \xi^{-1} x_0$. But the relation $\eta(x) \notin U$ implies that $x \notin C_m$ because $\eta(C_m) \subseteq U$, therefore $|x| > m$ and we have the following consequences:

$$
\begin{aligned}
\left|\xi^{-1} x_0\right| &> m && \text{because } x = \xi^{-1} x_0 \text{, then} \\
|\xi| &< \frac{|x_0|}{m} && \text{by algebraic manipulations, and} \\
|\xi| &< \varepsilon && \text{because } |x_0|\, m^{-1} < \varepsilon \text{ by construction of } m.
\end{aligned}
$$

This proves that $\eta \in V_\varepsilon$ and therefore $\psi(K)$ is a topological sub-group of $\widehat{K}$ isomorphic to K, both algebraically and topologically. This implies in particular that $\psi(K)$ is a locally compact sub-group of $\widehat{K}$ because K is locally compact, hence $\psi(K)$ must be a closed sub-group of $\widehat{K}$ by Proposition 2.1.3. Recall that the image of ψ is dense in $\widehat{K}$, the fact that it is also closed means that $\psi(K) = \widehat{K}$ and ψ is an isomorphism between K and its dual. □

Having already a canonical valuation on local fields, the next construction helps to find a canonical measure on them.

Remark 3.1.4. Since K is self-dual with respect to ψ, by Example 2.2.6, there is a canonical Haar measure μ on K associated to the character χ: the self-dual measure. Recall that μ is characterized by the property $\psi^\star \widehat{\mu} = \mu$ and the formula

$$
f(x) = \int_K \widehat{f}(\xi) \chi(\xi x) \, d\mu(\xi)
$$

valid for continuous, integrable functions f with integrable Fourier transform, where

$$
\widehat{f}(\xi) = \int_K f(x) \overline{\chi(\xi x)} \, d\mu(x) \, .
$$

Suppose that η is another non-trivial character of K, then there is a unique $a \in K^\times$ such that $\eta = \psi(a)$. One can ask how the self-dual measure changes if we exchange χ with η. Note that the character χ is recovered by the computation

$$
\chi(x) = \eta(a^{-1} x)
$$

for all $x \in K$, thus, the Fourier inversion formula associated to the character χ can be written in terms of the character η as

$$\begin{aligned}
f(x) &= \int_K \left[\int_K f(y)\overline{\chi(\xi y)}\, d\mu(y)\right] \chi(x\xi)\, d\mu(\xi) \\
&= \int_K \left[\int_K f(y)\overline{\eta(a^{-1}\xi y)}\, d\mu(y)\right] \eta(xa^{-1}\xi)\, d\mu(\xi) \\
&= \int_K \left[\int_K f(y)\overline{\eta(\omega y)}\, d\mu(y)\right] \eta(x\omega)\, d\mu(a\omega) && \text{by the substitution } \xi = a\omega \\
&= \int_K \left[\int_K f(y)\overline{\eta(\omega y)}\, d\mu(y)\right] \eta(x\omega) \cdot |a|\ d\mu(\omega) \\
&= \int_K \left[\int_K f(y)\overline{\eta(\omega y)} \cdot |a|^{\frac{1}{2}}\, d\mu(y)\right] \eta(x\omega) \cdot |a|^{\frac{1}{2}}\, d\mu(\omega)\,,
\end{aligned}$$

from which it can be inferred that $|a|^{\frac{1}{2}}\mu$ is the self-dual measure for the character η. On a general locally compact ring A which is isomorphic to its dual via a unitary character χ, we can repeat the same computation: if μ is the self-dual Haar measure for χ and $a \in A^\times$, then the self-dual measure for the character η defined by $\eta(x) = \chi(ax)$ for all $x \in A$ is $|a|^{\frac{1}{2}}\,\mu$, where $|\cdot|$ is the module of $A^\times$ acting on A.

Example 3.1.5 (case $K = \mathbb{R}$)**.** On the field of real numbers consider the usual measure dx that gives measure 1 to the interval $[0, 1]$ and the unitary character

$$e_\infty : \mathbb{R} \longrightarrow \mathbb{S}^1\,, \quad x \longmapsto e^{2\pi i x}\,.$$

From classical Fourier analysis, it can be deduced that the measure dx is self-dual with respect to the given character. Moreover, multiplication by a number $a \in \mathbb{R}^\times$ transforms the interval $[0, 1]$ to $[0, a]$ if $a > 0$ or to $[a, 0]$ if $a < 0$. In any case, it transforms a set of measure 1 to a set of measure $|a|_\infty$, thus the measure-theoretic module of $\mathbb{R}$ is the same as the usual absolute value.

Example 3.1.6 (case $K = \mathbb{C}$)**.** For the complex numbers, you can choose the character

$$\mathbb{C} \longrightarrow \mathbb{S}^1\,, \quad z \longmapsto e^{2\pi i(z+\overline{z})} = e_\infty(\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(z))$$

to give an isomorphism $\mathbb{C} \cong \widehat{\mathbb{C}}$. The self-dual measure for that identification is twice the usual Lebesgue measure of the plane and the module of z is $z\overline{z}$ for all $z \in \mathbb{C}$.

The above examples cover the cases of archimedean local fields. The next results deal with the non-archimedean case, starting with a lemma that describes characters for general groups that carry a totally disconnected topology like the non-archimedean local fields.

Lemma 3.1.7. *Let A be a* **LCA** *group with a basis of open neighbourhoods of the neutral element given by open sub-groups and suppose that $\chi : A \to \mathbb{C}^\times$ is a character. Then χ is locally constant with open kernel. If A is compact, then χ has finite image.*

Proof. Let U be an open neighbourhood of $1 \in \mathbb{C}^\times$ that does not contain sub-groups of the circle except for the trivial one, which is possible by Lemma 2.1.12. The pre-image of U by χ is an open neighbourhood of $0 \in A$, so there is an open sub-group B contained in it by hypothesis. The set

$\chi(B)$ is contained in U and is a sub-group of $\mathbb{C}^\times$, thus it must be trivial, i.e. χ factors through the quotient A/B. Any element $a \in A$ has an open neighbourhood $a + B$ on which χ is constant and equal to $\chi(a)$, so the character is locally constant. In particular, for a belonging to the kernel of χ we get an open neighbourhood $a + B$ that is all contained in $\ker(\chi)$, implying that χ has open kernel. If A is compact, then A/B is finite because it is compact and discrete. The image of χ is a group isomorphic to A/B, hence it must be finite. □

Corollary 3.1.8. *Let K be a non-archimedean local field and ψ a unitary, additive character of K. Then ψ is locally constant.*

Proof. Let $\mathcal{O}$ be the local sub-ring of K and $\pi \in \mathcal{O}$ a uniformizer. Pick an element x_0 of K and let n be the unique integer for which $|x_0| = |\pi|^n$. The open sub-group $A = \pi^n\mathcal{O}$ is a compact, open, sub-group of the additive group of K isomorphic to $\mathcal{O}$, it contains x_0 and the restriction

$$\chi : A \to \mathbb{S}^1\,,\ x \mapsto \psi(x)$$

is an additive, unitary character of A. The group A has a basis of open sub-groups, thus χ is locally constant with an open kernel. Let U be the open neighbourhood $x_0 + \ker\chi$ of x_0, then χ is constant and equal to $\chi(x_0)$ on U. Since U is also open in K, we have that ψ is constant on an open neighbourhood of an arbitrary element x_0 of K, i.e. ψ is locally constant. □

Proposition 3.1.9. *Let K be a non-archimedean local field and let ψ be a non-trivial, unitary, additive character. Then the group $\mathcal{O}^\perp$ of elements $\xi \in K$ such that $\psi(\xi x) = 1$ for all $x \in \mathcal{O}$ is a fractional ideal of K and it is the maximal one contained in the kernel of ψ. Moreover, the self-dual measure of K with respect to the character ψ is the unique Haar measure μ such that*

$$\mu(\mathcal{O}) \cdot \mu(\mathcal{O}^\perp) = 1\,.$$

Proof. The fractional ideals of K are all of the form $\pi^n\mathcal{O}$ for some integer n, where π is a uniformizer of K. Note that $\mathcal{O}^\perp$ is an $\mathcal{O}$-module: if $\xi \in \mathcal{O}^\perp$ and $y \in \mathcal{O}$ then $\psi(y\xi x) = 1$ for all $x \in \mathcal{O}$ because $y\mathcal{O} \subseteq \mathcal{O}$. There must be some integer n such that $\pi^m \notin \mathcal{O}^\perp$ for all integers $m < n$, otherwise $\mathcal{O}^\perp$ would be the whole field K, a condition that contradicts the non-triviality of ψ. Thus the minimal integer n for which $\pi^n \in \mathcal{O}^\perp$ exists and $\pi^n\mathcal{O} \subseteq \mathcal{O}^\perp$ by $\mathcal{O}$-linearity. The locally compact group $K/\mathcal{O}^\perp$ is isomorphic to the Pontryagin dual of the compact group $\mathcal{O}$, hence the quotient is discrete and $\mathcal{O}^\perp$ must be open in K. On the other hand, $\mathcal{O}^\perp$ is isomorphic to the Pontryagin dual of the discrete group $K/\mathcal{O}$, so it must also be compact. The fact that $\mathcal{O}^\perp$ is an open sub-group allows the construction of the open cover

$$\mathcal{O}^\perp = \bigcup_{\xi \in \mathcal{O}^\perp} \xi + \pi^n\mathcal{O}\,.$$

The compactness of $\mathcal{O}^\perp$ ensures that a finite sub-cover

$$\mathcal{O}^\perp = \bigcup_{\xi \in F} \xi + \pi^n\mathcal{O}\,,$$

for some finite subset F of $\mathcal{O}^\perp$, can be extracted from the open cover. $\mathcal{O}$-linearity implies that $\mathcal{O}^\perp$ is generated by the finite set $F \cup \{\pi^n\}$ and the minimality of n ensures that $\pi^n\mathcal{O} = \mathcal{O}^\perp$.

Now consider the kernel of ψ and suppose that $\mathfrak{a}$ is a fractional ideal of K contained in $\ker\psi$. If a is an element of $\mathfrak{a}$, then ax belongs to the fractional ideal for all $x \in \mathcal{O}$. In particular, since

$\mathfrak{a} \subseteq \ker \psi$, the element a satisfies $\psi(ax) = 1$ for all $x \in \mathcal{O}$, which is the defining condition to be an element of $\mathcal{O}^\perp$.

The last part of the statement is a specialization of Lemma 2.4.4. □

Definition 3.1.10. Let χ be a non-trivial character of a non-archimedean local field K. The *conductor* of χ is the maximal fractional ideal of K on which χ is trivial.

By Proposition 3.1.9, the conductor of χ is precisely $\mathcal{O}^\perp := \{x \in K : \chi(xy) = 1 \text{ for all } y \in \mathcal{O}\}$. If $\mathcal{I}$ is a fractional ideal of K, then its dual $\mathcal{I}^\perp := \{x \in K : \chi(xy) = 1 \text{ for all } y \in \mathcal{I}\}$ is equal to $\mathcal{O}^\perp \mathcal{I}^{-1}$, as can be verified by a direct calculation on generators of the ideals. Note that, when χ has conductor equal to $\mathcal{O}$, the self-dual measure μ associated with the character χ must be the unique one that satisfies $\mu(\mathcal{O}) = 1$ by the condition on the self-dual measure expressed in the last part of the statement of Proposition 3.1.9.

Example 3.1.11 (case $K = \mathbb{Q}_p$). Consider the field $\mathbb{Q}_p$ of p-adic numbers. The inclusion

$$\mathbb{Z}[p^{-1}] \hookrightarrow \mathbb{Q}_p$$

becomes surjective on the quotient $\mathbb{Q}_p/\mathbb{Z}_p$ and it has kernel $\mathbb{Z}$, so that

$$\frac{\mathbb{Z}[p^{-1}]}{\mathbb{Z}} \cong \frac{\mathbb{Q}_p}{\mathbb{Z}_p} .$$

Since the $\mathbb{Z}[p^{-1}]/\mathbb{Z}$ is naturally a sub-group of $\mathbb{R}/\mathbb{Z}$, we get a group homomorphism

$$e_p : \mathbb{Q}_p \longrightarrow \mathbb{S}^1$$

with kernel $\mathbb{Z}_p$ by composition with the isomorphism

$$\mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{S}^1 , \quad t + \mathbb{Z} \longmapsto e^{2\pi i t} .$$

It's easy to see that e_p is locally constant since, by definition, it is constant on the open set $x + \mathbb{Z}_p$ for any p-adic number x. The orthogonal $\mathbb{Z}_p^\perp$ consists of the elements $\xi \in \mathbb{Q}_p$ such that $e_p(\xi x) = 1$ for all $x \in \mathbb{Z}_p$. By construction of e_p, the latter condition is equivalent to $\xi\mathbb{Z}_p \subseteq \mathbb{Z}_p$ and this is true if and only if ξ is an element of $\mathbb{Z}_p$, so $\mathbb{Z}_p^\perp = \mathbb{Z}_p$. This fact makes clear that the self-dual measure of $\mathbb{Q}_p$ with respect to the character e_p must be the one for which $\mathbb{Z}_p$ has measure 1.

In this last example, we have the special property $\mathcal{O}^\perp = \mathcal{O}$, but this is not always true, as we can change the character (hence the group $\mathcal{O}^\perp$) by multiplications of non-zero elements of the field. Moreover, if K is a finite extension of $\mathbb{Q}_p$, it can happen that $\mathcal{O}^\perp \neq \mathcal{O}$ for any choice character, but at least it's possible to link the desirable property with the ramification of the extension $K/\mathbb{Q}_p$. Indeed the trace $\mathrm{Tr}_{K/\mathbb{Q}_p} : K \to \mathbb{Q}_p$ induces a non-degenerate bilinear form on K and the composition $\psi := e_p \circ \mathrm{Tr}_{K/\mathbb{Q}_p}$ defines a non-trivial, unitary character of K. An element $x \in K$ belongs to the conductor of ψ if and only if $e_p(\mathrm{Tr}_{K/\mathbb{Q}_p}(xy)) = 1$ for all $y \in \mathcal{O}$. Recall that the kernel of e_p is equal to $\mathbb{Z}_p$, therefore the conductor is equal to the set

$$\left\{ x \in K : \mathrm{Tr}_{K/\mathbb{Q}_p}(xy) \in \mathbb{Z}_p \text{ for all } y \in \mathcal{O} \right\} ,$$

which is, by definition, the inverse of the *different ideal* relative to the extension $K/\mathbb{Q}_p$. The theory of the different is treated, for example, in the book of Lang ([Lan70], Chapter III). The most important property for our current purpose is that the different of $K/\mathbb{Q}_p$ is equal to $\mathcal{O}$ if and only if the extension is unramified, which implies that $\mathcal{O} = \mathcal{O}^\perp$ in the unramified case. We conclude this section with the case of a local field of finite characteristic, which by the classification theorem is isomorphic to a field of Laurent series over a finite field.

Example 3.1.12 (Case $K = \mathbb{F}_q((T))$)**.** In this case, the local ring $\mathcal{O}$ of K is the ring of formal power series $\mathbb{F}_q[[T]]$. The residue map

$$\varrho : K \longrightarrow \mathbb{F}_q$$

that sands a series $f(T) = \sum_n a_n T^n$ to the coefficient a_{-1} is $\mathbb{F}_q$-linear and vanishes on $\mathbb{F}_q[[T]]$. Choose a character χ of the additive group of $\mathbb{F}_q$ that is non-trivial on $1 \in \mathbb{F}_q$, then $\psi := \chi \circ \varrho$ is a non-trivial unitary character of K that vanishes on the local ring. Let's compute the conductor of ψ. Take a series $f \in K$ with expansion $f(T) = \sum_n a_n T^n$ and suppose that $\psi(fg) = 1$ for all $g \in \mathbb{F}_q[[T]]$, in particular, $\psi(aT^n f(T)) = 1$ for all $a \in \mathbb{F}_q$ and all positive integer n. If there is an integer $n < 0$ such that the coefficient a_n in the expansion of f is non-zero then the series $a_n^{-1} T^{-1-n} f(T)$ would have residue 1, but this violates the condition $\psi(a_n^{-1} T^{-1-n} f(T)) = 1$. So it must be that $a_n = 0$ for all integers $n < 0$, which means that $f(T) \in \mathbb{F}_q[[T]]$. The calculation made shows that $\mathcal{O} = \mathcal{O}^\perp$ for the character ψ, therefore the self-dual measure of K is the unique one that gives measure 1 to the local ring.

3.2 Adèles of a global field

In the following section, we denote global fields by k and K. The symbol ν is used for places, the set of places being $\mathscr{P}$ or $\mathscr{P}_k, \mathscr{P}_K$ if it's necessary to highlight the field. k_ν denotes the completion with respect to the place ν. The symbol $\mathcal{O}_\nu$ indicates the valuation subring of k_ν when ν is non-archimedean, and π_ν is a uniformizer, namely a generator of the unique maximal ideal of $\mathcal{O}_\nu$. Let S vary in the family of finite sets of places of k that contain the set S_∞ of infinite places.

Definition 3.2.1. The ring of *adèles* of a global field k is the restricted direct product of the local fields k_ν for the compact and open sub-rings $\mathcal{O}_\nu$. It is denoted by $\mathbb{A}$ or $\mathbb{A}_k$ to specify the field, its elements are called *adèles*.

For the moment the definition of the adèles tells us only that we can collect together all the completions of the global field in a way that produces an object compatible with the theory of locally compact groups. In what follows we will progressively see how deep the link with the arithmetic of k is. We start by showing that $\mathbb{A}$ becomes an algebra over the global k through a diagonal embedding $k \to \mathbb{A}$ that is discrete and co-compact, meaning that the quotient $\mathbb{A}/k$ is compact. Once the embedding $k \hookrightarrow \mathbb{A}$ is established we will consider k as if it is a subset of its ring of adèles and we use normal letters like $a, b, \dots$ and $x, y, \dots$ to indicate elements of $\mathbb{A}$. If x is an adèle we use the subscript x_ν to indicate the component of x corresponding to the place ν. If S is a finite set of places containing the infinite ones, we denote by $\mathbb{A}_S$ or $\mathbb{A}_{k,S}$ the open sub-ring of adèles x subject to the condition $x_\nu \in \mathcal{O}_\nu$ for all places $\nu \notin S$.

We start by analysing the simple case of the field of rational numbers.

Example 3.2.2 (Case $k = \mathbb{Q}$)**.** Let x be a rational number. Since it is a quotient of two integers, the set of primes p for which $|x|_p > 1$ is finite, so x lies in the ring of p-adic integers for almost all primes p. With this in mind, it's easy to see that the diagonal map $\mathbb{Q} \to \mathbb{A}$ that sends a rational number x to the adèle $(x_\nu)_\nu$ defined by $x_\nu = x$ for all places ν is a well-defined homomorphism of rings. If we define the open set V as the set of adèles x such that for all places ν

$$\begin{cases} |x_\nu|_\nu \leq 1, & \text{if } \nu \text{ is finite,} \\ |x_\nu|_\nu < 1, & \text{if } \nu = \infty, \end{cases}$$

then $V \cap \mathbb{Q} = 0$, because a rational number has p-adic absolute bounded by 1 for all primes p if and only if it is an integer, and 0 is the only integer with archimedean absolute value strictly less than one. So for any $x \in \mathbb{Q}$ the open set $x + V$ meets $\mathbb{Q}$ only in x, thus the rationals form a discrete sub-group of $\mathbb{A}$. The additive group of $\mathbb{Q}$ with the discrete topology is a locally compact group and the embedding $\mathbb{Q} \to \mathbb{A}$ is a monomorphism in the category of locally compact groups. Its quotient $\mathbb{A}/\mathbb{Q}$ is again locally compact and now we prove that it is compact. The closure of $\overline{V}$ of V is the compact subset of $\mathbb{A}$ isomorphic to the product of the closed interval $[-1, 1]$ with $\prod_p \mathbb{Z}_p$. The image of $\overline{V}$ through the projection $\mathbb{A} \to \mathbb{A}/\mathbb{Q}$ is compact and if we can show that any adèles has a representative in $\overline{V}$ modulo $\mathbb{Q}$ we obtain the compactness of the quotient. Take an adèle x, it must belong to $\mathbb{A}_S$ for some finite set of places S that contains ∞. For each prime $p \in S \setminus \{\infty\}$ we have that x_p is has a p-adic expansion

$$x_p = \sum_{n \geq v(x_p)} a_n(x_p) p^n \,,$$

where $v(x_p)$ is a possibly negative integer depending on the p-adic valuation of x_p and $a_n(x_p) \in \{0, 1, \ldots, p\}$ for all n. We can form the rational number

$$y := \sum_{p \in S, p \neq \infty} y(p) \,,$$

where $y(p) := \sum_{v(x_p) \leq n < 0} a_n(x_p) p^n$ is the representative of x_p in $\mathbb{Z}[p^{-1}]$ through the surjective homomorphism

$$\mathbb{Z}[p^{-1}] \to \mathbb{Q}_p/\mathbb{Z}_p \,.$$

Not that $y(\ell)$ is a rational number that belongs to $\mathbb{Z}_p$ for any prime ℓ different from p, therefore, adding or subtracting the number $y(\ell)$ to a p-adic number does not change its class in the quotient $\mathbb{Q}_p/\mathbb{Z}_p$. From this, we get that

$$\begin{aligned} x_p - y &\equiv x_p - y(p) && \pmod{\mathbb{Z}_p} \\ &\equiv \sum_{n \geq 0} a_n(x_p) p^n && \pmod{\mathbb{Z}_p} \\ &\equiv 0 && \pmod{\mathbb{Z}_p}. \end{aligned}$$

By the above calculations, we get that the adèle $x - y = (x_\nu - y)_\nu$ belongs to $\mathbb{A}_\infty$, i.e. it is an adèle whose components corresponding to finite places have absolute value bounded by one. So far we have shown that any element of the quotient $\mathbb{A}/\mathbb{Q}$ has a representative in the sub-ring $\mathbb{A}_\infty$. Note that $\mathbb{A}_\infty \cap \mathbb{Q} = \mathbb{Z}$ because any rational number with p-adic absolute value smaller or equal to 1, for all primes p, must be an integer. We can find an integer n such that $|x_\infty - n|_\infty \leq 1$, hence $x - y - n$ lies in $\overline{V}$.

Similar calculations lead to the compactness of $\mathbb{A}_k/k$ when $k = \mathbb{F}_q(T)$. In general, the diagonal embedding

$$k \longrightarrow \mathbb{A}_k\,, \quad x \longmapsto (\dots, x, x, x, \dots)$$

is well-defined because, given $x \in k$, the set of plaes ν such that $|x|_\nu \neq 1$ is finite. For that condition to hold it's enough that for all $x \in K$ the set of places ν such that $|x|_\nu > 1$ is finite, as it would apply to both x and x^{-1} when $x \neq 0$. The finitness condition can be deduced by observing that k is the fraction field of the ring of integers $\mathcal{O}_k$ and the finite places correspond to the valuations induced by prime ideals of $\mathcal{O}_k$, so that, for all non-zero $x \in k$, the valuation of x is strictly larger than 1 precisely at those places that correspond to the primes that divide the denominator of x in a given fraction representation $x = ab^{-1}$, with $a, b \in \mathcal{O}_k$. Otherwise, one can rely on the following simple argument: consider a finite extension of fields K/k and let $|\cdot|$ be a non-archimedean valuation of K. Suppose that $x \in K$ satisfies $|x| > 1$ and let $f(X) = a_0 + a_1X + \dots + a_{n-1}X^{n-1} + X^n$ be a monic polynomial over k such that $f(x) = 0$. Then $x^n = -(a_0 + a_1x + \dots + a_{n-1}x^{n-1})$ and we can estimate the absolute value of x as follows:

$$\begin{aligned} |x|^n &= \left|a_0 + a_1x + \dots + a_{n-1}x^{n-1}\right| \\ &\leq \sup_{0\leq i<n} |a_i||x|^i \\ &\leq \sup_{0\leq i<n} |a_i||x|^{n-1} \end{aligned}$$

where $|x|^i \leq |x|^{n-1}$ because $|x| > 1$. Therefore we obtain an estimate

$$|x| \leq \sup_{0\leq i<n} |a_i|\,,$$

which tells as that $|x| \leq 1$ if $|a_i| \leq 1$ for all $i = 0, \dots, n-1$. The estimate has the following implication: if k is a field such that any element $a \in k$ satisfies $|a|_\nu \leq 1$ for almost all non-archimedean places $\nu \in \mathscr{P}_k$ (like the fields $\mathbb{Q}$ or $\mathbb{F}_q(T)$), then the same property lifts to the bigger field K.

Now we return to the setting where K is a general global field with adèle ring $\mathbb{A}$. We know that $K \subset \mathbb{A}$ through the diagonal embedding, as observed above.

Theorem 3.2.3. *The quotient $\mathbb{A}/K$ is compact and K is discrete in $\mathbb{A}$.*

Proof. The theorem is true for the fields $\mathbb{Q}$ and $\mathbb{F}_q(T)$. One strategy is to rely on these two cases using the fact that K is a finite, separable extension of them. Let k be the field of rational numbers or $\mathbb{F}_q(T)$ and suppose that K/k is a finite, separable extension of degree n. The lemma 3.2.4 below provides an isomorphism $\mathbb{A}_K \cong \mathbb{A}_k^n$ that identifies K with the discrete sub-group $k^n \subset \mathbb{A}_k^n$ and the quotient $\mathbb{A}_K/K$ with the product of n copies of the quotient $\mathbb{A}_k/k$, which is compact. $\square$

Lemma 3.2.4. *Let K/k be a finite, separable extension of global fields of degree n. Then any basis $u_1, \dots, u_n$ of K as k-vector space induces an isomorphism*

$$\Upsilon_{\mathbb{A}} : \mathbb{A}_k^n \to \mathbb{A}_K\,, \quad (x^{(1)}, \dots, x^{(n)}) \mapsto \sum_{i=1}^{n} x^{(i)} u_i$$

that fits in the commutative square

$$\begin{array}{ccc} \mathbb{A}_k^n & \xrightarrow{\Upsilon_{\mathbb{A}}} & \mathbb{A}_K \\ \uparrow & & \uparrow \\ k^n & \xrightarrow{\Upsilon} & K \end{array}$$

where Υ *is the* k*-linear isomorphism induced by the basis.*

Proof. If S is a finite set of places of k, denote by S' the set of places ν' of K lying over some place $\nu \in S$. The lemma is true if the restriction of $\Upsilon_{\mathbb{A}}$ on $\mathbb{A}_{k,S}^n$ induces an isomorphism

$$\Upsilon_S : \mathbb{A}_{k,S}^n \to \mathbb{A}_{K,S'}\,, \quad (x^{(1)}, \ldots, x^{(n)}) \mapsto \sum_{i=1}^n x^{(i)} u_i$$

for S larger and larger. This is further based on the following: let ν be a place of k and consider the product

$$\prod_{\nu'|\nu} K_{\nu'}$$

as a k_ν-algebra through the diagonal embedding (recall that $K_{\nu'}$ is a field-extension of k_ν if $\nu'|\nu$). Then the k_ν-linear map

$$k_\nu^n \to \prod_{\nu'|\nu} K_{\nu'}\,, \quad (x^{(1)}, \ldots, x^{(n)}) \mapsto \sum_{i=1}^n x^{(i)} u_i$$

is an isomorphism of topological vector spaces and, if ν is outside a large enough finite set S of places, the above map defines an isomorphism

$$\mathcal{O}_\nu \cong \prod_{\nu'|\nu} \mathcal{O}_{\nu'}$$

of topological $\mathcal{O}_\nu$-modules. This fact is the content of Proposition 4-39 in [RV99] and can also be found in Chapter II, Section 10 of [CF67]. Given that, it is just a matter of grouping the places ν' of K by the condition of dividing the same place of k and we obtain the isomorphism Υ_S by the chain of isomorphisms

$$\begin{aligned} \mathbb{A}_k^n &\cong \prod_{\nu \in S} k_\nu^n \times \prod_{\nu \notin S} \mathcal{O}_\nu^n \\ &\cong \prod_{\nu \in S} \prod_{\nu'|\nu} K_{\nu'} \times \prod_{\nu \notin S} \prod_{\nu'|\nu} \mathcal{O}_{\nu'} \\ &\cong \prod_{\nu' \in S'} K_{\nu'} \times \prod_{\nu' \notin S'} \mathcal{O}_{\nu'} \\ &\cong \mathbb{A}_{K,S'} \end{aligned}$$

□

Remark 3.2.5. Note that if $\mathbb{A}$ is the ring of adèles of a global field K then, for all finite sets of places S containing the infinite ones, you can recover the ring of S-integers as the intersection

$$\mathbb{A}_S \cap K = \mathcal{O}_{K,S} \,,$$

which is discrete in $\mathbb{A}_S$. You can also prove in general that

$$\mathbb{A} = \mathbb{A}_S + K$$

which implies that the quotient $\mathbb{A}/K$ is the same as the quotient $\mathbb{A}_S/\mathcal{O}_{K,S}$. Using the latter representation and the split exact sequence

$$0 \longrightarrow \textstyle\prod_{\nu \notin S} \mathcal{O}_\nu \longrightarrow \mathbb{A}_S \longrightarrow \textstyle\prod_{\nu \in S} K_\nu \longrightarrow 0 \,,$$

it's not hard to prove that the diagonal embedding

$$\mathcal{O}_{K,S} \longrightarrow \prod_{\nu \in S} K_\nu \,, \quad x \longmapsto (\dots, x, x, x, \dots)$$

is discrete with compact quotient. For S equal to the set of archimedean places you recover the well-known fact that the ring of integers of a number field form a lattice in a real vector space by the map

$$\mathcal{O}_K \longrightarrow \prod_\rho \mathbb{R} \times \prod_\sigma \mathbb{C} \,, \quad x \longmapsto (\dots, \rho(x), \dots, \sigma(x), \dots) \,,$$

where ρ ranges over real embeddings of the number field and σ over the complex embeddings up to conjugation.

3.3 Pontryagin duality for the adèles

In Section 3.1 we saw that the local fields are identified with their Pontryagin dual through the choice of a non-trivial character. The ring of adèles of a global field K has the same property but there are more subtleties in the selection of the character. Theorem 2.4.7 tells us that, for the adèles, there is an isomorphism

$$\widehat{\mathbb{A}} \cong \prod_{\nu \in \mathscr{P}_K}{}' \widehat{K}_\nu \,,$$

where the group on the right is the restricted direct product of the duals of the local fields K_ν with respect to the sub-groups of characters trivial on the local ring $\mathcal{O}_\nu$. By this isomorphism, it is possible to express a character ψ of the adèles as a product $\otimes_\nu \psi_\nu$ of local characters, namely characters ψ_ν of K_ν that are trivial on $\mathcal{O}_\nu$ for almost all places ν. If $\Psi_\nu : K_\nu \to \widehat{K}_\nu$ is the isomorphism induced by a non-trivial character ψ_ν of K_ν then there is a further isomorphism

$$\prod_\nu{}' (K_\nu, \mathcal{O}_\nu^\perp) \longrightarrow \prod_\nu{}' (\widehat{K}_\nu, \widehat{K_\nu/\mathcal{O}_\nu}) \,, \quad (\xi_\nu)_\nu \longmapsto \big(\Psi_\nu(\xi_\nu)\big)_\nu \,,$$

where the sub-groups of definition have been highlighted in the notation for the restricted direct products. The isomorphism Ψ_ν identifies the sub-group $\mathcal{O}_\nu^\perp = \big\{\xi \in K_\nu : \psi_\nu(\xi x) = 1 \text{ for all } x \in \mathcal{O}_\nu\big\}$

with the group of characters of K_ν that are trivial on $\mathcal{O}_\nu$. Using all these identifications we get an isomorphism

$$\prod_\nu{}' (K_\nu, \mathcal{O}_\nu^\perp) \longrightarrow \widehat{\mathbb{A}}\,, \quad (\xi_\nu)_\nu \longmapsto \otimes_\nu \Psi_\nu(\xi_\nu) \tag{3.1}$$

that makes the diagram

$$\begin{array}{ccc} \prod_\nu{}' (K_\nu, \mathcal{O}_\nu^\perp) & \longrightarrow & \widehat{\mathbb{A}} \\ & \searrow & \uparrow \\ & & \prod_\nu{}' (\widehat{K}_\nu, \widehat{K_\nu/\mathcal{O}_\nu}) \end{array}$$

commute. Therefore, the dual of $\mathbb{A}$ can be $\mathbb{A}$ itself as long as the isomorphism is induced by a tuple of non-trivial characters $(\psi_\nu)_\nu \in \prod_\nu{}' (\widehat{K}_\nu, \widehat{K_\nu/\mathcal{O}_\nu})$ such that $\mathcal{O}_\nu^\perp = \mathcal{O}_\nu$ for almost all ν. In that case, the isomorphism $\mathbb{A} \cong \widehat{\mathbb{A}}$ is the one associated with the character $\psi := \otimes_\nu \psi_\nu$, since the character $\otimes_\nu \Psi_\nu(\xi_\nu)$ send an adèle x to the product $\prod_\nu \psi_\nu(\xi_\nu x_\nu)$, which is equal to $\psi(\xi x)$ if ξ is an adèle. The problem now is to find characters ψ_ν of the local fields K_ν such that $\mathcal{O}_\nu = \mathcal{O}_\nu^\perp$ for almost all ν, but such examples were given at the end of Section 3.1 and we can make it explicit in the following example.

Example 3.3.1 (Dualizing character of the adèles)**.** Let K be a global field and suppose that it is separable and of finite degree over k, where k is the field of rational numbers if K has characteristic zero, otherwise k is the field of rational polynomials over a finite field. Let ν be a place of K and suppose it lies over the place υ of k. Now we construct a character ψ_ν of K_ν that has conductor $\mathcal{O}_\nu$ when the extension K_ν/k_υ is unramified (recall that this happens for almost all places of k).

(i) Suppose that $k = \mathbb{Q}$. If υ is a finite place corresponding to the prime $p \in \mathbb{Z}$, then define ψ_ν as the composition $e_p \circ \mathrm{Tr}_{K_\nu/\mathbb{Q}_p}$, where e_p is defined in Example 3.1.11. If $\upsilon = \infty$ define ψ_ν as the composition $\overline{e}_\infty \circ \mathrm{Tr}_{K_\nu/\mathbb{R}}$, where $\overline{e}_\infty(t) = e^{-2\pi i t}$ for all $t \in \mathbb{R}$. Let S be the finite set of places of $\mathbb{Q}$ that consists of ∞ and all finite places υ such that the corresponding prime p_υ ramifies in the extension K. Let S' be the finite set of places ν of K such that $\nu | \upsilon$ for some $\upsilon \in S$. Then, for all places ν in the complement of S' we have $\mathcal{O}_\nu^\perp = \mathcal{O}_\nu$ with respect to the character ψ_ν.

(ii) Suppose that $k = \mathbb{F}_p(T)$. For the place ∞, the local field k_∞ is the field of Laurent series $\mathbb{F}_p((T^{-1}))$ in the variable T^{-1}. Mimicking Example 3.1.12, define a unitary characters φ_∞ of k_∞ as follows:

$$\varphi_\infty(f) = e^{\frac{2\pi i}{p} a_{-1}} \qquad \text{for every } f(T) = \sum_{n \geq \mathrm{ord}_\infty(f)} a_n T^{-n}.$$

If υ is a finite place of k, let f_υ be the corresponding irreducible polynomial of $\mathbb{F}_p[T]$ and d_υ its degree. For every element x of k_υ there is a unique $g(T) \in k$ of the form

$$g(T) = \left(\frac{T^{d_\upsilon}}{f_\upsilon(T)} \right)^n \cdot \left(a_1 T^{-1} + c_2 T^{-2} + \cdots + c_{n d_\upsilon} T^{-n d_\upsilon} \right)$$

such that $x - g(T) \in \mathcal{O}_\upsilon$. Define

$$\varphi_\upsilon(x) = e^{\frac{2\pi i}{p} a_1}\,.$$

Then φ_υ is a non-trivial unitary character of k_υ with conductor $\mathcal{O}_\upsilon$. For the bigger field K, define

$$\psi_\nu = \varphi_\upsilon \circ \mathrm{Tr}_{K_\nu/k_\upsilon}$$

for every place υ of k and every place ν over υ. Then ψ_ν is a non-trivial unitary character of K_ν and its conductor is $\mathcal{O}_\nu$ for almost all places ν (see exercises 5,6 of [RV99] at the end of Chapter 7).

Define $\psi = \otimes_\nu \psi_\nu$ to get a character of $\mathbb{A}$ that induces an isomorphism $\mathbb{A} \cong \widehat{\mathbb{A}}$.

We have chosen the character of Example 3.3.1 in this way because the choice of ψ influences the position of the dual lattice

$$K^\perp := \{\xi \in \mathbb{A} : \psi(\xi x) = 1 \text{ for all } x \in K\}$$

and we would like to have $K = K^\perp$ inside the adèle ring. The latter property is achievable if ψ is a character of $\mathbb{A}$ trivial on the field K, namely a character in the image of the monomorphism

$$\widehat{\mathbb{A}/K} \longrightarrow \widehat{\mathbb{A}}\,,$$

as we will show.

Proposition 3.3.2. *The character $\psi \in \widehat{\mathbb{A}}$ defined in Example 3.3.1 is trivial on the field K.*

Proof. Suppose that K is a number field. Let $x \in K$ and consider its trace $\mathrm{Tr}_{K/\mathbb{Q}}(x) \in \mathbb{Q}$. We claim that

$$\mathrm{Tr}_{K/\mathbb{Q}}(x) = \sum_{\nu|\upsilon} \mathrm{Tr}_{K_\nu/\mathbb{Q}_\upsilon}(x) \quad \text{for all places } \upsilon \text{ of } \mathbb{Q}.$$

Let υ be any place of $\mathbb{Q}$, to each place ν of K lying over υ it corresponds a unique G_υ-orbit O_ν of field-embeddings σ from K to an algebraic closure $\overline{\mathbb{Q}}_\upsilon$ of $\mathbb{Q}_\upsilon$, where G_υ is the Galois group of $\overline{\mathbb{Q}}_\upsilon/\mathbb{Q}_\upsilon$. If one chooses an embedding $\sigma_\nu : K_\nu \to \overline{\mathbb{Q}}_\upsilon$, then the G_υ-orbit corresponding to the place ν is generated by $\sigma_\nu|_K$. We have

$$\mathrm{Tr}_{K/\mathbb{Q}}(x) = \sum_{\sigma \in \mathrm{Hom}_\mathbb{Q}(K, \overline{\mathbb{Q}}_\upsilon)} \sigma(x)$$

because the trace of x relative to the extension $K/\mathbb{Q}$ is the sum of all the congiugates of x inside an algebraic closure of $\mathbb{Q}$, and the embeddings $\sigma \in \mathrm{Hom}_\mathbb{Q}(K, \overline{\mathbb{Q}}_\upsilon)$ are precisely the embeddings of K in the algebraic closure of $\mathbb{Q}$ inside $\overline{\mathbb{Q}}_\upsilon$. On the other side, we can express the set $\mathrm{Hom}_\mathbb{Q}(K, \overline{\mathbb{Q}}_\upsilon)$ as the disjoint union of its orbits, and each orbit is made by the conjugates of $\sigma_\nu(x)$ inside $\overline{\mathbb{Q}}_\upsilon$, whose sum compute the trace of x relative to the extension of local fields $K_\nu/\mathbb{Q}_\upsilon$. Having observed this, we get

$$\begin{aligned}\mathrm{Tr}_{K/\mathbb{Q}}(x) &= \sum_{\sigma \in \mathrm{Hom}_\mathbb{Q}(K, \overline{\mathbb{Q}}_\upsilon)} \sigma(x) \\ &= \sum_{\nu|\upsilon} \sum_{\sigma \in O_\nu} \sigma(x) \\ &= \sum_{\nu|\upsilon} \mathrm{Tr}_{K_\nu/\mathbb{Q}_\upsilon}(x)\,.\end{aligned}$$

The consequence of the claim is that for all primes p

$$\begin{aligned}\prod_{\nu|p}\psi_\nu(x) &= \prod_{\nu|p} e_p\big(\operatorname{Tr}_{K_\nu/\mathbb{Q}_p}(x)\big)\\ &= e_p\left(\sum_{\nu|p}\operatorname{Tr}_{K_\nu/\mathbb{Q}_p}(x)\right)\\ &= e_p\big(\operatorname{Tr}_{K/\mathbb{Q}}(x)\big)\end{aligned}$$

and for the infinite place ∞

$$\begin{aligned}\prod_{\nu|\infty}\psi_\nu(x) &= \prod_{\nu|\infty} e_\infty\big(-\operatorname{Tr}_{K_\nu/\mathbb{R}}(x)\big)\\ &= e_\infty\left(-\sum_{\nu|\infty}\operatorname{Tr}_{K_\nu/\mathbb{R}}(x)\right)\\ &= e_\infty\big(-\operatorname{Tr}_{K/\mathbb{Q}}(x)\big),\end{aligned}$$

thus we can compute $\psi(x)$ in the following way:

$$\begin{aligned}\psi(x) &= \prod_{\nu\in\mathscr{P}_K}\psi_\nu(x)\\ &= \prod_{\upsilon\in\mathscr{P}_\mathbb{Q}}\prod_{\nu|\upsilon}\psi_\nu(x)\\ &= e_\infty\big(-\operatorname{Tr}_{K/\mathbb{Q}}(x)\big)\cdot\prod_{p\text{ prime}} e_p\big(\operatorname{Tr}_{K/\mathbb{Q}}(x)\big).\end{aligned}$$

The computation above makes clear that, if the statement is true for $K=\mathbb{Q}$ then it is true for all number fields, so now we proceed to check that

$$e_\infty(-t)\cdot\prod_{p\text{ prime}} e_p(t) = 1 \quad \text{for all } t\in\mathbb{Q}.$$

The equality is trivially satisfied if $t=0$, hence suppose that t is a non-zero rational and S the finite set primes p such that $|t|_p\neq 1$, so that the product over all primes reduces to a product over the primes of S. Recall that, by definition, $e_p(t)=e^{2\pi i t_p}$, where t_p is a rational contained in the ring $\mathbb{Z}[p^{-1}]$ such that $\big|t-t_p\big|_p\le 1$, thus

$$\begin{aligned}e_\infty(-t)\cdot\prod_{p\text{ prime}} e_p(t) &= e^{-2\pi i t}\cdot\prod_{p\in S} e^{2\pi i t_p}\\ &= e^{2\pi i(-t+\sum_{p\in S} t_p)}.\end{aligned}$$

We just have to verify that the rational number

$$z := -t + \sum_{p\in S} t_p$$

is an integer. For all prime ℓ we can bound the ℓ-adic absolute value of z as follows:

$$|z|_\ell \le \max\left\{\max_{p\in S, p\neq \ell}\left|t_p\right|_\ell \,,\, |t_\ell - t|_\ell\right\} \quad \text{for } \ell \in S\,,$$

$$|z|_\ell \le \max\left\{\max_{p\in S}\left|t_p\right|_\ell \,,\, |t|_\ell\right\} \quad \text{for } \ell \notin S\,.$$

If $\ell \in S$ then $\left|t_p\right|_\ell \le 1$ for $p \neq \ell$ because $t_p \in \mathbb{Z}[p^{-1}]$ and $|t_\ell - t|_\ell \le 1$ by definition of t_ℓ. If $\ell \notin S$ then $\left|t_p\right|_\ell \le 1$ for all $p \in S$ and $|t|_\ell \le 1$ by definition of S. Putting together both cases we have that $|z|_\ell \le 1$ for all primes ℓ, so z is an integer.

The proof for function fields is analogous (see Theorem 3 of [WW74], Chapter 4, §2). □

Theorem 3.3.3. *Let ψ be a non-trivial character of the adèles $\mathbb{A}$ that is trivial on the global field K. Then, ψ induces an isomorphism $\mathbb{A} \cong \widehat{\mathbb{A}}$ for wich $K^\perp = K$, i.e. for any other character $\chi \in \widehat{\mathbb{A}}$ trivial on K there exists a unique $\xi \in K$ such that $\chi(x) = \psi(\xi x)$ for all $x \in \mathbb{A}$.*

Proof. We begin by proving that if ψ induces an isomorphism $\mathbb{A} \cong \widehat{\mathbb{A}}$ then $K^\perp = K$. For all $\xi \in K$ the product ξx is contained in K for all $x \in K$, thus $K \subseteq K^\perp$ and $K^\perp$ is a K vector space. The isomorphism $\mathbb{A} \cong \widehat{\mathbb{A}}$ identifies $K^\perp$ with the group $\widehat{\mathbb{A}/K}$, when we consider the latter as a subgroup of $\mathbb{A}$. The quotient $\mathbb{A}/K$ is compact, hence its Pontryagin dual is discrete and therefore $K^\perp$ is discrete too. The quotient $K^\perp/K$ is a discrete sub-group of the compact group $\mathbb{A}/K$ and discrete sub-groups of compact groups are finite. Since the cardinality of K is infinite, there are no non-trivial K-vector spaces of finite cardinality. The quotient $K^\perp/K$ is a finite K-vector space, thus it must be trivial and this is equivalent to $K = K^\perp$. Now we prove that ψ induces an isomorphism $\mathbb{A} \cong \widehat{\mathbb{A}}$. Let $\tilde{\psi}$ be the character defined in the example 3.3.1, it is trivial on K and it induces a duality

$$\mathbb{A} \times \mathbb{A} \longrightarrow \mathbb{S}^1\,, \quad (\xi, x) \longmapsto \tilde{\psi}(\xi x)\,,$$

so there is a unique $\xi \in K$ such that $\psi(x) = \tilde{\psi}(\xi x)$ for all $x \in \mathbb{A}$. The element ξ must be non-zero because ψ is non-trivial, therefore $\xi \in \mathcal{O}_\nu^\times \cap K$ for almost all places ν of K. This last fact implies that the local character ψ_ν induces an identification $K_\nu \cong \widehat{K}_\nu$ for which $\mathcal{O}_\nu = \mathcal{O}_\nu^\perp$ for almost all places ν, because ψ_ν is obtained by $\tilde{\psi}_\nu$ via multiplication by ξ, an operation that does not change the conductor if $\xi \in \mathcal{O}^\times$, and the conductor of $\tilde{\psi}_\nu$ is equal, by construction, to the local ring for almost all places ν. Then ψ is a character of $\mathbb{A}$ that meets all the requirements for inducing an isomorphism $\mathbb{A} \cong \widehat{\mathbb{A}}$. □

The position of the global field K inside its ring of adèles $\mathbb{A}$ is fundamental for $\mathbb{A}$ to be more than just a collection of local data, moreover, the existence of the discrete and co-compact embedding $K \hookrightarrow \mathbb{A}$ it's almost enough to characterize both the global field and its adèle ring. In [Iwa53], Iwasawa has shown that, if a commutative topological ring R satisfies the following properties:

(i) R is semi-simple;

(ii) R is locally compact, non-compact, non-discrete;

(iii) R contains a discrete sub-field K such that the quotient R/K is compact;

then K is a global field and R is isomorphic to $\mathbb{A}_K$ as a K-algebra and as a topological ring.

3.4 The group of Idèles

In the previous section, we studied mainly the additive structure of the adèles and local fields, but the multiplicative group of such rings encodes important arithmetic properties of the global field. We know that the multiplicative group of a valued field has the sub-space topology, but this is not true for general topological rings. Let K be a global field, ν a place, K_ν the corresponding local field with local ring $\mathcal{O}_\nu$ and maximal ideal $\mathfrak{p}_\nu$ if ν is non-archimedean. The group of units $\mathcal{O}_\nu^\times$ of K_ν is an open and compact sub-group of $K_\nu^\times$ with a basis of compact open neighbourhoods of 1 given by the multiplicative sub-groups $1 + \mathfrak{p}_\nu^n$ for n natural number. We generally have the following description of the multiplicative group of local fields given by the image and the kernel of the module $|\cdot|_\nu$:

(i) The real numbers case: $\mathbb{R}^\times \cong \{\pm 1\} \times \mathbb{R}_+^\times$ and the group of positive real numbers $\mathbb{R}_+^\times$ is isomorphic to the additive group of $\mathbb{R}$ via the logarithm;

(ii) The complex numbers case: $\mathbb{C}^\times \cong \mathbb{S}^1 \times \mathbb{R}_+^\times$ via representation of a non-zero complex number in polar coordinates;

(iii) K_ν non-archimedean: $K_\nu^\times \cong \mathcal{O}_\nu^\times \times \mathbb{Z}$ where the factor $\mathbb{Z}$ comes from the fact that the image of $|\cdot|_\nu$ is a discrete free sub-group of rank 1 in the positive real line.

Recall that the functor that computes the group of units of rings is co-represented by the ring $\mathbb{Z}[X, Y]/(XY - 1)$ which is finitely presented, this means that the functor commutes with filtered colimits in addition to arbitrary limits of rings. The restricted direct product is a filtered colimit of a product of rings by definition, hence the group of units of the ring of adèles $\mathbb{A}$ is naturally identified with the restricted direct products of the groups $K_\nu^\times$ with respect to the sub-groups $\mathcal{O}_\nu^\times$. The identification

$$\mathbb{A}^\times \cong \prod_\nu{}' K_\nu^\times$$

is a priori algebraic, but $K_\nu^\times$ is a locally compact abelian group and $\mathcal{O}_\nu^\times$ is open and compact inside $K_\nu^\times$, therefore it makes sense to consider the topology of restricted direct products on the group $\mathbb{A}^\times$.

Definition 3.4.1. Let K be a global field and $\mathbb{A}$ its ring of adèles. The *group of idèles* of K is the group of invertible elements $\mathbb{A}^\times$ of the adèle ring. The structure of restricted direct product induces the topology of $\mathbb{A}^\times$. The invertible elements $x \in K$ viewed as idèles by the diagonal embedding are called *principal idèles.*

In this way, the group of idèles is a locally compact abelian group. Its topology is also the weakest that makes the map

$$\mathbb{A}^\times \longrightarrow \mathbb{A} \times \mathbb{A}\,, \quad x \longmapsto (x, x^{-1})$$

continuous, and it is different from the sub-space topology induced by the inclusion $\mathbb{A}^\times \subset \mathbb{A}$. If S is a finite set of places that contains S_∞, define the sub-group

$$(\mathbb{A}^\times)_S := \left\{x \in \mathbb{A}^\times : x_\nu \in \mathcal{O}_\nu^\times \text{ for all } \nu \notin S\right\}.$$

which is algebraically and topologically isomorphic to the product

$$\prod_{\nu \in S} K_\nu^\times \times \prod_{\nu \notin S} \mathcal{O}_\nu^\times\,.$$

Note that $(\mathbb{A}_S)^\times = (\mathbb{A}^\times)_S$, where on the left-hand side we have the group of units of the ring $\mathbb{A}_S$, so we can simplify the notation and write $\mathbb{A}_S^\times$ to indicate both. As a restricted direct product, the idèle group carries a Haar measure induced by a family of Haar measures $d^\times x_\nu$ defined on each group $K_\nu^\times$ in a way that the group of units $\mathcal{O}_\nu^\times$ gets measure 1 for almost all places ν. The multiplicative Haar measure of $K_\nu^\times$ can be constructed using an additive Haar measure dx_ν of the local field: define the multiplicative measure by $d^\times x_\nu := |x_\nu|_\nu^{-1}\, dx_\nu$, where this means that the integral of a continuous function f with compact support in $K_\nu^\times$ is

$$\int_{K_\nu^\times} f(x_\nu)\, d^\times x_\nu = \int_{K_\nu} f(x_\nu) \cdot |x_\nu|_\nu^{-1}\, dx_\nu \,.$$

The measure $d^\times x_\nu$ as a functional of the space $\mathscr{C}_c(K_\nu^\times)$ is given by the composition of

$$\mathscr{C}_c(K_\nu) \longrightarrow \mathbb{R}\,, \quad g(x_\nu) \longmapsto \int_{K_\nu} g(x_\nu)\, dx_\nu$$

after

$$\mathscr{C}_c(K_\nu^\times) \longrightarrow \mathscr{C}_c(K_\nu)\,, \quad f(x_\nu) \longmapsto f(x_\nu) \cdot |x_\nu|_\nu^{-1}\ ,$$

where x_ν is considered as the variable of the functions defined on the local field K_ν. The fact that for all $a \in K_\nu^\times$ the additive measure transforms by the rule $d(ax_\nu) = |a|_\nu\, dx_\nu$ ensures that $d^\times x_\nu$ is invariant by the group operation of $K_\nu^\times$. The additive measure of K_ν is fixed by the choice of a character $\psi \in \widehat{\mathbb{A}/K}$. To make the equality

$$\int_{\mathcal{O}_\nu^\times} d^\times x_\nu = 1$$

true for almost all places it is sufficient to rescale the multiplicative Haar measure: set

$$d^\times x_\nu := m_\nu \frac{dx_\nu}{|x_\nu|_\nu}\,,$$

where

$$\begin{cases} m_\nu = 1 & \text{for } \nu \text{ archimedean,} \\ m_\nu = \left(1 - q_\nu^{-1}\right)^{-1} & \text{for } \nu \text{ non-archimedean,} \end{cases}$$

and q_ν is the number of elements of the residue field of $\mathcal{O}_\nu$. The reason behind this choice is that for almost all places ν the additive measure of $\mathcal{O}_\nu$ is 1 and therefore the measure of $\mathcal{O}_\nu^\times$ is 1 minus the measure of the maximal ideal $\mathfrak{p}_\nu$ of $\mathcal{O}_\nu$, as $\mathcal{O}_\nu^\times$ is equal to the complement of the maximal ideal inside the local ring. The measure of $\mathfrak{p}_\nu$ is obtained by the fact that $\mathcal{O}_\nu$ is the union of q_ν cosets of $\mathfrak{p}_\nu$, implying that the measure of the local ring is q_ν times the measure of $\mathfrak{p}_\nu$. The measure of the idèles is the restricted direct product $d^\times x$ of the multiplicative measures $d^\times x_\nu$. The next discussion is about the module

$$|\cdot| : \mathbb{A}^\times \longrightarrow \mathbb{R}_+^\times\,,$$

also called *idèlic norm.*

Proposition 3.4.2. *The module of an idèle $a \in \mathbb{A}^\times$ is computed as the product of the local absolute values, i.e.*

$$|a| = \prod_\nu |a_\nu|_\nu\ ,$$

where $|a_\nu|_\nu = 1$ for almost all places.

Proof. The fact that $|a_\nu|_\nu = 1$ for almost all places is obvious from the definition of idèles, as a_ν must be a unit of $\mathcal{O}_\nu$ for almost all places ν and the group of units of the local ring is the kernel of $|\cdot|_\nu$. Let $f = \otimes_\nu f_\nu$ be a factorizable function of $\mathbb{A}$ which is also continuous, positive and integrable. On one side we have that

$$\int_{\mathbb{A}} f(a^{-1}x)\,dx = |a| \int_{\mathbb{A}} f(x)\,dx\,,$$

on the other

$$\begin{aligned}\int_{\mathbb{A}} f(a^{-1}x)\,dx &= \prod_\nu \int_{K_\nu} f_\nu(a_\nu^{-1}x_\nu)\,dx_\nu \\ &= \prod_\nu \left[|a_\nu|_\nu \cdot \int_{K_\nu} f_\nu(x_\nu)\,dx_\nu \right] \\ &= \left[\prod_\nu |a_\nu|_\nu \right] \cdot \int_{\mathbb{A}} f(x)\,dx\,.\end{aligned}$$

By comparing the two ways in which the integral of the function $x \mapsto f(a^{-1}x)$ is calculated we get

$$|a| \int_{\mathbb{A}} f(x)\,dx = \left[\prod_\nu |a_\nu|_\nu \right] \cdot \int_{\mathbb{A}} f(x)\,dx$$

and the fact that f has a non-zero integral concludes the proof. □

From the way the idèlic norm is related to the local absolute values, it's easy to see that it is a continuous homomorphism: by definition of restricted product topology, it is enough that $|\cdot|$ is continuous when restricted to $\mathbb{A}_S^\times$ for all S. For all $x \in \mathbb{A}_S^\times$, the module $|x|$ only depends on the components x_ν of x for $\nu \in S$, precisely

$$|x| = \prod_{\nu \in S} |x_\nu|_\nu\ ,$$

therefore we get the map $|\cdot|$ from the following composition of continuous maps:

$$\mathbb{A}_S^\times \longrightarrow \prod_{\nu \in S} K_\nu^\times\,, \quad x \longmapsto (x_\nu)_{\nu \in S}\,; \tag{3.2}$$

$$\prod_{\nu \in S} K_\nu^\times \longrightarrow \left(\mathbb{R}_+^\times\right)^S, \quad (x_\nu)_{\nu \in S} \longmapsto \left(|x_\nu|_\nu\right)_{\nu \in S}\,; \tag{3.3}$$

$$\left(\mathbb{R}_+^\times\right)^S \longrightarrow \mathbb{R}_+^\times\,, \quad (t_\nu)_{\nu \in S} \longmapsto \prod_{\nu \in S} t_\nu\,. \tag{3.4}$$

The map (3.2) is continuous because $\mathbb{A}_S^\times$ has the product topology, the map (3.3) is continuous because the valuation $|\cdot|_\nu$ is continuous on the local field K_ν and the map (3.4) is simply the multiplication of $\mathbb{R}$ which is continuous. The kernel of the idèlic norm is a closed sub-group of the idèles denoted by $\mathbb{A}^{\times,1}$, its elements are said *idèles of norm 1* and it plays a special role: the relation between the groups $K^\times$ and $\mathbb{A}^{\times,1}$ is a multiplicative analogue of the relation between K and its adèles.

Proposition 3.4.3. *Let K be a global field, $\mathbb{A}^\times$ its group of idèles and $\left|\mathbb{A}^\times\right|$ the image of the idèlic norm. The following holds:*

(i) *if K is a number field, then $\left|\mathbb{A}^\times\right| = \mathbb{R}_+^\times$, i.e. the idèlic norm is surjective;*

(ii) *if K is a function field, then $\left|\mathbb{A}^\times\right|$ is a free sub-group of $\mathbb{R}_+^\times$ of rank 1.*

Proof. If K is a number field, then there is a place ν of K such that $K_\nu = \mathbb{R}$ or $K_\nu = \mathbb{C}$. The group $K_\nu^\times$ is mapped canonically into the group of idèles and the idèlic norm on it is equal to the absolute value $|\cdot|_\nu$, which is surjective in the archimedean case. If K is a function field then it is an algebra over a finite field $\mathbb{F}_q$. Any local field K_ν is non-archimedean and it has a uniformizer π_ν. The value $|\pi_\nu|_\nu$ is the inverse of the cardinality of the residue field $\kappa_\nu := \mathcal{O}_\nu/\pi_\nu\mathcal{O}_\nu$. The latter is a finite field over $\mathbb{F}_q$, hence it has cardinality q^{d_ν}, where d_ν is the degree of the extension $\kappa_\nu/\mathbb{F}_q$. The image of the local valuation $|\cdot|_\nu$ is free and generated by q^{-d_ν}. By Proposition 3.4.2, the image of the idèlic norm is generated by $\left\{q^{-d_\nu} : \nu \in \mathscr{P}_K\right\}$ which is contained in the free sub-group $\left\{q^{-n} : n \in \mathbb{Z}\right\}$ isomorphic to $\mathbb{Z}$. Since all sub-groups of $\mathbb{Z}$ are free of rank 1, there must be a minimal positive integer d such that $\left|\mathbb{A}^\times\right|$ is generated by q^{-d}. □

Proposition 3.4.4. *There is a non-canonical isomorphism*

$$\mathbb{A}^\times \cong \left|\mathbb{A}^\times\right| \times \mathbb{A}^{\times,1}$$

of locally compact abelian groups. The isomorphism is induced by the choice of a continuous section of the idèlic norm, namely a morphism $\rho : \left|\mathbb{A}^\times\right| \to \mathbb{A}^\times$ of locally compact abelian groups satisfying $\left|\rho(t)\right| = t$ for all $t \in \left|\mathbb{A}^\times\right|$.

Proof. We first show that there are continuous sections of the idèlic norm. If the global field K is a function field over $\mathbb{F}_q$, then Proposition 3.4.3 implies that the idèlic norm takes values in the sub-group of the positive real numbers generated by $t := |\pi_\nu|_\nu$ for some place ν, where π_ν is a uniformizer of the local field K_ν. The map

$$\log_t|\cdot| : \mathbb{A}^\times \longrightarrow \mathbb{Z}\,, \quad x \longmapsto \log_t|x|$$

is a continuous, surjective homomorphism onto the discrete group $\mathbb{Z}$. This always has a section and it is trivially continuous. A section ρ is given by setting $\rho(t) = \iota_\nu(\pi_\nu)$, where ι_ν is the obvious map from the local field K_ν to the adèles of K. If K is a number field, then the idèlic norm is surjective, as stated in Proposition 3.4.3. Chose an archimedean place $\tilde{\nu}$ and for all $t \in \mathbb{R}_+^\times$ define $\rho(t)$ as the idèle x such that, for all places ν, the component

$$x_\nu = \begin{cases} t^{1/d} & \text{if } \nu = \tilde{\nu}, \\ 1 & \text{if } \nu \neq \tilde{\nu}, \end{cases}$$

where $d = 1$ if $K_{\tilde{\nu}} = \mathbb{R}$ and $d = 2$ if $K_{\tilde{\nu}} = \mathbb{C}$. In this way $\left|\rho(t)\right| = t$ for all positive real numbers t and ρ is continuous: take a basic open neighbourhood of $1 \in \mathbb{A}^\times$ of the form $V = \prod_\nu V_\nu$, where $V_\nu \subset K_\nu^\times$ is an open neighbourhood of the identity and $V_\nu = \mathcal{O}_\nu^\times$ for almost all places. Then

$$\rho^{-1}(V) = \left\{t \in \mathbb{R}_+^\times : t^{1/d} \in V_{\tilde{\nu}}\right\}\,,$$

which is open. Now, let K be a general global field and ρ a continuous section of the idèlic norm. Define the homomorphism

$$\left|\mathbb{A}^{\times}\right| \times \mathbb{A}^{\times,1} \longrightarrow \mathbb{A}^{\times}\,, \quad (t,x) \longmapsto \rho(t)\cdot x\,.$$

It is continuous because it is the product of continuous functions, and its inverse is the homomorphism

$$\mathbb{A}^{\times} \longrightarrow \left|\mathbb{A}^{\times}\right| \times \mathbb{A}^{\times,1}\,, \quad x \longmapsto \left(\, |x|\;,\; x\cdot\rho(|x|^{-1})\,\right),$$

which is also continuous because its components are: the first is the idèlic norm and the second is the product of $id_{\mathbb{A}^{\times}}$ with the composition of the continuous functions ρ and $|\cdot|^{-1}$. □

Now we analyse the relation between the global field and the idèles, starting with the next proposition, which is an analogue of the fact that the divisor of a meromorphic function defined on a compact Riemann surface has degree zero.

Proposition 3.4.5 (Product formula)**.** *Let K be a global field and denote by ν its places. Then*

$$\prod_{\nu} |x|_{\nu} = 1$$

for all $x \in K^{\times}$.

Proof. Suppose first that $K = \mathbb{Q}$ and let $x \in \mathbb{Q}$ be different from zero. We can suppose that x is positive because it does not change the valuation of x. Let

$$x = \prod_{\nu\neq\infty} p_{\nu}^{n_{\nu}}$$

be the factorization of x into a product of prime powers with integer exponents, where p_{ν} is the prime corresponding to the finite place ν. The valuation of x is computed as follows:

$$|x|_{\nu} = \begin{cases} x & \text{if } \nu = \infty\,, \\ p_{\nu}^{-n_{\nu}} & \text{otherwise}\,, \end{cases}$$

thus the product of the valuations over all places is

$$\begin{aligned} \prod_{\nu} |x|_{\nu} &= |x|_{\infty} \cdot \prod_{\nu\neq\infty} |x|_{\nu} \\ &= x \cdot \prod_{\nu\neq\infty} p_{\nu}^{-n_{\nu}} \\ &= x \cdot \left(\prod_{\nu\neq\infty} p_{\nu}^{n_{\nu}} \right)^{-1} \\ &= x\cdot x^{-1} \\ &= 1. \end{aligned}$$

The proof of the formula for the field of rational functions is analogous to the case of rational numbers. Suppose that $K = \mathbb{F}_q(T)$ and let $f(t) \in \mathbb{F}_q(T)$ be a non-zero rational function. Each finite place ν corresponds to an irreducible polynomial $f_\nu(T) \in \mathbb{F}_q[T]$ of degree d_ν, as it is described in Theorem 1.1.16. There is a factorization

$$f(T) = \prod_{\nu \neq \infty} f_\nu(T)^{n_\nu}$$

for a unique tuple of integers n_ν that are equal to zero for almost all finite places. Therefore the degree d of $f(T)$, namely the difference between the degrees of numerator and denominator of $f(T)$, for any given representation of $f(T)$ as a ratio of polynomials, is computed by

$$d = \sum_{\nu \neq \infty} d_\nu \cdot n_\nu \,. \tag{3.5}$$

For each place ν, the valuation of $f(T)$ at ν is

$$\big|f(T)\big|_\nu = \begin{cases} q^d & \text{if } \nu = \infty, \\ q^{-d_\nu \cdot n_\nu} & \text{otherwise,} \end{cases}$$

so the product formula

$$\prod_\nu \big|f(T)\big|_\nu = 1$$

holds by equation (3.5). Now let K be general. We can assume that K is a finite, separable extension of k, where k is the field of rational numbers or a field of rational functions over a finite field. Let $x \in K$ be non-zero, the valuation of x at the place ν of K is

$$|x|_\nu = \Big|N_{K_\nu/k_\upsilon}(x)\Big|_\upsilon \,,$$

where υ is the place of k satisfying $\nu|\upsilon$. Then

$$\prod_{\nu \in \mathscr{P}_K} |x|_\nu = \prod_{\upsilon \in \mathscr{P}_k} \prod_{\nu|\upsilon} \Big|N_{K_\nu/k_\upsilon}(x)\Big|_\upsilon \tag{3.6}$$

and we can fall back into the case of rational numbers or rational functions by a computation of the norm of the same nature as the trace computation done in the proof of Proposition 3.3.2. Let υ be any place of k, G_υ the absolute Galois group of k_υ and $\mathcal{E}$ the set of k-algebra embeddings of K into an algebraic closure $\overline{k}_\upsilon$ of the local field k_υ. For all places ν of K that lies over υ, let σ_ν be an embedding of the local field K_ν into the algebraic closure of k_υ and let $\mathcal{E}_\nu$ be the finite orbit of $\sigma_\nu|_K$ under the action of G_υ. The norm of x relative to the extension K/k is the product of all the conjugates $\sigma(x)$ of x for $\sigma \in \mathcal{E}$, while the norm of x relative to the extension K_ν/k_υ is the product of all the conjugates $\sigma(x)$ of x for $\sigma \in \mathcal{E}_\nu$. The set $\mathcal{E}$ is the disjoint union of its orbits $\mathcal{E}_\nu$ for ν varying in the set of places lying over υ, hence

$$N_{K/k}(x) = \prod_{\nu|\upsilon} N_{K_\nu/k_\upsilon}(x) \,. \tag{3.7}$$

The two equations (3.6) and (3.7) together implies that

$$\prod_{\nu \in \mathscr{P}_K} |x|_\nu = \prod_{\upsilon \in \mathscr{P}_k} \Big|N_{K/k}(x)\Big|_\upsilon \,.$$

Since $N_{K/k}(x) \in k$ and the product formula holds for k we get that the product formula holds for K too. □

Proposition 3.4.5 tells us that the multiplicative group $K^\times$ of the global field lies in the kernel $\mathbb{A}^{\times,1}$ of the idèlic norm.

Remark 3.4.6. The self-dual Haar measure of $\mathbb{A}$ depends on the identification of $\mathbb{A}$ with its dual in general, but the measure is unique if we assume that the duality $\mathbb{A} \cong \widehat{\mathbb{A}}$ is induced by a unitary character ψ trivial on the field. Indeed, if ψ' is another character with the same property, then there is a unique $\xi \in K$ such that $\psi'(x) = \psi(\xi x)$ for all adèles x (Theorem 3.3.3). Through the calculation that we did in Remark 3.1.4, one can see that the dual measure associated with ψ' is $dx' := |\xi|^{\frac{1}{2}}\,dx$, where dx is the self-dual measure associated with ψ. But $|\xi| = 1$ by the product formula and therefore $dx' = dx$.

Corollary 3.4.7. *The inclusion* $K^\times \subset \mathbb{A}^\times$ *is discrete.*

Proof. By the fact that the idèlic norm is continuous, we have that the set

$$U := \left\{x \in \mathbb{A}^\times : |x| < 1\right\}$$

is open in the group of idèles. It contains 1 and if x is a principal idèles different from 1, then $x-1$ is also a principal idèles. By Proposition 3.4.5, it must be $|x-1| = 1$ and in particular $x \notin U$. This means the sub-space topology induced by $\mathbb{A}^\times$ on $K^\times$ is discrete. ☐

The group $K^\times$ is discrete inside $\mathbb{A}^{\times,1}$ since the latter has the sub-space topology induced by $\mathbb{A}^\times$.

Definition 3.4.8. Let K be a global field and $\mathbb{A}$ its ring of adèles. The group $\mathbb{A}^\times/K^\times$ is called *idèle class group* of K or the group of *idèle classes*.

By the non-canonical decomposition of the idèles in Proposition 3.4.4, we have a decomposition of the idèle class group

$$\mathbb{A}^\times/K^\times \cong \left|\mathbb{A}^\times\right| \times \frac{\mathbb{A}^{\times,1}}{K^\times}$$

induced by the same section of the idèlic norm. The compactness of the group $\mathbb{A}^{\times,1}/K^\times$ is equivalent to the statement of Dirichlet's unit theorem plus the finiteness of the ideal class group of K. There is a nice exposition of the equivalence in B. Conrad's notes [Con]. They are based on [CF67], Chapter II, Sections 14-18. The compactness of $\mathbb{A}^{\times,1}/K^\times$ can be proved directly by showing that the topology of the group $\mathbb{A}^{\times,1}$ is the same as the one induced by the inclusion of $\mathbb{A}^{\times,1}$ into the adèles, contrary to the full group of idèles. Then it is just a matter of providing a compact subset W of $\mathbb{A}$ such that the projection

$$W \cap \mathbb{A}^{\times,1} \longrightarrow \mathbb{A}^{\times,1}/K^\times$$

is surjective (see the last lemma and the last theorem of [CF67], Chapter II, Section 16). This relies on an adèlic version of Minkowski's lemma, stating that there is a constant $C > 0$ such that whenever an adèles a satisfies $|a| > C$, there must be a $y \in K^\times$ with $|y|_\nu \leq |a_\nu|_\nu$ for all places ν. Then, the compact set W can be chosen to be

$$W := \left\{x \in \mathbb{A} : |x_\nu|_\nu \leq |a_\nu|_\nu \text{ for all places } \nu\right\}$$

for any adèles a satisfying $|a| > C$. The connection of the compactness of $\mathbb{A}^{\times,1}/K^\times$ with the two classical results of algebraic number theory derives from the fact that the group of fractional ideals of K is realized as a quotient of the idèles, where principal idèles correspond to principal ideals. Let us see some details of that. Denote by $\mathcal{J}(\mathcal{O}_K)$ the group of fractional ideals of the ring of integers of K. Set

$$\mathbb{A}_\infty^\times := \mathbb{A}_{S_\infty}^\times = \left\{a \in \mathbb{A}^\times : |a_\nu|_\nu = 1 \text{ for all finite place } \nu\right\}$$

for short. There is a map

$$\mathcal{L} : \mathbb{A}^\times \longrightarrow \mathcal{J}(\mathcal{O}_K)$$

defined by setting

$$\mathcal{L}(a) = \left\{x \in K : |xa_\nu|_\nu \leq 1 \text{ for all finite place } \nu\right\}$$

for all idèles a (note the similarity with the linear spaces of meromorphic functions associated with divisors). The map $\mathcal{L}$ is a surjective group homomorphism with kernel $\mathbb{A}_\infty^\times$, an open sub-group of the idèles, thus $\mathcal{L}$ induces an isomorphism of discrete groups

$$\frac{\mathbb{A}^\times}{\mathbb{A}_\infty^\times} \cong \mathcal{J}(\mathcal{O}_K)\,.$$

If a is a principal idèles, then an element x of the global field belongs to $\mathcal{L}(a)$ if and only if $ax \in \mathcal{O}_K$, which means that $\mathcal{L}(a)$ is equal to the principal ideal $\frac{1}{a}\mathcal{O}_K$. Therefore, $\mathcal{L}$ induces an epimorphism

$$\frac{\mathbb{A}^\times}{K^\times} \longrightarrow \mathcal{C}(\mathcal{O}_K) \tag{3.8}$$

onto the ideal class group $\mathcal{C}(\mathcal{O}_K)$. Using idèles belonging to the sub-group $\mathbb{A}_\infty^\times$ we can affect the idèlic norm without affecting the image of $\mathcal{L}$, then the induced map

$$\mathbb{A}^{\times,1}/K^\times \longrightarrow \mathcal{C}(\mathcal{O}_K) \tag{3.9}$$

is still an epimorphism. By knowing that $\mathbb{A}^{\times,1}$ is compact, we can infer the finiteness of the ideal class group because it is the continuous image of a compact group and it is discrete. Inspecting the compact kernel of the epimorphism (3.9) leads to a proof of Dirichlet's unit theorem.

3.5 Idèle Class Characters

In Chapter 4, we will construct L-functions from the analysis of a particular representation of the group of idèles made by generalized functions on the adèles. The one-dimensional sub-representations whose character is a continuous homomorphism $\omega : \mathbb{A}^\times \to \mathbb{C}^\times$ trivial on the group of principal are the fundamental objects on which the argument is based. For this reason, in the current section, we deal with the description of such characters: they turn out to be parametrized by couples (s, ω), where s is a complex number and ω comes from a unitary character of the compact group $\mathbb{A}^{\times,1}/K^\times$.

Definition 3.5.1. Let K be a global field and $\mathbb{A}$ its ring of adèles. Define Ω_K to be the group of continuous homomorphism $\omega : \mathbb{A}^\times \to \mathbb{C}^\times$ such that $\omega(K^\times) = 1$. Its elements are called *idèle class characters*. An idèle class character is said *unramified* if it is trivial on the group $\mathbb{A}^{\times,1}$.

Unramified idèle class characters factors through the image of the idèlic norm, hence they correspond to the continuous homomorphisms from $\mathbb{R}_+^\times$ to $\mathbb{C}^\times$ if K is a number field, and to the homomorphisms from $\mathbb{Z}$ to $\mathbb{C}^\times$ if K is a function field. In detail, if ω is unramified, then it fits in a commutative triangle

$$\begin{array}{ccc} & \left|\mathbb{A}^\times\right| & \\ {\scriptstyle |\cdot|}\nearrow & & \searrow{\scriptstyle \tilde{\omega}} \\ \mathbb{A}^\times & \xrightarrow{\quad\omega\quad} & \mathbb{C}^\times \end{array}$$

where $\tilde{\omega}$ is a character of the image of the idèlic norm. When K is a function field, the latter is generated by $t := |\pi_\nu|_\nu$ for some place ν and some uniformizer π_ν of the local field K_ν. The character $\tilde{\omega}$ is determined by its value $\tilde{\omega}(t)$ at t. Having observed this, we have that the homomorphism of groups

$$\mathbb{C} \longrightarrow \Omega_K\,, \quad s \longmapsto |\cdot|^s$$

parametrizes all unramified characters and it has kernel $\frac{2\pi i}{\log t}\mathbb{Z}$, so two unramified characters $|\cdot|^{s_1}$ and $|\cdot|^{s_2}$ are equal if and only if $s_1 - s_2$ is an integer multiple of $\frac{2\pi i}{\log t}$. When K is a number field, every unramified character ω decomposes through a character $\tilde{\omega}$ of $\mathbb{R}_+^\times$, making the diagram

$$\begin{array}{ccc} & \mathbb{R}_+^\times & \\ {\scriptstyle |\cdot|}\nearrow & & \searrow{\scriptstyle \tilde{\omega}} \\ \mathbb{A}^\times & \xrightarrow{\quad\omega\quad} & \mathbb{C}^\times \end{array}$$

commutes. Thus we only need to classify the characters of $\mathbb{R}_+^\times$ to obtain a complete description of the unramified idèle class characters. This is done in Lemma 3.5.2, which state that every homomorphism $\mathbb{R}_+^\times \to \mathbb{C}^\times$ is of the form $t \mapsto t^s$ for a unique $s \in \mathbb{C}$. The immediate consequence is that the map

$$\mathbb{C} \longrightarrow \Omega_K\,, \quad s \longmapsto |\cdot|^s$$

defines an isomorphism between the additive group of complex numbers and the group of unramified idèle class characters.

Lemma 3.5.2. *The map that sends a complex number s to the continuous homomorphism*

$$\mathbb{R}_+^\times \longrightarrow \mathbb{C}^\times\,, \quad t \longmapsto t^s$$

defines an isomorphism

$$\mathbb{C} \cong \mathrm{Hom}_{\mathbf{LCA}}(\mathbb{R}_+^\times, \mathbb{C}^\times)$$

from the additive group of the complex numbers to the group of characters of $\mathbb{R}_+^\times$.

Proof. Recall that $\mathbb{R}_+^\times \times \mathbb{S}^1 \cong \mathbb{C}^\times$ by expressing a complex number z in polar coordinates $z = ru$, with $r \in \mathbb{R}_+^\times$ and $u \in \mathbb{S}^1$, so

$$\mathrm{Hom}_{\mathbf{LCA}}(\mathbb{R}_+^\times, \mathbb{R}_+^\times) \times \widehat{\mathbb{R}_+^\times} \cong \mathrm{Hom}_{\mathbf{LCA}}(\mathbb{R}_+^\times, \mathbb{C}^\times)\,.$$

By the logarithm, $\mathbb{R}_+^\times$ is isomorphic to the additive group of the real numbers. The unitary characters of $\mathbb{R}$ are known: there is an isomorphism $\mathbb{R} \cong \widehat{\mathbb{R}}$ via the pairing e^{ibx} for $b, x \in \mathbb{R}$. The continuous endomorphisms of $\mathbb{R}$ are also known: they are the $\mathbb{R}$-linear maps, and by post-composition with the exponential map, we obtain all homomorphisms $\mathbb{R} \to \mathbb{R}_+^\times$. To obtain $\widehat{\mathbb{R}_+^\times}$ and the endomorphisms of $\mathbb{R}_+^\times$, use the logarithm $\log : \mathbb{R}_+^\times \to \mathbb{R}$ in the following way: a real number a goes to the homomorphism

$$\mathbb{R} \longrightarrow \mathbb{R}_+^\times , \quad x \longmapsto e^{ax} ,$$

which in turn goes to the homomorphism

$$\mathbb{R}_+^\times \longrightarrow \mathbb{R}_+^\times , \quad t \longmapsto t^a$$

via the change of variable $x = \log t$. In the same way, a real number b is sent to the homomorphism

$$\mathbb{R}_+^\times \longrightarrow \mathbb{R}_+^\times , \quad t \longmapsto t^{ib} .$$

This gives an isomorphism

$$\mathbb{R} \times \mathbb{R} \longrightarrow \mathrm{Hom}_{\mathbf{LCA}}(\mathbb{R}_+^\times, \mathbb{R}_+^\times) \times \widehat{\mathbb{R}_+^\times} , \quad (a,b) \longmapsto \big([t \mapsto t^a], [t \mapsto t^{ib}]\big)$$

from which we deduce that

$$\mathbb{C} \longrightarrow \mathrm{Hom}_{\mathbf{LCA}}(\mathbb{R}_+^\times, \mathbb{C}^\times) , \quad s \longmapsto [t \mapsto t^s]$$

is an isomorphism by the simple observation that $t^s = ru$ for $r = t^a$, $u = t^{ib}$ and $s = a + ib$. □

Corollary 3.5.3. *Every unramified idèle class character ω of a global field K is of the form*

$$\omega(x) = |x|^s , \quad \textit{for all } x \in \mathbb{A}^\times ,$$

for a complex number s. If K is a number field, the number s associated with ω is unique. If K is a function field, the complex number s is unique modulo integer multiples of $\frac{2\pi i}{\log t}$, where t is a generator of the image of the idèlic norm.

Now that we have a description of the unramified characters, we can use a non-canonical splitting $\mathbb{A}^\times \cong \left|\mathbb{A}^\times\right| \times \mathbb{A}^{\times,1}$ to decompose non-canonically the space Ω_K as a product of the complex plane (or a quotient of it in the function field case) with the Pontryagin dual of $\mathbb{A}^{\times,1}/K^\times$. We get the group of unitary characters because the quotient $\mathbb{A}^{\times,1}/K^\times$ is compact and any character from a compact group to $\mathbb{C}^\times$ is forced to be unitary, as we are going to see in Lemma 3.5.4.

Lemma 3.5.4. *Let A be a compact abelian group and $\omega : A \to \mathbb{C}^\times$ a continuous homomorphism. Then ω is a unitary character.*

Proof. Consider the composition of ω with the absolute value of $\mathbb{C}$. It is a continuous homomorphism from A to the group of positive real numbers. Compose it further with the natural logarithm and you obtain a continuous homomorphism from the compact group A to the additive group of the real numbers. Its image is a compact sub-group of $\mathbb{R}$, but any such group must be trivial because every non-zero real number generates an unbounded sub-group of $\mathbb{R}$ (a consequence of the archimedean nature of $\mathbb{R}$). Thus

$$\log\left\|\omega(a)\right\|_{\mathbb{C}} = 0 \quad \text{for all } a \in A,$$

which means that

$$\left\|\omega(a)\right\|_{\mathbb{C}} = 1 \quad \text{for all } a \in A,$$

proving that ω is unitary. □

Theorem 3.5.5. *Let ω be an idèle class character of the global field K. Let $\rho : \left|\mathbb{A}^\times\right| \to \mathbb{A}^\times$ be a continuous section of the idèlic norm. There is a unique unitary character χ of $\mathbb{A}^{\times,1}$ which is trivial on $K^\times$ and a unique unramified character $|\cdot|^s$, for some $s \in \mathbb{C}$, such that*

$$\omega(x) = \chi\big(\ \rho(|x|)^{-1}x\ \big) \cdot |x|^s \quad \textit{for all } x \in \mathbb{A}^\times .$$

Proof. Recall, by Proposition 3.4.4, that ρ induces an isomorphism

$$\left|\mathbb{A}^\times\right| \times \mathbb{A}^{\times,1} \longrightarrow \mathbb{A}^\times , \quad (t,a) \longmapsto \rho(t)\cdot a\,.$$

Define $\chi \in \widehat{\mathbb{A}^{\times,1}}$ as the restriction of ω on the sub-group $\mathbb{A}^{\times,1}$ of the idèles. It is trivial on $K^\times$ because $\omega(K^\times) = 1$ by definition. The composition $\omega \circ \rho \circ |\cdot|$ is a continuous homomorphism from $\mathbb{A}^\times$ to $\mathbb{C}^\times$ and it is trivial on $\mathbb{A}^{\times,1}$ because that group is the kernel of $|\cdot|$, therefore it is an unramified idèle class character and by Corollary 3.5.3 there is a complex number s such that

$$\omega\big(\rho(|x|)\big) = |x|^s \quad \text{for all } x \in \mathbb{A}^\times .$$

If we write an idèle x as $x = \rho(|x|) \cdot \rho(|x|)^{-1} \cdot x$, then

$$\begin{aligned}
\omega(x) &= \omega\big(\ \rho(|x|) \cdot \rho(|x|)^{-1} \cdot x\ \big) \\
&= \omega\big(\ \rho(|x|)\ \big) \cdot \omega\big(\ \rho(|x|)^{-1}x\ \big) \\
&= |x|^s \cdot \chi\big(\ \rho(|x|)^{-1}x\ \big) \\
&= \chi\big(\ \rho(|x|)^{-1}x\ \big) \cdot |x|^s \ ,
\end{aligned}$$

which is the expression required in the statement. If χ', s' is another couple satisfying

$$\omega(x) = \chi'\big(\ \rho(|x|)^{-1}x\ \big) \cdot |x|^{s'} \quad \text{for all } x \in \mathbb{A}^\times,$$

then for all $a \in \mathbb{A}^{\times,1}$

$$\begin{aligned}
\chi(a) &= \omega(a) \\
&= \chi'\big(\ \rho(|a|)^{-1}a\ \big) \cdot |a|^{s'} \\
&= \chi'(a)
\end{aligned}$$

and for all $t \in \left|\mathbb{A}^\times\right|$ we have

$$\begin{aligned}
t^s &= \left|\rho(t)\right|^s \\
&= \omega\Big(\ \rho\big(\ \left|\rho(t)\right|\ \big)\ \Big) \\
&= \omega\big(\ \rho(t)\ \big) \\
&= \chi'\big(\ \rho(\left|\rho(t)\right|)^{-1}\rho(t)\ \big) \cdot \left|\rho(t)\right|^{s'} \\
&= \chi'(1) \cdot t^{s'} \\
&= t^{s'}\,.
\end{aligned}$$

This proves the uniqueness of χ and $|\cdot|^s$. ◻

Theorem 3.5.5 describes the idèle class characters from a global point of view, but they have also a local structure given by the structure of the idèles as a restricted direct product. In Section 2.4 we described general characters of restricted direct products in terms of the characters of the factors via the two lemmas 2.4.6 and 2.4.5. In the particular case of the idèles, we have that a character $\omega : \mathbb{A}^\times \to \mathbb{C}^\times$ is associated with a tuple of *local* characters $\omega_\nu : K_\nu \to \mathbb{C}^\times$ which are trivial on $\mathcal{O}_\nu^\times$ for almost all places and the relation with ω is given by the formula

$$\omega(x) = \prod_\nu \omega_\nu(x_\nu)\,.$$

We now focus on the characters of local fields, the non-archimedean case first, so denote by K a local field with its local sub-ring $\mathcal{O}$, the maximal ideal is $\mathfrak{p}$, generated by π. Let $\omega : K^\times \to \mathbb{C}^\times$ be a character, and consider the restriction to the group of units $\mathcal{O}^\times$. It satisfies the hypothesis of Lemma 3.1.7, hence there is a sub-group of $\mathcal{O}^\times$ of the form $1 + \mathfrak{p}^n$, for some $n \in \mathbb{N}$, such that $\omega(1 + \mathfrak{p}^n) = 1$. As a consequence, ω is locally constant and $\omega(\mathcal{O}^\times)$ is a finite sub-group of $\mathbb{C}^\times$. If K is archimedean instead, then

$$K^\times \cong \begin{cases} \{\pm 1\} \times \mathbb{R}_+^\times & \text{if } K = \mathbb{R}, \\ \mathbb{S}^1 \times \mathbb{R}_+^\times & \text{if } K = \mathbb{C}, \end{cases}$$

therefore we can classify all the multiplicative characters ω of $K^\times$ because we know all characters of $\mathbb{R}_+^\times$ (Lemma 3.5.2), $\{\pm 1\}$ and $\mathbb{S}^1$. Indeed the last two groups are compact, so, by Lemma 3.5.4, their characters must be unitary and we know the dual groups $\widehat{\{\pm 1\}}$, $\widehat{\mathbb{S}^1}$: the unique non-trivial character of the group with two elements sends the generator to -1, while a character χ of the circle group, thought of as a subset of the complex plane, has the form $\chi(z) = z^n$ for all $z \in \mathbb{S}^1$ and a unique integer n depending only on χ. Note that the sub-groups $\{\pm 1\}$, $\mathbb{S}^1$ and $\mathcal{O}^\times$ are precisely the sub-groups of elements of absolute value 1 inside $\mathbb{R}$, $\mathbb{C}$ and a non-archimedean local field respectively.

Definition 3.5.6. Let K be a local field. Denote by $\mathfrak{u}$ the sub-group of $K^\times$ consisting of the elements x such that $|x| = 1$. A character ω of $K^\times$ is said *unramified* if $\omega(\mathfrak{u}) = 1$, otherwise it is said *ramified*. If K is non-archimedean and ω is ramified, define the *conductor* of ω as the minimal ideal $\mathfrak{f}$ of $\mathcal{O}$ such that $\omega(1 + \mathfrak{f}) = 1$.

Proposition 3.5.7. *Let K be a local field and ω an unramified character of $K^\times$. Then there is a complex number s such that*

$$\omega(x) = |x|^s \quad \textit{for all } x \in K^\times.$$

The number s is unique when K is archimedean, otherwise, it is unique modulo $\frac{2\pi i}{\log|\varpi|}\mathbb{Z}$, where ϖ is a uniformizer of K.

Proof. The character ω factors through the quotient group $K^\times/\mathfrak{u}$, which is isomorphic to the image of $|\cdot|$ and the image is equal to $\mathbb{R}_+^\times$ or a free sub-group of rank 1. The characters of such groups are completely understood and they are all of the forms t^s, where s is a complex number and t varies in the image of $|\cdot|$. Then, for all $x \in K^\times$, the value $\omega(x)$ depends only on the absolute value $t = |x|$ and it is equal to t^s. If K is archimedean, then s is unique, if K is non-archimedean, then $|\cdot|$ is determined by $t = |\varpi|$, where ϖ is a generator of the maximal ideal of $\mathcal{O}$, and $t^s = t^{s'}$ is equivalent to $(s - s') \cdot \log t$ belonging to the kernel of the complex exponential map e^z. ☐

Theorem 3.5.8. *Let ω be a character of a local field K and $|\cdot|$ the module of K. For K non-archimedean, let $\mathfrak{p}$ be the maximal ideal of its local ring and π a generator of $\mathfrak{p}$. Then ω is one of the following characters:*

(i) *if $K = \mathbb{R}$, there is a unique couple $(s, n) \in \mathbb{C} \times \{0, 1\}$ such that*

$$\omega(x) = x^{-n}|x|^{s+n} \quad \text{for all } x \in \mathbb{R}^\times;$$

(ii) *if $K = \mathbb{C}$, there is a unique couple $(s, n) \in \mathbb{C} \times \mathbb{N}$ such that*

$$\omega(x) = x^{-n}|x|^{s+\frac{n}{2}} \quad \text{for all } x \in \mathbb{C}^\times$$

or

$$\omega(x) = \overline{x}^{-n}|x|^{s+\frac{n}{2}} \quad \text{for all } x \in \mathbb{C}^\times,$$

where $\overline{x}$ is the complex conjugate of x;

(iii) *if K is non-archimedean, there is a character χ of $\mathcal{O}^\times$ and a complex number s such that*

$$\omega(x) = \chi\big(x\pi^{-\operatorname{ord}_\mathfrak{p}(x)}\big)|x|^s \quad \text{for all } x \in K^\times.$$

Proof. The restriction of ω on the subgroup $\mathfrak{u}$ is a unitary character χ.

Case (i): χ is determined by the image of $-1 \in \mathfrak{u}$. There is a unique integer $n \in \{0, 1\}$ such that

$$\chi(-1) = (-1)^{-n}$$

The character

$$\omega' : x \longmapsto \frac{x^n}{|x|^n}\omega(x)$$

is unramified because $\omega'(-1) = (-1)^n\chi(-1)$ and the right-hand side term is equal to 1 idipendently on n. Therefore, Proposition 3.5.7 implies that $\omega' = |\cdot|^s$ for a unique complex number s and

$$\begin{aligned}\omega(x) &= x^{-n}|x|^n\,\omega'(x)\\ &= x^{-n}|x|^n|x|^s\\ &= x^{-n}|x|^{s+n}.\end{aligned}$$

Case (ii): We have $\mathfrak{u} = \mathbb{S}^1$, so there is a unique integer n such that $\chi(x) = x^{-n}$ for all $x \in \mathfrak{u}$. For each non-zero complex number x, define

$$\tilde{x} := \frac{x}{|x|^{\frac{1}{2}}},$$

so that $x \mapsto \tilde{x}$ is a continuous homomorphism $\mathbb{C}^\times \to \mathfrak{u}$ which restricts to the identity on the sub-group $\mathfrak{u}$. Let ω' be the character defined by

$$\omega'(x) = \tilde{x}^n\omega(x) \quad \text{for all } x \in \mathbb{C}^\times,$$

then ω' is unramified and there is a unique $s \in \mathbb{C}$ such that $\omega' = |\cdot|^s$ by Proposition 3.5.7. One concludes that

$$\begin{aligned}\omega(x) &= \tilde{x}^{-n}\omega'(x)\\ &= \tilde{x}^{-n}|x|^s.\end{aligned}$$

similarly to case (i). Simple algebraic manipulations lead to the formulas of the statement once we have observed that $\tilde{x}^{-1}$ is equal to the complex conjugate of $\tilde{x}$.

Case (iii): note that $\mathrm{ord}_{\mathfrak{p}}(xy) = \mathrm{ord}_{\mathfrak{p}}(x) + \mathrm{ord}_{\mathfrak{p}}(y)$ for all $x, y \in K^\times$. Then, if we define $\tilde{x} := x\pi^{-\mathrm{ord}_{\mathfrak{p}}(x)}$ for all $x \in K^\times$, the map $x \mapsto \tilde{x}$ is a continuous homomorphism from $K^\times$ to the group $\mathfrak{u}$ that is the identiy on $\mathfrak{u}$. Therefore, the character that sends $x \in K^\times$ to $\chi(\tilde{x})^{-1}\omega(x)$ is unramified and the conclusion is obvious from Proposition 3.5.7. $\square$

We now return to the characters $\omega : \mathbb{A}_K^\times \to \mathbb{C}^\times$ for a global field K.

Definition 3.5.9. Suppose that $\omega = \prod_\nu \omega_\nu$ is a character of the idèles of K. It is said *unramified at* ν if ω_ν is unramified as a character of the local field K_ν.

Since $\mathbb{A}^\times$ is the restricted direct product of the groups $K_\nu^\times$ relative to the sub-groups $\mathcal{O}_\nu^\times$, every character ω of must satisfy $\omega_\nu(\mathcal{O}_\nu^\times) = 1$ for almost all ν, which is equivalent to saying that ω is unramified at almost all places.

A natural question to ask is if there is a relationship between the ramification of Definition 3.5.9 and the ramification of Definition 3.5.1. If ω is an idèle class character, the condition of being unramified at every place is equivalent to the existence of complex numbers s_ν such that $\omega_\nu = |\cdot|_\nu^{s_\nu}$, i.e.

$$\omega(x) = \prod_\nu |x_\nu|_\nu^{s_\nu} \qquad \text{for all } x \in \mathbb{A}^\times.$$

For ω to be unramified in the sense of Definition 3.5.1, the complex numbers s_ν should be all equal to each other. For $K = \mathbb{Q}$, this is true, but for K general it is far from being so. For example, characters of the ideal class groups induce idèle class characters which are unramified at every place but are not trivial on the group $\mathbb{A}^{\times,1}$. Let us see why: suppose that K is a number field with a non-trivial ideal class group and let χ be a non-trivial character of it. Since the ideal class group is an epimorphic image of the idèle class group via the map (3.8), there is a unique idèle class character ω making the triangle

$$\begin{array}{ccc} \mathbb{A}^\times/K^\times & \xrightarrow{\ \omega\ } & \mathbb{C}^\times \\ & \searrow \quad \nearrow_{\chi} & \\ & \mathcal{C}(\mathcal{O}_K) & \end{array}$$

commute. The character ω is not trivial on $\mathbb{A}^{\times,1}$ because the latter covers the ideal class group. However, all idèle classes represented by an idèle x satisfying $|x_\nu| = 1$ for every place ν are contained in the kernel of the projection

$$\mathbb{A}^\times/K^\times \longrightarrow \mathcal{C}(\mathcal{O}_K)\,,$$

thus ω_ν is unramified for every place ν.

Chapter 4

L-functions and Representation Theory of the Idèles

In the following chapter, we are going to study a particular representation of the idèles $\mathscr{S}'(\mathbb{A})$, obtained as the dual of the space of Schwartz functions on the adèles, and its 1-dimensional subrepresentations $\mathscr{S}'(\omega)$ attached to idèle class characters ω. The goal is to obtain the fundamental properties of a family of L-functions (the Hecke L-functions) from the analysis of the *zeta integral* $\zeta(\omega)$, an integral transform that belongs to the space $\mathscr{S}'(\omega)$. The L-function, completed with the archimedean factor, is obtained by evaluating $\zeta(\omega)$ on a nice Schwartz function f, while its functional equation will be a consequence of the existence of a functional equation for the zeta integral. The latter is established thanks to the self-duality of the lattice given by the global field inside its ring of adèles. This is the only completely 'global' feature of the zeta integral, while the rest is done by putting together local constructions and computations: given a global field K, we start from local representations $\mathscr{S}'(K_\nu)$ and the local zeta integrals ζ_ν to construct the corresponding global objects, following the exposition of Tate's thesis in [Kud04]. We follow Chapter 3 of [Bum97] for the functional equation of the global zeta integral.

4.1 Functional representations and the Fourier transform

In what follows, many definitions and results are simultaneously valid for local fields and the adèle ring of global fields, so we use k to indicate a local field, $\mathbb{A}$ for the ring of adèles of a global field K and $\Bbbk$ to indicate both k and $\mathbb{A}$. For example, $\Bbbk^\times$ stands for the multiplicative group $k^\times$ or the group of idèles. We use the following notation for the local field k:

- if k is non-archimedean we let $\mathcal{O}$ be its local ring and π a choice of a generator of the maximal ideal $\mathfrak{p}$;
- the residue field is denoted by κ and q is its cardinality;
- ψ is a non-trivial, unitary character of k inducing an identification $k \cong \widehat{k}$ for the additive group of k;
- dx is the self-dual measure with respect to ψ and $|\cdot|$ is the module;

- $d^\times x$ is the multiplicative Haar measure of $k^\times$, defined by the condition that $\mathcal{O}^\times$ is of measure 1 if k is non-archimedean, otherwise is defined to be simply $|x|^{-1}\,dx$ restricted on $k^\times$.

- if $k = K_\nu$ is a completion of a global field K at some ν we add everywhere a subscript ν in the notation given on previous points: $\mathcal{O}_\nu$, π_ν, $\mathfrak{p}_\nu$, κ_ν, q_ν and $|\cdot|_\nu$ are the corresponding notations for K_ν. The character ψ_ν and the self-dual measure dx_ν are understood as the local components of a character ψ and a measure dx of the adèle ring $\mathbb{A}_K$.

In the case of the ring $\mathbb{A}$, the character ψ inducing an isomorphism $\mathbb{A} \cong \widehat{\mathbb{A}}$ is assumed to be trivial on the global field, the self-dual measure dx is unique by Remark 3.4.6 and the measure $d^\times x$ of the idèles is the restricted product of the local measures $d^\times x_\nu = m_\nu |x|^{-1}\,dx_\nu$, where

$$m_\nu = \begin{cases} 1 & \text{if } \nu \text{ is archimedean,} \\ \dfrac{1}{1-q_\nu^{-1}} & \text{if } \nu \text{ is non-archimedean.} \end{cases}$$

If k is an archimedean local field, we write $\mathscr{S}(k)$ for the classical space of complex-valued Schwartz functions on k and $\mathscr{S}'(k)$ for its topological dual, i.e. the space of tempered distributions on k (see [Bon01], Chapter 9 for the definitions). If k is non-archimedean define $\mathscr{S}(k)$ to be the complex vector space of locally constant, complex-valued functions with compact support on k and $\mathscr{S}'(k)$ the space of $\mathbb{C}$-linear functionals on $\mathscr{S}(k)$. For $f \in \mathscr{S}(k)$ and $\lambda \in \mathscr{S}'(k)$, we use the notation

$$\langle \lambda\,, f \rangle := \lambda(f).$$

For the adèles, we define the Schwartz space $\mathscr{S}(\mathbb{A})$ as the $\mathbb{C}$-linear sub-space of $\mathscr{C}(\mathbb{A}) \cap L^1(\mathbb{A})$ generated by factorizable functions $f = \otimes_\nu f_\nu$ such that $f_\nu \in \mathscr{S}(K_\nu)$ for all ν, and f_ν is the characteristic function of $\mathcal{O}_\nu$ for almost all ν. Instead of specifying a linear topology on $\mathscr{S}(\mathbb{A})$ and taking the topological dual, we define directly $\mathscr{S}'(\mathbb{A})$ in terms of the local spaces $\mathscr{S}'(K_\nu)$, this will be enough for our purposes. $\mathscr{S}'(\mathbb{A})$ is the sub-space of $\mathbb{C}$-linear forms on $\mathscr{S}(\mathbb{A})$ generated by the standard linear forms $\lambda = \otimes_\nu \lambda$, where $\lambda_\nu \in \mathscr{S}'(K_\nu)$ and $\langle \lambda_\nu\,, 1_{\mathcal{O}_\nu} \rangle = 1$ for almost all ν, such that λ operates on a factorizable Schwartz function $f = \otimes_\nu f_\nu$ by

$$\langle \lambda\,, f \rangle = \prod_\nu \langle \lambda_\nu\,, f_\nu \rangle\,.$$

Now assume that $\Bbbk$ is a local field or the ring of adèles. Note that the multiplicative group $\Bbbk^\times$ acts on the additive group of $\Bbbk$ by multiplication, then the functional space $\mathscr{S}(\Bbbk)$ becomes a representation of $\Bbbk^\times$ if we define the action

$$\Bbbk^\times \times \mathscr{S}(\Bbbk) \to \mathscr{S}(\Bbbk)\,, \quad (a, f) \mapsto a \cdot f$$

by $(a \cdot f)(x) = f(xa)$ for all $x \in \Bbbk$. The rule

$$\langle a \cdot \lambda, f \rangle = \langle \lambda, a^{-1} \cdot f \rangle \qquad \text{for} \quad a \in \Bbbk^\times,\ \lambda \in \mathscr{S}'(\Bbbk),\ f \in \mathscr{S}(\Bbbk),$$

makes $\mathscr{S}'(\Bbbk)$ into a representation of $\Bbbk^\times$, precisely, it is the dual representation of $\mathscr{S}(\Bbbk)$. The map

$$\mathscr{S}(\Bbbk) \longrightarrow \mathscr{S}'(\Bbbk)\,, \quad g \longmapsto \langle g, \cdot \rangle\,,$$

where

$$\langle g, f\rangle := \int_{\Bbbk} g(x)f(x)\,dx$$

for all $f \in \mathscr{S}(\Bbbk)$, is not $\Bbbk^\times$-equivariant, unless we modify the action of $a \in \Bbbk^\times$ on the space $\mathscr{S}(\Bbbk)$ in such a way that the function $g(x)$ is sent to $|a|\,g(ax)$. The space $\mathscr{S}(\Bbbk)$ is contained in the space of L^1 functions of $\Bbbk$, hence the Fourier transform is well defined by the integral

$$\widehat{f}(\xi) = \int_{\Bbbk} f(x)\overline{\psi(x\xi)}\,dx\,,$$

for all functions $f \in \mathscr{S}(\Bbbk)$ and all elements $\xi \in \Bbbk$. It is well known that the Fourier transform of a Schwartz function defined on $\mathbb{R}$ or $\mathbb{C}$ is again in the Schwartz class. The same is true for non-archimedean local fields and the following lemma helps to see why.

Lemma 4.1.1. *Let k be a non-archimedean local field and let f be the characteristic function of the local ring $\mathcal{O}$ in k. Then f is a Schwartz function and $\widehat{f}$ is a constant multiple of the characteristic function of $\mathcal{O}^\perp$, the constant being the measure of $\mathcal{O}$.*

Proof. The function f is of Schwartz type because it is locally constant with support equal to the compact subset $\mathcal{O}$. Take an element $\xi \in k$ and lets compute the Fourier transform of f at ξ,

$$\begin{aligned}\widehat{f}(\xi) &= \int_{k} f(x)\overline{\psi(x\xi)}\,dx \\ &= \int_{\mathcal{O}} \overline{\psi(x\xi)}\,dx\,.\end{aligned}$$

Note that $\chi : \mathcal{O} \to \mathbb{C}^\times$ defined by $\chi(x) = \overline{\psi(x\xi)}$ for all $x \in \mathcal{O}$ is a unitary character of $\mathcal{O}$, so, by example 2.2.5, the integral of χ on the compact abelian group $\mathcal{O}$ equals the measure of it or the value 0 depending only on χ being trivial or not. The character χ is trivial if and only if $\xi \in \mathcal{O}^\perp$. Consequently, $\widehat{f}$ is the characteristic function of $\mathcal{O}^\perp$ multiplied by the measure of $\mathcal{O}$. □

Remark 4.1.2. Given this lemma, observe that, in the non-archimedean case, any Schwartz function f can be decomposed as

$$f(x) = \sum_r c_r \mathbb{1}_{\mathcal{O}}\left(\frac{xa - r}{\pi^m}\right) \quad \text{for all } x \in k,$$

where

$\mathbb{1}_X$ is the characteristic function of a set $X \subset k$,

a is some element of $k^\times$,

r runs in a finite set of representatives of the cosets of $\mathcal{O}/\mathfrak{p}^m$ for some positive integer m,

c_r are complex numbers.

This happens because any Schwartz function f has compact support and any compact set of k is contained inside $\pi^n\mathcal{O}$ for some integer n. The support of f is therefore contained in a compact subset of the form $\{x \in k : |x| \leq |a|\}$ for some $a \in k^\times$ and the function $f' : x \mapsto f(xa^{-1})$ is a locally constant function supported in $\mathcal{O}$. Since the latter is compact, there is a finite cover of clopen subsets of $\mathcal{O}$ such that f' is constant on each one of them. A basis of neighbourhoods of 0 is given by the positive powers of $\mathfrak{p}$. Take a sufficiently large positive integer m in such a way that the subsets $r + \mathfrak{p}^m$, for r ranging in a finite set of representatives of the classes in $\mathcal{O}/\mathfrak{p}^m$, define a refinement of the cover. The function f' is constant on $r + \mathfrak{p}^m$ for all r, so that we can write

$$f'(x) = \sum_r c_r \mathbb{1}_{r+\mathfrak{p}^m}(x)$$

with $c_r = f'(r)$. Finally, simple manipulations of the above equation lead to the formula for f.

Proposition 4.1.3. *The Fourier transform induces a $\mathbb{C}$-linear automorphism of the Schwartz space $\mathscr{S}(\mathbb{k})$.*

Proof. For an archimedean local field k, it is a classical result, if k is non-archimedean, by remark 4.1.2 we have to prove only that the Fourier transform of the function

$$f(x) = \mathbb{1}_{\mathcal{O}}\left(\frac{xa - r}{\pi^m}\right)$$

in the variable x, for m, a, r as above, is of Schwartz type. But this calculation is straightforward by the fact that we know the Fourier transform of $1_{\mathcal{O}}$ and how it behaves under linear transformations on the domain:

$$\widehat{f}(\xi) = c \cdot \left|a^{-1}\pi^m\right| \cdot \overline{\psi(\xi r a^{-1})} \cdot \mathbb{1}_{\mathcal{O}^\perp}(\xi \pi^m a^{-1})$$

for all $\xi \in k$, where c is the measure of $\mathcal{O}$. This function has compact support because of the product with the characteristic function. The local behaviour of $\widehat{f}$ is determined by the character $\xi \mapsto \overline{\psi(\xi r a^{-1})}$ of the group $a\pi^{-m}\mathcal{O}^\perp$, which has finite image and open kernel by Lemma 3.1.7. In particular, it's a locally constant function on an open, compact neighbourhood of the support of $\widehat{f}$, which is enough to conclude the proof.

Suppose that $\mathbb{k} = \mathbb{A}$ is the adèle ring of a global field K. By linearity, it is enough to check that a factorizable Schwartz function $f = \otimes_\nu f_\nu$ has Fourier transform in $\mathscr{S}(\mathbb{A})$. Let $\xi \in \mathbb{A}$, then

$$\begin{aligned}\widehat{f}(\xi) &= \int_{\mathbb{A}} f(x)\overline{\psi(\xi x)}\, dx \\ &= \prod_\nu \int_{K_\nu} f_\nu(x_\nu)\overline{\psi_\nu(\xi_\nu x_\nu)}\, dx_\nu\end{aligned}$$

because the product of the two factorizable functions f and $\overline{\psi}$ is again factorizable,

$$\int_{K_\nu} f_\nu(x_\nu)\overline{\psi_\nu(\xi_\nu x_\nu)}\, dx_\nu = \widehat{f}_\nu(\xi_\nu)$$

by definition, therefore

$$\begin{aligned}\prod_\nu \int_{K_\nu} f_\nu(x_\nu)\overline{\psi_\nu(\xi_\nu x_\nu)}\, dx_\nu &= \prod_\nu \widehat{f}_\nu(\xi_\nu) \\ &= \left[\otimes_\nu \widehat{f}_\nu\right](\xi)\,.\end{aligned}$$

The above calculations show that $\widehat{f}$ is factorizable and $\widehat{f} = \otimes_\nu \widehat{f}_\nu$. As we saw in the local field case, we have $\widehat{f}_\nu \in \mathscr{S}(K_\nu)$ for all ν. Let S be a finite set of places large enough to contain all the places for which ψ_ν has a conductor different from $\mathcal{O}_\nu$ and all the places for which f_ν is different from the characteristic function of $\mathcal{O}_\nu$. Then, for all $\nu \notin S$, the functions f_ν is the characteristic of $\mathcal{O}_\nu$ and the equality $\mathcal{O}_\nu^\perp = \mathcal{O}_\nu$ implies that also $\widehat{f}_\nu$ is the characteristic function of $\mathcal{O}_\nu$, as stated in Lemma 4.1.1. This proves that $\widehat{f} \in \mathscr{S}(\mathbb{A})$. □

Remark 4.1.4. Note that, in the proof of Proposition 4.1.3 for the adèlic case, we showed that the Fourier transform of a factorizable Schwartz function $f = \otimes_\nu f_\nu$ is of the same type, namely $\widehat{f} = \otimes_\nu \widehat{f}_\nu$ and $\widehat{f}_\nu = \mathbb{1}_{\mathcal{O}_\nu}$ for almost all places.

The Fourier transform of $\mathscr{S}'(\Bbbk)$ is defined by duality: if $\lambda \in \mathscr{S}'(\Bbbk)$ then its Fourier transform $\widehat{\lambda}$ is defined on Schwarz functions f by

$$\langle \widehat{\lambda}, f \rangle = \langle \lambda, \widehat{f} \rangle.$$

This is well-defined as it is the Fourier transform of a tempered distribution in the case of an archimedean local field, for a non-archimedean field there are no topological conditions to check. For the adèle ring $\mathbb{A}$, we have

$$\langle \lambda \,, \widehat{f} \rangle = \prod_\nu \langle \lambda_\nu \,, \widehat{f}_\nu \rangle$$

for all standard $\lambda = \otimes_\nu \lambda_\nu$ and $f = \otimes_\nu f_\nu$, which implies that $\widehat{\lambda} = \otimes_\nu \widehat{\lambda}_\nu$. If ν is a place such that ψ_ν has conductor $\mathcal{O}_\nu$ and $\langle \lambda_\nu \,, \mathbb{1}_{\mathcal{O}_\nu} \rangle = 1$, then $\widehat{\mathbb{1}_{\mathcal{O}_\nu}} = \mathbb{1}_{\mathcal{O}_\nu}$ and

$$\begin{aligned} \langle \widehat{\lambda}_\nu \,, \mathbb{1}_{\mathcal{O}_\nu} \rangle &= \langle \lambda_\nu \,, \widehat{\mathbb{1}_{\mathcal{O}_\nu}} \rangle \\ &= \langle \lambda_\nu \,, \mathbb{1}_{\mathcal{O}_\nu} \rangle \\ &= 1 \,, \end{aligned}$$

so $\widehat{\lambda}$ is standard. The above definition is compatible with the formula

$$\int_\Bbbk g(x) \widehat{f}(x) \, dx = \int_\Bbbk \widehat{g}(x) f(x) \, dx \,.$$

Lemma 4.1.5. *Let $f \in \mathscr{S}(\Bbbk)$ and $\lambda \in \mathscr{S}'(\Bbbk)$. Then, for any $a \in \Bbbk^\times$ we have*

$$\widehat{a \cdot f} = |a|^{-1} \left[a^{-1} \cdot \widehat{f} \right]$$

and

$$\widehat{a \cdot \lambda} = |a| \left[a^{-1} \cdot \widehat{\lambda} \right] .$$

Proof. The equality involving the function f is just an easy computation of the integral

$$\int_\Bbbk f(ax) \overline{\psi(\xi x)} \, dx \,,$$

wich is the Fourier tranform of $a \cdot f$ calculated at $\xi \in \Bbbk$. By a change of variable $x = a^{-1} y$ we get

$$\int_\Bbbk f(ax) \overline{\psi(\xi x)} \, dx = \int_\Bbbk f(y) \overline{\psi(\xi a^{-1} y)} \, |a|^{-1} \, dy \,,$$

where the right-hand side of the equality is the function $|a|^{-1}\,\widehat{f}$ calculated at ξa^{-1}. The case of λ is obtained by the previous one via duality: for all $f \in \mathscr{S}(\Bbbk)$ we have

$$\begin{aligned} \langle \widehat{a\cdot\lambda}\,, f\rangle &= \langle a\cdot\lambda\,, \widehat{f}\rangle \\ &= \langle \lambda\,, a^{-1}\cdot\widehat{f}\,\rangle \\ &= \langle \lambda\,, |a|\,\widehat{(a\cdot f)}\rangle \\ &= |a|\,\langle \widehat{\lambda}\,, a\cdot f\rangle \\ &= \Big\langle |a|\left[a^{-1}\cdot\widehat{\lambda}\,\right], f\Big\rangle\,. \end{aligned}$$

□

From Lemma 4.1.5 it's clear that the Fourier transform is not a $\Bbbk^\times$-equivariant automorphism. Still, it is an isomorphism of two closely related representations with underlying vector space $\mathscr{S}'(\Bbbk)$.

Definition 4.1.6. Let ω be character of $\Bbbk^\times$. Define the space of ω-*eigendistributions* as the $\mathbb{C}$-linear sub-space

$$\mathscr{S}'(\omega) := \left\{\lambda \in \mathscr{S}'(\Bbbk) : a\cdot\lambda = \omega(a)\lambda \quad \text{for all } a \in \Bbbk^\times\right\}$$

of $\mathscr{S}'(\Bbbk)$.

If s is a complex number and ω is a character of $\Bbbk^\times$, we use the pair (s,ω) to indicate the character $\omega|\cdot|^s$ of $\Bbbk^\times$. For example, the space $\mathscr{S}'(s,\omega)$ is made by all the distributions $\lambda \in \mathscr{S}'(\Bbbk)$ such that $a\cdot\lambda = \omega(a)|a|^s\,\lambda$ for all $a \in \Bbbk^\times$.

Theorem 4.1.7. *Let (s,ω) be a character of $\Bbbk^\times$. The Fourier transform induces an isomorphism*

$$\mathscr{S}'(s,\omega) \cong \mathscr{S}'(1-s,\omega^{-1})\,,$$

i.e., for each (s,ω)-eigendistribution λ, the Fourier transform $\widehat{\lambda}$ is an $(1-s,\omega^{-1})$-eigendistribution.

Proof. Let $\lambda \in \mathscr{S}'(s,\omega)$. We have to check that

$$a\cdot\widehat{\lambda} = \omega(a)^{-1}|a|^{1-s}\,\widehat{\lambda}$$

for all $a \in \Bbbk^\times$. Starting by the left-hand side, we have

$$a\cdot\widehat{\lambda} = |a|\,\widehat{a^{-1}\cdot\lambda}$$

by Lemma 4.1.5. Since λ is an (s,ω)-eigendistribution,

$$\begin{aligned} a^{-1}\cdot\lambda &= \omega(a^{-1})\left|a^{-1}\right|^s\lambda \\ &= \omega(a)^{-1}|a|^{-s}\,\lambda\,. \end{aligned}$$

By linearity of the Fourier transform, we get

$$\begin{aligned} |a|\,\widehat{a^{-1}\cdot\lambda} &= |a|\,\omega(a)^{-1}|a|^{-s}\,\widehat{\lambda} \\ &= \omega(a)^{-1}|a|^{1-s}\,\widehat{\lambda}\,, \end{aligned}$$

which proves that $\widehat{\lambda} \in \mathscr{S}'(1-s, \omega^{-1})$ and the Fourier transform define a linear map

$$\mathscr{S}'(s,\omega) \longrightarrow \mathscr{S}'(1-s,\omega^{-1}), \quad \lambda \longmapsto \widehat{\lambda}$$

This is an isomorphism because $(s,\omega) \mapsto (1-s,\omega^{-1})$ is an involution, the Fourier transform is invertible, its inverse is the Fourier transform relative to the unitary character ψ^{-1} and all calculations that we have made so far are valid also for ψ^{-1}, which does not change the self-dual measure dx and satisfies the same assumptions of ψ (like the non-triviality and the vanishing condition on the global field when $\mathbb{k} = \mathbb{A}$). □

The space $\mathscr{S}'(\omega)$ is the sum of all sub-representations of $\mathscr{S}'(\mathbb{k})$ isomorphic to the representation $\mathbb{C}(\omega)$, namely the 1-dimensional representation with character $\omega : \mathbb{k}^\times \to \mathbb{C}^\times$. A priori, it could be that $\mathscr{S}'(\omega) = 0$ for some ω, or $\mathscr{S}'(\omega)$ could have large dimension over $\mathbb{C}$. In Section 4.3, we will show that $\mathscr{S}'(\omega)$ has dimension 1 for all characters of a local field and all idèle class characters.

4.2 Zeta integrals

In the current section, we are going to define a functional on the space $\mathscr{S}(\mathbb{k})$ for each character or idèle class character ω. This functional is an integral that is not defined for all ω, but, when it does, it defines a non-zero ω-eigendistribution.

Definition 4.2.1. Let ω be a character of $\mathbb{k}^\times$. Define $\Re(\omega)$ as the unique real number such that $(-\Re(\omega), \omega)$ is a unitary character of $\mathbb{k}^\times$. Call it the *real part* of ω.

The intuition behind this definition comes from the fact that we can express a character ω in the form

$$\omega(x) = \chi(\tilde{x})|x|^s, \qquad \text{for } x \in \mathbb{k}^\times,$$

where s is a complex number, χ is a unitary character of the sub-group

$$\mathbb{k}^{\times,1} := \left\{x \in \mathbb{k}^\times : |x| = 1\right\},$$

and $x \mapsto \tilde{x}$ is a continuous group-homomorphism from $\mathbb{k}^\times$ to $\mathbb{k}^{\times,1}$ which is the identity on $\mathbb{k}^{\times,1}$ (it is possible by the two theorems 3.5.5 and 3.5.8). Since χ is unitary, we have that ω is unitary if and only if $\Re(s) = 0$, therefore $\Re(\omega) = \Re(s)$. Recall that s is unique up to the addition of a purely imaginary complex number, so its real part is uniquely determined by ω. The real part of ω is also the unique real number such that $|x|^{\Re(\omega)}$ is the length of the complex number $\omega(x)$ for all $x \in \mathbb{k}^\times$. Note that $\Re(s,\omega) = \Re(s) + \Re(\omega)$ for all complex numbers s.

Definition 4.2.2. Let s be a complex number, ω a character of $\mathbb{k}^\times$ and $f \in \mathscr{S}(\mathbb{k})$. Define the *zeta integral* of f associated with (s,ω) as

$$\zeta(s,\omega; f) := \int_{\mathbb{k}^\times} f(x)\omega(x)|x|^s \, d^\times x$$

when the integral exists.

Now we have to study the case of local fields separately from the case of adèles. We start by observing when $\zeta(\omega; f)$ is well-defined for a local field k.

Lemma 4.2.3. *Let $f \in \mathscr{S}(k)$ and ω a character of $k^\times$ with $\Re(\omega) > 0$. Then, the function*

$$k^\times \longrightarrow \mathbb{C}\,, \quad x \longmapsto f(x)\omega(x)$$

is integrable with respect to the measure $d^\times x$.

Proof. Let $\|\cdot\|$ be the standard norm of the complex plane, i.e. the norm defined by $\|z\|^2 = z \cdot \overline{z}$. We have to check that the function

$$\big\|f(x)\omega(x)\big\|$$

in the variable $x \in k^\times$ is $d^\times x$-integrable. Note that $\big\|\omega(x)\big\| = |x|^\sigma$, where $\sigma = \Re(\omega)$. The measure $d^\times x$ is a constant multiple of $|x|^{-1}\,dx$, therefore a positive function $g(x)$ is integrable for the measure $d^\times x$ if and only if $g(x)|x|^{-1}$ is integrable for the measure dx. We can split the integral as a sum of two integrals, one over the punctured ball

$$\dot{B} = \big\{x \in k^\times : |x| \leq 1\big\}$$

and one over its complement in $k^\times$, like this:

$$\int_{k^\times} \big\|f(x)\big\| \cdot |x|^\sigma \; d^\times x = \int_{\dot{B}} \big\|f(x)\big\| \cdot |x|^\sigma \; d^\times x + \int_{k^\times \setminus \dot{B}} \big\|f(x)\big\| \cdot |x|^\sigma \; d^\times x\,.$$

The integral

$$\int_{k^\times \setminus \dot{B}} \big\|f(x)\big\| \cdot |x|^\sigma \; d^\times x$$

is finite if and only if

$$\int_{k \setminus B} \big\|f(x)\big\| \cdot |x|^{\sigma-1} \; dx$$

is finite, where $B = \dot{B} \cup \{0\}$. The function $\big\|f(x)\big\| \cdot |x|^{\sigma-1}$ is continuous and integrable in the open set $k \setminus B$ because the Schwartz function f has compact support in the non-archimedean case and, for k archimedean, it remains integrable after multiplication by functions with polynomial growth at infinity, like $|x|^{\sigma-1}$. All these observations ensure that

$$\int_{k \setminus B} \big\|f(x)\big\| \cdot |x|^{\sigma-1} \; dx$$

is finite. Concerning the integral on B, we have

$$\int_{\dot{B}} \big\|f(x)\big\| \cdot |x|^\sigma \; d^\times x \leq \max_{x \in B} \big\|f(x)\big\| \int_{\dot{B}} |x|^\sigma \; d^\times x\,.$$

The maximum exists and is finite because f is continuous on k and B is compact. Then, it is enough to prove that

$$\int_{\dot{B}} |x|^\sigma \; d^\times x$$

is finite for $\sigma > 0$.

Suppose that $k = \mathbb{R}$. Then

$$\int_{\dot{B}} |x|^\sigma \; dx = 2 \int_0^1 x^{\sigma-1}\,dx\,,$$

which is finite for $\sigma > 0$.

Suppose that $k = \mathbb{C}$. Then we can express $x \in \mathbb{C}^\times$ in polar coordinates, which gives an isomorphism of locally compact groups

$$\varphi : \mathbb{R}_+^\times \times \mathbb{S}^1 \longrightarrow \mathbb{C}^\times\,, \quad (r,u) \longmapsto ru$$

Let $d^\times r\, du$ be the product of the Haar measures $d^\times r = r^{-1}dr$ of $\mathbb{R}_+^\times$ and du of $\mathbb{S}^1$, where dr is the standard measure of $\mathbb{R}$ and du is the unique Haar measure that gives measure 1 to the circle. By uniqueness of the Haar measure, there is a constant $m > 0$ such that the pull-back of $m\, d^\times r\, du$ by φ^{-1} is equal to $d^\times x$. Then

$$\begin{aligned}\int_{\dot{B}} |x|^\sigma \; d^\times x &= m \int_{(0,1)\times\mathbb{S}^1} r^{2\sigma}\, d^\times r\, du \\ &= m \left(\int_{(0,1)} r^{2\sigma} d^\times r \right) \cdot \left(\int_{\mathbb{S}^1} du \right) \\ &= m \int_{(0,1)} r^{2\sigma} d^\times r \\ &= m \int_0^1 r^{2\sigma-1}\, dr\,,\end{aligned}$$

and the last integral is finite if and only if $\sigma > 0$.

Suppose k is non-archimedean. Then $B = \mathcal{O}$ and $\mathcal{O}$ is the disjoint union of zero and all the sets $\pi^n\mathcal{O}^\times$ for $n \in \mathbb{N}$, where π is a uniformizer of k, so

$$\int_{\mathcal{O}\setminus\{0\}} |x|^\sigma \; d^\times x = \sum_{n=0}^{\infty} \int_{\pi^n\mathcal{O}^\times} |x|^\sigma \; d^\times x\,.$$

The homomorphism $|\cdot|$ is trivial on $\mathcal{O}^\times$ and the measure $d^\times x$ is invariant under multiplication of π, hence

$$\int_{\pi^n\mathcal{O}^\times} |x|^\sigma \; d^\times x = q^{-n\sigma} \int_{\mathcal{O}^\times} d^\times x\,,$$

where $|\pi| = q^{-1}$ and q is the cardinality of the residue field of k. Therefore

$$\int_{\mathcal{O}\setminus\{0\}} |x|^\sigma \; d^\times x = \left(\int_{\mathcal{O}^\times} d^\times x \right) \sum_{n=0}^{\infty} q^{-n\sigma}\,.$$

The series $1 + q^{-\sigma} + q^{-2\sigma} + q^{-3\sigma} + \dots$ is geometric, hence it converges if and only if $q^{-\sigma} < 1$ and this happens if and only if $\sigma > 0$. □

As examples, we make some explicit calculations of $\zeta(\omega; f)$ for specific functions f, which are useful later.

Example 4.2.4 (k non-archimedean, ω unramified)**.** Suppose that k is a non-archimedean local field and ω is trivial on the group of units $\mathcal{O}^\times$ and has a positive real part. Let f^o be the characteristic function of the local ring $\mathcal{O}$. It belongs to the space $\mathscr{S}(k)$, so we can compute its zeta integral:

$$\zeta(\omega; f^o) = \int_{\mathcal{O}\setminus\{0\}} \omega(x)\, d^\times x\,.$$

The domain of integration is the disjoint union

$$\mathcal{O} \setminus \{0\} = \bigcup_{n=0}^{\infty} \pi^n \mathcal{O}^\times \,,$$

therefore the integral splits into the infinite sum

$$\begin{aligned}
\int_{\mathcal{O}\setminus\{0\}} \omega(x)\, d^\times x &= \sum_{n=0}^{\infty} \int_{\pi^n \mathcal{O}^\times} \omega(x)\, d^\times x \\
&= \sum_{n=0}^{\infty} \int_{\mathcal{O}^\times} \omega(\pi^n x)\, d^\times x \\
&= \sum_{n=0}^{\infty} \omega(\pi)^n \int_{\mathcal{O}^\times} \omega(x)\, d^\times x
\end{aligned}$$

and $\omega(x) = 1$ for every $x \in \mathcal{O}^\times$, hence

$$\begin{aligned}
\zeta(\omega; f^o) &= \left(\int_{\mathcal{O}^\times} d^\times x \right) \sum_{n=0}^{\infty} \omega(\pi)^n \\
&= \frac{\int_{\mathcal{O}^\times} d^\times x}{1 - \omega(\pi)} \,.
\end{aligned}$$

If the Haar measure is normalized to give $\mathcal{O}^\times$ measure 1, the cardinality of the residue field is q, and s is a complex number for which $\omega = |\cdot|^s$, we get the familiar local Euler factor

$$\zeta(\omega; f^o) = \frac{1}{1 - q^{-s}} \,.$$

For a general unramified character ω of $k^\times$, we have that

$$\zeta(s, \omega; f^o) = \frac{1}{1 - \omega(\pi) q^{-s}}$$

for all complex number s in the half-plane $\Re(s) > -\Re(\omega)$. Note that the right-hand side of the equality is a holomorphic and non-vanishing function in the variable s for $\Re(s) > -\Re(\omega)$, and it extends to a meromorphic function on the entire complex plane with poles in the vertical line $\Re(\omega) + i\mathbb{R}$. Observe that we could have tested the zeta integral on the characteristic function of the group of units and would have got the simple value

$$\begin{aligned}
\zeta(\omega; \mathbb{1}_{\mathcal{O}^\times}) &= \int_{\mathcal{O}^\times} \omega(x)\, d^\times x \\
&= \int_{\mathcal{O}^\times} d^\times x.
\end{aligned}$$

Example 4.2.5 (k non-archimedean, ω ramified)**.** There is a complex number s with $\Re(s) = \Re(\omega)$ and a unitary character χ of $k^\times$ such that

$$\omega(x) = \chi(x) |x|^s$$

for all $x \in k^\times$, and χ is non-trivial on $\mathcal{O}^\times$. In this case, the zeta integral of the characteristic function of $\mathcal{O}$ is null because

$$\zeta(\omega; \mathbb{1}_\mathcal{O}) = \left(\int_{\mathcal{O}^\times} \chi(x)\, d^\times x \right) \sum_{n=0}^{\infty} q^{-ns}$$

by repeating the same calculations as in Example 4.2.4 and

$$\int_{\mathcal{O}^\times} \chi(x)\, d^\times x = 0\,,$$

since χ is non-trivial on $\mathcal{O}^\times$ (see Example 2.2.5). Define $f^o = \omega^{-1}\mathbb{1}_{\mathcal{O}^\times}$, then f^o is supported in the open, compact group $\mathcal{O}^\times$ and it is locally constant because $\omega(\mathcal{O}^\times)$ is a finite sub-group of $\mathbb{C}^\times$. Thus $f^o \in \mathscr{S}(k)$ and

$$\begin{aligned}\zeta(\omega; f^o) &= \int_{\mathcal{O}^\times} \omega(x)^{-1}\omega(x)\, d^\times x \\ &= \int_{\mathcal{O}^\times} d^\times x.\end{aligned}$$

When $\mathcal{O}^\times$ has measure 1 for $d^\times x$, we get

$$\zeta(\omega; f^o) = 1.$$

Example 4.2.6 ($k = \mathbb{R}$, ω unramified)**.** In the real case, if $\omega(-1) = 1$, then

$$\zeta(s, \omega; f) = 2 \int_0^{\infty} f(x)\omega(x)x^{s-1}\, dx$$

for all s with $\Re(s) > -\Re(\omega)$ and all even Schwartz functions f. The character ω has only the effect of translating the domain of the function in the variable s, there is no loss of generality if we assume $\omega = 1$. For all $x \in \mathbb{R}$, let

$$f^o(x) = e^{-\pi x^2}\,.$$

The function f^o is of Schvartz type and even, the zeta integral is

$$\zeta(s, 1; f^o) = 2 \int_0^{\infty} e^{-\pi x^2} x^{s-1}\, dx\,,$$

and with the change of variable $\pi x^2 = t$, we obtain

$$\zeta(s, 1; f^o) = \pi^{-\frac{s}{2}}\, \Gamma\left(\frac{s}{2}\right),$$

where $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$ is the Gamma function in the complex variable z.

Example 4.2.7 ($k = \mathbb{C}$, ω unramified)**.** As in Example 4.2.6, we can suppose that $\omega = 1$ for the unramified case. Let f^o be the function in the variable $x \in \mathbb{C}$

$$f^o(x) = e^{-2\pi|x|}\,.$$

We can write the measure $d^\times x$ as the pull-back of $m\, d^\times r\, du$ via the isomorphism $\mathbb{R}_+^\times \times \mathbb{S}^1 \cong \mathbb{C}^\times$, just like we did in the proof of Lemma 4.2.3 for $k = \mathbb{C}$. The constant m can be calculated in the following way: dx is twice the Lebesgue measure of the plane and $d^\times x = |x|^{-1}\, dx$, so

$$\int_{\dot{B}} |x|\ d^\times x = 2\pi\,,$$

where $\dot{B}$ is the punctured disk of radius 1 centred at the origin. By pull-back, we have the equality of the integrals

$$\int_{\dot{B}} |x|\, d^\times x = m \int_{(0,1)\times\mathbb{S}^1} r^2\, d^\times r\, du\,.$$

The left-hand side of the equality is 2π and the right-hand side is equal to $\frac{m}{2}$, thus $m = 4\pi$. We can also evaluate the zeta integral of f^o through the measure $4\pi\, d^\times r\, du$:

$$\begin{aligned}\zeta(s,1;f^o) &= \int_{\mathbb{C}^\times} e^{-2\pi|x|}|x|^s\ d^\times x \\ &= 4\pi \int_{\mathbb{R}_+^\times\times\mathbb{S}^1} e^{-2\pi r^2} r^{2s}\, d^\times r\, du\,.\end{aligned}$$

Since the integrand depends only on the variable r and the circle has measure 1 with respect to the measure du, we are left with

$$\zeta(s,1;f^o) = 4\pi \int_{\mathbb{R}_+^\times} e^{-2\pi r^2} r^{2s}\, d^\times r$$

and, after a change of variable $r = \frac{t}{\sqrt{2}}$,

$$\zeta(s,1;f^o) = 2\pi\cdot 2\int_{\mathbb{R}_+^\times} e^{-\pi t^2} t^{2s} 2^{-s}\, d^\times t\,.$$

One recognizes from the example of the real numbers that

$$2\int_{\mathbb{R}_+^\times} e^{-\pi t^2} t^{2s}\, d^\times t = \pi^{-s}\Gamma(s)\,,$$

hence

$$\zeta(s,1;f^o) = (2\pi)^{1-s}\Gamma(s)\,.$$

Example 4.2.8 (k archimedean, ω ramified)**.** If $k = \mathbb{R}$ and ω is ramified, then the character must be of the form

$$\omega(x) = x^{-1}|x|^{s+1} \qquad x \in \mathbb{R}^\times$$

by Theorem 3.5.8. The real part of ω is $\Re(s)$. In this case, we can define

$$f^o(x) = xe^{-\pi x^2}\,, \qquad x \in \mathbb{R}\,,$$

which is a Schwartz function, and we get

$$\begin{aligned}\zeta(\omega;f^o) &= 2\int_0^\infty e^{-\pi x^2} x^{s+1}\, d^\times x \\ &= \pi^{-\frac{s+1}{2}}\,\Gamma\left(\frac{s+1}{2}\right).\end{aligned}$$

The case $k = \mathbb{C}$ is similar. By Theorem 3.5.8, the character ω must be of the form

$$x^{-n}|x|^{s+\frac{n}{2}} \qquad \text{or} \qquad \overline{x}^{-n}|x|^{s+\frac{n}{2}}$$

for some $s \in \mathbb{C}$, $n \in \mathbb{N}$ and the functions

$$x^n e^{-2\pi x\overline{x}} \qquad \text{and} \qquad \overline{x}^n e^{-2\pi x\overline{x}}$$

belong to the Schwartz class. Let f^o be chosen between the two Schwartz functions above in such a way that the ramified part of ω is tamed, then

$$\zeta(\omega; f^o) = (2\pi)^{1-s-\frac{n}{2}}\Gamma(s + \frac{n}{2}).$$

Definition 4.2.9. For the choices of f^o made in the previous examples, define the *local L-function* associated with ω as the unique meromorphic function $L(s, \omega)$ in the complex variable s for which

$$L(s, \omega) = \zeta(s, \omega; f^o)$$

in the half-plane $\Re(s) > -\Re(\omega)$.

Note that, for every one of the examples we made above, the meromorphic function $L(s, \omega)$ is non-vanishing in the entire complex plane and is holomorphic in the half plane where the zeta integral is defined.

Theorem 4.2.10. *Let ω be a character of $k^\times$ with $\Re(\omega) > 0$. Then the zeta integral*

$$\zeta(\omega) : \mathscr{S}(k) \longrightarrow \mathbb{C}\,, \quad f \longmapsto \zeta(\omega; f)$$

is a non-zero ω-eigendistribution.

Proof. From Lemma 4.2.3, it is immediate to note that

$$\zeta(\omega) : \mathscr{S}(k) \longrightarrow \mathbb{C}\,, \quad f \longmapsto \zeta(\omega; f)$$

is a well-defined linear form on $\mathscr{S}(k)$ for all characters ω satisfying $\Re(\omega) > 0$. This is enough for $\zeta(\omega)$ to be in the space $\mathscr{S}'(k)$ for k non-archimedean. If the local field is archimedean, the zeta integral $\zeta(\omega)$ is continuous with respect to the family of semi-norms defining the topology of $\mathscr{S}(k)$. Indeed, a sequence $(f_n)_n$ of Schwartz functions converges to 0 if and only if

$$\sup_{x\in k}\left\|p(x) \cdot \partial f_n(x)\right\|$$

goes to zero as $n \to \infty$ for all ∂ and all $p(x)$, where ∂ is a composition of partial derivatives and $p(x)$ is a continuous function on k with polynomial growth at infinity. One can check that the sequence $\zeta(\omega; f_n)$ approaches zero by splitting the integral as in the proof of Lemma 4.2.3:

$$\zeta(\omega; f_n) = \int_B f_n(x)\omega(x)|x|^{-1}dx + \int_{k\setminus B} f_n(x)\omega(x)|x|^{-1}dx\,.$$

The integral over B is bounded by the sup-norm of f_n multiplied by the constant

$$\int_k |x|^{\Re(\omega)-1}\, dx\,,$$

hence it goes to zero as $n \to \infty$. The integral over the complement of B has no issues because $|x|^{-1}$ is continuous on $k \setminus B$. Moreover, it is bounded by the integral

$$\int_{k\setminus B} \|f_n(x)\| \cdot |x|^{\Re(\omega)-1} \, dx$$

which can be expressed as

$$\int_{k\setminus B} \|f_n(x)\| \cdot |x|^N \cdot |x|^{-N+\Re(\omega)-1} \, dx \,,$$

for all natural numbers N, leading to the inequality

$$\int_{k\setminus B} \|f_n(x)\| \cdot |x|^{\Re(\omega)-1} \, dx \leq \sup_{x\in k\setminus B} \left\| |x|^N f_n(x) \right\| \cdot \int_{k\setminus B} |x|^{-N+\Re(\omega)-1} \, dx \,.$$

The integral of $|x|^{-N+\Re(\omega)-1}$ over $k \setminus B$ is finite for N sufficiently large and the sup-norm

$$\sup_{x\in k\setminus B} \left\| |x|^N f_n(x) \right\|$$

goes to zero as $n \to \infty$. This provides the convergence $\zeta(\omega; f_n) \to 0$, hence $\zeta(\omega) \in \mathscr{S}'(k)$.

If $a \in k^\times$, then

$$\begin{aligned}
\langle a \cdot \zeta(\omega) \,, f \rangle &= \zeta(\omega; a^{-1} \cdot f) \\
&= \int_{k^\times} f(xa^{-1})\omega(x) \, d^\times x \\
&= \int_{k^\times} f(x)\omega(ax) \, d^\times x \\
&= \omega(a) \int_{k^\times} f(x)\omega(x) \, d^\times x \\
&= \langle \omega(a)\zeta(\omega) \,, f \rangle
\end{aligned}$$

for all functions $f \in \mathscr{S}(k)$, so $\zeta(\omega) \in \mathscr{S}'(\omega)$. To conclude, note that in the examples 4.2.4, 4.2.5, 4.2.6, 4.2.7 and 4.2.8, we gave a function $f^o \in \mathscr{S}(k)$ for each character ω such that $\zeta(\omega; f^o) \neq 0$, thus $\zeta(\omega)$ is a non-zero vector in the space $\mathscr{S}'(\omega)$. □

Remark 4.2.11. The zeta integral $\zeta(s, \omega; f)$ is understood as a functional operating on a Schwartz function f, but we can fix ω, f and let vary the complex number s in the half plane $\Re(s) > -\Re(\omega)$. Here, $\zeta(s, \omega; f)$ defines a holomorphic function: for all s_0 with real part $\Re(s_0) > \Re(\omega)$, we can express $|x|^s$ as a power series centered in s_0

$$|x|^s = |x|^{s_0} \sum_{n=0}^{\infty} \frac{\log^n |x|}{n!} (s - s_0)^n$$

and use it to obtain a power series representation of the zeta integral

$$\zeta(s, \omega; f) = \sum_{n=0}^{\infty} \frac{(s - s_0)^n}{n!} \int_{k^\times} f(x)\omega(x)|x|^{s_0} \log^n |x| \, d^\times x \,. \tag{4.1}$$

The verification that the sum can be interchanged with the integral and that the power series (4.1) has a positive radius of convergence is reduced to the estimate of the integral

$$\int_{k^\times} \frac{\left|\log|x|\right|_\infty^n}{n!} \left\|f(x)\right\| \cdot |x|^{\Re(s_0)+\Re(\omega)} \, d^\times x \,.$$

This is accomplished in a way similar to calculations made in the proof of Lemma 4.2.3. On the open set of elements x with $|x| > 1$ we can control the integral of

$$\frac{\log^n |x|}{n!} \left\|f(x)\right\| \cdot |x|^{\Re(s_0)+\Re(\omega)}$$

uniformly in n thanks to the properties of f as a Schwartz function. On the compact ball of elements x with $|x| \leq 1$ we can ignore $f(x)$ and calculate

$$\int_{0 \lhd |x| \leq 1} \frac{\left|\log|x|\right|_\infty^n}{n!} |x|^{\Re(s_0)+\Re(\omega)} \, d^\times x$$

explicitly. We would get a value bounded uniformly in n by a constant that depends on $\Re(s_0)+\Re(\omega)$.

Now, we can use the knowledge gained on the local zeta integral to analyse its global version. Let K be a global field with adèle ring $\mathbb{A}$ and denote the places of K by ν as always. Let $\omega = \otimes_\nu \omega_\nu$ be a character of the group $\mathbb{A}^\times$ and $f = \otimes_\nu f_\nu$ a standard function in the space $\mathscr{S}(\mathbb{A})$. Let $\zeta(s, \omega; f)$ be the global zeta integral. Since f is factorizable, $\zeta(s, \omega; f)$ is a product of integrals

$$\int_{\mathbb{A}^\times} f(x)\omega(x)|x|^s \, d^\times x = \prod_\nu \int_{K_\nu^\times} f_\nu(x_\nu)\omega_\nu(x_\nu)|x_\nu|_\nu^s \, d^\times x_\nu \,,$$

where the reader may recognize the local zeta integral

$$\zeta(s, \omega_\nu; f_\nu) = \int_{K_\nu^\times} f_\nu(x_\nu)\omega_\nu(x_\nu)|x_\nu|_\nu^s \, d^\times x_\nu$$

as the factor corresponding to the place ν. If necessary, a subscript ν will be added in the notation to emphasise when the zeta integral is global or local. We do the same for the L-functions:

$$L_\nu(s, \omega_\nu) = \begin{cases} \dfrac{1}{1 - \omega_\nu(\pi_\nu) q_\nu^{-s}} & \text{if } \nu \text{ is non-archimedean and } \omega_\nu \text{ is unramified,} \\ 1 & \text{if } \nu \text{ is non-archimedean and } \omega_\nu \text{ is ramified,} \\ \pi^{-\frac{s}{2}}\, \Gamma\left(\frac{s}{2}\right) & \text{if } K_\nu = \mathbb{R} \text{ and } \omega_\nu(x_\nu) = x_\nu^{-n},\ n \in \{0,1\}, \\ (2\pi)^{1-s}\Gamma(s) & \text{if } K_\nu = \mathbb{C} \text{ and } \omega_\nu(x_\nu) = x_\nu^{-n} \text{ or } \omega_\nu(x_\nu) = \overline{x}_\nu^{-n},\ n \in \mathbb{N}\,. \end{cases}$$

Definition 4.2.12. For a finite set of places S, define

$$L^S(s, \omega) := \prod_{\nu \notin S} L_\nu(s, \omega_\nu)$$

for s in the right half-plane $\Re(s) > 1 - \Re(\omega)$.

The condition $\Re(s) > 1 - \Re(\omega)$ is necessary to make the product converge absolutely. The function $L^S(s, \omega)$ is holomorphic and non-vanishing.

Theorem 4.2.13. *Let K be a global field and ω a character of its adèle ring $\mathbb{A}$. Suppose that $\Re(\omega) > 1$, then the global zeta integral*

$$\zeta(\omega) : \mathscr{S}(\mathbb{A}) \longrightarrow \mathbb{C}\,, \quad f \longmapsto \zeta(\omega; f)$$

defines a non-zero ω-eigendistribution.

Proof. By linearity of the integral, it is enough to test $\zeta(\omega)$ against a standard Schwartz function $f = \otimes_\nu f_\nu$. For all places ν, the real part of ω_ν is $\sigma := \Re(\omega)$ indipendently on ν. This is because $\omega(x)|x|^{-\sigma}$ is unitary and its local component at ν is $\omega_\nu(x_\nu)|x_\nu|_\nu^{-\sigma}$, which must also be unitary since it is the composition of the global unitary character with the natural embedding $K_\nu^\times \hookrightarrow \mathbb{A}^\times$. By hypothesis, $\sigma > 1$, hence, from Theorem 4.2.10, we know that the local zeta integrals $\zeta_\nu(\omega_\nu)$ are well-defined ω_ν-eigendistributions. Recall that, for almost all places ν of K, the local character ω_ν must be unramified and $f_\nu = \mathbb{1}_{\mathcal{O}_\nu}$. Let S be a finite set of places large enough to include:

all infinite places;

all the places ν such that $f_\nu \neq \mathbb{1}_{\mathcal{O}_\nu}$;

every place ν such that ω_ν is ramified.

Then

$$\begin{aligned}
\zeta(\omega; f) &= \prod_\nu \zeta_\nu(\omega_\nu; f_\nu) \\
&= \prod_{\nu \notin S} L_\nu(0, \omega_\nu) \cdot \prod_{\nu \in S} \zeta_\nu(\omega_\nu; f_\nu) \\
&= L^S(0, \omega) \prod_{\nu \in S} \zeta_\nu(\omega_\nu; f_\nu)
\end{aligned}$$

and $L^S(0, \omega)$ is finite and non-zero because $0 > 1 - \Re(\omega)$, hence $\lambda := L(0, \omega)^{-1}\zeta(\omega)$ is a well-defined standard distribution in the space $\mathscr{S}'(\mathbb{A})$ of the form $\otimes_\nu \lambda_\nu$, where

$$\lambda_\nu = \begin{cases} \zeta_\nu(\omega_\nu) & \text{if } \nu \in S, \\ L_\nu(0, \omega_\nu)^{-1}\zeta_\nu(\omega_\nu) & \text{if } \nu \notin S. \end{cases}$$

The property $\zeta(\omega) \in \mathscr{S}'(\omega)$ follow easely from the fact that $\zeta_\nu(\omega_\nu) \in \mathscr{S}'(\omega_\nu)$ for all ν, as we are going to see. If $a \in \mathbb{A}^\times$, we can enlarge S by including the places ν for which $a_\nu \notin \mathcal{O}_\nu^\times$, then $a_\nu^{-1} \cdot f_\nu = f_\nu$ for all $\nu \notin S$ because $\mathbb{1}_{\mathcal{O}_\nu}$ is invariant under the action of $\mathcal{O}_\nu^\times$. By this, $a^{-1} \cdot f$ is again a standard function of $\mathscr{S}(\mathbb{A})$, hence $\zeta(\omega; a^{-1} \cdot f)$ decomposes as a product.

$$\begin{aligned}
\langle a \cdot \zeta(\omega)\,, f\rangle &= \langle \zeta(\omega)\,, a^{-1} \cdot f\rangle && \text{by duality,}\\
&= \zeta(\omega; a^{-1} \cdot f)\\
&= \prod_{\nu} \zeta_\nu(\omega_\nu; a_\nu^{-1} \cdot f_\nu) && \text{because } a^{-1} \cdot f \text{ is factorizable,}\\
&= \prod_{\nu} \langle a_\nu \cdot \zeta_\nu(\omega_\nu)\,, f_\nu\rangle && \text{by duality again,}\\
&= \prod_{\nu} \langle \omega_\nu(a_\nu)\zeta_\nu(\omega_\nu)\,, f_\nu\rangle && \text{because } \zeta_\nu(\omega_\nu) \in \mathscr{S}'(\omega_\nu),\\
&= \langle \omega(a)\zeta(\omega)\,, f\rangle.
\end{aligned}$$

This is enough to prove that $\zeta(\omega) \in \mathscr{S}'(\omega)$. □

Observe that also in the global case, the zeta integral defines a non-vanishing holomorphic function

$$\zeta(s, \omega; f)$$

in the complex variable s belonging to the right half-plane $\Re(s) > 1 - \Re(\omega)$. This follows from Remark 4.2.11, as f is a linear combination of standard functions and, on a standard function, the zeta integral is a product

$$\zeta(s, \omega; \otimes_\nu f_\nu) = L^S(s, \omega) \prod_{\nu \in S} \zeta_\nu(s, \omega_\nu; f_\nu)$$

of a finite number of holomorphic functions, given a large enough finite set S of places.

4.3 Dimension of eigendistribution spaces

As anticipated at the end of Section 4.1, we will prove that the spaces of eigenditributions $\mathscr{S}'(\omega)$ are one-dimensional, following an argument of A. Weil presented in the chapter [Kud04] written by S. Kudla. The goal of the current section is to prove the following result: let $\Bbbk$ be a local field or a ring of adèles, ω a character of $\Bbbk^\times$ and $\mathscr{S}'(\omega)$ the sub-space of $\mathscr{S}'(\Bbbk)$ consisting of ω-eigendistributions,

Theorem 4.3.1. *The space $\mathscr{S}'(\omega)$ has dimension one as a vector space over $\mathbb{C}$.*

The strategy is to prove the local version of Theorem 4.3.1 since the result for the adèles can be deduced from the local case. Let k be a local field and ω a character of its multiplicative group. Note that the zeta integral $\zeta(\omega; f)$ would have been naturally well-defined if we had considered it on functions f with compact support in the open set $k \setminus \{0\}$. Indeed, if ω is unitary and $f \in \mathscr{C}_c(k^\times)$, the integral

$$\int_{k^\times} f(x)\omega(x)\, d^\times x$$

computes the Fourier transform of f at ω^{-1} relatively to the multiplicative group structure of $k^\times$. The first step toward the proof of Theorem 4.3.1 consists of passing from $\mathscr{S}(k)$ and $\mathscr{S}'(k)$ to spaces of functions defined on the locally compact abelian group $k^\times$ defining representations of it and try to obtain uniqueness results for the sub-representations isomorphic to $\mathbb{C}(\omega)$.

Definition 4.3.2. Denote by $\mathscr{D}(k^\times)$ the space of functions $f \in \mathscr{S}(k)$ with compact support in $k \setminus \{0\}$, called *compactly supported smooth functions* of $k^\times$. If k is non-archimedean, let $\mathscr{D}(k^\times)$ be endowed with the topology that makes all linear forms continuous. If k is archimedean, $\mathscr{D}(k^\times)$ is identified with the classical topological vector space of compactly supported smooth functions on the open set $\mathbb{R}^n \setminus \{0\}$, for $n = 1, 2$. Define the space of *distributions* on $k^\times$ as the topological dual $\mathscr{D}'(k^\times)$ of $\mathscr{D}(k^\times)$.

Given a distribution $\lambda \in \mathscr{S}'(k)$, we can consider its restriction to the sub-space $\mathscr{D}(k^\times)$, where it defines an element of $\mathscr{D}'(k^\times)$, denotet by $\lambda|_{k^\times}$. The spaces $\mathscr{D}(k^\times)$ and $\mathscr{D}'(k^\times)$ are representations of $k^\times$, the action being defined in the same way as for $\mathscr{S}(k)$ and $\mathscr{S}'(k)$. The space $\mathscr{D}'(\omega)$ makes sense and consists of the distributions $\lambda \in \mathscr{D}'(k^\times)$ such that $a \cdot \lambda = \omega(a)\lambda$ for all $a \in k^\times$. It is straightforward to note that the restriction of a distribution in $\mathscr{S}'(\omega)$ belongs to $\mathscr{D}'(\omega)$.

Lemma 4.3.3. *For every character ω of $k^\times$, the complex vector space $\mathscr{D}'(\omega)$ is one-dimensional and generated by the distribution*

$$\omega(x)\, d^\times x\,,$$

i.e., given any $\lambda \in \mathscr{D}'(\omega)$, there is a complex number c such that

$$\langle \lambda\,, f\rangle = c\int_{k^\times} f(x)\omega(x)\, d^\times x$$

for all functions $f \in \mathscr{D}(k^\times)$.

Proof. Let $\lambda \in \mathscr{D}'(\omega)$. If we pre-compose λ with the continuous, linear endomorphism

$$\mathscr{D}(k^\times) \longrightarrow \mathscr{D}(k^\times)\,, \quad f \longmapsto \omega^{-1} f\,,$$

given by point-wise multiplication of a function by the inverse of ω, we obtain a distribution $\tilde\lambda$ which is invariant by the action of $k^\times$, indeed, for all $a \in k^\times$ and all $f \in \mathscr{D}(k^\times)$,

$$\begin{aligned}
\langle a\cdot\tilde\lambda\,, f\rangle &= \langle \tilde\lambda\,, a^{-1}\cdot f\rangle && \text{by duality,}\\
&= \langle \lambda\,, \omega^{-1}(a^{-1}\cdot f)\rangle && \text{by definition of } \tilde\lambda,\\
&= \omega(a)^{-1}\langle \lambda\,, a^{-1}\cdot(\omega^{-1} f)\rangle && \text{because } \omega \text{ is multiplicative,}\\
&= \omega(a)^{-1}\langle a\cdot\lambda\,, \omega^{-1} f\rangle && \text{by duality again,}\\
&= \omega(a)^{-1}\omega(a)\langle \lambda\,, \omega^{-1} f\rangle && \text{since } \lambda \in \mathscr{D}'(\omega),\\
&= \langle \tilde\lambda\,, f\rangle,
\end{aligned}$$

thus $a\cdot\tilde\lambda = \tilde\lambda$ for all $a \in k^\times$. If we could prove that $\tilde\lambda$ operates on functions as the Haar measure multiplied by a constant complex number c, we would have

$$\begin{aligned}
\langle \lambda\,, f\rangle &= \langle \tilde\lambda\,, \omega f\rangle\\
&= c\int_{k^\times} f(x)\omega(x)\, d^\times x\,,
\end{aligned}$$

as desired. We can assume without loss of generality that $\omega = 1$ and proceed to prove that there is a constant $c \in \mathbb{C}$ such that

$$\langle \lambda\,, f\rangle = c\int_{k^\times} f(x)\, d^\times x \tag{4.2}$$

for all $f \in \mathscr{D}(k^\times)$. Note that, if λ extends to a $k^\times$-invariant, continuous functional on the space $\mathscr{C}_c(k^\times)$, then it defines a (complex-valued) Haar measure, hence it must be proportional to $d^\times x$ by the uniqueness of the Haar measure on locally compact abelian groups. In the non-archimedean case, one can prove that λ is of the form (4.2) by working directly with the functions $f \in \mathscr{D}(k^\times)$ expressed in the simple form of Remark 4.1.2. An argument following these lines can be found in the proof of Proposition 4.3.2 of [Bum97]. For k archimedean, we have that the additive group of k and the multiplicative group $k^\times$ are locally isomorphic, so λ can be pulled back locally on k. The fact that λ is invariant by the action of $k^\times$ implies that its local pull-backs are locally invariant by translation on k. Hence λ is locally constant on $k^\times$. In particular, all its derivatives are zero, implying that λ extends to $\mathscr{C}_c(k^\times)$. □

From Lemma 4.3.3, we have that for any $\lambda \in \mathscr{S}'(\omega)$, there is a complex number $c \in \mathbb{C}$ such that the restriction $\lambda|_{k^\times}$ is equal to $c\omega(x)\, d^\times x$. This means that

$$\langle \lambda\,, f\rangle = c\zeta(\omega; f)$$

for all Schwartz functions f whith compact support contained in $k^\times$. For other distributions $\lambda' \in \mathscr{S}'(\omega)$ with $\lambda'|_{k^\times} = c\omega(x)\, d^\times x$, we have that $\lambda - \lambda'$ is an ω-eigendistribution which vanishes on the space $\mathscr{D}(k^\times)$. The kernel of the restriction map from $\mathscr{S}'(k)$ to $\mathscr{D}'(k^\times)$ is the space of distributions supported at 0, denoted by $\mathscr{S}'_0(k)$, its intersection with $\mathscr{S}'(\omega)$ is denoted by $\mathscr{S}'_0(\omega)$. For k non-archimedean, $\mathscr{S}'_0(k)$ is generated by the Dirac delta distribution δ_0 wich sends a function $f \in \mathscr{S}(k)$ to $f(0)$. Indeed, if $\lambda \in \mathscr{S}'(k)$ is a non-zero distribution that vanishes on $\mathscr{D}(k^\times)$, then

$$\langle \lambda\,, f - f(0)g(0)^{-1}g\rangle = 0 \tag{4.3}$$

for all $f \in \mathscr{S}(k)$, where g is a fixed Schwartz function such that $\langle \lambda\,, g\rangle \neq 0$. This is because $f - f(0)g(0)^{-1}g$ is locally constant with compact support and vanishes in zero, hence it vanishes on a whole neighbourhood of it and this ensures that $f - f(0)g(0)^{-1}g$ belongs to $\mathscr{D}(k^\times)$. From Equation 4.3 we obtain

$$\lambda = g(0)^{-1}\langle \lambda\,, g\rangle\delta_0\,.$$

For k archimedean, it is known that the derivatives of δ_0 generate the space of distributions supported at 0.

Lemma 4.3.4. *The space $\mathscr{S}'_0(\omega)$ is at most 1-dimensional. Precisely,*

(i) *if k is non-archimedean, then $\mathscr{S}'_0(\omega) \neq 0$ if and only if ω is the trivial character, in which case $\mathscr{S}'_0(\omega)$ is generated by δ_0;*

(ii) *if $k = \mathbb{R}$, then $\mathscr{S}'_0(\omega) \neq 0$ if and only if ω is the character x^{-n} for some $n \in \mathbb{N}$, in which case $\mathscr{S}'_0(\omega)$ is generated by the n-th derivative of δ_0;*

(iii) *if $k = \mathbb{C}$, then $\mathscr{S}'_0(\omega) \neq 0$ if and only if ω is of the form $x^{-l}\overline{x}^{-m}$ for some $l, m \in \mathbb{N}$, in which case $\mathscr{S}'_0(\omega)$ is generated by $\partial^l\,\overline{\partial}^m\delta_0$, where ∂ and $\overline{\partial}$ are the partial derivative in the variables x and $\overline{x}$ respectively.*

Proof. If k is a non-archimedean local field, the space $\mathscr{S}'_0(k)$ is 1-dimensional and generated by δ_0. For all $a \in k^\times$ we have $a \cdot \delta_0 = \delta_0$ because the multiplication by a does not change the value of a Schwartz function at the origin, thus δ_0 is an ω-eigendistribution if and only if $\omega = 1$. Suppose $k = \mathbb{R}$. Let ∂ be the derivative operator of distributions associated with the derivative of smooth functions on the real line. For all non-zero real number a and each function $f \in \mathscr{S}(\mathbb{R})$, the n-th derivative of the function $f(a^{-1}x)$ in the variable x is $a^{-n}f^{(n)}(a^{-1}x)$, hence

$$\begin{aligned}\langle a \cdot \partial^n \delta_0 \,, f\rangle &= \langle \partial^n \delta_0 \,, a^{-1} \cdot f\rangle \\ &= (-1)^n \langle \delta_0 \,, a^{-n}(a^{-1} \cdot f^{(n)})\rangle \\ &= a^{-n}(-1)^n f^{(n)}(0) \\ &= \langle a^{-n}\partial^n \delta_0 \,, f\rangle.\end{aligned}$$

If $\lambda = \sum_{j=0}^{N} c_j \partial^j \delta_0$ is a general element of $\mathscr{S}_0'(\mathbb{R})$, we have that

$$a \cdot \lambda = \sum_{j=0}^{N} a^{-j} \partial^j \delta_0 \,,$$

thus the property $a \cdot \lambda = \omega(a)\lambda$ is satisfied for each $a \in \mathbb{R}^\times$ if and only if there is a $n \in \{0, \ldots, N\}$ such that $\omega(a) = a^{-n}$ for all $a \in \mathbb{R}^\times$ and $c_j = 0$ for all $j \neq n$. The case of $k = \mathbb{C}$ is analogous. □

The restriction of distributions produces an exact sequence of vector spaces

$$0 \longrightarrow \mathscr{S}_0'(\omega) \hookrightarrow \mathscr{S}'(\omega) \longrightarrow \mathscr{D}'(\omega) = \mathbb{C}\,\omega(x)\,d^\times x \,.$$

By lemmas 4.3.3 and 4.3.4, we deduce a bound on the dimension of $\mathscr{S}'(\omega)$: it has at most dimension 1 if $\mathscr{S}_0'(\omega) = 0$, and it has at most dimension 2 otherwise. Observe that $\mathscr{S}_0'(\omega) \neq 0$ never happens if $\Re(\omega) > 0$. In this domain, we gave a non-zero ω-eigendistribution, namely $\zeta(\omega)$, thus Theorem 4.3.1 is true for $\Re(\omega) > 0$. To complete the proof, we search for a holomorphic extension of

$$\zeta^o(s, \omega; f) := L(s, \omega)^{-1} \zeta(s, \omega; f)$$

to the whole complex plane and prove that the distribution $\omega(x)\,d^\times x$ does not extends to an ω-eigendistribution when $\mathscr{S}_0'(\omega) \neq 0$.

The non-archimedean case.

Let k be non-archimedean. The zeta integral is well-defined when the real part of the character ω is positive, and

$$L(\omega)^{-1}\zeta(\omega) = \zeta(\omega) - \omega(\pi)\zeta(\omega)$$

in case ω is unramified. Since $\zeta(\omega) \in \mathscr{S}'(\omega)$, the equality $\omega(\pi)\zeta(\omega) = \pi \cdot \zeta(\omega)$ holds. This mean that the distribution $\zeta^o(\omega) := L(\omega)^{-1}\zeta(\omega)$ operates on functions $f \in \mathscr{S}(k)$ in the following way:

$$\zeta^o(\omega; f) = \int_{k^\times} \Big[f(x) - f(\pi^{-1}x)\Big]\,\omega(x)\,d^\times x \,.$$

The function $f(x) - f(\pi^{-1}x)$ is locally constant with compact support, moreover, it vanishes on $x = 0$, thus on a whole neighbourhood of it. In other words, the image of the linear map

$$\mathscr{S}(k) \longrightarrow \mathscr{S}(k)\,, \quad f \longmapsto f - (\pi^{-1} \cdot f)$$

is contained in the smaller space $\mathscr{D}(k^\times)$, therefore it is immediate to check that the map

$$\zeta^o(\omega) : \mathscr{S}(k) \longrightarrow \mathbb{C}\,, \quad f \longmapsto \int_{k^\times} \Big[f(x) - f(\pi^{-1}x)\Big]\, \omega(x)\, d^\times x$$

is a well-defined ω-eigendistribution for every character ω. If $f^o = \mathbb{1}_{\mathcal{O}}$, the function $f - (\pi^{-1} \cdot f)$ is equal to $\mathbb{1}_{\mathcal{O}^\times}$, implying

$$\zeta^o(\omega; f^o) = \int_{\mathcal{O}^\times} \omega(x)\, d^\times x\,.$$

When ω is unramified, $\zeta^o(\omega; f^o)$ is equal to the measure of $\mathcal{O}^\times$, hence $\zeta^o(\omega)$ is a non-zero ω-eigendistribution for every unramified character ω. Moreover, the value $\zeta^o(\omega; f^o)$ does not depend on the unramified character ω, thus the holomorphic function $\zeta^o(s, \omega; f^o)$ in the variable s is constant and non-zero. The relation $\zeta(s,\omega) = L(s,\omega)\zeta^o(s,\omega)$ provides the meromorphic extension of $\zeta(s,\omega)$ to the whole complex plane, with the same poles of $L(s,\omega)$.

When the character ω is ramified, the integral of ω over $\mathcal{O}^\times$ is zero. From this fact, one obtains an extension of $\zeta(s,\omega)$ to $\Re(s) \leq -\Re(\omega)$ in a way that resembles the principal value of integrals with singularities on the real line. Define B_n as

$$B_n = \Big\{x \in k : |x| < q^{-n}\Big\}\,,$$

where q is the cardinality of the residue field of k. The family $\{B_n : n \in \mathbb{N}\}$ is a basis of open neighbourhoods $0 \in k$, hence, for every function $f \in \mathscr{S}(k)$, there is a positive integer n such that

$$f|_{B_n} = f(0)\mathbb{1}_{B_n}\,.$$

Suppose that ω is a ramified character with $\Re(\omega) > 0$, the zeta integral of ω and $\mathbb{1}_{B_n}$ is zero, indeed

$$\begin{aligned}\int_{B_n} \omega(x)\, d^\times x &= \sum_{j \geq n} \int_{\pi^j \mathcal{O}^\times} \omega(x)\, d^\times x \\ &= \left(\sum_{j\geq n} \omega(\pi)^j\right) \int_{\mathcal{O}^\times} \omega(x)\, d^\times x\,,\end{aligned}$$

the integral of ω on $\mathcal{O}^\times$ is zero by Example 2.2.5, while the sum is finite because $\omega(\pi)$ lies in the circle of radius $q^{-\Re(\omega)}$. Therefore, for any $f \in \mathscr{S}(k)$ there exists a $n \in \mathbb{N}$ such that

$$\int_{k^\times} f(x)\omega(x)\, d^\times x = \int_{k \setminus B_n} f(x)\omega(x)\, d^\times x$$

and for every integer $l > n$, we have

$$\begin{aligned}\int_{k\setminus B_l} f(x)\omega(x)\, d^\times x - \int_{k \setminus B_n} f(x)\omega(x)\, d^\times x &= f(0) \int_{B_n \setminus B_l} \omega(x)\, d^\times x \\ &= f(0) \left(\sum_{j=n}^{l} \omega(\pi)^j\right) \int_{\mathcal{O}^\times} \omega(x)\, d^\times x \\ &= 0\,.\end{aligned}$$

This last computation shows that the value of the integral

$$\int_{k\setminus B_j} f(x)\omega(x)\, d^\times x$$

stabilizes if j is large enough, additionally, it also makes sense for ramified character ω with $\Re(\omega) \leq 0$. Through these observations, we see that the distribution defined by

$$\zeta^o(\omega; f) := \lim_{n\to\infty} \int_{k\setminus B_n} f(x)\omega(x)\, d^\times x\,, \tag{4.4}$$

for all $f \in \mathscr{S}(k)$, is a well-defined ω-eigendistribution for every ramified character ω and it gives the holomorphic continuation of $\zeta^o(s,\omega)$ to all ramified characters. If one set $f^o = \omega^{-1}\mathbb{1}_{\mathcal{O}^\times}$, the equality

$$\zeta^o(s,\omega; f^o) = 1$$

holds for all $s \in \mathbb{C}$ and all ramified characters ω, so $\zeta^o(s,\omega)$ is a non-zero distribution.

It remains to prove that $\mathscr{S}'(\omega_0)$ is one-dimensional, where ω_0 is the trivial character of $k^\times$. The modified zeta integral $\zeta^o(\omega_0)$ defines a non-zero ω_0-eigendistribution which is trivial on the space $\mathscr{D}(k^\times)$: if f is a locally constant function with compact support in $k^\times$, then the integral $\zeta(\omega_0; f)$ is finite and

$$\begin{aligned}\zeta^o(\omega_0; f) &= \zeta(\omega_0; f) - \zeta(\omega_0; \pi^{-1}\cdot\omega_0)\\ &= \zeta(\omega_0; f) - \omega_0(\pi)\zeta(\omega_0;\omega_0)\\ &= 0.\end{aligned}$$

This means that $\zeta^o(\omega_0)$ belongs to the space $\mathscr{S}_0'(\omega_0)$, which is generated by δ_0 as we saw in Lemma 4.3.4. So $\zeta^o(\omega_0)$ is a constant multiple of the Dirac delta. A direct computation of $\zeta^o(\omega_0; \mathbb{1}_{\mathcal{O}})$ shows that the constant must be the measure of $\mathcal{O}^\times$, hence, up to rescaling the multiplicative Haar measure,

$$\zeta^o(\omega_0) = \delta_0\,.$$

We have to show that $\mathscr{S}'(\omega_0) = \mathscr{S}_0'(\omega_0)$ and this is obtained by showing that any distribution λ which has a non-zero restriction to $k^\times$ is not invariant by the action of $k^\times$. The argument provided in [Kud04] proceed as follows: let V be the pre-image of $\mathscr{D}'(\omega)$ under the restriction of distributions to $k^\times$. It is a two-dimensional vector space generated by δ_0 and any distribution λ such that $\lambda|_{k^\times} = c\, d^\times x$ for some non-zero constant $c \in \mathbb{C}$. The argument consists in showing that V is an indecomposable two-dimensional representation of $k^\times$, even more, the representation factors through the quotient $k^\times/\mathcal{O}^\times \cong \mathbb{Z}$ and it is isomorphic to the representation

$$\mathbb{Z} \longrightarrow \mathrm{GL}(2,\mathbb{C})\,, \quad n \longmapsto \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}. \tag{4.5}$$

This would prove that $\mathscr{S}'(\omega_0) = \mathscr{S}_0'(\omega_0)$ because $\mathscr{S}'(\omega_0)$ is a sum of trivial representations of $k^\times$ which is contained in the two-dimensional space V and contains $\mathscr{S}_0'(\omega_0)$. Since it can not be equal to the indecomposable representation V, it must be equal to the space generated by δ_0. Let us prove that V is isomorphic to the representation given in Equation 4.5 providing a basis $\{\delta_0, \lambda_0\}$ on which an element $a \in k^\times$ operates via the matrix

$$\begin{pmatrix} 1 & -\operatorname{ord}_{\mathfrak{p}}(a) \\ 0 & 1 \end{pmatrix}.$$

Let λ_0 be defined by

$$\langle \lambda_0 \, , f \rangle := \int_{k^\times} \big[f(x) - f(0) \mathbb{1}_{\mathcal{O}}(x) \big] \, d^\times x$$

for all $f \in \mathscr{S}(k)$. Its restriction to $k^\times$ is the distribution $d^\times x$, thus $\{\delta_0, \lambda_0\}$ is a basis of V. Let $a \in k^\times$ and consider the distribution $a \cdot \lambda_0$. Its restriction to $k^\times$ is again $d^\times x$ because $f(a^{-1}x) - f(0)\mathbb{1}_{\mathcal{O}}(a^{-1}) = f(a^{-1}x)$ for all functions $f \in \mathscr{D}(k^\times)$ and the integral of $f(a^{-1}x)$ over $k^\times$ is equal to that of $f(x)$. This implies that λ_0 and $a \cdot \lambda_0$ differs by a constant $c(a)$ times the Dirac delta, therefore a acts on the space V with basis $\{\delta_0, \lambda_0\}$ via the matrix

$$\begin{pmatrix} 1 & c(a) \\ 0 & 1 \end{pmatrix}.$$

The map $a \mapsto c(a)$ defines a homomorphism from $k^\times$ to the additive group of complex numbers, indeed

$$\begin{aligned} c(a_1 a_2)\delta_0 + \lambda_0 &= (a_1 a_2) \cdot \lambda_0 \\ &= a_1 \cdot (c(a_2)\delta_0 + \lambda_0) \\ &= c(a_2)\delta_0 + c(a_1)\delta_0 + \lambda_0 \, . \end{aligned}$$

This shows that V is a sub-representation of $\mathscr{S}'(k)$ and we have to determine the homomorphism c. The distribution λ_0 is invariant by the action of $\mathcal{O}^\times$ because $a^{-1}f - f(0)\mathbb{1}_{\mathcal{O}} = a^{-1} \cdot (f - f(0)\mathbb{1}_{\mathcal{O}})$ for all $a \in \mathcal{O}^\times$ and all $f \in \mathscr{S}(k)$, thus $c(\mathcal{O}^\times) = 0$ and the homomorphism c is determined by its value at π. The value $c(\pi)$ is obtained by the following computation:

$$\begin{aligned} c(\pi) &= \langle c(\pi)\delta_0 \, , \mathbb{1}_{\mathcal{O}} \rangle \\ &= \langle \pi \cdot \lambda_0 - \lambda_0 \, , \mathbb{1}_{\mathcal{O}} \rangle, \\ &= \int_{k^\times} \Big[\mathbb{1}_{\mathcal{O}}(\pi^{-1}x) - \mathbb{1}_{\mathcal{O}}(x) \Big] \, d^\times x - \int_{k^\times} \big[\mathbb{1}_{\pi\mathcal{O}}(x) - \mathbb{1}_{\mathcal{O}}(x) \big] \, d^\times x \\ &= - \int_{\mathcal{O}^\times} d^\times x \, , \end{aligned}$$

so, up to rescaling the Haar measure of $k^\times$, we have $c(\pi) = -1$.

At this point, Theorem 4.3.1 is proved for all non-archimedean local fields.

The archimedean case

Recall the classification of local characters given in Theorem 3.5.8. For $k = \mathbb{R}$, all characters are of the form $(s, \omega_n) := \omega_n |\cdot|^s$, where

$$\omega_n(x) = x^{-n}$$

for all $x \in \mathbb{R}^\times$ and $n \in \mathbb{Z}$. The representation of the character is not unique:

$$(s, \omega_n) = (s + 2, \omega_{n+2}) \, ,$$

and (s, ω_n), for $n = 0, 1$, parametize bijectively the whole space of characters. For $k = \mathbb{C}$, every character is of the form $(s, \omega_{n,m}) := \omega_{n,m} |\cdot|^s$, where

$$\omega_{n,m}(x) = x^{-n} \bar{x}^{-m}$$

for all $x \in \mathbb{C}^\times$, and $n, m \in \mathbb{Z}$. This representation of the complex characters is subject to the relation

$$(s, \omega_{n,m}) = (s+1, \omega_{n+1,m+1}),$$

and every character is of the form $(s, \omega_{n,m})$ for a unique triple (s, n, m) provided that $n, m \in \mathbb{N}$ and at least one between n and m is zero. In the real case, the zeta integral $\zeta(s, \omega_n)$ is a well-defined and non-zero (s, ω_n)-eigendistribution generating the space $\mathscr{S}'(s, \omega_n)$ when $\Re(s) > n$. The same is true in the complex case: for $\Re(s) > n + m$, the zeta integral $\zeta(s, \omega_{n,m})$ generates the space $\mathscr{S}'(s, \omega_{n,m})$. In the non-archimedean case, the derivation operator helps to extend the zeta integral beyond its natural domain. Let us see this for $k = \mathbb{R}$, as the case of $\mathbb{C}$ is analogous. Denote by ∂ the derivative of $\mathbb{R}$ with respect to the variable x. For every function $f \in \mathscr{S}(\mathbb{R})$ and every complex number s with $\Re(s) > n + 1$, the function

$$f(x)x^{-n}|x|^s$$

is continuously differentiable on $\mathbb{R}$ and goes to zero at infinity. Its derivative is

$$\partial\left[f(x)x^{-n}|x|^s\right] = f'(x)x^{-n}|x|^s \; - nf(x)x^{-(n+1)}|x|^s \; + sf(x)x^{-n}|x|^{s-1}\,\frac{x}{|x|}\,. \tag{4.6}$$

Integrating the functions of Equation (4.6) we obtain a relation between zeta integrals,

$$0 = \zeta(s+1, \omega_n; f') - n\zeta(s+1, \omega_{n+1}; f) + s\zeta(s-1, \omega_{n-1}; f)\,.$$

In therm of distributions, we have

$$\partial\zeta(s+1; \omega_n) = (s-n)\zeta(s-1, \omega_{n-1}) \tag{4.7}$$

using the fact that $(s-1, \omega_{n-1})$ and $(s+1, \omega_{n+1})$ represent the same character. Equation (4.7) is valid for $n \in \mathbb{Z}$ and $s \in \mathbb{C}$ satisfying $\Re(s) > n + 1$. Note that the left-hand side is well-defined for $\Re(s) > n - 1$ and we can use this to extend the zeta integral: for the trivial character ω_0,

$$\zeta(s, \omega_0) := \frac{1}{s}\,\partial\zeta(s+2, \omega_1) \qquad \text{for } -1 < \Re(s) \leq 0,$$

defines a meromorphic extension of $\zeta(s, \omega_0)$ to the right half-plane $\Re(s) > -1$ with possibly a simple pole at $s = 0$. For the character ω_1,

$$\zeta(s, \omega_1) := \frac{1}{s-1}\,\partial\zeta(s, \omega_0) \qquad \text{for } 0 < \Re(s) \leq 1,$$

defines a meromorphic extension of $\zeta(s, \omega_1)$ to $\Re(s) > 0$. By induction, we get a formula for the extension of $\zeta(s, \omega_0)$ to for $\Re(s) > -2N$, given $N \in \mathbb{N}$ arbitrarily large:

$$\zeta(s, \omega_0) = \left(\prod_{j=0}^{N-1} \frac{1}{(s+2j)(s+2j+1)}\right) \partial^{2N}\zeta(s+2N, \omega_0)\,.$$

This extends $\zeta(s, \omega_0)$ meromorphically to the whole complex plane with simple poles at $s = -2j$ for all $j \in \mathbb{N}$. There are no poles for $s = -1 - 2j$ because the simple poles of $(s+2j+1)^{-1}$ are canceled by the zeros of $\partial^{2N}\zeta(s+2N,\omega_0)$:

$$\begin{aligned}\left\langle \partial^{2N}\zeta(2N-2j-1,\omega_0)\,,f\right\rangle &= \int_{\mathbb{R}^\times} \partial^{2N}f(x)\,|x|^{2(N-j)-1}\ d^\times x\\ &= \int_{\mathbb{R}} \partial^{2N}f(x)\,x^{2(N-j-1)}\ dx\\ &= (2N-2j-2)!\int_{\mathbb{R}} \partial^{2(j+1)}f(x)\ dx\\ &= 0\,.\end{aligned}$$

For the zeta integral relative to the character ω_1, the relation $\zeta(s,\omega_1)=(s-1)^{-1}\partial\zeta(s,\omega_0)$ gives the meromorphic extension to the whole complex plane. Note again that the pole of $(s-1)^{-1}$ is canceled by the zero of $\partial\zeta(s,\omega_0)$ at $s=1$, indeed

$$\zeta(s,\omega_0;\partial f)=\int_{\mathbb{R}}\partial f(x)\,dx=0\,.$$

Note the overlapping between the poles of $\zeta(s,\omega_n)$, for $n=0,1$, and the poles of $\pi^{-s/2}\Gamma(s/2)$. Indeed, if we define

$$\begin{aligned}\zeta^o(s,\omega_0) &:= \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\zeta(s,\omega_0)\,,\\ \zeta^o(s,\omega_1) &:= \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\zeta(s,\omega_1)\,,\end{aligned}$$

we obtain two entire (distribution-valued) functions. Moreover

$$\zeta^o(s,\omega_n;f^o)=1$$

for all $s\in\mathbb{C}$, where

$$f^o(x)=x^n e^{-\pi x^2}\,,\qquad n=0,1\,.$$

Hence, the distribution $\zeta^o(s,\omega_n)$ is a non-zero vector of the space $\mathscr{S}'(s,\omega_n)$.

When $k=\mathbb{C}$, we have the following relations between the zeta integral and the derivatives $\partial,\bar\partial$:

$$\begin{aligned}\partial\zeta(s,\omega_{n,m}) &= (s-1-n)\zeta(s,\omega_{n+1,m})\\ \bar\partial\zeta(s,\omega_{n,m}) &= (s-1-m)\zeta(s,\omega_{n,m+1})\,.\end{aligned}$$

Using them we obtain the meromorphic extension of the zeta integral. Moreover, for $n,m\in\mathbb{N}$ with $n=0$ or $m=0$, the distribution

$$\zeta^o(s,\omega_{n,m}):=\frac{(2\pi)^{s-1}}{\Gamma(s)}\zeta(s,\omega_{n,m})$$

is entire and non-zero, as

$$\zeta^o(s,\omega_{n,m};f^o)=1$$

for

$$f^o(x)=x^n\bar{x}^m e^{-2\pi x\bar{x}}\,.$$

It remains to prove that $\mathscr{S}'(\omega) = \mathscr{S}'_0(\omega)$ in the cases of Lemma 4.3.4. We give the idea of the proof for the case $k = \mathbb{R}$ since the complex case is similar. Consider the trivial character ω_0. Let V_0 be the inverse image of $\mathscr{D}'(\omega_0)$ under the restriction morphism $\mathscr{S}'(\mathbb{R}) \to \mathscr{D}(\mathbb{R}^\times)$. The complex vector space V_0 is generated by the Dirac delta δ_0 and the extensions of $d^\times x$ to $\mathscr{S}'(\mathbb{R})$. Note that the distribution $d^\times x$ extends without effort to the space $\mathscr{S}(\mathbb{R}; \delta_0)$ of functions $f \in \mathscr{S}(\mathbb{R})$ such that $f(0) = 0$. This is because the smoothness of f implies that the function $f(x)|x|^{-1}$ is bounded in a neighbourhood of $x = 0$ when $f(0) = 0$, so the integral

$$\int_{\mathbb{R}^\times} f(x)\, d^\times x = \int_{\mathbb{R}} \frac{f(x)}{|x|}\, dx$$

is finite. If g is a fixed even bump-function of the real line with $g(0) = 1$, the distribution λ_0 defined by

$$\langle \lambda_0\,, f\rangle := \int_{\mathbb{R}^\times} \big[f(x) - f(0)g(x)\big]\; d^\times x\,, \qquad \text{for all } f \in \mathscr{S}(\mathbb{R})\,,$$

is a well-defined extension of $d^\times x$ from $\mathscr{S}(\mathbb{R}; \delta_0)$ to $\mathscr{S}(\mathbb{R})$. A basis of V_0 is given by $\{\delta_0, \lambda_0\}$ and we have to understand how $\mathbb{R}^\times$ acts on λ_0. Let a be a non-zero real number, then

$$\begin{aligned}\langle a\cdot\lambda_0\,, f\rangle &= \int_{\mathbb{R}^\times} \Big[f(a^{-1}x) - f(0)g(x)\Big]\; d^\times x \\ &= \int_{\mathbb{R}^\times} \big[f(x) - f(0)g(ax)\big]\; d^\times x\,,\end{aligned}$$

from which it follows that $a\cdot\lambda_0 = \lambda_0$ if $a = -1$ and

$$\langle a\cdot\lambda_0\,, f\rangle = \langle \lambda_0\,, f\rangle \qquad \text{for all } f \in \mathscr{S}(\mathbb{R}; \delta_0).$$

The space of distribution that is trivial on $\mathscr{S}(\mathbb{R}; \delta_0)$ is precisely the one generated by δ_0. Then

$$a\cdot\lambda_0 = \lambda_0 + c(a)\delta_0\,.$$

We have to prove that $c(a) \neq 0$. Consider the difference $a\cdot\lambda_0 - \lambda_0$. It acts on a function $f \in \mathscr{S}(\mathbb{R})$ by

$$\langle \lambda_0\,,\ (a^{-1}\cdot f)\ - f\ \rangle = \int_{\mathbb{R}} \frac{f(a^{-1}x) - f(x)}{|x|}\, dx\,.$$

Observe that the value of the integral depends only on $f(0)$ because $a\cdot\lambda_0 - \lambda_0 = c(a)\delta_0$. Choose a sequence $(f_j)_{j\in\mathbb{N}}$ of Schwartz functions f_j approximating the characteristic function $\mathbb{1}_{(-1,1)}$ of the interval $(-1, 1)$. Then

$$\begin{aligned}\lim_{j\to\infty} \int_{\mathbb{R}} \frac{f_j(a^{-1}x) - f_j(x)}{|x|}\, dx &= \int_{\mathbb{R}} \frac{\mathbb{1}_{(-1,1)}(a^{-1}x) - \mathbb{1}_{(-1,1)}(x)}{|x|}\, dx \\ &= 2\int_0^\infty \frac{\mathbb{1}_{(0,|a|)}(x) - \mathbb{1}_{(0,1)}(x)}{x}\, dx \\ &= \log(a^2)\end{aligned}$$

Therefore V_0 is a representation of $\mathbb{R}^\times$ isomorphic to the indecomposable 2-dimensional representation

$$\mathbb{R}^\times \longrightarrow \mathrm{GL}(2,\mathbb{C})\,, \quad a \longmapsto \begin{pmatrix} 1 & \log(a^2) \\ 0 & 1 \end{pmatrix}\,,$$

thus $\mathscr{S}'(\omega_0)$ must coincide with the 1-dimensional space generated by the Dirac delta δ_0.

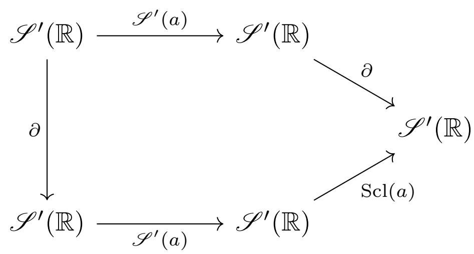
Now consider the character $\omega_1(x) = x^{-1}$. The space $\mathscr{D}'(\omega_1)$ is generated by the distribution $x^{-1}d^\times x$. The latter extends without effort to the space $\mathscr{S}(\mathbb{R}; \delta_0, \partial\delta_0)$ of functions $f \in \mathscr{S}(\mathbb{R})$ such that both f and its derivative vanishes at $x = 0$. Note that for the functions f in the latter space we have

$$\int_{\mathbb{R}^\times} \frac{f(x)}{x}\, d^\times x = \int_{\mathbb{R}^\times} \partial f(x)\, d^\times x\,,$$

so $\lambda_1 := -\partial\lambda_0$ is an extension of $x^{-1}d^\times x$ to the space $\mathscr{S}(\mathbb{R})$. It's easy to verify that for any $a \in \mathbb{R}^\times$ and any distribution $\lambda \in \mathscr{S}'(\mathbb{R})$, the the actions of a and ∂ on λ are related by

$$a(a \cdot \partial\lambda) = \partial(a \cdot \lambda)\,.$$

In other terms, the diagram



is commutative, where $\mathscr{S}'(a)$ is the action of a on $\mathscr{S}'(\mathbb{R})$ and $\mathrm{Scl}(a)$ is the scalar multiplication by a. From this, we can reconstruct the action of $\mathbb{R}^\times$ on λ_1 from the action of the same group on λ_0 as follows:

$$\begin{aligned} a \cdot \lambda_1 &= -a^{-1}\partial(a \cdot \lambda_0) \\ &= -a^{-1}\partial(\lambda_0 + \log(a^2)\delta_0) \\ &= a^{-1}\lambda_1 - a^{-1}\log(a^2)\partial\lambda_0 \\ &= \omega_1(a)\lambda_1 - \log(a^2)\omega_1(a)\partial\delta_0\,. \end{aligned}$$

Then $\lambda_1 \notin \mathscr{S}'(\omega_1)$ and the space $\mathscr{S}'(\omega_1)$ must be equal to the 1-dimensional space generated by $\partial\delta_0$. It should be clear how we can prove inductively that $\mathscr{S}'(\omega_n) = \mathbb{C}\,\partial^n\delta_0$ for all $n \in \mathbb{N}$.

The global case

Let K be a global field with adèles ring $\mathbb{A}$. Given a character of the idèles $\omega = \otimes_\nu \omega_\nu$, we have an ω_ν-eigendistribution $\zeta_\nu^o(s, \omega_\nu)$ for each place ν of K. For almost all ν, the value of the distribution $\zeta_\nu^o(s, \omega_\nu)$ against the function $\mathbb{1}_{\mathcal{O}_\nu}$ is 1, so

$$\zeta^o(s, \omega) := \bigotimes_\nu \zeta_\nu^o(s, \omega_\nu)$$

is a standard distribution in the space $\mathscr{S}'(s, \omega)$. To conclude the proof of Theorem 4.3.1, one observes that any distribution is a $\mathbb{C}$-linear combination of standard distributions $\lambda = \otimes_\nu \lambda_\nu$, and each λ_ν is determined by λ up to a scalar (see the discussion between Lemma 4.1 and Theorem

4.2 in [Kud04]). If λ belongs to $\mathscr{S}'(s,\omega)$, then each local distribution λ_ν belongs to $\mathscr{S}'(s,\omega_\nu)$. As a consequence, λ is a scalar multiple of $\zeta^o(s,\omega)$. For every ν, let f^o_ν be the function of the space $\mathscr{S}(K_\nu)$ defined in the examples 4.2.4, 4.2.5, 4.2.6, 4.2.7 and 4.2.8, associated with the local character ω_ν. It satisfies $\zeta^o_\nu(s,\omega_\nu;f^o)=1$ and for almost all places ν, the function f^o is equal to the characteristic function of the local ring $\mathcal{O}_\nu$. Let $f^o := \otimes_\nu f^o_\nu$. Then f^o is a standard Schwartz function of $\mathbb{A}$ and

$$\begin{aligned}\zeta^o(s,\omega;f^o) &= \prod_\nu \zeta^o_\nu(s,\omega_\nu;f^o_\nu)\\ &= 1\end{aligned}$$

for all $s\in\mathbb{C}$, which ensures that $\zeta^o(s,\omega)$ is a non-zero vector of $\mathscr{S}'(s,\omega)$.

4.4 The functional equation

We are now ready to approach the main application of the whole setting: the analytic continuation and functional equation of the zeta integral and L-functions attached to idèle class characters.

Let K be a global field, $\mathbb{A}$ its ring of adèles, ψ a non-trivial additive character of $\mathbb{A}$ which is trivial on K and dx the self-dual measure. All local data attached to a place ν of K are indicated with a sub-script ν as before. Let $\omega=\otimes_\nu\omega_\nu$ be an idèle class character. The distribution $\zeta^o_\nu(s,\omega_\nu)$ constructed in Section 4.3 for the local field K_ν, defines a basis vector for the one-dimensional vector space $\mathscr{S}'(s,\omega_\nu)$. The local zeta integral $\zeta_\nu(s,\omega_\nu)$ is also an (s,ω_ν)-eigendistribution, so it is a scalar multiple of $\zeta^o_\nu(s,\omega_\nu)$, precisely

$$\zeta_\nu(s,\omega_\nu)=L_\nu(s,\omega_\nu)\zeta^o_\nu(s,\omega_\nu)\,,$$

and this defines the meromorphic continuation of the local zeta integral to the left half-plane $\Re(s)\le\Re(\omega)$. The distribution

$$\zeta^o(s,\omega)=\bigotimes_\nu \zeta^o_\nu(s,\omega_\nu)$$

is an (s,ω)-eigendistribution. For $\Re(s)>1-\Re(\omega)$, the zeta integral $\zeta(s,\omega)$ is a well-defined, non-zero (s,ω)-eigendistribution. Moreover

$$\begin{aligned}\zeta(s,\omega) &= \bigotimes_\nu \zeta_\nu(s,\omega_\nu)\\ &= \bigotimes_\nu L_\nu(s,\omega_\nu)\zeta^o_\nu(s,\omega_\nu)\\ &= \left(\prod_\nu L_\nu(s,\omega_\nu)\right)\zeta^o(s,\omega)\,.\end{aligned}$$

For $\Re(s)>1-\Re(\omega)$, define the *completed global L-function* as the product

$$\Lambda(s,\omega):=\prod_\nu L_\nu(s,\omega_\nu)$$

of the local L-functions. It is the factor of proportionality between $\zeta(s,\omega)$ and $\zeta^o(s,\omega)$:

$$\zeta(s,\omega)=\Lambda(s,\omega)\zeta^o(s,\omega)\,. \tag{4.8}$$

At the local level, a functional equation appears due to the Fourier transform. Theorem 4.1.7 asserts that the Fourier transform induces an isomorphism between the spaces $\mathscr{S}'(s,\omega_\nu)$ and $\mathscr{S}'(1-s,\omega_\nu^{-1})$. Hence, the Fourier transform of $\zeta_\nu^o(1-s,\omega_\nu^{-1})$ is an element of $\mathscr{S}'(s,\omega_\nu)$, but this space is one-dimensional by Theorem 4.3.1. Therefore, there is a scalar $\varepsilon_\nu(s,\omega_\nu)$ such that

$$\zeta_\nu^o(\widehat{1-s,\omega_\nu^{-1}}) = \varepsilon_\nu(s,\omega_\nu)\zeta_\nu^o(s,\omega_\nu)\,.$$

The factor of proportionality is called *local epsilon factor* and it is an invertible holomorphic function in the variable s which can be calculated explicitly via

$$\varepsilon_\nu(s,\omega_\nu) = \zeta_\nu^o(1-s,\omega_\nu^{-1};\widehat{f_\nu^o})\,.$$

If S is a finite set of places containing every ν such that:

ν is infinite,

or the conductor of ψ_ν is different from $\mathcal{O}_\nu$,

or ω_ν is ramified,

then, for all $\nu \notin S$, the function f_ν^o is the characteristic function of $\mathcal{O}_\nu$, it is its own Fourier transform and

$$\begin{aligned}\varepsilon_\nu(s,\omega_\nu) &= \zeta_\nu^o(1-s,\omega_\nu^{-1};\mathbb{1}_{\mathcal{O}_\nu})\\ &= 1\,.\end{aligned}$$

The *global epsilon factor* is the product

$$\varepsilon(s,\omega) := \prod_\nu \varepsilon_\nu(s,\omega_\nu) \tag{4.9}$$

of the local ones. Since local epsilon factors are equal to 1 for almost all places, the product (4.9) involves only a finite number of terms. The definition of the adèlic distribution $\zeta^o(s,\omega)$ in terms of its local versions produces a functional equation of the form

$$\zeta^o(\widehat{1-s,\omega^{-1}}) = \varepsilon(s,\omega)\,\zeta^o(s,\omega). \tag{4.10}$$

The local epsilon factors may depend on the choice of the character ψ, as $\varepsilon_\nu(s,\omega)$ is an integral that depends on the local measure dx_ν and the function integrated is a Fourier transform which depends on the local character ψ_ν. If ψ' is another character coming from the group $\widehat{\mathbb{A}/K}$, then

$$\varepsilon_\nu(s,\omega_\nu,\psi'_\nu) = |a|_\nu^{s-\frac{1}{2}}\,\omega_\nu(a)\,\varepsilon_\nu(s,\omega_\nu,\psi_\nu)\,, \tag{4.11}$$

where $a \in K^\times$ is the unique principal idèles such that $\psi'(x) = \psi(ax)$ for all $x \in \mathbb{A}$. The global epsilon factor, instead, does not depend on the character ψ precisely because of relation (4.11):

$$\begin{aligned}\varepsilon(s,\omega,\psi') &= \prod_\nu \varepsilon_\nu(s,\omega_\nu,\psi'_\nu)\\ &= \prod_\nu \Big(|a|_\nu^{s-\frac{1}{2}}\,\omega_\nu(a)\,\varepsilon_\nu(s,\omega_\nu,\psi_\nu)\Big)\\ &= |a|^{s-\frac{1}{2}}\,\omega(a)\,\varepsilon(s,\omega,\psi)\\ &= \varepsilon(s,\omega,\psi)\,,\end{aligned}$$

where the last equality holds because the character ω is an idèle class character and the idèlic norm is trivial on $K^\times$. Since both $\Lambda(s,\omega)$ and $\zeta(s,\omega)$ are defined only in the right half-plane $\Re(s) > -\Re(\omega)$, the well-defined distribution $\zeta^o(s,\omega)$ with its functional equation wouldn't be enough to obtain the analytic continuation of the global L-function and its functional equation. The turning point comes from the existence of the analytic continuation of the zeta integral to the whole complex plane, together with the nicest possible functional equation for its Fourier transform. Precisely,

Theorem 4.4.1. *Let ω be an idèle class character. Then the zeta integral $\zeta(s,\omega)$ admits a meromorphic extension to all s in the complex plane except for simple poles which occur precisely when $\omega|\cdot|^s$ is equal to the trivial character or the idèlic norm. Moreover, the Fourier transform of the zeta integral satisfies the functional equation*

$$\widehat{\zeta(1-s,\,\omega^{-1})} = \zeta(s,\omega)\,. \tag{4.12}$$

Before presenting the proof of Theorem 4.4.1, let us look at its consequence. Equation (4.12) is an identity of distributions, so

$$\zeta(1-s,\,\omega^{-1};\widehat{f}) = \zeta(s,\omega;f) \tag{4.13}$$

for all $f \in \mathscr{S}(\mathbb{A})$. Consider the case $f = f^o$. On the right-hand side of Equation (4.13) we have

$$\zeta(s,\omega;f^o) = \Lambda(s,\omega)$$

by Equation (4.8) relating the global zeta integral with $\zeta^o(s,\omega)$, giving $\Lambda(s,\omega)$ the analytic continuation to the whole complex plane. On the left-hand side of Equation (4.13) we have

$$\begin{aligned}\zeta(1-s,\,\omega^{-1};\widehat{f^o}) &= \Lambda(1-s,\omega^{-1})\zeta^o(1-s,\omega^{-1};\widehat{f^o}) && \text{again by Equation (4.8),}\\ &= \Lambda(1-s,\omega^{-1})\varepsilon(s,\omega) && \text{by the functional equation (4.10) of } \zeta^o(s,\omega).\end{aligned}$$

Therefore, we obtain the following result for the completed global L-function:

Corollary 4.4.2. *Let ω be an idèle class character. Then the completed global L-function $\Lambda(s,\omega)$ admits an analytic extension to the whole complex plane, except for complex numbers s such that $\omega|\cdot|^s$ is equal to the trivial character or the idèlic norm, where $\Lambda(s,\omega)$ has a simple pole. Moreover, $\Lambda(s,\omega)$ satisfies the functional equation*

$$\Lambda(s,\omega) = \varepsilon(s,\omega)\Lambda(1-s,\,\omega^{-1})\,.$$

Proof of Theorem 4.4.1

The analytic continuation of the zeta integral with its functional equation is a deep consequence of the self-duality of the global field K inside its ring of adèles. The fact that ω is trivial on the group of principal idèles means that the zeta integral is really an integral over the idèle class group. If E is a Borel subset of $\mathbb{A}^\times$ such that

$$E \longrightarrow \mathbb{A}^\times/K^\times\,, \quad x \longmapsto xK^\times$$

is a bijection, then

$$\begin{aligned}\int_{\mathbb{A}^\times} f(x)\omega(x)|x|^s\, d^\times x &= \sum_{y\in K^\times}\int_E f(xy)\omega(xy)|xy|^s\, d^\times x\\ &= \int_E\left(\sum_{y\in K^\times} f(xy)\right)\omega(x)|x|^s\, d^\times x\,.\end{aligned}$$

Integration over E defines a Haar measure of $\mathbb{A}^\times/K^\times$ still denoted by $d^\times x$, so that we may write

$$\zeta(s,\omega;f) = \int_{\mathbb{A}^\times/K^\times} f^\times(x)\omega(x)|x|^s \, d^\times x \,,$$

where the variable x takes values in $\mathbb{A}^\times$ (mod $K^\times$) and $f^\times$ is defined by the sum

$$f^\times(x) = \sum_{y\in K^\times} f(xy)\,.$$

Let $\widehat{f}^\times$ be the function of $\mathbb{A}^\times/K^\times$ defined by

$$\widehat{f}^\times(x) = \sum_{y\in K^\times} \widehat{f}(xy)$$

for all x. The reason behind the existence of the functional equation of the zeta integral is the relation

$$f^\times(x) = -f(0) + |x|^{-1}\,\widehat{f}(0) + |x|^{-1}\,\widehat{f}^\times(x^{-1}) \tag{4.14}$$

implied by a Poisson summation formula for the adèles, as we are going to show. Equip K with the counting measure and $\mathbb{A}/K$ with the unique Haar measure μ for which $\mu(\mathbb{A}/K) = 1$. From what we observed at the end of Section 2.2, the Haar measure dx of $\mathbb{A}$ is equal to a positive scalar multiple of the measure constructed using those of K and $\mathbb{A}/K$: there is a constant m such that, for every $f \in \mathscr{C}_c(\mathbb{A})$, the integral of f over $\mathbb{A}$ is

$$\int_{\mathbb{A}} f(x)\,dx = m\int_{\mathbb{A}/K} f^\flat \, d\mu\,, \tag{4.15}$$

where $f^\flat$ is the continuous function of $\mathbb{A}/K$ defined by

$$f^\flat(x+K) = \sum_{y\in K} f(x+y) \qquad \text{for all } x \in K\,.$$

Equation 4.15 is true also for Schwartz functions. The constant m is the measure of a *fundamental domain* of $\mathbb{A}/K$, which is a Borel subset of $\mathbb{A}$ inducing a bijection with the quotient $\mathbb{A}/K$. This is often called the *volume* of $\mathbb{A}/K$. The Haar measure dx, which is the self-dual measure associated with the character ψ satisfying the hypothesis of Theorem 3.3.3, is special also in that $\mathbb{A}/K$ has volume 1 with respect to it. A way to see this is by Poisson summation formula 2.2.8:

$$\sum_{y\in K} f(y) = m\sum_{y\in K} \widehat{f}(y) \tag{4.16}$$

for all $f \in \mathscr{S}(\mathbb{A})$. The constant m appears because the Fourier transform of f is computed using the measure dx instead of the measure compatible with the exact sequence induced by $K \hookrightarrow \mathbb{A}$. Using Equation 4.16 a second time with $\widehat{f}$ in place of f, one obtains

$$\sum_{y\in K} f(y) = m^2 \sum_{y\in K} f(y)\,.$$

Since this holds for a sufficiently large family of functions f, we deduce that $m^2 = 1$, and $m = 1$ because m is positive. Hence, the Poisson summation formula for $\mathbb{A}$ has the following form,

$$\sum_{y\in K} f(y) = \sum_{y\in K} \widehat{f}(y). \tag{4.17}$$

The following result is an immediate consequence.

Proposition 4.4.3. *Let $f \in \mathscr{S}(\mathbb{A})$. Then*

$$\sum_{y\in K} f(xy) = |x|^{-1} \sum_{y\in K} \widehat{f}(x^{-1}y) \tag{4.18}$$

for all idèles x.

Proof. Apply the Poisson summation formula (4.17) to the function $x \cdot f$. □

It is straightforward to see that the formula (4.18) is equivalent to the relation (4.14) involving $f^\times$ and $\widehat{f}^\times$. Proposition 4.4.3 is often referred to as Riemann-Roch Theorem. Indeed, if K is the field of rational functions on a projective curve $\mathscr{C}$ defined over a finite field, the classical, geometric Riemann-Roch theorem can be proved for $\mathscr{C}$ using the identity (4.18) for a suitable function f (see Section 7.2 of [RV99]).

Returning to the integral

$$\int_{\mathbb{A}^\times/K^\times} f^\times(x)\omega(x)|x|^s \, d^\times x \,,$$

we can express it as a double integral by choosing a section

$$\rho : T \longrightarrow \mathbb{A}^\times/K^\times$$

of the idèlic norm, where T is the image of $|\cdot|$. The morphism ρ induces an isomorphism

$$T \oplus \frac{\mathbb{A}^{\times,1}}{K^\times} \cong \frac{\mathbb{A}^\times}{K^\times}\,, \quad (t,x) \longmapsto \rho(t)x\,,$$

which can be used for decomposing the measure of the idèle class group as a product of the measures of T and the compact group $\mathbb{A}^{\times,1}/K^\times$. If K is a number field, $T = \mathbb{R}_+^\times$ and we give it the measure $d^\times t = t^{-1}dt$, where dt is the usual Lebesgue measure of $\mathbb{R}$. If K is a function field, instead, the group T is a rank-one, free, discrete sub-group of $\mathbb{R}_+^\times$ and we use the symbol $d^\times t$ to indicate the counting measure on T. If K has characteristic p, then there is a positive power q of the prime p such that

$$\mathbb{Z} \longrightarrow T\,, \quad n \longmapsto q^{-n}$$

is an isomorphism. In both cases, define $d^{\times,1}x$ to be the unique Haar measure on $\mathbb{A}^{\times,1}/K^\times$ such that

$$\int_{\mathbb{A}^\times/K^\times} g(x)\, d^\times x = \int_T \left(\int_{\mathbb{A}^{\times,1}/K^\times} g(\rho(t)x)\, d^{\times,1}x \right) d^\times t$$

for all $g \in \mathscr{C}_c(\mathbb{A}^\times/K^\times)$. Given the classification of idèle class character by Theorem 3.5.5, we can suppose that ω is a unitary idèle class character induced by a unitary character χ of $\mathbb{A}^{\times,1}/K^\times$. Then, the zeta integral assumes the following form,

$$\zeta(s,\omega;f) = \int_T t^s \int_{\mathbb{A}^{\times,1}/K^\times} f^\times(\rho(t)x)\chi(x)\, d^{\times,1}x\, d^\times t\,.$$

Lemma 4.4.4. *Let ω be an idèle class character and f a function in the space $\mathscr{S}(\mathbb{A})$. For $s \in \mathbb{C}$, define $\zeta^{>}(s,\omega;f)$ to be the integral*

$$\zeta^{>}(s,\omega;f) := \int_{x\in\mathbb{A}^\times,\,|x|>1} f(x)\omega(x)|x|^s \; d^\times x\,.$$

Then $\zeta^{>}(s,\omega;f)$ is a holomorphic function in the variable s defined on the whole complex plane.

Proof. There is no loss of generality in assuming that ω is unitary. In the region of the idèles x with $|x| > 1$ we have

$$\left\| f(x)\omega(x)|x|^s \right\|_{\mathbb{C}} \leq \left\| f(x)|x|^{\max(\Re(s),2)} \right\|_{\mathbb{C}}$$

for all $s \in \mathbb{C}$. Then

$$\left\| \zeta^{>}(s,\omega;f) \right\|_{\mathbb{C}} \leq \int_{\mathbb{A}^\times} \left\| f(x)|x|^{\max(\Re(s),2)} \right\|_{\mathbb{C}} d^\times x \tag{4.19}$$

and the integral on the right-hand side of the inequality 4.19 is finite. For the holomorphicity, the argument is analogous to that of Remark 4.2.11. □

The zeta integral splits in the sum of two terms:

$$\begin{aligned}\zeta(s,\omega;f) &= \int_T t^s \int_{\mathbb{A}^{\times,1}/K^\times} f^\times(\rho(t)x)\chi(x)\, d^{\times,1}x\, d^\times t \\ &= \int_{t\in T,\; t\leq 1} t^s \int_{\mathbb{A}^{\times,1}/K^\times} f^\times(\rho(t)x)\chi(x)\, d^{\times,1}x\, d^\times t \quad + \quad \zeta^{>}(s,\omega;f)\,.\end{aligned}$$

By Lemma 4.4.4 we see that the obstruction to the analytic extension of $\zeta(s,\omega;f)$ is caused by the integral over the region consisting of the idèles with idèlic norm near zero. The relation (4.14) involving $f^\times$ is the key to getting around the singularity of the integral. In addition, it makes explicit the possible poles and the symmetry under the transformation $(s,\omega;f) \mapsto (1-s,\omega^{-1};\widehat{f})$. We are going to prove that the zeta integral has the following form:

$$\begin{aligned}\zeta(s,\omega;f) = \quad & \zeta^{>}(s,\omega;f) \quad + \quad \zeta^{>}(1-s,\omega^{-1};\widehat{f}) \\ & + \quad c(K)\left(\int_{\mathbb{A}^{\times,1}} f(x)\chi(x)\, d^\times x \; + \; \int_{\mathbb{A}^{\times,1}} \widehat{f}(x)\chi(x)^{-1}\, d^\times x\right) \\ & \qquad + \quad V(\chi)\left(P(s)f(0) + P(1-s)\widehat{f}(0)\right)\,,\end{aligned} \tag{4.20}$$

where

- χ is the restriction of ω to the compact group $\mathbb{A}^{\times,1}/K^\times$;
- $V(\chi)$ is the integral of χ, so that $V(\chi)$ is equal to zero when χ is non-trivial and equal to the measure of $\mathbb{A}^{\times,1}/K^\times$ otherwise;
- $c(K)$ is a constant that is equal to zero if K is a number field, otherwise $c(K) = \frac{1}{2}$;
- $P(s)$ is a meromorphic function with simple poles at the values of s for which $|\cdot|^s$ is trivial.

From Equation (4.20), it is easy to see that the zeta integral satisfies the properties stated in Theorem 4.4.1.

Lemma 4.4.5. *If ω is a unitary idèle class character and s is a complex number with $\Re(s) > 1$, then, the identity*

$$\int_{x\in\mathbb{A}^\times,\,|x|<1} f(x)\omega(x)|x|^s \, d^\times x = \\ \zeta^>(1-s,\omega^{-1};\widehat{f}) \quad - f(0)\int_{x\in\frac{\mathbb{A}^\times}{K^\times},\,|x|<1} \omega(x)|x|^s \, d^\times x \quad + \widehat{f}(0)\int_{x\in\frac{\mathbb{A}^\times}{K^\times},\,|x|<1} \omega(x)|x|^{s-1} \, d^\times x$$

holds for all $f \in \mathscr{S}(\mathbb{A})$.

Proof. Since $\omega|\cdot|^s$ is trivial on $K^\times$ we have

$$\int_{x\in\mathbb{A}^\times,\,|x|<1} f(x)\omega(x)|x|^s \, d^\times x = \int_{x\in\frac{\mathbb{A}^\times}{K^\times},\,|x|<1} f^\times(x)\omega(x)|x|^s \, d^\times x \,.$$

By proposition 4.4.3 we have that

$$\int_{x\in\frac{\mathbb{A}^\times}{K^\times},\,|x|<1} f^\times(x)\omega(x)|x|^s \, d^\times x = \int_{x\in\frac{\mathbb{A}^\times}{K^\times},\,|x|<1} \left[|x|^{-1}\,\widehat{f}^\times(x^{-1}) - f(0) + |x|^{-1}\,\widehat{f}(0)\right]\omega(x)|x|^s \, d^\times x \,.$$

Through the transformation $x \mapsto x^{-1}$ we obtain

$$\begin{aligned}\int_{x\in\frac{\mathbb{A}^\times}{K^\times},\,|x|<1} |x|^{-1}\,\widehat{f}^\times(x^{-1})\omega(x)|x|^s \, d^\times x &= \int_{x\in\frac{\mathbb{A}^\times}{K^\times},\,|x|>1} \widehat{f}^\times(x)\omega(x)^{-1}|x|^{1-s} \, d^\times x \\ &= \int_{x\in\mathbb{A}^\times,\,|x|>1} \widehat{f}(x)\omega(x)^{-1}|x|^{1-s} \, d^\times x \\ &= \zeta^>(1-s,\omega^{-1};\widehat{f})\end{aligned}$$

and the claim follows from this. □

Now we have to calculate the difference

$$\Delta(s,\omega;f) := \zeta(s,\omega;f) - \zeta^>(s,\omega;f) - \zeta^>(1-s,\omega^{-1};\widehat{f})$$

and show that it has the form given in Equation (4.20). The case of function fields has to be considered separately from the case of number fields because the integral over the region $|x| \le 1$ is not the same as the integral over the region $|x| < 1$ in the former case. Suppose that K is a function field of characteristic p and that $T = \left\{q^{-n} : n \in \mathbb{Z}\right\}$ is the image of the idèlic norm, where q is a power of the prime p. Then

$$\Delta(s,\omega;f) = \int_{\mathbb{A}^{\times,1}/K^\times} f^\times(x)\omega(x)\, d^\times x \\ - f(0)\int_{x\in\frac{\mathbb{A}^\times}{K^\times},\,|x|<1} \omega(x)|x|^s \, d^\times x \quad + \widehat{f}(0)\int_{x\in\frac{\mathbb{A}^\times}{K^\times},\,|x|<1} \omega(x)|x|^{s-1} \, d^\times x \,,$$

where the integral over $\mathbb{A}^{\times,1}/K^\times$ appears because, in the function field case, the subset of idèle classes x with $|x| = 1$ has non-zero measure inside the group $\mathbb{A}^\times/K^\times$. Consider the integral of $f^\times\omega$ over $\mathbb{A}^{\times,1}/K^\times$ and use Equation (4.14) on a half of it, so that one obtains

$$\int_{\mathbb{A}^{\times,1}/K^\times} f^\times(x)\omega(x)\,d^{\times,1}x = \frac{1}{2}\int_{\mathbb{A}^{\times,1}/K^\times} f^\times(x)\omega(x)\,d^{\times,1}x \;+\frac{1}{2}\int_{\mathbb{A}^{\times,1}/K^\times} \widehat{f}^\times(x)\omega(x)^{-1}\,d^{\times,1}x$$
$$-\frac{1}{2}f(0)\int_{\mathbb{A}^{\times,1}/K^\times}\omega(x)\,d^{\times,1}x \;+\frac{1}{2}\widehat{f}(0)\int_{\mathbb{A}^{\times,1}/K^\times}\omega(x)\,d^{\times,1}x$$

after a change of variables $x \mapsto x^{-1}$ in the integral where $\widehat{f}^\times$ appears. We now have the zeta integral decomposed as a sum of a symmetric term

$$\zeta^>(s,\omega;f) \;+\; \zeta^>(1-s,\omega^{-1};\widehat{f}) \;+\; \frac{1}{2}\int_{\mathbb{A}^\times} f(x)\omega(x)\,d^\times x \;+\; \frac{1}{2}\int_{\mathbb{A}^\times}\widehat{f}(x)\omega(x)^{-1}\,d^\times x \tag{4.21}$$

and a singular term. The singularity comes from the integral

$$\int_{x\in\frac{\mathbb{A}^\times}{K^\times},\,|x|<1}\omega(x)|x|^z\;d^\times x$$

where z is equal to s or $s-1$. As before, we can assume that $\omega(x) = \chi(x\rho(t)^{-1})$, where $t = |x|$, χ is a unitary character of $\mathbb{A}^{\times,1}/K^\times$ and $\rho : T \to \mathbb{A}^\times/K^\times$ is a section of the idèlic norm. Let

$$V(\chi) := \int_{\mathbb{A}^{\times,1}/K^\times}\chi(x)\,d^{\times,1}x\,,$$

then

$$\begin{aligned}\int_{x\in\frac{\mathbb{A}^\times}{K^\times},\,|x|<1}\omega(x)|x|^z\;d^\times x &= \int_{t<1} t^z\int_{\mathbb{A}^{\times,1}/K^\times}\chi(x)\,d^{\times,1}x\,d^\times t\\ &= \sum_{n=1}^{\infty} q^{-nz}V(\chi)\\ &= \frac{V(\chi)q^{-z}}{1-q^{-z}}\,.\end{aligned}$$

Therefore, the difference between the zeta integral and the term (4.21) is equal to

$$\frac{V(\chi)}{2}\left(-f(0) - 2f(0)\frac{q^{-s}}{1-q^{-s}} + \widehat{f}(0) + 2\widehat{f}(0)\frac{q^{1-s}}{1-q^{1-s}}\right)\,,$$

which becomes

$$\frac{V(\chi)}{2}\left(f(0)\frac{1+q^s}{1-q^s} + \widehat{f}(0)\frac{1+q^{1-s}}{1-q^{1-s}}\right) \tag{4.22}$$

after a simple algebraic manipulation. The expression (4.22) is symmetric, and we have

$$P(s) = \frac{1}{2}\cdot\frac{1+q^s}{1-q^s}$$

regarding the claimed symmetric formula (4.20) for the zeta integral. This proves the analytic continuation and the functional equation for the zeta integral of a function field. If K is a number field, then the image of the idèlic norm is $\mathbb{R}_+^\times$ and the difference

$$\Delta(s,\omega;f) = \zeta(s,\omega;f) - \zeta^>(s,\omega;f) - \zeta^>(1-s,\omega^{-1};\widehat{f})$$

is simpler:

$$\begin{aligned}
\Delta(s,\omega;f) &= -f(0)\int_{x\in\mathbb{A}^\times/K^\times} \omega(x)|x|^s\ d^\times x + \widehat{f}(0)\int_{x\in\mathbb{A}^\times/K^\times} \omega(x)|x|^{s-1}\ d^\times x \\
&= -f(0)\int_0^1 t^s V(\chi)\,\frac{dt}{t} + \widehat{f}(0)\int_0^1 t^{s-1} V(\chi)\,\frac{dt}{t} \\
&= -f(0)\frac{V(\chi)}{s} + \widehat{f}(0)\frac{V(\chi)}{s-1}\,,
\end{aligned}$$

where χ is the character of $\mathbb{A}^{\times,1}/K^\times$ inducing ω via a section $\rho : \mathbb{R}_+^\times \to \mathbb{A}^\times/K^\times$ of the idèlic norm, and $V(\chi)$ is its integral over the group $\mathbb{A}^{\times,1}/K^\times$. Finally, we get the following identity for the zeta integral in the case of a number field

$$\zeta(s,\omega;f) = \zeta^>(s,\omega;f) + \zeta^>(1-s,\omega^{-1};\widehat{f}) - V(\chi)\left(\frac{f(0)}{s} + \frac{\widehat{f}(0)}{1-s}\right) \tag{4.23}$$

and the main theorem is proved.

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