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STRONG GLOBAL DIMENSION AND PIECEWISE HEREDITARY ALGEBRAS

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Abstract

The goal of this thesis is to give a homological characterization of piecewise hereditary algebras. In particular we show that for a finite dimensional algebra Λ, it is equivalent to be piecewise hereditary or to have a finite strong global dimension. Furthermore we discuss in detail some prelimanary notions, such as derived categories or the representation theory of algebras, in order to make the thesis accessible for a wider audience.

Introduction

Homological dimensions are a useful tool to determine the complexity of abelian categories. In particular, when we focus on rings, these dimensions give us an indication of how complex modules over such rings are or give us informations about their derived categories. Take for example the global dimension of a ring R: we know that $\mathrm{gl.dim}\, R = 0$ means that R is semisimple or that $\mathrm{gl.dim}\, R = 1$ means that R is hereditary. Both these properties have consequences on the categories $\mathrm{mod}\, R$ and $D^b(R)$. Indeed, if we take a finite dimensional, hereditary algebra A, we know that $D^b(A)$ is particularly simple, in the sense that any complex is isomorphic a direct sum of shifts of its cohomologies. However, we have a class of algebras which are not hereditary, but still have a derived category which is "hereditary-like", meaning that we have an abelian, hereditary category $\mathcal{H}$ and a triangle equivalence $F : D^b(A) \to D^b(\mathcal{H})$, where A is one such algebra. We call these algebras *piecewise hereditary*. Because of the aforementioned equivalence, their derived categories are easy to deal with and thus we are interested in finding a way to classify all these algebras.

Following [6], we focused on a newer homological dimension, called *strong global dimension*, which was first introduced by Ringel. This dimension in particular measures the supremum of the length of indecomposable complexes of projectives A-modules in $K^b(A)$ and is shown to be bigger or equal than the normal global dimension.

The goal of this thesis is to follow [6] in order to prove that a finite dimensional algebra is piecewise hereditary if and only if its strong global dimension is finite. In order to be able to make [6] comprehensible for people without an extensive background in these topics, we try to cover several areas that are needed to understand it. Among these we focused mostly on the notion of derived category and on the representation theory of algebras.

The contents of this thesis are as follows. In the first chapter we cover the preliminaries notions needed to understand [6]. In particular we talk about the localization of categories, triangulated categories, derived categories, the representation theory of algebras and Yoneda extensions. The second chapter covers [6] more closely. First we discuss some technical results and then we introduce the concept of paths and of strong paths in triangulated categories. Finally we give a proof of the original claim. The thesis ends with some explicit computations. In particular we computed the strong global dimension for a class of algebras, based on linear, directed quivers and we found an algebra with finite global dimension and infinite strong global dimension, which is important to show the difference between these two homological dimensions.

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Chapter 1

Preliminaries

In this chapter we are going to introduce some concepts about derived categories and about the representation theory of algebras which are necessary to understand the central result of this thesis. This chapter though is not supposed to be intended as a thorough and complete dissertation on these topics, but rather as a collection of important definitions and results, necessary for understanding the topics of this thesis. For a more precise and systematic approach we refer to [8], [1] and [9], which are the main references used by the writer of this thesis.

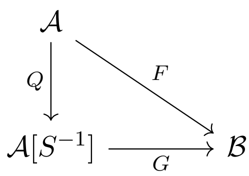
1.1 Localization of categories

In this section we will refer mainly to [8]. Any proof that will be omitted can be found in [8], unless otherwise stated.

Theorem 1.1.1. *Let $\mathcal{A}$ be a category and S be an arbitrary class of morphisms in $\mathcal{A}$. Then there exists a category $\mathcal{A}[S^{-1}]$ and a functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ such that*

i) $Q(s)$ is an isomorphism for every $s \in S$;

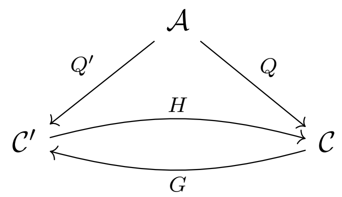
ii) For any category $\mathcal{B}$ and every functor $F : \mathcal{A} \to \mathcal{B}$ such that $F(s)$ is an isomorphism for any $s \in S$, there exists a unique functor $G : \mathcal{A}[S^{-1}] \to \mathcal{B}$ such that $F = G \circ Q$, i.e., we have the following commutative diagram of functors:



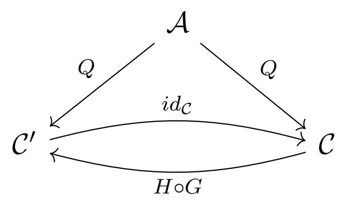
Moreover the category $\mathcal{A}[S^{-1}]$ is unique up to isomorphism.

The category $\mathcal{A}[S^{-1}]$ is called the *localization* of $\mathcal{A}$ with respect to S.

Proof. First we prove uniqueness. Assume we have two pairs $(\mathcal{C}, Q)$ and $(\mathcal{C}', Q')$ satisfying the conditions of the theorem. Then, the universal property implies the existence of the functors $G : \mathcal{C} \to \mathcal{C}'$ and $H : \mathcal{C}' \to \mathcal{C}$ such that $Q' = G \circ Q$ and $Q = H \circ Q'$, meaning that we would have the following commutative diagram of functors:



In particular this implies that $Q' = (G \circ H) \circ Q'$ and $Q = (H \circ G) \circ Q$. This leads us to the following commutative diagram of functors:



where $id_{\mathcal{C}}$ denotes the identity functor of $\mathcal{C}$. By the uniqueness of the factorization we must have $H \circ G = id_{\mathcal{C}}$. With analogous considerations we also get that $G \circ H = id_{\mathcal{C}'}$. Therefore H and G are isomorphisms of categories.

It remains to prove the existence of $\mathcal{A}[S^{-1}]$. First we put

$$\operatorname{Ob} \mathcal{A}[S^{-1}] = \operatorname{Ob} \mathcal{A}$$

It remains to define the morphisms in $\mathcal{A}[S^{-1}]$.

Consider two objects M and N in $\mathcal{A}$ (thus also in $\mathcal{A}[S^{-1}]$). Let $I_n = (0, 1, \ldots, n)$, $J_n = \{(i, i+1) : 0 \leq i \leq n-1\}$. We define a *path* of length n as

i) a map L of I_n into $\operatorname{Ob} \mathcal{A}$ such that $L_0 = M$ and $L_n = N$,

ii) a map Φ of J_n into the morphisms of $\mathcal{A}$ such that either $\Phi(i, i+1) = f_i : L_i \to L_{i+1}$ or $\Phi(i, i+1) = s_i : L_{i+1} \to L_i$ for some $s_i \in S$.

Paths can be represented diagrammatically by an oriented graph as

$$M \xrightarrow{f_0} L_1 \xrightarrow{f_1} \cdots \xleftarrow{s_{i-1}} L_i \xrightarrow{f_i} \cdots \xleftarrow{s_{n-1}} N$$

Now we define what an *elementary transformation* of a path is:

i) Switch of

$$\cdots L_{i-1} \xrightarrow{f_{i-1}} L_i \xrightarrow{f_i} L_{i+1} \cdots$$

and

$$\cdots L_{i-1} \xrightarrow{f_i \circ f_{i-1}} L_{i+1} \cdots$$

ii) Switch of

$$\cdots L \xrightarrow{s} P \xleftarrow{s} L \cdots$$

and

$$\cdots L \xrightarrow{id_L} L \cdots$$

iii) Switch of

$$\cdots L \xleftarrow{s} P \xrightarrow{s} L \cdots$$

and

$$\cdots L \xrightarrow{id_L} L \cdots$$

iv) Switch of

$$\cdots L \xrightarrow{id_L} L \xleftarrow{s} P \cdots$$

and

$$\cdots L \xleftarrow{s} P \cdots$$

Two paths between two objects M and N are said to be *equivalent* if one can be obtained from the other via a finite sequence of elementary transformations. This clearly defines an equivalence relation on the set of all paths between M and N.

Now we are finally able to define morphisms between two objects M and N in $\mathcal{A}[S^{-1}]$ as equivalence classes of paths between M and N. We also define the composition of paths as concatenation. It clearly induces a composition on equivalence classes. The identity morphism of an object M is given by the equivalence class of the path

$$M \xrightarrow{id_M} M$$

One can easily check that $\mathcal{A}[S^{-1}]$ is a category.

We also still have to define the functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$. We set $Q(M) = M$ for $M \in \mathrm{Ob}\,\mathcal{A}$. For a morphism $f : M \to N$ instead we define $Q(f)$ to be the path

$$M \xrightarrow{\quad f \quad} N$$

Clearly for $s \in S$ we have that $Q(s)$ is represented by

$$M \xrightarrow{\quad s \quad} N$$

and its inverse is the equivalence class of the path

$$N \xleftarrow{\quad s \quad} M$$

Hence $Q(s)$ is an isomorphism for all $s \in S$.

Now consider a functor $F : \mathcal{A} \to \mathcal{B}$ sending all morphisms in S into isomorphisms. We need to define $G : \mathcal{A}[S^{-1}] \to \mathcal{B}$ such that $F = G \circ Q$. We set G to be equal to F on objects. For a path P of length n between M and N instead we put

$$G(P) = G(\Phi(n-1, n)) \circ \cdots \circ G(\Phi(2,1)) \circ G(\Phi(1,0))$$

where

$$G(\Phi(i, i+1)) = \begin{cases} F(f_i), & \text{if } \Phi(i, i+1) = f_i : L_i \to L_{i+1} \\ F(s_i)^{-1}, & \text{if } \Phi(i, i+1) = s_i : L_{i+1} \to L_i \end{cases}$$

If a path P' is obtained from another path P by an elementary transformation, one can easily check that $G(P') = G(P)$. Therefore G is constant on the equivalence classes of paths. Hence it induces a map from $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$ into $\mathrm{Hom}_{\mathcal{B}}(G(M), G(N))$. Again it is easy to check that G, defined in this way, is a functor from $\mathcal{A}[S^{-1}]$ into $\mathcal{B}$ and that $G \circ Q = F$. Moreover, by construction G is uniquely determined by F. Therefore the pair $(\mathcal{A}[S^{-1}], Q)$ satisfies the conditions of the theorem. □

To make use of duality it is important to know what the opposite of a localized category $\mathcal{A}[S^{-1}]$ looks like and in particular if it can also be viewed as a localized category. In fact we are interested in the two following categories: $\mathcal{A}[S^{-1}]^{opp}$, i.e. the actual opposite of the localized category, and $\mathcal{A}^{opp}[S^{-1}]$, i.e. the localization with respect to S (viewed as a set of morphisms in $\mathcal{A}^{opp}$) of the opposite category $\mathcal{A}^{opp}$. Fortunately we have the following

Theorem 1.1.2. *Let $\mathcal{A}$ be a category and S be a class of morphisms in $\mathcal{A}$. Then the categories $\mathcal{A}[S^{-1}]^{opp}$ and $\mathcal{A}^{opp}[S^{-1}]$ are isomorphic.*

The construction we described in Theorem 1.1.1 is fairly general, however if S is an arbitrary class of morphisms, it is very hard to say anything useful about $\mathcal{A}[S^{-1}]$. That's why localization is mostly done with special types of classes of morphisms, called localizing classes, for which one can give a more manageable description of morphisms.

Definition 1.1.1. *Let $\mathcal{A}$ be a category. A class of morphisms S in $\mathcal{A}$ is called a* localizing class *if it has the following properties:*

LC1) For any object M in $\mathcal{A}$, the identity morphism id_M is in S;

LC2) If s, t are morphisms in S, then also their composition $s \circ t$ is in S (clearly given that such composition is possible);

LC3a) For any pair $f \in \operatorname{Mor} \mathcal{A}$ and $s \in S$, there exist $g \in \operatorname{Mor} \mathcal{A}$ and $t \in S$ such that the diagram

$$\begin{array}{ccc} K & \overset{g}{\dashrightarrow} & L \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ M & \xrightarrow{f} & N \end{array}$$

is commutative;

LC3b) For any pair $f \in \operatorname{Mor} \mathcal{A}$ and $s \in S$, there exist $g \in \operatorname{Mor} \mathcal{A}$ and $t \in S$ such that the diagram

$$\begin{array}{ccc} K & \overset{g}{\dashleftarrow} & L \\ {\scriptstyle t}\uparrow & & \uparrow{\scriptstyle s} \\ M & \xleftarrow{f} & N \end{array}$$

is commutative;

LC4) Let $f, g : M \to N$ be two morphisms. Then there exists s in S such that $s \circ f = s \circ g$ if and only if there exists t in S such that $f \circ t = g \circ t$.

It is clear that if S is a localizing class in $\mathcal{A}$, then it is also a localizing class in the opposite category $\mathcal{A}^{opp}$.

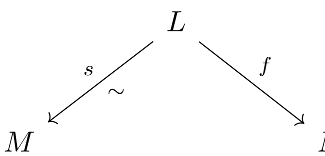
In this setting, we would like to be able to give a description of the morphisms in $\mathcal{A}[S^{-1}]$ which is more suitable for computations. First of all we notice that any morphism in $\mathcal{A}[S^{-1}]$ is represented as the composition of several morphisms $Q(s)^{-1}$, for $s \in S$, and $Q(f)$. Also, by (LC2) we know that $Q(t)^{-1} \circ Q(s)^{-1} = Q(s \circ t)^{-1}$ since both s and t and $s \circ t$ are in S and thus are invertible. Hence we can conclude that any morphism in $\mathcal{A}[S^{-1}]$ has the form

$$Q(f_1) \circ Q(s_1)^{-1} \circ Q(f_2) \circ Q(s_2)^{-1} \circ \cdots \circ Q(f_n) \circ Q(s_n)^{-1}$$

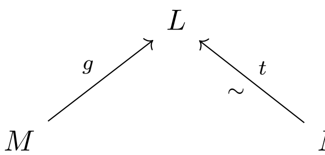
with $s_1, s_2, \ldots, s_n \in S$. Notice in particular that for all i, s_i and f_i have the same domain, exactly as it happens in (LC3a). Indeed, this property tells us that for any such morphisms f and $s \in S$, there exist g and $t \in S$ such that $f \circ t = s \circ g$. Clearly we also have $Q(f) \circ Q(t) = Q(s) \circ Q(g)$ which yields $Q(s)^{-1} \circ Q(f) = Q(g) \circ Q(t)^{-1}$. By induction on n on our previous representation of morphisms, we get that any morphism in $\mathcal{A}[S^{-1}]$ can be represented as $Q(f) \circ Q(s)^{-1}$ with $s \in S$. Analogously it can also be represented by $Q(s)^{-1} \circ Q(f)$ with $s \in S$.

With these considerations in mind we can give the following

Definition 1.1.2. *Let $\mathcal{A}$ be a category and S a localizing class of morphisms in $\mathcal{A}$. A* (left) roof *betwwen M and N is a diagram*



where s is in S and L is in $\operatorname{Ob}\mathcal{A}$. *The symbol $\sim$ denotes that that arrow is in S. Analogously we define a* (right) roof *between M and N as a diagram*

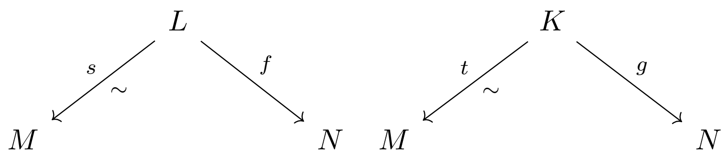


where t is in S.

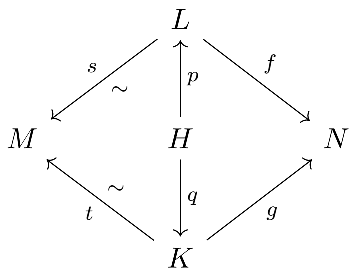
Clearly, going to $\mathcal{A}$ to the opposite category $\mathcal{A}^{opp}$ switches left roofs between M and N with right roofs between N and M. Therefore, it is enough to study the properties of left roofs.

From now on, when we will talk about roofs we will be referring to left roofs, always keeping in mind that analogous results can be obtained for right roofs.

We say that two roofs



are *equivalent* if there exists an object H in $\mathcal{A}$ and morphisms $p : H \to L$ and $q : H \to K$ such that the diagram



commutes and $s \circ p = t \circ q \in S$.

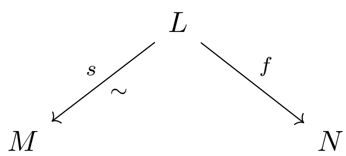
This definition is motivated by the fact that, using the same notation as above, $Q(p \circ s) = Q(p) \circ Q(s)$ is an isomorphism in $\mathcal{A}[S^{-1}]$. Indeed since $Q(s)$ is an isomorphism, this means that also $Q(p)$ is such and an analogous argument shows that $Q(q)$ is an isomorphism as well. Thus,

$$Q(f) \circ Q(s)^{-1} = Q(f) \circ Q(p) \circ Q(p)^{-1} \circ Q(s)^{-1} = Q(f \circ p) \circ Q(s \circ p)^{-1} =$$
$$= Q(g \circ q) \circ Q(t \circ q)^{-1} = Q(g) \circ Q(q) \circ Q(q)^{-1} \circ Q(t)^{-1} = Q(g) \circ Q(t)^{-1}$$

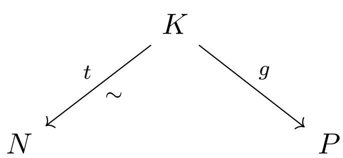
Notice that we can define an analogous notion of equivalent right roofs (we just require that there is a commutative diagram similar to the one for left roofs, but with p and q going into H).

Lemma 1.1.3. *The above relation on left roofs (and on right roofs) is an equivalence relation.*

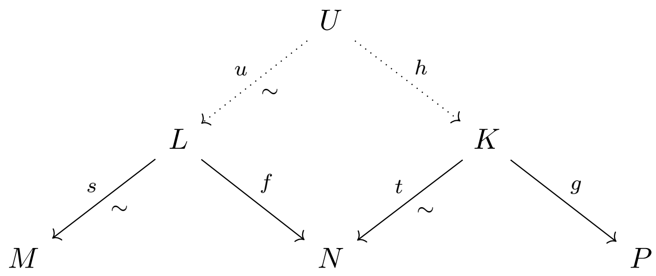
We also want to define a composition of equivalence classes of (left) roofs, since we said that roofs are a useful representation of morphisms in $\mathcal{A}[S^{-1}]$. To do this let



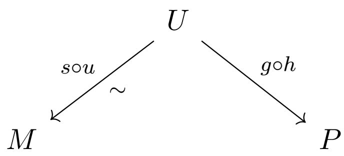
be a roof between M and N and



be a roof between N and P. Then, by (LC3a), there exists an object U and morphisms $u : U \to L$ in S and $h : U \to K$ such that



is a commutative diagram. It determines the roof

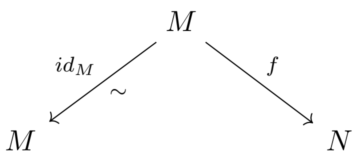


(⋆)

Lemma 1.1.4. *The equivalence class of the roof in* (⋆) *is independent of the choices of* U*,* u *and* h *and depends only on the equivalence classes of the first two roofs.*

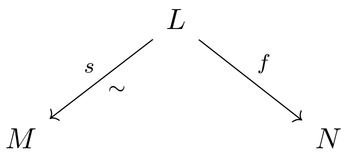
It follows that the above process defines a map from the product of the sets of equivalence classes of roofs between M and N and equivalence classes of roofs between N and P into the set of equivalence classes of roofs between M and P. We call this map the *composition of (left) roofs*. One can check that this composition is also associative.

Now we are able to give a complete description of the category $\mathcal{A}[S^{-1}]$ and of the functor $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ from Theorem 1.1.1 when S is a localizing class. In particular $\mathcal{A}[S^{-1}]$ is the category with $\operatorname{Ob}\mathcal{A}[S^{-1}] = \operatorname{Ob}\mathcal{A}$ and $\operatorname{Hom}_{\mathcal{A}[S^{-1}]}(M, N) = \{$ equivalence classes of roofs between M and $N\}$ and Q is the identity on objects and assigns to a morphism $f : M \to N$ the equivalence class of roofs attached to the roof



One can easily check that Q is a functor. Furthermore, it can be proven that if there is a functor $F : \mathcal{A} \to \mathcal{B}$ mapping morphisms in S into isomorphism, then F factors uniquely through the localized category $\mathcal{A}[S^{-1}]$. Thus we have a complete description of the localized category $\mathcal{A}[S^{-1}]$ that makes computations more manageable.

Now consider a category $\mathcal{A}$ with a localizing class S and let $\mathcal{B}$ be a subcategory of $\mathcal{A}$. Then, if $S_{\mathcal{B}} = S \cap \operatorname{Mor}\mathcal{B}$ forms a localizing class in $\mathcal{B}$, we have a natural functor $\mathcal{B}[S_{\mathcal{B}}^{-1}] \to \mathcal{A}[S^{-1}]$. This functor sends an object in $\mathcal{B}$ into itself and a morphism in $\mathcal{B}[S_{\mathcal{B}}^{-1}]$ represented by a roof



into the equivalence class of the same roof in $\mathcal{A}[S^{-1}]$.

Proposition 1.1.5. *Let $\mathcal{A}$ be a category, S a localizing class of morphisms in $\mathcal{A}$ and $\mathcal{B}$ a full subcategory of $\mathcal{A}$. Assume that the following conditions are satisfied:*

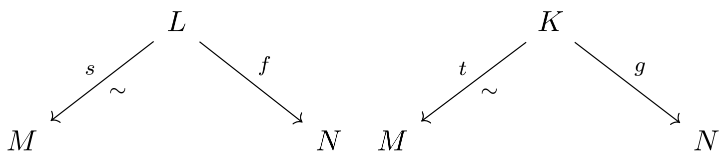
i) $S_{\mathcal{B}} = S \cap \operatorname{Mor} \mathcal{B}$ is a localizing class in $\mathcal{B}$;

ii) for each morphism $s : N \to M$ with $s \in S$ and $M \in \operatorname{Ob} \mathcal{B}$, there exists $u : P \to N$ such that $s \circ u \in S$ and $P \in \operatorname{Ob} \mathcal{B}$.

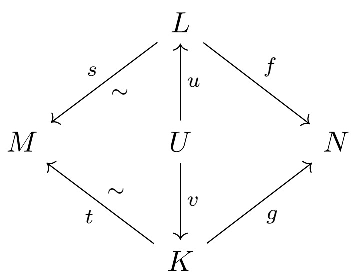
Then the natural functor $\mathcal{B}[S_{\mathcal{B}}^{-1}] \to \mathcal{A}[S^{-1}]$ is fully faithful.

Proof. Let M and N be two objects in $\mathcal{B}$. We have to show that the map $\operatorname{Hom}_{\mathcal{B}[S_{\mathcal{B}}^{-1}]}(M, N) \to \operatorname{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$ is a bijection.

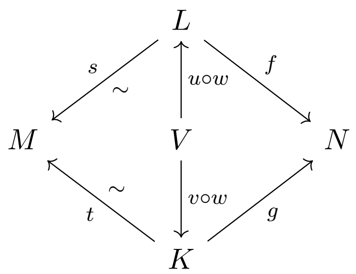
First we prove injectivity. Let



be two roofs representing morphisms in $\mathcal{B}[S_{\mathcal{B}}^{-1}]$ which determine the same morphism in $\mathcal{A}[S^{-1}]$. This implies that we have the following commutative diagram of roofs

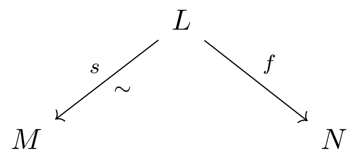


where $U \in \operatorname{Ob} \mathcal{A}$ and $s \circ v = t \circ u \in S$. Since $L, K \in \operatorname{Ob} \mathcal{B}$, by ii) there exist V in $\mathcal{B}$ and $w : V \to U$ such that $s \circ u \circ w = t \circ v \circ w \in S$. Hence, we get the diagram

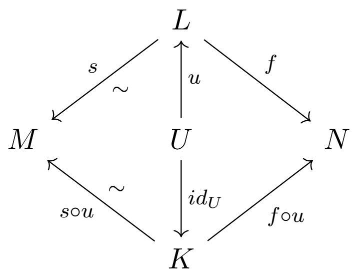


which clearly commutes. It follows that the above roofs determine the same morphism in $\mathcal{B}[S_{\mathcal{B}}^{-1}]$. Thus, the above map is an injection.

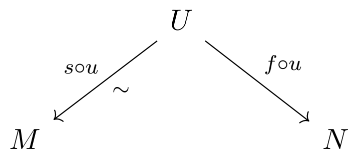
In order to show surjectivity, let



be a roof representing a morphism $\varphi \in \mathrm{Hom}_{\mathcal{A}[S^{-1}]}(M, N)$. By ii). there exist U in $\mathcal{B}$ and $u : U \to L$ in S such that $s \circ u \in S$. Therefore we have the following commutative diagram



which in particular implies that the roof



also represents φ. On the other hand, it determines also a morphism in $\mathrm{Hom}_{\mathcal{B}[S_{\mathcal{B}}^{-1}]}(M, N)$ which maps into φ, i.e. the map is surjective. $\square$

Therefore one can view $\mathcal{B}[S_{\mathcal{B}}^{-1}]$ as a full subcategory of $\mathcal{A}[S^{-1}]$.

If we now replace $\mathcal{A}$ with its opposite category, we get the following result.

Proposition 1.1.6. *Let $\mathcal{A}$ be a category, S a localizing class of morphisms in $\mathcal{A}$ and $\mathcal{B}$ a full subcategory of $\mathcal{A}$. Assume that the following conditions are satisfied:*

i) $S_{\mathcal{B}} = S \cap \operatorname{Mor} \mathcal{B}$ *is a localizing class in* $\mathcal{B}$*;*

ii) *for each morphism* $s : M \to N$ *with* $s \in S$ *and* $M \in \operatorname{Ob} \mathcal{B}$*, there exists* $u : N \to P$ *such that* $u \circ s \in S$ *and* $P \in \operatorname{Ob} \mathcal{B}$*.*

Then the natural functor $\mathcal{B}[S_{\mathcal{B}}^{-1}] \to \mathcal{A}[S^{-1}]$ *is fully faithful.*

We are mostly interested in localizing categories which are additive or abelian. For these kind of categories we have results analogous to Theorems 1.1.1 and 1.1.2.

Theorem 1.1.7. *Let* $\mathcal{A}$ *be an additive (respectively abelian) category and* S *a localizing class. There exists a unique additive (respectively abelian) category* $\mathcal{A}[S^{-1}]$ *and an additive (respectively exact) functor* $Q : \mathcal{A} \to \mathcal{A}[S^{-1}]$ *such that the pair* $(\mathcal{A}[S^{-1}], Q)$ *is the localization of* $\mathcal{A}$ *with respect to* S*. Furthermore we have once again that the categories* $\mathcal{A}[S^{-1}]^{opp}$ *and* $\mathcal{A}^{opp}[S^{-1}]$ *are isomorphic.*

This theorem in particular tells us that if $\mathcal{A}$ is an abelian category, S a localizing class and $\varphi : M \to N$ is a morphism in $\mathcal{A}[S^{-1}]$, then φ has a kernel and a cokernel.

Remark. When dealing with additive categories we can replace (LC4) in the definition of a localizing class with

LC4') Let $f : M \to N$ be a morphism. Then there exists $s \in S$ such that $s \circ f = 0$ if and only if there exists $t \in S$ such that $f \circ t = 0$

Indeed, since $\operatorname{Hom}_{\mathcal{A}}(M, N)$ is an abelian group, $s \circ f = s \circ g$ is equivalent to $s \circ (f - g) = 0$ and similarly $f \circ t = g \circ t$ is equivalent to $(f - g) \circ t = 0$. Therefore, if we replace f by $f - g$ in (LC4'), it becomes identical to (LC4).

To end this section we want to provide some results concerning morphisms in localized additive categories. In particular, from now on, $\mathcal{A}$ will denote an additive category and S will be a localizing class in $\mathcal{A}$.

Lemma 1.1.8. *Let* $\varphi : M \to N$ *be a morphism be a morphism in* $\mathcal{A}[S^{-1}]$ *represented by a roof*

$$M \xleftarrow[\sim]{\;s\;} L \xrightarrow{\;f\;} N$$

Then the following conditions are equivalent:7

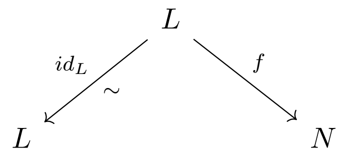
i) $\varphi = 0$*;*

ii) *There exists* $t \in S$ *such that* $f \circ t = 0$*;*

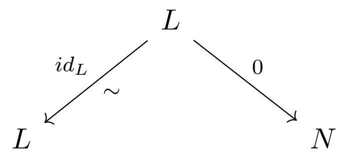
iii) There exists $t \in S$ such that $t \circ f = 0$.

Proof. Clearly ii) and iii) are equivalent conditions by (LC4').

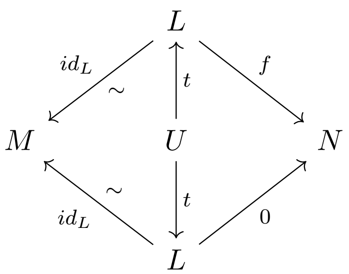
Now assume i) holds. Then $0 = Q(f) \circ Q(s)^{-1}$ and thus $Q(f) = 0$ since $Q(s)^{-1}$ is an isomorphism. Therefore the roof



represents the zero morphism in $\mathrm{Hom}_{\mathcal{A}[S^{-1}]}(L, N)$. We know that the zero morphism between L and N is also represented by the roof



This implies that these two roofs are equivalent, which means that there is $U \in \mathrm{Ob}\,\mathcal{A}$ and $t : U \to L$ such that the diagram



commutes and t is in S. This implies that there is $t \in S$ such that $f \circ t = 0$.

Conversely, if ii) holds, $f \circ t = 0$ and thus $Q(f) \circ Q(t) = 0$. Since $Q(t)$ is an isomorphism this means that $Q(f) = 0$ and thus $\varphi = Q(f) \circ Q(s)^{-1} = 0$, as we wanted. □

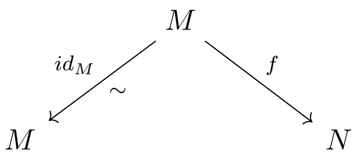
Corollary 1.1.9. *Let $f : M \to N$ be a morphism in $\mathcal{A}$. Then the following conditions are equivalent:*

i) $Q(f) = 0$;

ii) There exists $t \in S$ such that $f \circ t = 0$;

iii) There exists $t \in S$ such that $t \circ f = 0$.

Proof. The morphism $Q(f)$ is represented by the roof



Hence the result follows from Lemma 1.1.8. ☐

Corollary 1.1.10. *Let M be an object in $\mathcal{A}$. Then the following conditions are equivalent:*

i) $Q(M) = 0$*;*

ii) There exists an object N in $\mathcal{A}$ such that the zero morphism $N \to M$ is in S;

iii) There exists an object N in $\mathcal{A}$ such that the zero morphism $M \to N$ is in S.

Proof. We can easily see that ii) and iii) are equivalent just by switching to the opposite category.

Now assume $Q(M) = 0$. This implies that $Q(id_M) = 0$. Hence, by Corollary 1.1.9, there exists $s : N \to M$ in S such that $s = id_M \circ s = 0$. This implies ii).
Conversely if ii) holds, the zero morphism $Q(N) \to Q(M)$ is an isomorphism. This implies that $Q(M) = Q(N) = 0$, as we wanted. Indeed if the zero morphism $0 : Q(N) \to Q(M)$ is an isomorphism, then we have an inverse, say g, and we also know that $0 = ba$ where $a : Q(N) \to 0$ and $b : 0 \to Q(M)$. Furthermore we clearly have that ag is an inverse to b (and analogously gb is an inverse to a) since $bag = id_{Q(M)}$ by the definition of g and $agb : 0 \to 0$ needs to be the identity on 0. ☐

Finally, we have the following consequence of the above results.

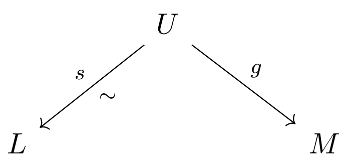
Lemma 1.1.11. *Let $f : M \to N$ be a morphism in $\mathcal{A}$. Then:*

i) If f is a monomorphism, $Q(f)$ is a monomorphism as well;

ii) If f is an epimorphism, $Q(f)$ is an epimorphism as well.

Proof. We can clearly see that i) and ii) are equivalent by switching from $\mathcal{A}$ to the opposite category $\mathcal{A}^{opp}$, therefore it suffices to prove i).

Let $\varphi : L \to M$ be a morphism in $\mathcal{A}[S^{-1}]$ such that $Q(f) \circ \varphi = 0$. The morphism φ is represented by a roof



and $\varphi = Q(g) \circ Q(s)^{-1}$. This implies that

$$0 = Q(f) \circ \varphi = Q(f) \circ Q(g) \circ Q(s)^{-1} = Q(f \circ g) \circ Q(s)^{-1}$$

and thus $Q(f \circ g) = 0$ since $Q(s)$ is an isomorphism. By Corollary 1.1.9 it follows that there exists $t \in S$ such that $f \circ g \circ t = 0$. But since f is a monomorphism, this implies that $g \circ t = 0$. By using 1.1.9 again, we see that this means that $Q(g) = 0$. Thus $\varphi = Q(g) \circ Q(s)^{-1} = 0$ and we proved that $Q(f)$ is a monomorphism. □

1.2 Triangulated categories

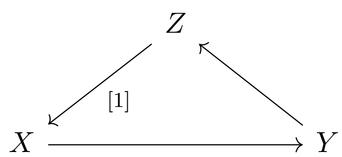
In this section we will refer mainly to [8]. Any proof that will be omitted can be found in [8], unless otherwise stated.

Let $\mathcal{C}$ be an additive category. Let $T : \mathcal{C} \to \mathcal{C}$ be an additive functor which is an automorphism of the category $\mathcal{C}$, meaning that T is bijective both on objects and on morphisms. We call such T the *translation functor* on $\mathcal{C}$. For an object X we will use the notation $T^n(X) = X[n]$ for any $n \in \mathbf{Z}$. An example of such a functor T is the shift functor on the category $\mathcal{C}(\mathcal{A})$ of complexes over an abelian category. We're going to give an intuitive description of it now, in order to have an idea of a non trivial such functor. Later on we are going to come back on it with more precise definitions. So for now, the shift functor is the functor sending a complex $X^\bullet = (X_i)_{i \in \mathbf{Z}}$ into the complex $X^\bullet[1]$ defined by $X[1]^i = X^{i+1}$ for all i.

A *triangle* in $\mathcal{C}$ is a diagram

$$X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$$

We are also going to represent triangles schematically as



A *morphism of triangles* is a commutative diagram

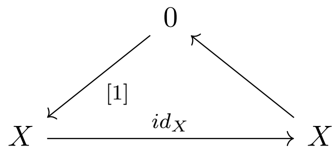
$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle T(u)} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array}$$

Such a morphism is an *isomorphism* of triangles if u, v and w are isomorphisms.

The category $\mathcal{C}$ is a *triangulated category* if it is equipped with a family of triangles called *distinguished triangles*, which satisfy the following properties:

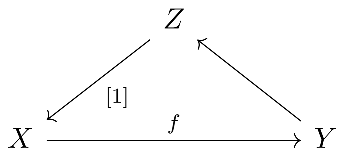
(TR1.a) Any triangle isomorphic to a distinguished triangle is a distinguished triangle.

(TR1.b) For any object X in $\mathcal{C}$,

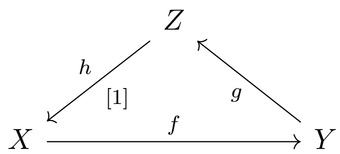


is a distinguished triangle.

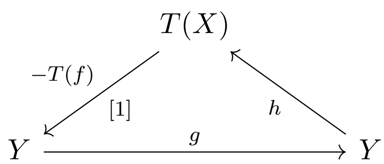
(TR1.c) For any morphism $f : X \to Y$ in $\mathcal{C}$, there exists a distinguished triangle



(TR2) The triangle



is distinguished if and only if the triangle



is distinguished.

(TR3) Let

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & & & \downarrow{\scriptstyle T(u)} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array}$$

be a diagram where the rows are distinguished triangles and the first square is commutative. Then there exists a morphism $w : Z \to Z'$ such that the diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle T(u)} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array}$$

is a morphism of distinguished triangles.

(TR4) Let f, g and $h = g \circ f$ be morphisms in $\mathcal{C}$. Then the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{a} & Z' & \longrightarrow & T(X) \\ \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle g} & & & & \downarrow{\scriptstyle T(id_X)} \\ X & \xrightarrow{h} & Z & \xrightarrow{b} & Y' & \longrightarrow & T(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id_Z} & & & & \downarrow{\scriptstyle T(f)} \\ Y & \xrightarrow{g} & Z & \xrightarrow{c} & X' & \longrightarrow & T(Y) \end{array}$$

where the rows are distinguished triangles can be completed to the diagram

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{a} & Z' & \longrightarrow & T(X) \\
\downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle T(id_X)} \\
X & \xrightarrow{h} & Z & \xrightarrow{b} & Y' & \longrightarrow & T(X) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id_Z} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle T(f)} \\
Y & \xrightarrow{g} & Z & \xrightarrow{c} & X' & \longrightarrow & T(Y) \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle id_{X'}} & & \downarrow{\scriptstyle T(a)} \\
Z' & \xrightarrow{u} & Y' & \xrightarrow{v} & X' & \xrightarrow{w} & T(Z')
\end{array}
$$

where all four rows are distinguished triangles and the vertical arrows are morphisms of triangles.

We will often refer to the second property as the *turning of triangles axiom*, and to the fourth property as the *octahedral axiom*.

Definition 1.2.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two triangulated categories. An additive functor $F : \mathcal{C} \to \mathcal{D}$ is called* graded *if $T \circ F$ is isomorphic to $F \circ T$.*

A graded functor $F : \mathcal{C} \to \mathcal{D}$ is called exact *if it maps distinguished triangles into distinguished triangles.*

If $F : \mathcal{C} \to \mathcal{D}$ is a graded functor, let η be the isomorphism of $F \circ T$ into $T \circ F$. If

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

is a triangle in $\mathcal{C}$ and we apply F to it, we get a diagram

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{F(h)} F(T(X)) \xrightarrow{\eta_X} T(F(X))$$

which is also a triangle since we can collapse the last two arrows into one.

Moreover if we have a morphism of triangles

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle T(u)} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X')
\end{array}
$$

and we apply F, we get the commutative diagram

$$\begin{array}{ccccccccc}
F(X) & \longrightarrow & F(Y) & \longrightarrow & F(Z) & \longrightarrow & F(T(X)) & \xrightarrow{\eta_X} & T(F(X)) \\
\downarrow{\scriptstyle F(u)} & & \downarrow{\scriptstyle F(v)} & & \downarrow{\scriptstyle F(w)} & & \downarrow{\scriptstyle F(T(u))} & & \downarrow{\scriptstyle T(F(u))} \\
F(X') & \longrightarrow & F(Y') & \longrightarrow & F(Z') & \longrightarrow & F(T(X')) & \xrightarrow{\eta_{X'}} & T(F(X'))
\end{array}$$

which is again a morphism of triangles, by collapsing the last two rectangles into one. Clearly if the original morphism is an isomorphism of triangles, so is the latter one, since functors preserve isomorphisms.

Now, if we have a triangulated category $(\mathcal{C}, T)$, we would like to know if we can give to its opposite category $\mathcal{C}^{opp}$ a compatible structure of a triangulated category. First of all we define the translation functor T^{opp} on $\mathcal{C}^{opp}$ as the inverse of the translation functor $X \mapsto T(X)$ on $\mathcal{C}$, i.e. we have $T^{opp}(X) = X[-1] = T^{-1}(X)$. If

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

is a distinguished triangle in $\mathcal{C}$, we declare

$$Z \xrightarrow{g} Y \xrightarrow{f} X \xrightarrow{T^{opp}(h)} T(Z)$$

to be a distinguished triangle in $\mathcal{C}^{opp}$.

Proposition 1.2.1. *The category $\mathcal{C}^{opp}$, equipped with the translation functor T^{opp} and with the family of distinguished triangles defined above, is a triangulated category.*

We will call $\mathcal{C}^{opp}$ the *opposite triangulated category* of $\mathcal{C}$.

Given a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

we can easily extend it to an infinite diagram

$$\cdots \xrightarrow{T^{-1}(h)} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X) \xrightarrow{T(f)} \cdots$$

Lemma 1.2.2. *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

be a distinguished triangle. Then the composition of any two consecutive morphisms in the triangle is equal to 0, i.e.

$$g \circ f = h \circ g = T(f) \circ h = 0$$

Proof. By (TR2) it is enough to show that $g \circ f = 0$ since for the other compositions we can just turn our triangle. Consider the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{id_X} & X & \longrightarrow & 0 & \longrightarrow & T(X) \\
\downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle f} & & & & \downarrow{\scriptstyle T(id_X)} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X)
\end{array}$$

By (TR1.b) the first row is also a distinguished triangle. Also by (TR3) we know that there is a morphism $u : 0 \to Z$ which completes the above diagram to the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{id_X} & X & \longrightarrow & 0 & \longrightarrow & T(X) \\
\downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle T(id_X)} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X)
\end{array}$$

which is a morphism of triangles. Since u is necessarily the zero morphism, from the commutativity of the middle square we conclude that $g \circ f = 0$. □

Now consider our triangulated category $\mathcal{C}$ and an additive functor $F : \mathcal{C} \to \mathcal{A}$, where $\mathcal{A}$ is an abelian category. For any distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

we have

$$F(g) \circ F(f) = 0$$

by 1.2.2. Moreover, the above long sequence of morphisms leads to the following complex

$$\cdots \xrightarrow{F(T^{-1}(h))} F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{F(h)} F(T(X)) \xrightarrow{F(T(f))} \cdots$$

of objects in $\mathcal{A}$.

Definition 1.2.2. *An additive functor $F : \mathcal{C} \to \mathcal{A}$ between a triangulated category $\mathcal{C}$ and an abelian category $\mathcal{A}$ is said to be a* cohomological functor *if for any distinguished triangle*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

we have an exact sequence

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

in $\mathcal{A}$. Therefore the above complex is exact.

Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Then for any object U in $\mathcal{C}$, it induces morphisms $f_* : \operatorname{Hom}_{\mathcal{C}}(U, X) \to \operatorname{Hom}_{\mathcal{C}}(U, Y)$ defined by $f_*(\varphi) = f \circ \varphi$ and $f^* : \operatorname{Hom}_{\mathcal{C}}(Y, U) \to \operatorname{Hom}_{\mathcal{C}}(X, U)$ given by $f^*(\psi) = \psi \circ f$.

Now let

$$X \xrightarrow{\quad f \quad} Y \xrightarrow{\quad g \quad} Z \xrightarrow{\quad h \quad} T(X)$$

be a distinguished triangle and U an object in $\mathcal{C}$. Then f, g and h induce morphisms in the following infinite sequences of abelian groups

$$\cdots \to \operatorname{Hom}_{\mathcal{C}}(U, X) \xrightarrow{\quad f_* \quad} \operatorname{Hom}_{\mathcal{C}}(U, Y) \xrightarrow{\quad g_* \quad} \operatorname{Hom}_{\mathcal{C}}(U, Z) \xrightarrow{\quad h_* \quad} \operatorname{Hom}_{\mathcal{C}}(U, T(X)) \longrightarrow$$

and

$$\cdots \xrightarrow{\quad T(f)^* \quad} \operatorname{Hom}_{\mathcal{C}}(T(X), U) \xrightarrow{\quad h^* \quad} \operatorname{Hom}_{\mathcal{C}}(Z, U) \xrightarrow{\quad g^* \quad} \operatorname{Hom}_{\mathcal{C}}(Y, U) \xrightarrow{\quad f^* \quad} \operatorname{Hom}_{\mathcal{C}}(X,$$

The next result tells us that these are long exact sequences of abelian groups.

Proposition 1.2.3. *Let U be an object in $\mathcal{C}$. Then*

i) The functor $X \mapsto \operatorname{Hom}_{\mathcal{C}}(U, X)$ from $\mathcal{C}$ to the category of abelian groups is a cohomological functor.

ii) The functor $X \mapsto \operatorname{Hom}_{\mathcal{C}}(X, U)$ from $\mathcal{C}^{opp}$ to the category of abelian groups is a cohomological functor.

Proof. Clearly i) and ii) are dual statements, so it suffices to prove i), i.e. we need to prove that $\operatorname{Im} f_* = \operatorname{Ker} g_*$. By 1.2.2 we already know that $\operatorname{Im} f_* \subseteq \operatorname{Ker} g_*$.

Assume now that $u : U \to Y$ is such that $g_*(u) = 0$, i.e. $g \circ u = 0$. Then we can consider the diagram

$$\begin{array}{ccccccc}
U & \xrightarrow{id_U} & U & \longrightarrow & 0 & \longrightarrow & T(U) \\
 & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle 0} & & \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & T(X)
\end{array}$$

where the middle square commutes and the rows are distinguished triangles. By turning both triangles we get the diagram

$$\begin{array}{ccccccc}
U & \longrightarrow & 0 & \longrightarrow & T(U) & \xrightarrow{-T(id_U)} & T(U) \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle T(u)} \\
Y & \xrightarrow{g} & Z & \longrightarrow & T(X) & \xrightarrow{-T(f)} & T(Y)
\end{array}$$

which we can complete by (TR3) to a morphism of distinguished triangles

$$\begin{array}{ccccccc}
U & \longrightarrow & 0 & \longrightarrow & T(U) & \xrightarrow{-T(id_U)} & T(U) \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle T(v)} & & \downarrow{\scriptstyle T(u)} \\
Y & \xrightarrow{g} & Z & \longrightarrow & T(X) & \xrightarrow{-T(f)} & T(Y)
\end{array}$$

By turning these triangles back, we get the morphism of distinguished triangles

$$\begin{array}{ccccccc}
U & \xrightarrow{id_U} & U & \longrightarrow & 0 & \longrightarrow & T(U) \\
\downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle T(v)} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & T(X)
\end{array}$$

Hence we constructed $v : U \to X$ such that $u = f \circ v = f_*(v)$. It follows that $u \in \operatorname{Im} f_*$. Hence, $\operatorname{Ker} g_* \subseteq \operatorname{Im} f_*$, and thus $\operatorname{Ker} g_* = \operatorname{Im} f_*$. □

Lemma 1.2.4. *Let*

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle T(u)} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X')
\end{array}$$

be a morphism of two distinguished triangles. If two morphisms between u, v and w are isomorphisms, then the third one is an isomorphism as well.

Proof. By turning the triangles we can assume without loss of generality that u and v are the two known isomorphisms. By 1.2.3, we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(Z',X) & \xrightarrow{u_*} & \mathrm{Hom}(Z',X') \\
\downarrow & & \downarrow \\
\mathrm{Hom}(Z',Y) & \xrightarrow{v_*} & \mathrm{Hom}(Z',Y') \\
\downarrow & & \downarrow \\
\mathrm{Hom}(Z',Z) & \xrightarrow{w_*} & \mathrm{Hom}(Z',Z') \\
\downarrow & & \downarrow \\
\mathrm{Hom}(Z',T(X)) & \xrightarrow{T(u)_*} & \mathrm{Hom}(Z',T(X')) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(Z',T(Y)) & \xrightarrow{T(v)_*} & \mathrm{Hom}(Z',T(Y'))
\end{array}$$

where both columns are exact and all horizontal arrows are isomorphisms, except possibly the middle one. Indeed since u and v are isomorphisms this means that also $T(u)$ and $T(v)$ are such, since $T(u^{-1}) \circ T(u) = T(u^{-1} \circ u) = id = T(u \circ u^{-1}) = T(u) \circ T(u^{-1})$. Clearly then the functor $\mathrm{Hom}(Z', -)$ sends them to other isomorphisms. By the five lemma the middle arrow is also an isomorphism. Therefore, there exists $a : Z' \to Z$ such that $w_*(a) = w \circ a = id_{Z'}$.

Analogously, by 1.2.3, we have the following commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(T(Y'), Z) & \xrightarrow{T(v)^*} & \mathrm{Hom}(T(Y)', Z) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(T(X'), Z) & \xrightarrow{T(u)^*} & \mathrm{Hom}(T(X), Z) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(Z', Z) & \xrightarrow{w^*} & \mathrm{Hom}(Z, Z) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(Y', Z) & \xrightarrow{v^*} & \mathrm{Hom}(Y, Z) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(X', Z) & \xrightarrow{u^*} & \mathrm{Hom}(X, Z)
\end{array}$$

where again both columns are exact and all horizontal arrows are isomorphisms, except possibly the middle one. Again by the five lemma the middle arrow is also an isomorphism. This means that there exists $b : Z' \to Z$ such that $w^*(b) = b \circ w = id_Z$. It follows that

$$b = b \circ (w \circ a) = (b \circ w) \circ a = a$$

Therefore w is an isomorphism, as wanted. □

Therefore, in the morphism

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & T(X) \\
\downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle id_Y} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle T(id_X)} \\
X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & T(X)
\end{array}$$

of two distinguished triangles based on $f : X \to Y$, the morphism $w : Z \to Z'$ is an isomorphism. It follows that the third vertex in a distinguished triangle is determined up to isomorphism. We call it a *cone* of f.

Lemma 1.2.5. *Let*

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow T(X)$$

be a distinguished triangle. Then the following statements are equivalent:

i) f is an isomorphism;

ii) $Z = 0$.

Proof. Consider the following morphism of distinguished triangles

$$\begin{array}{ccccccc} X & \xrightarrow{id_X} & X & \longrightarrow & 0 & \longrightarrow & T(X) \\ \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle T(id_X)} \\ X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & T(X) \end{array}$$

If $f : X \to Y$ is an isomorphism, then the first two vertical arrows are isomorphisms, therefore by Lemma 1.2.4 the third vertical arrow is an isomorphism as well, i.e. $Z = 0$.

Conversely, if $Z = 0$ the first and third vertical arrows are isomorphisms and by the same result $f : X \to Y$ is an isomorphism. □

The following result is a refinement of (TR3).

Proposition 1.2.6. *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

and

$$X' \xrightarrow{f} Y' \xrightarrow{g'} Z' \xrightarrow{h'} T(X')$$

be two distinguished triangles and $v : Y \to Y'$ be a morphism. Then we have the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\ \vdots{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \vdots{\scriptstyle w} & & \vdots{\scriptstyle T(u)} \\ X' & \xrightarrow{f} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X') \end{array}$$

and the following statements are equivalent:

i) $g' \circ v \circ f = 0$*;*

ii) there exists u such that the first square in the diagram is commutative;

iii) there exists w such that the second square in the diagram is commutative;

iv) there exist u and w such that the diagram is a morphism of triangles.

If these conditions are satisfied and $\mathrm{Hom}(X, Z'[-1]) = 0$*, the morphism u in ii) (resp. w in iii)) is unique.*

Proof. By Proposition 1.2.3 we have the following exact sequence

$$\mathrm{Hom}(X,Z'[-1]) \longrightarrow \mathrm{Hom}(X,X') \xrightarrow{f'_*} \mathrm{Hom}(X,Y') \xrightarrow{g'_*} \mathrm{Hom}(X,Z')$$

Therefore, if $g'_*(v\circ f) = g'\circ v\circ f = 0$, then $v\circ f \in \mathrm{Ker}\, g'_* = \mathrm{Im}\, f'_*$ and thus there is $u : X \to X'$ such that $v\circ f = f'(u) = f'\circ u$. Hence i) implies ii). Moreover, if $\mathrm{Hom}(X,Z'[-1]) = 0$, the morphism u is unique, since in this case f'_* would be a monomorphism by the exactness of the sequence.

Conversely, if ii) holds,

$$g'\circ v\circ f = g'\circ f'\circ u = 0$$

by Lemma 1.2.2, and i) holds. Analogously, by 1.2.3, we have the following exact sequence

$$\mathrm{Hom}(X[1],Z') \longrightarrow \mathrm{Hom}(Z,X') \xrightarrow{g^*} \mathrm{Hom}(Y,Z') \xrightarrow{f^*} \mathrm{Hom}(X,Z')$$

Therefore if $f^*(g'\circ v) = g'\circ v\circ f = 0$, there exists $w : Z \to Z'$ such that $g^*(w) = w\circ g = g'\circ v$, i.e. iii) holds. Moreover, if $\mathrm{Hom}(X[1],Z') = \mathrm{Hom}(X,Z'[-1]) = 0$, the morphism w is unique for similar reasons as before.

Conversely, if iii) holds,

$$g'\circ v\circ f = w\circ g\circ f = 0$$

again by 1.2.2, and i) holds.

Finally, if iv) holds, ii) a fortiori holds and if ii) implies iv) by (TR3). □

With similar arguments about the long exact sequence of homology one could prove the following.

Lemma 1.2.7. *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

and

$$X' \xrightarrow{f} Y' \xrightarrow{g'} Z' \xrightarrow{h'} T(X')$$

be two distinguished triangles. Then

$$X\oplus X' \xrightarrow{f\oplus f'} Y\oplus Y' \xrightarrow{g\oplus g'} Z\oplus Z' \xrightarrow{h\oplus h'} T(X\oplus X')$$

is a distinguished triangle.

In particular we are interested in the following consequences.

Corollary 1.2.8. *Let $\varepsilon_X : X \to X\oplus Y$ be the natural inclusion and $\pi_Y : X\oplus Y \to Y$ the natural projection. Then*

$$X \xrightarrow{\varepsilon_X} X \oplus Y \xrightarrow{\pi_Y} Y \xrightarrow{0} T(X)$$

is a distinguished triangle.

Proof. Clearly

$$X \xrightarrow{id_X} X \longrightarrow 0 \longrightarrow T(X)$$

and

$$Y \xrightarrow{id_Y} Y \longrightarrow 0 \longrightarrow T(Y)$$

are distinguished triangles by (TR1). By (TR2)

$$0 \longrightarrow Y \xrightarrow{id_Y} Y \longrightarrow T(0)$$

is also a distinguished triangle. Then the direct sum of the first and third distinguished triangles is also a distinguished triangle by Lemma 1.2.7. □

This result has the following converse.

Corollary 1.2.9. *Let*

$$X \xrightarrow{u} Z \xrightarrow{v} Y \xrightarrow{0} T(X)$$

be a distinguished triangle in $\mathcal{C}$. Then there exists an isomorphism $\varphi : X \oplus Y \to Z$.

Proof. By turning the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{\varepsilon_X} & X \oplus Y & \xrightarrow{\pi_X} & Y & \xrightarrow{0} & T(X) \\ \downarrow{\scriptstyle id_X} & & & & \downarrow{\scriptstyle id_Y} & & \downarrow{\scriptstyle id_{T(X)}} \\ X & \xrightarrow{u} & Z & \xrightarrow{v} & Y & \xrightarrow{0} & T(X) \end{array}$$

and using (TR2) and (TR3), we see that exists $\varphi : X \oplus Y \to Z$ such that it completes the above diagram to a morphism of triangles. By Lemma 1.2.4, φ is an isomorphism. □

Lemma 1.2.10. *Let $X \xrightarrow{f} Y \xrightarrow{\binom{u}{0}} Z_1 \oplus Z_2 \to T(X)$ be a distinguished triangle. Then the object Z_2 is a direct summand of $T(X)$ and in particular $Z_2 = 0$ whenever $\mathrm{Hom}(Z_2, T(X)) = 0$. Furthermore, if X is indecomposable and $Z_2 \neq 0$, then $u = 0$.*

Proof. Since the morphism $Y \to Z_1 \oplus Z_2$ is of the form $u \oplus 0 : B \oplus 0 \to Z_1 \oplus Z_2$, it follows that the given triangle is isomorphic to the direct sum of $X' \to Y \to Z_1 \to T(X')$ and

$$Z_2[-1] \to 0 \to Z_2 \xrightarrow{id} Z_2$$

The first part of the thesis follows immediately. Furthermore, if X is indecomposable and $Z_2 \neq 0$, f must be the zero morphism. □

Next we want to see what happens when you localize a triangulated category. For this let $\mathcal{C}$ be a triangulated category. A localizing class in $\mathcal{C}$ is said to be *compatible with triangulation* if it satisfies

LT1) For any morphism s, $s \in S$ if and only if $T(s) \in S$.

LT2) The diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & & & \downarrow{\scriptstyle T(s)} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array}$$

where rows are distinguished triangles, the first square is commutative and $s, t \in S$ can be completed to a morphism of triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle T(s)} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array}$$

where $p \in S$.

If we have such a triangulated category $\mathcal{C}$ and such a localizing class S, let $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ be the quotient functor. Since for any $s \in S$, $(Q \circ T)(s) = Q(T(s))$ is an isomorphism, the functor $Q \circ T$ factors through $\mathcal{C}[S^{-1}]$, i.e. we have the following commutative diagram of functors

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{T} & \mathcal{C} \\ \downarrow{\scriptstyle Q} & & \downarrow{\scriptstyle Q} \\ \mathcal{C}[S^{-1}] & \xrightarrow{T_S} & \mathcal{C}[S^{-1}] \end{array}$$

From the diagram it is clear that T_S is an automorphism of the category $\mathcal{C}[S^{-1}]$. We will often denote it just by T, by abuse of notation.

A triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$$

in $\mathcal{C}[S^{-1}]$ is *distinguished* if there exists a distinguished triangle

$$U \longrightarrow V \longrightarrow W \longrightarrow T(U)$$

in $\mathcal{C}$ and an isomorphism of triangles

$$\begin{array}{ccccccc}
U & \longrightarrow & V & \longrightarrow & W & \longrightarrow & T(U) \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle T(a)} \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X)
\end{array}$$

in $\mathcal{C}[S^{-1}]$.

We are now going to sum up some results about the localization of triangulated categories which are similar to some other results obtained in Section 1.1.

Theorem 1.2.11. *Let $\mathcal{C}$ be a triangulated category and S a localizing class in $\mathcal{C}$ compatible with triangulation. Then the category $\mathcal{C}[S^{-1}]$ is triangulated. The natural functor $Q : \mathcal{C} \to \mathcal{C}[S^{-1}]$ is exact.*

Furthermore, if $\mathcal{D}$ is another triangulated category and $F : \mathcal{C} \to \mathcal{D}$ an exact (resp. cohomological) functor such that $s \in S$ implies that $F(s)$ is an isomorphism in $\mathcal{D}$. Then there exists a unique functor $F_S : \mathcal{C}[S^{-1}] \to \mathcal{D}$ such that the diagram

$$\begin{array}{ccc}
\mathcal{C} & & \\
\downarrow{\scriptstyle Q} & \searrow{\scriptstyle F} & \\
\mathcal{C}[S^{-1}] & \xrightarrow{F_S} & \mathcal{D}
\end{array}$$

of functors commutes. The functor $F_S : \mathcal{C}[S^{-1}] \to \mathcal{D}$ is exact (resp. cohomological).

Finally we have that the categories $\mathcal{C}^{opp}[S^{-1}]$ and $\mathcal{C}[S^{-1}]^{opp}$ are isomorphic as triangulated categories.

1.3 Categories of complexes

In this section we will refer mainly to [8]. Any proof that will be omitted can be found in [8], unless otherwise stated.

Let $\mathcal{A}$ be an additive category. We will start by defining the *category of complexes of $\mathcal{A}$-objects*, which we are going to denote by $C(\mathcal{A})$. This is the category with $\operatorname{Ob} C(\mathcal{A}) = (X^i, d^i_X; i \in \mathbf{Z})$, where X^i is an object in $\mathcal{A}$ and $d^i_X : X^i \to X^{i+1}$ for all i. Furthermore we require that $d^i_X \circ d^{i-1}_X = 0$ for all i. Sometimes we are going to denote such complexes (X^i) by $X^\bullet$. Schematically we can view such complexes as a diagram

$$\cdots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \longrightarrow \cdots$$

A morphism of complexes $f^\bullet : X^\bullet \to Y^\bullet$ is a collection of morphisms $(f^i : X^i \to Y^i; i \in \mathbf{Z})$, such that $f^{i+1} \circ d_X^i = d_Y^i \circ f^i$ for all i. Schematically we can see morphisms as infinite commutative diagrams

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & X^{n-1} & \xrightarrow{d_X^{n-1}} & X^n & \xrightarrow{d_X^n} & X^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle f^{n+1}} & & \\ \cdots & \longrightarrow & Y^{n-1} & \xrightarrow{d_Y^{n-1}} & Y^n & \xrightarrow{d_Y^n} & Y^{n+1} & \longrightarrow & \cdots \end{array}$$

Now we are also able to give a precise definition for the *translation functor* $T : C(\mathcal{A}) \to C(\mathcal{A})$. Indeed T is the functor that sends a complex $X^\bullet$ to the complex $T(X^\bullet)$ such that

$$T(X^\bullet)^n = X^{n+1} \text{ and } d_{T(X)}^n = -d_X^{n+1}$$

for any $n \in \mathbf{Z}$ and sends a morphism of complexes $f : X \to Y$ to the morphism $T(f) : T(X) \to T(Y)$ given by $T(f)^n = f^{n+1}$ for any $n \in \mathbf{Z}$. Clearly T is an automorphism of the category $C(\mathcal{A})$. We are often going to use the notation $T^p(X^\bullet) = X^\bullet[p]$ and call this object the complex $X^\bullet$ *shifted to the left* p times.

The complex $\cdots \to 0 \to 0 \to \cdots$ is the zero object in $C(\mathcal{A})$. Furthermore if we have two complexes X and Y we can define their direct sum $X \oplus Y$ by $(X \oplus Y)^n = X^n \oplus Y^n$ with the natural differential $d_X \oplus d_Y$. Therefore we have the following

Lemma 1.3.1. *The category $C(\mathcal{A})$ is an additive category.*

We can also define an additive functor $C : \mathcal{A} \to C(\mathcal{A})$ by

$$C(X)^p = \begin{cases} X & \text{if } p = 0, \\ 0 & \text{if } p \neq 0; \end{cases} \qquad \text{and} \qquad d_{C(X)} = 0$$

for $X \in \operatorname{Ob} \mathcal{A}$, and

$$C(f)^p = \begin{cases} f & \text{if } p = 0, \\ 0 & \text{if } p \neq 0; \end{cases}$$

for any $f \in \operatorname{Mor} \mathcal{A}$.

Lemma 1.3.2. *The functor $C : \mathcal{A} \to C(\mathcal{A})$ is fully faithful.*

Hence $\mathcal{A}$ is isomorphic to the full subcategory of $C(\mathcal{A})$ consisting of complexes $X^\bullet$ with $X^p = 0$ for $p \neq 0$.

We say that a complex X is *bounded from above* (resp. *bounded from below*) if there exists $n_0 \in \mathbf{Z}$ such that $X^n = 0$ for $n > n_0$ (resp for $n < n_0$).

We will say that a complex is *bounded* if it is bounded from above and from below. We denote by $C^-(\mathcal{A})$ (resp. $C^+(\mathcal{A})$ and $C^b(\mathcal{A})$) the full subcategories of $C(\mathcal{A})$ consisting of complexes bounded from above (resp. bounded from below and bounded). Obviously all these subcategories are invariant for the action of the translation functor. Also, they are additive. In the following we are going to use the shorthand $C^*(\mathcal{A})$ to indicate any of the above categories.

We also have that the categories $C(\mathcal{A})^{opp}$ and $C(\mathcal{A}^{opp})$ are isomorphic and their isomorphism induces isomorphisms between the bounded subcategories $C^*(\mathcal{A})^{opp}$ and $C^*(\mathcal{A}^{opp})$.

Let $f : X \to Y$ be a morphism of complexes in $C(\mathcal{A})$. We will denote by $\mathrm{Hom}^{-1}(X, Y)$ the set of homotopies between X and Y, where an homotopy is $h = (h^i; i \in \mathbf{Z})$ where $h^i : X^i \to Y^{i-1}$ is a morphism in $\mathcal{A}$ for all i. We say that f is*homotopic to zero* if there exists a homotopy $h_f \in \mathrm{Hom}^{-1}(X, Y)$ such that

$$f = d_Y \circ h_f + h_f \circ d_X$$

We will denote by $\mathrm{Ht}(X, Y)$ the set of all morphisms in $\mathrm{Hom}_{C(\mathcal{A})}(X, Y)$ which are homotopic to zero.

Lemma 1.3.3. *The subset* $\mathrm{Ht}(X, Y)$ *is a subgroup* $\mathrm{Hom}_{C(\mathcal{A})}(X, Y)$

Proof. Clearly the zero morphism is in $\mathrm{Ht}(X, Y)$ with $h_0 = 0$. Now assume that $f, g \in \mathrm{Ht}(X, Y)$ with homotopies h_f and h_g. Then $h_f + h_g$ is a homotopy for $f + g$ thanks to the distributive property of the composition, thus $f + g \in \mathrm{Ht}(X, Y)$. Similarly if $f \in \mathrm{Ht}(X, Y)$ then $-h_f$ is a homotopy for $-f$, so we also have that $-f \in \mathrm{Ht}(X, Y)$. Hence we proved that $\mathrm{Ht}(X, Y)$ is indeed a subgroup. □

We say that two morphisms $f, g : X \to Y$ are *homotopic* if $f - g \in \mathrm{Ht}(X, Y)$ and denote this by $f \sim g$. Clearly $\sim$ is an equivalence relation on $\mathrm{Hom}_{C(\mathcal{A})}(X, Y)$.

Lemma 1.3.4. *Let X, Y and Z be in* $\mathrm{Ob}\, C(\mathcal{A})$*and let $f : X \to Y$ and $g : Y \to Z$ be two morphisms of complexes. If either f or g is homotopic to zero, then also $g \circ f$ is homotopic to zero.*

Proof. If $f \in \mathrm{Ht}(X, Y)$ with the homotopy h_f, then we have that

$$g \circ f = g \circ d_Y \circ h_f + g \circ h_f \circ d_X = d_z \circ g \circ h_f + g \circ h_f \circ d_X$$

Thus we have that $g \circ f \in \mathrm{Ht}(X, Z)$, since $g \circ h_f \in \mathrm{Hom}^{-1}(X, Z)$.

Analogously, if $g \in \mathrm{Ht}(Y, Z)$ with the homotopy h_g we have that

$$g \circ f = d_z \circ h_g \circ f + h_g \circ d_Y \circ f = d_Z \circ h_g \circ f + h_g \circ f \circ d_X$$

Thus we have once again that $g \circ f \in \mathrm{Ht}(X, Z)$, since $h_g \circ f \in \mathrm{Hom}^{-1}(X, Z)$. □

Now we are able to define the *homotopic* (or *stable*) *category of complexes of $\mathcal{A}$-objects* which we will denote by $K(\mathcal{A})$. This is the category with $\operatorname{Ob} K(\mathcal{A}) = \operatorname{Ob} C(\mathcal{A})$ and where the morphisms between two such objects are classes of homotopic morphisms. Lemma 1.3.4 ensures that the composition in $C(\mathcal{A})$ induces a well defined composition in $K(\mathcal{A})$, with in particular $[g]_\sim \circ [f]_\sim = [g \circ f]_\sim$. In the following we will also use the following convention: given a complex $X^\bullet = (X^i, d^i) \in K^b(\mathcal{A})$ we may consider a preimage $\bar{X}^\bullet = (\bar{X}^i, \bar{d}^i)$ in $C^b(\mathcal{A})$ without indecomposable null-homotopic summands. Clearly $\bar{X}^\bullet$ is uniquely determined by $X^\bullet$ up to isomorphism.

We also have that the zero object in $K(\mathcal{A})$ is the zero object in $C(\mathcal{A})$ and that for any two complexes in $K(\mathcal{A})$ we can define their direct sum as the direct sum in $C(\mathcal{A})$. Moreover the canonical inclusions and projections are just the homotopy classes of the corresponding morphisms in $C(\mathcal{A})$. Hence, this immediately leads to the following result.

Lemma 1.3.5. *The category $K(\mathcal{A})$ is an additive category.*

Lemma 1.3.6. *Let $f : X \to Y$ be a morphism of complexes. Then the following statements are equivalent:*

i) f is homotopic to zero;

ii) $T(f)$ is homotopic to zero.

Proof. If i) holds, then we have a homotopy h_f for f which is given by a family of morphisms $h_f^p : X^p \to Y^{p-1}$. Therefore we can also see h_f as a morphism in $h_{T(f)} \in \operatorname{Hom}^{-1}(X, Y)$. In this case we have

$$T(f)^p = f^{p+1} = d_Y^p \circ h_f^{p+1} + h_f^{p+2} \circ d_X^{p+1} = -d_{T(Y)}^{p-1} \circ h_{T(f)}^p - h_{T(f)}^{p+1} \circ d_{T(X)}^p$$

for all $p \in \mathbf{Z}$, i.e. $T(f)$ is homotopic to zero via the homotopy $-h_{T(f)}$.

The proof of the converse is analogous. □

Therefore, the translation functor T induces an isomorphism of $\operatorname{Hom}_{K(\mathcal{A})}(X, Y)$ onto $\operatorname{Hom}_{K(\mathcal{A})}(T(X), T(Y))$. It follows that T induces an automorphism of the category $K(\mathcal{A})$. We will again call it the *translation functor* and denote it by T.

As before, we define the full subcategories $K^+(\mathcal{A})$, $K^-(\mathcal{A})$ and $K^b(\mathcal{A})$ of complexes bounded from below, resp. bounded from above and bounded. Again we are going to indicate any of them with the shorthand $K^*(\mathcal{A})$. Clearly all these subcategories are once again invariant under the action of the translation functor.

Definition 1.3.1. *Let $0 \neq X^\bullet \in K^b(\mathcal{A})$ and let $\bar{X}^\bullet$ be its preimage in $C^b(\mathcal{A})$ as defined above. Let $r \leq s$ such that $\bar{X}^r \neq 0 \neq \bar{X}^s$ and $\bar{X}^i = 0$ for $i < r$ and $i > s$. These are well defined since $\bar{X}^\bullet$ is defined up to isomorphism of complexes in $C^b(\mathcal{A})$. We define the* length *of $X^\bullet$ as $\ell(X^\bullet) = s - r$.*

Now let $H : C(\mathcal{A}) \to K(\mathcal{A})$ be the natural functor which is the identity on objects and maps morphisms of complexes into their homotopy classes. This is clearly an additive functor which commutes with the translation functors. Moreover we have the additive functor $K = H \circ C : \mathcal{A} \to K(\mathcal{A})$.

Lemma 1.3.7. *The functor $K : \mathcal{A} \to K(\mathcal{A})$ is fully faithful.*

Proof. Let X and Y be two objects in $\mathcal{A}$. Then $K(X)$ and $K(Y)$ are complexes such that $K(X)^p = K(Y)^p = 0$ for all $p \neq 0$. Therefore any morphism in $\mathrm{Hom}^{-1}(X, Y)$ must be zero. In particular $\mathrm{Ht}(K(X), K(Y)) = 0$ and thus $\mathrm{Hom}_{K(\mathcal{A})}(X, Y) = \mathrm{Hom}_{C(\mathcal{A})}(X, Y)$. Now the statement follows from Lemma 1.3.2. □

Notice that we have again that $\mathcal{A}$ is isomorphic to the full subcategory of $K(\mathcal{A})$ consisting of complexes $X^\bullet$ with $X^p = 0$ for $p \neq 0$.

We also have once again that the categories $K(\mathcal{A})^{opp}$ and $K(\mathcal{A}^{opp})$ are isomorphic and their isomorphism induces isomorphisms between the bounded subcategories $K^*(\mathcal{A})^{opp}$ and $K^*(\mathcal{A}^{opp})$.

Assume now that $\mathcal{A}$ is an abelian category. For $p \in \mathbf{Z}$ and for any complex $X^\bullet$ in $C(\mathcal{A})$ we define

$$H^p(X^\bullet) = \mathrm{Ker}\, d_X^p / \mathrm{Im}\, d_X^{p-1}$$

which is an object in $\mathcal{A}$. If $f : X^\bullet \to Y^\bullet$ is a morphism of complexes, $f^p(\mathrm{Ker}_X^p) \subseteq \mathrm{Ker}\, d_Y^p$ and $f^p(\mathrm{Im}\, d_X^{p-1}) \subseteq \mathrm{Im}\, d_Y^{p-1}$ and therefore f induces a morphism $H^p(f) : H^p(X^\bullet) \to H^p(Y^\bullet)$. Hence we have that H^p is a functor from $C(\mathcal{A})$ into $\mathcal{A}$, which is clearly additive for all $p \in \mathbf{Z}$. The functors H^p, $p \in \mathbf{Z}$, are called the *cohomology* functors.

Clearly we have that

$$H^p(T(X^\bullet)) = \mathrm{Ker}\, d_{T(X)}^p / \mathrm{Im}\, d_{T(X)}^{p-1} = \mathrm{Ker}\, d_X^{p+1} / \mathrm{Im}\, d_X^p = H^{p+1}(X^\bullet)$$

and analogously $H^p(T(f)) = H^{p+1}(f)$. Therefore,

$$H^p = H^0 \circ T^p$$

for any $p \in \mathbf{Z}$, and thus it is enough to study the functor $H^0 : C(\mathcal{A}) \to \mathcal{A}$.

Lemma 1.3.8. *Let $f, g : X \to Y$ be two homotopic morphisms of complexes. Then $H^p(f) = H^p(g)$ for all $p \in \mathbf{Z}$.*

Proof. By the above remark it is enough to prove $H^0(f) = H^0(g)$.

Let h be the corresponding homotopy, then we have

$$f^0 - g^0 = d_Y^{-1} \circ h^0 + h^1 \circ d_X^0$$

This in particular implies that the restriction of $f^0 - g^0$ to $\operatorname{Ker} d_X^0$ agrees with the morphism $d_Y^{-1} \circ h^0$. Therefore the image of $f^0 - g^0 : \operatorname{Ker} d_X^0 \to Y^0$ is contained in $\operatorname{Im} d_Y^{-1}$. It follows that $f^0 - g^0$ induces the zero morphism from $\operatorname{Ker} d_X^0$ into $H^0(Y)$. Therefore $H^0(f) - H^0(g) = H^0(f-g) : H^0(X) \to H^0(Y)$ is the zero morphism. □

This last result in particular tells us that the functors $H^p : C(\mathcal{A}) \to \mathcal{A}$ induce functors $H^p : K(\mathcal{A}) \to \mathcal{A}$ which are again clearly additive. Moreover, they satisfy once again

$$H^p = H^0 \circ T^p$$

Our next goal is giving a triangulated structure to $K^*(\mathcal{A})$, where $\mathcal{A}$ is an additive category. We first start by defining the *cone* of a morphism $f : X \to Y$ in $C^*(\mathcal{A})$. This will be the complex (C_f, d_{C_f}) with $C_f^n = X^{n+1} \oplus Y^n$ and with $d_{C_f} : C_f^n \to C_f^{n+1}$ given by

$$d_{C_f}^n = \begin{bmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{bmatrix}$$

for any $n \in \mathbf{Z}$. One can easily check that $d_{C_f} \circ d_{C_f} = 0$, so that (C_f, d_{C_f}) is actually a complex in $C^*(\mathcal{A})$. We also have two natural morphisms $i_f : Y \to C_f$ given by the inclusions $Y^n \hookrightarrow X^{n+1} \oplus Y^n = C_f^n$ and $p_f : C_f \to T(X)$ given by the projections $C_f^n = X^{n+1} \oplus Y^n \twoheadrightarrow X^{n+1} = T(X)^n$. One can easily check that these actually determine morphisms of complexes, i.e. that they commute with the differentials.

We will call the diagram

$$X \xrightarrow{\quad f \quad} Y \xrightarrow{\quad i_f \quad} C_f \xrightarrow{\quad p_f \quad} T(X)$$

the *standard triangle* in $C^*(\mathcal{A})$ attached to f.

We say that a triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$$

in $K^*(\mathcal{A})$ is *distinguished* if it is isomorphic to the image of a standard triangle in $K^*(\mathcal{A})$.

Theorem 1.3.9. *The additive category $K^*(\mathcal{A})$ equipped with the translation functor T and the class of distinguished triangles in $K^*(\mathcal{A})$ is a triangulated category. Furthermore the categories $K^*(\mathcal{A})^{opp}$ and $K^*(\mathcal{A}^{opp})$ are isomorphic as triangulated categories.*

Now let $\mathcal{A}$ be an abelian category again. We have two important results concerning the cohomology functors H^p.

Theorem 1.3.10. *The functor $H^0 : K^*(\mathcal{A}) \to \mathcal{A}$ is a cohomological functor.*

This can be reformulated in the following way.

Corollary 1.3.11. *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

be a distinguished triangle in $K^*(\mathcal{A})$. Then

$$\cdots \to H^p(X) \xrightarrow{H^p(f)} H^p(Y) \xrightarrow{H^p(g)} H^p(Z) \xrightarrow{H^p(h)} H^{p+1}(X) \to \cdots$$

is exact in $\mathcal{A}$.

This exact sequence is called the *long exact sequence of cohomology* of the distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

1.4 Derived categories

In this section we will refer mainly to [8]. Any proof that will be omitted can be found in [8], unless otherwise stated.

Let $\mathcal{A}$ be an abelian category. Denote by $K^*(\mathcal{A})$ the corresponding homotopic category of complex with triangulated structure given in the last section.

Definition 1.4.1. *A morphism $f : X \to Y$ in $C^*(\mathcal{A})$ is called a* quasiisomorphism *if $H^p(f) : H^p(X) \to H^p(Y)$ is an isomorphism for all $p \in \mathbf{Z}$.*

If $f : X \to Y$ is a quasiisomorphism and $g : X \to Y$ is homotopic to f, then g is also a quasiisomorphism. Therefore we extend the definition of quasiisomorphism to $K^*(\mathcal{A})$ by saying that a morphism in $K^*(\mathcal{A})$ is a quasiisomorphism if all of its representatives are such. We will denote by S^* the class of all quasiisomorphisms in $K^*(\mathcal{A})$.

Definition 1.4.2. *An object X in $K^*(\mathcal{A})$ is called* acyclic *if $H^p(X) = 0$ for all $p \in \mathbf{Z}$.*

Lemma 1.4.1. *Let $f : X \to Y$ be a morphism in $K^*(\mathcal{A})$. Then the following conditions are equivalent:*

i) The morphism f is a quasiisomorphism.

ii) The cone of f is acyclic.

Proof. Let

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow T(X)$$

be a distinguished triangle based on f. By Corollary 1.3.11, we have the long exact sequence of cohomology

$$\cdots \to H^p(X) \xrightarrow{H^p(f)} H^p(Y) \to H^p(Z) \to H^{p+1}(X) \xrightarrow{H^{p+1}(f)} H^{p+1}(Y) \to \cdots$$

Hence, if f is a quasiisomorphism, then $H^p(f)$ and $H^{p+1}(f)$ are isomorphisms and thus $H^p(Z) = 0$ for all $p \in \mathbf{Z}$. Therefore Z is acyclic.

Conversely, if Z is acyclic, from the long exact sequence

$$\cdots \to H^{p-1}(Z) \longrightarrow H^p(X) \xrightarrow{H^p(f)} H^p(Y) \longrightarrow H^p(Z) \to \cdots$$

we conclude that $H^p(f)$ is an isomorphism for all $p \in \mathbf{Z}$, i.e. f is a quasiisomorphism. □

We have the following result.

Proposition 1.4.2. *The class S^* of all quasiisomorphisms in $K^*(\mathcal{A})$ is a localizing class compatible with triangulation.*

This leads to the following important definition.

Definition 1.4.3. *The localization of the category $K^*(\mathcal{A})$ with respect to the class S^* of all quasiisomorphisms is called the* derived category *of $\mathcal{A}$ and denoted by $D^*(\mathcal{A})$.*

By definition, the cohomological functor $H^0 : K^*(\mathcal{A}) \to \mathcal{A}$ maps quasiisomorphisms in $K^*(\mathcal{A})$ into isomorphisms in $\mathcal{A}$. Therefore, by Theorem 1.2.11, it induces a cohomological functor from $D^*(\mathcal{A})$ into $\mathcal{A}$. By abuse of notation, we denote it also by H^0. More explicitly, let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

be a distinguished triangle in $D^*(\mathcal{A})$. Then

$$\cdots \to H^p(X) \xrightarrow{H^p(f)} H^p(Y) \xrightarrow{H^p(g)} H^p(Z) \xrightarrow{H^p(h)} H^{p+1}(X) \to \cdots$$

is exact in $\mathcal{A}$. This exact sequence is called the *long exact sequence of cohomology* of the distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

We already know that we have canonical functors $C^*(\mathcal{A}) \to K^*(\mathcal{A}) \to D^*(\mathcal{A})$. Moreover, if we denote by $\tilde{S}^*$ the class of all quasiisomorphisms in $C^*(\mathcal{A})$, we see that any $s \in \tilde{S}^*$ induces an isomorphism in $D^*(\mathcal{A})$. By Theorem 1.1.1 we know that the above functor factors through the localization $C^*(\mathcal{A})[\tilde{S}^{*-1}]$, i.e. we have the following commutative diagram of functors

$$\begin{array}{ccc} C^*(\mathcal{A}) & \longrightarrow & K^*(\mathcal{A}) \\ \downarrow{\scriptstyle \tilde{Q}} & & \downarrow{\scriptstyle Q} \\ C^*(\mathcal{A})[\tilde{S}^{*-1}] & \xrightarrow{\iota} & D^*(\mathcal{A}) \end{array}$$

Theorem 1.4.3. *The functor $\iota : C^*(\mathcal{A})[\tilde{S}^{*-1}] \to D^*(\mathcal{A})$ in the above diagram is an isomorphism of categories.*

We also have the following result about the opposite category of a derived category.

Theorem 1.4.4. *The categories $D(\mathcal{A})^{opp}$ and $D(\mathcal{A}^{opp})$ are isomorphic as triangulated categories.*

We are now going to define the so-called truncation functors. Let A be a complex in $C(\mathcal{A})$ and $n \in \mathbf{Z}$. We define a complex $\tau_{\leq n}(A)$ as the subcomplex of A given by

$$\tau_{\leq n}(A)^p = \begin{cases} A^p, & \text{if } p < n \\ \operatorname{Ker} d^n, & \text{if } p = n \\ 0, & \text{if } p > n. \end{cases}$$

Let $i : \tau_{\leq n}(A) \to A$ be the canonical inclusion morphism. The next result follows immediately from the definition.

Lemma 1.4.5. *The morphism $H^p(i) : H^p(\tau_{\leq n}(A)) \to H^p(A)$ is an isomorphism for $p \leq n$ and is the zero morphism for $p > n$.*

We want to show that $\tau_{\leq n}$ induces a functor $D(\mathcal{A}) \to D(\mathcal{A})$. If $B \in \operatorname{Ob} C(\mathcal{A})$ is another complex and $f : A \to B$ is a morphism of complexes, then $d^n f^n = f^{n+1} d^n$ and therefore $f^n(\operatorname{Ker} d^n) \subseteq \operatorname{Ker} d^n$. It follows that f induces a morphism of complexes $\tau_{\leq n}(f) : \tau_{\leq n}(A) \to \tau_{\leq n}(B)$. Therefore $\tau_{\leq n} : C(\mathcal{A}) \to C(\mathcal{A})$ is an additive functor.

Now assume that $f, g : A \to B$ are homotopic morphisms of complexes, i.e. $f - g = dh + hd$ for some homotopy h. Then $\tau_{\leq n}(f)$ and $\tau_{\leq n}(g)$ are also homotopic, with the homotopy given by the restriction of h to $\tau_{\leq n}(A)$. This means that $\tau_{\leq n}$ also induces a functor $\tau_{\leq n} : K(\mathcal{A}) \to K(\mathcal{A})$.

Finally, we clearly have

$$H^p(\tau_{\leq n}(f)) = \begin{cases} H^p(f), & \text{if } p \leq n \\ 0, & \text{if } p > n. \end{cases}$$

Therefore, if $f : A \to B$ is a quasiisomorphism, then by 1.4.5 $H^p(f)$ is an isomorphism for $p \leq n$ between $H^p(\tau_{\leq n}(A))$ and $H^p(\tau_{\leq n}(B))$ and for $p > n$ the zero morphism is also an isomorphism between $H^p(\tau_{\leq n}(A)) = 0$ and $H^p(\tau_{\leq n}(B)) = 0$. Hence $\tau_{\leq n}(f)$ is a quasiisomorphism as well. It follows that $\tau_{\leq n}$ indeed induces a functor $\tau_{\leq n} : D(\mathcal{A}) \to D(\mathcal{A})$ and we will call it the *truncation functor* $\tau_{\leq n}$, as anticipated.

Analogously, given a complex A, we can define the complex $\tau_{\geq n}(A)$ as a quotient complex of A

$$\tau_{\geq n}(A)^p = \begin{cases} 0, & \text{if } p < n \\ \operatorname{Coker} d^{n-1}, & \text{if } p = n \\ A^p, & \text{if } p > n. \end{cases}$$

Let $q : A \to \tau_{\geq n}(A)$ be the canonical projection morphism. The next result follows immediately from the definition.

Lemma 1.4.6. *The morphism $H^p(q) : H^p(A) \to H^p(\tau_{\geq n}(A))$ is an isomorphism for $p \geq n$ and is the zero morphism for $p < n$.*

By some considerations analogous to the ones we made about $\tau_{\leq n}$, we can show that also $\tau_{\geq n}$ induces a functor $\tau_{\geq n} : D(\mathcal{A}) \to D(\mathcal{A})$ which we will call the *truncation functor* $\tau_{\geq n}$.

Notice in particular that both truncations functor send objects from $D^*(\mathcal{A})$ into other objects in $D^*(\mathcal{A})$. In particular, since the categories $D^*(\mathcal{A})$ are indeed full subcategories of $D(\mathcal{A})$ (we are going to prove this next), these truncation functors induce corresponding truncation functors in these categories, for which we will use the same notation.

We have a natural functor $K^-(\mathcal{A}) \to K(\mathcal{A})$ (resp. $K^+(\mathcal{A}) \to K(\mathcal{A})$), which induces a functor $D^-(\mathcal{A}) \to D(\mathcal{A})$ (resp. $D^+(\mathcal{A}) \to D(\mathcal{A})$). Moreover, the localizing class S^- (resp S^+) consists of all morphisms in S which are morphisms in $K^-(\mathcal{A})$ (resp. $K^+(\mathcal{A})$). Let now X and Y be two complexes. Assume that X is bounded from above (resp. from below). Let $s : Y \to X$ be a quasiisomorphism. Since X is bounded from above (resp. from below), there exists $n \in \mathbf{Z}$ such that $H^p(X) = 0$ for $p > n$ (resp. $p < n$). Thus, since s is a quasiisomorphism, we must also have $H^p(Y) = 0$ for $p > n$ (resp. $p < n$). Therefore, by Lemma 1.4.5 (resp. 1.4.6), $i : \tau_{\leq n}(Y) \to Y$ (resp. $q : Y \to \tau_{\geq n}(Y)$) is a quasiisomorphism. It follows that $s \circ i : \tau_{\leq n}(Y) \to X$ (resp. $q \circ s : X \to \tau_{\geq n}(Y)$) is a quasiisomorphism. Therefore, Proposition 1.1.5 (resp. 1.1.6) implies the following result.

Proposition 1.4.7. *The natural functors $D^-(\mathcal{A}) \to D(\mathcal{A})$ and $D^+(\mathcal{A}) \to D(\mathcal{A})$ are fully faithful, i.e. $D^-(\mathcal{A})$ and $D^+(\mathcal{A})$ are full subcategories of $D(\mathcal{A})$.*

Furthermore the natural functor $K^b(\mathcal{A}) \to K^+(\mathcal{A})$ induces a functor $D^b(\mathcal{A}) \to D^+(\mathcal{A})$. Moreover, the localizing class S^b consists of all morphisms in S^+ which are morphisms in $K^b(\mathcal{A})$. Let now X and Y be two complexes. Assume that X is bounded and that Y is bounded from below. Let s :

$Y \to X$ be a quasiisomorphism. Since X is bounded, there exists $n \in \mathbf{Z}$ such that $H^p(X) = 0$ for $p > n$. Thus, since s is a quasiisomorphism, we must also have $H^p(Y) = 0$ for $p > n$. Therefore, by Lemma 1.4.5, $i : \tau_{\leq n}(Y) \to Y$ is a quasiisomorphism. Moreover, $\tau_{\leq n}(Y)$ is a bounded complex. It follows that $s \circ i : \tau_{\leq n}(Y) \to X$ (resp. $q \circ s : X \to \tau_{\geq n}(Y)$) is a quasiisomorphism. Hence, by 1.1.5 the functor $D^b(\mathcal{A}) \to D^+(\mathcal{A})$ is fully faithful. When we combine this with pour previous result in 1.4.7 we see that the natural functor $D^b(\mathcal{A}) \to D(\mathcal{A})$ is fully faithful, i.e. $D^b(\mathcal{A})$ is a full subcategory of $D(\mathcal{A})$. This proves the following result.

Proposition 1.4.8. *The natural functor $D^b(\mathcal{A}) \to D(\mathcal{A})$ is fully faithful, i.e. $D^b(\mathcal{A})$ is a full subcategory of $D(\mathcal{A})$ equal to $D^-(\mathcal{A}) \cap D^+(\mathcal{A})$.*

We also have the following result about opposite categories.

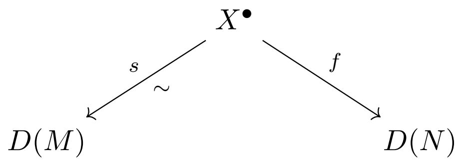
Theorem 1.4.9. *The isomorphism between $D(\mathcal{A})^{opp}$ and $D(\mathcal{A}^{opp})$ induces isomorphisms of triangulated categories between $D^*(\mathcal{A})^{opp}$ and $D^*(\mathcal{A}^{opp})$.*

We denote by $D : \mathcal{A} \to D^*(\mathcal{A})$ the natural functor which is the composition of the functor $K : \mathcal{A} \to K^*(\mathcal{A})$ and the quotient functor $Q : K^*(\mathcal{A}) \to D^*(\mathcal{A})$.

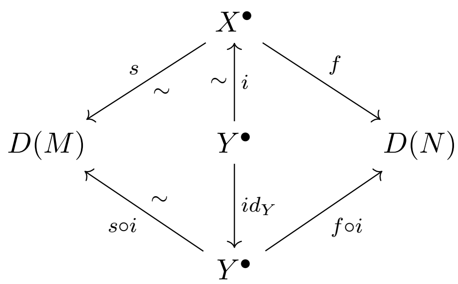
Theorem 1.4.10. *The functor $D : \mathcal{A} \to D^*(\mathcal{A})$ is fully faithful.*

Proof. Let M and N be objects in $\mathcal{A}$. Let $F : M \to N$ be a morphism in $\mathcal{A}$. Then $H^0(D(F)) = F$ and thus the mapping $\mathrm{Hom}_{\mathcal{A}}(M,N) \to \mathrm{Hom}_{D(\mathcal{A})}(D(M), D(N))$ is injective. Indeed if two morphisms $f, g : M \to N$ in $\mathcal{A}$ are mapped into the same morphism by D, then $f = H^0(D(f)) = H^0(D(g)) = g$.

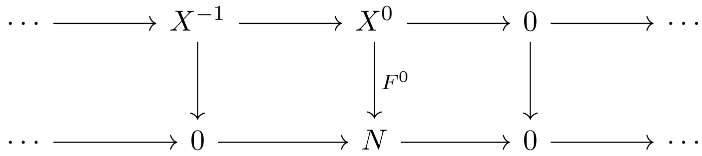
Now we prove the surjectivity of the map $\mathrm{Hom}_{\mathcal{A}}(M,N) \to \mathrm{Hom}_{D(\mathcal{A})}(D(M), D(N))$. For this, let $\varphi : D(M) \to D(N)$ be a morphism in $D(\mathcal{A})$. We can represent it by a roof



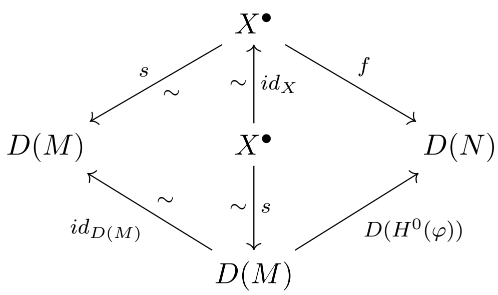
where $s : X^\bullet \to D(M)$ is a quasiisomorphism. It follows that $H^p(X^\bullet) = 0$ for $p \neq 0$. Therefore, by Lemma 1.4.5, $i : \tau_{\leq 0}(X^\bullet) \to X^\bullet$ is a quasiisomorphism. If we put $Y^\bullet = \tau_{\leq 0}(X^\bullet)$, the diagram



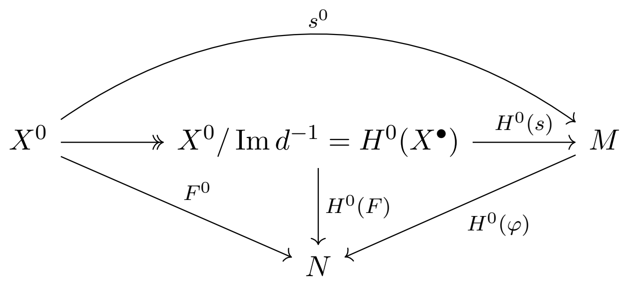
is commutative. This in particular implies that φ can be represented by a roof where $X^\bullet$ satisfies $X^p = 0$ for $p > 0$. Hence we have the commutative diagram



for a representative F of the homotopy class of f. Clearly all homotopies from $X^\bullet$ to $D(N)$ are zero. So this representative is unique. In addition, by the commutativity of the left square in the diagram we know that F^0 vanishes on $\operatorname{Im} d^{-1}$. Hence F^0 factors through $H^0(F) : H^0(X^\bullet) = X^0/\operatorname{Im} d^{-1} \to N$ and $H^0(F) = H^0(f) = H^0(\varphi) \circ H^0(s)$ since $\varphi = Q(f) \circ Q(s)^{-1}$ in $D(\mathcal{A})$. Therefore we have the following commutative diagram, which shows that $\varphi = D(H^0(\varphi))$.



Indeed commutativity of the left side of the diagram is clear. For the right side, we only need to check commutativity in degree 0, since for $p \neq 0$ we have that $D(N)^p = 0$. For this we have the following diagram



In particular both the right and the left triangle commute because of our previous considerations, thus also the outside diagram commutes. Thus we proved that our map $\operatorname{Hom}_{\mathcal{A}}(M, N) \to \operatorname{Hom}_{D(\mathcal{A})}(D(M), D(N))$ is surjective. □

Therefore, the full subcategory of $D^*(\mathcal{A})$ consisting of all complexes $X^\bullet$ such that $X^p = 0$ for $p \neq 0$ is isomorphic to $\mathcal{A}$.

We already know that for an abelian category $\mathcal{A}$, its category of complexes $C^*(\mathcal{A})$ is also abelian. Now let

$$0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0$$

be an exact sequence in $C^*(\mathcal{A})$. We can also consider the standard triangle

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{i_f} C_f \xrightarrow{p_f} X^\bullet[1]$$

attached to the monomorphism $f : X^\bullet \to Y^\bullet$. Let $m : C_f = X^\bullet[1] \oplus Y^\bullet \to Z^\bullet$ be the composition of the natural projection $q : X^\bullet[1] \oplus Y^\bullet \to Y^\bullet$ with $g : Y^\bullet \to Z^\bullet$. This clearly defines a morphism of complexes such that $m \circ i_f = g$.

Also, we can define a morphism of complexes $w : C_{id_X} \to C_f$ by

$$w^n = \begin{bmatrix} id_{X^{n+1}} & 0 \\ 0 & f^n \end{bmatrix}$$

This morphism is clearly a monomorphism and moreover we have

$$\operatorname{Im} w^n = X^{n+1} \oplus \operatorname{Im} f^n = X^{n+1} \oplus \operatorname{Ker} g^n = \operatorname{Ker} m^n$$

for any $n \in \mathbf{Z}$. Hence the sequence

$$0 \longrightarrow C_{id_X} \xrightarrow{w} C_f \xrightarrow{m} Z^\bullet \longrightarrow 0$$

is exact in $C^*(\mathcal{A})$.

Since we know that $K^*(\mathcal{A})$ is a triangulated category, by 1.2 $C_{id_x} = 0$ in $K^*(\mathcal{A})$, so in particular we have $H^p(C_{id_X}) = 0$ for any $p \in \mathbf{Z}$. Therefore, from the long exact sequence of cohomology attached to the above short exact sequence, we see that $H^p(m) : H^p(C_f) \to H^p(Z^\bullet)$ is an isomorphism for all $p \in \mathbf{Z}$, i.e. we have the following result.

Lemma 1.4.11. *The morphism $m : C_f \to Z^\bullet$ is a quasiisomorphism.*

This tells us in particular that the homotopy class of m is an isomorphism in $D^*(\mathcal{A})$. This leads to the following result.

Proposition 1.4.12. *Let*

$$0 \longrightarrow X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow 0$$

be an exact sequence in $C(\mathcal{A})$. Then it determines a distinguished triangle

$$X^\bullet \xrightarrow{f} Y^\bullet \xrightarrow{g} Z^\bullet \longrightarrow X^\bullet[1]$$

in $D(\mathcal{A})$.

Proof. By Lemma 1.4.11, the diagram

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f} & Y^\bullet & \xrightarrow{i_f} & C_f & \xrightarrow{p_f} & X^\bullet[1] \\ \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle id_Y} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle id_{X[1]}} \\ X^\bullet & \xrightarrow{f} & Y^\bullet & \xrightarrow{g} & Z^\bullet & \longrightarrow & X^\bullet[1] \end{array}$$

is an isomorphism of triangles in $D^*(\mathcal{A})$. Since the top triangle is distinguished, the lower one is also distinguished. □

Consider now a complex X of $\mathcal{A}$-objects and let $n \in \mathbf{Z}$. We have the following exact sequence of complexes

$$0 \longrightarrow \tau_{\leq n}(X) \longrightarrow X \longrightarrow Q \longrightarrow 0$$

Clearly Q is the kernel of the natural inclusion $i : \tau_{\leq n}(X) \to X$, i.e. we have

$$Q^p = \begin{cases} 0, & \text{if } p < n \\ \operatorname{Coim} d^n, & \text{if } p = n \\ X^p, & \text{if } p > n. \end{cases}$$

Therefore $H^p(Q) = 0$ for $p \leq n$ and $H^p(Q) = H^p(X)$ for $p > n$. If we consider the canonical projection $Q \to \tau_{\geq n+1}(Q) = \tau_{\geq n+1}(X)$, i.e. the commutative diagram

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & \operatorname{Coim} d^n & \longrightarrow & X^{n+1} & \longrightarrow & X^{n+2} & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \operatorname{Coker} d^n & \longrightarrow & X^{n+2} & \longrightarrow & \cdots \end{array}$$

we see that this morphism is a quasiisomorphism. Therefore, by Proposition 1.4.12, we have a distinguished triangle

$$\tau_{\leq n}(X) \xrightarrow{i} X \longrightarrow Q \longrightarrow \tau_{\leq n}(X)[1]$$

in $D(\mathcal{A})$. In particular, since Q is isomorphic to $\tau_{\geq n+1}(X)$ in $D(\mathcal{A})$ by the above considerations, this leads to a distinguished triangle

$$\tau_{\leq n}(X) \xrightarrow{i} X \xrightarrow{q} \tau_{\geq n+1}(X) \longrightarrow \tau_{\leq n}(X)[1]$$

This in particular proves the existence part of the following result.

Proposition 1.4.13. *For any complex X and any $n \in \mathbf{Z}$ there exists a unique morphism $h : \tau_{\geq n+1}(X) \to \tau_{\leq n}(X)[1]$ such that*

$$\tau_{\leq n}(X) \xrightarrow{\ i\ } X \longrightarrow Q \xrightarrow{\ h\ } \tau_{\leq n}(X)[1]$$

is a distinguished triangle in $D(\mathcal{A})$

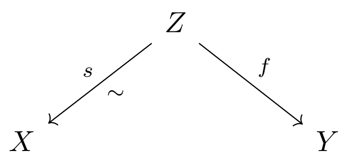
It remains to prove the uniqueness of h. It is a consequence of Proposition 1.2.6 and of the following lemma. Indeed if h, h' are two morphisms satisfying the requirements of the Proposition we have the following diagram

$$\begin{array}{ccccccc}
\tau_{\leq n}(X) & \xrightarrow{\ i\ } & X & \longrightarrow & Q & \xrightarrow{\ h\ } & \tau_{\leq n}(X)[1] \\
\downarrow{\scriptstyle id_{\tau_{\leq n}(X)}} & & \downarrow{\scriptstyle id_X} & & \downarrow{\scriptstyle id_Q} & & \downarrow{\scriptstyle id_{\tau_{\leq n}(X)[1]}} \\
\tau_{\leq n}(X) & \xrightarrow{\ i\ } & X & \longrightarrow & Q & \xrightarrow{\ h'\ } & \tau_{\leq n}(X)[1]
\end{array}$$

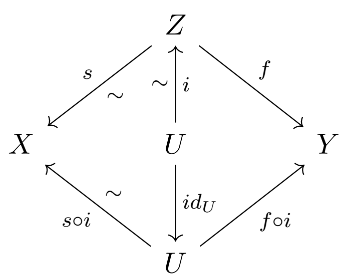
that can be completed to a morphism of distinguished triangles and we know that there is a unique way of completing it since $\tau_{\leq n}(X)^p = 0$ for $p \geq n+1$ and $\tau_{\geq n+1}(X)[-1]^p = 0$ for $p < n+1$ so by the following lemma $\mathrm{Hom}_{D(\mathcal{A})}(\tau_{\leq n}(X), \tau_{\geq n+1}(X)[-1]) = 0$ so by the commutativity of the right-most square we can deduce that $h = h'$.

Lemma 1.4.14. *Let X and Y be two complexes such that $X^p = 0$ for $p \geq n$ and $Y^p = 0$ for $p < n$. Then $\mathrm{Hom}_{D(\mathcal{A})}(X, Y) = 0$.*

Proof. Let φ be an element of $\mathrm{Hom}_{D(\mathcal{A})}(X, Y)$. Assume that it is represented by a roof



Since $H^p(X) = 0$ for $p \geq n$ and s is a quasiisomorphism, we see that for $p \geq n$, $H^p(Z) = 0$ as well. It follows that $i : \tau_{\leq n-1}(Z) \to Z$ is a quasi-isomorphism. Therefore, by putting $U = \tau_{\leq n-1}(Z)$ we have the following commutative diagram



In particular it shows that φ can be represented by a roof satisfying $Z^p = 0$ for $p \geq n$. In this case, f must be zero and thus φ is zero as well. □

Remark. An important consequence of this last lemma is that for any two objects X, Y in $\mathcal{A}$ we have that $\mathrm{Hom}_{D(\mathcal{A})}(X, Y[-1]) = 0$ by simply applying the last result with $n = 1$.

Now let

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

be a short exact sequence in $\mathcal{A}$. By Proposition 1.4.12, we have a distinguished triangle

$$D(L) \xrightarrow{D(f)} D(M) \xrightarrow{D(g)} D(N) \longrightarrow D(L)[1]$$

in $D^*(\mathcal{A})$. In this case, we have a stronger result.

Proposition 1.4.15. *There exists a unique morphism h such that*

$$D(L) \xrightarrow{D(f)} D(M) \xrightarrow{D(g)} D(N) \xrightarrow{h} D(L)[1]$$

is distinguished in $D^*(\mathcal{A})$.

Proof. The uniqueness of h is given by an argument similar to the one we gave to prove Proposition 1.4.13, considering that $\mathrm{Hom}_{D(\mathcal{A})}(D(L), D(N)[-1]) = 0$ by the above remark. □

1.5 Representation of algebras

In this section we will refer mainly to [1]. Any proof that will be omitted can be found in [1], unless otherwise stated.

The goal of this section is to give some useful results about algebras and then to introduce some elements of the representation theory of algebras. In particular we will mostly focus on finite dimensional k-algebras, where k denotes an algebraically closed field.

We start by recalling the standard duality between $\mathrm{mod}\, A$ and $\mathrm{mod}\, A^{op}$. This is the functor $D : \mathrm{mod}\, A \to \mathrm{mod}\, A^{op}$ that associates to each right A-module M the dual k-vector space $M^* = \mathrm{Hom}_k(M, k)$ endowed with the left A-module structure given by the formula $(a\varphi)(m) = \varphi(ma)$ for $\varphi \in \mathrm{Hom}_k(M, k)$, $a \in A$ and $m \in M$, and to each morphism $h : M \to N$ the dual morphism $D(h) = \mathrm{Hom}_k(h, k) : D(N) \to D(M)$, $\varphi \mapsto \varphi h$, of left A-modules. We denote its quasi-inverse also by $D : \mathrm{mod}\, A^{op} \to \mathrm{mod}\, A$. We have natural equivalences of functors $1_{\mathrm{mod}\, A} \cong D \circ D$ and $1_{\mathrm{mod}\, A^{op}} \cong D \circ D$.

Our first goal is to understand indecomposable modules over a k-algebra A. In order to study them we need to understand the role played by idempotent elements of A. An element $e \in A$ is called *idempotent* if $e^2 = e$. The idempotent e is called *central* if $ae = ea$ for all $a \in A$. Two idempotents $e_1, e_2 \in A$ are said to be *orthogonal* if $e_1 e_2 = e_2 e_1 = 0$. The idempotent e is called *primitive* if e cannot be written as a sum $e = e_1 + e_2$, where e_1 and e_2 are nonzero orthogonal idempotents of A.

Every algebra A has two trivial idempotents 0 and 1. Moreover if e is a nontrivial idempotent in A, then $1 - e$ is also a nontrivial idempotent, the idempotents e and $1 - e$ are orthogonal and there is a nontrivial right A-module decomposition $A_A = eA \oplus (1-e)A$. Conversely, if $A_A = M_1 \oplus M_2$ is a nontrivial A-module decomposition and $1 = e_1 + e_2$ with $e_i \in M_i$, then e_1 and e_2 are a pair of orthogonal idempotents of A, and $M_i = e_i A$ is indecomposable if and only if e_i is primitive.

If e is a central idempotent, then so is $1-e$ and thus eA and $(1-e)A$ are two-sided ideals. It can be easily shown that they have k-algebra structures (induced by the structure on A) with identity elements $e \in eA$ and $1 - e \in (1-e)A$ respectively. In this case the decomposition $A_A = eA \oplus (1-e)A$ is a direct product decomposition of the algebra A.

Since A is finite dimensional, the module A_A admits a direct sum decomposition $A_A = P_1 \oplus \cdots \oplus P_n$, where $P_1, \dots, P_n$ are indecomposable right ideals of A. It follows from the above discussion that $P_1 = e_1 A, \dots, P_n = e_n A$, where $e_1, \dots, e_n$ are primitive pairwise orthogonal idempotents of A such that $1 = e_1 + \cdots + e_n$. Conversely, every set of idempotents with the preceding properties induces a decomposition $A_A = P_1 \oplus \cdots \oplus P_n$ where the P_i are indecomposable right ideals of the form $P_i = e_i A$ for all $i = 1, \dots, n$. Such a decomposition is called an *indecomposable decomposition* of A and such a set $\{e_1, \dots, e_n\}$ is called a *complete set of primitive orthogonal idempotents* of A.

Definition 1.5.1. *An algebra A is said to be* connected *if A is not a direct product of two algebras. Equivalently, A is called connected if* 0 *and* 1 *are the only central idempotents of A.*

Now assume that $e \in A$ is an idempotent and that M is a right A-module. It is easy to check that the k-vector subspace eAe of A is a k-algebra with identity element e. Also, the k-vector subspace Me of M is a right eAe-module if we set $(me) \cdot (eae) = meae$ for all $m \in M$ and $a \in A$. In particular we have that Ae is a right eAe-module and eA is a left eAe-module. It follows that the k-vector space $\mathrm{Hom}_A(eA, M)$ is a right eAe-module with respect to the action $(\varphi \cdot eae)(x) = \varphi(eaex)$ for $x \in eA$, $a \in A$, $\varphi \in \mathrm{Hom}_A(eA, M)$. This leads us to the following fact, which will be used frequently.

Lemma 1.5.1. *Let A be a k-algebra, $e \in A$ be an idempotent and M be a right A-module. Then the following hold:*

i) The k-linear map

$$\theta_M : \mathrm{Hom}_A(eA, M) \to Me$$

defined by the formula $\varphi \mapsto \varphi(e)e$ for $\varphi \in \mathrm{Hom}_A(eA, M)$, is an isomorphism of right eAe-modules.

ii) *The isomorphism* $\theta_{eA} : \operatorname{End} eA \to eAe$ *of right* eAe*-modules induces an isomorphism of* k*-algebras.*

Proof. It is easy to see that θ_M is a homomorphism of right eAe-modules. We define a k-linear map $\theta'_M : Me \to \operatorname{Hom}_A(eA, M)$ by the formula

$$\theta'_M(me)(ea) = mea$$

for $a \in A$ and $m \in M$. Easy calculations show that $\theta'_M(me) : eA \to M$ is well-defined (i.e. does not depend on the choice of a in the presentation of ea) and that it is a homomorphism of A-modules. Moreover θ'_M is a homomorphism of eAe-modules. Let us check that θ'_M is an inverse of θ_M. Take $\varphi \in \operatorname{Hom}_A(eA, M)$. Then $\theta'_M(\theta_M(\varphi)) = \theta'_M(\varphi(e)e)$ and this is exactly the morphism mapping $ea \mapsto \varphi(e)ea = \varphi(eea) = \varphi(ea)$ as wanted. Furthermore take $me \in Me$. We have $\theta_M(\theta'_M(me)) = (\theta'_M(me)(e))e = me \cdot e = me$. It follows that θ_M is indeed an isomorphism. The statement ii) is just an easy consequence of i). □

Proposition 1.5.2. *Let* A *be an algebra and* $B = A/\operatorname{rad} A$*.*

i) *Every right ideal* I *of* B *is a direct sum of simple right ideals of the form* eB *for some primitive idempotent* $e \in B$*. In particular* B *is semisimple.*

ii) *If* $e \in A$ *is a primitive idempotent of* A*, then the* B*-module* $\operatorname{top} eA = eA/\operatorname{rad} eA$ *is simple.*

Proof. i) Let S be a nonzero right ideal of B contained in I that is of minimal dimension. Then S is a simple B-module and $S^2 \neq 0$ since this would mean that $0 \neq S \subseteq \operatorname{rad} B = \operatorname{rad}(A/\operatorname{rad} A) = 0$. Hence $S^2 = S$ and there exists $x \in S$ such that $xS \neq 0$. Thus $S = xS$ and therefore $x = xe$ for some nonzero $e \in S$. This means that the B-homomorphism $\varphi : S \to S$ given by $\varphi(y) = xy$ is not the zero morphism and thus it is an isomorphism by Schur's Lemma. Since $\varphi(e^2 - e) = x(e^2 - e) = xee - xe = xe - xe = 0$, it must be $e^2 - e = 0$, therefore $e \in S$ is a nonzero idempotent and we clearly have $S = eB$. It follows that $B = eB \oplus (1-e)B$ and $I = S \oplus (1-e)I$. Since $S \neq 0$ we have that $\dim_k (1-e)I < \dim_k I$ and we can assume by induction that i) is satisfied for $(1-e)I$ and therefore i) follows.

ii) The element $\overline{e} = e + \operatorname{rad} A$ is an idempotent of B and $\operatorname{top} eA \cong \overline{e}B$. Assume that $\overline{e}B$ is not simple. Then, by i) we have that $\overline{e}B = \overline{e_1}B \oplus \overline{e_2}B$, where $\overline{e_1}, \overline{e_2}$ are nonzero orthogonal idempotents such that $\overline{e} = \overline{e_1} + \overline{e_2}$. Since $\overline{e_1}^2 = (\overline{e} - \overline{e_2})\overline{e_1} = \overline{ee_1}$ we have that $\overline{e_1} = g_1 + \operatorname{rad} A$ for some $g_1 \in eA$. It can be computed that there is $m \in \mathbf{N}$ and $t \in A$ such that $e_1 = (gt)^m$ is an idempotent of A and $\overline{e_1} = e_1 + \operatorname{rad} A$.

In particular we need to take m such that $(\operatorname{rad} A)^m = 0$ and t such that $(g - g^2)^m - g^m = g^{m+1}t$. Now it follows that $\operatorname{top} eA = \overline{e}B = \overline{e_1}B \oplus \overline{e_2}B$. Since $g_1 \in eA$, $e_1 \in eA$ and thus $e_1A \subseteq eA$. then the decomposition $A_A = e_1A \oplus (1 - e_1)A$ induces the decomposition $eA = e_1A \oplus \{(1-e_1)A \cap eA\}$. Since e is a primitive idempotent in A, it follows that $eA = e_1A$. Hence $\overline{e}B = \operatorname{top} eA = \operatorname{top} e_1A = \overline{e_1}B$ and therefore $\overline{e_2}B = 0$, contrary to our assumption. Thus the module $\operatorname{top} eA$ is simple.

□

We recall the following result, which will be useful in the future.

Lemma 1.5.3. *Let A be a finite dimensional k-algebra. The following conditions are equivalent:*

i) A is a local algebra.

ii) The set of all noninvertible elements of A is a two-sided ideal.

iii) For any $a \in A$, one of the elements a or $1 - a$ is invertible.

iv) A has only two idempotents, 0 and 1.

v) The k-algebra $A/\operatorname{rad} A$ is isomorphic to k.

We will focus on some useful consequences of this lemma.

Corollary 1.5.4. *Let A be an arbitrary k-algebra and M a right A-module.*

i) If the algebra $\operatorname{End} M$ is local, then M is indecomposable.

ii) If M is finite dimensional and indecomposable, then the algebra $\operatorname{End} M$ is local and any A-module endomorphism of M is either nilpotent or an isomorphism.

Proof. i) If M decomposes as $M = X_1 \oplus X_2$ with both X_1 and X_2 nonzero, then there exist projections $\pi_i : M \to X_i$ and injections $\varepsilon_i : X_i \to M$ for $i = 1, 2$. In particular we know that $\varepsilon_1\pi_1 + \varepsilon_2\pi_2 = 1_M$. Since $\varepsilon_1\pi_1$ and $\varepsilon_2\pi_2$ are clearly nonzero idempotents in $\operatorname{End} M$, the algebra $\operatorname{End} M$ is not local by Lemma 1.5.3.

ii) Assume that M is finite dimensional and indecomposable. If $\operatorname{End} M$ is not local, by 1.5.3 there is a pair of nonzero idempotents e_1, $e_2 = 1 - e_1$ in $\operatorname{End} M$ and therefore $M \cong \operatorname{Im} e_1 \oplus \operatorname{Im} e_2$ is a nontrivial direct sum decomposition. Thus the algebra $\operatorname{End} M$ must be local. Furthermore, again by 1.5.3, any noninvertible A-module endomorphism $f : M \to M$ belongs to the radical of $\operatorname{End} M$ and therefore f is nilpotent, because

End M is finite dimensional and thus its radical is nilpotent. Indeed, if we set rad End $M = R$, since $\dim_k R < \infty$, the chain

$$R \supseteq R^2 \supseteq \cdots R^n \supseteq R^{n+1} \supseteq \cdots$$

becomes stationary. It follows that $R^m = (R^m)R$ for some m, and thus we have $R = 0$ by Nakayama's Lemma.

□

Corollary 1.5.5. *An idempotent $e \in A$ is primitive if and only if the algebra $eAe \cong \operatorname{End} eA$ has only two idempotents 0 and e, that is, the algebra eAe is local.*

Proof. We will first show that if eAe has a nontrivial idempotent, say efe for some $f \in A$, then $e \in A$ is not primitive. In this case indeed $e = efe + e(1-f)e$ and both efe and $e(1-f)e$ are nontrivial orthogonal idempotents in A as well, thus e is not primitive, as wanted. Conversely assume that End eA is local. By Corollary 1.5.4 we have that eA is indecomposable and thus e must be primitive. □

The following result is fundamental for the representation theory of finite dimensional algebras.

Theorem 1.5.6 (Unique decomposition theorem)**.** *Let A be a finite dimensionalo k-algebra.*

i) Every module M in mod A *has a decompostion $M \cong M_1 \oplus \cdots \oplus M_n$ where $M_1, \ldots, M_n$ are indecomposable modules and the endomorphism k-algebra* End M_i *is local for each $i = 1, \ldots, n$.*

ii) If $M \cong \bigoplus_{i=1}^n M_i \cong \bigoplus_{j=1}^m N_j$, where Mi and N_j are indecomposable, then $n = m$ and there exist a permutation σ of $\{1, \ldots, n\}$ such that $M_i \cong N_{\sigma(i)}$ for each $i = 1, \ldots, n$.

We know that if $A_A = e_1A \oplus \cdots e_n$ is a decomposition of A into indecomposable submodules (so in particular if $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents), then all projective A-modules P are of the form $P = (e_1A)^{m_1} \oplus \cdots (e_n)^{m_n}$ for some $m_1, \ldots, m_n \in \mathbf{N}$, since projective modules have to be summands of some free module. This, together with Proposition 1.5.2, leads to the following result.

Proposition 1.5.7. *Suppose that $A_A = e_1A \oplus \cdots e_n$ is a decomposition of A into indecomposable submodules.*

i) Every simple right A-module is isomorphic to one of the modules

$$S(1) = e_1A/\operatorname{rad} e_1A := \operatorname{top} e_1A, \ldots, S(n) = e_nA/\operatorname{rad} e_nA := \operatorname{top} e_nA$$

ii) *Every indecomposable projective right A-module is isomorphic to one of the modules*

$$P(1) = e_1 A, P(2) = e_2 A, \ldots, P(n) = e_n A$$

Moreover, $e_i A \cong e_j A$ *if and only if* $S(i) \cong S(j)$.

iii) *Dually, every indecomposable injective right A-module is isomorphic to one of the modules*

$$I(1) = D(Ae_1) \cong E(S(1)), \ldots, I(n) = D(Ae_n) \cong E(S(n))$$

where $E(S(j))$ *is an injective envelope of the simple module* $S(j)$.

Definition 1.5.2. *Let A be a k-algebra with a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents. The algebra A is called* basic *if* $e_i A \ncong e_j A$ *for all* $i \neq j$.

Proposition 1.5.8. *A finite dimensional k-algebra A is basic if and only if the algebra* $B = A/\operatorname{rad} A$ *is isomorphic to a product* $k \times k \times \cdots \times k$ *of copies of* k.

Definition 1.5.3. *Let A be a k-algebra with a complete set $\{e_1, \ldots, e_n\}$ of primitive orthogonal idempotents. A* basic algebra *associated to A is the algebra*

$$A^b = e_A A e_A$$

where $e_A = e_{j_1} + \cdots + e_{j_a}$ *and* $e_{j_1}, \ldots, e_{j_a}$ *are chosen such that* $e_{j_i} A \ncong e_{j_t} A$ *for* $i \neq t$ *and for all* $s = 1, \ldots, n$ *the module* $e_s A$ *is isomorphic to one of the modules* $e_{j_1} A, \ldots, e_{j_a} A$.

Lemma 1.5.9. *Let* $A^b = e_A A e_A$ *be a basic algebra associated to A. The algebra A^b does not depend on the choice of the sets* $e_1, \ldots, e_n$ *and* $e_{j_1}, \ldots e_{j_a}$, *up to a k-algebra isomorphism. Furthermore the algebra A^b is basic.*

This Lemma in particular tells us that for each algebra A there is up to isomorphism one basic algebra A^b associated with A.

Proof. We are just going to prove that A^b is basic. For this, assume that $\{e_1, \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents of A, $e_A = e_{j_1} + \cdots + e_{j_a}$ and $e_{j_1}, \ldots, e_{j_a}$ are chosen as in Definition 1.5.3. Then $e_{j_1}, \ldots, e_{j_a}$ are clearly orthogonal idempotents in A^b as well. Furthermore for all $t = 1, \ldots, a$ we have $e_{j_t} A^b e_{j_t} = e_{j_t} e_A A e_A e_{j_t} = e_{j_t} A e_{j_t}$. It follows from Corollary 1.5.5 that the algebra $\operatorname{End} e_{j_t} A^b \cong e_{j_t} A^b e_{j_t}$ is local, because $e_{j_t} A$ is indecomposable in $\operatorname{mod} A$. Hence e_{j_t} is a primitive idempotent of A^b. To show that the algebra A^b is basic, assume that $e_{j_t} A^b \cong e_{j_r} A^b$. We know that the multiplication map $m_{j_i} : e_{j_i} A^b \otimes e_A A \to e_{j_i} A$, $e_{j_i} x \otimes e_A a \mapsto e_{j_i} x e_A a$, is an A-module isomorphism for $i = 1, \dots, a$ (for more details see [1]). Thus we get A-module isomorphisms

$$e_{j_t}A \cong e_{j_t}A^b \otimes e_A A \cong e_{j_r}A^b \otimes e_A A \cong e_{j_r}A$$

and therefore $t = r$ by the choice of $e_{j_1}, \dots, e_{j_a}$ in 1.5.3. □

Proposition 1.5.10. *Let $A^b = e_A A e_A$ be a basic k-algebra associated with A. Then there are k-linear equivalences of categories quasi-inverse to each other between* $\operatorname{mod} A^b$ *and* $\operatorname{mod} A$.

This Proposition, and the equivalence it talks about, is going to be very important in the central chapter of this thesis. Indeed when dealing with categories such as $\operatorname{mod} A$ or $D^b(\operatorname{mod} A)$ for some algebra A, we can always suppose that A is basic because by switching A with A^b we would get equivalent categories and A^b is basic.

We are now interested in introducing quivers and the related terminology. Indeed they will play a central role in the theory of algebra representation.

Definition 1.5.4. *A* quiver *$Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two sets Q_0 (whose elements are called* points *or* vertices*) and Q_1 (whose elements are called* arrows*), and two maps $s, t : Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$, respectively.*

An arrow $\alpha \in Q_1$ of source $a = s(\alpha)$ and target $b = t(\alpha)$ is usually denoted by $\alpha : a \to b$. A quiver $Q = (Q_0, Q_1, s, t)$ is usually denoted by $Q = (Q_0, Q_1)$ or even just by Q.

It is also important to notice that a quiver is nothing but an oriented graph without any restriction as to the number of arrows between two points, to the existence of loops or oriented cyrcles.

A quiver Q is said to be *finite* if Q_0 and Q_1 are finite sets. A quiver Q is said to be *connected* if its underlying graph is a connected graph.

Now let $Q = (Q_0, Q_1, s, t)$ be a quiver and $a, b \in Q_0$. A *path* of length $\ell \geq 1$ from a to b is a sequence

$$(a \mid \alpha_1, \alpha_2, \dots, \alpha_\ell \mid b)$$

where $\alpha_k \in Q_1$ for all $1 \leq k \leq \ell$, and we have $s(\alpha_1) = a$, $t(\alpha_k) = s(\alpha_{k+1})$ for each $1 \leq k < \ell$, and finally $t(\alpha\ell) = b$. Such a path is denoted briefly by $\alpha_1\alpha_2 \dots \alpha_\ell$ and may be visualised as follows

$$a = a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} a_2 \longrightarrow \cdots \xrightarrow{\alpha_\ell} a_\ell = b$$

We denote by Q_ℓ the set of all paths in Q of length ℓ. We also agree to associate with each point $a \in Q_0$ a path of length $\ell = 0$, called the *trivial* or *stationary path* at a, and denoted by

$$\varepsilon_a = (a \mid\mid a)$$

Thus the paths of lengths 0 and 1 are in bijective correspondence with the elements of Q_0 and Q_1, respectively. A path of length $\ell \geq 1$ is called a *cycle* whenever its source and target coincide. A cycle of length 1 is called a *loop.* A quiver is called *acyclic* if it does not contain any cycle.

Definition 1.5.5. *Let Q be a quiver. The* path algebra kQ *of Q is the k-algebra whose underlying k-vector space has as its basis the set of all paths $(a \mid \alpha_1, \dots, \alpha_\ell \mid b)$ of length $\ell \geq 0$ in Q and such that the product of two basis vectors $(a \mid \alpha_1, , \dots, \alpha_\ell \mid b)$ and $(c \mid \beta_1, \dots, \beta_k \mid d)$ of kQ is defined by*

$$(a \mid \alpha_1, , \dots, \alpha_\ell \mid b) \cdot (c \mid \beta_1, \dots, \beta_k \mid d) = \delta_{bc}(a \mid \alpha_1, , \dots, \alpha_\ell, \beta_1, \dots, \beta_k \mid d)$$

where δ_{bc} denotes the Kronecker delta. In other words, the product of two paths $\alpha_1 \dots \alpha_\ell$ and $\beta_1 \dots \beta_k$ is equal to zero if $t(\alpha_\ell) \neq s(\beta_1)$ and is equal to the juxtaposition $\alpha_1 \dots \alpha_\ell \beta_1 \dots \beta_k$ of the two paths if $t(\alpha_\ell) = s(\beta_1)$. The product of basis elements is then extended to arbitrary elements of kQ by distributivity.

Clearly kQ is an associative algebra, since juxtaposition is clearly an associative operation.

Moreover, there is a direct sum decomposition

$$kQ = kQ_0 \oplus kQ_1 \oplus \cdots \oplus kQ_\ell \oplus \cdots$$

of the k-vector space kQ, where, for each $\ell \geq 0$, kQ_ℓ is the subspace of kQ generated by the set Q_ℓ of all paths of length ℓ.

Notice in particular that each stationary path $\varepsilon_a = (a \mid a)$ is clearly an idempotent of kQ, thus, if Q_0 is finite, $\sum_{a \in Q_0} \varepsilon_a$ is the identity on kQ. Indeed nif α is a path from b to c we have that $(\sum_{a \in Q_0} \varepsilon_a) \cdot \alpha = \varepsilon_b \alpha + \sum_{a \neq b} 0 = \alpha$ and similarly $\alpha \cdot (\sum_{a \in Q_0} \varepsilon_a) = \alpha \varepsilon_c + 0 = \alpha$. Furthermore it can be shown that the set $\{\varepsilon_a : a \in Q_0\}$ is a complete set of primitive orthogonal idempotents for kQ.

Lemma 1.5.11. *Let Q be a quiver and kQ be its path algebra. Then kQ is finite dimensional if and only if Q is finite and acyclic.*

Proof. If Q is infinite, then so is the basis of kQ, which is therefore infinite dimensional. Moreover, if $w = \alpha_1 \dots \alpha_\ell$ is a cycle in Q, then, for each $t \geq 0$, we have a basis vector $w^t = (\alpha_1 \dots \alpha_\ell)^t$, so kQ is again infinite dimensional. Conversely if Q is finite and acyclic, it contains only finitely many paths and thus kQ is finite dimensional. □

To prove the next lemma we are going to need a technical result that we will just state without proof. For a proof we refer as usual to [1].

Lemma 1.5.12. *Let A be an associative algebra with identity and assume that $\{e_1, \ldots, e_n\}$ is a (finite) complete set of primitive orthogonal idempotents. Then A is a connected algebra if and only if there is no nontrivial partition $I \dot{\cup} J$ of the set $\{1, \ldots, n\}$ such that $i \in I$ and $j \in J$ imply $e_i A e_j = 0 = e_j A e_i$.*

Lemma 1.5.13. *Let Q be a finite quiver. The path algebra kQ is connected if and only if Q is a connected quiver.*

Proof. Assume that Q is not connected and let Q' be a connected component of Q. Denote by Q'' the quiver with $Q''_0 = Q_0 \backslash Q'_0$ and with $Q''_1 = Q_1 \backslash Q'_1$. By hypothesis, neither Q' nor Q'' is empty. Thus consider $a \in Q'_0$ and $b \in Q''_0$. Since Q is not connected, an arbitrary path w in Q is entirely contained in either Q' or in (a connected component of) Q''. In the former case we have $w\varepsilon_b = 0$ and in the latter case we have $\varepsilon_a w = 0$, hence we always have $\varepsilon_a w \varepsilon_b = 0$. Thus $\varepsilon_a(kQ)\varepsilon_b = 0$. Similarly $\varepsilon_b(kQ)\varepsilon_a = 0$. By the previous result 1.5.12 this shows that kQ is not a connected algebra.

Conversely suppose that Q is connected but kQ is not. Again by 1.5.12, there exists a partition $Q_0 = Q'_0 \dot{\cup} Q''_0$ such that, if $x \in Q'_0$ and $y \in Q''_0$, then $\varepsilon_x(kQ)\varepsilon_y = 0 = \varepsilon_y(kQ)\varepsilon_x$. Since Q is connected, there exist $a \in Q'_0$ and $b \in Q''_0$ such that there is an arrow $\alpha : a \to b$ (or an arrow $\beta : b \to a$, but without loss of generality we can assume the previous case to hold). But now we have

$$\alpha = \varepsilon_a \alpha \varepsilon_b \in \varepsilon_a(kQ)\varepsilon_b = 0$$

which clearly is a contradiction and completes the proof of the lemma. □

Definition 1.5.6. *Let Q be a finite and connected quiver. The two-sided ideal of the path algebra kQ generated (as an ideal) by the arrows of Q is called the* arrow ideal *of kQ and denoted by R_Q (or simply by R).*

Note that there is a direct sum decomposition

$$R_Q = kQ_1 \oplus kQ_2 \oplus \cdots \oplus kQ_\ell \oplus \oplus \cdots$$

of the k-vector space R_Q. This shows in particular that the underlying k-vector space of R_Q is generated by all paths in Q of length $\ell \geq 1$. This implies that, for each $\ell \geq 1$,

$$R_Q^\ell = \bigoplus_{m \geq \ell} kQ_m$$

and therefore R_Q^ℓ is the ideal of kQ generated, as a k-vector space, by the set of all paths of length $\geq \ell$. In particular we have a k-vector space isomorphism $R_Q^\ell / R_Q^{\ell+1} \cong kQ_\ell$.

Proposition 1.5.14. *Let Q be a finite connected quiver which is also acyclic. Then* $\operatorname{rad} kQ = R_Q$ *and kQ is a finite dimensional basic algebra.*

Proof. We already know that kQ is finite dimensional by Lemma 1.5.11. Since Q is acyclic there exists a largest $\ell \geq 1$ such that Q contains a path of length ℓ. This in particular implies that any product of $\ell + 1$ arrows is zero, so $R_Q^{\ell+1} = 0$. Consequently R_Q is a nilpotent ideal and hence $R_Q \subseteq \operatorname{rad} kQ$. But we also know that kQ/R_Q has the finite set Q_0 as a basis and thus it is isomorphic to a product of copies of k, thus $R_Q = \operatorname{rad} kQ$. This in particular means that $kQ/\operatorname{rad} kQ = kQ/R_Q$ is also isomorphic to a product of copies of k and thus it follows from Proposition 1.5.8 that the algebra kQ is basic. □

Our goal now is to remove the hypothesis of a quiver Q being acyclic (thus leading to potentially infinite dimensional path algebras kQ) and to study the finite dimensional quotients of not necessarily finite dimensional path algebras.

Definition 1.5.7. *Let Q be a finite quiver and R_Q be the arrow ideal of the path algebra kQ. A two-sided ideal $\mathcal{I}$ of kQ is said to be* admissable *if there exists $m \geq 2$ such that*

$$R_Q^m \subseteq \mathcal{I} \subseteq R_Q^2$$

If $\mathcal{I}$ is an admissable ideal of kQ, the pair $(Q, \mathcal{I})$ is called a bound quiver*. The quotient algebra $kQ/\mathcal{I}$ is said to be a* bound quiver algebra*.*

It follows directly from the definition that an ideal $\mathcal{I}$ of kQ, contained in R_Q^2, is admissable if and only if it contains all paths whose length is large enough. It can be shown that this is the case if and only if , for each cycle σ in Q, there exists $s \geq 1$ such that $\sigma^s \in \mathcal{I}$. In particular, if Q is acyclic, any ideal contained in R_Q^2 is admissable.

Definition 1.5.8. *Let Q be a quiver. A* relation *in Q with coefficients in k is a k-linear combination of paths of length at least two having the same source and target. Thus a relation ρ is an element of kQ such that*

$$\rho = \sum_{i=1}^{m} \lambda_i w_i$$

where the λ_i are scalars (not all zero) and the w_i are paths in Q of length at least 2 such that all of their sources (or target, respectively) coincide.

The next few results show that if Q is a quiver and $\mathcal{I}$ is an admissable ideal of kQ, then the bound quiver algebra $kQ/\mathcal{I}$ has similar properties to a path algebra $k\tilde{Q}$ over an acyclic quiver $\tilde{Q}$.

Lemma 1.5.15. *Let Q be a finite quiver and $\mathcal{I}$ be an admissable ideal of kQ. The following statements hold.*

i) The set $\{e_a = \varepsilon_a + \mathcal{I} : a \in Q_0\}$ is a complete set of primitive orthogonal idempotents of the bound quiver algebra $kQ/\mathcal{I}$.

ii) The bound quiver algebra $kQ/\mathcal{I}$ is connected if and only if Q is a connected quiver.

iii) The bound quiver algebra $kQ/\mathcal{I}$ is finite dimensional.

Proof. We are just going to prove iii). Since $\mathcal{I}$ is admissable, there exists $m \geq 2$ such that $R^m \subseteq \mathcal{I}$, where R denotes the arrow ideal R_Q of kQ. This means that there exists a surjective algebra homomorphism $kQ/R^m \twoheadrightarrow kQ/\mathcal{I}$. Thus it suffices to prove that kQ/R^m is finite dimensional. But now, the residual classes of paths of length less than m form a basis of kQ/R^m as a k-vector space. Since there are only finitely many such paths, our statement follows. □

Lemma 1.5.16. *Let Q be a finite quiver. Every admissable ideal $\mathcal{I}$ of kQ is finitely generated.*

Proof. Let R be the arrow ideal of kQ and $m \geq 2$ an integer such that $R^m \subseteq \mathcal{I}$. We have a short exact sequence $0 \to R^m \to \mathcal{I} \to \mathcal{I}/R^m \to 0$ of kQ-modules. Hence, it suffices to show that R^m and $\mathcal{I}/R^m$ are finitely generated as kQ-modules. R^m is the kQ-module generated by the paths of length m. Since there are only finitely many such paths, R^m is finitely generated. On the other hand, $\mathcal{I}/R^m$ is an ideal of the finite dimensional algebra kQ/R^m (which is finite dimensional in view of 1.5.15 iii) since R^m is clearly an admissable ideal). Therefore $\mathcal{I}/R^m$ is a finite dimensional k-vector space, hence a finitely generated kQ-module. □

Corollary 1.5.17. *Let Q be a finite quiver and $\mathcal{I}$ be an admissable ideal of kQ. There exist a finite set of relations $\{\rho_1, \dots, \rho_m\}$ such that $\mathcal{I} = \langle \rho_1, \dots, \rho_m \rangle$.*

Proof. By 1.5.16, $\mathcal{I}$ has a finite generating set $\{\sigma_1, \dots, \sigma_t\}$. If the σ_i are not relations, then the set $\{\varepsilon_a \sigma_i \varepsilon_b : 1 \leq i \leq t; a, b \in Q_0\}$ is a finite set of relations generating $\mathcal{I}$. Indeed for any $1 \leq i \leq t$ we have that $\sigma_i = \sum_{a,b \in Q_0} \varepsilon_a \sigma_i \varepsilon_b$ and that $\varepsilon_a \sigma_i \varepsilon_b$ is either zero or a relation. □

Lemma 1.5.18. *Let Q be a finite quiver, R be the arrow ideal of kQ and $\mathcal{I}$ be an admissable ideal of kQ. Then $\operatorname{rad}(kQ/\mathcal{I}) = R_Q/\mathcal{I}$. Moreover, the bound quiver algebra $kQ/\mathcal{I}$ is basic.*

Proof. Since $\mathcal{I}$ is an admissable ideal of kQ, there exists $m \geq 2$ such that $R^m \subseteq \mathcal{I}$. Consequently, $(R/\mathcal{I})^m = 0$ and thus $R/\mathcal{I}$ is a nilpotent ideal of $kQ/\mathcal{I}$. On the other hand, the algebra $(kQ/\mathcal{I})/(R/\mathcal{I}) \cong kQ/R$ is isomorphic to a direct product of copies of k. Thus, by considerations analogous to the ones in the proof of Proposition 1.5.14, we have proved both of our assertions. □

Corollary 1.5.19. *For each $\ell \geq 1$, we have $\operatorname{rad}^\ell(kQ/\mathcal{I}) = (R_Q/\mathcal{I})^\ell$.*

It follows from the last two results that the k-vector space

$$\operatorname{rad}(kQ/\mathcal{I})/\operatorname{rad}^2(kQ/\mathcal{I}) = (R_Q/\mathcal{I})/(R_Q/\mathcal{I})^2 \cong R_Q/R_Q^2 \cong kQ_1$$

admits as basis the set $\overline{\alpha} + \operatorname{rad}^2(kQ/\mathcal{I})$, where $\overline{\alpha} = \alpha + kQ/\mathcal{I}$ and $\alpha \in Q_1$. This fact is going to be really important for the following part.

Now let A be a finite dimensional algebra over an algebraically closed field k. From the point of view of studying the representation theory of A, we can assume that A is basic and connected. Indeed if A is not basic, in view of Proposition 1.5.10 we can just consider A^b and if A is not connected, then $A \cong \bigoplus A_i$ for some connected algebras A_i and $\operatorname{mod}(A) \cong \bigoplus \operatorname{mod}(A_i)$. Our next goal is to understand under which hypotheses A is isomorphic to a bound quiver algebra $kQ/\mathcal{I}$, with Q a finite connected quiver and $\mathcal{I}$ an admissable ideal of kQ. We start by associating a finite quiver to each basic and connected finite dimensional algebra.

Definition 1.5.9. *Let A be a basic and connected finite dimensional k-algebra and $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of A. The* (ordinary) quiver *of A, denoted by Q_A, is defined as follows:*

i) The points of Q_A are the numbers $1, 2, \dots, n$, which are in bijective correspondence with the idempotents $e_1, e_2, \dots, e_n$.

ii) Given two points $a, b \in (Q_A)_0$, the arrows $\alpha : a \to b$ are in bijective correspondence with the vectors in a basis of the k-vector space $e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$.

Since A is finite dimensional, it follows that every vector space of the form $e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$ (with $a, b \in (Q_A)_0$) is also finite dimensional, thus Q_A is finite. Also, it can be shown that Q_A does not depend on the choice of a complete set of primitive orthogonal idempotents in A.

Lemma 1.5.20. *Let Q be a finite connected quiver, $\mathcal{I}$ be an admissable ideal of kQ, and $A = kQ/\mathcal{I}$. Then $Q_A = Q$.*

Proof. By Lemma 1.5.15 i), the set $\{e_a = \varepsilon_a + \mathcal{I} : a \in Q_0\}$ is a complete set of primitive orthogonal idempotents of $A = kQ/\mathcal{I}$. Thus the points of Q_A are in bijective correspondence with those of Q. On the other hand, by Corollary 1.5.19 and in particular by the remark following it, the arrows in Q are in bijective correspondence with a basis of the k-vector space $\operatorname{rad} A/\operatorname{rad}^2 A$. This implies in particular that the arrows from a to b in Q are in bijective correspondence with the vectors in a basis of the k-vector subspace $e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$, thus with the arrows from a to b in Q_A. □

Theorem 1.5.21. *Let A be a basic and connected finite dimensional k-algebra. Then there exists an admissable ideal $\mathcal{I}$ of kQ_A such that $A \cong kQ_A/\mathcal{I}$.*

Proof. We are just going to give a sketch of the proof. More precise details can be found in [1]. The main idea is to construct an algebra homomorphism $\varphi : kQ_A \to A$, then to show that it is surjective and that its kernel $\mathcal{I} := \operatorname{Ker}\varphi$ is an admissable ideal of kQ_A. We are just going to show how to construct φ.

For any point $a \in (Q_A)_0$ we define $\varphi(a) = e_a$. For each arrow $\alpha : i \to j$ in $(Q_A)_1$, first let $x_\alpha \in \operatorname{rad} A$ be chosen so that $\{x_\alpha + \operatorname{rad}^2 A \mid \alpha : i \to j\}$ forms a basis of $e_i(\operatorname{rad} A/\operatorname{rad}^2 A)e_j$. We define $\varphi(\alpha) = x_\alpha$ for all $\alpha \in (Q_A)_1$. Notice in particular that the elements $\varphi(a)$, $a \in (Q_A)_0$, form a complete set of primitive orthogonal idempotents in A and that for any $\alpha : a \to b$ we have $\varphi(a)\varphi(\alpha)\varphi(b) = e_a x_\alpha e_b = x_\alpha = \varphi(\alpha)$. This in particular ensures us that φ actually extends to a unique k-algebra homomorphism by the universal property of path algebras. □

We just saw how quivers help us in visualising finite dimensional algebras. Our next goal is to see how said quivers provide a convenient way to visualise any module over an algebra. For this we will need the following definition.

Definition 1.5.10. *Let Q be a finite quiver. A k-*linear representation*, or simply a* representation *M of Q is defined by the following data:*

i) To each point a in Q_0 is associated a k-vector space M_a.

ii) To each arrow $\alpha : a \to b$ in Q_1 is associated a k-linear map $\varphi_\alpha : M_a \to M_b$.

Such a representation is denoted as $M = (M_a, \varphi_\alpha)_{a \in Q_0, \alpha \in Q_1}$ or simply $M = (M_a, \varphi_\alpha)$. It is called finite dimensional *if each vector space M_a is finite dimensional.*

Let $M = (M_a, \varphi_\alpha)$ and $M' = (M'_a, \varphi'_\alpha)$ be two representations of Q. A *morphism* (of representations) $f : M \to M'$ is a family $f = (f_a)_{a \in Q_0}$ of k-linear maps $(f_a : M_a \to M'_a)_{a \in Q_0}$ that are compatible with the structure maps φ_α, that is, for each arrow $\alpha : a \to b$, we have $\varphi'_\alpha f_a = f_b \varphi_\alpha$. Equivalently, the following square needs to be commutative for any arrow $\alpha : a \to b$

$$\begin{array}{ccc} M_a & \xrightarrow{\varphi_\alpha} & M_b \\ \downarrow{\scriptstyle f_a} & & \downarrow{\scriptstyle f_b} \\ M'_a & \xrightarrow{\varphi'_\alpha} & M'_b \end{array}$$

Let $f = (f_a)_{a\in Q_0} : M \to M'$ and $g = (g_a)_{a\in Q_0} : M' \to M''$ be two morphisms of representations of Q. Their composition is defined to be the family $gf = (g_a f_a)_{a\in Q_0}$. It can be easily seen that gf is a morphism from M to M''.

We have thus defined a category $\mathrm{Rep}(Q)$ of k-linear representations of Q. We denote by $\mathrm{rep}(Q)$ the full subcategory of $\mathrm{Rep}(Q)$ consisting of the finite dimensional representations. In particular it can be shown that both of these categories are abelian k-categories.

Definition 1.5.11. *Let Q be a finite quiver and $M = (M_a, \varphi_\alpha)$ be a representation of Q. For any nontrivial path $w = \alpha_1\alpha_2 \dots \alpha_\ell$ from a to b in Q, we define the* evaluation *of M on the path w to be the k-linear map from M_a to M_b defined by*

$$\varphi_w = \varphi_{\alpha_\ell}\varphi_{\alpha_{\ell-1}} \cdots \varphi_{\alpha_2}\varphi_{\alpha_1}$$

This definitions extend to relations in Q. Indeed if $\rho = \sum_{i=1}^m \lambda_i w_i$ is a relation in Q we set

$$\varphi_\rho = \sum_{i=1}^m \lambda_i \varphi_{w_i}$$

We are now able to define the notion of representation of a bound quiver. Let thus Q be a finite quiver and $\mathcal{I}$ be an admissable ideal of kQ. A representation $M = (M_a, \varphi_\alpha)$ of Q is said to be *bound by* $\mathcal{I}$, or to *satisfy the relations* in $\mathcal{I}$, if we have

$$\varphi_\rho = 0, \qquad \text{for all relations } \rho \in \mathcal{I}$$

Since we know that $\mathcal{I}$ is a finitely generated by some relations (see 1.5.17) we can just check this condition on the generators. We denote by $\mathrm{Rep}_k(Q, \mathcal{I})$ (or by $\mathrm{rep}_k(Q, \mathcal{I})$) the full subcategory of $\mathrm{Rep}_k(Q)$ (or of $\mathrm{rep}_k(Q)$ respectively) consisting of the representations of Q bound by $\mathcal{I}$.

The next theorem will justify all this setup. Indeed our goal is to study the category $\mathrm{mod}\, A$, where A is a finite dimensional k-algebra, which we can assume to be basic and connected without loss of generality. We already saw that there exists a finite connected quiver Q_A and an admissable ideal $\mathcal{I}$ of kQ_A such that $A \cong kQ_A/\mathcal{I}$. We will now show that the category $\mathrm{mod}\, A$ of finitely generated A-modules is equivalent to the category $\mathrm{rep}_k(Q_A, \mathcal{I})$ of finite dimensional k-linear representations of Q_A bound by $\mathcal{I}$.

Theorem 1.5.22. *Let $A = kQ/\mathcal{I}$, where Q is a finite connected quiver and $\mathcal{I}$ is an admissable ideal of kQ. There exists a k-linear equivalence of categories*

$$\mathrm{Mod}\, A \xrightarrow{\ \simeq\ } \mathrm{Rep}_k(Q, \mathcal{I})$$

that restricts to an equivalence of categories

$$\operatorname{mod} A \xrightarrow{\simeq} \operatorname{rep}_k(Q, \mathcal{I})$$

Proof. We will show how to construct the equivalences $F : \operatorname{Mod} A \to \operatorname{Rep}_k(Q, \mathcal{I})$ and $G : \operatorname{Rep}_k(Q, \mathcal{I}) \to \operatorname{Mod} A$. We will not prove that they are indeed functors and that they are quasi-inverse to each other. From the constructions it will be clear that F and G restrict to equivalences $\operatorname{mod} A \to \operatorname{rep}_k(Q, \mathcal{I})$ and $\operatorname{Rep}_k(Q, \mathcal{I}) \to \operatorname{mod} A$ respectively.

First we construct F. Let M_A be an A-module. We define the k-linear representation $F(M) = (M_a, \varphi_\alpha)_{a \in Q_0, \alpha \in Q_1}$ of $(Q, \mathcal{I})$ as follows: if $a \in Q_0$, let $e_a = \varepsilon_a + \mathcal{I}$ be the corresponding primitive idempotent in $A = kQ/\mathcal{I}$, then set $M_a = Me_a$; if $\alpha : a \to b$ is in Q_1 and $\overline{\alpha} = \alpha + \mathcal{I}$ is its class modulo $\mathcal{I}$, define $\varphi_\alpha : M_a \to M_b$ by $\varphi_\alpha(x) = x\overline{\alpha}(= xe_a\overline{\alpha}e_b)$ for $x \in M_a$. Because M is an A-module, φ_a is a k-linear map. Also, $F(M)$ is bound by $\mathcal{I}$. Indeed let $\rho = \sum_{i=1}^m \lambda_i w_i$ be a relation from a to b in $\mathcal{I}$, where $w_i = \alpha_{i,1}\alpha_{i,2}\ldots\alpha_{i,\ell_i}$. Then we have

$$\begin{aligned}
\varphi_\rho(x) &= \sum_{i=1}^m \lambda_i \varphi_{w_i}(x) \\
&= \sum_{i=1}^m \lambda_i \varphi_{\alpha_{i,\ell_i}} \ldots \varphi_{\alpha_{i,1}}(x) \\
&= \sum_{i=1}^m \lambda_i (x\overline{\alpha}_{i,1} \ldots \overline{\alpha}_{i,\ell_i}) \\
&= x \cdot \sum_{i=1}^m \lambda_i (\overline{\alpha}_{i,1} \ldots \overline{\alpha}_{i,\ell_i}) \\
&= x \cdot \overline{\rho} = x \cdot 0 = 0
\end{aligned}$$

This defines the functor on the objects.

If $f : M_A \to M'_A$ is an A-module homomorphism, we define $F(f) : F(M) \to F(M')$ to be $(f_a)_{a \in Q_0}$, where f_a is the restriction of f to M_a. Indeed for any $a \in Q_0$ and $xe_a \in M_a = Me_a$, we have $f(xe_a) = f(xe_a^2) = f(xe_a)e_a \in M'e_a = M'_a$. It can be checked that this is indeed a morphism of representations.

Now we focus on the definition of G. If $M = (M_a, \varphi_\alpha)$ is an object of $\operatorname{Rep}_k(Q, \mathcal{I})$ we set $G(M) = \bigoplus_{a \in Q_0} M_a$ and we define an A-module structure on the k-vector space $G(M)$ as follows. Let $x = (x_a)_{a \in Q_0}$ be an element in $G(M)$. We need to define products of the form $x \cdot w = x \cdot w + \mathcal{I}$, where w is a path in Q. If $w = \varepsilon_a$ is the stationary path, we put

$$xw = x\varepsilon_a = x_a$$

If instead $w = \alpha_1\alpha_2 \ldots \alpha_\ell$ is a nontrivial path from a to b, we consider the k-linear map $\varphi_w : M_a \to M_b$ and we set

$$(xw)_c = \delta_{bc}\varphi_w(x_a)$$

where δ_{bc} denotes the Kronecker delta. In other words, xw is the element of $G(M)$ whose only nonzero coordinate is $(xw)_b = \varphi_w(x_a) \in M_b$. It can be checked that this definition is well-posed, i.e. that it does not depend on the representative of the class that we choose. This defines G on objects.

If $(f_a)_{a\in Q_0}$ is a morphism from $M = (M_a, \varphi_\alpha)$ to $M' = (M'_a, \varphi'_\alpha)$, we set $G(f) = \bigoplus_{a\in Q_0} f_a : G(M) \to G(M')$. It can be checked that this is indeed an A-module homomorphism.

Notice in particular that the second statement of the theorem follows from the fact that, since Q is finite, for a representation $M = (M_a, \varphi_\alpha)$ we have $\dim_k(\bigoplus_{a\in Q_0} M_a) < \infty$ if and only if $\dim_k M_a < \infty$ for all $a \in Q_0$. □

In the last part of this section we are going to define simple and projective representations and show some useful facts about them. For this part the main refernce will be [9].

Let Q be a finite connected quiver and let $\mathcal{I}$ be an admissable ideal of kQ. We will denote by A the bound quiver algebra $kQ/\mathcal{I}$.

Let i be a vertex of Q. We will define the following representations:

i) The *simple representation* at vertex i is defined by

$$S(i)_j = \begin{cases} k, & \text{if } i = j; \\ 0 & \text{otherwise} \end{cases}$$

with structure maps

$$\varphi_\alpha = 0 \qquad \text{for all arrows } \alpha$$

ii) The *projective representation* at vertex i is $P(i) = (P(i)_j, \varphi_\alpha)_{i\in Q_0, \alpha\in Q_1}$ where $P(i)_j$ is the k-vector space with basis the set of all $\overline{w} = w + \mathcal{I}$, with w a path from i to j and, for an arrow $\alpha : b \to c$, the structure map $\varphi_\alpha : P(i)_b \to P(i)_c$ is given by right multiplication by $\overline{\alpha} = \alpha + \mathcal{I}$.

We clearly have that for any $i \in Q_0$, the simple representation $S(i)$ has no proper subrepresentation, thus the representations $S(i)$ are simple objects in $\mathrm{rep}_k(Q, \mathcal{I})$. Furthermore, by applying the functor G described in the proof of Theorem 1.5.22, we can see that the representations $P(i)$ correspond to the indecomposable projective modules e_iA. Indeed $G(P(i))$ is the module whose underlying vector space is $\bigoplus_{j\in Q_0} P(i)_j$ which has a basis consisting of all residue classes $w+\mathcal{I}$ of paths w starting at i and thus is isomorphic to the vector space e_iA. Furthermore a residue class $w'+\mathcal{I}$ of a path w' acts on a basis element $w+\mathcal{I}$ of $G(P(i))$ by the formula $(w+\mathcal{I})\cdot(w'+\mathcal{I}) = \varphi_{w'}(w)$, which is given by the composition of paths ww'. Thus $G(P(i))$ is isomorphic to e_iA a an A-module for all $i \in Q_0$.

We also want to give a quick description of the indecomposable injective A-modules. By Proposition 1.5.7, a complete list of pairwise nonisomorphic indecomposable injective A-modules is given by $I(a) = D(Ae_a)$ for $a \in Q_0$, where D is the standard duality between right and left A-modules. We say that $I(a)$ is the indecomposable injective A-module corresponding to the point $a \in Q_0$.

Lemma 1.5.23. *Let $A = kQ/\mathcal{I}$ be the bound quiver algebra of $(Q, \mathcal{I})$.*

i) For any $a \in Q_0$, $S(a)$ viewed as an A-module is isomorphic to the top of the indecomposable projective A-module $e_a A$.

ii) The set $\{S(a) : a \in Q_0\}$ is a complete set of representatives of the isomorphism classes of the simple A-modules.

iii) Given $a \in Q_0$, the simple module $S(a)$ is isomorphic to the simple socle of $I(a)$.

Proof. We already saw that the $S(a)$, $a \in Q_0$, are simple representations and thus they represent simple modules. Furthermore, from the proof of Theorem 1.5.22 we have that $\mathrm{Hom}_A(e_a A, S(a)) \cong S(a)e_a \cong S(a)_a \neq 0$, so there exists a nonzero morphism from the indecomposable projective A-module $e_a A$ onto the simple A-module $S(a)$. This proves i), since by Proposition 1.5.2 we know that $e_a A$ has a simple top. On the other hand, if $a \neq b$, it is clear from their representation that $\mathrm{Hom}(S(a), S(b)) = 0$, so in particular $S(a) \ncong S(b)$. By Proposition 1.5.7 there exists a bijection between a complete set of primitive orthogonal idempotents and a complete set of pairwise nonisomorphic simple A-modules given by $e_a \mapsto \mathrm{top}\, e_a A$, so ii) follows.

For iii) we just need to notice that this statement is dual to the one in i). Thus we have the isomorphisms

$$\mathrm{soc}\, I(a) \cong P(a)/\operatorname{rad} P(a) \cong S(a)$$

of right A-modules. □

Lemma 1.5.24. *Let $A = kQ/\mathcal{I}$ be a bound quiver algebra. For every A-module M and $a \in Q_0$, the k-linear map from Lemma 1.5.1 induces isomorphisms of k-vector spaces*

$$\mathrm{Hom}_A(P(a), M) \cong Me_a \cong D\,\mathrm{Hom}_A(M, I(a))$$

Proof. We have the first isomorphism from Lemma 1.5.1. The second isomorphism is the composition

$$\begin{aligned} D\,\mathrm{Hom}_A(M, I(a)) &= D\,\mathrm{Hom}_A(M, D(Ae_a)) \cong D\,\mathrm{Hom}_{A^{op}}(Ae_a, DM) \\ &\cong D(e_a DM) \cong D(DM)e_a \cong Me_a \end{aligned}$$

□

As a consequence, we obtain an expression of the quiver of A in terms of the extensions between simple modules that will be useful later.

Theorem 1.5.25. *Let $A = kQ/\mathcal{I}$ be a bound quiver algebra and let $a, b \in Q_0$.*

i) There exists an isomorphism of k-vector spaces

$$\operatorname{Ext}^1_A(S(a), S(b)) \cong e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$$

ii) The number of arrows in Q from a to b is equal to the dimension $\dim_k \operatorname{Ext}^1_A(S(a), S(b))$ of $\operatorname{Ext}^1_A(S(a), S(b))$.

Proof. i) Let $\cdots \to P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} S \to 0$ be a minimal projective resolution of a simple module S. We wish to compute $\operatorname{Ext}^1_A(S, S')$ for some other simple module S. Using the definition of $\operatorname{Ext}^1_A(-, S')$ as a right derived functor, we consider the complex $\cdots \to P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \to 0$ to which we apply the functor $\operatorname{Hom}_A(-, S')$, thus obtaining the complex

$$\begin{aligned} 0 \to \operatorname{Hom}_A(P_0, S') &\xrightarrow{\operatorname{Hom}_A(p_1,S')} \operatorname{Hom}_A(P_1, S') \xrightarrow{\operatorname{Hom}_A(p_2,S')} \\ \operatorname{Hom}_A(P_2, S') &\xrightarrow{\operatorname{Hom}_A(p_3,S')} \operatorname{Hom}_A(P_3, S') \xrightarrow{\operatorname{Hom}_A(p_4,S')} \cdots \end{aligned}$$

We claim that the maps $\operatorname{Hom}_A(p_{i+1}, S') = 0$ for every $i \geq 0$. Let $f \in \operatorname{Hom}_A(P_i, S')$ be a nonzero homomorphism. By the definition of minimal projective resolution, we have that $\operatorname{Im} p_{i+1} = \operatorname{Ker} p_i \subseteq \operatorname{rad} P_i$, thus

$$\operatorname{Hom}_A(p_{i+1}, S')(f)(x) = (fp_{i+1})(x) \in f(\operatorname{Im} p_{i+1}) \subseteq f(\operatorname{rad} P_i) \subseteq \operatorname{rad} S' = 0$$

for any $x \in P_i$. Therefore $\operatorname{Hom}_A(p_{i+1}, S')(f) = 0$ and our claim follows. In particular, we get $\operatorname{Ext}^1_A(S, S') \cong \operatorname{Ker} \operatorname{Hom}_A(p_2, S')/\operatorname{Im} \operatorname{Hom}_A(p_1, S') \cong \operatorname{Hom}_A(P_1, S')$.

If $S = S(a)$ and we write $\operatorname{rad} P(a)/\operatorname{rad}^2 P(a) = \bigoplus_{c \in Q_0} S(c)^{n_c}$, a minimal projective resolution of $S(a)$ is of the form

$$\cdots \to \bigoplus_{c \in Q_0} P(c)^{n_c} \to P(a) \to S(a) \to 0$$

so that

$$\begin{aligned} \operatorname{Ext}^1_A(S(a), S(b)) &\cong \operatorname{Hom}_A(P_1, S(b)) \\ &\cong \operatorname{Hom}_A(\bigoplus_{c \in Q_0} P(c)^{n_c}, S(b)) \\ &\cong \operatorname{Hom}_A(\operatorname{rad} P(a)/\operatorname{rad}^2 P(a), S(b)) \end{aligned}$$

At this point we need to notice that we have an isomorphism

$$\operatorname{Hom}_A(\operatorname{rad} P(a)/\operatorname{rad}^2 P(a), S(b)) \cong \operatorname{Hom}_A(\operatorname{rad} P(a)/\operatorname{rad}^2 P(a), I(b))$$

This is true since $\operatorname{rad} P(a)/\operatorname{rad}^2 P(a)$ is semisimple, so its image in $I(b)$ must be semisimple. But since $I(b)$, by being the injective envelope of $S(b)$, is an essential extension of $S(b)$ the image of any morphism $\operatorname{rad} P(a)/\operatorname{rad}^2 P(a) \to I(b)$ must be contained in $S(b)$. Thus we have the following chain of isomorphisms.

$$\begin{aligned}
\operatorname{Ext}^1_A(S(a), S(b)) &\cong \operatorname{Hom}_A(\operatorname{rad} P(a)/\operatorname{rad}^2 P(a), S(b)) \\
&\cong \operatorname{Hom}_A(\operatorname{rad} P(a)/\operatorname{rad}^2 P(a), I(b)) \\
&\cong D\operatorname{Hom}_A(P(b), \operatorname{rad} P(a)/\operatorname{rad}^2 P(a)) \\
&\cong D\operatorname{Hom}_A(e_bA, e_a(\operatorname{rad} A/\operatorname{rad}^2 A)) \\
&\cong D(e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b) \\
&\cong e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b
\end{aligned}$$

ii) By definition, the number of arrows from a to b in the quiver Q is equal to $\dim_k(e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b)$. Then ii) follows from i). □

1.6 Yoneda extensions

The goal of this section is to show how to interpret the groups $\operatorname{Ext}_{\mathcal{A}}(X, Y)$ without having to go through projective (or injective) resolutions. In particular we will not need necessarily $\mathcal{A}$ to have enough projectives (or injectives). The main reference for this section will be [11] and [10].

Definition 1.6.1. *Let $\mathcal{A}$ be an abelian category and $A, B \in \operatorname{Ob}\mathcal{A}$. A* degree i Yoneda extension *of B by A is an exact sequence*

$$E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \cdots \to Z_0 \to B \to 0$$

in $\mathcal{A}$. We say that two Yoneda extensions E and E' of the same degreee are equivalent *if there exists a commutative diagram*

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & A & \longrightarrow & Z_{i-1} & \longrightarrow & \cdots & \longrightarrow & Z_0 & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle id} & & \uparrow & & & & \uparrow & & \uparrow{\scriptstyle id} & & \\
0 & \longrightarrow & A & \longrightarrow & Z''_{i-1} & \longrightarrow & \cdots & \longrightarrow & Z''_0 & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle id} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle id} & & \\
0 & \longrightarrow & A & \longrightarrow & Z'_{i-1} & \longrightarrow & \cdots & \longrightarrow & Z'_0 & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

where the middle row is a Yoneda extension as well.

It can be shown that the relation of the definition is an equivalence relation.

Definition 1.6.2. *Let $\mathcal{A}$ be an abelian category and let $i \in \mathbf{Z}$. For any two objects A, B of $\mathcal{A}$ we define the ith* extension group *of B by A to be*

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) = \{E : E \textit{ is a degree } i \textit{ Yoneda extension of } B \textit{ by } A\}/\sim$$

where $\sim$ denotes the equivalence relation between extensions defined above.

Notice that the word group in the definition is not casual. Indeed we can put a group structure on $\mathrm{Ext}^i_{\mathcal{A}}(A, B)$, with the so-called Baer sum.

Let $\mathcal{A}$ be an abelian category with objects A, B. Given a degree i Yoneda extension E of B by A we can naturally define a morphism in $\mathrm{Hom}_{D(\mathcal{A})}(B[0], A[i])$. We will denote this morphism by $\delta(E) = fs^{-1} : B[0] \to A[i]$, where s is the quasiisomorphism

$$(\cdots \to 0 \to A \to Z_{i-1} \to \cdots \to Z_0 \to 0 \to \cdots) \to B[0]$$

induced by the long exact sequence E and f is the morphism of complexes

$$(\cdots \to 0 \to A \to Z_{i-1} \to \cdots \to Z_0 \to 0 \to \cdots) \to A[i]$$

given by the identity in degree $-i$ and by the zero morphism otherwise. We will call $\delta(E) = fs^{-1}$ the *class* of the Yoneda extension. It turns out that this class characterizes the equivalence class of the Yoneda extension.

Lemma 1.6.1. *Let $\mathcal{A}$ be an abelian category with objects A, B. Any element in $\mathrm{Hom}_{D(\mathcal{A})}(B[0], A[i])$ is of the form $\delta(E)$ for some degree i Yoneda extension E of B by A. Furthermore if E and E' are two Yoneda extensions of the same degreee, then E is equivalent to E' if and only if $\delta(E) = \delta(E')$.*

Proof. Let $\xi : B[0] \to A[i]$ be an element of $\mathrm{Hom}_{D(\mathcal{A})}(B[0], A[i])$. We may write $\xi = f^\bullet s^{-1}$ for some quasiisomorphism $s : L^\bullet \to B[0]$ and a map $f^\bullet : L^\bullet \to A[i]$. By replacing $L^\bullet$ by its truncation $\tau_{\leq 0} L^\bullet$ we may assume that $L^j = 0$ for $j > 0$. We have the following situation

$$\begin{array}{ccccccccc} L^{-i-1} & \longrightarrow & L^{-i} & \longrightarrow & \cdots & \longrightarrow & L^0 & \longrightarrow B \longrightarrow & 0 \\ & & \big\downarrow f & & & & & & \\ & & A & & & & & & \end{array}$$

Then, by setting Z_{i-1} to be the pushout of the diagram $A \xleftarrow{f} L^{-i} \xrightarrow{d^{-i}} L^{-i+1}$ and $Z_j = L^{-j}$ for $j = i-2, \ldots, 0$, we obtain a degree i extension E of B by A. To show this, we want to prove that the sequence $0 \to A \to Z_{i-1} \to \cdots \to Z_0 \to B \to 0$ is exact. This would mean that this is indeed a Yoneda extension and that the complex $\cdots \to A \to Z_{i-1} \to \cdots \to Z_0 \to 0 \to \cdots$ is quasiisomorphic to the complex $L^\bullet$, so that the class $\delta(E)$ equals to ξ. We are going to show this just for modules.

First of all, we know that we have a commutative diagram

$$\begin{array}{ccc} L^{-i-1} & \xrightarrow{d^{-i-1}} & L^{-i} \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & A \end{array}$$

This shows that $fd^{-i-1} = 0$, so that f factors through $\operatorname{Coker} d^{-i-1} \cong L^{-i}/\operatorname{Im} d^{-i-1} \cong L^{-i}/\operatorname{Ker} d^{-i} \cong \operatorname{Im} d^{-i}2$, where the second isomorphism comes from the fact that $H^{-i}(L^\bullet) = 0$, because $L^\bullet$ is quasiisomorphic to $B[0]$. In particular we have that the complex $\widetilde{L}^\bullet$ given by $\cdots \to 0 \to \operatorname{Im} d^{-i} \to L^{-i+1} \to \cdots \to L^0 \to 0 \to \cdots$ is still quasiisomorphic to $B[0]$ and that $f^\bullet = \widetilde{f}^\bullet \circ \pi$, where π is the quasiisomorphism $L^\bullet \to \widetilde{L}^\bullet$. So in particular we have that our morphism $\xi : B \to A[i]$ can be represented by the roof $B \overset{s}{\leftarrow} \widetilde{L}^\bullet \xrightarrow{\widetilde{f}^\bullet} A[i]$.

Now consider the following diagram, where the top row is exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im} d^{-i} & \overset{\iota}{\hookrightarrow} & L^{-i+1} & \xrightarrow{d^{-i+1}} & L^{-i+2} & \xrightarrow{d^{-i+2}} & \cdots \\ & & \downarrow f & \ulcorner & \downarrow \alpha & & \| & & \\ 0 & \longrightarrow & A & \xrightarrow{\beta} & Z^{-i+1} & \xrightarrow{\gamma} & L^{-i+2} & \xrightarrow{d^{-i+2}} & \cdots \end{array}$$

In particular, α and β are the maps given by the pushout and γ is the unique map given by the universal property of pushouts such that $\gamma\beta = 0$ and $\gamma\alpha = d^{-i+1}$. Furthermore, since $d^{-i+2}\gamma\beta = 0 = d^{-i+2}\gamma\alpha$, the universal property of the pushout guarantees that $d^{-i+2}\gamma = 0$, so in particular the bottom row is also a complex.

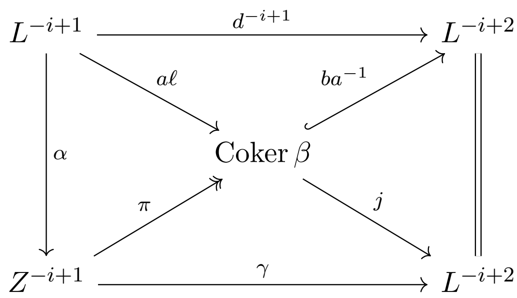
By the properties of pushouts we know that we have an epimorphism $\operatorname{Ker} \iota = 0 \twoheadrightarrow \operatorname{Ker} \beta$, so we have that β is a monomorphism and that the sequence E is exact at A. Also E is clearly exact at L^{-i+t} for any $3 \leq t \leq i$ since the top row is exact by hypothesis. Now it remains to see that $\operatorname{Ker} d^{-i+2} \subseteq \operatorname{Im} \gamma$ and $\operatorname{Ker} \gamma \subseteq \operatorname{Im} \beta$. For this, consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Im} d^{-i} & \overset{\iota}{\hookrightarrow} & L^{-i+1} & \xrightarrow{\ell} & \operatorname{Coker} \iota & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow \alpha & & \cong \downarrow a & & \\ 0 & \longrightarrow & A & \xrightarrow{\beta} & Z^{-i+1} & \xrightarrow{\pi} & \operatorname{Coker} \beta & \longrightarrow & 0 \end{array}$$

where we know that a is an isomorphism since the left square is a pushout square and thus it preserves cokernels. By the exactness of the top (long) sequence we have

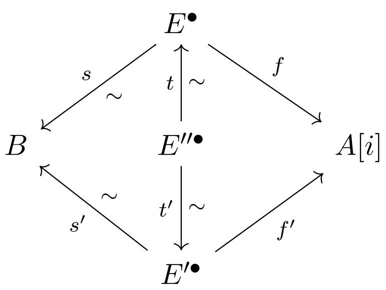
$$\operatorname{Coker} \iota \cong L^{-i+1}/\operatorname{Im} \iota \cong L^{-i+1}/\operatorname{Im} d^{-i} = L^{-i+1}/\operatorname{Ker} d^{-i+1} \cong \operatorname{Im} d^{-i+1} = \operatorname{Ker} d^{-i+2}$$

In particular this tells us that we can identify $\operatorname{Ker} d^{-i+2}$ with $\operatorname{Coker} \beta$. Notice also that $\gamma\beta = 0$, so γ factors through $\operatorname{Coker} \beta$, so we have the following diagram



We want to show that $j = ba^{-1}$, so that we can deduce that $\operatorname{Ker} d^{-i+2} \cong \operatorname{Coker}\beta \subseteq \operatorname{Im}\gamma$. For this we first need to notice that $\ell = a^{-1}\pi\alpha$. Then we have that $\gamma\alpha = d^{-i+1} = b\ell = ba^{-1}\pi\alpha$. Since we also have that $ba^{-1}\pi\beta = 0$ because $\pi\beta = 0$, we can conclude that $\gamma = ba^{-1}\pi$ by the uniqueness of the morphism given by the universal property. But now, since $\gamma = j\pi$ and π is an epimorphism, we can conclude that $j = ba^{-1}$, as wanted. Finally, we observe that if $x \in \operatorname{Ker}\gamma$, then $j\pi(x) = 0$. Since j is a monomorphism by the previous considerations this means that $\pi(x) = 0$, which in turn implies that $x \in \operatorname{Im}\beta$.

Now let E and E' be equivalent degree i Yoneda extensions of B by A with the same class. We have a diagram as in Definition 1.6.1 with an extension E'' in the middle. If we denote $E^\bullet = (\cdots \to 0 \to A \to Z_{i-1} \to \cdots \to Z_0 \to 0 \to \cdots)$, $E'^\bullet = (\cdots \to 0 \to A \to Z'_{i-1} \to \cdots \to Z'_0 \to 0 \to \cdots)$ and $E''^\bullet = (\cdots \to 0 \to A \to Z''_{i-1} \to \cdots \to Z''_0 \to 0 \to \cdots)$, then we have the following roof equivalence between $\delta(E) = fs^{-1}$ and $\delta(E') = f's'^{-1}$



where t and t' are the quasiisomorphisms from the equivalence diagram. Suppose instead that $E : 0 \to A \to Z_{i-1} \to Z_{i-2} \to \cdots \to Z_0 \to B \to 0$ and $E' : 0 \to A \to Z'_{i-1} \to Z'_{i-2} \to \cdots \to Z'_0 \to B \to 0$ are Yoneda extensions with the same class $\delta(E) = fs^{-1} = \delta(E') = f's'^{-1}$. Since $D(\mathcal{A})$ is the localization of $K(\mathcal{A})$ at the set of all quasiisomorphisms, this means that there exists a complex $L^\bullet$ and quasiisomorphisms

$$t : L^\bullet \to (\cdots \to 0 \to A \to Z_{i-1} \to \cdots \to Z_0 \to 0 \to \cdots)$$

and

$$t' : L^\bullet \to (\cdots \to 0 \to A \to Z'_{i-1} \to \cdots \to Z'_0 \to 0 \to \cdots)$$

such that $s \circ t = s' \circ t'$ and $f \circ t = f' \circ t'$. Let E'' be the degree i extension of B by A constructed from the pair $L^\bullet \to B[0]$ and $L^\bullet \to A[i]$ at the beginning of the proof. Then we clearly have a commutative diagram as the one in Definition 1.6.1, thus E is equivalent to E'. □

Remark. An important consequence of this Lemma is that for any abelian category $\mathcal{A}$ with objects A, B and for any $i \in \mathbf{Z}$ we have

$$\mathrm{Ext}^i_{\mathcal{A}}(A, B) = \mathrm{Hom}_{D(\mathcal{A})}(A, B[i]) = \mathrm{Hom}_{D(\mathcal{A})}(A[-i], B)$$

Chapter 2

Main result

The main goal of this chapter is to prove that a finite dimensional algebra Λ over a field k is piecewise hereditary if and only if its strong global dimension is finite. The result can be originally found in [6], together with most of the results in this Chapter. Another reference we use throughout this chapter is [3].

2.1 Setting

In this section we present some definitions and some results that will be central to prove our thesis.

Definition 2.1.1. *Let $\mathcal{H}$ be an abelian category. $\mathcal{H}$ is said to be a* hereditary category *if the functor $\mathrm{Ext}^2_{\mathcal{H}}(-,-)$ vanishes over $\mathcal{H}$, i.e. if $\mathrm{Ext}^2_{\mathcal{H}}(X,Y)=0$ for all objects X and Y in $\mathcal{H}$.*

An example of abelian, hereditary categories is given by the categories $\mathrm{mod}\, kQ$, where Q is a finite, acyclic quiver. A proof of this can be found in [9].

We now give a characterization of hereditary categories.

Proposition 2.1.1. *Let $\mathcal{A}$ be an abelian category. The following are equivalent:*

i) $\mathcal{A}$ is hereditary;

ii) $\mathrm{Ext}^n_{\mathcal{A}}(M,N)=0$ for all $n\geq 2$ and for all objects M, N of $\mathcal{A}$;

Proof. $i) \Rightarrow ii)$. Since $\mathcal{A}$ is hereditary we know that $\mathrm{Ext}^2_{\mathcal{A}}(-,-)$ vanishes. For $i>2$ write any class $\xi\in\mathrm{Ext}^i_{\mathcal{A}}(M,N)$ as $\delta(E)$, where E is a Yoneda extension

$$E: 0\to N\to Z_{i-1}\to Z_{i-2}\to\cdots\to Z_0\to M\to 0$$

This is possible as we already saw that for an element in $\mathrm{Ext}^i_{\mathcal{A}}(M, N)$ we have a unique degree i Yoneda extension of M by N, up to the equivalence relation discussed in 1.6.1. Now set $C = \mathrm{Ker}(Z_1 \to Z_0) = \mathrm{Im}(Z_2 \to Z_1)$. This means we can write $\delta(E)$ as the composition of $\delta(E')$ with $\delta(E'')$ where

$$E' : 0 \to C \to Z_1 \to Z_0 \to M \to 0$$

and

$$E'' := 0 \to N \to Z_{i-1} \to \cdots \to Z_2 \to C \to 0$$

Since $\delta(E') \in \mathrm{Ext}^2_{\mathcal{A}}(M, N) = 0$, then also $\delta(E) = 0$ and we can conclude.

On the other hand, ii) clearly implies i) so we are done. □

Now we can proceed by giving a useful description of the bounded derived category $D^b(\mathcal{H})$ of an abelian, hereditary category $\mathcal{H}$.

Lemma 2.1.2. *Let $\mathcal{H}$ be an abelian, hereditary category. Then any object K of its derived category $D^b(\mathcal{H})$ is isomorphic to the direct sum of its cohomologies in $D^b(\mathcal{H})$, in other words we have that:*

$$K \cong \bigoplus_{i \in \mathbf{Z}} H^i(K)[-i]$$

Proof. Before starting with the actual proof we need to notice that for $p \geq 2$ we have $\mathrm{Ext}^p_{\mathcal{H}}(H^i(K), H^j(K)) = 0$ for all $i, j \in \mathbf{Z}$ since $H^i(K)$ is an object in $\mathcal{H}$.

Now pick a, b such that $H^i(K) = 0$ for $i \notin [a, b]$. This is always possible since K is a bounded complex. We will proceed on induction over $b - a$. If $b - a = 0$ we have that K is quasi-isomorphic to 0, so $K \cong \bigoplus H^i(K)[-i]$. If $b - a > 0$ then we look at the distinguished triangle of truncations

$$\tau_{\leq b-1}K \to K \to H^b(K)[-b] \to (\tau_{\leq b-1}K)[1]$$

By Corollary 1.2.9, if the last arrow is zero, then $K \cong \tau_{\leq b-1}K \oplus H^b(K)[-b]$ and we conclude due to the inductive hypothesis. Again by induction we have that $\tau_{\leq b-1}K \cong \bigoplus H^i(\tau_{\leq b-1}K)[-i]$, which yields

$$\begin{aligned}
&\mathrm{Hom}_{D^b(\mathcal{H})}(H^b(K)[-b], (\tau_{\leq b-1}K)[1]) \cong \\
&\cong \bigoplus \mathrm{Hom}_{D^b(\mathcal{H})}(H^b(K)[-b], H^i(\tau_{\leq b-1}K)[1-i]) = \\
&= \bigoplus_{i<b} \mathrm{Ext}^{b-i+1}_{\mathcal{H}}(H^b(K), H^i(K))
\end{aligned}$$

Notice that the first isomorphism holds because $\tau_{\leq b-1}K \in D^b(\mathcal{H})$ and thus the direct sum is finite. By assumption the last direct sum is zero and our proof is complete. □

We are also going to need the following characterization of the derived category of an abelian, hereditary category $\mathcal{H}$, whose proof can be found in [3].

Theorem 2.1.3. *Let $\mathcal{D}$ be a triangulated category with a full additive subcategory $\mathcal{H}$. Then the following statements are equivalent:*

i) The category $\mathcal{H}$ is hereditary abelian with an equivalence $F : D^b(\mathcal{H}) \to \mathcal{D}$ of categories, which commutes with the translation functors and respects the canonical embedding of $\mathcal{H}$ into $D^b(\mathcal{H})$.

ii) $\mathcal{D}$ is equal to the smallest additive category which contains $\bigcup_{n\in\mathbf{Z}} \mathcal{H}[n]$ and is closed under isomorphisms (in which case we will write $\mathcal{D} = \mathrm{add}(\bigcup_{n\in\mathbf{Z}} \mathcal{A}[n])$) and $\mathrm{Hom}_{\mathcal{D}}(\mathcal{H}, \mathcal{H}[m]) = 0$ for $m < 0$.

Definition 2.1.2. *Let Λ be a finite dimensional algebra over a field k. Λ is called* piecewise hereditary *if there exists a hereditary, abelian category $\mathcal{H}$ such that the bounded derived categories $D^b(\mathrm{mod}\,\Lambda)$ and $D^b(\mathcal{H})$ are equivalent as triangulated categories.*

Note that we will often write $D^b(\Lambda)$ instead of $D^b(\mathrm{mod}\,\Lambda)$.

As one would expect, hereditary algebras are also piecewise hereditary. Indeed if H is a hereditary algebra, the category $\mathrm{mod}\,H$ can be easily shown to be hereditary. We will give a more precise proof of this in the last section of this thesis.

Example 2.1.4. We are interested in showing that we have algebras that are piecewise hereditary, but not hereditary. An example is the algebra $\Lambda = kQ/\mathcal{I}$, where Q is the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and $\mathcal{I} = \langle \alpha\beta \rangle$. It is well known (see for example [1]), that there is an equivalence of categories $D^b(\Lambda) \cong D^b(kQ)$ and we know that kQ is hereditary, since it is the path algebra of a finite, acyclic quiver. However, we also know that Λ is not hereditary itself, for example we know that $\mathrm{Ext}^2_\Lambda(S(1), S(3)) \neq 0$, so in particular Λ is piecewise hereditary, but not hereditary, as wanted. We will discuss and generalise this case in the last section of this thesis.

Definition 2.1.3. *Let $\mathcal{A}$ be an abelian category and $X \in C^b(\mathcal{A})$ be a bounded complex. By the boundness of X, there exist $r \leq s$ such that $X^r \neq 0 \neq X^s$ and $X^i = 0$ for $i < r$ or $i > s$. We define the* length *of X as $\ell(X) = s - r$. Notice that this definition extends to $K^b(\mathcal{A})$, since we have the convention that given $X \in K^b(\mathcal{A})$, we represent it with its unique (up to isomorphism) preimage in $C^b(\mathcal{A})$ without null-homotopic direct summands.*

Definition 2.1.4. *Let $\mathcal{A}$ be an abelian category. We define the* global dimension *of $\mathcal{A}$ as*

$$\operatorname{gl.dim}\mathcal{A} = \sup\{n \in \mathbf{N} \cup \{\infty\} : \exists \text{ objects } X, Y \text{ in } \mathcal{A} \text{ s.t. } \operatorname{Ext}^n_{\mathcal{A}}(X,Y) \neq 0\}$$

Morover, if we denote by $\mathcal{P}$ the full subcategory of $\mathcal{A}$ containing its projective objects, we can define the strong global dimension *of $\mathcal{A}$ as*

$$\operatorname{s.gl.dim}\mathcal{A} = \sup\left\{\ell(P^\bullet) : P^\bullet \text{ is indecomposable in } K^b(\mathcal{P})\right\}$$

where $\ell(P^\bullet)$ is the length of the complex $P^\bullet$ and $P^\bullet$ is said to be indecomposable if there are no nontrivial complexes $P_1^\bullet, P_2^\bullet$ such that $P^\bullet \cong P_1^\bullet \oplus P_2^\bullet$.

In the rest of this thesis we will just talk about the global dimension and the strong global dimension of categories of modules over a finite dimensional algebra Λ. Within this setup we will write ${}_\Lambda\mathcal{P}$ instead of just $\mathcal{P}$ as in the last definition.

We also recall an equivalent definition of global dimension for finite dimensional algebras, which will be useful later.

Theorem 2.1.5. *If Λ is a finite dimensional k-algebra, then*

$$\operatorname{gl.dim}\Lambda = \max\{\operatorname{pd} S : S \text{ is a simple right } \Lambda\text{-module}\}$$

Proof. Clearly "$\geq$" holds. For "$\leq$" we actually just need to show that for any module M we have $\operatorname{pd} M \leq \max\{\operatorname{pd} S : S \text{ is a simple right } \Lambda\text{-module}\} = n$. To do this we are going to proceed by induction on the length i of a composition series of M. Notice that this length is finite, since M is a finitely generated and Λ is a finite dimensional algebra.

If $i = 1$, then M is simple, so the previous inequality clearly holds. Assume now that our inequality holds for i. Let M be of length $i+1$ and let $M_i \subsetneq M_{i+1} = M$ be the penultimate term of the composition series, so that M/M_i is simple. In particular we have a short exact sequence $0 \to M_i \to M \to M/M_i \to 0$. The Horseshoe Lemma guarantees that the direct sum of the projective resolutions of M_i and M/M_i yields a projective resolution of M, which in particular is not longer than the previous ones. Thus, since by inductive hypothesis $\operatorname{pd} M_i \leq n$ and clearly $\operatorname{pd} M/M_i \leq n$, we also have that $\operatorname{pd} M \leq n$, as wanted. □

Definition 2.1.5. *A category $\mathcal{C}$ is said to be* Krull-Schmidt *if any object in $\mathcal{C}$ is isomorphic to a sum of indecomposable objects in $\mathcal{C}$ in a unique way up to isomorphism.*

Remark. All of the categories that we are interested in from now on are Krull-Schmidt, in particular the categories $\operatorname{mod}\Lambda$, $K^b({}_\Lambda\mathcal{P})$, $K^b(\Lambda)$ and $D^b(\Lambda)$ for a finite dimensional k-algebra Λ. We already showed that $\operatorname{mod}\Lambda$ is Krull-Schmidt in Theorem 1.5.6. For the other categories mentioned above, we refer to [12].

Lemma 2.1.6. *Let $\mathcal{A}$ be an additive Krull-Schmidt category. Let $X \in \mathcal{A}$ and $Y^\bullet \in K^b(\mathcal{A})$.*

a) *i) If $\mathrm{Hom}_{K^b(\mathcal{A})}(X[t], Y^\bullet) \neq 0$ for some $t \in \mathbf{Z}$, then $Y^{-t} \neq 0$;*
ii) If $\mathrm{Hom}_{K^b(\mathcal{A})}(Y^\bullet, X[t]) \neq 0$ for some $t \in \mathbf{Z}$, then $Y^{-t} \neq 0$.

b) If $0 \neq Y^\bullet = (Y^i, d^i) \in K^b(\mathcal{A})$ with $Y^r \neq 0 \neq Y^0$ for some $r \leq 0$ and $Y^i = 0$ for $i < r$ and for $i > 0$ then

i) $\mathrm{Hom}_{K^b(\mathcal{A})}(Y^\bullet, Y^r[-r]) \neq 0$ *and*
ii) $\mathrm{Hom}_{K^b(\mathcal{A})}(Y^0, Y^\bullet) \neq 0$.

Proof. Clearly both claims in *a*) hold. Indeed if we assume by contradiction that $Y^t = 0$ we would have that $\mathrm{Hom}_{C^b(\mathcal{A})}(X[t], Y^\bullet) = 0$ (resp. $\mathrm{Hom}_{C^b(\mathcal{A})}(Y^\bullet, X[t]) = 0$) since $X[t]^i = 0$ for $i \neq t$ and the only morphism in degree t would be the 0 morphism. Since for any two complexes B, D of $\mathcal{A}$ objects we have an epimorphism $\mathrm{Hom}_{C^b(\mathcal{A})}(B, D) \twoheadrightarrow \mathrm{Hom}_{K^b(\mathcal{A})}(B, D)$, this yields the desired contradiction, since $\mathrm{Hom}_{K^b(\mathcal{A})}(X[t], Y^\bullet) \neq 0$ (resp. $\mathrm{Hom}_{K^b(\mathcal{A})}(Y^\bullet, X[t]) \neq 0$) by hypothesis.

We will now show the first assertion in *b*). The second follows by dual considerations. Let $\pi^\bullet : Y^\bullet \to Y^r[-r]$ be defined by $\pi^r = id_{Y^r}$ and $\pi^j = 0$ for $j \neq r$. $\pi^\bullet$ is clearly a map of complexes. Now if $\mathrm{Hom}_{K^b(\mathcal{A})}(Y^\bullet, Y^r[-r]) = 0$, then $\pi^\bullet$ needs to be homotopic to 0. This would mean in particular that there is a map $h : Y^{r+1} \to Y^r$ with $hd^r = id_{Y^r}$. So $Y^\bullet$ has indecomposable null-homotopic direct summands in $C^b(\mathcal{A})$, in contrast to the convention agreed upon when we defined $K^b(\Lambda)$. Indeed since $hd^r = id_{Y^r}$, we have that d^r needs to be a monomorphism and we would have a split exact sequence $0 \to Y^r \to Y^{r+1} \to \mathrm{Coker}(d^r) \to 0$. This in particular means that $Y^{r+1} \cong Y^r \oplus \mathrm{Coker}(h)$, so we could write $Y^\bullet$ as the direct sum of the two complexes

$$\cdots \to 0 \to 0 \to \mathrm{Coker}(d^r) \to Y^{r+2} \to \ldots$$

and

$$\cdots \to 0 \to Y^r \xrightarrow{\cong} Y^r \to 0 \to \ldots$$

which is null-homotopic. □

In the rest of this section we will denote by $\mathcal{C}$ an additive triangulated k-category that is Krull-Schmidt and has the property that for all $X, Y \in \mathcal{C}$ the dimension of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is finite. For example, the category $K^b({}_\Lambda\mathcal{P})$, where Λ is a finite dimensional algebra, satisfies these assumptions. Moreover, when we will have to deal with maps $u : X \to E$ and $v : E \to Y$ with $E = \bigoplus_{i=1}^r E_i$ for some indecomposable E_i, we will use the following notation. First we call $\varepsilon_i : E_i \to E$ the canonical split monomorphisms and $\pi_i : E \to E_i$ the canonical split epimorphisms. Then for each $1 \leq i \leq r$ we define the map $u_i : X \to E_i$ by $u_i = \pi_i u$ and the map $v_i : E_i \to Y$ by $v_i = v\varepsilon_i$.

The next lemma collects some results about maps occuring in triangles.

Lemma 2.1.7. *i) The following are equivalent for a triangle*

$$X \xrightarrow{f} Y \xrightarrow{u} Z \xrightarrow{v} X[1]$$

a) f is a split monomorphism;

b) u is a split epimorphism;

c) $v = 0$.

ii) Let $f : X \to Y$ be nonzero and not invertible for X, Y indecomposable, and let

$$X \xrightarrow{f} Y \xrightarrow{u} \bigoplus_{i=1}^{r} Z_i \xrightarrow{v} X[1]$$

be a triangle with each Z_i indecomposable. Then the components u_i of u, as well as the components v_i of v, are nonzero and not invertible.

iii) If $r > 1$, in the situation of ii), we have that $v_i u_i$ is nonzero for each $i = 1, \dots, r$.

Proof. i) We will just show $a) \iff c)$. The proof for $b) \iff c)$ is similar.

Let f be a split monomorphism. Then there is $g : Y \to X$ such that $gf = id_X$. By applying $\mathrm{Hom}_{\mathcal{C}}(-, X)$ we get the following long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{C}}(Y, X) \xrightarrow{f^*} \mathrm{Hom}_{\mathcal{C}}(X, X) \xrightarrow{v[-1]^*} \mathrm{Hom}_{\mathcal{C}}(Z[-1], X) \to \cdots$$

Hence we see that $f^*(g) \in \mathrm{Ker}\, v[-1]^*$ by the exactness of the sequence. But $f^*(g) = gf = id_X$, thus we have $v[-1]^*(id_X) = v[-1] = 0$. Thus $v = 0$ since this is true whenever $v[i] = 0$ for some $i \in \mathbf{Z}$.

Conversely, if $v = 0$, from the same sequence from above, we see that f^* is surjective, thus there is $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$ such that $f^*(g) = gf = id_X$. Thus f is a split monomorphism, as wanted.

ii) Since f is not invertible, its cone is nonzero. Also we have $u = (u_1, \dots, u_r)$. Since $0 = u \circ f = (u_1 \circ f, \dots, u_r \circ f)$ and f is nonzero, it follows that the components u_i of u are not invertible. Similarly $v = (v_1, \dots, v_r)$ and $f[1] \circ v = 0$, thus the components v_i of v are not invertible. Furthermore, since f is nonzero, Lemma 1.2.10 implies that the components u_i of u are nonzero. By a dual argument we have that the components v_i of v are nonzero as well.

iii) Let us assume that there is $1 \leq i \leq r$ such that $v_i u_i$. So $0 = v_i u_i = v \varepsilon_i \pi_i u$. In particular we have the following situation

$$\begin{array}{ccccccc}
C_{u_i}[-1] & \longrightarrow & Y & \xrightarrow{\pi_i u} & Z_i & \longrightarrow & C_{u_i} \\
 & & \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varepsilon_i} & & \\
X & \xrightarrow{f} & Y & \xrightarrow{u} & \bigoplus_{i=1}^{r} Z_i & \xrightarrow{v} & X[1]
\end{array}$$

where the rows are triangles. Thus since $v\varepsilon_i\pi_i u = 0$, by Proposition 1.2.6 there is $\varphi_i : Y \to Y$ with $u\varphi_i = \varepsilon_i\pi_i u$. In particular we have that $u_j\varphi_i = \pi_j u\varphi_i = 0$ for $j \neq i$ (notice that since $r > 1$ there is such j) and $u_i\varphi_i = u_i$. By ii) we have that $u_j \neq 0$, so φ_i is neither an isomorphism (since $u_j\varphi_i = 0$ and $u_j \neq 0$) nor the zero morphisms (since $u_i\varphi_i = u_i$). Now Y is indecomposable, so we know that $\mathrm{End}_{\mathcal{C}}(Y)$ is local and finite dimensional, hence φ_i is nilpotent, which means that there is $n > 0$ such that $\varphi^n = 0$, in contrast to $u_i\varphi^n = u_i \neq 0$. Thus it must be $v_i u_i \neq 0$ for all i. □

Proposition 2.1.8. *Let $f : X \to Y$ be nonzero and not invertible for X, Y indecomposable, and let*

$$X \xrightarrow{f} Y \xrightarrow{u} C_f \xrightarrow{v} X[1]$$

be a triangle. If the induced map $f^* : \mathrm{Hom}_{\mathcal{C}}(Y, X[1]) \to \mathrm{Hom}_{\mathcal{C}}(X, X[1])$ is injective, then C_f is indecomposable.

Proof. Assume by contradiction that $C_f = C_1 \oplus C$ for C_1 indecomposable and $C \neq 0$. By Lemma 2.1.7 iii) we have that $v_1u_1 \neq 0$. Since f^* is injective we also have that $0 \neq f^*(v_1u_1) = v_1u_1f$. Now $u_1f = \pi_1uf = 0$ since $uf = 0$. Indeed we know that the composition of any two consecutive maps in a triangle is zero. So we obtain our desired contradiction and we can conclude that C_f must be indecomposable. □

This proposition has as an immediate consequence the following

Corollary 2.1.9. *Let $f : X \to Y$ be nonzero and not invertible for X, Y indecomposable. If $\mathrm{Hom}_{\mathcal{C}}(Y, X[1]) = 0$, then C_f is indecomposable.*

2.2 Paths in triangulated categories

The goal of this section is to prove some results concerned with shortening paths in a triangulated category $\mathcal{C}$ which we assume to be Krull-Schmidt and such that the dimension of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is finite for all $X, Y \in \mathcal{C}$. We will denote by $\mathrm{Ind}\,\mathcal{C}$ a complete class of representatives of indecomposable objects.

Following [3], a *path* in $\mathcal{C}$ is a sequence $X_0, \ldots, X_s$ of indecomposable objects in $\mathcal{C}$ such that either $\mathrm{Hom}_{\mathcal{C}}(X_i, X_{i+1}) \neq 0$ for $0 \leq i \leq s-1$ or $X_{i+1} = X_i[1]$. We will call a sequence $X_0, \ldots, X_s$ of indecomposable objects in $\mathcal{C}$ a *strong path* in $\mathcal{C}$, if $\mathrm{Hom}_{\mathcal{C}}(X_i, X_{i+1}) \neq 0$ for $0 \leq i \leq s-1$. We will say that these paths go from X_0 to X_s.

A subclass $\mathcal{U} \subseteq \mathrm{Ind}\,\mathcal{C}$ is called *path-closed* provided that for each path from X to Y, X lies in $\mathcal{U}$ if and only if so does Y. Clearly, a path-closed class $\mathcal{U}$ is closed under the translation functors [1] and [−1] and for any $X \in \mathcal{U}$, an indecomposable object Y necessarily lies in $\mathcal{U}$ whenever $\mathrm{Hom}_{\mathcal{C}}(X, Y) \neq 0$ or $\mathrm{Hom}_{\mathcal{C}}(Y, X) \neq 0$. We observe that whenever a class $\mathcal{U}$ is path-closed, so is its complement $\mathcal{V} = \mathrm{Ind}\,\mathcal{C} \setminus \mathcal{U}$.

Lemma 2.2.1. *Let $\mathcal{U} \subseteq \mathrm{Ind}\,\mathcal{C}$ be a path-closed class and let $\mathcal{V}$ be its complement. Set $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) to be the smallest additive category generated by the objects in $\mathcal{U}$ (resp. $\mathcal{V}$). Then both $\mathcal{C}_i$ are triangulated subcategories and $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ is their product.*

Proof. By their path-closedness, we have

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{C}_1, \mathcal{C}_2) = 0 = \mathrm{Hom}_{\mathcal{C}}(\mathcal{C}_2, \mathcal{C}_1) \qquad (*)$$

Then we have a decomposition $\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2$ of additive categories. This is clear for objects, since our category $\mathcal{C}$ is Krull-Schmidt by hypothesis. For morphisms instead, this decomposition makes sense exactly because of $(*)$. Indeed, if $f : X \to Y$ is a morphism in $\mathcal{C}$, it corresponds to a couple of morphisms (f_1, f_2) in $\mathrm{Mor}(\mathcal{C}_1 \times \mathcal{C}_2)$. To see this, we can write $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$ for some $X_i, Y_i \in \mathcal{C}_i$ and for $i = 1, 2$. By $(*)$ then we know that for $i = 1, 2$, X_i is mapped into Y_i by f, thus $f = (f_1 = f|_{X_1}, f_2 = f|_{X_2})$ in $\mathcal{C}_1 \times \mathcal{C}_2$.

Now recall that both $\mathcal{C}_i$ are closed under [1] and [−1]. To complete the proof, we take a morphism $u : A \to B$ in $\mathcal{C}_1$ and form a triangle in $\mathcal{C}$

$$A \xrightarrow{u} B \to C \to A[1]$$

If $C = C_1 \oplus C_2$, with $C_i \in \mathcal{C}_i$, then $C_2 = 0$ by Lemma 1.2.10. This proves that $\mathcal{C}_1$ is a triangulated subcategory. The proof for $\mathcal{C}_2$ is analogous. □

Definition 2.2.1. *A triangulated category $\mathcal{C}$ is called a* block *provided that it is nonzero and does not admit a proper decomposition into two triangulated subcategories.*

Lemma 2.2.2. *Let $\mathcal{C}$ be a triangulated category where for each indecomposable $X \in \mathcal{C}$, there exists an indecomposable object $Y \in \mathcal{C}$ such that there is a nonzero, noninvertible map $f : X \to Y$. Then each path in $\mathcal{C}$ can be refined to a strong path in $\mathcal{C}$.*

Proof. Let $X_0, \ldots, X_s$ be a path in $\mathcal{C}$. If $\mathrm{Hom}_{\mathcal{C}}(X_i, X_{i+1}) \neq 0$ for $0 \leq i \leq s-1$ we already have a strong path. Assume then that there is i such that $X_{i+1} = X_i[1]$. By hypothesis there is an indecomposable object $Y \in \mathcal{C}$ and a morphism $f : X_i \to Y$ which is nonzero and noninvertible. Consider now the following triangle:

$$X_i \xrightarrow{f} Y \xrightarrow{u} C_f \xrightarrow{v} X_i[1]$$

Then for each indecomposable summand C_j of C_f we obtain a strong path $X_i \to Y \to C_j \to X_i[1]$ and thus our original path can be refined to a strong path. Notice that Lemma 2.1.7 ii) ensures that the maps $u_j : Y \to C_j$ and $v_j : C_j \to X_i[1]$ are nonzero. □

The next lemma ensures we can use Lemma 2.2.2 within the setup we are working in.

Lemma 2.2.3. *Let Λ be a connected finite dimensional algebra which is not semi-simple. Then for each indecomposable object $X \in D^b(\Lambda)$ there exists an indecomposable object $Y \in D^b(\Lambda)$ and a nonzero, noninvertible map $f : X \to Y$.*

Proof. First we need to notice that since we are operating with the derived category $D^b(\Lambda)$ we can assume Λ to be basic. Indeed if this is not the case, by Proposition 1.5.10, we know that there is a basic algebra Λ^b associated to Λ such that $D^b(\Lambda)$ is equivalent to $D^b(\Lambda^b)$ and thus in this case we can simply use Λ^b instead of Λ.

Now let $X^\bullet \in D^b(\Lambda)$ be indecomposable. By applying the shift functor, if necessary, we may assume that $H^i(X^\bullet) = 0$ for $i > 0$, and that $H^0(X^\bullet) \neq 0$. Let $\tau_{\leq 0}X^\bullet$ be the complex defined by $(\tau_{\leq 0}X^\bullet)^i = 0$ for $i > 0$, $(\tau_{\leq 0}X^\bullet)^i = X^i$ for $i < 0$ and $(\tau_{\leq 0}X^\bullet)^0 = \operatorname{Ker} d^0$ with the induced differentials. Let $\tau_{>0}X^\bullet$ be the complex defined by $(\tau_{>0}X^\bullet)^i = X^i$ for $i > 1$, $(\tau_{>0}X^\bullet)^i = 0$ for $i \leq 0$ and $(\tau_{>0}X^\bullet)^1 = \operatorname{Coker} d^0$ with the induced differentials. Now we obtain the following triangle, where u is the morphism such that u^i is the inclusion map for all i

$$\tau_{\leq 0}X^\bullet \xrightarrow{u} X^\bullet \longrightarrow \tau_{>0}X^\bullet \longrightarrow (\tau_{\leq 0}X^\bullet)[1]$$

Since $\tau_{>0}X^\bullet$ is acyclic, the long exact sequence in cohomology shows that u is an isomorphism in $D^b(\Lambda)$, hence we may assume that $X^i = 0$ for $i > 0$ without loss of generality. This means that $H^0(X^\bullet) = \operatorname{Coker} d^{-1}$ and we obtain a map $f : X^\bullet \to \operatorname{Coker} d^{-1}[0]$ in $D^b(\Lambda)$ with f^0 the natural map $X^0 \to \operatorname{Coker} d^{-1}$. Considering cohomology we can see that that $H^0(f) \neq 0$, thus $f \neq 0$ in $D^b(\Lambda)$. In particular there exists an indecomposable direct summand Z of $\operatorname{Coker} d^{-1}$ and a nonzero map induced by f, say $f' : X^\bullet \to Z[0]$. If f' is noninvertible we are done. Thus we can reduce ourselves to the case that $X^\bullet$ is isomorphic in $D^b(\Lambda)$ to an indecomposable Λ-module Z. If Z is not simple, there exists a simple module S and a nonzero, noninvertible map $Z \to S$ (for example take for S one of the simple direct summands of $Z/radZ$ and for the map the canonical projection). So we may suppose Z to be simple. If now Z is not injective there exists an indecomposable injective I and a nonzero, noninvertible map $Z \to I$ (take for example the injective envelope with the canonical inclusion). So without loss of generality we can assume Z to be simple and injective. Now, since Λ is basic, connected and finite dimensional, by Theorem 1.5.21, we know that $\Lambda \cong kQ_\Lambda/I$, where Q_Λ is the quiver associated to Λ and I is an admissable ideal. Thus, since Z is simple, it must be of the form $Z = S(a)$ for some point a of Q_Λ. Also, since Z is injective, we have that $\mathrm{Ext}^1_\Lambda(S, Z) = 0$ for all Λ-modules S, which in particular means that there are no arrows starting from a in Q_Λ. Now assume that there are also no arrows with target a in Q_Λ, so that we have $\mathrm{Ext}^1_\Lambda(Z, S) = 0$ for all simple Λ-modules S. Since Λ is connected by hypothesis, the only possibility left is that Q_Λ is a quiver made of only one point, namely a. But in this case we would have that Λ is semi-simple, which contradicts our hypothesis. Thus there is an arrow in Q_Λ going from a to another point, say b. By Theorem 1.5.25 this implies that $\mathrm{Ext}^1_\Lambda(Z, S(b)) \neq 0$, where $S(b)$ is the simple module associated with the point b. But then we also have that $\mathrm{Hom}_{D^b(\Lambda)}(Z, S(b)[1]) \neq 0$. Clearly all maps in $\mathrm{Hom}_{D^b(\Lambda)}(Z, S(b)[1])$ are noninvertible, since by Lemma 1.4.14 $\mathrm{Hom}_{D^b(\Lambda)}(S(b)[1], Z) \cong \mathrm{Hom}_{D^b(\Lambda)}(S(b), Z[-1]) = 0$ so it would not be possible to have a nonzero inverse. □

Corollary 2.2.4. *Let Λ be a connected finite dimensional algebra which is not semi-simple. Then any path in $D^b(\Lambda)$ can be refined to a strong path in $D^b(\Lambda)$.*

Proof. This follows immediately from Lemmas 2.2.2 and 2.2.3. □

The following lemma is central to proving the main result of this section.

Lemma 2.2.5. *Let X_0, X_1, X_2, X_3 be a strong path in $\mathcal{C}$. Then there exists $0 \le t \le 1$, an indecomposable object $Y \in \mathcal{C}$ and a strong path $X_0[t] \to Y \to X_3$.*

Proof. Given X_0 and X_3 we may choose a strong path from X_0 to X_3 of minimal length. If this minimal length is less than three, there is an indecomposable object $Y \in \mathcal{C}$ and a strong path $X_0 \to Y \to X_3$ and we are done. So we may assume the minimal length for such a strong path to be three. We will show that in this case there is an indecomposable object $Y \in \mathcal{C}$ and a strong path $X_0[1] \to Y \to X_3$. To do this we choose among the strong paths of minimal length a strong path ω

$$X_0 \xrightarrow{f} X_1 \xrightarrow{g} X_2 \xrightarrow{h} X_3$$

such that $\dim_k \mathrm{Ker}\, g^*$ is minimal, where $g^* : \mathrm{Hom}_\mathcal{C}(X_2, X_1[1]) \to \mathrm{Hom}_\mathcal{C}(X_1, X_1[1])$ is the map induced by g. Notice in particular that g (as well as f and h) cannot be an isomorphism. Indeed if g were an isomorphism, we would have a nonzero morphism $gf : X_0 \to X_2$ and thus a path of length two, which would contradict the minimality of ω's length. Similar arguments show that we can suppose that also f and h are not isomorphisms. In particular "ω"'s minimality implies that $\mathrm{Hom}_{\mathcal{C}}(X_0, X_2) = 0 = \mathrm{Hom}_{\mathcal{C}}(X_1, X_3)$, since if any of them were nonzero we would already be able to have a strong path $X_0 \to X_2 \to X_3$ (or $X_0 \to X_1 \to X_3$) of length two.

Consider now the following triangle in $\mathcal{C}$

$$C_g[-1] \longrightarrow X_1 \xrightarrow{\ g\ } X_2 \longrightarrow C_g$$

obtained by rotating the distinguished triangle induced by g. Applying $\mathrm{Hom}_{\mathcal{C}}(X_0, -)$ and $\mathrm{Hom}_{\mathcal{C}}(-, X_3)$ to the triangle above we obtain the two following exact sequences

$$\mathrm{Hom}_{\mathcal{C}}(X_0, C_g[-1]) \to \mathrm{Hom}_{\mathcal{C}}(X_0, X_1) \to \mathrm{Hom}_{\mathcal{C}}(X_0, X_2) = 0$$
$$\mathrm{Hom}_{\mathcal{C}}(C_g, X_3) \to \mathrm{Hom}_{\mathcal{C}}(X_2, X_3) \to \mathrm{Hom}_{\mathcal{C}}(X_1, X_3) = 0$$

These sequences show that $\mathrm{Hom}_{\mathcal{C}}(X_0, C_g[-1]) \neq 0$ and $\mathrm{Hom}_{\mathcal{C}}(C_g, X_3) \neq 0$, since $\mathrm{Hom}_{\mathcal{C}}(X_0, X_1) \neq 0 \neq \mathrm{Hom}_{\mathcal{C}}(X_2, X_3)$ by the assumption that we have a path $X_0 \to X_1 \to X_2 \to X_3$. Note that we can also assume $\mathrm{Hom}_{\mathcal{C}}(X_1[1], X_3) = 0$, since otherwise we would have a strong path $X_0[1] \to X_1[1] \to X_3$.

At this point notice that if C_g is indecomposable we are done. Indeed given that we already showed that $\mathrm{Hom}_{\mathcal{C}}(X_0, C_g[-1]) \neq 0$ and $\mathrm{Hom}_{\mathcal{C}}(C_g, X_3) \neq 0$, if C_g is really indecomposable we would have a strong path $X_0[1] \to C_g \to X_3$. Thus the rest of the proof will be focused on showing that C_g is indeed indecomposable.

Assume to the contrary that C_g is not indecomposable, so that we have $C_g = \bigoplus_{i=1}^{r} C_i$ with C_i indecomposable and $r > 1$. We already know that $\mathrm{Hom}_{\mathcal{C}}(C_g, X_3) \neq 0$. Thus there is an indecomposable direct summand C_i of C_g such that $\mathrm{Hom}_{\mathcal{C}}(C_i, X_3) \neq 0$. Note that we can also assume $\mathrm{Hom}_{\mathcal{C}}(X_0, C_i[-1]) = 0$ since otherwise we would have a strong path $X_0[1] \to C_i \to X_3$.

The canonical split mono $\varepsilon_i : C_i \to C_g$ induces the following map of triangles:

$$\begin{array}{ccccccc}
X_1 & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & C_i & \xrightarrow{v_i} & X_1[1] \\
\| & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varepsilon_i} & & \| \\
X_1 & \xrightarrow{g} & X_2 & \xrightarrow{u} & C_g & \xrightarrow{v} & X_1[1]
\end{array}$$

Note that Y is defined by the property that $Y[1]$ is the cone of v_i and that φ exists since $\mathcal{C}$ is a triangulated category and as such (TR3) holds. Now if we apply $\mathrm{Hom}_{\mathcal{C}}(-, X_3)$ to the upper triangle we can see that $\mathrm{Hom}_{\mathcal{C}}(Y, X_3) \neq 0$ since we already know that $\mathrm{Hom}_{\mathcal{C}}(X_1[1], X_3) = 0$ and $\mathrm{Hom}_{\mathcal{C}}(C_i, X_3) \neq$

0. If we apply $\mathrm{Hom}_{\mathcal{C}}(X_0, -)$ to the same triangle instead, we get that $\mathrm{Hom}_{\mathcal{C}}(X_0, Y) \neq 0$ since we know that $\mathrm{Hom}_{\mathcal{C}}(X_0, C_i[-1]) = 0$ and $\mathrm{Hom}_{\mathcal{C}}(X_0, X_1) \neq 0$. If Y is indecomposable we get a strong path $X_0 \to Y \to X_3$, in contrast with the minimality of ω. Thus we can safely assume $Y = \bigoplus_{j=1}^{t} Y_j$ with Y_j indecomposable and $t > 1$. Since $\mathrm{Hom}_{\mathcal{C}}(Y, X_3) \neq 0$ there must be j such that $\mathrm{Hom}_{\mathcal{C}}(Y_j, X_3) \neq 0$. Recall now that g is not invertible and necessarily nonzero. Thus we can use Lemma 2.1.7 ii) to see that v_i, and thus $v_i[-1]$, is nonzero and not invertible. This means that we can again apply Lemma 2.1.7 ii) to the rotated triangle to infer that $\beta_j = \beta\varepsilon_j$ is nonzero and noninvertible. In particular this means that we have a strong path $X_0 \to X_1 \to Y_j \to X_3$ which is of minimal length by assumption.

Let us now apply the functor $\mathrm{Hom}_{\mathcal{C}}(-, X_1[1])$ to the commutative diagram of triangles from above. This yields the following commutative diagram of exact sequences of vector spaces:

$$
\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{C}}(C_g, X_1[1]) & \xrightarrow{u^*} & \mathrm{Hom}_{\mathcal{C}}(X_2, X_1[1]) & \xrightarrow{g^*} & \mathrm{Hom}_{\mathcal{C}}(X_1, X_1[1]) \\
\downarrow{\scriptstyle \varepsilon_i^*} & & \downarrow{\scriptstyle \varphi^*} & & \| \\
\mathrm{Hom}_{\mathcal{C}}(C_i, X_1[1]) & \xrightarrow{\beta^*} & \mathrm{Hom}_{\mathcal{C}}(Y, X_1[1]) & \xrightarrow{\alpha^*} & \mathrm{Hom}_{\mathcal{C}}(X_1, X_1[1])
\end{array}
$$

Since ε_i is a split monomorphism and $\mathrm{Hom}_{\mathcal{C}}(-, X_1[1])$ is a contravariant functor we infer that ε_i^* is a split epimorphism. Thus the above diagram gives us a surjective map $\mathrm{Ker}\, g^* \to \mathrm{Ker}\, \alpha^*$, namely the restriction of φ^* to the image of u^*, which implies $\dim \mathrm{Ker}\, g^* \geq \dim \mathrm{Ker}\, \alpha^*$.

The canonical split epimorphism $\pi_j : Y \to Y_j$ induces the following map of triangles:

$$
\begin{array}{ccccccc}
X_1 & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & C_i & \xrightarrow{v_i} & X_1[1] \\
\| & & \downarrow{\scriptstyle \pi_j} & & \downarrow & & \| \\
X_1 & \xrightarrow{\alpha_j} & Y_j & \longrightarrow & C_{\alpha_j} & \longrightarrow & X_1[1]
\end{array}
$$

Applying $\mathrm{Hom}_{\mathcal{C}}(-, X_1[1])$ to this commutative diagram of triangles yields the following commutative diagram of exact sequences of vector spaces:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Ker}\, \alpha_j^* & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Y_j, X_1[1]) & \xrightarrow{\alpha_j^*} & \mathrm{Hom}_{\mathcal{C}}(X_1, X_1[1]) \\
 & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \pi_j^*} & & \| \\
0 & \longrightarrow & \mathrm{Ker}\, \alpha^* & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Y, X_1[1]) & \xrightarrow{\alpha} & \mathrm{Hom}_{\mathcal{C}}(X_1, X_1[1])
\end{array}
$$

where ψ denotes the induced map. Since π_j is a split epimorphism and $\mathrm{Hom}_{\mathcal{C}}(-, X_1[1])$ is contravariant, we obtain that π_j^* is a split monomorphism. Thus ψ has to be a monomorphism as well. In particular we obtain that $\dim \mathrm{Ker}\, g^* \geq \dim \mathrm{Ker}\, \alpha^* \geq \dim \mathrm{Ker}\, \alpha_j^*$. By minimality of $\dim \mathrm{Ker}\, g^*$ we can infer that these inequalities are actually equalities, so in particular $\dim \mathrm{Ker}\, g^* = \dim \mathrm{Ker}\, \alpha^* = \dim \mathrm{Ker}\, \alpha_j^*$ and we can infer that the surjection $\mathrm{Ker}\, g^* \to \mathrm{Ker}\, \alpha^*$ is actually an isomorphism. Now from the diagram where φ^* was involved we get the following commutative square:

$$\begin{array}{ccc} \mathrm{Ker}\, \alpha^* & \hookrightarrow & \mathrm{Hom}_{\mathcal{C}}(Y, X_1[1]) \\ \downarrow{\scriptstyle \cong} & & \uparrow{\scriptstyle \varphi^*} \\ \mathrm{Ker}\, g^* & \hookrightarrow & \mathrm{Hom}_{\mathcal{C}}(X_2, X_1[1]) \end{array}$$

where the left arrow is the inverse of the aforementioned isomorphism $\mathrm{Ker}\, g^* \to \mathrm{Ker}\, \alpha^*$. In particular if $\varphi^*(f) = 0$ for some $f \in \mathrm{Hom}_{\mathcal{C}}(X_2, X_1[1])$ we have that $g^*(f) = \alpha^*(\varphi^* f) = 0$, so $f \in \mathrm{Ker}\, g^*$. Thus we can find a unique element f' in $\mathrm{Ker}\, \alpha^*$ sent to f by our isomorphism. But since the square is commutative and $\mathrm{Ker}\, \alpha^* \to \mathrm{Hom}_{\mathcal{C}}(Y, X_1[1])$ is injective we have necessarily $f' = 0$, thus φ^* is injective.

Finally let $s \neq i$. By Lemma 2.1.7 iii) we know that $v_s u_s \neq 0$. Thus $\varphi^*(v_s u_s) \neq 0$. But we also have $\varphi^*(v_s u_s) = v_s u_s \varphi = v_s \pi_s u \varphi = v_s \pi_s \varepsilon_i \beta = 0$ since $\pi_s \varepsilon_i = 0$ for $s \neq i$. So we obtained a contradiction, which shows that C_g is indecomposable, as wanted, and we can conclude our proof.

□

Theorem 2.2.6. *Let $X_0, \dots, X_s$ be a strong path in $\mathcal{C}$. Then there exists $0 \leq t \leq s-2$, an indecomposable object $Y \in \mathcal{C}$ and a strong path $X_0[t] \to Y \to X_s$.*

Proof. This follows by a repeated application of Lemma 2.2.5.

□

The next corollary shows how this result is useful for hereditary algebras.

Corollary 2.2.7. *Let Λ be a finite dimensional connected hereditary algebra which is not semi-simple. Let X, Y be indecomposable Λ-modules. Then there exists $0 \leq m \leq 2$, an indecomposable object $Z \in D^b(\Lambda)$ and a strong path $X[0] \to Z \to Y[m]$.*

Proof. First, we want to show that since Λ is a connected algebra, then there is a path from X to $Y[n]$ for some $n \in \mathbf{Z}$. Indeed, assume there is no such path. Then we can find a path-closed class in $D^b(\Lambda)$ such that X is in it and Y is not. Thus we have a nontrivial partition of $\mathrm{Ind}\, D^b(\Lambda)$ when taking the path-closed class $\mathcal{U}$ containing X and its complement. From Lemma 2.2.1 we have that $D^b(\Lambda)$ can be written as a product of triangulated subcategories

$\mathcal{D}_1 \times \mathcal{D}_2$. In particular, from this decomposition we have $\Lambda_\Lambda = (\Lambda_1, \Lambda_2)$. Thus we have that

$$\Lambda \cong \mathrm{End}_{\mathrm{mod}\,\Lambda}(\Lambda) = \mathrm{End}_{D^b(\Lambda)}(\Lambda) \cong \mathrm{End}_{\mathcal{D}_1\times\mathcal{D}_2}(\Lambda_1, \Lambda_2)$$

In particular, in $\mathrm{End}_{\mathcal{D}_1\times\mathcal{D}_2}(\Lambda_1, \Lambda_2)$ we have the central idempotents $(id_{\Lambda_1}, 0)$ and $(0, id_{\Lambda_2})$, so $\Lambda \cong \mathrm{End}_{\mathcal{D}_1\times\mathcal{D}_2}(\Lambda_1, \Lambda_2)$ is not connected, which would be a contradiction. Thus we showed that if we have a connected algebra Λ, its derived category is a block.

Furthermore, since Λ is not semi-simple, we may assume by Corollary 2.2.4 that we have in fact a strong path from $X[0]$ to $Y[n]$. By Theorem 2.2.6, there is $t \geq 0$, an indecomposable object $Z \in D^b(\Lambda)$ and a strong path $X[t] \to Z \to Y[n]$, which clearly would give us a strong path $X \to Z[-t] \to Y[n-t]$ after shifting. Notice that since Λ is hereditary, by Lemma 2.1.2, since Z is indecomposable there is an indecomposable module Z', such that $Z = Z'[i]$ for some $i \in \mathbf{Z}$. Thus, by having our path $X \to Z'[i-t] \to Y[n-t]$ we know that $\mathrm{Hom}_{D^b(\Lambda)}(X, Z'[i-t]) \neq 0$ and $\mathrm{Hom}_{D^b(\Lambda)}(Z'[i-t], Y[n-t]) \neq 0$. Hence we have $0 \leq i-t \leq 1$ and more importantly $0 \leq n-t \leq 2$. □

2.3 Piecewise hereditary algebras

In this section we are going to prove the main result of this chapter, i.e. that a finite dimensional algebra Λ over a field k is piecewise hereditary if and only if its strong global dimension is finite.

If Λ is piecewise hereditary, there exists a hereditary, abelian category $\mathcal{H}$ and a triangle equivalence $F : D^b(\Lambda) \to D^b(\mathcal{H})$. We denote by $\mathcal{U}_i$ the class of indecomposable Λ-modules X such that $F(X) \in \mathcal{H}[i]$. Clearly, by Lemma 2.1.2, every indecomposable object belongs to some $\mathcal{U}_i$.

The following two results are by [4].

Lemma 2.3.1. *Let $X \in \mathcal{U}_i$ and $Y \in \mathcal{U}_j$. Then $\mathrm{Hom}_\Lambda(X, Y) \neq 0$ implies $0 \leq j-i \leq 1$ and $\mathrm{Ext}^r_\Lambda(X, Y) \neq 0$ implies $0 \leq r+j-i \leq 1$.*

Proof. Since $X \in \mathcal{U}_i$ and $\mathcal{H}$ is hereditary, we have necessarily that $F(X) = X'[i]$ for some indecomposable object $X' \in \mathrm{Ob}\,\mathcal{H}$. Similarly $F(Y) = Y'[j]$ for some indecomposable object Y'. Then we have that

$$\mathrm{Hom}_\Lambda(X, Y) = \mathrm{Hom}_{D^b(\Lambda)}(X, Y) \cong \mathrm{Hom}_{D^b(\mathcal{H})}(X'[i], Y'[j]) = \mathrm{Ext}^{j-i}_{\mathcal{H}}(X', Y')$$

which is nonzero if and only if $0 \leq j-i \leq 1$. Note that the first equality comes from the fact that the canonical inclusion $\mathrm{mod}\,\Lambda \to D^b(\Lambda)$ is fully faithful.

Similarly we have

$$\begin{aligned}\mathrm{Ext}^r_\Lambda(X, Y) &= \mathrm{Hom}_{D^b(\Lambda)}(X, Y[r]) \\ &\cong \mathrm{Hom}_{D^b(\mathcal{H})}(X'[i], Y'[j+r]) = \mathrm{Ext}^{j+r-i}_{\mathcal{H}}(X', Y')\end{aligned}$$

which again is nonzero if and only if $0 \leq j + r - i \leq 1$. □

Lemma 2.3.2. *Each non-empty $\mathcal{U}_i$ contains simple Λ-modules.*

Proof. Let $X \in \mathcal{U}_i$ be of minimal dimension and suppose that X is not simple. Then there exists a non-split short exact sequence $0 \to S \to X \to X/S \to 0$, where S is a simple Λ-module. Indeed, if such sequence were to be split, it would imply that X would not be indecomposable, which would be a contradiction since $X \in \mathcal{U}_i$. By the previous Lemma, we have either $S \in \mathcal{U}_{i-1}$ or $S \in \mathcal{U}_i$, thus we can assume $S \in \mathcal{U}_{i-1}$. Let X' be an indecomposable summand of X/S such that $\mathrm{Ext}^1_\Lambda(X', S) \neq 0$. Notice that such summand exists, since the sequence from above constitutes a nonzero degree 1 Yoneda extension of X/S by S. Again, since X is of minimal dimension in $\mathcal{U}_i$, we infer $X' \in \mathcal{U}_{i+1}$. But then by the previous Lemma we must have $0 \leq 1 + (i-1) - (i+1) \leq 1$, which is a contradiction. □

Since in the algebras that we are interested in have finitely many simple modules, this last result assures us that we may assume that the triangle equivalence F is normalized in the sense that there exists $r \geq 0$ such that for an indecomposable Λ-module X we have $F(X) \in \bigcup_{i=0}^{r} \mathcal{H}[i]$ and that there exist indecomposable Λ-modules X, Y such that $F(X) \in \mathcal{H}[0]$ and $F(Y) \in \mathcal{H}[r]$. Note that such normalized equivalences are not unique and that the value of r may depend on the choice of $\mathcal{H}$. Moreover, by the Lemma above, for each $0 \leq i \leq r$ we have a simple Λ-module S_i, such that $F(S_i) \in \mathcal{H}[i]$. Thus, if we denote by n the cardinality of the set of all isomorphism classes of simple Λ-modules, we have $r + 1 \leq n$.

We are now ready to show the first part of our central result.

Proposition 2.3.3. *Let Λ be a finite dimensional piecewise hereditary algebra. Then* $\mathrm{s.gl.dim}\,\Lambda \leq \#\{$*isomorphism classes of simple* Λ*-modules*$\} + 1$. *In particular we have that* $\mathrm{s.gl.dim}\,\Lambda < \infty$.

Proof. Let $F : D^b(\Lambda) \to D^b(\mathcal{H})$ be a normalized equivalence. Let $P^\bullet = (P^i, d^i) \in K^b({}_\Lambda\mathcal{P})$ be indecomposable with length $\ell(P^\bullet) = t$. By applying the shift functor in $K^b({}_\Lambda\mathcal{P})$ if necessary, we may assume that $P^0 \neq 0$ and $P^i = 0$ for $i > 0$. As a consequence we also may assume that $P^i = 0$ for $i < -t$ and $P^{-t} \neq 0$. Since by our assumptions $P^\bullet$ has no indecomposable projective direct summands in $C^b({}_\Lambda\mathcal{P})$, we have that $\mathrm{Hom}_{K^b({}_\Lambda\mathcal{P})}(P^0, P^\bullet) \neq 0 \neq \mathrm{Hom}_{K^b({}_\Lambda\mathcal{P})}(P^\bullet, P^{-t}[t])$ by Lemma 2.1.6 b). Now, since F is normalized, we have that $F(P^{-t}[t]) \in \bigcup_{i=t}^{r+t} \mathcal{H}[i]$ and $F(P^0) \in \bigcup_{i=0}^{r} \mathcal{H}[i]$. Also, since $P^\bullet$ is indecomposable, there is $s \in \mathbf{Z}$ such that $F(P^\bullet) \in \mathcal{H}[s]$.

Now consider two indecomposable objects X and Y in $D^b(\mathcal{H})$ with

$$\mathrm{Hom}_{D^b(\mathcal{H})}(F(P^\bullet), X) \neq 0 \neq \mathrm{Hom}_{D^b(\mathcal{H})}(Y, F(P^\bullet))$$

We can assume that $X \in \mathcal{H}[i_X]$ and $Y \in \mathcal{H}[i_Y]$ for some $i_X, i_Y \in \mathbf{Z}$. Therefore, since $\mathcal{H}$ is hereditary we have that $i_X \in \{s, s+1\}$ and $i_Y \in \{s-1, s\}$. Indeed for a hereditary category $\mathcal{H}$ and objects $X, Y \in \mathcal{H}$ we have nonzero morphisms $X[i] \to Y[j]$ only if $0 \leq j - i \leq 1$ since $\mathrm{Hom}_{D^b(\mathcal{H})}(X[i], Y[j]) = \mathrm{Ext}^{j-i}_{\mathcal{H}}(X, Y)$ which is zero for $j - i \notin \{0, 1\}$. In particular, by Lemma 2.1.6 b) we have

$$\mathrm{Hom}_{K^b({}_\Lambda\mathcal{P})}(P^\bullet, P^{-t}[t]) \neq 0 \neq \mathrm{Hom}_{K^b({}_\Lambda\mathcal{P})}(P^0, P^\bullet)$$

thus we infer that

$$\mathrm{Hom}_{D^b(\mathcal{H})}(F(P^0), F(P^\bullet)) \neq 0 \neq \mathrm{Hom}_{D^b(\mathcal{H})}(F(P^\bullet), F(P^{-t}[t]))$$

Since $F(P^0) \in \bigcup_{i=0}^{r} \mathcal{H}[i]$, we have that $0 \leq s \leq r+1$. Similarly, $F(P^{-t}[t]) \in \bigcup_{i=t}^{r+t} \mathcal{H}[i]$ implies that $t-1 \leq s \leq r+t$. By combining these inequalities we conclude that $t-1 \leq s \leq r+1$ which yields $t \leq r+2$. This concludes the proof since it shows that the length of an indecomposable complex in $K^b({}_\Lambda\mathcal{P})$ is less or equal to the number of isomorphism classes of simple Λ-modules plus one, so in particular is finite. □

To show the reverse implication we first need to take a step back and prove that every complex bounded from above in $D^-(\Lambda)$ is quasiisomorphic to a complex in $K^-({}_\Lambda\mathcal{P})$. To do so we will give a generalization of the concept of projective resolution and then show that every complex admits a projective resolution in this new setup. This is a well-known result. We follow [2] to explain it.

Definition 2.3.1. *Let X be a bounded from above complex. A* projective resolution *of X is a complex $P \in K^-({}_\Lambda\mathcal{P})$ together with a quasiisomorphism $P \to X$.*

Definition 2.3.2. *Let X be a complex. A* Cartan-Eilenberg (projective) resolution *of X is a complex of complexes $P^{\bullet,\bullet} = \cdots \to P^{-2,\bullet} \to P^{-1,\bullet} \to P^{0,\bullet}$ together with a map of complexes $P^{0,\bullet} \to X$ such that*

i) if $X^q = 0$, then $P^{\bullet,q} = 0$

ii) for every q, the sequences

$$\cdots \to B^q(P^{-2,\bullet}) \to B^q(P^{-1,\bullet}) \to B^q(P^{0,\bullet}) \to B^q(X)$$

$$\cdots \to H^q(P^{-2,\bullet}) \to H^q(P^{-1,\bullet}) \to H^q(P^{0,\bullet}) \to H^q(X)$$

$$\cdots \to Z^q(P^{-2,\bullet}) \to Z^q(P^{-1,\bullet}) \to Z^q(P^{0,\bullet}) \to Z^q(X)$$

$$\cdots \to P^{-2,q} \to P^{-1,q} \to P^{0,q} \to X^q$$

are projective resolutions.

Proposition 2.3.4. *Every complex X in $K(\Lambda)$ has a Cartan-Eilenberg resolution.*

Proof. For every $q \in \mathbf{Z}$, fix projective resolutions of $B^q(X)$ and $H^q(X)$. By the horseshoe lemma, there is a projective resolution of $Z^q(X)$ fitting in a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_B^{-1,q} & \longrightarrow & P_Z^{-1,q} & \longrightarrow & P_H^{-1,q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_B^{0,q} & \longrightarrow & P_Z^{0,q} & \longrightarrow & P_H^{0,q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^q(X) & \longrightarrow & Z^q(X) & \longrightarrow & H^q(X) & \longrightarrow & 0
\end{array}$$

Similarly, since for every $q \in \mathbf{Z}$ $X^q \cong \operatorname{Ker} d^q \oplus \operatorname{Im} d^q$, we get a projective resolution of X^q fitting in a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_Z^{-1,q} & \longrightarrow & P^{-1,q} & \longrightarrow & P_B^{-1,q+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_Z^{0,q} & \longrightarrow & P^{0,q} & \longrightarrow & P_B^{0,q+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z^q(X) & \longrightarrow & X^q & \longrightarrow & B^{q+1}(X) & \longrightarrow & 0
\end{array}$$

From this construction we obtain maps $P^{\bullet,q} \to P_B^{\bullet,q+1} \to P_Z^{\bullet,q+1} \to P^{\bullet,q+1}$ which we will call d_h^q. These maps will be the horizontal differentials for our double complex $P^{\bullet,\bullet}$ which we claim to be a Cartan-Eilenberg resolution. Clearly $d_h \circ d_h = 0$ since we are composing two consecutive maps in an exact sequence. Also clearly $\cdots \to P^{-2,q} \to P^{-1,q} \to P^{0,q} \to X^q$ is a projective resolution for every q. We will just show that $\cdots \to Z^q(P^{-2,\bullet}) \to Z^q(P^{-1,\bullet}) \to Z^q(P^{0,\bullet}) \to Z^q(X)$ is a projective resolution for every q. The cases for $B^q(X)$ and $H^q(X)$ are analogous. Indeed, for every q, $Z^q(P^{i,\bullet})$ is the kernel of d_h^q and thus is isomorphic to $P_Z(i,q)$. Indeed since we have exact rows in the diagrams above, we have that d_h^q is the composition of two monomorphism after an epimorphism. Hence the kernel of d_h^q is isomorphic to the kernel of the epimorphism, which by the exactness of the rows is isomorphic to $P_Z(i,q)$. Clearly $\cdots \to P_Z^{-1,q} \to P_Z^{0,q} \to Z^q(X)$ is a projective resolution of $Z^q(X)$. □

Definition 2.3.3. *Let $P^{\bullet,\bullet}$ be a complex of complexes with differentials d_{vert} and d_{hor}. We define its* total complex $\mathrm{Tot}(P)$ *to be the complex defined by*

$$\mathrm{Tot}(P)^n = \bigoplus_{k+l=n} P^{k,l}$$

and whose differential is given by the linear combination

$$d^{\mathrm{Tot}} = d_{vert} + (-1)^{verticaldegree} d_{hor}$$

We give the following technical result without a proof. For more details, we refer to [11].

Lemma 2.3.5. *The total complex of a third quadrant double complex whose columns are acyclic is also acyclic.*

Notice that by third quadrant double complex we mean a double complex $P^{\bullet,\bullet}$ such that $P^{i,j} = 0$ whenever i or j are positive.

Proposition 2.3.6. *For every bounded above complex $X \in K^-(\Lambda)$ there is a triangle*

$$P \to X \to A \to P[1]$$

in $K(\Lambda)$ such that $P \in K^-({}_\Lambda\mathcal{P})$ and A is an acyclic complex. In particular, the long exact sequence of cohomology of this triangle tells us that the morphism $P \to X$ is a quasiisomorphism, i.e. P is a projective resolution of X.

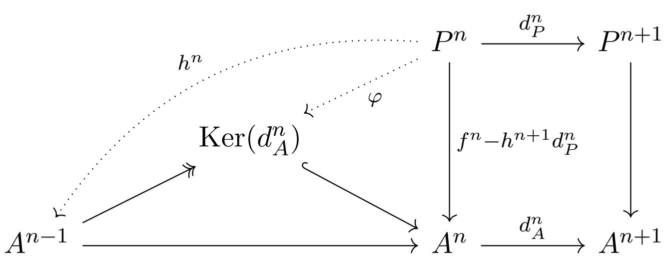
Proof. Let X be a bounded above complex. After applying the shift functor if necessary we can suppose that $X^n = 0$ for $n > 0$. Let $P^{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of X. By our hypothesis about X this is a third quadrant double complex, hence we can consider the total complex P of $P^{\bullet,\bullet}$ and the total complex A of the augmented double complex $P^{\bullet,\bullet} \to X$, shown in the next diagram.

$$
\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & P^{-2,-2} & \longrightarrow & P^{-2,-1} & \longrightarrow & P^{-2,0} \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & P^{-1,-2} & \longrightarrow & P^{-1,-1} & \longrightarrow & P^{-1,0} \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & P^{0,-2} & \longrightarrow & P^{0,-1} & \longrightarrow & P^{0,0} \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & X^{-2} & \longrightarrow & X^{-1} & \longrightarrow & X^{0}
\end{array}
$$

The augmentation map induces a map $P \to X$ and calculating the mapping cone of this morphism one can see that it is exactly A. Thus we have a triangle $P \to X \to A \to P[1]$ in $K(\Lambda)$. A is acyclic by 2.3.5. Also P (resp. A) is a bounded above complex because, since $P^{\bullet,\bullet}$ (resp. $P^{\bullet,\bullet} \to X$) is a third quadrant double complex, $P^n = 0$ (resp. $A^n = 0$). Furthermore, every component of P is a sum of projectives and hence it is projective itself, as wanted. □

Proposition 2.3.7. *Let $P \in K^-({}_\Lambda\mathcal{P})$ be a bounded above complex of projectives. Then* $\mathrm{Hom}_{K(\Lambda)}(P, A) = 0$ *for every acyclic complex A.*

Proof. Let $f : P \to A$ be a morphism of complexes. Without loss of generality we can suppose $P^n = 0$ for $n > 0$. We want to construct a homotopy h such that $f = d_A h + h d_P$ and we are going to do so inductively. Indeed for $n > 0$ we can consider $h^n : P^n \to A^{n-1}$ to be the zero morphism. Now suppose that we have h^{n+1}. We define h^n in the following way



where φ exists by the universal property of kernels, since

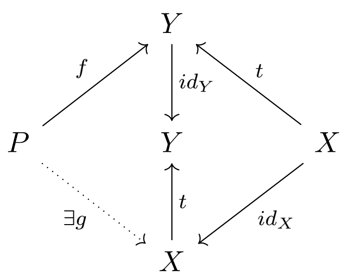
$$
\begin{aligned}
d_A^n(f^n - h^{n+1}d_P^n) &= d_A^n f^n - d_A^n h^{n+1} d_P^n = \\
&= d_A^n f^n - (f^{n+1} - h^{n+2}d_P^{n+1})d_P^n = d_A^n f^n - f^{n+1}d_P^n = 0
\end{aligned}
$$

and h^n exists by the projectivity of P^n. □

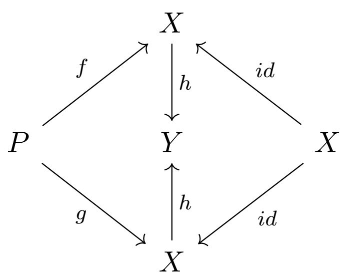
Proposition 2.3.8. *Let $P \in K^-({}_\Lambda\mathcal{P})$ be a bounded above complex of projectives. Then for every complex X, the canonical map $\mathrm{Hom}_{K(\Lambda)}(P, X) = \mathrm{Hom}_{D(\Lambda)}(P, X)$ is an isomorphism. In particular this tells us that the canonical inclusion functor $K^-({}_\Lambda\mathcal{P}) \to D^-(\Lambda)$ is fully faithful.*

Proof. First we need to notice that if $s : X \to Y$ is a quasiisomorphism, the canonical map $\mathrm{Hom}_{K(\Lambda)}(P, X) \to \mathrm{Hom}_{K(\Lambda)}(P, Y)$ is bijective. Indeed to show this we just need to consider the triangle $X \to Y \to \mathrm{cone}(s) \to X[1]$ and apply $\mathrm{Hom}_{K(\Lambda)}(P, -)$ to it. Then our claim follows from Proposition 2.3.7, since $\mathrm{cone}(s)$ is acyclic. We now prove the bijectivity of the map $\mathrm{Hom}_{K(\Lambda)}(P, X) = \mathrm{Hom}_{D(\Lambda)}(P, X)$.

Surjectivity. Consider a morphism $\varphi : P \to X$ in $D(\Lambda)$ represented by a right roof $P \xrightarrow{f} Y \xleftarrow{t} X$, where t is a quasiisomorphism. Then the following diagram shows that φ can be represented by a morphism $g : P \to X$ in $K(\Lambda)$ whose existence comes from the surjectivity of the map $\mathrm{Hom}_{K(\Lambda)}(P, X) \to \mathrm{Hom}_{K(\Lambda)}(P, Y)$.



Injectivity. If $f, g : P \to X$ in $K(\Lambda)$ represent the same map in $D(\Lambda)$, we have a diagram



Since h is a quasiisomorphism, from the injectivity of the map $\mathrm{Hom}_{K(\Lambda)}(P, X) \to \mathrm{Hom}_{K(\Lambda)}(P, Y)$ follows $f = g$. □

Theorem 2.3.9. *Let Λ be a finite dimensional algebra. Then $\mathrm{gl.dim}\,\Lambda$ is finite if and only if the canonical inclusion $K^b({}_\Lambda\mathcal{P}) \to D^b(\Lambda)$ is an equivalence of categories.*

Proof. By the proof of Proposition 2.3.8 the inclusion is fully faithful. Indeed if Y is not bounded we can just apply a suitable truncation functor. Hence we just need to show that $\operatorname{gl.dim} \Lambda$ is finite if an only if the canonical inclusion is essentially surjective.t

Suppose $\operatorname{gl.dim} \Lambda < \infty$. Consider an object $X \in D^b(\Lambda)$. By Proposition 2.3.6 we can construct a bounded above complex of projectives P which is quasiisomorphic to X. But then P is bounded also below by its construction as a total complex, since X is bounded and all the X^n have a finite projective resolution, by our hypothesis that $\operatorname{gl.dim} \Lambda < \infty$.

Conversely suppose that $\operatorname{gl.dim} \Lambda$ is infinite. Then, since Λ is a finite dimensional algebra, by Theorem 2.1.5 there is a simple module S of infinite projective dimension. But then S, viewed as an object in $D^b(\Lambda)$, cannot be quasiisomorphic to a complex in $K^b({}_\Lambda\mathcal{P})$ because it would make for a finite projective resolution. □

This last theorem in particular shows us that for any finite dimensional algebra Λ we have $\operatorname{gl.dim} \Lambda \leq \operatorname{s.gl.dim} \Lambda$. Indeed, for any indecomposable module $X \in \operatorname{mod} \Lambda$ viewed as an object in $D^-(\Lambda)$, we have that its projective resolution is indecomposable in $K^-({}_\Lambda\mathcal{P})$ by the equivalence of categories seen above and because the canonical inclusion $\operatorname{mod} \Lambda \to$. Thus, for any indecomposable Λ-module X we have that $\operatorname{proj.dim} X \leq \operatorname{s.gl.dim} \Lambda$ and the inequality holds also when we pass to the sup over all indecomposable Λ-modules. Moreover, if $X = \bigoplus X_i$ for some indecomposable X_i, we have that a projective resolution of X is given by the direct sum of the projective resolutions its components X_i. Hence our claim holds.

We need one last result from [3] in order to prove our main claim. We also need to introduce the following notation: if X is an indecomposable object in a triangulated category $\mathcal{C}$, we denote by $[X \to]$ the class of all indecomposable objects U in $\mathcal{C}$ with a path from X to U. Then $[X \to]$ is closed under the under the translation functor $[1]$. The complement of $[X \to]$ in $\operatorname{Ind} \mathcal{C}$ is closed under $[-1]$.

Theorem 2.3.10. *Let $\mathcal{D}$ be a triangulated category which is a block. Then the following are equivalent:*

i) The triangulated category $\mathcal{D}$ is equivalent to the derived category of an hereditary category.

ii) If X is an indecomposable object in $\mathcal{D}$, then there is no path from $X[1]$ to X.

iii) There is an indecomposable object X in $\mathcal{D}$ with no path from $X[1]$ to X.

Notice in particular that if we have a connected algebra Λ, its derived category $D^b(\Lambda)$ is a block, as we saw in the proof of Corollary 2.2.7, so this theorem works for this case, which is what we are really interested in.

Proof. "$i) \Rightarrow ii)$" is easy. Indeed since X is indecomposable in $\mathcal{D}$, by Lemma 2.1.2, it is of the form $X = A[n]$ for some indecomposable object A and $n \in \mathbf{Z}$. But then, since we cannot have nonzero morphism $A[n] \to B[n-1]$ for an indecomposable object B, we can deduce by induction on the length of paths that $[X \to] \subseteq \bigcup_{i \geq n} \mathcal{D}[i]$. In particular we have that $X[-1]$ does not belong to $[X \to]$.

"$ii) \Rightarrow iii)$" is trivial.

It remains to show "$iii) \Rightarrow i)$". For this we write $\mathcal{U} = [X \to]$ and $\mathcal{V} = \operatorname{Ind} D^b(\Lambda) \setminus \mathcal{U}$. We set $\mathcal{A}$ to be the smallest additive category containing $\mathcal{U} \cap \mathcal{V}[1]$ and which is closed under isomorphisms. We write $\mathcal{A} = \operatorname{add}(\mathcal{U} \cap \mathcal{V}[1])$. Our goal is to prove that $\mathcal{A}$ satisfies the conditions in Theorem 2.1.3 ii), so that we prove our claim.

The first step is showing that if we have $A \in \mathcal{A}$ and a nonzero morphism $u : A \to B$, then $B \notin \mathcal{U}[2]$. Assume on the contrary that $B \in \mathcal{U}[2]$ (so in particular $B \notin \mathcal{U}[i]$ for $i \geq 2$, since for $n \geq 0$ we have that $\mathcal{U}[n+1] \subseteq \mathcal{U}[n]$). In this case we have a path from $X[1]$ to $B[-1]$. By the facts that $A \notin \mathcal{U}[1]$ and that $\mathcal{U}[2] \subseteq \mathcal{U}[1]$, we infer that u is not an isomorphism, else we would have $A \in \mathcal{U}[2]$, contarily to what we just discussed. Then, by a dual version of Lemma 2.1.7 ii), we would get a path of length two from $B[-1]$ to A. This in turn would mean that we would have a path from $X[1]$ to A, which means that $A \in \mathcal{U}[1]$, a contradiction.

From this, we infer that in the above case, B lies in $\mathcal{A}$ or in $\mathcal{A}[1]$. Indeed if $B \in \mathcal{U}$, then $B \in \mathcal{A}$. Otherwise, we have $B \in \mathcal{U}[1]$. By what we just showed we also have $B \in \mathcal{V}[2]$, so $B \in \mathcal{A}[1]$. Indeed, if $B \notin \mathcal{V}[2]$, it means that we do have a path from $X \to B[-2]$, meaning that $B \in \mathcal{U}[2]$, contradicting what we just proved.

Now we need to show that $\mathcal{D} = \operatorname{add}(\bigcup_{n \in \mathbf{Z}} \mathcal{A}[n])$. We claim that each indecomposable object $Y \in \mathcal{D}$ is of the form $B[m]$ for some $B \in \mathcal{A}$ and $m \in \mathbf{Z}$. We start by observing that $X \in \mathcal{A}$ by assumption. Assume first that $Y \in \mathcal{U} = [X \to]$. Then there is a path $X = X_0, X_1, \ldots, X_t = Y$. By induction on the length of paths, we may assume that $X_{t-1} = A[n]$ for $A \in \mathcal{A}$ and some $n \in \mathbf{Z}$. If $Y = X_{t-1}[1]$, then $Y = A[n+1] \in \mathcal{A}[n+1]$ and we are done. If instead $\operatorname{Hom}_{\mathcal{D}}(X_{t-1}, Y) \neq 0$, we have equivalently that $\operatorname{Hom}_{\mathcal{D}}(A, Y[-n]) \neq 0$, so we infer by the previous step that $Y[-n]$ lies in $\mathcal{A}$ or in $\mathcal{A}[1]$. This proves the statement in this case.

For the general case, we already saw in the proof of Corollary 2.2.7, that in the case of Λ-modules (which is the case we are mostly interested in), given X, we always have an $n_Y \in \mathbf{N}$ and a path in $D^b(\Lambda)$ from X to $Y[n_Y]$ for any other Λ-module Y. In [3], one can find a more general proof of this for arbitrary triangulated categories which are blocks. In particular, in our case we have that $Y[n_Y] \in [X \rightarrow]$. By applying the above argument to $Y[n_Y]$, we proved our claim.

In order to complete our proof, we still need to check that

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}, \mathcal{A}[m]) = 0$$

for $m < 0$. We assume the contrary. Then let us take two indecomposable objects $A, B \in \mathcal{A}$ with $\mathrm{Hom}_{\mathcal{D}}(A, B[m]) \neq 0$. Then $B[m]$ lies in $\mathcal{U}$. On the other hand, $B[m]$ lies in $\mathcal{V}[m+1]$. Since $m+1 \leq 0$, we have $\mathcal{V}[m+1] \subseteq \mathcal{V}$ since $\mathcal{V}$ is closed under $[-1]$. We conclude that $B[m] \in \mathcal{U} \cap \mathcal{V}$, a contradiction. This completes the whole proof. □

Now we are finally ready to prove our main result.

Theorem 2.3.11. *Let Λ be a finite dimensional algebra. Then Λ is piecewise hereditary if and only if* $\mathrm{s.gl.dim}\,\Lambda < \infty$.

Proof. If Λ is piecewise hereditary, then $s.gl.dim\Lambda < \infty$ by Proposition 2.3.3.

Conversely assume that $\mathrm{s.gl.dim}\,\Lambda < \infty$. In particular by the considerations above, we have that $\mathrm{gl.dim} < \infty$ as well, so Theorem 2.3.9 tells us that $D^b(\Lambda) \cong K^b({}_\Lambda\mathcal{P})$. Assume by contradiction that Λ is not piecewise hereditary. Notice that we may assume that Λ is basic and connected and $\Lambda \ncong k$. Indeed if Λ is not connected then $\Lambda \cong \bigoplus_{i=1}^{t} \Lambda_i$ for some connected algebras Λ_i and some $t \in \mathbf{N}$ and, since $\mathrm{mod}\,\Lambda \cong \bigoplus_{i=1}^{t} \mathrm{mod}\,\Lambda_i$, if the theorem holds for the connected algebras Λ_i then it also holds for Λ. Moreover if Λ is not basic, by Proposition 1.5.10 there is a basic algebra associated with Λ, say Λ^b, such that the categories $\mathrm{mod}\,\Lambda$ and $\mathrm{mod}\,\Lambda^b$ are equivalent, which would a fortiori imply $D^b(\Lambda) \cong D^b(\Lambda^b)$, so in particular Λ and Λ^b have the same strong global dimension. Finally, if $\Lambda \cong k$ one can easily show that k is hereditary and thus piecewise hereditary.

Now we can resume the proof with our basic and connected algebra Λ. By Theorem 2.3.10 and Corollary 2.2.4, since Λ is not piecewise hereditary, we have that for each indecomposable object $X \in K^b({}_\Lambda\mathcal{P})$ there exists a strong path from $X[1]$ to X in $K^b({}_\Lambda\mathcal{P})$. Thus, just by shifting this path, we see that there is also a strong path from $X[2]$ to $X[1]$ and so on. In particular we can concatenate these paths in order to get a strong path in $K^b({}_\Lambda\mathcal{P})$ from $X[n]$ to X for each $n \geq 0$. By Theorem 2.2.6 we can shorten these paths, meaning that there exists $t \geq 0$, an indecomposable object $Q^\bullet_{n,t} \in K^b({}_\Lambda\mathcal{P})$ and a strong path

$$X[n+t] \to Q^\bullet_{n,t} \to X$$

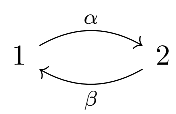
If we take $X = P[0]$ for an indecomposable projective Λ-module P, by using the considerations above, for each $n \geq 1$ we obtain an integer $t \geq 0$ and an indecomposable object $Q^{\bullet}_{n,t} \in K^b({}_{\Lambda}\mathcal{P})$, with $\mathrm{Hom}^b_K({}_{\Lambda}\mathcal{P})(P[n+t], Q^{\bullet}_{n,t}) \neq 0$ and $\mathrm{Hom}^b_K({}_{\Lambda}\mathcal{P})(Q^{\bullet}_{n,t}, P) \neq 0$. In particular, by Lemma 2.1.6 a) we have that $Q^{-(n+t)}_{n,t} \neq 0 \neq Q^0_{n,t}$. Hence $\ell(Q^{\bullet}_{n,t}) \geq n+t$. Thus we have constructed indecomposable complexes of arbitrary length in $K^b({}_{\Lambda}\mathcal{P})$, which contradicts the hypothesis that Λ has finite strong global dimension. Thus Λ is indeed piecewise hereditary. □

2.4 Examples

We are now going to finish this thesis by presenting some meaningful examples, in which we will use some notions discussed before.

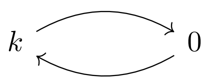
Example 2.4.1. The first example is given by hereditary algebras. In particular, given an hereditary algebra H, the category $\mathrm{mod}\, H$ is also hereditary. Indeed, since every H-module has a projective resolution of length at most 1, the functor $\mathrm{Ext}^i(-,-)$ vanishes for $i > 1$. In particular, hereditary algebras are also piecewise hereditary, as one would expect. Furthermore, for a hereditary algebra H we have $\mathrm{s.gl.dim}\, H \leq 1$. To show this, consider an indecomposable object P in $K^b({}_H\mathcal{P})$. Since $\mathrm{gl.dim}\, H \leq 1$, it will give rise to an indecomposable object P' in $D^b(H)$ under the equivalence discussed in Theorem 2.3.9. Then, by Lemma 2.1.2, there is an indecomposable H-module X such that $P' \cong X[i]$, for some $i \in \mathbf{Z}$. But, again by 2.3.9, this means that P is just the projective resolution of X shifted i times. Again, since H is hereditary, this means that $\ell(P) \leq 1$, so we showed our claim.

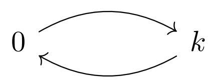
Example 2.4.2. For the second example we will consider a quiver algebra $A = kQ/\mathcal{I}$, where Q is the quiver



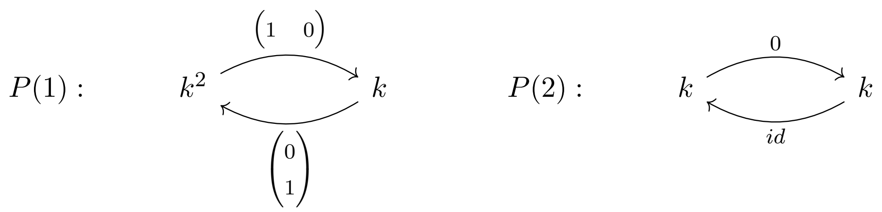
and $\mathcal{I}$ is the ideal generated by the path $\beta\alpha$. We will show that $\mathrm{gl.dim}\, A = 2$, but $\mathrm{s.gl.dim}\, A = \infty$, so in particular we infer that the algebra A is not piecewise hereditary.

First of all, recall that the simple A-modules are $S(1)$ and $S(2)$, given by the following representations

$S(1)$: 

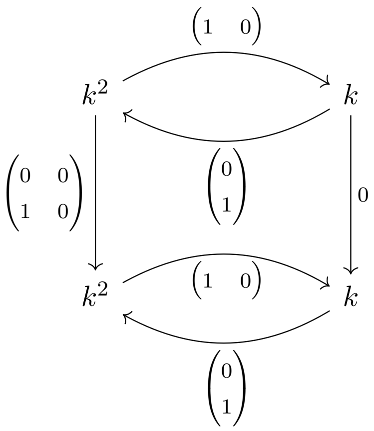
$S(2)$: 

Moreover, the indecomposable projectives A-modules are $P(1)$ and $P(2)$, given by



We start by computing the global dimension of A. By Theorem 2.1.5, we know that $\operatorname{gl.dim} A = \max_{i=1,2} \operatorname{pd} S(i)$, where pd denotes the projective dimension. Some easy computations yield $\cdots \to 0 \to P(2) \to P(1) \to S(1)$ as a projective resolution for $S(1)$ and $\cdots \to 0 \to P(2) \to P(1) \to P(2) \to S(2)$ as a projective resolution for $S(2)$. In particular, we infer that $\operatorname{gl.dim} A = 2$.

To show that $\operatorname{s.gl.dim} A = \infty$, let P_n be the complex with $(P_n)^i = P(1)$ for $-n \leq i \leq 0$ and 0 otherwise and with the differentials given by the map $d = \beta^* \alpha^* : P(1) \to P(1)$ induced by the path $\alpha\beta$. The following diagram describes d as a morphism of representations:



We clearly can see that $d^2 = 0$, so that the P_n are indeed complexes. We claim that P_n is indecomposable for every $n \in \mathbf{N}$. If this holds, then we have indecomposable complexes of projective of arbitrary length, so that $\operatorname{s.gl.dim} A = \infty$.

In order to show our claim we want to show that the algebras $\operatorname{End}_A(P_n)$ are local for all n. An endomorphism of P_n consists of n endomorphisms of $P(1)$ which commute with the differentials, hence we can start by studying $\operatorname{End}_A(P(1))$. A morphism of representation $P(1) \to P(1)$ is given by two linear maps $f_1 : k^2 \to k^2$ and $f_2 : k \to k$ such that $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \circ f_2 = f_1 \circ \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \end{pmatrix} \circ f_1 = f_2 \circ \begin{pmatrix} 1 & 0 \end{pmatrix}$. If we set $f_1 = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and $f_2 = \lambda$ for some $a, b, c, d, \lambda \in k$, the previous equations yield $\binom{c}{d} = \binom{0}{\lambda}$ and $(a, c) = (\lambda, 0)$ respectively, thus we have that $a = d = \lambda$ and $c = 0$. In particular, any endomorphism of $P(1)$ can be described by a matrix of the form $\begin{pmatrix} \lambda & 0 \\ b & \lambda \end{pmatrix}$ for some $\lambda, b \in k$. In order to understand the endomorphism algebra of P_n, take endomorphisms of $P(1)$, namely $f_i = \begin{pmatrix} \lambda_i & 0 \\ b_i & \lambda_i \end{pmatrix}$ for $-n \leq i \leq 0$ and impose that they commute with d. This in particular yields that $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \circ f_i = f_{i+1} \circ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, which in turn gives $\lambda_i = \lambda_{i+1}$. In particular we have $\mathrm{End}_{C^b(A)}(P_n) = \{f = (f_1, \dots, f_n) : \lambda_1 = \lambda_2 = \dots = \lambda_n := \lambda_f\}$, where $f_i \in \mathrm{End}_A(P(1))$ are of the form described above and all the operations are done componentwise. Notice that the $f \in \mathrm{End}_{C^b(A)}(P_n)$ is the identity if and only if $f_i = id$ for all $-n \leq i \leq 0$. Furthermore, we have that $f \in \mathrm{End}_{C^b(A)}(P_n)$ is invertible if and only if all the f_i are. One can easily see that any inverse to such an f is also a morphism of the same form. Consider then $f \in \mathrm{End}_{C^b(A)}(P_n)$ with a fixed λ_f. If $\lambda_f \neq 0$, then all of the f_i are invertible since their determinant is nonzero. Conversely, if $\lambda_f = 0$, then clearly $1 - f$ is invertible since $\lambda_{1-f} = 1 \neq 0$. Hence, by Lemma 1.5.3, $\mathrm{End}_{C^b(A)}(P_n)$ is local for every $n \in \mathcal{N}$. This implies that also $\mathrm{End}_{K^b(A)}(P_n)$ is local, since this is just a quotient of the local ring $\mathrm{End}_{C^b(A)}(P_n)$. Hence we have indecomposable complexes of projectives of arbitrary length in $K^b(A)$. In particular, as claimed, we have $\mathrm{s.gl.dim}\, A = \infty$.

Example 2.4.3. For the last example we need to introduce a new class of algebras. For this consider the quiver $\mathbf{A}_n$ with n vertices arranged as in the following diagram

$$\mathbf{A}_n : \qquad 1 \xrightarrow{\alpha} 2 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} n$$

We take into considerations the algebras $A(n,2) := k\mathbf{A}_n/\langle \alpha^2 \rangle$. We know by [5] that these algebras are piecewise hereditary, so that their strong global dimension is finite. We computed $\mathrm{s.gl.dim}\, A(n,2)$ for any $n \in \mathbf{N}$. To do so, we needed the following result by [7], which we will state without a proof.

Proposition 2.4.4. *Let $A = A(n,2)$ for some n.Let $P^\bullet \in K^b({}_A\mathcal{P})$ be an indecomposable complex. Let S be a simple A-module and $P(S)$ its projective cover. Then there is at most one i such that $P(S)$ is a direct summand of P^i.*

This in particular yields immediately the bound $\mathrm{s.gl.dim}\, A(n,2) \leq n-1$. Indeed, by 1.5.23 we know that for any $A(n,2)$ we have exactly n isomorphism classes of simple $A(n,2)$-modules. Our goal now is to find an indecomposable complex of length $n-1$, so that we can conclude that $\mathrm{s.gl.dim}\, A(n,2) = n-1$. To do this we consider the projective resolution of the simple module $S(1)$. Some easy computations show that $P(n) \to P(n-1) \to \dots \to P(2) \to P(1)$ is such a projective resolution. This in particular implies that the complex $P^\bullet = \dots \to 0 \to P(n) \to \dots \to P(1) \to 0 \to \cdots$

is indecomposable in $K^b({}_{A(n,2)}\mathcal{P})$. Since $\ell(P^\bullet) = n-1$, we are done. In particular we showed that $\operatorname{s.gl.dim} A(n,2) = n-1 = \operatorname{gl.dim} A(n,2)$, since this indecomposable complex arises from a projective resolution of a simple module.

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