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Department of Civil, Environmental and Architectural Engineering

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The singular free boundary in the thin obstacle problem for variable coefficients degenerate elliptic operator

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Abstract

In this thesis we study the singular part of the free boundary in the thin obstacle problem for some variable coefficient degenerate elliptic operator in the case of a zero thin obstacle. Our main objective was to establish the structure and regularity of the singular set. To prove it, new monotonicity formulas of Weiss and Monneau type were constructed that extend those of Garofalo-Petrosyan-Smit-Vega-Garcia to $a \in (0, 1)$.

Besides, to fully reveal the development of the approach we first presented the results regarding the singular free boundary for the thin obstacle problems of classical Laplacian and variable coefficient elliptic operators. In this way, we have understood how we can generalize the known results to reach our main goal.

The last preparing step was to study the paper of A. Banerjee, F. Buseghin and N. Garofalo, where the optimal interior regularity of the solution and smoothness of the regular part of the free boundary for our main degenerate problem were established.

Finally, we proved monotonicity of Almgren, Weiss and Monneau type which allowed to establish homogeneity, nondegeneracy, uniqueness, and continuous dependence of blowups at singular free boundary points.This, in turn, implies the main result.

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Additionally, I thank the Galilean School for an opportunity to become a part of an amazing community of very talented and extremely passionate about science students, which became a high motivation for me.

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Chapter 1

Introduction

Obstacle problems are a special class of variational problems in the field of the calculus of variations. Typically, one wants to minimize a given energy functional over a set of functions that live above a given obstacle. In this chapter we start from a discussion how these problems are tackled for a classical case, and later in the thesis we move to more complex degenerate elliptic problem with variable coefficients.

1.1 Classical obstacle problem

To provide the reader with a general idea of the type of problems we are interested in, we start with introducing the classical *obstacle problem*. In such problem one intends to understand the equilibrium configuration of an elastic membrane with fixed boundaries which is constrained to lie above a given obstacle. Suppose we are given a bounded open set $\Omega \subset \mathbb{R}^n$, and two smooth functions $\varphi : \bar{\Omega} \to \mathbb{R}$, and $g : \partial\Omega \to \mathbb{R}$, which satisfy $g \geq \varphi$ on $\partial\Omega$. We seek to minimize the Dirichlet integral

$$\mathcal{D}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx \tag{1.1.1}$$

among all configurations of function u which represents the vertical displacement of the membrane with prescribed boundary values $u|_{\partial\Omega} = g$, and constrained to remain above the obstacle φ (figure 1), in other words, on the closed convex set

$$\mathcal{K} = \{u \in W^{1,2} \mid u = g \text{ on } \partial\Omega, u \geq \varphi \text{ in } \Omega\}.$$

Solving this problem is equivalent to finding a function $u \in \mathcal{K}$ such that

$$\int_\Omega < \nabla u, \nabla(v-u) > dx \geq 0,$$

for every $v \in \mathcal{K}$. It is known [25] that there exists a unique solution $u \in W^{1,2}(\Omega)$ of such variational inequality.

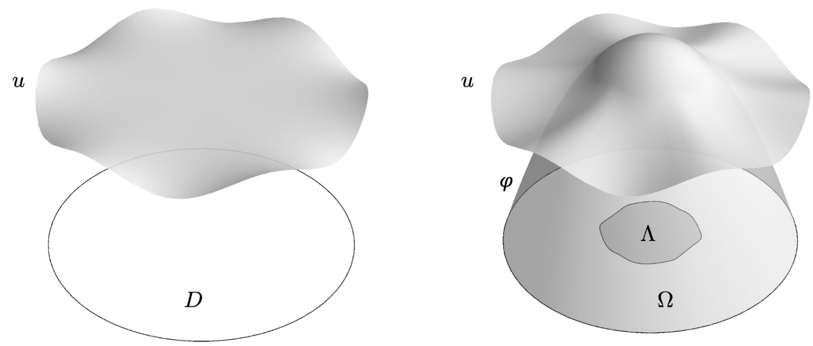

Figure 1. The free membrane (left) and the solution of the obstacle problem [27]

The choice of the class of functions as the Sobolev space $W^{1,2}(\Omega)$ is justified by the fact that in the calculus of variations searching for solution in apparently a natural but too narrow family often can give no result. The space $W^{1,2}(\Omega)$ is endowed with the inner product $< u, v >= \int_\Omega (uv + \nabla u \cdot \nabla v)dx$ and since the functional $\mathcal{D}(u)$ is continuous and strictly convex on the convex set $\mathcal{K}$, the existence and uniqueness of minimizers are guaranteed [12].

Definition 1.1.1. The set where u is above the obstacle φ is noted as

$$\Omega_\varphi(u) = \{x \in \Omega \mid u(x) > \varphi(x)\}$$

and, since one can prove that $u \in C(\Omega)$, it follows that $\Omega_\varphi(u)$ is open.

Definition 1.1.2. The set

$$\Lambda_\varphi(u) = \{x \in \Omega \mid u(x) = \varphi(x)\}$$

is called *coincidence set*. It is the part of the domain Ω where the solution u touches the obstacle φ (figure 1) and it is relatively close set in Ω.

Definition 1.1.3. The topological boundary of the coincidence set

$$\Gamma_\varphi(u) = \partial\Lambda_\varphi(u) = \partial\Omega_\varphi(u)$$

is called the *free boundary*.

The Euler Lagrange equation of the minimization problem is the following

$$\begin{cases} u \geq \varphi \text{ in } \Omega \\ u = g \text{ on } \partial\Omega \\ \Delta u \leq 0 \text{ in } \Omega \\ \Delta u = 0 \text{ in } \Omega_\varphi \end{cases}$$

which means that the solution u is superharmonic everywhere and harmonic when it is above the obstacle, i.e. in the set $\{u > \varphi\}$. It is seen from the standard variational arguments, namely having any perturbation of our solution $u + \varepsilon\eta$ we should get

$$\mathcal{D}(u+\varepsilon\eta) \geq \mathcal{D}(u) \quad \text{for some } \varepsilon \geq 0,\ \forall \eta \in C_0^\infty$$

$$\frac{1}{2}\int_\Omega |\nabla(u+\varepsilon\eta)|^2 \geq \frac{1}{2}\int_\Omega |\nabla u|^2$$

$$\frac{1}{2}\int_\Omega |\nabla u|^2 + \varepsilon\int_\Omega \nabla u \nabla\eta + \frac{1}{2}\varepsilon^2\int_\Omega |\nabla\eta|^2 \geq \frac{1}{2}\int_\Omega |\nabla u|^2$$

$$\int_\Omega \nabla u\nabla\eta + \frac{1}{2}\varepsilon\int_\Omega |\nabla\eta|^2 \geq 0$$

from where let $\varepsilon \to 0^+$ and obtain

$$\int_\Omega \nabla u \nabla\eta \geq 0 \quad \text{for } \forall\eta \in C_0^\infty. \tag{1.1.2}$$

To show that u is harmonic on the set Ω_φ, we recall that for $\varepsilon > 0$ small enough the perturbation of the solution $v = u + \varepsilon\eta$ should be still above the obstacle, i.e. $u + \varepsilon\eta \geq \varphi$, to be in the set $\mathcal{K}$. For $\{u > \varphi\}$ (1.1.2) holds for every test function η (and small enough ε), not necessarily nonnegative, so it must also hold for $-\eta$. From this we obtain

$$\int_\Omega \nabla u\nabla\eta = 0 \quad \text{for } \forall\eta \in C_0^\infty$$

which shows that u is weakly harmonic and by Weyl's lemma [33] we conclude that u is almost everywhere equal to a smooth harmonic function on Ω_φ. On the other hand, for the coincidence set $\{u = \varphi\}$ we can choose only nonnegative test function η for v to stay above the obstacle, i.e. $v = u + \varepsilon\eta = \varphi + \varepsilon\eta \geq \varphi$, which with (1.1.2) proves that u is weakly superharmonic.

Based on this reasoning we can rewrite our problem in another way as

$$\min\{-\Delta u, u - \varphi\} = 0 \tag{1.1.3}$$

subject to the boundary condition $u|_{\partial\Omega} = g(x)$

In principle we have two main questions for the obstacle problem. The former question is what is the optimal regularity of the solution u, and the latter one is how smooth is the free boundary Γ_φ.

Concernig the first question, it was first proven by Jens Frehse in 1972 [15] that the solution of the classical obstacle problem is $C^{1,1}_{loc}(\Omega)$, i.e. it has bounded second derivatives, if the obstacle is $C^{1,1}(\Omega)$. It can be seen from the heuristic observation that regularity can not exceed this result, using previous facts about harmonicity of the solution u and the definition of the coincidence set Λ_φ we know that

- $\Delta u = 0$ on $\Omega_\varphi(u)$
- $\Delta u = \Delta\varphi$ on $\Lambda_\varphi(u)$

Thus we see that Δu experiences a jump across the free boundary $\Gamma_\varphi(u)$ and therefore the second derivatives of u cannot be continuous in Ω.

The smoothness of the free boundary depends on the type of the boundary points which in the classical obstacle problem can be two types: regular and singular.

Definition 1.1.4. A free boundary point x_0 is called *regular* if there exists a small number $r_0 > 0$ such that for every $0 < r < r_0$ one has

$$cr^2 \leq \sup_{B_r(x_0)} (u - \varphi) \leq Cr^2,$$

for some universal constants $0 < c \leq C < \infty$ and $B_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}$, $r > 0$.

In other words, the function $u - \varphi$ exhibits quadratic growth near a regular free boundary point $x_0 \in \Gamma_\varphi(u)$.

Definition 1.1.5. A free boundary point x_0 is called *singular* if the coincidence set has vanishing n-dimensional density at x_0, i.e.,

$$\lim_{r\to 0^+} \frac{|\Lambda_\varphi(u) \cap B(x_0, r)|}{|B(x_0, r)|} = 0$$

To better demonstrate the difference let us consider two global solutions of the obstacle problem with $\varphi = 0$ (figure 2):

(i) $u(x) = \frac{1}{2}(x_n^+)^2$

(ii) $u(x) = \frac{1}{2}x_n^2$

In the first example the coincidence set $\Lambda(u) = \{u(x) = 0\} = \{(x', x_n) \in \mathbb{R}^n \mid x_n \leq 0\}$ and the free boundary $\Gamma(u)$ is the hyperplane $\{x_n = 0\}$ in $\mathbb{R}^n$. One can derive that all boundary points are regular and the contact set has positive Lebesgue measure at x_0, that is

$$\limsup_{r\to 0^+} \frac{|\{u(x) = 0\} \cap B_r(x_0)|}{|B_r(x_0)|} > 0$$

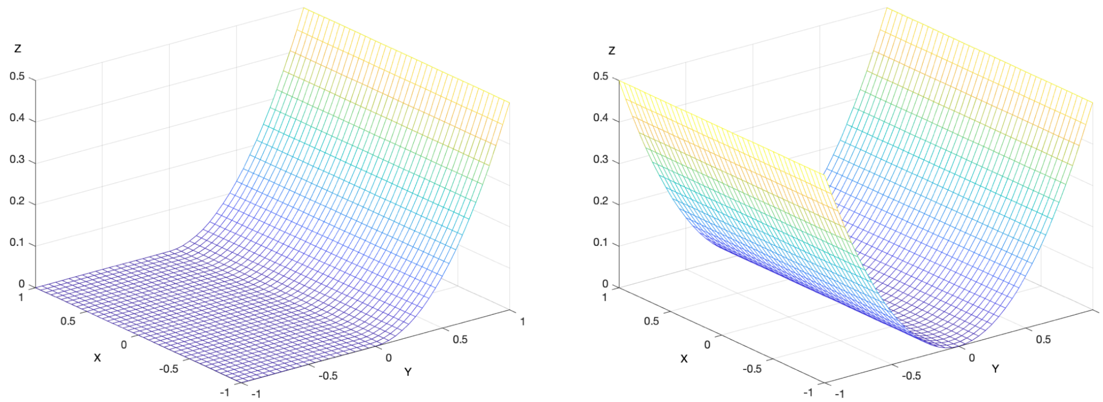

Figure 2. Examples of the solutions u with regular (left) and singular (right) free boundary

In the second example the coincidence set is the hyperplane $\Lambda(u) = \{u(x) = 0\} = \{x_n = 0\} = \Gamma(u)$ and thus $|\Lambda(u) \cap B_r(x_0)| = 0$ for every $r > 0$. Consequently, every free boundary point is a singular point for this solution.

Remark. If at a free boundary point x_0 the set $\Lambda_\varphi(u)$ has a cusp or a pinched bottleneck (figure 3), then x_0 is a singular point.

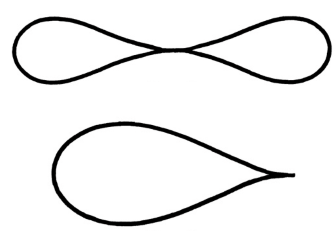

Figure 3. Singular points at cusps and pinched bottleneck [17]

Returning to the question of the smoothness of the free boundary, in [6] L. Caffarelli showed that if x_0 is a regular point (a point of the positive density for the coincidence set), then in a neighborhood of x_0 the free boundary is a $C^{1,\alpha}$ hypersurface and the solution u is C^2 up to the free boundary. In the same year D. Kinderlehrer and L. Nirenberg established the following result [24]: if the obstacle φ is real analytic, and one knows a priori that near $x_0 \in \Gamma_\varphi(u)$ the solution $u \in C^2$ up to $\Gamma_\varphi(u)$, and the free boundary is a C^1 hypersurface, then in fact it is real analytic. In this way, one can conclude that at any regular point the free boundary is real analytic.

Concerning the singular part of the free boundary, it is natural to consider a classification of points in singular set $\Sigma(U) \subset \Gamma_\varphi(U)$ based on the so-called dimension d_{x_0} of $\Sigma(U)$ at the a singular point $x_0 \in \Sigma(U)$ (which is defined in details later in the thesis). It is possible now to introduce the set

$$\Sigma_d(U) = \{x_0 \in \Sigma(U) \mid d_{x_0} = d\}, \quad d = 1, ..., n-2$$

In [9] L. Caffarelli established that $\Sigma_d(U)$ is locally contained in a d-dimensional manifold of class C^1. One year later, for the case $n = 2$, G. Weiss proved in [32] that C^1-regularity can be improved to $C^{1,\alpha}$ by means of an epiperimetric-type approach. These proofs are based on using the different monotonicity formulas, which more detailed discussed further in the work.

As mentioned at the beginning of the section, the basic motivating example of the obstacle problem is to find an equilibrium configuration of an elastic membrane whose boundary is held fixed at the height (descibed by the function g), and which is restricred to remain above a given obstacle φ (figure 4). As in [9] we can assume that the potential energy of such membrane is proportional to the area of this membrane which is given by the functional

$$\mathcal{A}(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}$$

and by the principle of minimal potential energy the equilibrium position can be find minimizing $\mathcal{A}$. By Taylor formula for small $|\nabla u|$ we obtain the approximation

$$\mathcal{A} \approx \int_\Omega 1 + \frac{|\nabla u|^2}{2}$$

And since constant 1 will not change the minimization problem, minimizing the area $\mathcal{A}$ is the same as minimizing the Dirichlet integral (1.1.1) over set $\mathcal{K}$

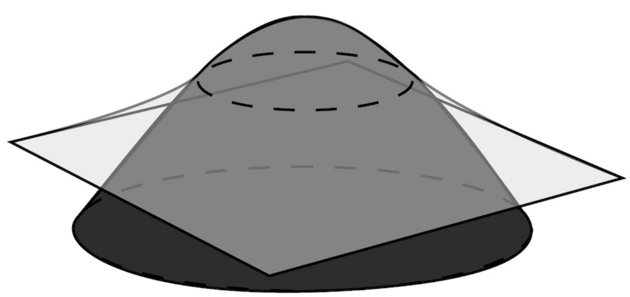

Figure 4. An elastic membrane with fixed boundaries (in this case they have quadratic shape, but it's not necessary) pulled on an obstacle (black)

1.2 The thin obstacle problem, the fractional laplacian

To move from the classical problem to the thin obstacle problem we start with a motivation which comes from the example of reverse osmosis of a saline concentration through the semipermeable membrane [12], [13]. Such membranes allow flow of fluid in one direction and block it in another one. The reverse osmosis is a mechanism which allow to remove low molecular weight solutes from the solvent. Applying the pressure, which exceeds the hydrostatic pressure in the salt solution side of the membrane, forces water to flow from the salt solution part to the other side, while the nature of the semipermeable membrane prevents the water from flowing back.

Mathematically, we let Ω be a region in $\mathbb{R}^n$ occupied by the pure solvent (figure 5, right) with the pressure field denoted by $u(x)$. We assume that $\mathcal{M} \subset \partial\Omega$ is our membrane with exterior unit normal ν. Combining the law of conservation of mass with Darcy's law, one finds that equilibrium configuration of the pressure is

$$\Delta u = 0 \quad \text{in } \Omega.$$

Let $\varphi(x)$ for $x \in \mathcal{M}$ be a fluid pressure applied on the outside of Ω. Then we have two possible scenarious

- $u(x) \le \varphi(x)$, when the solvent from the part with salt solution (figure 5, left) enters Ω
- $u(x) > \varphi(x)$, when the flux $\partial_\nu u = 0$

In the first case the flux $\partial_\nu u = -\lambda(u-h) > 0$, where $\lambda > 0$ is the permeability constant of the membrane. Given that the typical membrane thickness is negligible, in the infinite permeability case for $\lambda \to \infty$ we obtain following problem

$$u \ge \varphi, \quad \partial_\nu u \ge 0, \quad (u-\varphi)\partial_\nu u = 0 \quad \text{on } \mathcal{M}.$$

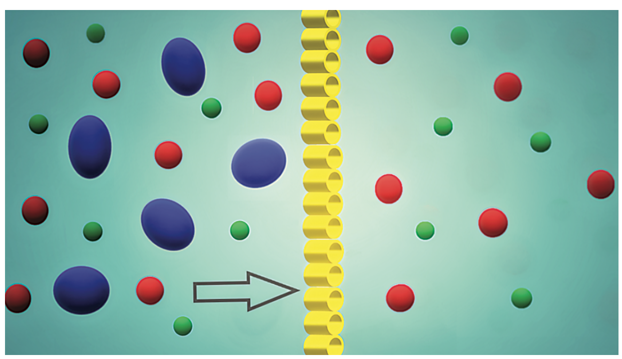

Figure 5. The process of the reverse osmosis [12]

Since the solution u is constrained to lie above φ not in the whole domain Ω as in the classical obstacle problem, but only on $\mathcal{M} \subset \mathbb{R}^{n-1}$, the function φ is referred as a *thin obstacle*, and the problem is called the *thin obstacle problem* (figure 6).

Another way to address the thin obstacle problem is by its local equivalence to a classical obstacle problem analogous to (1.1.3), but with the Laplacian replaced by the fractional Laplacian $(-\Delta)^s$, which was first introduced by M. Riesz in [28]:
Given a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$, find a function $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{cases} \min\{u-\varphi, (-\Delta)^s u\} = 0 & \mathbb{R}^n, \\ \lim_{|x|\to\infty} u(x) = 0 \end{cases} \tag{1.2.1}$$

where $0 < s < 1$ is the fractional power of the Laplacian and φ is the *obstacle*.

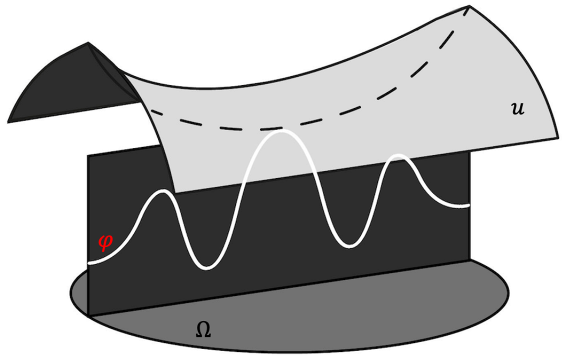

Figure 6. The thin obstacle problem

Definition 1.2.1. The *fractional Laplacian* of a function $u \in \mathcal{S}(\mathbb{R}^n)$ is the nonlocal operator in $\mathbb{R}^n$ defined by expression

$$(-\Delta)^s u(x) = \frac{\gamma(n,s)}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} dy$$

where $\gamma(n,s) > 0$ is a suitable normalisation constant given by

$$\gamma(n,s) = \frac{s2^s \Gamma(\frac{n}{2} + s)}{\pi^{\frac{n}{2}} \Gamma(1-s)}$$

By $\mathcal{S}(\mathbb{R}^n)$ we denoted the *Schwartz space* of rapidly decreasing functions, although larger classes can be allowed [16]. We recall that the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the space $C^\infty(\mathbb{R}^n)$ endowed with the metric topology

$$d(f,g) = \sum_{p=0}^{\infty} 2^{-p} \frac{\|f-g\|_p}{1 + \|f-g\|_p}$$

generated by the countable family of norms

$$\|f\|_p = \sup_{\alpha \leq p} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{\frac{p}{2}} |\partial^\alpha f(x)|, \quad p \in \mathbb{N} \cup \{0\}.$$

The pseudo-differential character of the fractional Laplacian is seen in the following identity [16]

$$\widehat{(-\Delta)^s} u(\xi) = (2\pi|\xi|)^{2s} \hat{u}(\xi)$$

where

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i <\xi, x>} f(x) dx$$

is a Fourier transform of a function f. Recall that the Fourier transform of a standard Laplacian is

$$\widehat{(-\Delta)f}(\xi) = (2\pi\xi)^2 \hat{f}(\xi)$$

since Laplacian is a linear operator.

As in the classical case we define the *coincidence set*

$$\Lambda_\varphi(u) = \{x \in \mathbb{R}^n : u(x) = \varphi(x)\}$$

and the *free boundary*

$$\Gamma_\varphi(u) = \partial\Lambda_\varphi(u).$$

We will denote them simply $\Lambda(u)$ and $\Gamma(u)$ when $\varphi = 0$.

From (1.2.1) in a similar way as it was obtained for the classical obstacle problem, it is possible to get that

- $(-\Delta)^s u = 0$ in the set $\{u > \varphi\}$
- $(-\Delta)^s u \geq 0$ in the whole $\mathbb{R}^n$

In this way, the problem (1.2.1) can be reformulated as following:
Given an obstacle φ, find a function u such that

$$u \geq \varphi, \quad (-\Delta)^s u \geq 0, \quad (u - \varphi)(-\Delta)^s u = 0 \quad \text{in } \mathbb{R}^n. \tag{1.2.2}$$

In [8] Caffarelli and Silvestre introduced a method to convert the global problem (1.2.1) in $\mathbb{R}^n$ into a local problem in $\mathbb{R}^{n+1}_+$. It was shown that if for a given $u \in \mathcal{S}(\mathbb{R}^n)$ one considers the function $U(x, y) : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ that solves the extension problem with the degenerate elliptic operator

$$\begin{cases} L_a U(x, y) = \operatorname{div}_{x,y}(|y|^a \nabla_{x,y} U) = 0, \quad a = 1 - 2s \\ U(x, 0) = u(x) \end{cases} \tag{1.2.3}$$

then one can use the weighted Dirichlet-to-Neumann relation to find the fractional Laplacian

$$(-\Delta)^s u(x) = -\frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)} \lim_{y\to 0^+} y^a \frac{\partial U}{\partial y} \tag{1.2.4}$$

and use the following proposition to obtain the connection with the global problem (1.2.1).

Proposition 1.2.2. *([8]) Let $U : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be a solution to $L_a U = 0$ such that for a given $R > 0$ one has for $|x| < R$*

$$\lim_{y\to 0^+} y^a \frac{\partial U}{\partial y}(x, y) = 0.$$

If we define $\tilde{U} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be even reflection in y, i.e.,

$$\tilde{U}(x,y) = \begin{cases} U(x,y) & y \geq 0, \\ U(x,-y) & y < 0, \end{cases}$$

then $\tilde{U}$ is a weak solution to the equation

$$\operatorname{div}_{x,y}(|y|^a \nabla_{x,y} \tilde{U}) = 0,$$

in $\{(x,y) \in \mathbb{R}^{n+1} \mid |x|^2 + y^2 < R^2\}$

This makes the nonlocal obstacle problem (1.2.1) equivalent to a problem for the extension operator L_a in one dimension up with an obstacle that is still defined in the lower dimensional manifold $M = \mathbb{R}^n \times \{0\}$. Thus the function $\tilde{U}(x,y)$ satisfies the *thin obstacle problem* in $\mathbb{R}^{n+1}$

$$\begin{cases} L_a\tilde{U} = \operatorname{div}(|y|^a \nabla_{x,y}\tilde{U}) = 0 \quad \text{in } \mathbb{R}^{n+1}_+ \cup \mathbb{R}^{n+1}_-, \\ \tilde{U}(x,-y) = \tilde{U}(x,y) \quad \text{for } x \in \mathbb{R}^n, y \in \mathbb{R}, \\ \tilde{U}(x,0) \geq \varphi(x) \quad \text{for } x \in \mathbb{R}^n, \\ -\lim_{y\to 0^+} y^a D_y\tilde{U}(x,y) \geq 0 \quad \text{for } x \in \mathbb{R}^n, \\ \lim_{y\to 0^+} y^a D_y\tilde{U}(x,y) = 0 \quad \text{in } \{\tilde{U}(x,0) > \varphi(x)\} \end{cases} \tag{1.2.5}$$

Assume that u is a solution of the nonlocal obstacle problem (1.2.1), or equivalently (1.2.2), and that U is the solution of the Dirichlet problem (1.2.3). It is clear that third equation of (1.2.5) is obtained from the second condition of (1.2.3) and first equation of (1.2.2). Moreover, fourth equation of (1.2.5) is derived from the Dirichlet-to-Neumann relation (1.2.4) and the second condition $(-\Delta)^s u \geq 0$ of (1.2.2), with $a = 1-2s$. While for the fifth equation is again used (1.2.4) and the fact that $(-\Delta)^s u = 0$ in the set $\{u > \varphi\}$.

From (1.2.5) we see that at every $x \in \mathbb{R}^n$ we must have

$$(\tilde{U}(x,0) - \varphi(x)) \lim_{y\to 0^+} y^a D_y\tilde{U}(x,y) = 0$$

which is called the *Signorini conditions*. Signorini himself was calling them the *ambiguous boundary conditions*, since at any given contact point either $\lim_{y\to 0^+} y^a D_y\tilde{U}(x,y) = 0$ or $\tilde{U}(x,0) = \varphi(x)$, but it is not known which one is satisfied.

The thin obstacle problem finds a lot of applications in different contexts, e.g. in optimal stopping problems for stochastic processes with jumps with an application in finance [11], the study of the regularity of minimizers of interaction energies in kinetic equations [10], the quasi-geostrophic equation in the geophysical fluid dynamics, etc. In linear elasticity the Signorini problem, first proposed by him in [29], consists of finding an equilibrium configuration of a spherical elastic body resting on a rigid, frictionless horizontal plane. The problem is of type (1.2.5) for $s = \frac{1}{2}$ and so $a = 0$. This is why such problems are usually called the *problems of* *the Signorini type.*

The main goal of these thesis is, in fact, to study the structure of the singular free boundary in the slightly more developed problem, that is the degenerate thin obstacle problem with variable coefficients satisfying minimal assumptions on the coefficients (which will be revealed in the subsequent chapters) with the zero obstacle: minimize the generalised Dirichlet energy

$$\min_{V\in\mathcal{K}}\int_{\Omega} < A(x)\nabla V, \nabla V > |y|^a dX \tag{1.2.6}$$

where V ranges in the closed convex set

$$\mathcal{K} = \{V \in W^{1,2}(\Omega) \mid V = g \text{ on } \partial\Omega \setminus \mathcal{M}, V \geq 0 \text{ on } \mathcal{M}\}.$$

Since the studied problem is of a local nature, we assume hereafter that $\Omega = \mathbb{B}_1^+$, and that the thin manifold which supports the obstacle is flat and given by $\mathcal{M} = B_1 \subset \partial\mathbb{B}_1^+$.

The main result, Theorem 4.0.2, rests on a Weiss and a Monneau type monotonicity formulas implying homogeneity, nondegeneracy, uniqueness, and continuous dependence of blowups at singular free boundary points. The proofs follow the outline of the analogous results in [19] for the case of the Laplacian and in [21] for the case of the Lipschitz variable coefficients operator, with changing in the reasanoning caused by the degenerate part.

The structure of the thesis: The rest of the thesis is organized as follows.
In Chapter 2, we introduce main results of [19] and [21] regarding the study of the main objective of the thesis - the structure of the singular free boundary, but for the particular cases of (1.2.6). We observe how new challenges, caused by moving from the Laplace Signorini problem to the Signorini type problem for the variable coefficient elliptic operator, have been overcome in [21] at every level of the study: Almgren, Weiss, Monneau type monotonicity formulas, a blowup analysis, growth lemmas and nondegeneracy of a solution, which, in turn, helped to characterize and study the configuration of the singular points.
In Chapter 3, we present the last paper [4] on the optimal interior regularity and the regularity of the regular free boundary of our main problem (1.2.6). We explain main ideas and show the results which we use further in Chapter 4.
And finally, in Chapter 4, following the argumentation discussed in Chapter 2, we derive the principal results for our degenerate generalised problem (1.2.6) which are necessary to prove the main result - Theorem 4.0.2.

Chapter 2

Known results

As we mentioned in the previous chapter, our main problem is

$$\begin{cases} L_a U = \operatorname{div}_{x,y}(|y|^a A(x) \nabla_{x,y} U) = 0 & \text{in } \mathbb{B}_1^+, \\ \min\{U(x,0), -\partial_y^a U(x,0)\} = 0 & \text{in } B_1. \end{cases} \tag{2.0.1}$$

In this chapter we discuss studies of this problem with the zero-obstacle on different levels:

- when $a = 0$, $A(x) = \mathbb{I}$ (constant coefficients), and so $L_a = \Delta_{x,y}$ as in [19]
- when $a = 0$, and so $L_a = \operatorname{div}_{x,y}(A(x)\nabla_{x,y})$ as in [21]

We will introduce the results for these problems at different steps of the study comparing how the reasoning changes when we go between these two problems. It will be a prelude for the discussion on the main problem (2.0.1) in the Chapter 4.

We note that for $x \in \mathbb{R}^n, y > 0$, we have indicated $X = (x, y) \in \mathbb{R}^{n+1}$, and let $|X| = \sqrt{|x|^2 + y^2}$. For $X_0 \in \mathbb{R}^{n+1}$ and $r > 0$, we denote $\mathbb{B}_r(X_0) = \{X = (x, y) \in \mathbb{R}^{n+1} \mid |X - X_0| < r\}$, and $\mathbb{S}_r(X_0) = \partial\mathbb{B}_r(X_0)$ the ball and the sphere of radius r centred at X_0 in the thick space. When $X_0 = 0$ we simply write $\mathbb{B}_r$ and $\mathbb{S}_r$, instead of $\mathbb{B}_r(0)$ and $\mathbb{S}_r(0)$.

We let $\mathbb{B}_r^+ = \{X = (x, y) \in \mathbb{B}_r \mid y > 0\}$, $\mathbb{B}_r^- = \{X = (x, y) \in \mathbb{B}_r \mid y < 0\}$ be the upper and lower parts of the thick ball, and indicate with $B_r = \{x \in \mathbb{R}^n, y = 0 \mid |x| < r\}$ the ball centered at 0 with radius r in the thin space $\mathbb{R}^n$ with the thin sphere $S_r = \{x \in \mathbb{R}^n, y = 0 \mid |x| = r\}$.

As before, we denote

$$\Omega = \{(x', 0) \in B_1 \mid u(x', 0) > 0\}$$

the set where the solution u is above the obstacle $\varphi = 0$ in the thin ball (the difference is that now the obstacle φ lives in the lower dimensional manifold), and by

$$\Lambda = \{(x', 0) \in B_1 \mid u(x', 0) = 0\} \tag{2.0.2}$$

the *coincidence set*, i.e. the set where u touches the obstacle, with the *free boundary* $\Gamma(u) = \partial\Lambda(u)$ (relative to the topology of B_1).

We extend the solution U of (2.0.1) to the whole ball $\mathbb{B}_1$ by extending the coefficients a_{ij} and the boundary datum g in the following way

1) $g(x,y) = g(x,-y)$,
2) $a_{ij}(x,y) = a_{ij}(x,-y)$ for $i,j < n+1$ or $i = j = n+1$,
3) $a_{i,n+1}(x,y) = a_{i,n+1}(x,-y) = 0$ for $i < n+1$.

Under these assumptions, if we extend U to the whole ball $\mathbb{B}_1$ as an even function with respect to y, then the extended function (which is denoted by U as well) will satisfy $D_yU(x,0) = 0$ at every point $(x,0) \in B_1$ where the derivative exists. Such normal derivative D_yU might not exist along coincidence set, where $D_y^+U = \lim_{y\to 0^+} D_yU(x,y)$ and $D_y^-U = \lim_{y\to 0^-} D_yU(x,y)$ might be different from each other.

We have the following alternative representation of our problem (2.0.1) after extending it as described before. Given a matrix valued function $x \mapsto A(x) = [a_{ij}(x)]$ in $\mathbb{B}_1$, we consider the problem of the minimizing generalized energy in the measure $|y|^a dX$

$$\min_{U\in\mathcal{K}} \int_{\mathbb{B}_1} < A(x)\nabla U(X), \nabla U(X) > |y|^a dX, \tag{2.0.3}$$

where we look for the solution U in the closed convex set

$$\mathcal{K} = \{U \in W^{1,2}(\mathbb{B}_1) \mid U = g \text{ on } \mathbb{S}_1, U(x,0) \geq 0 \text{ on } B_1\}.$$

We state the assumptions on the matrix-valued function $x \mapsto A(x) = [a_{ij}(x)]$:

(i) Symmetry: $a_{ij}(x) = a_{ji}(x)$ for $i,j = 1,...,n$ and $\forall x \in B_1$.

(ii) Ellipticity: there exist $\lambda > 0$ such that, for every $x \in B_1$ and $\xi \in \mathbb{R}^{n+1}$, one has

$$\lambda|\xi|^2 \leq < A(x)\xi, \xi > \leq \lambda^{-1}|\xi|^2. \tag{2.0.4}$$

(iii) The Lipschitz continuous and independent of y coefficients: the entries of $A(x) = [a_{ij}(x)]$ are in $W^{1,\infty}(\mathbb{B}_1)$, that is, one has for some $Q > 0$ and every $x, z \in B_1$

$$|a_{ij}(x) - a_{ij}(z)| \leq Q|x - z|. \tag{2.0.5}$$

We as well will need the following assumption on $A(x) = [a_{ij}(x)]$

$$a_{ij}(x) = \sum_{i,j=1}^{n} b_{ij}(x) e_i \otimes e_j + e_{n+1} \otimes e_{n+1}. \tag{2.0.6}$$

One can notice that (2.0.6) is equivalent to the following compatibility condition

$$a_{i,n+1}(x,0) = 0 \quad \text{in } B_1 \quad \text{for } i = 1,..,n \tag{2.0.7}$$

2.1 The Almgren Monotonicity

It is well-known that there are two crucial ingredients in the study of the thin obstacle problem (2.0.1): the monotonicity of the Almgren, Weiss and Monneau type formulas and the subsequent blowup analysis. The principal objective of this section is to present the approach of establishing the monotonicity of the Almgren formula for the Laplace Signorini problem and the Signorini type problem for the variable coefficient elliptic operator. First of all, let us introduce the conformal factor

$$\tilde{\mu}(X) =< A(x)\nabla r(X), \nabla r(X) >= \frac{< A(x)X, X >}{|X|^2}. \tag{2.1.1}$$

We next observe that when $A = \mathbb{I}_{n+1}$ we have $\tilde{\mu} \equiv 1$. From the assumption (2.0.4), we obtain that

$$\lambda \leq \tilde{\mu}(X) \leq \lambda^{-1}, \quad X \in \mathbb{B}_1. \tag{2.1.2}$$

We have also the following useful lemma.

Lemma 2.1.1. [21] *Suppose that* $A(0) = \mathbb{I}_{n+1}$. *Then, one has*

1. $\tilde{\mu}(0) = 1$,
2. $|1 - \tilde{\mu}(X)| \leq C|X|$,
3. $|\nabla\tilde{\mu}(X)| \leq C$,

where $C > 0$ *is some universal constant.*

Remark. When we say that a constant is universal, we mean that it depends exclusevely on n, on the ellipticity bound λ on $A(x)$ (2.0.4) and on the Lipschitz bound Q in (2.0.5) on the coefficients $a_{ij}(x)$.

We denote in the most general form the following necessary quantities. For any $r \in (0,1)$ the *height* function of U in $\mathbb{S}_r$ is defined as

$$H(U,r) = \int_{\mathbb{S}_r} U^2 \mu d\sigma, \tag{2.1.3}$$

where $\mu = \tilde{\mu}|y|^a$. We denote the *Dirichlet integral* of U in $\mathbb{B}_r$ by

$$D(U,r) = \int_{\mathbb{B}_r} < A(x)\nabla U, \nabla U > |y|^a dX. \tag{2.1.4}$$

And we define the *total energy* of U in $\mathbb{B}_r$ as

$$I(U,r) = \int_{\mathbb{S}_r} U < A\nabla U, \nu > |y|^a d\sigma. \tag{2.1.5}$$

It is shown later that $I(r) = D(r)$ when $\varphi = 0$.

In the original way the *Almgren frequency formula* is defined as following

$$N(U,r) = \frac{rD(r)}{H(r)}. \tag{2.1.6}$$

Henceforth, when the function U is fixed, we will write $H(r)$, $D(r)$, $I(r)$, $N(r)$ instead of $H(U,r)$, $D(U,r)$, $I(U,r)$, $N(U,r)$.

2.1.1 The Laplace Signorini problem

Let us now move the discussion to the more specific case of the problem (2.0.1). When $a = 0$ and $A(x) = \mathbb{I}_{n+1}$, the extension operator becomes $L_a = \Delta_{x,y}$. To simplify notations we move back to one dimension less, so the main problem (2.0.1) converts into

$$\begin{cases} \Delta u = 0, \text{ in } B_1^+, \\ \min\{u(x',0), -\partial_n u(x',0)\} = 0 \text{ in } B_1', \end{cases} \tag{2.1.7}$$

where we let $x' \in \mathbb{R}^{n-1}$ and denote a generic point in $\mathbb{R}^n$ as $x = (x', x_n)$. We also replaced the solution U with u for easier recognition of this particular case (2.1.7) of the main problem. We denote the thick ball as $B_1 = \{x \in \mathbb{R}^n \mid |x| < 1\}$ and indicate with $B_1^+ = B_1 \cap \{x_n > 0\}$, $B_1^- = B_1 \cap \{x_n < 0\}$ the upper and lower halfs of the ball, and with $B_1' = B_1 \cap \{x_n = 0\}$ the thin ball.

The solution to (2.1.7) extended to the whole ball B_1, as described in the beginning of this chapter, satisfies the following *Signorini* or *complementary conditions*

$$\Delta u = 0 \quad \text{in } B_1^+ \cup B_1^-, \tag{2.1.8}$$

$$u(x', -x_n) = u(x', x_n), \tag{2.1.9}$$

$$u \geq 0 \quad \text{in } B_1', \tag{2.1.10}$$

$$< \nabla u, \nu_+ > + < \nabla u, \nu_- > \geq 0 \quad \text{in } B_1', \tag{2.1.11}$$

$$u(< \nabla u, \nu_+ > + < \nabla u, \nu_- >) = 0 \quad \text{in } B_1', \tag{2.1.12}$$

$$\int_{B_1} < \nabla u, \nabla \eta > = \int_{\Lambda_\varphi(u)} (< \nabla u, \nu_+ > + < \nabla u, \nu_- >)\eta, \quad \eta \in C_0^\infty(B_1), \tag{2.1.13}$$

where we denoted with $\nu_{\pm} = \mp e_n$ the outer unit normals to $B_1^{\pm}$ on B_1'. Using (2.1.11) and (2.1.13) we see that, in particular, u is superharmonic in B_1. We denote by $\mathfrak{S}_{\mathfrak{o}}$ the class of functions satisfying the Signorini conditions (2.1.8) - (2.1.13) above.

Reguarding the regularity of the solution of the Laplace Signorini problem, we have the following theorem.

Theorem 2.1.2. [5] *Let u be the solution of the Signorini problem* (2.1.8) - (2.1.13) *in the ball B_1. Then there exists $\alpha > 0$ such that $u \in C^{1,\alpha}(B_1^{\pm} \cup B_1')$. The bounds on the $C^{1,\alpha}$ norm of u exclusively depend on $\|u\|_{L^2(B_1)}$.*

We also have the following important result

Proposition 2.1.3. [17] *Let $u \in \mathfrak{S}_{\mathfrak{o}}$. Then $u^{\pm}$ are subharmonic in B_1*

We need to find the first variations of the height and the Dirichlet integral that will help to establish the monotonicity of the Almgren functional (2.1.6). The proofs and calculations for the case of the Laplace Signorini problem are taken from [17] or [19].

We obtain from (2.1.3) that for our problem ($a = 0$, $A(x) = \mathbb{I}$) the *height* is

$$H(r) = \int_{S_r} u^2 \tag{2.1.14}$$

and from (2.1.4) the *Direchlet integral* is defined as

$$D(r) = \int_{B_r} |\nabla u|^2. \tag{2.1.15}$$

Hereafter, for a fixed $r \in (0,1)$ and $0 < \varepsilon < r$ we denote by $B_{r\varepsilon}^{+} = B_r \cap \{\varepsilon < x_n < r\}$, $B_{r\varepsilon}^{-} = B_r \cap \{-r < x_n < \varepsilon\}$, $S_{r\varepsilon}^{\pm} = S_r \cap \overline{B_{r\varepsilon}^{\pm}}$, $L_{r\varepsilon} = B_{r\varepsilon}^{\pm} \cap \{x_n = \pm\varepsilon\}$

Lemma 2.1.4. *Let $u \in \mathfrak{S}_{\mathfrak{o}}$. Then, for every $r \in (0,1)$ one has*

$$D(r) = \int_{S_r} u u_{\nu}.$$

Proof. Since, by (2.1.8), u is harmonic in $B_{r\varepsilon}^{\pm}$ we have in those sets that

$$\frac{1}{2}\Delta(u^2) = \frac{1}{2}\operatorname{div}(2u\nabla u) = |\nabla u|^2 + u\Delta u = |\nabla u|^2.$$

By the divergence theorem we have

$$\int_{B_{r\varepsilon}^{+}} |\nabla u|^2 = \frac{1}{2}\int_{B_{r\varepsilon}^{+}} \Delta(u^2) = \int_{S_{r\varepsilon}^{+}} u\partial_{\nu} u + \int_{L_{r\varepsilon}} u\partial_{\nu}^{+} u.$$

With the Theorem 2.1.2 we let $\varepsilon \to 0^+$ to conclude that

$$\int_{B_r^+} |\nabla u|^2 = \int_{S_r^+} u\partial_\nu u + \int_{B_r'} u\partial_\nu^+ u.$$

Repeating the argument for B_r^- and summing together we find that

$$\int_{B_r} |\nabla u|^2 = \int_{S_r} u\partial_\nu u + \int_{B_r'} u(\partial_\nu^+ u + \partial_\nu^- u) = \int_{S_r} u u_\nu,$$

where in the last step we used (2.1.12). □

Lemma 2.1.5. (First variation of the height) *Let $u \in \mathfrak{S}_{\mathfrak{o}}$. Then, for every $r \in (0,1)$ one has*

$$H'(r) = \frac{n-1}{r} H(r) + 2D(r).$$

Proof. Since outer normal to the sphere $\nu = \frac{x}{r}$ we have

$$\begin{aligned}
\int_{S_{r\varepsilon}^+} u^2 &= \frac{1}{r}\int_{S_{r\varepsilon}^+} < u^2 x, \nu >= \frac{1}{r}\int_{\partial B_{r\varepsilon}^+} < u^2 x, \nu > - \frac{1}{r}\int_{L_{r\varepsilon}^+} < u^2 x, \nu > \\
&\overset{\text{div.thm}}{=} \frac{1}{r}\int_{B_{r\varepsilon}^+} \operatorname{div}(u^2 x) - \frac{1}{r}\int_{L_{r\varepsilon}^+} < u^2 x, \nu >= \\
&= \frac{n}{r}\int_{B_{r\varepsilon}^+} u^2 + \frac{2}{r}\int_{B_{r\varepsilon}^+} u < x, \nabla u > + \varepsilon \int_{L_{r\varepsilon}^+} u^2,
\end{aligned}$$

where the sign of the last term changed due to the fact that $\nu_+ = -e_n$. Let $\varepsilon \to 0^+$ and use Theorem 2.1.2, we find

$$\int_{S_r} u^2 = \frac{n}{r}\int_{B_r^+} u^2 + \frac{2}{r}\int_{B_r^+} u < x, \nabla u > .$$

Adding to this equation the analogous one for B_r^-, we conclude

$$H(r) = \frac{1}{r}\left[n\int_{B_r} u^2 + 2\int_{B_r} u < x, \nabla u >\right]. \tag{2.1.16}$$

Using Cavalieri's principle, and differentiating (2.1.16) with respect to r we obtain

$$\begin{aligned}
H'(r) &= -\frac{1}{r^2}\left[n\int_{B_r} u^2 + 2\int_{B_r} u < x, \nabla u >\right] + \frac{1}{r}\left[n\int_{S_r} u^2 + 2\int_{S_r} u < x, \nabla u >\right] \\
&= -\frac{1}{r}H(r) + \frac{1}{r}\left[nH(r) + 2r\int_{S_r} u u_\nu\right] \\
&= \frac{n-1}{r}H(r) + 2D(r),
\end{aligned}$$

where in the second-to-last step we used that the outer normal to the sphere $\nu = \frac{x}{r}$ and $u_\nu =< \nabla u, \nu >$, and in the last step we applied the Lemma 2.1.4. $\square$

The following famous inequality will be useful in the computing the first variation of the Dirichlet integral. We also provide the proof.

Proposition 2.1.6. (Rellich identity) *Let $\Omega \subset \mathbb{R}^n$ be a piecewise C^1 domain. Given a function $u \in C^2(\overline{\Omega})$ one has*

$$\int_{\partial\Omega} |\nabla u|^2 < x, \nu >= (n-2)\int_\Omega |\nabla u|^2 + 2\int_{\partial\Omega} < x, \nabla u > u_\nu - 2\int_\Omega < x, \nabla u > \Delta u \quad (2.1.17)$$

Proof. First, we observe that

$$\begin{aligned} 2 < x, \nabla u > \Delta u &= 2\sum_{k=1}^n\sum_{i=1}^n x_k \frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_i^2} = \sum_{k=1}^n\sum_{i=1}^n \left[2x_k \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\right) - x_k \frac{\partial}{\partial x_k}\left(\frac{\partial u}{\partial x_i}\right)^2\right] \\ &= 2\sum_{k=1}^n\sum_{i=1}^n x_k \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\right) - < x, \nabla(|\nabla u|^2) > . \end{aligned} \quad (2.1.18)$$

Let us analyse terms in (2.1.18) separately

$$\begin{aligned} 2\int_\Omega \sum_{k=1}^n\sum_{i=1}^n x_k \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\right) &= 2\int_\Omega \sum_{k=1}^n\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}x_k\right) - 2\int_\Omega \sum_{k=1}^n\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}\delta_{i,k} \\ &= 2\int_{\partial\Omega} < x, \nabla u > u_\nu - 2\int_\Omega |\nabla u|^2. \end{aligned} \quad (2.1.19)$$

We obtained two terms for the final equation (2.1.17), now we consider the second term of (2.1.18)

$$\int_\Omega < x, \nabla(|\nabla u|)^2 >= \int_\Omega \mathrm{div}(|\nabla u|^2 x) - n\int_\Omega |\nabla u|^2 \overset{\text{div.thm}}{=} \int_{\partial\Omega} |\nabla u|^2 < x, \nu > -n\int_\Omega |\nabla u|^2. \quad (2.1.20)$$

If we combine (2.1.18), (2.1.19), (2.1.20), we obtain the desired result. $\square$

Lemma 2.1.7. (First variation of the Dirichlet integral) *Let $u \in \mathfrak{S}_\mathfrak{o}$. Then, for every $r \in (0,1)$ one has*

$$D'(r) = \frac{n-2}{r}D(r) + 2\int_{S_r} u_\nu^2.$$

Proof. We apply Proposition 2.1.6 to $\Omega = B^+_{r\varepsilon}$ and use (2.1.8)

$$\int_{\partial B^+_{r\varepsilon}} |\nabla u|^2 < x, \nu >= (n-2)\int_{B^+_{r\varepsilon}} |\nabla u|^2 + 2\int_{S^+_{r\varepsilon}} < x, \nabla u > u_\nu + 2\int_{L^+_{r\varepsilon}} < x, \nabla u > u_\nu. \tag{2.1.21}$$

On the other hand, we have

$$\begin{aligned}\int_{\partial B^+_{r\varepsilon}} |\nabla u|^2 < x, \nu > &= \int_{S^+_{r\varepsilon}} |\nabla u|^2 < x, \nu > + \int_{L^+_{r\varepsilon}} |\nabla u|^2 < x, \nu > \\ &= r\int_{S^+_{r\varepsilon}} |\nabla u|^2 - \varepsilon \int_{L^+_{r\varepsilon}} |\nabla u|^2.\end{aligned} \tag{2.1.22}$$

Combining (2.1.21) and (2.1.22) and using the Theorem 2.1.2, we let $\varepsilon \to 0^+$

$$\int_{S^+_r} |\nabla u|^2 = \frac{n-2}{r}\int_{B^+_r} |\nabla u|^2 + 2\int_{S^+_r} u^2_\nu + \frac{2}{r}\int_{B'_r} < x', \nabla_{x'} u > u^+_\nu. \tag{2.1.23}$$

Repeating the argument for $B^+_{r\varepsilon}$ and summing with (2.1.23), we obtain

$$D'(r) = \int_{S_r} |\nabla u|^2 = \frac{n-2}{r}\int_{B_r} |\nabla u|^2 + 2\int_{S_r} u^2_\nu + \frac{2}{r}\int_{B'_r} < x', \nabla_{x'} u > (u^+_\nu + u^-_\nu).$$

To see that the last integral vanishes it is enough to split it as follows

$$\begin{aligned}\int_{B'_r} < x', \nabla_{x'} u > (u^+_\nu + u^-_\nu) &= \int_{\Lambda(u)\cap B'_r} < x', \nabla_{x'} u > (u^+_\nu + u^-_\nu) \\ &+ \int_{\{(x',0)\in B'_r \mid u(x',0)>0\}} < x', \nabla_{x'} u > (u^+_\nu + u^-_\nu) = (I) + (II).\end{aligned}$$

On the $\Lambda(u) \cap B'_r$ we have that $u(x', 0) = 0$. Since in general, by (2.1.10), $u(x', 0) \geq 0$ in B_1, the function $x \mapsto u(x', 0)$ reaches its minimum value on $\Lambda(u) \cap B'_r$ and by Fermat's theorem we must have $\nabla_{x'} u = 0$, and therefore $< x', \nabla_{x'} u >= 0$. Thanks to the Theorem 2.1.2 the function $u^+_\nu + u^-_\nu$ is bounded in B'_1, thus we can conclude that $(I) = 0$.

On the set $\{(x', 0) \in B'_r \mid u(x', 0) > 0\}$ the condition (2.1.12) gives $u^+_\nu + u^-_\nu = 0$, and thus $(II) = 0$. And therefore the desired conclusion follows. □

We are ready to prove the main result of the section.

Theorem 2.1.8. (Almgren's monotonicity formula) *Let $u \in \mathfrak{S}_\mathfrak{o}$ in the ball $B_1 = \{x \in \mathbb{R}^n \mid |x| < 1\}$. Then the frequency of u*

$$r \mapsto N(r) = \frac{r\int_{B_r} |\nabla u|^2}{\int_{S_r} u^2}$$

is increasing in $(0,1)$*. Furthermore,* $N(u,r) \equiv k$ *if and only if* u *is homogeneous of degree* k *in* B_1*.*

Proof. To demonstrate the monotonicity of $N(r)$, it is enough to show that $r \mapsto \log N(r)$ is increasing. From (2.1.6), Lemma 2.1.5 and Lemma 2.1.7 we find

$$\begin{aligned}\frac{d}{dr}\log N(r) &= \frac{D'(r)}{D(r)} + \frac{1}{r} - \frac{H'(r)}{H(r)} = \frac{n-2}{r} + 2\frac{\int_{S_r} u_\nu^2}{D(r)} + \frac{1}{r} - \frac{n-1}{r} - 2\frac{D(r)}{H(r)} \\ &= 2\frac{\int_{S_r} u_\nu^2}{D(r)} - 2\frac{D(r)}{\int_{S_r} u^2} = 2\frac{\int_{S_r} u_\nu^2}{\int_{S_r} uu_\nu} - 2\frac{\int_{S_r} uu_\nu}{\int_{S_r} u^2}.\end{aligned}$$

The following Cauchy-Schwarz inequality

$$\left(\int_{S_r} uu_\nu\right)^2 \leq \int_{S_r} u^2 \int_{S_r} u_\nu^2$$

gives the desired result that $\frac{d}{dr}\log N(r) \geq 0$, i.e. the map $r \mapsto N(r)$ is increasing, that proves the first part of the theorem.

Now we assume that u is homogeneous of degree k. By the Euler's formula we have for any $x \in B_1$

$$< x, \nabla u(x) >= ku(x).$$

By Lemma 2.1.4

$$D(r) = \int_{S_r} uu_\nu = \frac{1}{r}\int_{S_r} u < x, \nabla u >= \frac{k}{r}\int_{S_r} u^2 = \frac{k}{r}H(r).$$

We infer that it must be $N \equiv k$ for $r \in (0,1)$.

Vice-versa, let us assume that $N \equiv k$ for $r \in (0,1)$, and we want to show that u is homogeneous of degree k in B_1 that is equivalent to show that the Euler's formula holds. Since we should have then $\frac{d}{dr}\log N(r) \equiv 0$ for $r \in (0,1)$, the above computation shows that we must have for any such r

$$\frac{\int_{S_r} u_\nu^2}{\int_{S_r} uu_\nu} - \frac{\int_{S_r} uu_\nu}{\int_{S_r} u^2} = 0.$$

This means that we have equality in the above application of the Cauchy-Schwarz inequality, and therefore for every $r \in (0,1)$ there exists $\alpha(r) \in \mathbb{R}$ such that $u_\nu = \alpha(r)u$ on S_r. This fact and Lemma 2.1.4 now

$$k \equiv N(r) = \frac{r\int_{S_r} uu_\nu}{\int_{S_r} u^2} = r\alpha(r),$$

hence we have $\alpha(r) = \frac{k}{r}$ and thus

$$u_\nu = \frac{1}{r} < x, \nabla u >= \frac{k}{r} u(x).$$

By the arbitrariness of $r \in (0,1)$ we conclude that u is homogeneous of degree k in the whole ball B_1. $\square$

2.1.2 The Signorini type problem for the variable coefficient elliptic operator

We consider the Signorini problem for a divergence-form elliptic operator with Lipschitz coefficients in the case of a zero thin obstacle

$$\begin{cases} L_a U = \operatorname{div}(A(x)\nabla U) = 0 \quad \text{in } \mathbb{B}_1^+, \\ \min\{U(x,0), -\partial_{n+1} U(x,0)\} = 0 \quad \text{in } B_1. \end{cases} \tag{2.1.24}$$

The solution of the problem (2.1.24) extended to the whole ball $\mathbb{B}_1$, as described in the beginning of this chapter, satisfies the following *Signorini conditions*:

$$L_a U = \operatorname{div}(A(x)\nabla U(x,y)) = 0 \quad \text{in } \mathbb{B}_1^+ \cup \mathbb{B}_1^- \tag{2.1.25}$$

$$U \geq 0 \quad \text{in } B_1 \tag{2.1.26}$$

$$< A\nabla U, \nu_+ > + < A\nabla U, \nu_- > \geq 0 \quad \text{in } B_1 \quad (\nu_\pm = \mp e_{n+1}) \tag{2.1.27}$$

$$U(< A\nabla U, \nu_+ > + < A\nabla U, \nu_- >) = 0 \quad \text{in } B_1 \tag{2.1.28}$$

$$\int_{\mathbb{B}_1} < A\nabla U, \nabla \eta >= \int_{B_1} (< A\nabla U, \nu_+ > + < A\nabla U, \nu_- >)\eta, \quad \eta \in C_0^\infty(\mathbb{B}_1) \tag{2.1.29}$$

We denote by $\mathfrak{S}_1$ the class of functions satisfying the Signorini conditions (2.1.25) - (2.1.29) above. The proofs and calculations for the case of the Signorini type problem (2.1.24) follow [21] and sometimes [18]. Although in the papers all calculations were made for $\mathbb{R}^n$, we will present them in $\mathbb{R}^{n+1}$ to make it easier to compare the results with whose of calculations for the main problem (2.0.1) presented in Chapter 4. We obtain from (2.1.3) and (2.1.4) that for the current problem ($a = 0$) the *height* function is defined as

$$H(r) = \int_{\mathbb{S}_r} U^2 \tilde{\mu} \tag{2.1.30}$$

and the *Dirichlet integral* is

$$D(r) = \int_{\mathbb{B}_r} < A\nabla U, \nabla U > . \tag{2.1.31}$$

For the case $\varphi = 0$ we have that

$$D(r) = \int_{\mathbb{S}_r} U < A\nabla U, \nu > . \tag{2.1.32}$$

Remark. By the results in [1], [2], under the assumption of Theorem 2.2.25, we know that weak solution U of (2.1.24) is in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{B}_1^{\pm} \cup B_1)$.

We recall the following first variation formula for the height [18]:

$$H'(r) = 2\int_{\mathbb{S}_r} U < A\nabla U, \nu > + \int_{\mathbb{S}_r} u^2 L|X|. \tag{2.1.33}$$

From this formula one can prove the following proposition.

Proposition 2.1.9. *Let $U \in \mathfrak{S}_1$. Assume that the normalisation hypothesis $A(0) = \mathbb{I}_{n+1}$ is in place. Under this assumption, for almost $0 < r < 1$ one has*

$$H'(r) - \frac{n}{r}H(r) - 2\int_{\mathbb{S}_r} U < A\nabla U, \nu >= O(1)H(r)$$

Theorem 2.1.10. (First variation of the Dirichlet integral) *Let $U \in \mathfrak{S}_1$. Suppose that the normalisation hypothesis $A(0) = \mathbb{I}_{n+1}$ is in place and that furthermore (2.0.7) is in force. Then, for almost every $r \in (0,1)$ one has*

$$D'(r) = 2\int_{\mathbb{S}_r} \frac{(< A\nabla U, \nu >)^2}{\tilde{\mu}} + \left(\frac{n-1}{r} + O(1)\right) D(r).$$

Using Proposition 2.1.9 and Theorem 2.1.10, the authors of [21] proves the following important result.

Theorem 2.1.11. *Let $U \in \mathfrak{S}_1$. Assume the hypothesis of the theorem Theorem 2.1.10 is satisfied. Then, there exists a universal constant $C > 0$ such that the function*

$$\tilde{N}(r) = e^{Cr}N(r) \tag{2.1.34}$$

is monotone nondecreasing in $(0,1)$. In particular, $\lim_{r\to 0^+} \tilde{N}(r) = \tilde{N}(0^+)$ exists. We conclude that $\lim_{r\to 0^+} N(r) = N(0^+)$ also exists, and equals $\tilde{N}(0^+)$

We do not show the proofs in this section, since they partly repeat the ideas of whose of the Laplace Signorini problem, previously shown, partly will be presented in the proofs for the generalised problem (2.0.1) in Chapter 4.

2.2 The blowup analysis and the regularity of the solution

The monotonicity of the Almgren frequency plays a fundamental role in controlling rescalings in the blowup analysis, presented in this section, which is required to establish the optimal regularity of solutions. Let $U \in \mathfrak{S}_1$ and assume that $X_0 = 0$ be a free boundary point. In the general form we define the *Almgren rescaling* for U at 0 as

$$\tilde{U}_r(x) := \frac{U(rX)}{d_r} \tag{2.2.1}$$

where

$$d_r = \left(\frac{H(r)}{r^{n+a}}\right)^{\frac{1}{2}} \tag{2.2.2}$$

We note the following important observation

$$\begin{aligned} H(r) &= \int_{\mathbb{S}_r} U^2 \tilde{\mu}|y|^a = r^{n+a} \int_{\mathbb{S}_1} U^2(rX)\tilde{\mu}(rX)|y|^a \\ &= r^{n+a} d_r^2 \int_{\mathbb{S}_1} \tilde{U}_r^2 \tilde{\mu}_r |y|^a = H(r) \int_{\mathbb{S}_1} \tilde{U}_r^2 \tilde{\mu}_r |y|^a \end{aligned}$$

where $\tilde{\mu}_r = \tilde{\mu}(rX)$. Therefore

$$\int_{\mathbb{S}_1} \tilde{U}_r^2 \tilde{\mu}_r |y|^a = 1 \tag{2.2.3}$$

The normalisation (2.2.3) is the main reason of the inroducing (2.2.1). Preliminarly, we need to introduce a classification of free boundary points. Since by Theorem 2.1.8, Theorem 2.1.11 the frequency is monotone and therefore it has a limit as $r \to 0^+$, a natural way of classifying the free boundary points is by means of the value of the frequency at the point in question.

Definition 2.2.1. Let $U \in \mathfrak{S}_1$ and assume that $0 \in \Gamma(U)$. We say that $0 \in \Gamma_k(U)$ if

$$N(U, 0^+) = \lim_{r \to 0^+} N(U, r) = k$$

When $X_0 \neq 0$ we say that $X_0 \in \Gamma_k(U)$ if $X_0 \in \Gamma(U)$ and

$$N(U(\cdot + X_0), 0^+) = k$$

The Almgren rescalings (2.2.1) and their homogeneous limits, Almgren blowups, play a key role in the establishing of the optimal interior regularity of the solution (Theorem 2.2.9, Theorem 2.2.25). Further, in the study of the smoothness of the singular points we will exploit another type of rescaling, the *homogeneous scaling of U of order k* (assuming that $U \in \mathfrak{S}_1$, with $0 \in \Gamma(U)$)

$$U_r(x) = \frac{U(rX)}{r^k}, \quad x \in B_{1/r}. \tag{2.2.4}$$

Remark. We let $U \in \mathfrak{S}_1$ in the assumptions, since $\mathfrak{S}_0$ is actually a subset of $\mathfrak{S}_1$ for $A(x) = \mathbb{I}$.

2.2.1 The Laplace Signorini problem

In our case $d_r = \left(\frac{H(r)}{r^{n-1}}\right)^{\frac{1}{2}}$, since $a = 0$ and we are in one dimension less then the main problem (2.0.1), i.e. in $\mathbb{R}^n$. First, we note that from (2.2.3), since $A(x) = \mathbb{I}_n$ and therefore $\tilde{\mu} = \tilde{\mu}_r = 1$ we have that

$$H(\tilde{u}_r, 1) = \int_{S_1} \tilde{u}_r^2 = 1 \tag{2.2.5}$$

Note that (2.2.5) means also

$$\|u\|_{L^2(\partial B_1)} = 1. \tag{2.2.6}$$

Since by definition of d_r

$$D(\tilde{u}_r, 1) = \int_{S_1} |\nabla \tilde{u}_r|^2 = \frac{r^{2-n}}{d_r^2} \int_{S_r} |\nabla u(rx)|^2 = \frac{rD(u,r)}{H(u,r)} = N(u,r)$$

we infer that

$$N(\tilde{u}_r, 1) = N(u, r). \tag{2.2.7}$$

In a similar way we can obtain that for every $r, \rho > 0$ such that $r\rho < 1$ one has

$$N(\tilde{u}_r, \rho) = N(u, r\rho) \tag{2.2.8}$$

Now we want to show how the monotonicity of the Almgren frequency helps to prove the following important result about an existence of limits of the Almgren rescalings.

Theorem 2.2.2. *Let $u \in \mathfrak{S}_0$ and suppose that $0 \in \Gamma(u)$. Assume that $N(u, 0^+) = k$. Given $r_j \to 0^+$, there exists a subsequence (which we will still denote by r_j) and a function $\tilde{u}_0 \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$ for some $\alpha \in (0, 1/2)$, such that $\tilde{u}_{r_j} \to \tilde{u}_0$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$. Such $\tilde{u}_0$ satisfies the condition* (2.1.8)-(2.1.13) *globally in $\mathbb{R}^n$, is homogeneous of degree k, and furthermore $\tilde{u}_0 \not\equiv 0$.*

Proof. Using (2.2.5), (2.2.7) and the monotonicity of $N(r)$ (Theorem 2.1.8), for $r \in (0, 1)$ one has

$$\int_{B_1} |\nabla \tilde{u}_r|^2 = N(\tilde{u}_r, 1) = N(u, r) \leq N(u, 1) < \infty. \tag{2.2.9}$$

Next we will need the following lemma

Lemma 2.2.3. (Trace inequality) *For $r > 0$ let $v \in W^{1,2}(B_r)$. Then, one has*

$$\frac{1}{r} \int_{B_r} v^2 \leq C(n) \left(\int_{S_r} v^2 + r \int_{B_r} |\nabla v|^2 \right)$$

Combining the trace inequality for $r = 1$ with (2.2.9) we find

$$\int_{B_1} \tilde{u}_r^2 \leq C(n) \left(\int_{S_1} \tilde{u}_r^2 + \int_{B_1} |\nabla \tilde{u}_r|^2 \right) \overset{(2.2.5)}{\leq} C(n)(1 + N(u,1)) < \infty$$

We conclude that there exists $C(u) > 0$, such that

$$\|\tilde{u}_r\|_{W^{1,2}(B_1)} \leq C(u) < \infty$$

Therefore there exists a function $\tilde{u}_0 \in W^{1,2}(B_1)$ and a sequence $r_j \to 0^+$ such that

$$\tilde{u}_{r_j} \to \tilde{u}_0 \text{ weakly in } W^{1,2}(B_1)$$

Since we know that the embedding $W^{1,2}(B_1) \hookrightarrow L^2(S_1)$ is compact, possibly passing to a subsequence we have a strong convergence

$$\tilde{u}_{r_j} \to \tilde{u}_0 \text{ in } L^2(S_1) \tag{2.2.10}$$

By Theorem 2.1.2 we also have for some $\beta \in (1, 1/2)$

$$\left\|\tilde{u}_{r_j}\right\|_{C^{1,\beta}(B^{\pm}_{1/2} \cup B'_{1/2})} \leq C(n, \|u\|_{W^{1,2}(B_1)})$$

By a standard diagonal process this implies the existence of a function $\tilde{u}_0$, and of a subsequence of $\{\tilde{u}_{r_j}\}_{j\in\mathbb{N}}$, such that for any $0 < \alpha < \beta$

$$\tilde{u}_{r_j} \to \tilde{u}_0 \text{ in } C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n_{\pm} \cup \mathbb{R}^{n-1}) \tag{2.2.11}$$

By change of variable it is seen that $\tilde{u}_r$ satisfies (2.1.8)-(2.1.13) in $B_{1/r}$. Passing the limit along the sequence r_j we conclude that $\tilde{u}_0$ is a global solution in $\mathbb{R}^n$. By Theorem 2.1.2 we know that $\tilde{u}_0 \in C^{1,\alpha}(B^{\pm}_1 \cup B'_1)$. And it is clear that $\tilde{u}_0$ is even in x_n. By (2.2.3) and (2.2.10) we have

$$1 \overset{(2.2.3)}{=} \int_{S_1} \tilde{u}^2_{r_j} \overset{(2.2.10)}{\to} \int_{S_1} \tilde{u}^2_0 \quad \text{as } j \to \infty \tag{2.2.12}$$

that conclude $\tilde{u}^2_0 \not\equiv 0$ on S_1.

We claim that if $\int_{S_r} \tilde{u}^2_0 = 0$ for some $0 < r < 1$, then $\tilde{u}^2_0 \equiv 0$ in B_1. Indeed, if $\int_{S_r} \tilde{u}^2_0 = 0$ then $\tilde{u}_0 = 0$ on S_r. Remember that $\int_{B_r} |\nabla \tilde{u}_0|^2 = \int_{S_r} \tilde{u}_0 < \nabla \tilde{u}_0, \nu >= 0$, so $\tilde{u}_0 \equiv c$ in B_r. Since $\tilde{u}_0 = 0$ on S_r, then $c \equiv 0$. But $\tilde{u}_0 = 0$ is harmonic in $B^+_1 \cup B^-_1$. Therefore, by the unique continuation property of harmonic functions, we have that $\tilde{u}_0 = 0$ in $B^{\pm}_1$, hence in B_1 (u is continuous in B_1), which contradicts (2.2.12).

Since $\int_{S_r} \tilde{u}^2_0 > 0$ we conclude

$$\begin{aligned} N(\tilde{u}_0, r) = r\frac{\int_{B_r} |\nabla \tilde{u}_0|^2}{\int_{S_r} \tilde{u}^2_0} &\overset{(2.2.11)}{=} \lim_{r_j \to 0^+} N(u_{r_j}, r) \\ &\overset{(2.2.8)}{=} \lim_{r_j \to 0^+} N(u, rr_j) = N(u, 0^+) = k \end{aligned}$$

It shows that frequency $N(\tilde{u}_0, \cdot) \equiv k$ is constant. By Theorem 2.1.8 we conclude that $\tilde{u}_0$ is homogeneous of degree k in B_1. $\square$

Definition 2.2.4. A function $\tilde{u}_0$ is called an *Almgren blowup* of the solution u at the free boundary point $0 \in \Gamma(u)$. Such global solution need not be unique.

Remark. Generally the blowups might be different over different subsequences $r = r_j \to 0^+$.

The next result is a growth lemma that plays a key role in the proof of the optimal regularity.

Lemma 2.2.5. (Growth lemma) *Let $u \in \mathfrak{S}_\mathfrak{o}$ in B_1 and assume that $0 \in \Gamma_k(u)$. If $N(0^+) \geq k$, then for $0 < r < 1/2$ one has*

$$\sup_{B_r} |u| \leq Cr^k,$$

for some universal constant $C > 0$

Proof. By the Almgren monotonicity theorem Theorem 2.1.8 we have that $N(r) \geq k$ for every $r \in (0,1)$. We note that the identity in Lemma 2.1.5 can be reformulated in the following way

$$r\frac{d}{dr}\log H(r) = \frac{rH'(r)}{H(r)} = n - 1 + 2\frac{rD(r)}{H(r)} = n - 1 + 2N(r) \tag{2.2.13}$$

And therefore from Theorem 2.1.8 we know that the function

$$r \mapsto r\frac{d}{dr}\log H(r)$$

is monotonically increasing for $r \in (0,1)$ and that

$$\frac{d}{dr}\log H(r) \geq \frac{n-1+2k}{r}.$$

Integrating from r to 3/4 we obtain

$$\log\frac{H(3/4)}{H(r)} \geq \log\left(\frac{3}{4r}\right)^{n-1+2k}$$

This gives for every $r \in (0, \frac{3}{4})$

$$H(r) \leq \left(\frac{4}{3}\right)^{n-1+2k} H(3/4)r^{n-1+2k} = Cr^{n-1+2k} \tag{2.2.14}$$

where $C = C(n, k, \|u\|_{L^2(S_{3/4})})$. Since $u^\pm$ are subharmonic functions in B_1 (Proposition 2.1.3), we obtain for every $r \in (0, 1/2)$

$$\sup_{B_r} u^\pm \leq C(n)\left(\frac{1}{r^{n-1}}\int_{S_{3r/2}}(u^\pm)^2\right)^{\frac{1}{2}} \leq C(u)r^k$$

which implies the desired conclusion □

Corollary 2.2.5.1. *Let $u \in \mathfrak{S}_{\mathfrak{o}}$ in B_1 and assume that $0 \in \Gamma_k(u)$. If $N(u, 0^+) \geq k$, the there exists a universal constant $C = C(n, k, \|u\|_{L^2(S_{3/4})})$ such that for $r \in (0, 3/4)$ one has*

$$D(r) \leq Cr^{n-2+2k}. \tag{2.2.15}$$

Proof. Integrating (2.2.14), we obtain for $r \in (0, 3/4)$

$$\int_{B_r} u^2 \leq Cr^{n+2k}, \tag{2.2.16}$$

where $C = C(n, k, \|u\|_{L^2(S_{3/4})})$. The desired conclusion will follow from the following Caccioppoli type inequality

$$D\left(\frac{r}{2}\right) \leq \frac{C}{r^2} \int_{B_r} u^2,$$

the proof of which can be found in the Corollary 2.11 [17]. □

Remark. If $x_0 \in \Gamma_k(u)$ then, in view of Theorem 2.2.2, the homogeneity of any Almgren blowup $\tilde{u}_0$ of $u(\cdot + x_0)$ at zero is precisely k. Thanks to Theorem 2.1.2, we must have $k \geq 1 + \alpha > 1$, for some $\alpha \in (0, 1/2)$.

The following theorem contains the basic information needed to establish the optimal regularity.

Theorem 2.2.6. (Minimal homogeneity) *Let $u \in \mathfrak{S}_{\mathfrak{o}}$, with $0 \in \Gamma_k(u)$. Then, we must have $k \geq 3/2$.*

The proof follows from Theorem 5.2 [30] which states that if the thin coincidence set of the Almgren blowup $\tilde{u}_0$ is convex, then $x' \mapsto \tilde{u}_0(x', 0)$ must be in $C^{1,\frac{1}{2}}_{\text{loc}}$. The convexity of the coincidence set, in turn, follows from the following result.

Theorem 2.2.7. *Let $u \in \mathfrak{S}_{\mathfrak{o}}$, with $0 \in \Gamma_k(u)$ and $1 < k < 2$. Then, for any $e \in \mathbb{R}^{n-1} \times \{0\}$ we have*

$$\partial_{ee}\tilde{u}_0(x', 0) \geq 0.$$

The minimal homogeneity allows to establish the following maximal growth of the solution near free boundary points.

Theorem 2.2.8. (Improved growth lemma) *Let $u \in \mathfrak{S}_{\mathfrak{o}}$ in B_1 and assume that $0 \in \Gamma_k(u)$. Then, for $0 < r < 1/2$ one has*

$$\sup_{B_r} |u| \leq Cr^{3/2}$$

for some universal constant $C > 0$.

The proof follows from Theorem 2.2.6 and Lemma 2.2.5. With Theorem 2.2.8 it is possible to establish the main result of the section.

Theorem 2.2.9. (Optimal regularity for zero obstacle) [3] *Let u be the solution of the Signorini problem* (2.0.1) *with the zero obstacle. Then $u \in C^{1,\frac{1}{2}}(B^{\pm}_{\frac{1}{2}} \cup B'_{\frac{1}{2}})$. More precisely, we have*

$$\|u\|_{C^{1,\frac{1}{2}}(B^{\pm}_{\frac{1}{2}} \cup B'_{\frac{1}{2}})} \leq C(n, \|u\|_{W^{1,2}(B_1)})$$

We continue with the following result about the asymptotic behavior of the homogeneous scalings.

Proposition 2.2.10. (Existence of the homogeneous blowups) *Let $u \in \mathfrak{S}_0$ and $0 \in \Gamma(u)$. Suppose that $N(u, 0^+) \geq 0$. Given $r_j \to 0$, there exists a subsequence (which we will still denote by r_j) and for any $\alpha \in (0, 1/2)$ a function $u_0 \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n_{\pm} \cup \mathbb{R}^{n-1})$ for some $\alpha \in (0, 1/2)$, such that $u_{r_j} \to u_0$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n_{\pm} \cup \mathbb{R}^{n-1})$. Such $\tilde{u}_0$ satisfies the condition* (2.1.8)-(2.1.13) *globally in $\mathbb{R}^n$.*

The proof is similar to the proof of the first part of the Theorem 2.2.2. Using (2.2.16) and (2.2.15), one can prove that $\{u_{r_j}\}_{j \in \mathbb{N}}$ is uniformly bounded in $W^{1,2}(B_1)$. Combining the estimate of Theorem 2.1.2 and the boundness of the sequence in $W^{1,2}(B_1)$, we can obtain the convergence in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n_{\pm} \cup \mathbb{R}^{n-1})$ for any $\alpha \in (0, 1/2)$ and passing through the limit we conclude that u_0 is a global solution.

Remark. We undrerline that Proposition 2.2.10 do not guarantee that the limit u_0 is non zero, and also whether u_0 is a homogeneous function of degree k. But these facts are actually truthful and can be proven with the use of the Weiss functional (theorem 3.12 [17]), so we can state the following definition.

Definition 2.2.11. Let $u \in \mathfrak{S}_0$ and $0 \in \Gamma(u)$. Any function u_0 as in Proposition 2.2.10 is called a k-homogeneous blowup of u at zero.

2.2.2 The Signorini type problem for the variable coefficient elliptic operator

The idea of the blowup analysis for the problem (2.1.24) is similar to that for the Sinorini problem (2.1.7). However, we would like to present the proofs for complpetness, and also because the blowup analysis of the generalised problem (2.0.1) in the Chapter 4 strongly relies on these derivations. To begin we note that that in our case $d_r = \left(\frac{H(r)}{r^n}\right)^{\frac{1}{2}}$ and that (2.2.3) gives for our problem

$$\int_{\mathbb{S}_r} \tilde{U}_r^2 \tilde{\mu} = 1 \tag{2.2.17}$$

Lemma 2.2.12. *Let $U \in \mathfrak{S}_1$ and define $A_r(x) = A(rx)$. Then, both the functions $\tilde{U}_r$ and U_r are even in $x_{n+1} = y$ and solve the thin obstacle problem* (2.1.25)-(2.1.29) *in $\mathbb{B}_{1/r}$ for the operator $L_r = \operatorname{div}(A_r \nabla)$*

Proof. The lemma will be proved only for $\tilde{U}_r$, for U_r it establishes similarly if we replace $\tilde{U}_r$ with U_r. By change of variable one can show that $\tilde{U}_r$ satisfy (2.1.25)-(2.1.28) for the operator L_r. To prove (2.1.29), let $\eta \in C_0^\infty(\mathbb{B}_{1/r})$, a change of variable leads to

$$\int_{\mathbb{B}_{1/r}} < A_r \nabla \tilde{U}_r, \nabla \eta >= -2 \int_{B_{1/r}} (a_{n+1,n+1})_r \eta D_y^+ \tilde{U}_r. \tag{2.2.18}$$

This proves the lemma. □

Remark. When we consider $\tilde{U}_r$ and U_r, it is important to remember that the operator being considered is $L_r = \operatorname{div}(A_r \nabla)$. Therefore to avoid confusions, the functions $N(r)$, $\tilde{N}(r)$ will be denoted $N_{L_r}(r)$, $\tilde{N}_{L_r}(r)$ or $N_L(r)$, $\tilde{N}_L(r)$ depending on which operator we refer to, L_r or L. If no operator is indicated, it is understood to be L.

Lemma 2.2.13. *Let* $U \in \mathfrak{S}_1$ *and suppose* $0 \in \Gamma(U)$*. Then* $N_{L_r}(\tilde{U}_r, 1) = N_L(U, r)$*.*

Proof. The result follows from the following calculations

$$\begin{aligned} N_{L_r}(\tilde{U}_r, 1) &= \frac{\int_{\mathbb{B}_1} < A_r \nabla \tilde{U}_r, \nabla \tilde{U}_r >}{\int_{\mathbb{S}_1} \tilde{U}_r^2 \tilde{\mu}_r} = \frac{r^2 \int_{\mathbb{B}_1} < A(rx) \nabla U(rX), \nabla U(rX) >}{\int_{\mathbb{S}_1} U^2(rX) \tilde{\mu}(rX)} \\ &= \frac{r \int_{\mathbb{B}_r} < A \nabla U, \nabla U >}{\int_{\mathbb{S}_r} U^2 \tilde{\mu}} = N_L(U, r) \end{aligned}$$

□

The next lemma combines Theorem 2.1.11 about the monotonicty of the generalised frequency $\tilde{N}(U, r)$ with the previous lemma to prove a uniform boundness of the Almgren scalings in $W^{1,2}$ norm.

Lemma 2.2.14. *Let* $U \in \mathfrak{S}_1$ *and suppose* $0 \in \Gamma(U)$*. Assume that the normalisation hypothesis* $A(0) = \mathbb{I}_{n+1}$ *is in place and that furthermore* (2.0.7) *is in force. Given* $r_j \to 0$ *the sequence* $\{\tilde{U}_{r_j}\}_{j \in \mathbb{N}}$ *is uniformly bounded in* $W^{1,2}(\mathbb{B}_1)$

Proof. By (2.0.6), Lemma 2.2.13 we obtain

$$\begin{aligned} \int_{\mathbb{B}_1} |\nabla \tilde{U}_r|^2 &\leq \lambda^{-1} \int_{\mathbb{B}_1} < A_r \nabla \tilde{U}_r, \nabla \tilde{U}_r >= \lambda^{-1} D_{L_r}(\tilde{U}_r, 1) \overset{(2.2.17)}{=} \lambda^{-1} N_{L_r}(\tilde{U}_r, 1) \\ &= \lambda^{-1} N_L(U, r) = \lambda^{-1} e^{-Cr} \tilde{N}_L(U, r) \leq \lambda^{-1} \tilde{N}_L(U, 1) < \infty, \end{aligned}$$

where in the last step we used Theorem 2.1.11. Furthermore, by (2.0.6) and (2.2.17) we have

$$\int_{\mathbb{S}_1} \tilde{U}_r^2 \leq \lambda^{-1} \int_{\mathbb{S}_1} \tilde{U}_r^2 \tilde{\mu}_r = \lambda^{-1}.$$

Combining these estimates with the following trace inequality, which is valid for any function $v \in W^{1,2}(\mathbb{B}_r)$

$$\frac{1}{r}\int_{\mathbb{B}_r} v^2 \leq C(n)\left(\int_{\mathbb{S}_r} v^2 + r\int_{\mathbb{B}_r} |\nabla v|^2\right)$$

we conclude that $\left\|\tilde{U}_r\right\|_{W^{1,2}(\mathbb{B}_1)} < \infty$ □

Lemma 2.2.15. *Let $U \in \mathfrak{S}_1$ and suppose that $0 \in \Gamma(U)$. Assume that $A(0) = \mathbb{I}_{n+1}$ and that furthermore* (2.0.7) *is in force. Given $r_j \to 0$, there exists a subsequence (which we will still denote by r_j), and for any $\alpha \in (0, \frac{1}{2})$, a function $\tilde{U}_0 \in C^{1,\alpha}_{loc}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$, such that $\tilde{U}_{r_j} \to \tilde{U}_0$ in $C^{1,\alpha}_{loc}(\mathbb{R}^{n+1}_\pm \cup \mathbb{R}^n)$. Such $\tilde{U}_0$ is a global solution of the Signorini problem* (2.1.25)-(2.1.29) *in $\mathbb{R}^{n+1}$ with $A \equiv \mathbb{I}_{n+1}$, and we have $\tilde{U}_0 \not\equiv 0$*

Proof. In [18] it was proved that $U \in C^{1,1/2}_{\text{loc}}(\mathbb{B}^\pm_1 \cup B_1)$ with

$$\|U\|_{C^{1,1/2}(\mathbb{B}^\pm_{1/2} \cup B_{1/2})} \leq C(n, \lambda, Q, \|U\|_{W^{1,2}(\mathbb{B}_1)}).$$

Given $r_j \searrow 0$, consider the sequence $\{\tilde{U}_{r_j}\}_{j\in\mathbb{N}}$. By Lemma 2.2.14, such a sequence is bounded in $W^{1,2}(\mathbb{B}_1)$. For any $\alpha \in (0, 1/2)$, by a standard diagonal process we obtain the convergence of a subsequence of the functions $\tilde{U}_{r_j}$ to a function $\tilde{U}_0$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^{n+1}_\pm \cup \mathbb{R}^n)$. Passing through the limit in (2.2.18), we conclude that such $\tilde{U}_0$ is a global solution of (2.1.25)-(2.1.29) with $A = \mathbb{I}_{n+1}$, and that $\tilde{U}_0$ is even in $x_{n+1} = y$. By (2.2.17) we obtain

$$1 = \int_{\mathbb{S}_1} \tilde{U}^2_{r_j}\tilde{\mu}_r \to \int_{\mathbb{S}_1} \tilde{U}^2_0$$

and therefore $\tilde{U}_0 \not\equiv 0$. □

Definition 2.2.16. We will call the function $\tilde{U}_0$ as in Lemma 2.2.15 an *Almgren blowup* of function U at zero.

Lemma 2.2.17. *Let $U \in \mathfrak{S}$, $0 \in \Gamma(u)$ and suppose that $A(0) = \mathbb{I}_{n+1}$ and that furthermore* (2.0.7) *is in force. Let $\tilde{U}_0$ be an Almgren blowup of U at zero. If $N(0^+) = \lim_{r\to 0^+} N(r)$ exists, then $\tilde{U}_0$ is homogeneous function of degree $k = N(0^+)$.*

Proof. Fix $r > 0$, and consider a sequence r_j where $\tilde{U}_{r_j} \to \tilde{U}_0$ in $C^{1,\alpha}_{loc}(\mathbb{R}^{n+1}_\pm \cup \mathbb{R}^n)$ as in Lemma 2.2.15. Thus, we obtain

$$N_{L_{r_j}}(\tilde{U}_{r_j}, r) \to N_\Delta(\tilde{U}_0, r).$$

On the other hand, since $N_L(U, 0^+)$ exists, then

$$N_L(U, rr_j) \to N_L(U, 0^+).$$

Reasoning like in Lemma 2.2.13 we can obtain

$$N_L(U, rr_j) = N_{L_{r_j}}(\tilde{U}_{r_j}, r).$$

Passing through the limit as $j \to \infty$ in this equality, we infer

$$N_\Delta(\tilde{U}_0, r) \equiv N_L(U, 0^+)$$

which leads to the desired conclusion by the property of the Almgren monotonicity functional. □

Before we continue to analyse the asymptotic behaviour of the homogeneous rescalings, we want to present the following growth estimates.

Lemma 2.2.18. *Under that $A(0) = \mathbb{I}_{n+1}$ and that furthermore* (2.0.7) *is in force, suppose that $N(0^+) \geq k$. Then, for $r \in (0,1) one has$*

$$H(r) \leq \tilde{C} r^{n+2k} \tag{2.2.19}$$

where $\tilde{C} = e^C H(1)$ as in Theorem 2.1.11.

The proof is based on Proposition 2.1.9 and Theorem 2.1.11. The next result is the growth estimate for the solution U.

Lemma 2.2.19. *Under that $A(0) = \mathbb{I}_{n+1}$ and that furthermore* (2.0.7) *is in force, suppose that $N(0^+) \geq k$. Then, there exists a universal constant $C > 0$, depending also on k, such that for every $X \in \mathbb{B}_{1/2}$, one has*

$$|U(x)| \leq C|X|^k. \tag{2.2.20}$$

Proof. Integrating (2.2.19) and using (2.1.2), we obtain for $r \in (0,1)$

$$\int_{\mathbb{B}_r} U^2 \leq C_1 r^{n+1+2k} \tag{2.2.21}$$

where $C_1 = C_1(\tilde{C}, n, k)$. Since $Lu^{\pm} \geq 0$ in $\mathbb{B}_1$, we can apply the following theorem.

Theorem 2.2.20. (theorem 8.17, [23])
Consider the operator

$$Lu = D_i(a^{ij}(x) D_j u + b^i(x) u) + c^i(x) D_i u + d(x) u$$

whose coefficients are assumed to be measurable functions on a domain $\Omega \subset \mathbb{R}^n$
Let the operator L be strictly elliptic in Ω, that is, there exist a positive number λ such that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \forall x \in \Omega, \xi \in \mathbb{R}^n$$

Assume that L has bounded coefficients, that is, for some Λ and $\nu \geq 0$ we have for all $x \in \Omega$

$$\begin{aligned}&\sum |a^{ij}(x)|^2 \leq \Lambda^2,\\ &\lambda^{-2}\sum(|b^i(x)|^2 + |c^i(x)|^2) + \lambda^{-1}|d(x)| \leq \nu^2\end{aligned}$$

and suppose that $f^i \in L^q(\Omega), i = 1, ..., n,\ g \in L^{q/2}(\Omega)$ for some $q > n$. Then if u is a $W^{1,2}(\Omega)$ subsolution (supersolution) of equation

$$Lu = g + D_i f^i \quad \text{in}\Omega$$

we have for any ball $B_{2R}(y) \subset \Omega$ and $p > 1$,

$$\sup_{B_R(y)} u\ (-u) \leq C(R^{-n/p}\|u^+\ (u^-)\|_{L^p(B_{2R}(y))} + k(R))$$

where $C = C(n, \Lambda/\lambda, \nu R, q, p)$

It allows us to infer the existence of $c = c(n, \lambda) > 0$ such that $\mathbb{B}_{2R}(X) \subset \mathbb{B}_1$, then

$$\sup_{\mathbb{B}_R(X)} U^+ \leq cR^{-(n+1)/2}\|U^+\|_{L^2(\mathbb{B}_{2R}(X))}. \tag{2.2.22}$$

We pick now $X \in \mathbb{B}_{1/2}$, and let $R = |X|/2$. Clearly, $\mathbb{B}_{2R}(X) \subset \mathbb{B}_{4R} \subset \mathbb{B}_1$. Applying (2.2.22)

$$\begin{aligned}U^+(X) &\leq cR^{-(n+1)/2}\left(\int_{\mathbb{B}_{2R}(X)}(U^+)^2\right)^{1/2} \leq cR^{-(n+1)/2}\left(\int_{\mathbb{B}_{4R}}(U^+)^2\right)^{1/2}\\ &\leq cR^{-(n+1)/2}\left(\int_{\mathbb{B}_{4R}}U^2\right)^{1/2} \leq \tilde{C}R^{-(n+1)/2}R^{(n+1+2k)/2} = \tilde{C}|X|^k,\end{aligned}$$

where in the second-to-last inequality we have used (2.2.21) above. Since a similar result holds for u^-, we have reached the desired conclusion. □

Lemma 2.2.21. *Assume that $A(0) = \mathbb{I}_{n+1}$ and that furthermore* (2.0.7) *is in force, and suppose $N(0^+) \geq k$. Then, there exists a universal constant $C > 0$ such that for $r \in (0, 1)$, we have*

$$D(r) \leq C^* r^{n-1+2k}. \tag{2.2.23}$$

Similar to the case of the Signorini problem of the Laplacian, the proof is based on the estimate (2.2.21) and the Caccioppoli-type inequality.

Lemma 2.2.22. *Let $U \in \mathfrak{S}$, $0 \in \Gamma(U)$, and assume that $A(0) = \mathbb{I}_{n+1}$ and that furthermore* (2.0.7) *is in force. Suppose that $N(0^+) \geq k$. Given $r_j \to 0$, there exists a subsequence (which we will still denote by r_j), and for any $\alpha \in (0, \frac{1}{2})$, a function $U_0 \in C^{1,\alpha}_{loc}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$, such* *that* $U_{r_j} \to U_0$ *in* $C^{1,\alpha}_{loc}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$. *Such* U_0 *is a global solution of the Signorini problem* (2.1.25)-(2.1.29) *in* $\mathbb{R}^n$ *with* $A \equiv \mathbb{I}_n$.

Proof. Consider the family $\{U_{r_j}\}_{j\in\mathbb{N}}$. By (2.2.21)

$$\int_{\mathbb{B}_1} U_{r_j}^2 = r_j^{-2k} \int_{\mathbb{B}_1} U^2(r_j X) = r_j^{-(n+1+2k)} \int_{\mathbb{B}_{r_j}} U^2 \le C_1.$$

Similarly, using (2.2.23)

$$\int_{\mathbb{B}_1} |\nabla U_{r_j}|^2 = r_j^{2-2k} \int_{\mathbb{B}_1} |\nabla U_{r_j}(r_j X)|^2 = r_j^{-n+1-2k} \int_{\mathbb{B}_{r_j}} |\nabla U|^2 \le \lambda^{-1} C^*.$$

We infer that $\{U_{r_j}\}_{j\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}(\mathbb{B}_1)$. As proved in [18], $U \in C^{1,1/2}_{\text{loc}}(\mathbb{B}_1^\pm \cup B_1)$ with

$$\|U\|_{C^{1,1/2}(\mathbb{B}^\pm_{1/2} \cup B_{1/2})} \le C(n, \lambda, Q, \|U\|_{W^{1,2}(\mathbb{B}_1)})$$

By a standard diagonal process, for any $\alpha \in (0, 1/2)$ we obtain the convergence of a subsequence of the functions U_{r_j} to a function U_0 in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^{n+1}_\pm \cup \mathbb{R}^n)$. Passing through the limit in (2.2.18), we conclude that such a U_0 is a global solution of (2.1.25)-(2.1.29) with $A = \mathbb{I}_{n+1}$, and that U_0 is even in $x_{n+1} = y$. □

Definition 2.2.23. We will call the function U_0 in Lemma 2.2.22 a *homogeneous blowup* of U at zero.

Lemma 2.2.24. (Minimal homogeneity) *Assume that* $A(0) = \mathbb{I}_{n+1}$ *and that furthermore* (2.0.7) *is in force. Then* $N(0^+) = \frac{3}{2}$ *or* $N(0^+) \ge 2$

With this theorem about minimal homogeneity, it is possible to improve the growth Lemma 2.2.19 and use it to establish the following important result.

Theorem 2.2.25. (Optimal regularity) *Suppose that the coefficients of the matrix-valued function* $A(x)$ *are Lipschitz continuous. Let* U *be the solution of the problem* (2.1.24)*, with the thin obstacle* $\varphi \in C^{1,1}(B_1)$*, and let* $0 \in \Gamma_\varphi(U)$*. Then* $U \in C^{1,1/2}(\mathbb{B}^+_{1/2} \cup B_{1/2})$.

2.3 The Weiss type functional and the regular free boundary

Theorem 2.2.6 and Lemma 2.2.24 are crucial components in the proving the optimal regularity of the solution, but it does not tell us much about classification of the free boundary points. We will discuss it in this section.

Definition 2.3.1. Let $U \in \mathfrak{S}_1$. We say that a free boundary point X_0 is *regular* if at such point the frequency takes it lowest possible value, i.e. $N(u(\cdot + X_0), 0^+) = \frac{3}{2}$. In other words, the *regular set* is $\Gamma_{3/2}(U)$

Since our main task is to study singular set of the free boundary of the problem (2.0.1), we will just briefly discuss the strategy of the proofs of the regularity of regular points. The Almgren monotonicity formula is not suitable for understanding the regularity of the free boundary. Instead, we introduce the following powerful tool.

Definition 2.3.2. For $k > 0$ let $\varphi = 0$ and define the *Weiss type functional* as

$$W_k(U,r) = W_k(r) = \frac{1}{r^{n+a-1+2k}}\left[\int_{\mathbb{B}_r} < A(x)\nabla U, \nabla U > |y|^a dX - \frac{k}{r}\int_{\mathbb{S}_r} U^2\tilde{\mu}|y|^a\right]$$

It is important to note that if U is homogeneous of degree k, in other words $N(r) \equiv k$, then

$$W_k(r) = \frac{1}{r^{n+a-1+2k}}D(r) - \frac{1}{r^{n+a-1+2k}}H(r) = \frac{H(r)}{r^{n+a+2k}}\left[N(r) - k\right] \tag{2.3.1}$$

Thus, in a way, the functional $W_k(r)$ measures the discrepancy of U from being homogeneous of degree k.

Remark. In the case when $\varphi \neq 0$ there is slightly different approach of defining the Weiss type functional, which will be discussed in the Chapter 3.

The strategy is to prove that the boundness of $W_k(r)$, that its limit exists as $r \to 0^+$ and that it is equal to 0, and finally prove that the finctional $W_k(r)$ is monotone nondecreasing. These results allow to show that the homogeneous blowup of U at zero U_0 is actually homogeneous of degree k. However, since the calculations for both problems and for the main problem of these thesis are in some way (not completely) repeating itself, we refer the reader to the Chapter 4, where all calculations are presented for the most generalised case (2.0.1).

At the same time, we want to point out the following important results for both problems (2.1.7), (2.1.24), which states that the regular set of the free boundary is locally $C^{1,\beta}$ $(n-2)$-dimensional surface. It can be obtained by combining the Weiss type monotonicity with the suitable epiperimetric inequality.

Theorem 2.3.3. ([3]) *Let $u \in \mathfrak{S}_o$ in B_1, with $x_0 \in \Gamma_{3/2}(u)$. Then, there exists $\eta_0 > 0$, depending on x_0, such that, after a possible rotation of coordinate axes in $\mathbb{R}^{n-1}$, one has $B'_{\eta_0}(x_0) \cap \Gamma(u) \subset \Gamma_{3/2}(u)$, and*

$$B'_{\eta_0}(x_0) \cap \Lambda(u) = B'_{\eta_0}(x_0) \cap \{x_{n-1} \leq g(x_1, ..., x_{n-2})\}$$

for $g \in C^{1,\beta}(\mathbb{R}^{n-2})$ with a universal exponent $\beta \in (0,1)$.

Denote as $\mathcal{R}_\varphi(U)$ the regular part of the free boundary $\Gamma_\varphi(U)$

Theorem 2.3.4. ([20]) *Suppose that the coefficients of the matrix-valued function $A(x)$ are Lipschitz continuous. Let U be the solution of the problem (2.1.24), with the thin obstacle $\varphi \in C^{1,1}(B_1)$, and let $x_0 \in \mathcal{R}_\varphi(U)$. Then there exists $\eta_0 > 0$, depending on x_0, such that, after a possible rotation of coordinate axis in $\mathbb{R}^n$, one has $B_{\eta_0>0} \cap \Gamma_\varphi(U) \subset \mathcal{R}_\varphi(U)$ and*

$$B_{\eta_0>0} \cap \Lambda_\varphi(U) = B_{\eta_0>0} \cap \{x_n \leq g(x_1, ..., x_{n-1})\}$$

for $g \in C^{1,\beta}(\mathbb{R}^{n-1})$ with a universal exponent $\beta \in (0,1)$

2.4 The Monneau type monotonicity formula

In this section we discuss another type of the monotonicity formulas suited for the study of the singular points of the free boundary. The original formula was used by R. Monneau in [26] for the study of the singular points in the classical obstacle problem. Later in [19] N. Garofalo and A.Petrosyan established a generalization of the one-parameter Monneau-type monotonicity formulas for solutions of (2.1.7), which they developed later for the problem (2.1.24) together with M. Smit Vega Garcia in [21]. Since in the Chapter 4 we present the derivation of the Monneau type monotonicity formula for the generalised problem (2.0.1), in this section we confine ourselves to the definition and description of the main results regarding this formula.

Definition 2.4.1. We will indicate with $\mathfrak{P}_k^+(\mathbb{R}^{n+a+1})$ the class of all non zero homogeneous polynomials p_k of degree k in $\mathbb{R}^{n+1}$, such that

$$\operatorname{div}(|y|^a \nabla p_k) = 0, \quad p_k(x,0) \geq 0, \quad p_k(x,-y) = p_k(x,y)$$

Definition 2.4.2. *(Monneau type Monotonicity formula)*
Let U be the solution of (2.0.1) and let $0 \in \Gamma_k(U)$. For any $p_k \in \mathfrak{P}_k^+(\mathbb{R}^{n+a+1})$ we define

$$\begin{aligned} M_k(r) = M_k(U,p_k,r) &= \frac{1}{r^{n+a+2k}} \int_{\mathbb{S}_r} (U-p_k)^2 \tilde{\mu} |y|^a \\ &= \frac{1}{r^{n+a+2k}} \int_{\mathbb{S}_r} (U-p_k)^2 \mu \end{aligned} \tag{2.4.1}$$

In further discussion we will need the "almost" monotonicity of the Monneau type functional that could be established by proving the following formula

$$\frac{d}{dr}(M_k(r) + Cr) \geq \frac{2W_k(r)}{r},$$

for some universal constant $C > 0$. Using the results of the "almost" monotonicity of the Weiss type functional, the desired result follows.

Remark. For the Signorini problem for the Laplacian it is possible to achieve the strong (not "almost") monotonicity of the Monneau and Weiss type functionals [19].

The Monneau type functional then is used in the proofs of the degeneracy of the solution and the uniqueness of the homogeneus blowups at a singular point.

2.5 The singular free boundary

In this section we introduce and study the singular points of the free boundary, which constitute the main objective of these thesis. Let us start with the relevant definition.

Definition 2.5.1. Let $U \in \mathfrak{S}_1$. We say that 0 is a singular point of the free boundary $\Gamma(U)$, if

$$\lim_{r\to 0^+} \frac{\mathcal{H}^n(\Lambda(U)\cap B_r)}{\mathcal{H}^n(B_r)} = 0,$$

where $\mathcal{H}^n$ the n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$. We denote by $\Sigma(U)$ the subset of singular points of $\Gamma(U)$. We also denote

$$\Sigma_k(U) = \Sigma(U) \cap \Gamma_k(U).$$

Note that in terms of the rescaling (2.2.1) the condition $0 \in \Sigma(U)$ is equivalent to

$$\lim_{r\to 0^+} \mathcal{H}^n(\Lambda(\tilde{U}_r)\cap B_1) = 0 \tag{2.5.1}$$

2.5.1 The Laplace Signorini problem

The following results give a characterization of the singular points via the value of the frequency $N(r)$ and the type of the blowup. It is important to follow the proofs, since they change from one problem to another, also because, for example, while the blowups for the problems (2.1.7), (2.1.24) are harmonic functions in a weak sense and so we can use the Liouville theorem to prove that they are harmonic also in the strong sense, the blowups of the problem (2.0.1) are not harmonic but satisfy $\operatorname{div}(|y|^a \nabla p_k) = 0$ and we should take it into account.

Theorem 2.5.2. (Characterisation of singular points) *Let $u \in \mathfrak{S}_0$ and $0 \in \Gamma_k(U)$. Then the following statements are equivalent:*

(i) $0 \in \Sigma_k(u)$.

(ii) Any Almgren blowup $\tilde{u}_0$ of u at the origin is a nonzero homogeneous polynomial $p_k \in \mathfrak{P}_k^+(\mathbb{R}^n)$.

(iii) $k = 2m$ for some $m \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii) The rescalings (2.2.1) satisfy

$$\Delta \tilde{u}_r = 2(\partial_{x_n}\tilde{u}_r)\mathcal{H}^{n-1}\Big|_{\Lambda(\tilde{u}_r)} \text{ in } \mathcal{D}'(B_1). \tag{2.5.2}$$

Since $|\nabla \tilde{u}_r|$ are locally uniformly bounded in B_1 (Theorem 2.2.2) and $\lim_{r\to 0^+}\mathcal{H}^n(\Lambda(\tilde{u}_r)\cap B_1) = 0$, (2.5.2) implies that $\Delta\tilde{u}_r$ converges weakly to 0 in $\mathcal{D}'(B_1)$ and therefore any Almgren

blowup $\tilde{u}_0$ must be harmonic in $\mathcal{D}'(B_1)$. On the other hand, by Theorem 2.2.2 $\tilde{u}_0$ is homogeneous in B_1 and therefore can be extended to $\mathbb{R}^n$ and will have at most a polynomial at infinity.
By Liouville theorem we conclude that $\tilde{u}_0$ must be homogeneous of degree k harmonic polynomial p_k. Also by Theorem 2.2.2 we have that $p_k \not\equiv 0$. And finally, the properties of u imply that $p_k(x',0) \geq 0$ for all $x' \in \mathbb{R}^{n-1}$ and $p_k(x',-x_n) = p_k(x',x_n)$ for all $x \in \mathbb{R}^n$.

(ii) $\Rightarrow$ (iii) Let p_k be a blowup of u at the origin. If k is odd, the nonnegativity of p_k on $\mathbb{R}^{n-1} \times \{0\}$ implies that p_k vanishes identically on $\mathbb{R}^{n-1} \times \{0\}$. On the other hand, from the even symmetry we also have that $\partial_{x_n} p_k \equiv 0$ on $\mathbb{R}^{n-1} \times \{0\}$. Since p_k is harmonic in $\mathbb{R}^n$, the Cauchy-Kovalevskaya theorem implies that $p_k \equiv 0$ $\mathbb{R}^n$, that contradicts to the assumption.

(iii) $\Rightarrow$ (ii) The proof follows from the following Liouville type result.

Lemma 2.5.3. *Let v be a k-homogenenous global solution of the thin obstacle problem in $\mathbb{R}^n$ with $k = 2m$, $m \in \mathbb{N}$. Then v is a homogeneous harmonic polynomial.*

Lemma 2.5.4. *Let $v \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies $\Delta v \leq 0$ in $\mathbb{R}^n$ and $\Delta v = 0$ in $\mathbb{R}^n \setminus \{x_n = 0\}$. If v is homogeneous of degree $k = 2m$, $m \in \mathbb{N}$, then $\Delta v = 0$ in $\mathbb{R}^n$.*

Proof. By assumption, $\mu = \Delta v$ is nonpositive measure supported in $\{x_n = 0\}$. We claim that $\mu = 0$. Let P be a $2m$-homogeneous harmonic polynomial, which is positive on $\{x_n = 0\} \setminus \{0\}$. And consider $\psi \in C_0^\infty(0,\infty)$ with $\psi \geq 0$ and $\Psi(x) = \psi(|x|)$. Then we have

$$\begin{aligned} -<\mu, \Psi P> &= -<\Delta v, \Psi P> = \int_{\mathbb{R}^n} <\nabla v, \nabla(\Psi P)> = \int_{\mathbb{R}^n} \Psi <\nabla v, \nabla P> + P <\nabla v, \nabla \Psi> \\ &= \int_{\mathbb{R}^n} -\Psi v \Delta P - v <\nabla \Psi, \nabla P> + P <\nabla v, \nabla \Psi> \\ &= \int_{\mathbb{R}^n} \frac{\psi'(|x|)}{|x|} <x, \nabla P> v + \frac{\psi'(|x|)}{|x|} <x, \nabla v> P = 0 \end{aligned}$$

where in the second-to-last step we used harmonicity of P and in the last step the homogeneity of v and P. This implies that μ is supported at the origin, hence $\mu = c\delta_0$, where δ_0 is the Dirac's delta and $c \geq 0$. But μ is $2m-2$ homogeneous (since it is a second derivative of of a $2m$-homogeneous function) and δ_0 is $(-n)$-homogeneous. Since $n, m \in \mathbb{N}$, the only option is $c = 0$, and so $\mu = 0$. $\square$

The Almgren blowup $\tilde{u}_0$ satisfies the assumptions of the lemma, thus we obtained the desired result.

(ii) $\Rightarrow$ (i) Suppose that 0 is not a singular point and that over some sequence $r = r_j \to 0^+$, we have $\mathcal{H}^{n-1}(\Lambda(\tilde{u}_r) \cap B_1') \geq \delta > 0$. Taking a subsequence if necessary, we may assume that $\tilde{u}_{r_j}$ converges to a blowup $\tilde{u}_0$. We claim that $\mathcal{H}^{n-1}(\Lambda(\tilde{u}_0) \cap B_1') \geq \delta > 0$. Indeed, otherwise there exists an open set U in $\mathbb{R}^{n-1}$ with $\mathcal{H}^{n-1}(U) < \delta$ so that $\Lambda(\tilde{u}_0) \cap B_1' \subset U$. Then for large j we must have $\Lambda(\tilde{u}_{r_j}) \cap B_1' \subset U$, which is a contradiction, since $\mathcal{H}^{n-1}(\Lambda(\tilde{u}_{r_j}) \cap \overline{B_1'}) \geq \delta > \mathcal{H}^{n-1}(U)$.

But then u_0 vanishes identically on $\mathbb{R}^{n-1} \times \{0\}$ and consequently on $\mathbb{R}^n$ by the Cauchy-Kovalevskaya theorem. $\square$

Remark. One can show that the homogeneous blowup u_0 satisfies the reasoning of the proof of (i) $\Rightarrow$ (ii). Thus, we know that if $0 \in \Sigma_k(u)$, then, any homogeneous blowup of u at the origin is a homogeneous polynomial $p_k \in \mathfrak{P}_k^+(\mathbb{R}^n) \cap \{0\}$.

Lemma 2.5.5. (Nondegeneracy at singular points) *Let $u \in \mathfrak{S}_\mathfrak{o}$ and $0 \in \Sigma_k(u)$. There exists $c > 0$, possibly depending on u, such that for $r \in (0,1)$*

$$\sup_{S_r} |u(x)| \geq cr^k. \tag{2.5.3}$$

Proof. see Lemma 1.5.2 in [19]. $\square$

Remark. To continue we have to note that the Almgren's frequency functional (2.1.6) can be defined at any point $x_0 \in \Gamma(u)$ by simply translating that point to the origin

$$N^{x_0}(r,u) = \frac{r\int_{B_r(x_0)} |\nabla u|^2}{\int_{S_r(x_0)} u^2},$$

where $r > 0$ is such that $B_r(x_0) \Subset B_1$. Since for every fixed $r \in (0,1)$ the function $x \mapsto H^{x_0}(u,r) = \int_{S_r(x_0)} u^2$ is continuous, (2.2.13) shows that $x_0 \mapsto N^{x_0}(u,r)$ is continuous. On the other hand, if $x_0 \in \Gamma(u)$ we have the monotonicity of $r \mapsto N^{x_0}(u,r)$ that

$$N^{x_0}(u,0^+) = \inf_{0<r<1} N^{x_0}(u,r).$$

Since the infimum of a family of continuous functions is upper continuous, we can conclude that on $\Gamma(u)$ the function $x_0 \mapsto N^{x_0}(u,r)$ is upper semicontinuous.

Lemma 2.5.6. ($\Sigma_k(u)$ is F_σ) *For any $u \in \mathfrak{S}_\mathfrak{o}$, the set $\Sigma_k(u)$ is of type F_σ, i.e., it is a union of countably many closed sets.*

Proof. Consider sets

$$E_j = \{x_0 \in \Sigma_k(U) \cap \overline{B_{1-1/j}} \mid \frac{1}{j}\rho^k \leq \sup_{S_\rho} |u(x)| \leq j\rho^k, \text{ for } 0 < r < 1 - |x_0|\} \tag{2.5.4}$$

By Lemma 2.2.5 and Lemma 2.5.5 we have that

$$\Sigma_k(u) = \sum_{j=1}^{\infty} E_j,$$

that is $\Sigma_k(u)$ is a union of countably many sets, so we need to prove that E_j is closed.

Let $x_0 \in \overline{E_j}$, then it is clear that $x_0 \in \overline{B_{1-1/j}}$ and the estimate of (2.5.4) holds. Thus it suffices to prove that $x_0 \in \Sigma_k(u)$. It follows from the upper semicontinuity of $N^{x_0}(u,r)$, so $N^{x_0}(u,0^+) \geq k$, and the fact that if $N^{x_0}(u,0^+) > k$, then for small enough ball we obtain the contradiction with the estimate of (2.5.4). □

Theorem 2.5.7. (Uniqueness of the homogeneous blowup at singular points) *Let $u \in \mathfrak{S}_\mathfrak{o}$ and $0 \in \Sigma_k(u)$. Then there exists a unique nonzero $p_k \in \mathfrak{P}_k^+(\mathbb{R})^n$ such that the homogeneous scalings* (2.2.4) *u_r converges in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n_\pm \cap \mathbb{R}^{n-1})$ to p_k.*

The proof uses the results of the blowup analysis and the Monneau type monotonicity. This result jointly with the Monneau formula helps to proof the following important theorem.

Theorem 2.5.8. (Continuous dependence of the blowups) *Let $u \in \mathfrak{S}_\mathfrak{o}$. For $x_0 \in \Sigma_k(u)$ denote by $p_k^{x_0}$ the homogeneous blowup of u at x_0 as in Theorem* 2.5.7*, so that*

$$u(x) = p_k^{x_0}(x - x_0) + o(|x - x_0|^k).$$

Then, the mapping $x_0 \mapsto p_k^{x_0}$ from $\Sigma_k(u)$ to $\mathfrak{P}_k^+(\mathbb{R}^n)$ is continuous. Moreover, for any compact $K \subset \Sigma_k(u) \cap B_1$ there exists a modulus of continuity σ_K, with $\sigma_K(0^+) = 0$, such that

$$|u(x) - p_k^{x_0}(x - x_0)| \leq \sigma_K(|x - x_0|)|x - x_0|^k \tag{2.5.5}$$

for any $x_0 \in K$

Definition 2.5.9. Given a singular point $x_0 \in \Sigma_k(u)$ we define the *dimension* of $\Sigma_k(u)$ at x_0 to be

$$d_k^{x_0} := \dim\{\xi \in \mathbb{R}^{n-1} \mid \ <\xi, \nabla_{x'} p_k^{x_0}(x', 0)> = 0 \text{ for all } x' \in \mathbb{R}^{n-1}\}. \tag{2.5.6}$$

By the Cauchy-Kovalevskaya theorem since $p_k^{x_0} \not\equiv 0$ on $\mathbb{R}^{n-1} \times \{0\}$ one has that $0 \leq d_k^{x_0} \leq n-2$. For $d = 0, 1, ..., n-2$ we define

$$\Sigma_k^d(u) := \{x_0 \in \Sigma_k(u) \mid d_k^{x_0} = d\}$$

Theorem 2.5.10. *Let $u \in \mathfrak{S}_\mathfrak{o}$. Then $\Sigma_k(u) = \Gamma_k(u)$ for $k = 2m$, $m \in \mathbb{N}$, and every set $\Sigma_k^d(u)$, $d = 0, 1, ..., n-2$ is contained in a countable union of d-dimensional C^1 manifolds.*

Proof. $\Sigma_k(u) = \Gamma_k(u)$ for $k = 2m$, $m \in \mathbb{N}$ is proved in Theorem 2.5.2. The following proof is based on two classical results, that is Whitney's extension theorem and the implicit function theorem.

<u>Step 1</u> (Whitney's extension) Let $K = E_j$ defined in (2.5.4), remember E_j is compact. We can write the polynomial $p_k^{x_0}$ that represents the unique homogeneous blowup of u at x_0 as

$$p_k^{x_0} = \sum_{|\alpha|=k} \frac{a_\alpha(x_0)}{\alpha!} x^\alpha.$$

The coefficients $a_\alpha(x)$ are continuous by Theorem 2.5.8 and, since $u(x) = 0$ on $\Sigma_k(u) \subset \Gamma(u)$, we have

$$|p_k^{x_0}(x - x_0)| \leq \sigma(|x - x_0|)|x - x_0|^k, \text{ for } x, x_0 \in K$$

where $\sigma = \sigma_k$. For any multiindex α, with $|\alpha| \leq k$, we define for $x \in \Sigma_k(u)$

$$f_\alpha(x) = \begin{cases} 0, & |\alpha| < k \\ a_\alpha(x), & |\alpha| = k \end{cases}$$

With the following lemma (compatibility condition) we apply the Whitney's extension theorem and conclude that there exists $F \in C^k(\mathbb{R}^n)$ such that $\partial^\alpha F = f_\alpha$, for all $|\alpha| \leq k$.

Lemma 2.5.11. *For any $x_0, x \in K$*

$$f_\alpha(x) = \sum_{|\beta| \leq k - \alpha} \frac{f_{\alpha+\beta}(x_0)}{\beta!}(x - x_0)^\beta + R_\alpha(x, x_0),$$

where for some modulis of continuity $\sigma_\alpha = \sigma_\alpha^K$

$$|R_\alpha(x, x_0)| \leq \sigma_\alpha(|x - x_0|)|x - x_0|^{k-|\alpha|}$$

Proof. see [19] □

Step 2 (Implicit function theorem) Suppose $x_0 \in \Sigma_k^d(u) \cap E_j$. From (2.5.6) we know that there are $n - 1 - d$ linearly independent vectors $\nu_i \in \mathbb{R}^{n-1}$ such that

$$< \nu_i, \nabla_{x'} p_k^{x_0} > \neq 0, \ i = 1, ..., n - 1 - d.$$

This implies that there exists multiindex β of order $k - 1$ such that

$$< \nu_i, \nabla_{x'}(\partial^\beta p_k^{x_0}) > \neq 0, \ i = 1, ..., n - 1 - d.$$

that is from the previous step

$$< \nu_i, \nabla_{x'}(\partial^\beta F(x_0) > \neq 0, \ i = 1, ..., n - 1 - d.$$

On the other hand,

$$\Sigma_k^d(u) \cap E_j \subset \bigcap_{i=1}^{n-1-d} \{\partial^\beta F = 0\}$$

Hence, the implicit function theorem implies that $\Sigma_k^d(u) \cap E_j$ is contained in the d-dimensional manifold in a neighbourhood of x_0. Since $\Sigma_k(u) = \bigcup_{j=1\infty} E_j$, the theorem holds. □

2.5.2 The Signorini type problem for the variable coefficient elliptic operator

We proceed by proving analogous results but for the variable coefficient problem (2.1.24).

Theorem 2.5.12. (Characterisation of singular points) *Let $U \in \mathfrak{S}_1$, where we have $0 \in \Gamma_k(U)$ for $k > \frac{3}{2}$. The following statements are equivalent*

(i) $0 \in \Sigma_k(U)$

(ii) Any Almgren blowup $\tilde{U}_0$ of U at the origin is a nonzero homogeneous polynomial $p_k \in \mathfrak{P}_k^+(\mathbb{R}^{n+1})$

(iii) $k = 2m$, for some $m \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii) As in Theorem 2.5.2 we first need to prove that the rescaling (2.2.1) is weakly solves $L_a \tilde{U}_r = 0$ in $\mathbb{B}_1$, which will leads us through limit that the Almgren blowup $\tilde{U}_0$ is weakly harmonic in $\mathbb{B}_1$ (since $A(0) = \mathbb{I}_{n+1}$) and therefore by the Caccioppoli-Weyl lemma it is possible to conclude that the Almgren blowup is a classical harmonic function in $\mathbb{B}_1$.

By (2.2.18), for $r \in (0,1)$ and any $\eta \in C_0^\infty(\mathbb{B}_1)$, we find

$$\int_{\mathbb{B}_1} < A_r \nabla \tilde{U}_r, \nabla \eta >= -2 \int_{B_1 \cap \Lambda(\tilde{U}_r)} (a_{n+1,n+1})_r D_y^+ \tilde{U}_r \eta \tag{2.5.7}$$

By Lemma 2.2.14, $\nabla \tilde{u}_r$ are uniformly bounded in $\mathbb{B}_1$ and, thus, from (2.5.1) we obtain that the Almgren rescalings solve our problem weakly, then the desired conclusion follows as descibed in the beginning.

(ii) $\Rightarrow$ (iii) follows as in Theorem 2.5.2, since the set $\mathfrak{P}_k^+$ is the same for both problems.

(iii) $\Rightarrow$ (ii) follows as in Theorem 2.5.2, since both lemmas are applied to the Almgren blowup $\tilde{U}_0$, which in the case of the variable coefficient elliptic operator satisfies the thin obstacle problem for Laplacian (since $A(0) = \mathbb{I}_{n+1}$).

(ii) $\Rightarrow$ (i) follows as in Theorem 2.5.2. □

Lemma 2.5.13. *Let $U \in \mathfrak{S}_1$ with $0 \in \Sigma_k(U)$. Then, any homogeneous blowup of U at the origin (as in Lemma* 2.2.22*) is a homogeneous polynomial $p_k \in \mathfrak{P}_k^+(\mathbb{R}^n) \cup \{0\}$.*

Proof. Notice that (2.5.7) holds for the homogeneous rescaling (2.2.4). As proven in Lemma 2.2.22, $\{U_r\}_{r<1}$ is uniformly bounded in $W^{1,2}(\mathbb{B}_1)$, and by assumption

$$\lim_{r \to 0^+} \mathcal{H}^n(\Lambda(U_r) \cap B_1) = 0.$$

The proof then follows as in (i) $\Rightarrow$ (ii) in Theorem 2.5.12. □

Lemma 2.5.14. ($\Sigma_k(U)$ is F_σ) *For any $U \in \mathfrak{S}_1$, the set $\Sigma_k(U)$ is of type F_σ.*

The proof follows that of Lemma 2.5.6 but for the sets E_j defined as

$$E_j = \{X_0 \in \Sigma_k(U) \cap \overline{\mathbb{B}_{1-1/j}} \mid \frac{1}{j}\rho^k \leq \sup_{\mathbb{S}_\rho} |U_{x_0}(X)| \leq j\rho^k, \text{ for } 0 < r < \lambda^{-1}(1 - |X_0|)\}$$

As in the case of the Laplace Signorini problem with the appropriate Monneau type monotonicity formula it is possible to prove the nondegeneracy of the solution, the uniqueness of the homogeneous blowups first and then use it to establish continuous dependance of the blowups.

Remark. It is important to note the discussion in the chapter 7 of [21], where the authors introduced the following important normalisation of the solution: for a generic point $X_0 = (x_0, 0) \in \Gamma(U)$ one has

$$\begin{aligned}
&U_{x_0}(z) = U(x_0 + A^{1/2}(x_0)z),\\
&A_{x_0}(z) = A^{-1/2}(x_0)A(x_0 + A^{1/2}(x_0)z)A^{-1/2}(x_0)
\end{aligned}$$

under which U_{x_0} solves (2.1.24) corresponding to the new matrix A_{x_0} and moreover $0 \in \Gamma(U_{x_0})$ and $A_{x_0}(0) = \mathbb{I}_{n+1}$, thus, these conditions in the assumptions of previous results can be dropped.

With help of all these results authors of [21] proved their main theorem about the regularity of the singular part of the free boundary.

Theorem 2.5.15. (Structure of the singular set) *Let $U \in \mathfrak{S}_1$. Then, $\Gamma_k(U) = \Sigma_k(u)$ for $k = 2m$, $m \in \mathbb{N}$. Moreover, every set $\Sigma_k^d(U)$, $d \in \{0, ..., n-1\}$, is contained in a countable union of d-dimensional C^1 manifolds.*

The proof follows the proof of Theorem 2.5.10 with appropriate modifications.

Chapter 3

The regularity of the solution and the regular free boundary

Our work in these thesis is connected with the results of the recent article [4] in the same way as the paper [21] was connected with the results of [18], but for the case $a = 0$, so the opearator was $L = \mathrm{div}(A(x)\nabla\cdot)$. In the paper [18], N.Garofalo and M. Smit Vega Garcia proved the optimal interior regularity of the solution in the non-zero obstacle problem for $L = \mathrm{div}(A(x)\nabla\cdot)$. After that N.Garofalo, A. Petrosyan and M. Smit Vega Garcia in [21] studied the singular free boundary of the same operator but for the zero-obstacle problem.

Similarly, in the article [4] A.Banerjee, F.Buseghin and N.Garofalo established the optimal interior regularity of the solution of (3.0.1) (the non-zero thin obstacle problem of the variable coefficients degenerate operator) and besides that the $C^{1,\gamma}$ smoothness of the regular part of the free boundary. And we, in turn, would like to investigate the singular points of the same degenerate operator, but for the zero-obstacle.

In this chapter we would like to present the main results and lines of argumantations of [4], if it is necessary, since our goal is strongly relies on them, and also some of the ideas and proofs of [18], if the results of [4] depend on them or if they are needed in the subsequent calculations.

Let us introduce the lower-dimensional (or thin) obstacle problem for a class of degenerate elliptic equations with variable coefficients with non-zero obstacle. We consider the *thick* space $\mathbb{R}^{n+1}$ with generic variable $X = (x, y)$, where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$. The *thin* space $\mathbb{R}^n \times \{0\}$ will be simply identified with $\mathbb{R}^n$. As before, we assume that $X \to A(x) = [a_{ij}(x)]$ in (3.0.1) is uniformly elliptic, symmetric matrix-valued function with Lipschitz continuous independent of y coefficients and satisfy (2.0.6) and, thus, (2.0.7).

Given a number $a \in (-1, 1)$, and a function φ in B_1, known as the *thin obstacle* since it is defined in the thin set $B_1 \times \{0\} \subset \mathbb{R}^n \times \{0\}$, we consider the problem of finding a function U in $\mathbb{B}_1^+ \cup B_1$ such that:

$$\begin{cases} L_a U = \mathrm{div}_X(y^a A(x) \nabla_X U(x,y)) = 0 \ \text{ in } \mathbb{B}_1^+, \\ \min\{U(x,0) - \varphi(x), -\partial_y^a U(x,0)\} = 0 \ \text{ on } B_1, \end{cases} \tag{3.0.1}$$

where we have defined

$$\partial_y^a U(x,0) \stackrel{\text{def}}{=} \lim_{y \to 0^+} y^a \partial_y U(x,y). \tag{3.0.2}$$

The presence of the weight $y^a = \mathrm{dist}(X, \{y = 0\})^a$ makes the problem *degenerate*. For the sake of notation simplicity we will write hereafter div and ∇ for respectively div_X and ∇_X.

The set $\Lambda_\varphi(U) = \{x \in B_1 \mid U(x,0) = \varphi(x)\}$, as usually, is called the *coincidence set* and its topological boundary (relative to the topology of B_1) is referred to as the free boundary and is denoted by $\Gamma_\varphi(U)$.

Remark. In the applications of the divergence theorem to the domains $\mathbb{B}_1^+$, $\mathbb{B}_1^-$ the orientation of the outer unit normal is opposite to that used in (3.0.2). In this respect, we explicitly note that if we denote by ν_+ and ν_- the outer unit normal to the boundary of the upper and lower half-ball, respectively, then in B_1 we have from (2.0.6): $A(x)\nu_\pm = \mp e_{n+1}$. Therefore, we have as well

$$\lim_{y \to 0^+} y^a < \nabla U, A(x)\nu_+ >= -\partial_y^a U(x,0), \tag{3.0.3}$$

$$\lim_{y \to 0^-} y^a < \nabla U, A(x)\nu_- >= -\partial_y^a U(x,0). \tag{3.0.4}$$

It is also known that the local problem (3.0.1) is equavalent to the following nonlocal obstacle problem

$$\min\{(-\mathrm{div}(B(x)\nabla))^s u, u - \varphi\} = 0, \quad 0 < s < 1,$$

where the matrix-valued function $B(x)$ is connected to $A(x)$ by expression (2.0.6) and the connection between parameter a and parameter s is given by $a = 1 - 2s$.

3.1 Some estimates and regularity results

As we already mentioned, the main objective of [4] was to prove the optimal regularity regularity of the solution and the $C^{1,\gamma}$ local smoothness of the regular part of the free boundary in (3.0.1). It is known, that the first step to achieve that is to obtain the fundamental initial results of the Hölder continuity up to the thin set of the solution U, its weighted Neumann derivative $y^a \partial_y U$ and that of $\nabla_x U$. After you have it, the next thing to do is to develop suitable monotonicity formulas which will play a critical role in the blowup analysis. Thus the main result of the first part of the paper became the following two regularity theorem.

Theorem 3.1.1. *Let U be a solution to*

$$\begin{cases} \operatorname{div}(y^a A(x)\nabla U) = y^a f \ \text{ in } \mathbb{B}_1^+, \\ \min\{U(x,0), -\partial_y^a U(x,0)\} = 0 \ \text{ on } B_1, \end{cases} \tag{3.1.1}$$

with $a \geq 0$ and $f \in L^\infty(\overline{\mathbb{B}_1^+})$. Then there exists $\beta > 0$ such that $U, y^a U_y \in C^\beta(\overline{\mathbb{B}_{\frac{1}{2}}^+})$.

Remark. By subtracting off the obstacle φ from the solution, the homogeneous thin obstacle problem with non zero obstacle (3.0.1) is reduced to the non-homogeneous thin obstacle problem with zero obstacle (3.1.1).

Theorem 3.1.2. *Let $a \in [0,1)$ and U be a solution of (3.0.1) with an obstacle $\varphi \in C^{1,1}$. Then $\nabla_x U \in C^\alpha(\overline{\mathbb{B}_{\frac{1}{2}}^+})$ for some $\alpha > 0$.*

To achieve these results the authors used the following theorem from [31], which we use as well in a sequel, thus we introduce it here.

Theorem 3.1.3. (Theorem 1.2, [31]) *Let V be an even in y weak solution to the following degenerate problem*

$$\operatorname{div}(|y|^a A(x)\nabla V) = y^a f,$$

where $f \in L^\infty(\mathbb{B}_1)$. Then $V \in C^{1,\alpha}_{loc}(\mathbb{B}_1)$ for any $\alpha \in (0,1)$ and the following estimate holds

$$\|V\|_{C^{1,\alpha}(\mathbb{B}_{\frac{1}{2}})} \leq C(\|V\|_{L^2(\mathbb{B}_1, |y|^a dX)} + \|f\|_{L^\infty(\mathbb{B}_1)}),$$

where $C > 0$ depends also on α.

Remark. We should note that these results are obtained for $a \in [0;1)$, and not for $a \in (-1;1)$ as in Chapter 2.

3.2 Monotonicity formulas

After all necessary regularity tools are ready, it is the time to move to the next essential ingredient in the study of the thin obstacle problem which is the Almgren type monotonicity functional. Different from the previous section all the results in this section are valid for $a \in (-1,1)$. Let U be the solution to (3.1.1), then after an even reflection in y across $\{y=0\}$ we obtain that U solves the following problem in the distributional sense

$$\begin{cases} \operatorname{div}(|y|^a A(x)\nabla U) = |y|^a f + 2\partial_y^a U \mathcal{H}^n(\{y=0\}), \\ U\partial_y^a U \equiv 0. \end{cases} \tag{3.2.1}$$

First, the authors of [4] calculate the first variation of height

Lemma 3.2.1. *The function $H(r)$ is absolutely continuous and for a.e. $r \in (0,1)$ one has*

$$H'(r) = 2I(r) + \int_{\mathbb{S}_r} U^2 L_a|X|. \tag{3.2.2}$$

Therefore, at every point $r \in (0,1)$ where (3.2.2) *holds and $H(r) \neq 0$, we have*

$$\frac{H'(r)}{H(r)} = 2\frac{I(r)}{H(r)} + \frac{\int_{\mathbb{S}_r} U^2 L_a|X|}{H(r)} \tag{3.2.3}$$

and show a relation which reflects the dependence between the total energy and the Dirichlet energy

Lemma 3.2.2. *For every $r \in (0,1)$ we have*

$$I(r) = D(r) + \int_{\mathbb{B}_r} Uf|y|^a. \tag{3.2.4}$$

that we present since we use them in the succeeding chapter and also in the explanation why it is necessary to introduce the following quantities.

As first described in [18] (for the case with $a = 0$), one of the main challenges in the proving the monotonicity of the Almgren type formula came from the (3.2.3), namely

$$\frac{r}{2}\frac{d}{dr}\log H(r) = \frac{r}{2}\frac{H'(r)}{H(r)} = \frac{rI(r)}{H(r)} + \frac{r}{2}\frac{\int_{\mathbb{S}_r} U^2 L_a|X|}{H(r)} \tag{3.2.5}$$

at points where $H(r) \neq 0$. When $L = \Delta$ (in $\mathbb{R}^{n+1}$), we have $L|X| = \frac{n}{|x|}$, $\mu \equiv 1$, and thus considering the Almgren type monotonicity function as $F(r) = \frac{rI(r)}{H(r)}$, we can obtain

$$\frac{r}{2}\frac{d}{dr}\log H(r) = F(r) + \frac{n}{2}$$

and if the obstacle $\varphi = 0$, we have $D(r) = I(r)$ and therefore $F(r) = N(r) = \frac{rD(r)}{H(r)}$. Thus the monotonicity of $\frac{r}{2}\frac{d}{dr}\log H(r)$ is equivalent to that of the frequency $F(r)$, or $N(r)$, if we are in the case of the zero-obstacle problem.

The problem with the term $\frac{\int_{\mathbb{S}_r} U^2 L_a|X|}{H(r)}$ in (3.2.3) arises when we are not anymore in the classical Laplacian case, but we work with a variable coefficient operator. To restrict this term that creates obstructions in the monotonicities' proofs the following quantities were introduced

Definition 3.2.3. Let U be a solution of (3.0.1). Consider the function $G : (0,1] \to (0,\infty)$

defined for any $r \in (0,1]$ by

$$G(r) = \begin{cases} \frac{\int_{\mathbb{S}_r} U^2 L_a|X|}{\int_{\mathbb{S}_r} U^2 \mu(X)} & \text{if } H(r) \neq 0, \\ \frac{n+a}{r} & \text{if } H(r) = 0. \end{cases} \tag{3.2.6}$$

Definition 3.2.4. The function $\psi : (0,1] \to (0,\infty)$ is defined by the Cauchy problem

$$\begin{cases} \frac{d}{dr} \log \psi(r) = \frac{\psi'(r)}{\psi(r)}, = G(r) & \text{if } r \in (0,1), \\ \psi(1) = 1 \end{cases} \tag{3.2.7}$$

Definition 3.2.5. The function $\sigma : (0,1] \to (0,\infty)$ defined by the Cauchy problem

$$\begin{cases} \frac{\sigma'(r)}{\sigma(r)} - \frac{\psi'(r)}{\psi(r)} + \frac{n-1+a}{r} = 0 & \text{if } r \in (0,1), \\ \sigma(1) = 1, \end{cases} \tag{3.2.8}$$

which, as you can see from the first glance, allow to replace our problematic term with $\frac{n-1+a}{r} + \frac{\sigma'(r)}{\sigma(r)}$, where first term is analogous to the term we get for $L = \Delta$ but for the fractional dimension $N = n+1+a$ and the second term helps to compensate the errors created by introducing $\psi(r)$ in the calculations of the Theorem 3.2.12.

Let us note again that when $L = \Delta$ (in $\mathbb{R}^{n+1}$), we have $L|x| = \frac{n}{|x|}$, $\mu \equiv 1$, and therefore $G(r) = \frac{n}{r}$. In our case instead we have the following result.

Lemma 3.2.6. *There exists a universal constant $\beta \geq 0$ such that for any $r \in (0,1)$:*

$$\frac{n+a}{r} - \beta \leq G(r) \leq \frac{n+a}{r} + \beta.$$

Similarly, when $L = \Delta$ (in $\mathbb{R}^{n+1}$), we have $\psi(r) = r^n$, $\sigma(r) = r$, and for our problem the following is true.

Lemma 3.2.7. *There exists a universal constant $\beta \geq 0$ such that if $r \in (0,1)$ one has*

$$\frac{n+a}{r} - \beta \leq \frac{d}{dr} \log(\psi(r)) \leq \frac{n+a}{r} + \beta$$

and therefore

$$e^{-\beta(1-r)} r^{n+a} \leq \psi(r) \leq e^{\beta(1-r)} r^{n+a}.$$

This implies, in particular, that $\psi(0^+) = 0$. For the function $\sigma(r)$ we have $\sigma(r) = \frac{\psi(r)}{r^{n-1+a}}$, and so

$$e^{-\beta(1-r)} r \leq \sigma(r) \leq e^{\beta(1-r)} r$$

for $0 < r < 1$. In particular, $\sigma(0^+) = 0$.

Lemma 3.2.8. *There exists a universal constant r_0 such that the function $r \mapsto \sigma(r)$ is increasing on $(0, r_0)$.*

The next result is important for the proof of the optimal regularity of the solution to connect two types of frequencies $\tilde{N}(0^+)$ and $N(0^+)$.

Lemma 3.2.9. (Lemma 4.9, [4])
One has for $r \in (0,1)$

$$\left| \frac{\sigma(r)}{r} - \alpha^{\pm} \right| \leq \beta e^{\beta} r,$$

where $\alpha^- = \liminf_{r\to 0^+} \frac{\sigma(r)}{r}$ and $\alpha^+ = \limsup_{r\to 0^+} \frac{\sigma(r)}{r}$.
In particular, we have $\alpha^+ = \alpha^-$, and thus, in particular, it exists

$$\alpha \stackrel{\text{def}}{=} \lim_{r\to 0^+} \frac{\sigma(r)}{r} > 0.$$

Given $\delta \in (0,1)$ and universal constant $r_0 > 0$ (which will also depend on δ) we introduce the sets

$$\Lambda_{r_0} = \{r \in (0, r_0) \mid H(r) > \psi(r) r^{3+\delta}\}, \tag{3.2.9}$$

$$\Gamma_{r_0} = \{r \in (0, r_0) \mid H(r) > e^{-\beta} r^{3+\delta+n+a}\}. \tag{3.2.10}$$

Lemma 3.2.10. *One has the inclusion $\Lambda_{r_0} \subset \Gamma_{r_0}$. In particular, $H(r) \neq 0$ for every $r \in \Lambda_{r_0}$.*

Following [18] the authors of [4] introduce

$$M(r) = \frac{H(r)}{\psi(r)}, \qquad J(r) = \frac{I(r)}{\psi(r)} \tag{3.2.11}$$

and define the *generalised frequency* as

$$\Phi(r) = \frac{\sigma(r) J(r)}{M(r)}. \tag{3.2.12}$$

Next we see the result of the first variation of the Dirichlet integral.

Theorem 3.2.11. *For almost every $r \in (0,1)$ one has*

$$D'(r) = 2 \int_{\mathbb{S}_r} \frac{(< A(x)\nabla U, \nu >)^2}{\tilde{\mu}} |y|^a + \left(\frac{n+a-1}{r} + O(1) \right) D(r) - \frac{2}{r} \int_{\mathbb{B}_r} < Z, \nabla U > f |y|^a, \tag{3.2.13}$$

where $Z = \frac{A(x)X}{\tilde{\mu}(X)}$.

And these results help to prove the monotonicity of generalised and truncated Almgren type frequencies.

Theorem 3.2.12. *Assume that $U(0) = 0$. Given $\delta \in (0, 1)$, there exist universal constants $r_0, K, K' > 0$ such that the function $r \mapsto e^{Kr^{\frac{1-\delta}{2}}}\Phi(r)$ is non-decreasing on Γ_{r_0}. Precisely, for every $r \in \Gamma_{r_0}$ we have*

$$\frac{d}{dr}\log\Phi(r) = \frac{\Phi'(r)}{\Phi(r)} \geq -\frac{K'}{r^{\frac{1+\delta}{2}}}.$$

Theorem 3.2.13. *Assume U(0) = 0. With r_0, K as in Theorem 3.2.12 corresponding to some choice of $\delta \in (0, 1)$, we have that*

$$r \mapsto N^*(r) \overset{\text{def}}{=} \frac{\sigma(r)}{2} e^{Kr^{\frac{1-\delta}{2}}} \frac{d}{dr}\log\max\{M(r), r^{3+\delta}\} \tag{3.2.14}$$

is non-decreasing in $(0, r_0)$. In particular, $N(0^+)$ exists.

We also need to define the following quantity

$$\tilde{N}^*(r) \overset{\text{def}}{=} \frac{r}{\sigma(r)}N(r). \tag{3.2.15}$$

Corollary 3.2.13.1. *Let $\tilde{N}(r)$ be defined as in (3.2.15). Then $\tilde{N}(0^+)$ exists.*

3.3 The growth lemma, the optimal regularity

Next we have a growth theorem Theorem 3.3.1 which was proved using the monotonicity results from the previous sections (choosing δ such that $3 + \delta > 3 - a$) and the known blowup analysis [18], [7]:

Theorem 3.3.1. *Let U be a solution to (3.1.1) and let $X_0 = (x_0, 0) \in \Gamma(U)$. Then we have that*

$$U(X) \leq C|X - X_0|^{\frac{3-a}{2}}, \tag{3.3.1}$$

for some universal constant C,

which with the estimates and regularity results obtained before helps to reach the first goal of the paper [4], that is

Theorem 3.3.2. *Assume $0 \leq a < 1$. Let U be a solution to (3.0.1) with $\varphi \in C^{1,1}$. Then $\nabla_x U \in C^{\frac{1-a}{2}}(\overline{\mathbb{B}^+_{\frac{1}{2}}})$ and $y^a U_y \in C^{\frac{1+a}{2}}(\overline{\mathbb{B}^+_{\frac{1}{2}}})$. In particular, $U(\cdot, 0) \in C^{\frac{3-a}{2}}(B_{\frac{1}{2}})$.*

3.4 The Weiss Type formula, the regular free boundary

Finally the Weiss type monotonicity formula was obtained, but we should note that it works only for one value of homogeneity $k = \frac{3-a}{2}$ since it is only one needed in the study of the smoothness of the regular set (Theorem 3.4.2). While in our subsequent reasoning we will need the monotonicity of the general Weiss type functional, which is computed in Theorem 4.3.3.

Theorem 3.4.1. *(Weiss Type Monotonicity Formula)*
Given a solution U to to (3.1.1)*, such that $0 \in \Gamma_{\frac{3-a}{2}}(U)$, define*

$$W(U,r) = W(r) = \frac{\sigma(r)}{r^{3-a}}\{J(r) - \frac{3-a}{2r}M(r)\}. \tag{3.4.1}$$

There exist universal constants $C, r_0 > 0$, depending on $\|f\|_{L^\infty(\mathbb{B}_1)}$, such that for any $0 < r < r_0$ one has:

$$\frac{d}{dr}(W(U,r) + Cr^{\frac{1+a}{2}}) \geq \frac{2}{r^{n+2}} \int_{\mathbb{S}_r} \left(\frac{< A\nabla U, nu >}{\sqrt{\tilde{\mu}}} - \frac{3-a}{2r}\tilde{\mu}U \right)^2 |y|^a. \tag{3.4.2}$$

In particular, there exists $C > 0$ such that the function $r \mapsto W(U,r) + Cr^{\frac{1+a}{2}}$ is monotone increasing and therefore the limit $W(U, 0^+) := \lim_{r\to 0^+} W(U,r)$ exists.

Combining the Weiss type monotonicity with the epiperimetric inequality, the authors of [4] reached their second main result.

Theorem 3.4.2. *Suppose that $0 \leq a < 1$ and let U be as in Theorem* 3.3.2*. Then $\Gamma_\varphi^{\frac{3-a}{2}}(U)$ is a relatively open subset of $\Gamma_\varphi(U)$. After possibly a translation and rotation of the coordinate axes in the thin space $\mathbb{R}^n \times \{0\}$, the set $\Gamma_\varphi^{\frac{3-a}{2}}(U)$ is locally given as a graph*

$$x_n = g(x_1, ...x_{n-1})$$

with $g \in C^{1,\gamma}$.

Chapter 4

The structure of the singular free boundary

In the paper [4], which we discussed in the previous chapter, the authors reached results about the optimal interior regularity of the solution (Theorem 3.3.2) and the smoothness of the regular part of free boundary (Theorem 3.4.2) in the thin obstacle problem with non-zero obstacle (3.0.1). In this chapter we study the structure of the singular free boundary in the same problem but with the zero obstacle. Thus, our zero-obstacle problem is

$$\begin{cases} L_a U = \operatorname{div}(y^a A(x)\nabla U(x,y)) = 0 \text{ in } \mathbb{B}_1^+, \\ \min\{U(x,0), -\partial_y^a U(x,0)^+\} = 0 \text{ on } B_1, \end{cases} \tag{4.0.1}$$

where for $x \in \mathbb{R}^n, y > 0$, we have indicated $X = (x,y) \in \mathbb{R}^{n+1}$. The set

$$\Lambda(U) = \{x \in B_1 \mid U(x,0) = 0\}$$

is as usual called the *coincidence set* and its topological boundary (relative to the topology of B_1)

$$\Gamma(U) = \partial\Lambda(U)$$

is referred to as the *free boundary*.

The extended to the whole ball solution of problem (4.0.1) (as in Chapter 2), the function U satisfies the following *Signorini* or *complementary conditions*:

$$L_a U = \operatorname{div}(|y|^a A(x)\nabla U(x,y)) = 0 \quad \text{in } \mathbb{B}_1^+ \cup \mathbb{B}_1^-, \tag{4.0.2}$$

$$U \geq 0 \quad \text{in } B_1, \tag{4.0.3}$$

$$\begin{aligned} &\lim_{y\to 0}[(< \nabla U, A(x)\nu_+ > + < A\nabla U, A(x)\nu_- >)|y|^a] \\ &\qquad = -2\partial_y^a U(x,0) \geq 0 \quad \text{in } B_1 \quad (\nu_\pm = \mp e_{n+1}), \end{aligned} \tag{4.0.4}$$

$$U\lim_{y\to 0}[(< A\nabla U,\nu_+ > + < A\nabla U,\nu_- >)|y|^a] \\ = -2U\partial_y^a U(x,0) = 0 \quad \text{in } B_1, \tag{4.0.5}$$

$$\int_{\mathbb{B}_1} < A\nabla U,\nabla\eta > |y|^a = -2\int_{B_1}\partial_y^a U(x,0)\eta, \quad \eta\in C_0^\infty(\mathbb{B}_1), \tag{4.0.6}$$

The equations (4.0.2)-(4.0.5) follow straightforward from the problem (4.0.1) and (3.0.2)-(3.0.4). To prove the equation (4.0.6), we denote $\mathbb{B}_{1\varepsilon}^+ = \mathbb{B}_1\cap\{\varepsilon < y < 1\}$, $\mathbb{S}_{1\varepsilon}^+ = \mathbb{S}_1\cap\overline{\mathbb{B}_{1\varepsilon}^+}$, $L_{1\varepsilon} = \mathbb{B}_{1\varepsilon}^+\cap\{y=\varepsilon\}$ for some $0<\varepsilon<1$ and observe that

$$\begin{aligned}\int_{\mathbb{B}_{1\varepsilon}^+} < A\nabla U,\nabla\eta > |y|^a &\overset{\text{div.thm}}{=} -\int_{\mathbb{B}_{1\varepsilon}^+}\operatorname{div}(|y|^a A\nabla U)\eta + \int_{\partial\mathbb{B}_{1\varepsilon}^+} < A\nabla U,\nu > \eta|y|^a \\ &\overset{(4.0.1)}{=} \int_{\mathbb{S}_{1\varepsilon}^+} < A\nabla U,\nu > \eta|y|^a + \int_{L_{1\varepsilon}} < A\nabla U,\nu_+ > \eta|y|^a = \\ &= \int_{L_{1\varepsilon}} < A\nabla U,\nu_+ > \eta|y|^a,\end{aligned}$$

where in the last step we used the fact that $\eta\in C_0^\infty(\mathbb{B}_1)$. In fact, we can replace ε with y now, since $L_{1\varepsilon}\subset\{y=\varepsilon\}$, i.e. $y=\varepsilon$ in $L_{1\varepsilon}$. Now letting $y\to 0^+$ we obtain

$$\int_{\mathbb{B}_1^+} < A\nabla U,\nabla\eta > |y|^a = -\int_{B_1}\partial_y^a U(x,0)\eta.$$

Repeating the calculations for $\mathbb{B}_1^-$ and sum all together we obtain (4.0.6).

Definition 4.0.1. We denote by $\mathfrak{S}$ the class of solutions of the normalised Signorini problem (4.0.2) - (4.0.6).

Let $U\in\mathfrak{S}$, we denote by $\Gamma_k(U)$ the set of free boundary points $X_0\in\Gamma(U)$ where frequency $\tilde{N}^{x_0}(0^+) = k$ (see (4.1.5)). Given $U\in\mathfrak{S}$, we say that $X_0=(x_0,0)\in\Gamma(U)$ is a *singular point* of the free boundary if

$$\lim_{r\to 0^+}\frac{\mathcal{H}^n(\Lambda(U_{x_0})\cap B_r)}{\mathcal{H}^n(B_r)} = 0.$$

The following is the main result of the chapter and the thesis.

Theorem 4.0.2. (Structure of the singular set)
Let $U\in\mathfrak{S}$. Then $\Gamma_k(U)=\Sigma_k(U)$ for $k=2m$, $m\in\mathbb{N}$. Moreover, every set

$$\Sigma_k^d(U) := \{X_0\in\Sigma_k(U) \mid d_k^{X_0} = d\}, \quad d = \{0,...,n-1\}$$

is contained in a countable union of d-dimensional C^1 manifolds.

4.1 The monotonicty of the Almgren frequency

The *frequency* of U in $\mathbb{B}_r$ is given by

$$N(U,r) = \frac{rD(r)}{H(r)} \tag{4.1.1}$$

Under the assumption of Theorem 3.3.2 above, we know that the weak solution $U \in \mathfrak{S}$ is in $C^{1,\alpha}_{\text{loc}}(\mathbb{B}_1^{\pm} \cup B_1)$. Consequently, all derivatives are classical in the ensuing computations.. Remember that from Lemma 3.2.2 in a case with the obstacle $\varphi = 0$ and therefore homogeneous system (4.0.1), meaning $f = 0$, we have

$$D(r) = I(r) = \int_{\mathbb{S}_r} U < A(x)\nabla U, \nu > |y|^a \tag{4.1.2}$$

and from Theorem 3.2.11, also since $f = 0$, we have the following first variation of the Dirichlet integral

$$D'(r) = 2\int_{\mathbb{S}_r} \frac{(< A(x)\nabla U, \nu >)^2}{\tilde{\mu}}|y|^a + \left(\frac{n+a-1}{r} + O(1)\right) D(r). \tag{4.1.3}$$

Concerning the first variation of the height function, even though we have Lemma 3.2.1, we prefer to show the proof of the following proposition.

Proposition 4.1.1. *Assume that the normalization hypothesis $A(0) = \mathbb{I}_{n+1}$ is in place. Under this assumption, for almost every $r \in (0,1)$ one has*

$$H'(r) = 2I(r) + \left(\frac{n+a}{r} + O(1)\right) H(r). \tag{4.1.4}$$

Proof. By changing of variable $X = (x,y) = (r\omega, r\xi) = rZ$, $Z = (\omega,\xi) \in \mathbb{S}_1 \subset \mathbb{R}^n$ in the integral $H(r)$, and using the fact that $|y|^a d\sigma(X) = r^{n+a}|\xi|^a d\sigma(Z)$, we find that

$$H(r) = \int_{\mathbb{S}_r} U^2(X)\mu(X) = r^{n+a}\int_{\mathbb{S}_1} U^2(rZ)\tilde{\mu}(rZ)|\xi|^a = r^{n+a}\int_{\mathbb{S}_1} U^2(rZ) < A(rx)\nu,\nu > |\xi|^a.$$

Differentiating under the integral sign in the above formula, we obtain

$$\begin{aligned}
H'(r) =& \frac{n+a}{r} r^{n+a}\int_{\mathbb{S}_1} U^2(rZ)\tilde{\mu}(rZ)|\xi|^a + 2r^{n+a}\int_{\mathbb{S}_1} U(rZ) < \nabla U(rZ), Z > \tilde{\mu}(rZ)|\xi|^a + \\
&+ r^{n+a}\int_{\mathbb{S}_1} U^2(rZ) < \nabla\tilde{\mu}(rZ), Z > |\xi|^a \\
=& \frac{n+a}{r}\int_{\mathbb{S}_r} U^2(X)\mu(X) + 2\int_{\mathbb{S}_r} U(X) < \nabla U(X), \frac{X}{r} > \mu|y|^a \\
&+ \int_{\mathbb{S}_r} U^2(X) < \nabla\tilde{\mu}(X), \frac{X}{r} > |y|^a,
\end{aligned}$$

$$\begin{aligned} H'(r) &= \frac{n+a}{r}H(r) + 2\int_{\mathbb{S}_r} U < A(x)\nabla U, \nu > |y|^a + \int_{\mathbb{S}_r} U^2 < \nabla\tilde{\mu}, \nu > |y|^a \\ &= \frac{n+a}{r}H(r) + 2I(r) + O(1)\int_{\mathbb{S}_r} U^2|y|^a, \end{aligned}$$

where in the last step we used the definition (2.1.5) and also (3) in the Lemma 2.1.1. Consequently, we obtain

$$\begin{aligned} H'(r) &= \frac{n+a}{r}H(r) + 2I(r) + O(1)\frac{1}{\tilde{\mu}}\int_{\mathbb{S}_r} U^2\mu \\ &\overset{(2.1.2)}{=} \frac{n+a}{r}H(r) + 2I(r) + O(1)H(r). \end{aligned}$$

□

With these results we can now prove the monotonicity of the frequency.

Theorem 4.1.2. (Monotonicity of the adjusted frequency)
Assume that the normalization hypothesis $A(0) = \mathbb{I}_{n+1}$ is in place, and that furthermore (2.0.7) *is in force. Then, there exists a universal constant $C > 0$ such that the function*

$$\tilde{N}(r) \overset{\text{def}}{=} e^{Cr}N(r) \tag{4.1.5}$$

is monotone nondecreasing in $(0, 1)$. *In particular, the limit* $\lim_{r\to 0^+}\tilde{N}(r) = \tilde{N}(0^+)$ *exists. We conclude that* $\lim_{r\to 0^+} N(r) = N(0^+)$ *also exists, and equals* $\tilde{N}(0^+)$.

Proof. From (4.1.4) and (4.1.3), we have for almost every $r \in (0, 1)$

$$\begin{aligned} \frac{d}{dr}\log N(r) &= \frac{D'(r)}{D(r)} + \frac{1}{r} - \frac{H'(r)}{H(r)} \\ &= 2\frac{\int_{\mathbb{S}_r}\frac{(<A(x)\nabla U,\nu>)^2}{\tilde{\mu}}|y|^a}{D(r)} + \frac{n+a-1}{r} + O(1) + \frac{1}{r} \\ &\quad - 2\frac{\int_{\mathbb{S}_r} U < A(x)\nabla U, \nu > |y|^a}{H(r)} - \frac{n+a}{r} - O(1) \\ &= 2\frac{\int_{\mathbb{S}_r}\frac{(<A(x)\nabla U,\nu>)^2}{\tilde{\mu}}|y|^a}{D(r)} - 2\frac{\int_{\mathbb{S}_r} U < A(x)\nabla U, \nu > |y|^a}{H(r)} + O(1) \end{aligned}$$

By the following Cauchy-Schwarz inequality

$$\begin{aligned} \left[\int_{\mathbb{S}_r} U < A(x)\nabla U, \nu > |y|^a\right]^2 &= \left[\int_{\mathbb{S}_r}\left(U\tilde{\mu}^{\frac{1}{2}}|y|^{\frac{a}{2}}\right)\left(\frac{< A(x)\nabla U, \nu >}{\tilde{\mu}^{\frac{1}{2}}}|y|^{\frac{a}{2}}\right)\right]^2 \le \\ &\le \int_{\mathbb{S}_r} U^2\mu \cdot \int_{\mathbb{S}_r}\frac{(< A(x)\nabla U, \nu >)^2}{\tilde{\mu}}|y|^a \end{aligned}$$

and (4.1.2), we obtain that

$$\frac{d}{dr}\log N(r) \geq -C,$$

for some universal constant $C > 0$, that gives

$$\frac{d}{dr}\log \tilde{N}(r) = \frac{d}{dr}\log N(r) + C \geq 0$$

which implies the desired result. □

4.2 The growth lemmas

Lemma 4.2.1. *Assume that the normalization hypothesis $A(0) = \mathbb{I}_{n+1}$ is in place, and that furthermore* (2.0.7) *is in force. Suppose that $N(0^+) \geq k$. Then, for $r \in (0,1)$ one has*

$$H(r) \leq Cr^{n+a+2k}, \tag{4.2.1}$$

where $C > 0$ is a universal constant.

Proof. To obtain (4.2.1) it is enough to follow the reasoning in the Lemma 4.1 of [21], but for "dimension" $n + a + 1$, since we have Proposition 4.1.1 and Theorem 4.1.2. □

We note the following gap of frequency follows from Theorem 5.7 of [7], which concerns the degree of homogeneous global solutions to the constant coefficient Signorini problem ($A(x) = \mathbb{I}_{n+1}$) with zero obstacle.

Lemma 4.2.2. (Frequency gap) *Let $0 \in \Gamma(U)$ and assume that $A(0) = \mathbb{I}$. Then either $\tilde{N}(0^+) = \frac{3-a}{2}$ or $\tilde{N}(0^+) \geq \frac{3+\delta}{2}$.*

Thus, combining Lemma 4.2.2 with Theorem 3.3.1, we obtain the following improved growth estimate of the solution.

Lemma 4.2.3. *Assume the hypothesis of Lemma* 4.2.1*, there exists a universal constant $C > 0$, depending on k, such that for every $X \in \mathbb{B}_{1/2}$, one has*

$$|U(X)| \leq C|X - X_0|^k. \tag{4.2.2}$$

Lemma 4.2.4. *Assume the hypothesis of Lemma* 4.2.1*. Then, there exists a universal constant $C^* > 0$ such that*

$$D(r) \leq C^* r^{n+a-1+2k}. \tag{4.2.3}$$

Proof. We will follow the reasoning of the Lemma 2.2.21. The desired conclusion will follow from

$$\int_{\mathbb{B}_r} U^2|y|^a \leq \int_0^r H(r) \overset{(4.2.1)}{\leq} Cr^{n+a+1+2k} \tag{4.2.4}$$

and the following Caccioppoli-type inequality

$$D\left(\frac{r}{2}\right) \leq \frac{C_2}{r^2}\int_{\mathbb{B}_r} U^2|y|^a, \tag{4.2.5}$$

which holds for $r \in (0,1)$, for a universal constant $C_2 > 0$.

To prove (4.2.5) let $\alpha \in C_0^\infty(\mathbb{B}_r)$ be such that $0 \leq \alpha \leq 1$, $\alpha \equiv 1$ on $\mathbb{B}_{r/2}$, and $|\nabla\alpha| \leq C/r$. Define $h = \alpha^2 U$, then

$$\begin{aligned}\int_{\mathbb{B}_r^+} < A(x)\nabla U, \nabla h > |y|^a &= -\int_{\mathbb{B}_r^+} \mathrm{div}(A\nabla U|y|^a)h + \int_{\mathbb{S}_r^+} < A(x)\nabla U, \nu > h|y|^a \\ &\quad + \int_{B_r} < A(x)\nabla U, \nu_+ > h|y|^a \\ &= \int_{B_r} < A(x)\nabla U, \nu_+ > h|y|^a,\end{aligned}$$

where we used (4.0.1) and the fact that $\alpha \in C_0^\infty(\mathbb{B}_r)$.

Thus repeating the argument for $\mathbb{B}_r^-$ we obtain

$$\int_{\mathbb{B}_r} < A(x)\nabla U, \nabla h > |y|^a = \int_{B_r} \alpha^2 U(< A(x)\nabla U, \nu_+ > + < A(x)\nabla U, \nu_- >)|y|^a dH^n \overset{(4.0.5)}{=} 0$$

Therefore

$$\begin{aligned}\int_{\mathbb{B}_r} < A(x)\nabla U, \nabla(\alpha^2 U) > |y|^a &= \int_{\mathbb{B}_r} \alpha^2 < A(x)\nabla U, \nabla U > |y|^a \\ &\quad + 2\int_{\mathbb{B}_r} \alpha U < A(x)\nabla U, \nabla\alpha > |y|^a = 0,\end{aligned}$$

which implies

$$\int_{\mathbb{B}_r} \alpha^2 < A(x)\nabla U, \nabla U > |y|^a \leq 2\left(\int_{\mathbb{B}_r} U^2 < A(x)\nabla\alpha, \nabla\alpha > |y|^a\right)^{\frac{1}{2}}\left(\int_{\mathbb{B}_r} \alpha^2 < A(x)\nabla U, \nabla U > |y|^a\right)^{\frac{1}{2}},$$

$$\left(\int_{\mathbb{B}_r} \alpha^2 < A(x)\nabla U, \nabla U > |y|^a\right)^{\frac{1}{2}} \leq 2\left(\int_{\mathbb{B}_r} U^2 < A(x)\nabla\alpha, \nabla\alpha > |y|^a\right)^{\frac{1}{2}} \leq 2\left(\frac{C_3^2}{r^2}\int_{\mathbb{B}_r} U^2\tilde{\mu}|y|^a\right)^{\frac{1}{2}}.$$

By (4.2.1)

$$\int_{\mathbb{B}_r} \alpha^2 < A(x)\nabla U, \nabla U > |y|^a \leq \frac{4C_3}{r}H(r) \leq C^* r^{n+a-1+2k}.$$

Since $\alpha \equiv 1$ on $\mathbb{B}_{r/2}$

$$\int_{\mathbb{B}_r} \alpha^2 < A(x)\nabla U, \nabla U > |y|^a = \int_{\mathbb{B}_{r/2}} < A(x)\nabla U, \nabla U > |y|^a + \int_{\mathbb{B}_r/\mathbb{B}_{r/2}} \alpha^2 < A(x)\nabla U, \nabla U > |y|^a$$
$$\geq \int_{\mathbb{B}_{r/2}} < A(x)\nabla U, \nabla U > |y|^a = D\left(\frac{r}{2}\right).$$

This gives (4.2.3). □

4.3 A one parameter family of Weiss type Monotonicity Formula

In this section we introduce a generalisation of the Weiss type functional in [19], and prove its basic monotonicity property.

Definition 4.3.1. *(Weiss-type monotonicity formula)*
For $k > 0$, we define

$$\begin{aligned} W_k(r) &= \frac{1}{r^{n+a-1+2k}} \left[\int_{\mathbb{B}_r} < A(x)\nabla U, \nabla U > |y|^a dX - \frac{k}{r} \int_{\mathbb{S}_r} U^2 \tilde{\mu} |y|^a \right] \\ &= \frac{1}{r^{n+a-1+2k}} \left[D(r) - \frac{k}{r} H(r) \right]. \end{aligned} \tag{4.3.1}$$

Since in our case the obstacle $\varphi(x) = 0$, we know that $I(r) = D(r)$, while it is not true for the general case of the non-zero obstacle problem. Using that and the fact that $\frac{\sigma(r)}{\psi(r)} = \frac{1}{r^{n+a-1}}$, we can obtain the generalised formula

$$W_k(r) = W_k(U, r) = \frac{\sigma(r)}{r^{2k}} \left[J(r) - \frac{k}{r} M(r) \right] = \frac{\sigma(r)}{r^{2k}\psi(x)} \left[I(r) - \frac{k}{r} H(r) \right], \tag{4.3.2}$$

that is compatible with the Weiss Type formula of [4] (which was defined just for one value of $k = \frac{3-a}{2}$, see (3.4.1)).

Lemma 4.3.2. *Suppose the normalization hypothesis $A(0) = \mathbb{I}_{n+1}$ is in place, and that furthermore (2.0.7) is in force. If $N(0^+) \geq k$, then there exist $\bar{C} > 0$ such that $|W_k(r)| \leq \bar{C}$ for every $0 < r < 1$. If instead $N(0^+) = k$, then $W_k(0^+) = \lim_{r \to 0^+} W_k(r)$.*

Proof. The fact that $W_k(r)$ is bounded follows from its definition (4.3.1) and boundness of the height $H(r)$ (4.2.1) and the Dirichlet integral $D(r)$ (4.2.3).
We know from the Theorem 4.1.2 that $\tilde{N}(0^+)$ exists and that

$$\tilde{N}(0^+) = N(0^+) = \lim_{r \to 0^+} \frac{rD(r)}{H(r)}.$$

Let us rewrite the definition of the Weiss type formula (4.3.1) in this form

$$W_k(0^+) = \lim_{r\to 0^+}\left(\frac{H(r)}{r^{n+a+2k}}\left[\frac{rD(r)}{H(r)} - k\right]\right) = \lim_{r\to 0^+}\left(\frac{H(r)}{r^{n+a+2k}}\left(N(r)-k\right)\right).$$

And since we know, that the limit $\tilde{N}(0^+)$ exists by previous reasoning and that $H(r)/r^{n+a+2k}$ is bounded by (4.2.1), the second conclusion follows. $\square$

Further we prove the "almost monotonicity" property of the functional W_k

Theorem 4.3.3. . *Assume that the normalization hypothesis $A(0) = \mathbb{I}_{n+1}$ is in place, and that furthermore* (2.0.7) *is in force. Suppose that $N(0^+) \geq k$. Then there exist a universal constant $C > 0$ such that*

$$\frac{d}{dr}(W_k(r) + Cr) \geq \frac{2}{r^{n+a+1+2k}}\int_{\mathbb{S}_r}\left(\frac{< A(x)\nabla U, X >}{\sqrt{\tilde{\mu}}} - k\sqrt{\tilde{\mu}}U\right)^2 |y|^a. \tag{4.3.3}$$

As a consequence of (4.3.3)*, the function $r \mapsto W_k(r) + Cr$ is monotone non-decreasing, and therefore it has a limit as $r \to 0^+$. As a consequence, also the limit $W_k(0^+) = \lim_{r\to 0^+} W_k(r)$ exists and is finite.*

Proof. Let us now calculate derivative of $W_k(r)$

$$\begin{aligned}\frac{d}{dr}W_k(r) &= \frac{\sigma(r)}{r^{2k}}\left(J'(r) + \frac{k}{r^2}M(r) - \frac{k}{r}M'(r)\right) + \left(\frac{\sigma'(r)}{r^{2k}} - \frac{2k\sigma(r)}{r^{2k+1}}\right)\left(J(r) - \frac{k}{r}M(r)\right)\\ &= \frac{\sigma(r)}{r^{2k}}\left[\left(\frac{\sigma'(r)}{\sigma(r)} - \frac{2k}{r}\right)\left(J(r) - \frac{k}{r}M(r)\right) + \left(J'(r) + \frac{k}{r^2}M(r) - \frac{k}{r}M'(r)\right)\right].\end{aligned} \tag{4.3.4}$$

Now we can substitute in (4.3.4) the expression for $J'(r)$ from the theorem 4.19 of [4]

$$\begin{aligned}J'(r) = &\left(\frac{n-1+a}{r} - \frac{\psi'(r)}{\psi(r)} + O(1)\right)J(r) + \frac{2}{\psi(r)}\int_{\mathbb{S}_r}\frac{(< A(x)\nabla U, \nu >)^2}{\tilde{\mu}}|y|^a +\\ &+ \frac{1}{\psi(r)}\left[\int_{\mathbb{S}_r} Uf|y|^a - \left(\frac{n-1+a}{r} + O(1)\right)\int_{\mathbb{B}_r} Uf|y|^a - \frac{2}{r}\int_{\mathbb{B}_r} < Z, \nabla U > f|y|^a\right],\end{aligned}$$

taking into account that $f \equiv 0$, since we are in the zero-obstacle case. We use also (3.2.7), (3.2.8)

$$\begin{aligned}\frac{d}{dr}W_k(r) = &\frac{\sigma(r)}{r^{2k}}\left[\left(\frac{\psi'(r)}{\psi(r)} - \frac{n-1+a}{r} - \frac{2k}{r}\right)J(r) - \frac{k}{r}\left(\frac{\psi'(r)}{\psi(r)} - \frac{n-1+a}{r} - \frac{2k}{r}\right)M(r)\right] +\\ &+ \frac{\sigma(r)}{r^{2k}}\left[\left(\frac{n-1+a}{r} - \frac{\psi'(r)}{\psi(r)} + O(1)\right)J(r) + \frac{2}{\psi(r)}\int_{\mathbb{S}_r}\frac{(< A(x)\nabla U, \nu >)^2}{\tilde{\mu}}|y|^a\right] +\end{aligned}$$

$$+\frac{\sigma(r)}{r^{2k}}\left[\frac{k}{r^2}M(r)-\frac{k}{r}M'(r)\right].$$

From the proof of the theorem 4.19 of [4] we also have that $M'(r)=2J(r)$

$$\begin{aligned}\frac{d}{dr}W_k(r)=&\frac{\sigma(r)}{r^{2k}}\left[\left(-\frac{2k}{r}+O(1)\right)J(r)-\frac{2k}{r}J(r)+\frac{2}{\psi(r)}\int_{\mathbb{S}_r}\frac{(<A(x)\nabla U,\nu>)^2}{\tilde{\mu}}|y|^a\right]+\\&+\frac{\sigma(r)}{r^{2k}}\left[\frac{k}{r^2}M(r)\left(1-\left(r\frac{\psi'(r)}{\psi(r)}-(n-1+a)-2k\right)\right)\right]\\=&\frac{\sigma(r)}{r^{2k}}\left[\left(-\frac{4k}{r}+O(1)\right)J(r)+\frac{2}{\psi(r)}\int_{\mathbb{S}_r}\frac{(<A(x)\nabla U,\nu>)^2}{\tilde{\mu}}|y|^a\right]+\\&+\frac{\sigma(r)}{r^{2k}}\left[\frac{k}{r^2}M(r)\left(-r\frac{\psi'(r)}{\psi(r)}+n+a+2k\right)\right].\end{aligned}$$

From (3.2.7) and the Lemma 3.2.6 we know that

$$\frac{\psi'(r)}{\psi(r)}=\frac{n+a}{r}+O(1),$$

so

$$r\frac{\psi'(r)}{\psi(r)}=n+a+O(r)$$

and we have

$$\frac{d}{dr}W_k(r)=\frac{\sigma(r)}{r^{2k}}\left[\left(-\frac{4k}{r}+O(1)\right)J(r)+\frac{2}{\psi(r)}\int_{\mathbb{S}_r}\frac{(<A(x)\nabla U,\nu>)^2}{\tilde{\mu}}|y|^a+\frac{2k^2}{r^2}M(r)-\frac{O(1)}{r}M(r)\right].\tag{4.3.5}$$

Let us calculate the following

$$\begin{aligned}\int_{\mathbb{S}_r}\left(\frac{<A(x)\nabla U,\nu>}{\sqrt{\tilde{\mu}}}-\frac{k}{r}\sqrt{\tilde{\mu}}U\right)^2|y|^a=&\int_{\mathbb{S}_r}\frac{(<A(x)\nabla U,\nu>)^2}{\tilde{\mu}}|y|^a+\frac{k^2}{r^2}\int_{\mathbb{S}_r}U^2\tilde{\mu}|y|^a\\&-\frac{2k}{r}\int_{\mathbb{S}_r}U<A(x)\nabla U,\nu>|y|^a\\=&\int_{\mathbb{S}_r}\frac{(<A(x)\nabla U,\nu>)^2}{\tilde{\mu}}|y|^a+\frac{k^2}{r^2}H(r)-\frac{2k}{r}I(r)\\=&\int_{\mathbb{S}_r}\frac{(<A(x)\nabla U,\nu>)^2}{\tilde{\mu}}|y|^a+\frac{k^2}{r^2}H(r)-\frac{2k}{r}D(r),\end{aligned}\tag{4.3.6}$$

where in the last step we used that $D(r)=I(r)$, when the obstacle is equal to 0.

Then substitute (4.3.6) in (4.3.5) and obtain

$$
\begin{aligned}
\frac{d}{dr}W_k(r) &= \frac{2\sigma(r)}{r^{2k}\psi(r)}\left[\int_{\mathbb{S}_r}\left(\frac{< A(x)\nabla U,\nu >}{\sqrt{\tilde{\mu}}} - \frac{k}{r}\sqrt{\tilde{\mu}}U\right)^2 |y|^a + O(1)I(r) - \frac{O(1)}{r}H(r)\right] \\
&= \frac{2}{r^{n+a-1+2k}}\left[\int_{\mathbb{S}_r}\left(\frac{< A(x)\nabla U,\nu >}{\sqrt{\tilde{\mu}}} - \frac{k}{r}\sqrt{\tilde{\mu}}U\right)^2 |y|^a + O(1)D(r) - \frac{O(1)}{r}H(r)\right] \\
&= \frac{2}{r^{n+a-1+2k}}\left[\int_{\mathbb{S}_r}\left(\frac{< A(x)\nabla U,\nu >}{\sqrt{\tilde{\mu}}} - \frac{k}{r}\sqrt{\tilde{\mu}}U\right)^2 |y|^a\right] + \frac{O(1)D(r)}{r^{n+a-1+2k}} - \frac{O(1)H(r)}{r^{n+a+2k}},
\end{aligned}
$$

where in the second-to-last step we used the fact that $\frac{\sigma(r)}{\psi(r)} = \frac{1}{r^{n+a-1}}$. No we use (4.2.1), (4.2.3)

$$
\begin{aligned}
\frac{d}{dr}W_k(r) &= \frac{2}{r^{n+a-1+2k}}\int_{\mathbb{S}_r}\left(\frac{< A(x)\nabla U,\nu >}{\sqrt{\tilde{\mu}}} - \frac{k}{r}\sqrt{\tilde{\mu}}U\right)^2 |y|^a + O(1) \\
&= \frac{2\sigma(r)}{r^{2k}\psi(r)}\int_{\mathbb{S}_r}\left(\frac{< A(x)\nabla U,\nu >}{\sqrt{\tilde{\mu}}} - \frac{k}{r}\sqrt{\tilde{\mu}}U\right)^2 |y|^a + O(1) \\
&= \frac{2\sigma(r)}{r^{2k}\psi(r)}\int_{\mathbb{S}_r}\left(\frac{< A(x)\nabla U,\nu >}{\sqrt{\tilde{\mu}}} - \frac{k}{r}\sqrt{\tilde{\mu}}U\right)^2 |y|^a + C^+ - C^-
\end{aligned}
$$

where $C^+, C^- > 0$ are universal constants
In this way we get for $C > 0$

$$
\frac{d}{dr}W_k(r) \geq \frac{2\sigma(r)}{r^{2k}\psi(r)}\int_{\mathbb{S}_r}\left(\frac{< A(x)\nabla U,\nu >}{\sqrt{\tilde{\mu}}} - \frac{k}{r}\sqrt{\tilde{\mu}}U\right)^2 |y|^a - C,
$$

$$
\frac{d}{dr}(W_k(r) + Cr) \geq \frac{2}{r^{n+a-1+2k}}\int_{\mathbb{S}_r}\left(\frac{< A(x)\nabla U,\nu >}{\sqrt{\tilde{\mu}}} - \frac{k}{r}\sqrt{\tilde{\mu}}U\right)^2 |y|^a,
$$

$$
\frac{d}{dr}(W_k(r) + Cr) \geq \frac{2}{r^{n+a+1+2k}}\int_{\mathbb{S}_r}\left(\frac{< A(x)\nabla U,X >}{\sqrt{\tilde{\mu}}} - k\sqrt{\tilde{\mu}}U\right)^2 |y|^a,
$$

which means

$$
\frac{d}{dr}(W_k(r) + Cr) \geq 0. \tag{4.3.7}
$$

□

4.4 A one parameter family of Monneau-type Monotonicity Formula

The goal of this section is to establish a generalization of the one-parameter Monneau-type monotonicity formulas that was obtained in [19] for solutions of (4.0.1) and to prove its monotonicity. For the function p, we define

$$\begin{aligned}\Psi_p(r) &= \frac{1}{r^{n+a+2k-1}}\left[\int_{\mathbb{B}_r} |\nabla p|^2|y|^a - \frac{k}{r}\int_{\mathbb{S}_r} p^2|y|^a\right] \\ &= \frac{1}{r^{n+a+2k-1}}\int_{\mathbb{B}_r} |\nabla p|^2|y|^a - \frac{k}{r^{n+a+2k}}\int_{\mathbb{S}_r} p^2|y|^a.\end{aligned} \tag{4.4.1}$$

We consider polynomials p_k are homogeneous of degree k such that $p_k(x,0) \geq 0$ and $\operatorname{div}(|y|^a\nabla p_k) = 0$. By the divergence theorem, we have

$$\begin{aligned}\Psi_{p_k}(r) &= \frac{1}{r^{n+a+2k-1}}\left[\int_{\mathbb{S}_r} p_k < \nabla p_k, \nu > |y|^a - \frac{k}{r}\int_{\mathbb{S}_r} p_k^2|y|^a\right] \\ &= \frac{1}{r^{n+a+2k}}\int_{\mathbb{S}_r} p_k(< \nabla p_k, X > -kp_k)|y|^a = 0.\end{aligned}$$

Definition 4.4.1. *(Monneau-type monotonicity formula)*

$$\begin{aligned}M_k(r) = M_k(U, p_k, r) &= \frac{1}{r^{n+a+2k}}\int_{\mathbb{S}_r} (U-p_k)^2\tilde{\mu}|y|^a \\ &= \frac{\sigma(r)}{r^{2k+1}\psi(r)}\int_{\mathbb{S}_r} (U-p_k)^2\mu.\end{aligned} \tag{4.4.2}$$

Theorem 4.4.2. (Monneau-type monotonicity formula)
Let $U \in \mathfrak{S}$ and assume that $0 \in \Gamma_k(U)$. Suppose that the normalization hypothesis $A(0) = \mathbb{I}_{n+1}$ is in place, and that furthermore (2.0.7) *is in force. Then, there exists a universal constant $\tilde{C} > 0$ such that*

$$\frac{d}{dr}(M_k(r) + \tilde{C}r) \geq \frac{2W_k(r)}{r}. \tag{4.4.3}$$

Proof. Let $w = U - p_k$, so that

$$M_k(r) = \frac{1}{r^{n+a+2k}}\int_{\mathbb{S}_r} w^2\mu. \tag{4.4.4}$$

Then its derivative will be

$$M_k'(r) = -\frac{n+a+2k}{r^{n+a+1+2k}}\int_{\mathbb{S}_r} w^2\mu + \frac{1}{r^{n+a+2k}}\frac{d}{dr}\int_{\mathbb{S}_r} w^2 < A(x)\nabla r, \nu > |y|^a,$$

since $\nabla r = \frac{X}{|X|} = \nu$. Thus, by the divergence theorem

$$\begin{aligned}\frac{d}{dr}\int_{\mathbb{S}_r} w^2 < A(x)\nabla r, \nu > |y|^a &= \frac{d}{dr}\int_{\mathbb{B}_r} \mathrm{div}(w^2 A(x)\nabla r|y|^a) \\ &= \frac{d}{dr}\left[\int_{\mathbb{B}_r} 2w < \nabla w, A(x)\nabla r > |y|^a + \int_{\mathbb{B}_r} w^2 \mathrm{div}(A\nabla r|y|^a)\right] \\ &= \left[2\int_{\mathbb{S}_r} w < A(x)\nabla w, \nabla r > |y|^a + \int_{\mathbb{S}_r} w^2 L_a|X|\right].\end{aligned}$$

At this point we can exploit the following result.

Lemma (Lemma 4.1, [4])
For $r \neq 0$ one has

$$L_a r = \mathrm{div}(|y|^a A(x)\nabla r) = \frac{n+a}{r}|y|^a + O(|y|^a).$$

In particular, $L_a r \in L^1(\mathbb{B}_1)$.

and in this way, since $X = \nabla r \cdot |X| = \nabla r \cdot r$, obtain that

$$\begin{aligned}M_k'(r) &= -\frac{n+a+2k}{r^{n+a+1+2k}}\int_{\mathbb{S}_r} w^2\mu + \frac{2}{r^{n+a+1+2k}}\int_{\mathbb{S}_r} w < A(x)\nabla w, X > |y|^a + \\ &\quad + \frac{1}{r^{n+a+2k}}\int_{\mathbb{S}_r} w^2\left(\frac{n+a}{r} + O(|y|^a)\right) \\ &= -\frac{n+a+2k}{r^{n+a+1+2k}}\int_{\mathbb{S}_r} w^2\mu + \frac{2}{r^{n+a+1+2k}}\int_{\mathbb{S}_r} w < A(x)\nabla w, X > |y|^a \\ &\quad + \frac{n+a}{r^{n+a+1+2k}}\int_{\mathbb{S}_r} w^2|y|^a + \frac{1}{r^{n+a+2k}}\int_{\mathbb{S}_r} w^2 O(|y|^a) \\ &= -\frac{2k}{r^{n+a+1+2k}}\int_{\mathbb{S}_r} w^2\mu + \frac{2}{r^{n+a+1+2k}}\int_{\mathbb{S}_r} w < A(x)\nabla w, X > |y|^a + \\ &\quad + \frac{n+a}{r^{n+a+1+2k}}\int_{\mathbb{S}_r} w^2(1-\tilde{\mu})|y|^a + O(1),\end{aligned}$$

where in the last step we used (4.2.2) and the fact that polynomial p_k is homogeneous of degree k, so $w = U - p_k = O(r^k)$ and therefore

$$\int_{\mathbb{S}_r} w^2|y|^a = O(r^{n+k+a}).$$

Now, from (2) of the Lemma 2.1.1 it is clear that

$$\int_{\mathbb{S}_r} w^2(1-\tilde{\mu})|y|^a = O(r^{n+2k+1+a}).$$

and therefore

$$M_k'(r) = -\frac{2k}{r^{n+a+1+2k}} \int_{\mathbb{S}_r} w^2\mu + \frac{2}{r^{n+a+1+2k}} \int_{\mathbb{S}_r} w < A(x)\nabla w, X > |y|^a + O(1). \qquad (4.4.5)$$

From the definition of the Weiss-type formula (4.3.1), we know

$$W_k(r) = \frac{1}{r^{n+a-1+2k}} \left[\int_{\mathbb{B}_r} < A(x)\nabla U, \nabla U > |y|^a - \frac{k}{r} \int_{\mathbb{S}_r} U^2\mu \right].$$

Remember that $\Psi_{p_k}(r) = 0$

$$\begin{aligned} W_k(r) = W_k(r) - \Psi_{p_k}(r) =& \frac{1}{r^{n+a-1+2k}} \int_{\mathbb{B}_r} < A(x)\nabla U, \nabla U > |y|^a - \frac{k}{r^{n+a+2k}} \int_{\mathbb{S}_r} U^2\mu - \\ &- \frac{1}{r^{n+a+2k-1}} \int_{\mathbb{B}_r} |\nabla p_k|^2 |y|^a + \frac{k}{r^{n+a+2k}} \int_{\mathbb{S}_r} p_k^2 |y|^a \\ \overset{A(0)=\mathbb{I}}{=}& \frac{1}{r^{n+a-1+2k}} \int_{\mathbb{B}_r} (< A(x)\nabla U, \nabla U > - < A(x)\nabla p_k, \nabla p_k >)|y|^a + \\ &+ \frac{1}{r^{n+a-1+2k}} \int_{\mathbb{B}_r} < (A(x) - A(0))\nabla p_k, \nabla p_k > |y|^a - \\ &- \frac{k}{r^{n+a+2k}} \int_{\mathbb{S}_r} (U^2 - p_k^2)\tilde{\mu}|y|^a + \frac{k}{r^{n+a+2k}} \int_{\mathbb{S}_r} p_k^2(1-\tilde{\mu})|y|^a. \end{aligned}$$

Using the assumption on $A(x)$ (2.0.5) and the fact that p_k is homogeneous of degree k

$$\int_{\mathbb{B}_r} < (A(x) - A(0)) > \nabla p_k, \nabla p_k > |y|^a = O(r^{n+2k+a}).$$

By (2) of the Lemma 2.1.1 and the homogeneity of p_k again, we obtain

$$\int_{\mathbb{S}_r} p_k^2(1-\tilde{\mu})|y|^a = O(r^{n+2k+1+a}).$$

Thus

$$\begin{aligned} W_k(r) = \frac{1}{r^{n+a-1+2k}} \int_{\mathbb{B}_r} (< A(x)\nabla U, \nabla U > - < A(x)\nabla p_k, \nabla p_k >)|y|^a - \\ - \frac{k}{r^{n+a+2k}} \int_{\mathbb{S}_r} (U^2 - p_k^2)\tilde{\mu}|y|^a + O(r). \end{aligned}$$

Having

$$
\begin{aligned}
&< A(x)\nabla w, \nabla w > +2 < A(x)\nabla w, \nabla p_k >=< A(x)(\nabla U - \nabla p_k), (\nabla U - \nabla p_k) > + \\
&+ 2 < A(x)(\nabla U - \nabla p_k), \nabla p_k >=< A(x)\nabla U, \nabla U > + < A(x)\nabla p_k, \nabla p_k > - \\
&- 2 < A(x)\nabla U, \nabla p_k > +2 < A(x)\nabla U, \nabla p_k > -2 < A(x)\nabla p_k, \nabla p_k > \\
&=< A(x)\nabla U, \nabla U > - < A(x)\nabla p_k, \nabla p_k >
\end{aligned}
$$

and

$$w^2 + 2p_k w = (U - p_k)^2 + 2p_k(U - p_k) = U^2 - 2Up_k + p_k^2 + 2Up_k - 2p_k^2 = U^2 - p_k^2,$$

we get

$$
\begin{aligned}
W_k(r) = \frac{1}{r^{n+a-1+2k}} \int_{\mathbb{B}_r} (< A(x)\nabla w, \nabla w > +2 < A(x)\nabla w, \nabla p_k >)|y|^a - \\
- \frac{k}{r^{n+a+2k}} \int_{\mathbb{S}_r} (w^2 + 2p_k w)\mu + O(r).
\end{aligned}
$$

Using the divergence theorem

$$
\begin{aligned}
\int_{\mathbb{B}_r} < A(x)\nabla w, \nabla p_k > |y|^a &= \int_{\mathbb{S}_r} w < A(x)\nabla p_k, \nu > |y|^a - \int_{\mathbb{B}_r} w\mathrm{div}(|y|^a A(x)\nabla p_k) \\
&= \int_{\mathbb{S}_r} w < A(x)\nabla p_k, \nu > |y|^a - \int_{\mathbb{B}_r} wL_a p_k,
\end{aligned}
\tag{4.4.6}
$$

then let $B(x) = [b_{ij}(x)] = A(x) - A(0)$ (remember by assumption $A(0) = \mathbb{I}_{n+1}$)

$L_a p_k = \mathrm{div}(|y|^a A(x)\nabla p_k) = \mathrm{div}(|y|^a \nabla p_k) + \mathrm{div}(|y|^a B(x)\nabla p_k) = D_i(b_{ij})D_j p_k |y|^a + b_{ij} D_i(|y|^a D_j p_k)$.

Using (2.0.5) and the fact that p_k is homogeneous of degree k, for almost every $x \in \mathbb{B}_r$ we obtain that $L_a p_k = O(r^{k+a-1})$. Then

$$\int_{\mathbb{B}_r} wL_a p_k = O(r^{n+2k+a}). \tag{4.4.7}$$

Use (4.4.6) and (4.4.7)

$$
\begin{aligned}
W_k(r) =& \frac{1}{r^{n+a-1+2k}} \int_{\mathbb{B}_r} < A(x)\nabla w, \nabla w > |y|^a + \frac{2}{r^{n+a+2k}} \int_{\mathbb{S}_r} w < A(x)\nabla p_k, X > |y|^a - \\
&- \frac{k}{r^{n+a+2k}} \int_{\mathbb{S}_r} (w^2 + 2p_k w)\mu + O(r).
\end{aligned}
$$

Let us analyse the first integral at the left hand side

$$
\begin{aligned}
\int_{\mathbb{B}_r} < A(x)\nabla w, \nabla w > |y|^a &\overset{\text{div.thm}}{=} -\int_{\mathbb{B}_r} w \operatorname{div}(A(x)\nabla w|y|^a) + \int_{\mathbb{S}_r} w < A(x)\nabla w, \nu > |y|^a \\
&\quad + \int_{B_r} w(< A(x)\nabla w, \nu_+ > + < A(x)\nabla w, \nu_- >)|y|^a \\
&\overset{(4.0.1)}{=} \int_{\mathbb{B}_r} w\ L_a p_k + \frac{1}{r}\int_{\mathbb{S}_r} w < A(x)\nabla w, X > |y|^a \\
&\quad + \int_{B_r} w(< A(x)\nabla w, \nu_+ > + < A(x)\nabla w, \nu_- >)|y|^a \\
&\overset{(4.4.7)}{=} \int_{B_r} w(< A(x)\nabla w, \nu_+ > + < A(x)\nabla w, \nu_- >)|y|^a \\
&\quad + \frac{1}{r}\int_{\mathbb{S}_r} w < A(x)\nabla w, X > |y|^a + O(r^{n+a+2k}).
\end{aligned}
$$

Now we want to prove that

$$
\int_{B_r} w(< A(x)\nabla w, \nu_+ > + < A(x)\nabla w, \nu_- >)|y|^a \leq 0. \tag{4.4.8}
$$

First of all, notice that since we have, by (4.0.5), that

$$
\begin{aligned}
&\int_{B_r} w(< A(x)\nabla w, \nu_+ > + < A(x)\nabla w, \nu_- >)|y|^a = \\
&= -\int_{B_r} U(< A(x)\nabla p_k, \nu_+ > + < A(x)\nabla p_k, \nu_- >)|y|^a \\
&\qquad + \int_{B_r} p_k(< A(x)\nabla p_k, \nu_+ > + < A(x)\nabla p_k, \nu_- >)|y|^a \\
&\qquad - \int_{B_r} p_k(< A(x)\nabla U, \nu_+ > + < A(x)\nabla U, \nu_- >)|y|^a \\
&= -\int_{B_r} p_k(< A(x)\nabla U, \nu_+ > + < A(x)\nabla U, \nu_- >)|y|^a
\end{aligned}
$$

Where the last step is due

$$
(< A(x)\nabla p_k, \nu_+ > + < A(x)\nabla p_k, \nu_- >) = \frac{1}{r}(- < A(x)\nabla p_k, X > + < A(x)\nabla p_k, X >) = 0,
$$

since $\nu_+ = -X/|X|$, $\nu_- = X/|X|$.

Now (4.4.8) follows from $p_k \geq 0$ and (4.0.4). In this way we obtain

$$\frac{1}{r^{n+a-1+2k}} \int_{\mathbb{B}_r} < A(x)\nabla w, \nabla w > |y|^a \leq \frac{1}{r^{n+a+2k}} \int_{\mathbb{S}_r} w < A(x)\nabla w, X > |y|^a + O(r)$$

and finally

$$\begin{aligned}
W_k(r) \leq & \frac{1}{r^{n+a+2k}} \int_{\mathbb{S}_r} w < A(x)\nabla w, X > |y|^a + \frac{2}{r^{n+a+2k}} \int_{\mathbb{S}_r} w < A(x)\nabla p_k, X > |y|^a - \\
& - \frac{k}{r^{n+a+2k}} \int_{\mathbb{S}_r} (w^2 + 2p_k w)\mu + O(r) \\
= & \frac{1}{r^{n+a+2k}} \left[\int_{\mathbb{S}_r} w < A(x)\nabla w, X > |y|^a - k \int_{\mathbb{S}_r} w^2 \mu \right] + O(r) + \\
& + \frac{2}{r^{n+a+2k}} \int_{\mathbb{S}_r} w < A(x)\nabla p_k, X > |y|^a - \frac{2k}{r^{n+a+2k}} \int_{\mathbb{S}_r} p_k w \mu \\
\overset{(4.4.5)}{=} & \frac{rM_k'(r)}{2} + \frac{2}{r^{n+a+2k}} \int_{\mathbb{S}_r} w < A(x)\nabla p_k, X > |y|^a - \frac{2k}{r^{n+a+2k}} \int_{\mathbb{S}_r} p_k w \mu + O(r).
\end{aligned}$$

Remember that, by assumption, $A(0) = \mathbb{I}_{n+1}$, so

$$\begin{aligned}
& \frac{2}{r^{n+a+2k}} \int_{\mathbb{S}_r} w < A(x)\nabla p_k, X > |y|^a - \frac{2k}{r^{n+a+2k}} \int_{\mathbb{S}_r} p_k w \mu = \\
& = \frac{2}{r^{n+a+2k}} \int_{\mathbb{S}_r} w < (A(x) - A(0)) > \nabla p_k, X > |y|^a + \\
& \quad + \frac{2}{r^{n+a+2k}} \int_{\mathbb{S}_r} w(< \nabla p_k, X > - k p_k)|y|^a + \\
& \quad + \frac{2k}{r^{n+a+2k}} \int_{\mathbb{S}_r} p_k w (1 - \tilde{\mu})|y|^a = O(r),
\end{aligned}$$

where we again we used the homogeneity of p_k, (4.2.2) and (2) of the Lemma 2.1.1.

In this way, we obtain

$$\begin{aligned}
W_k(r) &\leq \frac{rM_k'(r)}{2} + O(r), \\
M_k'(r) &\geq \frac{2W_k(r)}{r} - \tilde{C}, \\
\frac{d}{dr}\left(M_k'(r) + \tilde{C}r \right) &\geq \frac{2W_k(r)}{r}.
\end{aligned}$$

□

Corollary 4.4.2.1. *Under the assumptions of Theorem* 4.4.2*, we have*

$$\frac{d}{dr}\left(M_k(r) + C^{**}r\right) \geq 0, \tag{4.4.9}$$

where C^{**} is a universal constant.
In particular, the limit $M_k(0^+) = \lim_{r\to 0^+} M_k(r)$ exists.

Proof. From Lemma 4.3.2 we know that $W_k(0^+)$ exists and that it equals to 0 since by assumption $0 \in \Gamma_k(U)$ (in other words, that $\tilde{N}(0^+) = k$). Then by (4.3.7), there exist a constant $C > 0$ such that $W_k(r) + Cr$ is monotone nondecreasing and consequently

$$W_k(r) + Cr \geq W_k(0^+) = 0.$$

This implies that

$$W_k(r) \geq -Cr$$

and therefore by (4.4.3)

$$\frac{d}{dr}\left(M_k(r) + \tilde{C}r\right) \geq -2C.$$

The conclusion now follows

$$\frac{d}{dr}\left(M_k(r) + C^{**}r\right) \geq 0.$$

□

4.5 The blowup analysis

We will follow the procedure of the blow up analysis in the chapter 6 of [21] (also Subsection 2.2.2 in the thesis) and since we almost completely repeat it we do not divide our reasoning in theorems and lemmas, but we will just make some remarks about how calculations in [21] changes if we apply it to our degenerate problem (4.0.1).

We begin by reminding following quantity

$$d_r = \left(\frac{H(r)}{r^{n+a}}\right)^{\frac{1}{2}} \tag{4.5.1}$$

where $H(r)$ as in (2.1.3). We know that in [4] the authors used $d_r = (M(r))^{\frac{1}{2}}$, but for our purposes it is enough to define d_r analogously to [21], [19] (since we are in the zero-obstacle case).

Remark. Notice that from (4.2.1) we have

$$d_r = \left(\frac{H(r)}{r^{n+a}}\right)^{\frac{1}{2}} = O(r^k). \tag{4.5.2}$$

Now we define Almgren and homogeneous scalings

Definition 4.5.1. We define the *Almgren scalings* of U as follows

$$\tilde{U}_r(X) = \frac{U(rX)}{d_r}, \quad x \in \mathbb{B}_{1/r}. \tag{4.5.3}$$

Definition 4.5.2. The *homogeneous scaling* of U are defined in the following way

$$U_r(X) = \frac{U(rX)}{r^k}, \quad x \in \mathbb{B}_{1/r}. \tag{4.5.4}$$

Next we make the following observation

$$H(r) = \int_{\mathbb{S}_r} U^2 \tilde{\mu} |y|^a = r^n \int_{\mathbb{S}_1} U^2(rX)\tilde{\mu}(rX)|ry|^a = r^{n+a} d_r^2 \int_{\mathbb{S}_1} \tilde{U}_r^2 \tilde{\mu}_r |y|^a = r^{n+a} d_r^2 \int_{\mathbb{S}_1} \tilde{U}_r^2 \mu_r,$$

where $\tilde{\mu}_r(X) = \tilde{\mu}(rX)$ and $\mu_r(X) = \mu(rX)$. Then by (4.5.1)

$$\int_{\mathbb{S}_1} \tilde{U}_r^2 \mu_r = \frac{H(r)}{r^{n+a} d_r^2} = \frac{H(r) r^{n+a}}{r^{n+a} H(r)} = 1. \tag{4.5.5}$$

As in Lemma 2.2.12, we can show that both functions $\tilde{U}_r$ and U_r defined in (4.5.3),(4.5.4) are even in y and solve the Signorini conditions (4.0.2)-(4.0.6) in $\mathbb{B}_{1/r}$ for the operator $L_{a,r} = \operatorname{div}(|y|^a A_r \nabla)$, where $A_r(x) = A(rx)$.
It is straightforward that $\tilde{U}_r$ and U_r satisfy (4.0.2)-(4.0.5) and that they are even, we will show the logic on proving that $\tilde{U}_r$ and U_r satisfy (4.0.3) in $\mathbb{B}_{1/r}$, namely

$$U(x,0) \geq 0 \quad \text{for } (x,0) \in B_1,$$

then

$$U(rx,0) \geq 0 \quad \text{for } (rx,0) \in B_1. \Rightarrow (x,0) \in B_{1/r}$$

Since d_r and r^k do not depend on X and both are bigger than 0, we have

$$\tilde{U}_r(x,0) = \frac{U(rX)}{d_r} \geq 0 \quad \text{and} \quad U_r(x,0) = \frac{U(rX)}{r^k} \geq 0 \quad \text{for } (x,0) \in B_{1/r}.$$

Given $\eta \in C_0^\infty(\mathbb{B}_{1/r})$, (4.0.6) is follows by following change of variables. First, remember that $\nabla(\eta(X/r)) = \frac{\nabla\eta(X/r)}{r}$ then

$$\frac{1}{d_r} \int_{\mathbb{B}_{1\varepsilon}} < A(x)\nabla U(x), \frac{\nabla\eta(X/r)}{r} > |y|^a = r^{n-1} \int_{\mathbb{B}_{\varepsilon\frac{1}{r}}} < A(rx)\frac{r\nabla U(rX)}{d_r}, \frac{\nabla\eta(X) >}{r} |ry|^a$$

$$= r^{n+a-2} \int_{\mathbb{B}_{\varepsilon\frac{1}{r}}} < A_r(x)\nabla\tilde{U}_r(X), \nabla\eta(X) > |y|^a \overset{\text{div.thm}}{=} -r^{n+a-2} \int_{\mathbb{B}_{\varepsilon\frac{1}{r}}} \operatorname{div}(|y|^a A_r \nabla\tilde{U}_r)\eta +$$

$$
\begin{aligned}
+r^{n+a-2}\int_{\partial\mathbb{B}_{\varepsilon\frac{1}{r}}} < A_r\nabla\tilde{U}_r,\nu > \eta|y|^a = r^{n+a-2}\int_{\mathbb{S}_{\varepsilon\frac{1}{r}}} < A_r\nabla\tilde{U}_r,\nu > \eta|y|^a+ \\
+r^{n+a-2}\int_{L_{\varepsilon\frac{1}{r}}} (< A_r\nabla\tilde{U}_r,\nu_+ > + < A_r\nabla\tilde{U}_r,\nu_- >)\eta|y|^a \\
= r^{n+a-2}\int_{L_{\varepsilon\frac{1}{r}}} (< A_r\nabla\tilde{U}_r,\nu_+ > + < A_r\nabla\tilde{U}_r,\nu_- >)\eta|y|^a,
\end{aligned}
$$

where in the second-to-last step we used the fact that Almgren rescaled function $\tilde{U}_r$ solves the problem $L_{a,r}\tilde{U}=0$, and in the last step the fact that $\eta \in C_0^\infty(\mathbb{B}_{1/r})$. Also we observe that

$$
\begin{aligned}
&\frac{1}{d_r}\int_{L_{1\varepsilon}} (< A(x)\nabla U(X),\nu_+ > + < A(x)\nabla U(X),\nu_- >)\eta(X/r)|y|^a \\
&= r^{n-2}\int_{L_{\varepsilon\frac{1}{r}}} (< A(rx)\frac{r\nabla U(rX)}{d_r},\nu_+ > + < A(rx)\frac{r\nabla U(rX)}{d_r},\nu_- >)\eta(X)|ry|^a \\
&= r^{n+a-2}\int_{L_{\varepsilon\frac{1}{r}}} (< A_r(x)\nabla\tilde{U}_r,\nu_+ > + < A_r(x)\nabla\tilde{U}_r,\nu_- >)\eta(X)|y|^a
\end{aligned}
$$

and therefore by (4.0.6) for U and letting $\varepsilon \to 0^+$

$$
\int_{\mathbb{B}_{1/r}} < A_r\nabla\tilde{U}_r,\nabla\eta > |y|^a = -2\int_{B_{1/r}} \eta\partial_y^a\tilde{U}_r(x,0), \tag{4.5.6}
$$

since U is extended to the whole ball evenly in y.

Same logic is valid for the homogeneous rescaled functions U_r.

As in Lemma 2.2.13 we make following calculations for $r \in (0;1)$

$$
\begin{aligned}
N_{L_a,r}(\tilde{U}_r,1) &= \frac{\int_{\mathbb{B}_1} < A_r\nabla\tilde{U}_r,\nabla\tilde{U}_r > |y|^a}{\int_{\mathbb{S}_1}\tilde{U}_r^2\mu_r} \\
&= \frac{r^2d_r^2}{d_r^2}\frac{\int_{\mathbb{B}_1} < A(rx)\nabla U(rX),\nabla U(rX) > |y|^a}{\int_{\mathbb{S}_1} U^2(rX)\tilde{\mu}(rX)|y|^a} \\
&= \frac{r^2\cdot r^{n+a}}{r^{n+a+1}}\frac{\int_{\mathbb{B}_r} < A(x)\nabla U(X),\nabla U(X) > |y|^a}{\int_{\mathbb{S}_1} U^2(X)\mu(X)} \\
&= \frac{rD(r)}{H(r)} = N_L(U,r).
\end{aligned} \tag{4.5.7}
$$

As in Lemma 2.2.14 we can prove that the sequence $\{\tilde{U}_{r_j}\}_{j\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}(\mathbb{B}_1,|y|^a dX)$

By (4.5.5),(4.5.7) and ellipticity of A, we have for $r = r_j$

$$\begin{aligned}\int_{\mathbb{B}_1} |\nabla \tilde{U}_r|^2 |y|^a &\le \lambda^{-1} \int_{\mathbb{B}_1} < A_r \nabla \tilde{U}_r, \nabla \tilde{U}_r > |y|^a = \lambda^{-1} D_{L_r}(\tilde{U}_r, 1)\\ &\overset{(4.5.4),(4.5.5)}{=} \lambda^{-1} N_{L_{a,r}}(\tilde{U}_r, 1) = \lambda^{-1} N_L(U, r)\\ &= \lambda^{-1} e^{-Cr} \tilde{N}_L(U, r) \le \lambda^{-1} \tilde{N}_L(U, r) \le \lambda^{-1} \tilde{N}_L(U, 1),\end{aligned} \tag{4.5.8}$$

where in the last inequality we use monotonicity of $\tilde{N}_L(U, r)$ and that $e^{-Cr} < 1$ for $r \in (0, 1)$

$$\int_{\mathbb{S}_1} \tilde{U}_r^2 |y|^a \le \lambda^{-1} \int_{\mathbb{S}_1} \tilde{U}_r^2 \tilde{\mu}_r |y|^a \overset{(4.5.5)}{=} \lambda^{-1}. \tag{4.5.9}$$

At this point we use the trace inequality for functions in the Sobolev space $W^{1,2}(\mathbb{B}_1, |y|^a dX)$

Lemma 4.5.3. (Lemma 14.4, [16])
For $r > 0$ let $U \in W^{1,2}(\mathbb{B}_1, |y|^a dX)$. There exists a constant $C = C(n, a) > 0$ such that

$$\int_{\mathbb{S}_r} U^2 |y|^a \le C \left(\frac{1}{r} \int_{\mathbb{B}_r} U^2 |y|^a + r \int_{\mathbb{B}_r} |\nabla U|^2 |y|^a \right) \tag{4.5.10}$$

and

$$\frac{1}{r} \int_{\mathbb{B}_r} U^2 |y|^a \le C \left(\int_{\mathbb{S}_r} U^2 |y|^a + r \int_{\mathbb{B}_r} |\nabla U|^2 |y|^a \right). \tag{4.5.11}$$

Combining (4.5.8),(4.5.9) with (4.5.11) we can conclude that

$$\left\| \tilde{U}_{r_j} \right\|_{W^{1,2}(\mathbb{B}_1, |y|^a dX)} < \infty.$$

But since we are going to use the estimate from Theorem 3.1.3, we need the uniform boundedness of $\{\tilde{U}_{r_j}\}_{j \in \mathbb{N}}$ just in $L^2(\mathbb{B}_1, |y|^a dX)$, which is also easily follows from (4.5.11), and we have

$$\left\| \tilde{U}_{r_j} \right\|_{L^2(\mathbb{B}_1, |y|^a dX)} < \infty.$$

We follow Lemma 2.2.15. We start by observing that, as was proved in the Theorem 3.1.3, $U \in C^{1,\alpha}_{loc}(\mathbb{B}_1)$ for any $\alpha \in (0, 1)$ with

$$\|U\|_{C^{1,\alpha}(\mathbb{B}_{\frac{1}{2}})} \le C \|U\|_{L^2(\mathbb{B}_1, |y|^a dX)},$$

since $f \equiv 0$ in our situation.
Given $r_j \searrow 0$ consider the sequence $\{\tilde{U}_{r_j}\}_{j \in \mathbb{N}}$. By previous calculations, such a sequence is uniformly bounded in $L^2(\mathbb{B}_1, |y|^a dX)$. For any $\alpha \in (0, 1)$, by a standard diagonal process we obtain convergence in $C^{1,\alpha}(\mathbb{R}^{n+1}_{\pm} \cup \mathbb{R}^n)$ to a function $\tilde{U}_0$ of a subsequence of the functions $\tilde{U}_{r_j}$. Passing to the limit in (4.5.6), we conclude that $\tilde{U}_0$ is a global solution to the Signorini

problem (4.0.2)-(4.0.6) with $A \equiv \mathbb{I}_{n+1}$. And $\tilde{U}_0$ is even in y.

Definition 4.5.4. We call the function $\tilde{U}_0$ an *Almgren blowup* of the solution U at zero.

Now to prove the result similar to Lemma 2.2.17 it is enough to follow the lemma. As in Lemma 2.2.22 we first prove that $\{U_{r_j}\}_{j\in\mathbb{N}}$ is uniformly bounded in $L^2(\mathbb{B}_1, |y|^a dX)$.
By (4.2.4),

$$\int_{\mathbb{B}_1} U_{r_j}^2 |y|^a = r_j^{-2k} \int_{\mathbb{B}_1} U(r_j X)^2 |y|^a = r_j^{-(n+a+1+2k)} \int_{\mathbb{B}_{r_j}} U^2 |y|^a \overset{(4.2.4)}{\leq} C.$$

Moreover as proved in the Theorem 3.1.3, $U \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{B}_1)$ with

$$\|U\|_{C^{1,\alpha}(\mathbb{B}_{\frac{1}{2}})} \leq C\|U\|_{L^2(\mathbb{B}_1,|y|^a dX)}$$

By a standart diagonal process, for any $\alpha \in (0,1)$, we obtain convergence in $C^{1,\alpha}(\mathbb{R}^{n+1}_{\pm} \cup \mathbb{R}^n)$ to a function U_0 of a subsequence of the functions U_{r_j}. Passing to the limit in (4.5.6), which also holds for U_r, we conclude that U_0 is a global solution to the Signorini problem (4.0.2)-(4.0.6) with $A \equiv \mathbb{I}_{n+1}$. And U_0 is even in y.

Definition 4.5.5. We call the function U_0 a *homogeneous blowup* of the solution at zero.

Remark. We note that , unlike what happens for the Almgren blowups, it is not guaranteed that a homogeneous blowup will be nonzero. Further this fact will be proved using the nondegeneracy of a solution (Section 4.7).

To prove that U_0 is homogeneous of degree k is enough to follow exactly Lemma 6.12 of [21] but for fractional dimension $N = n + a + 1$ and using the generalised Weiss type formula (4.3.3). For completeness and since we did not present the proofs of this type of results for the Laplace Signorini problem and the Signorini problem for variable coefficient operator in Chapter 2, we present it now for our generalised problem (4.0.1).

Proposition 4.5.6. *Let U_0 be a homogeneous blowup. Then U_0 is a homogeneous function of degree $k = N(0^+)$.*

Proof. Let $r \in (0, R)$. For a fixed r_j we integrate (4.3.3) in Theorem 4.3.3 over the interval $[r_j r; r_j R]$, obtaining

$$\begin{aligned}
&W_k(r_j R, U) - W_k(r_j r, U) + C r_j (R - r) \\
&\quad \geq 2 \int_{r_j r}^{r_j R} \frac{1}{t^{n+a+1+2k}} \int_{\mathbb{S}_t} \left(\frac{< A(x)\nabla U, X >}{\sqrt{\tilde{\mu}}} - k\sqrt{\tilde{\mu}} U \right)^2 |y|^a d\sigma dt \\
&\quad = 2 r_j \int_r^R \frac{1}{(r_j s)^{n+a+1+2k}} \int_{\mathbb{S}_{r_j s}} \left(\frac{< A(x)\nabla U, X >}{\sqrt{\tilde{\mu}}} - k\sqrt{\tilde{\mu}} U \right)^2 |y|^a d\sigma ds
\end{aligned}$$

$$= 2r_j^{n+1} \int_r^R \frac{1}{(r_j s)^{n+a+1+2k}} \int_{\mathbb{S}_s} \left(\frac{< A(r_j x)\nabla U(r_j X), r_j X >}{\sqrt{\tilde{\mu}(r_j X)}} - k\sqrt{\tilde{\mu}(r_j X)} U(r_j X) \right)^2 |r_j y|^a d\sigma ds$$

$$= \frac{2}{r_j^{2k}} \int_r^R \frac{1}{s^{n+a+1+2k}} \int_{\mathbb{S}_s} \left(\frac{< A_{r_j}(x)\nabla U_{r_j}(X), X >}{\sqrt{\tilde{\mu}_{r_j}(X)}} r_j^k - k\sqrt{\tilde{\mu}_{r_j}(X)} U_{r_j}(X) r_j^k \right)^2 |y|^a d\sigma ds \quad .$$

$$= 2\int_{\mathbb{B}_R \setminus \mathbb{B}_r} \frac{1}{|x|^{n+2k}} \left(\frac{< A_{r_j}(x)\nabla U_{r_j}(X), X >}{\sqrt{\tilde{\mu}_{r_j}(X)}} - k\sqrt{\tilde{\mu}_{r_j}(X)} U_{r_j}(X) \right)^2 |y|^a$$

By Theorem 4.3.3, we know that $W(0^+)$ exists and, by Lemma 4.3.2, that $W(0^+) = 0$ (since we assume that $N(0^+) = 0$). So we take the limit as $r_j \to 0$ and remember that $A(0) = \mathbb{I}_{n+1}$ and $\mu_{r_j}(X) \to 1$, since $\mu(0) = 1$. In this way, from the fact that U_{r_j} converges to U_0 in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^{n+1}_{\pm} \cup \mathbb{R}^n)$, letting $j \to \infty$ in the above inequality, we infer that the latter converges to

$$0 \geq 2\int_{\mathbb{B}_R \setminus \mathbb{B}_r} \frac{1}{|x|^{n+2k}} (< U_0, X > -kU_0)^2.$$

By the arbitrariness of $0 < r < R < \infty$, we conclude that U_0 is homogeneous of degree k in $\mathbb{R}^{n+1}$. □

4.6 Characterization of the singular boundary

Theorem 4.6.1. (Characterization of singular points) *Let $U \in \mathfrak{S}$, where we have $0 \in \Gamma_k(U)$ and $N(U, 0^+) = k$. The following statements are equivalent:*

(i) $0 \in \Sigma_k(U)$.

(ii) Any Almgren blowup $\tilde{U}_0$ of U at the origin is a nonzero homogenenous polynomial $p_k \in \mathfrak{P}_k^+(\mathbb{R}^{n+a+1})$.

(iii) $k = 2m$, for some $m \in \mathbb{N}$.

Proof. The proof follows the reasonings of the Theorem 2.5.2, Theorem 2.5.12 and Proposition 4.4. of [22].

By (4.5.6), for $0 < r < 1$ and any $\eta \in C_0^\infty(\mathbb{B}_1)$, we have

$$\int_{\mathbb{B}_1} < A_r \nabla \tilde{U}_r, \nabla \eta > |y|^a = -2\int_{B_1 \cap \Lambda(\tilde{U}_r)} \eta \partial_y^a \tilde{U}_r(x, 0) \tag{4.6.1}$$

We know from the Section 4.5 that $|y|^a \partial_y \tilde{U}_r$ is uniformly bounded in $\mathbb{B}_1$. This fact and (2.5.1) allow to conclude that

$$\lim_{r \to 0^+} \int_{B_1 \cap \Lambda(\tilde{U}_r)} \eta \partial_y^a \tilde{U}_r = 0$$

which means that $L_{a,r}\tilde{U}_r$ converges weakly to 0. On the other hand,

$$\int_{\mathbb{B}_1} < A_r \nabla \tilde{U}_r, \nabla \eta > |y|^a \to \int_{\mathbb{B}_1} < \nabla \tilde{U}_0, \nabla \eta > |y|^a.$$

By Theorem 12.17 (Nonlocal Caccioppoli-Cimmino-Weyl lemma) in [16], we conclude that any blowup $\tilde{U}_0$ satisfies $\operatorname{div}(|y|^a \tilde{U}_0) = 0$ in $\mathbb{B}_1$. Since $\tilde{U}_0$ is homogeneous, then $\operatorname{div}(|y|^a \tilde{U}_0) = 0$ in $\mathbb{R}^{n+1}$, because it means that

$$\tilde{U}_0(X) = \tilde{U}_0 \left(\frac{X}{\|X\|} \|X\| \right) = \|X\|^k \tilde{U}_0 \left(\frac{X}{\|X\|} \right)$$

and since $X/\|X\| \in \mathbb{S}_1$, it is enough to describe $\tilde{U}_0 \left(\frac{X}{\|X\|} \right)$ on the unit sphere $\mathbb{S}_1$ to know its behaviour in $\mathbb{R}^{n+1}$. Therefore, by Lemma 5.3 in [7], $\tilde{U}_0$ is a polynomial of degree k satisfying $\operatorname{div}(|y|^a \tilde{U}_0) = 0$. The properties of $\tilde{U}_r$ imply the missing conditions (see Section 4.5).

(ii) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (i) The proof repeats the reasoning in Theorem 2.5.2 and Proposition 4.4. of [22] $\square$

Lemma 4.6.2. *Let $U \in \mathfrak{S}$ with $0 \in \Sigma_k(U)$. Then, any homogeneous blowup of U at the origin is a homogeneous polynomial $p_k \in \mathfrak{P}_k^+(\mathbb{R}^{n+1} \cap \{0\})$.*

The proof follows Lemma 2.5.13.

4.7 Nondegeneracy

In this section, we use the Almgren and Monneau monotonicity formulas to prove a nondegeneracy property, Lemma 4.7.2 below. This allows us to show the uniqueness of homogeneous blowups, that such a blowup cannot vanish identically, and moreover, the continuous dependence of the blowups at the singular points. Now we want to follow the line of chapter 10 of [21] and start with the lower bound of $H(r)$ which is used to prove the nondegeneracy property.

Lemma 4.7.1. *Let $U \in \mathfrak{S}$ with $0 \in \Gamma_k(U)$. Then for every $\varepsilon > 0$ there exist $r_\varepsilon \in (0,1)$ and a universal constant $C_\varepsilon > 0$ (depending also on U) such that for every $0 < r < r_\varepsilon$ one has*

$$H(r) \geq C_\varepsilon r^{n+a+2k+\varepsilon} \tag{4.7.1}$$

Proof. Let us start from the formula for the first variation of height (4.1.4) that we obtained before. With use of it and the fact that for the case of the zero-obstacle we have $I(r) = D(r)$ we find

$$\frac{d}{dr} \log H(r) = \frac{H'(r)}{H(r)} = 2\frac{D(r)}{H(r)} + \frac{n+a}{r} + O(1).$$

Remember the definition of the frequency (4.1.1) and that the limit $N(0^+) = k$ exists (Theorem 4.1.2)

$$r\frac{d}{dr}\log H(r) = n + a + 2k + 2(N(r) - k) + O(r). \tag{4.7.2}$$

We see that for every $\varepsilon > 0$, there exists $r_\varepsilon > 0$ small enough such that

$$\frac{d}{dr}\log H(r) \le \frac{n + a + 2k + \varepsilon}{r} = (n + a + 2k + \varepsilon)\frac{d}{dr}\log r.$$

After integrating from r to r_ε, we have

$$\log H(r_\varepsilon) - \log H(r) = \log\frac{H(r_\varepsilon)}{H(r)} \le (n + a + 2k + \varepsilon)(\log r_\varepsilon - \log r) = (n + a + 2k + \varepsilon)\log\frac{r_\varepsilon}{r}.$$

And since the logarithm is a monotone function

$$\frac{H(r_\varepsilon)}{H(r)} \le \left(\frac{r_\varepsilon}{r}\right)^{n+a+2k+\varepsilon},$$

that gives the result for $C_\varepsilon = H(r_\varepsilon)/r^{n+a+2k+\varepsilon}$. □

Lemma 4.7.2. (Nondegeneracy)
Let $U \in \mathfrak{S}$, with $0 \in \Sigma_k(U)$, and suppose that the normalization hypothesis $A(0) = \mathbb{I}_{n+1}$ is in place, and that furthermore (2.0.7) *is in force. Then there exists universal $c > 0$ and $r^* \in (0, 1)$ such that for $0 < r < r^*$, one has*

$$\sup_{\mathbb{S}_r} |U(X)| \ge c r^k. \tag{4.7.3}$$

Proof. To prove the lemma we follow Lemma 10.2 in [21]. We argue by contradiction and suppose that (4.7.3) does not hold. Then, there exists a sequence $r_j \to 0$ such that

$$\frac{\sup_{\mathbb{S}_r}|U(X)|}{r_j^k} \to 0.$$

This implies, in particular,

$$\frac{d_{r_j}}{r_j^k} = \left(\frac{1}{r_j^{n+a+2k}}\int_{\mathbb{S}_{r_j}} U^2\tilde{\mu}|y|^a\right)^{\frac{1}{2}} = o(1), \tag{4.7.4}$$

where we used (4.2.2) and (1) of Lemma 2.1.1.

Consider now the sequence of Almgren scalings $\tilde{U}_{r_j}(X) = U(r_jX)/d_{r_j}$, with $j \in \mathbb{N}$. By previous results, we know that there exists $q_k \in \mathfrak{P}_k^+(\mathbb{R}^{n+a+1})$, such that $\tilde{U}_{r_j} \to q_k$ on $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^{n+1}_\pm \cup \mathbb{R}^n)$ as $j \to \infty$. Corollary 4.4.2.1 implies that $M_k(U, q_k, 0^+)$ exists. Thus, we can use the sequence r_j to compute such a limit, that is,

$$M_k(U, q_k, 0^+) = \lim_{j\to\infty} M_k(U, q_k, r_j).$$

By definition of the Monneau type monotonicity formula (4.4.2)

$$M_k(U, q_k, r_j) = \frac{1}{r^{n+a+2k}} \int_{\mathbb{S}_{r_j}} (U - q_k)^2 \tilde{\mu}|y|^a = \frac{1}{r_j^{n+a+2k}\psi(r)} \int_{\mathbb{S}_{r_j}} (U^2 - 2Up_k + q_k^2)\mu.$$

Then (4.7.4) gives

$$\frac{1}{r_j^{n+a+2k}} \int_{\mathbb{S}_{r_j}} U^2\mu \to 0. \tag{4.7.5}$$

On the other hand, the Lebesgue dominated convergence theorem gives

$$\lim_{j\to\infty} \frac{1}{r_j^{n+a+2k}} \int_{\mathbb{S}_{r_j}} q_k^2\tilde{\mu}|y|^a = \lim_{j\to\infty} \int_{\mathbb{S}_1} q_k^2(X)\tilde{\mu}(r_jX)|y|^a = \tilde{\mu}(0) \int_{\mathbb{S}_1} q_k^2|y|^a = \int_{\mathbb{S}_1} q_k^2|y|^a < \infty, \tag{4.7.6}$$

where we used a change of variables, homogeneity of p_k and the fact that $\mu(0) = 1$.
We infer from this that

$$\begin{aligned} 0 \le \frac{1}{r_j^{n+a+2k}} \int_{\mathbb{S}_{r_j}} |Uq_k|\mu &\le \frac{1}{r_j^{n+a+2k}} \left(\int_{\mathbb{S}_{r_j}} U^2\mu\right)^{\frac{1}{2}} \left(\int_{\mathbb{S}_{r_j}} q_k^2\mu\right)^{\frac{1}{2}} \\ &= \left(\frac{1}{r_j^{n+a+2k}} \int_{\mathbb{S}_{r_j}} U^2\mu\right)^{\frac{1}{2}} \left(\int_{\mathbb{S}_1} q_k^2(X)\tilde{\mu}(r_jX)|y|^a\right)^{\frac{1}{2}} \to 0, \end{aligned}$$

since the second factor is bounded by (4.7.6) and the first factor goes to 0 as $j \to \infty$ by (4.7.5).
In this way, we have

$$M_k(U, q_k, 0^+) = \lim_{j\to\infty} M_k(U, q_k, r_j) = \lim_{j\to\infty} \frac{1}{r_j^{n+a+2k}} \int_{\mathbb{S}_{r_j}} q_k^2\mu = \int_{\mathbb{S}_1} q_k^2|y|^a. \tag{4.7.7}$$

By (4.7.7) and homogeneity of p_k, we infer that for every $r \in (0, 1)$

$$M_k(U, q_k, 0^+) = \frac{1}{r^{n+a+2k}} \int_{\mathbb{S}_r} q_k^2|y|^a. \tag{4.7.8}$$

Since according to Corollary 4.4.2.1 the function $r \to M_k(r) + C^{**}r$ is monotone non decreasing, we have

$$\frac{1}{r^{n+a+2k}}\int_{\mathbb{S}_r}(U-q_k)^2\mu + C^{**}r \geq M_k(U,q_k,0^+) = \frac{1}{r^{n+a+2k}}\int_{\mathbb{S}_r} q_k^2|y|^a.$$

Equivalently, we have

$$\frac{1}{r^{n+a+2k}}\int_{\mathbb{S}_r}(U^2-2Uq_k)\mu + \frac{1}{r^{n+a+2k}}\int_{\mathbb{S}_r} q_k^2\tilde{\mu}|y|^a + C^*r \geq \int_{\mathbb{S}_r} q_k^2|y|^a,$$

$$\frac{1}{r^{n+a+2k}}\int_{\mathbb{S}_r}(U^2-2Uq_k)\mu \geq -C^*r + \frac{1}{r^{n+a+2k}}\int_{\mathbb{S}_r} q_k^2(1-\tilde{\mu})|y|^a.$$

Use the definition of the Almgren scaling (4.5.3) and homogeneity of q_k

$$\frac{1}{r^{2k}}\int_{\mathbb{S}_1}(d_r^2\tilde{U}_r^2 - 2d_r r^k\tilde{U}_r q_k)\tilde{\mu}(rX)|y|^a \geq -C^*r + \int_{\mathbb{S}_1} q_k^2(1-\tilde{\mu}(rX))|y|^a.$$

Multiply by $\frac{r^k}{d_r} > 0$

$$\int_{\mathbb{S}_1}\left(\frac{d_r}{r^k}\tilde{U}_r^2 - 2\tilde{U}_r q_k\right)\tilde{\mu}(rX)|y|^a \geq -C^*\frac{r^{k+1}}{d_r} + \frac{r^k}{d_r}\int_{\mathbb{S}_1} q_k^2(1-\tilde{\mu}(rX))|y|^a. \tag{4.7.9}$$

Now we claim that the right hand side goes to 0 as $r \to 0$. Indeed, by the definition of d_r (4.5.1) and (4.7.1), for any $\varepsilon > 0$ there exist $r_\varepsilon, C_\varepsilon > 0$ such that for $0 < r < r_\varepsilon$, we have

$$\frac{r^{2k+2}}{d_r^2} = \frac{r^{2k+2+n+a}}{H(r)} \leq \frac{r^{n+a+2k+2}}{C_\varepsilon r^{n+a+2k+\varepsilon}} = \frac{r^{2-\varepsilon}}{C_\varepsilon}.$$

We infer that

$$\frac{r_j^{k+1}}{d_r} \leq \frac{r_j^{1-\varepsilon/2}}{\sqrt{C_\varepsilon}} \to 0 \quad \text{as } j\to\infty.$$

And thus $r_j^{k+1}/d_r \to 0$ as $j \to \infty$.

By (2) of Lemma 2.1.1 above, we obtain for a universal constant $C > 0$

$$\Big|\int_{\mathbb{S}_1} q_k^2(1-\tilde{\mu}(rX))|y|^a\Big| \leq \sup_{X\in\mathbb{S}_1}|1-\tilde{\mu}(rX)|\int_{\mathbb{S}_1} q_k^2|y|^a \leq Cr\int_{\mathbb{S}_1} q_k^2|y|^a.$$

Therefore, we have as $j\to\infty$

$$\Big|\frac{r_j^k}{d_{r_j}}\int_{\mathbb{S}_1} q_k^2(1-\tilde{\mu}(rX))|y|^a\Big| \leq C\frac{r_j^{k+1}}{d_{r_j}}\int_{\mathbb{S}_1} q_k^2|y|^a \to 0.$$

Since the integral is bounded and the ratio $r_j^{k+1}/d_r \to 0$.

At the end, if we let $r = r_j \to 0$ in (4.7.9), by (4.7.4), the fact that $\tilde{\mu}(0) = 1$ and $\tilde{U}_r \to q_k$, we obtain

$$-2 \int_{\mathbb{S}_1} q_k^2 |y|^a \geq 0.$$

Since $q_k \not\equiv 0$, we have reached a contradiction. □

Remark. We note that by Lemma 4.7.2 it becomes clear that U_0 is nonzero. Indeed, if U_0 were zero, then there would exists j_0 such that for all $j > j_0$ $\sup_{\overline{B_1}} |U_{r_j}| \leq c/2$, and therefore

$$\sup_{\overline{B_1}} |U(rX)| = \sup_{\overline{B_{r_j}}} |U(X)| \leq \frac{cr_j^k}{2},$$

which gives contradiction with Lemma 4.7.2. This fact together with the reasoning about the homogeneous blowup in Section 4.5, Proposition 4.5.6, Lemma 4.6.2 shows that if $U_r \to U_0$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^{n+1}_{\pm} \cup \mathbb{R}^n)$ for $r = r_j \to 0^+$, then $U_0 \in \mathfrak{P}_k^+(\mathbb{R}^{n+a+1})$.

The following two results follow exactly the reasoning of Theorem 10.5 and 10.7 in [21], but for completeness and since we did not show these type of calculations in Chapter 2, we will present them here for generalised problem (4.0.1).

Theorem 4.7.3. (Uniqueness of the homogeneous blowup at singular point) *Let $U \in \mathfrak{S}$ and assume that $0 \in \Sigma_k(U)$. Suppose that the normalization hypothesis $A(0) = \mathbb{I}_{n+1}$ is in place, and that furthermore* (2.0.7) *is in force. Then, there exists a unique $p_k \in \mathfrak{P}_k^+(\mathbb{R}^{n+a+1})$ such that the homogeneous scalings U_r converge in $C^{1,\alpha}_{loc}(\mathbb{R}^{n+1}_{\pm} \cup \mathbb{R}^n)$ to p_k.*

Proof. The conclusions in the Section 4.5 about the homogeneous blowups, Lemma 4.6.2 and the previous Remark guarantee the existence of such a polynomial, so it is left to prove the uniqueness.

Let $U_{r_j} \to U_0$ in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^{n+1}_{\pm} \cup \mathbb{R}^n)$ for a certain sequence $r_j \to 0^+$. By the previous Remark we know that $U_0 \in \mathfrak{P}_k^+(\mathbb{R}^{n+a+1})$. Then we can apply Corollary 4.4.2.1 using as $p_k = U_0$, so $M_k(U, U_0, 0^+)$ exists and can be computed as

$$M_k(U, U_0, 0^+) = \lim_{r_j \to 0^+} M_k(U, U_0, r_j) = \lim_{r_j \to 0^+} \int_{\mathbb{S}_1} (U_{r_j} - U_0)^2 \tilde{\mu}(r_j X) |y|^a = 0,$$

since, by the homogeneity of U_0, we have

$$\begin{aligned} M_k(U, U_0, r) &= \frac{1}{r^{n+a+2k}} \int_{\mathbb{S}_r} (U - U_0)^2 \tilde{\mu} |y|^a = \frac{1}{r^{2k}} \int_{\mathbb{S}_1} (U(rX) - U_0(rX))^2 \tilde{\mu}(r_j X) |y|^a \\ &= \int_{\mathbb{S}_1} \left(\frac{U(rX)}{r^k} - r^k \frac{U_0(X)}{r^k} \right)^2 \tilde{\mu}(r_j X) |y|^a = \int_{\mathbb{S}_1} (U_r - U_0)^2 \tilde{\mu}(r_j X) |y|^a. \end{aligned}$$

Since $M_k(U, U_0, 0^+) = 0$, we have that for any $r \to 0^+$, not just for the sequence $r_j \to 0^+$, that

$$M_k(U, U_0, r) = \int_{\mathbb{S}_1} (U_r - U_0)^2 \tilde{\mu}(r_j X)|y|^a \to 0.$$

Therefore, if U_0' is a limit of $U_{r_j'}$ over another sequence $r_j' \to 0^+$, then

$$\begin{aligned} 0 \le \int_{\mathbb{S}_1} (U_0 - U_0')^2 |y|^a &\le \lambda^{-1} \int_{\mathbb{S}_1} (U_0 - U_0')^2 \tilde{\mu}(r_j X)|y|^a \\ &\le \lambda^{-1} \int_{\mathbb{S}_1} (U_0 - U_{r_j})^2 \tilde{\mu}(r_j X)|y|^a + \lambda^{-1} \int_{\mathbb{S}_1} (U_{r_j'} - U_0)^2 \tilde{\mu}(r_j X)|y|^a \to 0 \end{aligned}$$

This implies that $U_0 = U_0'$ in $\mathbb{S}_1$, and since U and U_0 are homogeneous of degree k, they must coincide in $\mathbb{R}^{n+1}$. □

Theorem 4.7.4. (Continuous dependence of the blowups) *Let $U \in \mathfrak{S}$. Given $X_0 = (x_0, 0) \in \Sigma_k(U)$, with $k > (3-a)/2$, denote by $p_k^{x_0}$ the homogeneous blowup of U at $X_0 = (x_0, 0)$ as in Theorem* 4.7.3*, so that*

$$U(X) = p_k^{x_0}(A^{-1/2}(x_0)(X - X_0)) + o(|A^{-1/2}(x_0)(X - X_0)|^k).$$

Then, the mapping $X_0 \to p_k^{x_0}$ from $\Sigma_k(U)$ to $\mathfrak{P}_k^+(\mathbb{R}^{n+a+1})$ is continuous. Moreover, for any k compact $K \in \Sigma_k(U) \cap \mathbb{B}_1$, there exists a modulus of continuity σ_K, with $\sigma_K(0^+) = 0$, such that

$$|U(X) - p_k^{x_0}(A^{-1/2}(x_0)(X - X_0))| \le \sigma_K(|A^{-1/2}(x_0)(X - X_0)|)|A^{-1/2}(x_0)(X - X_0)|^k, \quad (4.7.10)$$

for any $X_0 \in K$

Proof. $\mathfrak{P}_k^+(\mathbb{R}^{n+a+1})$ is a convex subset of the finite dimensional vector space of all k-homogeneous polynomials, and so all norms are equivalent. We endow such space with the norm of $L^2(\mathbb{S}_1, |y|^a d\sigma)$.

Given $X_0 = (x_0, 0) \in \Sigma_k(U)$ and $\varepsilon > 0$ small enough, there $r_\varepsilon = r_\varepsilon(X_0) > 0$ such that

$$M_k^{x_0}(U, p_k^{x_0}, r_\varepsilon) \stackrel{\text{def}}{=} M_k(U_{x_0}, p_k^{x_0}, r_\varepsilon) = \frac{1}{r_\varepsilon^{n+a+2k}} \int_{\mathbb{S}_{r_\varepsilon}} (U_{x_0} - p_k^{x_0})^2 \tilde{\mu}|y|^a < \varepsilon,$$

where $U_{x_0}(X) = U(x_0 + A^{1/2}(x_0)x, y)$ solves (4.0.1) coresponding to the matrix $A_{x_0}(x, y) = A^{-\frac{1}{2}}(x_0)A(x_0 + A^{\frac{1}{2}}(x_0)x)A^{-\frac{1}{2}}(x_0)$. This implies that there exists $\delta_\varepsilon = \delta_\varepsilon(X_0) > 0$ such that if $Z_0 = (z_0, 0) \in \Sigma_k(U) \cap \mathbb{B}_{\delta_\varepsilon}(X_0)$, then

$$M_k^{z_0}(U, p_k^{x_0}, r_\varepsilon) = \frac{1}{r_\varepsilon^{n+a+2k}} \int_{\mathbb{S}_{r_\varepsilon}} (U_{z_0} - p_k^{x_0})^2 \tilde{\mu}|y|^a < 2\varepsilon.$$

Since, by Corollary 4.4.2.1, $M_k^{z_0}(U, p_k^{x_0}, \cdot) + C^{**}r$ is monotone nondecreasing, we conclude that for r_ε small enough, $M_k^{z_0}(U, p_k^{x_0}, r) < 3\varepsilon$, with $0 < r < r_\varepsilon$. Letting $r \to 0^+$, we obtain

$$M_k^{z_0}(U, p_k^{x_0}, 0^+) = \int_{\mathbb{S}_1} (p_k^{z_0} - p_k^{x_0})^2 |y|^a \leq 3\varepsilon,$$

which concludes the first part of the theorem. To prove the second part, we notice that for $|Z_0 - X_0| < \delta_\varepsilon$ and $0 < r < r_\varepsilon$,

$$\|U_{z_0} - p_k^{z_0}\|_{L^2(\mathbb{S}_r, |y|^a d\sigma)} \leq \|U_{z_0} - p_k^{x_0}\|_{L^2(\mathbb{S}_r, |y|^a d\sigma)} + \|p_k^{x_0} - p_k^{z_0}\|_{L^2(\mathbb{S}_r, |y|^a d\sigma)} \leq 2(3\varepsilon)^{1/2} r^{(n+a)/2+k} \lambda^{-1/2}.$$

Integrating in r, this also gives an estimate for solid integrals:

$$\|U_{z_0} - p_k^{z_0}\|_{L^2(\mathbb{B}_{2r}, |y|^a d\sigma)} \leq C\varepsilon^{1/2} r^{(n+a+1)/2+k}.$$

To proceed, we know notice that

$$\begin{aligned} L_{z_0} p_k^{z_0} &= \operatorname{div}(|y|^a A_{z_0} \nabla p_k^{z_0}) = \operatorname{div}(|y|^a (A_{z_0} - \mathbb{I}_{n+1}) \nabla p_k^{z_0}) + \operatorname{div}(|y|^a \nabla p_k^{z_0}) = \operatorname{div}(|y|^a (A_{z_0} - \mathbb{I}_{n+1}) \nabla p_k^{z_0}) \\ &= \nabla A_{z_0} |y|^a \nabla p_k^{z_0} + (A_{z_0} - A(0)) D_i(|y|^a D_j p_k^{z_0}). \end{aligned}$$

and hence, by (2.0.7) and homogeneity of $p_k^{z_0}$, $|L_{z_0} p_k^{z_0}| \leq C r^{k+a-1}$ in $\mathbb{B}_{2r}$. This then imply, by the statement of the problem (4.0.1) and linearity of L_{z_0}, that

$$|L_{z_0}(U_{z_0} - p_k^{z_0})| \leq C r^{k+a-1} \quad \text{in } \mathbb{B}_{2r},$$

and consequently, by the interior $L^\infty - L^2$ estimates [14], we obtain

$$\begin{aligned} \|U_{z_0} - p_k^{z_0}\|_{L^\infty(\mathbb{B}_r)} &\leq C r^{-(n+a+1)/2} \|U_{z_0} - p_k^{z_0}\|_{L^2(\mathbb{B}_{2r})} + C r^{k+1} \\ &= C r^{-(n+a+1)/2} \|U_{z_0} - p_k^{z_0}\|_{L^2(\mathbb{B}_{2r})} + C r^{k+1} \\ &\leq C\varepsilon^{1/2} r^k + C r^{k+1}. \end{aligned}$$

Rescaling, this gives

$$\|U_{z_0,r} - p_k^{z_0}\|_{L^\infty(\mathbb{B}_1)} \leq C(\varepsilon^{1/2} + r) \leq C_\varepsilon, \tag{4.7.11}$$

for $r \leq r_\varepsilon$ small, and $C_\varepsilon \to 0$ as $\varepsilon \to 0$, where $U_{z_0,r} := U_{z_0}(rX)/r^k$. Now, convering the compact $K \subset \Sigma_k(U) \cap \mathbb{B}_1$ with finitely many balls $\mathbb{B}_{\delta_\varepsilon(x_0^i)}(x_0^i)$ for some $x_0^i \in K$, $i = 1, ..., N$, we conclude that (4.7.11) holds for all $z_0 \in K$ with $r < r_\varepsilon^K := \min\{r_\varepsilon(x_0^i) \mid i = 1, ..., N\}$. $\square$

From this point we can follow the reasoning of the Section 2.5 to achieve the main result, Theorem 4.0.2.

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