Control of the induced earthquakes

Master Thesis

Student: Lucas Davron

Advisors: Pierre Lissy*, Swann Marx†, Davide Barilari‡

2023

*Université Paris-Dauphine

†École Centrale Nantes

‡Università degli studi di Padova

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1 Introduction to the research topic

1.1 Control of the induced earthquakes

It is now a scientific fact that human activities can induce earthquakes (see [9] for a thorough review). Owing to the online database [24], oil and gas industry together with geothermy and water reservoir impoundment are the anthropogenic activities which are accountable for the largest part of the known induced earthquakes.

The oil and gas industry processes may induce earthquakes at the stage of extraction. This is mostly the case with hydraulic fracturing (*a.k.a.* fracking), when disposing waste water, or even when disposing back water that was already present in the extraction site, see [20] for more on the earthquakes induced by the oil and gas industry. These phenomenons are all linked with the injection of a fluid in the ground. Fluid injection causing earthquakes is refered to as injection-induced earthquakes or fluid-injection-induced earthquakes. This Mémoire is devoted to the mathematical analysis of a model for fluid-injection-induced earthquakes.

Our model is taken from [11] and is depicted in Figure (1). It bears a simplified structure, no-

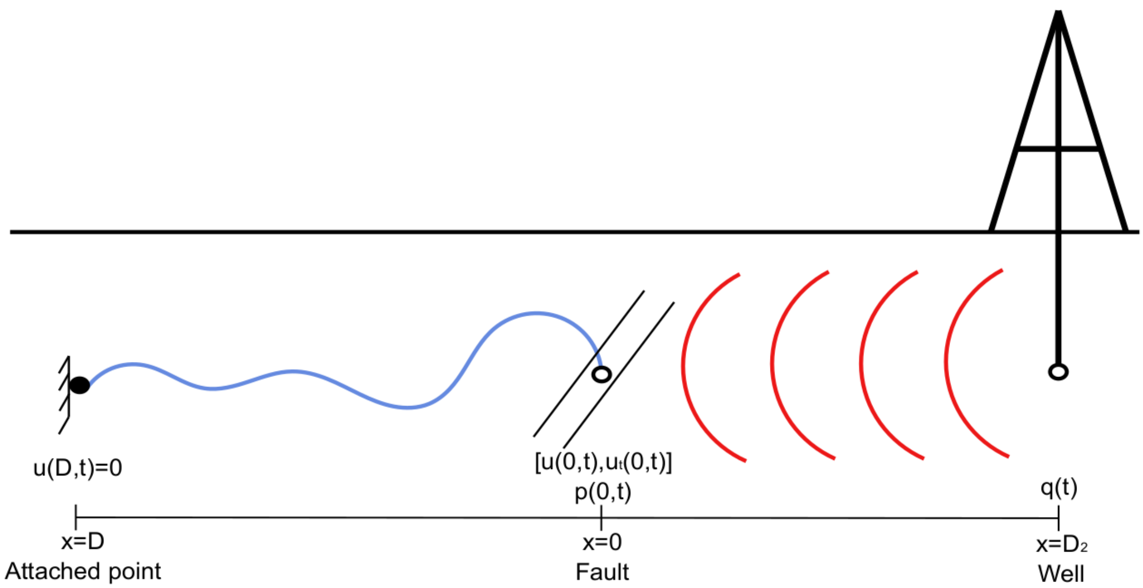

Figure 1: Modelization of the control system, at the right q represents injected water through a well, in red the generated pression travells toward the sismic fault, in blue the displacement spreads from the fault until it reaches $x = D$ which is an attached point. Taken from [11]

tably because it is 1D. The corresponding system of equations is (see [11])

$$\left\{\begin{array}{ccc} u_{tt} & = & c^2 u_{xx}, & -D_1 < x < 0, \\ u_x(t,0) & = & \frac{\mu(u(t,0),u_t(t,0),t)-\mu_0}{G}\sigma' - \frac{\mu(u(t,0),u_t(t,0),t)}{G}p(t,0), & \\ u(t,-D_1) & = & 0, & \\ p_t & = & dp_{xx}, & 0 < x < D_2, \\ p_x(t,0) & = & 0, & \\ p_x(t,D_2) & = & q+\varphi, & \end{array}\right. \tag{1}$$

where for mathematical convenience, we changed notation of [11] and Figure (1) from $D < 0$ to $-D_1$. In these equations, $t \geq 0$ is the time variable, $-D_1 < x < D_2$ is the space variable, $u = u(t,x)$ is the displacement of earth' crust, $p = p(t,x)$ is the pressure, $q = q(t)$ is the injected fluid, φ is a noise, μ is an unknown function, and c, d, G, μ_0, σ' are physical constants.

To the best of our knowledge, very little is known about this model. Most of the undertaken work has been focusing on closely related ODE models, see [22] and [23]. For instance it is not clear if the system (1) is controllable or stabilizable. Moreover in [11] the authors design controls decoupling the system. Namely they aim at controlling the wave subsystem

$$\left\{\begin{array}{ccc} u_{tt} & = & c^2 u_{xx}, & -D_1 < x < 0, \\ u_x(t,0) & = & \frac{\mu(u(t,0),u_t(t,0),t)-\mu_0}{G}\sigma' - \frac{\mu(u(t,0),u_t(t,0),t)}{G}p(t,0), & \\ u(t,-D_1) & = & 0, & \end{array}\right.$$

interpretting $p(t,0)$ as a control, and to separately control the heat subsystem

$$\left\{\begin{array}{ccc} p_t & = & dp_{xx}, \quad 0 < x < D_2, \\ p_x(t,0) & = & 0, \\ p_x(t,D_2) & = & q+\varphi. \end{array}\right.$$

This motivates the study of the coupling mechanism in (1). To this aim we will simplify this model into

$$\left\{\begin{array}{ccc} u_{tt}(t,x) & = & c^2 u_{xx}(t,x), & -D_1 < x < 0, \\ u_x(t,0) & = & p(t,0), & \\ u(t,-D_1) & = & 0, & \\ p_t(t,x) & = & dp_{xx}(t,x), & 0 < x < D_2, \\ p_x(t,0) & = & 0, & \\ p_x(t,D_2) & = & q(t), & \end{array}\right. \tag{2}$$

and aim at study its controllability.

1.2 Control of PDEs via duality methods

Consider the system

$$\Sigma(A,B) : \left\{\begin{array}{ccc} \dot{z} & = & Az + Bu, \\ z(0) & = & z^0, \end{array}\right. \tag{3}$$

where

- A is an unbounded operator (the dynamics) on a Hilbert space X (the state space),

- $B \in \mathcal{L}_c(U; D(A^*)^*)$ is the control operator and U is the control space (another Hilbert space),
- $z = z(t)$ is the quantity of interest, $0 \leq t \leq T$ is the time variable, $0 < T < \infty$ is the horizon time,
- $z^0 \in X$ is the initial condition,
- $u \in L^2(0, T; U)$ is the control.

Assume that (3) is well-posed, in a sense to be clarified later, so that for any $z^0 \in X$ and $u \in L^2(0, T; U)$ there exists a unique solution $z = z(t)$ to (3).

Definition 1.2.0.1. *(Exact controllability)*
The system (3) is said exactly controllable (at time T) if for any initial data $z^0 \in X$, for any terminal data $z^T \in X$, there exists a control $u \in L^2(0, T; U)$ which steers z^0 to z^T at time T. That is such that the solution z of (3) solves

$$z(T) = z^T.$$

To show directly that (3) is controllable is a hard task because we actually have to exhibit a control. Rather, using duality theory it is possible to show that controllability is equivalent to the obtention of some inequality. The latter is much more appealing, for we may apply all our favourite tools from real, complex, and functionnal analysis.

The solution z of (3) is given by the Duhamel formula

$$z(t) = e^{tA} z^0 + \int_0^t e^{(t-s)A} B u(s) ds,$$

hence we have the closed formula for the final state

$$z(T) = e^{TA} z^0 + \int_0^T e^{(T-s)A} B u(s) ds.$$

We introduce the operator

$$F_T : \begin{cases} L^2(0, T; U) & \longrightarrow & X \\ u & \longmapsto & \int_0^T e^{(T-s)A} B u(s) ds \end{cases}$$

which is such that the final state reads

$$z(T) = e^{TA} z^0 + F_T u.$$

The following Theorem is an example of duality theory appliyed to the control of (3).

Theorem 1.2.0.2. *Under the standing assumptions, the following are equivalent:*

1. *The system (3) is exactly controllable,*
2. *The map $F_T : L^2(0, T; U) \to H$ is onto,*

3. The system is initial time observable:

$$\exists c > 0, \quad \forall \varphi^0 \in H, \quad \|\varphi^0\|^2 \leq c \int_0^T \|B^* S_t^* \varphi^0\|^2 dt.$$

For a proof of this result we refer to [4, Theorem 2.4.2]. The inequality of the third point is referred as an "initial time observability for the adjoint system" inequality, or for short "initial time observability".

The system (3) is called an abstract linear control system, or a control system for short. From a formal point of view, it looks closely like a distributed contorl system. Those distributed control systems are controlable systems where the control takes place inside of the physical domain. To illustrate this, consider the following controled heat equation

$$\left\{ \begin{array}{rcl} z_t(t,x) & = & z_{xx}(t,x) + 1_\omega(x)u(t,x), \\ z_x(t,0) & = & 0, \\ z_x(t,L) & = & 0, \end{array} \right. \tag{4}$$

where $0 < L < \infty$, $\omega \subset (0,L)$ is a non empty and open subset, and $u = u(t,x)$ is the control. This models the evolution of the temperature of a metal string which does not exchange energy with the exterior. Note that at each time t, the controller is able to choose the heat source at any $x \in \omega$. For this reason (4) is called a distributed control system. Another situation is when the controler is only capable of choosing the heat source at one extremity of the string, for instance

$$\left\{ \begin{array}{rcl} z_t(t,x) & = & z_{xx}(t,x), \\ z_x(t,0) & = & u(t), \\ z_x(t,L) & = & 0. \end{array} \right. \tag{5}$$

The equations now look different because the control does not take place in the abstract ODE, but rather at the boundary. Such a system (5) is more suited for the applications and can be theoretically categorized as a boundary control system

$$\Sigma(L,G) : \left\{ \begin{array}{rcl} \dot{z} & = & Lz, \\ Gz & = & u, \\ z(0) & = & z^0, \end{array} \right. \tag{6}$$

where L is the dynamics operator and G is the boundary control operator. Comparing (4) and (5), the intuition tells us that (4) should be the easiest system to control, since the control acts on more space. In the next Section we explain how to deal with such boundary control systems.

2 Generalities

In this Section we establish all the mathematical framework to investigate the controllability of (2) and introduce some notations. This Section is rather theoretical and the confident reader is invited to directly go to Section (3).

We will use the following conventions:

- For theoretical purposes, all vector spaces are complex but we search for real valued solutions to our PDEs.
- Scalar products are denoted $(\cdot,\cdot)_H$ while duality pairing are denoted $\langle\cdot,\cdot\rangle_{X,Y}$. We will take them both linear on the first argument and anti-linear on the second (see below for an explanation).
- The word "isomorphism" should be understood as in the category where the objects live, which should be clear from the context. Categories are refered by bold letters. The superscript "∞" indicates that morphisms are allowed to increase the norm. For instance, $\mathbf{Ban}^\infty$ stands for the category of $\mathbb{C}$-Banach spaces with morphisms the linear and continuous maps. Therefore, in $\mathbf{Ban}^\infty$ the isomorphisms are the linear, continuous and bijective maps (and not the surjective linear isometries). The superscript "1" means that on the contrary morphisms do not increase the norm, and thus $\mathbf{Ban}^1$-isomorphisms are isometric.
- If X, Y are complex vector spaces we denote by $\mathcal{L}(X;Y)$ the set of linear maps from X to Y. When both X and Y are equipped with a topology we refer to the set of continuous and linear maps $X \to Y$ as $\mathcal{L}_c(X;Y)$.

2.1 Realizations of the anti-dual space

We begin by recalling the notion of realisations of the anti-dual space[1]. In what follows H is an arbitrary complex Hilbert space and H' denotes it's anti-dual space, the set of all anti-linear and continuous maps $H \to \mathbb{C}$. Considering the anti-dual space in place of the dual space is harmless in practice and has the theoretical advantage that the Riesz isomorphism

$$R_H : \left\{ \begin{array}{ccc} u & \longmapsto & (u,\cdot)_H \\ H & \longrightarrow & H' \end{array} \right.$$

is $\mathbb{C}$-linear, hence an isomorphism. This is indeed harmless because it amounts to add a complex conjugation to every function representing linear functionnals.

Definition 2.1.0.1. *A realization of the anti-dual space of H is a couple (H^*, Φ) where H^* is a complex Hilbert space and $\Phi : H^* \to H'$ is an isomorphism.*

Note that in the above definition, H' is endowed with its Hilbert structure, which exists in view of the Riesz identification theorem.

Remark 2.1.0.2. *In the above definition one cannot forget Φ, because in practice all the Hilbert spaces are separable infinite dimensional, hence isomorphic to each others.*

[1] We refer to [1, Chapitre III] for this notion. Note that herein the author deals with real Hilbert spaces, which is not suitable for spectral theory.

We shall note that H always has two realizations of its anti-dual. The first one is trivial, $(H', \mathrm{Id}_{H'})$. The second one is given by the Riesz isomorphism $R_H : H \to H'$, which makes (H, R_H) a realization of the anti-dual space of H. In the latter case we say that H is a pivot space. We adopt the convention to use the H' notation for the "true" anti-dual (the one made by functionnals) and H^* for a realization of the anti-dual space (which could be H'...).

Observe that we can put H^* and H in duality via

$$\langle h^*, h \rangle_{H^*,H} := \langle \Phi h^*, h \rangle_{H',H}$$

which defines a duality pairing between H and H^*. The knowledge of the map Φ is contained in this duality bracket which allows one to essentially "forget" Φ.

The following Proposition generalizes the concept of an adjoint operator (in Hilbert spaces), allowing one to take advantage of a specific realization of the anti-dual space.

Proposition 2.1.0.3. *Let H_1, H_2 be complex Hilbert spaces together with respective anti-dual realizations H_1^* and H_2^*. Let also $T \in \mathcal{L}_c(H_1; H_2)$ and the formal definition*

$$\langle T^* h_2^*, h_1 \rangle_{H_1^*,H_1} = \langle h_2^*, T h_1 \rangle_{H_2^*,H_2}, \quad \forall h_2^* \in H_2^*, \quad \forall h_1 \in H_1.$$

We then have

1. *This defines $T^* \in \mathcal{L}_c(H_2^*; H_1^*)$ such that*

$$\|T\|_{\mathcal{L}_c(H_1;H_2)} = \|T^*\|_{\mathcal{L}_c(H_2^*;H_1^*)}.$$

2. *If both realizations are trivial we recover the definition of the adjoint in* **TVS***:*

$$T^* \in \mathcal{L}_c(H_2'; H_1'), \quad \langle T^* \varphi_2, h_1 \rangle_{H_1',H_1} = \langle \varphi_2, T h_1 \rangle_{H_2',H_2}.$$

3. *If both spaces are pivot, we recover the definition of the adjoint in* **Hilb***:*

$$T^* \in \mathcal{L}_c(H_2; H_1), \quad (T^* h_2, h_1)_{H_1} = (h_2, T h_1)_{H_2}.$$

2.2 Anti-dual with respect to a pivot

We recall the meaning of

$$V \subset H = H' \subset V^*$$

where H, V are fixed complex Hilbert spaces with $V \subset H$ continuous dense.

Remark 2.2.0.1. *The notation $V \subset H$ means that V is a linear subspace of H, that has in addition a Hilbert structure in its own right. This hypothesis is not anecdotic as in practice all the Hilbert spaces are separable infinite dimensional, hence isomorphic to each others (in* **Hilb***). Therefore, in order to establish a consistant theory we ought to keep track of the embedding $V \hookrightarrow H$. This is contained in the identification of V as a subset of H.*

Let us first recall a result on completion.

Proposition 2.2.0.2. *(Completion of a normed space)*
Let X be a normed space, then it has a completion in the sense that there exists some couple $(\hat{X}, i)$ with

- *$\hat{X}$ is a Banach space,*
- *$i : X \to \hat{X}$ linear isometric dense map.*

It should be noted that the completion of a normed space is not unique, but only unique up to an isomorphism (it solves a universal property). This situation will transpose for realizations of the anti-dual with respect to a pivot.

The following result explains how to produce a realization of the anti-dual of V that exploits $V \subset H$.

Theorem 2.2.0.3. *The map*

$$\Phi : \left\{ \begin{array}{ccc} H & \longrightarrow & V' \\ h & \longmapsto & (h, \cdot)_H \end{array} \right.$$

is linear, injective and dense. Moreover,

$$\|h\|_* := \sup_{\substack{v \in V \\ \|v\|_V = 1}} |(h, v)_H|$$

is a norm on H. Therefore, for all $(\hat{H}, i)$ completion of $(H, \|\cdot\|_*)$, denoting $\hat{\Phi} : \hat{H} \to V'$ the linear and continuous extension of Φ, we get that $\hat{\Phi}$ is a $\mathbf{Ban}^1$ *isomorphism and $\hat{H}$ is hilbertizable*[2]*. With such a Hilbert structure, $(\hat{H}, \hat{\Phi})$ is a realisation of the anti-dual of V.*

Proof. The map Φ is trivially a well defined linear map from H to $\mathcal{L}(V; \mathbb{C})$. It is in fact V' valued because for any $h \in H$ and $v \in V$ there holds

$$|(\Phi h)v| = |(h, v)_H| \leq \|h\|_H \|v\|_H \leq c\|h\|_H \|v\|_V$$

where $c > 0$ is such that

$$\forall w \in V, \quad \|w\|_H \leq c\|w\|_V.$$

It is also clearly a linear map. The injectivity then follows from the density of $V \subset H$. The density comes from a famous corollary of Hahn-Banach (see [2, Corollary 1.8]) which asserts that $\Phi(H) \subset V'$ is dense if (and only if) for any $\xi \in V''$,

$$[\forall \varphi \in \Phi(H), \quad \langle \xi, \varphi \rangle_{V'', V'} = 0] \Longrightarrow \xi = 0.$$

For such a ξ, being V reflexive (it is Hilbert), there exists $v_0 \in V$ such that

$$\forall \varphi \in V', \quad \langle \xi, \varphi \rangle_{V'', V'} = \langle \varphi, v_0 \rangle_{V', V}.$$

Therefore, the hypothesis on ξ means that for any $h \in H$,

$$0 = \langle \xi, \Phi h \rangle_{V'', V'} = \langle \Phi h, v_0 \rangle_{V', V} = (h, v_0)_H.$$

[2]We say that a Banach space is hilbertizable when its norm is equivalent to a norm deriving from a scalar product. This is equivalent to being $\mathbf{Ban}^1$-isomorphic to a Hilbert space.

Then $v_0 = 0$ and $\xi = 0$, which shows the density of $\Phi(H) \subset V'$.

It is then elementary to check that $\|\cdot\|_*$ is a norm on H. Pick $(\hat{H}, i)$ any completion of $(H, \|\cdot\|_*)$, then $\Phi : (H, \|\cdot\|_*) \to V'$ is linear and continuous, it is isometric by definition of $\|\cdot\|_*$. It thus has a linear and continuous extension $\hat{\Phi} \in \mathcal{L}_c(\hat{H}; V')$. Then $\hat{\Phi} : \hat{H} \to V'$ is linear and isometric. We claim it is furthermore surjective: indeed $\hat{\Phi}$ has closed range (it is classical that a coercive linear operator has closed range), hence

$$\operatorname{Range} \hat{\Phi} \supset \operatorname{Cl}_{V'} \operatorname{Range} \hat{\Phi} \supset \operatorname{Cl}_{V'} \Phi(H) = V'.$$

Then $\hat{\Phi} : \hat{H} \to V'$ is bijective, hence a $\mathbf{Ban}^1$ isomorphism.

Up to now in this proof V' has been invoked as a Banach space, endowed of the operator norm (denoted $\|\cdot\|_{V'}$), and $\hat{\Phi} : \hat{H} \to V'$ is a $\mathbf{Ban}^1$ isomorphism but not yet a **Hilb** isomorphism. Both the Banach spaces $\hat{H}$ and V' are hilbertizable, hence they both possess scalar products inducing their norms. For then the map $\hat{\Phi} : \hat{H} \to V'$ is a **Hilb** isomorphism because being linear and isometric is enough to respect the scalar product. □

Such a $(\hat{H}, \hat{\Phi})$ is called the anti-dual space of V with respect of the pivot H and will often be denoted V^*. Observe that V^* is not uniquely defined, though it is unique up to isomorphism since it solves some universal property. This means that when it is not explicitely said, all results concerning V^* should begin by fixing a realization of V^*.

Note that V^* is endowed of the norm

$$\|u\|_{V^*} := \|\Phi u\|_{V'}$$

which is such that

$$\forall u \in H, \quad \|u\|_{V^*} = \sup_{\substack{v \in V \\ \|v\|_{V=1}}} |(u, v)_H| \leq c\|u\|_H.$$

Proposition 2.2.0.4. *(First non trivial applications)*

1.

$$\forall u \in H, \quad \forall \varphi \in V, \quad \langle u, \varphi \rangle_{V^*,V} = (u, \varphi)_H.$$

2.

$$H = \{u \in V^* : \exists c > 0, \quad \forall \varphi \in V, \quad |\langle u, \varphi \rangle_{V^*,V}| \leq c\|\varphi\|_H\}.$$

3.

$$V = \{u \in V^* : \exists c > 0, \quad \forall \varphi \in V, \quad |\langle u, \varphi \rangle_{V^*,V}| \leq c\|\varphi\|_{V^*}\}.$$

This Proposition shows why V^* is a convenient choice of realization of the anti-dual space of V, when H is a "nice" Hilbert space.

2.3 Unbounded operators and $D(A^*)^*$

In this Subsection we perform several constructions on operators using the notion of anti-dual space with respect to a pivot. Most of the material is taken from [25, Section 2].

2.3.1 Unbounded operators

Throughout this Subsubsection we assume that A is a closable, densely defined, unbounded[3] operator on a Hilbert space X and refer to [15, Sections 2.1, 2.4] for elementary spectral theory considerations.

Since A is densely defined and closable, it has a well defined adjoint A^* which is also a densely defined and closed unbounded operator on X with domain $D(A^*)$ (see [15][Section 2.4]. We will endow $D(A^*)$ with the graph norm

$$\|u\|_{D(A^*)}^2 := \|u\|_X^2 + \|A^*u\|_X^2$$

which makes it a Hilbert space. We let $D(A^*)^*$ be a realisation of the anti-dual of $D(A^*)$ with respect to X.

Lemma 2.3.1.1. *1. A has an extension $\mathcal{L}_c(X; D(A^*)^*)$ that we call $\hat{A}$.*

2. $\hat{A}$ is a densely defined unbounded operator on $D(A^*)^*$.

3. If $\lambda \in \rho(A)$, then $\lambda \in \rho(\hat{A})$ and

$$(\lambda - \hat{A})^{-1}\Big|_X = (\lambda - A)^{-1}.$$

In particular, $\rho(A) \subset \rho(\hat{A})$.

Proof. 1. It is enough to show that

$$\exists c > 0, \quad \forall z \in D(A), \quad \|Az\|_{D(A^*)^*} \leq c\|z\|_X.$$

We then write for arbitrary $z \in D(A)$

$$\begin{aligned} \|Az\|_{D(A^*)^*} &= \sup_{\substack{\varphi \in D(A^*) \\ \|\varphi\|_{D(A^*)=1}}} |(Az, \varphi)_X| \\ &= \sup_{\substack{\varphi \in D(A^*) \\ \|\varphi\|_{D(A^*)=1}}} |(z, A^*\varphi)_X| \\ &\leq \|z\|_X. \end{aligned}$$

2. The fact that $\hat{A}$ is an unbounded operator on $D(A^*)^*$ is trivial. The fact that it is densely defined follows from the definition of $D(A^*)^*$ as a completion of $(X, \|\cdot\|_{D(A^*)^*})$.

3. Assume there exists $\lambda \in \rho(A)$, we have to show that $(\lambda - \hat{A}) : X \to D(A^*)^*$ is bijective and its inverse is $\mathcal{L}_c(D(A^*)^*)$. We claim that $(\lambda - A)^{-1}$ has a $\mathcal{L}_c(D(A^*)^*; X)$ extension, this follows

[3] This is a slightly weaker assumption than the one used in [25], which is that A is a densely defined operator with nonempty resolvant set.

from the following computation, which holds for any $\varphi \in X$

$$\begin{aligned}\|(\lambda - A)^{-1}\varphi\|_X &= \sup_{\substack{u\in X\\ \|u\|_X=1}} |((\lambda - A)^{-1}\varphi, u)_X| \\ &= \sup_{\substack{u\in X\\ \|u\|_X=1}} |(\varphi, (\overline{\lambda} - A^*)^{-1}u)_X| \\ &\leq \sup_{\substack{\psi\in D(A^*)\\ \|\psi\|_{D(A^*)}\leq\|(\overline{\lambda}-A^*)^{-1}\|_{\mathcal{L}_c(X;D(A^*))}}} |(\varphi,\psi)_X| \\ &= \|(\overline{\lambda} - A^*)^{-1}\|_{\mathcal{L}_c(X;D(A^*))}\|\varphi\|_{D(A^*)^*}\end{aligned}$$

where we have used in the second equality that, since $\lambda \in \rho(A)$ and A is a closable densely defined operator, we have

$$\overline{\lambda} \in \rho(A^*), \quad [(\lambda - A)^{-1}]^* = (\overline{\lambda} - A^*)^{-1}.$$

Then by continuity and density, the extension of $(\lambda - A)^{-1}$ is the inverse of $\lambda - \hat{A}$.

□

We now give a precise meaning to the genuine equations

$$\langle \hat{A}x, \varphi\rangle_{D(A^*)^*,D(A^*)} = (x, A^*\varphi)_X, \quad D(A^*)^{**} = D(A^*).$$

Proposition 2.3.1.2. *Let A be a closable densely defined operator on X and $D(A^*)^*$ be any realization of the anti-dual space of $D(A^*)$ with respect to X.*

1. *We have*
$$\forall x \in X, \quad \forall\varphi \in D(A^*), \quad \langle \hat{A}x, \varphi\rangle_{D(A^*)^*,D(A^*)} = (x, A^*\varphi)_X.$$

2. *The space $D(A^*)$ is a realization of the anti-dual space of $D(A^*)^*$.*

Proof. 1. The equation to show is trivial if $x \in D(A)$, in view of Proposition (2.2.0.4). By continuity of $\hat{A} : X \to D(A^*)^*$ and density of $D(A) \subset X$ we conclude that the equation holds for any $x \in X$.

2. Fix $\Phi : D(A^*)^* \to D(A^*)'$ the **Hilb**-isomorphism such that $(D(A^*)^*, \Phi)$ is a realization of the anti-dual space of $D(A^*)$ with respect to X. We then have that

$$R_{D(A^*)^*}\Phi^{-1}R_{D(A^*)} : D(A^*) \to [D(A^*)^*]'$$

is a **Hilb**-isomorphism, by composition.

□

2.3.2 Semi-groups

We now assume that A generates a strongly continuous semi-group on X that we denote $(S_t)_{t\geq0}$.

Lemma 2.3.2.1. *1. For any $t \geq 0$, the map $S_t : X \to X$ has a $\mathcal{L}_c(D(A^*)^*)$ extension denoted $\hat{S}_t$.*

2. *The induced family* $(\hat{S}_t)_{t\geq 0}$ *is a strongly continuous semi-group on* $D(A^*)^*$*.*

3. *For any* $t \geq 0$ *we have*

$$\forall u \in D(A^*)^*, \quad \forall \varphi \in D(A^*), \quad \langle u, S_t^* \varphi \rangle_{D(A^*)^*, D(A^*)} = \langle \hat{S}_t u, \varphi \rangle_{D(A^*)^*, D(A^*)}.$$

4. *The generator of* $(\hat{S}_t)_{t\geq 0}$ *is* $\hat{A}$*, with domain* X*.*

Proof. 1. Let $x \in X$, we have

$$\begin{aligned} \|S_t x\|_{D(A^*)^*} &= \sup_{\substack{\varphi \in D(A^*) \\ \|\varphi\|_{D(A^*)=1}}} |(S_t x, \varphi)_X| \\ &= \sup_{\substack{\varphi \in D(A^*) \\ \|\varphi\|_{D(A^*)=1}}} |(x, S_t^* \varphi)_X| \\ &\leq \|S_t^*\|_{\mathcal{L}_c(D(A^*))} \|x\|_{D(A^*)^*} \end{aligned} \tag{7}$$

where we have used that $S_t^* \in \mathcal{L}_c(D(A^*))$. Indeed, for any $\varphi \in D(A^*)$ we have $S_t^* \varphi \in D(A^*)$ and

$$\begin{aligned} \|S_t^* \varphi\|_{D(A^*)}^2 &= \|S_t^* \varphi\|_X^2 + \|A^* S_t^* \varphi\|_X^2 \\ &\leq \|S_t^*\|_{\mathcal{L}_c(X)}^2 \|\varphi\|_X^2 + \|S_t^* A^* \varphi\|_X^2 \\ &\leq \|S_t\|_{\mathcal{L}_c(X)}^2 \left(\|\varphi\|_X^2 + \|A^* \varphi\|_X^2 \right) \\ &= \|S_t\|_{\mathcal{L}_c(X)}^2 \|\varphi\|_{D(A^*)}^2. \end{aligned} \tag{8}$$

2. The semi-group property is easy to get from density and continuity. We are left to show the strong continuity:

$$\forall x \in D(A^*)^*, \quad \hat{S}_t x \xrightarrow[t \to 0^+]{D(A^*)^*} x.$$

To this aim we first show that

$$\exists M > 0, \quad \forall 0 \leq t \leq 1, \quad \|\hat{S}_t\|_{\mathcal{L}_c(D(A^*)^*)} \leq M.$$

This is from (7) and (8), which yield

$$\|\hat{S}_t\|_{\mathcal{L}_c(D(A^*)^*)} \leq \|S_t^*\|_{\mathcal{L}_c(D(A^*))} \leq \|S_t\|_{\mathcal{L}_c(X)}.$$

Now we fix $x \in D(A^*)^*$ and $(x_j)_{j=0}^{\infty}$ a sequence of X converging to x in $D(A^*)^*$. We obtain

$$\begin{aligned} \limsup_{t \to 0^+} \|\hat{S}_t x - x\|_{D(A^*)^*} &\leq \inf_j \limsup_{t \to 0^+} \left(\|\hat{S}_t x - \hat{S}_t x_j\|_{D(A^*)^*} + \|\hat{S}_t x_j - x_j\|_{D(A^*)^*} + \|x_j - x\|_{D(A^*)^*} \right) \\ &\leq \inf_j \limsup_{t \to 0^+} \|\hat{S}_t x - \hat{S}_t x_j\|_{D(A^*)^*} \\ &\leq \inf_j \limsup_{t \to 0^+} \|\hat{S}_t\|_{\mathcal{L}_c(D(A^*)^*)} \|x - x_j\|_{D(A^*)^*} \\ &= 0. \end{aligned}$$

3. The equality is trivial when $u \in X$ and we conclude by density and continuity.

4. We let Λ be the generator of the strongly continuous semi-group $(\hat{S}_t)_{t\geq 0}$ on $D(A^*)^*$. We show that $\Lambda = (X, \hat{A})$. For any $x \in D(A)$, we have

$$\frac{S_t x - x}{t} \xrightarrow[t\to 0^+]{X} Ax,$$

hence

$$\frac{\hat{S}_t x - x}{t} \xrightarrow[t\to 0^+]{D(A^*)^*} \hat{A}x,$$

and we deduce $(D(A), \hat{A}) \subset \Lambda$. Now to show that $(X, \hat{A}) = \Lambda$ we first claim that

$$(X, \hat{A}) \subset \overline{(D(A), \hat{A})} \tag{9}$$

where the closure is taken in $D(A^*)^*$. Indeed, let $x \in X$, we show that there exists a sequence (x_j) of $D(A)$ such that

$$x_j \xrightarrow[j\to\infty]{D(A^*)^*} x, \quad \hat{A}x_j \xrightarrow[j\to\infty]{D(A^*)^*} \hat{A}x.$$

We will take (x_j) any sequence of $D(A)$ that goes to x for the X topology, which exists as A is a densely defined operator. We then obviously have

$$x_j \xrightarrow[j\to\infty]{D(A^*)^*} x$$

and because $\hat{A} \in \mathcal{L}_c(X; D(A^*)^*)$ we have

$$\hat{A}x_j \xrightarrow[j\to\infty]{D(A^*)^*} \hat{A}x.$$

This shows (9), and by closedness of Λ we obtain

$$(X, \hat{A}) \subset \overline{(D(A), \hat{A})} \subset \overline{\Lambda} = \Lambda.$$

We then show that the operator $(X, \hat{A})$ is a generator on $D(A^*)^*$. To this aim we observe that it is conjugated with A (seen here as an unbounded operator on X) as follows. For a fixed $\lambda \in \rho(A)$, which exists as A is a generator, the map

$$\lambda - \hat{A} : X \to D(A^*)^*$$

is a **Ban**$^\infty$-isomorphism in view of Lemma (2.3.1.1). It makes the following diagram commute

$$\begin{array}{ccc} X & \xrightarrow{\hat{A}} & D(A^*)^* \\ {\scriptstyle \lambda-\hat{A}}\uparrow & & \downarrow{\scriptstyle (\lambda-\hat{A})^{-1}} \\ D(A) & \xrightarrow{A} & X \end{array}$$

as shown by $(\lambda - \hat{A})(D(A)) = X$ and for any $\varphi \in D(A)$,

$$(\lambda - \hat{A})^{-1}\hat{A}(\lambda - \hat{A})\varphi = A\varphi.$$

Therefore, as it is classical that an operator that is conjugated with a generator is itself a generator, $(X, \hat{A})$ is a generator on $D(A^*)^*$.

We conclude the proof using a maximality argument, namely we will show that if P, Q are generators on a Hilbert space H such that $P \subset Q$, then in fact $P = Q$. To this aim let $x \in D(Q)$, we have $x \in D(P)$ as soon as the quantity

$$\frac{e^{tP}x - x}{t}, \quad t > 0,$$

has a limit in H as $t \to 0^+$. We will show that

$$\forall t \geq 0, \quad \forall x_0 \in D(Q), \quad e^{tP}x_0 = e^{tQ}x_0. \tag{10}$$

Indeed, if $x_0 \in D(P)$ the two above curves are both $C([0,\infty); D(Q)) \cap C^1([0,\infty); H)$ and such that

$$\frac{d}{dt}e^{tP}x_0 = Pe^{tP}x_0 = Qe^{tP}x_0, \quad \frac{d}{dt}e^{tQ}x_0 = Qe^{tP}x_0.$$

Hence they are classical solutions of the abstract ODE

$$\begin{cases} \dot{x}(t) &= Qx(t), \\ x(0) &= x_0, \end{cases}$$

and by uniqueness they agree. Thus

$$\forall t \geq 0, \quad \forall x_0 \in D(P), \quad e^{tP}x_0 = e^{tQ}x_0.$$

Now fix $t \geq 0$ and $x \in D(Q)$, let (x_j) be a sequence of $D(P)$ going to x in H, we pass to the limit in

$$\forall j \in \mathbb{N}, \quad e^{tP}x_j = e^{tQ}x_j,$$

to obtain (10). Finally, for any $t > 0$ and $x \in D(Q)$ we obtain

$$\frac{e^{tQ}x - x}{t} = \frac{e^{tP}x - x}{t} \to Px,$$

which shows $x \in D(P)$ and $Px = Qx$, as desired.

□

2.4 Control systems

In this Subsection we adress the well posedness and controllability for control systems (3), we will follow the lines of [4]. Fix U, X complex Hilbert spaces, A generator on X and $B \in \mathcal{L}_c(U; D(A^*)^*)$. We write B^* the the adjoint of B where U is a pivot space and $D(A^*)^*$ has realization of its dual $D(A^*)$. In other words: $B^* \in \mathcal{L}_c(D(A^*); U)$.

2.4.1 Well-posedness

Definition 2.4.1.1. *The operator B is an admissible control operator for A if*

$$\exists 0 < T < \infty, \quad \exists c > 0, \quad \forall z \in D(A^*), \quad \int_0^T \|B^* S_t^* z\|_U^2 dt \leq c\|z\|_X^2.$$

It is classical to show that if the above condition is satisfied for a fixed $0 < T < \infty$, then it is also satisfied for any $0 < T < \infty$. From now on we assume that B is admissible, we can then for any $0 < T < \infty$ extend by linearity and continuity the map

$$\begin{cases} z & \longmapsto & [t \mapsto B^* S_{T-\cdot}^* z] \\ (D(A^*), \|\cdot\|_X) & \longrightarrow & L^2(0,T;U) \end{cases}$$

on the whole of X. We will denote by the same symbol this extension.

Definition 2.4.1.2. *Let $z_0 \in X$ and $u \in L^2(0,T;U)$. A transposition solution of the control system* (3) *is a map $z \in C([0,T];X)$ such that*

$$\forall \tau \in [0,T], \quad \forall \varphi^\tau \in X, \quad (z(\tau), \varphi^\tau)_X - (z^0, S_\tau^* \varphi^\tau)_X = \int_0^\tau (u(t), B^* S_{\tau-t}^* \varphi^\tau)_U dt.$$

Observe that this definition only requires the knowledge of B^*.

Theorem 2.4.1.3. *Assume that A is a generator and that $B \in \mathcal{L}_c(U; D(A^*)^*)$ is an admissible control operator for A. Then* (3) *is well-posed in the sensee that for any $0 < T < \infty$, $z^0 \in X$ and $u \in L^2(0,T;U)$, it has a unique transposition solution on $[0,T]$. Moreover, denoting $z \in C([0,T];X)$ the corresponding solution, we have an estimation of the form*

$$\forall 0 < T < \infty, \quad \exists c > 0, \quad \forall z^0 \in X, \quad \forall u \in L^2(0,T;U), \quad \|z\|_{C([0,T];X)} \leq c\left(\|z^0\|_H + \|u\|_{L^2(0,T;U)}\right).$$

For a proof of this result, see [4, Theorem 2.37]. Note that one can then build the input-output map

$$\Xi : \begin{cases} L^2_{loc}([0,\infty);U) \times X & \longrightarrow & C([0,\infty);X) \\ (u, z_0) & \longmapsto & z(\cdot) \end{cases}$$

which is linear continuous when all the involved spaces are endowed of their natural Fréchet structure.

We now turn to the derivation of an explicit formula for the solution.

Proposition 2.4.1.4. *The solution of* (3) *is given by the extended Duhamel formula*

$$\forall t \geq 0, \quad z(t) = S_t z^0 + \int_0^t \hat{S}_{t-s} B u(s) ds,$$

where in particular

$$\left[t \mapsto \int_0^t \hat{S}_{t-s} B u(s) ds\right] \in C([0,\infty);X).$$

Proof. Let $\varphi \in D(A^*)$, observe that a priori

$$\left[t \mapsto \int_0^t \hat{S}_{t-s} B u(s) ds\right] \in C([0,\infty); D(A^*)^*),$$

and compute for any $0 \leq t < \infty$

$$\begin{aligned}
\langle z(t), \varphi \rangle_{D(A^*)^*, D(A^*)} &= (z(t), \varphi)_X \\
&= (z^0, S_t^* \varphi)_X + \int_0^t (u(s), B^* S_{t-s}^* \varphi)_U ds \\
&= (S_t z^0, \varphi)_X + \int_0^t \langle \hat{S}_{t-s} B u(s), \varphi \rangle_{D(A^*)^*, D(A^*)} ds \\
&= \langle S_t z^0, \varphi \rangle_{D(A^*)^*, D(A^*)} + \left\langle \int_0^t \hat{S}_{t-s} B u(s) ds, \varphi \right\rangle_{D(A^*)^*, D(A^*)}.
\end{aligned}$$

This shows the extended Duhamel formula, finally to get

$$\left[t \mapsto \int_0^t \hat{S}_{t-s} B u(s) ds\right] \in C([0,\infty); X),$$

a first method is to invoke that

$$z(\cdot), \left[t \mapsto S_t z^0\right] \in C([0,\infty); X),$$

hence the difference

$$\left[t \mapsto \int_0^t \hat{S}_{t-s} B u(s) ds\right] = z(\cdot) - \left[t \mapsto S_t z^0\right]$$

is $C([0,\infty); X)$. A more concrete way is to observe that

$$t \mapsto \int_0^t \hat{S}_{t-s} B u(s) ds$$

takes values in X because B is admissible and in view of Proposition (2.2.0.4). Then we can adapt the arguments of [4], in the existence part from the proof of Theorem 2.37, to show the continuity. □

We will refer to a system that satisfies the hypotheses of Theorem (2.4.1.3) to a well-posed system.

2.4.2 Controllability

We define exact, null and approximate controllability for control systems (3). We fix a finite horizon time $0 < T < \infty$.

Definition 2.4.2.1. *(Exact controllability)*
The system $\Sigma(A, B)$ is said exactly controllable at time T if for any $z^0 \in X$, for any $z^T \in X$, there exists a control $u \in L^2(0,T;U)$ such that the solution z solves

$$z(T) = z^T.$$

Definition 2.4.2.2. *(Approximate controllability)*
The system $\Sigma(A,B)$ is said approximatly controllable at time T if

$$\forall z^0, z^T \in X, \quad \forall \epsilon > 0, \quad \exists u \in L^2(0,T;U), \quad \|z^T - z(T)\|_X < \epsilon.$$

Definition 2.4.2.3. *(Null controllability)*
The system $\Sigma(A,B)$ is said null controllable at time T if

$$\forall z^0 \in X, \quad \exists u \in L^2(0,T;U), \quad z(T) = 0.$$

We now state the caracterizations of controllability derived by the control theory by duality, we refer to [4] for proofs of these criterions.

Theorem 2.4.2.4.

1. *The system $\Sigma(A,B)$ is exactly controllable at time T if and only if the adjoint system is initial time observable:*

$$\exists c > 0, \quad \forall \varphi \in D(A^*), \quad \|\varphi\|_X^2 \leq c \int_0^T \|B^* S_t^* \varphi\|_U^2 dt.$$

2. *The system $\Sigma(A,B)$ is null controllable at time T if and only if the adjoint system is final time observable:*

$$\exists c > 0, \quad \forall \varphi \in D(A^*), \quad \|S_T^* \varphi\|_X^2 \leq c \int_0^T \|B^* S_t^* \varphi\|_U^2 dt.$$

3. *The system $\Sigma(A,B)$ is approximately controllable at time T if and only if the adjoint system has the unique continuation property:*

$$\forall \varphi \in X, \quad B^* S_t^* \varphi \equiv 0 \Longrightarrow \varphi = 0.$$

Observe that for the approximate controllability, the symbol $B^* S_t^* \varphi$ stands for the extension of the map

$$\begin{cases} z & \longmapsto & [t \mapsto B^* S_\cdot^* z] \\ (D(A^*), \|\cdot\|_X) & \longrightarrow & L^2(0,T;U) \end{cases}$$

evaluated at the vector φ.

2.5 Boundary control systems

We now give a precise framework for boundary control systems (6), we follow the lines and notations of [25, Section 10]. Let U, Z, X be complex Hilbert spaces modelizing respectively the control space, the domain of A and the state space. We assume that $Z \subset X$ continuously and densly.

Definition 2.5.0.1. *A boundary control system on (U,Z,X) is a pair (L,G) where*

$$L \in \mathcal{L}_c(Z;X), \quad G \in \mathcal{L}_c(Z;U),$$

are such that

- *G is onto,*

- $\operatorname{Null} G \subset X$ *is dense* ,
- *The unbounded operator A on X defined by*

$$D(A) = \operatorname{Null} G, \quad Az = Lz,$$

is closable densely defined[4].

Recall that $D(A^*)$ is a Hilbert space with the graph norm, and that $D(A^*) \subset X$ is dense, so that we are allowed to fix $D(A^*)^*$ a realisation of the anti-dual of $D(A^*)$ with respect to the pivot X.

We can apply all the results of Subsection (2.3) and extend A as $\hat{A} \in \mathcal{L}_c(X; D(A^*)^*)$ which is the generator of the extended semi-group $(\hat{S}_t)_{t\geq 0}$ that is strongly continuous on $D(A^*)^*$.

The following theorem gives a canonical way of transforming a boundary control problem (6) into a control problem (3) and is the cental result of this paragraph.

Theorem 2.5.0.2. *There exists a unique $B \in \mathcal{L}_c(U; D(A^*)^*)$ such that*

$$L = \hat{A} + BG.$$

Proof. We begin by the uniqueness, assume that there exists two such operators B, denote them B_1 and B_2. Then from

$$\hat{A} + B_1 G = L = \hat{A} + B_2 G$$

we deduce

$$(B_1 - B_2)G = 0.$$

Being G surjective we get $B_1 = B_2$ hence the uniqueness.

We now turn to the existence, note that G is surjective hence it has a right inverse $R \in \mathcal{L}_c(U; Z)$ (see [2, Theorem 2.12]). We then consider

$$B := (L - \hat{A})R \in \mathcal{L}_c(U; D(A^*)^*)$$

which is such that

$$BG = (L - \hat{A})RG = (L - \hat{A})(RG - \mathrm{Id}_Z) + L - \hat{A}. \tag{11}$$

Now observe that

$$G(RG - \mathrm{Id}_Z) = 0,$$

so that

$$\operatorname{Range}(RG - \mathrm{Id}_Z) \subset \operatorname{Null} G = D(A),$$

and because $L = \hat{A}$ in $D(A)$ we obtain

$$(L - \hat{A})(RG - \mathrm{Id}_Z) = 0.$$

Coming back to (11) we obtain

$$BG = L - \hat{A}$$

as required. ☐

[4] Again this is a slightly weaker assumption than what is done in [25, Section 10].

With such a result we formally have

$$\begin{cases} \dot{z} &= Lz, \\ Gz &= u, \end{cases} \Longrightarrow \dot{z} = \hat{A}z + Bu$$

which was our original goal. It is straightforward to show that this implication is true at least for classical solutions, as stated in the following Proposition.

Proposition 2.5.0.3. *Let $0 < T < \infty$, $z^0 \in Z$ and $u \in L^2(0,T;U)$. Assume that there exists $z \in C([0,T];Z) \cap C^1([0,T];X)$ classical solution of (6), then z is a classical solution of*

$$\begin{cases} \dot{z} &= \hat{A}z + Bu, \\ z(0) &= z^0, \end{cases}$$

This shows that

$$\dot{z} = \hat{A}z + Bu$$

is a consistent weak formulation of

$$\begin{cases} \dot{z} &= Lz, \\ Gz &= u, \end{cases}$$

The next lemma is a key tool when doing explicit computations.

Lemma 2.5.0.4. *For any $D(A^*)^*$ realization of the anti-dual of $D(A^*)$ with respect to the pivot X, there exists a realisation of the dual of $D(A^*)^*$ such that*

$$[D(A^*)^*]^* = D(A^*),$$

and

$$B^* \in \mathcal{L}_c(D(A^*);U)$$

solves

$$\forall z \in Z, \quad \forall \varphi \in D(A^*), \quad (Lz, \varphi)_X = (z, A^*\varphi)_X + (Gz, B^*\varphi)_U. \tag{12}$$

Proof. We let $\Phi : D(A^*)^* \to D(A^*)'$ be the **Hilb**-isomorhpsim making $D(A^*)^*$ a realization of the anti-dual space of $D(A^*)$ with respect to the pivot X. As in Proposition (2.3.1.2) we consider

$$\Psi := R_{D(A^*)^*}\Phi^{-1}R_{D(A^*)}$$

which makes $D(A^*)$ a realization of the anti-dual space of $D(A^*)^*$. Setting U as a pivot we get $B^* \in \mathcal{L}_c(D(A^*);U)$ and for any $z \in Z$ and $\varphi \in D(A^*)$ we get

$$\begin{aligned} (Lz, \varphi)_X &= ((\hat{A} + BG)z, \varphi)_X \\ &= \langle \hat{A}z, \varphi \rangle_{D(A^*)^*, D(A^*)} + \langle BGz, \varphi \rangle_{D(A^*)^*, D(A^*)} \\ &= (z, A^*\varphi)_X + \langle BGz, \varphi \rangle_{D(A^*)^*, D(A^*)}. \end{aligned}$$

Therefore we are left to show that

$$\langle BGz, \varphi \rangle_{D(A^*)^*, D(A^*)} = (Gz, B^*\varphi)_U,$$

on the one hand we have

$$\begin{aligned}
(B^*\varphi, Gz)_U &= \langle B^*\varphi, Gz\rangle_{U^*,U} \\
&= \langle \varphi, BGz\rangle_{(D(A^*)^*)^*, D(A^*)^*} \\
&= \langle \Psi\varphi, BGz\rangle_{(D(A^*)^*)', D(A^*)^*} \\
&= \langle R_{D(A^*)^*}\Phi^{-1}R_{D(A^*)}\varphi, BGz\rangle_{(D(A^*)^*)', D(A^*)^*} \\
&= (\Phi^{-1}R_{D(A^*)}\varphi, BGz)_{D(A^*)^*} && (\langle R_H u, v\rangle_{H',H} = (u,v)_H) \\
&= (R_{D(A^*)}\varphi, \Phi BGz)_{D(A^*)'} \\
&= \overline{(\Phi BGz, R_{D(A^*)}\varphi)_{D(A^*)'}} \\
&= \overline{\langle \Phi BGz, \varphi\rangle_{D(A^*)', D(A^*)}} && ((\varphi, R_H u)_{H'} = \langle \varphi, u\rangle_{H',H})
\end{aligned}$$

and the other hand we have by definition of the duality pairing $\langle \cdot, \cdot\rangle_{D(A^*)^*, D(A^*)}$ that

$$\langle BGz, \varphi\rangle_{D(A^*)^*, D(A^*)} = \langle \Phi BGz, \varphi\rangle_{D(A^*)', D(A^*)}$$

which concludes the proof. □

Remark 2.5.0.5. *Because G is surjective, the formula* (12) *allows one to completely determine B^*. It is remarkable that B depends on the choice of $D(A^*)^*$, but the above formula for B^* only depends on L and A^*. It is this version of B^* that we will use.*

2.6 An abstract point of view

We will developp a rather abstract point of view on boundary control systems allowing one to completely identify them as control systems. As a useful application we will see a proper way to get rid of the constants in (2).

2.6.1 Conjugation of unbounded operators

For starters we recall a result about the conjugation of unbounded operators.

Lemma 2.6.1.1. *Let A be an unbounded operator on a Banach space X and B an unbounded operator on a Banach space Y. Assume that there exists $T : X \to Y$ a $\mathbf{Ban}^\infty$ isomorphism such that $T(D(A)) = D(B)$ and $B = TAT^{-1}$. Then*

1. *$\rho(A) = \rho(B)$ and for any $\lambda \in \rho(A)$, $(\lambda - B)^{-1} = T(\lambda - A)^{-1}T^{-1}$.*
2. *A and B are simultaneously densely defined (resp. closed, resp. generators).*
3. *$T : D(A) \to D(B)$ is a $\mathbf{NS}^\infty$ isomorphism.*

It seems desirable to be able to take the conjugate of $B = TAT^{-1}$, which can be done as summed up in the following result.

Lemma 2.6.1.2. *Suppose X and Y are Hilbert and let A and B be densely defined operators respectively on X and Y. Let also $T : X \to Y$ be linear and continuous. Assume that*

$$T(D(A)) = D(B), \quad BT = AT.$$

We then have the following.

1. $T^* : D(B^*) \to D(A^*)$ *is well defined, solves the equation* $T^* B^* = A^* T^*$.
2. $T^* \in \mathcal{L}_c(D(B^*); D(A^*))$.
3. *If* $T : X \to Y$ *was a* **Ban**$^\infty$ *(resp.* **Hilb***) isomorphism, then so is* $T^* : D(B^*) \to D(A^*)$.

We finally extend these results to duals with respect to pivot spaces.

Lemma 2.6.1.3. *Suppose* X *and* Y *are Hilbert and let* A *and* B *be closable densely defined operators respectively on* X *and* Y*. Assume that*

$$T(D(A)) = D(B), \quad BT = AT,$$

for $T : X \to Y$ *linear and continuous. We then have*

1. T *has a* $\mathcal{L}_c(D(A^*)^*; D(B^*)^*)$ *extension denoted* $\hat{T}$.
2. *If* $T : X \to Y$ *was a* **Ban**$^\infty$ *(resp.* **Hilb***) isomorphism, then so is* $\hat{T} : D(A^*)^* \to D(B^*)^*$.

2.6.2 The category of control systems

We define the category of the control system **CS** as follows. An object is a quadruplet (U, X, A, B^*) where U, X are complex Hilbert spaces, A is a densely defined operator on X and $B^* \in \mathcal{L}_c(D(A^*); U)$[5]. A morphism $(U_1, X_1, A_1, B_1^*) \to (U_2, X_2, A_2, B_2^*)$ is a couple (Φ, Ψ), where

- $\Phi \in \mathcal{L}_c(X_1; X_2)$ is such that $\Phi(D(A_1)) \subset D(A_2)$ and the diagram

  $$\begin{array}{ccc} D(A_1) & \xrightarrow{A_1} & X_1 \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ D(A_2) & \xrightarrow{A_2} & X_2 \end{array} \qquad (13)$$

  commutes in **Set**.

- $\Psi \in \mathcal{L}_c(U_1; U_2)$ is such that the diagram

  $$\begin{array}{ccc} D(A_1^*) & \xrightarrow{B_1^*} & U_1 \\ {\scriptstyle \Phi^*}\uparrow & & \downarrow{\scriptstyle \Psi} \\ D(A_2^*) & \xrightarrow{B_2^*} & U_2 \end{array}$$

  commutes in **Set**.

Note that in the last diagram, $\Phi^* : D(A_2^*) \to D(A_1^*)$ is a well defined linear and continuous map owing to Lemma (2.6.1.2).

The composition of morphisms in **CS** is the standard composition of functions, it is straightforward that it is associative, and hence **CS** is a well defined category. For short and when there is no

[5] B^* is a formal notation, we do not suppose that B^* is the adjoint of an operator $B \in \mathcal{L}_c(U; D(A^*)^*)$ because we do not want the choice of $D(A^*)^*$ to interfer

possible ambiguity we will call $\Sigma(A, B)$ such a control system.

If (U, X, A, B^*) is a control system we say that it is well-posed if A generates a semi-group and B^* is admissible in finite time. Observe that the admissibility in finite time requires only the knowledge of B^*, and not of B, hence this is well defined. We define the full sub-category of **CS** whose objects are the well-posed control systems, we call it **wpCS**.

Lemma 2.6.2.1. *1. The isomorphisms of* **CS** *are the morphisms (Φ, Ψ) such that both Φ and Ψ are bijections. Moreover, if $(\Phi, \Psi) : (U_1, X_1, A_1, B_1^*) \to (U_2, X_2, A_2, B_2^*)$ is a* **CS***-isomorphism then*

- *$\Phi : X_1 \to X_2$ is a* $\mathbf{Ban}^\infty$ *isomorphism,*
- *$\Phi : D(A_1) \to D(A_2)$ is a* $\mathbf{NS}^\infty$ *isomorphism,*
- *$\Phi^* : D(A_2^*) \to D(A_1^*)$ is a* $\mathbf{Ban}^\infty$ *isomorphism.*

2. Two control systems that are isomorphic in **CS** *are simultaneously well-posed. In other words,* **wpCS** *is stable by the isomorphisms of* **CS***.*

3. If $(\Phi, \Psi) : (U_1, X_1, A_1, B_1^*) \to (U_2, X_2, A_2, B_2^*)$ is a **wpCS***-isomorphism, then $\Phi : D(A_1^*)^* \to D(A_2^*)^*$ is a* $\mathbf{Ban}^\infty$ *isomorphism.*

Given a well-posed control system $\Sigma(A, B)$, as shown in Theorem (2.4.1.3), there exists the input-output map

$$\Xi_{\Sigma(A,B)} : \left\{ \begin{array}{ccc} L^2_{loc}([0,\infty); U) \times X & \longrightarrow & C([0,\infty); X) \\ (u, z_0) & \longmapsto & z(\cdot) \end{array} \right.$$

which is, in theory, the only thing we need to define the controllability of the system. We can now explicitely say how the morphisms of **wpCS** act on the input-output map.

Proposition 2.6.2.2. *If $(\Phi, \Psi) : \Sigma(A_1, B_1) \to \Sigma(A_2, B_2)$ is a* **wpCS***-morphism, if z is the solution of*

$$\left\{ \begin{array}{ccc} \dot{z} & = & A_1 z + B_1 u, \\ z(0) & = & z^0, \end{array} \right.$$

then $\xi := \Phi z$ is the solution of

$$\left\{ \begin{array}{ccc} \dot{\xi} & = & A_2 \xi + B_2 \Psi u, \\ \xi(0) & = & \Phi z^0. \end{array} \right.$$

In other words,

$$\Xi_{\Sigma(A_2,B_2)}(\Psi u, \Phi z^0) = \Phi \Xi_{\Sigma(A_1,B_1)}(u, z^0).$$

From the this formula we may inspect how the morphisms of **CS** transport the controllability.

Proposition 2.6.2.3. *Assume that $(\Phi, \Psi) : \Sigma(A_1, B_1) \to \Sigma(A_2, B_2)$ is a* **wpCS***-morphism and fix $0 < T < \infty$.*

1. If $\Phi : X_1 \to X_2$ is surjective and $\Sigma(A_1, B_1)$ is exactly controllable in time T, then so is $\Sigma(A_2, B_2)$.

2. If $\Phi : X_1 \to X_2$ is a $\mathbf{Ban}^\infty$*-isomorphism and $\Sigma(A_1, B_1)$ is approximately (resp. null) controllable in time T, then so is $\Sigma(A_2, B_2)$.*

In particular we obtain that the controllability at a fixed time T is unchanged by **CS**-isomorphism.

2.6.3 The category of boundary control systems

Definition 2.6.3.1. *The category of boundary control systems* **BCS** *has objects* (U, Z, X, L, G) *where*

- U, Z, X *are complex Hilbert spaces with* $Z \subset X$ *continuously and as sets,*
- $L \in \mathcal{L}_c(Z; X)$ *and* $G \in \mathcal{L}_c(Z; U)$.

We will denote such an object $\Sigma(L, G)$ *when there is no possible confusion. A morphism* $(U_1, Z_1, X_1, L_1, G_1) \to (U_2, Z_2, X_2, L_2, G_2)$ *is a couple* (Φ, Ψ) *where*

- $\Phi \in \mathcal{L}_c(X_1; X_2)$ *is such that* $\Phi(Z_1) \subset Z_2$
- $\Psi \in \mathcal{L}_c(U_1; U_2)$,
- *The diagrams*

$$\begin{array}{ccc} Z_1 & \xrightarrow{L_1} & X_1 \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ Z_2 & \xrightarrow{L_2} & X_2 \end{array} \qquad \begin{array}{ccc} Z_1 & \xrightarrow{G_1} & U_1 \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Psi} \\ Z_2 & \xrightarrow{G_2} & X_2 \end{array}$$

  commute in **Set**.

The composition of morphisms is the natural one.

If $\Sigma(L, G)$ is a boundary control system we can define the unbounded operator on X

$$D(A) = \operatorname{Null} G, \quad Az = Lz.$$

At this point it is merely an unbounded operator. We say that a boundary control system $\Sigma(L, G)$ is admissible when it satisfies the hypotheses of Theorem (2.5.0.2), namely when A is closable densely defined and G is surjective. We consider the full sub-category of **BCS** made of these admissible boundary control systems, we call it **admBCS**.

As for **CS** it is straightforward to check that the isomorphisms of **BCS** are the morphisms (Φ, Ψ) such that both $\Phi : X_1 \to X_2$ and $\Psi : U_1 \to U_2$ are bijective. It is then easy to check that two isomorphic boundary control systems are simultaneously admissible.

We now consider the functor

$$F : \left\{ \begin{array}{ccc} \textbf{admBCS} & \longrightarrow & \textbf{CS} \\ (U, Z, X, L, G) & \longmapsto & (U, X, A, B^*) \\ (\Phi, \Psi) & \longmapsto & (\Phi, \Psi) \end{array} \right.$$

where B^* is given by Theorem (2.5.0.2).

Theorem 2.6.3.2. *The functor* F *is faithfull. Moreover if* $F(\Sigma(L, G))$ *is a well-posed control system, for any* $u \in L^2(0, T; U)$ *and* $z_0 \in X$*, if there exists* $z(\cdot) \in C^1([0, T]; X) \cap C([0, T]; Z)$ *classical solution of* (6)*, we have*

$$\forall 0 \le t \le T, \quad z(t) = \Xi_{F(\Sigma(L,G))}(u, z_0)(t).$$

This Theorem motivates the understanding of boundary control systems as control systems, and we will always assimilate the mathematical object $\Sigma(L, G)$ with $\Sigma(A, B^*)$. In particular we won't define again the notions of well-posedness, the input-output map, or the controllability of a boundary control systems: those are already defined in the super-category **CS**.

Remark 2.6.3.3. *It is possible to make the definition of F independent of the axiom of the choice, defining it on objects by its graph*

$$\{((U, Z, X, L, G), (U, X, A, \Gamma)) : \exists D(A^*)^*, \quad \exists B \in \mathcal{L}_c(U; D(A^*)^*), \quad \Gamma = B^*, \quad L = \hat{A} + BG\}.$$

Remark 2.6.3.4. *The question of the functor F being full is not clear but interesting. We leave it as an open question.*

At this point we have two notions of boundary control system, the one from the corresponding Subsection of this document (taken from [25]); and the one of this Subsection. In order not to make any confusion we will always refer to a boundary control system as in this Subsection as a **BCS** object.

3 Well posedness of the system

In this Section we adress the well-posedness of (2).

3.1 Choice of the functionnal setting

We have to choose the objects U, Z, X, L, G so that the hypotheses of Theorem (2.4.1.3) are met with the induced operators A, B^*. We will choose the most natural functionnal setting for (2). Let us first introduce some notations: for any $-\infty < a < b < +\infty$ we let

$$H^1_{(a)}(a,b) = \{u \in H^1(a,b) : u(a) = 0\},$$

and we observe that for any $u \in H^1_{(a)}$

$$|u(x)| = |u(x) - u(a)| = \left|\int_x^a u'(t)dt\right| \leq c\int_a^b |u'(t)|^2 dt.$$

So on the set $H^1_{(a)}(a,b)$ the H^1 norm is equivalent to

$$\|u\|^2_{H^1_{(a)}} = \int_a^b |u'(t)|^2 dt.$$

We then endow $H^1_{(a)}$ with the associated Hilbert space structure. We will also denote

$$H^2_{NH}(a,b) = \{u \in H^2(a,b) : u_x(a) = u_x(b) = 0\}, \quad H^2_{(a)} = \{u \in H^2(a,b) : u_x(a) = 0\}.$$

We let

$$X_{D_1,D_2} = H^1_{(-D_1)}(-D_1,0) \times L^2(-D_1,0) \times L^2(0,D_2)$$

be the state space, and

$$Z_{D_1,D_2} = \left\{(u,v,p) \in (H^2 \cap H^1_{(-D_1)})(-D_1,0) \times H^1_{(-D_1)}(-D_1,0) \times H^2(0,D_2) : u_x(0) = p(0), \quad p_x(0) = 0\right\},$$

endowed with the subspace Hilbert structure, on which we define the operator

$$L\begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} v \\ c^2 u_{xx} \\ dp_{xx} \end{pmatrix}.$$

Let also

$$G\begin{pmatrix} u \\ v \\ p \end{pmatrix} = p_x(D_2), \quad U = \mathbb{C}.$$

At this point, the modelization data $(\mathbb{C}, Z_{D_1,D_2}, X_{D_1,D_2}, L, G)$ makes a boundary control system (a **BCS** object) on $(U, Z_{D_1,D_2}, X_{D_1,D_2})$, that we call $\Sigma(c,d,D_1,D_2)$. We will first reduce the study to the case $c = d = 1$.

Lemma 3.1.0.1. *Fix the constants* $c, d, D_1, D_2 > 0$, *then* $\Sigma(c,d,D_1,D_2)$ *is equivalent to* $\Sigma\left(1, 1, cD_1, \sqrt{d}D_2\right)$ *in* **BCS**.

Proof. Consider the maps

$$\Phi : \left\{ \begin{array}{ccc} (u,v,p) & \longmapsto & (u(\cdot/c), v(\cdot/c), p(\cdot/\sqrt{d})) \\ Z_{D_1,D_2} & \longrightarrow & Z_{cD_1,\sqrt{d}D_2} \end{array} \right. , \quad \Psi = \sqrt{d}\,\mathrm{Id}_{\mathbb{C}}\,.$$

It is clear that (Φ, Ψ) meets the 2 first points of the definition of a **BCS** morphism with

$$X_2 = X_{cD_1,\sqrt{d}D_2},$$

and that Φ and Ψ are both $\mathbf{Ban}^{\infty}$-isomorphisms. Therefore, $\Sigma(c, d, D_1, D_2)$ is equivalent to the object of **BCS**

$$\Sigma := (\mathbb{C}, Z_{cD_1,\sqrt{d}D_2}, X_{cD_1,\sqrt{d}D_2}, \Phi L \Phi^{-1}, \sqrt{d} G \Phi^{-1}).$$

We observe that

$$\Phi L \Phi^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{d^2}{dx^2} & 0 & 0 \\ 0 & 0 & \frac{d^2}{dx^2} \end{pmatrix}, \quad \Psi G \Phi^{-1}(u,v,p) = p_x(0)$$

and therefore

$$\Sigma(c,d,D_1,D_2) \sim \Sigma = \Sigma\left(1, 1, cD_1, \sqrt{d}D_2\right).$$

□

In view of this Lemma, we consider the system

$$\left\{ \begin{array}{rcll} u_{tt}(t,x) & = & u_{xx}(t,x), & -D_1 < x < 0, \\ u_x(t,0) & = & p(t,0), & \\ u(t,-D_1) & = & 0, & \\ p_t(t,x) & = & p_{xx}(t,x), & 0 < x < D_2, \\ p_x(t,0) & = & 0, & \\ p_x(t,D_2) & = & q(t), & \end{array} \right. \tag{14}$$

with arbitrary $D_1, D_2 > 0$. Consider the state space

$$X = H^1_{(-D_1)}(-D_1, 0) \times L^2(-D_1, 0) \times L^2(0, D_2),$$

together with

$$Z = \left\{(u,v,p) \in (H^2 \cap H^1_{(-D_1)})(-D_1,0) \times H^1_{(-D_1)}(-D_1,0) \times H^2(0,D_2) : u_x(0) = p(0), \quad p_x(0) = 0\right\}.$$

The control space is $\mathbb{C}$ and the boundary control system is

$$L \begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} \\ p_{xx} \end{pmatrix}, \quad G \begin{pmatrix} u \\ v \\ p \end{pmatrix} = p_x(D_2).$$

We now wish to make the **BCS** object $\Sigma(L, G)$ admissible and such that $F\Sigma(L, G)$ is well-posed (an object of **wpCS**). The operator A is defined by

$$D(A) = \mathrm{Null}\, G = \{(u,v,p) \in H^2 \cap H^1_{(-D_1)} \times H^1_{(-D_1)} \times H^2_{NH} : u_x(0) = p(0)\}, \quad A \begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} \\ p_{xx} \end{pmatrix}.$$

In view of Theorem (2.4.1.3), the system (14) is well-posed as soon as

1. G is surjective,
2. A is a generator on X,
3. B is an admissible contol operator.

The rest of this Section is devoted to the obtention of these facts.

Proposition 3.1.0.2. *The map G is surjective and the set* $\operatorname{Null} G \subset X$ *is dense.*

Proof. Since G is linear and its codomain has dimension 1, it is enough to check that $G \neq 0$. To this aim, we introduce $p \in C^\infty[0, D_2]$ such that $p(0) = p_x(0) = 0$ and $p_x(D_2) = 1$. Then $(0, 0, p) \in Z$ and $G(u, v, p) = 1$.

To check density fix $(u, v, p) \in X$ and consider any function ϕ such that

$$\begin{cases} \phi & \in & C_c^\infty(-D_1, 0], \\ \phi(0) & = & 1, \\ \phi_x(0) & = & 0. \end{cases}$$

Then consider approximations

$$C_c^\infty(-D_1, 0) \ni v_j \xrightarrow[j\to\infty]{L^2(-D_1,0)} v, \quad C_c^\infty(0, D_2) \ni p_j \xrightarrow[j\to\infty]{L^2(0,D_2)} p$$

and observe that

$$u - u(0)\phi \in H_0^1(-D_1, 0).$$

So there is a sequence (u_j) of $C_c^\infty(-D_1, 0)$ that goes to $u - u(0)\phi$ in the H^1 norm and moreover it is easy to check

$$\begin{pmatrix} u_j + u(0)\phi \\ v_j \\ p_j \end{pmatrix} \in \operatorname{Null} G$$

so that $\operatorname{Null} G$ is dense in X. □

3.2 Spectral analysis

We will investigate the spectrum of A and show that the eigenvectors of A (properly rescaled) form a Riesz basis of X.

3.2.1 Spectrum of A

Lemma 3.2.1.1. *(Spectrum of A)*
The operator A has point spectrum

$$\sigma_p(A) = \{\pm\lambda_m^h : m \in \mathbb{N}\} \cup \{\lambda_n^p : n \in \mathbb{N}\}, \quad \lambda_m^h := i\left(m + \frac{1}{2}\right)\frac{\pi}{D_1}, \quad \lambda_n^p = -\left(\frac{n\pi}{D_2}\right)^2,$$

and all these eigenvalues are simple.

For any $m \in \mathbb{N}$, $\mathrm{Null}(\lambda_m^h - A)$ *is generated by* $V_m^{h,+} = (u_m^h, \lambda_m^h u_m^h, 0)$, *where*

$$u_m^h(x) = \sqrt{\frac{2}{D_1}} \cos\left(\left(m + \frac{1}{2}\right)\frac{\pi}{D_1}x\right).$$

For any $m \in \mathbb{N}$, $\mathrm{Null}(-\lambda_m^h - A)$ *is generated by* $V_m^{h,-} = (u_m^h, -\lambda_m^h u_m^h, 0)$.

For any $n \in \mathbb{N}$, $\mathrm{Null}(\lambda_n^p - A)$ *is generated by* $V_n^p = (u_n^p, \lambda_n^p u_n^p, p_n^p)$, *where*

$$p_n^p(x) = \begin{cases} \sqrt{\frac{2}{D_2}} \cos\left(\frac{n\pi}{D_2}x\right), & \text{if} \quad n \neq 0, \\ \frac{1}{\sqrt{D_2}}, & \text{if} \quad n = 0, \end{cases} \tag{15}$$

and u_n^p *is the solution of*

$$\begin{cases} u_{xx} &= (\lambda_n^p)^2 u, \\ u(-D_1) &= 0, \\ u_x(0) &= p_n^p(0). \end{cases} \tag{16}$$

The suprescript "p" stands for parabolic and "h" stands for hyperbolic. We will often refer to these two branches of eigenvalues, and the corresponding eigenvectors, with those adjectives.

Proof. We begin by showing that for all $n \in \mathbb{N}$, (16) has exactly one solution. For $n \in \mathbb{N}^*$ observe that the equation

$$\xi^2 = (\lambda_n^p)^2$$

has two distinct real solutions $\xi = \pm\lambda_n^p$, because $\lambda_n^p \neq 0$. Therefore, (16) has a solution if and only if the following system has a solution in $(\alpha, \beta) \in \mathbb{C}^2$:

$$\begin{cases} \alpha e^{-D_1\lambda_n^p} + \beta e^{D_1\lambda_n^p} &= 0, \\ \alpha\lambda_n^p - \beta\lambda_n^p &= p_n^p(0), \end{cases} \tag{17}$$

where recal that p_n^p is defined by (15). More precisely, the solutions u of (16) are in bijection with the solutions (α, β) of (17). Therefore, to show that (16) has a unique solution is equivalent to showing that the system (17) has a unique solution. The determinant of the latter is

$$\begin{vmatrix} e^{-D_1\lambda_n^p} & e^{D_1\lambda_n^p} \\ \lambda_n^p & -\lambda_n^p \end{vmatrix} = -\lambda_n^p(e^{-D_1\lambda_n^p} + e^{D_1\lambda_n^p})$$

which is non zero, hence (17) has a unique solution. In case $n = 0$ we have $\lambda_0^p = 0$ and (16) becomes

$$\begin{cases} u_{xx} &= 0, \\ u(-D_1) &= 0, \\ u_x(0) &= \frac{1}{\sqrt{D_2}}, \end{cases}$$

which has the unique solution

$$u(x) = \frac{1}{\sqrt{D_2}}(x + D_1).$$

Now we reason by analysis and synthesis to show the claimed point spectrum and eigenspaces. Assume first that there exists $\lambda \in \sigma_p(A)$, let $V = (u, v, p)$ be an associated eigenvector. We will distinguish two cases.

If $p \neq 0$, there exists a non-zero solution of

$$\begin{cases} p_{xx} &= \lambda p, \\ p_x(D_2) &= 0, \\ p_x(0) &= 0, \end{cases}$$

hence there exists $n \in \mathbb{N}$ such that

$$\lambda = -\left(\frac{n\pi}{D_2}\right)^2 = \lambda_n^p,$$

and, up to a rescaling of V,

$$p(x) = \begin{cases} \sqrt{\frac{2}{D_2}} \cos\left(\frac{n\pi}{D_2} x\right), & \text{if} \quad n \neq 0, \\ \frac{1}{\sqrt{D_2}}, & \text{if} \quad n = 0, \end{cases}$$
$$= p_n^p(x).$$

Therefore, u solves (16) and $v = \lambda u$.

If $p = 0$ then $u \neq 0$ (otherwise $V = 0$) and u solves

$$\begin{cases} u_{xx} &= \lambda^2 u \\ u(-D_1) &= 0 \\ u_x(0) &= 0 \end{cases}$$

hence there exists $m \in \mathbb{N}$ such that

$$\lambda^2 = -\left(\left(m + \frac{1}{2}\right)\frac{\pi}{D_1}\right)^2$$

and

$$\lambda = \pm i \left(m + \frac{1}{2}\right)\frac{\pi}{D_1} = \pm \lambda_m^h.$$

In any cases $\lambda = \lambda_m^h$ or $\lambda = -\lambda_m^h$ we have (up to a rescaling of V) $u = u_m^h$ and $v = \pm\lambda_m^h u_m^h$, as required.

Up to now we have shown that

$$\sigma_p(A) \subset \{\pm\lambda_m^h : m \in \mathbb{N}\} \cup \{\lambda_n^p : n \in \mathbb{N}\}$$

and that for any

$$\lambda \in \sigma_p(A) \cap \left(\{\pm\lambda_m^h : m \in \mathbb{N}\} \cup \{\lambda_n^p : n \in \mathbb{N}\}\right),$$

the claimed vector line is well defined (because (16) then has a unique solution) and the corresponding eigenspace is contained in the claimed vector line.

To conclude the proof we are left to show that for any

$$\lambda \in \{\pm\lambda_m^h : m \in \mathbb{N}\} \cup \{\lambda_n^p : n \in \mathbb{N}\},$$

the claimed generator (*i.e.* $V_m^{h,\pm}$ or V_n^p depending on λ) is indeed an eigenvector (associated to λ). At this point this is trivial. □

3.2.2 Bases in Hilbert spaces

Prior to show that the eigenvectors of A, properly rescaled, make a Riesz basis of X we recall some general facts concerning the theory of bases in Hilbert spaces. In what follows X is an arbitrary Hilbert space.

Definition 3.2.2.1. *A Riesz basis of X is a sequence $(e_n)_{n=0}^{\infty}$ of X which is the image of a Hilbert basis of X by a bounded invertible linear map $X \to X$.*

Definition 3.2.2.2. *A sequence $(e_n)_{n=0}^{\infty}$ of X is said ω-independent if for any complex sequence $(c_n)_{n=0}^{\infty}$, if*

$$0 < \sum_{n=0}^{\infty} |c_n|^2 \|e_n\|_X^2 < \infty, \tag{18}$$

then $\sum c_n e_n$ cannot converge to 0.

Remark 3.2.2.3. *In case the sequence $(e_n)_{n=0}^{\infty}$ is almost normalized, i.e.*

$$0 < \inf_n \|e_n\|_X \leq \sup_n \|e_n\|_X < \infty,$$

the condition (18) *is equivalent to*

$$0 < \sum_{n=0}^{\infty} |c_n|^2 < \infty.$$

Thus, ω-independence boils down to: for any $(c_n)_{n=0}^{\infty} \in \ell^2(\mathbb{N})$, if the series $\sum c_n e_n$ converges to 0, then (c_n) is the null series.

Definition 3.2.2.4. *Two sequences $(e_n)_{n=0}^{\infty}$ and $(f_n)_{n=0}^{\infty}$ of X are said quadratically close if*

$$\sum_{n=0}^{\infty} \|e_n - f_n\|_X^2 < \infty.$$

A usefull result to prove that a given sequence is a Riesz basis is the following.

Theorem 3.2.2.5. *([10, Theorem 2.3 of Chapter 6])*
If $(e_n)_{n=0}^{\infty}$ is a sequence of X that is ω-independent and quadratically close to a Riesz basis of X, then $(e_n)_{n=0}^{\infty}$ is itself a Riesz basis of X.

Up to this point a basis of X is a *sequence*, yet the spectrum of A is naturally indexed by $\{1, 2, 3\} \times \mathbb{N}$ as it has three (countable infinite) branches. This motivates the adaptation of the above tools to families of vectors. Until the end of this Subsubsection we fix I a countable infinite set.

Definition 3.2.2.6. *A family $(e_i)_{i\in I}$ is a Riesz basis of X if there exists some $\varphi : \mathbb{N} \to I$ bijection which makes $(e_{\varphi(n)})_{n=0}^{\infty}$ a Riesz basis of X.*

It is clear that if $(e_i)_{i\in I}$ is a Riesz basis of X, then for *any* $\varphi : \mathbb{N} \to I$ bijection the sequence $(e_{\varphi(n)})_{n=0}^{\infty}$ is a Riesz basis of X.

We can then adapt all the above theory for families of vectors.

Definition 3.2.2.7. *We say that $(e_i)_{i\in I}$ is ω-independent if for any family of scalars $(c_i)_{i\in I}$, for any bijection $\varphi : \mathbb{N} \to I$, if*

$$0 < \sum_{n=0}^{\infty} |c_{\varphi(n)}|^2 \|e_{\varphi(n)}\|_X^2 < \infty$$

then $\sum c_{\varphi(n)} e_{\varphi(n)}$ cannot converge to 0.

Definition 3.2.2.8. *Two families $(e_i)_{i\in I}$ and $(f_i)_{i\in I}$ of X are quadratically close if*

$$\sum_{i\in I} \|e_i - f_i\|_X^2 < \infty.$$

Corollary 3.2.2.9. *The family $(e_i)_{i\in I}$ is a Riesz basis of X if and only if it is ω-independent and quadratically close to a Riesz basis of X.*

3.2.3 Riesz basis property for the eigenvectors of A

We look for equivalents of the norm of these eigenvectors, we first deal with the parabolic component:

$$\|V_n^p\|_X^2 = \|u_n^p\|_{H^1_{(-D_1)}}^2 + \|\lambda_n^p u_n^p\|_{L^2}^2 + \|p_n^p\|_{L^2}^2 = \left\| \frac{d}{dx} u_n^p \right\|_{L^2}^2 + |\lambda_n^p|^2 \|u_n^p\|_{L^2}^2 + 1$$

where an easy computation brings a closed formula for u_n^p for $n \geq 1$ (recall that this is the solution of (16))

$$u_n^p(x) = \frac{1}{\lambda_n^p} \sqrt{\frac{2}{D_2}} \left(\frac{e^{\lambda_n^p x}}{1 + e^{-2D_1\lambda_n^p}} - \frac{e^{-\lambda_n^p x}}{e^{2D_1\lambda_n^p} + 1} \right).$$

Now write

$$\alpha_n(x) = \frac{e^{\lambda_n^p x}}{1 + e^{-2D_1\lambda_n^p}}, \quad \beta_n(x) = \frac{e^{-\lambda_n^p x}}{e^{2D_1\lambda_n^p} + 1}$$

and compute

$$\|\alpha_n\|_{L^2}^2 = \frac{1}{|1 + e^{-2D_1\lambda_n^p}|^2} \frac{1 - e^{-2D_1\lambda_n^p}}{2\lambda_n^p}.$$

Remembering that $\lambda_n^p \to -\infty$ we get

$$e^{-2D_1\lambda_n^p} \to \infty$$

hence

$$\|\alpha_n\|_{L^2}^2 \sim \frac{1}{2|\lambda_n^p|e^{-2D_1\lambda_n^p}}$$

which vanishes. Similarly we obtain

$$\|\beta_n\|_{L^2}^2 \sim \frac{1}{2|\lambda_n^p|}$$

which also vanishes. Note that

$$\|\alpha_n\|_{L^2}^2 = o(\|\beta_n\|_{L^2}^2)$$

hence by Cauchy-Schwartz

$$\|\alpha_n - \beta_n\|_{L^2}^2 = \|\alpha_n\|_{L^2}^2 - 2\Re(\alpha_n, \beta_n)_{L^2} + \|\beta_n\|_{L^2}^2 \sim \|\beta_n\|_{L^2}^2 \sim \frac{1}{2|\lambda_n^p|}.$$

Then

$$\|u_n^p\|_{L^2}^2 = \frac{1}{|\lambda_n^p|^2}\frac{2}{D_2}\|\alpha_n - \beta_n\|_{L^2}^2 \sim \frac{1}{D_2}\frac{1}{|\lambda_n^p|^3} \tag{19}$$

which vanishes. The remaining computations for the parabolic component are now easy:

$$\begin{aligned}\left\|\frac{d}{dx}u_n^p\right\|_{L^2}^2 &= \left|\frac{1}{\lambda_n^p}\sqrt{\frac{2}{D_2}}\right|^2 \left\|\frac{\lambda_n^p e^{\lambda_n^p x}}{1+e^{-2D_1\lambda_n^p}} + \frac{\lambda_n^p e^{-\lambda_n^p x}}{e^{2D_1\lambda_n^p}+1}\right\|_{L^2}^2 \\ &= \frac{2}{D_2}\|\alpha_n + \beta_n\|_{L^2}^2 \\ &\sim \frac{1}{D_2}\frac{1}{|\lambda_n^p|} \end{aligned} \tag{20}$$

which also vanishes, hence

$$\|V_n^p\|_X \sim 1. \tag{21}$$

Now for the hyperbolic eigenvectors we readily obtain

$$\|V_m^{h,\pm}\|_X^2 = 2|\lambda_m^h|^2. \tag{22}$$

Then define

$$\forall n \in \mathbb{N}, \quad V_{1,n} = V_n^p, \quad V_{2,n} = \frac{1}{\lambda_n^h}V_n^{h,+}, \quad V_{3,n} = \frac{1}{\lambda_n^h}V_n^{h,-}$$

and consider the candidate family to be a Riesz basis $(V_i)_{i\in\{1,2,3\}\times\mathbb{N}}$. Observe that in view of the asymptotics (21) and (22), and since the vectors $V_n^{h,\pm}, V_n^p$ never vanish, the family $(V_i)_{i\in\{1,2,3\}\times\mathbb{N}}$ is almost normalized.

Theorem 3.2.3.1. *The family* $(V_i)_{i\in\{1,2,3\}\times\mathbb{N}}$ *is a Riesz basis of* X*.*

Proof. Step 1: We show that $(V_i)_{i\in\{1,2,3\}\times\mathbb{N}}$ is ω-independent.

We fix $\phi : \mathbb{N} \to \{1,2,3\} \times \mathbb{N}$ bijective, we assume that for some sequence $(\alpha_n)_{n=0}^\infty \in \ell^2(\mathbb{N})$ there holds

$$\sum_{n=0}^{N} \alpha_n V_{\phi(n)} \xrightarrow[N\to\infty]{X} 0,$$

and we show that $(\alpha_n)_{n=0}^{\infty}$ is constant to 0. Note that this is enough in view of Remark (3.2.2.3). We explicitly write down for fixed $N \in \mathbb{N}$,

$$\sum_{n=0}^{N} \alpha_n V_{\phi(n)} = \sum_{\substack{n=0\\ \phi(n)_1=1}}^{N} \alpha_n \begin{pmatrix} u^p_{\phi(n)_2} \\ \lambda^p_{\phi(n)_2} u^p_{\phi(n)_2} \\ p^p_{\phi(n)_2} \end{pmatrix} + \sum_{\substack{n=0\\ \phi(n)_1=2}}^{N} \alpha_n \begin{pmatrix} u^h_{\phi(n)_2}/\lambda^h_{\phi(n)_2} \\ u^h_{\phi(n)_2} \\ 0 \end{pmatrix} + \sum_{\substack{n=0\\ \phi(n)_1=3}}^{N} \alpha_n \begin{pmatrix} u^h_{\phi(n)_2}/\lambda^h_{\phi(n)_2} \\ -u^h_{\phi(n)_2} \\ 0 \end{pmatrix}. \tag{23}$$

Take the third coordinate of the above sequence and send N to $+\infty$ to obtain

$$\sum_{\substack{n=0\\ \phi(n)_1=1}}^{N} \alpha_n p^p_{\phi(n)_2} \xrightarrow[N\to\infty]{L^2(0,D_2)} 0.$$

Since $(p^p_n)_{n=0}^{\infty}$ is a Hilbert basis of $L^2(0, D_2)$, we obtain

$$\forall n \in \mathbb{N}, \quad \phi(n)_1 = 1 \Longrightarrow \alpha_n = 0,$$

which means that the parabolic component of $(\alpha_n)_{n=0}^{\infty}$ vanishes. Therefore (23) becomes

$$\forall N \in \mathbb{N}, \quad \sum_{n=0}^{N} \alpha_n V_{\phi(n)} = \sum_{\substack{n=0\\ \phi(n)_1=2}}^{N} \alpha_n \begin{pmatrix} u^h_{\phi(n)_2}/\lambda^h_{\phi(n)_2} \\ u^h_{\phi(n)_2} \\ 0 \end{pmatrix} + \sum_{\substack{n=0\\ \phi(n)_1=3}}^{N} \alpha_n \begin{pmatrix} u^h_{\phi(n)_2}/\lambda^h_{\phi(n)_2} \\ -u^h_{\phi(n)_2} \\ 0 \end{pmatrix}.$$

Further for any $N \in \mathbb{N}$, the two first coordinate of the right hand side of the above equation are given by

$$\sum_{\substack{n=0\\ \phi(n)_1=2}}^{N} \alpha_n \left(\frac{u^h_{\phi(n)_2}}{\lambda^h_{\phi(n)_2}}, u^h_{\phi(n)_2} \right) + \sum_{\substack{n=0\\ \phi(n)_1=3}}^{N} \alpha_n \left(\frac{u^h_{\phi(n)_2}}{\lambda^h_{\phi(n)_2}}, -u^h_{\phi(n)_2} \right) \tag{24}$$

which goes to $(0,0)$ in $H^1_{(-D_1)} \times L^2$. Define the bijective sequences $(p_n)_{n=0}^{\infty}$ and $(q_n)_{n=0}^{\infty}$ of $\mathbb{N}$ by

$$\forall n \in \mathbb{N}, \quad \phi(p_n) = (2, n), \quad \phi(q_n) = (3, n).$$

Fix $n \in \mathbb{N}$, take the L^2-scalar product of the first coordinate of (24) against u^h_n and let $N \to \infty$ to discover

$$\frac{\alpha_{p_n}}{\lambda^h_n} + \frac{\alpha_{q_n}}{\lambda^h_n} = 0,$$

because $(u^h_n)_{n=0}^{\infty}$ is a Hilbert basis of $L^2(-D_1, 0)$. Doing the same with the second coordinate we obtain that

$$\alpha_{p_n} - \alpha_{q_n} = 0,$$

hence the couple $(\alpha_{p_n}, \alpha_{q_n})$ solves

$$\begin{cases} \frac{\alpha_{p_n}}{\lambda^h_n} + \frac{\alpha_{q_n}}{\lambda^h_n} &= 0, \\ \alpha_{p_n} - \alpha_{q_n} &= 0, \end{cases}$$

whence $\alpha_{p_n} = \alpha_{q_n} = 0$. This means that the the hyperbolic components of $(\alpha_n)_{n=0}^{\infty}$ vanishes, hence $(\alpha_n)_{n=0}^{\infty}$ is null, and $(V_i)_{i\in\{1,2,3\}\times\mathbb{N}}$ is ω-independent.

Step 2: We show that $(V_i)_{i\in\{1,2,3\}\times\mathbb{N}}$ is quadratically close to the family

$$\left\{(u_m^h,\lambda_m^h,0):m\in\mathbb{N}\right\}\cup\left\{(u_m^h,-\lambda_m^h,0):m\in\mathbb{N}\right\}\cup\{(0,0,p_n^p):n\in\mathbb{N}\}$$

where the indexation by $\{1,2,3\}\times\mathbb{N}$ is obvious.

To this aim, note that the two hyperbolic branches are equal hence they are quadratically close. Moreover the parabolic branches solve

$$\sum_{n=0}^{\infty}\|V_n^p-(0,0,p_n^p)\|_X^2=\sum_{n=0}^{\infty}\|u_n^p\|_{H^1_{(-D_1)}}^2+\|\lambda_n^p u_n^p\|_{L^2}^2$$

where, in view of the asymptotics (19) and (20),

$$\|u_n^p\|_{H^1_{(-D_1)}}^2\sim\frac{1}{D_2}\frac{1}{|\lambda_n^p|},\quad \|\lambda_n^p u_n^p\|_{L^2}^2\sim\frac{1}{D_2}\frac{1}{|\lambda_n^p|},$$

which are summable, hence the quadratic closedness.

Step 3: We show that the family

$$\left\{\left(\frac{u_m^h}{\lambda_m^h},u_m^h,0\right):m\in\mathbb{N}\right\}\cup\left\{\left(\frac{u_m^h}{\lambda_m^h},-u_m^h,0\right):m\in\mathbb{N}\right\}\cup\{(0,0,p_n^p):n\in\mathbb{N}\}$$

is a Riesz basis of X.

We first reduce the problem as follows. Observe that in arbitrary Hilbert spaces H_1,H_2, if $(v_n)_{n=0}^{\infty}$ (resp. $(w_n)_{n=0}^{\infty}$) is a Riesz basis of H_1 (resp. of H_2), then the family

$$\{(v_n,0):n\in\mathbb{N}\}\cup\{(0,w_n):n\in\mathbb{N}\}$$

is a Riesz basis of $H_1\times H_2$. Thus, to show the claim it is enough to show that

$$\left\{\left(\frac{u_m^h}{\lambda_m^h},u_m^h\right):m\in\mathbb{N}\right\}\cup\left\{\left(\frac{u_m^h}{\lambda_m^h},-u_m^h\right):m\in\mathbb{N}\right\}$$

is a Riesz basis of $H^1_{(-D_1)}\times L^2$, and that $(p_n^p)_{n=0}^{\infty}$ is a Riesz basis of L^2. The second point is trivial as $(p_n^p)_{n=0}^{\infty}$ is a Hilbert basis of L^2. Now for the first point, we will use a characterization of Riesz bases, namely we will show that the family is total and that there exists $C_1,C_2>0$ constants such that for any $N\in\mathbb{N}$ and $\alpha_0^{\pm},...,\alpha_N^{\pm}\in\mathbb{C}$, there holds the frame type inequalities

$$C_1\sum_{m=0}^{N}\left(\left|\alpha_m^+\right|^2+\left|\alpha_m^+\right|^2\right)\le\left\|\sum_{m=0}^{N}\left\{\alpha_m^+\left(\frac{u_m^h}{\lambda_m^h},u_m^h\right)+\alpha_m^-\left(\frac{u_m^h}{\lambda_m^h},-u_m^h\right)\right\}\right\|_{H^1_{(-D_1)}\times L^2}^2 \tag{25}$$

$$\le C_2\sum_{m=0}^{N}\left(\left|\alpha_m^+\right|^2+\left|\alpha_m^+\right|^2\right). \tag{26}$$

This characterization is classical and we refer to [3, Theorem 3.6.6] for a proof. Let us first show that the family is total, let (u, v) be in its orthogonal, we then fix $(\alpha_m)_{m=0}^{\infty}, (\beta_m)_{m=0}^{\infty}$ the two square summable sequences such that

$$u = \sum_{m=0}^{\infty} \alpha_m u_m^h, \quad v = \sum_{m=0}^{\infty} \beta_m u_m^h,$$

where the two series converge simply in L^2, which is possible as $(u_m^h)_{m=0}^{\infty}$ is a Hilbert basis of L^2. Actually, because $u \in H^1_{(-D_1)}$ we even have better: the series $\sum \alpha_m u_m^h$ converges in $H^1_{(-D_1)}$. Therefore, for any $m \in \mathbb{N}$ we obtain

$$\begin{aligned}
0 &= \left((u, v), \left(\frac{u_m^h}{\lambda_m^h}, u_m^h \right) \right)_{H^1_{(-D_1)} \times L^2} \\
&= \left(u_x, \frac{d}{dx} \frac{u_m^h}{\lambda_m^h} \right)_{L^2} + (v, u_m^h)_{L^2} \\
&= \sum_{n=0}^{\infty} \frac{\alpha_n}{\lambda_m^h} \left(\frac{d}{dx} u_n^h, \frac{d}{dx} u_m^h \right)_{L^2} + \beta_m \\
&= -\lambda_m^h \alpha_m + \beta_m
\end{aligned}$$

where we have used that the sequence $\left(\frac{d}{dx} u_n^h\right)_{n=0}^{\infty}$ is orthogonal in L^2 with

$$\left\| \frac{d}{dx} u_n^h \right\|_{L^2}^2 = \left[\left(n + \frac{1}{2} \right) \frac{\pi}{D_1} \right]^2 = -(\lambda_n^h)^2.$$

Similarly, using that (u, v) is also orthogonal to $\left(\frac{u_m^h}{\lambda_m^h}, -u_m^h \right)$ we obtain

$$-\lambda_m^h \alpha_m - \beta_m = 0,$$

hence

$$\alpha_m = \beta_m = 0,$$

whence the totality of the family, since $m \in \mathbb{N}$ was arbitrary.

It remains only to show the frame-type inequalities (25)-(26), to this aim we compute

$$\left\|\sum_{m=0}^{N}\left\{\alpha_m^+\left(\frac{u_m^h}{\lambda_m^h},u_m^h\right)\ +\ \alpha_m^-\left(\frac{u_m^h}{\lambda_m^h},-u_m^h\right)\right\}\right\|^2_{H^1_{(-D_1)}\times L^2}=$$

$$=\left\|\left(\sum_{m=0}^{N}\left(\frac{\alpha_m^+}{\lambda_m}+\frac{\alpha_m^-}{\lambda_m}\right)u_m^h,\sum_{m=0}^{N}(\alpha_m^+-\alpha_m^-)u_m^h\right)\right\|^2_{H^1_{(-D_1)}\times L^2}$$

$$=\left\|\sum_{m=0}^{N}\left(\frac{\alpha_m^+}{\lambda_m}+\frac{\alpha_m^-}{\lambda_m}\right)\frac{d}{dx}u_m^h\right\|^2_{L^2}+\left\|\sum_{m=0}^{N}(\alpha_m^+-\alpha_m^-)u_m^h\right\|^2_{L^2}$$

$$=\sum_{m=0}^{N}\left|\frac{\alpha_m^+}{\lambda_m}+\frac{\alpha_m^-}{\lambda_m}\right|^2|\lambda_m^h|^2+\sum_{m=0}^{N}|\alpha_m^+-\alpha_m^-|^2$$

$$=2\sum_{m=0}^{N}(|\alpha_m^+|^2+|\alpha_m^-|^2),$$

which shows that (25)-(26) hold with $C_1=C_2=2$. □

3.3 Semi-group generation

In this Subsection we show that A generates a C_0 semi-group on X. We note that with our choices for $(X,D(A),A)$ it is not trivial that A is a generator. Indeed we do not know if the operator is dissipative in view of

$$\forall\begin{pmatrix}u\\v\\p\end{pmatrix}\in D(A),\quad\left(A\begin{pmatrix}u\\v\\p\end{pmatrix},\begin{pmatrix}u\\v\\p\end{pmatrix}\right)_X=v(0)p(0)-\int_0^{D_2}p_x^2,$$

where the right hand side seems not bounded with respect to $\|\cdot\|_X$.

In [26] the authors study a boundary control system that is very close to (14), and overcome this lack of dissipativity by differentiating the boundary conditions, implementing the result in the domain of the operator, and augmenting the order of the Sobolev spaces making X. Because of the boundary condition

$$p_x(t,0)=0,$$

if one wants to adapt this method to get dissipativity for A, one also needs to implement the condition

$$u_{xx}(0)=0,$$

to get $(u_x,p)\in H^2(-D_1,D_2)$ (according to the notations of the authors). We will show that A is a generator without adding boundary conditions, using the Riesz spectral structure of A. We call a Riesz spectral operator on X a closed densely defined operator whose eigenvalues are simple, with finitely many accumulation point, and for which there exists a Riesz basis of X made of eigenvectors. We refer to [5, Section 3.2] for a proof of the following Theorem.

Theorem 3.3.0.1. *Let A be a Riesz spectral operator on X. Then A is a generator if and only if*

$$\sup_{\lambda \in \sigma_p(A)} \Re\lambda < \infty.$$

As a direct consequence of the spectral study of A we obtain the following.

Corollary 3.3.0.2. *The operator A is a generator.*

3.4 Admissibility of B

We begin by deriving a closed formula for $B^* \in \mathcal{L}_c(D(A^*); \mathbb{C})$. To this aim we will first compute A^*.

Lemma 3.4.0.1. *The operator A^* is given by*

$$D(A^*) = \{(f, g, h) \in H^2_{(0)} \cap H^1_{(-D_1)} \times H^1_{(-D_1)} \times H^2_{(D_2)} : g(0) + h_x(0) = 0\},$$

and

$$A^*(f, g, h) = (-g, -f_{xx}, h_{xx}).$$

Proof. We first show that

$$D(A^*) \subset \{(f, g, h) \in H^2_{(0)} \cap H^1_{(-D_1)} \times H^1_{(-D_1)} \times H^2_{(D_2)} : g(0) + h_x(0) = 0\}.$$

Let $(f, g, h) \in D(A^*)$, we *a priori* have

$$(f, g, h) \in X = H^1_{(-D_1)} \times L^2 \times L^2$$

and there exists a constant $c > 0$ such that

$$\forall (u, v, p) \in D(A), \qquad \left| \int_{-D_1}^{0} v_x f_x + \int_{-D_1}^{0} u_{xx} g + \int_0^{D_2} p_{xx} h \right| = |(A(u, v, p), (f, g, h))_X| \leq c \left(\|u\|_{H^1} + \|v\|_{L^2} + \|p\|_{L^2} \right).$$

Let $p \in C_c^\infty(0, D_2)$, we have $(0, 0, p) \in D(A)$ hence the above estimation becomes

$$\left| \int_0^{D_2} p_{xx} h \right| \leq c\|p\|_{L^2}.$$

Therefore we obtain $p \in H^2$ by elliptic regularity (see [15, Lemme A.4]). Furthermore, for any $p \in H^2(0, D_2)$ such that $p(0) = p_x(0) = p_x(D_2) = 0$ we obtain $(0, 0, p) \in D(A)$ so that by integration by part

$$c\|p\|_{L^2} \geq \left| \int_0^{D_2} p_{xx} h \right| = \left| \int_0^{D_2} h_{xx} p - h_x(D_2) p(D_2) \right|.$$

It is then classical to infer that $h_x(D_2) = 0$.

Next for arbitrary $v \in C_c^\infty(-D_1, 0)$, $(0, v, 0) \in D(A)$ hence

$$c\|v\|_{L^2} \geq \left| \int_{-D_1} v_x f_x \right| = \left| -\int_{-D_1}^{0} v_{xx} f \right|.$$

Thus $f_{xx} \in L^2$ and $f \in H^2$. Now for any $v \in H^1_{(-D_1)}$ we get $(0, v, 0) \in D(A)$ and

$$c\|v\|_{L^2} \geq \left| \int_{-D_1} v_x f_x \right| = \left| - \int_{-D_1}^0 v_{xx} f + f_x(0)v(0) \right|.$$

Again it is straightforward to check that necessarily $f_x(0) = 0$.

We now have to be more cautious for g, for any $u \in H^2$ with $u(-D_1) = u_x(0) = 0$ we have $(u, 0, 0) \in D(A)$ hence

$$\left| \int_{-D_1}^0 u_{xx} g \right| \leq c\|u\|_{H^1_{(-D_1)}} = c\|u_x\|_{L^2}. \tag{27}$$

Observe that the map

$$T : \begin{cases} \alpha & \longmapsto & \left[x \mapsto \int_{-D_1}^x \alpha(t)dt \right] \\ H^1_{(0)} & \longrightarrow & H^2_{(0)} \cap H^1_{(-D_1)} \end{cases}$$

is well defined, hence for any $\alpha \in H^1_{(0)}$ we may apply (27) with $u = T\alpha$ to discover

$$\left| \int_{-D_1}^0 \alpha_x g \right| \leq c\|\alpha\|_{L^2}.$$

This shows that $g \in H^1$ and using integration by part, we obtain for any $\alpha \in H^1_{(0)}$ and $u = T\alpha$ that $(u, 0, 0) \in D(A)$ hence by the above estimate

$$c\|\alpha\|_{L^2} \geq \left| \int_{-D_1}^0 \alpha_x g \right| = \left| -g(-D_1)\alpha(-D_1) - \int_{-D_1}^0 g_x \alpha \right|.$$

We then deduce that $g(-D_1) = 0$ as required.

We now obtain that $g(0) + h_x(0) = 0$, we start again from the definition of $(f, g, h) \in D(A^*)$ to obtain, using integration by parts, for any $(u, v, p) \in D(A)$,

$$\begin{aligned} c\left(\|u\|_{H^1} + \|v\|_{L^2} + \|p\|_{L^2}\right) &\geq |(A(u, v, p), (f, g, h))_X| \\ &= \left| \int_{-D_1}^0 v_x f_x + \int_{-D_1}^0 u_{xx} g + \int_0^{D_2} p_{xx} h \right| \\ &= \left| - \int_{-D_1}^0 f_{xx} v + g(0)u_x(0) - \int_{-D_1}^0 g_x u_x + h_x(0)p(0) + \int_0^{D_2} h_{xx} p \right| \\ &= \left| - \int_{-D_1}^0 f_{xx} v - \int_{-D_1}^0 g_x u_x + \int_0^{D_2} h_{xx} p + (g(0) + h_x(0))p(0) \right| \qquad (28) \end{aligned}$$

which shows that indeed $g(0) + h_x(0) = 0$.

Now for any $(f, g, h) \in D(A^*)$ we have

$$(f, g, h) \in \{(f, g, h) \in H^2_{(0)} \cap H^1_{(-D_1)} \times H^1_{(-D_1)} \times H^2_{(D_2)} : g(0) + h_x(0) = 0\}.$$

So for any $(u,v,p) \in D(A)$, using (28) which also holds without absolute values, we obtain

$$(A(u,v,p),(f,g,h))_X = ((-g,-f_{xx},h_{xx}),(u,v,p))_X,$$

so that

$$\forall (f,g,h) \in D(A^*), \quad A^*(f,g,h) = (-g,-f_{xx},h_{xx}).$$

Conversely, if

$$(f,g,h) \in \{(f,g,h) \in H^2_{(0)} \cap H^1_{(-D_1)} \times H^1_{(-D_1)} \times H^2_{(D_2)} : g(0) + h_x(0) = 0\},$$

the equality (28) brings $(f,g,h) \in D(A^*)$, which proves the formula for A^*. ☐

Proposition 3.4.0.2. *We have, for any $(f,g,h) \in D(A^*)$,*

$$B^* \begin{pmatrix} f \\ g \\ h \end{pmatrix} = h(D_2).$$

Proof. In view of (12), we have

$$\begin{aligned} \left(G \begin{pmatrix} f \\ g \\ h \end{pmatrix}, B^* \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right)_{\mathbb{C}} &= \left(L \begin{pmatrix} u \\ v \\ p \end{pmatrix}, \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right)_X - \left(\begin{pmatrix} u \\ v \\ p \end{pmatrix}, A^* \begin{pmatrix} f \\ g \\ h \end{pmatrix} \right)_X \\ &= \overline{h(D_2)} p_x(D_2) \\ &= \left(G \begin{pmatrix} f \\ g \\ h \end{pmatrix}, h(D_2) \right)_{\mathbb{C}} \end{aligned}$$

for all

$$\begin{pmatrix} u \\ v \\ p \end{pmatrix} \in Z, \quad \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in D(A^*).$$

We conclude by surjectivity of G. ☐

Now the admissibility of B can be stated as follows: there exists $c > 0$ such that for all $\varphi_0 = (f_0, g_0, h_0) \in D(A^*)$, the classical solution $\varphi = S_t^* \varphi_0 = (f,g,h)$ of the adjoint system

$$\left\{ \begin{array}{ccc} f_t & = & -g, \\ g_t & = & -f_{xx}, \\ h_t & = & h_{xx}, \\ f(t,-D_1) & = & 0, \\ f_x(t,0) & = & 0, \\ h_x(t,D_2) & = & 0, \\ g(t,0) + h_x(t,0) & = & 0, \\ f(0,x) & = & f_0(x), \\ g(0,x) & = & g_0(x), \\ h(0,x) & = & h_0(x), \end{array} \right. \tag{29}$$

solves

$$\int_0^T |h(t,D_2)|^2 \le c \left(\|f_0\|^2_{H^1_{(-D_1)}} + \|g_0\|^2_{L^2} + \|h_0\|^2_{L^2} \right).$$

Proposition 3.4.0.3. *The operator B is admissible for A.*

Proof. Let (f, g, h) be a classical solution of the above system, then f is a classical solution of the hyperbolic subsystem

$$\begin{cases} f_{tt} &= f_{xx}, \\ f(t, -D_1) &= 0, \\ f_x(t, 0) &= 0, \\ f(0, x) &= f_0(x), \\ f_t(0, x) &= g_0(x). \end{cases} \tag{30}$$

From the hidden regularity for this wave problem (see [4, Theorem 2.53] or Theorem (A.0.0.1) from the Appendix)

$$\|f(\cdot, 0)\|_{H^1(0,T)} \lesssim \|f_0\|_{H^1} + \|g_0\|_{L^2}.$$

Moreover, denoting $q(t) = -f_t(t, 0) \in C[0, T]$ we have h a classical solution of

$$\begin{cases} h_t &= h_{xx}, \\ h_x(t, D_2) &= 0, \\ h_x(t, 0) &= q(t), \\ h(0, x) &= h_0(x). \end{cases} \tag{31}$$

Thus it is enough to show that, for any $q \in L^2(0, T)$ and $h_0 \in H^2_{(0)}$, any strong solution h of (31) is such that

$$\int_0^T |h(t, D_2)|^2 \lesssim \|q(t)\|^2_{L^2(0,T)} + \|h_0\|^2_{L^2(0,D_2)}. \tag{32}$$

We put (31) in the realm of Theorem (2.4.1.3) introducing

$$\mathcal{X} = L^2(0, D_2), \quad \mathcal{L} = \frac{d^2}{dx^2}, \quad \mathcal{Z} = \{h \in H^2 : h_x(0) = 0\}, \quad \mathcal{G}h = h_x(0), \quad \mathcal{U} = \mathbb{C}.$$

It is elementary to check that such objects satisfy the hypotheses of the Theorem (2.4.1.3), with semi group

$$\mathcal{S}_t h = \sum_{n=0}^{\infty} (h, v_n)_{\mathcal{X}} e^{\mu_n t} v_n,$$

where

$$\mu_n = -\left(\frac{n\pi}{D_1}\right)^2,$$

and

$$v_n(x) \propto \begin{cases} \cos(\sqrt{-\mu_n} x), & \text{if} \quad n \in \mathbb{N}^*, \\ 1, & \text{if} \quad n = 0, \end{cases}$$

is the eigenbasis associated to the generator $\mathcal{A}$. It is also elementary to check that the extended semi-group $(\hat{\mathcal{S}}_t)_{t \geq 0}$ on $D(\mathcal{A}^*)^*$ solves

$$\forall h \in D(\mathcal{A}^*)^*, \quad \mathcal{S}_t h = \sum_{n=0}^{\infty} \langle h, v_n \rangle_{D(\mathcal{A}^*)^*, D(\mathcal{A}^*)} e^{\mu_n t} v_n.$$

Moreover the associated in-domain control operator $\mathcal{B}$ is such that $\mathcal{B}^* = \delta_0$ where δ_{x_0} denotes the Dirac mass as x_0. Now it is elementary to obtain (32) when $q = 0$. Therefore, from the extended Duhamel formula we can assume without loss of generality that $h_0 = 0$. In this case

$$\begin{aligned} h(t) &= \int_0^t \hat{\mathcal{S}}_{t-s}\mathcal{B}q(s)ds \\ &= \int_0^t \sum_{n=0}^{\infty} \langle \mathcal{B}q(s), v_n\rangle e^{\mu_n(t-s)} v_n ds \\ &= \int_0^t \sum_{n=0}^{\infty} q(s)v_n(0)e^{\mu_n(t-s)} v_n ds \\ &= \int_0^t q(s)\sum_{n=0}^{\infty} v_n(0)e^{\mu_n(t-s)} v_n ds. \end{aligned}$$

Let

$$\vartheta(x) = \sum_{n=0}^{\infty} x^{n^2} \in C^{\infty}(-1,1),$$

which is known as a Jacobi function. It is known that (see [19, problem 36, Part Two, Chapter 1])

$$\vartheta(x) \underset{1^-}{\sim} \frac{\sqrt{\pi}}{2\sqrt{1-x}}.$$

Then

$$\mu(s) := \sum_{n=0}^{\infty} e^{\mu_n s} = \sum_{n=0}^{\infty} \left(e^{-\left(\frac{\pi}{D_1}\right)^2 s}\right)^{n^2} = \vartheta\left(e^{-\left(\frac{\pi}{D_1}\right)^2 s}\right) \underset{0^+}{\sim} \frac{D_1}{2}\frac{1}{\sqrt{s}}$$

shows that $\mu \in L^1_{loc}[0,\infty)$. In particular, being $(v_n)_{n=0}^{\infty}$ bounded in $C[0, D_2]$,

$$\int_0^t \left\| q(s)\sum_{n=0}^{\infty} v_n(0)e^{\mu_n(t-s)} v_n \right\|_{C[0,D_2]} ds \lesssim \int_0^t |q(s)||\mu(t-s)|ds < \infty$$

whence

$$\int_0^t v_n(0)q(s)\sum_{n=0}^{\infty} e^{\mu_n(t-s)} v_n ds$$

is an integral that converges in $L^1(0,T;C[0,D_2])$. In view of the Hille's theorem for bounded operators (see [6, Proposition 1.2.2]) we obtain

$$h(t, D_2) = \delta_{D_2}h(t) = \int_0^t q(s)\delta_{D_2}\sum_{n=0}^{\infty} v_n(0)e^{\mu_n(t-s)} v_n ds$$

since δ_{D_2} is a linear and continuous functionnal on $C[0, D_2]$. We can then pass δ_{D_2} into the series because, for any fixed $0 < s < t$, it converges in H^2_{NH} (this is an easy computation). Finally we compute

$$\begin{aligned}
\int_0^T |h(t,D_2)|^2 dt &\lesssim \int_0^T \left| \int_0^t \sum_{n=0}^{\infty} q(s) e^{\mu_n (t-s)} ds \right|^2 dt \\
&= \int_0^T \left| \int_0^t q(s)\mu(t-s) ds \right|^2 dt \\
&\leq \int_0^T \int_0^t \mu(\sigma) d\sigma \int_0^t q(s)^2 \mu(t-s) ds dt && \text{(Jensen)} \\
&\leq \left(\int_0^T \mu(\sigma) d\sigma \right) \times \int_0^T \int_s^T q(s)^2 \mu(t-s) dt ds && \text{(Fubini)} \\
&\leq \left(\int_0^T \mu(\sigma) d\sigma \right)^2 \times \int_0^T q(s)^2 ds
\end{aligned}$$

which yields the admissibility of B. □

Remember that up to now we have proved that (14) generates a well-posed boundary control system. However since $D_1, D_2 > 0$ were arbitrary we have the following.

Corollary 3.4.0.4.

1. *For every $D_1, D_2 > 0$, the system* (14) *is a well-posed boundary control system.*
2. *For every $c, d, D_1, D_2 > 0$, the system* (2) *is a well-posed boundary control system.*

4 Controllability of the system

We now tackle the controllability of (14), we acknowledge that the output map formally writes

$$B^* S_t^*(f_0, g_0, h_0) = h(t, D_2).$$

4.1 Null controllability

We will show that (14) is not null controllable, in arbitrary time and with arbitrary Sobolev exponent on the output (see (33) below for a precise statement).

Remember that in view of Theorem (2.4.2.4), null controllability is equivalent to final time observability:

$$\forall (f_0, g_0, h_0) \in D(A^*), \quad \|f(T)\|^2_{H^1_{(-D_1)}} + \|g(T)\|^2_{L^2} + \|h(T)\|^2_{L^2} \lesssim \int_0^T |h(t, D_2)|^2 dt$$

where $(f, g, h) = (f(t), g(t), h(t)) = S_t^*(f_0, g_0, h_0)$ is the solution of

$$\frac{d}{dt}(f, g, h) = A^*(f, g, h), \quad (f(0), g(0), h(0)) = (f_0, g_0, h_0).$$

In [26] the authors study the null controllability for a coupled model very close to (14). To contradict the final time observability they test it with against hyperbolic eigenvectors of A^*, we will follow the same idea.

We recall that A^* is defined on the state space

$$X = H^1_{(-D_1)}(-D_1, 0) \times L^2(-D_1, 0) \times L^2(0, D_2)$$

by

$$D(A^*) = \{(f, g, h) \in H^2 \cap H^1_{(-D_1)} \times H^1_{(-D_1)} \times H^2 : f_x(0) = h_x(D_2) = g(0) + h_x(0) = 0\},$$

and

$$A^* \begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} -g \\ -f_{xx} \\ h_{xx} \end{pmatrix}.$$

Theorem 4.1.0.1. *For any* $0 < T < \infty$*,* $D_1 > 0$*,* $D_2 > 0$ *and* $N \in \mathbb{N}$*, we have*

$$\inf_{\substack{(f_0, g_0, h_0) \in D(A^*) \\ S_T^*(f_0, g_0, h_0) \neq 0}} \frac{\|h(\cdot, D_2)\|_{H^N(0,T)}}{\|S_T^*(f_0, g_0, h_0)\|_X} = 0. \tag{33}$$

Proof. Fix $N \in \mathbb{N}$ and $T, D_1, D_2 > 0$. For any $n \in \mathbb{N}$, we consider

$$\lambda_n = i\left(n + \frac{1}{2}\right)\frac{\pi}{D_1}, \quad f_n(x) = \sqrt{\frac{2}{D_1}} \cos\left(\left(n + \frac{1}{2}\right)\frac{\pi}{D_1} x\right), \quad g_n = -\lambda_n f_n,$$

and

$$h_n(x) = -\sqrt{\frac{2}{D_1}}\xi_n \left(\frac{e^{\xi_n x}}{e^{2\xi_n D_2} - 1} + \frac{e^{-\xi_n x}}{1 - e^{-2\xi_n D_2}} \right), \quad \xi_n = \sqrt{\left(n + \frac{1}{2}\right) \frac{\pi}{D_1}} e^{i\pi/4}.$$

We check that $(f_n, g_n, h_n) \in D(A^*)$ and that $A^*(f_n, g_n, h_n) = \lambda_n(f_n, g_n, h_n)$. First h_n is well defined because

$$e^{\pm 2\xi_n D_2} - 1 = 0 \iff \pm 2\sqrt{\left(n + \frac{1}{2}\right) \frac{\pi}{D_1}} e^{i\pi/4} D_2 \in 2i\pi\mathbb{Z} \iff \sqrt{\left(n + \frac{1}{2}\right) \frac{\pi}{D_1}} e^{i\pi/4} D_2 \in i\pi\mathbb{Z}$$

where the right hand side is not possible considering the arguments of those complex numbers. Further we obviously have

$$f_n, g_n \in H^2(-D_1, 0), \quad g_n \in H^2(0, D_2),$$

and f_n solves

$$\begin{cases} (f_n)_{xx} &= \lambda_n^2 f_n, \\ f_n(-D_1) &= 0, \\ (f_n)_x(0) &= 0. \end{cases}$$

Therefore, to get $(f_n, g_n, h_n) \in D(A^*)$ and $A^*(f_n, g_n, h_n) = \lambda_n(f_n, g_n, h_n)$ we are left to check that h_n solves

$$\begin{cases} (h_n)_{xx} &= \lambda_n h_n, \\ (h_n)_x(0) &= -g_n(0), \\ (h_n)_x(D_1) &= 0. \end{cases}$$

The differential equation comes from $\xi_n^2 = \lambda_n$ and we check the boundary conditions:

$$\begin{aligned} (h_n)_x(0) &= -\sqrt{\frac{2}{D_1}}\xi_n \left(\frac{\xi_n}{e^{2\xi_n D_2} - 1} + \frac{-\xi_n}{1 - e^{-2\xi_n D_2}} \right) \\ &= -\lambda_n \sqrt{\frac{2}{D_1}} \left(\frac{e^{-\xi_n D_2}}{e^{\xi_n D_2} - e^{-\xi_n D_2}} - \frac{e^{\xi_n D_2}}{e^{\xi_n D_2} - e^{-\xi_n D_2}} \right) \\ &= \lambda_n \sqrt{\frac{2}{D_1}} \\ &= -g_n(0), \end{aligned}$$

and

$$\begin{aligned} (h_n)_x(D_2) &= -\sqrt{\frac{2}{D_1}}\xi_n \left(\frac{\xi_n e^{\xi_n D_2}}{e^{2\xi_n D_2} - 1} + \frac{-\xi_n e^{-\xi_n D_2}}{1 - e^{-2\xi_n D_2}} \right) \\ &= -\lambda_n \sqrt{\frac{2}{D_1}} \left(\frac{1}{e^{\xi_n D_2} - e^{-\xi_n D_2}} - \frac{1}{e^{\xi_n D_2} - e^{-\xi_n D_2}} \right) \\ &= 0. \end{aligned}$$

Thus (f_n, g_n, h_n) is indeed an eigenvector of A^* associated to λ_n and

$$\forall t \geq 0, \quad S_t^*(f_n, g_n, h_n) = e^{\lambda_n t}(f_n, g_n, h_n).$$

We will denote

$$\forall t \geq 0, \quad (f_n(t), g_n(t), h_n(t)) = S_t^*(f_n, g_n, h_n),$$

and acknowledge that $f_n(t)$ never vanishes. We will show that

$$\frac{\|h_n(\cdot, D_2)\|^2_{H^N(0,T)}}{\|f_n(T)\|^2_{H^1_{(-D_1)}}} \xrightarrow[n\to\infty]{} 0, \tag{34}$$

which is enough to conclude. On the one hand, being λ_n purely imaginary, we have

$$\|f_n(T)\|^2_{H^1_{(-D_1)}} = \|(f_n)_x\|^2_{L^2} = \left(\left(n + \frac{1}{2}\right)\frac{\pi}{D_1}\right)^2 \sim \left(\frac{n\pi}{D_1}\right)^2.$$

On the other hand,

$$\begin{aligned}
\|h_n(\cdot, D_2)\|^2_{H^N(0,T)} &= \sum_{k=0}^{N} \int_0^T \left|\frac{\partial^k h_n}{\partial t^k}(t, D_2)\right|^2 dt \\
&= \sum_{k=0}^{N} \int_0^T \left|\lambda_n^k e^{\lambda_n t} h_n(D_2)\right|^2 dt \\
&= \sum_{k=0}^{N} |\lambda_n|^{2k} T |h_n(D_2)|^2 \\
&= T|h_n(D_2)|^2 \frac{|\lambda_n|^{2N+2} - 1}{|\lambda_n|^2 - 1} \underset{n\infty}{\sim} T|h_n(D_2)|^2 \left(\frac{n\pi}{D_1}\right)^{2N}.
\end{aligned}$$

We get an equivalent for $|h_n(D_2)|$, prior to do so observe that

$$\Re\xi_n = \sqrt{\left(n + \frac{1}{2}\right)\frac{\pi}{D_1}}\frac{\sqrt{2}}{2} \to \infty,$$

hence

$$e^{-\xi_n D_2} \to 0.$$

Now

$$\begin{aligned}
|h_n(D_2)| &= \sqrt{\frac{2}{D_1}}\sqrt{\left(n + \frac{1}{2}\right)\frac{\pi}{D_1}}\left|\frac{e^{\xi_n D_2}}{e^{2\xi_n D_2} - 1} + \frac{e^{-\xi_n D_2}}{1 - e^{-2\xi_n D_2}}\right| \\
&\underset{n\infty}{\sim} \sqrt{\frac{2}{D_1}}\sqrt{\frac{n\pi}{D_1}} 2|e^{-\xi_n D_2}| \\
&\underset{n\infty}{\sim} 2\frac{\sqrt{2n\pi}}{D_1}\exp\left(-\sqrt{\frac{n\pi}{D_1}}\frac{\sqrt{2}}{2}D_2\right),
\end{aligned}$$

hence

$$\begin{aligned}\frac{\|h_n(\cdot,D_2)\|^2_{H^N(0,T)}}{\|f_n(T)\|^2_{H^1_{(-D_1)}}} &\underset{n\infty}{\sim} \frac{T|h_n(D_2)|^2\left(\frac{n\pi}{D_1}\right)^{2N}}{\left(\frac{n\pi}{D_1}\right)^2}\\ &\underset{n\infty}{\sim} T\left(\frac{n\pi}{D_1}\right)^{2N-2} 8\frac{n\pi}{D_1^2}\exp\left(-\sqrt{\frac{2n\pi}{D_1}}D_2\right)\\ &= \frac{8T}{D_1}\left(\frac{n\pi}{D_1}\right)^{2N-1}\exp\left(-\sqrt{\frac{2n\pi}{D_1}}D_2\right)\\ &\xrightarrow[n\to\infty]{} 0.\end{aligned}$$

□

Corollary 4.1.0.2. *1. For any $T, D_1, D_2 > 0$, the system* (14) *is not null controllable at time T.*

2. For every $c, d, D_1, D_2, T > 0$, the system (2) *is not null controllable at time T. Moreover,* (2) *also satisfies* (33)*.*

Proof. 1. This is an immediate consequence of (33), which is stronger than the contradiction of the initial time observability for the adjoint system:

$$\inf_{\substack{(f_0,g_0,h_0)\in D(A^*)\\ S_T^*(f_0,g_0,h_0)\neq 0}} \frac{\|h(\cdot,D_2)\|_{L^2(0,T)}}{\|S_T^*(f_0,g_0,h_0)\|_X} = 0.$$

2. Let $c, d, D_1, D_2, T > 0$, in view of Lemma (3.1.0.1) we have $\Sigma(c,d,D_1,D_2)$ equivalent to $\Sigma\left(1,1,cD_1,\sqrt{d}D_2\right)$ in **BCS** via

$$\Phi : \left\{\begin{array}{ccc} (u,v,p) & \longmapsto & (u(\cdot/c), v(\cdot/c), p(\cdot/\sqrt{d})) \\ Z_{D_1,D_2} & \longrightarrow & Z_{cD_1,\sqrt{d}D_2}\end{array}\right. \quad , \quad \Psi = \sqrt{d}\,\mathrm{Id}_{\mathbb{C}}\,.$$

Denote $(\mathcal{S}_t)_{t\geq 0}$ the semi-group induced by $\Sigma(c,d,D_1,D_2)$, $\mathcal{A}$ the induced generator and $\mathcal{B}$ the induced control operator. We then have

$$\mathcal{S}_t = \Phi S_t \Phi^{-1}, \quad \mathcal{B} = \Phi B \Psi^{-1},$$

so that for any $\mathcal{V}_0 \in D(\mathcal{A}^*)$, we have

$$\mathcal{S}_T^*\mathcal{V}_0 \neq 0 \Longleftrightarrow S_T^*\Phi^*\mathcal{V}_0 \neq 0.$$

Now for such $\mathcal{V}_0$ we obtain

$$\begin{aligned}\frac{\|\mathcal{B}^*\mathcal{S}_\cdot^*\mathcal{V}_0\|_{H^N(0,T)}}{\|\mathcal{S}_T^*\mathcal{V}_0\|_{X_{D_1,D_2}}} &= \frac{\|(\Psi^{-1})^*B^*S_\cdot^*\Phi^*\mathcal{V}_0\|_{H^N(0,T)}}{\|(\Phi^{-1})^*S_T^*\Phi^*\mathcal{V}_0\|_{X_{D_1,D_2}}}\\ &\leq \|(\Psi^{-1})^*\|_{\mathcal{L}_c(\mathbb{C})}\|\Phi^*\|_{\mathcal{L}_c(X_{cD_1,\sqrt{d}D_2};X_{D_1,D_2})}\frac{\|B^*S_\cdot^*\Phi^*\mathcal{V}_0\|_{H^N(0,T)}}{\|S_T^*\Phi^*\mathcal{V}_0\|_{X_{cD_1,\sqrt{d}D_2}}}.\end{aligned}$$

Finally, using that $\Phi^*\mathcal{S}_T^* = S_T^*\Phi^*$ and that $\Phi^* : D(\mathcal{A}^*) \to D(A^*)$ is a bijection we arrive to

$$\{\Phi^*\mathcal{V}_0 : \mathcal{V}_0 \in D(\mathcal{A}^*), \quad \mathcal{S}_T^*\mathcal{V}_0 \neq 0\} = \{V_0 : V_0 \in D(A^*), \quad S_T^* V_0 \neq 0\},$$

whence

$$\begin{aligned}
\inf_{\substack{\mathcal{V}_0 \in D(\mathcal{A}^*) \\ \mathcal{S}_T^*\mathcal{V}_0 \neq 0}} \frac{\|\mathcal{B}^*\mathcal{S}_\cdot^*\mathcal{V}_0\|_{H^N(0,T)}}{\|\mathcal{S}_T^*\mathcal{V}_0\|_{X_{D_1,D_2}}} &\lesssim \inf_{\substack{\mathcal{V}_0 \in D(\mathcal{A}^*) \\ \mathcal{S}_T^*\mathcal{V}_0 \neq 0}} \frac{\|B^*S_\cdot^*\Phi^*\mathcal{V}_0\|_{H^N(0,T)}}{\|S_T^*\Phi^*\mathcal{V}_0\|_{X_{cD_1,\sqrt{d}D_2}}} \\
&= \inf_{\substack{V_0 \in D(A^*) \\ S_T^* V_0 \neq 0}} \frac{\|B^*S_\cdot^*V_0\|_{H^N(0,T)}}{\|S_T^*V_0\|_{X_{cD_1,\sqrt{d}D_2}}} \\
&= 0.
\end{aligned}$$

□

Coming back to (34), we observe that this vanishing limit is stronger than the mere negation of the null controlability, which is (33). We conclude this Subsection stating precisely this stronger result, to this aim we will briefly introduce additional theoretical background in control systems theory.

Definition 4.1.0.3. *Let V, W be closed and linear subspaces of X, we say that the V-component of $\Sigma(A, B)$ is null controlable at time T with initial data in W if*

$$\forall z_0 \in W, \quad \exists u \in L^2(0,T;U), \quad \Pi_V z(T) = 0,$$

where $\Pi_V : X \to X$ stands for the orthogonal projection onto V.

Note that the concept of "V-component" is equivalent to the introduction of an output operator, giving rise to what is broadly denoted $\Sigma(A, B, \Pi_V)$. Using duality theory it is elementary to obtain the following result.

Proposition 4.1.0.4. *The V-component of the system $\Sigma(A, B)$ is null-controlable at time T with initial data in W if and only if*

$$\exists c > 0, \quad \forall z \in V, \quad \|\Pi_W S_T^* z\|_X \leq c\|F_T^* z\|_{L^2(0,T;U)}. \tag{35}$$

Proof. Assume that the V-component of the system $\Sigma(A, B)$ is null-controlable at time T with initial data in W. We then have for any $z_0 \in W$ the existence of $u \in L^2(0,T;U)$ such that

$$0 = \Pi_V z(T) = \Pi_V S_T z_0 + \Pi_V F_T u.$$

This shows that

$$\operatorname{Range} \Pi_V S_T \Pi_W \subset \operatorname{Range} \Pi_V F_T$$

hence, owing to the Douglas Lemma and being the orthogonal projections self-adjoint operators,

$$\exists c > 0, \quad \forall z \in X, \quad \|\Pi_W S_T^* \Pi_V z\|_X \leq c\|F_T^* \Pi_V z\|_{L^2(0,T;U)}.$$

The last assertion is clearly equivalent to (35), which shows the direct implication.

The proof of the converse implication can be obtained reversing the used arguments since they are in fact all equivalences. □

Now let
$$W = H^1_{(-D_1)} \times \{0\} \times \{0\}, \quad V = \mathrm{Cl}\,\mathrm{Span}\{(f_n, g_n, h_n) : n \in \mathbb{N}\},$$
and observe that V is one of the two branches of hyperbolic eigenvectors of A^*. Indeed, one can show that
$$\sigma_p(A^*) = \sigma_p(A) = \{\pm\lambda^h_m : m \in \mathbb{N}\} \cup \{\lambda^p_n : n \in \mathbb{N}\},$$
and that
$$\mathrm{Null}(\lambda^h_m - A^*) = \mathrm{Span}(f_n, g_n, h_n).$$
Moreover, we say that (35) has a defect of infinite order when it does not hold whenever the $L^2(0,T;U)$ norm is replaced by the $H^N(0,T;U)$ pseudo-norm on $L^2(0,T;U)$.

Corollary 4.1.0.5. *1. For any $T, D_1, D_2 > 0$, the V-component of (14) is not null controllable at time T, with initial data in W. This moreover happens with a defect of infinite order.*

2. For every $c, d, D_1, D_2, T > 0$, the same holds for (2).

4.2 Approximate controllability

Recall that approximate controllability is equivalent to the injectivity of the linear and continuous extension of the map
$$\begin{cases} \varphi & \longmapsto & B^* S_t^* \varphi \\ (D(A^*), \|\cdot\|_X) & \longrightarrow & L^2(0,T;U) \end{cases}$$
We will denote by Θ_T this $\mathcal{L}_c(X; L^2(0,T;U))$ extension.

4.2.1 Unique continuation for a heat problem

The injectivity Θ_T has the following "unique continuation" interpretation: for any $(f_0, g_0, h_0) \in X$, if (f, g, h) is the solution of the adjoint system (29) starting from (f_0, g_0, h_0), then if it solves
$$h(t, D_2) \equiv 0,$$
in $L^2(0,T)$, we have $(f_0, g_0, h_0) = (0,0,0)$. Since the output operator Θ_T a priori bears information about the parabolic component $h = h(t)$, this motivates the the study of the heat problem
$$\begin{cases} h_t & = & h_{xx}, \\ h(t, D_2) & = & 0, \\ h_x(t, D_2) & = & 0, \end{cases} \tag{36}$$
and we wish to show that its solutions are necessarily constant to 0. This is tedious as it is an ill-posed PDE problem. We will use regularity theory for the heat equation in 1D, as summed up in the following result.

Lemma 4.2.1.1. *(Caloric regularity in 1D)*
Let $0 < T, L < \infty$ be fixed, assume that $u \in \mathcal{D}'((0,T) \times (0,L))$ is a distributionnal solution of the heat equation
$$u_t = u_{xx}.$$
Then $u \in C^\infty((0,T)\times(0,L))$ is a classical solution of the heat equation. Moreover, for any $0 < t < T$ we have $u(t, \cdot)$ analytic on $(0, L)$.

Proof. The heat operator

$$\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$$

is hypoelliptic on $\mathbb{R}^2$ (see [18, Sections 8 and 9]). This gives us that from

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u = 0 \in C^\infty((0,T)\times(0,L)),$$

we in fact have

$$u \in C^\infty((0,T)\times(0,L)).$$

Note carefully that this means that u can be assimilated with a $C^\infty((0,T)\times(0,L))$ function, say ϕ, so that $u = T_\phi$ where T_ϕ stands for the distribution associated to the locally integrable function ϕ. But since the distributionnal calculus generalizes the standard calculus we obtain in $\mathcal{D}'((0,T)\times(0,L))$ that

$$T_{\phi_t-\phi_{xx}} = \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) T_\phi = \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) u = 0.$$

We then get $\phi_t - \phi_{xx} = 0$ in $L^1_{loc}((0,T)\times(0,L))$, hence almost everywhere. By continuity, ϕ is a classical solution of the heat equation on $(0,T)\times(0,L)$.

For the analycity in space we refer to [17, Theorem 1, Chapter 6]. □

We can now show the unique continuation property for the parabolic component of the adjoint system (29).

Lemma 4.2.1.2. *Let $(f_0, g_0, h_0) \in X$ be such that $\Theta_T(f_0, g_0, h_0) = 0$. Then $h \equiv 0$.*

Proof. Step 0: We first give a heuristic. Since h solves (36), we obtain that for any $0 < t_0 < T$, the anlytic map $\phi(x) = h(t_0, x)$ is such that all its derivatives at D_2 vanishes, owing to the computation

$$\phi''(D_2) = \frac{\partial^2}{\partial x^2} h(t_0, D_2) = \frac{\partial}{\partial t} h(t_0, D_2) = \frac{\partial}{\partial t} 0 = 0$$

and an induction argument. Then $\phi \equiv 0$ and $h \equiv 0$. However, this requires regularity for h.

Step 1: We extract in (29) the information we will need concerning h.

We keep in mind that h solves the Cauchy problem

$$\left\{\begin{array}{ccc} h_t &=& h_{xx}, \\ h_x(t,0) &=& -f_t(t,0), \\ h_x(t,D_2) &=& 0, \\ h(0,x) &=& h_0(x), \end{array}\right. \tag{37}$$

which does not fit the Hille-Yosida theory (the boundary conditions are not homogeneous), whereas we will need this framework. Therefore we will only use this system at the formal level.

We acknowledge that $h \in C([0,T];L^2(0,D_2))$, $h(0) = h_0$ and that for any $0 \leq t \leq T$, we have

$$\int_0^t h(s)ds \in H^2(0,D_2), \quad \frac{d}{dx}\Big|_{x=D_2} \int_0^t h(s)ds = 0, \quad h(t) - h_0 = \frac{d^2}{dx^2}\int_0^t h(s)ds. \tag{38}$$

Moreover we claim that $h_t = h_{xx}$ in $\mathcal{D}'((0,T)\times(0,D_2))$. Indeed we first obtain it for $(f_0, g_0, h_0) \in D(A^*)$ because then (f,g,h) is a strong solution of (29). Next for $(f_0,g_0,h_0) \in X$ we pick an approximating sequence $((f_0^j, g_0^j, h_0^j))_{j=0}^\infty$ of $D(A^*)$ such that

$$(f_0^j, g_0^j, h_0^j) \xrightarrow[j\to\infty]{X} (f_0, g_0, h_0).$$

In particular we obtain that the solution (f^j, g^j, h^j) of (29) with initial data (f_0^j, g_0^j, h_0^j) solves

$$(f^j, g^j, h^j) \xrightarrow[j\to\infty]{C([0,T];X)} (f, g, h)$$

which is enough to pass to the limit $j \to \infty$ in the weak formulation

$$\forall \varphi \in C_c^\infty((0,T)\times(0,D_2)), \quad \forall j \in \mathbb{N}, \quad -\int_0^T\int_0^{D_2} h^j \varphi_t = \int_0^T\int_0^{D_2} h^j \varphi_{xx}.$$

Thus $h_t = h_{xx}$ in $\mathcal{D}'((0,T)\times(0,D_2))$. By regularity theory, we therefore have (up to a modification on a null set) that h is a $C^\infty((0,T)\times(0,D_2))$ caloric function and $h(t_0,\cdot)$ is analytic on $(0,D_2)$, for any $0 < t_0 < T$.

Step 2: We re-arrange the system (37) so that it fits the Hille-Yosida theory.

We first move away from all the space and time boundaries, except for $x = D_2$ (this is where we will need regularity theory). Fix $\epsilon > 0$ small so that the intervals

$$(\epsilon, T-\epsilon), \quad (\epsilon, D_2)$$

are well defined and non empty. Second we use a lift for h in (37) to exchange the non-homogeneous boundary condition with a source term. We consider the functions

$$\psi(t) = h_x(t,\epsilon) \in C^\infty[\epsilon, T-\epsilon], \quad h_\epsilon(x) = h(\epsilon, x) \in L^2(\epsilon, D_2)$$

which are boundary and initial datum such that formally

$$\begin{cases} h_t &= h_{xx}, \\ h_x(t,\epsilon) &= \psi(t), \\ h_x(t,D_2) &= 0, \\ h(\epsilon,x) &= h_\epsilon(x), \end{cases} \tag{39}$$

on $(\epsilon, T-\epsilon)\times(\epsilon, D_2)$. We consider the lift

$$k(t,x) = \frac{1}{2(\epsilon - D_2)}\psi(t)(x-D_2)^2 \in C^\infty([\epsilon, T-\epsilon]\times[\epsilon, D_2])$$

which is such that defining the functions

$$u = h - k, \quad F = k_{xx} - k_t, \quad u_\epsilon(x) = h_\epsilon(x) - k(\epsilon, x),$$

we formally deduce from (39) the system

$$\begin{cases} u_t &= u_{xx} + F, \\ u_x(t,\epsilon) &= 0, \\ u_x(t, D_2) &= 0, \\ u(\epsilon, x) &= u_\epsilon(x), \end{cases} \tag{40}$$

on $(\epsilon, T-\epsilon) \times (\epsilon, D_2)$. Note that given the regularity of the boundary and initial datum, *i.e.*,

$$F \in C^\infty([\epsilon, T-\epsilon] \times [\epsilon, D_2]), \quad u_\epsilon \in L^2(\epsilon, D_2),$$

the problem (40) is actually well posed within the Hille-Yosida theory: it has a unique mild solution $u \in C([\epsilon, T-\epsilon]; L^2(\epsilon, D_2))$.

We now show that u, which is a priori $C([\epsilon, T-\epsilon]; L^2(\epsilon, D_2))$, is a mild solution of (40). First observe that for any $\epsilon \le t \le T - \epsilon$,

$$\begin{aligned} \int_\epsilon^t u(s)ds &= \int_\epsilon^t [h(s) - k(s)]ds \\ &= \int_\epsilon^t h(s)ds - \int_\epsilon^t k(s)ds \\ &= \int_0^t h(s)ds - \int_0^\epsilon h(s)ds - \int_\epsilon^t k(s)ds \end{aligned}$$

which is a sum of three $H^2(\epsilon, D_2)$ functions, hence is $H^2(\epsilon, D_2)$. Moreover we get, being u and k smooth on $(0,T) \times (0, D_2)$, that

$$\frac{d}{dx}\Big|_{x=\epsilon} \int_\epsilon^t u(s)ds = \int_\epsilon^t \frac{d}{dx}\Big|_{x=\epsilon} u(s)ds = \int_\epsilon^t [h_x(s,\epsilon) - \psi(s)]ds = 0$$

in view of the definition of ψ. For the other initial condition we use again the mild formulation of (29) (see (38)) to obtain

$$\frac{d}{dx}\Big|_{x=D_2} \int_\epsilon^t u(s)ds = \frac{d}{dx}\Big|_{x=D_2} \left\{ \int_0^t h(s)ds - \int_0^\epsilon h(s)ds - \int_\epsilon^t k(s)ds \right\} = 0 - 0 - \int_\epsilon^t k_x(s, D_2)ds = 0.$$

We are left to check the mild formulation of the abstract ODE, which is

$$u(t) - u_\epsilon = \frac{d^2}{dx^2} \int_\epsilon^t u(s)ds + \int_\epsilon^t F(s)ds.$$

Replacing u and F by their definitions, this amounts to

$$h(t) - k(t) - h(\epsilon) + k(\epsilon) = \frac{d^2}{dx^2} \int_\epsilon^t [h(s) - k(s)]ds + \int_\epsilon^t [k_{xx}(s) - k_t(s)]ds,$$

which is equivalent to

$$h(t) - h(\epsilon) = \frac{d^2}{dx^2} \int_\epsilon^t h(s)ds.$$

This last equation holds true evaluating the mild formulation in (38) at times t and ϵ, and taking the difference. This shows that u is a mild solution in (40).

Step 3: We apply the theory of analytic generators to obtain regularity for h.

In view of the previous step, we have u mild solution of (40). Note that this problem reads as an abstract ODE with a source term, with generator the Neumann Laplacian. The latter generator is analytic, hence

$$u \in C^\infty((\epsilon, T - \epsilon]; H^2(\epsilon, D_2)).$$

We refer to [16, Theorem 11.44] for a precise statement of the regularity theory of analytic semigroups. Therefore,

$$h = k + u \in C^\infty((\epsilon, T - \epsilon]; H^2(\epsilon, D_2)).$$

Note that $\epsilon > 0$ is arbitrarily small, so that we also have

$$h \in C^\infty([\epsilon, T - \epsilon]; H^2(\epsilon, D_2)).$$

Step 4: We show that in the system (36), the boundary conditions at $x = D_2$ are solved pointwise by h.

From the previous Step we obtain

$$h \in C^\infty([\epsilon, T - \epsilon]; H^2(\epsilon, D_2))$$

and since the trace maps

$$\left\{ \begin{array}{ccc} \alpha & \longmapsto & \alpha(D_2) \\ H^2(\epsilon, D_2) & \longrightarrow & \mathbb{C} \end{array} \right. , \quad \left\{ \begin{array}{ccc} \alpha & \longmapsto & \alpha_x(D_2) \\ H^2(\epsilon, D_2) & \longrightarrow & \mathbb{C} \end{array} \right.$$

are continuous, we obtain that the maps

$$t \mapsto h(t, D_2), \quad t \mapsto h_x(t, D_2),$$

are well-defined for any $t \in [\epsilon, T - \epsilon]$ and furthermore of class C^∞. Moreover, we deduce that h solves the boundary condition

$$h(t, D_2) = h_x(t, D_2) = 0$$

everywhere in $t \in [\epsilon, T - \epsilon]$. Indeed, h_t and h_{xx} are two functions in

$$C^\infty([\epsilon, T - \epsilon]; L^2(\epsilon, D_2)) \cap C^\infty((\epsilon, T - \epsilon) \times (\epsilon, D_2))$$

which agree pointwise on $(\epsilon, T - \epsilon) \times (\epsilon, D_2)$. Therefore for any $\varphi \in C_c^\infty((\epsilon, T - \epsilon) \times (\epsilon, D_2])$, we multiply the equality $h_t = h_{xx}$ by φ and integrate over $(\epsilon, T - \epsilon) \times (\epsilon, D_2)$ to discover, after integrations by part, that

$$\int_\epsilon^{T-\epsilon} \{\varphi(t, D_2) h_x(t, D_2) + \varphi_x(t, D_2) h(t, D_2)\} \, dt = 0.$$

From the above integral vanishing, we obtain that the functions $h(t, D_2)$ and $h_x(t, D_2)$ are almost everywhere null on $(\epsilon, T-\epsilon)$. Being continuous on this intervall, they both vanish everywhere on $(\epsilon, T-\epsilon)$.

<u>Step 5:</u> Now we consider the reflexion of h through the $\{x = D_2\}$ axis:

$$\vartheta(t,x) = \begin{cases} h(t,x), & \text{if} \quad \epsilon < t < T-\epsilon, \quad \epsilon < x \leq D_2, \\ h(t, 2D_2 - x), & \text{if} \quad \epsilon < t < T-\epsilon, \quad D_2 \leq x < 2D_2 - \epsilon, \end{cases}$$

and claim that it is a distributionnal solution of the heat equation on $(\epsilon, T-\epsilon) \times (\epsilon, 2D_2 - \epsilon)$.

First observe that ϑ is well defined in $C([\epsilon, T-\epsilon] \times [\epsilon, 2D_2 - \epsilon])$ because

$$h \in C^\infty([\epsilon, T-\epsilon]; H^2(\epsilon, D_2)) \subset C([\epsilon, T-\epsilon] \times [\epsilon, D_2]).$$

Further, fix $\varphi \in C_c^\infty((\epsilon, T-\epsilon) \times (\epsilon, 2D_2 - \epsilon))$ and compute

$$\int_\epsilon^{T-\epsilon} \int_\epsilon^{2D_2-\epsilon} \vartheta(\varphi_t + \varphi_{xx}) = \int_\epsilon^{T-\epsilon} \int_\epsilon^{D_2} \vartheta(\varphi_t + \varphi_{xx}) + \int_\epsilon^{T-\epsilon} \int_{D_2}^{2D_2-\epsilon} \vartheta(\varphi_t + \varphi_{xx}).$$

On the one hand,

$$\begin{aligned}
\int_\epsilon^{T-\epsilon} \int_\epsilon^{D_2} \vartheta(\varphi_t + \varphi_{xx}) &= \int_\epsilon^{T-\epsilon} \int_\epsilon^{D_2} h(\varphi_t + \varphi_{xx}) \\
&= \int_\epsilon^{D_2} \{h(T-\epsilon, x)\varphi(T-\epsilon, x) - h(\epsilon, x)\varphi(\epsilon, x)\}\, dx - \int_\epsilon^{D_2} \int_\epsilon^{T-\epsilon} h_t \varphi + \\
&+ \int_\epsilon^{T-\epsilon} \{h(t, D_2)\varphi_x(t, D_2) - h(t,\epsilon)\varphi_x(t,\epsilon) - h_x(t, D_2)\varphi(t, D_2) + h_x(t,\epsilon)\varphi(t,\epsilon)\}\, dt + \\
&+ \int_\epsilon^{T-\epsilon} \int_\epsilon^{D_2} h_{xx} \varphi \\
&= 0.
\end{aligned}$$

On the other hand, the term

$$\begin{aligned}
\int_\epsilon^{T-\epsilon} \int_{D_2}^{2D_2-\epsilon} \vartheta(\varphi_t + \varphi_{xx}) &= \int_\epsilon^{T-\epsilon} \int_{D_2}^{2D_2-\epsilon} h(t, 2D_2 - x)(\varphi_t(t,x) + \varphi_{xx}(t,x)) \\
&= \int_\epsilon^{T-\epsilon} \int_\epsilon^{D_2} h(t,x) \{\varphi_t(t, 2D_2 - x) + \varphi_{xx}(t, 2D_2 - x)\}\, dx dt
\end{aligned}$$

also vanishes by similar computations. This shows that

$$\vartheta_t = \vartheta_{xx} \quad \text{in} \quad \mathcal{D}'((\epsilon, T-\epsilon) \times (\epsilon, 2D_2 - \epsilon)).$$

<u>Step 6:</u> We conclude by making the heuristic of Step 0 rigorous.

Now $\vartheta \in C([\epsilon, T-\epsilon] \times [\epsilon, 2D_2 - \epsilon])$ is a caloric distribution on $(\epsilon, T-\epsilon) \times (\epsilon, 2D_2 - \epsilon)$. By regularity theory it is in fact a $C^\infty((\epsilon, T-\epsilon)\times(\epsilon, 2D_2-\epsilon))$ caloric function. Moreover, for a fixed $\epsilon < t_0 < T-\epsilon$ the function $\phi(x) = \vartheta(t_0, x)$ is analytic on $(\epsilon, 2D_2-\epsilon)$. We then claim that

$$\forall n \in \mathbb{N}, \quad \phi^{(n)}(D_2) = 0.$$

Observe that this claim would be enough to conclude the proof of the Lemma. Indeed if this is the case, being ϕ analytic we get $\phi \equiv 0$ on $(\epsilon, 2D_2-\epsilon)$, hence $u \equiv 0$ on $\{t_0\}\times(\epsilon, D_2)$, and being $\epsilon < t_0 < T-\epsilon$ and $0 < \epsilon \ll 1$ aritrary we get $h \equiv 0$.

To show that all derivatives of ϕ vanish at D_2, we first deal with the odd derivatives and observe that for any $n \in \mathbb{N}$,

$$\lim_{x\to D_2^-} \phi^{(2n+1)}(x) = \phi^{(2n+1)}(D_2) = \lim_{x\to D_2^+} \phi^{(2n+1)}(x). \tag{41}$$

On the one hand, if $\epsilon < x < D_2$ we have

$$\phi^{(2n+1)}(x) = \frac{\partial^{2n+1}}{\partial x^{2n+1}} h(t_0, x),$$

and thus the quantity

$$\frac{\partial^{2n+1}}{\partial x^{2n+1}} h(t_0, x), \quad \epsilon < x < D_2,$$

has a limit as $x \to D_2^-$. On the other hand, if $D_2 < x < 2D_2 - \epsilon$ we have

$$\phi^{(2n+1)}(x) = -\frac{\partial^{2n+1}}{\partial x^{2n+1}} h(t_0, 2D_2 - x),$$

and thus the quantity

$$\frac{\partial^{2n+1}}{\partial x^{2n+1}} h(t_0, 2D_2 - x), \quad D_2 < x < 2D_2 - \epsilon,$$

also has a limit as $x \to D_2^+$. By composition of the limits, we obtain

$$\lim_{x\to D_2^+} \frac{\partial^{2n+1}}{\partial x^{2n+1}} h(t_0, 2D_2 - x) = \lim_{h\to D_2^-} \frac{\partial^{2n+1}}{\partial x^{2n+1}} h(t_0, x)$$

and thus (41) brings

$$\phi^{(2n+1)}(D_2) = \lim_{x\to D_2^-} \frac{\partial^{2n+1}}{\partial x^{2n+1}} h(t_0, x) = \lim_{x\to D_2^+} \frac{\partial^{2n+1}}{\partial x^{2n+1}} h(t_0, 2D_2 - x) = -\phi^{(2n+1)}(D_2),$$

hence $\phi^{(2n+1)}(D_2) = 0$. For the even derivatives we formally write

$$\phi^{(2n)}(D_2) = \lim_{x\to D_2^-} \frac{\partial^{2n}}{\partial x^{2n}} h(t_0, x) = \lim_{x\to D_2^-} \frac{\partial^{n}}{\partial t^{n}} h(t_0, x) = \frac{d^n}{dt^n} h(t_0, D_2) = \frac{d^n}{dt^n} 0 = 0.$$

We observe that the third equality has to be justified, it will follow from the fact that for arbitrary $k \in \mathbb{N}$, and compact intervalls I, J, the map

$$\Phi : \begin{cases} C^k(I; C(J)) & \longrightarrow & C(J; C^k(I)) \\ (t \mapsto (x \mapsto \alpha(t,x))) & \longmapsto & (x \mapsto (t \mapsto \alpha(t,x))) \end{cases}$$

is well defined with

$$\frac{d^k}{dt^k}\bigg|_t [(\Phi\alpha)x] = [\alpha^{(k)}(t)](x).$$

With this fact, we obtain that

$$h \in C^\infty([\epsilon, T-\epsilon]; H^2(\epsilon, D_2)) \subset C([\epsilon, D_2]; C^n[\epsilon, T-\epsilon])$$

hence

$$h(\cdot, x) \xrightarrow[x\to D_2^-]{C^n[\epsilon, T-\epsilon]} h(\cdot, D_2) = 0.$$

This concludes the proof. □

4.2.2 Approximate controllability dictated by the waves

Prior to state and prove the approximate controllability for (14) we acknowledge the following elementary result.

Lemma 4.2.2.1. *The boundary control system*

$$\left\{\begin{array}{rcll} f_{tt} & = & f_{xx}, & -D_1 < x < 0, \\ f(t,-D_1) & = & 0, & \\ f_x(t,0) & = & q(t), & \\ f(0,x) & = & f_0(x), & \\ f_t(0,x) & = & g_0(x), & \end{array}\right. \tag{42}$$

is approximately controllable in any time $T \geq 2D_1$, and not approximately controllable for any time $0 < T < 2D_1$.

This result is not easy to find in the litterature and to the best of our knowledge, only the case $T > 2D_1$ was clearly stated and proved (see [4, Proposition 2.60]). For the sake of completeness we shall give a proof of this Lemma.

Proof. It is elementary to show that (42) is an admissible and well-posed boundary control system, with output operator

$$\mathcal{B}^*\mathcal{S}_t^*(f_0, g_0) = f_t(t,0).$$

Further, we obtain for any $(f_0, g_0) \in H^1_{(-D_1)} \times L^2(-D_1, 0)$ and almost every $t \in (0,T)$ that

$$f_t(t,0) = \sqrt{\frac{2}{D_1}} \sum_{n=0}^{\infty} \left\{ -(f_0, c_n)\sqrt{-\lambda_n}\sin(\sqrt{-\lambda_n}t) + (g_0, c_n)\cos(\sqrt{-\lambda_n}t) \right\} \tag{43}$$

where the series converges in $L^2_{loc}(\mathbb{R})$ and where we used the notations of Lemma (A.0.0.1).

On the one hand, assume that $T \geq 2D_1$, let $(f_0, g_0) \in H^1_{(-D_1)} \times L^2(-D_1, 0)$ be such that $f_t(t,0) \equiv 0$ in $L^2(0,T)$, from

$$\forall n \in \mathbb{N}, \quad \sqrt{-\lambda_n} = \left(n + \frac{1}{2}\right)\frac{\pi}{D_1}$$

we infer that all the sines and cosines appearing in the series making $f_t(t,0)$ in (43) solve

$$\forall t \in \mathbb{R}, \quad \sin(\sqrt{-\lambda_n}(t+2D_1)) = -\sin(\sqrt{-\lambda_n}t), \quad \cos(\sqrt{-\lambda_n}(t+2D_1)) = -\cos(\sqrt{-\lambda_n}t).$$

Thus the series in (43) is an $L^2_{loc}(\mathbb{R})$ function that is almost everywhere null on $\mathbb{R}$. We re arrange this as

$$\sum_{n=0}^{\infty}(f_0,c_n)\sqrt{-\lambda_n}\sin(\sqrt{-\lambda_n}t) = \sum_{n=0}^{\infty}(g_0,c_n)\cos(\sqrt{-\lambda_n}t)$$

almost everywhere on $\mathbb{R}$. The above function is therefore almost everywhere odd and even, hence vanishes. Being

$$\sqrt{\frac{2}{D_1}}\cos(\sqrt{-\lambda_n}t), \quad n \in \mathbb{N},$$

a Hilbert basis of $L^2(0,D_1)$ we conclude that

$$\forall n \in \mathbb{N}, \quad (g_0,c_n) = 0,$$

hence $g_0 = 0$. Similarly we conclude that

$$\forall n \in \mathbb{N}, \quad (f_0,c_n)\sqrt{-\lambda_n} = 0,$$

and being the $(\sqrt{-\lambda_n})_{n=0}^{\infty}$ never vanishing, we also have $f_0 = 0$. Hence the approximate controlability.

We now assume that $0 < T < 2D_1$ and show that (42) is not approximately controlable at time T. To this aim, we claim that denoting for $n \in \mathbb{Z}_-^*$,

$$e^{i\sqrt{-\lambda_n}t} = e^{-i\sqrt{-\lambda_{-n}}t},$$

the family $\{e^{i\sqrt{-\lambda_n}t}\}_{n\in\mathbb{Z}}$ is such that there exists $(a_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ for which

$$\phi := \sum_{n\in\mathbb{Z}} a_n e^{i\sqrt{-\lambda_n}t} \in L^2(0,2D_1)$$

is defined through an unconditionnaly convergent series, has its support contained in $[T,2D_1]$ and is not constant to 0. Assume for a moment that this is true and write

$$\begin{aligned}
\phi &= a_0 e^{i\sqrt{-\lambda_0}t} + \sum_{n=1}^{\infty}\left\{a_n e^{i\sqrt{-\lambda_n}t} + a_{-n}e^{-i\sqrt{-\lambda_n}t}\right\} \\
&= b_0\cos(\sqrt{-\lambda_0}t) + ic_0\sin(\sqrt{-\lambda_0}t) + \sum_{n=1}^{\infty}\left\{b_n\cos(\sqrt{-\lambda_n}t) + ic_n\sin(\sqrt{-\lambda_n}t)\right\} \\
&= \sum_{n=0}^{\infty}\left\{b_n\cos(\sqrt{-\lambda_n}t) + ic_n\sin(\sqrt{-\lambda_n}t)\right\},
\end{aligned}$$

in $L^2(0,2D_1)$, setting

$$\forall n \geq 1, \quad b_n = a_n + a_{-n}, \quad c_n = a_n - a_{-n},$$

and

$$b_0 = c_0 = a_0.$$

This makes $(b_n)_{n=0}^{\infty}$ and $(c_n)_{n=0}^{\infty}$ square summable. At this point, the equations

$$\forall n \in \mathbb{N}, \quad \begin{cases} -(f_0, c_n)\sqrt{-\lambda_n} & = & ic_n \\ (g_0, c_n) & = & b_n \end{cases}$$

define $f_0 \in H^1_{(-D_1)}(-D_1, 0)$ and $g_0 \in L^2(-D_1, 0)$. They cannot both vanish since then $a_n = 0$ for any $n \in \mathbb{Z}$ and $\phi \equiv 0$. We then consider (f_0, g_0) as an initial condition of (30), denote $f = f(t)$ its mild solution, and from (43) we deduce that $f_t(t, 0) \equiv 0$ almost everywhere on $(0, T)$ whereas $(f_0, g_0) \neq 0$, hence the lack of approximate controllability.

We are left to show that there exists $(a_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ for which

$$\phi := \sum_{n\in\mathbb{Z}} a_n e^{i\sqrt{-\lambda_n}t} \in L^2(0, 2D_1)$$

is defined through an unconditionnaly convergent series, has its support contained in $[T, 2D_1]$ and is not constant to 0. We will do so in several steps.

Step 0: We first give a heuristic that will be usefull for later.

For fixed $u \in L^2(0, 2D_1)$ and $(u_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, we formally have

$$\sum_{n\in\mathbb{Z}} u_n e^{i\sqrt{-\lambda_n}t} = u \iff \sum_{n\geq 0} u_n e^{i\frac{n\pi}{D_1}t} e^{i\frac{\pi}{2D_1}t} + \sum_{n>0} u_n e^{-i\frac{n\pi}{D_1}t} e^{-i\frac{\pi}{2D_1}t} = u \tag{44}$$

$$\iff \sum_{n\geq 0} u_n e^{i\frac{(n+1)\pi}{D_1}t} + \sum_{n<0} u_n e^{i\frac{n\pi}{D_1}t} = u e^{i\frac{\pi}{2D_1}t} \tag{45}$$

$$\Longrightarrow u e^{i\frac{\pi}{2D_1}t} \in \{1\}^{\perp, L^2(0,2D_1)} \tag{46}$$

$$\iff u \in \{e^{i\frac{\pi}{2D_1}t}\}^{\perp, L^2(0,2D_1)}. \tag{47}$$

and we define the Hilbert space

$$H := \{e^{i\frac{\pi}{2D_1}t}\}^{\perp, L^2(0,2D_1)}.$$

Step 1: We show that the family $\{e^{i\sqrt{-\lambda_n}t}\}_{n\in\mathbb{Z}}$ is orthogonal in $L^2(0, 2D_1)$.

If $n \neq m$ are in $\mathbb{N}$ we compute

$$
\begin{aligned}
(e_n, e_m)_{L^2(0,2D_1)} &= \int_0^{2D_1} e_n(t)\overline{e_m(t)}dt \\
&= \frac{1}{2D_1}\int_0^{2D_1} e^{i\sqrt{-\lambda_n}t}e^{-i\sqrt{-\lambda_m}t}dt \\
&= \frac{1}{2D_1}\int_0^{2D_1} e^{i(\sqrt{-\lambda_n}-\sqrt{-\lambda_m})t}dt \\
&= 0,
\end{aligned}
$$

as $\sqrt{-\lambda_n} \neq \sqrt{-\lambda_m}$ being the sequence $(\sqrt{-\lambda_n})_{n=0}^{\infty}$ injective. If $n \neq m$ are in $\mathbb{Z}_-^+$ we similarly obtain that e_n and e_m are orthogonal. The last case is when $n \in \mathbb{N}$ and $m \in \mathbb{Z}_-^+$, for which we compute for any $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$,

$$
\begin{aligned}
(e_n, e_{-m})_{L^2(0,2D_1)} &= \int_0^{2D_1} e_n(t)\overline{e_{-m}(t)}dt \\
&= \frac{1}{2D_1}\int_0^{2D_1} e^{i\sqrt{-\lambda_n}t}e^{i\sqrt{-\lambda_m}t}dt \\
&= \frac{1}{2D_1}\int_0^{2D_1} e^{i(\sqrt{-\lambda_n}+\sqrt{-\lambda_m})t}dt \\
&= 0,
\end{aligned}
$$

being $(\sqrt{-\lambda_n})_{n=0}^{\infty}$ positive. Hence the orthogonality.

Step 2: We show that the family $\{e^{i\sqrt{-\lambda_n}t}\}_{n\in\mathbb{Z}}$ is a Riesz basis of H.

Note that the family is valued in H because of (44)-(47). Since the family has constant non-zero norm, it remains only to show that it is total in H (see *e.g.* [3, Theorem 3.6.6]). Let $u \in H$, because of (46) and (47) we have

$$
ue^{i\frac{\pi}{2D_1}t} \in \{1\}^{\perp, L^2(0,2D_1)}.
$$

It is clear that the family $\{e^{in\frac{\pi}{D_1}t}\}_{n\in\mathbb{Z}^*}$ is a Riesz basis in $\{1\}^{\perp, L^2(0,2D_1)}$, hence the existence of $(u_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that (45) holds, where the two series converge unconditionnaly. Because the multiplication by $e^{i\frac{\pi}{2D_1}t}$ is a linear and continuous bijection on $L^2(0, 2D_1)$, we deduce that (45) implies (44), where the series also converge unconditionnally. This shows that the series

$$
\sum_{n\in\mathbb{Z}} u_n e^{i\sqrt{-\lambda_n}t}
$$

converges unconditionnally and equals u, which is thus in the closed span of $\{e^{i\sqrt{-\lambda_n}t}\}_{n\in\mathbb{Z}}$. This shows that the family is a Riesz basis of H.

Step 3: We conclude the argument exhibiting ϕ.

We write that

$$\{u \in L^2(0, 2D_1) : \operatorname{supp} u \subset [T, 2D_1]\} \cap H = \{u \in L^2(0, 2D_1) : \operatorname{supp} u \subset [T, 2D_1]\} \cap \{e^{i\frac{\pi}{2D_1}t}\}^{\perp, L^2(0,2D_1)}$$
$$\supset \{e^{-i\frac{\pi}{2D_1}t}\}^{\perp, L^2(T,2D_1)},$$

where the smallest of these vector spaces is non reduced to zero by dimension considerations, since $T < 2D_1$. Therefore there exists $\phi \in H \setminus \{0\}$ that has support in $[T, 2D_1]$. In view of the previous Step, it decomposes as

$$\phi = \sum_{n\in\mathbb{Z}} a_n e^{i\sqrt{-\lambda_n}t}, \quad (a_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}),$$

where the series converges unconditionnally, as required. □

We are now able to completely decide the approximate controllability of the system.

Theorem 4.2.2.2. *Fix the constants $c, d, D_1, D_2, T > 0$. We have*

1. *The system $\Sigma(D_1, D_2)$ (see (14)) is approximately controllable at time T if and only if $T \geq 2D_1$.*
2. *The system $\Sigma(c, d, D_1, D_2)$ (see (2)) is approximately controllable at time T if and only if $T \geq 2cD_1$.*

Proof. The second assertion is a consequence of the first assertion, being

$$\Sigma(c, d, D_1, D_2) \sim \Sigma(1, 1, cD_1, \sqrt{d}D_2)$$

and owing to Proposition (2.6.2.3). Hence there is only necessity of proving the first assertion.

Assume first that $T \geq 2D_1$, we show that the system $\Sigma(D_1, D_2)$ is approximately controllable in time T. To this aim let $(f_0, g_0, h_0) \in X$ be such that $\Theta_T(f_0, g_0, h_0) = 0$, in view of Lemma (4.2.1.2) we have $h \equiv 0$, so $h_0 = 0$. Moreover, as $f = f(t)$ is the mild solution of (30), as shown in Lemma (4.2.2.1) we have $f_0 = g_0 = 0$. Therefore the approximate controllability of $\Sigma(D_1, D_2)$ at time T.

We now assume that $0 < T < 2D_1$ and show that $\Sigma(D_1, D_2)$ is not null controllable. Since the hyperbolic subystem (42) is not approximately controlable at time T, there exists $(f_0, g_0) \in H^1_{(-D_1)} \times L^2$ not null such that the mild solution $f = f(t)$ of (30) solves $f_t(t, 0) = 0$ almost everywhere on $(0, T)$. We then choose $V_0 = (f_0, g_0, 0)$ as an initial condition of the adjoint system (29), it is non zero and the parabolic component $h = h(t)$ solves

$$\begin{cases} h_t &= h_{xx}, \\ h_x(t, 0) &= 0, \\ h_x(t, D_2) &= 0, \\ h(0, x) &= 0, \end{cases}$$

in the mild sense. By uniqueness, $h \equiv 0$, thus $h(t, D_2) \equiv 0$, and being $V_0 \neq 0$ the system $\Sigma(D_1, D_2)$ is not approximately controlable at time T. □

A Regularity for a wave problem

Prior to understand that the operator A is Riesz spectral, our method to show that it is a generator was based on the (b) characterization of the generation property given by Theorem 6.7 of [12]. This characterization teaches us that a densely defined operator A is a generator if and only if $\rho(A) \neq \emptyset$ and for all initial condition $z_0 \in D(A)$ and $0 < T < \infty$, the abstract ODE

$$\begin{cases} \dot{z} & = & Az \\ z(0) & = & z_0 \end{cases}$$

has a unique classical solution on $[0, T]$. For our operator A it is straightforward to see that the only non trivial part is to obtain the regularity for the wave subproblem

$$\begin{cases} u_{tt} & = & u_{xx} \\ u_x(t,0) & = & q(t) \\ u(t,-D_1) & = & 0 \\ u(0,x) & = & u^0(x) \\ u_t(0,x) & = & v^0(x) \end{cases} \tag{48}$$

with $q(t) = p(t, 0)$. This motivated the following result on regularity for (48) that we leave here as an additional material.

Theorem A.0.0.1. *1. For any*

$$u^0 \in H^1_{(-D_1)}(-D_1, 0), \quad v^0 \in L^2(-D_1, 0), \quad q \in L^2(0, T)$$

there exists a unique $u \in L^2(0, T; H^1_{(-D_1)}) \cap H^1(0, T; L^2)$ weak solution[6] *of (48).*

2. In fact, this solution u is $C([0, T]; H^1_{(-D_1)}) \cap C^1([0, T]; L^2)$ and $u(\cdot, 0) \in H^1(0, T)$ with

$$\|u(\cdot, 0)\|_{H^1(0,T)} + \|u\|_{C([0,T];H^1_{(-D_1)}) \cap C^1([0,T];L^2)} \lesssim \|u^0\|_{H^1} + \|v^0\|_{L^2} + \|q\|_{L^2}$$

3. If

$$u^0 \in H^2, \quad v^0 \in H^1, \quad q \in H^1, \quad u^0_x(0) - q(0) = u^0(-D_1) = v^0(-D_1) = 0$$

the weak solution is

$$C^2([0, T]; L^2) \cap C^1([0, T]; H^1) \cap C([0, T]; H^2)$$

and classical.

We emphasize that, to the best of our knowledge, regularity of point 3 has not been done yet. Our main reference is [4], Theorem 2.53, which is inspired from [7], Chapter 3, Section 8, and only reaches point 2 of Theorem (A.0.0.1). To our knowledge, the best regularity theory for $q \in H^1$ can be found in [13], Theorem 3.1, which is weaker than what is claimed here, but works in a more general framework.

[6] in a sense that will be precised in the proof

Proof. The proof is quite long and technical, we divide it into several steps. The reader which is already confident with points 1 and 2 is invited to directly go to the Step 10 of the proof.

Step 1: Classical solutions for the homogeneous system

We introduce the wave operator

$$\mathcal{X} = H^1_{(-D_1)} \times L^2(-D_1, 0), \quad D(\mathcal{A}) = H^2_{(0)} \cap H^1_{(-D_1)} \times H^1_{(-D_1)}, \quad \mathcal{A}(u, v) = (v, u_{xx}).$$

This operator is skew-adjoint hence it generates a semi-group that is denotes $e^{t\mathcal{A}}$. We can compute explicitely the mild solution of

$$\begin{cases} u_{tt} &= u_{xx} \\ u_x(t,0) &= 0 \\ u(t,-D_1) &= 0 \\ u(0,x) &= u^0(x) \\ u_t(0,x) &= v^0(x) \end{cases}$$

for $(u^0, v^0) \in \mathcal{X}$, it is given by

$$\pi_1 e^{t\mathcal{A}}(u^0, v^0) = u(t,x) = \sum_{n=0}^{\infty} \left[(u^0, c_n) \cos(\sqrt{-\lambda_n} t) + \frac{(v^0, c_n)}{\sqrt{-\lambda_n}} \sin(\sqrt{-\lambda_n} t) \right] c_n(x)$$

where

$$\lambda_n = -\left(n + \frac{1}{2}\right)^2 \left(\frac{\pi}{D_1}\right)^2$$

and

$$c_n(x) \propto \cos(\sqrt{-\lambda_n} x) = \sqrt{\frac{2}{D_1}} \cos(\sqrt{-\lambda_n} x)$$

makes a Hilbert basis of $L^2(-D_1, 0)$. Also $\pi_1 : \mathcal{X} \to H^1_{(-D_1)}$ is the projection onto the first factor. From the semi-group theory,

$$(u^0, v^0) \in \mathcal{X} \Longrightarrow u \in C([0,T]; H^1_{(-D_1)}) \cap C^1([0,T]; L^2)$$

and

$$(u^0, v^0) \in D(\mathcal{A}) \Longrightarrow u \in C([0,T]; H^2_{(0)}) \cap C^1([0,T]; H^1_{(-D_1)}) \cap C^2([0,T]; L^2).$$

Furthermore, in view of Theorem 11.16 of [16], for any

$$f \in W^{1,1}(0,T; \mathcal{X}), \quad (u^0, v^0) \in D(\mathcal{A})$$

there exists a unique mild solution u to

$$\begin{cases} u_{tt} &= u_{xx} + f \\ u_x(t,0) &= 0 \\ u(t,-D_1) &= 0 \\ u(0,x) &= u^0(x) \\ u_t(0,x) &= v^0(x) \end{cases}$$

and it is moreover classical:

$$u \in C^2([0,T];L^2) \cap C^1([0,T];H^1_{(-D_1)}) \cap C([0,T];H^2_{(0)} \cap H^1_{(-D_1)}).$$

It is given by the Duhamel formula

$$u = \pi_1 e^{t\mathcal{A}}(u^0, v^0) + \int_0^t \pi_1 e^{(t-s)\mathcal{A}} f(s) ds$$

where $\pi_1 : X \to H^1_{(-D_1)}$ stands for the projection onto the first factor.

Step 2: Lifted system

For any

$$(u^0, v^0) \in D(\mathcal{A}), \quad q \in W^{3,1}(0,T)$$

such that $q(0) = 0$ we have the following compatibility condition

$$u^0_x(0) = 0 = q(0).$$

We consider

$$k(t,x) := (x + D_1)q(t) \in C^2([0,T];L^2) \cap C^1([0,T];H^1_{(-D_1)}) \cap C([0,T];H^2 \cap H^1_{(-D_1)})$$

which is such that, at least formally, u is a solution of the non homogeneous system (48) if and only if $w := u - k$ solves the homogeneous system with a source term

$$\left\{ \begin{array}{ccc} w_{tt} & = & w_{xx} - k_{tt} \\ w_x(t,0) & = & 0 \\ w(t,-D_1) & = & 0 \\ w(0,x) & = & u^0(x) - k(0,x) \\ w_t(0,x) & = & v^0(x) - k_t(0,x) \end{array} \right. \tag{49}$$

Observe that

$$u^0(x) - k(0,x) = u^0(x) \in H^2_{(0)} \cap H^1_{(-D_1)}, \quad v^0(x) - k_t(0,x) = v^0(x) - (x + D_1)\dot{q}(0) \in H^1_{(-D_1)}$$

and

$$k_{tt} = (x + D_1)\ddot{q} \in W^{1,1}(0,T;H^1_{(-D_1)})$$

brings the source term

$$(0, -k_{tt}) \in W^{1,1}(0,T;\mathcal{X}).$$

Thus from the previous step there exists

$$w \in C^2([0,T];L^2) \cap C^1([0,T];H^1_{(-D_1)}) \cap C([0,T];H^2_{(0)} \cap H^1_{(-D_1)})$$

classical solution of (49), whence

$$u := w + k \in C^2([0,T];L^2) \cap C^1([0,T];H^1_{(-D_1)}) \cap C([0,T];H^2 \cap H^1_{(-D_1)})$$

is a classical solution of (48).

Step 3: A weak formulation

Let

$$(u^0, v^0) \in D(\mathcal{A}), \quad q \in W^{3,1}(0,T), \quad q(0) = 0$$

and u the associated classical solution of (48), for $\phi \in C^1([0,T] \times [-D_1, 0])$ such that

$$\forall t \in [0,T], \quad \phi(t, -D_1) = 0$$

we multiply $u_{tt} = u_{xx}$ by ϕ, integrate over $[0,\tau] \times [-D_1, 0]$, perform some integrations by part[7], and discover

$$\int_{-D_1}^0 \phi(\tau,x)u_t(\tau,x)dx - \int_{-D_1}^0 \phi(0,x)v^0(x)dx - \int_0^\tau \int_{-D_1}^0 \phi_t(t,x)u_t(t,x)dxdt \tag{50}$$

$$= \int_0^\tau \phi(t,0)q(t)dt - \int_0^\tau \int_{-D_1}^0 \phi_x(t,x)u_x(t,x)dxdt \tag{51}$$

This motivates the following definition of a weak solution for (48): it is $u \in L^2(0,T; H^1_{(-D_1)}) \cap H^1(0,T; L^2)$ such that $u(0) = u^0$ and for all ϕ as above, (50) holds for almost every $\tau \in [0,T]$. Observe that we essentially proved that a classical solution is a weak solution.

Step 4: Energy estimates

Let

$$(u^0, v^0) \in D(\mathcal{A}), \quad q \in W^{3,1}(0,T), \quad q(0) = 0$$

and u the associated classical solution of (48), consider the energy

$$E(t) = \frac{1}{2}\int_{-D_1}^0 (u_t^2(t,x) + u_x(t,x)^2)dx \in C^1[0,T]$$

which is such that

$$\dot{E}(t) = u_t(t,0)q(t).$$

We now choose the test function

$$\phi(t,x) = (x + D_1)u_x(t,x),$$

which is legitimate because by density and continuity the weak formulations holds for

$$\phi \in L^1(0,T; H^1_{(-D_1)}) \cap W^{1,1}(0,T; L^2),$$

and use (50) with integrations by part to discover, for any $\tau \in [0,T]$,

$$\int_{-D_1}^0 (x+D_1)u_x(\tau,x)u_t(\tau,x)dx - \int_{-D_1}^0 (x+D_1)u_x^0(x)v^0(x)dx + \int_0^\tau E(t)dt = \frac{D_1}{2}\int_0^\tau \{u_t(t,0)^2 + q(t)^2\}dt \tag{52}$$

[7] We deal with the $u_{tt}\phi$ term using parabolic integration by part. We refer to [6], section 2.5, for more details and proofs.

Now write the Young inequality

$$
\begin{aligned}
E(t) &= E(0) + \int_0^t \dot{E}(\tau)d\tau \\
&\leq E(0) + \frac{\epsilon}{2}\int_0^t u_t(\tau,0)^2 d\tau + \frac{1}{2\epsilon}\int_0^t q(\tau)^2 d\tau
\end{aligned}
$$

for an $\epsilon > 0$ to be chosen suitably, depending only on D_1. From (52) we get

$$
\frac{D_1}{2}\int_0^\tau u_t(t,0)^2 dt = \int_0^\tau E(t)dt + \int_{-D_1}^0 (x+D_1)u_x(\tau,x)u_t(\tau,x)dx - \int_{-D_1}^0 (x+D_1)u_x^0(x)v^0(x)dx - \frac{D_1}{2}\int_0^\tau q(t)^2 dt \tag{53}
$$

$$
\leq \int_0^\tau E(t)dt + D_1 E(\tau) + D_1\|u_x^0\|_{L^2}\|v^0\|_{L^2} \tag{54}
$$

hence for a suitabe choice of ϵ, we get

$$
E(t) \leq C(\|u_x^0\|_{L^2}^2 + \|v^0\|_{L^2}^2) + C\int_0^\tau E(t)dt + \frac{1}{2}E(t) + C\int_0^t q(\tau)^2 d\tau
$$

for some $C > 0$ depending only on D_1. In view of the integral version of the Grönwall lemma

$$
x(t) \leq B(t) + \int_0^t a(s)x(s)ds + x_0 \Longrightarrow x(t) \leq e^{A(t)}x_0 + B(t)
$$

we get

$$
E(t) \leq C\left(\|u_x^0\|_{L^2}^2 + \|v^0\|_{L^2}^2 + \|q\|_{L^2}^2\right) \tag{55}
$$

for a larger C, depending only on T and D_1. Observe that using (55) in (53) we obtain for a larger constant $C = C(T, D_1)$,

$$
\|u_t(\cdot,0)\|_{L^2}^2 \leq C\left(\|u_x^0\|_{L^2}^2 + \|v^0\|_{L^2}^2 + \|q\|_{L^2}^2\right). \tag{56}
$$

Step 5: Compactness

For a sequence of smooth approximation of the data (u^0, v^0, q) we build the corresponding classical solutions and then use compactness arguments in order to find a limit point. Let

$$
(u^0, v^0) \in H^1_{(-D_1)} \times L^2, \quad q \in L^2(0,T)
$$

and approximations

$$
C_c^\infty(-D_1, 0] \ni u_j^0 \xrightarrow[j\to\infty]{H^1} u^0, \quad C_c^\infty(-D_1,0) \ni v_j^0 \xrightarrow[j\infty]{L^2} v^0, \quad C_c^\infty(0,T) \ni q_j \xrightarrow[j\infty]{L^2} q.
$$

By step 2, being

$$
\frac{d}{dx}u_j^0(0) = 0 = q_j(0),
$$

we let for all j,

$$u_j \in C^2([0,T];L^2) \cap C^1([0,T];H^1_{(-D_1)}) \cap C([0,T];H^2 \cap H^1_{(-D_1)})$$

classical solution of (48) with data (u_j^0, v_j^0, q_j). Then by (55), the sequence (u_j) is bounded in

$$C^1([0,T];L^2) \cap C([0,T];H^1_{(-D_1)}).$$

By the Aubin-Simon lemma (see Corollary 4 of [21]) the inclusion

$$L^\infty(0,T;H^1_{(-D_1)}) \cap H^1(0,T;L^2) \hookrightarrow C([0,T];L^2)$$

is compact whence up to a subsequence (not relabeled) we get the existence of $u \in C([0,T];L^2)$ such that

$$u_j \xrightarrow[j\to\infty]{C([0,T];L^2)} u.$$

Next using the Banach-Alaoglu's theorem[8] we further get

$$u_j \xrightarrow[j\to\infty]{L^\infty(0,T;H^1_{(-D_1)})-w-*} u, \quad (u_j)_t \xrightarrow[j\to\infty]{L^\infty(0,T;L^2)-w-*} u_t.$$

Also from (56), Rellich-Kondrachov theorem[9], and Kakutani's theorem (see Theorem 3.18 of [2]), there exists $v \in H^1(0,T)$ such that

$$u_j(\cdot,0) \xrightarrow[j\to\infty]{H^1_w \cap C([0,T])} v.$$

<u>Step 6: Passing to the limit in the weak formulation</u>

We show that u built as in the previous step is a weak solution. Observe that it belongs to the correct space $L^2(0,T;H^1_{(-D_1)}) \cap H^1(0,T;L^2)$ and that

$$u(0) \leftarrow u_j(0) = u_j^0 \xrightarrow[j\infty]{L^2} u^0.$$

Hence u is a weak solution if and only if it solves the weak formulation. Let ϕ as in the weak formulation, we obviously have

$$\int_{-D_1}^0 \phi(0,x)v_j^0(x)dx \to \int_{-D_1}^0 \phi(0,x)v^0(x)dx$$

and since $\phi(\cdot,0) \in L^2(0,T)$ we have for all $0 \le \tau \le T$

$$\int_0^\tau \phi(t,0)q_j(t)dt \to \int_0^\tau \phi(t,0)q(t)dt.$$

[8]This can be seen applying Corollary 3.30 from [2]. To check the hypotheses of this result we use Theorem 1.4.1 and Corollary 1.3.2 of [6].

[9]This can be seen as a consequence of the Ascoli-Arzelà theorem together with $H^1(0,T) \hookrightarrow C^{0,1/2}[0,T]$ that yields

$$H^1(0,T) \hookrightarrow C^{0,1/2}[0,T] \hookrightarrow\hookrightarrow C[0,T].$$

Alternatively, Theorem 12.61 from [14] is a more general result.

Now observe that for any τ,

$$\psi \mapsto \int_0^\tau \int_{-D_1}^0 \phi_x(t,x)\psi(t,x)dxdt$$

is linear continuous on $L^2(0,\tau;L^2)$. Therefore

$$\int_0^\tau \int_{-D_1}^0 \phi_x(t,x)(u_j)_x(t,x)dxdt \to \int_0^\tau \int_{-D_1}^0 \phi_x(t,x)u_x(t,x)dxdt$$

and similarly

$$\int_0^\tau \int_{-D_1}^0 \phi_x(t,x)(u_j)_t(t,x)dxdt \to \int_0^\tau \int_{-D_1}^0 \phi_x(t,x)u_t(t,x)dxdt.$$

We are left with the term

$$\psi_j(\tau) := \int_{-D_1}^0 \phi(\tau,x)(u_j)_t(\tau,x)dx.$$

Since all the other terms of the weak formulation are $C[0,T]$ and converge pointwise, ψ_j is $C[0,T]$ and converges pointwise, denote ψ the limit. The convergence is dominated in $L^\infty(0,T)$:

$$\forall \tau \in [0,T], \quad |\psi_j(\tau)| \leq \|\phi(\tau,\cdot)\|_{L^2}\|(u_j)_t(t)\|_{L^2} \leq \|\phi\|_{H^1(0,T;L^2)}\|u_j\|_{C^1([0,T];L^2)}$$

hence it happens in $L^2(0,T)$. Now since

$$(u_j)_t \xrightarrow[j\infty]{L^2_w(0,T;L^2)} u$$

we have for any $\xi \in L^2(0,T)$:

$$\int_0^T \psi(t)\xi(t) \leftarrow \int_0^T \psi_j(t)\xi(t)dt = \int_0^T \int_{-D_1}^0 \phi(t,x)(u_j)_t(t,x)\xi(t)dxdt \to \int_0^T \int_{-D_1}^0 \phi(t,x)u_t(t,x)\xi(t)dxdt$$

from which we deduce

$$\psi(t) = \int_{-D_1}^0 \phi(t,x)u_t(t,x)dx$$

almost everywhere and

$$\int_{-D_1}^0 \phi(\tau,x)(u_j)_t(\tau,x)dx \to \int_{-D_1}^0 \phi(t,x)u_t(t,x)dx$$

almost everywhere, as required. This shows the existence of a weak solution.

Step 7: Uniqueness of the weak solution

We let $u \in L^2(0,T;H^1_{(-D_1)}) \cap H^1(0,T;L^2)$ be any weak solution of

$$\begin{cases} u_{tt} &= u_{xx} \\ u_x(t,0) &= 0 \\ u(t,-D_1) &= 0 \\ u(0,x) &= 0 \\ u_t(0,x) &= 0 \end{cases}$$

and we show that $u = 0$. We have for any $\phi \in C^1([0,T] \times [-D_1, 0])$ such that $\phi(\cdot, -D_1) \equiv 0$ that for almost every $\tau \in [0,T]$,

$$\int_{-D_1}^{0} u_t(\tau, x)\phi(\tau, x)dx - \int_0^\tau \int_{-D_1}^{0} u_t(t,x)\phi_t(t,x)dxdt = -\int_0^\tau \int_{-D_1}^{0} u_x(t,x)\phi_x(t,x)dxdt.$$

If $\phi(T) = 0$ we obtain

$$\int_0^T \int_{-D_1}^{0} u_t(t,x)\phi_t(t,x)dxdt = \int_0^T \int_{-D_1}^{0} u_x(t,x)\phi_x(t,x)dxdt \tag{57}$$

and therefore, by density and continuity, the above equality holds for any $\phi \in C_c^\infty(0,T; H^1_{(-D_1)})$. We let $V = H^1_{(-D_1)}$, V^* be a realization of the anti-dual space of V with respect to the pivot L^2, and define

$$\frac{d^2}{dx^2} : V \to V^*, \quad \left\langle \frac{d^2}{dx^2} u, v \right\rangle_{V^*,V} := -\int_{-D_1}^{0} u_x v_x$$

which is a linear and continuous operator, but does not generalize the standard distributional calculus. Then we can write

$$u_{tt} = \frac{d^2}{dx^2} u \quad \text{in} \quad \mathcal{D}'(0,T;V^*).$$

Observe that from

$$u \in L^2(0,T;V)$$

we get

$$\frac{d^2}{dx^2} u \in L^2(0,T;V^*)$$

hence

$$u \in H^2(0,T;V^*).$$

As a first application we show that $u_t(0)$, which has a meaning, vanishes. To this aim fix $\phi \in C_c^\infty([0,T);V)$ and write

$$\begin{aligned} -\langle u_t(0), \phi(0)\rangle_{V^*,V} &= [\langle u_t(t), \phi(t)\rangle_{V^*,V}]_0^T \\ &= \int_0^T \{\langle u_{tt}, \phi\rangle_{V^*,V} + \langle u_t, \phi'\rangle_{V^*,V}\} \\ &= \int_0^T \left\langle \frac{d^2}{dx^2} u, \phi \right\rangle_{V^*,V} + \int_0^T \int_{-D_1}^{0} u_x \phi_x \\ &= 0 \end{aligned}$$

where in the third equality we used (57) and (2.2.0.4). We now show the uniqueness, let $0 < t_r < T$ arbitrary and

$$\phi(t) = \begin{cases} -\int_t^{t_r} u(s)ds & \text{if} \quad 0 \le t < t_r \\ 0 & \text{if} \quad t_r \le t \le T \end{cases}$$

which is $H^1(0,T;V)$ with

$$\phi'(t) = \begin{cases} u(s) & \text{if} \quad 0 \le t < t_r \\ 0 & \text{if} \quad t_r \le t \le T \end{cases}$$

We use this ϕ as a test function in $u_{tt} = \frac{d^2}{dx^2}u$ to discover that on the one hand

$$\int_0^T \left\langle \frac{d^2}{dx^2}u, \phi \right\rangle_{V^*,V} = \int_0^{t_r} \left\langle \frac{d^2}{dx^2}u, \phi \right\rangle_{V^*,V} = -\int_0^{t_r} (u_x, \phi_x)_{L^2} = -\int_0^{t_r} (\phi_{tx}, \phi_x)_{L^2}$$

while on the other hand,

$$\begin{aligned}\int_0^T \langle u_{tt}, \phi\rangle_{V^*,V} &= \int_0^{t_r} \langle u_{tt}, \phi\rangle_{V^*,V} \\ &= (u_t(t_r), \phi(t_r))_{L^2} - (u_t(0), \phi(0))_{L^2} - \int_0^{t_r} \langle u_t, \phi_t\rangle_{V^*,V} \\ &= -\int_0^{t_r} \langle u_t, \phi_t\rangle_{V^*,V} \\ &= -\int_0^{t_r} \langle u_t, u\rangle_{V^*,V} = -\int_0^{t_r} (u_t, u)_{L^2}\end{aligned}$$

so that in view of the Lions-Magenes lemma (see for instance [14], Theorem 8.60) and the Schwartz symmetry of second derivatives for $\phi \in H^1(0,T;H^1)$,

$$\begin{aligned}0 &= \int_0^{t_r} \{(u, u_t)_{L^2} - (\phi_{tx}, \phi_x)_{L^2}\} \\ &= \int_0^{t_r} \frac{d}{dt}\frac{1}{2}\{\|u\|_{L^2}^2 - \|\phi_x\|_{L^2}^2\} \\ &= \frac{\|u(t_r)\|_{L^2}^2 + \|\phi(0)\|_{L^2}^2}{2} \geq \frac{\|u(t_r)\|_{L^2}^2}{2}.\end{aligned}$$

Thus $u(t_r) = 0$, and since t_r is arbitrary in $[0,T]$ we get $u \equiv 0$ whence the uniqueness.

Step 8: Proof of the continuity of point 2

We start by showing the regularity of the weak solution. From Step 5 and 6, for any initial data $(u^0, u^1, q) \in H^1_{(-D_1)} \times L^2 \times L^2$ there exists a weak solution $u \in L^\infty(0,T;H^1_{(-D_1)}) \cap W^{1,\infty}(0,T;L^2)$ with trace $v \in H^1(0,T)$ (in the sense that for the constructed approximation, the trace converges to v). We show that it turns out to be $C([0,T];H^1_{(-D_1)}) \cap C^1([0,T];L^2)$.

As in the previous Step we can obtain

$$u \in W^{2,\infty}(0,T;(H^1_0)^*) \quad \text{and} \quad u_{tt} = \frac{d^2}{dx^2}u \quad \text{in} \quad \mathcal{D}'(0,T;(H^1_0)^*).$$

In view of Lemma 8.1 from [7],

$$u_t \in L^\infty(0,T;L^2) \cap C([0,T];(H^1_0)^*) \subset C([0,T];L^2 - w).$$

Moreover we note

$$u \in L^\infty(0,T;H^1_{(-D_1)}) \cap C([0,T];L^2) \subset C([0,T];H^1_{(-D_1)} - w)$$

so that the map

$$t \mapsto E(t) := \frac{1}{2}\{\|u_t(t)\|_{L^2}^2 + \|u_x(t)\|_{L^2}^2\}$$

is everywhere defined on $[0,T]$, bounded and lower semi continuous. We show that E is upper semi continuous at 0^+: let $0 < t_r < T$, we have approximations

$$u_j \xrightarrow[j\infty]{L^\infty(0,t_r;H^1_{(-D_1)})-w-*} u, \quad (u_j)_t \xrightarrow[j\infty]{L^\infty(0,t_r;L^2)-w-*} u_t$$

which allow

$$\begin{aligned}
\operatorname*{ess\ sup}_{0\le t\le t_r} E(t) &= \operatorname*{ess\ sup}_{0\le t\le t_r} \frac{\|u_x(t)\|_{L^2}^2 + \|u_t(t)\|_{L^2}^2}{2} \\
&= \|(u_x,u_t)\|^2_{L^\infty(0,t_r;L^2)\times L^\infty(0,t_r;L^2)} \\
&\le \liminf_j \|((u_j)_x,(u_j)_t)\|^2_{L^\infty(0,t_r;L^2)\times L^\infty(0,t_r;L^2)} \\
&= \liminf_j \operatorname*{ess\ sup}_{0\le t\le t_r} \left\{ \frac{\|(u_j^0)_x\|_{L^2}^2 + \|u_j^1\|_{L^2}^2}{2} + \int_0^t u_j(s,0)q_j(s)ds \right\} \\
&\le \frac{\|u_x^0\|_{L^2}^2 + \|u^1\|_{L^2}^2}{2} + \int_0^{t_r} |v(s)q(s)|ds
\end{aligned}$$

where the second equality must be seen as the definition of a norm on $L^\infty(0,t_r;L^2)\times L^\infty(0,t_r;L^2)$, which is elementary to be shown equivalent to the standard one. Next it is straightforward to show that for a lower semi continuous function, essential suprema are classical suprema, thus

$$\sup_{0\le t\le t_r} E(t) = \operatorname*{ess\ sup}_{0\le t\le t_r} E(t) \le \frac{\|u_x^0\|_{L^2}^2 + \|u^1\|_{L^2}^2}{2} + \int_0^{t_r} |v(t)q(t)|dt \underset{t_r\to 0^+}{\longrightarrow} E(0),$$

E is lower semi continuous at 0^+ and E is continuous at 0^+.

We can then easily deduce that E is right continuous on $[0,T)$: for any $0 \le t_0 < T$ we have $(u(t_0),u_t(t_0)) \in H^1_{(-D_1)} \times L^2$ and the restriction of u to $[t_0,T]$ is a weak solution of

$$\begin{cases} u_{tt} = u_{xx} & t_0 < t < T \\ u_x(t,0) = q(t) & t_0 < t < T \\ u(t,-D_1) = 0 & t_0 < t < T \\ u(t_0,x) = u(t_0)(x) & \\ u_t(t_0,x) = u_t(t_0)(x) & \end{cases}$$

whose energy $\tilde{E}$ is right continuous at t_0, doing again the same proof using approximations. Since $\tilde{E} = E$ on $[t_0,T]$, E is right continuous at t_0.

We also have the left continuity for free: the map $w : t \mapsto u(T-t)$ is a solution of the problem

$$\begin{cases} w_{tt} = w_{xx} \\ w_x(t,0) = q(T-t) \\ w(t,-D_1) = 0 \\ w(0,x) = u(T)(x) \\ w_t(0,x) = u_t(T)(x) \end{cases}$$

hence the reversed energy $E(T-t)$ is right continuous, hence E is left continuous.

This allows to show the regularity: let $t_0 \in [0,T]$ and (t_j) be a sequence of $[0,T]$ with limit t_0, using the continuity of E and the weak continuity of $u_x, u_t : [0,T] \to L^2$,

$$\begin{aligned}\frac{1}{2}\left(\|u_t(t_j)-u_t(t_0)\|_{L^2}^2+\|u_x(t_j)-u_x(t_0)\|_{L^2}^2\right) &= E(t_j)+E(t_0)-(u_t(t_j),u_t(t_0))_{L^2}-(u_x(t_j),u_x(t_0))_{L^2}\\ &\to E(t_0)+E(t_0)-(u_t(t_0),u_t(t_0))_{L^2}-(u_x(t_0),u_x(t_0))_{L^2}\\ &=0.\end{aligned}$$

Step 9: Proof of the estimations of point 3

Let u be the weak solution corresponding to data (u^0, v^0, q), consider approximations u_j^0, v_j^0, q_j, u_j as in Step 5, then the sequence (u_j) has a converging subsequence, denote $\tilde{u}$ the limit:

$$u_{\sigma(j)} \xrightarrow[j\infty]{L^\infty(0,T;H^1_{(-D_1)})-w-*} \tilde{u}, \quad (u_{\sigma(j)})_t \xrightarrow[j\infty]{L^\infty(0,T;L^2)-w-*} \tilde{u}_t$$

for some extraction σ. As shown in Step 6, $\tilde{u}$ is a weak solution to the problem with data (u^0, v^0, q), hence $\tilde{u} = u$. By classical point-set topology we infer that the whole sequence (u_j) converges to u. Then by weak-$*$ lower semi-continuity of the norm

$$\begin{aligned}\|u\|_{C([0,T];H^1_{(-D_1)})\cap C^1([0,T];L^2)} &\lesssim \|u\|_{L^\infty(0,T;H^1_{(-D_1)})}+\|u_t\|_{L^\infty(0,T;L^2)}\\ &\le \liminf_{j\infty}\left\{\|u_j\|_{L^\infty(0,T;H^1_{(-D_1)})}+\|(u_j)_t\|_{L^\infty(0,T;L^2)}\right\}\\ &\lesssim \liminf_{j\infty}\left\{\|u_j^0\|_{H^1}+\|v_j^0\|_{L^2}+\|q_j\|_{L^2}\right\}\\ &= \|u^0\|_{H^1}+\|v^0\|_{L^2}+\|q\|_{L^2}.\end{aligned}$$

We repeat the same arguments to get

$$(u_j)_t(\cdot,0) \xrightarrow[j\infty]{H^1_w} u_t(\cdot,0)$$

and the required estimations.

Step 10: Deduction of a candidate

We recall that up to this Step, we have established the existence, uniqueness, and regularity of a weak solution for (48) as in point 3 of Lemma (**??**). The goal is now to show that with improved data we can actually get more regularity solution for the solution by computing it explicitely.

We deduce a formula by formal computations. Let

$$(u^0, u^1) \in D(\mathcal{A}), \quad q \in W^{3,1}(0,T), \quad q(0) = 0$$

and denote u the associated solution. From the Duhamel formula in Step 1 we get, forgetting that

$$q(0) = 0$$

the following:

$$\begin{aligned}
u(t,x) - (x+D_1)q(t) &= \pi_1 e^{t\mathcal{A}}(u^0 - (x+D_1)q(0), u^1 - (x+D_1)\dot{q}(0)) + \int_0^t \pi_1 e^{(t-s)\mathcal{A}}(0, -(x+D_1)\ddot{q}(s))ds \\
&= \sum_{n=0}^{\infty}\left[(u^0 - (x+D_1)q(0), c_n)\cos(\sqrt{-\lambda_n}t) + \frac{(u^1 - (x+D_1)\dot{q}(0), c_n)}{\sqrt{-\lambda_n}}\sin(\sqrt{-\lambda_n}t)\right]c_n(x) \\
&+ \int_0^t \sum_{n=0}^{\infty} \frac{(-(x+D_1)\ddot{q}(s), c_n)}{\sqrt{-\lambda_n}}\sin(\sqrt{-\lambda_n}(t-s))c_n(x)ds \\
&= \sum_{n=0}^{\infty}\left[(u^0 - (x+D_1)q(0), c_n)\cos(\sqrt{-\lambda_n}t) + \frac{(u^1, c_n)}{\sqrt{-\lambda_n}}\sin(\sqrt{-\lambda_n}t)\right]c_n(x) \\
&+ \sum_{n=0}^{\infty} -\frac{((x+D_1), c_n)}{\sqrt{-\lambda_n}}\left[\dot{q}(0)\sin(\sqrt{-\lambda_n}t) + \int_0^t \ddot{q}(s)\sin(\sqrt{-\lambda_n}(t-s))ds\right]c_n(x) \\
&= \sum_{n=0}^{\infty}\left[(u^0 - (x+D_1)q(0), c_n)\cos(\sqrt{-\lambda_n}t) + \frac{(u^1, c_n)}{\sqrt{-\lambda_n}}\sin(\sqrt{-\lambda_n}t)\right]c_n(x) \\
&+ \sqrt{\frac{2}{D_1}}\sum_{n=0}^{\infty}\left[q(t) - q(0)\cos(\sqrt{-\lambda_n}t) - \sqrt{-\lambda_n}\int_0^t \sin(\sqrt{-\lambda_n}(t-s))q(s)ds\right]\frac{c_n(x)}{\lambda_n}
\end{aligned}$$

where we have used

$$((x+D_1), c_n) = -\sqrt{\frac{2}{D_1}}\frac{1}{\lambda_n}$$

and

$$\int_0^t \ddot{q}(s)\sin(\sqrt{-\lambda_n}(t-s))ds = -\sin(\sqrt{-\lambda_n}t)\dot{q}(0) + \sqrt{-\lambda_n}q(t) - \sqrt{-\lambda_n}q(0)\cos(\sqrt{-\lambda_n}t) + \lambda_n\int_0^t \sin(\sqrt{-\lambda_n}(t-s))q(s)ds.$$

Next from

$$(x+D_1)q(t) = q(t)\sum_{n=0}^{\infty}((x+D_1), c_n)c_n(x) = -q(t)\sqrt{\frac{2}{D_1}}\sum_{n=0}^{\infty}\frac{1}{\lambda_n}c_n(x)$$

and

$$\int_0^t \sin(\sqrt{-\lambda_n}(t-s))q(s)ds = \frac{q(t)}{\sqrt{-\lambda_n}} - \frac{q(0)}{\sqrt{-\lambda_n}}\cos(\sqrt{-\lambda_n}t) - \int_0^t \frac{\cos(\sqrt{-\lambda_n}(t-s))}{\sqrt{-\lambda_n}}\dot{q}(s)ds$$

we can again simplify the expression for u to obtain

$$\begin{aligned} u(t,x) &= \sum_{n=0}^{\infty}\left[(u^0-(x+D_1)q(0),c_n)\cos(\sqrt{-\lambda_n}t)+\frac{(u^1,c_n)}{\sqrt{-\lambda_n}}\sin(\sqrt{-\lambda_n}t)\right]c_n(x) \\ &+\sqrt{\frac{2}{D_1}}\sum_{n=0}^{\infty}\left[q(t)-\int_0^t\cos(\sqrt{-\lambda_n}(t-s))\dot{q}(s)ds\right]\frac{c_n(x)}{-\lambda_n} \\ &:= w+f \end{aligned} \tag{58}$$

Step 11: The candidate is a weak solution

Now, for arbitrary

$$u^0\in H^2\cap H^1_{(-D_1)},\quad u^1\in H^1_{(-D_1)},\quad q\in H^1(0,T),\quad u_x(0)=q(0)$$

we check that the previous candidate u defined by (58) is a weak solution. We start by showing it is well defined:

$$u\in H^1(0,T;L^2)\cap L^2(0,T;H^1_{(-D_1)}).$$

Observe that in (58),

$$w=\pi_1 e^{t\mathcal{A}}(u^0-(x+D_1)q(0),u^1)\in C^2([0,T];L^2)\cap C^1([0,T];H^1_{(-D_1)})\cap C([0,T];H^2_{(0)}\cap H^1_{(-D_1)})$$

so it is sufficient to show that

$$f\in H^1(0,T;L^2)\cap L^2(0,T;H^1_{(-D_1)})$$

where f is, up to now, defined by (58) as a formal series. We will show that the series f converges in both spaces, that is we will show that the series defined by

$$\sum f_n\frac{c_n}{-\lambda_n},\quad f_n(t):=q(t)-\int_0^t\cos(\sqrt{-\lambda_n}(t-s))\dot{q}(s)ds$$

converges in $H^1(0,T;L^2)$ and in $L^2(0,T;H^1_{(-D_1)})$. Observe that $f_n\in C^2[0,T]$ with

$$f_n'(t)=\sqrt{-\lambda_n}\int_0^t\sin(\sqrt{-\lambda_n}(t-s))\dot{q}(s)ds,\quad f_n''(t)=-\lambda_n\int_0^t\cos(\sqrt{-\lambda_n}(t-s))\dot{q}(s)ds.$$

Hence the series f converges in $H^1(0,T;L^2)$ if and only if

$$\sum_{n=0}^{\infty}\frac{\|f_n\|^2_{H^1(0,T)}}{\lambda_n^2}<\infty$$

We can see that the above inequality holds true bounding f_n and f_n' in $L^\infty(0,T)$:

$$\|f_n\|_{L^\infty(0,T)}\lesssim\|q\|_{H^1(0,T)},\quad \|f_n'\|_{L^\infty(0,T)}\lesssim\sqrt{-\lambda_n}\|q\|_{H^1(0,T)}$$

so that

$$\sum_{n=0}^{\infty} \frac{\|f_n\|_{H^1(0,T)}^2}{\lambda_n^2} \lesssim \|q\|_{H^1(0,T)}^2 \sum_{n=0}^{\infty} \frac{1}{-\lambda_n}$$

and invoking

$$0 < -\lambda_n \sim \left(\frac{\pi}{D_1}\right)^2 n^2$$

we deduce the convergence of the series.

Further to get the convergence in $L^2(0,T;H^1_{(-D_1)})$ we use the orthogonality of (c_n) in $H^1_{(-D_1)}$ with

$$\|c_n\|_{H^1_{(-D_1)}}^2 = -\lambda_n$$

so that the series converges in $L^2(0,T;H^1_{(-D_1)})$ if and only if

$$\sum_{n=0}^{\infty} \frac{\|f_n\|_{L^2(0,T)}^2}{-\lambda_n} < \infty$$

which follows from the above computations.

We then check that u is a weak solution of (48), we observe w is the weak solution of

$$\begin{cases} w_{tt} &= w_{xx} \\ w_x(t,0) &= 0 \\ w(t,-D_1) &= 0 \\ w(0,x) &= u^0(x) - (x+D_1)q(0) \\ w_t(0,x) &= u^1 \end{cases}$$

so that, by superposition, u is the weak solution of (48) if and only if f is a weak solution of

$$\begin{cases} f_{tt} &= f_{xx} \\ f_x(t,0) &= q(t) \\ f(t,-D_1) &= 0 \\ f(0,x) &= (x+D_1)q(0) \\ f_t(0,x) &= 0 \end{cases} \tag{59}$$

To check this, observe we have the correct initial condition

$$f(0) = \sqrt{\frac{2}{D_1}} \sum_{n=0}^{\infty} q(0) \frac{c_n(x)}{-\lambda_n} = (x+D_1)q(0)$$

so we are left to check weak formulation. Let ϕ as in the weak formulation, we wish that for almost every $\tau \in [0,T]$,

$$\int_{-D_1}^0 \phi(\tau,x) f_t(\tau,x)dx - \int_0^\tau \int_{-D_1}^0 \phi_t(t,x) f_t(t,x)dxdt + \int_0^\tau \int_{-D_1}^0 \phi_x(t,x) f_x(t,x)dxdt = \int_0^\tau \phi(t,0)q(t)dt.$$

To this aim we observe that because the series making f converges in $H^1(0,T;L^2)$ we can intervert the time derivative and the series to obtain in $L^2(0,T;L^2)$

$$f_t(\tau,x) = -\sqrt{\frac{2}{D_1}}\sum_{n=0}^{\infty}\left[\frac{\sin(\sqrt{-\lambda_n}\tau)q(0)}{\sqrt{-\lambda_n}} + \int_0^\tau \cos(\sqrt{-\lambda_n}(\tau-s))q(s)ds\right]c_n(x).$$

Similarly

$$f_x(\tau,x) = -\sqrt{\frac{2}{D_1}}\sum_{n=0}^{\infty}\left[q(\tau) - \int_0^\tau \cos(\sqrt{-\lambda_n}(\tau-s))\dot{q}(s)ds\right]\frac{C_n(x)}{\sqrt{-\lambda_n}}$$

where

$$C_n(x) := -\frac{1}{\sqrt{-\lambda_n}}\frac{d}{dx}c_n.$$

Further we compute

$$\begin{aligned}\int_0^\tau\int_{-D_1}^0 \phi_x(t,x)f_x(t,x)dxdt &= \int_0^\tau\int_{-D_1}^0 -\sqrt{\frac{2}{D_1}}\sum_{n=0}^{\infty}\left[q(t) - \int_0^t \cos(\sqrt{-\lambda_n}(t-s))\dot{q}(s)ds\right]\frac{C_n(x)}{\sqrt{-\lambda_n}}\phi_x(t,x)dxdt \\ &= -\sqrt{\frac{2}{D_1}}\sum_{n=0}^{\infty}\int_0^\tau\int_{-D_1}^0\left[q(t) - \int_0^t \cos(\sqrt{-\lambda_n}(t-s))\dot{q}(s)ds\right]\frac{C_n(x)}{\sqrt{-\lambda_n}}\phi_x(t,x)dxdt \\ &= \sqrt{\frac{2}{D_1}}\sum_{n=0}^{\infty}\int_0^\tau\int_{-D_1}^0\left[q(t) - \int_0^t \cos(\sqrt{-\lambda_n}(t-s))\dot{q}(s)ds\right]c_n(x)\phi(t,x)dxdt\end{aligned}$$

where the interversion series-integral is justified because of the L^2 convergence and the last equality is an integration by part in space. With very similar considerations we obtain

$$\int_0^\tau\int_{-D_1}^0 \phi_t(t,x)f_t(t,x)dxdt = \sqrt{\frac{2}{D_1}}\sum_{n=0}^{\infty}\left\{\begin{array}{l}\frac{1}{\sqrt{-\lambda_n}}\int_0^\tau \sin(\sqrt{-\lambda_n}(t-s))\dot{q}(s)ds \times \int_{-D_1}^0 \phi(\tau,x)c_n(x)dx \\ -\int_0^\tau\int_0^t \cos(\sqrt{-\lambda_n}(t-s))\dot{q}(s)ds\int_{-D_1}^0 \phi(t,x)c_n(x)dxdt\end{array}\right\}$$

and

$$\int_{-D_1}^0 \phi(\tau,x)f_t(\tau,x)dx = \sqrt{\frac{2}{D_1}}\sum_{n=0}^{\infty}\int_{-D_1}^0\int_0^t\frac{\sin(\sqrt{-\lambda_n}(t_s))}{\sqrt{-\lambda_n}}\dot{q}(s)ds\phi(\tau,x)c_n(x)dx$$

which allows one to check the weak formulation. Hence f is the weak solution of (59), and u is the weak solution of (48).

Step 11: Additionnal regularity for the candidate

We now use the representation formula (58) to show that when

$$u^0 \in H^2, \quad v^0 \in H^1, \quad q \in H^1, \quad u_x^0(0) - q(0) = u^0(-D_1) = v^0(-D_1) = 0$$

we have

$$u \in C^2([0,T];L^2) \cap C^1([0,T];H^1) \cap C([0,T];H^2).$$

We again use the fact that w is in the above space, hence it is enough to check that

$$f \in C^2([0,T];L^2) \cap C^1([0,T];H^1) \cap C([0,T];H^2).$$

We will first show that $f \in C^2([0,T];L^2)$, we begin by showing that $f \in H^2(0,T;L^2)$. To this aim we will show that there exists $C = C(T, D_1) > 0$ such that

$$\forall \alpha \in L^2(0,T), \quad \forall t \in [0,T], \quad \sum_{n=0}^{\infty} \left| \int_0^t \cos(\sqrt{-\lambda_n}(t-s))\alpha(s)ds \right|^2 \leq C\|\alpha\|_{L^2(0,t)}^2. \tag{60}$$

Let $\alpha \in L^2(0,T)$, $0 \leq t \leq T$ and

$$\beta(s) := \begin{cases} \alpha(s) & \text{if} \quad 0 \leq s \leq t \\ 0 & \text{if} \quad s \in \mathbb{R} \setminus [0,t] \end{cases}$$

which is such that

$$\beta \in L_c^2(\mathbb{R}), \quad \|\beta\|_{L^2(\mathbb{R})} = \|\alpha\|_{L^2(0,t)}.$$

We also introduce $N \in \mathbb{N}^*$ such that

$$T \leq N D_1.$$

We then have

$$\begin{aligned}
\sum_{n=0}^{\infty} \left| \int_0^t \cos(\sqrt{-\lambda_n}(t-s))\alpha(s)ds \right|^2 &= \sum_{n=0}^{\infty} \left| \int_0^t \cos(\sqrt{-\lambda_n}(t-s))\beta(s)ds \right|^2 \\
&= \sum_{n=0}^{\infty} \left| \int_0^t \cos(\sqrt{-\lambda_n}s)\beta(t-s)ds \right|^2 \\
&= \sum_{n=0}^{\infty} \left| \int_0^{ND_1} \cos(\sqrt{-\lambda_n}s)\beta(t-s)ds \right|^2 \qquad (61) \\
&= \sum_{n=0}^{\infty} \left| \sum_{k=0}^{N-1} \int_{kD_1}^{(k+1)D_1} \cos(\sqrt{-\lambda_n}s)\beta(t-s)ds \right|^2 \\
&\leq \sum_{n=0}^{\infty} N \sum_{k=0}^{N-1} \left| \int_{kD_1}^{(k+1)D_1} \cos(\sqrt{-\lambda_n}s)\beta(t-s)ds \right|^2 \\
&= N \sum_{k=0}^{N-1} \sum_{n=0}^{\infty} \left| \int_{kD_1}^{(k+1)D_1} \cos(\sqrt{-\lambda_n}s)\beta(t-s)ds \right|^2 \qquad (62)
\end{aligned}$$

where the inequality arises from the convexity of the square on $\mathbb{R}$, and the last equality is the Fubini-Tonelli theorem for non-negative summands. We remember that

$$\cos(\sqrt{-\lambda_n}s) = \sqrt{\frac{D_1}{2}} c_n(s)$$

where (c_n) is a Hilbert basis of $L^2(-D_1, 0)$. It is easy to see that (c_n) is also a Hilbert basis of $L^2(0, D_1)$, for instance because the cosine is an even function. Thereore the Parseval (see *e.g.* [2] Corollary 5.10) formula yields

$$\sum_{n=0}^{\infty} \left| \int_0^{D_1} \cos(\sqrt{-\lambda_n}s)\beta(t-s)dsds \right|^2 = \frac{D_1}{2} \|\beta(t-\cdot)\|_{L^2(0,D_1)}^2 \leq \frac{D_1}{2} \|\alpha\|_{L^2(0,t)}^2.$$

This shows how to control the summand $k = 0$ in (62), we now explain how to control the term $k = 1$. With a change of variable and the definition of λ_n we have

$$\int_{D_1}^{2D_1} \cos(\sqrt{-\lambda_n} s)\beta(t-s)ds = \int_0^{D_1} \cos(\sqrt{-\lambda_n}(s+D_1))\beta(t-s-D_1)ds$$
$$= (-1)^{n+1} \int_0^{D_1} \sin(\sqrt{-\lambda_n} s)\beta(t-s-D_1)ds$$

and the functions

$$s_n(x) := \sin(\sqrt{-\lambda_n} x), \quad n \in \mathbb{N}$$

are orthogonal in $L^2(0, D_1)$ with

$$\|s_n\|^2_{L^2(0,D_1)} = \frac{D_1}{2}.$$

Hence the Bessel inequality (see e.g. 5.26 of Chapter 5 from [8]) yields

$$\sum_{n=0}^{\infty} \left| \int_{D_1}^{2D_1} \cos(\sqrt{-\lambda_n} s)\beta(t-s)ds \right|^2 = \sum_{n=0}^{\infty} \left| \int_0^{D_1} s_n(s)\beta(t-s-D_1)ds \right|^2$$
$$\leq \frac{D_1}{2} \|\beta(t - \cdot - D_1)\|^2_{L^2(0,D_1)}$$
$$\leq \frac{D_1}{2} \|\alpha\|^2_{L^2(0,t)}.$$

By induction we see that this inequality holds for any $k = 0, ..., N-1$ and (60) is true with

$$C = N^2 \frac{D_1}{2}$$

where N only depends on T, D_1, as desired.

Next observe that the series representing f converges in $H^2(0, T; L^2)$ if and only if

$$\sum_{n=0}^{\infty} \frac{\|f_n''\|^2_{L^2(0,T)}}{\lambda_n^2} < \infty$$

which we check using (60)

$$\sum_{n=0}^{\infty} \frac{\|f_n''\|^2_{L^2(0,T)}}{\lambda_n^2} = \sum_{n=0}^{\infty} \int_0^T \left| \int_0^t \cos(\sqrt{-\lambda_n}(t-s))\dot{q}(s)ds \right|^2 dt$$
$$= \int_0^T \sum_{n=0}^{\infty} \left| \int_0^t \cos(\sqrt{-\lambda_n}(t-s))\dot{q}(s)ds \right|^2 dt$$
$$\lesssim \int_0^T \|\dot{q}\|^2_{L^2(0,t)} dt \lesssim \|q\|^2_{H^1(0,T)}.$$

We now derive several consequence concerning f, first $f \in H^2(0,T;L^2)$ and we can compute f_{tt} differentiating term by terms the series representing f. Second f_{tt} has the following representative, labeled abusing notation,

$$f_{tt}(t) = \sqrt{\frac{2}{D_1}} \sum_{n=0}^{\infty} \left[\int_0^t \cos(\sqrt{-\lambda_n}(t-s))\dot{q}(s)ds \right] c_n$$

which is such that for any $t \in [0,T]$, the series converges in L^2 (same application of (60)). We will show that this representative is $C([0,T];L^2)$. We let $0 \leq t < t+h \leq T$ and we compute

$$\begin{aligned} \|f''(t+h) - f''(t)\|_{L^2}^2 &\lesssim \sum_{n=0}^{\infty} \frac{|f_n''(t+h) - f_n''(t)|^2}{\lambda_n^2} \\ &= \sum_{n=0}^{\infty} \left| \int_0^{t+h} \cos(\sqrt{-\lambda_n}(t+h-s)\dot{q}(s)ds - \int_0^t \cos(\sqrt{-\lambda_n}(t-s)\dot{q}(s)ds \right|^2 \end{aligned}$$

where denoting α the extension of $\dot{q}$ by 0 outside of $[t,t+h]$ we get

$$\begin{aligned} \int_0^{t+h} \cos(\sqrt{-\lambda_n}(t+h-s)\dot{q}(s)ds - \int_0^t \cos(\sqrt{-\lambda_n}(t-s)\dot{q}(s)ds &= \int_0^t \cos(\sqrt{-\lambda_n}s)(\alpha(t+h-s) - \alpha(t-s))ds \\ &+ \int_t^{t+h} \cos(\sqrt{-\lambda_n}s)\alpha(t+h-s)ds. \end{aligned}$$

In view of (60) the series associated to the first term of the right hand side is controlled, up to a constant, by

$$\|\alpha(t+h-\cdot) - \alpha(t-\cdot)\|_{L^2(0,t)}^2 = \|\alpha(\cdot+h) - \alpha(\cdot)\|_{L^2(0,t)}^2 \underset{h\to 0^+}{\longrightarrow} 0$$

by continuity of the translations in $L^2(0,t)$. We deal with the remaining term writting

$$\int_t^{t+h} \cos(\sqrt{-\lambda_n}s)\alpha(t+h-s)ds = \int_0^{ND_1} \cos(\sqrt{-\lambda_n}s)\alpha(t+h-s)ds$$

and we observe that coming back to (61), the associated series is controlled by (up to a constant)

$$\|\alpha(t+h-\cdot)\|_{L^2(t,t+h)}^2 = \|\alpha(\cdot)\|_{L^2(t,t+h)}^2 \underset{h\to 0^+}{\longrightarrow} 0.$$

Therefore, $f'' : [0,T] \to L^2$ is right continuous, we can adapt the above arguments to show that f'' is also left continuous, whence $f \in C([0,T];L^2)$ and $f \in C^2([0,T];L^2)$.

To conclude on the regularity of f we remember that

$$f_{tt} = f_{xx} \quad \text{in} \quad \mathcal{D}'(0,T;(H_0^1)^*)$$

whence

$$f_{xx} \in C([0,T];L^2)$$

which brings

$$f \in C([0,T];H^2)$$

by elliptic regularity. Finally $f \in C^1([0,T];H^1)$ is obtained from the formal equality

$$f_{tx} = -\sum_{n=0}^{\infty} \left[\int_0^t \sin(\sqrt{-\lambda_n}(t-s))\dot{q}(s)ds \right] s_n(x) \in C([0,T];L^2)$$

which can be made rigorous adapting the above arguments and in view of (60).

<u>Step 12: The candidate is a classical solution</u>

To finish the proof of the Lemma we are left to show that u is a classical solution of (48). Note that $u(0) = u^0$ and we can show that $u_t(0) = v^0$ testing

$$u_{tt} = u_{xx} \quad \text{in} \quad \mathcal{D}'(0,T;(H_0^1)^*)$$

against an arbitrary $\phi \in C_c^\infty([0,T);H_0^1)$ to discover

$$\begin{aligned}
-\langle u_t(0), \phi(0)\rangle_{V^*,V} &= [\langle u_t(t), \phi(t)\rangle_{V^*,V}]_0^T \\
&= \int_0^T \{\langle u_{tt}, \phi\rangle_{V^*,V} + \langle u_t, \phi'\rangle_{V^*,V}\} \\
&= \int_0^T \left\langle \frac{d^2}{dx^2}u, \phi \right\rangle_{V^*,V} + \int_0^T \int_{-D_1}^0 u_t\phi_t \\
&= -\int_0^T \int_{-D_1}^0 u_x\phi_x + \int_0^T \int_{-D_1}^0 u_t\phi_t \\
&= -\int_{-D_1}^0 \phi(0,x)v^0(x)dx = -\langle v^0, \phi(0)\rangle_{V^*,V}
\end{aligned}$$

Now observe the PDE $u_{tt} = u_{xx}$ holds in the distributional sense and both functions are $C([0,T];L^2)$, hence it holds in $C([0,T];L^2)$. Finally we check that $u_x(t,0) = q(t)$. Test $u_{tt} = u_{xx}$ against an arbitrary test function ϕ as in the weak formulation, to discover for any $\tau \in [0,T]$

$$\begin{aligned}
\int_0^\tau \phi(t,0)q(t)dt - \int_0^\tau \int_{-D_1}^0 \phi_x u_x &= \int_{-D_1}^0 \phi(\tau,x)u_t(\tau,x)dx - \int_{-D_1}^0 \phi(0,x)v^0(x)dx - \int_0^\tau \int_{-D_1}^0 \phi_t u_t dxdt \\
&= \int_0^\tau \phi(t,0)u_x(t,0)dt - \int_0^\tau \int_{-D_1}^0 \phi_x u_x
\end{aligned}$$

which allows

$$\int_0^\tau \phi(t,0)q(t)dt = \int_0^\tau \phi(t,0)u_x(t,0)dt.$$

Differentiate this equality with respect to τ and take a test function ϕ such that $\phi(t,0) \equiv 1$ to get $u_x(t,0) = q(t)$. $\square$

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