UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Corso di Laurea Magistrale in Fisica

Tesi di Laurea

General relativistic effects
in large scale galaxy clustering

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"I believe in the possible. I believe small though we are, insignificant we may be, we can reach a full understanding of the Universe. You were right, when you said you felt small looking up at all that out there. We are very, very small, but we are profoundly capable of very, very big things."

"Hawking", BBC film (2004).

Abstract

The size of galaxy redshift surveys has constantly increased over the latest years in terms of solid angle and redshift coverage; therefore, the future large-volume galaxy surveys (e.g. SKA, Euclid) will allow us to measure galaxy clustering on scales comparable with the Hubble radius. For such large scales and at high redshifts, general relativistic effects alter the observed galaxy number overdensity through projection onto our past lightcone. They consist mainly in redshift space distortions and gravitational lensing convergence; however, further corrections may now be measured and should be taken into account, including Doppler, standard and integrated Sachs-Wolfe effect, and time delay contributions. These corrections lead to new terms in wide-angle two-point correlation functions, going beyond the plane-parallel and Newtonian approach.

The aim of this thesis is to investigate and compute the full general relativistic expression for the angular correlation functions, including all redshift-space distortions, wide-angle, lensing and gravitational potential effects on linear scales. In particular, this angular power spectrum is written in order to generalise the 2-FAST algorithm recently developed in literature, which circumvents the direct integration of highly oscillating spherical Bessel functions. The Limber's approximation is also applied for large multipole moments ℓ in order to give a simplified expression of the angular power spectrum, more suitable for further numerical integration.

Contents

Chapter 1

Introduction

1.1 The standard ΛCDM model

The development of general relativity in the last century by Einstein provided a compelling theory of the Universe [1]. This powerful theory is assumed to be the correct theory of gravity on cosmological scales by the *standard model* of Big Bang cosmology, socalled because it provides the simplest concordance with the following properties, both theoretical and observational (see Fig.1.2 and 1.3):

- the expansion exhibited by the Hubble diagram;
- light elements abundances, in accord with the Big Bang nucleosynthesis;
- the blackbody radiation left from the first 400000 years, that is the Cosmic Microwave Background radiation (CMB) (see Fig.1.1);
- the large scale structure of galaxies distribution.

The most reliable model is called ΛCDM because it provides:

- a cosmological constant Λ, which has the modern meaning of quantum vacuum energy, and in this cosmological model is associated with the dark energy in order to explain the accelerated expansion of the Universe;
- Cold Dark Matter (CDM), which is a form of non-baryonic matter necessary to take into account the gravitational effects observated in large scale structures (e.g. galaxies rotation curves, gravitational lensing of light by galaxy clusters).

Then, ΛCDM assumes: the *cosmological principle*, which provides that on sufficiently large scales the Universe looks homogeneous (invariant under translations) and isotropic (invariant under rotations); a "flat" spatial geometry, and an expanding spacetime (these concepts will be further explained in the following sections). In this chapter we are going to start with the basic assumption of the ΛCDM as valid, because it is consistent with the latest experimental observations (e.g. Planck [2]), thus allowing to reliably constrain cosmological parameters. Then we will explore the homogeneous background Universe, focusing on the background Friedmann-Robertson-Walker metric and the concept of expansion of the spacetime. After, the inhomogeneous Universe will be presented, because at smaller scales galaxies appear distributed in a pattern that is not homogeneous any more (see Fig.1.13); we are going to see how matter density fluctuations generated in the early Universe were "frozen" at the time of decoupling,

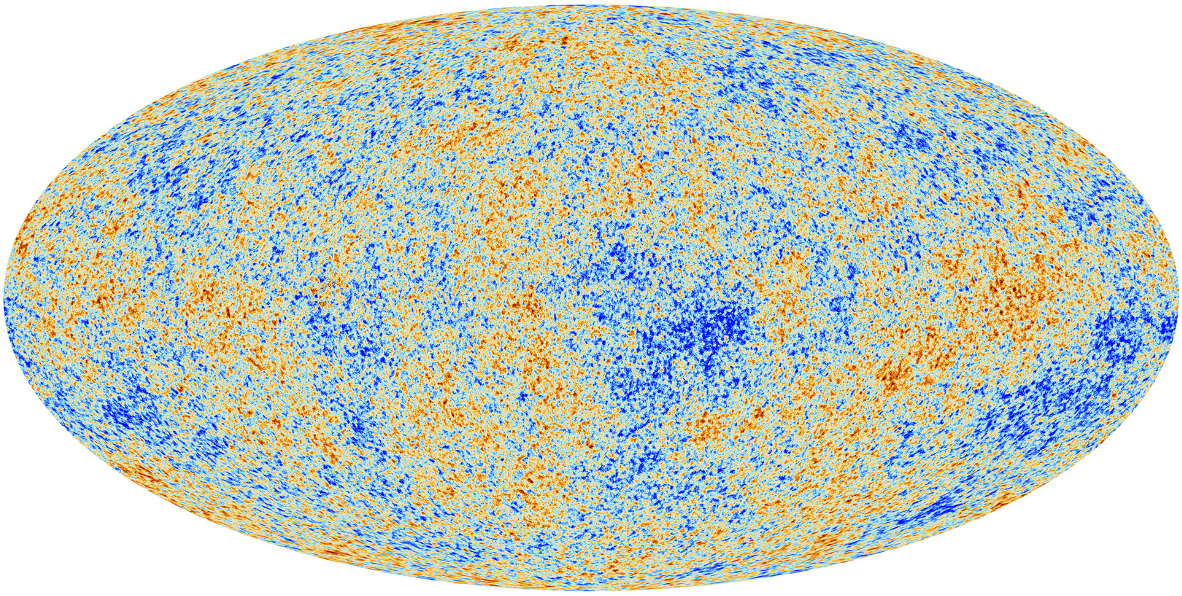


Figure 1.1: *The anisotropies of the Cosmic Microwave Background (CMB) as observed by ESA's Planck. The CMB is a snapshot of the oldest light in our Universe, imprinted on the sky when the Universe was just 380 000 years old. It shows tiny temperature fluctuations that correspond to regions of slightly different densities, representing the seeds of all future structure: the stars and galaxies of today. Copyright: ESA and the Planck Collaboration [3].*

when CMB was produced and photons began to propagate freely; so CMB anisotropies are a sort of "photograph" of that primordial epoch, and in principle they allow us to gain information about the early phases of the Universe, and to study the pattern of the galaxies distribution itself. The evolution of the inhomogeneities is studied with the perturbed Einstein equations, which involve both *geometry* (through the metric) and *matter density* (through the stress-energy tensor). For small fluctuations the *linear perturbation theory* can be used and will be presented. Through Einstein equation themselves, metric perturbations are connected to the matter density fluctuations, which are involved in the statistical analysis.

First of all, we are going to summarize the stages in the evolution of the Universe, as predicted by the ΛCDM model.

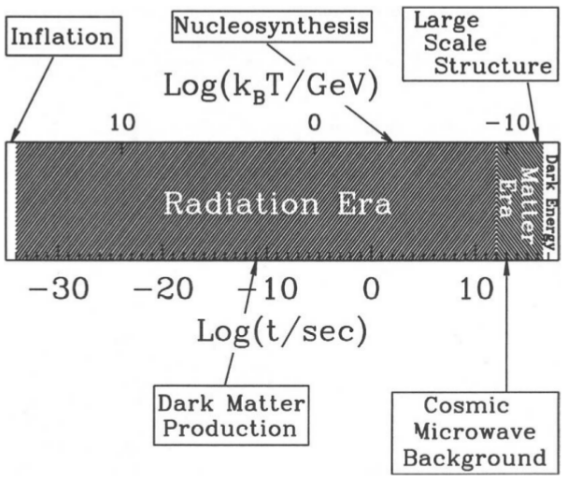


Figure 1.2: *A brief history of the Universe, where any epoch can be characterized by the time since the Big Bang (bottom scale) or by the temperature (top scale). Image from Ref.[1].*

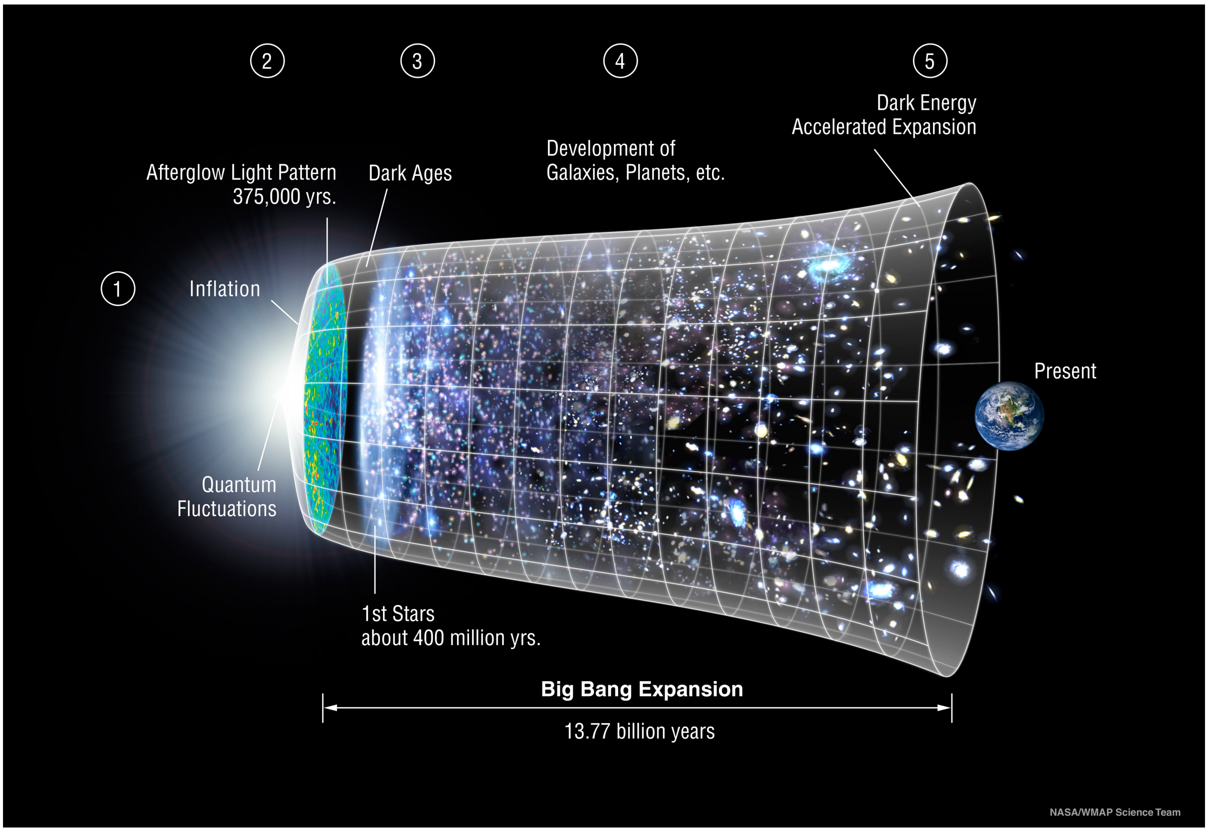


Figure 1.3: *The evolution of the Universe according the standard ΛCDM cosmological model. Image Credit: Adaptation of original NASA WMAP Science Team image [4].*

1.1.1 A brief history of the Universe

About 13,8 billion years ago the Universe was in a hot, dense and nearly uniform state. With the Big Bang, the expansion began. The early Universe was composed of a mixture of photons and matter, tightly coupled together as a plasma, which evolution is approximately pictured in Fig.1.3.

1) At very early time ($\sim 10^{-32}$ s) the mechanism of exponential expansion given by the cosmic inflation established the initial conditions: homogeneity, isotropy, and flatness [5, 6]. This epoch was dominated by radiation, and baryonic matter consisted in free electrons and atomic nuclei with photons bouncing between them, forming the primordial plasma. Further, the microscopic quantum fluctuations were amplified to density perturbations, propagating through the plasma collisionally as a sound wave; this produced under- and overdensities in the plasma, with simultaneous density changes in both matter and radiation. CDM was not involved in these pressure-induced oscillation, but acted gravitationally, thus enhancing or negating the acoustic pattern for photons and baryons [7]. This phenomenon is known as Baryon Acoustic Oscillations (BAO).

2) A bit less than 400000 years after the Big Bang, expansion let the Universe become cool enough ($\sim$ 3000 K) to allow protons capturing electrons, and thus forming neutral hydrogens atoms, and soon after helium (recombination). At that time, photons didn't interact any more with charged particles since they had been bounded in neutral atoms; therefore, photons began to propagate freely, being decoupled from matter. Perturbations then didn't propagate as acoustic

waves any more, but the pattern that existed at that time became "frozen". [8] It was the origin of the CMB, that fills the Universe still at present time and that has been detected with a redshift $z \sim 1100$. Measures have been performed by COBE, WMAP and Planck (see Fig. 1.4 for a comparison [9], and the following subsection).

3) The period after recombination is known also as the "Dark Ages", because then the Universe was largely neutral and therefore unobservable throughout most of the electromagnetic spectrum. During this epoch, cold dark matter began to collapse gravitationally in overdense regions; then, the gravitational collapse of baryonic matter into these CDM halos occurred, and the era of the "Cosmic Dawn" began with the formation of the first radiation sources, such as stars; then the radiation produced by these objects reionized the intergalactic medium.

4) Due to gravity, cosmic structures growed and merged, forming a large cosmic web of dark matter density. The abundance of luminous galaxies traced the statistics of the underlying matter density. The largest bound objects that can be observed are clusters or superclusters of galaxies and, despite this arrangement, the BAO correlation length is retained as "frozen" when the CMB was born.

5) The Universe continuously expands. The negative pressure associated with the cosmological constant, which is related to the dark energy in the ΛCDM model, becomes more and more relevant with respect to the opposing gratitational forces, so the expansion of the Universe becomes accelerated [4].

1.1.2 ΛCDM consistency

In this subsection we are going to show why the ΛCDM model is consistent with experimental observations and data, and can for this reason be assumed in this thesis work. Many experiments have been performed over the last decades with the aim of measuring and studying the CMB spectrum, through which cosmological parameters can be constrained [9]. The first one was the Cosmic Background Explorer (COBE), launched by NASA in 1989, which discovered that averaging the CMB spectrum across the whole sky, it was substantially that of a black body at a temperature of 2,73 K, even though it showed very small fluctuations of the order of 10^{-5}. Later in 2001 NASA's second generation space mission, the Wilkinson Microwave Anisotropy Probe (WMAP) was launched in order to study in a more detailed way these tiny fluctuations, imprinted when photons and matter decoupled (as seen before). WMAP's results helped to determine the proportions of the fundamental constituents of the Universe and to establish the standard model of cosmology prevalent today. The last experiment is ESA's Planck, launched in 2009, with a wider frequency range in more bands and a higher sensitivity than WMAP, as shown in Fig.1.4 which reports the difference between WMAP and Planck. Planck is a space-based observatory observing the Universe at wavelengths between 0.3 mm and 11.1 mm (corresponding to frequencies between 27 GHz and 1 THz), broadly covering the far-infrared, microwave, and high frequency radio domains. The results that we are going to report are from one of the latest articles of Planck collaboration [2], without discussing in details the methods of analysis but only explaining the cosmological parameters constrained and the final results; more details can be found in all the articles of the Planck collaboration. The ΛCDM model is assumed, with purely adiabatic scalar primordial perturbations with a power-law spectrum. Three

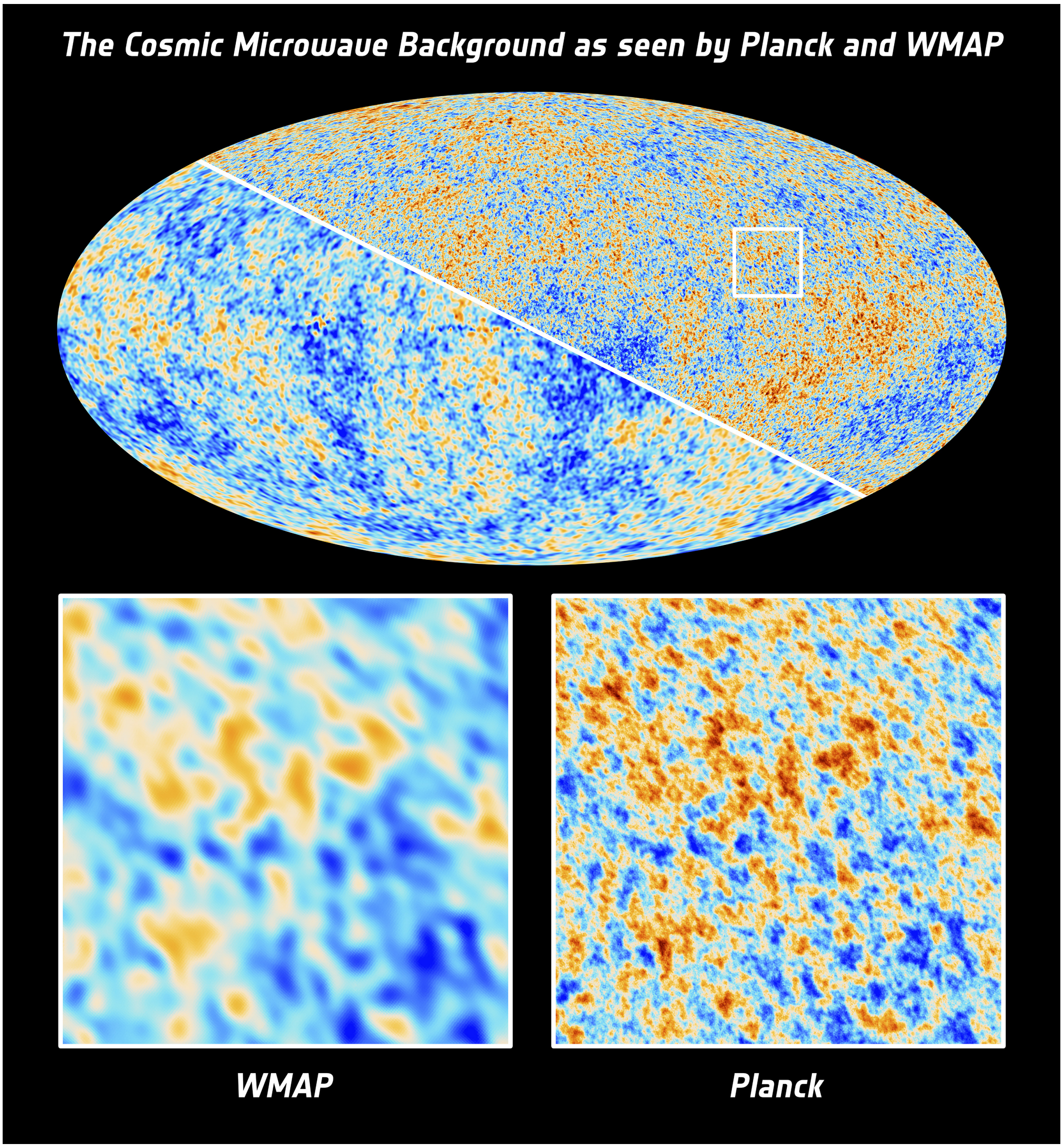


Figure 1.4: *This image shows the Cosmic Microwave Background as seen by ESA's Planck satellite (upper right half) and by its predecessor, NASA's Wilkinson Microwave Anisotropy Probe (lower left half). With greater resolution and sensitivity over nine frequency channels, Planck has delivered the most precise image so far of the Cosmic Microwave Background. The Planck image is based on data collected over the first 15.5 months of the mission; the WMAP image is based on nine years of data. Copyright: ESA and the Planck Collaboration; NASA / WMAP Science Team [10].*

neutrinos species are assumed, approximated as two massless states and a single massive neutrino of mass [1]$m_\nu = 0.06$ eV. Then the following features are set: flat priors on the baryon density $\omega_b = \Omega_b h^2$, cold dark matter density $\omega_c = \Omega_c h^2$, an approximation to the observed angular size of the sound horizon at recombination θ_{MC} (as reported below), the reionization optical depth τ, the initial super-horizon amplitude of curvature perturbations A_s at [2]$k = 0.05\,\mathrm{Mpc}^{-1}$, and the primordial spectral index n_s (see Ref.[2, 12, 13]). We report here an interesting plot of the constraints on parameters of the ΛCDM model (Fig.1.5), and a table of its main parameters as extrapolated by the various Planck analyses (Fig.1.6). But first we want to recap their meaning [13]:

- $\omega_b = \Omega_b h^2$ is the baryon density today;
- $\omega_c = \Omega_c h^2$ is the cold dark matter density today;
- $100\,\theta_{MC}$ is a 100 × approximation of the sound horizon at recombination to the observed angular size r_*/D_A (used in CosmoMC, see Ref.[14, 15]);
- is Thomson scattering optical depth due to reionization;
- is the log power of the primordial curvature perturbations ($k_0 = 0.05\,\mathrm{Mpc}^{-1}$);
- n_S is the scalar spectrum power-law index ($k_0 = 0.05\,\mathrm{Mpc}^{-1}$);
- H_0 is the current expansion rate in km s^{-1} Mpc^{-1};
- Ω_Λ is the dark energy density divided by the critical density today;
- Ω_m is the matter density (including massive neutrinos) today divided by the critical density;
- 'Age' is the age of the Universe today (in Gyr);
- σ_8 measures the amplitude of the (linear) power spectrum on the scale of 8 h^{-1}Mpc;
- z_{re} is the redshift at which the Universe is half reionized.

What we can see in general in Fig.1.5 [2] is that likelihoods all overlap in a consistent region of parameter space. A likelihood function[3] is a measure of the plausibility of a parameter value of a model describing a given data set. Planck results consist indeed in these likelihoods for both polarization (E) and temperature (T) of CMB, and their combination EE, TE, TT can be seen right in Fig.1.5.
By focusing on two crucial parameters as H_0 and Ω_m, Fig.1.7 shows a very tight constrain on H_0 with the full Planck CMB power spectrum data and a ΛCDM model [2]. Further, the base-ΛCDM model assumes that the spatial hypersurfaces are flat; this prediction can be tested to high accuracy by the combination of CMB and BAO data

[1] 1 eV $= 1.9 \cdot 10^{-19}$ Joule.

[2] 1 Mpc $= 10^6$ pc, where pc stand for parsec. 1 parsec is defined as the distance at which 1 Astronomical Unit subtends an angle of 1 second of arc (1/3600 of a degree). 1 pc = 3.26 light year = 31×10^{12} km [11].

[3] Given a data set of N independent measured values x_i, f a probability density function and θ the value of the parameter on which f depends, the expression $\mathscr{L}(x_1, x_2, ...x_N; \theta) = \prod_{i=1}^{N} f(x_i; \theta)$ is the *likelihood function* and represents the probability density to obtain a certain value θ of the parameter, given the experimental data set. The most consistent value of the theoretical model is assumed to be the one that maximizes the likelihood function [16].

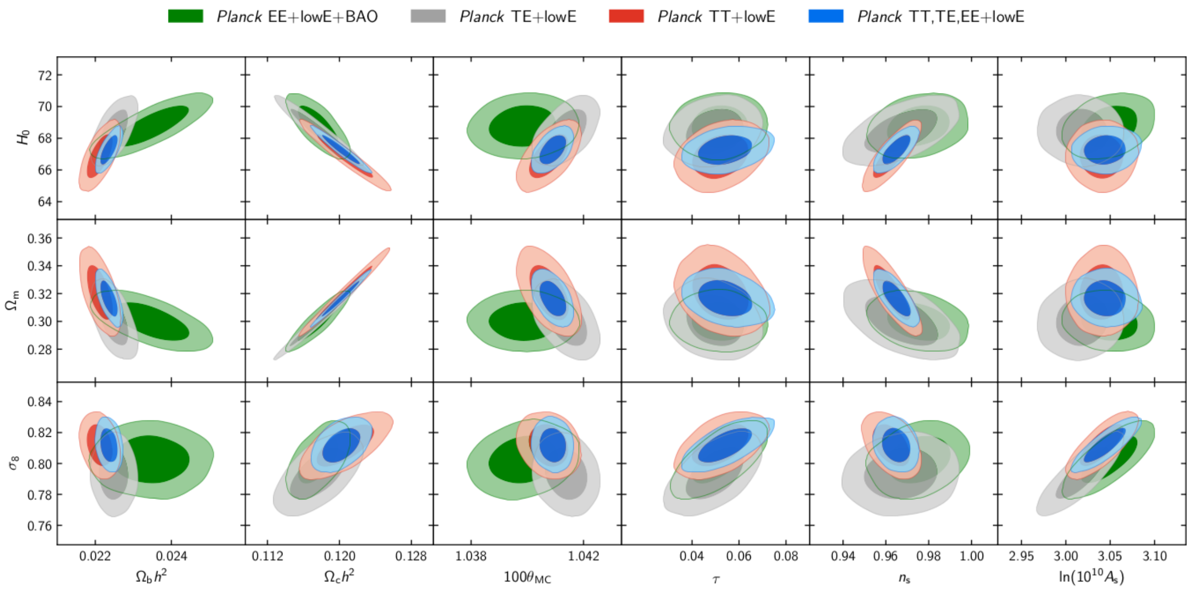


Figure 1.5: *Constraints on parameters of the base-ΛCDM model from the separate from the separate Planck EE, TE and TT high-ℓ spectra combined with low-ℓ polarization (lowE) and, in the case of EE, also with BAO, compared to the joint result using Planck TT,TE,EE+lowE. Parameters on the bottom axis are Planck's sampled MCMC parameters with flat priors, and parameters on the left axis are derived parameters (with H_0 in $kms^{-1}Mpc^{-1}$). Contours contain 68% and 95% of the probability [2].*

Parameter	TT+lowE 68% limits	TE+lowE 68% limits	EE+lowE 68% limits	TT,TE,EE+lowE 68% limits	TT,TE,EE+lowE+lensing 68% limits	TT,TE,EE+lowE+lensing+BAO 68% limits
$\Omega_b h^2$	0.02212 ± 0.00022	0.02249 ± 0.00025	0.0240 ± 0.0012	0.02236 ± 0.00015	0.02237 ± 0.00015	0.02242 ± 0.00014
$\Omega_c h^2$	0.1206 ± 0.0021	0.1177 ± 0.0020	0.1158 ± 0.0046	0.1202 ± 0.0014	0.1200 ± 0.0012	0.11933 ± 0.00091
$100\theta_{MC}$	1.04077 ± 0.00047	1.04139 ± 0.00049	1.03999 ± 0.00089	1.04090 ± 0.00031	1.04092 ± 0.00031	1.04101 ± 0.00029
τ	0.0522 ± 0.0080	0.0496 ± 0.0085	0.0527 ± 0.0090	$0.0544^{+0.0070}_{-0.0081}$	0.0544 ± 0.0073	0.0561 ± 0.0071
$\ln(10^{10}A_s)$	3.040 ± 0.016	$3.018^{+0.020}_{-0.018}$	3.052 ± 0.022	3.045 ± 0.016	3.044 ± 0.014	3.047 ± 0.014
n_s	0.9626 ± 0.0057	0.967 ± 0.011	0.980 ± 0.015	0.9649 ± 0.0044	0.9649 ± 0.0042	0.9665 ± 0.0038
H_0 [km s^{-1} Mpc^{-1}] . .	66.88 ± 0.92	68.44 ± 0.91	69.9 ± 2.7	67.27 ± 0.60	67.36 ± 0.54	67.66 ± 0.42
Ω_Λ	0.679 ± 0.013	0.699 ± 0.012	$0.711^{+0.033}_{-0.026}$	0.6834 ± 0.0084	0.6847 ± 0.0073	0.6889 ± 0.0056
Ω_m	0.321 ± 0.013	0.301 ± 0.012	$0.289^{+0.026}_{-0.033}$	0.3166 ± 0.0084	0.3153 ± 0.0073	0.3111 ± 0.0056
$\Omega_m h^2$	0.1434 ± 0.0020	0.1408 ± 0.0019	$0.1404^{+0.0034}_{-0.0039}$	0.1432 ± 0.0013	0.1430 ± 0.0011	0.14240 ± 0.00087
$\Omega_m h^3$	0.09589 ± 0.00046	0.09635 ± 0.00051	$0.0981^{+0.0016}_{-0.0018}$	0.09633 ± 0.00029	0.09633 ± 0.00030	0.09635 ± 0.00030
σ_8	0.8118 ± 0.0089	0.793 ± 0.011	0.796 ± 0.018	0.8120 ± 0.0073	0.8111 ± 0.0060	0.8102 ± 0.0060
$S_8 \equiv \sigma_8(\Omega_m/0.3)^{0.5}$.	0.840 ± 0.024	0.794 ± 0.024	$0.781^{+0.052}_{-0.060}$	0.834 ± 0.016	0.832 ± 0.013	0.825 ± 0.011
$\sigma_8\Omega_m^{0.25}$	0.611 ± 0.012	0.587 ± 0.012	0.583 ± 0.027	0.6090 ± 0.0081	0.6078 ± 0.0064	0.6051 ± 0.0058
z_{re}	7.50 ± 0.82	$7.11^{+0.91}_{-0.75}$	$7.10^{+0.87}_{-0.73}$	7.68 ± 0.79	7.67 ± 0.73	7.82 ± 0.71
$10^9 A_s$	2.092 ± 0.034	2.045 ± 0.041	2.116 ± 0.047	$2.101^{+0.031}_{-0.034}$	2.100 ± 0.030	2.105 ± 0.030
$10^9 A_s e^{-2\tau}$	1.884 ± 0.014	1.851 ± 0.018	1.904 ± 0.024	1.884 ± 0.012	1.883 ± 0.011	1.881 ± 0.010
Age [Gyr]	13.830 ± 0.037	13.761 ± 0.038	$13.64^{+0.16}_{-0.14}$	13.800 ± 0.024	13.797 ± 0.023	13.787 ± 0.020

Figure 1.6: *Parameter 68% intervals for the base-ΛCDM model from Planck CMB power spectra, in combination with CMB lensing reconstruction and BAO. The top group of six rows are the base parameters, which are sampled in the MCMC analysis with flat TT priors. The bottom group lists derived parameters [2].*

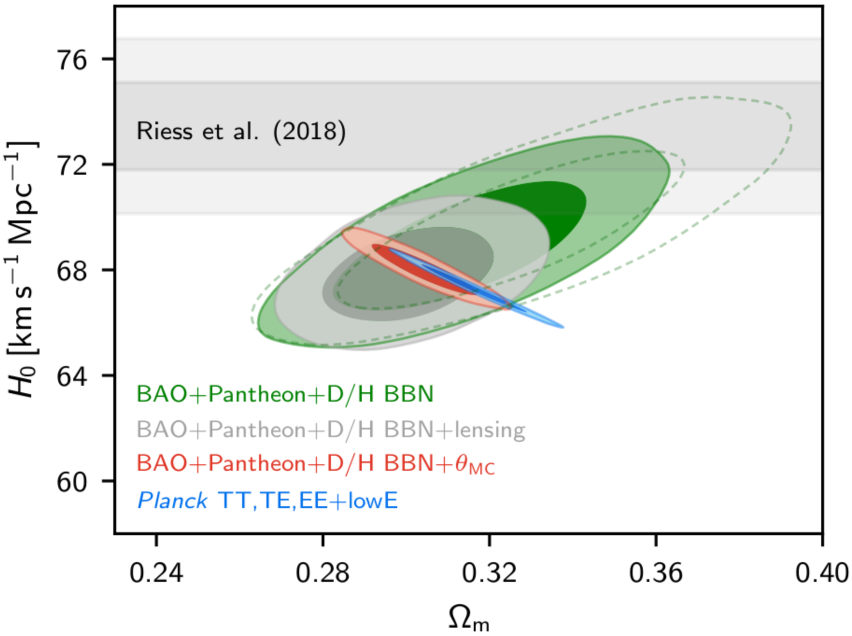


Figure 1.7: *Inverse distance-ladder constraints on the Hubble parameter H_0 and Ω_m in the base-ΛCDM model, compared to the result from the full Planck CMB power spectrum data. BAO constrains the ratio of the sound horizon at the epoch of baryon drag and the distances; the sound horizon depends on the baryon density, which is constrained by the conservative prior of $\Omega_b h^2 = 0.0222 \pm 0.0005$. Adding Planck CMB lensing constrains the matter density, or adding a conservative Planck CMB "BAO" measurement ($100\theta_{MC} = 1.0409 \pm 0.0006$) gives a tight constraint on H_0, comparable to that from the full CMB data set [2].*

(the CMB alone suffers from a geometric degeneracy, which is weakly broken with the addition of CMB lensing). Such a result can be seen in Fig. 1.8, where $\Omega_K \equiv 1 - \Omega_m - \Omega_\Lambda$ is the curvature density [1]; this plot shows a very consistent constrain with a flat Universe [2].

Finally, we want to report an important test on the equation of state, assuming that dark energy is responsible of the accelerated expansion of the Universe. If the dark energy (DE) is a generic dynamical fluid, its equation of state parameter $w \equiv p/\rho$ (where p and ρ are the spatially-averaged background DE pressure and density) will be in general a function of time. In order to test it, the following equation of state is adopted: $w(a) = w_0 + (1 - a)w_a$, with w_0 and w_a assumed to be constants, and ΛCDM predicts $w_0 = -1$ and $w_a = 0$. Without going into details of the analysis that has been done, marginalized contours of the posterior distributions for w_0 and w_a are shown in Fig.1.9. A wide volume of dynamical dark-energy parameter space would be allowed using only Planck data, but most of it corresponds to phantom models with very high values of H_0, which are inconsistent with the late-time evolution constrained by SNe and BAO data. The tightest constraints are found for the data combination Planck TT,TE,EE+lowE+lensing+BAO+SNe; the difference in [4] χ^2 between the best-fit DE and ΛCDM models for this data combination is only $\Delta\chi^2 = -1.4$. Numerical results for cosmological parameters constrained from this analysis are reported in Fig.1.10.[2]

In conclusion, assuming the ΛCDM model, cosmological parameters are proved to be tightly constrained by the various Planck analyses with a sufficient confidence level. For

[4] Given a data set of N measures, if o_i is the experimental value of a quantity and e_i is the expectation value, then $\chi^2 \equiv \sum_{i=1}^{N} \frac{o_i - e_i}{e_i}$.

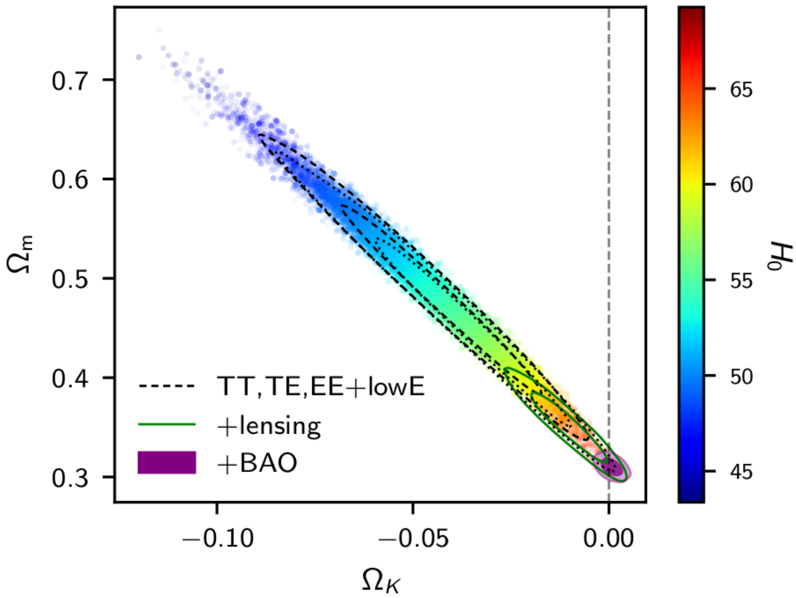


Figure 1.8: *Constraints on a non-flat universe as a minimal extension to the base-ΛCDM model. Points show samples from the Planck TT,TE,EE+lowE chains coloured by the value of the Hubble parameter and with transparency proportional to the sample weight. Dashed lines show the corresponding 68% and 95% confidence contours that close away from the flat model (vertical line), while dotted lines are the equivalent contours from the alternative CamSpec likelihood. The solid dashed line shows the constraint from adding Planck lensing, which pulls the result back towards consistency with flat (within 2 σ). The filled contour shows the result of also adding BAO data, which makes the full joint constraint very consistent with a flat Universe [2].*

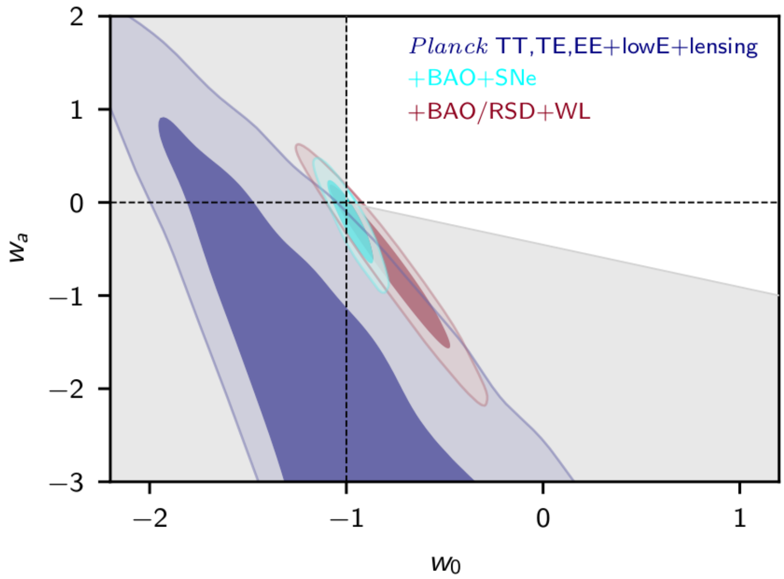


Figure 1.9: *Marginalized posterior distributions of the (w_0, w_a) parameters for various data combinations. The tightest constraints come from the combination Planck TT,TE,EE+lowE+lensing+SNe+BAO and are compatible with ΛCDM. Using Planck TT,TE,EE+lowE+lensing alone is considerably less constraining and allows for an area in parameter space that corresponds to large values of the Hubble constant [12]. The dashed lines indicate the point corresponding to the ΛCDM model [2].*

Parameter	*Planck*+SNe+BAO	*Planck*+BAO/RSD+WL
w_0	-0.961 ± 0.077	-0.76 ± 0.20
w_a	$-0.28^{+0.31}_{-0.27}$	$-0.72^{+0.62}_{-0.54}$
H_0 [km s^{-1}Mpc^{-1}]	68.34 ± 0.83	66.3 ± 1.8
σ_8	0.821 ± 0.011	$0.800^{+0.015}_{-0.017}$
S_8	0.829 ± 0.011	0.832 ± 0.013
$\Delta\chi^2$	-1.4	-1.4

Figure 1.10: *Marginalized values and 68% confidence limits for cosmological parameters obtained by combining Planck TT,TE,EE+lowE+lensing with other data sets, assuming the* $(w_0,\, w_a)$ *parameterization of an equation of state given by* $w(a) = w_0 + (1-a)w_a$. *The* $\Delta\chi^2$ *values for best fits are computed with respect to the* Λ*CDM best fits computed from the corresponding data set combination [2].*

this reason we can assume ΛCDM in this thesis work.

Notation and conventions

Natural units will be used, in which the speed of light and Planck's constant are set equal to one: $c = \hbar = 1$. Therefore, lenght and time have the same units. The metric signature is $(-,+,+,+)$, so that $ds^2 = -dt^2 + d\boldsymbol{x}^2$ in Minkowski. The Greek indices $\mu, \nu, ...$ run from 0 to 3. whole the Latin indices $i, k, ...$ stands for spatial indices. Bold font denotes spatial three-vectors (e.g. $\boldsymbol{x}$). Einstein summation convention will be used.

1.2 The homogeneous Universe

If we look further and further to our Universe, we can notice that on sufficiently large scales the clumpy distribution of galaxies becomes more and more *isotropic* (see Fig. 1.13), that means indipendent from the direction, according to the *cosmological principle*: the Universe should appear isotropic to any inertial observer; therefore, if it is isotropic in any point of spacetime, it will also be *homogeneous*, that means independent of position. For these reasons, for sufficiently large scales ($>$ 600 Mpc), we can approximate the Universe as perfectly homogeneous and isotropic. Before talking about geometry, we want to recall some basic concepts about the expansion.

1.2.1 Expansion

Experimental evidence like the Hubble diagram tells us that the Universe is expanding, therefore in the past the distance between two cosmological objects was smaller than it is at present time. This fact is mathematically described by the *scale factor* a, whose present value is $a_0 \equiv 1$; it relates the physical, proper distance between two points of spacetime with their comoving distance that remains unchanged with the expansion:

$$d(t) = a(t)d_0 \,, \tag{1.1}$$

where $d(t)$ is the proper distance at time t, and d_0 is the distance at reference time t_0. If we imagine a grid that expands, the points on that grid mantain their coordinates, thus their comoving distance doesn't change, while the proper distance is affected by the expansion and therefore evolve with time proportionally to the scale factor. The

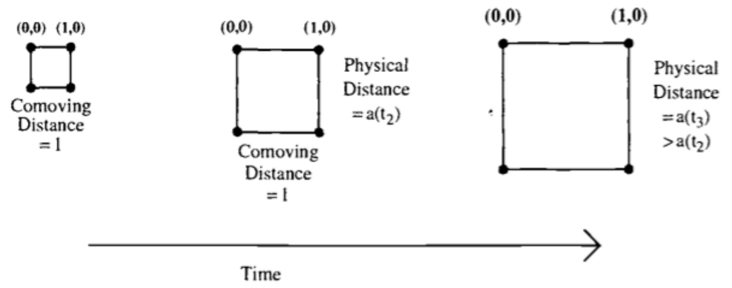


Figure 1.11: *If we imagine to set a grid in space, the comoving distance between two points on it remains constant with the expansion of the Universe, while the physical distance changes and gets larger with time, proportionally to the scale factor [1].*

way in which the scale factors change with time is quantified by the Hubble rate:

$$H(t) \equiv \frac{\dot{a}}{a} , \tag{1.2}$$

where $\cdot = d/dt$.[1] According to the final Planck 2018 results, the value for the Hubble rate today is $H_0 = (67.66 \pm 0.42)$ (km/s)/Mpc [2].
Further, general relativity provides the connection between this evolution and the energy density in the Universe; it is given by the first Friedmann equation:

$$H^2(t) = \frac{8\pi G}{3}\left[\rho(t) + \frac{\rho_c - \rho_0}{a^2(t)}\right] , \tag{1.3}$$

where the subscript 0 stand for the value of the quantity today, $\rho(t)$ is the energy density, and G is the universal gravitational constant. The critical density is defined as

$$\rho_c \equiv \frac{3H_0^2}{8\pi G} . \tag{1.4}$$

1.2.2 Friedmann-Lemaître-Robertson-Walker metric

The features of homogeneity and isotropy of the background spacetime can be represented by a specific kind of metric tensor, which is a fundamental tool used in general relativity incorporating the geometric nature of gravity. In a four dimensions spacetime, the invariant line element is:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu , \tag{1.5}$$

where the first coordinate is time-like ($dx^0 = dt$), and the last three are spatial. The metric $g_{\mu\nu}$ is symmetric, so in principle has four diagonal and six off-diagonal components. It provides the connection between the values of the coordinates and the more physical measure of the invariant interval ds^2 [1]. But what is the metric describing an expanding Universe? As we could see in Fig.1.11, when two grid points move away from each other, the distance between them is proportional to the scale factor $a(t)$. If the comoving distance today is x_0, the physical distance between the two points at some earlier time t was $a(t)x_0$. Therefore, in a flat Universe, the metric then is similar to Minkowski (diagonal), except for the fact that each distance must be multiplied by the scale factor [1]. This suggests that, for an expanding flat Universe, the metric in comoving coordinates $x^\mu = (t, x, y, z)$ is:

$$ds^2 = -dt^2 + a^2(t)\delta_{ij} dx^i dx^j = -dt^2 + a^2(t)\big(dx^2 + dy^2 + dz^2\big) . \tag{1.6}$$

However, it will be more convenient to use the conformal time τ, defined as

$$d\tau = \frac{dt}{a(t)} , \tag{1.7}$$

in a way that the background FRW metric can be written as

$$ds^2 = a^2(t)\left(-d\tau^2 + \delta_{ij}dx^i dx^j\right) = a^2(t)\left(-d\tau^2 + dx^2 + dy^2 + dz^2\right) . \tag{1.8}$$

Moreover, using conformal time, the derivative with respect to it is defined as $' = d/d\tau = ad/dt = a\,\cdot$, so the Hubble rate becomes

$$\mathcal{H}(t) \equiv \frac{a'}{a} = \frac{a\dot{a}}{a} = \dot{a} = aH(t) . \tag{1.9}$$

This quantity will be largely used in the next chapters. The aim of this section was only to introduce the FRW metric describing geometrically a homogeneous and isotropic background Universe.

1.3 The inhomogeneous Universe

1.3.1 Cosmological perturbations theory

In order to understand how primordially-generated fluctuations in matter and radiation grew into galaxies and clusters of galaxies due to self-gravity, it is necessary to deal with inhomogeneities. As long as perturbations remain relatively small, we can treat them in perturbation theory, and the growth of the fluctuations can be solved linearly. The aim of the cosmological perturbation theory is to relate the physics of the early universe (e.g. inflation) to CMB anisotropy and large-scale structure, and to provide the initial conditions for numerical simulations of structure formation. In the early universe, gravitational perturbations were inflated to wavelengths beyond the horizon at the end of the inflationary epoch; then, for each given length scale, they re-enter the horizon at a later time when the horizon has grown to the size of the fluctuations (e.g. see Ref.[17]). Newtonian gravity can be used on small scales, well inside the Hubble radius, and for non relativistic matter (e.g. cold dark matter and baryons after decoupling). However, for scales of order $\mathrm{O}(1/H^2)$ and larger than that (super-horizon, before the horizon crossing time) and for relativistic fluids (like photons and neutrinos) relativistic effects arise, so a full general relativistic treatment is required.

The goal of the theory of spacetime perturbations is to find approximate solutions of some field equations (Einstein equations), that means findind small deviations from a known exact background solution, that in our case is FRW metric. However, in general relativity, we have to perturb not only fields in a given geometry (e.g. $T_{\mu\nu}$, $G_{\mu\nu}$), but geometry itself ($g_{\mu\nu}$) [18]. Then the socalled *gauge problem* arise, since the concept itself of cosmological perturbation in general relativity presents ambiguity.

1.3.2 The gauge problem

In the perturbation theory of general relativity we must consider the perturbed spacetime M close to a simple and symmetric background spacetime that we already know (see Fig. 1.12) [19]. Then, we consider a physical quantity represented by the generic tensor T (e.g. $g_{\mu\nu}$, $T_{\mu\nu}$), and we define:

- T_0 : the value of the tensor in the unperturbed background spacetime M_0;
- T : the value of the tensor in the physical perturbed spacetime M;

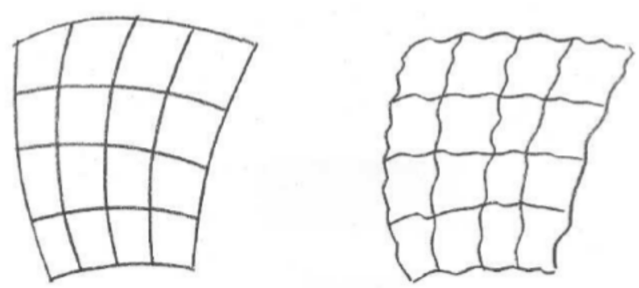


Figure 1.12: *On the left side, the background spacetime M_0; on the right side, the perturbed spcetime M.*

- $\Delta T = T - T_0$: the perturbation of the tensor.

If we want to compare two tensors in differential geometry, we have to consider them at the same point in spacetime. But T and T_0 are defined in two different spacetimes, that entails an intrinsic problem in defining the perturbation itself. In order to fix it, we have to establish a one-to-one corrispondence (a map, a diffeomorfism) between the points in the background spacetime M_0 and the points in the physical perturbed spacetime M: this is called a *gauge choice*. A change of this map corresponds to a *gauge transformation*, and in principle we can choose it in an arbitrary way, so that the value of the perturbation of the tensor is arbitrary too at any given spacetime point unless it is gauge invariant. This is the kernel of the *gauge problem* [18].

1.3.3 General definition of a gauge transformation

We now want to practically build a gauge transformation ϕ_λ, with λ a real parameter. We consider a family of spacetimes M_λ, each one identified by the value of the parameter; if $\lambda = 0$ we recover the background M_0. Want we want to do is the establish a family of one-to-one maps from M_0 to M, for each value of λ. Therefore we have to:

- fix a coordinate system x^μ in M_0;
- take a vector $\xi^\mu(x)$;
- introduce the parameter λ is a way that: $\xi^\mu = dx^\mu/d\lambda$.

A one-to-one correspondence ϕ_λ carries the coordinates x^μ over M_λ, defining a gauge choice. A change in this correspondence, keeping the background coordinates fixed, is a gauge transformation [18].
Now, we want first to consider a point P with coordinates $x^\mu(P)$ in M_0, and its corrispondent point $O = \phi_\lambda(P)$ in M_λ. We can also think O as the point of M_λ corresponding to a different point Q in the background using a different gauge ψ_λ: then $O = \psi_\lambda(Q) = \phi_\lambda(P)$. Therefore, the gauge transformation can actually be seen as a one-to-one correspondence between different points in the background; it can be built going from P in M_0 to O in M_λ through ϕ_λ, and then "going back" from O to Q through ψ_λ^{-1}, in a way that $Q = \Phi_\lambda(P) = \psi_\lambda^{-1}\big(\phi_\lambda(P)\big)$. Then we have the coordinates of Q that can be written as $x^\mu(Q) = \Phi_\lambda\big(x^\mu(P)\big)$. In linear theory, the action of Φ_λ can be represented with that of a one parameter group of transformations, associated with the vector ξ^μ defined before. Therefore, for a very small value of λ, the gauge transformation can be written as

$$x^\mu(Q) = x^\mu(P) + \lambda \xi^\mu\big(x(P)\big) \,, \tag{1.10}$$

which is known as the "active" approach where we are not changing coordinates. However, a "passive" approach is also possible reading (1.10) as a passive coordinate transformation. Isolating x^μ and doing an expansion at first order in λ, we get:

$$x^\mu(P) = x^\mu(Q) - \lambda\xi^\mu\big(x(P)\big) \simeq x^\mu(Q) - \lambda\xi^\mu\big(x(Q)\big) + O(\xi^2)\,. \tag{1.11}$$

Therefore we can introduce a new coordinate system y^μ in a way that:

$$y^\mu(Q) \equiv x^\mu(Q) - \lambda\xi^\mu\big(x(Q)\big)\,. \tag{1.12}$$

For $\lambda = 1$ an ordinary coordinate transformation is found. Given these definitions, how do objects as scalars, vectors and tensors transform?

1.3.4 Explicit transformation laws

We want now to build the explicit expressions of the gauge transformations for a generic scalar, vector and tensor. In order to do that, we start with a vector and then we will generalize the results. Consider a vectorial field Z with components Z^μ in the coordinate system x^μ fixed. The components of the new tensor $\tilde{Z}^\mu\big(x^\mu(P)\big)$ corresponds to the components Z'^μ that the old tensor Z assumes in the new coordinates, using the "passive" approach for a gauge transformation. Then:

$$\tilde{Z}^\mu(x(P)) \equiv Z'^\mu\big(y(Q)\big) = \left.\frac{\partial y^\mu}{\partial x^\nu}\right|_{x(Q)} Z^\nu\big(x(Q)\big)\,. \tag{1.13}$$

We are defining a sort of "dragging law" of the tensor from the point Q to the point P. Using the fact that

$$y^\mu = x^\mu - \lambda\xi^\mu \quad\Longrightarrow\quad \frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu - \lambda\frac{\partial \xi^\mu}{\partial x^\nu}\,,$$

we have

$$\tilde{Z}^\mu\big(x(P)\big) = Z^\mu\big(x(Q)\big) - \lambda\frac{\partial \xi^\mu}{\partial x^\nu} Z^\nu\big(x(Q)\big)\,. \tag{1.14}$$

Doing an expansion $x^\mu(Q) = x^\mu(P) + \lambda\xi^\mu(x(P))$ at first order in λ, we get:

$$\begin{aligned}\tilde{Z}^\mu\big(x(P)\big) &\simeq Z^\mu\big(x(P)\big) + \frac{\partial Z^\mu}{\partial x^\nu}\big(\lambda\xi^\nu\big(x(P)\big)\big) - \lambda\frac{\partial \xi^\mu}{\partial x^\nu} Z^\nu\big(x(P)\big) + O(\xi^2) \\ &\simeq Z^\mu\big(x(P)\big) + \lambda\mathcal{L}_\xi Z^\mu\big(x(P)\big)\,,\end{aligned} \tag{1.15}$$

where $\mathcal{L}_\xi Z^\mu$ is the Lie derivative of the vector Z^μ, defined as

$$\mathcal{L}_\xi Z^\mu = \frac{\partial Z^\mu}{\partial x^\nu}\xi^\nu - \frac{\partial \xi^\mu}{\partial x^\nu} Z^\nu\,. \tag{1.16}$$

If $\lambda = 1$, the socalled "Lie dragging" can be defined, that tells us how the vector transforms under a gauge transformation:

$$\tilde{Z}^\mu = Z^\mu + \mathcal{L}_\xi Z^\mu\,, \tag{1.17}$$

and $\tilde{Z}$ is called the *pull-back* of Z.
This result can be generalized to a generic tensor T of arbitrary rank, for which a generic gauge tranformation is defined as:

$$\tilde{T} = T + \mathcal{L}_\xi T\,. \tag{1.18}$$

Lie dragging of a tensor perturbation

Since our final goal is dealing with cosmological perturbations, we are interested in how a perturbation of a generic tensor transforms under a generic gauge transformation. We preliminary define the tensor T for the first gauge choice, and the tensor $\tilde{T}$ for the second different gauge choice. Then in principle two perturbations can be defined:

- $\Delta T = T - T_0$ in the first gauge
- $\Delta \tilde{T} = \tilde{T} - T_0$ in the second gauge

where T_0 is the tensor in the background. Then

- $T = T_0 + \Delta T$ in the first gauge
- $\tilde{T} = T_0 + \Delta \tilde{T}$ in the second gauge

Using both the latest definitions and (1.18), we find that:

$$\tilde{T} = T + \mathcal{L}_\xi T = T_0 + \Delta T + \mathcal{L}_\xi T = T_0 + \Delta \tilde{T} \quad \Longrightarrow \quad \Delta \tilde{T} = \Delta T + \mathcal{L}_\xi T \,.$$

At first order in perturbations, since $\mathcal{L}_\xi T$ is already at first order, we can write T_0 instead of T, getting the generic gauge transformation for the perturbation of an arbitrary tensor at first order:

$$\Delta \tilde{T} = \Delta T + \mathcal{L}_\xi T_0 \,. \tag{1.19}$$

Lie derivatives

We summarize here the explicit expressions of the Lie derivatives of the three objects we work with. Knowing that $\xi^\mu = dx^\mu/d\lambda$, and that ; is the symbol for the covariant derivative while , stands for partial derivative (which here coindice), we can write the following expressions.

- **scalar** S:
$$\mathcal{L}_\xi S = S_{;\mu}\xi^\mu = \frac{\partial S}{\partial x^\mu}\xi^\mu \,; \tag{1.20}$$

- **vector** V:
$$\mathcal{L}_\xi V_\mu = V_{\mu;\lambda}\xi^\lambda + \xi^\lambda_{;\mu} V_\lambda \equiv V_{\mu,\lambda}\xi^\lambda + \xi^\lambda_{,\mu} V_\lambda \tag{1.21}$$
$$\mathcal{L}_\xi V^\mu = V^\mu_{;\lambda}\xi^\lambda - \xi^\mu_{;\lambda} V^\lambda \equiv V^\mu_{,\lambda}\xi^\lambda - \xi^\mu_{,\lambda} V^\lambda \,; \tag{1.22}$$

- **tensor** T:
$$\mathcal{L}_\xi T_{\mu\nu} = T_{\mu\nu;\lambda}\xi^\lambda + \xi^\lambda_{;\mu} T_{\lambda\nu} + \xi^\lambda_{;\nu} T_{\mu\lambda} \equiv T_{\mu\nu,\lambda}\xi^\lambda + \xi^\lambda_{,\mu} T_{\lambda\nu} + \xi^\lambda_{,\nu} T_{\mu\lambda} \,. \tag{1.23}$$

In the specific case in which the tensor is the metric $g_{\mu\nu}$, the previous expression can be lightened due to its properties of symmetry and torsionless ($g^{\mu\nu}_{,\lambda} = 0$). Then we have:

$$\mathcal{L}_\xi g_{\mu\nu} = \left(\xi^\lambda g_{\lambda\nu}\right)_{;\mu} + \left(\xi^\lambda g_{\lambda\mu}\right)_{;\nu} = \xi_{\nu;\mu} + \xi_{\mu;\nu} \,. \tag{1.24}$$

1.3.5 Perturbed flat FRW Universe

If we want to perturb Einstein equaions, we fist have to consider the perturbation of the background Friedmann-Robertson-Walker metric (1.6). Calling r the order of perturbation ($r = 1$ means linear), the components of the perturbed spatially flat FRW metric are:

$$g_{00} = -a^2(\tau)\left[1 + 2\sum_{r=1}^{+\infty}\frac{1}{r!}A^{(r)}\right] , \tag{1.25}$$

$$g_{0i} = g_{i0} = a^2(\tau)\sum_{r=1}^{+\infty}\frac{1}{r!}B_i^{(r)} , \tag{1.26}$$

$$g_{ij} = a^2(\tau)\left\{\left[1 - 2\sum_{r=1}^{+\infty}\frac{1}{r!}D^{(r)}\right]\delta_{ij} + \sum_{r=1}^{+\infty}\frac{1}{r!}E_{ij}^{(r)}\right\} , \tag{1.27}$$

where $E_i^{i(r)} = 0$, τ is the conformal time, $A^{(r)} = A^{(r)}(\boldsymbol{x}, \tau)$ are called the *lapse functions*, and $B_i^{(r)}(\boldsymbol{x}, \tau)$ are called the *shift functions*.

Scalar, vector and tensor components

It is often used a decomposition on perturbations into the so-called scalar, vecor and tensor parts.

- The *scalar* (or longitudinal) components are relatex to scalar potentials, like $D^{(r)}$ and $A^{(r)}$.
- The *vector* parts are those related to transverse (divergence-free or solenoidal) vector fields. For example:

$$B_i^{(r)} = \underbrace{\partial_i B_\star^{(r)}}_{\text{scalar}} + \underbrace{B_i^{(r)\perp}}_{\text{vector}} , \tag{1.28}$$

  where $B_i^{(r)\perp}$ is a solenoidal vector, thus $\partial^i B_i^{(r)\perp} = 0$.

- The *tensor* parts are related to transverse traceless tensors. They appear for example in the decomposition of the traceless part of the spatial metric:

$$E_{ij} = \underbrace{\Delta_{ij}E_\star^{(r)}}_{\text{scalar}} + \underbrace{\partial_i E_j^{(r)} + \partial_j E_i^{(r)}}_{\text{vector}} + \underbrace{E_{ij}^{(r)\perp}}_{\text{tensor}} , \tag{1.29}$$

  where $\partial_i E_{ij}^{\perp} = 0$ and

$$\Delta_{ij} = \partial_i\partial_j - \frac{1}{3}\nabla^2\delta_{ij} . \tag{1.30}$$

Perfect fluid

In perturbing Einstein equation $G_{\mu\nu} = 8\pi G_{\mu\nu}$, we also have to deal with the stress-energy tensor and its components. For a perfect fluid

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} , \tag{1.31}$$

where:

- ρ is the energy density, that can be written as

$$\rho = \rho_0 + \sum_{r=1}^{+\infty} \frac{1}{r!} \delta\rho^{(r)} \; ; \tag{1.32}$$

- $p = p(\rho, S)$ is the isotropic pressure (S here is entropy), for which the perturbation is given by

$$\delta p = \underbrace{\underbrace{\left.\frac{\partial p}{\partial \rho}\right|_S}_{c_s^2} \delta\rho}_{\text{adiabatic part}} + \underbrace{\left.\frac{\partial p}{\partial S}\right|_\rho \delta S}_{\text{non adiabatic part}} \quad . \tag{1.33}$$

So $\delta p = c_s^2 \delta\rho + \delta p_{\text{non adiabatic}}$, where c_s is the adiabatic speed of sound of perturbations;

- u_μ is the four-velocity of a fluid element, that can be written as

$$u^\mu = \frac{1}{a} \left[\delta_0^\mu + \sum_{r=1}^{+\infty} \frac{1}{r!} v^\mu_{(r)} \right] , \tag{1.34}$$

where the first addend corresponds to the background four-velocity in FRW, comoving with the cosmic expansion (Hubble flow), while the second addend represents the pecurial velocity of the fluid element. Moreover, the normalization $u_\mu u^\mu = -1$ sussists, and allows to connect $v^0_{(r)}$ to the lapse perturbation $A^{(r)}$; in particular, at first order $v^0_{(1)} = -A^{(1)}$. Finally, for the spatial component v^i of the peculiar velocity the usual decomposition can be done:

$$w^i = \partial^i v_\star + v^{i\,\perp} \quad \text{with} \quad \partial^i v_i^\perp = 0 \; ; \tag{1.35}$$

- $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. It can be easily proved that $h_{\mu\nu} u^\nu = 0$, so $h_{\mu\nu}$ is a projector on hypersurfaces which are orthogonal to the four-velocity u_μ.

First order gauge transformations

As we saw in the previous subsections, gauge transformations are determined by the vector ξ^μ. Its time and spatial parts can be written as

$$\xi^0 = \alpha \, , \tag{1.36}$$

$$\xi^i = \partial^i \beta + d^i \, , \tag{1.37}$$

where α and β are scalars, and d^i is a vector with the property $\partial_i d^i = 0$. Starting from these definitions, and using (1.19) with the metric perturbations in a way that:

$$\delta \tilde{g}_{\mu\nu} = \delta g_{\mu\nu} + \mathcal{L}_\xi \bar{g}_{\mu\nu} \, , \tag{1.38}$$

where $\bar{g}_{\mu\nu}$ is the backgroud FRW metric, the following gauge transformations at first order can be calculated (we are omitting the label (1)):

$$\tilde{A} = A + \alpha' + \frac{a'}{a}\alpha \;\; , \tag{1.39}$$

$$\tilde{B}_i = B_i - \alpha_{,i} + \beta'_{,i} + d'_i \;\; , \tag{1.40}$$

$$\tilde{D} = D - \frac{1}{3}\nabla^2\beta - \frac{a'}{a}\alpha \;\; , \tag{1.41}$$

$$\tilde{E}_{ij} = E_{ij} + 2\Delta_{ij}\beta + d_{i,j} + d_{j,i} \;\; , \tag{1.42}$$

where $' = d/d\tau$. Using (1.20), we have for the density perturbation

$$\delta\tilde{\rho} = \delta\rho + \mathcal{L}_\xi \rho_0 = \delta\rho + \rho_0' \alpha \; , \tag{1.43}$$

while, using (1.21), we have for the four-velocity

$$\delta\tilde{u}^\mu = \delta u^\mu + \mathcal{L}_\xi u_0^\mu \; , \tag{1.44}$$

where $u_0^\mu = \delta_0^\mu / a(\tau)$ is the background value in FRW (comoving). Then:

$$\tilde{v}^0 = v^0 - \frac{a'}{a}\alpha - \alpha' \; , \tag{1.45}$$

$$\tilde{v}^i = v^i - \beta'^{\,,i} - d^{i\,'} \; . \tag{1.46}$$

Important gauge in cosmology

Fixing a gauge means choosing a specific form for ξ^μ, and so for the perturbations which depend on it. Here we present the main gauge used in cosmology.

Poisson (conformal-Newtonian)

The *Poisson gauge*, also known as *conformal Newtonian* or *longitudinal*, consists in setting:

$$\begin{aligned} B_\star &= 0 \; , \\ E_\star &= 0 \; , \\ E^i_\perp &= 0 \; . \end{aligned} \tag{1.47}$$

It is also known as *orthogonal - zero shear* gauge, since this particular quantity $\sigma = -B_\star + E_\star'/2$ called shear is zero in this gauge.

Synchronous

The *synchronous* is one of the most used gauge in cosmology, and consists in setting:

$$\begin{aligned} B_\star &= 0 \; , \\ B^i_\perp &= 0 \; , \\ A &= 0 \; . \end{aligned} \tag{1.48}$$

It is also labeled as *time-orthogonal* because the first two conditions imply $g_{0i} = 0$. Through this gauge choice we can define the proper time T of observers with fixed spatial coordinates; in the hypothesis of $c = 1$, if $A = 0$, we have that $dT = \left(a^2(\tau)d\tau^2\right)^{1/2}$ so $dT = a(\tau)d\tau = dt$ with t cosmic time in FRW. In this way all the observers with the same spatial coordinates (who "lie" on the same hypersurface) are synchronised. However, there is a residual degree of freedom because a function for the spatial coordinates has not been fixed.

Comoving

The *comoving* gauge is socalled because the peculiar velocity of the fluid element is zero:

$$v_i = 0 \quad \Longrightarrow \quad v_\star = v^i_\perp = 0 \; , \tag{1.49}$$

that means that it is comoving with the cosmic expansion.

Synchronous-Comoving

If we fix both the last two gauges with all the conditions listed above (1.48) and (1.49):

$$\begin{aligned} A &= 0 \\ B_i &= 0 \\ v_i &= 0 \,, \end{aligned} \tag{1.50}$$

we obtain the *synchronous − comoving* gauge. (1.50) implies that

$$v_\star + B_\star = 0 \,, \tag{1.51}$$

which is a condition of orthogonality of the hypersurfaces at constant τ. In order to prove that, we introduce the four-vector $N^\mu = (1 - A, -B^i)/a$, unitary time-like and orthogonal to space-like hypersurfaces at constant τ. If we make N^μ and u^μ covariant:

$$\begin{aligned} N_\mu &= -a\big(1 + A, 0\big) \,, \\ u_\mu &= a\big[-(1 + A), v_i + B_i\big] \,. \end{aligned}$$

We can notice that in the synchronous-comoving gauge they coincide, so the four-velocity is orthogonal to $\tau = constant$ hypersurfaces. This gauge is what we will use in the analysis of the following chapters, because it allows us to synchronize observers on the same spacelike hypersurface, and to make them comoving with the cosmic expansion. It is actually possible right because we are adopting the ΛCDM model and its features.

Uniform energy density

The uniform energy density gauge consists in fixing:

$$\delta\rho = 0 \,. \tag{1.52}$$

Then

$$\delta\tilde{\rho} = \delta\rho + \rho_0'\alpha = 0 \quad \Longrightarrow \quad \alpha = -\frac{\delta\rho}{\rho_0'} \,. \tag{1.53}$$

At constant τ, we are selecting spatial hypersurfaces at constant ρ.

Gauge invariant scalars

Two gauge invariant and linearly indipendent scalar quantities can be built from the metric pertubations:

$$2\Psi = 2A + 2B_\star' + 2\frac{a'}{a}B_\star - E_\star'' - \frac{a'}{a}E_\star' \,, \tag{1.54}$$

$$2\Phi = -2D - \frac{1}{3}\nabla^2\chi_\star + 2\frac{a'}{a}B_\star - \frac{a'}{a}E_\star' \,. \tag{1.55}$$

These quantities are the *Bardeen potentials* [20]. In Poisson (or *zero shear*) gauge they reduce to $\Psi = A$ and $\Phi = -D$.

1.3.6 Metric and the matter density perturbation

We have dealt with the metric perturbations, which have a purely geometric nature. However, through perturbed Einstein equations, they can be connected with the matter density perturbation $\delta\rho$. We are not going to report all the perturbed Einstein equations, but we will take only the results we need from the straightforward article by Ma and Bertschinger [17]. First of all, we can start from this expression of the spatial perturbation:

$$\delta g_{ij}(\tau, \boldsymbol{x}) = 2D(\tau, \boldsymbol{x})\delta_{ij} + 2\left(\partial_i\partial_j - \frac{1}{3}\nabla^2\delta_{ij}\right)E(\tau, \boldsymbol{x}) \,, \tag{1.56}$$

where the perturbed FRW metric used in the previous sections has been conformally transformed and replaced with $g_{\mu\nu}/a^2$ in order to get rid of the factor a^2; what conformal transformations are and why they are possible will be explained in Chapter 2. Since in Chapter 3 we are going to use the Fourier transform of δ_m, it is convenient to shift to the Fourier space where the metric perturbation becomes

$$\delta g_{ij}(\tau, \boldsymbol{k}) = 2D(\tau, \boldsymbol{k})\delta_{ij} - 2\left(\boldsymbol{k}_i\boldsymbol{k}_j - \frac{1}{3}k^2\delta_{ij}\right)E(\tau, \boldsymbol{k}) \,. \tag{1.57}$$

However, what we will discuss now is based on the results explained by Ma and Bertschinger in [17], therefore we have to compare for a moment our and their notation. They use such a metric perturbation in Fourier space:

$$\delta g_{ij}(\tau, \boldsymbol{k}) = \frac{\boldsymbol{k}_i\boldsymbol{k}_j}{k^2}h(\tau, \boldsymbol{k}) + 6\left(\frac{\boldsymbol{k}_i\boldsymbol{k}_j}{k^2} - \frac{1}{3}\delta_{ij}\right)\eta(\tau, \boldsymbol{k}) \,. \tag{1.58}$$

These perturbations h and η can be related to our D and E by comparing (1.57) and (1.58):

$$D(\tau, \boldsymbol{k}) = \frac{h(\tau, \boldsymbol{k})}{6} \,, \tag{1.59}$$

$$E(\tau, \boldsymbol{k}) = -\frac{h(\tau, \boldsymbol{k}) + 6\eta(\tau, \boldsymbol{k})}{2k^2} \,. \tag{1.60}$$

If we consider the energy-momentum conservation equation ${T^{\mu\nu}}_{;\mu} = 0$ and take the $\mu = 0$ part, we get the the continuity equation for the matter energy density. If we perturb it, we get in the synchronous gauge for non relativistic matter:

$$\delta_m'(\tau, \boldsymbol{k}) = -\frac{1}{2}h'(\tau, \boldsymbol{k}) \,. \tag{1.61}$$

We can now consider the linearized Einstein equations in Fourier space, and take the *0i* part (Eqs.(21b) and (22) in Ref.[17]); something peculiar happens in the synchronous-comoving gauge:

$$k^2\eta'(\tau, \boldsymbol{k}) = 4\pi i G a^2 k^j \delta T^0_j \propto u = 0 \,, \tag{1.62}$$

since the peculiar velocity u of comoving observers is zero in this gauge, as we also said in the previous chapter. Therefore we can get an expression of $E'(\tau, \boldsymbol{k})$, starting from (1.60) and then using (1.61) to substitute h' with δ_m':

$$E'(\tau, \boldsymbol{k}) = -\frac{h'(\tau, \boldsymbol{k})}{2k^2} = \frac{\delta_m'(\tau, \boldsymbol{k})}{k^2} \,. \tag{1.63}$$

Furthermore, we can relate δ'_m with δ_m through the linearized Boltzman equation for cold dark matter density perturbation, knowing that $\eta' = 0$:

$$\delta'_m(\tau, \boldsymbol{k}) = -ikv(\tau, \boldsymbol{k}) = -ik \cdot \frac{ifaH}{k}\delta_m(\tau, \boldsymbol{k}) = aHf\delta_m(\tau, \boldsymbol{k}) \,, \tag{1.64}$$

with $v(\tau, \boldsymbol{k})$ as the velocity field of matter in linear theory. The parameter f is the dimensionless linear growth rate and is defined as

$$f \equiv \frac{d \ln D_1}{d \ln a} \,, \tag{1.65}$$

where D_1 is the growing mode of δ_m in a way that

$$\delta_m(\boldsymbol{x}, z) = \delta(\boldsymbol{x}, 0)\frac{D_1(z)}{D(0)} \,. \tag{1.66}$$

We can now substitute (1.64) into (1.63) and get

$$E'(\tau, \boldsymbol{k}) = \frac{aHf}{k^2}\delta_m(\tau, \boldsymbol{k}) = \frac{H}{1+z}\frac{f}{k^2}\delta_m(\tau, \boldsymbol{k}) \,. \tag{1.67}$$

We can do more by considering the Fourier transform expression of δ_m, in a way that $\delta_m(\boldsymbol{x}, z) = \int d^3k/(2\pi)^3\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\delta_m(\boldsymbol{k}, \boldsymbol{z})$, and by noticing that applying the operator ∇^2 on both sides of this expression, we get $\nabla^2\delta_m(\boldsymbol{x}, z) = -k^2\delta_m(\boldsymbol{x}, z)$. Therefore in Fourier space the action of the operator arises to be $\nabla^{-2} = -1/k^2$ and, substituting it into (1.67), we find this final expression:

$$E' = -\frac{H}{1+z}f\nabla^{-2}\delta_m \,. \tag{1.68}$$

From this result, the second and the third derivative of E can be obtained: [5]

$$E'' = -\frac{H^2}{(1+z)^2}\left(\frac{3}{2}\Omega_m - f\right)\nabla^{-2}\delta_m \,, \tag{1.69}$$

$$E''' = -3\frac{H^3}{(1+z)^3}\Omega_m\big(f-1\big)\nabla^{-2}\delta_m \,. \tag{1.70}$$

We still have to explicit the dependence of the parameter $\phi = D - \frac{1}{3}\nabla^2 E$ on the matter density contrast δ_m; this can be done with the 00 part of the linearized Einstein equation ((21a) in [17]), knowing that $\phi(\tau, \boldsymbol{k}) = -\eta(\tau, \boldsymbol{k})$:

$$\begin{aligned}\phi &= -\frac{1}{2k^2}\big(aHh' + 8\pi Ga^2\delta T^0_0\big) = -\frac{1}{2k^2}\left[aH(-2\delta'_m) + 8\pi Ga^2\bar{\rho}_m\delta_m\right] \\ &= \frac{1}{k^2}\left[a^2H^2f\delta_m + 4\pi Ga^2\frac{3H^2}{8\pi G}\Omega_m\delta_m\right] = \frac{1}{k^2}a^2H^2\left(f + \frac{3}{2}\Omega_m\right)\delta_m \,.\end{aligned}$$

At last we can state that:

$$\phi = -\frac{H^2}{(1+z)^2}\left(f + \frac{3}{2}\Omega_m\right)\nabla^{-2}\delta_m \,. \tag{1.71}$$

Finally we have the useful expressions of E', E'', E''' and ϕ, which will appear in the galaxy overdensity in terms of the matter density contrast. We are going to use them

[5]From the time evolution of the linear density contrast and the definition of Ω_m.

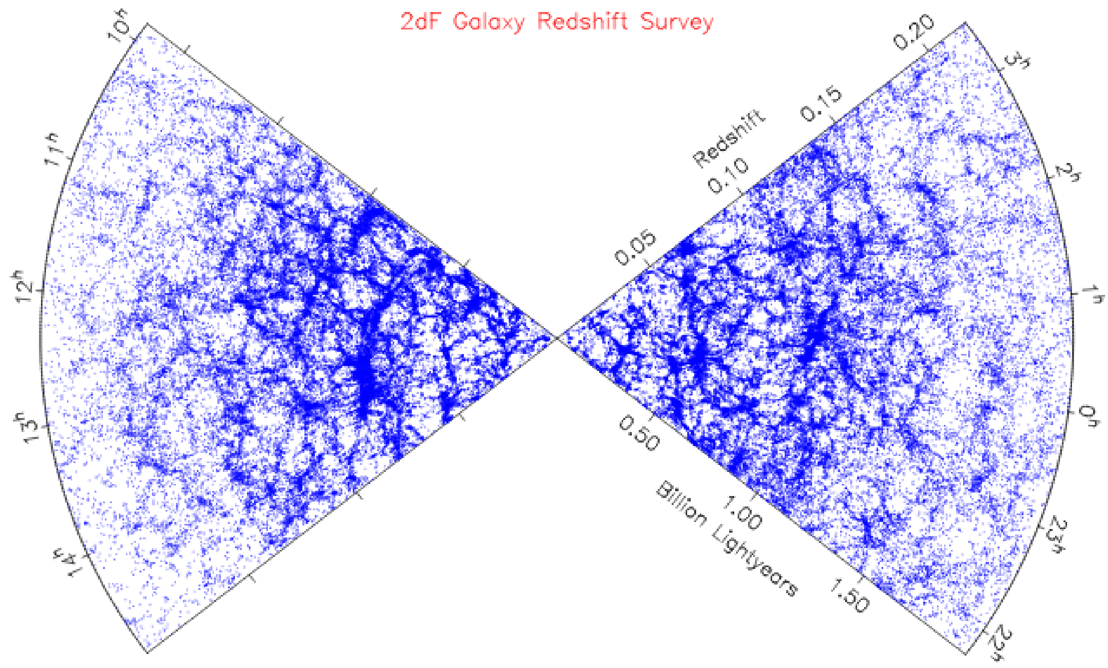


Figure 1.13: *3D map of galaxy positions from the 2dF galaxy redshift survey. Note that redshift 0.15 is at a comoving distance of 600 Mpc. Image credit: 2dF [22].*

in calculations in Chapter 3.

An excursus on metric and matter perturbations has been done, in order to introduce the tools necessary to face with calculations in Chapter 2, where we will entirely compute the galaxy overdensity Δ_g, and in Chapter 3 with the angular power spectrum. What we are still missing is the concept of galaxy clustering, which is the technique studied and adopted in this thesis.

1.3.7 Galaxy clustering

Galaxies are not randomly distributed but they tend to crowd together in *clusters* and even *superclusters* due to gravity; they surround large areas with very few galaxies, the socalled *voids*, in a way that on the largest scales the distribution is similar to soap bubbles, far from the homogeneous and isotropic Universe that we assume in the cosmological principle. However, if we smooth the pattern on large scales ($\sim 100Mpc$), it starts to look much more homogeneous (Fig.1.13). An interesting challenge is to understand what is the largest structure in the universe, or in other words at what scale do galaxies or clusters appear to be randomly distributed. We know that the pattern in which galaxies are distributed today derives from the initial distribution of matter in the early Universe; therefore the knowledge of the large-scale distribution and clustering of galaxies today is a key mean to test cosmological theories.
Galaxy clustering is related to the 3D distribution of galaxies at present time, measured from the angular positions of galaxies in the sky and the redshifts of the galaxies. It allows the direct measurement of the cosmic expansion $[H(z)]$ through baryon acoustic oscillations (BAO), the growth history of cosmic large scale structure $[f(z)]$, and the redshift-space distortions (RSD) on large scales [21]. Clustering is understood and measured in terms of statistics, in particular of correlation functions.

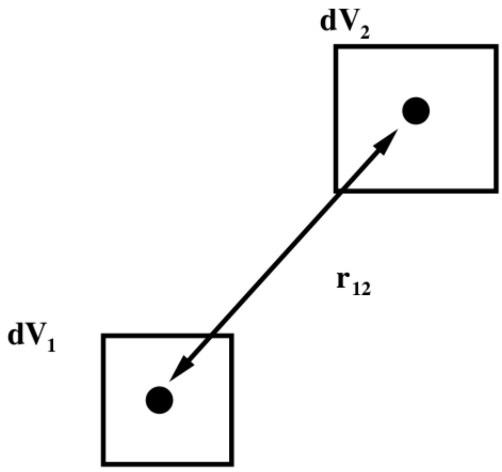


Figure 1.14: *The two-point correlation function describes the excess probability, compared with a random distribution of galaxies, of finding a galaxy in an element of volume dV_2 at distance $\boldsymbol{r}_{12}$ away from a galaxy in dV_1 [23].*

1.3.8 Correlation function and power spectrum

The fields of fluctuation, which in our case will be the galaxy number density perturbations, need proper statistical tools to be analysed with a clustering technique: they are the N-point correlation functions and their Fourier transforms, as power spectra, bispectra, etc... However, in our analysis we will focus only on the two-point correlation function and the corrispondent power spectrum $P(k)$ (even though the final point of our analysis will be actually the angular power spectrum C_ℓ that will be introduced in Chapter 3).

Consider any random field $\delta(\boldsymbol{x})$ with zero mean $\langle\delta(\boldsymbol{x})\rangle$, just like the density fluctuations we work with. The probability of realising some field configuration is a functional $P[\delta(\boldsymbol{x})]$ of the density fluctuation. Correlators of fields are expectation values of products of fields at different spatial points. The spatial *two-point correlation function* or *autocorrelation* is defined as the excess probability, compared with that expected for a random distribution, of finding a pair of galaxies at a separation $\boldsymbol{r}_{12}$ (see Fig.1.14):

$$dP = \bar{n}_g^2\big(1+\xi(\boldsymbol{r}_{12})\big)dV_1 dV_2 \ , \tag{1.72}$$

where $\bar{n}_g$ is the mean galaxy number density, and the two-point correlation function is the mean value of the product of the two fluctuations evaluated at different spatial points $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{r}_{12}$, respectively at time t and t':

$$\xi(\boldsymbol{x}) \equiv \langle\delta(\boldsymbol{x},t)\delta(\boldsymbol{x}+\boldsymbol{r}_{12},t')\rangle \ . \tag{1.73}$$

At this point we can expand the fluctuation in Fourier space:

$$\delta(\boldsymbol{x},t) = \frac{1}{(2\pi)^3}\int d^3k e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\delta(\boldsymbol{k},t) \ , \tag{1.74}$$

and define the *power spectrum* $P(k)$ of δ as:

$$\langle\delta(\boldsymbol{k},t)\delta(\boldsymbol{k}',t')\rangle \equiv (2\pi)^3 P(\boldsymbol{k})\delta_D^3(\boldsymbol{k}+\boldsymbol{k}') \ . \tag{1.75}$$

Here the angular brackets denote an average over the whole distribution, and $\delta_D^3()$ is the Dirac delta function which constrains $\boldsymbol{k}'=-\boldsymbol{k}$. The power spectrum represents the "spread", or the variance, of the distribution: if there are many under- and overdense regions, it will be large, otherwise if the distribution is smooth it will be small [1]. It

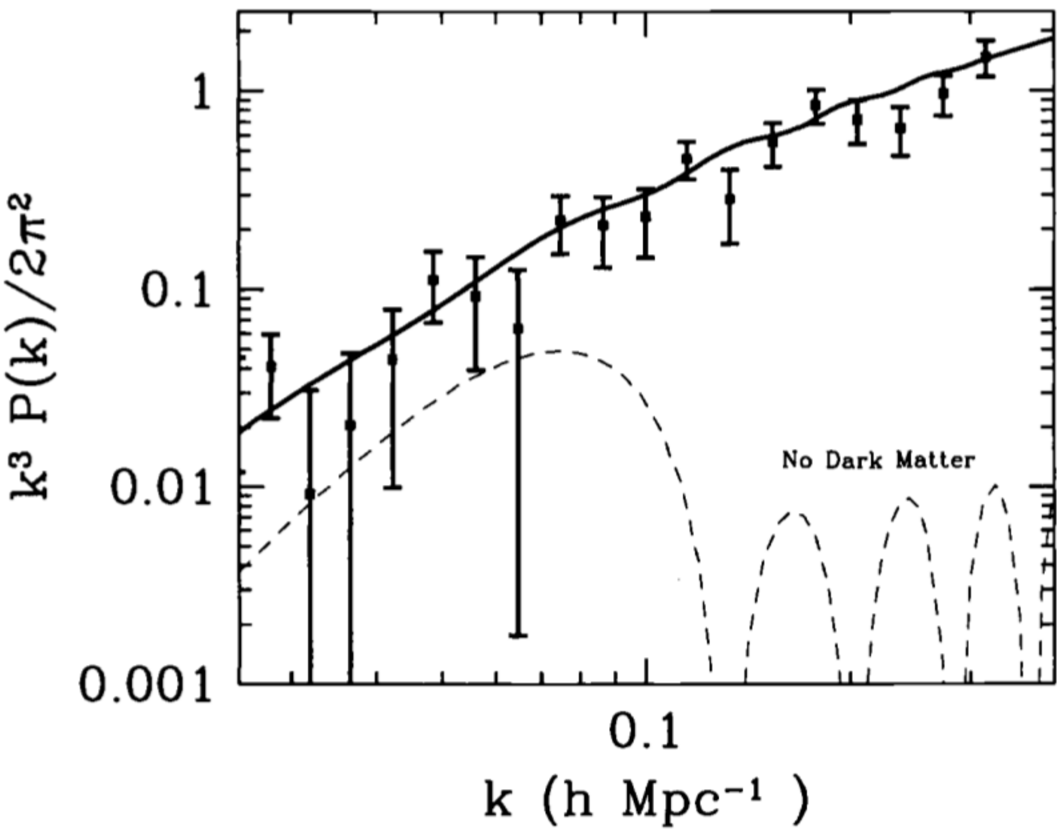


Figure 1.15: *On large scales, the variance $\Delta^2 \equiv k^3 P(k)/2\pi^2$ is smaller than unity, so the distribution is smooth. The solid line is the theoretical prediction from a model in which the universe contains dark matter, a cosmological constant, with perturbations generated by inflation. The dashed line is a theory with only baryons and no dark matter. Data come from the PSCz survey (Saunders et ai, 2000) as analyzed by Hamilton and Tegmark (2001) [1].*

can be easily proven that the power spectrum is the Fourier transform of the two-point correlation function:

$$\begin{aligned} \xi(\boldsymbol{x}) \equiv \langle \delta(\boldsymbol{x}, t)\delta(\boldsymbol{x} + \boldsymbol{r}_{12}, t')\rangle &= \langle \frac{1}{(2\pi)^6} \int d^3k e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \int d^3k' e^{i\boldsymbol{k}'\cdot(\boldsymbol{x}+\boldsymbol{r}_{12})} \delta(\boldsymbol{k}, t)\delta(\boldsymbol{k}', t')\rangle \\ &= \frac{1}{(2\pi)^6} \int d^3k e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \int d^3k' e^{i\boldsymbol{k}'\cdot(\boldsymbol{x}+\boldsymbol{r}_{12})} (2\pi)^3 P(k) \delta^3_D(\boldsymbol{k} + \boldsymbol{k}') \\ &= \frac{1}{(2\pi)^3} \int d^3k e^{-i\boldsymbol{k}\cdot\boldsymbol{r}_{12}} P(k) \, . \end{aligned} \tag{1.76}$$

The shape and amplitude of the power spectrum of density fluctuations contain information about both the amount and nature of matter in the universe (see Fig.1.15).

Chapter 2

Galaxy overdensity: a general relativistic expression

One of the key tools to study the large scale structure of the Universe and costrain the cosmological model are *galaxy redshift surveys*, which are sets of source positions along with the measured redshifts [24] (e.g. in Fig.2.1). The size of galaxy catalogues has constantly increased over the last fourty years, both in terms of solid angle and redshift coverage, and sampling rates. The latest generation of surveys will allow us to measure galaxy clustering on more and more larger scales, comparable with the Hubble radius [26] (see Fig.2.2 and 2.3). Galaxies that we observe are located at a time coordinate in which their world line intersects our past light cone. By detecting photons that travel from their source position in our past light cone towards us, we can infer the comoving location of a galaxy from two basic observables:

- the *direction* of the photon $\hat{\mathbf{n}}$, which is a unit vector that points to the apparent position of the galaxy in the sky. Indeed, in absence of distortions the trajectory of the photon is a straight line that starts from the galaxy that emitted it and ends at the observer's position on Earth;
- the measured *redshift* of the galaxy. It can be easily related to the distance of the object from the Earth using Hubble's law; therefore, combining the redshift with the angular position, a 3D map of the galaxy distribution over a section of the sky can be constructed.

The three dimensional maps that can be built from the previous information are based on the assumption of an unperturbed Friedmann-Robertson-Walker universe with a fixed set of cosmological parameters. However, these redshift space maps gives a distorted picture due to the presence of inhomogeneities in real spacetime. Indeed, the presence of such perturbations superimposed to a FRW background deflects the null geodesics of the photons emitted by galaxies. Therefore, the observable quantities for a galaxy that differ in the two cases of an unperturbed and perturbed universe are: the redshift, the angular position on the sky and the flux in any given waveband. These are relativistic effects which arise on cosmological scales because we make our observations in our past light cone, and consist mainly in the socalled redshift space distortions and gravitational lensing convergence. Anyway several additional corrections, which are tipically suppressed on small scales, might be detected in nowadays surveys even on distances comparable with the Hubble radius. At linear order, these corrections include Doppler, standard and integrated Sachs-Wolfe terms, and (Shapiro) time delay contributions. A reliable model of large scale clustering should include all the previous

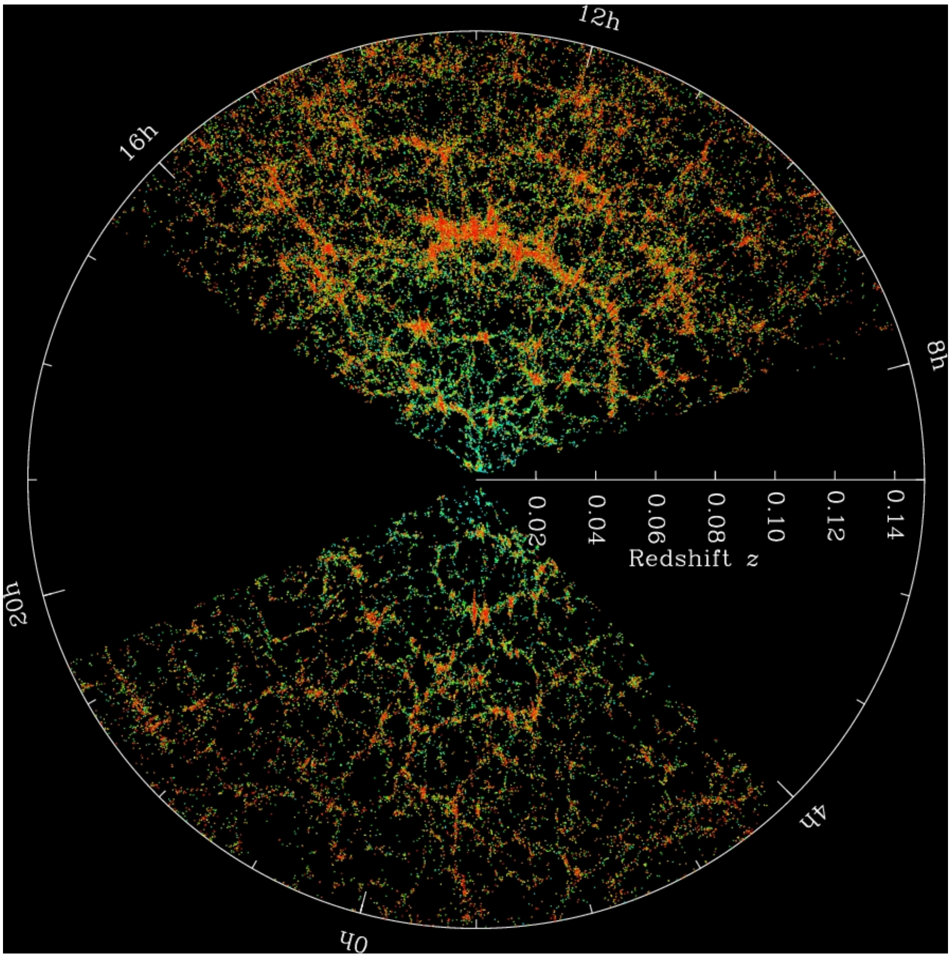
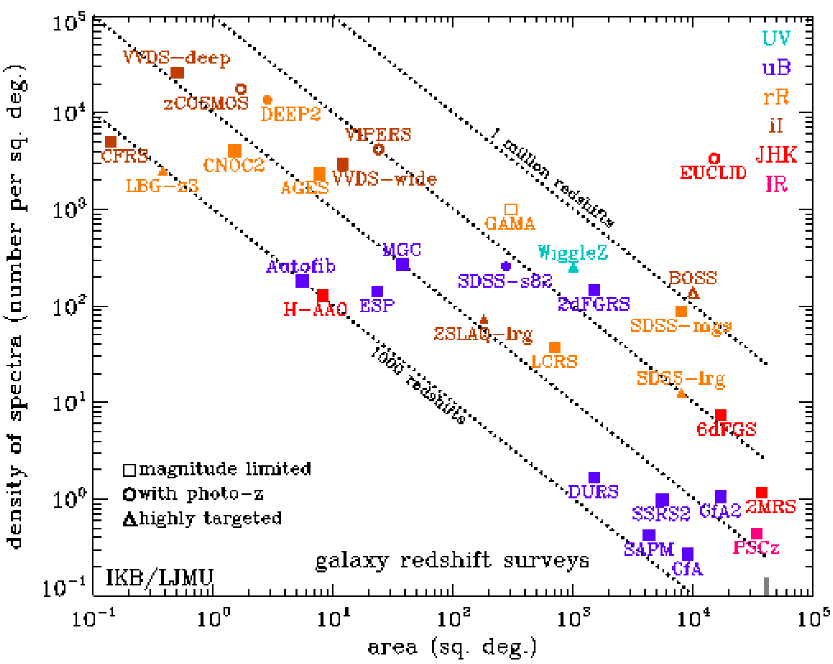


Figure 2.1: *The SDSS's map of the Universe. Each dot is a galaxy; the colour is the g-r colour of that galaxy. Image credit: M. Blanton and SDSS [25].*

Figure 2.2: *Comparison between galaxy redshift surveys: squares represent predominantly magnitude-limited surveys; circles represent surveys involving colour cuts for photometric redshift selection; triangles represent highly targeted surveys. Filled symbols show completed surveys. Surveys are colour coded according to selection wavelength. The dotted lines correspond to surveys of 1000,* 10^4, 105^5 *and* 10^6 *galaxies. Image credit: Ivan K. Baldry, Liverpool JMU [27].*



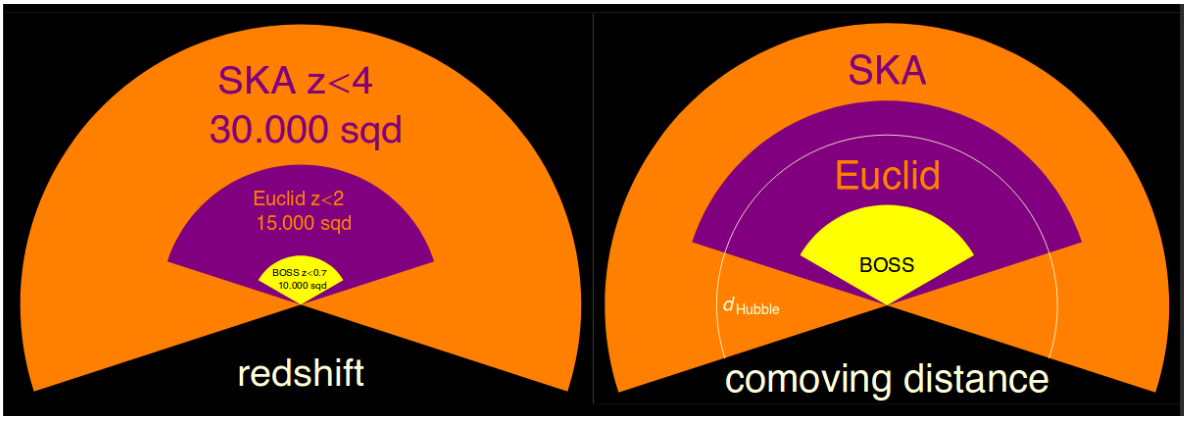


Figure 2.3: *Comparison of the sky coverage of the most recent surveys: BOSS, EUCLID and SKA, where "sqd" stands for square degrees. Image credit: Roy Maartens and SKA.*

modifications. For this reason, a full explicit expression of the galaxy overdensity Δ_g will be calculated in this chapter, including all the general relativistic contributions at first order in perturbation theory; this expression will be given in terms of the metric perturbations, the redshift distortions and the bias parameters. We will see that Δ_g includes three main addends: a local term (relative to the source), a weak lensing and a time delay contribution.
We must highlight that the relation between the galaxy number density fluctuation and the underlying matter fluctuation δ_m is non trivial on cosmological scales, because the luminous matter does non exactly corresponds to the dark matter: there is a local *bias* we must take account of, and that must be defined properly. In order to do that, we need to choose an appropriate frame where the baryon velocity perturbation vanishes; in this way the baryon rest frame coincides with the CDM rest frame, and in ΛCDM the synchronous-comoving gauge defines such a frame up to the second order [28]. In this gauge, galaxy and matter overdensities are gauge invariant ant their relation becomes linear; therefore we will do our calculations and considerations under this peculiar gauge choice.

2.1 Perturbation of the photon geodesics

2.1.1 The metric

Assuming a flat Friedmann-Robertson-Walker background, we consider linear perturbations in the synchronous-comoving gauge for the reason just mentioned in the introduction of this chapter. *Synchronous* means that the proper time of all the observers coincides with the comoving time, so they actually lie in costant time hypersurfaces which are equivalent to constant age hypersurfaces. *Comoving* means that their peculiar velocities with respect to the background are set to zero, so they are moving together with the background expansion; moreover, constant time hypersurfaces are orthogonal to the four-velocity of the cosmic fluid. Therefore, the *perturbed metric* we obtain is

$$ds^2 = a^2(\tau)\{-d\tau^2 + [(1+2D)\delta_{ij} + 2E_{ij}]dx^i dx^j\} \,, \tag{2.1}$$

where τ is the conformal time, D is a scalar perturbation of the spatial diagonal part of the metric, and E_{ij} is a traceless ($E^i_i = 0$) and transverse ($\partial^i E_{ij} = 0$) spatial perturbation; it includes in principle pure scalar, vectorial and tensorial components, but we

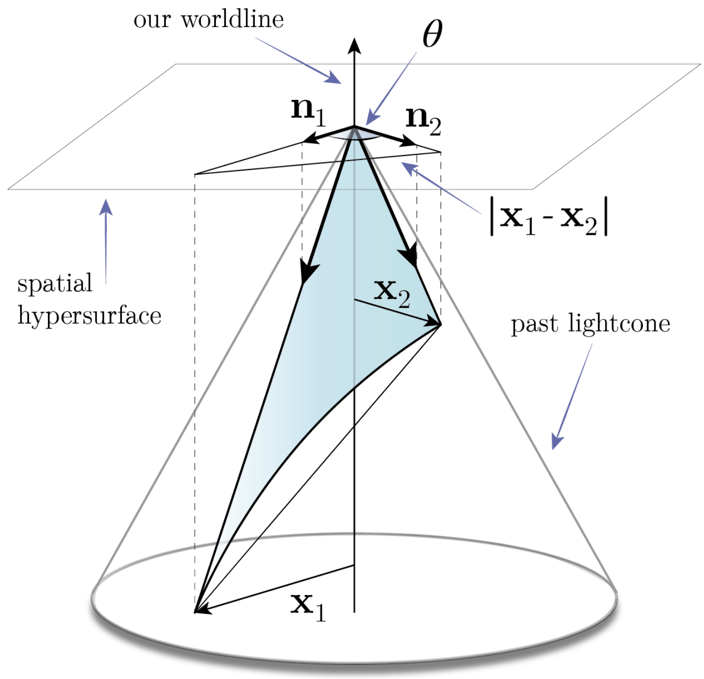


Figure 2.4: *Position of a pair of galaxies on the past lightcone, which differs from its projection on the spatial hypersurface for a wide angle* θ [29].

consider only the scalar one E which is related to E_{ij} via the operator Δ_{ij}, so that:

$$E_{ij} = \Delta_{ij} E = \left(\partial_i \partial_j - \frac{1}{3} \nabla^2 \delta_{ij} \right) E \, . \tag{2.2}$$

We can do this because in first-order perturbation theory the scalar, vector, and tensor parts do not couple to each other, but evolve independently. This is the reason why we can treat them separately, studying the scalar perturbations as if the vector and tensor ones were absent. The total evolution of the full perturbation is just a linear superposition of the independent evolution of the scalar, vector, and tensor components. However, we will focus only on the scalar perturbations because they couple to density and pressure perturbations and exhibit gravitational instability: overdense regions grow more overdense, thus they are responsible for the formation of structure in the universe from small initial perturbations [19].
After defining the perturbed metric $g_{\mu\nu}$ in Eq.(2.1), we perform the conformal transformation $\hat{g}_{\mu\nu} = g_{\mu\nu}/a^2$ in order to get rid of the factor a^2 and simplify calculations that will follow. We are able to do that due to an important fact: *null curves* or light-like geodesics, along which photons propagate in the expanding Universe, *are invariant under conformal transformations* [30]. This means that, if $x^\mu(\lambda)$ is a curve that is null with respect to $g_{\mu\nu}$, it will also be null with respect to $\hat{g}_{\mu\nu}$. This property can be easily demonstrated, starting from the definition of a *null curve*: it is a geodesic whose tangent vector is a light-like vector everywhere along it, that means

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \, . \tag{2.3}$$

Switching to the conformally related metric:

$$\hat{g}_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \frac{1}{a^2} g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \, . \tag{2.4}$$

Therefore, it is clear that null geodesics defined by the metric $g_{\mu\nu}$ remain so also if they are defined by the conformally related metric $\hat{g}_{\mu\nu}$. As a consequence, it can be said that conformal transformations leave light cones invariant. For this reason, studying the photon geodesics in conformal coordinates $x^\mu(\chi)$ is equivalent to studying those in the "old" coordinates $x^\mu(\lambda)$ defined by $g_{\mu\nu}$; however, the affine parameter λ is transformed to another affine parameter χ, in a way that

$$d\lambda/d\chi \propto a^2 \, . \tag{2.5}$$

From this moment we will work with the conformally transformed metric

$$\hat{g}_{\mu\nu} = \frac{g_{\mu\nu}}{a^2(\tau)} \, . \tag{2.6}$$

All the coordinates we will use (spatial and temporal), momenta and frequencies are from now on the conformal ones.

2.1.2 The photon geodesic

The motion of a *photon* in the observed frame is characterized by two observable quantities:

- the *direction* $\tilde{n}$ of motion measured by the observer on Earth, which is related to the apparent position of the source (a galaxy in our case);
- the observed redshift $\tilde{z}$.

First of all we can assigne a position $\tilde{\boldsymbol{x}}$ to the galaxy (our source), so that

$$\tilde{\boldsymbol{x}} = \tilde{\chi}\hat{\boldsymbol{n}} \, , \tag{2.7}$$

where $\tilde{\boldsymbol{x}}$ is the observed position of the galaxy, $\hat{\boldsymbol{n}}$ is the observed direction of the photon, and $\tilde{\chi}$ is the comoving distance from the observer of events located along the geodesic in the unperturbed spacetime, and moreover defines the observed-redshift relation $\tilde{\chi}(z)$. The null geodesic from a galaxy observed to us can be described in terms of the following conformal spacetime coordinates:

$$\bar{x}^\mu(\chi) = (\tau_0 - \chi, \hat{\boldsymbol{n}}\chi) \, , \tag{2.8}$$

where τ_0 is the present day value (at observation) of the conformal time. In the unperturbed universe Eq.(2.8) is a straight line. We deduce that:

$$\frac{d\bar{x}^\mu}{d\chi} = (-1, \hat{\boldsymbol{n}}) \, . \tag{2.9}$$

At this point, as it can be seen in Fig.2.5, we want to define a map from the real space to the redshift one, in a way that:

$$x^\mu = \bar{x}^\mu + \Delta x^\mu(\hat{\boldsymbol{n}}, \tilde{z}) \, , \tag{2.10}$$

where x^μ is the actual comoving position located at distance χ along the direction $n^i = x^i/\chi$ and $\bar{x}^\mu$ is the apparent position. Writing that $\chi_e = \tilde{\chi} + \delta\chi$, where the subscript e stands for "evaluated at emission" (at the source) and $\tilde{\chi}$ and is the observed comoving distance, and then perturbing x^μ around $\bar{x}^\mu$, we obtain at linear order:

$$\begin{aligned} x^\mu(\chi_e) &= \bar{x}^\mu(\chi_e) + \delta x^\mu(\chi_e) \\ &= \bar{x}^\mu(\tilde{\chi}) + \frac{d\bar{x}^\mu}{d\tilde{\chi}}\delta\chi + \delta x^\mu(\tilde{\chi}) \, . \end{aligned} \tag{2.11}$$

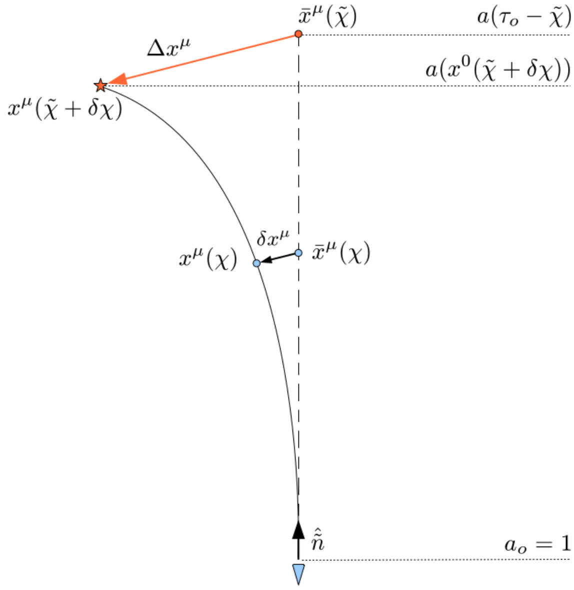


Figure 2.5: *Representation of the perturbed photon geodesics, where the observer is located at the bottom and sees the photon arriving at the direction $\hat{n}$. The solid line is the actual geodesic from the source (indicated by the star) to the observer, while the dashed line is the apparent unperturbed geodesic (a straight path) which is inferred from the observed direction $\hat{n}$ and which traces back to an apparent source position (the circle) that is different from the real one [31].*

Comparing expressions (2.10) and (2.11), we deduce that:

$$\Delta x^0(\tilde{\chi}) = \underbrace{\frac{d\bar{x}^0}{d\tilde{\chi}}}_{-1}\delta\chi + \delta x^0(\tilde{\chi}) = -\delta\chi + \delta x^0(\tilde{\chi}) \, , \tag{2.12}$$

$$\Delta x^i(\tilde{\chi}) = \underbrace{\frac{d\bar{x}^i}{d\tilde{\chi}}}_{\hat{n}^i}\delta\chi + \delta x^0(\tilde{\chi}) = \hat{n}^i\delta\chi + \delta x^i(\tilde{\chi}) \, . \tag{2.13}$$

It can be noticed that the first addend of (2.13) corresponds to the change in the affine parameter, while the second addend comes from the perturbation of the photon path [26].
The aim of this subsection is to compute the explicit expression of all the terms in (2.12) and (2.13), and in order to that we need to see how metric perturbations alter null geodesic. We consider a photon emitted at a certain galaxy position $\boldsymbol{x_E}$, which arrives at the observer position $\tilde{\boldsymbol{x}}_o = (0,0,0)$ from a direction $\hat{\boldsymbol{n}}$ with a redshift $\tilde{z}$ measured at $\tilde{\boldsymbol{x}}_o$ and given by

$$1 + \tilde{z} = \frac{(u_\mu p^\mu)|_e}{(u_\mu p^\mu)|_o} \, . \tag{2.14}$$

In this formula u_μ is the covariant four-velocity of the matter fluid (we are modelling the matter content of the Universe as a collissionless fluid), p^μ is the photon four-momentum, the subscript e stands for "evaluated at the source", and the subscript o stands for "evaluated at the observer". The expressions for δx^μ and $\delta\chi$ can be derived by perturbing the photon geodesic around the FRW solution. From (2.11) we deduce that

$$x^\mu(\chi) = \bar{x}^\mu(\chi) + \delta x^\mu(\chi) \, , \tag{2.15}$$

and we can write that

$$\frac{dx^\mu}{d\chi} = (-1 + \delta\nu, \hat{\boldsymbol{n}} + \delta\boldsymbol{e}) \, , \tag{2.16}$$

where the corrections are defined as:

- the fractional frequency perturbation $\delta\nu \equiv d\delta x^0/d\chi$;

- the fractional perturbation to the photon momentum $\delta e^i \equiv d\delta x^i/d\chi$.

Then we consider the *unperturbed geodesic equation*

$$\frac{d^2x^\mu}{d\chi^2} = -\hat{\Gamma}^\mu_{\nu\rho}\frac{dx^\nu}{d\chi}\frac{dx^\rho}{d\chi} \ , \tag{2.17}$$

that describes the motion of a free particle which is subjected only to the gravitational interaction, and thus to the effect of only the geometry of spacetime. $\hat{\Gamma}^\mu_{\nu\rho}$ are the Christoffel symbols computed from the metric $\hat{g}_{\mu\nu}$ (2.6) conformally transformed, that are defined as

$$\hat{\Gamma}^\mu_{\nu\rho} = \frac{1}{2}\hat{g}^{\mu\sigma}\big(\partial_\nu \hat{g}_{\rho\sigma} + \partial_\rho \hat{g}_{\nu\sigma} - \partial_\sigma \hat{g}_{\nu\rho}\big) \ . \tag{2.18}$$

In the background, the Christoffel symbols $\hat{\Gamma}^\mu_{\nu\rho}$ computed from $\hat{g}_{\mu\nu}$ that we are going to use are different from $\Gamma^\mu_{\nu\rho}$ computed from the metric $g_{\mu\nu}$ (2.1). Indeed, for the metric $g_{\mu\nu}$ the non null Christoffel symbols are

$$\Gamma^0_{00} = \frac{a'}{a} \quad , \quad \Gamma^i_{0j} = \frac{a'}{a}\delta^i_j \quad , \quad \Gamma^0_{00} = \frac{a'}{a}\delta_{ij} \ , \tag{2.19}$$

while for the metric $\hat{g}_{\mu\nu}$ we can easily see that all Christoffel symbols $\hat{\Gamma}^\mu_{\nu\rho}$ are null, since the scale factor $a(\tau)$ disappear with the conformal transformation; this fact simplifies further calculations.
We can now perturb the geodesic equation at the first order:

$$\frac{d^2\delta x^\mu}{d\chi^2} = -\delta\hat{\Gamma}^\mu_{\nu\rho}\frac{dx^\nu}{d\chi}\frac{dx^\rho}{d\chi} - 2\hat{\Gamma}^\mu_{\nu\rho}\frac{d\delta x^\nu}{d\chi}\frac{dx^\rho}{d\chi} \ , \tag{2.20}$$

and we can notice that the second addend in the rhs is zero due to what we have just said above. We need to know the perturbed Christoffel symbols $\delta\Gamma^\mu_{\nu\rho}$ in order to explicit the time and spatial components of (2.20). They can be computed as:

$$\delta\hat{\Gamma}^\mu_{\nu\rho} = \frac{1}{2}\Big[\delta\hat{g}^{\mu\sigma}\big(\partial_\nu \bar{\hat{g}}_{\rho\sigma} + \partial_\rho \bar{\hat{g}}_{\nu\sigma} - \partial_\sigma \bar{\hat{g}}_{\nu\rho}\big) + \bar{\hat{g}}^{\mu\sigma}\big(\partial_\nu \delta\hat{g}_{\rho\sigma} + \partial_\rho \delta\hat{g}_{\nu\sigma} - \partial_\sigma \delta\hat{g}_{\nu\rho}\big)\Big] \ , \tag{2.21}$$

where $\bar{\hat{g}}_{\mu\nu}$ is the background metric, the inverse background metric is given by $\bar{\hat{g}}^{\mu\nu} = diag(-1, \delta^{ij})$, and its first order perturbations $\delta\hat{g}^{\mu\nu}$ is given by

$$\delta\hat{g}^{\mu\nu} = -\bar{\hat{g}}^{\mu\rho}\bar{\hat{g}}^{\nu\sigma}\delta\hat{g}_{\rho\sigma} \ . \tag{2.22}$$

The only non-zero perturbed Chrystoffel symbols are[1,2]

$$\delta\hat{\Gamma}^0_{ij} = D'\delta_{ij} + E'_{ij} \ , \tag{2.23}$$
$$\delta\hat{\Gamma}^i_{0j} = D'\delta^i_j + E^{i\prime}_j \ , \tag{2.24}$$
$$\delta\hat{\Gamma}^i_{jk} = \partial_j D\ \delta^i_k + \partial_j E^i_k + \partial_k D\ \delta^i_j + \partial_k E^i_j - \partial^i D\ \delta_{jk} - \partial^i E_{jk} \ . \tag{2.25}$$

We are now able to compute the *perturbed geodesic equations*.

- $\mu = 0$:

$$\frac{d\delta\nu}{d\chi} = -\big(D' + E_\parallel{}'\big) \ ; \tag{2.26}$$

[1]0 indicates the time componet, while $i, j, k = 1, 2, 3$ stands for the spatial components.
[2]Prime $'$ represents the derivative with respect to τ.

- $\mu = i$:

$$\frac{d\delta e^i}{d\chi} = -2\frac{d}{d\chi}\Big(D\hat{n}^i + E^i_j\hat{n}^j\Big) + \partial^i D + \partial^i E_{jk}\hat{n}^j\hat{n}^k \tag{2.27}$$

$$= -2\frac{d}{d\chi}\Big(D\hat{n}^i + E^i_j\hat{n}^j\Big) + \partial^i D + \partial^i E_\parallel - \frac{2}{\chi}\Big(E^i_k\hat{n}^k - E_\parallel\hat{n}^i\Big) . \tag{2.28}$$

Calculations, which can be found in details in appendix A, are done using (2.20), the fact that

$$\frac{d}{d\chi} = \partial_\parallel - \partial_\tau , \tag{2.29}$$

and all the definitions of the projections along and perpendicular to the line of sight of the derivative operators, vectors, and tensors which can also be found in appendix A. We can obtain the perturbations to the photon momentum by integrating these expressions along the line of sight, and in order to do this we have to fix the conditions at the observer's position $\chi = 0$. The reference frame of the observer is built with an orthonormal tetrad e^a_μ, which is a local basis for the tangent space in spacetime. In this way, by defining p^μ as the observed photon momentum and $p = p_i p^i$ its spatial modulus, we can compute components of the observed photon direction as

$$\hat{n}^a = \frac{e^a_\mu p^\mu}{p} . \tag{2.30}$$

Using the orthonormality condition

$$\hat{g}^{\mu\nu} e^a_\mu e^b_\nu = \eta^{ab} , \tag{2.31}$$

we can first get the components of the tetrad. We find that[3]:

$$e^0_\mu = (-1, 0, 0, 0) , \tag{2.32}$$

$$e^i_\mu = (0, (1 + D_o)\delta^i_j + E^i_{oj}) . \tag{2.33}$$

At this point the photon momentum perturbations can be obtained by perturbing the orthonormality conditions (2.31). We start with

$$\big(\bar{\hat{g}}^{\mu\nu} + \delta\hat{g}^{\mu\nu}\big)\big(e^a_\mu + \delta e^a_\mu\big)\big(e^b_\nu + \delta e^b_\nu\big) = \eta^{ab} , \tag{2.34}$$

and we take the 00 equation:

$$\big(\bar{\hat{g}}^{00} + \delta\hat{g}^{00}\big)\big(e^0_0 + \delta e^0_0\big)^2 + \big(g^{ij}_{(0)} + \delta g^{ij}\big)\big(e^0_i + \delta e^0_i\big)\big(e^0_j + \delta e^0_j\big) = \eta^{00} = -1 .$$

By substituting the expressions of e^a_μ and dropping the terms of orders higher than the first one, we get

$$\delta e^0_0 = \delta\nu_0 = 0 . \tag{2.35}$$

Then, by perturbing the other condition:

$$\big(e^j_i + \delta e^j_i\big)\big(e^i_j + \delta e^i_j\big) = 1 ,$$

and neglecting higher order terms as before, we get

$$\delta e^i_0 = -\big(D_o\hat{n}^i + E^i_{oj}\hat{n}^j\big) . \tag{2.36}$$

[3]The notation "o" stands for the quantity evaluated at the observer's position.

Detailed calculations can be found in appendix A. We are now ready to integrate (2.26) along the line of sight:

$$\int_0^{\chi}\left[\frac{d\delta e^i}{d\chi}+2\frac{d}{d\chi}\Big(D\hat{n}^i+E^i_j\hat{n}^j\Big)\right]d\chi'=\int_0^{\chi}\left[\partial^i D+\partial^i E_{\parallel}-\frac{2}{\chi}\Big(E^i_k\hat{n}^k-E_{\parallel}\hat{n}^i\Big)\right]d\chi' \tag{2.37}$$

$$\begin{aligned}&\delta e^i(\chi)-\delta e^i(0)+2\Big(D\hat{n}^i+E^i_j\hat{n}^j\Big)_{\chi}-2\Big(D\hat{n}^i+E^i_j\hat{n}^j\Big)_o\\&=\int_0^{\chi}\left[\partial^i D+\partial^i E_{\parallel}-\frac{2}{\chi}\Big(E^i_k\hat{n}^k-E_{\parallel}\hat{n}^i\Big)\right]d\chi'\end{aligned}$$

$$\begin{aligned}\delta e^i(\chi)=&-2\Big(D\hat{n}^i+E^i_j\hat{n}^j\Big)_{\chi}+\Big(D\hat{n}^i+E^i_j\hat{n}^j\Big)_o\\&+\int_0^{\chi}\left[\partial^i D+\partial^i E_{\parallel}-\frac{2}{\chi}\Big(E^i_k\hat{n}^k-E_{\parallel}\hat{n}^i\Big)\right]d\chi' .\end{aligned} \tag{2.38}$$

Since $\delta e^i = d\delta x^i/d\chi$, δx^i can be obtained by integrating over the line of sight from 0 and $\tilde{\chi}$. The double integral can be simplified with a trick (see Fig.10.5 in [1]) which reduces it to a single integral of the quantity inside multiplied by $\tilde{\chi}-\chi$:

$$\begin{aligned}\delta x^i=\int_0^{\tilde{\chi}}\delta e^i d\chi=\tilde{\chi}\Big(D\hat{n}^i+E^i_j\hat{n}^j\Big)_o+\int_0^{\tilde{\chi}}\Bigg[&-2\Big(D\hat{n}^i+E^i_j\hat{n}^j\Big)\\&+(\tilde{\chi}-\chi)\Big(\partial^i D+\partial^i E_{\parallel}-\frac{2}{\chi}\Big(E^i_k\hat{n}^k-E_{\parallel}\hat{n}^i\Big)\Big)\Bigg]d\chi .\end{aligned} \tag{2.39}$$

At this point (2.27) can be integrated too:

$$\int_0^{\tilde{\chi}}\frac{d\delta\nu}{d\chi}d\chi=-\int_0^{\tilde{\chi}}\big(D'+E_{\parallel}{}'\big)d\chi \quad\Longrightarrow\quad \delta\nu(\tilde{\chi})=-\int_0^{\tilde{\chi}}\big(D'+E_{\parallel}{}'\big) . \tag{2.40}$$

Since $\delta\nu = d\delta x^0/d\chi$, by integrating over the line of sight as before we get:

$$\delta x^0=\int_0^{\tilde{\chi}}\delta\nu\, d\chi=-\int_0^{\tilde{\chi}}d\chi\big(\tilde{\chi}-\chi\big)\big(D'+E_{\parallel}{}'\big) . \tag{2.41}$$

The physical interpretation of the frequency shift $\delta\nu$ embodies the Doppler, Sachs-Wolfe and integrated Sachs-Wolfe effects.

2.1.3 Redshift

Now an important physical quantity will be introduced, because it is the key observable quantity we work with, and that will be related to the affine parameter at emission: we are talking about the redshift $z(\chi)$. We consider a photon emitted from a galaxy (the source), which is moving in direction $\hat{\boldsymbol{n}}$ (hence it is seen under the direction $-\hat{\boldsymbol{n}}$ from the observer). Given $\hat{n}^{\mu}=a^{-2}(-1+\delta\hat{n}^0,\hat{\boldsymbol{n}}+\delta\hat{\boldsymbol{n}})=a^{-2}(-1+\delta\nu,\hat{\boldsymbol{n}}+\delta\hat{\boldsymbol{n}})$, and the comoving observer's four-velocity $u_{\mu}=(a,0,0,0)$ in the synchronous-comoving gauge, the redshift along the perturbed photon geodesic is given by:

$$1+z(\chi)=\frac{(u_{\mu}\hat{n}^{\mu})_e}{(u_{\mu}\hat{n}^{\mu})_o} , \tag{2.42}$$

where "e" stands for "evaluated at the emission (source) position" and "o" for "evaluated at the observer's position". Setting $a_o = 1$, and

$$\delta z(\tilde{\chi}) \equiv -\delta\nu(\tilde{\chi}) = \int_0^{\tilde{\chi}} d\chi \big(D' + E_{\|}{}'\big) \,, \tag{2.43}$$

as the redshift perturbation, we compute that:

$$1 + z(\chi) = \frac{\big(-1 + \delta\nu(\chi)\big) a^{-1}(x^0(\chi))}{-1} = \frac{1 + \delta z(\chi)}{a(x^0(\chi))} \,, \tag{2.44}$$

where $x^0(\chi) = \tau$ is the conformal time. For a given source observed at redshift $\tilde{z}$, Eq. (2.44) is an implicit relation for the affine parameter χ_e at emission:

$$1 + z(\chi_e) = 1 + \tilde{z} \,, \tag{2.45}$$

which gives a definition of the space-time location of the source

$$x^\mu_{source} = x^\mu(\chi_e) \,. \tag{2.46}$$

The redshift $\bar{z}(\chi_e)$ that would have been observed for the same source without any perturbations along the line of sight is given by

$$1 + \bar{z}(\chi_e) = \frac{1}{a(x^0(\chi_e))} \,. \tag{2.47}$$

Therefore at $\chi = \chi_e$ we get

$$1 + \tilde{z} = \big(1 + \bar{z}\big)\big(1 + \delta z(\chi_e)\big) \,. \tag{2.48}$$

At zero order we have $\chi_e = \tilde{\chi}$ and $z(\chi) = \bar{z}(\chi)$, but at first order we can make an expansion around $\chi_e = \tilde{\chi} + \delta\chi$. First af all we can notice that, from expressions (2.10) and (2.11), we have:

$$x^\mu(\chi_e) = \bar{x}^\mu(\tilde{\chi}) + \Delta x^\mu(\tilde{\chi}) \,. \tag{2.49}$$

Then, we can perform the expansion of $a(x^0(\chi_e))$:

$$\begin{aligned} a(x^0(\chi_e)) &= a(\bar{x}^0(\tilde{\chi}) + \Delta x^0(\tilde{\chi})) \\ &= a(\bar{x}^0(\tilde{\chi})) + \frac{da(\bar{x}^0)}{dx^0}\Delta x^0(\tilde{\chi}) \\ &= a(\bar{x}^0(\tilde{\chi}))\Bigg[1 + \underbrace{\frac{1}{a(\bar{x}^0)}\frac{da(\bar{x}^0)}{dx^0}}_{\frac{a'(\bar{x}^0)}{a(\bar{x}^0)} = \mathcal{H}(\bar{x}^0) = aH(\bar{x}^0)} \Delta x^0(\tilde{\chi})\Bigg] \\ &= a(\bar{x}^0(\tilde{\chi}))\Big[1 + aH(\bar{x}^0)\Delta x^0(\tilde{\chi})\Big] \,. \end{aligned} \tag{2.50}$$

Now the expression (2.12) for Δx^0 can be inserted, getting to:

$$a(x^0(\tilde{\chi})) = a(\bar{x}^0(\tilde{\chi}))\Big\{1 + aH(\bar{x}^0)\Big[-\delta\chi + \delta x^0(\tilde{\chi})\Big]\Big\} \,. \tag{2.51}$$

At first order in the perturbations, its reciprocal is

$$\frac{1}{a(x^0(\tilde{\chi}))} = \frac{1}{a(\bar{x}^0(\tilde{\chi}))}\Big\{1 - aH(\bar{x}^0)\Big[-\delta\chi + \delta x^0(\tilde{\chi})\Big]\Big\} \,. \tag{2.52}$$

Therefore the expression (2.48) becomes:

$$1+\tilde{z} = \big(1+\tilde{z}\big)\Big\{1-\big(aH\big)_{\tilde{z}}\Big[\delta x^0(\tilde{\chi})-\delta\chi\Big]+\delta z(\tilde{\chi})\Big\}\,. \tag{2.53}$$

We can now give the connection between the affine parameter and the observed redshift by solving (2.53) for $\delta\chi$:

$$\delta x^0-\delta\chi = \frac{\delta z}{\big(aH\big)_{\tilde{z}}} = \frac{1+\tilde{z}}{H(\tilde{z})}\delta z \implies \delta\chi = \delta x^0 - \frac{1+\tilde{z}}{H(\tilde{z})}\delta z\,. \tag{2.54}$$

Finally, projections along and perpendicular to the line of sight can be computed starting from the expression (2.13) of Δx^i, and using the definitions for the projections that can be found in appendix A:

$$\Delta x_\parallel = \Delta x^i\,\hat{n}_i = \delta\chi + \delta x^i(\tilde{\chi})\,\hat{n}_i = \delta x^0 - \frac{1+\tilde{z}}{H(\tilde{z})}\delta z + \delta x_\parallel\,, \tag{2.55}$$

$$\Delta x^i_\perp = (\delta^i_j - \hat{n}^i\hat{n}_j)\Delta x^i = \hat{n}_i\delta\chi + \delta x^i - \hat{n}^i\hat{n}_j\hat{n}^i\delta\chi - \hat{n}^i\hat{n}_j\delta x^j = \delta x^i - \hat{n}^i\delta x_\parallel\,. \tag{2.56}$$

More explicit expressions of these displacements will be computed later, but we can give here a useful expression of δz using the fact that $E_{ij} = \partial_i\partial_j E - \frac{1}{3}\nabla^2\delta_{ij}E$, that $d/d\chi + {}' = \partial_\parallel$, and the definition of $E_\parallel$ which can be found in appendix A.

$$\begin{aligned}
\delta z &= \int_0^{\tilde{\chi}} d\chi\big(D' + \hat{n}^i\hat{n}^j E'_{ij}\big) = \int_0^{\tilde{\chi}} d\chi\Big(D' + \partial^2_\parallel E' - \frac{1}{3}\nabla^2 E'\Big)\\
&= \int_0^{\tilde{\chi}} d\chi\Big[D' - \frac{1}{3}\nabla^2 E' + \Big(\frac{d}{d\chi}+\frac{d}{d\tau}\Big)\partial_\parallel E'\Big]\\
&= \big[\partial_\parallel E'\big]_0^{\tilde{\chi}} + \int_0^{\tilde{\chi}} d\chi\Big[D' - \frac{1}{3}\nabla^2 E' + \Big(\frac{d}{d\chi}+\frac{d}{d\tau}\Big)E''\Big]\\
&= \big[\partial_\parallel E' + E''\big]_0^{\tilde{\chi}} + \int_0^{\tilde{\chi}} d\chi\Big(D' - \frac{1}{3}\nabla^2 E' + E'''\Big)\,.
\end{aligned}$$

Then we can see that

$$\delta z = \big[\partial_\parallel E' + E''\big]_o^s + \delta z_{ISW}\,, \tag{2.57}$$

where "o" stands for "evaluated at the observer's position", "s" stands for "evaluated at the source position", and

$$\delta z_{ISW} = \int_0^{\tilde{\chi}} d\chi\Big(D' - \frac{1}{3}\nabla^2 E' + E'''\Big) \tag{2.58}$$

corresponds to the integrated Sachs-Wolfe contribution in the synchronous-comoving gauge. This claim can be easily explained if we take the general expression for the ISW term

$$\delta z_{ISW} = \int_0^{\tilde{\chi}} d\chi\big(\Phi+\Psi\big)'\,, \tag{2.59}$$

where Φ and Ψ are the gauge invariant Bardeen potentials [20, 31]. They are special combinations of metric perturbations that do not transform under a change of coordinates, therefore it's useful to express physical quantities in terms of them in order to avoid gauge problems. Starting from the metric $\hat{g}_{\mu\nu}$, in the synchronous comoving gauge we are adopting, they assume the form [19]:

$$\Psi = E''\,, \tag{2.60}$$

$$\Phi = D - \frac{1}{3}\nabla^2 E\,. \tag{2.61}$$

Therefore the expression (2.58) is immediately obtained by taking the derivative with respect to τ of the sum of the two previous potentials, and then of course by integrating it along the line of sight.

2.2 Perturbation of the galaxy number density

After perturbing photon trajectories, we are ready to study how the observed galaxy number density is measured and to compute its perturbation. The starting point the number of galaxies N within a volume $\tilde{V}$, which can be defined starting from the true spatial position $\mathbf{x}$ given in terms of the observed coordinates $\tilde{\mathbf{x}}$ and the galaxy number density $n_g(x^\mu)$:

$$N = \int_{\tilde{V}} \sqrt{-g(x^\alpha)} n_g(x^\alpha) \epsilon_{\mu\nu\rho\sigma} u^\mu(x^\alpha) \frac{\partial x^\nu}{\partial \tilde{x}^1} \frac{\partial x^\rho}{\partial \tilde{x}^2} \frac{\partial x^\sigma}{\partial \tilde{x}^3} d^3\tilde{\mathbf{x}} \,, \tag{2.62}$$

where $u^\mu = (1/a, 0, 0, 0)$ is the observer's four-velocity in the synchronous-comoving gauge, and the quantity inside the integral is the number density we want to perturb. Therefore:

$$N = \int_{\tilde{V}} \sqrt{-g(x^\alpha)} n_g(x^\alpha) \frac{1}{a(x^0)} \epsilon_{ijk} \frac{\partial x^i}{\partial \tilde{x}^1} \frac{\partial x^j}{\partial \tilde{x}^2} \frac{\partial x^k}{\partial \tilde{x}^3} d^3\tilde{\mathbf{x}} = \int_{\tilde{V}} \sqrt{-g(x^\alpha)} n_g(x^\alpha) \frac{1}{a(x^0)} \left| \frac{\partial x^i}{\partial \tilde{x}^j} \right| d^3\tilde{\mathbf{x}} \,. \tag{2.63}$$

Here below we compute the perturbations for these three terms:
$\sqrt{-g(x)}$, n_g, $\left|\frac{\partial x^i}{\partial \tilde{x}^j}\right|$.

2.2.1 Metric determinant g and Jacobian

Given the unperturbed square root of the metric determinant $\sqrt{-\bar{g}} = a^4$, we can write that:

$$\sqrt{-g} = \sqrt{-\bar{g}} + \delta\sqrt{-g} \,. \tag{2.64}$$

Knowing that $g = \exp(\mathrm{Tr}(\ln(g_{\mu\nu})))$, it's easy to compute $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-\bar{g}}\, g\, g^{\mu\nu}\delta g_{\mu\nu}$, thus:

$$\sqrt{-g} = a^4 + \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu} = a^4\left[1 + \frac{1}{2} g^{\mu\nu}\delta g_{\mu\nu}\right] = a^4\left[1 + \frac{1}{2}\delta g^\mu_\mu\right] \,. \tag{2.65}$$

The expansion of the Jacobian at the first order in the displacement $\Delta\mathbf{x}$ is:

$$\left|\frac{\partial x^i}{\partial \tilde{x}^j}\right| = \left|\delta^i_j + \frac{\partial \Delta x^i}{\partial \tilde{x}^j}\right| = 1 + \frac{\partial \Delta x^i}{\partial \tilde{x}^i} \,. \tag{2.66}$$

2.2.2 Comoving galaxy number density

We can now consider the comoving galaxy number density $a^3 n_g$ and start to perturb it:

$$a^3(x^0) n_g(x^0, \mathbf{x}) = a^3(x^0) \bar{n}_g(x^0) \left[1 + \delta_g(\mathbf{x}, x^0)\right] \,, \tag{2.67}$$

where δ_g is the galaxy number density perturbation and x^0 is the conformal time, connected to the redshift through (2.47). Now we want to make an expansion around

$\bar{x}^0$ at the first order as already done in section 2.1.3, by expanding all the terms inside the expression (2.67):

$$a^3(x^0) = a^3(\bar{x}^0 + \Delta x^0) = a^3(\bar{x}^0) + \frac{da^3(\bar{x}^0)}{dx^0}\Delta x^0 \,, \tag{2.68}$$

$$\bar{n}_g(x^0) = \bar{n}_g(\bar{x}^0 + \Delta x^0) = \bar{n}_g(\bar{x}^0) + \frac{d\bar{n}_g(\bar{x}_0)}{dx^0}\Delta x^0 \,. \tag{2.69}$$

Therefore at first order in perturbations we find:

$$\begin{aligned}
a^3(x^0)n_g(x^0,\mathbf{x}) &= \left[a^3(\bar{x}^0) + \frac{da^3(\bar{x}^0)}{dx^0}\Delta x^0\right]\left[\bar{n}_g(\bar{x}^0) + \frac{d\bar{n}_g(\bar{x}_0)}{dx^0}\Delta x^0\right]\left[1+\delta_g(\mathbf{x},x^0)\right] \\
&= \left[a^3(\bar{x}^0)\bar{n}_g(\bar{x}^0) + a^3(\bar{x}^0)\frac{d\bar{n}_g(\bar{x}_0)}{dx^0}\Delta x^0 + \bar{n}_g(\bar{x}^0)\frac{da^3(\bar{x}^0)}{dx^0}\Delta x^0\right]\left[1+\delta_g(\mathbf{x},x^0)\right] \\
&= a^3(\bar{x}^0)\bar{n}_g(\bar{x}^0)\left[1 + \underbrace{\frac{1}{\bar{n}_g}\frac{d\bar{n}_g(\bar{x}_0)}{dx^0}}_{\frac{d\ln\bar{n}_g(\bar{x}^0)}{dx^0}}\Delta x^0 + \underbrace{\frac{1}{a^3(\bar{x}^0)}\frac{da^3(\bar{x}^0)}{dx^0}}_{\frac{d\ln a^3(\bar{x}^0)}{dx^0}}\Delta x^0\right]\left[1+\delta_g(\mathbf{x},x^0)\right] \\
&= a^3(\bar{x}^0)\bar{n}_g(\bar{x}^0)\left[1 + \frac{d\ln\big(a^3\bar{n}_g(x^0)\big)}{dx^0}\Delta x^0\right]\left[1+\delta_g(\mathbf{x},x^0)\right] \\
&= a^3(\bar{x}^0)\bar{n}_g(\bar{x}^0)\left[1+\delta_g(\mathbf{x},x^0)\right] + \frac{d\big(a^3\bar{n}_g(\bar{x}^0)\big)}{dx^0}\Delta x^0 \,.
\end{aligned} \tag{2.70}$$

At this point we can write this expression in terms of the observed redshift $\tilde{z}$, reminding that:

- $\Delta x^0 = \delta x^0 - \delta\chi = \frac{1+\tilde{z}}{H(\tilde{z})}\delta z$ from (2.12) and (2.54);
- $\frac{d}{dx^0} = -H(\tilde{z})\frac{d}{dz}\big|_{z=\tilde{z}}$ from (2.47).

By doing these substitutions we have[4]:

$$\begin{aligned}
a^3(z)n_g(\mathbf{x},\bar{z}) &= a^3(\tilde{z})\bar{n}_g(\tilde{z})\left[1+\delta_g(\tilde{\mathbf{x}})\right] - H(\tilde{z})\frac{d\big(a^3\bar{n}_g\big)}{dz}\bigg|_{z=\tilde{z}}\frac{1+\tilde{z}}{H(\tilde{z})}\delta z \\
&= a^3(\tilde{z})\bar{n}_g(\tilde{z})\left[1+\delta_g(\tilde{\mathbf{x}})\right] - \big(1+\tilde{z}\big)\frac{d\big(a^3\bar{n}_g\big)}{dz}\bigg|_{z=\tilde{z}}\delta z \,.
\end{aligned} \tag{2.71}$$

The comoving galaxy number density gives N when integrated over the volume $\tilde{V}$

$$\int_{\tilde{V}} a^3(\tilde{z})\tilde{n}_g(\tilde{\mathbf{x}},\tilde{z})d^3\tilde{\mathbf{x}} = N \,. \tag{2.72}$$

Therefore we can compare the quantity inside the integral of (2.63) with the final expression of $a^3\bar{n}_g$ (2.71), with the insertion of the perturbed expressions of $\sqrt{-\hat{g}}$ and of the Jacobian:

$$a^3(\tilde{z})\tilde{n}_g(\tilde{\mathbf{x}},\tilde{z}) = \sqrt{-g(\mathbf{x},\bar{z})}n_g(\mathbf{x},\bar{z})\frac{1}{a(\bar{z})}\left|\frac{\partial x^i}{\partial\tilde{x}^j}\right| = a^4(\bar{z})\left(1+\frac{1}{2}\delta g^\mu_\mu\right)n_g(\mathbf{x},\bar{z})\frac{1}{a(\bar{z})}\left(1+\frac{\partial\Delta x^i}{\partial\tilde{x}^i}\right) . \tag{2.73}$$

[4] Note that the distinction between $\delta g(\tilde{\mathbf{x}})$ and $\mathbf{x}$ is second order.

We can now insert the expansions (2.68) of $a^3(\bar{z})$ and (2.69) $n_g(\bar{z})$ with the previous substitutions related to $\tilde{z}$:

$$
\begin{aligned}
a^3(\tilde{z})\tilde{n}_g(\tilde{\mathbf{x}},\tilde{z}) &= \left(1+\frac{1}{2}\delta g^\mu_\mu\right)\left[a^3(\tilde{z})-(1+\tilde{z})\frac{da^3}{dz}\Big|_{\tilde{z}}\delta z\right] \\
&\quad\times\left[\bar{n}_g(\tilde{z})-(1+\tilde{z})\frac{d\bar{n}_g}{dz}\Big|_{\tilde{z}}\delta z\right]\Big[1+\delta_g(\tilde{\mathbf{x}})\Big]\left(1+\frac{\partial\Delta x^i}{\partial\tilde{x}^i}\right) \\
&= a^3(\tilde{z})\bar{n}_g(\tilde{z})\Bigg[1-\underbrace{\frac{1}{\bar{n}_g(\tilde{z})}\frac{d\bar{n}_g}{dz}\Big|_{\tilde{z}}}_{\frac{d\ln(\bar{n}_g(z))}{dz}\big|_{\tilde{z}}}(1+\tilde{z})\delta z-\underbrace{\frac{1}{a^3(\tilde{z})}\frac{da^3}{dz}\Big|_{\tilde{z}}}_{\frac{d\ln(a^3(z))}{dz}\big|_{\tilde{z}}}(1+\tilde{z})\delta z\Bigg] \\
&\quad\times\underbrace{\left(1+\frac{1}{2}\delta g^\mu_\mu\right)\Big(1+\delta_g(\tilde{\mathbf{x}})\Big)\left(1+\frac{\partial\Delta x^i}{\partial\tilde{x}^i}\right)}_{\simeq 1+\delta_g(\tilde{x})+\frac{\partial\Delta x^i}{\partial\tilde{x}^i}+\frac{1}{2}\delta g^\mu_\mu},
\end{aligned}
$$

where we discarded high order terms as usual. In the end we find:

$$
\begin{aligned}
\frac{\tilde{n}_g(\tilde{\mathbf{x}},\tilde{z})}{\bar{n}_g(\tilde{z})} &= \left(1-\frac{d\ln(a^3\bar{n}_g)}{dz}\Big|_{\tilde{z}}(1+\tilde{z})\delta z\right)\left(1+\delta_g(\tilde{\mathbf{x}})+\frac{\partial\Delta x^i}{\partial\tilde{x}^i}+\frac{1}{2}\delta g^\mu_\mu\right) \\
&= 1+\delta_g(\tilde{\mathbf{x}})+\frac{\partial\Delta x^i}{\partial\tilde{x}^i}+\frac{1}{2}\delta g^\mu_\mu-\frac{d\ln(a^3\bar{n}_g)}{dz}\Big|_{\tilde{z}}\underbrace{(1+\tilde{z})}_{\frac{1}{a(\tilde{z})}}\delta z\,. \qquad (2.74)
\end{aligned}
$$

At this point we can remind that $1+\tilde{z}=1/a(\tilde{z})$, therefore $d/dz=-a^2\,d/da$, thus:

$$
\begin{aligned}
\frac{\tilde{n}_g(\tilde{\mathbf{x}},\tilde{z})}{\bar{n}_g(\tilde{z})} &= 1+\delta_g(\tilde{\mathbf{x}})+\frac{\partial\Delta x^i}{\partial\tilde{x}^i}+\frac{1}{2}\delta\hat{g}^\mu_\mu-a(\tilde{z})\frac{d\ln(a^3\bar{n}_g)}{dz}\Big|_{\tilde{z}}\delta z \\
&= 1+\delta_g(\tilde{\mathbf{x}})+\frac{\partial\Delta x^i}{\partial\tilde{x}^i}+\frac{1}{2}\delta\hat{g}^\mu_\mu-\frac{d\ln(a^3\bar{n}_g)}{d\ln a}\Big|_{\tilde{z}}\delta z\,. \qquad (2.75)
\end{aligned}
$$

We can now define the quantity

$$
b_e\equiv\frac{d\ln(a^3\bar{n}_g)}{d\ln a}\Big|_{\tilde{z}}=-(1+\tilde{z})\frac{d\ln(a^3\bar{n}_g)}{dz}\Big|_{\tilde{z}}\,. \qquad (2.76)
$$

Further, starting from the metric (2.1) we can find that

$$
\frac{1}{2}\delta g^\mu_\mu=\frac{1}{2}(2D\times 3)=3D\,. \qquad (2.77)
$$

Then it can be computed that

$$
\frac{\partial\Delta x^i}{\partial\tilde{x}^i}=\partial_\parallel\Delta x_\parallel+2\frac{\Delta x_\parallel}{\tilde{\chi}}-2\hat{\kappa}\,, \qquad (2.78)
$$

where $\hat{\kappa}$ is the coordinate convergence, which is related to the effect of lensing (distortions on the directions which are perpendicular to the line of sight) and is here defined as

$$
\hat{\kappa}=-\frac{1}{2}\partial_{\perp i}\Delta x^i_\perp\,. \qquad (2.79)
$$

The expression (2.78) is obtained by using the property (A.14) which can be found in appendix A and that leads to

$$
\frac{\partial\Delta x^i}{\partial\tilde{x}^i}=\partial_\parallel\Delta x_\parallel+\partial_{\perp i}\Delta x^i_\perp+\Delta x_\parallel\partial_i\hat{n}^i\,,
$$

where the third terms gives $\Delta x_\parallel \cdot \frac{2}{\tilde{\chi}}$ according to the property (A.6) of appendix A. We are now ready to return to the expression (2.74) of the ratio between $\tilde{n}_g(\tilde{\mathbf{x}}, \tilde{z})$ and $\tilde{n}_g(\tilde{z})$:

$$\frac{\tilde{n}_g(\tilde{\mathbf{x}}, \tilde{z})}{\bar{n}_g(\tilde{z})} = 1 + 3D + \delta_g + b_e \delta z + \partial_\parallel \Delta x_\parallel + 2\frac{\Delta x_\parallel}{\tilde{\chi}} - 2\hat{\kappa} \,. \tag{2.80}$$

2.2.3 Explicit expression of $\tilde{n}_g(\tilde{x}, \tilde{z})/\tilde{n}_g(\tilde{z})$

$\partial_\parallel \Delta x_\parallel$

The expression (2.80) can be further explicited. We start with the computation of $\partial_\parallel \Delta x_\parallel$, for which we need to derive the following expression:

$$\Delta x_\parallel = -\int_0^{\tilde{\chi}} d\chi \big(D + E_\parallel\big) - \frac{1+\tilde{z}}{H(\tilde{z})} \underbrace{\int_0^{\tilde{\chi}} d\chi \big(D' + E'_\parallel\big)}_{\delta z} \,. \tag{2.81}$$

Reminding that at the first order $\partial_\parallel = \partial/\partial\tilde{\chi}$, we can calculate:

$$\partial_\parallel \Delta x_\parallel = -\big(D + E_\parallel\big)\big|_{\tilde{\chi}} - \frac{\partial}{\partial\tilde{\chi}}\left(\frac{1+\tilde{z}}{H(\tilde{z})}\right)\bigg|_{\chi=\tilde{\chi}} \delta z - \frac{1+\tilde{z}}{H(\tilde{z})}\big(D' + E'_\parallel\big)\big|_{\tilde{\chi}} \,. \tag{2.82}$$

It's easy to see that:

$$\tilde{z}(\chi_e) = \frac{1}{a(\tilde{x}^0(\chi_e))} - 1 \implies \frac{d\tilde{z}}{d\chi}\bigg|_{\tilde{\chi}} = -\underbrace{\frac{da}{d\tilde{x}^0}}_{a'} \underbrace{\frac{d\tilde{x}^0}{d\tilde{\chi}}}_{-1} \frac{1}{a^2(\tilde{x}^0(\chi_e))}\bigg|_{\tilde{\chi}} = \frac{a'}{a^2}\bigg|_{\tilde{z}} = H(\tilde{z}) \,.$$

Besides we can notice that $\partial/\partial\tilde{\chi} = (d\tilde{z}/d\tilde{\chi})\partial/\partial\tilde{z} = H(\tilde{z})\partial/\partial\tilde{z}$. Therefore:

$$\partial_\parallel \Delta x_\parallel = -\big(D + E_\parallel\big)\big|_{\tilde{\chi}} - \delta z H(\tilde{z}) \frac{d}{dz}\left(\frac{1+\tilde{z}}{H(\tilde{z})}\right) - \frac{1+\tilde{z}}{H(\tilde{z})}\big(D' + E'_\parallel\big)\big|_{\tilde{\chi}} \,. \tag{2.83}$$

Another useful expression is:

$$\partial_\parallel \Delta x_\parallel = -\big(D + E_\parallel\big)\big|_{\tilde{\chi}} - \left[1 - \frac{1+\tilde{z}}{H(\tilde{z})}\frac{dH(\tilde{z})}{d\tilde{z}}\right]\delta z - \frac{1+\tilde{z}}{H(\tilde{z})}\big(D' + E'_\parallel\big)\big|_{\tilde{\chi}} \,, \tag{2.84}$$

where a total derivative has been used to substitute the term in $d/d\tilde{z}$. This expression will be inserted in the final form of $\tilde{n}_g(\tilde{\mathbf{x}}, \tilde{z})/\tilde{n}_g(\tilde{z})$.

Convergence $\hat{\kappa}$

We go on with the explicit derivation of $\hat{\kappa}$, starting from the expression:

$$\Delta x^i_\perp = \tilde{\chi}\big(E^i_j \hat{n}^j - E_\parallel \hat{n}^i\big)_o + \int_0^{\tilde{\chi}} d\chi \left[-2\frac{\tilde{\chi}}{\chi}\big(E^i_j \hat{n}^j - E_\parallel \hat{n}^i\big) + \big(\tilde{\chi} - \chi\big)\partial^i_\perp\big(D + E_\parallel\big)\right] . \tag{2.85}$$

The computation of $\hat{\kappa}$ can now be performed [31]:

$$\begin{aligned}\hat{\kappa} = -\frac{1}{2}\partial_{\perp i}\Delta x^i = & \underbrace{-\frac{1}{2}\partial_{\perp i}\Big[\tilde{\chi}\big(E^i_j \hat{n}^j - E_\parallel \hat{n}^i\big)_o\Big]}_{(1)} \underbrace{+\partial_{\perp i}\int_0^{\tilde{\chi}} d\chi \frac{\tilde{\chi}}{\chi}\big(E^i_j \hat{n}^j - E_\parallel \hat{n}^i\big)}_{(2)} \\ & \underbrace{-\frac{1}{2}\nabla^2_\perp \int_0^{\tilde{\chi}} d\chi \frac{\tilde{\chi}}{\chi}\big(\tilde{\chi} - \chi\big)\big(D + E_\parallel\big)}_{(3)} \,, \end{aligned} \tag{2.86}$$

where in the term (3) the extra factor $\tilde{\chi}/\chi$ comes from the fact that $\partial_\perp^i$ has been moved outside the integral, so it acts at radius $\tilde{\chi}$ rather than χ; the subscript o stands for "evaluated at the observer's position". These terms will be further explicited one by one. In the third one (3) we can notice that $D + E_\parallel = D - \frac{1}{3}\nabla^2 E + \partial_\parallel^2 E$, since $E_{ij} = (\partial_i\partial_j - \frac{1}{3}\nabla^2\delta_{ij})$. In the second (2):
$\hat{n}^j E_j^i - \hat{n}^i E_\parallel = \hat{n}^j\big(\partial_j\partial^i - \frac{1}{3}\delta_j^i\nabla^2\big)E - \hat{n}^i\big(\partial_\parallel^2 - \frac{1}{3}\nabla^2\big)E = \big(\partial_\parallel\partial^i - \partial_\parallel\hat{n}^i\partial_\parallel\big)E = \partial_\parallel\partial_\perp^i E = \partial_\perp^i\partial_\parallel E - \frac{1}{\chi}\partial_\perp^i E$ (see the commutation properties in appendix A).
Therefore the second and the third terms can be written as:

$$
\begin{aligned}
(2)+(3) &= \partial_{\perp i}\int_0^{\tilde{\chi}} d\chi\frac{\tilde{\chi}}{\chi}\left(\partial_\perp^i\partial_\parallel E - \frac{1}{\chi}\partial_\perp^i E\right) - \frac{1}{2}\nabla_\perp^2\int_0^{\tilde{\chi}} d\chi\frac{\tilde{\chi}}{\chi}(\tilde{\chi}-\chi)\left(D - \frac{1}{3}\nabla^2 E + \partial_\parallel^2 E\right)\\
&= \nabla_\perp^2\int_0^{\tilde{\chi}} d\chi\left[\frac{\tilde{\chi}^2}{\chi^2}\partial_\parallel E - \frac{\tilde{\chi}^2}{\chi^3}E - \frac{1}{2}\frac{\tilde{\chi}}{\chi}(\tilde{\chi}-\chi)\left(D - \frac{1}{3}\nabla^2 E + \partial_\parallel^2 E + E'' - E''\right)\right]\\
&= \underbrace{-\frac{1}{2}\int_0^{\tilde{\chi}} d\chi\frac{\chi}{\tilde{\chi}}(\tilde{\chi}-\chi)\nabla_\perp^2\left(D - \frac{1}{3}\nabla^2 E + E''\right)}_{\kappa}\\
&\quad + \frac{1}{2}\nabla_\perp^2\int_0^{\tilde{\chi}} d\chi\left[\left(\tilde{\chi} - \frac{\tilde{\chi}^2}{\chi}\right)\left(\partial_\parallel^2 E - E'' + 2\frac{\tilde{\chi}^2}{\chi^2}\partial_\parallel E - 2\frac{\tilde{\chi}^2}{\chi^3}E\right)\right].
\end{aligned}
$$

Here we defined

$$
\kappa = -\frac{1}{2}\int_0^{\tilde{\chi}} d\chi\frac{\chi}{\tilde{\chi}}(\tilde{\chi}-\chi)\nabla_\perp^2\left(D - \frac{1}{3}\nabla^2 E + E''\right). \tag{2.87}
$$

Then, by using the fact that $' = \partial_\parallel - d/d\chi$, we can compute that: $\partial_\parallel^2 E - E'' = \partial_\parallel^2 E - (E')' = \partial_\parallel^2 E - (\partial_\parallel E - dE/d\chi)' = \partial_\parallel^2 E - \partial_\parallel^2 E + d(\partial_\parallel E)/d\chi + d(\partial_\parallel E)/d\chi - d^2E/d\chi^2 = d(2\partial_\parallel E - dE/d\chi)/d\chi$. If we do these substitutions:

$$
\begin{aligned}
(2)+(3) &= \kappa + \frac{1}{2}\nabla_\perp^2\int_0^{\tilde{\chi}} d\chi\left[\left(\tilde{\chi} - \frac{\tilde{\chi}^2}{\chi}\right)\frac{d}{d\chi}\left(2\partial_\parallel E - \frac{d}{d\chi}E\right) + 2\frac{\tilde{\chi}^2}{\chi^2}\partial_\parallel E - 2\frac{\tilde{\chi}^2}{\chi^3}E\right)\right]\\
&= \kappa + \frac{1}{2}\nabla_\perp^2\int_0^{\tilde{\chi}} d\chi\left\{\left(\frac{\tilde{\chi}^2}{\chi} - \tilde{\chi}\right)\frac{d^2E}{d\chi^2} + \frac{d}{d\chi}\left[\left(\tilde{\chi} - \frac{\tilde{\chi}^2}{\chi}\right)2\partial_\parallel E\right]\right.\\
&\quad \left.\underbrace{-\frac{d}{d\chi}\left(\tilde{\chi} - \frac{\tilde{\chi}^2}{\chi}\right)}_{-\frac{\tilde{\chi}^2}{\chi^2}}\cdot 2\partial_\parallel E + 2\frac{\tilde{\chi}^2}{\chi^2}\partial_\parallel E - 2\frac{\tilde{\chi}^2}{\chi^3}E\right)\right\}.
\end{aligned}
$$

It can be easily noticed that there is a divergence for $\chi = 0$, but this unconvenient can be skipped by evaluating the integral from ϵ to $\tilde{\chi}$, with ϵ very small but positive, thus cancelling divergences. This is possible because we are looking to regions well outside the horizon, so we don't care about small scales.

$$
(2)+(3) = \kappa + \nabla_\perp^2\left\{\left(\tilde{\chi} - \frac{\tilde{\chi}^2}{\chi}\right)\partial_\parallel E\right\}\Bigg|_\epsilon^{\tilde{\chi}} + \frac{1}{2}\nabla_\perp^2\int_\epsilon^{\tilde{\chi}} d\chi\left[\left(\frac{\tilde{\chi}^2}{\chi} - \tilde{\chi}\right)\frac{d^2E}{d\chi^2} - 2\frac{\tilde{\chi}^2}{\chi^3}E\right].
$$

Since we are considering small $\epsilon > 0$, we can do an expansion of $\partial_\parallel E$ at the first order in ϵ:
$\partial_\parallel E(\epsilon) = \hat{n}^i\partial_i E(0) + \hat{n}^i\partial_i\frac{d}{d\chi}\big|_\epsilon E(0)\epsilon = \hat{n}^i\partial_i E(0) + \hat{n}^i\hat{n}^j\partial_i\partial_j E(0)\epsilon - \hat{n}^i\partial_i E'(0)\epsilon + O(\epsilon^2)$.

We can substitute this expression in the term in braces {}:

$$
\begin{aligned}
&\left\{\left(\tilde{\chi}-\frac{\tilde{\chi}^2}{\chi}\right)\partial_\parallel E\right\}\Bigg|_\epsilon^{\tilde{\chi}} = -\left(\tilde{\chi}-\frac{\tilde{\chi}^2}{\epsilon}\right)\left(\hat{n}^i\partial_i E(0)+\hat{n}^i\hat{n}^j\partial_i\partial_j E(0)\epsilon-\hat{n}^i\partial_i E'(0)\epsilon\right)\\
&= -\left(\tilde{\chi}-\frac{\tilde{\chi}^2}{\epsilon}\right)\hat{n}^i\partial_i E(0) - \left(\tilde{\chi}\hat{n}^i\hat{n}^j\partial_i\partial_j E(0)\epsilon - \tilde{\chi}^2\hat{n}^i\hat{n}^j\partial_i\partial_j E(0) - \tilde{\chi}\hat{n}^i\partial_i E'(0)\epsilon\right.\\
&\quad \left. + \tilde{\chi}^2\hat{n}^i\partial_i E'(0)\right) \longrightarrow -\left(\tilde{\chi}-\frac{\tilde{\chi}^2}{\epsilon}\right)\hat{n}^i\partial_i E(0) + \tilde{\chi}^2\hat{n}^i\hat{n}^j\partial_i\partial_j E(0) - \tilde{\chi}^2\hat{n}^i\partial_i E'(0)\\
&\qquad\qquad \text{for } \epsilon\to 0\,.
\end{aligned}
$$

We want to know how the operator $\nabla_\perp^2$ acts. First of all, if we consider a fixed measured distance $\tilde{\chi}$, we are considering points of emission as on a sphere around us; therefore we can assert that $\nabla_\perp^2 = \frac{1}{\tilde{\chi}^2}\nabla_\Omega^2$, where ∇_Ω^2 acts only on the angular coordinates θ, ϕ and is defined as

$$
\nabla_\Omega^2 = \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\,. \tag{2.88}
$$

If we remember the expression of the operator L^2 (with L as the angular momentum) in terms of the spherical coordinates, we can recognize that $\nabla_\Omega^2|\ell m\rangle \propto -L|\ell m\rangle = -\ell(\ell+1)|\ell m\rangle$. For this reason, when we have dipoles ($\ell = 1$) the operator $\nabla_\perp^2$ pulls down a factor $-2/\tilde{\chi}^2$, while when we have quadrupoles ($\ell = 2$) it pulls down a factor $-6/\tilde{\chi}^2$. Therefore for (2) + (3) we find:

$$
\begin{aligned}
(2)+(3) = \kappa &+ \left(\frac{2}{\tilde{\chi}}-\frac{2}{\epsilon}\right)\hat{n}^i\partial_i E(0) - 6\hat{n}^i\hat{n}^j\partial_i\partial_j E(0) + 2\hat{n}^i\partial_i E'(0)\\
&+\frac{1}{2}\nabla_\perp^2\int_\epsilon^{\tilde{\chi}} d\chi\left[\left(\frac{\tilde{\chi}^2}{\chi}-\tilde{\chi}\right)\frac{d^2E}{d\chi^2} - 2\frac{\tilde{\chi}^2}{\chi^3}E\right]\,.
\end{aligned}
$$

We can now treat the last term with two integrations by part:

$$
\begin{aligned}
&\int_\epsilon^{\tilde{\chi}} d\chi\left[\left(\frac{\tilde{\chi}^2}{\chi}-\tilde{\chi}\right)\frac{d^2E}{d\chi^2} - 2\frac{\tilde{\chi}^2}{\chi^3}E\right]\\
&= \left[\left(\frac{\tilde{\chi}^2}{\chi}-\tilde{\chi}\right)\frac{dE}{d\chi}\right]\Bigg|_\epsilon^{\tilde{\chi}} - \int_\epsilon^{\tilde{\chi}} d\chi\left[-\frac{\tilde{\chi}^2}{\chi^2}\frac{dE}{d\chi}\right] - \int_\epsilon^{\tilde{\chi}} d\chi\left(2\frac{\tilde{\chi}^2}{\chi^3}E\right)\\
&= -\left(\frac{\tilde{\chi}^2}{\epsilon}-\tilde{\chi}\right)\frac{dE}{d\chi}(\epsilon) + \frac{\tilde{\chi}^2}{\chi^2}E\Bigg|_\epsilon^{\tilde{\chi}} + \int_\epsilon^{\tilde{\chi}} d\chi\left(2\frac{\tilde{\chi}^2}{\chi^3}E\right) - \int_\epsilon^{\tilde{\chi}} d\chi\left(2\frac{\tilde{\chi}^2}{\chi^3}E\right)\\
&= -\left(\frac{\tilde{\chi}^2}{\epsilon}-\tilde{\chi}\right)\frac{dE}{d\chi}(\epsilon) + E(\tilde{\chi}) - \frac{\tilde{\chi}^2}{\epsilon^2}E(\epsilon)\,.
\end{aligned}
$$

By inserting this expression in the previous one we find:

$$
\begin{aligned}
(2)+(3) = \;& \kappa + \left(\frac{2}{\tilde{\chi}}-\frac{2}{\epsilon}\right)\hat{n}^i\partial_i E(0) - 6\hat{n}^i\hat{n}^j\partial_i\partial_j E(0) + 2\hat{n}^i\partial_i E'(0)\\
&+\frac{1}{2}\nabla_\perp^2\left[-\left(\frac{\tilde{\chi}^2}{\epsilon}-\tilde{\chi}\right)\frac{dE}{d\chi}(\epsilon) - \frac{\tilde{\chi}^2}{\epsilon^2}E(\epsilon)\right] + \frac{1}{2}\nabla_\perp^2 E(\tilde{\chi})\,.
\end{aligned}
$$

Another expansion for small ϵ around the observer's position can be done for the quantity in square brackets [ ]:

$$
\begin{aligned}
&-\left(\frac{\tilde{\chi}^2}{\epsilon}-\tilde{\chi}\right)\frac{dE}{d\chi}(\epsilon)-\frac{\tilde{\chi}^2}{\epsilon^2}E(\epsilon)= \\
&=-\left(\frac{\tilde{\chi}^2}{\epsilon}-\tilde{\chi}\right)\left[\frac{dE}{d\chi}(0)+\epsilon\frac{d^2E}{d\chi^2}(0)\right]-\frac{\tilde{\chi}^2}{\epsilon^2}\left[E(0)+\epsilon\frac{dE}{d\chi}(0)+\epsilon^2\frac{d^2E}{d\chi^2}(0)\right] \\
&=-\left(\frac{\tilde{\chi}^2}{\epsilon}-\tilde{\chi}\right)\left[\hat{n}^i\partial_iE(0)-E'(0)+\epsilon\hat{n}^i\hat{n}^j\partial_i\partial_jE(0)-2\epsilon\hat{n}^i\partial_iE'(0)+\epsilon E''(0)\right] \\
&\quad-\frac{\tilde{\chi}^2}{\epsilon^2}\left[E(0)+\epsilon\hat{n}^i\partial_iE(0)-\epsilon E'(0)+\frac{1}{2}\epsilon^2\hat{n}^i\hat{n}^j\partial_i\partial_jE(0)-\epsilon^2\hat{n}^i\partial_iE'(0)-\frac{1}{2}\epsilon^2E''(0)\right].
\end{aligned}
$$

Again the operator $\nabla_\perp^2$ pulls down a factor $-2/\tilde{\chi}^2$ for dipoles and $-6/\tilde{\chi}^2$ for quadrupoles. At this point, by doing all the calculations, we find that the terms with ϵ in the denominator cancels out, the terms in ϵ and ϵ^2 goes to zero since we take the limit for small ϵ, and the other terms gather in the following way:

$$
\begin{aligned}
\frac{1}{2}\nabla_\perp^2\left[-\left(\frac{\tilde{\chi}^2}{\epsilon}-\tilde{\chi}\right)\frac{dE}{d\chi}(\epsilon)-\frac{\tilde{\chi}^2}{\epsilon^2}E(\epsilon)\right]&=\kappa+\frac{1}{2}\nabla_\perp^2E(\tilde{\chi})+\frac{1}{\tilde{\chi}}\hat{n}^i\partial_iE(0) \\
&\quad-\frac{3}{2}\hat{n}^i\hat{n}^j\partial_i\partial_jE(0)-\hat{n}^i\partial_iE'(0)\,.
\end{aligned}
$$

The term (1) in (2.86) is indipendent from $\tilde{\chi}$ and gives

$$
(1)=\frac{3}{2}\hat{n}^i\hat{n}^j\partial_i\partial_jE(0)\,,
$$

which cancels with the third term in the previous expression. Therefore, by substituting the final expressions of (1), (2), (3) into (2.86), we get this final expression of $\hat{\kappa}$:

$$
\hat{\kappa}=\kappa+\frac{1}{2}\nabla_\perp^2E(\tilde{\chi})+\frac{1}{\tilde{\chi}}\hat{n}^i\partial_iE(0)-\hat{n}^i\partial_iE'(0)\,. \tag{2.89}
$$

Final result for $\tilde{n}_g(\tilde{\mathbf{x}},\tilde{z})/\tilde{n}_g(\tilde{z})$

We are now ready to give a final explicit form of the ratio between $\tilde{n}_g(\tilde{\mathbf{x}},\tilde{z})$ and $\tilde{n}_g(\tilde{z})$, by substituting all the terms we computed:

$$
\begin{aligned}
\frac{\tilde{n}_g(\tilde{\mathbf{x}},\tilde{z})}{\bar{n}_g(\tilde{z})}&=1+3D+\delta_g+b_e\delta z-\big(D+E_\parallel\big)\big|_{\tilde{\chi}}-\left[1-\frac{1+\tilde{z}}{H(\tilde{z})}\frac{dH(\tilde{z})}{d\tilde{z}}\right]\delta z-\frac{1+\tilde{z}}{H(\tilde{z})}\big(D'+E_\parallel'\big)\big|_{\tilde{\chi}} \\
&\quad-\frac{2}{\tilde{\chi}}\left\{\int_0^{\tilde{\chi}}d\chi\big(D+E_\parallel\big)+\frac{1+\tilde{z}}{H(\tilde{z})}\delta z\right\}-2\left\{\kappa+\frac{1}{2}\nabla_\perp^2E(\tilde{\chi})+\frac{1}{\tilde{\chi}}\hat{n}^i\partial_iE(0)-\hat{n}^i\partial_iE'(0)\right\}.
\end{aligned}
$$

But this expression is quite unpractical because we can't clearly see the physical meaning of all the terms. First of all it's better to introduce the quantity

$$
\phi\equiv D-\frac{1}{3}\nabla^2E\,, \tag{2.90}
$$

which removes the residual spatial gauge modes that emerge in the synchronous-comoving gauge. Further, we can notice that it's a physical significant parameter because it corresponds to the Bardeen potential Φ (2.61) in the synchronous-comoving gauge. Then:

$$\frac{\tilde{n}_g(\tilde{\mathbf{x}},\tilde{z})}{\bar{n}_g(\tilde{z})} = 1+\delta_g+b_e\delta z+2\phi+\frac{2}{3}\nabla^2 E-\left(\partial_\parallel^2 E-\frac{1}{3}\nabla^2 E\right)$$
$$-\left[1-\frac{1+\tilde{z}}{H(\tilde{z})}\frac{dH(\tilde{z})}{d\tilde{z}}+\frac{2}{\tilde{\chi}}\frac{1+\tilde{z}}{H(\tilde{z})}\right]\delta z-\frac{2}{\tilde{\chi}}\int_0^{\tilde{\chi}} d\chi\big(D+E_\parallel\big)$$
$$-\frac{1+\tilde{z}}{H(\tilde{z})}\big(D'+E_\parallel'\big)\big|_{\tilde{\chi}}-2\kappa-\underbrace{\nabla_\perp^2 E}_{\nabla^2 E-\partial_\parallel^2 E-\frac{2}{\tilde{\chi}}\partial_\parallel E}-\frac{2}{\tilde{\chi}}\hat{n}^i\partial_i E(0)+2\hat{n}^i\partial_i E'(0)$$
$$=1+\delta_g+b_e\delta z+2\phi+\frac{2}{\tilde{\chi}}\big[\partial_\parallel E-\partial_\parallel E(o)\big]-\left[1-\frac{1+\tilde{z}}{H(\tilde{z})}\frac{dH(\tilde{z})}{d\tilde{z}}+\frac{2}{\tilde{\chi}}\frac{1+\tilde{z}}{H(\tilde{z})}\right]\delta z$$
$$-\frac{2}{\tilde{\chi}}\int_0^{\tilde{\chi}} d\chi\big(\phi+\partial_\parallel^2 E\big)-\frac{1+\tilde{z}}{H(\tilde{z})}\big(D'+E_\parallel'\big)\big|_{\tilde{\chi}}-2k+2\partial_\parallel E'(0)\,.$$

We can now simplify the integral using again the fact that $\partial_\parallel=\frac{d}{d\chi}+{}'$:

$$\int_0^{\tilde{\chi}} d\chi\left[\phi+\left(\frac{d}{d\chi}+\frac{d}{d\tau}\right)\partial_\parallel E\right]=\big[\partial_\parallel E\big]_0^{\tilde{\chi}}+\int_0^{\tilde{\chi}} d\chi\left[\phi+\left(\frac{d}{d\chi}+\frac{d}{d\tau}\right)E'\right]$$
$$=\big[\partial_\parallel E+E'\big]_0^{\tilde{\chi}}+\int_0^{\tilde{\chi}} d\chi\big(\phi+E''\big)$$
$$=\partial_\parallel E-\partial_\parallel E(0)+E'-E'(0)+\int_0^{\tilde{\chi}} d\chi\big(\phi+E''\big)\,.$$

Further, we can notice that $D'+E_\parallel'=D'+\partial_\parallel^2 E'-\frac{1}{3}\nabla^2 E'=\partial_\parallel^2 E'+\phi'$. By substituting all of these expressions we find that:

$$\frac{\tilde{n}_g(\tilde{\mathbf{x}},\tilde{z})}{\bar{n}_g(\tilde{z})}=1+\delta_g+b_e\delta z+2\phi-\left[1-\frac{1+\tilde{z}}{H(\tilde{z})}\frac{dH(\tilde{z})}{d\tilde{z}}+\frac{2}{\tilde{\chi}}\frac{1+\tilde{z}}{H(\tilde{z})}\right]\delta z-\frac{2}{\tilde{\chi}}\big[E'-E'(0)\big]$$
$$-\frac{2}{\tilde{\chi}}\int_0^{\tilde{\chi}} d\chi\big(\phi+E''\big)-\frac{1+\tilde{z}}{H(\tilde{z})}\big(\partial_\parallel^2 E'+\phi'\big)-2k+2\partial_\parallel E'(0)\,. \tag{2.91}$$

2.3 Observed galaxy density perturbation

In a galaxy survey we usually measure the galaxy number density $\tilde{n}_g(\tilde{\boldsymbol{x}},\tilde{z})$ at a certain direction $\hat{\boldsymbol{n}}$ (which is linked to the observed position $\tilde{\boldsymbol{x}}=\tilde{\chi}\hat{\boldsymbol{n}}$) and with a certain redshift $\tilde{z}$, and then we refer it to the average number density $\bar{n}_g(\tilde{z})$ at a fixed observed redshift. Then we define the observed galaxy density fluctuations as

$$\tilde{\delta}_g(\tilde{\boldsymbol{x}})=\frac{\tilde{n}_g(\tilde{\boldsymbol{x}},\tilde{z})-\bar{n}_g(\tilde{z})}{\bar{n}_g(\tilde{z})}=\frac{\tilde{n}_g(\tilde{\boldsymbol{x}},\tilde{z})}{\bar{n}_g(\tilde{z})}-1\,. \tag{2.92}$$

Therefore from (2.91) we deduce that:

$$\tilde{\delta}_g(\tilde{\boldsymbol{x}})=\delta_g+b_e\delta z-\frac{1+\tilde{z}}{H(\tilde{z})}\partial_\parallel^2 E'-\left[1-\frac{1+\tilde{z}}{H(\tilde{z})}\frac{dH(\tilde{z})}{d\tilde{z}}+\frac{2}{\tilde{\chi}}\frac{1+\tilde{z}}{H(\tilde{z})}\right]\delta z+2\phi$$
$$-\frac{2}{\tilde{\chi}}\big[E'-E'(o)\big]-\frac{2}{\tilde{\chi}}\int_0^{\tilde{\chi}} d\chi\big(\phi+E''\big)-2\kappa-\frac{1+\tilde{z}}{H(\tilde{z})}\phi'+2\partial_\parallel E'(o)\,. \tag{2.93}$$

If we look at this formula, we can see that there are many different contributions: the first two terms are a gauge invariant expression of the intrinsic galaxy density perturbation, the third one is the standard redshift distortion, the terms in square brackets and ϕ contain the volume distortion due to the redshift perturbation, the terms with E' contain the volume distortion due to the metric perturbations, and finally there are the contributions from the time delay, lensing convergence and Doppler effect. However, a clearer description of the physical meaning of all these terms will be done a the end of the chapter, after considering the issue of the magnification and galaxy bias.

2.4 Magnification bias

Until this moment only the intrinsic physical properties of the galaxies, such as their redshift and their position, have been considered as contributions to $\tilde{\delta}_g(\tilde{\boldsymbol{x}})$. However, what we measure also depends on the apparent flux from the source, in a way that the luminosity distance [5] D_L differs from the mean luminosity distance $\bar{D}_L$. But the luminosity distance is related to the angular diameter [6] D_A [31]:

$$D_L = \left(1 + \tilde{z}\right)^2 D_A \,. \tag{2.94}$$

Therefore the magnification parameter can be introduced as

$$\mathcal{M} \equiv \frac{D_A^{-2}}{\bar{D}_A^{-2}(\tilde{z})} = \frac{D_L^{-2}}{\bar{D}_L^{-2}(\tilde{z})} \,, \tag{2.95}$$

which corresponds to the solid angle or flux perturbation with respect to a source at the same $\tilde{z}$ in an unperturbed universe. Since it has mean value 1, its perturbation can be written as

$$\delta\mathcal{M} \equiv \mathcal{M} - 1 \,. \tag{2.96}$$

At this point we can define the factor $\mathcal{Q}$ which expresses the connection between the observed number density of galaxies and the magnification:

$$\mathcal{Q} = \left.\frac{\partial \ln \tilde{n}_g}{\partial \ln \mathcal{M}}\right|_{\tilde{z}} \,, \tag{2.97}$$

in a way that the observed galaxy overdensity can be written as

$$\tilde{\delta}_g = \tilde{\delta}_g^{\text{no magnification}} + \mathcal{Q}\delta\mathcal{M} \,, \tag{2.98}$$

where $\tilde{\delta}_g^{\text{no magnification}}$ is given by (2.93), and $\mathcal{Q}$ represents the magnification bias. Our aim is now to find the dependence of $\delta\mathcal{M}$ on the metric perturbations, in order to give an explicit expression of this contribution to $\tilde{\delta}_g$. First of all we must express the angular diameter distance in terms of the geometrical quantities we know, so we consider:

- a reference frame generated by an orthonormal basis $\{\hat{n}^i, \hat{a}^i, \hat{b}^i\}$, where $\hat{n}^i$ is the direction of observation and $\hat{a}^i, \hat{b}^i$ span the plane perpendicular to it;

- a unit purely spatial vector $\hat{\ell}^\mu$ which points away from the observer in a way that $u_\mu \hat{\ell}^\mu = 0$;

[5] D_L can be defined as $L/4\pi F$, where L is the luminosity and F is the flux [1].
[6] $D_A = D/\theta$ where D is the real spatial distance, while θ is the angular distance [32].

- the spatial part of the tangent vector to the past light cone $L^\nu = \partial x^\nu / \partial \tilde{\chi}\big|_{\hat{\boldsymbol{n}}}$, which is parallel to $\hat{\ell}^\nu$. Then, by defining $u_\sigma = (-a, 0, 0, 0)$ and $u^k = 0$ for the comoving observer's four velocity, we can write that

$$\hat{\ell}^\mu = \frac{L^\nu}{L^\sigma u_\sigma} + u^\nu = \frac{L^\nu}{a(x^0) L^0} = \frac{1}{a(x^0)} \frac{\partial x^\nu / \partial \tilde{\chi}}{-\partial x^0 / \partial \tilde{\chi}} \,. \tag{2.99}$$

An expression for D_A can be obtained from the area perpendicular to the line of sight and spanned by $\hat{a}^i$ and $\hat{b}^i$ on the past light cone:

$$D_A^2 = \sqrt{-g(x^\alpha)} \epsilon_{\mu\nu\rho\sigma} u^\mu \hat{\ell}^\nu \frac{\partial x^\rho}{\partial \hat{n}^i} \frac{\partial x^\sigma}{\partial \hat{n}^j} \hat{a}^i \hat{b}^j \,. \tag{2.100}$$

From (2.95) we deduce that:

$$\begin{aligned}
\mathcal{M}^{-1} = \frac{D_A^2}{[\bar{a}(\tilde{\chi})]^2 \tilde{\chi}^2} &= \frac{1}{[\bar{a}(\tilde{\chi})]^2 \tilde{\chi}^2} \sqrt{-g(x^\alpha)} \epsilon_{\mu\nu\rho\sigma} u^\mu \hat{\ell}^\nu \frac{dx^\rho}{d\tilde{x}^\alpha} \underbrace{\frac{\partial \tilde{x}^\alpha}{\partial \hat{n}^i}}_{\tilde{\chi}\delta_i^\alpha} \frac{dx^\sigma}{d\tilde{x}^\beta} \underbrace{\frac{\partial \tilde{x}^\beta}{\partial \hat{n}^j}}_{\tilde{\chi}\delta_j^\beta} \hat{a}^i \hat{b}^j \\
&= \frac{\sqrt{-g(x^\alpha)}}{[\bar{a}(\tilde{\chi})]^2} \epsilon_{\mu\nu\rho\sigma} u^\mu \hat{\ell}^\nu \frac{\partial x^\rho}{\partial \tilde{x}^i} \frac{\partial x^\sigma}{\partial \tilde{x}^j} \hat{a}^i \hat{b}^j \,.
\end{aligned} \tag{2.101}$$

Using (2.99), collapsing the Levi-Civita symbol to three dimensions and considering that $\{\hat{n}^i, \hat{a}^i, \hat{b}^i\}$ is an orthonormal basis, we get:

$$\mathcal{M}^{-1} = \frac{\sqrt{-g(x^\sigma)}}{-[\bar{a}(\tilde{\chi}) a(x^0)]^2 \partial x^0 / \partial \tilde{\chi}} \epsilon_{\alpha\beta\gamma} u^\mu \hat{\ell}^\nu \frac{\partial x^\alpha}{\partial \tilde{x}^k} \frac{\partial x^\beta}{\partial \tilde{x}^i} \frac{\partial x^\gamma}{\partial \tilde{x}^k} \hat{n}^k \hat{a}^i \hat{b}^j = \frac{\sqrt{-g(x^\sigma)}}{-[\bar{a}(\tilde{\chi}) a(x^0)]^2 \partial x^0 / \partial \tilde{\chi}} \left| \frac{\partial x^i}{\partial x^j} \right| . \tag{2.102}$$

At this point we can introduce the first order expansions of the various factors. We have already computed that:

$$\begin{aligned}
\sqrt{-g(x^\sigma)} &= a^4 (1 + 3D) \,, \\
\left| \frac{\partial x^i}{\partial x^j} \right| &= 1 + \frac{\partial \Delta x^i}{\partial \tilde{x}^i} \,.
\end{aligned}$$

Then we can use the null condition:

$$\begin{aligned}
L^\mu L_\mu &= 0 \\
g_{00} \left(L^0\right)^2 + g_{ij} L^i L^j &= 0 \\
-\left(\frac{\partial x^0}{\partial \tilde{\chi}}\right)^2 + \Big[(1 + 2D)\delta_{ij} + 2E_{ij}\Big] \frac{\partial x^i}{\partial \tilde{\chi}} \frac{\partial x^j}{\partial \tilde{\chi}} &= 0 \\
-\left(\frac{\partial x^0}{\partial \tilde{\chi}}\right)^2 + \Big[1 + 2D + 2E_\parallel\Big] \left(\frac{dx_\parallel}{d\tilde{\chi}}\right)^2 &= 0 \,.
\end{aligned}$$

Therefore, at the first order:

$$-\frac{\partial x^0}{\partial \tilde{\chi}} = \Big(1 + D + E_\parallel\Big) \frac{dx_\parallel}{d\tilde{\chi}} = 1 + D + E_\parallel + \partial_\parallel \Delta x_\parallel \,. \tag{2.103}$$

All of these expressions can be put into (2.102):

$$
\begin{aligned}
\mathcal{M}^{-1} &= \frac{a^4(x^0)\big(1+3D\big)}{\bar{a}^2(\tilde{\chi})a^2(x^0)\big(1+D+E_\parallel+\partial_\parallel\Delta x_\parallel\big)}\left(1+\frac{\partial\Delta x^i}{\partial\tilde{x}^i}\right)\\
&= \left[\frac{a(x^0)}{\bar{a}(\tilde{\chi})}\right]^2\big(1+3D\big)\big(1-D-E_\parallel-\partial_\parallel\Delta x_\parallel\big)\left(1+\frac{\partial\Delta x^i}{\partial\tilde{x}^i}\right)\\
&= \left[\frac{a(x^0)}{\bar{a}(\tilde{\chi})}\right]^2\big(1+2D-E_\parallel-\partial_\parallel\Delta x_\parallel\big)\left(1+\partial_\parallel\Delta x_\parallel+\frac{2\Delta x_\parallel}{\tilde{\chi}}-2\hat{\kappa}\right)\\
&= \left[\frac{a(x^0)}{\bar{a}(\tilde{\chi})}\right]^2\left(1+2D-E_\parallel+\frac{2\Delta x_\parallel}{\tilde{\chi}}-2\hat{\kappa}\right).
\end{aligned}
$$

It's now easy to obtain the reciprocal of this quantity:

$$
\mathcal{M} = \left[\frac{\bar{a}(\tilde{\chi})}{a(x^0)}\right]^2\left(1-2D+E_\parallel-\frac{2\Delta x_\parallel}{\tilde{\chi}}+2\hat{\kappa}\right). \tag{2.104}
$$

We can notice that

$$
1+\tilde{z} = (1+\bar{z})(1+\delta z) \;\;\rightarrow\;\; 1+\delta z = \frac{1+\tilde{z}}{1+\bar{z}} = \frac{a(x^0)}{a(\tilde{\chi})}\,,
$$

therefore:

$$
\left[\frac{\bar{a}(\tilde{\chi})}{a(x^0)}\right]^2 \simeq 1-2\delta z\,.
$$

So at the first order we get:

$$
\mathcal{M} = \big(1-2\delta z\big)\left(1-2D+E_\parallel-\frac{2\Delta x_\parallel}{\tilde{\chi}}+2\hat{\kappa}\right) = 1-2D+E_\parallel-\frac{2\Delta x_\parallel}{\tilde{\chi}}+2\hat{\kappa}-2\delta z\,. \tag{2.105}
$$

In the end, since the perturbation of the magnification is $\delta\mathcal{M} = \mathcal{M}-1$, we get for it:

$$
\delta\mathcal{M} = -2\delta z-2D+E_\parallel-\frac{2\Delta x_\parallel}{\tilde{\chi}}+2\hat{k}\,. \tag{2.106}
$$

Many contributions can be recognized:

- $-2\delta z-2D$ is associated with the isotropic conversion from coordinate distances to physical distances [31];
- $E_\parallel$ is associated with the anisotropy of the coordinate system;
- $-2\Delta x_\parallel/\tilde{\chi}$ is associated to bringing the source closer to or farther from the observer [31];
- $2\hat{\kappa}$ is the convergence term.

We can now attempt to give a more explicit expression of $\delta\mathcal{M}$ by expanding the last two terms using (2.81), (2.87) and (2.86).

$$
\begin{aligned}
\delta\mathcal{M} &= -2\delta z - 2D + E_\parallel + \frac{2}{\tilde\chi}\int_0^{\tilde\chi} d\chi\big(D + E_\parallel\big) + \frac{2}{\tilde\chi}\frac{1+\tilde z}{H(\tilde z)}\delta z \\
&\quad + 2\kappa + \underbrace{\nabla_\perp^2 E}_{\nabla^2 E - \partial_\parallel^2 E - \frac{2}{\tilde\chi}\partial_\parallel E} + \underbrace{\frac{2}{\tilde\chi}\hat n^i\partial_i E(0)}_{\frac{2}{\tilde\chi}\partial_\parallel E(0)} \underbrace{-2\hat n^i\partial_i E'(0)}_{-2\partial_\parallel E'(0)} \\
&= -2D + E_\parallel + \nabla^2 E - \partial_\parallel^2 E - \frac{2}{\tilde\chi}\Big[\partial_\parallel E - \partial_\parallel E(0)\Big] \\
&\quad + 2\kappa - 2\partial_\parallel E'(0) + \left[-2 + \frac{2}{\tilde\chi}\frac{1+\tilde z}{H(\tilde z)}\right]\delta z + \frac{2}{\tilde\chi}\int_0^{\tilde\chi} d\chi\big(D + E_\parallel\big)\,.
\end{aligned}
$$

It can be noticed that, given $\phi = D - \frac{1}{3}\nabla^2 E$, we can write the following equivalence:

$$
D + E_\parallel = \phi + \partial_\parallel^2 E\,.
$$

Indeed it is easy to see that $D + E_\parallel = D + \hat n^i \hat n^j E_{ij} = D + \partial_\parallel^2 E - \frac{1}{3}\nabla^2 E = \phi + \partial_\parallel^2 E$. Therefore the four terms at the beginning of the expression of $\delta\mathcal{M}$ can be written as:

$$
\begin{aligned}
-2D + E_\parallel + \nabla^2 E - \partial_\parallel^2 E &= -3D + \nabla^2 E + D + E_\parallel - \partial_\parallel^2 E \\
&= -3\phi + \phi + \partial_\parallel^2 E - \partial_\parallel^2 E = -2\phi\,.
\end{aligned}
$$

Moreover, the integral in $\delta\mathcal{M}$ can be treated as follows:

$$
\begin{aligned}
\int_0^{\tilde\chi} d\chi\big(D + E_\parallel\big) &= \int_0^{\tilde\chi} d\chi\big(\phi + \partial_\parallel^2 E\big) = \int_0^{\tilde\chi}\left[\phi + \partial_\parallel\left(E' + \frac{dE}{d\chi}\right)\right] \\
&= \int_0^{\tilde\chi}\left(\phi + E'' + \frac{dE'}{d\chi} + \frac{dE'}{d\chi} + \frac{d^2E}{d\chi^2}\right) \\
&= \int_0^{\tilde\chi} d\chi\big(\phi + E''\big) + \int_0^{\tilde\chi} d\chi\frac{d}{d\chi}\left(2E' + \frac{dE}{d\chi}\right) \\
&= \int_0^{\tilde\chi} d\chi\big(\phi + E''\big) + \left[2E' + \frac{dE}{d\chi}\right]_0^{\tilde\chi} \\
&= \int_0^{\tilde\chi} d\chi\big(\phi + E''\big) + \Big[2E' + \partial_\parallel E - E'\Big]_0^{\tilde\chi} \\
&= \int_0^{\tilde\chi} d\chi\big(\phi + E''\big) + E' + \partial_\parallel E - E'(0) - \partial_\parallel E(0)\,.
\end{aligned}
$$

Substituting al these expressions into $\delta\mathcal{M}$, we ge the final expression for it:

$$
\begin{aligned}
\delta\mathcal{M} =& -2\phi - \frac{2}{\tilde\chi}\partial_\parallel E + \frac{2}{\tilde\chi}\partial_\parallel E(0) + 2\kappa - 2\partial_\parallel E'(0) + \left[-2 + \frac{2}{\tilde\chi}\frac{1+\tilde z}{H(\tilde z)}\right]\delta z \\
& + \frac{2}{\tilde\chi}\int_0^{\tilde\chi} d\chi\big(\phi + E''\big) + \frac{2}{\tilde\chi}\Big(E' - E'(0)\Big) + \frac{2}{\tilde\chi}\partial_\parallel E - \frac{2}{\tilde\chi}\partial_\parallel E(0)\,.
\end{aligned}
$$

At last we have:

$$
\begin{aligned}
\delta\mathcal{M} =& -2\phi + 2\kappa + \left[-2 + \frac{2}{\tilde\chi}\frac{1+\tilde z}{H(\tilde z)}\right]\delta z + \frac{2}{\tilde\chi}\Big(E' - E'(0)\Big) \\
& - 2\partial_\parallel E'(0) + \frac{2}{\tilde\chi}\int_0^{\tilde\chi} d\chi\big(\phi + E''\big)\,.
\end{aligned}
\tag{2.107}
$$

2.5 Galaxy bias

The aim of this subsection is to find a relation between the intrinsic galaxy overdensity δ_g in the synchronous-comoving gauge and the matter and metric perturbations for large scales, thus considering only the linear terms. Since we are dealing with large scales, we must focus on the main features of such an environment for a given galaxy:

- its mean density;
- the evolutionary stage (proper time, the linear growth factor).

In order to represents these quantities, we can consider:

- a spatial volume V centered around the space-time point x^μ_P on a constant age hypersurface represented by $t_U = constant$, where t_U is the proper time of comoving observers since the Big Bang. This volume is assumed to be large enough for linear perturbation theory;
- the number of galaxies N_g within V, which depends on the mass M enclosed in it and the age of the universe t_U in that volume (that is kept fixed).

A general expression of N_g is

$$N_g = F\big(M, t_U, x^\mu_P\big) \ , \tag{2.108}$$

with F as a generic function of the just mentioned variables. Now a coordinate system $(\tau, \boldsymbol{x})$ is considered, together with linear perturbations of the quantities we deal with. Given $\bar{\rho}_m$ as the average physical matter density in the background and $\delta\rho$ as its linear perturbation, the physical matter density ρ can be expressed as:

$$\rho = \bar{\rho}_m + \delta\rho = \bar{\rho}_m\left[1 + \frac{\delta\rho}{\bar{\rho}_m}\right] = \bar{\rho}_m\big[1 + \delta\ln\rho\big] \ . \tag{2.109}$$

Since $\rho_m \sim a^{-3}$, we can notice that:

$$\frac{d\ln\bar{\rho}_m}{d\tau} = \frac{d\ln a^{-3}}{d\tau} = \frac{1}{a^{-3}}(-3a^{-4})\underbrace{\frac{da}{d\tau}}_{a'} = -3\frac{a'}{a} = -3aH \ . \tag{2.110}$$

The mass M enclosed within V can then be written as:

$$M = \int_V \rho = \bar{\rho}_m(\tau)\big[1 + \delta\ln\rho\big]V = \bar{\rho}_m(\tau)\big[1 + \delta_m - 3aH\delta_\tau\big]V \ , \tag{2.111}$$

where we split the perturbation in two contributions:

- δ_m is the matter density perturbation on a constant-coordinate-time hypersurface $\tau = const.$;
- $-\delta_\tau$ is the time-coordinate displacement on a hypersurface at constant proper time $t_U = const.$, in a way that:

$$a(\tau)\big[\tau - \delta\tau(\boldsymbol{x})\big] = t_U = constant \ . \tag{2.112}$$

Therefore $-3aH$ represents the shift from a constant-age hypersurface to a constant-coordinate-time hypersurface.

At this point, defining $\bar{n}_g$ as the average physical galaxy number density on a constant-coordinate-time hypersurface, we can do the same analysis for N_g:

$$N_g = \int_V n_g = \bar{n}_g(\tau)\big[1 + \delta \ln n_g\big]V = \bar{n}_g(\tau)\big[1 + \delta_g + b_{ep} aH\delta\tau\big]V \,, \tag{2.113}$$

where the two terms in the perturbations are:

- δ_g as the galaxy number density perturbation at $\tau = const.$;
- $$b_{ep} = \frac{d\ln \bar{n}_g}{d\ln a} \,, \tag{2.114}$$

  and the whole term $b_{ep}aH\delta\tau$ represents the galaxy density perturbation on a constant age-hypersurface.

Expressions (2.108) and (2.113) can now be compared. We first make the following definitions:

- the average of the function F on hypersurfaces at fixed t_U

  $$\bar{F}(M; t_U) \equiv \langle F(M; t_U; x^\mu_P)\rangle_{t_U} \,; \tag{2.115}$$

- the bias

  $$b \equiv \left.\frac{\partial \ln \bar{F}(M; t_U)}{\partial \ln M}\right|_{\bar{\rho}_m V} = \left.\frac{\partial \bar{F}(\rho_V V; t_U)}{\partial \ln \rho_V}\right|_{\bar{\rho}_m} \,, \tag{2.116}$$

  where ρ_V is the average matter density at $t_U = const.$ within the volume V;

- the stochastic contribution to galaxy density

  $$\epsilon(x^\mu) \equiv \frac{F(M; t_U; x^\mu)}{\bar{F}(M; t_U)} - 1 \,. \tag{2.117}$$

The comparison of the two expressions of N_g can now be made:

$$\begin{aligned} N_g &= F(M; t_U; x^\mu_P) \\ &= \bar{F}(\rho_m V; t_U)\big[1 + b(\delta_m - 3aH\delta\tau) + \epsilon\big] \\ &= \bar{n}_g\big[1 + \delta_g + b_{ep}aH\delta\tau\big]V \,. \end{aligned} \tag{2.118}$$

In the background we found that, for $\delta_m \to 0$ and $\delta\tau \to 0$: $\bar{n}_g(\tau)V = \bar{F}(\bar{\rho}_m V; a\tau)$. From (2.118) we deduce that the galaxy density perturbation is:

$$\delta_g(x^\mu) = b(\tau)\big[\delta_m(x^\mu) - 3aH(\tau)\delta\tau(x^\mu)\big] - b_{ep}(\tau)aH(\tau)\delta\tau(x^\mu) + \epsilon(x^\mu) \,. \tag{2.119}$$

If we observe this expression, the following considerations can be done:

⋆ on subhorizon scales ($\lambda \ll aH$) the term $aH\delta\tau$ is negligible with respect to δ_m, so (2.119) becomes a linear relation

$$\delta_g(x^\mu) = b\delta_m + \epsilon \,; \tag{2.120}$$

- ⋆ in the synchronous-comoving gauge, with $t_U = a\tau$ everywhere and $\delta\tau = 0$, expression (2.119) becomes linear as just mentioned but for all scales; it is right the case we dealt with in calculations in the previous sections. Inserting (2.120) (with eventually ϵ set to zero) into (2.93), we obtain an expression of the observed galaxy overdensites completely in terms of the metric and matter perturbations, and of two peculiar parameters b and b_e;
- ⋆ In this section we used the physical quantities, but it is better to deal with the comoving ones because we are doing our analysis in the synchronous-comoving gauge. Therefore, from now on we will consider the comoving galaxy number density $a^3 n_g$ and a new parameter b_e instead of b_{ep} defined as

$$b_e \equiv \frac{d\ln(a^3\bar{n}_g)}{d\ln a} \; . \tag{2.121}$$

2.5.1 Gauge invariance

The bias parameters b and b_e are gauge invariant. We could expect such a thing because they are in principle observable:

- given a change δ_m in the average mass density within a certain volume V, b represents how much the galaxy density changes as δ_g in response to δ_m;
- b_e tells us how much the average number density of galaxies changes with the age of the universe.

The quantities we are using generally change with the coordinate time. In order to check the gauge-invariance of the bias parameters, we study the effects of a time-coordinate transformation from τ to $\breve{\tau}$:

$$\tau \;\rightarrow\; \breve{\tau} = \tau + T \; , \tag{2.122}$$

where in general $T = T(\tau, \boldsymbol{x})$. For a generic scalar function f there is no change under this transformation; therefore, given the background value $\bar{f}$ and the scalar perturbation δf, f is given by both:

$$f = \bar{f}(\tau) + \delta f(\tau, \boldsymbol{x}) = \bar{f}(\breve{\tau}) + \breve{\delta} f(\breve{\tau}, \breve{\boldsymbol{x}}) \; . \tag{2.123}$$

The explicit expression of the scalar perturbation in the new coordinates in terms of the old ones is:

$$\breve{\delta} f = \delta f + \bar{f}(\tau) - \bar{f}(\tau + T) = \delta f - \frac{d\bar{f}(\tau)}{d\tau} T \; . \tag{2.124}$$

This formula can ben applied to the perturbations of the matter density, of the redshift and of the galaxy number density:

$$\breve{\delta}_m = \delta_m - \frac{d\ln\rho_m}{d\tau} T = \delta_m + 3aHT \; , \tag{2.125}$$

$$\breve{\delta} z = \delta z - \frac{d\ln(1+\bar{z})}{d\tau} T = \delta z + aHT \; , \tag{2.126}$$

$$\breve{\delta}_g = \delta_g - \frac{d\ln(a^3\bar{n}_g)}{d\tau} T = \delta_g - b_e aHT \; , \tag{2.127}$$

where in the first line we used (2.110), in the second line

$$\frac{d\ln(1+\bar{z})}{d\tau} = \frac{d\ln(a^{-1}(\bar{x}^0(\tau)))}{d\tau} = -\frac{1}{a}\frac{da}{d\tau} = -\frac{a'}{a} = -aH \; ,$$

and in the third line

$$\frac{d\ln(a^3\bar{n}_g)}{d\tau} = \frac{d\ln(a^3\bar{n}_g)}{d\ln a}\frac{d\ln a}{d\tau} = \frac{d\ln(a^3\bar{n}_g)}{d\ln a}\frac{1}{a}\frac{da}{d\tau} = b_e\frac{a'}{a} = b_e aH \ .$$

We can notice that the quantity $\delta_g + b_e\delta z$ which appears in (2.93) is gauge-invariant, indipendent of any bias choice. Then, we can also perform the transformation of (2.119) at fixed b and b_e (at first order):

$$\breve{\delta}_g = b\big[\breve{\delta}_m - 3aH\breve{\delta\tau}\big] - b_e aH\breve{\delta\tau} + \epsilon \tag{2.128}$$
$$= b\big[\delta_m + 3aHT - 3aH(\delta\tau + T)\big] - b_e aH(\delta\tau + T) + \epsilon \tag{2.129}$$
$$= \delta_g - b_e aHT \ , \tag{2.130}$$

where we used Eq.(2.119). We perfectly recovered the gauge transformed expression of δ_g with fixed bias, and in this sense they must be gauge-invariant in order to assure that δ_g itself is gauge-invariant.

2.6 Final expression of the observed galaxy overdensity

What we have done until this moment can now be summarized: we first have computed the expression for the observed galaxy overdensity (2.93); then we have added the contribution of the magnification bias $\mathcal{Q}\delta\mathcal{M}$, thus calculating the explicit expression of $\delta\mathcal{M}$; at last we have introduced the concept of galaxy bias, and found that in the synchronous-comoving gauge it becomes a linear relation. At this point we need to gather all of these results in order to give a final expression Δ_g for the observed galaxy overdensity in terms of the metric perturbations and of the bias parameters. We will do this in the synchronous-comoving gauge, exploiting some advantages that arise from this choice. We can start by writing explicitly the sum of all the terms we have just explained; therefore, the final observed galaxy overdensity Δ_g will be written as

$$\Delta_g = \tilde{\delta}_g + \mathcal{Q}\delta\mathcal{M} \ . \tag{2.131}$$

The first addend $\tilde{\delta}_g$ corresponds to the expression (2.93), and within it the following quantities must be explicited:

- the term δ_g is substituted with the linear bias relation for the synchronous-comoving gauge
$$\delta_g = b\delta_m \ , \tag{2.132}$$
as we saw in the previous section (but setting ϵ to zero);

- as regards ϕ, we will use this quantity in order to remove any residual spatial gauge mode. But the important property we want to highlight here is linked to its time derivative ϕ', since it appears in the $i0$ Einstein's equation:
$$\phi' = -4\pi G\delta T^i_0 \propto u^i = 0 \ , \tag{2.133}$$
where δT^i_0 is the $i0$ component of the perturbed stress-energy tensor and u^i is the peculiar velocity of the observer, which is zero in the synchronous-comoving gauge. Therefore, everytime we see a term with ϕ' we can drop it;

- δz is expressed through (2.57) which takes also account of the Integrated Sachs-Wolfe effect. It is right in the ISW term that we can use the property of the previous point, because we notice that in the synchronous-comoving gauge, since $\phi' = 0$:

$$\delta z_{ISW}^{sc} = \int_0^{\tilde{\chi}} d\chi \Big(D' - \frac{1}{3}\nabla^2 E' + E''' \Big) = \int_0^{\tilde{\chi}} d\chi \Big(\phi' + E''' \Big) = \int_0^{\tilde{\chi}} d\chi E''' \; ;$$

- the convergence κ is explicited with the expression (2.87), but here is used in terms of [7] ϕ.

The second addend is the magnification contribution, where $\mathcal{Q}$ is given by (2.97) and $\delta\mathcal{M}$ is given by (2.107); also in this case δz and κ are explicited as just written above. We are going to write the explicit expression of Δ_g by writing all the previous terms together, but dropping the unobservable constant contributions from the perturbations evaluated at the observer's position and exploiting $\phi' = 0$:

$$\begin{aligned}
\Delta_g &= b\delta_m + b_e \delta z - \frac{1+\tilde{z}}{H(\tilde{z})}\partial_\parallel^2 E' - \left[1 - \frac{1+\tilde{z}}{H(\tilde{z})}\frac{dH(\tilde{z})}{d\tilde{z}} + \frac{2}{\tilde{\chi}}\frac{1+\tilde{z}}{H(\tilde{z})} \right] \delta z \\
&\quad + 2\phi - \frac{2}{\tilde{\chi}} E' - \frac{2}{\tilde{\chi}} \int_0^{\tilde{\chi}} d\chi \Big(\phi + E'' \Big) - 2\kappa \\
&\quad + \mathcal{Q} \Big\{ -2\phi + 2\kappa + \left[-2 + \frac{2}{\tilde{\chi}}\frac{1+\tilde{z}}{H(\tilde{z})} \right] \delta z + \frac{2}{\tilde{\chi}} E' + \frac{2}{\tilde{\chi}} \int_0^{\tilde{\chi}} d\chi \Big(\phi + E'' \Big) \Big\} \\
&= b\delta_m + b_e \Big\{ \partial_\parallel E' + E'' + \int_0^{\tilde{\chi}} d\chi E''' \Big\} - \frac{1+\tilde{z}}{H(\tilde{z})}\partial_\parallel^2 E' \\
&\quad - \left[1 - \frac{1+\tilde{z}}{H(\tilde{z})}\frac{dH(\tilde{z})}{d\tilde{z}} + \frac{2}{\tilde{\chi}}\frac{1+\tilde{z}}{H(\tilde{z})} \right] \Big\{ \partial_\parallel E' + E'' + \int_0^{\tilde{\chi}} d\chi E''' \Big\} + 2\phi - \frac{2}{\tilde{\chi}} E' \\
&\quad - \frac{2}{\tilde{\chi}} \int_0^{\tilde{\chi}} d\chi (\phi + E'') + \nabla_\perp^2 \int_0^{\tilde{\chi}} d\chi \frac{\tilde{\chi}}{\chi} (\tilde{\chi} - \chi) \Big(\phi + E'' \Big) \\
&\quad - \mathcal{Q} \nabla_\perp^2 \int_0^{\tilde{\chi}} d\chi \frac{\tilde{\chi}}{\chi} (\tilde{\chi} - \chi) \Big(\phi + E'' \Big) \\
&\quad + \mathcal{Q} \Big\{ -2\phi + \left[-2 + \frac{2}{\tilde{\chi}}\frac{1+\tilde{z}}{H(\tilde{z})} \right] \Big\{ \partial_\parallel E' + E'' + \int_0^{\tilde{\chi}} d\chi E''' \Big\} \\
&\quad + \frac{2}{\tilde{\chi}} E' + \frac{2}{\tilde{\chi}} \int_0^{\tilde{\chi}} d\chi \Big(\phi + E'' \Big) \Big\} .
\end{aligned}$$

These terms can be organised more conveniently by gathering all of these different contributions into three bigger ones, in a way that:

$$\Delta_g = \Delta_S + \Delta_K + \Delta_I \; , \tag{2.134}$$

[7] However, the operator $\nabla_\perp^2$ is moved outside the integral thus having an extra factor $\tilde{\chi}^2/\chi^2$ inside.

where the three terms result:

$$\Delta_S = b\delta_m + \left[b_e - \big(1+2\mathcal{Q}\big) + \frac{(1+\tilde{z})}{H(\tilde{z})}\frac{dH(\tilde{z})}{d\tilde{z}} - \frac{2}{\tilde{\chi}}\big(1-\mathcal{Q}\big)\frac{(1+\tilde{z})}{H(\tilde{z})}\right]\Big(\partial_\parallel E' + E''\Big)$$
$$- \frac{1+\tilde{z}}{H(\tilde{z})}\partial_\parallel^2 E' + \frac{2}{\tilde{\chi}}\big(1-\mathcal{Q}\big)\big(\tilde{\chi}\phi - E'\big)\,, \tag{2.135}$$

$$\Delta_K = \big(1-\mathcal{Q}\big)\nabla_\perp^2 \int_0^{\tilde{\chi}} d\chi \frac{\tilde{\chi}}{\chi}\big(\tilde{\chi}-\chi\big)\Big(\phi + E''\Big)\,, \tag{2.136}$$

$$\Delta_I = -\frac{2}{\tilde{\chi}}\big(1-\mathcal{Q}\big)\int_0^{\tilde{\chi}} d\chi \Big(\phi + E''\Big)$$
$$+ \left[b_e - \big(1+2\mathcal{Q}\big) + \frac{(1+\tilde{z})}{H(\tilde{z})}\frac{dH(\tilde{z})}{d\tilde{z}} - \frac{2}{\tilde{\chi}}\big(1-\mathcal{Q}\big)\frac{(1+\tilde{z})}{H(\tilde{z})}\right]\int_0^{\tilde{\chi}} d\chi E'''\,. \tag{2.137}$$

- Δ_S is a **local** term evaluated at the *source* which includes the galaxy density perturbation, the redshift distortion and the volume distortion caused by the redshift perturbation. It substantially contains the Newtonian local terms plus some general relativity corrections.
- Δ_K is the **weak lensing** convergence integral along the line of sight, and is the same that arises in the Newtonian context.
- Δ_I is a **time delay** integral along the line of sight, and is a pure GR correction.

Chapter 3

Angular power spectrum

A full explicit expression of the galaxy number density perturbation Δ_g (2.134), which takes account of all the possible non negligible corrections at large scales, has been just given in the previous chapter. But why do we need such a quantity?
Galaxies' overdensities become part of statistical tools used by cosmologist to extract information about the early phases of the universe (e.g. primordial non gaussianity), to test cosmological models and general relativity itself, which effects becomes important at large scales: we are talking about two-point statistics, that deals with correlation functions and power spectra. These tools must reflect the underlying simmetries of the large scale structure of the Universe, which is statistically homogeneous and isotropic in standard cosmology. For example, for galaxy redshift surveys, the three dimensional redshift-dependent Fourier *power spectrum* $P(k, z)$ (which is the two-point correlation function in Fourier space) is not well defined: it works for the flat-sky approximation, instead with large scales there are problems. This quantity is not directly accessible, because when we observe the large scale structure of the Universe we lose the full spatial simmetry. Indeed, by making observations only within our past light cone, at each cosmological distance we observe the large-scale structure at a different time. Therefore, the time evolution of large scale structure genuinely breaks the homogeneity along the radial direction (the line of sight) [33]. We have no more a three dimensional spatial simmetry, but we are left with a spherical simmetry on a two dimensional sky where two angular cooordinates can now be used. In such an environment we must switch from the power spectrum $P(k, z)$ to a more suitable quantity for large scales: the *angular power spectrum* C_ℓ, which is the two-point correlation function in the spherical harmonic space. The aim of this chapter is to calculate the C_ℓ coefficients for the full explicit expression of Δ_g (2.134). In particular, the different contributions from Δ_S, Δ_K and Δ_I will be computed, thus distinguishing various categories of coefficients relative to the local, the weak lensing and the time delay terms, but also to the mixed ones where these contributions are combined together (in the product of the Δs). The starting point of all the calculations are the expressions (2.135), (2.136) and (2.137) of the three contributions just mentioned. We can notice that, except for the first addend in Δ_S, the others don't show the explicit dependence in the matter density perturbation δ_m, which will be crucial to involve the definition of the power spectrum $P(k)$ in calculations. These other addends are basically made of big coefficients depending only on the redshift z multiplied for expressions of ϕ, E' and its derivatives. Therefore the results of subsection 1.3.6 in Chapter 1 will be used to connect the metric perturations with the density ones. However, first of all we are going to give a brief overview on the concept of spherical harmonic expansion of our observables.

3.1 Spherical harmonic expansion

The observable we work with is the galaxy density perturbation $\Delta_g = \Delta_g(\hat{\boldsymbol{n}}, z)$[1]. Since we measure it on the two dimensional sky, the spherical simmetry allows us to make an expansion of this quantity in spherical harmonics $Y_{\ell m}(z)$ with respect to its angular position on the sky for every redshift z:

$$\Delta_g(\hat{\boldsymbol{n}}, z) = \sum_{\ell m} \Delta_{\ell m}(z) Y_{\ell m}(\hat{\boldsymbol{n}}) \; , \tag{3.1}$$

where the coefficients $\Delta_{\ell m}$ are given by

$$\Delta_{\ell m}(z) = \langle \ell m | \Delta_g \rangle = \int d\Omega_{\boldsymbol{n}} \langle \ell m | \hat{\boldsymbol{n}} \rangle \langle \hat{\boldsymbol{n}} | \Delta_g \rangle = \int d\Omega_{\hat{\boldsymbol{n}}} Y^*_{\ell m}(\hat{\boldsymbol{n}}) \Delta_g(\hat{\boldsymbol{n}}, z) \; . \tag{3.2}$$

We can briefly prove that (3.2) is correct by substituting (3.1) in it:

$$\Delta_{\ell m}(z) = \int d\Omega_{\hat{\boldsymbol{n}}} Y^*_{\ell m}(\hat{\boldsymbol{n}}) \sum_{\ell' m'} \Delta_{\ell' m'}(z) Y_{\ell' m'}(\hat{\boldsymbol{n}}) \; .$$

This gives an identity due to the property

$$\int d\Omega_{\hat{\boldsymbol{n}}} Y^*_{\ell m}(\hat{\boldsymbol{n}}) Y_{\ell' m'}(\hat{\boldsymbol{n}}) = \delta_{\ell \ell'} \delta_{m m'} \; , \tag{3.3}$$

which kills the summation and gives $\Delta_{\ell m}(z)$, identical to the left hand side. Since our observables are based on the measure of the redshift z, there will be some errors connected to such a measure which must be taken into account; but, most importantly, we have to filter our observables within the angular coverage of the survey, while in principle we are integrating over all scales in the previous definitions. For these reasons $\Delta_g(\hat{\boldsymbol{n}}, z)$ must be averaged over a window function $W(z)$ normalised to one. If we want to express it in comoving distance space, we can write that $W(\chi) = W(z) a H(z)$ since from (2.54) we can deduce that $\chi(z) = \int_0^z dz'/aH(z')$. Then the normalisation of W is:

$$\int_0^\infty d\chi W(\chi) = \int_0^\infty d\chi a H(z) W(z) = \int_0^\infty dz W(z) = 1 \; . \tag{3.4}$$

We can now express $\Delta_g(\hat{\boldsymbol{n}}, z)$ averaged over W

$$\Delta_g(\hat{\boldsymbol{n}}) = \int_0^\infty d\chi W(\chi) \Delta_g(\hat{\boldsymbol{n}}, z) \; , \tag{3.5}$$

in a way that the quantity $\Delta_g(\hat{\boldsymbol{n}})$ is defined and will be used from now on in calculations. By putting it into (3.2) we get the expression

$$\Delta_{\ell m} = \int d\Omega_{\hat{\boldsymbol{n}}} Y^*_{\ell m}(\hat{\boldsymbol{n}}) \int_0^\infty d\chi W(\chi) \Delta_g(\hat{\boldsymbol{n}}, z) \tag{3.6}$$

for the coefficients of the expansions we will work with. Since $\Delta_g(\hat{\boldsymbol{n}}, z)$ will be computed in the momentum space, we write down here the plane wave expansion in spherical harmonics

$$e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = 4\pi \sum_{\ell m} i^\ell j_\ell(kx) Y^*_{\ell m}(\hat{\boldsymbol{k}}) Y_{\ell m}(\hat{\boldsymbol{x}}) \; , \tag{3.7}$$

[1]From now on z will be used instead of $\tilde{z}$ to simplify the notation.

where j_ℓ are the spherical Bessel function, that will be largely used later. The quantities we have just presented are used to compute the explicit expression of the angular power spectrum C_ℓ, which is a set of coefficients in the expansion of the two-point correlation function in spherical harmonics:

$$\xi(\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, z_1, z_2) = \langle \Delta_g(\hat{\boldsymbol{n}}_1, z_1)\Delta_g(\hat{\boldsymbol{n}}_2, z_2)\rangle = \sum_{\ell m} C_\ell(z_1, z_2) Y_{\ell m}(\hat{\boldsymbol{n}}_1) Y^*_{\ell m}(\hat{\boldsymbol{n}}_2) \, . \tag{3.8}$$

However, this definition is not straightforward to use practically, so these C_ℓ coefficients are better computed with another definition. If uncorrelated $\Delta_{\ell m}$ coefficients and isotropy are assumed, in the observed sky we have that:

$$\delta_{\ell\ell'}\delta_{mm'}C_\ell = \langle \Delta^*_{\ell m}\Delta_{\ell' m'}\rangle \, , \tag{3.9}$$

where the C_ℓ are specified by the cosmological theory and the symbol $\langle\,\rangle$ means averaging over many sky realizations of the field Δ_g. However, one actually observes a unique sky realization, and an estimator $\hat{C}_\ell$ of the angular power spectrum can thus be constructed by averaging over m the value of $|\Delta_{\ell m}|^2$:

$$\hat{C}_\ell = \frac{1}{2\ell+1}\sum_{m=-\ell}^{m} |\Delta_{\ell m}|^2 \, . \tag{3.10}$$

It's trivial to show that this estimator $\hat{C}_\ell$ coincides in our case with C_ℓ which appear in (3.9):

$$\hat{C}_\ell = \frac{1}{2\ell+1}\sum_{m=-\ell}^{m} \Delta^*_{\ell m}\Delta_{\ell m} = \frac{1}{2\ell+1}\sum_{m=-\ell}^{m} C_\ell = \frac{1}{2\ell+1}(2\ell+1)C_\ell = C_\ell \, . \tag{3.11}$$

Therefore, it is sufficient to find $\Delta_{\ell m}$ and its complex coniugate and then multiply them in order to compute C_ℓ; of course, it must be considered that there will be a large number of $\Delta_{\ell m}$ coefficients according to the GR effect we are considering, and for each one there will be various addends, so the calculation in not so straightforward. These C_ℓ can then be compared with their measured values in galaxy surveys, and can so be used to make the tests nominated in the introduction. In performing calculations, some particular integrals where the power spectrum $P(k)$ and a product of spherical Bessel functions will arise. They will be also defined in a more complex way, but the basic expression we are going to find is

$$w_{\ell,\ell'}(\chi,\chi') \equiv \frac{2}{\pi}\int_0^\infty dk\, k^2 P(k) j_\ell(k\chi) j_{\ell'}(k\chi') \, , \tag{3.12}$$

where χ and χ' are the comoving distances at two different epochs. These integrals are hard to compute, because spherical Bessel functions have a highly oscillatory behaviour when integrating in the $k \to \infty$ limit; nevertheless numerical methods can be used to give an estimation of them. However, in this discussion we are going to just explicit $w_{\ell\ell'}$ whenever we recognize such an integral, and we are going to define other similar integrals in order to express the C_ℓ coefficients in terms of a sum over them.

3.2 Explicit expression of $\Delta^g_{\ell m}$

As we have already seen in the previous chapter, the full expression of the galaxy density perturbation Δ_g is made of three addends Δ_S, Δ_K and Δ_I. The aim of this section is to compute the coefficients $\Delta^g_{\ell m}$ (3.2) that appear in the expansion (3.1) with all of the previous contributions; they will be needed in the C_ℓ calculations of the next section.

3.2.1 The local term $\Delta^S_{\ell m}$

In this subsection the coefficient $\Delta^S_{\ell m}$ will be calculated starting from the expression (2.135) of Δ_S:

$$\Delta_S = b\delta_m + \left[b_e - \big(1+2\mathcal{Q}(z)\big) + \frac{(1+z)}{H(z)}\frac{dH(z)}{dz} - \frac{2}{\chi}\big(1-\mathcal{Q}(z)\big)\frac{(1+z)}{H(z)}\right]\big(\partial_\parallel E' + E''\big)$$
$$- \frac{1+z}{H(z)}\partial^2_\parallel E' + \frac{2}{\chi}\big(1-\mathcal{Q}(z)\big)\big(\chi\phi - E'\big) \,.$$

It can be rewritten subtituting the expressions of E', E'' and ϕ as functions of δ_m (see the subsection 1.3.6 in Chapter 1):

$$\begin{aligned}\Delta_S = \quad & b\delta_m \\ & + \left[b_e - \big(1+2\mathcal{Q}(z)\big) + \frac{(1+z)}{H(z)}\frac{dH(z)}{dz} - \frac{2}{\chi}\big(1-\mathcal{Q}(z)\big)\frac{(1+z)}{H(z)}\right]\left(-\frac{H(z)}{1+z}f(z)\right)\partial_\parallel\nabla^{-2}\delta_m \\ & + \left[b_e - \big(1+2\mathcal{Q}(z)\big) + \frac{(1+z)}{H(z)}\frac{dH(z)}{dz} - \frac{2}{\chi}\big(1-\mathcal{Q}(z)\big)\frac{(1+z)}{H(z)}\right] \\ & \times \left[-\frac{H^2(z)}{(1+z)^2}\left(\frac{3}{2}\Omega_m - f(z)\right)\right]\nabla^{-2}\delta_m \\ & - \frac{1+z}{H(z)}\left(-\frac{H(z)}{1+z}f(z)\right)\partial^2_\parallel\nabla^{-2}\delta_m \\ & + 2\big(1-\mathcal{Q}(z)\big)\left[-\frac{H^2(z)}{(1+z^2)}\left(f(z) + \frac{3}{2}\Omega_m\right)\right]\nabla^{-2}\delta_m \\ & - \frac{2}{\chi}\big(1-\mathcal{Q}(z)\big)\left(-\frac{H(z)}{1+z}f(z)\right)\nabla^{-2}\delta_m \,.\end{aligned}$$

We find such an expression

$$\begin{aligned}\Delta_S = {} & b\delta_m - A(z)\mathcal{H}(z)f(z)\partial_\parallel\nabla^{-2}\delta_m + f(z)\partial^2_\parallel\nabla^{-2}\delta_m - A(z)\mathcal{H}^2(z)\left(\frac{3}{2}\Omega_m - f(z)\right)\nabla^{-2}\delta_m \\ & - 2\big(1-\mathcal{Q}(z)\big)\mathcal{H}^2(z)\left(\frac{3}{2}\Omega_m + f(z)\right)\nabla^{-2}\delta_m + \frac{2}{\chi}\big(1-\mathcal{Q}(z)\big)\mathcal{H}(z)f(z)\nabla^{-2}\delta_m \,,\end{aligned} \tag{3.13}$$

where we have defined:

$$A(z) = b_e - \big(1+2\mathcal{Q}(z)\big) + \frac{(1+z)}{H(z)}\frac{dH(z)}{dz} - \frac{2}{\chi}\big(1-\mathcal{Q}(z)\big)\frac{(1+z)}{H(z)} \,, \tag{3.14}$$

$$\mathcal{H}(z) = \frac{H(z)}{1+z} \,. \tag{3.15}$$

Expression (3.13) can be simplified by introducing parameters $\alpha(z)$, $\beta(z)$ and $\gamma(z)$ defined in Eq.(20), (21) and (22) in Ref.[29]:

$$\alpha(z) = -\chi(z)\frac{H(z)}{(1+z)}\left[b_e(z) - 1 - 2\mathcal{Q}(z) + \frac{3}{2}\Omega_m(z) - \frac{2}{\chi(z)}\Big[1 - \mathcal{Q}(z)\Big]\frac{(1+z)}{H(z)}\right] , \tag{3.16}$$

$$\beta(z) = \frac{f(z)}{b(z)} , \tag{3.17}$$

$$\begin{aligned}\gamma(z) = \frac{H(z)}{(1+z)}\Bigg\{ &\frac{H(z)}{(1+z)}\left[\beta(z) - \frac{3}{2}\frac{\Omega_m(z)}{b(z)}\right]b_e(z) + \frac{3}{2}\frac{H(z)}{(1+z)}\beta(z)\Big[\Omega_m(z) - 2\Big] \\ &- \frac{3}{2}\frac{H(z)}{(1+z)}\frac{\Omega_m(z)}{b(z)}\left[1 - 4\mathcal{Q}(z) + \frac{3}{2}\Omega_m(z)\right] + \frac{3}{\chi(z)}\Big[1 - \mathcal{Q}(z)\Big]\frac{\Omega_m(z)}{b(z)}\Bigg\} . \end{aligned}\tag{3.18}$$

$\alpha(z)$ is a generalization of the Newtonian expression, $\beta(z)$ has the same form as in the Newtonian analysis and $\gamma(z)$ is a new term arising from general relativistic corrections [29]. An important equivalence will be used in order to connect the big expressions of z which appear in Eq.(3.13) to α, β and γ:

$$\frac{(1+z)}{H(z)}\frac{dH(z)}{dz} = \frac{3}{2}\Omega_m . \tag{3.19}$$

It is easy to prove it starting from the Friedmann equation for a flat background:

$$H^2 = \frac{8}{3}\pi G\big(\bar{\rho}_m + \rho_\Lambda\big) , \tag{3.20}$$

where $\bar{\rho}_m$ is the mean matter density, for which $\bar{\rho}_m = \bar{\rho}_{m\,0}(1+z)^3$, and ρ_Λ is the energy density associated with the cosmological constant, and is constant itself (we are neglecting the other contributions to ρ, e.g. barions, radiation,...) If we take the derivative with respect to z:

$$2H\frac{dH}{dz} = \frac{8}{3}\pi G\frac{d\bar{\rho}_m}{dz} , \tag{3.21}$$

we need to compute the derivative of the matter density: $d\bar{\rho}_m/dz = \bar{\rho}_{m_0}3(1+z)^2 = 3a\bar{\rho}_m$. At this point we can insert what we have just found into the previous equation:

$$\frac{dH}{dz} = \frac{8\pi G}{6H}\frac{d\bar{\rho}_m}{dz} = \frac{8\pi G}{6H}3a\bar{\rho}_m . \tag{3.22}$$

Therefore[2]:

$$\frac{(1+z)}{H(z)}\frac{dH(z)}{dz} = \frac{1}{aH(z)}\frac{8\pi G}{6H}3a\bar{\rho}_m = \underbrace{\frac{8\pi G\bar{\rho}_m}{3H^2}}_{\frac{\bar{\rho}_m}{\rho_c}=\Omega_m}\frac{3}{2} = \frac{3}{2}\Omega_m .$$

This relation is important because it allows us to use it in the definition of $A(z)$ (3.14), and therefore, gathering the similar terms in the expression (3.13), we can write them in terms of α, β and γ. The following equivalences are found:

- $$-A(z)\mathcal{H}^2(z)\left(\frac{3}{2}\Omega_m - f\right) - 2\big(1-\mathcal{Q}(z)\big)\mathcal{H}^2(z)\left(\frac{3}{2}\Omega_m + f\right) + \frac{2}{\chi}\big(1-\mathcal{Q}(z)\big)\mathcal{H}(z)f = b(z)\gamma(z) ; \tag{3.23}$$
- $$-A(z)\mathcal{H}(z)f = \frac{1}{\chi}b(z)\alpha(z)\beta(z) . \tag{3.24}$$

[2] $\rho_c = 3H^2/8\pi G$

Expression (3.13) can thus be written as

$$\Delta_S = b\delta_m + \sum_{i=1}^{3} F_i\left(\chi, \frac{\partial}{\partial\chi}, \frac{\partial^2}{\partial\chi^2}\right)\nabla^{-2}\delta_m \ , \tag{3.25}$$

where:

$$F_1 = \frac{1}{\chi} b(z)\alpha(z)\beta(z)\partial_\parallel \ ; \tag{3.26}$$

$$F_2 = f(z)\partial_\parallel^2 \ ; \tag{3.27}$$

$$F_3 = b(z)\gamma(z) \ . \tag{3.28}$$

We are now ready to compute $\Delta^S_{\ell m}$ with the formula [3](3.6):

$$\begin{aligned}
\Delta^S_{\ell m} &= \int d\Omega_{\hat{\boldsymbol{n}}} Y^*_{\ell m}(\hat{\boldsymbol{n}}) \int_0^\infty d\chi W(\chi)\Delta_S(\hat{\boldsymbol{n}}, z) \\
&= \int d\Omega_{\hat{\boldsymbol{n}}} Y^*_{\ell m}(\hat{\boldsymbol{n}}) \int_0^\infty d\chi W(\chi)\left[b\delta_m + \sum_{i=1}^{3} F_i\nabla^{-2}\delta_m\right] \\
&= \Delta^S_{\ell m}① + \Delta^S_{\ell m}② \ ,
\end{aligned}$$

where

$$\Delta^S_{\ell m}① = \int d\Omega_{\hat{\boldsymbol{n}}} Y^*_{\ell m}(\hat{\boldsymbol{n}}) \int_0^\infty d\chi W(\chi) b\delta_m \tag{3.29}$$

$$\Delta^S_{\ell m}② = \int d\Omega_{\hat{\boldsymbol{n}}} Y^*_{\ell m}(\hat{\boldsymbol{n}}) \int_0^\infty d\chi W(\chi) \sum_{i=1}^{3} F_i\nabla^{-2}\delta_m \ . \tag{3.30}$$

These two terms need to be explicited in order to compute C^S_ℓ in the end.

First addend of $\boldsymbol{\Delta^S_{\ell m}}$

In this subsection we are going to focus on the first addend $\Delta^S_{\ell m}①$. First of all, we want to write $b\delta_m$ in the Fourier space:

$$b\delta_m(\hat{\boldsymbol{n}}, z) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{n}} b(z)\delta_m(\hat{\boldsymbol{k}}, z) \ . \tag{3.31}$$

Since $e^{i\boldsymbol{k}\cdot\boldsymbol{n}} = 4\pi\sum_{\ell m} i^\ell j_\ell(k\chi)Y^*_{\ell m}(\hat{\boldsymbol{k}})Y_{\ell m}(\hat{\boldsymbol{n}})$ from (3.7), by substituting it on the previous expression we find

$$b\delta_m(\hat{\boldsymbol{n}}, z) = \int \frac{d^3k}{(2\pi)^3} 4\pi \sum_{\ell m} i^\ell j_\ell(k\chi)Y^*_{\ell m}(\hat{\boldsymbol{k}})Y_{\ell m}(\hat{\boldsymbol{n}}) b(z)\delta_m(\hat{\boldsymbol{k}}, z) \ . \tag{3.32}$$

Now (3.32) can be inserted in (3.29):

$$\Delta^S_{\ell m}① = \int d\Omega_{\hat{\boldsymbol{n}}} Y^*_{\ell m}(\hat{\boldsymbol{n}}) \int_0^\infty d\chi W(\chi) \int \frac{d^3k}{(2\pi)^3} 4\pi \sum_{\ell' m'} i^{\ell'} j_{\ell'}(k\chi)Y^*_{\ell' m'}(\hat{\boldsymbol{k}})Y_{\ell' m'}(\hat{\boldsymbol{n}}) b(z)\delta_m(\hat{\boldsymbol{k}}, z) \ .$$

Using $\int d\Omega_{\hat{\boldsymbol{n}}} Y^*_{\ell m}(\hat{\boldsymbol{n}})Y_{\ell' m'}(\hat{\boldsymbol{n}}) = \delta_{\ell\ell'}\delta_{mm'}$ (property (3.3)) we kill the summation, having at last:

$$\Delta^S_{\ell m}① = \int_0^\infty d\chi W(\chi) \int \frac{d^3k}{2\pi^2} i^\ell j_\ell(k\chi)Y^*_{\ell m}(\hat{\boldsymbol{k}}) b(z)\delta_m(\hat{\boldsymbol{k}}, z) \ . \tag{3.33}$$

[3] We will omit the arguments of F_i to simplify the notation

Second addend of $\Delta^S_{\ell m}$

We can now pass on the second addend $\Delta^S_{\ell m}②$. The first step is to write $\sum_i F_i \nabla^{-2}\delta_m$ in Fourier space. Operators F_i acts on $\nabla^{-2}\delta_m$ and are composed of coefficients which depend only by z multiplied for derivatives with respect to χ. Indeed, $\partial_\parallel$ and $\partial^2_\parallel$ appear, but in this context they are basically derivatives along the line of sight, therefore they can be replaced with $\partial/\partial\chi$ and $\partial^2/\partial\chi^2$. These derivatives will act on $\nabla^{-2}\delta_m$, and in particular on the spherical Bessel functions $j_\ell(k\chi)$ which depend on χ. For these reasons we can Fourier transform only $\nabla^{-2}\delta_m$. If we define $\varphi(\hat{\boldsymbol{n}}, z) = \nabla^{-2}\delta_m(\hat{\boldsymbol{n}}, z)$, we can write the Fourier transform as:

$$\begin{aligned}
\varphi(\hat{\boldsymbol{n}}, z) &= \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{n}} \varphi(\hat{\boldsymbol{k}}, z) \\
&= \int \frac{d^3k}{(2\pi)^3} 4\pi \sum_{\ell m} i^\ell j_\ell(k\chi) Y^*_{\ell m}(\hat{\boldsymbol{k}}) Y_{\ell m}(\hat{\boldsymbol{n}}) \varphi(\hat{\boldsymbol{k}}, z) \\
&= \int \frac{d^3k}{(2\pi)^3} 4\pi \sum_{\ell m} i^\ell j_\ell(k\chi) Y^*_{\ell m}(\hat{\boldsymbol{k}}) Y_{\ell m}(\hat{\boldsymbol{n}}) \nabla^{-2}\delta_m(\hat{\boldsymbol{k}}, z)\,. \qquad (3.34)
\end{aligned}$$

Now (3.34) can be inserted in (3.30):

$$\begin{aligned}
\Delta^S_{\ell m}② &= \int d\Omega_{\hat{\boldsymbol{n}}} Y^*_{\ell m}(\hat{\boldsymbol{n}}) \int_0^\infty d\chi W(\chi) \\
&\quad \times \sum_{a=1}^{5} F_a \int \frac{d^3k}{(2\pi)^3} 4\pi \sum_{\ell' m'} i^{\ell'} j_{\ell'}(k\chi) Y^*_{\ell' m'}(\hat{\boldsymbol{k}}) Y_{\ell' m'}(\hat{\boldsymbol{n}}) \nabla^{-2}\delta_m(\hat{\boldsymbol{k}}, z) \\
&= \int_0^\infty d\chi W(\chi) \int \frac{d^3k}{2\pi^2} i^\ell \sum_{a=1}^{3} F_a j_\ell(k\chi) Y^*_{\ell m}(\hat{\boldsymbol{k}}) \nabla^{-2}\delta_m(\hat{\boldsymbol{k}}, z)\,. \qquad (3.35)
\end{aligned}$$

It can be noticed that $\Delta^S_{\ell m}②$ is made of three addends, which must be calculated separately.

F_1 term

We call $\Delta^S_{\ell m}②_I$ the term in F_1 and the first addend of $\Delta^S_{\ell m}②$:

$$\begin{aligned}
\Delta^S_{\ell m}②_I &= \int_0^\infty d\chi W(\chi) \int \frac{d^3k}{2\pi^2} i^\ell F_1 j_\ell(k\chi) Y^*_{\ell m}(\hat{\boldsymbol{k}}) \nabla^{-2}\delta_m(\hat{\boldsymbol{k}}, z) \\
&= \int_0^\infty d\chi W(\chi) \int \frac{d^3k}{2\pi^2} i^\ell \frac{1}{\chi} b(z)\alpha(z)\beta(z) \partial_\parallel j_\ell(k\chi) Y^*_{\ell m}(\hat{\boldsymbol{k}}) \nabla^{-2}\delta_m(\hat{\boldsymbol{k}}, z)\,. \qquad (3.36)
\end{aligned}$$

Now we can notice that:

- ∇^{-2} acts on $\delta_m(\hat{\boldsymbol{k}}, z)$ in Fourier space by pulling down a factor $-1/k^2$;
- $\partial_\parallel$ can be replaced by $\partial/\partial\chi$, which acts on the spherical Bessel function $j_\ell(k\chi)$. A recurrence relation can be used to calculate such a derivative:

$$\frac{d}{dx} j_\ell(x) = \frac{\ell}{2\ell+1} j_{\ell-1}(x) - \frac{\ell+1}{2\ell+1} j_{\ell+1}(x)\,. \qquad (3.37)$$

Therefore in our case we find:

$$\frac{\partial}{\partial\chi} j_\ell(k\chi) = k\left[\frac{\ell}{2\ell+1} j_{\ell-1}(k\chi) - \frac{\ell+1}{2\ell+1} j_{\ell+1}(k\chi)\right]. \qquad (3.38)$$

Further, we can lighten the notation by calling:

$$f_{-1} = \frac{\ell}{2\ell+1} \qquad , \qquad f_1 = -\frac{\ell+1}{2\ell+1} \ . \tag{3.39}$$

So, using the property (3.38) with definition (3.39), this final expression for the F_1 term turns out:

$$\begin{aligned}\Delta^S_{\ell m}②_I &= \int_0^\infty d\chi W(\chi) \int \frac{d^3k}{2\pi^2 k}\left[-\frac{1}{\chi}b(z)\alpha(z)\beta(z)i^\ell\right]\left[f_{-1}j_{\ell-1}(k\chi) + f_1 j_{\ell+1}(k\chi)\right] \\ &\qquad \times Y^*_{\ell m}(\hat{\boldsymbol{k}})\delta_m(\hat{\boldsymbol{k}}, z) \\ &= \int_0^\infty d\chi W(\chi)\left[-\frac{1}{\chi}b(z)\alpha(z)\beta(z)i^\ell\right] \int \frac{d^3k}{2\pi^2 k}\left[f_{-1}j_{\ell-1}(k\chi) + f_1 j_{\ell+1}(k\chi)\right] \\ &\qquad \times Y^*_{\ell m}(\hat{\boldsymbol{k}})\delta_m(\hat{\boldsymbol{k}}, z) \ ,\end{aligned} \tag{3.40}$$

where the terms which don't depend on k has been pulled out of the integral in k in the second line.

F_2 term

The term $\Delta^S_{\ell m}②_{II}$ in F_3 presents a second derivative $\partial^2_\parallel$, which here corresponds to $\partial^2/\partial\chi^2$ and acts always on $j_\ell(k\chi)$. Such a derivative can be computed using another recurrence relation:

$$\frac{d^2}{dx^2}j_\ell(x) = f_{-2}j_{\ell-2}(x) + f_0 j_\ell(x) + f_2 j_{\ell+2}(x) \ , \tag{3.41}$$

where

$$f_{-2} = \frac{\ell(\ell-1)}{(2\ell-1)(2\ell+1)} \quad , \quad f_0 = -\frac{2\ell^2+2\ell-1}{(2\ell-1)(2\ell+3)} \quad , \quad f_2 = \frac{(\ell+1)(\ell+2)}{(2\ell+1)(2\ell+3)} \ . \tag{3.42}$$

In our case we have:

$$\frac{\partial^2}{\partial\chi^2}j_\ell(k\chi) = k^2\left[f_{-2}j_{\ell-2}(k\chi) + f_0 j_\ell(k\chi) + f_2 j_{\ell+2}(k\chi)\right] \ . \tag{3.43}$$

The term in F_3 can be written explicitely:

$$\begin{aligned}\Delta^S_{\ell m}②_{III} &= \int_0^\infty d\chi W(\chi) \int \frac{d^3k}{2\pi^2} i^\ell F_2 j_\ell(k\chi) Y^*_{\ell m}(\hat{\boldsymbol{k}}) \nabla^{-2}\delta_m(\hat{\boldsymbol{k}}, z) \\ &= \int_0^\infty d\chi W(\chi) \int \frac{d^3k}{2\pi^2} i^\ell f(z) \frac{\partial^2}{\partial\chi^2} j_\ell(k\chi) Y^*_{\ell m}(\hat{\boldsymbol{k}}) \nabla^{-2}\delta_m(\hat{\boldsymbol{k}}, z) \\ &= \int_0^\infty d\chi W(\chi) \int \frac{d^3k}{2\pi^2} i^\ell \left[-f(z)\right]\left[f_{-2}j_{\ell-2}(k\chi) + f_0 j_\ell(k\chi) + f_2 j_{\ell+2}(k\chi)\right] \\ &\qquad \times Y^*_{\ell m}(\hat{\boldsymbol{k}})\delta_m(\hat{\boldsymbol{k}}, z) \ .\end{aligned} \tag{3.44}$$

$\boldsymbol{F_3}$ term

The term $\Delta^S_{\ell m}②_{III}$ in F_4 is simple to handle in a few steps:

$$\begin{aligned}
\Delta^S_{\ell m}②_{IV} &= \int_0^\infty d\chi W(\chi) \int \frac{d^3k}{2\pi^2} i^\ell F_3 j_\ell(k\chi) Y^*_{\ell m}(\hat{\boldsymbol{k}}) \nabla^{-2} \delta_m(\hat{\boldsymbol{k}}, z) \\
&= \int_0^\infty d\chi W(\chi) \int \frac{d^3k}{2\pi^2} i^\ell b(z)\gamma(z) j_\ell(k\chi) Y^*_{\ell m}(\hat{\boldsymbol{k}}) \nabla^{-2} \delta_m(\hat{\boldsymbol{k}}, z) \\
&= \int_0^\infty d\chi W(\chi) \big[-b(z)\gamma(z)\big] i^\ell \int \frac{d^3k}{2\pi^2 k^2} j_\ell(k\chi) Y^*_{\ell m}(\hat{\boldsymbol{k}}) \delta_m(\hat{\boldsymbol{k}}, z) \,.
\end{aligned} \tag{3.45}$$

Final expression for the second addend of $\Delta^S_{\ell m}$

A final expression for $\Delta^S_{\ell m}②$, which is the second addend of $\Delta^S_{\ell m}$, can now be given by summing over all the three contributions just calculated:

$$\begin{aligned}
\Delta^S_{\ell m}② &= \Delta^S_{\ell m}②_I + \Delta^S_{\ell m}②_{II} + \Delta^S_{\ell m}②_{III} \\
&= \int_0^\infty d\chi W(\chi) i^\ell \Bigg\{ -\frac{1}{\chi} b(z)\alpha(z)\beta(z) \int \frac{1}{2\pi^2} \frac{d^3k}{k} \Big[f_{-1} j_{\ell-1}(k\chi) + f_1 j_{\ell+1}(k\chi) \Big] \\
&\qquad - f(z) \int \frac{d^3k}{2\pi^2} \Big[f_{-2} j_{\ell-2}(k\chi) + f_0 j_\ell(k\chi) + f_2 j_{\ell+2}(k\chi) \Big] \\
&\qquad - b(z)\gamma(z) \int \frac{1}{2\pi^2} \frac{d^3k}{k^2} j_\ell(k\chi) \,.
\end{aligned} \tag{3.46}$$

Finally, the full expression for the $\Delta^S_{\ell m}$ is given by the sum of (3.33) and (3.46):

$$\begin{aligned}
\Delta^S_{\ell m} &= \int_0^\infty d\chi W(\chi) \int \frac{d^3k}{2\pi^2} i^\ell j_\ell(k\chi) Y^*_{\ell m}(\hat{\boldsymbol{k}}) b(z) \delta_m(\hat{\boldsymbol{k}}, z) \\
&\quad - \int_0^\infty d\chi W(\chi) i^\ell \Bigg\{ b(z)\gamma(z) \int \frac{1}{2\pi^2} \frac{d^3k}{k^2} j_\ell(k\chi) \\
&\qquad + \frac{1}{\chi} b(z)\alpha(z)\beta(z) \int \frac{1}{2\pi^2} \frac{d^3k}{k} \Big[f_{-1} j_{\ell-1}(k\chi) + f_1 j_{\ell+1}(k\chi) \Big] \\
&\qquad + f(z) \int \frac{d^3k}{2\pi^2} \Big[f_{-2} j_{\ell-2}(k\chi) + f_0 j_\ell(k\chi) + f_2 j_{\ell+2}(k\chi) \Big] \Bigg\} Y^*_{\ell m}(\hat{\boldsymbol{k}}) \delta_m(\hat{\boldsymbol{k}}, z) \,.
\end{aligned} \tag{3.47}$$

3.2.2 The weak lensing term $\boldsymbol{\Delta^K_{\ell m}}$

In this subsection the coefficient $\Delta^K_{\ell m}$ will be computed starting from the expression (2.136). However, we will use χ as the observed distance and $\tilde{\chi}$ as integration variable:

$$\Delta_K = \big[1 - \mathcal{Q}(z)\big] \nabla^2_\perp \int_0^\chi d\tilde{\chi} \frac{\chi}{\tilde{\chi}} \big(\chi - \tilde{\chi}\big) \big(\phi + E''\big) \,.$$

Knowing that the expressions of E'' and ϕ as function of δ_m are [Eq.(1.69) and (1.71)]:

$$\begin{aligned}
E'' &= -\frac{H^2(z)}{(1+z)^2} \left[\frac{3}{2} \Omega_m(z) - f(z) \right] \nabla^{-2} \delta_m \,, \\
\phi &= -\frac{H^2(z)}{(1+z)^2} \left[\frac{3}{2} \Omega_m(z) + f(z) \right] \nabla^{-2} \delta_m \,,
\end{aligned}$$

we find that

$$\begin{aligned}\Delta_K &= -\big[1-\mathcal{Q}(z)\big]\nabla_\perp^2\int_0^\chi d\tilde\chi\frac{\chi}{\tilde\chi}\big(\chi-\tilde\chi\big)\frac{H^2(\tilde z)}{(1+\tilde z)^2}\left[\frac{3}{2}\Omega_m(\tilde z)+f(\tilde z)+\frac{3}{2}\Omega_m(\tilde z)-f(\tilde z)\right]\\ &\qquad\times\nabla^{-2}\delta_m(\tilde{\hat{\boldsymbol{n}}},\tilde z)\\ &= -\big[1-\mathcal{Q}(z)\big]\nabla_\perp^2\int_0^\chi d\tilde\chi\frac{\chi}{\tilde\chi}\big(\chi-\tilde\chi\big)3\Omega_m(\tilde z)\mathcal{H}^2(\tilde z)\nabla^{-2}\delta_m(\tilde{\hat{\boldsymbol{n}}},\tilde z)\,.\end{aligned} \tag{3.48}$$

Now, using Eq.(3.6) combined with the Fourier transform (3.34) of $\nabla^{-2}\delta_m(\hat{\mathbf{n}},z)$, and using the following properties by order:

$$\nabla_\perp^2 Y_{\ell m} = \frac{1}{\chi^2}\nabla_\Omega^2 Y_{\ell m} = -\frac{\ell(\ell+1)}{\chi^2}Y_{\ell m}\,, \tag{3.49}$$

$$\int d\Omega_{\hat{\mathbf{n}}}Y^*_{\ell m}(\hat{\mathbf{n}})Y_{\ell' m'}(\hat{\mathbf{n}}) = \delta_{\ell\ell'}\delta_{mm'}\,, \tag{3.50}$$

we can find the $\Delta^K_{\ell m}$ coefficient with the following steps:

$$\begin{aligned}\Delta^K_{\ell m} &= \int d\Omega_{\hat{\boldsymbol{n}}}Y^*_{\ell m}(\hat{\boldsymbol{n}})\int_0^\infty d\chi W(\chi)\Delta_K(\hat{\boldsymbol{n}},z)\\ &= -\int d\Omega_{\hat{\boldsymbol{n}}}Y^*_{\ell m}(\hat{\boldsymbol{n}})\int_0^\infty d\chi W(\chi)\big[1-\mathcal{Q}(z)\big]\nabla_\perp^2\int_0^\chi d\tilde\chi\frac{\chi}{\tilde\chi}\big(\chi-\tilde\chi\big)3\Omega_m(\tilde z)\mathcal{H}^2(\tilde z)\\ &\qquad\times\nabla^{-2}\delta_m(\tilde{\hat{\boldsymbol{n}}},\tilde z)\\ &= -\int d\Omega_{\hat{\boldsymbol{n}}}Y^*_{\ell m}(\hat{\boldsymbol{n}})\int_0^\infty d\chi W(\chi)\big[1-\mathcal{Q}(z)\big]\nabla_\perp^2\int_0^\chi d\tilde\chi\frac{\chi}{\tilde\chi}\big(\chi-\tilde\chi\big)3\Omega_m(\tilde z)\mathcal{H}^2(\tilde z)\\ &\qquad\times\int\frac{d^3k}{(2\pi)^3}4\pi\sum_{\ell'm'}i^{\ell'}j_{\ell'}(k\tilde\chi)Y^*_{\ell'm'}(\hat{\boldsymbol{k}})Y_{\ell'm'}(\hat{\boldsymbol{n}})\left(-\frac{1}{k^2}\right)\delta_m(\hat{\boldsymbol{k}},\tilde z)\\ &= \int d\Omega_{\hat{\boldsymbol{n}}}Y^*_{\ell m}(\hat{\boldsymbol{n}})\int_0^\infty d\chi W(\chi)\big[1-\mathcal{Q}(z)\big]\frac{1}{\chi^2}\int_0^\chi d\tilde\chi\frac{\chi}{\tilde\chi}\big(\chi-\tilde\chi\big)3\Omega_m(\tilde z)\mathcal{H}^2(\tilde z)\\ &\qquad\times\int\frac{d^3k}{2\pi^2k^2}\sum_{\ell'm'}i^{\ell'}j_{\ell'}(k\tilde\chi)Y^*_{\ell'm'}(\hat{\boldsymbol{k}})\underbrace{\nabla_\Omega^2Y_{\ell'm'}(\hat{\boldsymbol{n}})}_{-\ell'(\ell'+1)Y_{\ell'm'}(\hat{\boldsymbol{n}})}\delta_m(\hat{\boldsymbol{k}},\tilde z)\\ &= -\ell(\ell+1)\int_0^\infty d\chi W(\chi)\big[1-\mathcal{Q}(z)\big]\int_0^\chi d\tilde\chi\frac{\big(\chi-\tilde\chi\big)}{\chi\tilde\chi}3\Omega_m(\tilde z)\mathcal{H}^2(\tilde z)\\ &\qquad\times\int\frac{d^3k}{2\pi^2k^2}i^{\ell}j_\ell(k\tilde\chi)Y^*_{\ell m}(\hat{\boldsymbol{k}})\delta_m(\hat{\boldsymbol{k}},\tilde z)\,.\end{aligned} \tag{3.51}$$

3.2.3 The time delay term $\Delta^I_{\ell m}$

The coefficient $\Delta^I_{\ell m}$ will be now computed starting from the expression (2.137):

$$\begin{aligned}\Delta_I = &-\frac{2}{\chi}\big[1-\mathcal{Q}(z)\big]\int_0^\chi d\tilde\chi\Big(\phi+E''\Big)\\ &+\left[b_e-\big(1+2\mathcal{Q}(z)\big)+\frac{(1+\tilde z)}{H(\tilde z)}\frac{dH(\tilde z)}{d\tilde z}-\frac{2}{\tilde\chi}\big(1-\mathcal{Q}(z)\big)\frac{(1+\tilde z)}{H(\tilde z)}\right]\int_0^\chi d\tilde\chi E'''\,.\end{aligned} \tag{3.52}$$

Expressions (1.71), (1.69) and (1.70) can be introduced, and $A(z)$ can be written in place of the expression multiplying the integral in E''', according to definition (3.14):

$$\begin{aligned}\Delta_I =& -\frac{2}{\chi}\big[1-\mathcal{Q}(z)\big] \\ &\times \int_0^\chi d\tilde\chi\bigg[-\frac{H^2(\tilde z)}{(1+\tilde z)^2}\bigg(\frac{3}{2}\Omega_m(\tilde z)+f(\tilde z)\bigg)-\frac{H^2(\tilde z)}{(1+\tilde z)^2}\bigg(\frac{3}{2}\Omega_m(\tilde z)-f(\tilde z)\bigg)\bigg]\nabla^{-2}\delta_m \\ &+A(z)\int_0^\chi d\tilde\chi\bigg[-3\frac{H^3(\tilde z)}{(1+\tilde z)^3}\Omega_m(\tilde z)\big(f(\tilde z)-1\big)\bigg]\nabla^{-2}\delta_m\,. \end{aligned} \tag{3.53}$$

Since $A(z) = -b(z)\alpha(z)\beta(z)/\chi f(z)H(z) = -\alpha(z)/\chi H(z)$, we have:

$$\begin{aligned}\Delta_I =& \frac{2}{\chi}\big[1-\mathcal{Q}(z)\big]\int_0^\chi d\tilde\chi\; 3\Omega_m(\tilde z)\mathcal{H}^2(\tilde z)\nabla^{-2}\delta_m \\ &+\frac{1}{\chi}\frac{\alpha(z)}{H(z)}\int_0^\chi d\tilde\chi\; 3\Omega_m(\tilde z)\mathcal{H}^3(\tilde z)\big(f(\tilde z)-1\big)\nabla^{-2}\delta_m\,. \end{aligned} \tag{3.54}$$

Now, proceeding in the same way of the other terms, we compute:

$$\begin{aligned}\Delta^I_{\ell m} =& \int d\Omega_{\hat{\boldsymbol n}} Y^*_{\ell m}(\hat{\boldsymbol n})\int_0^\infty d\chi W(\chi)\Delta_I(\hat{\boldsymbol n},z) \\ =& \int d\Omega_{\hat{\boldsymbol n}} Y^*_{\ell m}(\hat{\boldsymbol n})\int_0^\infty d\chi W(\chi) \\ &\times\bigg\{-\frac{2}{\chi}\big[1-\mathcal{Q}(z)\big]\int_0^\chi d\tilde\chi\; 3\Omega_m(\tilde z)\mathcal{H}^2(\tilde z)\int\frac{d^3k}{2\pi^2k^2}\sum_{\ell'm'} i^{\ell'}j_{\ell'}(k\tilde\chi)Y^*_{\ell'm'}(\hat{\boldsymbol k})Y_{\ell'm'}(\hat{\boldsymbol n})\delta_m(\hat{\boldsymbol k},\tilde z) \\ &-\frac{1}{\chi}\frac{\alpha(z)}{H(z)}\int_0^\chi d\tilde\chi\; 3\Omega_m(\tilde z)\mathcal{H}^3(\tilde z)\big[f(\tilde z)-1\big]\int\frac{d^3k}{2\pi^2k^2}\sum_{\ell'm'} i^{\ell'}j_{\ell'}(k\tilde\chi)Y^*_{\ell'm'}(\hat{\boldsymbol k})Y_{\ell'm'}(\hat{\boldsymbol n})\delta_m(\hat{\boldsymbol k},\tilde z)\bigg\} \\ =& \int_0^\infty d\chi W(\chi)\bigg\{-\frac{2}{\chi}\big[1-\mathcal{Q}(z)\big]\int_0^\chi d\tilde\chi\; 3\Omega_m(\tilde z)\mathcal{H}^2(\tilde z)\int\frac{d^3k}{2\pi^2k^2}i^{\ell}j_{\ell}(k\tilde\chi)Y^*_{\ell m}(\hat{\boldsymbol k})\delta_m(\hat{\boldsymbol k},\tilde z) \\ &-\frac{1}{\chi}\frac{\alpha(z)}{H(z)}\int_0^\chi d\tilde\chi\; 3\Omega_m(\tilde z)\mathcal{H}^3(\tilde z)\big[f(\tilde z)-1\big]\int\frac{d^3k}{2\pi^2k^2}i^{\ell}j_{\ell}(k\tilde\chi)Y^*_{\ell m}(\hat{\boldsymbol k})\delta_m(\hat{\boldsymbol k},\tilde z)\bigg\}\,. \end{aligned} \tag{3.55}$$

3.3 Angular power spectrum C_ℓ

Once that the three contributions of $\Delta^g_{\ell m}$ have been computed, the expression of the angular power spectrum C_ℓ can be found. Indeed, using (3.9):

$$\begin{aligned}\delta_{\ell\ell'}\delta_{mm'}C_\ell =& \langle\Delta^*_{\ell m}\Delta_{\ell'm'}\rangle \\ =& \langle\big(\Delta^{S*}_{\ell m}+\Delta^{K*}_{\ell m}+\Delta^{I*}_{\ell m}\big)\big(\Delta^{S}_{\ell'm'}+\Delta^{K}_{\ell'm'}+\Delta^{I}_{\ell'm'}\big)\rangle \\ =& \underbrace{\Delta^{S*}_{\ell m}\Delta^{S}_{\ell'm'}}_{C^S_\ell}+\underbrace{\Delta^{S*}_{\ell m}\Delta^{K}_{\ell'm'}}_{(a)}+\underbrace{\Delta^{S*}_{\ell m}\Delta^{I}_{\ell'm'}}_{(b)} \\ &+\underbrace{\Delta^{K*}_{\ell m}\Delta^{K}_{\ell'm'}}_{C^K_\ell}+\underbrace{\Delta^{K*}_{\ell m}\Delta^{S}_{\ell'm'}}_{(c)}+\underbrace{\Delta^{K*}_{\ell m}\Delta^{I}_{\ell'm'}}_{(d)} \\ &+\underbrace{\Delta^{I*}_{\ell m}\Delta^{I}_{\ell'm'}}_{C^K_\ell}+\underbrace{\Delta^{I*}_{\ell m}\Delta^{S}_{\ell'm'}}_{(e)}+\underbrace{\Delta^{I*}_{\ell m}\Delta^{K}_{\ell'm'}}_{(f)}\,. \end{aligned} \tag{3.56}$$

It can be noticed that 9 terms arise in the full expression of C_ℓ:

- $\Delta^{S*}_{\ell m}\Delta^{S}_{\ell' m'}$, $\Delta^{K*}_{\ell m}\Delta^{K}_{\ell' m'}$ and $\Delta^{I*}_{\ell m}\Delta^{I}_{\ell' m'}$, which are the unmixed local, weak lensing and time delay contributions;

- the mixed terms [from (a) to (f)] in which the local, lensing and time delay contributions in the galaxy density perturbation are related through their mixed product. In the following subsections all these terms will be computed, in order to give a full final explicit expression of the C_ℓ coefficients. However, new mathematical definitions are needed.

3.4 New integral definitions

In C_ℓ calculations many different integrals will appear, which contain the product of the power spectrum $P(k)$, spherical Bessel functions for different ℓ, and various power of k, all integrated in the Fourier space. First of all, we start with the definition used also in [33] and already named at the beginning of the chapter:

$$w_{\ell,\ell'}(\chi_1,\chi_2) \equiv \frac{2}{\pi}\int_0^\infty dk\, k^2 P(k) j_\ell(k\chi_1) j_{\ell'}(k\chi_2)\,, \tag{3.57}$$

where we used the notation $\chi_{1,2}$ in order to avoid any confusion later. However, since in this analysis where all the contributions to the angular power spectrum will be considered, also integrals with different powers of k will arise, we need a generalization of the previous definition. We define:

$$w^\nu_{\ell,\ell'}(\chi_1,\chi_2) \equiv \frac{2}{\pi}\int dk\, k^2 \frac{P(k)}{k^\nu} j_\ell(k\chi_1) j_{\ell'}(k\chi_2)\,. \tag{3.58}$$

But we can do more, since also particular sums of these integrals will appear. Therefore the following definitions will be introduced, which are a generalization of those which appear in [33] [from (G5) to (G8)], with f_{-2}, f_0, f_2 previously defined in (3.42):

$$w^\nu_{\ell,00}(\chi_1,\chi_2) \equiv w^\nu_{\ell,\ell}(\chi_1,\chi_2)\,; \tag{3.59}$$

$$w^\nu_{\ell,02}(\chi_1,\chi_2) \equiv \begin{pmatrix} f_{-2} & f_0 & f_2 \end{pmatrix} \begin{pmatrix} w^\nu_{\ell,\ell-2}(\chi_1,\chi_2) \\ w^\nu_{\ell,\ell}(\chi_1,\chi_2) \\ w^\nu_{\ell,\ell+2}(\chi_1,\chi_2) \end{pmatrix}\,; \tag{3.60}$$

$$w^\nu_{\ell,20}(\chi_1,\chi_2) \equiv \begin{pmatrix} f_{-2} & f_0 & f_2 \end{pmatrix} \begin{pmatrix} w^\nu_{\ell-2,\ell}(\chi_1,\chi_2) \\ w^\nu_{\ell,\ell}(\chi_1,\chi_2) \\ w^\nu_{\ell+2,\ell}(\chi_1,\chi_2) \end{pmatrix}\,; \tag{3.61}$$

$$w^\nu_{\ell,22}(\chi_1,\chi_2) \equiv \begin{pmatrix} f_{-2} & f_0 & f_2 \end{pmatrix} \begin{pmatrix} w^\nu_{\ell-2,\ell-2}(\chi_1,\chi_2) & w^\nu_{\ell-2,\ell}(\chi_1,\chi_2) & w^\nu_{\ell-2,\ell+2}(\chi_1,\chi_2) \\ w^\nu_{\ell,\ell-2}(\chi_1,\chi_2) & w^\nu_{\ell,\ell}(\chi_1,\chi_2) & w^\nu_{\ell,\ell+2}(\chi_1,\chi_2) \\ w^\nu_{\ell+2,\ell-2}(\chi_1,\chi_2) & w^\nu_{\ell+2,\ell}(\chi,\chi') & w^\nu_{\ell+2,\ell+2}(\chi_1,\chi_2) \end{pmatrix} \begin{pmatrix} f_{-2} \\ f_0 \\ f_2 \end{pmatrix}\,. \tag{3.62}$$

Further, these new definitions will be used:

$$w^{\nu}_{\ell,01}(\chi_1,\chi_2) \equiv \begin{pmatrix} f_{-1} & f_1 \end{pmatrix} \begin{pmatrix} w^{\nu}_{\ell,\ell-1}(\chi_1,\chi_2) \\ w^{\nu}_{\ell,\ell+1}(\chi_1,\chi_2) \end{pmatrix} ; \tag{3.63}$$

$$w^{\nu}_{\ell,10}(\chi_1,\chi_2) \equiv \begin{pmatrix} f_{-1} & f_1 \end{pmatrix} \begin{pmatrix} w^{\nu}_{\ell-1,\ell}(\chi_1,\chi_2) \\ w^{\nu}_{\ell+1,\ell}(\chi_1,\chi_2) \end{pmatrix} ; \tag{3.64}$$

$$w^{\nu}_{\ell,11}(\chi_1,\chi_2) \equiv \begin{pmatrix} f_{-1} & f_1 \end{pmatrix} \begin{pmatrix} w^{\nu}_{\ell-1,\ell-1}(\chi_1,\chi_2) & w^{\nu}_{\ell-1,\ell+1}(\chi_1,\chi_2) \\ w^{\nu}_{\ell+1,\ell-1}(\chi_1,\chi_2) & w^{\nu}_{\ell+1,\ell+1}(\chi_1,\chi_2) \end{pmatrix} \begin{pmatrix} f_{-1} \\ f_1 \end{pmatrix} ; \tag{3.65}$$

$$w^{\nu}_{\ell,12}(\chi_1,\chi_2) \equiv \begin{pmatrix} f_{-1} & f_1 \end{pmatrix} \begin{pmatrix} w^{\nu}_{\ell-1,\ell-2}(\chi_1,\chi_2) & w^{\nu}_{\ell-1,\ell}(\chi_1,\chi_2) & w^{\nu}_{\ell-1,\ell+2}(\chi_1,\chi_2) \\ w^{\nu}_{\ell+1,\ell-2}(\chi_1,\chi_2) & w^{\nu}_{\ell+1,\ell}(\chi_1,\chi_2) & w^{\nu}_{\ell+1,\ell+2}(\chi_1,\chi_2) \end{pmatrix} \begin{pmatrix} f_{-2} \\ f_0 \\ f_2 \end{pmatrix} ; \tag{3.66}$$

$$w^{\nu}_{\ell,21}(\chi_1,\chi_2) \equiv \begin{pmatrix} f_{-2} & f_0 & f_2 \end{pmatrix} \begin{pmatrix} w^{\nu}_{\ell-2,\ell-1}(\chi_1,\chi_2) & w^{\nu}_{\ell-2,\ell+1}(\chi_1,\chi_2) \\ w^{\nu}_{\ell,\ell-1}(\chi_1,\chi_2) & w^{\nu}_{\ell,\ell+1}(\chi_1,\chi_2) \\ w^{\nu}_{\ell+2,\ell-1}(\chi_1,\chi_2) & w^{\nu}_{\ell+2,\ell+1}(\chi_1,\chi_2) \end{pmatrix} \begin{pmatrix} f_{-1} \\ f_1 \end{pmatrix} . \tag{3.67}$$

3.5 C^S_ℓ coefficient for the local term

After finding the explicit expression of $\Delta^S_{\ell m} = \Delta^S_{\ell m}① + \Delta^S_{\ell m}②$ as sum of (3.33) and (3.46), we are ready to calculate the C^S_ℓ coefficient for the local term of the galaxy density perturbation.

$$\begin{aligned} \delta_{\ell\ell'}\delta_{mm'}C^S_\ell = \langle \Delta^{S*}_{\ell m}\Delta^S_{\ell' m'}\rangle &= \langle \left(\Delta^{S*}_{\ell m}① + \Delta^{S*}_{\ell m}②\right)\left(\Delta^S_{\ell' m'}① + \Delta^S_{\ell' m'}②\right)\rangle \\ &= \underbrace{\Delta^{S*}_{\ell m}①\Delta^S_{\ell' m'}①}_{(i)} + \underbrace{\Delta^{S*}_{\ell m}①\Delta^S_{\ell' m'}②}_{(ii)} + \underbrace{\Delta^{S*}_{\ell m}②\Delta^S_{\ell' m'}①}_{(iii)} \\ &\quad + \underbrace{\Delta^{S*}_{\ell m}②\Delta^S_{\ell' m'}②}_{(iv)} . \end{aligned} \tag{3.68}$$

We can notice that there is a sum of four terms, called for simplicity (i), (ii), (iii), (iv), and we are going to compute all of them by order.

Term (*i*)

$$
\begin{aligned}
(i) &= \Delta^{S*}_{\ell m}① \Delta^{S}_{\ell' m'}① \\
&= \int_0^\infty d\chi_1 W(\chi_1) \int \frac{d^3k}{2\pi^2} (-i)^\ell j_\ell(k\chi_1) Y_{\ell m}(\hat{\boldsymbol{k}}) b(z_1) \delta^*_m(\hat{\boldsymbol{k}}, z_1) \\
&\quad \times \int_0^\infty d\chi_2 W(\chi_2) \int \frac{d^3k'}{2\pi^2} i^{\ell'} j_{\ell'}(k'\chi_2) Y^*_{\ell' m'}(\hat{\boldsymbol{k}}') b(z_2) \delta_m(\hat{\boldsymbol{k}}', z_2) \\
&= \int_0^\infty d\chi_1 W(\chi_1) \int_0^\infty d\chi_2 W(\chi_2) \int \frac{d^3k}{2\pi^2} (-i)^\ell j_\ell(k\chi_1) Y_{\ell m}(\hat{\boldsymbol{k}}) b(z_1) \\
&\qquad \times \int \frac{d^3k'}{2\pi^2} i^{\ell'} j_{\ell'}(k'\chi_2) Y^*_{\ell' m'}(\hat{\boldsymbol{k}}') b(z_2) \underbrace{\delta^*_m(\hat{\boldsymbol{k}}, z_1) \delta_m(\hat{\boldsymbol{k}}', z_2)}_{(2\pi)^3 P(k') \delta^{(3)}(\boldsymbol{k}' - \boldsymbol{k})} \\
&= \int_0^\infty d\chi_1 W(\chi_1) \int_0^\infty d\chi_2 W(\chi_2) \frac{2}{\pi} \int dk\, k^2 (-i)^\ell i^{\ell'} j_\ell(k\chi_1) j_{\ell'}(k\chi_2) \\
&\qquad \times \underbrace{\int d\Omega_{\hat{\mathbf{k}}} Y_{\ell m}(\hat{\boldsymbol{k}}) Y^*_{\ell' m'}(\hat{\boldsymbol{k}}')}_{\delta_{\ell\ell'} \delta_{mm'}} b(z_1) b(z_2) P(k) \\
&= \int_0^\infty d\chi_1 W(\chi_1) \int_0^\infty d\chi_2 W(\chi_2) \underbrace{\frac{2}{\pi} \int dk\, k^2 j_\ell(k\chi_1) j_\ell(k\chi_2) P(k)}_{w^0_{\ell,00}(\chi_1,\chi_2)} b(z_1) b(z_2) \,,
\end{aligned}
$$

where we introduced $w^0_{\ell,00}(\chi_1, \chi_2)$ according to definition (3.59). In the end, for the first addend of the C^S_ℓ local coefficient we find:

$$
(i) = \Delta^{S*}_{\ell m}① \Delta^{S}_{\ell' m'}① = \int_0^\infty d\chi_1 W(\chi_1) \int_0^\infty d\chi_2 W(\chi_2) b_1 b_2 w^0_{\ell,00}(\chi_1, \chi_2) \,, \tag{3.69}
$$

where for simplicity $b_1 = b(z_1)$ and $b_2 = b(z_2)$.

Term (*ii*)

$$
\begin{aligned}
(ii) &= \Delta^{S*}_{\ell m}① \Delta^{S}_{\ell' m'}② \\
&= \int_0^\infty d\chi_1 W(\chi_1) \int \frac{d^3k}{2\pi^2} (-i)^\ell j_\ell(k\chi_1) Y_{\ell m}(\hat{\boldsymbol{k}}) b(z_1) \delta^*_m(\hat{\boldsymbol{k}}, z_1) \\
&\quad \times \int_0^\infty d\chi_2 W(\chi_2) i^{\ell'} \Bigg\{ -b(z_2)\gamma(z_2) \int \frac{1}{2\pi^2} \frac{d^3k'}{k'^2} j_{\ell'}(k'\chi_2) \\
&\quad - \frac{1}{\chi_2} b(z_2)\alpha(z_2)\beta(z_2) \int \frac{1}{2\pi^2} \cdot \frac{d^3k'}{k'} \left[f'_{-1} j_{\ell'-1}(k'\chi_2) + f'_1 j_{\ell'+1}(k'\chi_2) \right] \\
&\quad - f(z_2) \int \frac{d^3k'}{2\pi^2} \left[f'_{-2} j_{\ell'-2}(k'\chi_2) + f'_0 j_{\ell'}(k'\chi_2) + f'_2 j_{\ell'+2}(k'\chi_2) \right] \Bigg\} Y^*_{\ell' m'}(\hat{\boldsymbol{k}}') \delta_m(\hat{\boldsymbol{k}}', z_2) \,.
\end{aligned}
$$

Here $f'_{0,\pm1,\pm2} = f_{0,\pm1,\pm2}(\ell')$ and $f(z_2) = b(z_2)\beta(z_2)$ according to definition (3.17). Using again the fact that

$$
\delta^*_m(\hat{\boldsymbol{k}}, z_1) \delta_m(\hat{\boldsymbol{k}}', z_2) = (2\pi)^3 P(k') \delta^{(3)}(\boldsymbol{k}' - \boldsymbol{k}) \,, \tag{3.70}
$$

$$
\int d^3k = \int dk\, k^2 \int d\Omega_{\hat{\boldsymbol{k}}} \,, \tag{3.71}
$$

$$
\int d\Omega_{\hat{\mathbf{n}}} Y_{\ell m}(\hat{\boldsymbol{k}}) Y^*_{\ell' m'}(\hat{\boldsymbol{k}}') = \delta_{\ell\ell'} \delta_{mm'} \,, \tag{3.72}
$$

we find:

$$
\begin{aligned}
(ii) &= \Delta^{S*}_{\ell m}①\Delta^{S}_{\ell' m'}② \\
&= \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\Bigg\{-b(z_1)b(z_2)\gamma(z_2)\underbrace{\frac{2}{\pi}\int dk j_\ell(k\chi_1)j_\ell(k\chi_2)P(k)}_{w^2_{\ell,00}(\chi_1,\chi_2)} \\
&\quad -\frac{1}{\chi_2}b(z_1)b(z_2)\alpha(z_2)\beta(z_2)\underbrace{\frac{2}{\pi}\int dk\, kj_\ell(k\chi_1)\Big[f_{-1}j_{\ell-1}(k\chi_2)+f_1 j_{\ell+1}(k\chi_2)\Big]P(k)}_{w^1_{\ell,01}(\chi_1,\chi_2)} \\
&\quad -b(z_1)b(z_2)\beta(z_2)\underbrace{\frac{2}{\pi}\int dk\, k^2 j_\ell(k\chi)\Big[f_{-2}j_{\ell'-2}(k\chi_2)+f_0 j_\ell(k\chi_2)+f_2 j_{\ell+2}(k\chi_2)\Big]P(k)}_{w^0_{\ell,02}(\chi_1,\chi_2)}\Bigg\},
\end{aligned}
$$

where we recognized particular integrals defined in the previous section. Therefore for the second addend of the local C^S_ℓ coefficient we get:

$$
\begin{aligned}
(ii) = \Delta^{S*}_{\ell m}①\Delta^{S}_{\ell' m'}② &= -\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2) \\
&\times b_1 b_2\Big[\gamma_2\, w^2_{\ell,00}(\chi_1,\chi_2)+\frac{1}{\chi_2}\alpha_2\beta_2\, w^1_{\ell,01}(\chi_1,\chi_2)+\beta_2\, w^0_{\ell,02}(\chi_1,\chi_2)\Big], \qquad (3.73)
\end{aligned}
$$

where for simplicity the quantities denoted with the subscripts $1,2$ are computed in $z_{1,2}$.

Term (*iii*)

$$
\begin{aligned}
(iii) &= \Delta^{S*}_{\ell m}②\Delta^{S}_{\ell' m'}① \\
&= \int_0^\infty d\chi_1 W(\chi_1) i^\ell\Bigg\{-b(z_1)\gamma(z_1)\int\frac{1}{2\pi^2}\frac{d^3k}{k^2}j_\ell(k\chi_1) \\
&\qquad -\frac{1}{\chi_1}b(z_1)\alpha(z_1)\beta(z_1)\int\frac{1}{2\pi^2}\frac{d^3k}{k}\Big[f_{-1}j_{\ell-1}(k\chi_1)+f_1 j_{\ell+1}(k\chi_1)\Big] \\
&\qquad -f(z_1)\int\frac{d^3k}{2\pi^2}\Big[f_{-2}j_{\ell-2}(k\chi_1)+f_0 j_\ell(k\chi_1)+f_2 j_{\ell+2}(k\chi_1)\Big]\Bigg\}Y^*_{\ell m}(\hat{\boldsymbol{k}})\delta_m(\hat{\boldsymbol{k}},z_1) \\
&\quad\times\int_0^\infty d\chi_2 W(\chi_2)\int\frac{d^3k'}{2\pi^2}i^{\ell'}j_{\ell'}(k'\chi_2)Y^*_{\ell' m'}(\hat{\boldsymbol{k}}')b(z_2)\delta_m(\hat{\boldsymbol{k}}',z_2) \\
&= \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\Bigg\{-b(z_1)b(z_2)\gamma(z_1)\underbrace{\frac{2}{\pi}\int dk j_\ell(k\chi_1)j_\ell(k\chi_2)P(k)}_{w^2_{\ell,00}(\chi_1,\chi_2)} \\
&\quad -\frac{1}{\chi_1}b(z_1)b(z_2)\alpha(z_1)\beta(z_1)\underbrace{\frac{2}{\pi}\int dk\, k\Big[f_{-1}j_{\ell-1}(k\chi_1)+f_1 j_{\ell+1}(k\chi)\Big]j_\ell(k\chi_2)P(k)}_{w^1_{\ell,10}(\chi_1,\chi_2)} \\
&\quad -b(z_1)\beta(z_1)b(z_2)\underbrace{\frac{2}{\pi}\int dk\, k^2\Big[f_{-2}j_{\ell-2}(k\chi_1)+f_0 j_\ell(k\chi_1)+f_2 j_{\ell+2}(k\chi_1)\Big]j_\ell(k\chi_2)P(k)}_{w^0_{\ell,20}(\chi_1,\chi_2)}\Bigg\}.
\end{aligned}
$$

Therefore in the end we have for the third addend of the local C_ℓ^S coefficient:

$$
\begin{aligned}
(iii) = \Delta^{S*}_{\ell m}②\Delta^{S}_{\ell' m'}① = &-\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2) \\
&\times b_1 b_2\left[\gamma_1 w^2_{\ell,00}(\chi_1,\chi_2)+\frac{1}{\chi_1}\alpha_1\beta_1 w^1_{\ell,10}(\chi_1,\chi_2)+\beta_1 w^0_{\ell,20}(\chi_1,\chi_2)\right], \qquad (3.74)
\end{aligned}
$$

where the quantities denoted with the subscripts $1,2$ still design that they are valuated at $z_{1,2}$.

Term (*iv*)

$$
\begin{aligned}
(iv) &= \Delta^{S*}_{\ell m}②\Delta^{S}_{\ell' m'}② \\
&= \int_0^\infty d\chi_1 W(\chi_1) i^\ell\Bigg\{-b(z_1)\gamma(z_1)\int\frac{1}{2\pi^2}\frac{d^3k}{k^2}j_\ell(k\chi_1) \\
&\qquad -\frac{1}{\chi_1}b(z_1)\alpha(z_1)\beta(z_1)\int\frac{1}{2\pi^2}\frac{d^3k}{k}\Big[f_{-1}j_{\ell-1}(k\chi_1)+f_1 j_{\ell+1}(k\chi_1)\Big] \\
&\qquad - f(z_1)\int\frac{d^3k}{2\pi^2}\Big[f_{-2}j_{\ell-2}(k\chi_1)+f_0 j_\ell(k\chi_1)+f_2 j_{\ell+2}(k\chi_1)\Big]\Bigg\}Y_{\ell m}(\hat{\boldsymbol{k}})\delta^*_m(\hat{\boldsymbol{k}},z_1) \\
&\times\int_0^\infty d\chi_2 W(\chi_2) i^{\ell'}\Bigg\{-b(z_2)\gamma(z_2)\int\frac{1}{2\pi^2}\frac{d^3k'}{k'^2}j_{\ell'}(k'\chi_2) \\
&\qquad -\frac{1}{\chi_2}b(z_2)\alpha(z_2)\beta(z_2)\int\frac{1}{2\pi^2}\cdot\frac{d^3k'}{k'}\Big[f'_{-1}j_{\ell'-1}(k'\chi_2)+f'_1 j_{\ell'+1}(k'\chi_2)\Big] \\
&\qquad - f(z_2)\int\frac{d^3k'}{2\pi^2}\Big[f'_{-2}j_{\ell'-2}(k'\chi_2)+f'_0 j_{\ell'}(k'\chi_2)+f'_2 j_{\ell'+2}(k'\chi_2)\Big]\Bigg\}Y^*_{\ell' m'}(\hat{\boldsymbol{k}}')\delta_m(\hat{\boldsymbol{k}}',z_2)\,.
\end{aligned}
$$

All the calculations, using always the same properties of the previous paragraphs, give:

$$
\begin{aligned}
(iv) = &\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\times\\
&\Bigg\{ b(z_1)b(z_2)\gamma(z_1)\gamma(z_2)\underbrace{\frac{2}{\pi}\int\frac{dk}{k^2}j_\ell(k\chi_1)j_\ell(k\chi_2)P(k)}_{w^4_{\ell,00}(\chi_1,\chi_2)}\\
&+\frac{1}{\chi_2}b(z_1)b(z_2)\alpha(z_2)\beta(z_2)\gamma(z_1)\underbrace{\frac{2}{\pi}\int\frac{dk}{k}j_\ell(k\chi_1)\Big[f_{-1}j_{\ell-1}(k\chi_2)+f_1 j_{\ell+1}(k\chi_2)\Big]P(k)}_{w^3_{\ell,01}(\chi_1,\chi_2)}\\
&+b(z_1)\gamma(z_1)b(z_2)\beta(z_2)\underbrace{\frac{2}{\pi}\int dk j_\ell(k\chi_1)\Big[f_{-2}j_{\ell-2}(k\chi_2)+f_0 j_\ell(k\chi_2)+f_2 j_{\ell+2}(k\chi_2)\Big]P(k)}_{w^2_{\ell,02}(\chi_1,\chi_2)}\\
&+\frac{1}{\chi_1}b(z_1)b(z_2)\alpha(z_1)\beta(z_1)\gamma(z_2)\underbrace{\frac{2}{\pi}\int\frac{dk}{k}\Big[f_{-1}j_{\ell-1}(k\chi_1)+f_1 j_{\ell+1}(k\chi_1)\Big]j_\ell(k\chi_2)P(k)}_{w^3_{\ell,10}(\chi_1,\chi_2)}\\
&+\frac{1}{\chi_1\chi_2}b(z_1)b(z_2)\alpha(z_1)\alpha(z_2)\beta(z_1)\beta(z_2)\\
&\quad\times\underbrace{\frac{2}{\pi}\int dk\Big[f_{-1}j_{\ell-1}(k\chi_1)+f_1 j_{\ell+1}(k\chi_1)\Big]\Big[f_{-1}j_{\ell-1}(k\chi_2)+f_1 j_{\ell+1}(k\chi_2)\Big]P(k)}_{w^2_{\ell,11}(\chi_1,\chi_2)}\\
&+\frac{1}{\chi}b(z_1)\alpha(z)\beta(z_1)b(z_2)\beta(z_2)\\
&\quad\times\underbrace{\frac{2}{\pi}\int dk\,k\Big[f_{-1}j_{\ell-1}(k\chi_1)+f_1 j_{\ell+1}(k\chi_1)\Big]\Big[f_{-2}j_{\ell-2}(k\chi_2)+f_0 j_\ell(k\chi_2)+f_2 j_{\ell+2}(k\chi_2)\Big]P(k)}_{w^1_{\ell,12}(\chi_1,\chi_2)}\\
&+b(z_2)\gamma(z_2)b(z_1)\beta(z_1)\underbrace{\frac{2}{\pi}\int dk\Big[f_{-2}j_{\ell-2}(k\chi_1)+f_0 j_\ell(k\chi_1)+f_2 j_{\ell+2}(k\chi_1)\Big]j_\ell(k\chi_2)P(k)}_{w^2_{\ell,20}(\chi_1,\chi_2)}\\
&+\frac{1}{\chi_2}b(z_2)\alpha(z_2)\beta(z_2)b(z_1)\beta(z_1)\\
&\quad\times\underbrace{\frac{2}{\pi}\int dk\,k\Big[f_{-2}j_{\ell-2}(k\chi_1)+f_0 j_\ell(k\chi_1)+f_2 j_{\ell+2}(k\chi_1)\Big]\Big[f_{-1}j_{\ell-1}(k\chi_2)+f_1 j_{\ell+1}(k\chi_2)\Big]P(k)}_{w^1_{\ell,21}(\chi_1,\chi_2)}\\
&+b(z_1)\beta(z_1)b(z_2)\beta(z_2)\,\frac{2}{\pi}\int dk\,k^2\Big[f_{-2}j_{\ell-2}(k\chi_1)+f_0 j_\ell(k\chi_1)+f_2 j_{\ell+2}(k\chi_1)\Big]\\
&\qquad\times\underbrace{\Big[f_{-2}j_{\ell-2}(k\chi_2)+f_0 j_\ell(k\chi_2)+f_2 j_{\ell+2}(k\chi_2)\Big]P(k)}_{w^0_{\ell,22}(\chi_1,\chi_2)}\Bigg\}.
\end{aligned}
$$

Therefore, by designing with subscripts $1, 2$ all the quantities computed in $z_{1,2}$, we find for the fourth addend of the local C_ℓ^S coefficient:

$$
\begin{aligned}
(iv) =& \Delta^{S*}_{\ell m} ② \Delta^{S}_{\ell' m'} ② \\
=& \int_0^\infty d\chi_1 W(\chi_1) \int_0^\infty d\chi_2 W(\chi_2) \\
& \times b_1 b_2 \Big\{ \gamma_1\gamma_2 w^4_{\ell,00}(\chi_1,\chi_2) + \frac{1}{\chi_1}\alpha_1\beta_1\gamma_2 w^3_{\ell,10}(\chi_1,\chi_2) + \frac{1}{\chi_2}\alpha_2\beta_2\gamma_1 w^3_{\ell,01}(\chi_1,\chi_2) \\
& \quad + \gamma_1\beta_2 w^2_{\ell,02}(\chi_1,\chi_2) + \gamma_2\beta_1 w^2_{\ell,20}(\chi_1,\chi_2) + \frac{1}{\chi_1\chi_2}\alpha_1\alpha_2\beta_1\beta_2 w^2_{\ell,11}(\chi_1,\chi_2) \\
& \quad + \frac{1}{\chi_1}\alpha_1\beta_1\beta_2 w^1_{\ell,12}(\chi_1,\chi_2) + \frac{1}{\chi_2}\alpha_2\beta_2\beta_1 w^1_{\ell,21}(\chi_1,\chi_2) + \beta_1\beta_2 w^0_{\ell,22}(\chi_1,\chi_2) \Big\} ,
\end{aligned}
\tag{3.75}
$$

where all the definitions of the integrals that has been replaced with the "w" are those from (3.58) to (3.67).

3.5.1 Final complete expression of C_ℓ^S

We are now ready to gather all the addends that we have computed in the previous subsections. The final expression for the C_ℓ^S coefficient for the local term of the galaxy density perturbation is given by the sum of the previous terms (i),(ii),(iii) and (iv):

$$
\begin{aligned}
C_\ell^S =& \int_0^\infty d\chi_1 W(\chi_1) \int_0^\infty d\chi_2 W(\chi_2)\, b_1 b_2 \times \\
& \Big\{ w^0_{\ell,00}(\chi_1,\chi_2) - \beta_2 w^0_{\ell,02}(\chi_1,\chi_2) - \beta_1 w^0_{\ell,20}(\chi_1,\chi_2) + \beta_1\beta_2 w^0_{\ell,22}(\chi_1,\chi_2) \\
& - \frac{1}{\chi_2}\alpha_2\beta_2 w^1_{\ell,01}(\chi_1,\chi_2) - \frac{1}{\chi_1}\alpha_1\beta_1 w^1_{\ell,10}(\chi_1,\chi_2) + \frac{1}{\chi_1}\alpha_1\beta_1\beta_2 w^1_{\ell,12}(\chi_1,\chi_2) \\
& + \frac{1}{\chi_2}\alpha_2\beta_2\beta_1 w^1_{\ell,21}(\chi_1,\chi_2) - \big[\gamma_1 + \gamma_2\big] w^2_{\ell,00}(\chi_1,\chi_2) + \gamma_1\beta_2 w^2_{\ell,02}(\chi_1,\chi_2) \\
& + \gamma_2\beta_1 w^2_{\ell,20}(\chi_1,\chi_2) + \frac{1}{\chi_1\chi_2}\alpha_1\alpha_2\beta_1\beta_2 w^2_{\ell,11}(\chi_1,\chi_2) + \frac{1}{\chi_2}\alpha_2\beta_2\gamma w^3_{\ell,01}(\chi_1,\chi_2) \\
& + \frac{1}{\chi_1}\alpha_1\beta_1\gamma_2 w^3_{\ell,10}(\chi_1,\chi_2) + \gamma_1\gamma_2 w^4_{\ell,00}(\chi_1,\chi_2) \Big\} .
\end{aligned}
\tag{3.76}
$$

3.6 C_ℓ^K coefficient for the lensing term

We are now ready to compute C_ℓ^K by inserting (3.51) into (3.9).

$$\begin{aligned}
\delta_{\ell\ell'}\delta_{mm'}C_\ell^K = \langle \Delta_{\ell m}^{K*}\Delta_{\ell' m'}^K\rangle = \big[\ell(\ell+1)\big]\big[\ell'(\ell'+1)\big]\int_0^\infty d\chi_1 W(\chi_1)\big[1-\mathcal{Q}(z_1)\big] \\
\times \int_0^{\chi_1} d\tilde\chi_1 \frac{(\chi_1-\tilde\chi_1)}{\chi_1\tilde\chi_1}3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1)\int \frac{d^3k}{2\pi^2k^2}i^\ell j_\ell(k\tilde\chi_1)Y_{\ell m}^*(\hat{\boldsymbol{k}})\delta_m(\hat{\boldsymbol{k}},\tilde z_1) \\
\times \int_0^\infty d\chi_2 W(\chi_2)\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2} d\tilde\chi_2 \frac{(\chi_2-\tilde\chi_2)}{\chi_2\tilde\chi_2}3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2) \\
\times \int \frac{d^3k'}{2\pi^2k'^2}i^{\ell'} j_{\ell'}(k'\tilde\chi_2)Y_{\ell' m'}^*(\hat{\boldsymbol{k}}')\delta_m(\hat{\boldsymbol{k}}',\tilde z_2)\,. \qquad (3.77)
\end{aligned}$$

By using the same properties of the previous sections, as $\int d\Omega_{\hat{\mathbf{n}}} Y_{\ell m}(\hat{\boldsymbol{k}})Y_{\ell' m'}^*(\hat{\boldsymbol{k}}') = \delta_{\ell\ell'}\delta_{mm'}$ and $\delta_m^*(\hat{\boldsymbol{k}},z_1)\delta_m(\hat{\boldsymbol{k}}',z_2) = (2\pi)^3P(k')\delta^{(3)}(\boldsymbol{k}'-\boldsymbol{k})$, we obtain:

$$\begin{aligned}
C_\ell^K = \ell^2(\ell+1)^2\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\big[1-\mathcal{Q}(z_1)\big]\big[1-\mathcal{Q}(z_2)\big] \\
\times \int_0^{\chi_1} d\tilde\chi_1 \frac{(\chi_1-\tilde\chi_1)}{\chi_1\tilde\chi_1}3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1)\int_0^{\chi_2} d\tilde\chi_2 \frac{(\chi_2-\tilde\chi_2)}{\chi_2\tilde\chi_2}3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2) \\
\times \underbrace{\frac{2}{\pi}\int \frac{dk}{k^2}j_\ell(k\tilde\chi_1)j_\ell(k\tilde\chi_2)P(k)}_{w_{\ell,00}^4(\tilde\chi_1,\tilde\chi_2)}\,.
\end{aligned}$$

Therefore the final expression for the C_ℓ^K coefficient results:

$$\begin{aligned}
C_\ell^K = \ell^2(\ell+1)^2\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\big[1-\mathcal{Q}(z_1)\big]\big[1-\mathcal{Q}(z_2)\big] \\
\times \int_0^{\chi_1} d\tilde\chi_1 \frac{(\chi_1-\tilde\chi_1)}{\chi_1\tilde\chi_1}3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1)\int_0^{\chi_2} d\tilde\chi_2 \frac{(\chi_2-\tilde\chi_2)}{\chi_2\tilde\chi_2}3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2)w_{\ell,00}^4(\tilde\chi_1,\tilde\chi_2)\,. \qquad (3.78)
\end{aligned}$$

3.7 C_ℓ^I coefficient for the time delay term

The C_ℓ^I coefficient can now be obtained in the same way of the previous contributions:

$$\begin{aligned}
\delta_{\ell\ell'}\delta_{mm'}C_\ell^I = \langle \Delta_{\ell m}^{I*}\Delta_{\ell' m'}^I\rangle = \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2) \\
\times\Big\{-\frac{2}{\chi_1}\big[1-\mathcal{Q}(z_1)\big]\int_0^{\chi_1} d\tilde\chi_1\,3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1)\int \frac{d^3k}{2\pi^2k^2}i^\ell j_\ell(k\tilde\chi_1)Y_{\ell m}^*(\hat{\boldsymbol{k}})\delta_m(\hat{\boldsymbol{k}},\tilde z_1) \\
-\frac{1}{\chi_1}\frac{\alpha(z_1)}{H(z_1)}\int_0^{\chi_1} d\tilde\chi_1\,3\Omega_m(\tilde z_1)\mathcal{H}^3(\tilde z_1)\big[f(\tilde z_1)-1\big]\int \frac{d^3k}{2\pi^2k^2}i^\ell j_\ell(k\tilde\chi_1)Y_{\ell m}^*(\hat{\boldsymbol{k}})\delta_m(\hat{\boldsymbol{k}},\tilde z_1)\Big\} \\
\times\Big\{-\frac{2}{\chi_2}\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2} d\tilde\chi_2\,3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2)\int \frac{d^3k'}{2\pi^2k'^2}i^{\ell'} j_{\ell'}(k'\tilde\chi_2)Y_{\ell' m'}^*(\hat{\boldsymbol{k}}')\delta_m(\hat{\boldsymbol{k}}',\tilde z_2) \\
-\frac{1}{\chi_2}\frac{\alpha(z_2)}{H(z_2)}\int_0^{\chi_2} d\tilde\chi_2\,3\Omega_m(\tilde z_2)\mathcal{H}^3(\tilde z_2)\big[f(\tilde z_2)-1\big]\int \frac{d^3k'}{2\pi^2k'^2}i^{\ell'} j_{\ell'}(k'\tilde\chi_2)Y_{\ell' m'}^*(\hat{\boldsymbol{k}}')\delta_m(\hat{\boldsymbol{k}}',\tilde z_2)\Big\}\,.
\end{aligned}$$

Doing all the products and using all the properties mentioned in the previous sections, the following expression is found:

$$
\begin{aligned}
C_\ell^I = &\int_0^\infty d\chi_1 W(\chi_1) \int_0^\infty d\chi_2 W(\chi_2) \\
\times &\Bigg\{ \frac{4}{\chi_1\chi_2}\big[1-\mathcal{Q}(z_1)\big]\big[1-\mathcal{Q}(z_2)\big] \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1) \int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2) \\
&\qquad\qquad \times \underbrace{\frac{2}{\pi}\int \frac{dk}{k^2} j_\ell(k\tilde\chi_1) j_\ell(k\tilde\chi_2) P(k)}_{w^4_{\ell,00}(\tilde\chi_1,\tilde\chi_2)} \\
+ &\frac{2}{\chi_1\chi_2}\big[1-\mathcal{Q}(z_1)\big]\frac{\alpha(z_2)}{H(z_2)} \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1) \int_0^{\chi_2} d\tilde\chi_2 3\, \Omega_m(\tilde z_2)\mathcal{H}^3(\tilde z_2)\big[f(\tilde z_2)-1\big] \\
&\qquad\qquad \times \underbrace{\frac{2}{\pi}\int \frac{dk}{k^2} j_\ell(k\tilde\chi_1) j_\ell(k\tilde\chi_2) P(k)}_{w^4_{\ell,00}(\tilde\chi_1,\tilde\chi_2)} \\
+ &\frac{2}{\chi_1\chi_2}\big[1-\mathcal{Q}(z_2)\big]\frac{\alpha(z_1)}{H(z_1)} \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^3(\tilde z_1)\big[f(\tilde z_1)-1\big] \int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2) \\
&\qquad\qquad \times \underbrace{\frac{2}{\pi}\int \frac{dk}{k^2} j_\ell(k\tilde\chi_1) j_\ell(k\tilde\chi_2) P(k)}_{w^4_{\ell,00}(\tilde\chi_1,\tilde\chi_2)} \\
+ &\frac{1}{\chi_1\chi_2}\frac{\alpha(z_1)\alpha(z_2)}{H(z_1)H(z_2)} \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^3(\tilde z_1)\big[f(\tilde z_1)-1\big] \int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^3(\tilde z_2)\big[f(\tilde z_2)-1\big] \\
&\qquad\qquad \times \underbrace{\frac{2}{\pi}\int \frac{dk}{k^2} j_\ell(k\tilde\chi_1) j_\ell(k\tilde\chi_2) P(k)}_{w^4_{\ell,00}(\tilde\chi_1,\tilde\chi_2)} \Bigg\}\,.
\end{aligned}
$$

Therefore the final expression of C_ℓ^I is:

$$
\begin{aligned}
C_\ell^I = &\int_0^\infty d\chi_1 W(\chi_1) \int_0^\infty d\chi_2 W(\chi_2)\, \frac{1}{\chi_1\chi_2} \\
\times &\Bigg\{ 4\big[1-\mathcal{Q}(z_1)\big]\big[1-\mathcal{Q}(z_2)\big] \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1) \int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2) \\
&+ 2\big[1-\mathcal{Q}(z_1)\big]\frac{\alpha(z_2)}{H(z_2)} \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1) \int_0^{\chi_2} d\tilde\chi_2 3\, \Omega_m(\tilde z_2)\mathcal{H}^3(\tilde z_2)\big[f(\tilde z_2)-1\big] \\
&+ 2\big[1-\mathcal{Q}(z_2)\big]\frac{\alpha(z_1)}{H(z_1)} \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^3(\tilde z_1)\big[f(\tilde z_1)-1\big] \int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2) \\
+ &\frac{\alpha(z_1)\alpha(z_2)}{H(z_1)H(z_2)} \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^3(\tilde z_1)\big[f(\tilde z_1)-1\big] \int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^3(\tilde z_2)\big[f(\tilde z_2)-1\big] \Bigg\} \\
&\times w^4_{\ell,00}(\tilde\chi_1,\tilde\chi_2)\,.
\end{aligned} \tag{3.79}
$$

3.8 C_ℓ mixed terms

Besides the "pure" unmixed contributions that we have just computed, other terms are needed in order to give the final total expression of C_ℓ: they are named from (a) to (f) in the expression (3.98), and will be explicitely calculated in the following paragraphs. However, a specification on the definition of the power spectrum is needed now, because the density perturbations $\delta(\hat{\mathbf{k}}, z)$ that will appear are actually in the form (1.66), so the power spectrum $P(k)$ where the dependency on the redshift has been omitted results to be actually:

$$P(k, z_1, z_2) = P(k)\frac{D(z_1)D(z_2)}{D^2(0)}\ . \tag{3.80}$$

Previously, none or both the $D(z)$ factors were inside the integrals in $\tilde{\chi}$, so we didn't need to specify such a definition. In the mixed terms, instead, some $\delta(\hat{\mathbf{k}}, z)$ will be inside the integrals in $\tilde{\chi}$ and others will not, so that when the power spectrum will arise, one $D(z)$ factor will be integrated and the other will not. In order to keep the notation simple, we have not inserted before and we will not insert now the $D(z)$ factors, but we keep in mind that when we find mixed expressions as $w^{\nu}_{\ell,00}(\tilde{\chi}_1, \chi_2)$, $w^{\nu}_{\ell,00}(\chi_1, \tilde{\chi}_2)$ and so on, we are omitting the Ds.

(a) $\boldsymbol{\Delta^{S*}_{\ell m}\Delta^{K}_{\ell' m'}}$

$$\begin{aligned}
&\Delta^{S*}_{\ell m}\Delta^{K}_{\ell' m'} = \big(\Delta^{S*}_{\ell m}①+\Delta^{S*}_{\ell m}②\big)\Delta^{K}_{\ell' m'}\\
&=\Bigg\{\int_0^\infty d\chi_1 W(\chi_1)\int\frac{d^3k}{2\pi^2}(-i)^\ell j_\ell(k\chi_1)Y_{\ell m}(\hat{\boldsymbol{k}})b(z_1)\delta^*_m(\hat{\boldsymbol{k}}, z_1)\\
&\qquad-\int_0^\infty d\chi_1 W(\chi_1)(-i)^\ell\Bigg\{b(z_1)\gamma(z_1)\int\frac{1}{2\pi^2}\frac{d^3k}{k^2}j_\ell(k\chi_1)\\
&\qquad-\frac{1}{\chi_1}b(z_1)\alpha(z_1)\beta(z_1)\int\frac{1}{2\pi^2}\frac{d^3k}{k}\Big[f_{-1}j_{\ell-1}(k\chi_1)+f_1 j_{\ell+1}(k\chi_1)\Big]\\
&\qquad-b(z_1)\beta(z_1)\int\frac{d^3k}{2\pi^2}\Big[f_{-2}j_{\ell-2}(k\chi_1)+f_0 j_\ell(k\chi_1)+f_2 j_{\ell+2}(k\chi_1)\Big]\Bigg\}Y_{\ell m}(\hat{\boldsymbol{k}})\delta^*_m(\hat{\boldsymbol{k}}, z_1)\Bigg\}\\
&\times\Bigg\{-\ell'(\ell'+1)\int_0^\infty d\chi_2 W(\chi_2)\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2}d\tilde{\chi}_2\frac{(\chi_2-\tilde{\chi}_2)}{\chi_2\tilde{\chi}_2}3\Omega_m(\tilde{z}_2)\mathcal{H}^2(\tilde{z}_2)\\
&\qquad\times\int\frac{d^3k'}{2\pi^2k'^2}i^{\ell'}j_{\ell'}(k'\tilde{\chi}_2)Y^*_{\ell' m'}(\hat{\boldsymbol{k}}')\delta_m(\hat{\boldsymbol{k}}', \tilde{z}_2)\Bigg\}\ .
\end{aligned}\tag{3.81}$$

Doing all the products and using the same properties and definitions of the previous sections, we compute:

$$
\begin{aligned}
\Delta^{S*}_{\ell m}\Delta^{K}_{\ell' m'} = &\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\,\ell(\ell+1)\big[1-\mathcal{Q}(z_2)\big] \\
&\times \int_0^{\chi_2} d\tilde{\chi}_2 \frac{\left(\chi_2-\tilde{\chi}_2\right)}{\chi_2\tilde{\chi}_2} 3\Omega_m(\tilde{z}_2)\mathcal{H}^2(\tilde{z}_2) \\
&\times \Bigg\{ -b(z_1)\underbrace{\frac{2}{\pi}\int dk j_\ell(k\chi_1) j_\ell(k\tilde{\chi}_2)P(k)}_{w^2_{\ell,00}(\chi_1,\tilde{\chi}_2)} + b(z_1)\gamma(z_1)\underbrace{\frac{2}{\pi}\int \frac{dk}{k^2} j_\ell(k\chi_1) j_\ell(k\tilde{\chi}_2)P(k)}_{w^4_{\ell,00}(\chi_1,\tilde{\chi}_2)} \\
&+ \frac{1}{\chi_1} b(z_1)\alpha(z_1)\beta(z_1)\underbrace{\frac{2}{\pi}\int \frac{dk}{k}\Big[f_{-1} j_{\ell-1}(k\chi_1) + f_1 j_{\ell+1}(k\chi_1)\Big] j_\ell(k\tilde{\chi}_2)P(k)}_{w^3_{\ell,10}(\chi_1,\tilde{\chi}_2)} \\
&+ b(z_1)\beta(z_1)\underbrace{\frac{2}{\pi}\int dk\Big[f_{-2} j_{\ell-2}(k\chi_1) + f_0 j_\ell(k\chi_1) + f_2 j_{\ell+2}(k\chi_1)\Big] j_\ell(k\tilde{\chi}_2)P(k)}_{w^2_{\ell,20}(\chi_1,\tilde{\chi}_2)} \Bigg\} .
\end{aligned}
\tag{3.82}
$$

The first mixed term (a) results:

$$
\begin{aligned}
\Delta^{S*}_{\ell m}\Delta^{K}_{\ell' m'} = &\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\,\ell(\ell+1)\big[1-\mathcal{Q}(z_2)\big] \\
&\times \int_0^{\chi_2} d\tilde{\chi}_2 \frac{\left(\chi_2-\tilde{\chi}_2\right)}{\chi_2\tilde{\chi}_2} 3\Omega_m(\tilde{z}_2)\mathcal{H}^2(\tilde{z}_2) \\
&\times \Bigg\{ -b_1 w^2_{\ell,00}(\chi_1,\tilde{\chi}_2) + b_1\gamma_1 w^4_{\ell,00}(\chi_1,\tilde{\chi}_2) + \frac{1}{\chi_1} b_1\alpha_1\beta_1 w^3_{\ell,10}(\chi_1,\tilde{\chi}_2) \\
&+ b_1\beta_1 w^2_{\ell,20}(\chi_1,\tilde{\chi}_2) \Bigg\} ,
\end{aligned}
\tag{3.83}
$$

where the quantities with subscript 1 are evaluated in z_1.

(b) $\boldsymbol{\Delta^{S*}_{\ell m}\Delta^{I}_{\ell' m'}}$

$$
\begin{aligned}
\Delta^{S*}_{\ell m}\Delta^{I}_{\ell' m'} &= \big(\Delta^{S*}_{\ell m}① + \Delta^{S*}_{\ell m}②\big)\Delta^{K}_{\ell' m'}\\
&= \Bigg\{\int_0^\infty d\chi_1 W(\chi_1)\int \frac{d^3k}{2\pi^2}(-i)^\ell j_\ell(k\chi_1)Y_{\ell m}(\hat{\boldsymbol{k}})b(z_1)\delta^*_m(\hat{\boldsymbol{k}},z_1)\\
&\quad - \int_0^\infty d\chi_1 W(\chi_1)(-i)^\ell\Bigg\{b(z_1)\gamma(z_1)\int\frac{1}{2\pi^2}\frac{d^3k}{k^2}j_\ell(k\chi_1)\\
&\quad - \frac{1}{\chi_1}b(z_1)\alpha(z_1)\beta(z_1)\int\frac{1}{2\pi^2}\frac{d^3k}{k}\Big[f_{-1}j_{\ell-1}(k\chi_1)+f_1 j_{\ell+1}(k\chi_1)\Big]\\
&\quad - b(z_1)\beta(z_1)\int\frac{d^3k}{2\pi^2}\Big[f_{-2}j_{\ell-2}(k\chi_1)+f_0 j_\ell(k\chi_1)+f_2 j_{\ell+2}(k\chi_1)\Big]\Bigg\}Y_{\ell m}(\hat{\boldsymbol{k}})\delta^*_m(\hat{\boldsymbol{k}},z_1)\Bigg\}\\
&\times\int_0^\infty d\chi_2 W(\chi_2)\Bigg\{-\frac{2}{\chi_2}\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2}d\tilde{\chi}_2\, 3\Omega_m(\tilde{z}_2)\mathcal{H}^2(\tilde{z}_2)\\
&\qquad\qquad\times\int\frac{d^3k'}{2\pi^2k'^2}i^{\ell'}j_{\ell'}(k'\tilde{\chi}_2)Y^*_{\ell' m'}(\hat{\boldsymbol{k}}')\delta_m(\hat{\boldsymbol{k}}',\tilde{z}_2)\\
&-\frac{1}{\chi_2}\frac{\alpha(z_2)}{H(z_2)}\int_0^{\chi_2}d\tilde{\chi}_2\,3\Omega_m(\tilde{z}_2)\mathcal{H}^3(\tilde{z}_2)\big[f(\tilde{z}_2)-1\big]\int\frac{d^3k'}{2\pi^2k'^2}i^{\ell'}j_{\ell'}(k'\tilde{\chi}_2)Y^*_{\ell' m'}(\hat{\boldsymbol{k}}')\delta_m(\hat{\boldsymbol{k}}',\tilde{z}_2)\Bigg\}\,.
\end{aligned}
\tag{3.84}
$$

Then, doing all the products:

$$
\begin{aligned}
\Delta^{S*}_{\ell m}\Delta^{I}_{\ell' m'} &= \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\\
&\times\Bigg\{-b(z_1)\frac{2}{\chi_2}\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2}d\tilde{\chi}_2\,3\Omega_m(\tilde{z}_2)\mathcal{H}^2(\tilde{z}_2)\underbrace{\frac{2}{\pi}\int dk\, j_\ell(k\chi_1)j_\ell(k\tilde{\chi}_2)P(k)}_{w^2_{\ell,00}(\chi_1,\tilde{\chi}_2)}\\
&- b(z_1)\frac{1}{\chi_2}\frac{\alpha(z_2)}{H(z_2)}\int_0^{\chi_2}d\tilde{\chi}_2\,3\Omega_m(\tilde{z}_2)\mathcal{H}^3(\tilde{z}_2)\big[f(\tilde{z}_2)-1\big]\underbrace{\frac{2}{\pi}\int dk\, j_\ell(k\chi_1)j_\ell(k\tilde{\chi}_2)P(k)}_{w^2_{\ell,00}(\chi_1,\tilde{\chi}_2)}\\
&+ b(z_1)\gamma(z_1)\frac{2}{\chi_2}\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2}d\tilde{\chi}_2\,3\Omega_m(\tilde{z}_2)\mathcal{H}^2(\tilde{z}_2)\underbrace{\frac{2}{\pi}\int \frac{dk}{k^2} j_\ell(k\chi_1)j_\ell(k\tilde{\chi}_2)P(k)}_{w^4_{\ell,00}(\chi_1,\tilde{\chi}_2)}\\
&+ b(z_1)\gamma(z_1)\frac{1}{\chi_2}\frac{\alpha(z_2)}{H(z_2)}\int_0^{\chi_2}d\tilde{\chi}_2\,3\Omega_m(\tilde{z}_2)\mathcal{H}^3(\tilde{z}_2)\big[f(\tilde{z}_2)-1\big]\underbrace{\frac{2}{\pi}\int \frac{dk}{k^2} j_\ell(k\chi_1)j_\ell(k\tilde{\chi}_2)P(k)}_{w^4_{\ell,00}(\chi_1,\tilde{\chi}_2)}\\
&+ \frac{2}{\chi_1\chi_2}b(z_1)\alpha(z_1)\beta(z_1)\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2}d\tilde{\chi}_2\,3\Omega_m(\tilde{z}_2)\mathcal{H}^2(\tilde{z}_2)\\
&\quad\times\underbrace{\frac{2}{\pi}\int\frac{dk}{k}\Big[f_{-1}j_{\ell-1}(k\chi_1)+f_1 j_{\ell+1}(k\chi_1)\Big]j_\ell(k\tilde{\chi}_2)P(k)}_{w^3_{\ell,10}(\chi_1,\tilde{\chi}_2)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\chi_1\chi_2}\frac{b(z_1)\alpha(z_1)\alpha(z_2)\beta(z_1)}{H(z_2)}\int_0^{\chi_2}d\tilde{\chi}_2\,3\Omega_m(\tilde{z}_2)\mathcal{H}^3(\tilde{z}_2)\big[f(\tilde{z}_2)-1\big]\\
&\quad\times\underbrace{\frac{2}{\pi}\int\frac{dk}{k}\Big[f_{-1}j_{\ell-1}(k\chi_1)+f_1j_{\ell+1}(k\chi_1)\Big]j_\ell(k\tilde{\chi}_2)P(k)}_{w^3_{\ell,10}(\chi_1,\tilde{\chi}_2)}\\
&+b(z_1)\beta(z_1)\frac{2}{\chi_2}\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2}d\tilde{\chi}_2\,3\Omega_m(\tilde{z}_2)\mathcal{H}^2(\tilde{z}_2)\\
&\quad\times\underbrace{\frac{2}{\pi}\int dk\Big[f_{-2}j_{\ell-2}(k\chi_1)+f_0j_\ell(k\chi_1)+f_2j_{\ell+2}(k\chi_1)\Big]j_\ell(k\tilde{\chi}_2)P(k)}_{w^2_{\ell,20}(\chi_1,\tilde{\chi}_2)}\\
&+\frac{1}{\chi_2}\frac{b(z_1)\alpha(z_2)\beta(z_1)}{H(z_2)}\int_0^{\chi_2}d\tilde{\chi}_2\,3\Omega_m(\tilde{z}_2)\mathcal{H}^3(\tilde{z}_2)\big[f(\tilde{z}_2)-1\big]\\
&\quad\times\left.\underbrace{\frac{2}{\pi}\int dk\Big[f_{-2}j_{\ell-2}(k\chi_1)+f_0j_\ell(k\chi_1)+f_2j_{\ell+2}(k\chi_1)\Big]j_\ell(k\tilde{\chi}_2)P(k)}_{w^2_{\ell,20}(\chi_1,\tilde{\chi}_2)}\right\}.
\end{aligned}
\tag{3.85}
$$

The second mixed term results:

$$
\begin{aligned}
\Delta^{S*}_{\ell m}\Delta^{I}_{\ell'm'}=&\int_0^\infty d\chi_1W(\chi_1)\int_0^\infty d\chi_2W(\chi_2)\,b_1\left\{\frac{2}{\chi_2}\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2}d\tilde{\chi}_2\,3\Omega_m(\tilde{z}_2)\mathcal{H}^2(\tilde{z}_2)\right.\\
&\left.+\frac{1}{\chi_2}\frac{\alpha(z_2)}{H(z_2)}\int_0^{\chi_2}d\tilde{\chi}_2\,3\Omega_m(\tilde{z}_2)\mathcal{H}^3(\tilde{z}_2)\big[f(\tilde{z}_2)-1\big]\right\}\\
&\times\left\{-w^2_{\ell,00}(\chi_1,\tilde{\chi}_2)+\gamma_1w^4_{\ell,00}(\chi_1,\tilde{\chi}_2)+\frac{1}{\chi_1}\alpha_1\beta_1w^3_{\ell,10}(\chi_1,\tilde{\chi}_2)+\beta_1w^2_{\ell,20}(\chi_1,\tilde{\chi}_2)\right\}.
\end{aligned}
\tag{3.86}
$$

(c) $\Delta^{K*}_{\ell m}\Delta^{S}_{\ell'm'}$

$$
\begin{aligned}
\Delta^{K*}_{\ell m}\Delta^{S}_{\ell'm'}&=\Delta^{K*}_{\ell m}\big(\Delta^{S}_{\ell'm'}①+\Delta^{S}_{\ell'm'}②\big)\\
&=\left\{-\ell(\ell+1)\int_0^\infty d\chi_1W(\chi_1)\big[1-\mathcal{Q}(z_1)\big]\int_0^{\chi_1}d\tilde{\chi}_1\frac{(\chi_1-\tilde{\chi}_1)}{\chi_1\tilde{\chi}_1}3\Omega_m(\tilde{z}_1)\mathcal{H}^2(\tilde{z}_1)\right.\\
&\qquad\left.\times\int\frac{d^3k}{2\pi^2k^2}(-i)^\ell j_\ell(k\tilde{\chi}_1)Y_{\ell m}(\hat{\boldsymbol{k}})\delta^*_m(\hat{\boldsymbol{k}},\tilde{z}_1)\right\}\\
&\times\left\{\int_0^\infty d\chi_2W(\chi_2)\int\frac{d^3k'}{2\pi^2}i^{\ell'}j_{\ell'}(k'\chi_2)Y^*_{\ell'm'}(\hat{\boldsymbol{k}}')b(z_2)\delta_m(\hat{\boldsymbol{k}}',z_2)\right.\\
&-\int_0^\infty d\chi W(\chi_2)i^{\ell'}\left\{b(z_2)\gamma(z_2)\int\frac{1}{2\pi^2}\frac{d^3k'}{k'^2}j_{\ell'}(k'\chi_2)\right.\\
&+\frac{1}{\chi_2}b(z_2)\alpha(z_2)\beta(z_2)\int\frac{1}{2\pi^2}\frac{d^3k'}{k'}\Big[f'_{-1}j_{\ell'-1}(k'\chi_2)+f'_1j_{\ell'+1}(k'\chi_2)\Big]\\
&\left.\left.+f(z_2)\int\frac{d^3k'}{2\pi^2}\Big[f'_{-2}j_{\ell'-2}(k'\chi_2)+f'_0j_{\ell'}(k'\chi_2)+f'_2j_{\ell'+2}(k'\chi_2)\Big]\right\}Y^*_{\ell'm'}(\hat{\boldsymbol{k}}')\delta_m(\hat{\boldsymbol{k}}',z_2)\right\}.
\end{aligned}
\tag{3.87}
$$

Doing all the products we find:

$$
\begin{aligned}
\Delta^{K*}_{\ell m}\Delta^{S}_{\ell' m'} &= -\ell(\ell+1)\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\\
&\times\big[1-\mathcal{Q}(z_1)\big]\int_0^{\chi_1} d\tilde\chi_1 \frac{\big(\chi_1-\tilde\chi_1\big)}{\chi_1\tilde\chi_1}3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1)\\
&\Bigg\{ b(z_2)\underbrace{\frac{2}{\pi}\int dk j_\ell(k\tilde\chi_1)j_\ell(k\chi_2)P(k)}_{w^2_{\ell,00}(\tilde\chi_1,\chi_2)}\\
&- b(z_2)\gamma(z_2)\underbrace{\frac{2}{\pi}\int \frac{dk}{k^2} j_\ell(k\tilde\chi_1)j_\ell(k\chi_2)P(k)}_{w^4_{\ell,00}(\tilde\chi_1,\chi_2)}\\
&- \frac{1}{\chi_2}b(z_2)\alpha(z_2)\beta(z_2)\underbrace{\frac{2}{\pi}\int \frac{dk}{k} j_\ell(k\tilde\chi_1)\Big[f_{-1}j_{\ell-1}(k\chi_2)+f_1 j_{\ell+1}(k\chi_2)\Big]P(k)}_{w^3_{\ell,01}(\tilde\chi_1,\chi_2)}\\
&- b(z_2)\beta(z_2)\underbrace{\frac{2}{\pi}\int dk j_\ell(k\tilde\chi_1)\Big[f_{-2}j_{\ell-2}(k\chi_2)+f_0 j_\ell(k\chi_2)+f_2 j_{\ell+2}(k\chi_2)\Big]P(k)}_{w^2_{\ell,02}(\tilde\chi_1,\chi_2)}\Bigg\}\,.
\end{aligned}
\tag{3.88}
$$

The third mixed term results

$$
\begin{aligned}
\Delta^{K*}_{\ell m}\Delta^{S}_{\ell' m'} &= \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\\
&\times \ell(\ell+1)b_2\big[1-\mathcal{Q}(z_1)\big]\int_0^{\chi_1} d\tilde\chi_1 \frac{\big(\chi_1-\tilde\chi_1\big)}{\chi_1\tilde\chi_1}3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1)\\
&\times\left\{-w^2_{\ell,00}(\tilde\chi_1,\chi_2)+\gamma_2 w^4_{\ell,00}(\tilde\chi_1,\chi_2)+\frac{1}{\chi_2}\alpha_2\beta_2 w^3_{\ell,01}(\tilde\chi_1,\chi_2)+\beta_2 w^2_{\ell,02}(\tilde\chi_1,\chi_2)\right\}.
\end{aligned}
\tag{3.89}
$$

(d) $\boldsymbol{\Delta^{K*}_{\ell m}\Delta^{I}_{\ell' m'}}$

$$
\begin{aligned}
\Delta^{K*}_{\ell m}\Delta^{I}_{\ell' m'} = &\Bigg\{-\ell(\ell+1)\int_0^\infty d\chi_1 W(\chi_1)\big[1-\mathcal{Q}(z_1)\big]\int_0^{\chi_1} d\tilde\chi_1 \frac{\big(\chi_1-\tilde\chi_1\big)}{\chi_1\tilde\chi_1}3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1)\\
&\quad\times\int \frac{d^3k}{2\pi^2k^2}(-i)^\ell j_\ell(k\tilde\chi_1)Y_{\ell m}(\hat{\boldsymbol{k}})\delta^*_m(\hat{\boldsymbol{k}},\tilde z_1)\Bigg\}\\
\times&\Bigg\{\int_0^\infty d\chi_2 W(\chi_2)\Bigg\{-\frac{2}{\chi_2}\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2)\\
&\qquad\times\int \frac{d^3k'}{2\pi^2k'^2}i^{\ell'} j_{\ell'}(k'\tilde\chi_2)Y^*_{\ell' m'}(\hat{\boldsymbol{k}}')\delta_m(\hat{\boldsymbol{k}}',\tilde z_2)\\
-&\frac{1}{\chi_2}\frac{\alpha(z_2)}{H(z_2)}\int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^3(\tilde z_2)\big[f(\tilde z_2)-1\big]\int \frac{d^3k'}{2\pi^2k'^2}i^{\ell'} j_{\ell'}(k'\tilde\chi_2)Y^*_{\ell' m'}(\hat{\boldsymbol{k}}')\delta_m(\hat{\boldsymbol{k}}',\tilde z_2)\Bigg\}\Bigg\}\,.
\end{aligned}
\tag{3.90}
$$

With some calculations we find:

$$\begin{aligned}
\Delta^{K*}_{\ell m}\Delta^{I}_{\ell' m'} &= \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\,\ell(\ell+1)\big[1-\mathcal{Q}(z_1)\big] \\
&\times \int_0^{\chi_1} d\tilde\chi_1 \frac{\big(\chi_1-\tilde\chi_1\big)}{\chi_1\tilde\chi_1} 3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1)\bigg\{\frac{2}{\chi_2}\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2) \\
&\quad + \frac{1}{\chi_2}\frac{\alpha(z_2)}{H(z_2)}\int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^3(\tilde z_2)\big[f(\tilde z_2)-1\big]\bigg\}\underbrace{\frac{2}{\pi}\int\frac{dk}{k^2}j_\ell(k\tilde\chi_1)j_\ell(k\tilde\chi_2)P(k)}_{w^4_{\ell,00}(\tilde\chi_1,\tilde\chi_2)}\,.
\end{aligned} \tag{3.91}$$

Therefore the fourth mixed term is:

$$\begin{aligned}
\Delta^{K*}_{\ell m}\Delta^{I}_{\ell' m'} &= \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\,\ell(\ell+1)\big[1-\mathcal{Q}(z_1)\big] \\
&\times \int_0^{\chi_1} d\tilde\chi_1 \frac{\big(\chi_1-\tilde\chi_1\big)}{\chi_1\tilde\chi_1} 3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1)\bigg\{\frac{2}{\chi_2}\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2) \\
&\quad + \frac{1}{\chi_2}\frac{\alpha(z_2)}{H(z_2)}\int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^3(\tilde z_2)\big[f(\tilde z_2)-1\big]\bigg\} w^4_{\ell,00}(\tilde\chi_1,\tilde\chi_2)\,.
\end{aligned} \tag{3.92}$$

(e) $\boldsymbol{\Delta^{I*}_{\ell m}\Delta^{S}_{\ell' m'}}$

$$\begin{aligned}
&\Delta^{I*}_{\ell m}\Delta^{S}_{\ell' m'} = \Delta^{I*}_{\ell m}\big(\Delta^{S}_{\ell' m'}① + \Delta^{S}_{\ell' m'}②\big) \\
&= \bigg\{\int_0^\infty d\chi_1 W(\chi_1)\bigg\{-\frac{2}{\chi_1}\big[1-\mathcal{Q}(z_1)\big]\int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1) \\
&\qquad\qquad \times\int\frac{d^3k}{2\pi^2k^2} i^\ell j_\ell(k\tilde\chi_1)Y_{\ell m}(\hat{\boldsymbol{k}})\delta^*_m(\hat{\boldsymbol{k}},\tilde z_1) \\
&- \frac{1}{\chi_1}\frac{\alpha(z_1)}{H(z_1)}\int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^3(\tilde z_1)\big[f(\tilde z_1)-1\big]\int\frac{d^3k}{2\pi^2k^2} i^\ell j_\ell(k\tilde\chi_1)Y_{\ell m}(\hat{\boldsymbol{k}})\delta_m(\hat{\boldsymbol{k}},\tilde z_1)\bigg\}\bigg\} \\
&\times\bigg\{\int_0^\infty d\chi_2 W(\chi_2)\int\frac{d^3k'}{2\pi^2} i^{\ell'} j_{\ell'}(k'\chi_2)Y^*_{\ell' m'}(\hat{\boldsymbol{k}}')b(z_2)\delta_m(\hat{\boldsymbol{k}}',z_2) \\
&- \int_0^\infty d\chi W(\chi_2) i^{\ell'}\bigg\{b(z_2)\gamma(z_2)\int\frac{1}{2\pi^2}\frac{d^3k'}{k'^2} j_{\ell'}(k'\chi_2) \\
&+ \frac{1}{\chi_2}b(z_2)\alpha(z_2)\beta(z_2)\int\frac{1}{2\pi^2}\frac{d^3k'}{k'}\Big[f'_{-1}j_{\ell'-1}(k'\chi_2) + f'_1 j_{\ell'+1}(k'\chi_2)\Big] \\
&+ f(z_2)\int\frac{d^3k'}{2\pi^2}\Big[f'_{-2}j_{\ell'-2}(k'\chi_2) + f'_0 j_{\ell'}(k'\chi_2) + f'_2 j_{\ell'+2}(k'\chi_2)\Big]\bigg\}Y^*_{\ell' m'}(\hat{\boldsymbol{k}}')\delta_m(\hat{\boldsymbol{k}}',z_2)\bigg\}\,.
\end{aligned} \tag{3.93}$$

All the calculations give:

$$
\begin{aligned}
\Delta^{I*}_{\ell m}\Delta^{S}_{\ell' m'} = &\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\bigg\{\frac{2}{\chi_1}\big[1-\mathcal{Q}(z_1)\big]\int_0^{\chi_1} d\tilde{\chi}_1\, 3\Omega_m(\tilde{z}_1)\mathcal{H}^2(\tilde{z}_1)\\
&+\frac{1}{\chi_1}\frac{\alpha(z_1)}{H(z_1)}\int_0^{\chi_1} d\tilde{\chi}_1\, 3\Omega_m(\tilde{z}_1)\mathcal{H}^3(\tilde{z}_1)\big[f(\tilde{z}_1)-1\big]\bigg\}\\
\times&\bigg\{-b(z_2)\underbrace{\frac{2}{\pi}\int dk j_\ell(k\tilde{\chi}_1)j_\ell(k\chi_2)P(k)}_{w^2_{\ell,00}(\tilde{\chi}_1,\chi_2)}+b(z_2)\gamma(z_2)\underbrace{\frac{2}{\pi}\int\frac{dk}{k^2}j_\ell(k\tilde{\chi}_1)j_\ell(k\chi_2)P(k)}_{w^4_{\ell,00}(\tilde{\chi}_1,\chi_2)}\\
+&\frac{1}{\chi_2}b(z_2)\alpha(z_2)\beta(z_2)\underbrace{\frac{2}{\pi}\int\frac{dk}{k}j_\ell(k\tilde{\chi}_1)\Big[f_{-1}j_{\ell-1}(k\chi_2)+f_1 j_{\ell+1}(k\chi_2)\Big]P(k)}_{w^3_{\ell,01}(\tilde{\chi}_1,\chi_2)}\\
+&b(z_2)\beta(z_2)\underbrace{\frac{2}{\pi}\int dk j_\ell(k\tilde{\chi}_1)\Big[f_{-2}j_{\ell-2}(k\chi_2)+f_0 j_\ell(k\chi_2)+f_2 j_{\ell+2}(k\chi_2)\Big]P(k)}_{w^2_{\ell,02}(\tilde{\chi}_1,\chi_2)}\bigg\}.
\end{aligned}
\tag{3.94}
$$

Thus the fifth mixed term is:

$$
\begin{aligned}
\Delta^{I*}_{\ell m}\Delta^{S}_{\ell' m'} = &\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\bigg\{\frac{2}{\chi_1}\big[1-\mathcal{Q}(z_1)\big]\int_0^{\chi_1} d\tilde{\chi}_1\, 3\Omega_m(\tilde{z}_1)\mathcal{H}^2(\tilde{z}_1)\\
&+\frac{1}{\chi_1}\frac{\alpha(z_1)}{H(z_1)}\int_0^{\chi_1} d\tilde{\chi}_1\, 3\Omega_m(\tilde{z}_1)\mathcal{H}^3(\tilde{z}_1)\big[f(\tilde{z}_1)-1\big]\bigg\}b_2\\
&\times\bigg\{-w^2_{\ell,00}(\tilde{\chi}_1,\chi_2)+\gamma_2 w^4_{\ell,00}(\tilde{\chi}_1,\chi_2)+\frac{1}{\chi_2}\alpha_2\beta_2 w^3_{\ell,01}(\tilde{\chi}_1,\chi_2)+\beta_2 w^2_{\ell,02}(\tilde{\chi}_1,\chi_2)\bigg\}.
\end{aligned}
\tag{3.95}
$$

(f) $\boldsymbol{\Delta^{I*}_{\ell m}\Delta^{K}_{\ell' m'}}$

$$
\begin{aligned}
\Delta^{I*}_{\ell m}\Delta^{K}_{\ell' m'} = &\Bigg\{\int_0^\infty d\chi_1 W(\chi_1)\bigg\{-\frac{2}{\chi_1}\big[1-\mathcal{Q}(z_1)\big]\int_0^{\chi_1} d\tilde{\chi}_1\, 3\Omega_m(\tilde{z}_1)\mathcal{H}^2(\tilde{z}_1)\\
&\qquad\qquad\qquad\times\int\frac{d^3k}{2\pi^2k^2}i^\ell j_\ell(k\tilde{\chi}_1)Y_{\ell m}(\hat{\boldsymbol{k}})\delta^*_m(\hat{\boldsymbol{k}},\tilde{z}_1)\\
&-\frac{1}{\chi_1}\frac{\alpha(z_1)}{H(z_1)}\int_0^{\chi_1} d\tilde{\chi}_1\, 3\Omega_m(\tilde{z}_1)\mathcal{H}^3(\tilde{z}_1)\big[f(\tilde{z}_1)-1\big]\int\frac{d^3k}{2\pi^2k^2}i^\ell j_\ell(k\tilde{\chi}_1)Y_{\ell m}(\hat{\boldsymbol{k}})\delta_m(\hat{\boldsymbol{k}},\tilde{z}_1)\bigg\}\Bigg\}\\
&\times\bigg\{-\ell'(\ell'+1)\int_0^\infty d\chi_2 W(\chi_2)\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2} d\tilde{\chi}_2\frac{(\chi_2-\tilde{\chi}_2)}{\chi_2\tilde{\chi}_2}3\Omega_m(\tilde{z}_2)\mathcal{H}^2(\tilde{z}_2)\\
&\quad\times\int\frac{d^3k'}{2\pi^2k'^2}i^{\ell'} j_{\ell'}(k'\tilde{\chi}_2)Y^*_{\ell' m'}(\hat{\boldsymbol{k}}')\delta_m(\hat{\boldsymbol{k}}',\tilde{z}_2)\bigg\}\\
&=\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\bigg\{\frac{2}{\chi_1}\big[1-\mathcal{Q}(z_1)\big]\int_0^{\chi_1} d\tilde{\chi}_1\, 3\Omega_m(\tilde{z}_1)\mathcal{H}^2(\tilde{z}_1)\\
&\quad+\frac{1}{\chi_1}\frac{\alpha(z_1)}{H(z_1)}\int_0^{\chi_1} d\tilde{\chi}_1\, 3\Omega_m(\tilde{z}_1)\mathcal{H}^3(\tilde{z}_1)\big[f(\tilde{z}_1)-1\big]\bigg\}\,\ell(\ell+1)\\
&\times\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2} d\tilde{\chi}_2\frac{(\chi_2-\tilde{\chi}_2)}{\chi_2\tilde{\chi}_2}3\Omega_m(\tilde{z}_2)\mathcal{H}^2(\tilde{z}_2)\underbrace{\frac{2}{\pi}\int\frac{dk}{k^2}j_\ell(k\tilde{\chi}_1)j_\ell(k\tilde{\chi}_2)P(k)}_{w^4_{\ell,00}(\tilde{\chi}_1,\tilde{\chi}_2)}\,. \qquad (3.96)
\end{aligned}
$$

Therefore the sixth mixed term results:

$$
\begin{aligned}
\Delta^{I*}_{\ell m}\Delta^{K}_{\ell' m'} = &\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\bigg\{\frac{2}{\chi_1}\big[1-\mathcal{Q}(z_1)\big]\int_0^{\chi_1} d\tilde{\chi}_1\, 3\Omega_m(\tilde{z}_1)\mathcal{H}^2(\tilde{z}_1)\\
&+\frac{1}{\chi_1}\frac{\alpha(z_1)}{H(z_1)}\int_0^{\chi_1} d\tilde{\chi}_1\, 3\Omega_m(\tilde{z}_1)\mathcal{H}^3(\tilde{z}_1)\big[f(\tilde{z}_1)-1\big]\bigg\}\ell(\ell+1)\\
&\times\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2} d\tilde{\chi}_2\frac{(\chi_2-\tilde{\chi}_2)}{\chi_2\tilde{\chi}_2}3\Omega_m(\tilde{z}_2)\mathcal{H}^2(\tilde{z}_2)\, w^4_{\ell,00}(\tilde{\chi}_1,\tilde{\chi}_2)\,. \qquad (3.97)
\end{aligned}
$$

3.9 Total C_ℓ

The total C_ℓ coefficient consists in the sum of the three unmixed contributions (3.76), (3.78), (3.79) and of the mixed terms (3.83), (3.86), (3.89), (3.92), (3.95), (3.97):

$$
\begin{aligned}
C_\ell = & \int_0^\infty d\chi_1 W(\chi_1) \int_0^\infty d\chi_2 W(\chi_2) \Bigg\{ \\
& b_1 b_2 \Bigg\{ w^0_{\ell,00}(\chi_1,\chi_2) - \beta_2 w^0_{\ell,02}(\chi_1,\chi_2) - \beta_1 w^0_{\ell,20}(\chi_1,\chi_2) + \beta_1\beta_2 w^0_{\ell,22}(\chi_1,\chi_2) \\
& - \frac{1}{\chi_2}\alpha_2\beta_2 w^1_{\ell,01}(\chi_1,\chi_2) - \frac{1}{\chi_1}\alpha_1\beta_1 w^1_{\ell,10}(\chi_1,\chi_2) + \frac{1}{\chi_1}\alpha_1\beta_1\beta_2 w^1_{\ell,12}(\chi_1,\chi_2) \\
& + \frac{1}{\chi_2}\alpha_2\beta_2\beta_1 w^1_{\ell,21}(\chi_1,\chi_2) - \left[\gamma_1+\gamma_2\right] w^2_{\ell,00}(\chi_1,\chi_2) + \gamma_1\beta_2 w^2_{\ell,02}(\chi_1,\chi_2) \\
& + \gamma_2\beta_1 w^2_{\ell,20}(\chi_1,\chi_2) + \frac{1}{\chi_1\chi_2}\alpha_1\alpha_2\beta_1\beta_2 w^2_{\ell,11}(\chi_1,\chi_2) + \frac{1}{\chi_2}\alpha_2\beta_2\gamma w^3_{\ell,01}(\chi_1,\chi_2) \\
& + \frac{1}{\chi_1}\alpha_1\beta_1\gamma_2 w^3_{\ell,10}(\chi_1,\chi_2) + \gamma_1\gamma_2 w^4_{\ell,00}(\chi_1,\chi_2)\Bigg\} + \Bigg\{ \ell^2(\ell+1)^2\left[1-\mathcal{Q}(z_1)\right]\left[1-\mathcal{Q}(z_2)\right] \\
& \times \int_0^{\chi_1} d\tilde\chi_1 \frac{(\chi_1-\tilde\chi_1)}{\chi_1\tilde\chi_1} 3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1) \int_0^{\chi_2} d\tilde\chi_2 \frac{(\chi_2-\tilde\chi_2)}{\chi_2\tilde\chi_2} 3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2) \\
& + \frac{1}{\chi_1\chi_2}\Bigg[4\left[1-\mathcal{Q}(z_1)\right]\left[1-\mathcal{Q}(z_2)\right] \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1) \int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2) \\
& \quad + 2\left[1-\mathcal{Q}(z_1)\right]\frac{\alpha(z_2)}{H(z_2)} \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1) \int_0^{\chi_2} d\tilde\chi_2 3\,\Omega_m(\tilde z_2)\mathcal{H}^3(\tilde z_2)\left[f(\tilde z_2)-1\right] \\
& \quad + 2\left[1-\mathcal{Q}(z_2)\right]\frac{\alpha(z_1)}{H(z_1)} \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^3(\tilde z_1)\left[f(\tilde z_1)-1\right] \int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2) \\
& + \frac{\alpha(z_1)\alpha(z_2)}{H(z_1)H(z_2)} \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^3(\tilde z_1)\left[f(\tilde z_1)-1\right] \int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^3(\tilde z_2)\left[f(\tilde z_2)-1\right]\Bigg] \\
& + \ell(\ell+1)\left[1-\mathcal{Q}(z_1)\right] \int_0^{\chi_1} d\tilde\chi_1 \frac{(\chi_1-\tilde\chi_1)}{\chi_1\tilde\chi_1} 3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1) \\
& \times \left[\frac{2}{\chi_2}\left[1-\mathcal{Q}(z_2)\right] \int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2) + \frac{1}{\chi_2}\frac{\alpha(z_2)}{H(z_2)} \int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^3(\tilde z_2)\left[f(\tilde z_2)-1\right]\right\}\Bigg] \\
& + \ell(\ell+1)\left[1-\mathcal{Q}(z_2)\right] \int_0^{\chi_2} d\tilde\chi_2 \frac{(\chi_2-\tilde\chi_2)}{\chi_2\tilde\chi_2} 3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2)\; w^4_{\ell,00}(\tilde\chi_1,\tilde\chi_2) \\
& \times \left[\frac{2}{\chi_1}\left[1-\mathcal{Q}(z_1)\right] \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1) + \frac{1}{\chi_1}\frac{\alpha(z_1)}{H(z_1)} \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^3(\tilde z_1)\left[f(\tilde z_1)-1\right]\right]\Bigg\} \\
& \quad \times w^4_{\ell,00}(\tilde\chi_1,\tilde\chi_2) \\
& + \ell(\ell+1)\left[1-\mathcal{Q}(z_2)\right] b_1 \int_0^{\chi_2} d\tilde\chi_2 \frac{(\chi_2-\tilde\chi_2)}{\chi_2\tilde\chi_2} 3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2) \\
& \times \left\{-w^2_{\ell,00}(\chi_1,\tilde\chi_2) + \gamma_1 w^4_{\ell,00}(\chi_1,\tilde\chi_2) + \frac{1}{\chi_1}\alpha_1\beta_1 w^3_{\ell,10}(\chi_1,\tilde\chi_2) + \beta_1 w^2_{\ell,20}(\chi_1,\tilde\chi_2)\right\} \\
& + b_1\left\{\frac{2}{\chi_2}\left[1-\mathcal{Q}(z_2)\right] \int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2) + \frac{1}{\chi_2}\frac{\alpha(z_2)}{H(z_2)} \int_0^{\chi_2} d\tilde\chi_2\, 3\Omega_m(\tilde z_2)\mathcal{H}^3(\tilde z_2)\left[f(\tilde z_2)-1\right]\right\} \\
& \times \left\{-w^2_{\ell,00}(\chi_1,\tilde\chi_2) + \gamma_1 w^4_{\ell,00}(\chi_1,\tilde\chi_2) + \frac{1}{\chi_1}\alpha_1\beta_1 w^3_{\ell,10}(\chi_1,\tilde\chi_2) + \beta_1 w^2_{\ell,20}(\chi_1,\tilde\chi_2)\right\} \\
& + \ell(\ell+1)\left[1-\mathcal{Q}(z_1)\right] b_2 \int_0^{\chi_1} d\tilde\chi_1 \frac{(\chi_1-\tilde\chi_1)}{\chi_1\tilde\chi_1} 3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1) \\
& \times \left\{-w^2_{\ell,00}(\tilde\chi_1,\chi_2) + \gamma_2 w^4_{\ell,00}(\tilde\chi_1,\chi_2) + \frac{1}{\chi_2}\alpha_2\beta_2 w^3_{\ell,01}(\tilde\chi_1,\chi_2) + \beta_2 w^2_{\ell,02}(\tilde\chi_1,\chi_2)\right\} \\
& + b_2\left\{\frac{2}{\chi_1}\left[1-\mathcal{Q}(z_1)\right] \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1) + \frac{1}{\chi_1}\frac{\alpha(z_1)}{H(z_1)} \int_0^{\chi_1} d\tilde\chi_1\, 3\Omega_m(\tilde z_1)\mathcal{H}^3(\tilde z_1)\left[f(\tilde z_1)-1\right]\right\} \\
& \times \left\{-w^2_{\ell,00}(\tilde\chi_1,\chi_2) + \gamma_2 w^4_{\ell,00}(\tilde\chi_1,\chi_2) + \frac{1}{\chi_2}\alpha_2\beta_2 w^3_{\ell,01}(\tilde\chi_1,\chi_2) + \beta_2 w^2_{\ell,02}(\tilde\chi_1,\chi_2)\right\}\Bigg\}\,.
\end{aligned}
\tag{3.98}
$$

Chapter 4

Limber's approximation

The exact expression of the angular power spectrum (3.98) is given in terms of several integrals which must be evaluated numerically, but this task is often very difficult and time consuming because at high multipoles ℓ and large arguments $k\chi$ the spherical Bessel functions $j_\ell(k\chi)$ rapidly oscillate; their highly oscillatory behaviour delays the convergence of the numerical integration. For this reason the Limber approximation has been often used in recent papers (e.g. in Ref.[33]) as a tecnique to simplify calculations. It is a powerful method to estimate the magnitude and understand the analytic dependence of the projected power spectra. Its implementation requires two basic assumptions:

- working with small angular separations or with large multipole moments ℓ; we will focus on the latest aspect.
- some of the functions which are integrated are more slowly varying than others; in our case, we assume that the power spectrum $P(k)$ which appears inside the $w^{\nu}_{\ell,jj'}$ integrals varies more much more slowly than the spherical Bessel functions $j_\ell(k\chi)$.

In the Limber's approximation, the spherical Bessel function is replaced by a delta function:

$$j_\ell(k\chi) \longrightarrow \sqrt{\frac{\pi}{2a}}\delta_D(a - k\chi) \qquad , \qquad \text{where} \quad a = \ell + \frac{1}{2} \,. \tag{4.1}$$

This definition is taken from Ref.[34], which calls the parameter a as ν, as is usually done in literature. Here we change this notation because ν has already been used in the $w^{\nu}_{\ell,jj}$ definitions, and this can bring misunderstandings in further calculations. Therefore, we will use a as Limber parameter in order to avoid confusion. We will further demonstrate that using (4.1) the integral $w^0_{\ell,00}$, defined in (3.58) and (3.59), becomes

$$w^0_{\ell,00}(\chi_1, \chi_2) \approx \frac{\delta_D(\chi_1 - \chi_2)}{\chi_1^2} P\left(\frac{a}{\chi_1}\right) \,, \tag{4.2}$$

and we will generalize this result to the case $w^{\nu}_{\ell,jj'}$, with generic values of ν and j, j'. The Dirac delta suggests that this tecnique reduces the number of integrals, thus making calculations simpler. Before applying such an approximation to our results, we want to ensure that it is consistent by looking at the work of Jeong et al. in Ref.[33]: they compare the numerical estimation computed with the 2-FAST and CAMB [35] algorithms with the results from the Limber's approximation for different values of a (which they call ν). Watching Fig.4.1, it can be seen that: in the case $a = \ell$ the largest

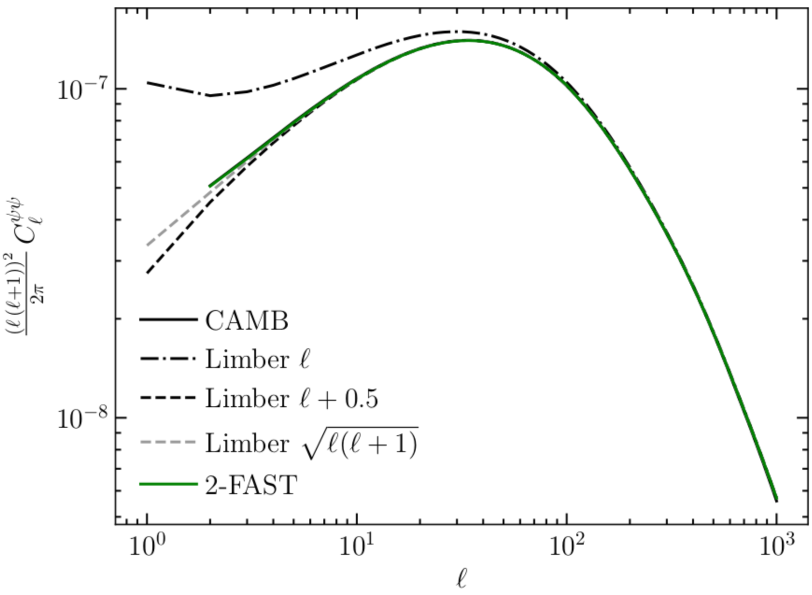


Figure 4.1: *Here the lensing potential angular power spectrum is plotted [33]. In solid black is the result from CAMB, in dashed-dotted black the Limber's approximation with $a = \ell$ ($a = \nu$ in [33]), in dashed black with $a = \ell + 1/2$, in dashed gray with $a = \sqrt{\ell(\ell+1)}$, and in solid green the 2-FAST method presented in Ref.[33] .*

deviation happens for small ℓ, while with the more proper choice of $a = \ell + 0.5$ [36] the approximation works much better up to $\ell \simeq 10$; a further improvement has been given by choosing $a = \sqrt{\ell(\ell+1)}$, but we will not use it because it would complicate calculations too much. Fig.4.1 then shows that the Limber's approximation reproduces the exact calculation for large multipole moments $\ell > 100$, while for larger angular scales (smaller ℓ) there is a deviation. Therefore, by assuming the previous two hypothesis, we are able to apply the Limber's approximation to all our $w^\nu_{\ell,jj'}$ definitions, in order to give an easier expression of C_ℓ.

4.1 The case $\ell = \ell'$, $\nu = 0$

The first step is to demonstrate Eq.(4.2) using (4.1). It is useful to recap the definition of $w^\nu_{\ell,\ell} \equiv w^\nu_{\ell,00}$ for $\nu = 0$, to which the approximation is applied in Ref.[34] and [36], and that is our starting point:

$$w^0_{\ell,00}(\chi_1,\chi_2) \equiv w^0_{\ell,\ell}(\chi_1,\chi_2) = \frac{2}{\pi}\int dk\, k^2 P(k) j_\ell(k\chi_1) j_\ell(k\chi_2) \,.$$

First of all, we want to insert approximation (4.1) in the previous formula:

$$\begin{aligned} w^0_{\ell,\ell}(\chi_1,\chi_2) &\approx \frac{2}{\pi}\int dk\, k^2 P(k)\sqrt{\frac{\pi}{2a}}\delta_D(a-k\chi_1)\sqrt{\frac{\pi}{2a}}\delta_D(a-k\chi_2) \\ &\approx \int dk\, k^2 P(k)\frac{1}{a}\delta_D(a-k\chi_1)\delta_D(a-k\chi_2) \,. \end{aligned} \tag{4.3}$$

At this point, we want to write the Dirac delta functions in a different way using the property:

$$\delta_D[g(k)] = \sum_i \frac{\delta_D(k-k_i)}{|g'(k_i)|} \,, \tag{4.4}$$

where k_i are the zeros of $g(k)$. In our case $g(k) = a - k\chi$, $g'(k) = -\chi$ and $k_i = a/\chi$, therefore we find that

$$\delta_D(a-k\chi) = \frac{\delta_D\left(a - \frac{k}{\chi}\right)}{\chi} \,. \tag{4.5}$$

This result can be inserted in (4.3):

$$w^0_{\ell,\ell}(\chi_1,\chi_2) \approx \int dkk^2 P(k) \frac{1}{a} \frac{\delta_D\left(a - \frac{k}{\chi_1}\right)}{\chi_1} \frac{\delta_D\left(a - \frac{k}{\chi_2}\right)}{\chi_2}$$
$$\approx \frac{a^2}{\chi_1^2} \frac{1}{a} P\left(\frac{a}{\chi_1}\right) \frac{1}{\chi_1\chi_2} \delta_D\left(\frac{a}{\chi_1} - \frac{a}{\chi_2}\right) . \tag{4.6}$$

We have almost proved (4.2); the last step is to write the Dirac delta function as $\delta_D(\chi_1 - \chi_2)$, using the same property (4.4) mentioned before. Here we consider a $g(\chi) = a(\chi_1 - \chi_2)/\chi_1\chi_2$, with $g'(\chi) = -a/\chi_1^2$, so the delta becomes

$$\delta_D\left(\frac{a}{\chi_1} - \frac{a}{\chi_2}\right) = \frac{\delta_D(\chi_1 - \chi_2)}{\left| - \frac{a}{\chi_1^2}\right|} = \chi_1^2 \frac{\delta_D(\chi_1 - \chi_2)}{a} . \tag{4.7}$$

Therefore, since the Dirac delta constrains $\chi_1 = \chi_2$, we find

$$w^0_{\ell,\ell}(\chi_1,\chi_2) \approx \frac{a}{\chi_1^2} P\left(\frac{a}{\chi_1}\right) \frac{1}{\chi_1^2} \chi_1^2 \frac{\delta_D(\chi_1 - \chi_2)}{a}$$
$$\approx \frac{\delta_D(\chi_1 - \chi_2)}{\chi_1^2} P\left(\frac{a}{\chi_1}\right) , \tag{4.8}$$

which is exactly the formula (4.2).

4.2 The case $\ell \neq \ell'$, generic ν

Our aim is to generalize the formula (4.2) in the case when $j \neq j'$ and so $\ell \neq \ell'$, varying also the value of the index ν in order to find the approximated expressions of the $w^\nu_{\ell,jj'}(\chi_1,\chi_2)$ integrals that appear in the expression of C_ℓ. We have to use the approximation (4.1) in the definition of $w^\nu_{\ell,\ell'}(\chi_1,\chi_2)$, which represents the tipical addend in $w^\nu_{\ell,jj'}(\chi_1,\chi_2)$. Therefore, step by step, we find:

$$w^\nu_{\ell,\ell'}(\chi_1,\chi_2) = \frac{2}{\pi} \int dk\, k^2 \frac{P(k)}{k^\nu} j_\ell(k\chi_1) j_\ell(k\chi_2)$$
$$\approx \frac{2}{\pi} \int dk\, k^2 \frac{P(k)}{k^\nu} \sqrt{\frac{\pi}{2a}} \delta_D(a - k\chi_1) \sqrt{\frac{\pi}{2a'}} \delta_D(a' - k\chi_2) , \tag{4.9}$$

where $a = \ell + 0.5$ and $a' = \ell' + 0.5$. Now, using again the property (4.4) for the Dirac delta functions, we find:

$$w^\nu_{\ell,\ell'}(\chi_1,\chi_2) \approx \int dk \frac{k^2}{k^\nu} P(k) \frac{1}{\sqrt{aa'}} \frac{\delta_D\left(k - \frac{a}{\chi_1}\right)}{\chi_1} \frac{\delta_D\left(k - \frac{a'}{\chi_2}\right)}{\chi_2} \tag{4.10}$$
$$\approx \left(\frac{a}{\chi_1}\right)^2 \left(\frac{\chi_1}{a}\right)^\nu P(k) \frac{1}{\sqrt{aa'}} \frac{1}{\chi_1\chi_2} \delta_D\left(\frac{a}{\chi_1} - \frac{a'}{\chi_2}\right) . \tag{4.11}$$

Using again (4.4) with $g(\chi_1) = (a/\chi_1) - (a'/\chi_2)$ and $g'(\chi_1) = -a/\chi_1^2$, we have:

$$w^\nu_{\ell,\ell'}(\chi_1,\chi_2) \approx \left(\frac{a}{\chi_1}\right)^2 \left(\frac{\chi_1}{a}\right)^\nu P(k) \frac{1}{\sqrt{aa'}} \frac{1}{\chi_1^2} \frac{\chi_1^2}{a} \delta_D(\chi_1 - \chi_2)$$
$$\approx \frac{\delta_D(\chi_1 - \chi_2)}{\chi_1^{2-\nu}\, a^\nu} \sqrt{\frac{a}{a'}} P\left(\frac{a}{\chi_1}\right) . \tag{4.12}$$

Symmetrization

However, the generalization we have just found is not symmetric, since we "used" only one Dirac delta function to replace the integral in k. Therefore, we want to write the products of the two deltas in (4.10) as:

$$\delta_D\left(k-\frac{a}{\chi_1}\right)\delta_D\left(k-\frac{a'}{\chi_2}\right)=\frac{1}{2}\left[\delta_D\left(k-\frac{a}{\chi_1}\right)\delta_D\left(k-\frac{a'}{\chi_2}\right)+\delta_D\left(k-\frac{a'}{\chi_2}\right)\delta_D\left(k-\frac{a}{\chi_1}\right)\right]. \tag{4.13}$$

In this way (4.10) becomes

$$\begin{aligned}
&w^{\nu}_{\ell,\ell'}(\chi_1,\chi_2)\\
&\approx\int dk\,\frac{k^2}{k^{\nu}}P(k)\frac{1}{\sqrt{aa'}}\frac{1}{\chi_1\chi_2}\frac{1}{2}\left[\delta_D\left(k-\frac{a}{\chi_1}\right)\delta_D\left(k-\frac{a'}{\chi_2}\right)+\delta_D\left(k-\frac{a'}{\chi_2}\right)\delta_D\left(k-\frac{a}{\chi_1}\right)\right]\\
&\approx\frac{1}{2}\left(\frac{a}{\chi_1}\right)^{2-\nu}P\left(\frac{a}{\chi_1}\right)\frac{1}{\sqrt{aa'}}\frac{1}{\chi_1\chi_2}\underbrace{\delta_D\left(\frac{a}{\chi_1}-\frac{a'}{\chi_2}\right)}_{\frac{\chi_1^2}{a}\delta_D(\chi_1-\chi_2)}\\
&+\frac{1}{2}\left(\frac{a'}{\chi_2}\right)^{2-\nu}P\left(\frac{a'}{\chi_2}\right)\frac{1}{\sqrt{aa'}}\frac{1}{\chi_1\chi_2}\underbrace{\delta_D\left(\frac{a'}{\chi_2}-\frac{a}{\chi_1}\right)}_{\frac{\chi_2^2}{a}\delta_D(\chi_2-\chi_1)}.
\end{aligned}\tag{4.14}$$

Doing the calculations and with the constrain $\chi_1=\chi_2$ we find the following symmetric formula:

$$w^{\nu}_{\ell,\ell'}(\chi_1,\chi_2)\approx\frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}a^{\nu}}\sqrt{\frac{a}{a'}}P\left(\frac{a}{\chi_1}\right)+\frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}a'^{\nu}}\sqrt{\frac{a'}{a}}P\left(\frac{a'}{\chi_1}\right), \tag{4.15}$$

where $a=\ell+1/2$ and $a'=\ell'+1/2$.
This is the starting point to compute the Limber's approximation of all the $w^{\nu}_{\ell,jj'}$ integrals, in order to give that of the C_ℓ coefficients in the end.

4.3 Limber's approximation of $w^{\nu}_{\ell,jj'}$ integrals

$w^{\nu}_{\ell,00}(\chi_1,\chi_2)$

$$\begin{aligned}
\boldsymbol{w^{\nu}_{\ell,00}(\chi_1,\chi_2)}&=w^{\nu}_{\ell,\ell}(\chi_1,\chi_2)\\
&=\frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}a^{\nu}}P\left(\frac{a}{\chi_1}\right)+\frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}a^{\nu}}P\left(\frac{a}{\chi_1}\right)\\
&=\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}a^{\nu}}P\left(\frac{a}{\chi_1}\right),
\end{aligned}\tag{4.16}$$

with

$$a=\ell+\frac{1}{2}. \tag{4.17}$$

$\boldsymbol{w^{\nu}_{\ell,02}(\chi_1,\chi_2)}$

$$w^{\nu}_{\ell,\ell-2}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}a^{\nu}}\sqrt{\frac{a}{a-2}}P\left(\frac{a}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-2)^{\nu}}\sqrt{\frac{a-2}{a}}P\left(\frac{a-2}{\chi_1}\right)\ ; \tag{4.18}$$

$$w^{\nu}_{\ell,\ell+2}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}a^{\nu}}\sqrt{\frac{a}{a+2}}P\left(\frac{a}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+2)^{\nu}}\sqrt{\frac{a+2}{a}}P\left(\frac{a+2}{\chi_1}\right)\ ; \tag{4.19}$$

$$\begin{aligned}\boldsymbol{w^{\nu}_{\ell,02}(\chi_1,\chi_2)} &= f_{-2}w^{\nu}_{\ell,\ell-2}(\chi_1,\chi_2) + f_0 w^{\nu}_{\ell,\ell}(\chi_1,\chi_2) + f_2 w^{\nu}_{\ell,\ell+2}(\chi_1,\chi_2)\\ &= \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}}\frac{\sqrt{a}}{a^{\nu}}\left[\frac{f_{-2}}{\sqrt{a-2}} + \frac{2f_0}{\sqrt{a}} + \frac{f_2}{\sqrt{a+2}}\right]P\left(\frac{a}{\chi_1}\right)\\ &+ \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}}\frac{1}{\sqrt{a}}\left[f_{-2}\frac{\sqrt{a-2}}{(a-2)^{\nu}}P\left(\frac{a-2}{\chi_1}\right) + f_2\frac{\sqrt{a+2}}{(a+2)^{\nu}}P\left(\frac{a+2}{\chi_1}\right)\right]\ ,\end{aligned} \tag{4.20}$$

where the quantities a, $f_{0,\pm 2}$ have been defined in (4.17) and (3.42).

$\boldsymbol{w^{\nu}_{\ell,20}(\chi_1,\chi_2)}$

$$w^{\nu}_{\ell-2,\ell}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-2)^{\nu}}\sqrt{\frac{a-2}{a}}P\left(\frac{a-2}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}a^{\nu}}\sqrt{\frac{a}{a-2}}P\left(\frac{a}{\chi_1}\right)\ ; \tag{4.21}$$

$$w^{\nu}_{\ell+2,\ell}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+2)^{\nu}}\sqrt{\frac{a+2}{a}}P\left(\frac{a+2}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}a^{\nu}}\sqrt{\frac{a}{a+2}}P\left(\frac{a}{\chi_1}\right)\ ; \tag{4.22}$$

$$\begin{aligned}\boldsymbol{w^{\nu}_{\ell,20}(\chi_1,\chi_2)} &= f_{-2}w^{\nu}_{\ell-2,\ell}(\chi_1,\chi_2) + f_0 w^{\nu}_{\ell,\ell}(\chi_1,\chi_2) + f_2 w^{\nu}_{\ell+2,\ell}(\chi_1,\chi_2)\\ &= \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}}\frac{\sqrt{a}}{a^{\nu}}\left[\frac{f_{-2}}{\sqrt{a-2}} + \frac{2f_0}{\sqrt{a}} + \frac{f_2}{\sqrt{a+2}}\right]P\left(\frac{a}{\chi_1}\right)\\ &+ \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}}\frac{1}{\sqrt{a}}\left[f_{-2}\frac{\sqrt{a-2}}{(a-2)^{\nu}}P\left(\frac{a-2}{\chi_1}\right) + f_2\frac{\sqrt{a+2}}{(a+2)^{\nu}}P\left(\frac{a+2}{\chi_1}\right)\right]\ ,\end{aligned} \tag{4.23}$$

which results equal to (4.20), as we expected using a symmetric approach.

$\boldsymbol{w^{\nu}_{\ell,22}(\chi_1,\chi_2)}$

$$w^{\nu}_{\ell-2,\ell-2}(\chi_1,\chi_2) = \frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-2)^{\nu}} P\left(\frac{a-2}{\chi_1}\right)\,; \tag{4.24}$$

$$w^{\nu}_{\ell+2,\ell+2}(\chi_1,\chi_2) = \frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+2)^{\nu}} P\left(\frac{a+2}{\chi_1}\right)\,; \tag{4.25}$$

$$w^{\nu}_{\ell-2,\ell+2}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-2)^{\nu}}\sqrt{\frac{a-2}{a+2}} P\left(\frac{a-2}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+2)^{\nu}}\sqrt{\frac{a+2}{a-2}} P\left(\frac{a+2}{\chi_1}\right)\,; \tag{4.26}$$

$$w^{\nu}_{\ell+2,\ell-2}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+2)^{\nu}}\sqrt{\frac{a+2}{a-2}} P\left(\frac{a+2}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-2)^{\nu}}\sqrt{\frac{a-2}{a+2}} P\left(\frac{a-2}{\chi_1}\right)\,. \tag{4.27}$$

Also here we can notice that (4.26) and (4.27) are equal. At this point we have all the elements to compute:

$$\begin{aligned} w^{\nu}_{\ell,22}(\chi_1,\chi_2) &= (f_{-2})^2 w^{\nu}_{\ell-2,\ell-2}(\chi_1,\chi_2) + f_{-2}f_0 w^{\nu}_{\ell-2,\ell}(\chi_1,\chi_2) + f_{-2}f_2 w^{\nu}_{\ell-2,\ell+2}(\chi_1,\chi_2) \\ &\quad + f_0 f_{-2} w^{\nu}_{\ell,\ell-2}(\chi_1,\chi_2) + (f_0)^2 w^{\nu}_{\ell,\ell}(\chi_1,\chi_2) + f_0 f_2 w^{\nu}_{\ell,\ell+2}(\chi_1,\chi_2) \\ &\quad + f_2 f_{-2} w^{\nu}_{\ell+2,\ell-2}(\chi_1,\chi_2) + f_2 f_0 w^{\nu}_{\ell+2,\ell}(\chi_1,\chi_2) + (f_2)^2 w^{\nu}_{\ell+2,\ell+2}(\chi_1,\chi_2)\,. \end{aligned}$$

With some calculations we find the expression:

$$\begin{aligned} \boldsymbol{w^{\nu}_{\ell,22}(\chi_1,\chi_2)} &= \frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}}\left[\frac{f_{-2}}{\sqrt{a-2}} + \frac{f_0}{\sqrt{a}} + \frac{f_2}{\sqrt{a+2}}\right] \\ &\quad \times \left[f_{-2}\frac{\sqrt{a-2}}{(a-2)^{\nu}} P\left(\frac{a-2}{\chi_1}\right) + f_0\frac{\sqrt{a}}{a^{\nu}} P\left(\frac{a}{\chi_1}\right) + f_2\frac{\sqrt{a+2}}{(a+2)^{\nu}} P\left(\frac{a+2}{\chi_1}\right)\right]. \end{aligned} \tag{4.28}$$

$\boldsymbol{w^{\nu}_{\ell,01}(\chi_1,\chi_2)}$

$$w^{\nu}_{\ell,\ell-1}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}a^{\nu}}\sqrt{\frac{a}{a-1}} P\left(\frac{a}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-1)^{\nu}}\sqrt{\frac{a-1}{a}} P\left(\frac{a-1}{\chi_1}\right)\,; \tag{4.29}$$

$$w^{\nu}_{\ell,\ell+1}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}a^{\nu}}\sqrt{\frac{a}{a+1}} P\left(\frac{a}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+1)^{\nu}}\sqrt{\frac{a+1}{a}} P\left(\frac{a+1}{\chi_1}\right)\,; \tag{4.30}$$

$$\begin{aligned} \boldsymbol{w^{\nu}_{\ell,01}(\chi_1,\chi_2)} &= f_{-1} w^{\nu}_{\ell,\ell-1}(\chi_1,\chi_2) + f_1 w^{\nu}_{\ell,\ell+1}(\chi_1,\chi_2) \\ &= \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}}\frac{\sqrt{a}}{a^{\nu}}\left[\frac{f_{-1}}{\sqrt{a-1}} + \frac{f_1}{\sqrt{a+1}}\right] P\left(\frac{a}{\chi_1}\right) \\ &\quad + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}}\frac{1}{\sqrt{a}}\left[f_{-1}\frac{\sqrt{a-1}}{(a-1)^{\nu}} P\left(\frac{a-1}{\chi_1}\right) + f_1\frac{\sqrt{a+1}}{(a+1)^{\nu}} P\left(\frac{a+1}{\chi_1}\right)\right]. \end{aligned} \tag{4.31}$$

$w^{\nu}_{\ell,10}(\chi_1,\chi_2)$

$$w^{\nu}_{\ell-1,\ell}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-1)^{\nu}}\sqrt{\frac{a-1}{a}}P\left(\frac{a-1}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}a^{\nu}}\sqrt{\frac{a}{a-1}}P\left(\frac{a}{\chi_1}\right)\,; \tag{4.32}$$

$$w^{\nu}_{\ell+1,\ell}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+1)^{\nu}}\sqrt{\frac{a+1}{a}}P\left(\frac{a+1}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}a^{\nu}}\sqrt{\frac{a}{a+1}}P\left(\frac{a}{\chi_1}\right)\,; \tag{4.33}$$

$$\begin{aligned}
\boldsymbol{w^{\nu}_{\ell,10}(\chi_1,\chi_2)} &= f_{-1}w^{\nu}_{\ell-1,\ell}(\chi_1,\chi_2) + f_1 w^{\nu}_{\ell+1,\ell}(\chi_1,\chi_2) \\
&= \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}}\frac{\sqrt{a}}{a^{\nu}}\left[\frac{f_{-1}}{\sqrt{a-1}} + \frac{f_1}{\sqrt{a+1}}\right]P\left(\frac{a}{\chi_1}\right) \\
&+ \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}}\frac{1}{\sqrt{a}}\left[f_{-1}\frac{\sqrt{a-1}}{(a-1)^{\nu}}P\left(\frac{a-1}{\chi_1}\right) + f_1\frac{\sqrt{a+1}}{(a+1)^{\nu}}P\left(\frac{a+1}{\chi_1}\right)\right]\,,
\end{aligned} \tag{4.34}$$

which is the same expression of (4.31).

$w^{\nu}_{\ell,11}(\chi_1,\chi_2)$

$$w^{\nu}_{\ell-1,\ell-1}(\chi_1,\chi_2) = \frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-1)^{\nu}}P\left(\frac{a-1}{\chi_1}\right)\,; \tag{4.35}$$

$$w^{\nu}_{\ell+1,\ell+1}(\chi_1,\chi_2) = \frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+1)^{\nu}}P\left(\frac{a+1}{\chi_1}\right)\,; \tag{4.36}$$

$$w^{\nu}_{\ell-1,\ell+1}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-1)^{\nu}}\sqrt{\frac{a-1}{a+1}}P\left(\frac{a-1}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+1)^{\nu}}\sqrt{\frac{a+1}{a-1}}P\left(\frac{a+1}{\chi_1}\right)\,; \tag{4.37}$$

$$w^{\nu}_{\ell+1,\ell-1}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+1)^{\nu}}\sqrt{\frac{a+1}{a-1}}P\left(\frac{a+1}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-1)^{\nu}}\sqrt{\frac{a-1}{a+1}}P\left(\frac{a-1}{\chi_1}\right)\,. \tag{4.38}$$

Also here we can notice that (4.37) and (4.38) are equivalent. At this point we can compute:

$$\begin{aligned}
w^{\nu}_{\ell,11}(\chi_1,\chi_2) &= (f_{-1})^2 w^{\nu}_{\ell-1,\ell-1}(\chi_1,\chi_2) + f_{-1}f_1 w^{\nu}_{\ell-1,\ell+1}(\chi_1,\chi_2) \\
&\quad + f_1 f_{-1} w^{\nu}_{\ell+1,\ell-1}(\chi_1,\chi_2) + (f_1)^2 w^{\nu}_{\ell+1,\ell+1}(\chi_1,\chi_2)\,.
\end{aligned}$$

With some calculations we find the expression:

$$\begin{aligned}
\boldsymbol{w^{\nu}_{\ell,11}(\chi_1,\chi_2)} &= \frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}}\left[\frac{f_{-1}}{\sqrt{a-1}} + \frac{f_1}{\sqrt{a+1}}\right] \\
&\times\left[f_{-1}\frac{\sqrt{a-1}}{(a-1)^{\nu}}P\left(\frac{a-1}{\chi_1}\right) + f_1\frac{\sqrt{a+1}}{(a+1)^{\nu}}P\left(\frac{a+1}{\chi_1}\right)\right]\,.
\end{aligned} \tag{4.39}$$

$w^{\nu}_{\ell,12}(\chi_1,\chi_2)$

$$w^{\nu}_{\ell-2,\ell-1}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-2)^{\nu}}\sqrt{\frac{a-2}{a-1}}P\left(\frac{a-2}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-1)^{\nu}}\sqrt{\frac{a-1}{a-2}}P\left(\frac{a-1}{\chi_1}\right)\,; \tag{4.40}$$

$$w^{\nu}_{\ell-2,\ell+1}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-2)^{\nu}}\sqrt{\frac{a-2}{a+1}}P\left(\frac{a-2}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+1)^{\nu}}\sqrt{\frac{a+1}{a-2}}P\left(\frac{a+1}{\chi_1}\right)\,; \tag{4.41}$$

$$w^{\nu}_{\ell+2,\ell-1}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+2)^{\nu}}\sqrt{\frac{a+2}{a-1}}P\left(\frac{a+2}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-1)^{\nu}}\sqrt{\frac{a-1}{a+2}}P\left(\frac{a-1}{\chi_1}\right)\,; \tag{4.42}$$

$$w^{\nu}_{\ell+2,\ell+1}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+2)^{\nu}}\sqrt{\frac{a+2}{a+1}}P\left(\frac{a+2}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+1)^{\nu}}\sqrt{\frac{a+1}{a+2}}P\left(\frac{a+1}{\chi_1}\right)\,; \tag{4.43}$$

$$w^{\nu}_{\ell-1,\ell-2}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-1)^{\nu}}\sqrt{\frac{a-1}{a-2}}P\left(\frac{a-1}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-2)^{\nu}}\sqrt{\frac{a-2}{a-1}}P\left(\frac{a-2}{\chi_1}\right)\,; \tag{4.44}$$

$$w^{\nu}_{\ell-1,\ell+2}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-1)^{\nu}}\sqrt{\frac{a-1}{a+2}}P\left(\frac{a-1}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+2)^{\nu}}\sqrt{\frac{a+2}{a-1}}P\left(\frac{a+2}{\chi_1}\right)\,; \tag{4.45}$$

$$w^{\nu}_{\ell+1,\ell-2}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+1)^{\nu}}\sqrt{\frac{a+1}{a-2}}P\left(\frac{a+1}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a-2)^{\nu}}\sqrt{\frac{a-2}{a+1}}P\left(\frac{a-2}{\chi_1}\right)\,; \tag{4.46}$$

$$w^{\nu}_{\ell+1,\ell+2}(\chi_1,\chi_2) = \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+1)^{\nu}}\sqrt{\frac{a+1}{a+2}}P\left(\frac{a+1}{\chi_1}\right) + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}(a+2)^{\nu}}\sqrt{\frac{a+2}{a+1}}P\left(\frac{a+2}{\chi_1}\right)\,. \tag{4.47}$$

The simmetry of the various couples of formulas is evident also here. Putting all these elements together, we can find:

$$\begin{aligned} w^{\nu}_{\ell,12}(\chi_1,\chi_2) &= f_{-1}f_{-2}w^{\nu}_{\ell-1,\ell-2}(\chi_1,\chi_2) + f_{-1}f_0 w^{\nu}_{\ell-1,\ell}(\chi_1,\chi_2) + f_{-1}f_2 w^{\nu}_{\ell-1,\ell+2}(\chi_1,\chi_2) \\ &\quad + f_1 f_{-2}w^{\nu}_{\ell+1,\ell-2}(\chi_1,\chi_2) + f_1 f_0 w^{\nu}_{\ell+1,\ell}(\chi_1,\chi_2) + f_1 f_2 w^{\nu}_{\ell+1,\ell+2}(\chi_1,\chi_2)\,. \end{aligned}$$

Doing all the calculations, we have:

$$\begin{aligned} \boldsymbol{w^{\nu}_{\ell,12}(\chi_1,\chi_2)} &= \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}}\left[\frac{f_{-1}}{\sqrt{a-1}} + \frac{f_1}{\sqrt{a+1}}\right] \\ &\quad \times \left[f_{-2}\frac{\sqrt{a-2}}{(a-2)^{\nu}}P\left(\frac{a-2}{\chi_1}\right) + f_0\frac{\sqrt{a}}{a^{\nu}}P\left(\frac{a}{\chi_1}\right) + f_2\frac{\sqrt{a+2}}{(a+2)^{\nu}}P\left(\frac{a+2}{\chi_1}\right)\right] \\ &+ \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}}\left[\frac{f_{-2}}{\sqrt{a-2}} + \frac{f_0}{\sqrt{a}} + \frac{f_2}{\sqrt{a+2}}\right] \\ &\quad \times \left[f_{-1}\frac{\sqrt{a-1}}{(a-1)^{\nu}}P\left(\frac{a-1}{\chi_1}\right) + f_1\frac{\sqrt{a+1}}{(a+1)^{\nu}}P\left(\frac{a+1}{\chi_1}\right)\right]\,. \end{aligned} \tag{4.48}$$

$\boldsymbol{w^{\nu}_{\ell,21}(\chi_1,\chi_2)}$

We already have all the elements in order to compute the following expression:

$$\begin{aligned} w^{\nu}_{\ell,21}(\chi_1,\chi_2) = & f_{-2}f_{-1}w^{\nu}_{\ell-2,\ell-1}(\chi_1,\chi_2) + f_{-2}f_1 w^{\nu}_{\ell-2,\ell+1}(\chi_1,\chi_2) + f_0 f_{-1} w^{\nu}_{\ell,\ell-1}(\chi_1,\chi_2) \\ & + f_0 f_1 w^{\nu}_{\ell,\ell+1}(\chi_1,\chi_2) + f_2 f_{-1} w^{\nu}_{\ell+2,\ell-1}(\chi_1,\chi_2) + f_2 f_1 w^{\nu}_{\ell+2,\ell+1}(\chi_1,\chi_2) \, . \end{aligned}$$

Doing all the calculations we find exactly the same expression of (4.48):

$$\begin{aligned} \boldsymbol{w^{\nu}_{\ell,21}(\chi_1,\chi_2)} = & \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}}\left[\frac{f_{-1}}{\sqrt{a-1}} + \frac{f_1}{\sqrt{a+1}}\right] \\ & \times \left[f_{-2}\frac{\sqrt{a-2}}{(a-2)^{\nu}}P\left(\frac{a-2}{\chi_1}\right) + f_0\frac{\sqrt{a}}{a^{\nu}}P\left(\frac{a}{\chi_1}\right) + f_2\frac{\sqrt{a+2}}{(a+2)^{\nu}}P\left(\frac{a+2}{\chi_1}\right)\right] \\ & + \frac{1}{2}\frac{\delta_D(\chi_1-\chi_2)}{\chi_1^{2-\nu}}\left[\frac{f_{-2}}{\sqrt{a-2}} + \frac{f_0}{\sqrt{a}} + \frac{f_2}{\sqrt{a+2}}\right] \\ & \times \left[f_{-1}\frac{\sqrt{a-1}}{(a-1)^{\nu}}P\left(\frac{a-1}{\chi_1}\right) + f_1\frac{\sqrt{a+1}}{(a+1)^{\nu}}P\left(\frac{a+1}{\chi_1}\right)\right] . \end{aligned} \tag{4.49}$$

4.4 Approximated C_ℓ coefficients

We are now able to apply the Limber's approximation to all the contributions of the C_ℓ coefficients: the local, lensing and time delay terms, and the mixed terms.

4.4.1 Approximated local term C_ℓ^S

Substituting the approximated expressions of $w^\nu_{\ell,jj'}$ into (3.76), and leaving the dependence in $a = \ell + 1/2$, we find:

$$
\begin{aligned}
C_\ell^S = \Delta^{S*}_{\ell m}\Delta^S_{\ell' m'} = & \int_0^\infty d\chi_1 W(\chi_1) \int_0^\infty d\chi_2 W(\chi_2)\, b_1 b_2\, \delta_D(\chi_1-\chi_2)\times \\
& \left\{ \frac{1}{\chi_1^2}\left[P\left(\frac{a}{\chi_1}\right) - \frac{1}{2}(\beta_1+\beta_2)\sqrt{a}\left(\frac{f_{-2}}{\sqrt{a-2}} + \frac{2f_0}{\sqrt{a}} + \frac{f_2}{\sqrt{a+2}}\right) P\left(\frac{a}{\chi_1}\right)\right.\right. \\
& \quad - \frac{1}{2}(\beta_1+\beta_2)\frac{1}{\sqrt{a}}\left(f_{-2}\sqrt{a-2}P\left(\frac{a-2}{\chi_1}\right) + f_2\sqrt{a+2}P\left(\frac{a+2}{\chi_1}\right)\right) \\
& \quad + \beta_1\beta_2\left(\frac{f_{-2}}{\sqrt{a-2}} + \frac{f_0}{\sqrt{a}} + \frac{f_2}{\sqrt{a+2}}\right) \\
& \qquad \left. \times \left(f_{-2}\sqrt{a-2}P\left(\frac{a-2}{\chi_1}\right) + f_0\sqrt{a}P\left(\frac{a}{\chi_1}\right) + f_2\sqrt{a+2}P\left(\frac{a+2}{\chi_1}\right)\right)\right] \\
& + \frac{1}{\chi_1}\left[-\frac{1}{2}\left(\frac{\alpha_1\beta_1}{\chi_1} + \frac{\alpha_2\beta_2}{\chi_2}\right)\frac{1}{\sqrt{a}}\left(\frac{f_{-1}}{\sqrt{a-1}} + \frac{f_1}{\sqrt{a+1}}\right)P\left(\frac{a}{\chi_1}\right)\right. \\
& \quad - \frac{1}{2}\left(\frac{\alpha_1\beta_1}{\chi_1} + \frac{\alpha_2\beta_2}{\chi_2}\right)\frac{1}{\sqrt{a}}\left(\frac{f_{-1}}{\sqrt{a-1}}P\left(\frac{a-1}{\chi_1}\right) + \frac{f_1}{\sqrt{a+1}}P\left(\frac{a+1}{\chi_1}\right)\right) \\
& \quad + \frac{1}{2}\left(\frac{\alpha_1\beta_1\beta_2}{\chi_1} + \frac{\alpha_2\beta_2\beta_1}{\chi_2}\right)\left(\frac{f_{-1}}{\sqrt{a-1}}P\left(\frac{a-1}{\chi_1}\right) + \frac{f_1}{\sqrt{a+1}}P\left(\frac{a+1}{\chi_1}\right)\right) \\
& \qquad \times \left(\frac{f_{-2}}{\sqrt{a-2}} + \frac{f_0}{\sqrt{a}} + \frac{f_2}{\sqrt{a+2}}\right) \\
& \quad + \frac{1}{2}\left(\frac{\alpha_1\beta_1\beta_2}{\chi_1} + \frac{\alpha_2\beta_2\beta_1}{\chi_2}\right)\left(\frac{f_{-1}}{\sqrt{a-1}} + \frac{f_1}{\sqrt{a+1}}\right) \\
& \qquad \left. \times \left(\frac{f_{-2}}{\sqrt{a-2}}P\left(\frac{a-2}{\chi_1}\right) + \frac{f_0}{\sqrt{a}}P\left(\frac{a}{\chi_1}\right) + \frac{f_2}{\sqrt{a+2}}P\left(\frac{a+2}{\chi_1}\right)\right)\right] \\
& + \left[-(\gamma_1+\gamma_2)\frac{1}{a^2}P\left(\frac{a}{\chi_1}\right) + \frac{1}{2}(\gamma_1\beta_2+\gamma_2\beta_1)\frac{\sqrt{a}}{a^2}\left(\frac{f_{-2}}{\sqrt{a-2}} + \frac{2f_0}{\sqrt{a}} + \frac{f_2}{\sqrt{a+2}}\right)P\left(\frac{a}{\chi_1}\right)\right. \\
& \quad + \frac{1}{2}(\gamma_1\beta_2+\gamma_2\beta_1)\frac{1}{\sqrt{a}}\left(f_{-2}\frac{\sqrt{a-2}}{(a-2)^2}P\left(\frac{a-2}{\chi_1}\right) + f_2\frac{\sqrt{a+2}}{(a+2)^2}P\left(\frac{a+2}{\chi_1}\right)\right) \\
& \quad + \frac{\alpha_1\alpha_2\beta_1\beta_2}{\chi_1\chi_2}\left(f_{-1}\frac{\sqrt{a-1}}{(a-2)^2}P\left(\frac{a-1}{\chi_1}\right) + f_1\frac{\sqrt{a+1}}{(a+1)^2}P\left(\frac{a+1}{\chi_1}\right)\right) \\
& \qquad \left. \times \left(\frac{f_{-1}}{\sqrt{a-1}} + \frac{f_1}{\sqrt{a+1}}\right)\right] \\
& + \chi_1\left[\frac{1}{2}\left(\frac{\alpha_1\beta_1\gamma_2}{\chi_1} + \frac{\alpha_2\beta_2\gamma_1}{\chi_2}\right)\frac{\sqrt{a}}{a^3}\left(\frac{f_{-1}}{\sqrt{a-1}} + \frac{f_1}{\sqrt{a+1}}\right)P\left(\frac{a}{\chi_1}\right)\right. \\
& \quad \left. + \frac{1}{2}\left(\frac{\alpha_1\beta_1\gamma_2}{\chi_1} + \frac{\alpha_2\beta_2\gamma_1}{\chi_2}\right)\frac{1}{\sqrt{a}}\left(f_{-1}\frac{\sqrt{a-1}}{(a-1)^3}P\left(\frac{a-1}{\chi_1}\right) + f_1\frac{\sqrt{a+1}}{(a+1)^3}P\left(\frac{a+1}{\chi_1}\right)\right)\right] \\
& \left. + \chi_1^2\frac{\gamma_1\gamma_2}{a^4}P\left(\frac{a}{\chi_1}\right)\right\}. \qquad (4.50)
\end{aligned}
$$

Expliciting the Dirac delta and the dependence on ℓ, we find:

$$
\begin{aligned}
C_\ell^S = \Delta^{S*}_{\ell m}\Delta^S_{\ell' m'} = \int_0^\infty d\chi_1 W(\chi_1)\, b_1^2 \Bigg\{ \frac{1}{\chi_1^2}\Bigg[P\left(\frac{\ell+\frac{1}{2}}{\chi_1}\right) \\
- \beta_1\sqrt{\ell+\frac{1}{2}}\left(\frac{f_{-2}}{\sqrt{\ell-\frac{3}{2}}} + \frac{2f_0}{\sqrt{\ell+\frac{1}{2}}} + \frac{f_2}{\sqrt{\ell+\frac{5}{2}}}\right) P\left(\frac{\ell+\frac{1}{2}}{\chi_1}\right) \\
- \beta_1\frac{1}{\sqrt{\ell+\frac{1}{2}}}\left(f_{-2}\sqrt{\ell-\frac{3}{2}}P\left(\frac{\ell-\frac{3}{2}}{\chi_1}\right) + f_2\sqrt{\ell+\frac{5}{2}}P\left(\frac{\ell+\frac{5}{2}}{\chi_1}\right)\right) \\
+ \beta_1^2\left(\frac{f_{-2}}{\sqrt{\ell-\frac{3}{2}}} + \frac{f_0}{\sqrt{\ell+\frac{1}{2}}} + \frac{f_2}{\sqrt{\ell+\frac{5}{2}}}\right) \\
\times\left(f_{-2}\sqrt{\ell-\frac{3}{2}}P\left(\frac{\ell-\frac{3}{2}}{\chi_1}\right) + f_0\sqrt{\ell+\frac{1}{2}}P\left(\frac{\ell+\frac{1}{2}}{\chi_1}\right) + f_2\sqrt{\ell+\frac{5}{2}}P\left(\frac{\ell+\frac{5}{2}}{\chi_1}\right)\right)\Bigg] \\
- \frac{\alpha_1\beta_1}{\sqrt{\ell+\frac{1}{2}}}\left[\left(\frac{f_{-1}}{\sqrt{\ell-\frac{1}{2}}} + \frac{f_1}{\sqrt{\ell+\frac{3}{2}}}\right)P\left(\frac{\ell+\frac{1}{2}}{\chi_1}\right) + \frac{f_{-1}}{\sqrt{\ell-\frac{1}{2}}}P\left(\frac{\ell-\frac{1}{2}}{\chi_1}\right) + \frac{f_1}{\sqrt{\ell+\frac{3}{2}}}P\left(\frac{\ell+\frac{3}{2}}{\chi_1}\right)\right] \\
+ \alpha_1\beta_1^2\Bigg[\left(\frac{f_{-2}}{\sqrt{\ell-\frac{3}{2}}} + \frac{f_0}{\sqrt{\ell+\frac{1}{2}}} + \frac{f_2}{\sqrt{\ell+\frac{5}{2}}}\right)\left(\frac{f_{-1}}{\sqrt{\ell-\frac{1}{2}}}P\left(\frac{\ell-\frac{1}{2}}{\chi_1}\right) + \frac{f_1}{\sqrt{\ell+\frac{3}{2}}}P\left(\frac{\ell+\frac{3}{2}}{\chi_1}\right)\right) \\
+ \left(\frac{f_{-1}}{\sqrt{\ell-\frac{1}{2}}} + \frac{f_1}{\sqrt{\ell+\frac{3}{2}}}\right)\left(\frac{f_{-2}}{\sqrt{\ell-\frac{3}{2}}}P\left(\frac{\ell-\frac{3}{2}}{\chi_1}\right) + \frac{f_0}{\sqrt{\ell+\frac{1}{2}}}P\left(\frac{\ell+\frac{1}{2}}{\chi_1}\right) + \frac{f_2}{\sqrt{\ell+\frac{5}{2}}}P\left(\frac{\ell+\frac{5}{2}}{\chi_1}\right)\right)\Bigg] \\
- \frac{2\gamma_1}{\left(\ell+\frac{1}{2}\right)^2}P\left(\frac{\ell+\frac{1}{2}}{\chi_1}\right) + \gamma_1\beta_1\frac{\sqrt{\ell+\frac{1}{2}}}{\left(\ell+\frac{1}{2}\right)^2}\left(\frac{f_{-2}}{\sqrt{\ell-\frac{3}{2}}} + \frac{2f_0}{\sqrt{\ell+\frac{1}{2}}} + \frac{f_2}{\sqrt{\ell+\frac{5}{2}}}\right)P\left(\frac{\ell+\frac{1}{2}}{\chi_1}\right) \\
+ \frac{\gamma_1\beta_1}{\sqrt{\ell+\frac{1}{2}}}\left(f_{-2}\frac{\sqrt{\ell-\frac{3}{2}}}{\left(\ell-\frac{3}{2}\right)^2}P\left(\frac{\ell-\frac{3}{2}}{\chi_1}\right) + f_2\frac{\sqrt{\ell+\frac{5}{2}}}{\left(\ell+\frac{5}{2}\right)^2}P\left(\frac{\ell+\frac{5}{2}}{\chi_1}\right)\right) \\
+ \frac{\alpha_1^2\beta_1^2}{\chi_1^2}\left(f_{-1}\frac{\sqrt{\ell-\frac{1}{2}}}{\left(\ell-\frac{1}{2}\right)^2}P\left(\frac{\ell-\frac{1}{2}}{\chi_1}\right) + f_1\frac{\sqrt{\ell+\frac{3}{2}}}{\left(\ell+\frac{3}{2}\right)^2}P\left(\frac{\ell+\frac{3}{2}}{\chi_1}\right)\right)\left(\frac{f_{-1}}{\sqrt{\ell-\frac{1}{2}}} + \frac{f_1}{\sqrt{\ell+\frac{3}{2}}}\right) \\
+ \alpha_1\beta_1\gamma_1\frac{\sqrt{\ell+\frac{1}{2}}}{\left(\ell+\frac{1}{2}\right)^3}\Bigg[\left(\frac{f_{-1}}{\sqrt{\ell-\frac{1}{2}}} + \frac{f_1}{\sqrt{\ell+\frac{3}{2}}}\right)P\left(\frac{\ell+\frac{1}{2}}{\chi_1}\right) \\
+ \frac{1}{\sqrt{\ell+\frac{1}{2}}}\left(f_{-1}\frac{\sqrt{\ell-\frac{1}{2}}}{\left(\ell-\frac{1}{2}\right)^3}P\left(\frac{\ell-\frac{1}{2}}{\chi_1}\right) + f_1\frac{\sqrt{\ell+\frac{3}{2}}}{\left(\ell+\frac{3}{2}\right)^3}P\left(\frac{\ell+\frac{3}{2}}{\chi_1}\right)\right)\Bigg] + \chi_1^2\,\frac{\gamma_1^2}{\left(\ell+\frac{1}{2}\right)^4}P\left(\frac{\ell+\frac{1}{2}}{\chi_1}\right)\Bigg\}.
\end{aligned}
\tag{4.51}
$$

4.4.2 Approximated weak lensing term C_ℓ^K

Substituting the approximated expressions of $w^{\nu}_{\ell,jj'}$ into (3.78) as before, knowing that $a=\ell+1/2$, we have:

$$\begin{aligned}
C_\ell^K &= \ell^2(\ell+1)^2\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\big[1-Q(z_1)\big]\big[1-Q(z_2)\big]\\
&\times\int_0^{\chi_1} d\tilde{\chi_1}\frac{\chi_1-\tilde{\chi_1}}{\chi_1\tilde{\chi_1}}3\Omega_m(\tilde{\chi_1})\mathcal{H}^2(\tilde{z_1})\int_0^{\chi_2} d\tilde{\chi_2}\frac{\chi_2-\tilde{\chi_2}}{\chi_2\tilde{\chi_2}}3\Omega_m(\tilde{\chi_2})\mathcal{H}^2(\tilde{z_2})\\
&\times\frac{\tilde{\chi_1}^2}{a^4}\delta_D(\tilde{\chi_1}-\tilde{\chi_2})P\left(\frac{a}{\tilde{\chi_1}}\right)\\
&= \ell^2(\ell+1)^2\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\big[1-Q(z_1)\big]\big[1-Q(z_2)\big]\\
&\times\int_0^{\chi_1} d\tilde{\chi_1}\frac{(\chi_1-\tilde{\chi_1})^2}{(\chi_1\tilde{\chi_1})^2}9\Omega_m^2(\tilde{\chi_1})\mathcal{H}^4(\tilde{z_1})\frac{\tilde{\chi_1}}{a^4}P\left(\frac{a}{\tilde{\chi_1}}\right)\,.
\end{aligned}\tag{4.52}$$

Writing explicitely the dependence in ℓ, we find the following expression:

$$\begin{aligned}
C_\ell^K &= \frac{\ell^2(\ell+1)^2}{\left(\ell+\frac{1}{2}\right)^2}\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\big[1-Q(z_1)\big]\big[1-Q(z_2)\big]\\
&\times\int_0^{\chi_1} d\tilde{\chi_1}\frac{(\chi_1-\tilde{\chi_1})^2}{\chi_1^2}9\Omega_m^2(\tilde{\chi_1})\mathcal{H}^4(\tilde{z_1})P\left(\frac{\ell+\frac{1}{2}}{\tilde{\chi_1}}\right)\,.
\end{aligned}\tag{4.53}$$

4.4.3 Approximated time delay term C_ℓ^I

Expression (3.79) becomes:

$$\begin{aligned}
C_\ell^I &= \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\,\frac{1}{\chi_1\chi_2}\\
&\times\Bigg\{4\big[1-\mathcal{Q}(z_1)\big]\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_1} d\tilde{\chi_1}\,3\Omega_m(\tilde{z_1})\mathcal{H}^2(\tilde{z_1})\int_0^{\chi_2} d\tilde{\chi_2}\,3\Omega_m(\tilde{z_2})\mathcal{H}^2(\tilde{z_2})\\
&\quad+2\big[1-\mathcal{Q}(z_1)\big]\frac{\alpha(z_2)}{H(z_2)}\int_0^{\chi_1} d\tilde{\chi_1}\,3\Omega_m(\tilde{z_1})\mathcal{H}^2(\tilde{z_1})\int_0^{\chi_2} d\tilde{\chi_2}3\,\Omega_m(\tilde{z_2})\mathcal{H}^3(\tilde{z_2})\big[f(\tilde{z_2})-1\big]\\
&\quad+2\big[1-\mathcal{Q}(z_2)\big]\frac{\alpha(z_1)}{H(z_1)}\int_0^{\chi_1} d\tilde{\chi_1}\,3\Omega_m(\tilde{z_1})\mathcal{H}^3(\tilde{z_1})\big[f(\tilde{z_1})-1\big]\int_0^{\chi_2} d\tilde{\chi_2}\,3\Omega_m(\tilde{z_2})\mathcal{H}^2(\tilde{z_2})\\
&+\frac{\alpha(z_1)\alpha(z_2)}{H(z_1)H(z_2)}\int_0^{\chi_1} d\tilde{\chi_1}\,3\Omega_m(\tilde{z_1})\mathcal{H}^3(\tilde{z_1})\big[f(\tilde{z_1})-1\big]\int_0^{\chi_2} d\tilde{\chi_2}\,3\Omega_m(\tilde{z_2})\mathcal{H}^3(\tilde{z_2})\big[f(\tilde{z_2})-1\big]\Bigg\}\\
&\quad\times\frac{\tilde{\chi_1}^2}{a^4}\delta_D(\tilde{\chi_1}-\tilde{\chi_2})P\left(\frac{a}{\tilde{\chi_1}}\right)\,.
\end{aligned}\tag{4.54}$$

Therefore, since $a=\ell+1/2$, due to the Dirac delta we find:

$$\begin{aligned}
C_\ell^I &= \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\,\frac{1}{\chi_1\chi_2}\\
&\Bigg\{4\big[1-\mathcal{Q}(z_1)\big]\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_1} d\tilde{\chi_1}\,9\Omega_m^2(\tilde{z_1})\mathcal{H}^4(\tilde{z_1})\\
&+2\left[\big[1-\mathcal{Q}(z_1)\big]\frac{\alpha(z_2)}{H(z_2)}+\big[1-\mathcal{Q}(z_2)\big]\frac{\alpha(z_1)}{H(z_1)}\right]\int_0^{\chi_1} d\tilde{\chi_1}\,9\Omega_m^2(\tilde{z_1})\mathcal{H}^5(\tilde{z_1})\big[f(\tilde{z_1})-1\big]\\
&+\frac{\alpha(z_1)\alpha(z_2)}{H(z_1)H(z_2)}\int_0^{\chi_1} d\tilde{\chi_1}\,9\Omega_m^2(\tilde{z_1})\mathcal{H}^6(\tilde{z_1})\big[f(\tilde{z_1})-1\big]^2\Bigg\}\frac{\tilde{\chi_1}^2}{\left(\ell+\frac{1}{2}\right)^4}P\left(\frac{\ell+\frac{1}{2}}{\tilde{\chi_1}}\right)\,.
\end{aligned}\tag{4.55}$$

4.4.4 Approximated mixed terms

Always remembering that $a = \ell + 1/2$, we find the approximated expressions of the mixed terms in the following subsections.

(a) $\boldsymbol{\Delta^{S*}_{\ell m}\Delta^{K}_{\ell' m'}}$

$$
\begin{aligned}
&\Delta^{S*}_{\ell m}\Delta^{K}_{\ell' m'} = \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\\
&\times \ell(\ell+1)\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2} d\tilde{\chi}_2 \frac{(\chi_2-\tilde{\chi}_2)}{\chi_2\tilde{\chi}_2}3\Omega_m(\tilde{z}_2)\mathcal{H}^2(\tilde{z}_2)\\
&\times\Bigg\{-\frac{b_1}{a^4}\delta_D(\chi_1-\tilde{\chi}_2)P\left(\frac{a}{\chi_1}\right)+\frac{b_1\gamma_1\chi_1^2}{a^4}\delta_D(\chi_1-\tilde{\chi}_2)P\left(\frac{a}{\chi_1}\right)\\
&+\frac{b_1\alpha_1\beta_1}{\chi_1}\Bigg[\frac{\chi_1}{2}\delta_D(\chi_1-\tilde{\chi}_2)\frac{\sqrt{a}}{a^3}\left(\frac{f_{-1}}{\sqrt{a-1}}+\frac{f_1}{\sqrt{a+1}}\right)P\left(\frac{a}{\chi_1}\right)\\
&+\frac{\chi_1}{2}\delta_D(\chi_1-\tilde{\chi}_2)\frac{1}{\sqrt{a}}\left(f_{-1}\frac{\sqrt{a-1}}{\sqrt{(a-1)^3}}P\left(\frac{a-1}{\chi_1}\right)+f_1\frac{\sqrt{a+1}}{(a+1)^3}P\left(\frac{a+1}{\chi_1}\right)\right)\Bigg]\\
&+b_1\beta_1\Bigg[\frac{1}{2}\delta_D(\chi_1-\tilde{\chi}_2)\frac{\sqrt{a}}{a^2}\left(\frac{f_{-2}}{\sqrt{a-2}}+\frac{2f_0}{\sqrt{a}}+\frac{f_2}{\sqrt{a+2}}\right)P\left(\frac{a}{\chi_1}\right)\\
&+\frac{1}{2}\delta_D(\chi_1-\tilde{\chi}_2)\frac{1}{\sqrt{a}}\left(f_{-2}\frac{\sqrt{a-2}}{(a-2)^2}P\left(\frac{a-2}{\chi_1}\right)+f_2\frac{\sqrt{a+2}}{(a+2)^2}P\left(\frac{a+2}{\chi_1}\right)\right)\Bigg]\Bigg\}.
\end{aligned}
\tag{4.56}
$$

Then:

$$
\begin{aligned}
&\Delta^{S*}_{\ell m}\Delta^{K}_{\ell' m'} = \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\\
&\times \ell(\ell+1)\big[1-\mathcal{Q}(z_2)\big]\frac{(\chi_2-\chi_1)}{\chi_2\chi_1}3\Omega_m(z_1)\mathcal{H}^2(z_1)b_1\\
&\times\Bigg\{-\frac{1}{\left(\ell+\frac{1}{2}\right)^4}P\left(\frac{\ell+\frac{1}{2}}{\chi_1}\right)+\frac{\gamma_1\chi_1^2}{\left(\ell+\frac{1}{2}\right)^4}P\left(\frac{\ell+\frac{1}{2}}{\chi_1}\right)\\
&+\frac{\alpha_1\beta_1}{2}\Bigg[\frac{\sqrt{a}}{a^3}\left(\frac{f_{-1}}{\sqrt{a-1}}+\frac{f_1}{\sqrt{a+1}}\right)P\left(\frac{a}{\chi_1}\right)\\
&+\frac{1}{\sqrt{\ell+\frac{1}{2}}}\left(f_{-1}\frac{\sqrt{\ell-\frac{1}{2}}}{\sqrt{\left(\ell-\frac{1}{2}\right)^3}}P\left(\frac{\ell-\frac{1}{2}}{\chi_1}\right)+f_1\frac{\sqrt{\ell+\frac{3}{2}}}{(\ell+\frac{3}{2})^3}P\left(\frac{\ell+\frac{3}{2}}{\chi_1}\right)\right)\Bigg]\\
&+\frac{\beta_1}{2}\Bigg[\frac{\sqrt{\ell+\frac{1}{2}}}{\left(\ell+\frac{1}{2}\right)^2}\left(\frac{f_{-2}}{\sqrt{\ell-\frac{3}{2}}}+\frac{2f_0}{\sqrt{\ell+\frac{1}{2}}}+\frac{f_2}{\sqrt{\ell+\frac{5}{2}}}\right)P\left(\frac{\ell+\frac{1}{2}}{\chi_1}\right)\\
&+\frac{1}{\sqrt{\ell+\frac{1}{2}}}\left(f_{-2}\frac{\sqrt{\ell-\frac{3}{2}}}{\left(\ell-\frac{3}{2}\right)^2}P\left(\frac{\ell-\frac{3}{2}}{\chi_1}\right)+f_2\frac{\sqrt{\ell+\frac{5}{2}}}{\left(\ell+\frac{5}{2}\right)^2}P\left(\frac{\ell+\frac{5}{2}}{\chi_1}\right)\right)\Bigg]\Bigg\}.
\end{aligned}
\tag{4.57}
$$

(b) $\boldsymbol{\Delta^{S*}_{\ell m}\Delta^{I}_{\ell' m'}}$

$$
\begin{aligned}
\Delta^{S*}_{\ell m}\Delta^{I}_{\ell' m'} &= \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\, b_1 \bigg\{\frac{2}{\chi_2}\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2} d\tilde{\chi}_2\, 3\Omega_m(\tilde{z}_2)\mathcal{H}^2(\tilde{z}_2) \\
&+ \frac{1}{\chi_2}\frac{\alpha(z_2)}{H(z_2)}\int_0^{\chi_2} d\tilde{\chi}_2\, 3\Omega_m(\tilde{z}_2)\mathcal{H}^3(\tilde{z}_2)\big[f(\tilde{z}_2)-1\big]\bigg\}\bigg\{-\frac{\delta_D(\chi_1-\tilde{\chi}_2)}{a^2}P\Big(\frac{a}{\chi_1}\Big) \\
&+ \frac{\gamma_1\chi_1^2}{a^4}\delta_D(\chi_1-\tilde{\chi}_2)P\Big(\frac{a}{\chi_1}\Big) + \frac{\alpha_1\beta_1}{\chi_1}\bigg[\frac{\chi_1}{2}\delta_D(\chi_1-\tilde{\chi}_2)\frac{\sqrt{a}}{a^3}\Big(\frac{f_{-1}}{\sqrt{a-1}}+\frac{f_1}{a+1}\Big)P\Big(\frac{a}{\chi_1}\Big) \\
&+ \frac{\chi_1}{2}\delta_D(\chi_1-\tilde{\chi}_2)\frac{1}{\sqrt{a}}\bigg(f_{-1}\frac{\sqrt{a-1}}{(a-1)^3}P\Big(\frac{a-1}{\chi_1}\Big)+f_1\frac{\sqrt{a+1}}{(a+1)^3}P\Big(\frac{a+1}{\chi_1}\Big)\bigg)\bigg] \\
&+ \frac{\beta_1}{2}\bigg[\delta_D(\chi_1-\tilde{\chi}_2)\frac{\sqrt{a}}{a^2}\Big(\frac{f_2}{\sqrt{a-2}}+\frac{2f_0}{\sqrt{a}}+\frac{f_2}{\sqrt{a+2}}\Big)P\Big(\frac{a}{\chi_1}\Big) \\
&+ \delta_D(\chi_1-\tilde{\chi}_2)\frac{1}{\sqrt{a}}\bigg(f_{-2}\frac{\sqrt{a-2}}{(a-2)^2}P\Big(\frac{a-2}{\chi_1}\Big)+f_2\frac{\sqrt{a+2}}{(a+2)^2}P\Big(\frac{a+2}{\chi_1}\Big)\bigg)\bigg]\bigg\}. \qquad (4.58)
\end{aligned}
$$

Then:

$$
\begin{aligned}
\Delta^{S*}_{\ell m}\Delta^{I}_{\ell' m'} &= \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\, b_1 \bigg\{\frac{1}{\chi_2}\big[3\Omega_m(z_1)\mathcal{H}^2(z_1)\big]\Big[2\big[1-\mathcal{Q}(z_2)\big] \\
&+ \frac{\alpha(z_2)}{H(z_2)}\mathcal{H}(z_1)\big[f(z_1)-1\big]\bigg\}\bigg\{-\frac{1}{\big(\ell+\frac12\big)^2}P\bigg(\frac{\ell+\frac12}{\chi_1}\bigg)+\frac{\gamma_1\chi_1^2}{\big(\ell+\frac12\big)^4}P\bigg(\frac{\ell+\frac12}{\chi_1}\bigg) \\
&+ \frac{\alpha_1\beta_1}{2}\bigg[\frac{\sqrt{\ell+\frac12}}{\big(\ell+\frac12\big)^3}\bigg(\frac{f_{-1}}{\sqrt{\ell-\frac12}}+\frac{f_1}{\sqrt{\ell+\frac32}}\bigg)P\bigg(\frac{\ell+\frac12}{\chi_1}\bigg) \\
&\qquad + \frac{1}{\sqrt{\ell+\frac12}}\bigg(f_{-1}\frac{\sqrt{\ell-\frac12}}{\big(\ell-\frac12\big)^3}P\bigg(\frac{\ell-\frac12}{\chi_1}\bigg)+f_1\frac{\sqrt{\ell+\frac32}}{\big(\ell+\frac32\big)^3}P\bigg(\frac{\ell+\frac32}{\chi_1}\bigg)\bigg)\bigg] \\
&+ \frac{\beta_1}{2}\bigg[\frac{\sqrt{\ell+\frac12}}{\big(\ell+\frac12\big)^2}\bigg(\frac{f_2}{\sqrt{\ell-\frac32}}+\frac{2f_0}{\sqrt{\ell+\frac12}}+\frac{f_2}{\sqrt{\ell+\frac52}}\bigg)P\bigg(\frac{\ell+\frac12}{\chi_1}\bigg) \\
&\quad + \frac{1}{\sqrt{\ell+\frac12}}\bigg(f_{-2}\frac{\sqrt{\ell-\frac32}}{\big(\ell-\frac32\big)^2}P\bigg(\frac{\ell-\frac32}{\chi_1}\bigg)+f_2\frac{\sqrt{\ell+\frac52}}{\big(\ell+\frac52\big)^2}P\bigg(\frac{\ell+\frac52}{\chi_1}\bigg)\bigg)\bigg]\bigg\}. \qquad (4.59)
\end{aligned}
$$

(c) $\Delta^{K*}_{\ell m}\Delta^{S}_{\ell' m'}$

$$
\begin{aligned}
\Delta^{K*}_{\ell m}\Delta^{S}_{\ell' m'} &= \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2) \\
&\times \ell(\ell+1)b_2\big[1-\mathcal{Q}(z_1)\big]\int_0^{\chi_1} d\tilde{\chi}_1 \frac{(\chi_1-\tilde{\chi}_1)}{\chi_1\tilde{\chi}_1} 3\Omega_m(\tilde{z}_1)\mathcal{H}^2(\tilde{z}_1) \\
&\times \Bigg\{-\frac{\delta_D(\tilde{\chi}_1-\chi_2)}{a^2}P\Big(\frac{a}{\tilde{\chi}_1}\Big) + \frac{\gamma_2\tilde{\chi}_1^2}{a^4}\delta_D(\tilde{\chi}_1-\chi_2)P\Big(\frac{a}{\tilde{\chi}_1}\Big) \\
&+ \frac{\alpha_2\beta_2}{\chi_2}\Bigg[\frac{\tilde{\chi}_1}{2}\delta_D(\tilde{\chi}_1-\chi_2)\frac{\sqrt{a}}{a^3}\Big(\frac{f_{-1}}{\sqrt{a-1}}+\frac{f_1}{a+1}\Big)P\Big(\frac{a}{\tilde{\chi}_1}\Big) \\
&\qquad + \frac{\tilde{\chi}_1}{2}\delta_D(\tilde{\chi}_1-\chi_2)\frac{1}{\sqrt{a}}\Bigg(f_{-1}\frac{\sqrt{a-1}}{(a-1)^3}P\Big(\frac{a-1}{\tilde{\chi}_1}\Big) + f_1\frac{\sqrt{a+1}}{(a+1)^3}P\Big(\frac{a+1}{\tilde{\chi}_1}\Big)\Bigg)\Bigg] \\
&+ \frac{\beta_2}{2}\Bigg[\delta_D(\tilde{\chi}_1-\chi_2)\frac{\sqrt{a}}{a^2}\Big(\frac{f_2}{\sqrt{a-2}}+\frac{2f_0}{\sqrt{a}}+\frac{f_2}{\sqrt{a+2}}\Big)P\Big(\frac{a}{\tilde{\chi}_1}\Big) \\
&\qquad + \delta_D(\tilde{\chi}_1-\chi_2)\frac{1}{\sqrt{a}}\Bigg(f_{-2}\frac{\sqrt{a-2}}{(a-2)^2}P\Big(\frac{a-2}{\tilde{\chi}_1}\Big) + f_2\frac{\sqrt{a+2}}{(a+2)^2}P\Big(\frac{a+2}{\tilde{\chi}_1}\Big)\Bigg)\Bigg]\Bigg\}.
\end{aligned}
\tag{4.60}
$$

Then:

$$
\begin{aligned}
\Delta^{K*}_{\ell m}\Delta^{S}_{\ell' m'} &= \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2) \\
&\times \ell(\ell+1)b_2\big[1-\mathcal{Q}(z_1)\big]\frac{(\chi_1-\chi_2)}{\chi_1\chi_2} 3\Omega_m(z_2)\mathcal{H}^2(z_2) \\
&\times \Bigg\{-\frac{1}{\left(\ell+\frac{1}{2}\right)^2}P\Big(\frac{\ell+\frac{1}{2}}{\chi_2}\Big) + \frac{\gamma_2\chi_2^2}{\left(\ell+\frac{1}{2}\right)^4}P\Big(\frac{\ell+\frac{1}{2}}{\chi_2}\Big) \\
&+ \frac{\alpha_2\beta_2}{2}\Bigg[\frac{\sqrt{\ell+\frac{1}{2}}}{\left(\ell+\frac{1}{2}\right)^3}\Bigg(\frac{f_{-1}}{\sqrt{\ell-\frac{1}{2}}}+\frac{f_1}{\ell+\frac{3}{2}}\Bigg)P\Big(\frac{\ell+\frac{1}{2}}{\chi_2}\Big) \\
&\qquad + \frac{1}{\sqrt{\ell+\frac{1}{2}}}\Bigg(f_{-1}\frac{\sqrt{\ell-\frac{1}{2}}}{\left(\ell-\frac{1}{2}\right)^3}P\Big(\frac{\ell-\frac{1}{2}}{\chi_2}\Big) + f_1\frac{\sqrt{\ell+\frac{3}{2}}}{\left(\ell+\frac{3}{2}\right)^3}P\Big(\frac{\ell+\frac{3}{2}}{\chi_2}\Big)\Bigg)\Bigg] \\
&+ \frac{\beta_2}{2}\Bigg[\frac{\sqrt{\ell+\frac{1}{2}}}{\left(\ell+\frac{1}{2}\right)^2}\Bigg(\frac{f_2}{\sqrt{\ell-\frac{3}{2}}}+\frac{2f_0}{\sqrt{\ell+\frac{1}{2}}}+\frac{f_2}{\sqrt{\ell+\frac{5}{2}}}\Bigg)P\Big(\frac{\ell+\frac{1}{2}}{\chi_2}\Big) \\
&\qquad + \frac{1}{\sqrt{\ell+\frac{1}{2}}}\Bigg(f_{-2}\frac{\sqrt{\ell-\frac{3}{2}}}{\left(\ell-\frac{3}{2}\right)^2}P\Big(\frac{\ell-\frac{3}{2}}{\chi_2}\Big) + f_2\frac{\sqrt{\ell+\frac{5}{2}}}{\left(\ell+\frac{5}{2}\right)^2}P\Big(\frac{\ell+\frac{5}{2}}{\chi_2}\Big)\Bigg)\Bigg]\Bigg\}.
\end{aligned}
\tag{4.61}
$$

(d) $\boldsymbol{\Delta^{K*}_{\ell m}\Delta^{I}_{\ell' m'}}$

$$\Delta^{K*}_{\ell m}\Delta^{I}_{\ell' m'} = \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\,\ell(\ell+1)\big[1-\mathcal{Q}(z_1)\big]$$
$$\times\int_0^{\chi_1} d\tilde\chi_1\frac{\big(\chi_1-\tilde\chi_1\big)}{\chi_1\tilde\chi_1}3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1)\bigg\{\frac{2}{\chi_2}\big[1-\mathcal{Q}(z_2)\big]\int_0^{\chi_2} d\tilde\chi_2\,3\Omega_m(\tilde z_2)\mathcal{H}^2(\tilde z_2)$$
$$+\frac{1}{\chi_2}\frac{\alpha(z_2)}{H(z_2)}\int_0^{\chi_2} d\tilde\chi_2\,3\Omega_m(\tilde z_2)\mathcal{H}^3(\tilde z_2)\big[f(\tilde z_2)-1\big]\bigg\}\frac{\tilde\chi_1^2}{a^4}\delta_D(\tilde\chi_1-\tilde\chi_2)P\bigg(\frac{a}{\tilde\chi_1}\bigg)\,. \quad (4.62)$$

Then:

$$\Delta^{K*}_{\ell m}\Delta^{I}_{\ell' m'} = \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\,\ell(\ell+1)\big[1-\mathcal{Q}(z_1)\big]$$
$$\times\int_0^{\chi_1} d\tilde\chi_1\frac{\big(\chi_1-\tilde\chi_1\big)}{\chi_1\tilde\chi_1}3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1)\bigg\{\frac{1}{\chi_2}3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1)\bigg[2\big[1-\mathcal{Q}(z_2)\big]$$
$$+\frac{\alpha(z_2)}{H(z_2)}\mathcal{H}(\tilde z_1)\big[f(\tilde z_1)-1\big]\bigg]\bigg\}\frac{\tilde\chi_1^2}{\big(\ell+\frac{1}{2}\big)^4}P\bigg(\frac{\ell+\frac{1}{2}}{\tilde\chi_1}\bigg)\,. \quad (4.63)$$

(e) $\boldsymbol{\Delta^{I*}_{\ell m}\Delta^{S}_{\ell' m'}}$

$$\Delta^{I*}_{\ell m}\Delta^{S}_{\ell' m'} = \int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\,b_2\bigg\{\frac{2}{\chi_1}\big[1-\mathcal{Q}(z_1)\big]\int_0^{\chi_1} d\tilde\chi_1\,3\Omega_m(\tilde z_1)\mathcal{H}^2(\tilde z_1)$$
$$+\frac{1}{\chi_1}\frac{\alpha(z_1)}{H(z_1)}\int_0^{\chi_1} d\tilde\chi_1\,3\Omega_m(\tilde z_1)\mathcal{H}^3(\tilde z_1)\big[f(\tilde z_1)-1\big]\bigg\}$$
$$\times\bigg\{-\frac{\delta_D(\tilde\chi_1-\chi_2)}{a^2}P\bigg(\frac{a}{\tilde\chi_1}\bigg)+\frac{\gamma_2\tilde\chi_1^2}{a^4}\delta_D(\tilde\chi_1-\chi_2)P\bigg(\frac{a}{\tilde\chi_1}\bigg)$$
$$+\frac{\alpha_2\beta_2}{\chi_2}\bigg[\frac{\tilde\chi_1}{2}\delta_D(\tilde\chi_1-\chi_2)\frac{\sqrt{a}}{a^3}\bigg(\frac{f_{-1}}{\sqrt{a-1}}+\frac{f_1}{a+1}\bigg)P\bigg(\frac{a}{\tilde\chi_1}\bigg)$$
$$+\frac{\tilde\chi_1}{2}\delta_D(\tilde\chi_1-\chi_2)\frac{1}{\sqrt{a}}\bigg(f_{-1}\frac{\sqrt{a-1}}{(a-1)^3}P\bigg(\frac{a-1}{\tilde\chi_1}\bigg)+f_1\frac{\sqrt{a+1}}{(a+1)^3}P\bigg(\frac{a+1}{\tilde\chi_1}\bigg)\bigg)\bigg]$$
$$+\frac{\beta_2}{2}\bigg[\delta_D(\tilde\chi_1-\chi_2)\frac{\sqrt{a}}{a^2}\bigg(\frac{f_2}{\sqrt{a-2}}+\frac{2f_0}{\sqrt{a}}+\frac{f_2}{\sqrt{a+2}}\bigg)P\bigg(\frac{a}{\tilde\chi_1}\bigg)$$
$$+\delta_D(\tilde\chi_1-\chi_2)\frac{1}{\sqrt{a}}\bigg(f_{-2}\frac{\sqrt{a-2}}{(a-2)^2}P\bigg(\frac{a-2}{\tilde\chi_1}\bigg)+f_2\frac{\sqrt{a+2}}{(a+2)^2}P\bigg(\frac{a+2}{\tilde\chi_1}\bigg)\bigg)\bigg]\bigg\}\,. \quad (4.64)$$

Then:

$$\begin{aligned}
\Delta^{I*}_{\ell m}\Delta^{S}_{\ell' m'} = &\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\, b_2\bigg\{\frac{2}{\chi_1}\big[1-\mathcal{Q}(z_1)\big]3\Omega_m(z_2)\mathcal{H}^2(z_2) \\
&\quad + \frac{1}{\chi_1}\frac{\alpha(z_1)}{H(z_1)}3\Omega_m(z_2)\mathcal{H}^3(z_2)\big[f(z_2)-1\big]\bigg\} \\
&\times\left\{-\frac{1}{\left(\ell+\frac{1}{2}\right)^2}P\left(\frac{\ell+\frac{1}{2}}{\chi_2}\right)+\frac{\gamma_2\chi_2^2}{\left(\ell+\frac{1}{2}\right)^4}P\left(\frac{\ell+\frac{1}{2}}{\chi_2}\right)\right. \\
&+\frac{\alpha_2\beta_2}{2}\left[\frac{\sqrt{\ell+\frac{1}{2}}}{\left(\ell+\frac{1}{2}\right)^3}\left(\frac{f_{-1}}{\sqrt{\ell-\frac{1}{2}}}+\frac{f_1}{\ell+\frac{3}{2}}\right)P\left(\frac{\ell+\frac{1}{2}}{\chi_2}\right)\right. \\
&\qquad\left.+\frac{1}{\sqrt{\ell+\frac{1}{2}}}\left(f_{-1}\frac{\sqrt{\ell-\frac{1}{2}}}{\left(\ell-\frac{1}{2}\right)^3}P\left(\frac{\ell-\frac{1}{2}}{\chi_2}\right)+f_1\frac{\sqrt{\ell+\frac{3}{2}}}{\left(\ell+\frac{3}{2}\right)^3}P\left(\frac{\ell+\frac{3}{2}}{\chi_2}\right)\right)\right] \\
&+\frac{\beta_2}{2}\left[\frac{\sqrt{\ell+\frac{1}{2}}}{\left(\ell+\frac{1}{2}\right)^2}\left(\frac{f_2}{\sqrt{\ell-\frac{3}{2}}}+\frac{2f_0}{\sqrt{\ell+\frac{1}{2}}}+\frac{f_2}{\sqrt{\ell+\frac{5}{2}}}\right)P\left(\frac{\ell+\frac{1}{2}}{\chi_2}\right)\right. \\
&\qquad\left.\left.+\frac{1}{\sqrt{\ell+\frac{1}{2}}}\left(f_{-2}\frac{\sqrt{\ell-\frac{3}{2}}}{\left(\ell-\frac{3}{2}\right)^2}P\left(\frac{\ell-\frac{3}{2}}{\chi_2}\right)+f_2\frac{\sqrt{\ell+\frac{5}{2}}}{\left(\ell+\frac{5}{2}\right)^2}P\left(\frac{\ell+\frac{5}{2}}{\chi_2}\right)\right)\right]\right\}. \qquad (4.65)
\end{aligned}$$

(f) $\Delta^{I*}_{\ell m}\Delta^{K}_{\ell' m'}$

$$\begin{aligned}
\Delta^{I*}_{\ell m}\Delta^{K}_{\ell' m'} = &\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\ell(\ell+1)\big[1-\mathcal{Q}(z_2)\big]\times \\
&\left\{\frac{2}{\chi_1}\big[1-\mathcal{Q}(z_1)\big]\int_0^{\chi_1}d\tilde{\chi}_1\, 3\Omega_m(\tilde{z}_1)\mathcal{H}^2(\tilde{z}_1)+\frac{1}{\chi_1}\frac{\alpha(z_1)}{H(z_1)}\int_0^{\chi_1}d\tilde{\chi}_1\, 3\Omega_m(\tilde{z}_1)\mathcal{H}^3(\tilde{z}_1)\big[f(\tilde{z}_1)-1\big]\right\} \\
&\times\int_0^{\chi_2}d\tilde{\chi}_2\frac{(\chi_2-\tilde{\chi}_2)}{\chi_2\tilde{\chi}_2}3\Omega_m(\tilde{z}_2)\mathcal{H}^2(\tilde{z}_2)\,\frac{\tilde{\chi}_1^2}{a^4}\delta_D(\tilde{\chi}_1-\tilde{\chi}_2)P\left(\frac{a}{\tilde{\chi}_1}\right). \qquad (4.66)
\end{aligned}$$

Then:

$$\begin{aligned}
\Delta^{I*}_{\ell m}\Delta^{K}_{\ell' m'} = &\int_0^\infty d\chi_1 W(\chi_1)\int_0^\infty d\chi_2 W(\chi_2)\ell(\ell+1)\big[1-\mathcal{Q}(z_2)\big]\times \\
&\left\{\frac{2}{\chi_1}\big[1-\mathcal{Q}(z_1)\big]\int_0^{\chi_1}d\tilde{\chi}_1\, 3\Omega_m(\tilde{z}_1)\mathcal{H}^2(\tilde{z}_1)+\frac{1}{\chi_1}\frac{\alpha(z_1)}{H(z_1)}\int_0^{\chi_1}d\tilde{\chi}_1\, 3\Omega_m(\tilde{z}_1)\mathcal{H}^3(\tilde{z}_1)\big[f(\tilde{z}_1)-1\big]\right\} \\
&\times\frac{(\chi_2-\tilde{\chi}_1)}{\chi_2\tilde{\chi}_1}3\Omega_m(\tilde{z}_1)\mathcal{H}^2(\tilde{z}_1)\,\frac{\tilde{\chi}_1^2}{\left(\ell+\frac{1}{2}\right)^4}P\left(\frac{\ell+\frac{1}{2}}{\tilde{\chi}_1}\right). \qquad (4.67)
\end{aligned}$$

Chapter 5

Conclusion

5.1 Overview

In this thesis project a general relativistic approach to large scale galaxy clustering has been studied. *General relativity* is necessary because for relativistic fluids and for scales of order $O(1/H)$ and larger relativistic effects arise and can't be neglected. *Galaxy clustering* is a statistical analysis of the distribution of galaxies at present time, measured from the angular positions of galaxies in the sky and the redshifts of the galaxies.
We started from the basic assumption of the *ΛCDM model*, which is consistent with the latest experimental observation (e.g. Planck), allowing to reliably constrain cosmological parameters.
After giving the necessary theoretical tools, we focused on our observational data: the *direction* and the *redshift* of the photon from a certain galaxy. We *perturbed photon geodesics* in order to take into account of the relativistic effects of distortion of the trajectory. Then we computed the *comoving galaxy number density* as a function of the observed position and redshift, and finally we found the expression for the *observed galaxy overdensity* Δ_g, considering also the effect of magnification. Our observable Δ_g has three main contributions: the *local* Δ_S (evaluated at the source), the *weak lensing* Δ_K and the *time delay* term Δ_I.
Then we got into the statistical issue, using the *two points correlation function* with the observables Δ_g just computed. In particular, due to the spherical symmetry, the expansion in the spherical harmonic space has been performed, thus shifting from the power spectrum $P(k)$ to the *angular power spectrum* C_ℓ. These coefficients are computed in the redshift space for all the three contributions of local, weak lensing and time delay, including also all the mixed terms deriving from the product of the two deltas. In order to include all of these terms, *new* definitions for the angular correlation functions $w^{\nu}_{\ell,jj'}$ have been introduced, in order to generalise that reported in [33]. The final results are very big and complex coefficients where integrals of the product of spherical Bessel functions appear; their highly oscillatory nature represents a challenge in the direct integration, since there are convergence problems for large multipoles ℓ and for large arguments $k\chi$.
A simplification that can be applied to the full analytic expression of the C_ℓ coefficients is the *Limber's approximation*, assuming large multipoles ℓ and that the power spectrum $P(k)$ varies much more slowly than the spherical Bessel functions in their product inside the various integrals. This approximation allows to simplify the final results obtained previously.

5.2 Future perspectives

One possible operative approach is to compute the $w^{\nu}_{\ell,jj'}$ integrals (both approximated or not) numerically with some algorithm that can circumvent the direct integration of the highly oscillating spherical Bessel functions. Recently the 2-FAST algorithm has been developed with that goal [33]; it employs the FFTLOG transformation of the power spectrum to divide the computation of the integrals into: coefficients depending on $P(k)$, and integrations independent on $P(k)$ of basis functions multiplied by spherical Bessel functions. For these latter integrals analytical expressions are used, in terms of particular functions for which recursion provides a fast and accurate evaluation.[33]
This algorithm has only been applied to the $w^{0}_{\ell,jj'}$ integrals but not to the generalization $w^{\nu}_{\ell,jj'}$ where an extra factor $1/k^{\nu}$ is involved inside the integrals. A straightforward challenge would be to apply the 2FAST algorithm to the generalization and results reported in this thesis, in order to have plots that can show in a more explicit way the behaviour of the various analytic expressions found, and in order to compare these results, which include all the general relativistic effects, with those already presented in literature. In general, besides the 2FAST algorithm and the others that have preceded it, a numerical approach is needed. Due to the complexity of the computation, an important issue would be also to evaluate the efficiency of the algorithm in terms of computational cost.

Appendix A

Mathematical tools

A.1 Useful definitions and properties

In our analysis we often prefer to work with projected quantities, in order to identify the various contributes along and perpendicular with respect to the line of sight. Imagine to consider a three dimensional cartesian reference frame in which:

- the z axis connects us (the observers) to the starting point of the photon we detect, thus defining the so-called *line of sight* which is basically identified with the direction $\hat{\boldsymbol{n}}$;

- the x-y plane is perpendicular to the line of sight and passes through the source position.

Many quantities we work with, such as derivative operators, vectors and tensors, can be defined and projected along or perpendicular to the line of sight. If we consider a generic spatial vector A^i and a tensor B^{ij}, the following projections can be defined:

$$A_{\parallel} \equiv \hat{n}_i A^i \qquad\qquad A^i_{\perp} \equiv A^i - A_{\parallel} \equiv (\delta^i_j - \hat{n}^i \hat{n}_j) A^j \tag{A.1}$$

$$B_{\parallel} \equiv \hat{n}_i \hat{n}_j B^{ij} \qquad\qquad B_{\perp} \equiv B - B_{\parallel} \equiv (\delta_{ij} - \hat{n}_i \hat{n}_j) B^{ij} \tag{A.2}$$

The same can be done for the derivative operators:

$$\partial^i_{\parallel} \equiv \hat{n}^i \hat{n}^j \partial_j \tag{A.3}$$

$$\partial_{\parallel} \equiv \hat{n}^i \partial_i \tag{A.4}$$

$$\partial^i_{\perp} \equiv \partial^i - \partial^i_{\parallel} \equiv (\delta^{ij} - \hat{n}^i \hat{n}^j) \partial_j \tag{A.5}$$

At this point, many useful relations can be specified:

$$\partial_j \hat{n}^i = \tilde{\chi}^{-1}\big(\delta^i_j - \hat{n}^i \hat{n}_j\big) \tag{A.6}$$

Demonstration of (A.6): $\hat{\boldsymbol{n}} = \tilde{\chi}^{-1} \tilde{\boldsymbol{x}}$, so $\partial_j \hat{n}^i = \tilde{\chi}^{-1} \delta^i_j - \tilde{\chi}^{-2} \partial_j \tilde{\chi}\, \tilde{x}^i$. In order to calculate $\partial_j \tilde{\chi}$ we can observe that: $\frac{\partial \tilde{\chi}}{\partial x^j} \cdot \frac{\partial x^j}{\partial \tilde{\chi}} = 1 \;\rightarrow\; \partial_j \tilde{\chi} \cdot \tilde{n}^j = 1 \;\Rightarrow\; \partial_j \tilde{\chi} = \tilde{n}_j$ since $|\tilde{\boldsymbol{n}}|^2 = 1$. Therefore $\partial_j \hat{n}^i = \tilde{\chi}^{-1}\big(\delta^i_j - \partial_j \tilde{\chi}\, \tilde{n}^i\big) = \tilde{\chi}^{-1}\big(\delta^i_j - \tilde{n}^i\, \tilde{n}_j\big)$.

$$\big[\partial_i, \hat{n}_j\big] = \partial_i \hat{n}_j = \tilde{\chi}^{-1}\big(\delta_{ij} - \hat{n}_i \hat{n}_j\big) \tag{A.7}$$

Demonstration of (A.7): $\big[\partial_i, \hat{n}_j\big] = \partial_i \hat{n}_j - \hat{n}_j \partial_i = \tilde{\chi}^{-1}\big(\delta_{ij} - \hat{n}_i \hat{n}_j\big) - \hat{n}_j \partial_i \hat{n}^j \hat{n}_j = \tilde{\chi}^{-1}\Big(\big(\delta_{ij} - \hat{n}_i \hat{n}_j\big) - \hat{n}_j \big(\delta^j_i - \hat{n}_i \hat{n}^j\big) \hat{n}_j\Big) = \tilde{\chi}^{-1}\big(\delta_{ij} - \hat{n}_i \hat{n}_j - \hat{n}_i \hat{n}_j + \hat{n}_i \hat{n}_j\big) = \tilde{\chi}^{-1}\big(\delta_{ij} - \hat{n}_i \hat{n}_j\big) = \partial_i \hat{n}_j$.

$$\big[\partial_{\parallel}, \hat{n}_i\big] = 0 \tag{A.8}$$

Demonstration of (A.8): $\left[\partial_\parallel, \hat{n}_i\right] = \left[\hat{n}^j \partial_j, \hat{n}_i\right] = \hat{n}^j \left[\partial_j, \hat{n}_i\right] + \left[\hat{n}^j, \hat{n}_i\right]\partial_j = \hat{n}^j \Big(\tilde{\chi}^{-1}\big(\delta_{ij} - \hat{n}_i\hat{n}_j\big)\Big) + 0 = \tilde{\chi}^{-1}\big(\hat{n}_i - \hat{n}_i\big) = 0$

$$\left[\partial_i, \partial_\parallel\right] = \tilde{\chi}^{-1}\partial_{\perp i} \tag{A.9}$$

Demonstration of (A.9): $\left[\partial_i, \partial_\parallel\right] = \left[\partial_i, \hat{n}^j\partial_j\right] = \left[\partial_i, \hat{n}^j\right]\partial_j + \left[\partial_i, \partial_j\right]\hat{n}^j = \tilde{\chi}^{-1}\big(\delta_i^j - \hat{n}_i\hat{n}^j\big)\partial_j + 0 = \tilde{\chi}^{-1}\big(\partial_i - \partial_{\parallel i}\big) = \tilde{\chi}^{-1}\partial_{\perp i}$.

$$\left[\partial_\parallel, \partial_{\parallel j}\right] = \left[\partial_{\parallel i}, \partial_{\parallel j}\right] = 0 \tag{A.10}$$

Demonstration of (A.10): $\left[\partial_\parallel, \partial_{\parallel j}\right] = \left[\partial_\parallel, \hat{n}_j\partial_\parallel\right] = \left[\partial_\parallel, \hat{n}_j\right]\partial_\parallel + \left[\partial_\parallel, \partial_\parallel\right]\hat{n}_j = 0$ because of (A.8). $\left[\partial_{\parallel i}, \partial_{\parallel j}\right] = \left[\hat{n}_i\partial_\parallel, \hat{n}_j\partial_\parallel\right] = \left[\hat{n}_i\partial_\parallel, \hat{n}_j\right]\partial_\parallel + \left[\hat{n}_i\partial_\parallel, \partial_\parallel\right]\hat{n}_j = \big(\hat{n}_i\partial_\parallel\hat{n}_j - \hat{n}_j\hat{n}_i\partial_\parallel\big)\partial_\parallel + \left[\partial_{\parallel i}, \partial_\parallel\right]\hat{n}_j$. The first term is zero due to (A.8): $\hat{n}_i\partial_\parallel\hat{n}_j - \hat{n}_j\hat{n}_i\partial_\parallel = \hat{n}_i\hat{n}_j\partial_\parallel - \hat{n}_i\hat{n}_j\partial_\parallel = 0$. The second term is also zero due to the first part of this demonstration. Therefore $\left[\partial_{\parallel i}, \partial_{\parallel j}\right] = 0$.

$$\left[\partial_{\perp i}, \partial_\parallel\right] = \tilde{\chi}^{-1}\partial_{\perp i} \tag{A.11}$$

Demonstration of (A.11): $\left[\partial_{\perp i}, \partial_\parallel\right] = \left[\partial_i - \partial_{\parallel i}, \partial_\parallel\right] = \left[\partial_i, \partial_\parallel\right] - \left[\partial_{\parallel i}, \partial_\parallel\right] = \tilde{\chi}^{-1}\partial_{\perp i} - 0 = \tilde{\chi}^{-1}\partial_{\perp i}$ by using (A.9).
We can also project the Laplacian operator $\nabla^2 = \partial_i\partial^i$, thus defining:

$$\partial_\parallel^2 \equiv \partial_{\parallel i}\partial_\parallel^i = \partial_\parallel\partial_\parallel \tag{A.12}$$

$$\nabla_\perp^2 \equiv \partial_{\perp i}\partial_\perp^i = \nabla^2 - \partial_\parallel^2 - \frac{2}{\chi}\partial_\parallel \tag{A.13}$$

Demonstration of (A.1): $\nabla_\perp^2 = \partial_{\perp i}\partial_\perp^i = \big(\partial_i - \hat{n}_i\partial_\parallel\big)\big(\partial^i - \hat{n}^i\partial_\parallel\big) = \partial_i\partial^i - \partial_i\hat{n}^i\partial_\parallel - \hat{n}_i\partial_\parallel\partial^i + \hat{n}_i\partial_\parallel\hat{n}^i\partial_\parallel = \nabla^2 - \tilde{\chi}^{-1}\big(\delta_i^i - \hat{n}^i\hat{n}_i\big)\partial_\parallel - \partial_\parallel\hat{n}_i\partial^i + \hat{n}_i\hat{n}^i\partial_\parallel^2 = \nabla^2 - \tilde{\chi}^{-1}(3-1)\partial_\parallel - \partial_\parallel^2$.

$$\partial_i A^i = \partial_\parallel A_\parallel + \partial_{\perp i}A_\perp^i + A_\parallel\partial_i\hat{n}^i \tag{A.14}$$

$$\partial_\parallel\hat{n}^i = \hat{n}^i\partial_{\perp i} = 0 \tag{A.15}$$

A.2 Derivation of Eq. (2.26) and Eq. (2.27)

(2.26) : $\frac{d^2\delta x^0}{d\chi^2} = -\delta\hat{\Gamma}^0_{\nu\rho}\frac{dx^\nu}{d\chi}\frac{dx^\rho}{d\chi} - 2\hat{\Gamma}^0_{\nu\rho}\frac{d\delta x^\nu}{d\chi}\frac{dx^\rho}{d\chi}$

The second term is zero due to the fact that the unperturbed Christoffel symbols of the conformally transformed metric we are using are zero. Therefore: $\frac{d\delta\nu}{d\chi} = -\delta\hat{\Gamma}^0_{ij}\frac{dx^i}{d\chi}\frac{dx^j}{d\chi} = -\big(D'\delta_{ij} + E'_{ij}\big)\hat{n}^i\hat{n}^j = -\big(D' + E'_\parallel\big)$.

(2.27) : $\frac{d^2\delta x^i}{d\chi^2} = -\delta\hat{\Gamma}^i_{\nu\rho}\frac{dx^\nu}{d\chi}\frac{dx^\rho}{d\chi} - 2\hat{\Gamma}^i_{\nu\rho}\frac{d\delta x^\nu}{d\chi}\frac{dx^\rho}{d\chi}$

The second term is always zero, so:

$$\frac{de^i}{d\chi} = -2\delta\hat{\Gamma}^i_{0j}\frac{dx^0}{d\chi}\frac{dx^j}{d\chi} - \delta\hat{\Gamma}^i_{jk}\frac{dx^j}{d\chi}\frac{dx^k}{d\chi} = -2\big(D'\delta^i_j + E^{i\prime}_j\big)(-1)\hat{n}^j - \big(\partial_j D\,\delta^i_k + \partial_j E^i_k + \partial_k D\,\delta^i_j + \partial_k E^i_j - \partial^i D\,\delta_{jk} - \partial^i E_{jk}\big)\hat{n}^j\hat{n}^k = \overbrace{2D'\hat{n}^i}^{1)} + \underbrace{2E^{i\prime}_j\hat{n}^j}_{2)} \overbrace{-\partial_\parallel D\hat{n}^i}^{1)} \underbrace{-\partial_j E^i_k\hat{n}^j\hat{n}^k}_{2)} \overbrace{-\partial_\parallel D\hat{n}^i}^{1)} \underbrace{-\partial_k E^i_j\hat{n}^j\hat{n}^k}_{2)} + \partial^i D + \partial^i E_\parallel.$$

1) $2D'\hat{n}^i - 2\partial_\parallel D\hat{n}^i = -2\frac{d}{d\chi}\big(D\hat{n}^i\big)$ because $\partial_\parallel - {}' = \frac{d}{d\chi}$

2) $2E^{i\prime}_j\hat{n}^j - \partial_j E^i_k\hat{n}^j\hat{n}^k - \partial_k E^i_j\hat{n}^j\hat{n}^k = 2E^{i\prime}_j\hat{n}^j - \big(\hat{n}^j\partial_j + 2\tilde{\chi}^{-2}\big)E^i_k\hat{n}^k - \big(\hat{n}^k\partial_k + 2\tilde{\chi}^{-2}\big)E^i_j\hat{n}^j = 2E^{i\prime}_j\hat{n}^j - 2\partial_\parallel E^i_j\hat{n}^j - \frac{2}{\tilde{\chi}}\big(E^i_k\hat{n}^k + E^i_j\hat{n}^j\hat{n}_i\hat{n}^i\big) = 2\big(E^{i\prime}_j - \partial_\parallel E^i_j\big)\hat{n}^j - \frac{2}{\tilde{\chi}}\big(E^i_k\hat{n}^k + E_\parallel\hat{n}^i\big) = -2\frac{d}{d\chi}\big(E^i_j\hat{n}^j\big) - \frac{2}{\tilde{\chi}}\big(E^i_k\hat{n}^k + E_\parallel\hat{n}^i\big)$

Now the expressions 1 and 2 can be inserted in the computation of (2.27): $\frac{d\delta e^i}{d\chi} = -2\frac{d}{d\chi}\big(D\hat{n}^i\big) - 2\frac{d}{d\chi}\big(E^i_j\hat{n}^j\big) - \frac{2}{\tilde{\chi}}\big(E^i_k\hat{n}^k + E_\parallel\hat{n}^i\big) + \partial^i D + \partial^i E_\parallel = -2\frac{d}{d\chi}\big(D\hat{n}^i + E^i_j\hat{n}^j\big) + \partial^i D + \partial^i E_\parallel - \frac{2}{\tilde{\chi}}\big(E^i_k\hat{n}^k + E_\parallel\hat{n}^i\big)$.

A.3 Derivation of Eq. (2.35) and Eq. (2.36)

(2.35) : $\big(\bar{\hat{g}}^{00} + \delta\hat{g}^{00}\big)\big(e^0_0 + \delta e^0_0\big)^2 + \big(\bar{\hat{g}}^{ij} + \delta\hat{g}^{ij}\big)\big(e^0_i + \delta e^0_i\big)\big(e^0_j + \delta e^0_j\big) = \eta^{00} = -1$

$\big(\bar{\hat{g}}^{00}+\delta\hat{g}^{00}\big)\big((e^0_0)^2+(\delta e^0_0)^2+2e^0_0\delta e^0_0\big)+\big(\bar{\hat{g}}^{ij}+\delta\hat{g}^{ij}\big)\big(e^0_ie^0_j+e^0_i\delta e^0_j+e^0_j\delta e^0_i+\delta e^0_i\delta e^0_j\big) = -1$

By substituting $e^0_i, e^0_j = 0$ and dropping the terms of order higher than the first, we get: $\big(e^0_0\big)^2\big(\bar{\hat{g}}^{00} + \delta\hat{g}^{00}\big)^2 + 2\bar{\hat{g}}^{00}e^0_0\delta e^0_0 = -1$

$-1 + 2\delta e^0_0 = -1 \implies \delta e^0_0 = 0$

(2.36) : $\big(e^j_i + \delta e^j_i\big)\big(e^i_j + \delta e^i_j\big) = 1$

$e^j_ie^i_j + e^j_i\delta e^i_j + e^i_j\delta e^j_i + \delta e^i_j\delta e^j_i = 1$

By substituting the expression of e^i_j and dropping higher order terms, we get:

$\big((1+D)\delta^j_i + E^j_i\big)\big((1+D)\delta^i_j + E^i_j\big) + \big((1+D)\delta^j_i + E^j_i\big)\delta e^i_j + \big((1+D)\delta^i_j + E^i_j\big)\delta e^j_i = 1$

$(1 + D)^2\delta^i_i + 2(1 + D)E^i_i + 2(1 + D)\delta e^i_i = 1$

$2D\delta^i_i + 2E^i_i + 2(1 + D)\delta e^i_i = 0$

$\delta e^i_i = -\frac{D\delta^i_i+E^i_i}{1+D} \simeq (D\delta^i_i + E^i_i)(1 - D) \simeq -(D\delta^i_i + E^i_i)$

After projecting this expression along the line of sight we obtain: $\delta e^i_o = -(D_o\hat{n}^i + E^i_{oj}\hat{n}^j)$

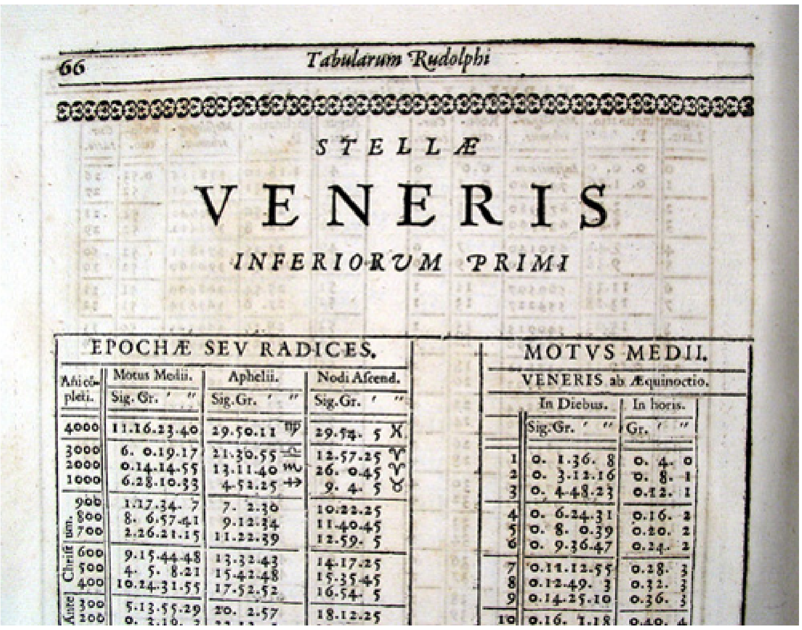
Appendix B

Sky maps

Modern astronomical surveys are the result of a process that has its roots in antiquity and that it is worth to overview [37]. People have observed the sky for centuries, trying to understand how the Universe around us works. The stars and planets were the first to be noticed in the night sky to the naked eye, and from the most ancient times humans realised that certain celestial events repeated at regular intervals, thus marking time and helping with agriculture. Following naked-eye observations, sky charts were produced and became essential for navigation and trade; this feature of sky surveys still survives today, as in the U.S. Naval Observatory[1].
The earliest sky surveys were records of the positions and motions of stars and planets. Over 5,000 years ago, people produced these catalogues in Egypt, China, Central America and Mesopotamia, recording what they saw on stone tablets or temple walls, or building giant structures that aligned with specific astronomical events (as Stonehenge). The first known star catalog contains 800 stars and was created in China in about 350 B.C. by Shih Shen. Maps of the universe improved from 600 B.C. to 400 A.D., when Greek philosophers and astronomers began to develop theories about how the Universe works: they were able to make predictions for the motion and the size of the Sun, the Moon and the planets just with detailed observations of the sky and the use of geometry. Further, the development of trigonometry allowed the calculation of distances to planet and stars (Hipparchus). These ideas culminated in the Ptolemaic system, by Ptolemy.

[1]https://www.usno.navy.mil/USNO

Figure B.1: *An ancient Chinese star chart from ca. 940 A.D. Copyright c 1997, The British Library Board, British Library, Or.8210/S.3226 [37].*


66 Tabularum Rudolphi

STELLÆ

VENERIS

INFERIORUM PRIMI

EPOCHÆ SEV RADICES.

Anni completi.	Motus Medii. Sig. Gr. ′ ″	Aphelii. Sig.Gr. ′ ″	Nodi Ascend. Sig.Gr. ′ ″
4000	11.16.23.40	29.50.11 ♍	29.54. 5 ♓
3000	6. 0.19.17	21.30.55 ♎	12.57.25 ♈
2000	0.14.14.55	13.11.40 ♏	26. 0.45 ♈
1000	6.28.10.33	4.52.25 ♐	9. 4. 5 ♉
900	1.17.34. 7	7. 2.30	10.22.25
800	8. 6.57.41	9.12.34	11.40.45
700	2.26.21.15	11.22.39	12.59. 5
600	9.15.44.48	13.32.43	14.17.25
500	4. 5. 8.21	15.42.48	15.35.45
400	10.24.31.55	17.52.52	16.54. 5
300	5.13.55.29	20. 2.57	18.12.25

MOTUS MEDII.

VENERIS ab Æquinoctio.

	In Diebus. Sig.Gr. ′ ″	In horis. Gr. ′ ″
1	0. 1.36. 8	0. 4. 0
2	0. 3.12.16	0. 8. 1
3	0. 4.48.23	0.12. 1
4	0. 6.24.31	0.16. 2
5	0. 8. 0.39	0.20. 2
6	0. 9.36.47	0.24. 2
7	0.11.12.55	0.28. 3
8	0.12.49. 3	0.32. 3
9	0.14.25.10	0.36. 3

Figure B.2: *The Rudolphine Tables (Tabulae Rudolphinae), named after Rudolf II, consist of a star catalogue and planetary tables published by Johannes Kepler in 1627, using some observational data collected by Tycho Brahe (1546–1601). Although it was not immediately recognised, the positions predicted in this work were generally around thirty times better than those of previous and competing tables.*

Then, for more than 1,000 years astronomy had a deadlock. During this time, Islamic and Hindi astronomers made a significant progress in understanding the sky, so the old astronomical knowledge began to be rediscovered as the works of the ancient Greeks returned through Arabic translations. Copernicus proposed his theory that provided the Earth rotating on its axis and revolving around the Sun along with all the other planets. Meanwhile, astronomical observatories were established in Europe, like the one located on a Danish island (Uraniborg) where Tycho Brahe and Johannes Kepler compiled the most accurate and complete astronomical observations to that time, cataloguing around a thousand of stars. In Italy Galileo Galilei used his telescope for the first time, to see astronomical objects that no one had ever seen before to the naked eye. Since Isaac Newton formulated his theory of the universal gravitation in 1687, Kepler's observations and Newton's laws have been the milestones of astronomy for nearly two hundred years.

In the late nineteenth century the invention of the camera and the spectrograph enabled to produce the first permanent records of the sky. Photographic plates could be also exposed for long periods, allowing astronomers to see fainter objects at greater distances. By the 1930's, they understood that many of these new objects were actually other galaxies containing trillions of stars. At this point a new objective arised: studying distant galaxies. In order to do that, more faint galaxies needed to be found, so systematic photographic surveys of the sky began to be taken. A new telescope, the Schmidt camera, was employed to photograph larger areas of the sky at once; it was used at Palomar Observatory (California) in 1936 to search for supernovae. The first complete unbiased survey was produced in 1949 (POSS-I). Over the decades photographic emulsions improved and a new survey of the northern sky was produced in 1980s (POSS-II).
With the development of computer and digital images, the plates from the photographic surveys began to be scanned to create digital pictures that anyone can download. Moreover, the birth of astronomical observatories at other wavelengths (radio: FIRST; X-ray: RASS; infrared: 2MASS) revealed completely new views of the sky. The introduction of modern electronic detectors (like CCD) has brought a consistently higher sensitivity

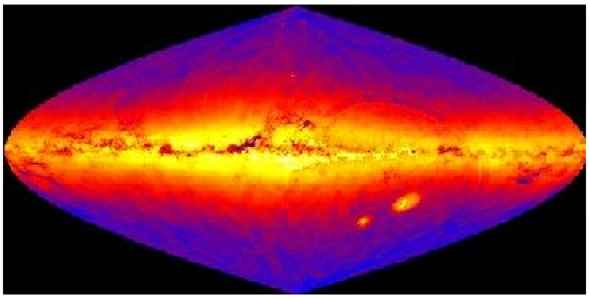


Figure B.3: *A map of the whole sky, based on digitized photographic plates from the Palomar and UK 48" Schmidt telescopes (Courtesy USNO) [37].*

with respect to that provided by photographic plates; then, astronomers can take digital pictures, process and store the data they collect thanks to fast computers and large data storage systems. But what do they actually look for now?

The astronomical objects we can detect are not only galaxies, but also clusters of galaxies, streams and clumps of gas, and dark matter. Astronomers' task is to know where to find them, how they interact and their evolution over time. Therefore many modern sky surveys have been produced in the last century in order to map the Universe over larger areas, greater depths, and over an increasing range of wavelengths. Some of the most recent surveys have been mentioned in this thesis.

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