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Raney extensions of frames as pointfree T_0 spaces

Relatrice: Prof.ssa Maria Emilia Maietti
Laureanda: Anna Laura Suarez (n. matricola 2072502)
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Abstract

Pointfree topology is an approach to topology that takes the notion of open set as primitive, rather than that of point. There exists a dual adjunction between the category of topological spaces and that of complete lattices satisfying a certain distributivity law. These are called *frames*, and in pointfree topology they are regarded as topological spaces. One of the functors involved in the adjunction mentioned above assigns to each topological space its ordered collection of open sets, which is always a frame. The fixpoints of this adjunction are *spatial* frames on one side and *sober* spaces on the other. All Hausdorff spaces are sober, and all sober spaces are T_0. One of the great advantages of pointfree topology is that the category of frames is algebraic, unlike that of sober spaces. In this category, we can prove constructively results that classically require some choice principle, such as the Tychonoff Theorem or the Hofmann-Mislove Theorem (useful in domain theory). This is why the fact that the category of frames is an extension, and not a perfect representation, of that of sober spaces is an advantage; and this is why in pointfree topology we work in this category, and not in the smaller one of spatial frames.

In this thesis, we propose to extend the dual adjunction at the core of pointfree topology in order to capture not only sober spaces but all T_0 spaces. A similar approach had already been outlined: Raney duality establishes a dual equivalence of categories between the category of T_0 spaces and that of the so-called Raney algebras. But Raney algebras are not a generalization of T_0 spaces; they are a perfect representation. Therefore, rather than working in a pointfree setting, we are working in an order-theoretical rephrasing of the classical point-set approach. With our approach, we obtain a representation of T_0 spaces that is entirely pointfree, and a category that generalizes that of T_0 spaces in the same way frames generalize sober ones. We call the pointfree structures we introduce *Raney extensions.* In this thesis, we prove new results in pointfree topology using Raney extensions. In particular, we see how various structures that have captured researchers' interest in pointfree topology recently are actually special cases of Raney extensions. This holds, for example, for the concept of a canonical extension of a frame. Primarily, therefore, we will use Raney extensions to prove new results regarding these structures.

Contents

Introduction

This thesis presents some new results in the field of *pointfree topology*. The main idea behind pointfree topology is that the study of topology should take the lattice of the open sets, not the points, as being the primitive notion. The first foundational result connecting lattice theory with topology is Stone's Representation Theorem, see [49]. It states that spaces which are compact, Hausdorff, and zero-dimensional are completely determined by their Boolean algebras of clopen sets. Thus, for these spaces, lattice theory can replace topology. A topological space is not in general completely determined by its lattice of open sets, but nonetheless the study of this lattice can give information on the space, and this was the approach followed by Wallman, for example in [52]. Later, Ehresmann and his student Bénabou (see [20] and [12]) adopted the more radical approach to replace topological spaces with complete lattices satisfying a certain distributivity law as their main object of study. This is the current approach to pointfree topology. For texts on this, we refer the reader to Johnstone's famous book [32], or to the more recent book [40].

The current approach considers a dual adjunction $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm}^{op} : \mathsf{pt}$ between the category **Top** of topological spaces and the category **Frm** of *frames*, complete lattices where the infinitary distributivity law $\bigvee_i (b \vee a_i) = (\bigvee_i a_i) \vee b$ holds. The functor Ω assigns to each space its ordered collection of open sets. The main idea in pointfree topology is to study topology working in $\mathbf{Frm}^{op}$ as if it were a category of pointfree topological spaces. When working in the category $\mathbf{Frm}^{op}$, usually, its objects are called *locales*. The dual adjunction restricts to a dual equivalence between *spatial* frames and *sober* spaces. Sober spaces, then, are completely determined by their frame of opens. This means that, even if we do not commit to replacing spaces with frames, if we restrict ourselves to sober spaces, topology can be replaced by frame theory without loss of information. This is not very restrictive: for example, all Hausdorff spaces are sober.

Because not all locales are spatial, locales generalize sober spaces, and this is an important strength of pointfree topology. Isbell, in [28], points out that pointfree topology has several advantages with respect to classical point-set topology. For a discussion of these advantages, we refer the reader to [31] and [41]. One example is that the category of frames is algebraic, and this means that we can construct frames from generators and relations. Another advantage comes from the fact that in pointfree topology often we can prove constructively results which in the usual setting require choice principles. Such an example is the pointfree version of Tychonoff's Theorem, stating that the product of compact spaces is compact. Another example is the Hofmann-Mislove Theorem, a result connecting sobriety in topology with domain theory, of which the localic version is proven in [32].

In the category of topological spaces, regular monomorphisms are exactly the subspace inclusions. In the category of locales, regular monomorphisms are called *sublocales*, and these are the pointfree counterparts of subspaces. The pointfree theory of spaces has some interesting features that contrast decidedly with point-set topology. Consider, for instance, that Isbell's famous Density Theorem (see [28]) shows that every locale has the smallest dense sublocale and, in fact, in the localic setting dense sublocales are closed under arbitrary intersections. Furthermore, the ordered collection $\mathsf{S}(L)$ of sublocales of a locale is the opposite of a locale, meaning that in pointfree topology the collection of all subspaces has a topological structure. This is in contrast with the usual setting, where the collection of all subspaces of a space X are just the powerset of X. The fact that for a locale $\mathsf{S}(L)^{op}$ is a locale means that this construction can be iterated. This is studied in [53].

Purpose of this thesis

In this thesis, we introduce an extension of the classical dual adjunction between frames and spaces at the core of pointfree topology. We study an extension of the category of frames which captures all the T_0 spaces. A duality for T_0 spaces already exists, it is *Raney duality*, as illustrated in [14]. Here, rather than mapping a space X to the frame $\Omega(X)$ of its open sets, we map it to the embedding $\Omega(X) \subseteq \mathcal{U}(X)$ of its open sets into the lattice saturated[1] sets. The limitation of Raney duality is that, on the algebraic side, our objects are all of the form $(\Omega(X), \mathcal{U}(X))$ for some space X, and this means that this category does not generalize T_0 spaces in the way that frames generalize sober spaces. In order to gain a more pointfree perspective, we consider as objects of our category pairs (L, C) where C is a coframe and $L \subseteq C$ is a frame which meet-generates C and such that the embedding preserves the frame operations together with strongly exact meets[2], and the frame L meet-generates C. Let us illustrate some of the themes which we can contribute to explore with the tool of Raney extensions.

The study of separation axioms in pointfree topology has been quite active, and recently the results on the matter have been published in a book, see [43]. In pointfree topology, the T_1

[1] *Saturated sets* are intersections of open sets.

[2] *Strongly exact meets* are the pointfree version of those intersections of open sets which are open. Because a meet of a collection $\{U_i : i \in I\}$ of opens in general is calculated as the interior of $\bigcap_i U_i$, these are exactly the meets that are preserved by the embedding $\Omega(X) \subseteq \mathcal{U}(X)$.

axiom has several different translations, the most common ones being *fitness* and *subfitness*, see for example Chapter V of [40] or Chapter II of [43]. In point-set topology, T_1 spaces are characterized by all their subspaces being saturated, and this means that for a space X the embedding of the opens into the saturated sets is $\Omega(X) \subseteq \mathcal{P}(X)$. This leads us to defining a Raney extension (L, C) to be T_1 if and only if C is Boolean. We prove that a frame admits a T_1 Raney extension if and only if it is subfit, giving a precise sense in which subfitness is the weakest possible frame version of the T_1 axiom. We also explore Raney versions of sublocales, and characterize T_1 Raney extensions in terms of these.

Another important separation axiom in pointfree topology is the T_D axiom. This was first introduced in [2], and it is stronger than T_0 and weaker than T_1. For the importance of this axiom in pointfree topology, see [11]. The axiom T_D is a mirror image of sobriety in the following sense. A space X is sober if and only if there can be no nontrivial subspace inclusion $i : X \subseteq Y$ such that $\Omega(i)$ is an isomorphism; and a space X is T_D if and only if there can be no nontrivial subspace inclusion $i : Y \subseteq X$ such that $\Omega(i)$ is an isomorphism. In [11], the $\mathsf{pt}_D(L)$ spectrum of a frame is introduced, an alternative to the classical spectrum which is always a T_D space. In this thesis, we add to this, showing that for a frame L the possible spectra of a Raney extension over L are the interval $[\mathsf{pt}_D(L), \mathsf{pt}(L)]$ of the powerset of the classical spectrum $\mathsf{pt}(L)$ of the frame L.

Recently, the collection $\mathsf{S}_{\mathfrak{c}}(L)$ has received quite a lot of attention in pointfree topology. See [8], [42], [36], [6], [4]. This is the collection of joins of closed sublocales, and it is a frame such that the subset inclusion $\mathsf{S}_{\mathfrak{c}}(L) \subseteq \mathsf{S}(L)$ preserves the frame operations. Functoriality of the assignment $L \mapsto \mathsf{S}_{\mathfrak{c}}(L)$ is explored in [6] for subfit frames, and the authors identify some ways of restricting the objects so that all frame morphisms $f : L \to M$ lift to frame morphisms. We use some of our results for Raney extensions to find conditions on morphisms $f : L \to M$ between arbitrary frames (not necessarily subfit) to lift, and find that a morphism lifts if and only if for an exact meet $\bigwedge_i x_i \in L$ we have both that $\bigwedge_i f(x_i)$ is exact and that it equals $f(\bigwedge_i x_i)$.

An important structure in pointfree topology is the subcoframe $\mathsf{S}_{\mathfrak{o}}(L) \subseteq \mathsf{S}(L)$ of *fitted* sublocales, that is, intersections of open sublocales. See, for instance, [19], [36]. The structures $\mathsf{S}_{\mathfrak{c}}(L)$ and $\mathsf{S}_{\mathfrak{o}}(L)$ are compared in [36] and in [35]. In this thesis we show that, for a frame L, $\mathsf{S}_{\mathfrak{o}}(L)$ and $\mathsf{S}_{\mathfrak{c}}(L)$ are, respectively, the largest and smallest Raney extensions, showing that these two structures have universal properties which are dual to each other.

Finally, another axiom which has been studied quite extensively is *scatteredness*. Simmons, in [48], characterizes the scattered spaces X as being those such that $\mathsf{S}(\Omega(X))$ is a Boolean algebra, and these turn out to be exactly the spaces such that subspaces of X are perfectly represented by the sublocales of $\Omega(X)$. Scatteredness for a frame L is defined in [45] and [46] as the property that $\mathsf{S}(L)$ is Boolean. The relation between scatteredness and the T_D property is studied in [39]. In [4], the authors characterize the frames for which $\mathsf{S}_{\mathfrak{c}}(L) = \mathsf{S}(L)$ as those subfit frames such that they are scattered. Here, we show that subfit frames which are scattered coincide with those subfit frames with unique Raney extensions. Following the work in [48] and [37], it is known that scattered spatial frames are exactly those frames such that there is a perfect correspondence between $\mathsf{S}(L)$ and the collection of subspaces of the spectrum $\mathsf{pt}(L)$. We will see that for a spatial Raney extension (L, C) to satisfy the analogue of this it suffices for L to be a fit frame.

Outline of the thesis

The purpose of **Section 1** is to give context for the results in the rest of the thesis. We give the necessary background in the field of pointfree topology and frame theory.

In **Section 2** we define Raney extensions. We shall see that every frame has the largest and the smallest Raney extension, that is, that for a Raney extension (L, C), we always have two surjections

$$(L, \mathsf{Filt}_{\mathcal{SE}}(L)^{op}) \twoheadrightarrow (L, C) \twoheadrightarrow (L, \mathsf{Filt}_{\mathcal{E}}(L)^{op}),$$

where $\mathsf{Filt}_{\mathcal{SE}}(L)$ is the frame of strongly exact filters and $\mathsf{Filt}_{\mathcal{E}}(L)$ that of exact filters. We then introduce the category **Raney** of Raney extensions. For Raney extensions (L, C) and (M, D), we characterize those frame maps $f : L \to M$ which can be lifted to maps of Raney extensions $(L, C) \to (M, D)$. In particular, we explore conditions for the assignment $L \mapsto (L, \mathsf{Filt}_{\mathcal{E}}(L)^{op})$ to be functorial. We come show that frame maps $f : L \to M$ which can be extended to frame maps $\mathsf{Filt}_{\mathcal{E}}(L) \to \mathsf{Filt}_{\mathcal{E}}(M)$ are exactly the *exact* ones.

In **Section 3**, where we define the dual adjunction between the category **Raney** and that of topological spaces. We define the spectrum of a Raney extension, and we come to proving that for a frame L, the spectra of all possible Raney extensions on it are the interval $[\mathsf{pt}_D(L), \mathsf{pt}(L))]$ of the powerset of $\mathsf{pt}(L)$, where $\mathsf{pt}_D(L)$ is the T_D spectrum of L. We look at a Raney version of sobriety, and for each Raney extension we define its *sobrification*, i. e. its sober coreflection in **Raney**. We look at an analogue of the T_D and of the T_1 axiom, too. We prove that a frame has a T_1 Raney extension if and only if it is subfit.

Section 4 is devoted to canonical extensions of frames seen as Raney extensions. In this short section, we show that canonical extensions of frames are kinds of Raney extensions. We show that the canonical extension of a pre-spatial frame, as defined in [29], is the free Raney extension over it which is *algebraic*, that is, such that the coframe component is generated by its compact elements.

In **Section 5** we look at the fact that several structures in pointfree topology are Raney extensions. We study the fact that every Raney extension corresponds to some subcolocale of the coframe $\mathsf{S}_{\mathfrak{o}}(L)$ of fitted sublocales of a frame. We characterize several collections of filters of a frame L as kernels of certain sublocales of L.

Finally, in **Section 6**, we begin to outline a theory of pointfree subspaces of Raney extensions. We simply regard these as subcolocales of the coframe component of the Raney extension. We will revisit the T_1 axiom and characterize it in terms of pointfree Raney subspaces. Finally, we use this notion of pointfree subspace of a Raney extension to explore the relation between pointfree and point-set subspaces.

1 Background

We first recall some background on point-free topology. For more information on this subject, we refer the reader to Johnstone's book [30] or to more recent book [40] by Picado and Pultr.

1.1 Frames as spaces

A *frame* is a complete lattice L satisfying

$$a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\}$$

for all $a \in L$ and $B \subseteq L$. Frames form a category **Frm**, whose morphisms are functions preserving arbitrary joins (including the bottom element 0) and finite meets (including the top element 1). Because for each a the map $a \wedge -$ preserves arbitrary joins, it has a right adjoint $a \to -$, making L a complete Heyting algebra. This right adjoint is called the *Heyting operator*, or the *Heyting implication*. In particular, the *pseudocomplement* of an $a \in L$ is the element $\neg a = a \to 0$ and it can be characterized as the largest $b \in L$ such that $a \wedge b = 0$. Even though all frames are complete Heyting algebras, morphisms of frames do not in general preserve the Heyting operator.

Example 1.1. *To see that frame maps in general do not preserve the Heyting implication, it suffices to consider the map $f : C_3 \to C_2$ from the three-element to the two-element chain, defined as $f(1) = 1$, $f(0) = 0$, and $f(a) = 0$, where $a \in C_3$ is the middle element. This preserves all meets and all joins. However, in C_3 we have that $\neg a = 0$, and $f(\neg a) = 0 \neq 1 = \neg f(a)$.*

Morphisms of frames preserve all joins, and as such they have right adjoints. For a frame map $f : L \to M$, we will denote as f_* its right adjoint. Concretely, this acts as $m \mapsto \bigvee\{x \in L : f(x) \leq m\}$. In this thesis, we will also consider the category **CoFrm**. This is the category whose objects are *coframes*, complete lattices where the distributivity law dual to that of frames holds, and whose morphisms are functions preserving finite joins and arbitrary meets. In a coframe C we have the *co-Heyting operator*, or *difference*, defined for two elements $c, d \in C$ as $c \backslash d = \bigwedge\{x \in C : c \leq x \vee d\}$. The *supplement* of an element $c \in C$ is the element $c^* = 1 \backslash c$. Morphisms of coframes, as they preserve all meets, have left adjoints. For a coframe map $f : C \to D$, we denote as f^* its left adjoint.

Frames as spaces

Given a topological space X, its lattice of open sets $\Omega(X)$ is always a frame, and this assignment is the object part of a functor $\Omega : \mathbf{Top} \to \mathbf{Frm}^{op}$ which sends a continuous map $f : X \to Y$ to the preimage map $f^{-1} : \Omega(Y) \to \Omega(X)$. For a topological space $\Omega(X)$, the arbitrary joins in the frame $\Omega(X)$ correspond to set-theoretical unions, whereas the finite meets correspond to set-theoretical intersections. Because a frame is a complete lattice, this means that in $\Omega(X)$ infinite meets exist, too. The meet of a collection $U_i \in \Omega(X)$ is computed as the interior of $\bigcap_i U_i$.

The correspondence between frames and topological spaces at the core of pointfree topology is an adjunction $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm}^{op} : \mathsf{pt}$ with $\Omega \dashv \mathsf{pt}$. Let us now describe the functor pt. For a frame L, an element $p \in L$ is said to be *prime* if whenever $x \wedge y \leq p$ for $x, y \in L$, then $x \leq p$ or $y \leq p$. The collection of all primes of L will be denoted by $\mathsf{pt}(L)$. This will be the set of points of the space associated with the frame L.

In order to topologize the set $\mathsf{pt}(L)$, we consider a map $\varphi_L : L \to \mathcal{P}(\mathsf{pt}(L))$ defined for each $a \in L$ as $\varphi_L(a) = \{p \in \mathsf{pt}(L) : a \nleq p\}$. By exploiting primality of the elements of $\mathsf{pt}(L)$, it is easy to see that for $a, b, a_i \in L$

1. $\varphi_L(a \wedge b) = \varphi_L(a) \cap \varphi_L(b)$,

2. $\varphi_L(\bigvee_i a_i) = \bigcup_i \varphi_L(a_i)$.

Thus, the elements of the form $\varphi_L(a)$ for some $a \in L$ form a topology on $\mathsf{pt}(L)$, and in fact we define the *spectrum* of a frame L to be the set of primes $\mathsf{pt}(L)$ topologized in this way. It remains to define the assignment on morphisms. For a frame map $f : L \to M$, we note that the right adjoint f_* maps prime elements to prime elements. Therefore, we may extend the definition of pt to morphisms as $f \mapsto f_*$, thus obtaining a functor $\mathsf{pt} : \mathbf{Frm}^{op} \to \mathbf{Top}$. We have the following theorem.

Theorem 1.1. *There is an adjunction* $\Omega : \mathbf{Top} \leftrightarrows \mathbf{Frm}^{op} : \mathsf{pt}$ *with* $\Omega \dashv \mathsf{pt}$. *This adjunction is idempotent, and for each space the unit of adjunction is the map*

$$\begin{aligned} \psi_X : X &\to \mathsf{pt}(\Omega(X)), \\ x &\mapsto X \backslash cl(\{x\}). \end{aligned}$$

For each frame L the counit of adjunction, seen as a map in **Frm**, *is defined as*

$$\begin{aligned} \varphi_L : L &\to \Omega(\mathsf{pt}(L)), \\ a &\mapsto \varphi_L(a). \end{aligned}$$

For a frame L, the map $\varphi_L : L \to \Omega(\mathsf{pt}(L))$ is always a surjective frame map, by definition of the topology on its spectrum. Frame maps are isomorphisms whenever they are bijective, and this gives us the characterization of the fixpoints in **Frm** as the frames for which $\varphi_L(a)$ is injective. We say that a frame L is *spatial* when for $a, b \in L$ such that $a \nleq b$ there is some $p \in \mathsf{pt}(L)$ such that $b \leq p$ and $a \nleq p$. We call the map φ_L the *spatialization* of the frame L. The map ψ_X, instead, is not in general injective or surjective. It turns out that this is a bijection, and thus a homeomorphism, precisely when the space X is *sober*: that is, when every irreducible closed set is the closure $cl(\{x\})$ of a unique point x. Sobriety is an axiom stronger than T_0, weaker than T_2, and incomparable with T_1. We call the map ψ_X the *sobrification* of the space X.

Alternative definitions of the spectrum of a frame

There is also an alternative view of the spectrum of a frame. It is well-known that prime elements of a frame are in bijective correspondence with *completely prime filters*, that is, filters inaccessible by any join. We note that for a completely prime filter $P \subseteq L$ we have $\bigvee(L \backslash P) \notin P$, and as P is a filter, the element $\bigvee L \backslash P$ must additionally be prime. The assignment $P \mapsto \bigvee L \backslash P$ is in fact the bijection mentioned above. One may then identify the points of a frame L

with the collection $\mathsf{Filt}_{\mathcal{CP}}(L)$ of its completely prime filters. The sobrification map of a space X, under this identification, becomes the map

$$x \mapsto N(x),$$

assigning to each point the collection of its open neighborhoods, the so-called *neighborhood filter*. We see every point of a space as being encoded in $\Omega(X)$ as the collection of its open neighborhoods. If we view $\mathsf{Filt}_{\mathcal{CP}}(L)$ as the collection of points of the frame L, the opens are the sets of the form $\{P \in \mathsf{Filt}_{\mathcal{CP}}(L) : a \in P\}$ for some $a \in L$.

There is a third, equivalent way of defining the spectrum of a frame L. Since every frame map $f : L \to 2$ preserves all joins, the set $\{x \in L : f(x) = 1\}$ is a completely prime filter of L. This, in fact, establishes a bijection between maps $f : L \to 2$ and completely prime filters of L. When we view the set $\mathbf{Frm}(L, 2)$ as the set of points of L, the opens being the sets of the form $\{f \in \mathbf{Frm}(L, 2) : f(a) = 1\}$. This view of the spectrum exhibits 2 as the dualizing object for the adjunction between frames and spaces. On the space side, we have the Sierpinski space $S = \{0, 1\}$, whose collection of opens is $\{\emptyset, \{1\}, S\}$. We observe that continuous maps $\mathbf{Top}(X, S)$ may be identified with the opens of X by considering the preimage of $\{1\}$.

1.2 Sublocales

In this thesis, we will work in the category **Frm**. It is sometimes custom in pointfree topology to work in a category isomorphic to $\mathbf{Frm}^{op}$, in order to have a covariant correspondence with **Top**. This is the category **Loc** of locales. The category **Loc** is defined as the category whose objects are frames, which are referred to as *locales* when adopting this approach. The morphisms of **Loc** are the right adjoints to frame maps. Hence, a frame map $f : L \to M$ will correspond to the morphism $f_* : M \to L$ in **Loc**. In the category of topological spaces, subspace inclusions are, up to isomorphism, the regular monomorphisms. This is the motivation behind the definition of a *sublocale* as a regular monomorphism in **Loc**. It turns out that a morphism $f_* : M \to L$ in **Loc** is a regular monomorphism precisely when $f : L \to M$ is a frame surjection. Frame surjections are, in fact, both the regular epimorphisms and the extremal epimorphisms in **Frm**. Even when working with frames, the term *sublocale* is still used. We follow Picado and Pultr in [40] in defining a *sublocale* of a frame L to be a subset $S \subseteq L$ such that:

1. It is closed under all meets;

2. Whenever $s \in S$ and $x \in L$ we have $x \to s \in S$.

These requirements are equivalent to stating that $S \subseteq L$ is a regular monomorphism in **Loc**. Observe that the collection of sublocales of a frame is closed under all intersections. It is a great advantage to be able to work with sublocales as concrete subsets of frames. Consider, for instance, the concept of density. A frame surjection $s : L \twoheadrightarrow S$ is *dense* if $s(x) = 0$ implies $x = 0$, this definition follows naturally from the fact that this condition holds for the dualization $\Omega(i) : \Omega(Y) \to \Omega(X)$ of a subspace inclusion $i : X \subseteq Y$ if and only if the subspace is dense. A frame surjection being dense is equivalent to the corresponding sublocale $S \subseteq L$ containing 0, and this gives us an immediate proof of Isbell's theorem.

Theorem 1.2 (Isbell's Density Theorem, see [28]). *Every frame has the smallest dense sublocale, and this is the intersection of all its sublocales which contain* 0.

We also have the following useful fact.

Lemma 1.3. *If S and T are sublocales of L such that $S \subseteq T$, then S is a sublocale of T.*

The family $\mathsf{S}(L)$ of all sublocales of L ordered by inclusion is a coframe. Meets in $\mathsf{S}(L)$ are set-theoretical intersections. The top element is L and the bottom element is $\{1\}$. The frame $\mathsf{S}(L)^{op}$ is always zero-dimensional. Since $\mathsf{S}(L)$ is a coframe, it has a difference operation on it. We shall freely use some its properties, for example the ones listed below.

1. $S \backslash T \subseteq S$;
2. $S \backslash T = 0$ if and only if $S \subseteq T$;
3. $(\bigvee_i S_i) \backslash T = \bigvee_i (S_i \backslash T)$;
4. $S \backslash \bigcap_i S_i = \bigvee_i (S \backslash S_i)$;
5. $(S \backslash T) \backslash R = (S \backslash R) \backslash T$;

for each $S, T, R, S_i \in \mathsf{S}(L)$. A comprehensive list of its properties may be found in [22].

Open and closed sublocales

We now focus on the pointfree versions of open and closed subspaces. For a space X and a subspace inclusion $i : U \subseteq X$ where U is open, we know that the opens of U are the intersections of the opens of X with U. The functor Ω maps the inclusion to the frame surjection $- \cap U : \Omega(X) \to \Omega(U)$. By definition of the Heyting implication, the right adjoint i_* of this map acts as $Z \mapsto U \to Z$. This means that the sublocale of $\Omega(X)$ corresponding to this map, $i_*[\Omega(U)]$, is

$$\{U \to V : V \in \Omega(X)\} \subseteq \Omega(X).$$

For the inclusion of a closed set $C \subseteq X$, the frame map associated with it is $- \cap C : \Omega(X) \to \Omega(C)$. Its right adjoint is $Z \mapsto Z \cup C^c$. The sublocale of $\Omega(X)$ determined by this map is then

$$\{V \cup C^c : V \in \Omega(X)\} = \uparrow C^c \subseteq \Omega(X).$$

This is the motivation behind the following definitions. For each $a \in L$, there are an *open sublocale* and a *closed sublocale* associated with it. These are, respectively,

$$\mathfrak{o}(a) = \{a \to b : b \in L\}, \qquad \mathfrak{c}(a) = \uparrow a.$$

Open and closed sublocales behave like open and closed subspaces in many respects; we list a few of them below.

Proposition 1.4. *For every frame L and $a, b, a_i \in L$ we have*

1. $\mathfrak{o}(1) = L$ *and* $\mathfrak{o}(0) = \{1\}$;
2. $\mathfrak{c}(1) = \{1\}$ *and* $\mathfrak{c}(0) = L$;
3. $\bigvee_i \mathfrak{o}(a_i) = \mathfrak{o}(\bigvee_i a_i)$ *and* $\mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$;
4. $\bigcap_i \mathfrak{c}(a_i) = \mathfrak{c}(\bigwedge_i a_i)$ *and* $\mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)$;
5. *The elements* $\mathfrak{o}(a)$ *and* $\mathfrak{c}(a)$ *are complements of each other in* $\mathsf{S}(L)$*: we have* $\mathfrak{o}(a) \cap \mathfrak{c}(a) =$ *and* $\mathfrak{o}(a) \vee \mathfrak{c}(a) = L$;
6. $\mathfrak{c}(a) \subseteq \mathfrak{o}(b)$ *if and only if* $a \vee b = 1$, *and* $\mathfrak{o}(a) \subseteq \mathfrak{c}(b)$ *if and only if* $a \wedge b = 0$.

The ordered collection of closed sublocales of a frame L is a subcoframe of $\mathsf{S}(L)$. In fact, this coframe is anti-isomorphic to L. Open sublocales form a subframe of $\mathsf{S}(L)$, and this is isomorphic to L. The sublocales which are intersections of open sublocales are called *fitted*.

Proposition 1.5. *Every sublocale S can be written in canonical form as an intersection of finite joins of open and closed sublocales, as follows:*

$$S = \bigcap \{\mathfrak{o}(a) \vee \mathfrak{c}(b) : S \subseteq \mathfrak{o}(a) \vee \mathfrak{c}(b)\}.$$

Boolean sublocales

For a complete lattice L and a subset $S \subseteq L$, we denote as $\mathcal{M}(S)$ its closure under arbitrary meets and $\mathcal{J}(S)$ its closure under arbitrary joins. For a frame L and each $a \in L$, there is a sublocale $\mathfrak{b}(a) = \{b \to a : b \in L\}$ which turns out to be Boolean. This is the smallest sublocale containing the element a. The smallest dense sublocale of a frame L, then, is the Boolean sublocale $\mathfrak{b}(0)$, called its *Booleanization*. This coincides with the set of all regular elements of L, that is, those elements of the form $\neg a$ for some $a \in L$. Despite what the name might suggest, this is not a Boolean reflection of the frame L, and in fact the assignment of a frame to its Booleanization is not in general functorial, as studied in [10]. Every Boolean sublocale of L is of the form $\mathfrak{b}(a)$ for some $a \in L$. A special role is played by the Boolean sublocales from prime elements.

Lemma 1.6. *For a frame L and prime elements $p, p_i \in L$, and $x \in L$, and $S \subseteq L$ a sublocale, we have:*

1. $x \to p = p$ *if and only if* $x \nleq p$,
2. $\mathsf{pt}(\mathfrak{o}(a)) = \mathsf{pt}(\mathfrak{c}(a))^c$,
3. $\mathfrak{b}(p) = \{p, 1\}$,
4. $\bigvee_i \mathfrak{b}(p_i) = \mathcal{M}(\{p_i : i \in I\})$,
5. $\mathsf{pt}(S) = \mathsf{pt}(L) \cap S$,
6. $\mathsf{pt}(\mathsf{S}(L)^{op}) = \{\mathfrak{b}(p) : p \in \mathsf{pt}(L)\}$.

Sublocales of the form $\mathfrak{b}(p)$ for some prime $p \in L$ are usually referred to as *two-element sublocales*, as they are the only sublocales which contain exactly two elements (the only sublocale containing only one element is $\{1\}$). For a frame L, we may identify its spatialization with a sublocale of L. A frame is spatial if and only if all its elements are meets of primes. By the universal property of the spatialization, then, this is the largest sublocale of L all whose elements are meets of primes. By point (4) of Lemma 1.6 above, this is $\bigvee\{\mathfrak{b}(p) : p \in L\}$. By point (4) of the same lemma, for any sublocale $S \subseteq L$, its spatialization sublocale is $\bigvee\{\mathfrak{b}(p) : p \in \mathrm{pt}(L) \cap S\}$. The spatial sublocales coincide with the joins of two-element sublocales. We denote as $\mathsf{S}_{sp}(L)$ the collection of the spatial sublocales of L. The inclusion $\mathsf{S}_{sp}(L)^{op} \subseteq \mathsf{S}(L)^{op}$ is a sublocale inclusion. By point (6) of Lemma 1.6 above, it is the spatialization sublocale.

Notable subcollections of $\mathsf{S}(L)$

In our work, we will look at several subcollections of $\mathsf{S}(L)$. We have already mentioned the frame $\mathsf{S}_{\mathfrak{c}}(L)$ of joins of closed sublocales, the coframe $\mathsf{S}_{\mathfrak{o}}(L)$ of fitted sublocales, and the coframe $\mathsf{S}_{sp}(L)$ of spatial sublocales. Additionally, we will look at the Booleanization $\mathsf{S}_b(L) \subseteq \mathsf{S}(L)$. This consists of the collection of *smooth* sublocales, that is, sublocales of the form $\bigvee_i \mathfrak{c}(x_i) \cap \mathfrak{o}(y_i)$. These are studied in [1], and they coincide with the joins of complemented sublocales. We also consider the collection $\mathsf{S}_k(L)$ of joins of compact sublocales, where a sublocale $S \subseteq L$ is said to be *compact* if for each directed collection $D \subseteq L$ we have $S \subseteq \bigvee\{\mathfrak{o}(d) : d \in D\}$ implies $S \subseteq \mathfrak{o}(d)$ for some $d \in D$.

1.3 Saturated sets and fitted sublocales

For a T_0 topological space X we have the *specialization order*, an order $\leq$ on its points defined as $x \leq y$ whenever $x \in U$ implies $y \in U$ for all open sets $U \subseteq X$. For a space X, we denote as $\mathcal{U}(X)$ the lattice of its upsets (upper-closed sets) under the specialization order.

Proposition 1.7. *For a T_0 topological space X, a subset is an upset in the specialization order if and only if it is saturated.*

Proof. It is clear that a saturated subset is an upset. For the converse, suppose that $Y \subseteq X$ is an upset. Suppose that $x \in X$ is such that whenever $Y \subseteq U$ for an open U we have $x \in U$. Suppose towards contradiction that $x \notin Y$. As Y is an upset, this means that for all $y \in Y$ we cannot have $y \leq x$, that is, there is some open U_y with $y \in U_y$ and $x \notin U_y$. Consider $\bigcup_{y \in Y} U_y$. We have both $Y \subseteq \bigcup_y U_y$ and $x \notin \bigcup_y U_y$, and this is a contradiction. □

We observe that a space is T_1 if and only if the specialization order on it is discrete. Let us see what the specialization order translates to in the various definitions of the spectrum of a frame.

1. The collection $\mathsf{pt}(L)$ comes equipped with the order inherited from L. This is opposite to the specialization order, as for primes $p, q \in \mathsf{pt}(L)$ we have that $p \leq q$ implies that $a \nleq q$ implies $a \nleq p$ for all $a \in L$. Hence, $q \in \varphi_L(a)$ implies $p \in \varphi_L(a)$.

2. For the spectrum $\mathsf{Filt}_{\mathcal{CP}}(L)$ of completely prime filters, the specialization order is subset inclusion.

3. For the collection of frame morphisms to 2, the specialization order coincides with the point-wise order.

The following is an important theorem by Hofmann and Mislove. A filter of a frame L is *Scott-open* if it is not accessible by directed joins. We call $\mathsf{Filt}_{\mathcal{SO}}(L)$ the ordered collection of Scott-open filters of a frame L.

Theorem 1.8. *([27], Theorem 2.16) If the Prime Ideal Theorem holds, then for each sober space X there is an anti-isomorphism between $\mathsf{Filt}_{\mathcal{SO}}(\Omega(X))$ and the ordered collection of compact saturated sets of X, assigning to each filter F the set $\bigcap F$.*

The pointfree version of this theorem does not rely on choice principles, and is due to Johnstone.

Theorem 1.9. *([32], Lemma 3.4) For a frame L, there is an anti-isomorphism between $\mathsf{Filt}_{\mathcal{SO}}(L)$ and the ordered collection of compact fitted sublocales of L, assigning to each filter F the sublocale $\bigcap\{\mathfrak{o}(f) : f \in F\}$.*

Subfitness and fitness

A frame is *subfit* if whenever $x, y \in L$ are such that $x \nleq y$, there is some $u \in L$ such that $x \vee u = 1$ and $y \vee u \neq 1$. A frame is *fit* if whenever $x, y \in L$ are such that $x \nleq y$, there is some $u \in L$ such that $x \vee u = 1$ and $u \to y \neq y$. Both properties have been used as pointfree analogues of the T_1 axiom. It is apparent that both fitness and subfitness are separation properties for sublocales if we consider the following characterization. A frame is subfit if and only if, whenever $\mathfrak{c}(y) \nsubseteq \mathfrak{c}(x)$ for $x, y \in L$, there is some $u \in L$ such that

$$\begin{cases} \mathfrak{c}(x) \subseteq \mathfrak{o}(u) \\ \mathfrak{c}(y) \nsubseteq \mathfrak{o}(u). \end{cases}$$

On the other hand, a frame is fit if and only if whenever $\mathfrak{b}(y) \nsubseteq \mathfrak{c}(x)$ for $x, y \in L$, there is some $u \in L$ such that

$$\begin{cases} \mathfrak{c}(x) \subseteq \mathfrak{o}(u) \\ \mathfrak{b}(y) \nsubseteq \mathfrak{o}(u). \end{cases}$$

We have the following characterizations of fitness and subfitness.

Theorem 1.10. *For a frame L, the following are equivalent.*

1. *L is subfit.*

2. *Every open sublocale of L is a join of closed sublocales.*

3. *For each $a \in L$ we have $\mathfrak{o}(a) = \bigvee\{\mathfrak{c}(b) : b \vee a = 1\}$.*

Theorem 1.11. *For a frame L, the following are equivalent.*

1. *L is fit.*

2. *Every closed sublocale of L is fitted.*

3. *For each $a \in L$ we have $\mathfrak{c}(a) = \bigcap\{\mathfrak{o}(b) : b \vee a = 1\}$.*

4. *All sublocales of L are fitted.*

5. *All sublocales of L are fit frames.*

Exact and strongly exact filters

For a frame L, we call $\mathsf{S}_{\mathfrak{o}}(L)$ the coframe of its fitted sublocales. For a fitted sublocale $\bigcap_i \mathfrak{o}(a_i)$, its supplement is $\bigvee_i \mathfrak{c}(a_i)$. We have that the collection $\{F^* : F \in \mathsf{S}_{\mathfrak{o}}(L)\}$ is a frame, and it coincides with the collection $\mathsf{S}_{\mathfrak{c}}(L)$ of joins of closed sublocales. Both the collections $\mathsf{S}_{\mathfrak{o}}(L)$ and $\mathsf{S}_{\mathfrak{c}}(L)$ are isomorphic to ordered collections of filters of L. For every frame L we have an adjunction $ker \dashv fitt$ defined as

$$\begin{aligned} &fitt : \mathsf{Filt}(L)^{op} \to \mathsf{S}(L) : ker \\ &\{x \in L : S \subseteq \mathfrak{o}(x)\} \leftarrow\!\shortmid S, \\ &F \mapsto \bigcap\{\mathfrak{o}(f) : f \in F\}. \end{aligned}$$

This maximally restricts to an anti-isomorphism between fitted sublocales and strongly exact filters (see [36]). Recall that a filter is *strongly exact* if it is closed under strongly exact meets, and that a meet $\bigwedge_i x_i$ is *strongly exact* if for all $y \in L$ we have that $x_i \to y = y$ implies $(\bigwedge_i x_i) \to y = y$. We call $\mathsf{Filt}_{\mathcal{SE}}(L)$ the ordered collection of strongly exact filters. This is a frame where meets are computed as intersections, and additionally it is a sublocale of $\mathsf{Filt}(L)$. The following is shown in [36].

Lemma 1.12. *For a filter $F \subseteq L$, the following are equivalent.*

1. *F is strongly exact.*
2. *$\bigcap_{f \in F} \mathfrak{o}(f) \subseteq \mathfrak{o}(x)$ is equivalent to $x \in F$.*
3. *F is a fixpoint of the adjunction $ker \dashv fitt$.*
4. *$F = \{x \in L : S \subseteq \mathfrak{o}(x)\}$ for some sublocale $S \subseteq L$.*

A meet $\bigwedge_i x_i$ of a frame L is *exact* if for every $a \in L$ we have $(\bigwedge_i x_i) \vee a = \bigwedge_i (x_i \vee a)$. Exact filters are studied in [36]. There, it is also shown that the exact filters form a frame, and in particular the frame $\mathsf{Filt}_{\mathcal{E}}(L)$ of exact filters is a sublocale of $\mathsf{Filt}_{\mathcal{SE}}(L)$. The main theorem that we will need is the following.

Theorem 1.13. *We have an isomorphism of coframes*

$$\begin{aligned} fitt : \mathsf{Filt}_{\mathcal{SE}}(L)^{op} &\cong \mathsf{S}_{\mathfrak{o}}(L), \\ F &\mapsto \bigcap \{\mathfrak{o}(f) : f \in F\}. \end{aligned}$$

We also have an isomorphism of frames

$$\begin{aligned} cl : \mathsf{Filt}_{\mathcal{E}}(L) &\cong \mathsf{S}_{\mathfrak{c}}(L), \\ F &\mapsto \bigvee \{\mathfrak{c}(f) : f \in F\}. \end{aligned}$$

The two isomorphisms are connected by the following diagram:

$$\begin{array}{ccc} \mathsf{Filt}_{\mathcal{SE}}(L)^{op} & \xrightarrow{cl_{\mathcal{E}}} & \mathsf{Filt}_{\mathcal{E}}(L)^{op} \\ {\scriptstyle ker}\uparrow & & \uparrow{\scriptstyle cl} \\ \mathsf{S}_{\mathfrak{o}}(L) & \xrightarrow{(-)^*} & \mathsf{S}_{\mathfrak{c}}(L)^{op}, \end{array}$$

where $cl_{\mathcal{E}}(F) = \bigcap\{G \in \mathsf{Filt}_{\mathcal{E}}(L) : F \subseteq G\}$.

Yet another characterization of subfit frames relates this property with the frame $\mathsf{S}_{\mathfrak{c}}(L)$.

Theorem 1.14. *A frame* L *is subfit if and only if* $\mathcal{S}_{\mathfrak{c}}(L)$ *is a Boolean algebra. When this holds,* $\mathcal{S}_{\mathfrak{c}}(L) \subseteq \mathsf{S}(L)$ *is the Booleanization sublocale.*

1.4 Canonical extensions: Stone duality and frames

Stone duality

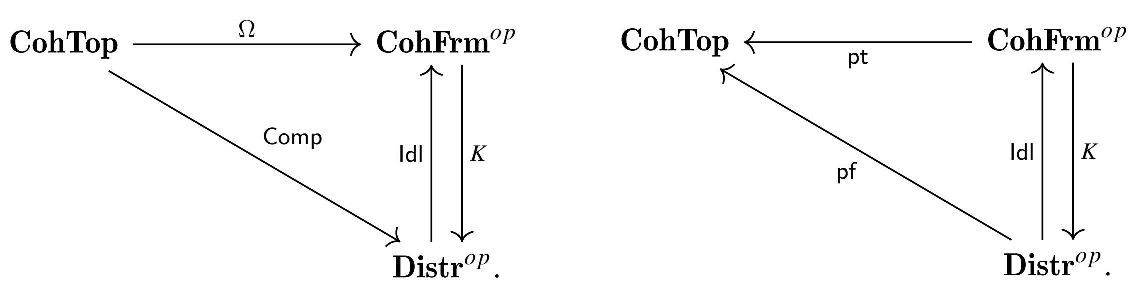
Stone's landmark Representation Theorem, proven in 1936 (see [49]), establishes what nowadays we call a dual equivalence between the category **Bool** of Boolean algebras and the category **Stone** of *Stone spaces*, topological spaces X which are compact, Hausdorff, and zero-dimensional. The functor in one direction is $\mathsf{Clop} : \mathbf{Stone} \to \mathbf{Bool}^{op}$, and it assigns to a Stone space its Boolean algebra of clopen sets. We observe that, under continuous maps, preimages of clopen sets are clopen, and furthermore the preimage map preserves intersections and unions. Thus, the assignment $f \mapsto f^{-1}$ is a well-defined assignment $\mathsf{Mor}(\mathbf{Stone}) \to \mathsf{Mor}(\mathbf{Bool}^{op})$; this is the extension of Clop to morphisms. In the other direction we have the functor $\mathsf{pf} : \mathbf{Bool}^{op} \to \mathbf{Stone}$, assigning to each Boolean algebra B the collection $\mathsf{pf}(B)$ of its prime filters, suitably topologized. This functor, too, sends morphisms to their corresponding preimage maps. Stone duality relies on the Prime Ideal Theorem. Intuitively: as the Stone dual of a Boolean algebra is a space of *prime filters*, in order for the assignment $B \mapsto \mathsf{pf}(B)$ to preserve enough information we need there to be sufficiently many prime filters.

The importance and impact of this theorem is hard to overstate. Among other things, it was the starting point of a large variety of results connecting ordered structures and logical theories to topological spaces. Stone later proved (see [50]) that the category **Distr** of distributive lattices is equivalent to the category **CohTop** of coherent spaces. A space is *coherent* if it is sober, compact, and if the compact open elements are closed under finite intersections, and generate the whole topology. Note that morphisms in **CohTop** are not the continuous maps, but those continuous maps $f : X \to Y$ such that $f^{-1}(K)$ is compact whenever $K \subseteq Y$ is compact. The equivalence generalizes his previous result. In one direction, we have the functor $\mathsf{Comp} : \mathbf{CohTop} \to \mathbf{Distr}^{op}$ assigning to each space the ordered collection of its compact elements. In the other direction, we have a functor $D \mapsto \mathsf{pf}(D)$, once again assigning to a distributive lattice the collection of its prime filters, suitably topologized.

Remark 1.2. *The duality of coherent spaces has been re-interpreted in several ways. Priestley (see [47]) proved that* $\mathbf{Distr}^{op}$ *is equivalent to a category of spaces equipped with an order, with morphisms being continuous monotone maps. This is usually known as* Priestley duality. *An alternative approach is to regard the duals of distributive lattices as being certain* bitopological spaces, *for this approach see [38] and [15]. The advantage of this approach is that we do not have to restrict the morphisms or equip topological spaces with extra structure. Yet another approach is to regard the duals of distributive lattices as being* Pervin spaces, *special kinds of uniform spaces, as introduced in [44]. For an illustration of this approach and the connection with the bitopological one, see [18].*

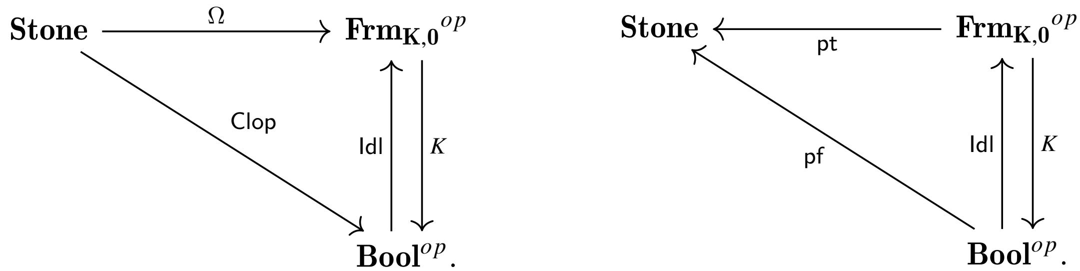
We now illustrate how these dualities are tied to the dual adjunction between frames and spaces. For a frame L, an element $a \in L$ is *compact* if whenever $D \subseteq L$ is a directed family, $a \leq \bigvee D$ implies $a \leq d$ for some $d \in D$. We say that a frame L is *compact* if the top element 1 is compact. A frame L is *coherent* if its compact elements form a sublattice of L which join-generates it. In particular, 1 is a compact element, and so coherent frames are compact. Coherent frames are also characterized by them being isomorphic to $\mathsf{Idl}(L)$ for some distributive lattice D. This, in fact, is a functor $\mathsf{Idl} : \mathbf{Distr} \to \mathbf{CohFrm}$ from the category of coherent frames to that of distributive lattices. Note that $\mathbf{CohFrm}$ is not a full subcategory of $\mathbf{Frm}$, as its morphisms are those frame morphisms $f : L \to M$ where $f(a)$ is compact whenever a is compact. The functor Idl is, in fact, part of an equivalence of categories, and its inverse is the functor $K : \mathbf{CohFrm} \to \mathbf{Distr}$ sending each frame to its sublattice of compact elements.

Proposition 1.15. *The following commute, up to natural isomorphism.*



A frame is *zero-dimensional* if it is join-generated by its complemented elements. The equivalence $K : \mathbf{CohFrm} \cong \mathbf{Distr} : \mathsf{Idl}$ restricts to an equivalence between the category $\mathbf{Frm_{K,0}}$ compact, zero-dimensional frames and that of Boolean algebras.

Proposition 1.16. *The following commute, up to natural isomorphism.*



Canonical extensions

Canonical extensions for Boolean algebras were introduced by Jónsson and Tarski (see [33] and [34]) in dealing with Boolean algebras with operators. The main idea behind canonical extensions of Boolean algebras is that they are pointfree representations of the embedding of the lattice of clopen sets of a Stone space into the powerset of this space. In fact, if we assume the Prime Ideal Theorem, the canonical extension of a Boolean algebra is the powerset of its Stone spectrum. Without assuming the Prime Ideal Theorem, the canonical extension of a Boolean algebra may be seen as the Booleanization of $\mathcal{U}(\mathsf{Filt}(B)\backslash\{B\})$ – or, equivalently, the Booleanization of $\mathcal{U}(\mathsf{Idl}(B))$ – as shown in [13]. Canonical extensions have also been introduced for distributive lattices. On this topic, we refer the reader to [25], [24], and [26]. For a distributive lattice, the canonical extension represents the embedding of the lattice of compact opens into the saturated sets. The canonical extension of a distributive lattice, provided the Prime Ideal Theorem holds, is isomorphic to the lattice of saturated sets of the corresponding coherent space.

We may ask ourselves what is the canonical extension of a frame, that is, what is the algebraic representation of the embedding of the frame of opens of an *arbitrary* space into the lattice of saturated sets. We shall see that our answer to this is that the question is so general that there is no unique way of extending a frame in such a way. In [29], the question of what is the canonical extension of a frame is tackled for locally compact frames. The canonical extension of a frame is defined as follows. It is a monotone map $f^\delta : L \to L^\delta$ to a complete lattice L^δ such that the following two properties hold:

1. Density: every element of L^δ is a join of elements in $\{\bigwedge f[F] : F \in \mathsf{Filt}_{\mathcal{SO}}(L)\}$;

2. Compactness: for every Scott-open filter F we have $\bigwedge f[F] \leq f(a)$ implies $a \in F$, for each $a \in L$.

That the two properties above are satisfied by the embedding $\Omega(X) \subseteq \mathcal{U}(X)$, for a sober space X, follows from the Hofmann-Mislove Theorem (Theorem 1.8 in this work). In [29], the canonical extension of a frame is proven to be unique. Let us see the theorem more in detail. Here, $\mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L))$ denotes the closure of $\mathsf{Filt}_{\mathcal{SO}}(L)$ under meets in the frame $\mathsf{Filt}(L)$, namely, set-theoretical intersections.

Theorem 1.17. *([29], Theorem 4.2) For a frame L, its canonical extension is unique, up to isomorphism. This is the map*

$$L \to \mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L))^{op},$$
$$a \mapsto \bigcap\{F \in \mathsf{Filt}_{\mathcal{SO}}(L) : a \in F\}.$$

We also have the following fact. A frame L is *pre-spatial* if whenever $a \nleq b$ there is a Scott-open filter containing a and omitting b, for all $a, b \in L$.

Proposition 1.18. *([29], Proposition 5.1) The map $f^\delta : L \to L^\delta$ is an injection if and only if the starting frame is pre-spatial.*

Our work stems from a generalization of this notion of canonical extension. We require that a Raney extension is an embedding into a coframe, that it preserves the frame operations, that the frame L meet-generates this coframe, and we further ask for it to preserve strongly exact meets. In that respect, our definition is more restrictive. But we do not regard the behavior of $\Omega(X) \subseteq \mathcal{U}(X)$ with respect to Scott-open filters as being a defining property of this embedding. In fact, we do not impose any other additional restrictions apart from the properties we mentioned. This is why a frame admits several Raney extensions.

2 Raney extensions

For a complete lattice C, we say that $L \subseteq C$ is a *subframe* of C if L equipped with the inherited order is a frame, and if the embedding $L \subseteq C$ preserves all joins and finite meets. A *Raney extension* is a pair (L, C) such that C is a coframe and L is a subframe of C such that:

- The frame L meet-generates C;
- The embedding $L \subseteq C$ preserves strongly exact meets.

. We will sometimes use the expression *Raney extension* to refer to the coframe component of the pair, and for a pair (L, C) we will say that this is a Raney extension of L, or that it is a Raney extension *over* L. The following is the motivating example behind the introduction of Raney extensions.

Example 2.1. *For a topological space X, the pair $(\Omega(X), \mathcal{U}(X))$ is a Raney extension. That strongly exact meets are preserved is the content of Proposition 5.3 of [3].*

For a collection $\mathcal{F} \subseteq \mathsf{Filt}(L)$, we introduce the following two properties.

1. $\mathcal{F}$-density: the collection $\{\bigwedge F : F \in \mathcal{F}\}$ join-generates C;
2. $\mathcal{F}$-compactness: whenever $\bigwedge F \leq a$ we also have $a \in F$ for every $F \in \mathcal{F}$ and every $a \in L$.

We say that a Raney extension is $\mathcal{F}$-canonical if and only if it is both $\mathcal{F}$-dense and $\mathcal{F}$-compact. The name comes from the fact that, in an $\mathcal{F}$-canonical Raney extension, every element can be written in a unique (canonical) way as a meet of a filter in $\mathcal{M}(\mathcal{F})$. Existence and uniqueness of weaker versions of $\mathcal{F}$-canonical Raney extensions are well-known, and these results stem from the theory of polarities by Birkhoff (see [17]). For a general version of the existence and uniqueness result for general polarities, see for instance Section 2 of [23], see [24] for its application to distributive lattices. From particularizing the analysis of [24] to the case where we start from a frame, we directly obtain the following.

Theorem 2.1. *(see for example [24], in particular Remark 2.8) For a frame L and a collection $\mathcal{F} \subseteq L$ of its filters containing the principal ones, there is a unique injective monotone map $f^{\mathcal{F}} : L \to L^{\mathcal{F}}$ to a complete lattice $L^{\mathcal{F}}$ which is $\mathcal{F}$-canonical. Concretely, this is the embedding $L \subseteq \mathcal{M}(\mathcal{F})^{op}$ of the principal filters into $\mathcal{F}$. This embedding preserves the frame operations.*

We now wish to refine this result to our case, where we have as an extra requirement that f ought to preserve strongly exact meets and that $L^{\mathcal{F}}$ ought to be a coframe. An extension as required will not exist for all choices of $\mathcal{F}$. In our account, we will consider several different choices of $\mathcal{F}$. The first example that we shall see corresponds to the collection $\mathsf{Filt}_{\mathcal{SO}}(L)$ of Scott-open filters. For brevity, in the following we will refer to $\mathsf{Filt}_{\mathcal{SO}}(L)$-canonicity simply as $\mathcal{SO}$-canonicity, and analogously for all other similarly denoted collections of filters.

Example 2.2. *For a sober space X, the pair $(\Omega(X), \mathcal{U}(X))$ is a $\mathcal{SO}$-canonical Raney extension, as already observed in Example 3.5 of [29]. In fact, for every Scott-open filter F we have that the* *intersection $\bigcap F$ is a compact saturated set, by the Hofmann-Mislove theorem (Theorem 1.8). It is a straightforward calculation that this implies $\mathcal{SO}$-canonicity. Recall that the Hofmann-Mislove theorem is dependent on the Prime Ideal Theorem. As we shall see, this means that $\mathcal{SO}$-canonicity of the pair $(\Omega(X), \mathcal{U}(X))$, too, is dependent on this. We will prove that if we replace* Scott-open *by* completely prime*, we have an analogous result which does not rely on the Prime Ideal Theorem.*

Every Raney extension (L, C) may be seen as a lattice embedding of a distributive lattice L^{op} into a frame C^{op}. The universal property of the ideal completion of a distributive lattice means that there is a coframe map $\mathsf{F}(f) : \mathsf{Filt}(L)^{op} \to C$ which extends this embedding. This means that all Raney extensions of a frame L are coframe quotients of $\mathsf{Filt}(L)^{op}$. For a Raney extension (L, C), we may consider the collection of filters which are fixpoints of the adjunction $\mathsf{F}(f)^* \dashv \mathsf{F}(f)$. We will abbreviate it as $C^* \subseteq \mathsf{Filt}(L)^{op}$. For a collection of filters $\mathcal{F}$ and an arbitrary filter G of a frame, we introduce the following closure operator in $\mathsf{Filt}(L)$, defined for any arbitrary filter G as

$$\mathrm{cl}_{\mathcal{F}}(G) = \bigcap \{F \in \mathcal{F} : G \subseteq F\}.$$

In case $\mathcal{F} = \mathsf{Filt}_{\mathcal{SO}}(L)$, as an abbreviation we will denote this closure operator as $\mathrm{cl}_{\mathcal{SO}}$, and similarly for similarly named collections of filters. Additionally, for a Raney extension (L, C) and for $x \in C$, we will denote $\uparrow x \cap L$ simply as $\uparrow^L x$.

Lemma 2.2. *For a Raney extension (L, C), we have an adjunction*

$$\bigwedge : \mathsf{Filt}(L)^{op} \leftrightarrows C : \uparrow^L,$$

which restricts to a pair of mutually inverse isomorphisms

$$\bigwedge : C^* \leftrightarrows C : \uparrow^L.$$

These are also isomorphisms of Raney extensions $\bigwedge : (L, C^*) \leftrightarrows (L, C) : \uparrow^L$.

Proof. Let $e : L \subseteq C$ be the subset embedding and consider the adjunction $\mathsf{F}(e) : \mathsf{Filt}(L)^{op} \leftrightarrows C : \mathsf{F}(e)^*$. Because $\mathsf{F}(e)$ is a surjection, all elements of C must be fixpoints, and the fixpoints in $\mathsf{Filt}(L)$ are the filters in C^*. Therefore, the adjunction restricts to an isomorphism $\mathsf{F}(e)^* : C \leftrightarrows C^* : \mathsf{F}(e)$. We have that $\mathsf{F}(e)(F) = \bigwedge F$ for every filter $F \subseteq L$. Let us check that the left adjoint is as required. Applying the general definition of the left adjoint of a $\bigwedge$-preserving morphism to our case yields the following (the first meet is computed in $\mathsf{Filt}(L)^{op}$):

$$\mathsf{F}(e)^*(c) = \bigwedge \{F \in \mathsf{Filt}(L) : c \leq \bigwedge F\} = \uparrow^L x.$$

It remains to show that the isomorphisms $\uparrow^L : C \leftrightarrows C^* : \bigwedge$ restrict correctly to the frame components. We notice that for $a \in L$, indeed, the filter $\uparrow^L a$ is a principal filter of a. Similarly, for a principal filter $\uparrow a \subseteq L$ it is easy to check that $a = \bigwedge \uparrow^L a$. Finally, it is well-known that the embedding $a \mapsto \uparrow a$ of L into $\mathsf{Filt}(L)^{op}$ preserves the frame operations, hence so does its inverse. ◻

Proposition 2.3. *For any Raney extension (L, C) and any collection $\mathcal{F} \subseteq \mathsf{Filt}(L)$,*

1. *(L, C) is $\mathcal{F}$-dense if and only if $C^* \subseteq \mathcal{M}(\mathcal{F})$;*
2. *(L, C) is $\mathcal{F}$-compact if and only if $\mathcal{F} \subseteq C^*$.*

In particular, (L, C) is $\mathcal{F}$-canonical if and only if $\mathcal{M}(\mathcal{F}) = C^*$, and every Raney extension (L, C) is C^*-canonical.

Proof. In light of the isomorphism given by 2.2, the claim about $\mathcal{F}$-density is clear. To see the equivalence stated in the second claim, we observe that for any filter $F \subseteq L$ we always have $F \subseteq \uparrow^L \bigwedge F$. For any collection $\mathcal{F} \subseteq \mathsf{Filt}(L)$ it is the case that for all $F \in \mathcal{F}$ we have the reverse set inclusion if and only if the Raney extension is $\mathcal{F}$-compact. But this is also equivalent to having that all filters in $\mathcal{F}$ are fixpoints of $\uparrow^L \dashv \bigwedge$, i.e. them being elements of C^*. ◻

Lemma 2.4. *Suppose that (L, C) is a Raney extension. All the following hold.*

- *C^* contains all principal filters;*
- *$C^* \subseteq \mathsf{Filt}(L)^{op}$ is a subcolocale inclusion;*
- *All filters in C^* are strongly exact.*

Proof. For the first item, we only notice that it is clear that $a = \bigwedge \uparrow^L a$ for all $a \in L$. For the second item, it is known that whenever we have a frame surjection $s : L \twoheadrightarrow S$ the inclusion $s_*[S] \subseteq L$ is a sublocale inclusion. Let us show the third item. Suppose that $F \in C^*$, and that $x_i \in F$ is a family such that the meet $\bigwedge_i^L x_i$, as calculated in L, is strongly exact. By definition of Raney extension, this meet is preserved by the embedding $e : L \subseteq C$. This means that $\bigwedge_i^L x_i = \bigwedge_i x_i$, where the second meet is computed in C. Therefore, since $\bigwedge F \leq x_i$ for all $i \in I$, we also have $\bigwedge F \leq \bigwedge_i x_i$. Since $F \in C^*$, we have $F = \uparrow^L \bigwedge F$, and so $\bigwedge_i^L x_i \in F$. ◻

In light of Proposition 2.3, Lemma 2.4 above is telling us that, for a frame L, the existence of an $\mathcal{F}$-canonical Raney extension requires three properties: (1) that $\mathcal{M}(\mathcal{F})$ contains all principal filters, (2) that $\mathcal{M}(\mathcal{F}) \subseteq \mathsf{Filt}(L)$ is a sublocale inclusion, (3) that $\mathcal{M}(\mathcal{F}) \subseteq \mathsf{Filt}_{\mathcal{SE}}(L)$. We shall now see that these are also sufficient conditions.

Theorem 2.5. *For any collection $\mathcal{F}$ such that:*

1. *$\mathcal{M}(\mathcal{F})$ contains all principal filters;*
2. *$\mathcal{M}(\mathcal{F}) \subseteq \mathsf{Filt}(L)$ is a sublocale inclusion;*
3. *$\mathcal{M}(\mathcal{F}) \subseteq \mathsf{Filt}_{\mathcal{SE}}(L)$;*

the $\mathcal{F}$-canonical Raney extension exists. It is unique, up to isomorphism. Concretely, it is described as the structure coming from the theory of polarities, that is, the pair $(L, \mathcal{M}(\mathcal{F})^{op})$.

Proof. We will first show that for a collection $\mathcal{F} \subseteq \mathsf{Filt}(L)^{op}$ satisfying the three properties above the pair $(L, \mathcal{M}(\mathcal{F})^{op})$ is a Raney extension. We know from Theorem 2.1 that $L \subseteq \mathcal{M}(\mathcal{F})^{op}$ preserves the frame operations (and this is also easy to check), and that L meet-generates the coframe component. We show that the embedding $L \subseteq \mathcal{M}(\mathcal{F})^{op}$ preserves strongly exact meets. Suppose that $x_i \in L$ is a family such that their meet $\bigwedge_i^L x_i$ is strongly exact. As all filters in $\mathcal{M}(\mathcal{F})$ are strongly exact, any such filter which contains $\uparrow x_i$ for all $i \in I$ must also contain $\bigwedge_i^L x_i$. This means that in the frame $\mathcal{M}(\mathcal{F})$ the least upper bound of the family $\{\uparrow x_i : i \in I\}$ is the principal filter $\uparrow \bigwedge_i^L x_i$. This means that the meet $\bigwedge_i^L x_i$ is preserved. Uniqueness follows from Theorem 2.1. □

2.1 Notable examples of Raney extensions

We now introduce the concrete constructions at the center of this work. We would now like to introduce for a generic frame L several collections $\mathcal{F}$ of filters, and study $\mathcal{F}$-canonical extensions for each instance. Since the collection $\mathsf{Filt}(L)$ is a frame, there is a Heyting operation $\to$ on it. In the following, whenever we write $F \to G$ for two filters F and G, it will be understood that we are referring to this operation. Notice that for a frame L and for $a, b \in L$ we have

$$\uparrow a \to \uparrow b = \{x \in L : b \leq x \vee a\}.$$

Lemma 2.6. *A filter is exact if and only if it is the intersection of filters of the form $\uparrow a \to \uparrow b$ for some $a, b \in L$. In particular, if F is an exact filter,*

$$F = \bigcap \{a \to b : b \leq a \vee f \text{ for all } f \in F\}.$$

Proof. Straightforward calculations show that each filter of the form $\uparrow a \to \uparrow b$ is exact, and this together with the fact that exact filters are closed under intersections shows one part of the claim. Let us consider, now, an arbitrary exact filter F. If $x \in F$ and if $b \leq a \vee f$ for all $f \in F$ then in particular $x \in \uparrow a \rightarrow \uparrow b$. For the other inclusion, suppose that whenever $b \leq a \vee f$ for all $f \in F$ we also have that $b \leq a \vee x$. We claim that $x = \bigwedge(\uparrow x \cap F) = \bigwedge\{x \vee f : f \in F\} =: m$, and that this is an exact meet. Notice that we have $m \leq x \vee f$ for all $f \in F$, and by hypothesis on x this implies that $m \leq x$. Therefore, $x = m$. It remains to show that this meet is exact. Let $y \in L$. To show exactness of the meet $\bigwedge\{x \vee f : f \in F\}$ it suffices to show that

$$\bigwedge\{x \vee f \vee y : f \in F\} \leq x \vee y,$$

given that $m = x$. Notice that for all $f \in F$ we have

$$\bigwedge\{x \vee f \vee y : f \in F\} \leq f \vee y \vee x$$

By hypothesis on x, this implies

$$\bigwedge\{x \vee f \vee y : f \in F\} \leq x \vee y \vee x = x \vee y,$$

and this completes the proof. □

We say that a filter is *regular* if it is a regular element in the frame of filters (that is, if it is of the form $F \rightarrow \{1\}$ for some filter F). Let us call $\mathsf{Filt}_{\mathcal{R}}(L)$ the ordered collection of regular filters. Note that $\mathsf{Filt}_{\mathcal{R}}(L) \subseteq \mathsf{Filt}(L)$ is the Booleanization of the frame of $\mathsf{Filt}(L)$.

Proposition 2.7. *The regular filters coincide with the intersections of filters of the form* $\{x \in L : x \vee a = 1\}$ *for some* $a \in L$.

Proof. We claim that for a principal filter $\uparrow a$ we have $\uparrow a \rightarrow \{1\} = \{x \in L : x \vee a = 1\}$. The pseudocomplement, in fact, is calculated as

$$\uparrow a \rightarrow \uparrow 1 = \{x \in L : 1 \leq a \vee x\}.$$

An arbitrary pseudocomplement is of the form $F \rightarrow \{1\}$ for some filter F, and because Heyting implication reverses all joins on the first component, this is the intersection of all the filters $\uparrow f \rightarrow \{1\}$ for $f \in F$. □

From the characterization in Proposition 2.7 above, together with that of exact filters in Lemma 2.6, it is immediate that every regular filter is exact. As sublocales are closed under all meets this gives a sublocale inclusion $\mathsf{Filt}_{\mathcal{R}}(L) \subseteq \mathsf{Filt}_{\mathcal{E}}(L)$. Let us now look at how the other collections of filters we are interested in are related between each other.

Lemma 2.8. *Suppose that for a frame L the subcollection $S \subseteq L$ is such that $x \to s \in \mathcal{M}(S)$ for all $x \in L$ and $s \in S$. Then, the inclusion $\mathcal{M}(S) \subseteq L$ is a sublocale.*

Proof. Suppose that for a frame L the subcollection $S \subseteq L$ is such that $x \to s \in \mathcal{M}(S)$ for all $x \in L$. Then, as Heyting implication preserves meets on the second component, we have that the collection $\mathcal{M}(S)$ is stable under the operation $x \to -$, as for a collection $T \subseteq S$ we have

$$x \to \bigwedge T = \bigwedge \{x \to t : t \in T\},$$

and this is in $\mathcal{M}(S)$ by our hypotheses. □

Lemma 2.9. *For a frame L, if a collection $\mathcal{F} \subseteq \mathsf{Filt}(L)$ is stable under the operation ${\uparrow}x \to -$ for each $x \in L$, the inclusion $\mathcal{M}(\mathcal{F}) \subseteq \mathsf{Filt}(L)$ is a sublocale.*

Proof. Suppose that $\mathcal{F} \subseteq \mathsf{Filt}(L)$ is stable under ${\uparrow}x \to -$. Because Heyting implication reverses all joins on the first component, for every $F \in \mathcal{F}$ we have

$$G \to F = \bigcap \{{\uparrow}g \to F : g \in G\}.$$

Therefore the conditions of Lemma 2.8 are satisfied, and this means that the inclusion $\mathcal{M}(\mathcal{F}) \subseteq \mathsf{Filt}(L)$ is a sublocale. □

Lemma 2.10. *The collection $\mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L))$ is a sublocale of $\mathsf{Filt}(L)$.*

Proof. By Lemma 2.9, it suffices to prove that the collection $\mathsf{Filt}_{\mathcal{SO}}(L)$ is stable under the operation ${\uparrow}x \to -$ for all $x \in L$. Let F be a Scott-open filter and let $x \in L$. We have that ${\uparrow}x \to F = \{y \in L : x \vee y \in F\}$. Let $D \subseteq L$ be a directed family, and suppose that $x \vee \bigvee D \in F$. Then, as F is Scott-open, there is $d \in D$ such that $y \vee d \in F$, which means $d \in {\uparrow}x \to F$. □

Lemma 2.11. *The collection $\mathcal{M}(\mathsf{Filt}_{\mathcal{CP}}(L))$ is a sublocale of $\mathsf{Filt}(L)$.*

Proof. By Lemma 2.9, it suffices to show that whenever P is a completely prime filter and $x \in L$, the filter $\{y \in L : x \vee y \in F\} \in P$ is completely prime. It is a straightforward calculation to show that this filter is indeed inaccessible by arbitrary joins. □

The following is a consequence of Theorem 1.9

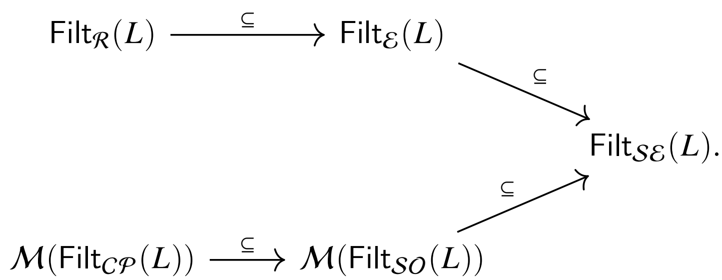
Theorem 2.12. *Every Scott-open filter is strongly exact.*

We immediately deduce the following.

Lemma 2.13. *We have a sublocale inclusion $\mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L)) \subseteq \mathsf{Filt}_{\mathcal{SE}}(L)$.*

We come to the following theorem, which gathers the results we have got so far on the notable collections of filters. By Lemma 1.3, we obtain the following.

Theorem 2.14. *For any frame L, we have the following poset of sublocale inclusions:*



By Theorem 2.5, for any of the collections $\mathcal{F}$ of filters above it is sufficient that it contains all principal filters for $(L, \mathcal{F}^{op})$ to be a Raney extension, where we have identified each element of L with the principal filter it determines. Thus, $(L, \mathsf{Filt}_{\mathcal{SE}}(L)^{op})$ and $(L, \mathsf{Filt}_{\mathcal{E}}(L)^{op})$ are Raney extensions for any frame L. Let us study when the other collections in the diagram above define Raney extensions similarly. In the following, for a frame L and a collection of filters $\mathcal{F}$, we say that it is *$\mathcal{F}$-separable* if whenever $a, b \in L$ are such that $a \nleq b$ then there is $F \in \mathcal{F}$ such that $a \in F$ and $b \notin F$.

Lemma 2.15. *For a frame L and a collection of filters $\mathcal{F} \subseteq \mathsf{Filt}(L)$, we have that $\mathcal{M}(\mathcal{F})$ containing all principal filters is equivalent to the property of L being $\mathcal{F}$-separable.*

Proof. The equivalence holds because the property of being $\mathcal{F}$-separable may be rephrased as having $\mathrm{cl}_{\mathcal{F}}(\uparrow a) \subseteq \uparrow a$ for all $a \in L$, and because the reverse set inclusion always holds. □

Proposition 2.16. *For a frame L we have the following.*

- *L is pre-spatial if and only if $\mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L))$ contains all principal filters.*
- *L is spatial if and only if $\mathcal{M}(\mathsf{Filt}_{\mathcal{CP}}(L))$ contains all principal filters.*
- *L is subfit if and only if $\mathsf{Filt}_{\mathcal{R}}(L)$ contains all principal filters.*

Proof. Notice that all three statements are special cases of Lemma 2.15, with $\mathcal{F}$ chosen to be the collection $\mathsf{Filt}_{\mathcal{SO}}(L)$ in the first case, the collection $\mathsf{Filt}_{\mathcal{CP}}(L)$ in the second, and the collection of filters of the form $\{x \in L : x \vee a = 1\}$ in the third. Indeed, closing the collection of filters of the last form under intersections yields $\mathsf{Filt}_{\mathcal{R}}(L)$, by Proposition 2.7. □

We now prove a point-set version of the result of Theorem 1.13 on strongly exact filters, with subsets of the spectrum of L replacing sublocales of L. In the following φ is the spatialization map of the frame L.

Theorem 2.17. *For a frame L, there is an adjunction*

$$\mathit{fitt}_{sp} : \mathsf{Filt}(L)^{op} \leftrightarrows \mathcal{P}(\mathsf{pt}(L)) : \mathit{ker}_{sp}$$
$$F \mapsto \bigcap \varphi[F],$$
$$\{a \in L : X \subseteq \varphi(a)\} \leftarrow\!\shortmid X,$$

with $\mathit{ker}_{sp} \dashv \mathit{fitt}_{sp}$ *which maximally restricts to an isomorphism* $\mathcal{M}(\mathsf{Filt}_{\mathcal{CP}}(L))^{op} \cong \mathcal{U}(\mathsf{pt}(L))$.

Proof. Let $X \subseteq \mathsf{pt}(L)$ and $F \in \mathsf{Filt}(L)$. We have that $F \subseteq \mathit{ker}_{sp}(X)$ if and only if for all $f \in F$ we have $X \subseteq \varphi(f)$, and this holds if and only if for each prime $p \in X$ and all $f \in F$ we have $f \nleq p$. This holds if and only if $X \subseteq \mathit{fitt}_{sp}(F)$. It is clear that the fixpoint on the $\mathcal{P}(\mathsf{pt}(L))$ side are the saturated sets, as any saturated set can be written as $\bigcap \varphi[F]$ for some filter $F \subseteq L$. On the other side, first we show that any fixpoint is an intersection of completely prime filters. We observe that for $X \subseteq \mathsf{pt}(L)$ the set $\{a \in L : X \subseteq \varphi(a)\}$ is the intersection $\bigcap\{L \setminus {\downarrow} p : p \in X\}$. On the other hand, given a family $L \setminus {\downarrow} p_i$ of completely prime filters, this is $\{a \in L : \{p_i : i \in I\} \subseteq \varphi(a)\}$. □

Let us now consider the motivating example behind the definition of Raney extension.

Lemma 2.18. *For every topological space X the pair* $(\Omega(X), \mathcal{U}(X))$ *is* $\mathcal{CP}$*-dense.*

Proof. Let X be a topological space. Consider some family $U_i \subseteq X$ of open sets. We will show that the filter ${\uparrow}^{\Omega(X)} \bigcap_i U_i$ is an intersection of completely prime filters. It suffices to notice that the filter may be written as

$$\bigcap\{N(x) : x \in \bigcap_i U_i\},$$

where $N(x)$ denotes the neighborhood filter of a point $x \in X$. □

Proposition 2.19. *A topological space X is sober if and only if* $(\Omega(X), \mathcal{U}(X))$ *is* $\mathcal{CP}$*-compact. In particular, a space X is sober if and only if the pair* $(\Omega(X), \mathcal{U}(X))$ *is a* $\mathcal{CP}$*-canonical Raney extension.*

Proof. Suppose that X is a sober space, and let $P \subseteq \Omega(X)$ be an intersection of completely prime filters. We have to show that $P \subseteq U$ implies that $U \in P$ for every open $U \subseteq X$. By sobriety, each completely prime filter is the neighborhood filter of some point, and so we may assume

$$P = \bigcap \{N(x) : x \in Y\} = \{U \in \Omega(X) : Y \subseteq U\}$$

for some subset $Y \subseteq X$. Notice that an open set contains Y precisely when it contains its saturation $\bigcap\{U \in \Omega(X) : Y \subseteq U\} = \bigcap P$. Therefore $\bigcap P \subseteq U$ implies $U \in P$ for every open $U \subseteq X$, and this means that we have $\mathcal{CP}$-compactness. Finally, assume that for some space X the pair $(\Omega(X), \mathcal{U}(X))$ is $\mathcal{CP}$-compact. Let $P \subseteq \Omega(X)$ be a completely prime filter. We prove that there is $x \in \bigcap P$ such that $\bigcap P = \uparrow x$. Observe that for all $y \in \bigcap P$ we have $\uparrow y \subseteq \bigcap P$. Towards contradiction, assume that for all $y \in \bigcap P$ we have that $\bigcap P \nsubseteq \uparrow y$. This means that for all $y \in \bigcap P$ there is an open $U_y \subseteq X$ such that $y \in U_y$ and $\bigcap P \nsubseteq U_y$. As P is completely prime, this implies that $\bigcup\{U_y : y \in \bigcap P\} \notin P$. By $\mathcal{CP}$-compactness, this implies that $\bigcap P \nsubseteq \bigcup_y U_y$. This is a contradiction, as for each $y \in \bigcap P$ we have $\uparrow y \subseteq U_y$, and taking unions on both sides yields $\bigcap P \subseteq \bigcup_y U_y$. Thus, X must be sober. The last part of the claim follows by combining this characterization of sobriety with Lemma 2.18. □

2.2 The category of Raney extensions

It is now time to consider Raney extensions as objects of a category. A morphism $f : (L, C) \to (M, D)$ is a coframe map $f : C \to D$ such that whenever $a \in L$ we have $f(a) \in M$ and such that the restriction $f|_L : L \to M$ is a frame map. We call **Raney** the category of Raney extensions with Raney maps. We would now like to explore the question of when we can extend a certain assignment $\mathsf{Obj}(\mathbf{Frm}) \to \mathsf{Obj}(\mathbf{Raney})$ to a functor. The question of when frame maps can be extended to maps of Raney extensions amounts to when frame maps can be lifted to maps between collections of filters. For frames L and M, and for sublocales $\mathcal{F} \subseteq \mathsf{Filt}(L)$ and $\mathcal{G} \subseteq \mathsf{Filt}(M)$ such that they contain all the principal filters, we say that a morphism $f : L \to M$ *lifts* to a morphism $F : \mathcal{F}^{op} \to \mathcal{G}^{op}$ when a coframe morphism F exists such that the following square commutes:

$$\begin{array}{ccc} \mathcal{F}^{op} & \xrightarrow{F} & \mathcal{G}^{op} \\ \uparrow & & \uparrow \\ L & \xrightarrow{f} & M. \end{array}$$

If such lifting exists then by commutativity of the diagram it has to be defined as $F \mapsto \mathrm{cl}_{\mathcal{G}}(f[F])$. As $\mathcal{G}$ is closed under arbitrary intersections, this map always exists. It is, however, not a coframe map in general.

Lemma 2.20. *Suppose that we have coframes C and D, such that $L \subseteq C$ is a subframe that meet-generates C. Suppose that a map $f : C \to D$ preserves all meets, as well as all the joins of L. Then it is a coframe map.*

Proof. To show that it is a coframe map, it suffices to show that it preserves finite joins. Consider $x, y \in C$. We have $x = \bigwedge_i x_i$ and $y = \bigwedge_j y_j$ for families $x_i, y_j \in L$. Then

$$f(x \vee y) = f(\bigwedge_i x_i \vee \bigwedge_j y_j) = \bigwedge_{i,j} f(x_i) \vee f(y_j) = f(x) \vee f(y). \qquad \square$$

Lemma 2.21. *Suppose that we have a frame map $f : L \to M$ and that $\mathcal{F} \subseteq \mathsf{Filt}(L)$ and $\mathcal{G} \subseteq \mathsf{Filt}(M)$ are sublocale inclusions where $\mathcal{F}$ and $\mathcal{G}$ contain all principal filters. If the map $\mathrm{cl}_{\mathcal{G}}(f[-]) : \mathcal{F}^{op} \to \mathcal{G}^{op}$ preserves all meets, then it is a coframe map.*

Proof. It suffices to observe that we are in the situation described by Lemma 2.20. This is because the map $\mathrm{cl}_{\mathcal{G}}(f[-])$ lifts a frame map $f : L \to M$, and so its restriction to L is the composition of this join-preserving map with the join-preserving inclusion $M \subseteq \mathcal{M}^{op}$. $\square$

For a poset map to preserve all meets, it suffices for it to have a left adjoint. Let us consider $\mathrm{cl}_{\mathcal{G}}(f[-]) : \mathcal{F} \to \mathcal{G}$ as a frame map, so that the order is simply set inclusion, as $\mathrm{cl}_{\mathcal{G}}(F) \subseteq G$ if and only if $f[F] \subseteq G$, and this holds if and only if $F \subseteq f^{-1}(G)$. The issue with the preimage map is that it is not guaranteed to map filters of $\mathcal{G}$ to filters of $\mathcal{F}$.

Lemma 2.22. *Suppose that $f : L \to M$ is a frame map and that we have sublocales $\mathcal{F} \subseteq \mathsf{Filt}(L)$ and $\mathcal{G} \subseteq \mathsf{Filt}(M)$ containing all principal filters. Then there is a frame map F making the following commute:*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{F} & \mathcal{G} \\ \uparrow & & \uparrow \\ L^{op} & \xrightarrow{f} & M^{op}. \end{array}$$

if and only if $f^{-1}(F) \in \mathcal{G}$ for every $F \in \mathcal{F}$. If this map exists, it is $\mathrm{cl}_{\mathcal{G}}(f[-])$.

More generally, given a map $f : L \to M$ of frames we ask when is it that we can extend it to a map $f_C : (L, C) \to (M, D)$ of Raney extensions. If we have a coframe morphism $f_{C^*} : C^* \to D^*$ such that $f_{C^*}(\uparrow a) = \uparrow f(a)$ for every $a \in L$, then we also have a coframe morphism $f_C : C \to D$ such that the following commutes. Recall that the morphisms $\bigwedge$ are isomorphisms:

$$\begin{array}{ccc} C & \xrightarrow{f_C} & D \\ {\scriptstyle \bigwedge(\cong)}\uparrow & & {\scriptstyle \bigwedge(\cong)}\uparrow \\ C^* & \xrightarrow{f_{C^*}} & D^* \\ \uparrow & & \uparrow \\ L^{op} & \xrightarrow{f} & M^{op}. \end{array}$$

Proposition 2.23. *A frame map $f : L \to M$ can be extended to a map between Raney extensions $f_C : (L, C) \to (M, D)$ if and only if preimages of filters in D^* are in C^*. In case this holds, the map is defined as*

$$(L, C) \to (M, D),$$
$$c \mapsto \bigwedge f[\uparrow^L c].$$

Proof. It remains to show that this definition is equivalent to the definition in 2.22, in the sense that the two definitions are the same when we identify elements of D with their isomorphic image under $\uparrow^L : D \cong D^*$. We show, then, that $\mathrm{cl}_{\mathcal{F}}(f[\uparrow^L c]) = \uparrow^L \bigwedge f[\uparrow^L c]$. Expanding definitions

$$\begin{aligned} \mathrm{cl}_{\mathcal{F}}(f[\uparrow^L c]) &= \bigcap\{\uparrow^L d : d \in D, f[\uparrow^L c] \subseteq \uparrow^L d\} \\ &= \bigcap\{\uparrow^L d : d \in D, d \leq f(a) \text{ for all } a \in \uparrow^L c\} \\ &= \uparrow^L \bigvee\{d \in D : d \leq f(a) \text{ for all } a \in \uparrow^L c\} \\ &= \uparrow^L \bigwedge f[\uparrow^L c]. \end{aligned}$$ □

We apply Proposition 2.23 above to some of the Raney extensions that we have seen.

Lemma 2.24. *Any frame morphism $f : L \to M$ lifts to a coframe morphism*

$$f_{\mathcal{SE}} : \mathsf{Filt}_{\mathcal{SE}}(L)^{op} \to \mathsf{Filt}_{\mathcal{SE}}(M)^{op}.$$

Proof. By Proposition 2.23, a frame morphism $f : L \to M$ lifts as required if preimages of strongly exact filters are strongly exact. Suppose, then, that $F \subseteq M$ is strongly exact. Suppose that $x_i \in L$ is a family such that the meet $\bigwedge_i x_i$ is strongly exact, such that $f(x_i) \in F$. Because all frame morphisms preserve strongly exact meets, as well as strong exactness of meets, we have $\bigwedge_i f(x_i) = f(\bigwedge_i x_i) \in F$, as desired. □

Lemma 2.25. *Any frame morphism $f : L \to M$ between spatial frames lifts to a coframe morphism*

$$f_{\mathcal{CP}} : \mathcal{M}(\mathsf{Filt}_{\mathcal{CP}}(L))^{op} \to \mathcal{M}(\mathsf{Filt}_{\mathcal{CP}}(M))^{op}.$$

Proof. By Proposition 2.22, it suffices to show that if $f : L \to M$ is a frame map, preimages of completely prime filters are completely prime. Suppose, than, that $P \subseteq M$ is a completely prime filter, and that $f(\bigvee_i x_i) \in P$. We then have $\bigvee_i f(x_i) \in P$, therefore $f(x_i) \in P$ for some $i \in I$. Indeed, then, $f^{-1}(P)$ is completely prime. □

Lemma 2.26. *Any frame morphism $f : L \to M$ between pre-spatial frames lifts to a coframe morphism $f_{\mathcal{SO}} : \mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L))^{op} \to \mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(M))^{op}$.*

Proof. By Proposition 2.22, it suffices to show that for a frame morphism $f : L \to M$ between pre-spatial frames preimages of Scott-open filters are Scott-open. Suppose that $F \subseteq L$ is a Scott-open filter, and that $\{x_i : i \in I\} \subseteq L$ is a directed family such that $f(\bigvee_i x_i) = \bigvee f(x_i) \in F$. Observe that the family $\{f(x_i) : i \in I\}$ is directed, and so by Scott-openness of F we must have $f(x_i) \in F$ for some $i \in I$, as desired. □

It is clear that there is a forgetful functor $\pi_1 : \mathbf{Raney} \to \mathbf{Frm}$ which forgets about the second component of the extension. We know already that for a frame L the Raney extension $(L, \mathsf{Filt}_{\mathcal{SE}}(L)^{op})$ is in a certain sense the largest. We observe that the assignment $L \mapsto (L, \mathsf{Filt}_{\mathcal{SE}}(L)^{op})$ on objects can be extended to a functor by mapping each frame morphism to the morphism $f_{\mathcal{SE}}$ whose existence is established by Lemma 2.24. Let us see that this is the left adjoint of π_1.

Theorem 2.27. *For a frame L, the pair $(L, \mathsf{Filt}_{\mathcal{SE}}(L)^{op})$ is the free Raney extension over it. In particular, the category of frames is a full coreflective subcategory of* **Raney***.*

Proof. Suppose that we have a frame map $f : L \to M$. Let (M, D) be a Raney extension. By the characterization in Theorem 2.5 we have $D^* \subseteq \mathsf{Filt}_{\mathcal{SE}}(M)^{op}$, and by Lemma 2.24, preimages of strongly exact filters are strongly exact. Therefore, preimages of elements in D^* are in $\mathsf{Filt}_{\mathcal{SE}}(L)$. By Proposition 2.23, the frame map lifts as required. □

Remark 2.3. *In light of the isomorphism fitt* : $\mathsf{Filt}_{\mathcal{SE}}(L) \cong \mathsf{S}_{\mathfrak{o}}(L)$ *in Theorem 1.13, this suggests that the embedding* $\mathfrak{o} : L \hookrightarrow \mathsf{S}_{\mathfrak{o}}(L)$ *is to be interpreted as the most general way to extend a frame to a coframe of saturated sets. This comes from the requirement that a Raney extension should be such that* $L \subseteq C$ *preserves strongly exact meets. One can check that if we drop this requirement the equivalent of this result is that the free coframe generated by a frame (such that the embedding preserves the frame operations) is simply* $\mathsf{Filt}(L)$.

2.3 The assignment $L \mapsto (L, \mathsf{Filt}_{\mathcal{E}}(L)^{op})$

We already have seen that each frame has the largest Raney extension. Every frame also has the smallest Raney extension.

Lemma 2.28. *For a frame L the collection* $\mathsf{Filt}_{\mathcal{E}}(L)$ *is the smallest sublocale of* $\mathsf{Filt}(L)$ *containing all the principal filters.*

Proof. Let $\mathcal{S} \subseteq \mathsf{Filt}(L)$ be the smallest sublocale containing all the principal filters. For any $x, y \in L$, we must have $\uparrow x \to \uparrow y \in \mathcal{S}$. As sublocales are closed under all meets, all intersections of filters of the form $\uparrow x \to \uparrow y$ must be in $\mathcal{S}$. Therefore, by the characterization in Lemma 2.6, $\mathsf{Filt}_{\mathcal{E}}(L) \subseteq \mathcal{S}$. ◻

The following result is already implicitly in the literature. In [7], Theorem 3.7, it is shown that for a meet-semilattice S the smallest frame generated by it is $\mathcal{J}^e(S)$, the collection of all downsets which are closed under those joins of S that distribute over all finite meets. Recently the same result has been re-proven for frames with bases of meet-semilattices in [16].

Proposition 2.29. *For all Raney extensions* (L, C) *we have a surjection*

$$(L, C) \to (L, \mathsf{Filt}_{\mathcal{E}}(L)^{op}),$$
$$c \mapsto cl_{\mathcal{E}}(\uparrow^L c).$$

Proof. The collection C^* contains all principal filters, and so $\mathsf{Filt}_{\mathcal{E}}(L) \subseteq C^*$ by Lemma 2.28. Then by Proposition 2.23 we have a surjection $c \mapsto \mathrm{cl}_{\mathcal{E}}(\uparrow^L c)$. ◻

We finally reach the following boundary conditions for Raney extensions.

Theorem 2.30. *For a frame L, the collection of Raney extensions with base L is isomorphic to the section* $[\mathsf{Filt}_{\mathcal{E}}(L), \mathsf{Filt}_{\mathcal{SE}}(L)]$ *of the coframe of sublocales of* $\mathsf{Filt}_{\mathcal{SE}}(L)$.

Proof. That every Raney extension belongs to the section $[\mathsf{Filt}_{\mathcal{E}}(L), \mathsf{Filt}_{\mathcal{SE}}(L)]$ follows from Theorem 2.27 and Proposition 2.29. Suppose that we have a sublocale $\mathcal{F} \subseteq \mathsf{Filt}_{\mathcal{SE}}(L)$ such that $\mathsf{Filt}_{\mathcal{E}}(L) \subseteq \mathcal{F}$. By Lemma 2.22, there are two surjections

$$(L, \mathsf{Filt}_{\mathcal{SE}}(L)^{op}) \xrightarrow{\mathrm{cl}_{\mathcal{F}}} (L, \mathcal{F}^{op}) \xrightarrow{\mathrm{cl}_{\mathcal{E}}} (L, \mathsf{Filt}_{\mathcal{E}}(L)^{op}). \qquad \square$$

Remark 2.4. *We have seen in Remark 2.3 that the embedding* $\mathfrak{o} : L \hookrightarrow \mathsf{S}_{\mathfrak{o}}(L)$ *is the free Raney extension over L. Recall (Theorem 1.13) that the frame* $\mathsf{Filt}_{\mathcal{E}}(L)$ *is isomorphic to the frame* $\mathsf{S}_{\mathfrak{c}}(L)$*. Therefore, the embedding* $\mathfrak{c} : L \hookrightarrow \mathsf{S}_{\mathfrak{c}}(L)^{op}$ *has the dual property that it is the smallest Raney extension over* L*.*

The assignment $L \mapsto (L, \mathsf{Filt}_{\mathcal{E}}(L)^{op})$ is not always functorial, as shown in [6]). In [6], the authors study conditions under which every morphism $f : L \to M$ between subfit frames lifts to a morphism $\mathsf{S}_{\mathfrak{c}}(L) \to \mathsf{S}_{\mathfrak{c}}(M)$. We will now seek to reach a natural restriction on the category **Frm** so that Proposition 2.29 may be refined to the existence of a right adjoint to π_1 for a certain subcategory of **Raney**. In the following, for a frame morphism $f : L \to M$, we will say that it is *exact* if whenever the meet of a family $\{x_i : i \in I\} \subseteq L$ is exact, so is the meet of $\{f(x_i) : i \in I\}$, and furthermore $\bigwedge_i f(x_i) = f(\bigwedge_i x_i)$.

Lemma 2.31. *A morphism* $f : L \to M$ *is exact if and only if preimages of exact filters are exact. This holds if and only if the morphism can be extended to a morphism*

$$f_{\mathcal{E}} : (L, \mathsf{Filt}_{\mathcal{E}}(L)^{op}) \to (M, \mathsf{Filt}_{\mathcal{E}}(M)^{op}).$$

Proof. Suppose that $f : L \to M$ is an exact frame map, and that $G \subseteq M$ is an exact filter. Suppose that $\bigwedge_i x_i \in L$ is an exact meet such that $f(x_i) \in G$. By exactness of this map, the meet $\bigwedge_i f(x_i)$ is exact and so $\bigwedge_i f(x_i) \in G$. Again, by exactness of f, we have $\bigwedge_i f(x_i) = f(\bigwedge_i x_i)$. Indeed, then, $\bigwedge_i x_i \in f^{-1}(G)$. Conversely, suppose that we have a frame map $f : L \to M$ such that it is not exact. This means that either we have an exact meet $\bigwedge_i x_i \in L$ such that it is not preserved by f, or we have an exact meet $\bigwedge_i x_i \in L$ such that $\bigwedge_i f(x_i)$ is not exact. We consider these two cases in turn. In the first case, we consider the principal filter $\uparrow \bigwedge_i f(x_i)$. This is exact, as it is closed under all meets. We notice that by our hypothesis $f(\bigwedge_i x_i)$ is not an element of this filter. Let us call F the preimage of this filter. We have that $x_i \in F$ but $\bigwedge_i x_i \in F$, and so F is not exact. In the second case, consider an exact meet $\bigwedge_i x_i \in L$ such that $\bigwedge_i f(x_i)$ is not exact. In particular, let $y \in M$ be such that $\bigwedge_i (f(x_i) \vee y) \not\leq (\bigwedge_i f(x_i)) \vee y$. We now consider the exact filter

$$\uparrow y \to \uparrow \bigwedge_i (f(x_i) \vee y) = \{m \in M : \bigwedge_i (f(x_i) \vee y) \leq y \vee m\}.$$

That this is an exact filter follows from the characterization of Lemma 2.6. Let F be the preimage of this filter. We have that $x_i \in F$ for each $i \in I$. We claim that $\bigwedge_i x_i \notin F$. This follows from the fact that by our hypothesis $\bigwedge_i(f(x_i) \vee y) \nleq (\bigwedge_i f(x_i)) \vee y$ and $f(\bigwedge_i x_i) \leq \bigwedge_i f(x_i)$. □

Let us call $\mathbf{Frm}_{\mathcal{E}}$ the subcategory of $\mathbf{Frm}$ determined by restricting the morphisms to those preserving exactness of meets. Let us also call $\mathbf{Raney}_{\mathcal{E}}$ the subcategory of $\mathbf{Raney}$ determined by the morphisms such that their restriction to the frame component preserves exactness of meets.

Theorem 2.32. *The forgetful functor* $\pi_1 : \mathbf{Raney}_{\mathcal{E}} \to \mathbf{Frm}_{\mathcal{E}}$ *has a right adjoint, and this is the functor acting on objects as* $L \mapsto (L, \mathsf{Filt}_{\mathcal{E}}(L)^{op})$ *and acting on morphisms as* $f \mapsto f_{\mathcal{E}}$.

Proof. Suppose that we have an exact frame map $f : L \to M$, and that (L, C) is a Raney extension. By Lemma 2.31, as f is exact, preimages of filters in $\mathsf{Filt}_{\mathcal{E}}(M)$ are in $\mathsf{Filt}_{\mathcal{E}}(L)$. Furthermore, C^* must contain all principal filters, and so by Lemma 2.28 this implies that $\mathsf{Filt}_{\mathcal{E}}(L) \subseteq C^*$. Then, preimages of exact filters of M are in C^*. By Proposition 2.23, then, we have a map of Raney extensions $(L, C) \to (M, \mathsf{Filt}_{\mathcal{E}}(M)^{op})$. □

3 Generalizing Raney duality

3.1 Spectra of Raney extensions

We now extend Raney duality, as illustrated in [14], to our setting. For a Raney extension (L, C) we define $\mathsf{pt}_R(C)$ to be the collection of its completely join-prime elements and we topologize it as follows. We first define the function $\varphi_{(L,C)} : C \to \mathcal{P}(\mathsf{pt}_R(C))$ as

$$\varphi_{(L,C)}(a) = \{x \in \mathsf{pt}_R(C) : x \leq a\}.$$

It is easy to see that the following two properties hold:

1. $\varphi_{(L,C)}(\bigwedge_i a_i) = \bigcap_i \varphi_{(L,C)}(a_i)$,
2. $\varphi_{(L,C)}(\bigvee_i a_i) = \bigcup_i \varphi_{(L,C)}(a_i)$,

for each family $a_i \in L$. This implies that the elements of the form $\varphi_{(L,C)}(a)$ for $a \in L$ form a topology. We refer to the topological space we obtain as $\mathsf{pt}_R(L, C)$, and we call it the *spectrum* of the Raney extension (L, C). By property 1 it also follows that the elements of the form $\varphi_{(L,C)}(c)$ with $c \in C$ are the saturated sets of this space.

Lemma 3.1. *For a Raney extension (L, C), an element $x \in C$ is completely join-prime if and only if $\uparrow^L x$ is a completely prime filter.*

Proof. It is immediate that if $x \in C$ is completely join-prime then $\uparrow^L x$ is completely prime. For the converse, suppose that we have $x \in C$ such that $\uparrow^L x$ is completely prime. Suppose that $x \le \bigvee D$ for $D \subseteq C$. This means that $\uparrow^L \bigvee D \subseteq \uparrow^L x$. Observe that $\uparrow^L \bigvee D = \bigcap\{\uparrow^L d : d \in D\}$. As $\uparrow^L x$ is assumed to be completely prime, there must be some $d \in D$ such that $\uparrow^L d \subseteq \uparrow^L x$. This implies that $x \le d$. □

Lemma 3.2. *For a morphism $f : (L, C) \to (M, D)$ of Raney extensions, we have that if $x \in \mathsf{pt}_R(D)$ then $f^*(x) \in \mathsf{pt}_R(C)$.*

Proof. By Lemma 3.1, it suffices to show that for a morphism $f : (L, C) \to (M, D)$ of Raney extensions, if $x \in \mathsf{pt}_R(D)$ then $\uparrow^L f^*(x)$ is a completely prime filter of L. If $f^*(x) \le \bigvee A$ for $A \subseteq L$, then as f respects the frame operations of L, and because $f^* \dashv f$, we have that $x \le \bigvee\{f(a) : a \in A\}$. Since x is completely join-prime, there is some $a \in A$ such that $x \le f(a)$, that is $f^*(x) \le a$. □

Lemma 3.3. *The assignment $\mathsf{pt}_R : (L, C) \mapsto \mathsf{pt}_R(L, C)$ is the object part of a functor pt_R : $\mathbf{Raney}^{op} \to \mathbf{Top}$ which acts on morphisms as $f \mapsto f^*$.*

Proof. That every morphism is mapped to a well-defined function between the set of points follows from Lemma 3.2. Continuity follows from the fact that the f^*-preimage of some $\varphi(a)$ for $a \in L$ is, expanding definitions,

$$\{x \in \mathsf{pt}_R(D) : f^*(x) \le a\} =$$
$$\{x \in \mathsf{pt}_R(D) : x \le f(a)\} = \varphi(f(a)),$$

and this set is indeed open in $\mathsf{pt}_R(D)$ as by definition of Raney morphism $f(a) \in M$. □

For a topological space X we define $\Omega_R(X)$ as the pair $(\Omega(X), \mathcal{U}(X))$. The assignment $X \mapsto \Omega_R(X)$ is the object part of a functor Ω_R : $\mathbf{Top} \to \mathbf{Raney}^{op}$ which acts on morphisms as $f \mapsto f^{-1}$, this fact is easy to check from basic set-theoretical properties of preimages. The following follows from the definition of the topologizing map $\varphi_{(L,C)}$.

Lemma 3.4. *For every Raney extension (L, C) the assignment $c \mapsto \varphi_{(L,C)}(c)$ is a surjective map of Raney extensions $(L, C) \to \Omega_R(\mathsf{pt}_R(L, C))$.*

The map we have just defined will be the evaluation at an object of the natural transformation $\Omega_R \circ \mathsf{pt}_R \Rightarrow 1_{\mathbf{Raney}^{op}}$. Let us now define the other natural transformation $1_{\mathbf{Top}} \Rightarrow \mathsf{pt}_R \circ \Omega_R$.

Lemma 3.5. *For every topological space X the map $\psi_X : X \to \mathsf{pt}_R(\Omega_R(X))$ defined as $x \mapsto \uparrow x$ is a continuous map.*

Proof. That the map is well-defined follows from the observation that the completely join-prime elements of $\mathcal{U}(X)$ are precisely the principal upsets. Let us now abbreviate $\varphi_{(\Omega(X),\mathcal{U}(X))}$ as simply φ. For continuity, we observe that the ψ_X-preimage of an open set $\varphi(U)$ is the set $\{x \in X : \uparrow x \in \varphi(U)\} = U$. □

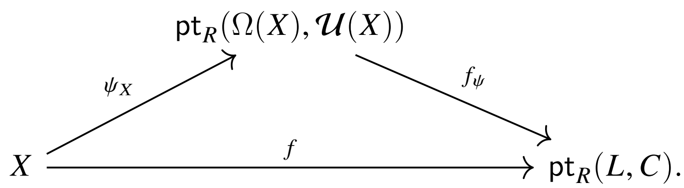
Remark 3.1. *In Lemma 3.4 above, as usual $\mathcal{U}(\mathsf{pt}_R(L, C))$ denotes the ordered collection of saturated sets of the space $\mathsf{pt}_R(L, C)$. Note that these sets are not upsets according to the order inherited from C, as the order or C is the opposite of the specialization order in $\mathsf{pt}_R(L)$. The situation is analogous for frames, as seen in the discussion after Proposition 1.7.*

Theorem 3.6. *The pair $(\Omega_R, \mathsf{pt}_R)$ constitutes an idempotent adjunction* $\mathbf{Top} \leftrightarrows \mathbf{Raney}^{op}$.

Proof. We claim that the two maps defined in Lemmas 3.4 and 3.5 are the required natural transformations, as defined for each component. Suppose that (L, C) is a Raney extension, and that there is a space X such that there is a map of Raney extensions $f : (L, C) \to (\Omega(X), \mathcal{U}(X))$. We denote the topologizing map simply as φ. Let us define a map f_φ such that the following commutes:

$$\begin{array}{ccc} & (\varphi[L], \varphi[C]) & \\ {}^{\varphi}\nearrow & & \searrow^{f_\varphi} \\ (L, C) & \xrightarrow{\quad f \quad} & (\Omega(X), \mathcal{U}(X)). \end{array}$$

We observe that on the coframe component the map φ is such that for all $c, d \in C$ we have $\varphi(c) \subseteq \varphi(d)$ if and only if for every completely join-prime element $x \in C$ we have $x \le c$ implies $x \le d$. The map f, then, factors through this map if and only if $f(c) \le f(d)$ whenever the second condition holds for $c, d \in C$. Suppose, then, that this holds. Suppose, now, that $y \in f(c)$. This implies that $\uparrow y \subseteq f(c)$, and so $f^*(\uparrow y) \le d$. As by Lemma 3.2 the element $f^*(\uparrow y)$ is completely join-prime, we have $f^*(\uparrow y) \le d$, that is, $y \in f(d)$. For spaces, consider a space X and a Raney extension (L, C), and suppose that there is a continuous map $f : X \to \mathsf{pt}_R(L, C)$. We show that there is a map f_ψ making the following commute.



For a completely join-prime element $\uparrow x$, we define $f_\psi(\uparrow x) = f(x)$. Routine calculations show that this map is continuous. $\square$

Proposition 3.7. *The fixpoints of the adjunction $\Omega_R \dashv \mathsf{pt}_R$ in* **Top** *are precisely the T_0 spaces. The map ψ_X is the T_0 reflection for each space X.*

Proof. For a space X, we consider the map $\psi_X : X \to \mathsf{pt}_R(\Omega_R(X))$ defined as $x \mapsto \uparrow x$. By Lemma 3.5 this is a continuous map. As all completely join-prime elements of $\mathcal{U}(X)$ are of the form $\uparrow x$ for some $x \in X$, it follows that the map is always surjective. The only way it can fail to be a homeomorphism, then, is when it is not injective. In general, $\uparrow x = \uparrow y$ for $x, y \in X$ means that $x \in U$ if and only if $y \in U$ for all opens $U \in \Omega(X)$. Therefore, the map is a homeomorphism precisely when the space is T_0. To check that this is a T_0 reflection, notice that any map $f : X \to Y$ to a T_0 space must identify all points identified by ψ_X, and that the topology on $\mathsf{pt}_R(\Omega_R(X))$ is the coarsest one that makes ψ_X continuous. $\square$

Let us now look at the fixpoints of the adjunction $\Omega_R \dashv \mathsf{pt}_R$ in the category **Raney**.

Proposition 3.8. *A Raney extension (L, C) is a fixpoint of the $\Omega_R \dashv \mathsf{pt}_R$ adjunction if and only if the coframe C is join-generated by its completely join-prime elements.*

Proof. Suppose that a Raney extension (L, C) is join-generated by its completely join-prime elements. Consider the spatialization map

$$\varphi_{(L,C)} : (L, C) \to (\Omega(\mathsf{pt}_R(L, C)), \mathcal{U}(\mathsf{pt}_R(L, C))).$$

This is a map of Raney extensions, and by its definition it is always a surjection. Therefore, it is an isomorphism if and only if it is injective. By definition of the map $\varphi_{(L,C)}$, we have that it is injective exactly when for $c, d \in C$ we have that $c \nleq d$ implies that $x \leq c$ and $x \nleq d$ for some completely join-prime element $x \in C$. $\square$

Because of Proposition 3.8 above, we define a Raney extension (L, C) to be *spatial* if the coframe C is join-generated by its completely join-prime elements. We note that Raney duality, as illustrated in [14], is a restriction of the dual adjunction $\Omega_R \dashv \mathsf{pt}_R$.

Proposition 3.9. *For a sublocale $\mathcal{F} \subseteq \mathsf{Filt}(L)$ containing all principal filters we have*

$$\mathsf{pt}_R(L, \mathcal{F}^{op}) = \mathcal{F} \cap \mathsf{Filt}_{C\mathcal{P}}(L).$$

The opens of this space are the sets of the form $\{P \in \mathsf{Filt}_{C\mathcal{P}}(L) \cap \mathcal{F} : a \in P\}$ for some $a \in L$.

Proof. The first part of the statement follows from the characterization of completely join-prime elements in Lemma 3.1: adapting the result to our case we see that, in the coframe $\mathcal{F}^{op}$, for a filter $F \in \mathcal{F}$,

$$\uparrow^L F = \{\uparrow a : a \in F\},$$

this collection is the filter F itself, under the identification of elements of L with their principal filters. For the second part of the statement, it suffices to unravel the definition of the topology on $\mathsf{pt}_R(L, \mathcal{F}^{op})$. ◻

Lemma 3.10. *A Raney extension (L, C) is spatial if and only if $C^* \subseteq \mathcal{M}(C^* \cap \mathsf{Filt}_{C\mathcal{P}}(L))$.*

Proof. Because of the isomorphism $\uparrow^L : C \cong C^*$, a Raney extension (L, C) is spatial precisely when all elements of C^* are intersections of completely join-prime elements in C^*, as by Lemma 3.9 we have $\mathsf{pt}_R(C^*) = \mathsf{Filt}_{C\mathcal{P}}(L) \cap C^*$. ◻

Theorem 3.11. *For a spatial frame L, the pair $(L, \mathcal{M}(\mathsf{Filt}_{C\mathcal{P}}(L))^{op})$ is the free spatial Raney extension over it. In particular, the category of spatial frames is a full coreflective subcategory of that of spatial Raney extensions.*

Proof. The assignment $L \mapsto (L, \mathcal{M}(\mathsf{Filt}_{C\mathcal{P}}(L))^{op})$ from objects of **spFrm** to objects of **Raney** can be extended to morphisms as $f \mapsto f_{C\mathcal{P}}$, as shown in Lemma 2.25. The assignment, then, is functorial. Let us show that it is left adjoint to $\pi_1 : \mathbf{spRaney} \to \mathbf{spFrm}$. Suppose that we have a map $f : L \to M$ between spatial frames, and that (M, C) is a spatial Raney extension. By spatiality, we must have $C^* \subseteq \mathcal{M}(\mathsf{Filt}_{C\mathcal{P}}(M))$, by Proposition 3.10. By Proposition 2.25, preimages under f of completely prime filters are completely prime. This means that preimages of filters in C^* are in $\mathcal{M}(\mathsf{Filt}_{C\mathcal{P}}(L))$. By Proposition 2.31, we have a morphism $(L, \mathcal{M}(\mathsf{Filt}_{C\mathcal{P}}(L))^{op}) \to (M, C)$ which extends the frame map $f : L \to M$. ◻

We are now in a condition to analyze the spectra of the examples of Raney extensions seen before. Recall that for a frame L a prime $p \in L$ is *covered* if whenever $\bigwedge_i x_i = p$ for some family $x_i \in L$ then $x_i = p$ for some $i \in I$. In [11], the authors define the T_D *spectrum* of a frame L to be the collection of covered primes of a frame, with the subspace topology inherited from $\mathsf{pt}(L)$. This space is denoted as $\mathsf{pt}_D(L)$. This turns out to always be a T_D space.

Lemma 3.12. *For a frame L, for any $a \in L$ the meet $\bigwedge\{x \in L : a < x\}$ is exact.*

Proof. Let L be a frame and let $a \in L$. Let us consider the meet $\bigwedge\{x \in L : a < x\}$. Let $b \in L$. We claim that $\bigwedge\{x \vee b : a < x\} \leq \bigwedge\{x \in L : a < x\} \vee b$. We consider two cases. First, let us assume that $b \leq a$. If this is the case, then $b \leq x$ whenever $a < x$, and so both the left hand side and the right hand side equal $\bigwedge\{x \in L : a < x\}$. Now, let us assume instead that $b \nleq a$. This is equivalent to saying that $a < a \vee b$. This means that we have the chain of inequalities

$$\bigwedge\{x \vee b : a < x\} \leq a \vee b \leq \bigwedge\{x \in L : a < x\} \vee b. \qquad \square$$

Lemma 3.13. *A completely prime filter $L \backslash {\downarrow} p$ is exact if and only if the prime p is covered.*

Suppose that the completely prime filter $L \backslash {\downarrow} p$ is exact. To show that the prime p is covered, we prove that $\bigwedge\{x \in L : p < x\} \nleq p$. By Lemma 3.12, the meet on the right-hand side is exact. The result follows by our assumption that $L \backslash {\downarrow} p$ is closed under exact meets. For the converse, we suppose that p is a covered prime and that $x_i \nleq p$ for the members of some family $\{x_i : i \in I\}$ such that their meet is exact. We then have that $x_i \vee p \neq p$ for every $i \in I$, and as p is covered, this implies that $\bigwedge_i (x_i \vee p) \neq p$. By exactness of the meet $\bigwedge_i x_i$, we also have $(\bigwedge_i x_i) \vee p \neq p$, that is $\bigwedge_i x_i \nleq p$, as required.

Lemma 3.14. *For any frame L, the spectrum of $(L, \mathsf{Filt}_{\mathcal{E}}(L)^{op})$ is homeomorphic to the space $\mathsf{pt}_D(L)$. The spectrum of $(L, \mathsf{Filt}_{\mathcal{SE}}(L)^{op})$ is homeomorphic to $\mathsf{pt}(L)$.*

Proof. By Proposition 3.9, the points of $(L, \mathsf{Filt}_{\mathcal{E}}(L)^{op})$ are the completely prime filters which are also exact. By Lemma 3.18 these are the filters of the form $L \backslash {\downarrow} p$ for some covered prime $p \in L$. Indeed, then, we have a bijection between the points of $\mathsf{pt}_R(L, \mathsf{Filt}_{\mathcal{E}}(L)^{op})$ and those of $\mathsf{pt}_D(L)$. This is a restriction of the standard homeomorphism between the spectrum $\mathsf{pt}(L)$ and its space of completely prime filters, and so it is a homeomorphism. For $(L, \mathsf{Filt}_{\mathcal{SE}}(L)^{op})$, it suffices to notice that since all completely prime filters are strongly exact, $\mathsf{Filt}_{\mathcal{CP}}(L) \cap \mathsf{Filt}_{\mathcal{SE}}(L) = \mathsf{Filt}_{\mathcal{CP}}(L)$, and the result then follows from Proposition 3.9. $\square$

The following results may be seen as point-set analogues of Theorem 2.30.

Lemma 3.15. *For a Raney extension (L, C) we have subspace inclusions*

$$\mathsf{pt}_D(L) \subseteq \mathsf{pt}_R(L, C) \subseteq \mathsf{pt}(L).$$

Proof. In the following proof, we identify prime elements with the completely prime filters that they determine. Suppose that (L, C) is a Raney extension. We have $\mathsf{Filt}_{\mathcal{E}}(L) \subseteq C^* \subseteq \mathsf{Filt}_{\mathcal{SE}}(L)$, by Theorem 2.5 and by Lemma 2.28. Therefore, we also have

$$\mathsf{Filt}_{\mathcal{CP}}(L) \cap \mathsf{Filt}_{\mathcal{E}}(L) \subseteq \mathsf{Filt}_{\mathcal{CP}}(L) \cap C^* \subseteq \mathsf{Filt}_{\mathcal{CP}}(L) \cap \mathsf{Filt}_{\mathcal{SE}}(L).$$

By Proposition 3.9, this means that we have a chain of subspace inclusions

$$\mathsf{pt}_R(L, \mathsf{Filt}_{\mathcal{E}}(L)^{op}) \subseteq \mathsf{pt}_R(L, C) \subseteq \mathsf{pt}_R(L, \mathsf{Filt}_{\mathcal{SE}}(L)^{op}).$$

The result follows from Lemma 3.14. □

We introduce the closure operator $\mathcal{S}$ on a frame L: for a subset $X \subseteq L$ we define $\mathcal{S}(X)$ to be the smallest sublocale of L such that it contains X. We will now consider the case in which our frame is $\mathsf{Filt}(L)$ for some frame L.

Lemma 3.16. *For a frame L and a subset $\mathcal{X} \subseteq \mathsf{Filt}(L)$ we have that $\mathcal{S}(\mathcal{X})$ is the set $\mathcal{M}(\{\uparrow a \to F : a \in L, F \in \mathcal{X}\})$.*

Proof. As sublocales are stable under left implication and closed under all meets, we have the inclusion $\mathcal{M}(\{\uparrow a \to F : a \in L, F \in \mathcal{X}\}) \subseteq \mathcal{S}(\mathcal{X})$. For the reverse inclusion, it suffices to show that the set $\mathcal{M}(\{\uparrow a \to F : a \in L, F \in \mathcal{X}\})$ is a sublocale. It is clearly closed under all meets. For stability under left implication, it suffices to observe that for a filter $G \in \mathsf{Filt}(L)$, and for $F \in \mathcal{X}$, we have $G \to F = \bigcap\{\uparrow g \to F : g \in G\}$. □

Theorem 3.17. *The spectra of Raney extensions over L coincide with the interval*

$$[\mathsf{pt}_D(L), \mathsf{pt}(L)]$$

of the powerset of $\mathsf{pt}(L)$.

Proof. In the following proof, we identify primes and completely prime filters, in particular by Lemma 3.13 we identify $\mathsf{pt}_D(L)$ with $\mathsf{Filt}_{\mathcal{E}}(L) \cap \mathsf{Filt}_{\mathcal{CP}}(L)$. That for a Raney extension (L, C) its spectrum is contained in the $[\mathsf{pt}_D(L), \mathsf{pt}(L)]$ interval is the content of Lemma 3.15. Conversely, suppose that we have a collection of completely prime filters $\mathcal{P} \subseteq \mathsf{Filt}(L)$ such that $\mathsf{Filt}_{\mathcal{E}}(L) \cap \mathsf{Filt}_{\mathcal{CP}}(L) \subseteq \mathcal{P}$. We now consider the Raney extension $(L, \mathcal{S}(\mathcal{P} \cup L)^{op})$. It is clear that all the elements of P are points of this Raney extension. Let us show the reverse set inclusion. Suppose that there is a completely prime filter F such that $F \in \mathcal{S}(\mathcal{P} \cup L)$. By Lemma 3.16, this means that

$$F \in \mathcal{M}(\{\uparrow a \to P : P \in \mathcal{P}\} \cup \{\uparrow a \to \uparrow x : a, x \in L\}).$$

Because F is a completely prime filter, it is completely prime in the frame $\mathsf{Filt}(L)$, and so we must have

$$F \in \{\uparrow a \to P : P \in \mathcal{P}\} \cup \{\uparrow a \to \uparrow x : a, x \in L\}.$$

First, suppose that $F \in \{\uparrow a \to P : P \in \mathcal{P}\}$. Each element $\uparrow a \to P$ in this set must equal either P or L, as completely prime filters are prime elements of $\mathsf{Filt}(L)$. This means that F must be a completely prime filter in $\mathcal{P}$, already. Secondly, suppose $F \in \{\uparrow a \to \uparrow x : a, x \in L\}$. Then, by the characterization in Lemma 2.6, the filter F must be exact. But by Lemma 3.13 this means that the prime associated with it is covered, and so this filter is in $\mathsf{pt}_D(L) \subseteq \mathcal{P}$, already. Indeed, then, $\mathsf{pt}_R(L, \mathcal{S}(\mathcal{P} \cup L)^{op}) = \mathcal{P}$, as desired. □

Remark 3.2. *It may be surprising that the spectrum* $\mathsf{pt}_R(L, C)$ *does not contain all points of* $\mathsf{pt}(L)$*, as this may be seen as a spectrum construction that forgets about too much information. However, it is the coframe* C *that ought to be seen, alone, as the ordered structure of which we are taking the points. The frame* L *(just like in Raney duality) is nothing but a carrier of information on how to topologize such set of points. Furthermore, from the result above we may see that this is what makes Raney extensions more expressive than frames: if all points of* L *were points of* $\mathsf{pt}_R(L, C)$*, then Raney extensions would only be able to capture the sober spaces.*

We shall now refine the result above to the case of subfit frames. We call $\mathsf{maxpt}(L)$ the collection of maximal primes of a frame L, equipped with the subspace topology inherited from $\mathsf{pt}(L)$.

Lemma 3.18. *Let* L *be a frame. A prime* $p \in L$ *is maximal if and only if* $L \backslash \downarrow p$ *is a regular filter.*

Proof. Suppose that we have a maximal prime $p \in L$. Because it is maximal, we have $\uparrow p = \{p, 1\}$. We claim that the completely prime filter $L \backslash \downarrow p$ is its pseudocomplement in the frame of filters. Indeed, we have $L \backslash \downarrow p \cap \{1, p\} = \{1\}$. Furthermore, if for a filter F we have $F \cap \{1, p\} = \{1\}$ then $p \notin F$, and so for $f \in F$ we must have $f \not\leq p$. For the converse, suppose that we have a prime $p \in L$ such that $L \backslash \downarrow p$ is a regular filter. By Proposition 2.7, this is the intersection of a collection of filters of the form $\{x \in L : x \vee a = 1\}$ for some $a \in L$. As $L \backslash \downarrow p$ is completely prime, it must be $\{x \in L : x \vee a = 1\}$ for some $a \in L$. This means that for all $x \in L$ the conditions $x \leq p$ and $x \vee a \neq 1$ are equivalent. In particular, because the filter is not all of L (as it is completely prime), we must have $a \leq p$ since $a \vee a = a \neq 1$. This means that if $x \nleq p$ we have $x \vee p = 1$ for all $x \in L$, and this means that p must be maximal. □

Proposition 3.19. *For a subfit frame L, the spectrum of the Raney extension $(L, \mathsf{Filt}_{\mathcal{R}}(L)^{op})$ is homeomorphic to the T_1 space $\mathsf{maxpt}(L)$.*

Proof. Suppose that L is a subfit frame. We claim that all its exact filters are regular. By Lemma 2.16 we have that $\mathsf{Filt}_{\mathcal{R}}(L)$ contains all principal filters, and so by Lemma 2.28 we must have $\mathsf{Filt}_{\mathcal{E}}(L) \subseteq \mathsf{Filt}_{\mathcal{R}}(L)$. The reverse inclusion holds for all frames. By Lemma 3.9, then, the points of $(L, \mathsf{Filt}_{\mathcal{E}}(L)^{op})$ are the regular completely prime filters, which by Lemma 3.18 are those corresponding to maximal primes of L. The fact that this is a homeomorphism comes from the fact that this is a restriction of the standard homeomorphism between the spectrum $\mathsf{pt}(L)$ and its space of completely prime filters. The space $\mathsf{maxpt}(L)$ is a T_1 space, since whenever $p, q \in \mathsf{maxpt}(L)$ we have both $p \nleq q$ and $q \nleq p$ by maximality, and so the open set $\{a \in L : a \nleq p\}$ contains q and omits p, and the open set $\{a \in L : a \nleq q\}$ contains p and omits q. □

3.2 Sobriety, the T_1 axiom, and the T_D axiom

Following the authors of [14] (see for example Theorem 4.8), we define a Raney extension (L, C) to be *sober* if every completely prime filter $P \subseteq L$ is $\uparrow^L x$ for some $x \in C$. Notice that if this holds we must have $x = \bigwedge P$ and $x \in \mathsf{pt}_R(C)$. Let us prove some characterizations for sobriety of Raney extensions.

Proposition 3.20. *The following are equivalent for a Raney extension (L, C).*

1. *(L, C) is sober.*
2. *(L, C) is $\mathcal{CP}$-compact.*
3. *The inclusion $\mathsf{pt}_R(L) \subseteq \mathsf{pt}(L)$ given by $x \mapsto \bigvee\{a \in L : x \nleq a\}$ is a homeomorphism.*

Proof. The equivalence of the first two conditions is clear by Lemma 2.2. Suppose that the Raney extension (L, C) is sober. This means that all prime elements $p \in L$ are such that $L \setminus \downarrow p = \{a \in L : x \leq a\}$ for some element $x \in C$, and by Lemma 3.1 this must be completely join-prime. This implies that for this same $x \in \mathsf{pt}_R(L)$ we have $p = \bigvee\{a \in L : x \nleq a\}$. Finally, suppose that the third condition holds, and that $p \in L$ is prime. We will show that

$\uparrow^L \bigwedge(L\setminus\downarrow p) \subseteq L\setminus\downarrow p$. By our hypothesis, there must be some completely join-prime element $x \in C$ such that $p = \bigvee\{a \in L : x \not\leq a\}$. Therefore, since x is completely join-prime, $x \not\leq p$, and this implies that whenever $a \in L$ is such that $x \leq a$ then $a \not\leq p$. This means that $x \leq \bigwedge(L\setminus\downarrow p)$, and so whenever $a \in \uparrow^L \bigwedge(L\setminus\downarrow p)$, we also have $x \leq a$ and as a consequence $x \in L\setminus\downarrow p$. Indeed, then, $\uparrow^L \bigwedge(L\setminus\downarrow p) \subseteq L\setminus\downarrow p$. □

We work towards proving that any Raney extension admits a *sobrification*, a completion to a sober space. For a Raney extension (L, C) we call a map $\sigma : S(L, C) \to (L, C)$ of the category **Raney** a *sobrification* if $S(L, C)$ is sober, and if whenever $f : (M, D) \to (L, C)$ is a morphism from a sober Raney extension, we have a commuting diagram

$$\begin{array}{ccc} S(L,C) & \xrightarrow{\sigma} & (L,C). \\ {\scriptstyle f_\sigma}\uparrow & \nearrow_{f} & \\ (M,D) & & \end{array}$$

Theorem 3.21. *For a Raney extension (L, C), the map*

$$\sigma : (L, \mathcal{M}(C^* \cup \mathsf{Filt}_{\mathcal{CP}}(L))^{op}) \to (L, C)$$
$$F \mapsto \bigwedge F$$

is its sobrification.

Proof. Observe that, as $C^* \subseteq \mathsf{Filt}(L)$ is a sublocale and the elements of $\mathsf{Filt}_{\mathcal{CP}}(L)$ are prime elements of $\mathsf{Filt}(L)$, we have that $\mathcal{M}(C^*\cup\mathsf{Filt}_{\mathcal{CP}}(L))$ is a sublocale. First, we show that $(L, \mathcal{M}(C^*\cup \mathsf{Filt}_{\mathcal{CP}}(L))^{op})$ is sober. The completely join-prime elements of $\mathcal{M}(C^* \cup \mathsf{Filt}_{\mathcal{CP}}(L))^{op}$ are the completely prime filters, by Lemma 3.9. Indeed, under the identification of L with $\{\uparrow x : x \in L\}$, every completely prime filter $P \subseteq L$ is $\{\uparrow x : \uparrow x \subseteq P\}$ for P itself. This Raney extension is then sober. By definition, we have $C^* \subseteq \mathcal{M}(C^* \cup \mathsf{Filt}_{\mathcal{CP}}(L))$, by Proposition 2.23 it means that we have a surjective map of Raney extensions

$$(L, \mathcal{M}(C^* \cup \mathsf{Filt}_{\mathcal{CP}}(L))^{op}) \twoheadrightarrow (L, C)$$
$$F \mapsto \bigwedge F.$$

Let us show that this map has the required universal property. Suppose that $f : (M, D) \to (L, C)$ is a Raney map from a sober Raney extension. We then have a frame map $f|_M : M \to L$. By Proposition 2.23, to show that the map lifts it suffices to show that the preimage of each filter in $\mathsf{Filt}_{\mathcal{CP}}(L)$ as well as each filter in C^* is in D^*. For filters in C^*, this holds because there is a map $f : (M, D) \to (L, C)$. For a completely prime filter $P \subseteq L$, recall that by Lemma 2.25 we have $f^{-1}(P) \in \mathsf{Filt}_{\mathcal{CP}}(M)$, and by Lemma 3.20 we also have $\mathsf{Filt}_{\mathcal{CP}}(M) \subseteq D^*$. Thus, the desired universal property is satisfied. □

Let us now compare sobriety with spatiality for Raney extensions.

Lemma 3.22. *A Raney extension (L, C) is sober and spatial if and only if it is $\mathcal{CP}$-canonical.*

Proof. It follows from Propositions 3.20 and 3.10 that a Raney extension (L, C) is sober and spatial if and only if $C^* = \mathcal{M}(\mathsf{Filt}_{\mathcal{CP}}(L))$. By Proposition 2.3, this holds if and only if the Raney extension is $\mathcal{CP}$-canonical. □

Proposition 3.23. *A spatial frame L admits a unique sober and spatial Raney extension, up to isomorphism. This is $(L, \mathcal{U}(\mathsf{pt}(L)))$.*

Proof. By Lemma 3.22, when a sober and spatial Raney extension exists, it is unique, up to isomorphism. If L is a spatial frame, then $(L, \mathcal{M}(\mathsf{Filt}_{\mathcal{CP}}(L))^{op})$ is a Raney extension. By Lemma 2.17, this is isomorphic to $(L, \mathcal{U}(\mathsf{pt}(L)))$. □

Let us now look at the Raney version of the T_D axiom. We say that a Raney extension (L, C) is T_D if for every completely join-prime element $x \in \mathsf{pt}_R(C)$ the filter $\uparrow^L x$ is exact. We shall use the following characterization of T_D spaces.

Proposition 3.24. *([11], Proposition 2.3.2) A space X is T_D if and only if all elements of the form $X \setminus \{x\}$ are covered primes in $\Omega(X)$.*

Proposition 3.25. *The following are equivalent for a Raney extension (L, C).*

1. *(L, C) is T_D.*

2. *The space $\mathsf{pt}_R(L, C)$ is the T_D spectrum $\mathsf{pt}_D(L)$.*

3. *We have $\mathsf{Filt}_{\mathcal{CP}}(L) \cap C^* \subseteq \mathsf{Filt}_{\mathcal{E}}(L)$.*

Proof. The equivalence between (1) and (2) is clear. Suppose that (L, C) is a T_D Raney extension. We first consider the isomorphic Raney extension (L, C^*). Its points are the elements in $\mathsf{Filt}_{\mathcal{CP}}(L) \cap C^*$, by Lemma 3.9. As the map $\uparrow^L : C \to C^*$ is an isomorphism, these are precisely the filters of the form $\uparrow^L x$ for some completely join-prime element $x \in C$. Because (L, C) is

T_D, all these are exact. Suppose, now, that (3) holds. This means that whenever $P \subseteq L$ is a completely prime filter in C^*, it is exact. Because we have mutually inverse isomorphisms $\uparrow^L : C \leftrightarrows C^* : \bigwedge$, all completely prime filters of C^* being exact means that all completely join-prime elements of C are such that $\uparrow^L x$ is exact. Thus, the Raney extension (L, C) is T_D, by definition. □

We obtain the following result relating the T_D axiom for Raney extensions and classical T_D-spatiality of frames.

Proposition 3.26. *A Raney extension (L, C) is T_D and sober if and only if every prime of L is covered.*

Proof. In the following, we use the characterization of completely prime exact filters of Lemma 3.13. By the characterizations in Propositions 3.20 and 3.25, a Raney extension (L, C) is sober and T_D if and only if both $\mathsf{Filt}_{\mathcal{CP}}(L) \cap C^* \subseteq \mathsf{Filt}_{\mathcal{E}}(L)$ and $\mathsf{Filt}_{\mathcal{CP}}(L) \subseteq C^*$. These two conditions imply that $\mathsf{Filt}_{\mathcal{CP}}(L) \subseteq \mathsf{Filt}_{\mathcal{E}}(L)$. Conversely, if $\mathsf{Filt}_{\mathcal{CP}}(L) \subseteq \mathsf{Filt}_{\mathcal{E}}(L)$, then because $C^* \cap \mathsf{Filt}_{\mathcal{CP}}(L) \subseteq \mathsf{Filt}_{\mathcal{E}}(L)$ the Raney extension is T_D by the characterization in Proposition 3.25. On the other hand, we also have $\mathsf{Filt}_{\mathcal{CP}}(L) \subseteq \mathsf{Filt}_{\mathcal{E}}(L) \subseteq C^*$. □

Let us now look at the T_1 axiom. A topological space is T_1 if and only if all subspaces are intersections of open subspaces. Motivated by this, we define a Raney extension (L, C) to be T_1 if and only if C is a Boolean algebra.

Lemma 3.27. *For a frame L, the Booleanization $\mathfrak{b}(0)$ is maximal among the Boolean sublocales, meaning that for each $x \in L$ we have that $\mathfrak{b}(0) \subseteq \mathfrak{b}(x)$ implies $x = 0$.*

Proof. If we have $\mathfrak{b}(0) \subseteq \mathfrak{b}(x)$ then we must have that $0 \in \mathfrak{b}(x)$, and this means that $a \to x = 0$ for some $a \in L$. But the assignment $a \to -$ is inflationary, and so $x \leq 0$. □

Theorem 3.28. *For a frame L, the following are equivalent.*

1. *L is subfit.*
2. *All exact filters of L are regular.*
3. *$(L, \mathsf{Filt}_{\mathcal{E}}(L)^{op})$ is a T_1 Raney extension.*
4. *There exists a T_1 Raney extension (L, C).*

5. *There is a unique T_1 Raney extension on L, up to isomorphism. This is $(L, \mathsf{Filt}_{\mathcal{R}}(L)^{op})$.*

Proof. Suppose that L is a subfit frame. By Proposition 2.16, all principal filters are regular filters. By Lemma 2.28, this implies that $\mathsf{Filt}_{\mathcal{E}}(L) \subseteq \mathsf{Filt}_{\mathcal{R}}(L)$, and as the reverse set inclusion holds for every frame, the desired result is proven. Now, suppose that we have $\mathsf{Filt}_{\mathcal{E}}(L) \subseteq \mathsf{Filt}_{\mathcal{R}}(L)$. This implies that $(L, \mathsf{Filt}_{\mathcal{E}}(L)^{op}) = (L, \mathsf{Filt}_{\mathcal{R}}(L)^{op})$. Indeed, by Proposition 2.7 the coframe $\mathsf{Filt}_{\mathcal{R}}(L)^{op}$ is a Boolean algebra. It is clear that condition (3) implies condition (4). Let us show that (4) implies (5). If (L, B) is a Raney extension such that B is Boolean, as B^* contains all principal filters, we must have $\mathsf{Filt}_{\mathcal{E}}(L) \subseteq B^*$. We also have $\mathsf{Filt}_{\mathcal{R}}(L) \subseteq \mathsf{Filt}_{\mathcal{E}}(L)$, as this holds for all frames, and so by Lemma 3.27 we must have $\mathsf{Filt}_{\mathcal{R}}(L) = \mathsf{Filt}_{\mathcal{E}}(L) = B^*$. Now, suppose that (5) holds. Then, $(L, \mathsf{Filt}_{\mathcal{R}}(L)^{op})$ is a Raney extension. This means that all principal filters are regular, and so by Proposition 2.16 the frame L must be subfit. ◻

Let us call $\mathbf{sfRaney}_{\mathcal{E}}$ the full subcategory of $\mathbf{Raney}_{\mathcal{E}}$ determined by those objects such that the frame component is subfit.

Proposition 3.29. *The category of T_1 Raney extensions is a full reflective subcategory of $\mathbf{sfRaney}_{\mathcal{E}}$. In particular, the reflector acts on objects as $(L, C) \mapsto (L, \mathsf{Filt}_{\mathcal{R}}(L)^{op})$.*

Proof. The assignment in the claim is functorial, as for subfit frames, by Theorem 3.28, regular filters coincide with the exact ones, and by Lemma 2.31 exact morphisms lifts to coframes of exact filters. Suppose that L is a subfit frame, and that we have a map $f : (L, C) \to (M, B)$ in $\mathbf{sfRaney}_{\mathcal{E}}$, such that (M, B) is a T_1 Raney extension. By Lemma 3.28, M is subfit and $B^* = \mathsf{Filt}_{\mathcal{R}}(M)^{op}$. Since f is exact, preimages of filters in $\mathsf{Filt}_{\mathcal{R}}(M)$ are in $\mathsf{Filt}_{\mathcal{R}}(L)$. Furthermore, as C^* contains all principal filters, $\mathsf{Filt}_{\mathcal{E}}(L) = \mathsf{Filt}_{\mathcal{R}}(L) \subseteq C^*$, and this means that preimages of filters in B^* are in C^*. By Proposition 2.23, the map f lifts as required. ◻

4 Canonical extensions as Raney extensions

We now look at the notion of canonical extension of a frame from [29], and see how it relates to Raney extensions. We will also look at how the Prime Ideal Theorem relates to some of our results. In [21] the Prime Ideal Theorem – **PIT** hereon – is shown to be equivalent to the statement that every pre-spatial frame is also spatial. A classical duality result is that spatiality of coherent frames is equivalent to the Prime Ideal Theorem. The so-called *Strong Prime Element Theorem* – which we will abbreviate as **SPET** – states that for every complete distributive lattice D, and any Scott-open filter $F \subseteq D$, for every element $a \in D$ not in F there is a prime element $p \in D$ above a with $p \notin F$. In [9] (Proposition 1) it is shown that **PIT** implies **SPET**. It is also known that **SPET** implies **PIT**. For a pre-spatial frame L, we will call its *canonical extension* the Raney extension

$$(L, \mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L))^{op}).$$

This is indeed a Raney extension, by Lemma 2.16. For a Raney extension (L, C), we say that an element $c \in C$ is *compact* if, for every directed collection $x_i \in L$, we have that $c \leq \bigvee_i x_i$ implies that $c \leq x_i$ for some $i \in I$. We say that a Raney extension (L, C) is *algebraic* if every element of c is the join of compact elements.

Lemma 4.1. *A Raney extension (L, C) is algebraic if and only if $C^* \subseteq \mathcal{M}(C^* \cap \mathsf{Filt}_{\mathcal{SO}}(L))$.*

Proof. Notice that an element $x \in C$ is compact if and only if the filter $\uparrow^L x$ is Scott-open. Because we have an isomorphism $\uparrow^L : C \cong C^*$, this means that the Raney extension (L, C) is algebraic of and only every filter in C^* is an intersection of Scott-open filters of the form $\uparrow^L x$ for some $x \in C$. □

Lemma 4.2. *A frame admits an algebraic Raney extension if and only if it is pre-spatial.*

Proof. First, we observe that if a frame admits an algebraic Raney extension this means that principal filters must all be intersections of Scott-open filters, by Lemma 4.1. By Proposition 2.16 the frames with this property are exactly the pre-spatial ones. For a pre-spatial frame L, an algebraic Raney extension is $(L, \mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L))^{op})$. □

We are now ready to characterize canonical extensions of frames as free algebraic Raney extensions.

Theorem 4.3. *For a pre-spatial frame L, its canonical extension is the free algebraic Raney extension over it. That is, whenever we have a frame map $L \to M$ and (M, C) is an algebraic Raney extension, the map f can be extended to a map of Raney extensions*

$$(L, \mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L))^{op}) \to (M, C).$$

Proof. The assignment is functorial, by Proposition 2.26. Suppose that L is a pre-spatial frame, and that (M, C) is an algebraic Raney extension. Suppose that we have a frame map $f : L \to M$. Consider the canonical extension $(L, \mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L))^{op})$. We have that, as (M, C) is algebraic, $C^* \subseteq \mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(M))^{op}$, by Lemma 4.1. By Lemma 2.26, preimages of Scott-open filters are Scott-open. Then, preimages of filters in C^* are in $\mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L))$. By Proposition 2.23 this means that there is a map of Raney extensions $(L, \mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L))^{op}) \to (M, C)$ extending the frame map $f : L \to M$. ◻

We introduce the following notion in order to highlight, with the following proposition, a parallel between canonical extensions and spatial, sober Raney extensions. We say that a space X is *post-sober* if every proper Scott-open filter of its frame of opens is $\{U \in \Omega(X) : F \subseteq U\}$ for some compact saturated set F. We notice that post-sobriety is stronger than sobriety. This holds because for a post-sober space X, a completely prime filter $P \subseteq \Omega(X)$ is Scott-open, then it must be the set of neighborhoods for some $F \in \mathcal{U}(X)$. As P is completely prime, F must be a completely join-prime element of $\mathcal{U}(X)$. It must then be of the form $\uparrow x$ for some $x \in X$. The motivation behind this name is to highlight a parallel with pre-spatiality for frames. A pre-spatial frame is spatial if and only if the Prime Ideal Theorem holds. A sober space is post-sober if and only if the Prime Ideal Theorem holds (as we shall see with the next results).

Example 4.1. *As we will see, given* ***PIT*** *sobriety implies post-sobriety. If we do not assume this, this implication does not hold. Let B be a Boolean algebra which contains at least a prime filter, such that we have a proper filter $F \subseteq B$ which is not contained in any prime filter. Let us now consider the Stone dual of B, namely, the space $\mathsf{pf}(B)$ of prime filters of B. It is known that Scott-open filters of $\Omega(\mathsf{pf}(B))$ are in bijective correspondence with proper filters of $\{\varphi(b) : b \in B\}$, where φ is the topologizing map. The correspondence maps a proper filter of $\varphi[B]$ to the filter generated by it. Let us then consider the filter generated by $\varphi[F]$. By our hypothesis, $\bigcap\{\varphi(f) : f \in F\}$ is empty. As the space $\mathsf{pf}(B)$ is nonempty by hypothesis, this means that $\varphi[F]$ cannot be the neighborhood filter of any compact saturated set.*

We say that a Raney extension (L, C) is *post-sober* if every Scott-open filter $F \subseteq L$ is $\uparrow^L c$ for some $c \in C$, for which, necessarily, $c = \bigwedge F$. Observe that a Raney extension is post-sober if and only if it is $\mathcal{SO}$-compact. Furthermore, a space X is post-sober if and only if $(\Omega(X), \mathcal{U}(X))$ is a post-sober Raney extension.

Proposition 4.4. *A space is post-sober if and only if $(\Omega(X), \mathcal{U}(X))$ is the canonical extension of $\Omega(X)$.*

Proof. If X is post-sober, $(\Omega(X), \mathcal{U}(X))$ is a post-sober Raney extension, hence $\mathcal{SO}$-compact. Because it is spatial, it is also $\mathcal{CP}$-dense, and as completely prime filters are Scott-open it is also $\mathcal{SO}$-dense. Conversely, if X is a space such that $(\Omega(X), \mathcal{U}(X))$ is a canonical extension, in particular this Raney extension is $\mathcal{SO}$-compact, hence post-sober. ◻

Lemma 4.5. *The Prime Ideal Theorem is equivalent to the statement that every Scott-open filter is an intersection of completely prime filters.*

Proof. We need to show that every Scott-open filter being an intersection of completely prime filters is equivalent to **SPET**. Suppose that **SPET** holds, and that L is a frame and $F \subseteq L$ a Scott-open filter. Suppose, towards contradiction, that there is some $a \notin F$ such that $a \in P$ whenever P is a completely prime filter with $F \subseteq P$. By **SPET**, there is a prime element $p \in L$ with $a \leq p$ and $p \notin F$. The completely prime filter $L \backslash {\downarrow} p$ contains F but not a, and this is a contradiction. Conversely, suppose that every Scott-open filter is in $\mathcal{M}(\mathsf{Filt}_{\mathcal{CP}}(L))$. Let $F \subseteq L$ be a Scott-open filter, and suppose that $a \notin F$. There has to be a prime $p \in L$ such that $F \subseteq L \backslash {\downarrow} p$ and such that $a \notin L \backslash {\downarrow} p$. □

Proposition 4.6. *The following are equivalent.*

1. *The Prime Ideal Theorem holds.*
2. *We have* $\mathsf{Filt}_{\mathcal{SO}}(L) \subseteq \mathcal{M}(\mathsf{Filt}_{\mathcal{CP}}(L))$ *for every frame* L.
3. *Sober Raney extensions are post-sober.*
4. *Sober spaces are post-sober.*
5. *For a sober space* X*, the canonical extension of its frame of opens is* $(\Omega(X), \mathcal{U}(X))$.

Proof. That (1) and (2) are equivalent follows from Lemma 4.5. If (2) holds, all sober Raney extensions are post-sober by definition. Suppose, now, that (3) holds. For a sober space X, the Raney extension $(\Omega(X), \mathcal{U}(X))$ is sober, as each completely prime filter of $\Omega(X)$ is ${\uparrow}^L({\uparrow} x)$ for some $x \in X$. Therefore, $(\Omega(X), \mathcal{U}(X))$ is post-sober, by hypothesis. If (5) and (6) are equivalent by Proposition 4.4. Finally, (5) implies (1) by Example 4.1. □

Finally, we essentially re-prove, using Raney extensions, the result in [13] that the canonical extension of a Boolean algebra B is the Booleanization of $\mathcal{U}(\mathsf{Idl}(B))$.

Proposition 4.7. *([29], Proposition 8.1) For a coherent frame* L*, its canonical extension is the canonical extension of the distributive lattice* $K(L)$.

Lemma 4.8. *For a compact, zero-dimensional frame* L*, we have* $\mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L)) \subseteq \mathsf{Filt}_{\mathcal{R}}(L)$.

Proof. Let L be a compact, zero-dimensional frame. We claim

$$F = \bigcap\{a \in L : a \vee \neg k = 1 : k \in K(L) \cap F\}.$$

Because L is compact and zero-dimensional, the elements of $K(L)$ coincide with the complemented elements. So, for $k \in K(L)$, the condition $a \vee \neg k = 1$ is equivalent to $k \leq a$. Let $f \in F$. Let $\{k_i : i \in I\}$ be the collection $\downarrow f \cap K(L)$; by zero-dimensionality $f = \bigvee_i k_i$. As this is a directed collection, $k_i \in F$ for some $i \in I$. For the reverse inclusion, suppose that $\neg k \vee f = 1$ for some $k \in K(L) \cap F$. This means that $\neg\neg k = k \leq f$, hence $f \in F$. □

Lemma 4.9. *For a compact, zero-dimensional frame L, we have $\mathsf{Filt}_{\mathcal{R}}(L) \subseteq \mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L))$.*

Proof. In light of Proposition 2.7, we show that every filter of the form $\{x \in L : x \vee a = 1\}$ for some $a \in A$ is Scott-open. By zero-dimensionality and compactness, for $x, a \in L$, we have $x \vee a = 1$ if and only if $x \vee k = 1$ for some $k \in \downarrow a \cap K(L)$. This is equivalent to having that x is in the filter generated by $\{\neg k : k \in \downarrow a \cap K(L)\}$, which is Scott-open by compactness of each $\neg k$. □

Proposition 4.10. *Let L be a compact, zero-dimensional frame. Its canonical extension is*

$$(L, \mathsf{Filt}_{\mathcal{R}}(L)^{op}).$$

This is also the canonical extension of the Boolean algebra $K(L)$.

Proof. The first part of the claim follows from Lemmas 4.8 and 4.9. The second part of the claim follows from Proposition 4.7, and the fact that $\mathsf{Filt}_{\mathcal{R}}(L)$ is the Booleanization of $\mathsf{Filt}(L)$. □

5 Constructions in pointfree topology as Raney extensions

Because of there being isomorphism between the collection of strongly exact filters and that of fitted sublocales (see Theorem 1.13), many of the collections of filters that we have seen in the previous sections correspond to subcollections of the coframe of fitted sublocales. In the following, we use the closure operator $cl_{\mathfrak{o}}$ on the collection of sublocales, defined as $S \mapsto \bigcap\{\mathfrak{o}(x) : S \subseteq \mathfrak{o}(x)\}$ for all sublocales $S \subseteq L$. The following is a direct consequence of the fact that the map $ker : \mathsf{S}(L) \to \mathsf{Filt}(L)^{op}$, as it is a left adjoint, preserves all joins.

Lemma 5.1. *For a collection $\mathcal{S} \subseteq \mathsf{S}(L)$ of sublocales, we have that $ker[\mathcal{J}(\mathcal{S})] = \mathcal{M}(ker[\mathcal{S}])$.*

Lemma 5.2. *For a frame L, and for $x, y \in L$, we have*

$$ker(\mathfrak{c}(x) \cap \mathfrak{o}(y)) = \uparrow x \to \uparrow y,$$

with the Heyting implication $\to$ computed in the frame of filters.

Proof. For this, it suffices to unravel definitions, using the fact that open and closed sublocales are mutual complements. We have $ker(\mathfrak{c}(x) \cap \mathfrak{o}(y)) = \{z \in L : \mathfrak{o}(y) \subseteq \mathfrak{o}(x) \vee \mathfrak{o}(z)\} = \{z \in L : y \leq x \vee z\}$. □

Proposition 5.3. *For a frame, its exact filters are precisely the kernels of smooth sublocales. In particular, there is an isomorphism*

$$ker : cl_{\mathfrak{o}}[\mathsf{S}_b(L)] \cong \mathsf{Filt}_{\mathcal{E}}(L)^{op}.$$

Proof. Let $\mathcal{F} \subseteq \mathsf{Filt}(L)$ be the collection of filters of the form $\uparrow x \to \uparrow y$ for some $x, y \in L$. By Lemma 5.2,

$$\mathcal{F} = ker[\{\mathfrak{c}(x) \cap \mathfrak{o}(y) : x, y \in L\}].$$

We have $\mathcal{M}(\mathcal{F}) = \mathsf{Filt}_{\mathcal{E}}(L)$ by Lemma 2.6, and on the other hand smooth sublocales are those of the form $\bigvee_i \mathfrak{c}(x_i) \cap \mathfrak{o}(y_i)$ for $x_i, y_i \in L$. So, Lemma 5.1 gives us the desired result. Finally, the rest of the claim follows from the fact that exact filters are strongly exact, and so they are fixpoints of the adjunction $ker \dashv fitt$. □

Proposition 5.4. *For a frame, its regular filters are precisely the kernels of joins of closed sublocales. In particular, there is an isomorphism*

$$ker : cl_{\mathfrak{o}}[\mathsf{S}_c(L)] \cong \mathsf{Filt}_{\mathcal{R}}(L)^{op}.$$

Proof. By Proposition 1.4, for $x, y \in L$ we have that $\mathfrak{c}(y) \subseteq \mathfrak{o}(x)$ if and only if $x \vee y = 1$. This immediately gives us that the kernels of closed sublocales coincide with the regular filters. The first part of the claim follows from Lemma 5.1. The rest of the claim follows from the fact that all regular filters are strongly exact, so they are fixpoints of the adjunction $ker \dashv fitt$. □

Proposition 5.5. *For a frame, the intersections of its completely prime filters are precisely the kernels of spatial sublocales. In particular, there is an isomorphism*

$$ker : cl_{\mathfrak{o}}[\mathsf{S}_{sp}(L)] \cong \mathcal{M}(\mathsf{Filt}_{\mathcal{CP}}(L))^{op}.$$

Proof. By Lemma 5.1, it suffices to show that the kernels of the two-element sublocales $\mathfrak{b}(p)$ coincide with the completely prime filters. For an element $x \in L$, indeed, we have $\mathfrak{b}(p) \subseteq \mathfrak{o}(x)$ if and only if $x \to p = p$, and by Lemma 1.6 this holds if and only if $x \nleq p$, that is, $x \in L \setminus {\downarrow} p$. The rest of the claim follows from the fact that all intersections of completely prime filters are strongly exact. ◻

Proposition 5.6. *For a frame, the intersections of its Scott-open filters are precisely the kernels of joins of compact sublocales. In particular, there is an isomorphism*

$$ker : cl_{\mathfrak{o}}[\mathsf{S}_k(L)] \cong \mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L))^{op}.$$

Proof. By Theorem 1.9, Scott-open filters coincide with the kernels of compact fitted sublocales. We claim that a sublocale of L is compact and fitted if and only if it is $cl_{\mathfrak{o}}(K)$ for some compact sublocale K. It is clear that every compact fitted sublocale is of this form. Conversely, suppose that K is a compact sublocale. We claim that $cl_{\mathfrak{o}}(K)$ is still compact. This holds because $K \subseteq \mathfrak{o}(x)$ if and only if $cl_{\mathfrak{o}}(K) \subseteq \mathfrak{o}(x)$ for all $x \in L$. Therefore, Scott-open filters are exactly the kernels of compact sublocales. The rest of the claim follows from Lemma 5.1, and the fact that Scott-open filters are strongly exact, thus fixpoints of $ker \dashv fitt$. ◻

We gather our results in the following theorem.

Theorem 5.7. *For any frame L, we have the following poset of sublocale inclusions:*

$$\begin{array}{ccccc}
cl_{\mathfrak{o}}[\mathsf{S}_c(L)] & \xrightarrow{\subseteq} & cl_{\mathfrak{o}}[\mathsf{S}_b(L)] & & \\
 & & & \searrow^{\subseteq} & \\
 & & & & \mathsf{S}_{\mathfrak{o}}(L). \\
 & & & \nearrow_{\subseteq} & \\
cl_{\mathfrak{o}}[\mathsf{S}_{sp}(L)] & \xrightarrow{\subseteq} & cl_{\mathfrak{o}}[\mathsf{S}_k(L)] & &
\end{array}$$

Proof. The result follows from Propositions 5.3, 5.4, 5.5, and 5.6; as well as Theorem 2.14. ◻

Fit frames are characterized by all their sublocales being fitted. This means that several of the results in this section become stronger in this particular case.

Lemma 5.8. *If L is a fit frame, the adjunction $ker \dashv fitt$ restricts to the following isomorphisms.*

- $fitt : \mathcal{M}(\mathsf{Filt}_{\mathcal{CP}}(L)) \cong \mathsf{S}_{sp}(L)$.

- $fitt : \mathcal{M}(\mathsf{Filt}_{\mathcal{SO}}(L)) \cong \mathsf{S}_k(L)$.

Proof. This follows from Propositions 5.5 and 5.6, and from the fact that, for a fit frame, the map $cl_{\mathfrak{o}}$ is the identity. ◻

Theorem 5.9. *For a fit frame L, we have the following diagram of subcolocale inclusions.*

$$\begin{array}{ccccc} & & \mathsf{S}_b(L) = \mathsf{S}_{\mathfrak{c}}(L) & & \\ & & & \overset{\subseteq}{\searrow} & \\ & & & & \mathsf{S}_{\mathfrak{o}}(L) = \mathsf{S}(L). \\ & & & \overset{\subseteq}{\nearrow} & \\ \mathsf{S}_{sp}(L) & \xrightarrow{\subseteq} & \mathsf{S}_k(L) & & \end{array}$$

Proof. This follows from Lemma 5.8 above, and Theorem 5.7. ◻

5.1 Scatteredness: frames with unique Raney extensions

A frame is said to be *scattered* if the coframe $\mathsf{S}(L)$ is Boolean. As proven in [5], a frame is subfit and scattered if and only if all its sublocales are joins of closed sublocales. Subfit scattered frames are also fit, and so $\mathsf{S}(L) = \mathsf{S}_{\mathfrak{o}}(L)$, from which we obtain that for L subfit and scattered the equality $\mathsf{Filt}_{\mathcal{R}}(L) = \mathsf{Filt}_{\mathcal{E}}(L) = \mathsf{Filt}_{\mathcal{SE}}(L)$ holds (see Proposition 3.28).

Proposition 5.10. *For a subfit frame L, the following are equivalent.*

1. *The frame L is scattered.*
2. $\mathsf{Filt}_{\mathcal{SE}}(L) = \mathsf{Filt}_{\mathcal{E}}(L)$.
3. *The frame has a unique Raney extension, up to isomorphism.*
4. *We have $\mathsf{S}_{\mathfrak{o}}(L) = \mathsf{S}_{\mathfrak{c}}(L)$.*
5. *The frame has a unique Raney extension, up to isomorphism, and this is $(L, \mathsf{S}(L))$.*

Proof. Suppose that L is a scattered subfit frame. Since every sublocale of $\mathsf{S}(L)$ is complemented, all fitted sublocales of L are fittings of complemented sublocales. Therefore,

$$\mathsf{Filt}_{\mathcal{SE}}(L) \subseteq \mathsf{Filt}_{\mathcal{E}}(L),$$

by Proposition 5.3. Let us show that (2) implies (3). The inclusion $\mathsf{Filt}_{\mathcal{E}}(L) \subseteq \mathsf{Filt}_{\mathcal{SE}}(L)$ holds for every frame. Now, suppose that in L every strongly exact filter is exact. For any Raney extension (L, C), we must have $\mathsf{Filt}_{\mathcal{E}}(L) \subseteq C^* \subseteq \mathsf{Filt}_{\mathcal{SE}}(L)$. Our assumption, then, implies $\mathsf{Filt}_{\mathcal{E}}(L) = C^* = \mathsf{Filt}_{\mathcal{SE}}(L)$. This proves (3), by the isomorphism $\uparrow^L : (L, C) \cong (L, C^*)$: it means that whenever (L, C) and (L, D) are such that $C^* = D^*$, they must be isomorphic. Suppose, now, that L has a unique Raney extension, up to isomorphism. As L is subfit, this must be a Boolean extension, by Theorem 3.28. The pair $(L, \mathsf{S}_{\mathfrak{o}}(L))$ is always a Raney extension. Therefore, $\mathsf{S}_{\mathfrak{o}}(L)$ is Boolean. As $\mathsf{S}_{\mathfrak{o}}(L)$ is a subcoframe of $\mathsf{S}(L)$, this means that in $\mathsf{S}(L)$ every fitted sublocale has a complement, which is itself a fitted sublocale. In particular, all joins of closed sublocales are fitted and so $\mathsf{S}_{\mathfrak{c}}(L) \subseteq \mathsf{S}_{\mathfrak{o}}(L)$. Finally, recall that the lattice $\mathsf{S}_b(L)$ of joins of complemented sublocales is $\mathsf{S}_{\mathfrak{c}}(L)$ for subfit frames. We then also have the reverse set inclusion $\mathsf{S}_{\mathfrak{o}}(L) \subseteq \mathsf{S}_{\mathfrak{c}}(L)$. If (4) holds, by subfitness we have that $(L, \mathsf{S}_{\mathfrak{o}}(L))$ is a Boolean extension. Since this is the largest Raney extension, all its Raney extensions must be Boolean. By Theorem, 3.28, when Boolean extensions exist they are unique. Suppose, finally, that (5) holds. Because all fit frames have a Boolean extension, by Theorem 3.28, $\mathsf{S}(L)$ must be Boolean, and so L is scattered. ◻

6 Subcolocales of Raney extensions: pointfree subspaces

We now introduce several analogues of types of sublocales of pointfree topology. Let (L, C) be a Raney extension. We define a *subcolocale* of (L, C) to be simply a subcolocale of C. For a Raney extension (L, C), let us denote as $\mathsf{RS}(L, C)$ the ordered collection of all its Raney subcolocales. This is nothing but the coframe $\mathsf{S}(C^{op})$ of classical sublocales of C^{op}. The motivation behind the following names will be obvious if we consider that a Raney extension (L, C) represents the embedding $L \subseteq C$ of the frame open sets into the coframe of saturated sets.

- For an element $a \in L$ we call $\mathfrak{ro}(a)$ the subcolocale $\downarrow a \subseteq C$, and we say that the subcolocales of this form are *Raney-open.*
- For an element $c \in C$ we call $\mathfrak{rf}(c)$ the subcolocale $\downarrow c \subseteq C$, and we say that subcolocales of this form are *Raney-fitted.*
- For an element $a \in L$ we call $\mathfrak{rc}(a)$ the subcolocale $\{c \backslash a : c \in C\}$, and we say that the subcolocales of this form are *Raney-closed.*

- For an element $c \in C$, we call $\mathfrak{rjc}(c)$ the sublocale $\bigvee\{\mathfrak{rc}(a) : c \leq a, a \in L\}$.
- For an element $c \in C$ we call $\mathfrak{rb}(c)$ the subcolocale $\{c\backslash d : d \in C\}$, and we say that subcolocales of this form are *Boolean.*

We have the following facts, direct translations of the standard facts of pointfree topology in Proposition 1.4.

Proposition 6.1. *For every Raney extension (L, C) and $a, b, a_i \in C$ we have*

1. $\mathfrak{ro}(1) = C$ *and* $\mathfrak{ro}(0) = \{0\}$;
2. $\mathfrak{rc}(1) = \{0\}$ *and* $\mathfrak{rc}(0) = C$;
3. $\bigcap_i \mathfrak{rf}(c_i) = \mathfrak{rf}(\bigwedge_i c_i)$ *and* $\mathfrak{rf}(c) \vee \mathfrak{rf}(d) = \mathfrak{rf}(c \vee d)$;
4. $\bigvee_i \mathfrak{rjc}(c_i) = \mathfrak{rjc}(\bigwedge_i c_i)$ *and* $\mathfrak{rjc}(c) \cap \mathfrak{rjc}(d) = \mathfrak{rjc}(c \vee d)$;
5. *The elements* $\mathfrak{rf}(c)$ *and* $\mathfrak{rjc}(c)$ *are complements of each other in* $\mathsf{RS}(L, C)$;
6. $\mathfrak{rf}(c) \subseteq \mathfrak{rjc}(d)$ *if and only if* $c \wedge d = 0$, *and* $\mathfrak{rjc}(c) \subseteq \mathfrak{rjc}(d)$ *if and only if* $d \vee c = 1$.

We also have that the coframe $\mathsf{RS}_{\mathfrak{rf}}(L, C)$ of Raney-fitted subcolocales is isomorphic to C, and that the frame $\mathsf{RS}_{\mathfrak{rjc}}(L, C)$ of join of Raney-closed subcolocale is anti-isomorphic to it.

Proposition 6.2. *The collection of Raney-open subcolocales together with that of joins of Raney-closed subcolocales generate* $\mathsf{RS}(L, C)$, *in the sense that for each subcolocale $S \subseteq C$ we have:*

$$S = \bigcap\{\mathfrak{ro}(a) \vee \mathfrak{rjc}(c) : S \subseteq \mathfrak{ro}(a) \vee \mathfrak{rjc}(c)\}.$$

Proof. By Proposition 1.5, adapted to our terminology, for each subcolocale $S \subseteq C$ we have

$$S = \bigcap\{\mathfrak{rf}(c) \vee \mathfrak{rjc}(d) : S \subseteq \vee\mathfrak{rf}(c) \vee \mathfrak{rjc}(d)\},$$

and as $\mathfrak{rf}(c) = \bigcap\{\mathfrak{ro}(a) : a \in L, c \leq a\}$, and by the coframe distributivity of $\mathsf{RS}(L, C)$, the result follows. □

Remark 6.1. *One may ask what the connections are between sublocales of a frame L and subcolocales of some Raney extension (L, C) over it. A first step in answering this question is the consideration that because of $\mathsf{S}_{\mathfrak{o}}(L)$ and $\mathsf{S}_{\mathfrak{c}}(L)^{op}$ being, respectively, the largest and the smallest Raney extensions of a frame L, we have a diagram in* **CoFrm** *as follows.*

$$
\begin{array}{ccc}
\mathsf{RS}_{\mathfrak{rf}}(L,C) & \xrightarrow{(-)^*(\cong)} & \mathsf{RS}_{\mathfrak{rjc}}(L,C)^{op} \\
\uparrow & & \downarrow \\
\mathsf{S}_{\mathfrak{o}}(L) & \xrightarrow{(-)^*} & \mathsf{S}_{\mathfrak{c}}(L)^{op}
\end{array}
$$

We leave it as an open question to give an exhaustive account of the connections between classical sublocales of L and Raney subcolocale of a Raney extension (L, C).

We shall now characterize the T_1 property in terms of subcolocales.

Proposition 6.3. *For a Raney extension (L, C), the following are equivalent.*

1. *(L, C) is T_1.*
2. *The map $\mathfrak{rf} : C \to \mathsf{RS}(L, C)$ is an isomorphism.*
3. *All subcolocales of (L, C) are fitted.*
4. *All subcolocales of (L, C) are joins of closed subcolocales.*
5. *All joins of closed subcolocales of (L, C) are fitted.*

Proof. Suppose, first, that (L, C) is a T_1 Raney extension. Because C is Boolean, the frame C^{op} is Boolean. Therefore, the map $\mathfrak{c} : C^{op} \to \mathsf{S}(L)$ is an isomorphism. This equals the map $\mathfrak{rf} : C \to \mathsf{RS}(L, C)$. Therefore, all subcolocales of (L, C) are fitted. By taking complements, it is clear that (3) and (4) are equivalent. It is clear that (3) implies (5). If all joins of closed subcolocales of (L, C) are fitted, for any $c \in C$, we have $\mathfrak{rjc}(c) = \mathfrak{rjc}(d)^*$ for some $d \in C$, and this implies that each $c \in C$ is complemented, hence (L, C) is T_1. □

Remark 6.2. *A more categorical approach would be to consider for a Raney extension (L, C) the surjections $(L, C) \twoheadrightarrow (M, D)$. We have not explored this direction yet, and we leave this, too, as an open question. An initial difficulty is that, for instance, for $c \in C$ it is not in general the case that the coframe surjection $x \mapsto x \wedge c$ (corresponding to $\mathfrak{rf}(c)$ in our approach above) preserves the joins of L.*

6.1 Subspaces and sublocales for Raney extension

In [51], it is proven that for every frame L we have a diagram as below, in the category of coframes. The map $\mathrm{sp} : \mathsf{S}(L) \to \mathsf{S}_{sp}(L)$ maps each sublocale $S \subseteq L$ to its spatialization $\mathcal{M}(\mathsf{pt}(S)) \subseteq L$. This turns out to be the spatialization map of the frame $\mathsf{S}(L)^{op}$. The collection $\mathrm{sob}[\mathcal{P}(\mathsf{pt})(L)]$ is the ordered collection of sober subspaces of $\mathsf{pt}(L)$, which is known to be a coframe. The upper horizontal arrow is an isomorphism.

$$\begin{array}{ccc} \mathsf{S}_{sp}(L) & \xrightarrow{\mathsf{pt}(\cong)} & \mathrm{sob}[\mathcal{P}(\mathsf{pt}(L))] \\ {\scriptstyle \mathrm{sp}}\uparrow & & \downarrow \\ \mathsf{S}(L) & \xrightarrow{\mathsf{pt}} & \mathcal{P}(\mathsf{pt}(L)) \end{array}$$

We now seek to adapt the diagram to our case. As all T_0 spaces are fixpoints of our adjunction, and as all subspaces of T_0 spaces are T_0, in our case the analogue of the inclusion of the coframe of sober subspaces into the powerset is simply the identity.

For a Raney extension (L, C), we say that a subcolocale $S \subseteq C$ is *Raney-spatial* if $S = \bigvee\{\mathfrak{rb}(x) : x \in \mathsf{pt}_R(C) \cap S\}$. For a Raney extension (L, C), we denote as $\mathsf{RS}_{sp}(L, C)$ the ordered collection of Raney-spatial subcolocales of C. We call the assignment $S \mapsto \bigvee\{\mathfrak{b}(x) : x \in S \cap \mathsf{pt}_R(C)\}$ the *Raney-spatialization* of the subcolocale S.

Remark 6.3. *We note that Raney-spatiality on a subcolocale $S \subseteq C$ is a condition stronger than having $S = \bigvee\{\mathfrak{rb}(x) : x \in \mathsf{pt}_R(S)\}$. In fact, while it is true that every element in $\mathsf{pt}_R(C) \cap S$ is also completely join-prime in the coframe S, the converse does not hold. We leave it as an open question to develop a theory of spatial subcolocales of Raney extensions.*

Lemma 6.4. *In the coframe $\mathsf{RS}(L, C)$, the completely join-prime elements are the subcolocales of the form $\mathfrak{b}(x) = \{0, x\}$ for some completely join-prime element $x \in C$.*

Proof. It is clear that all subcolocales of that form are completely join-prime. For the converse, suppose that a subcolocale S is not of the form $\mathfrak{b}(x)$ for some $x \in \mathsf{pt}_R(C)$. Then, there are $x, y \in S$, with $x \not\leq y$, and both distinct from 0. We have $S \subseteq \mathfrak{rf}(y) \vee \mathfrak{rjc}(y) = C$. However, we have $x \notin \mathfrak{rf}(y)$, and $y \notin \mathfrak{rjc}(y)$. Therefore, S is not completely join-prime. ◻

Lemma 6.5. *The inclusion of Raney-spatial subcolocales into all subcolocales $\mathsf{RS}_{sp}(L, C) \subseteq \mathsf{RS}(L, C)$ is a subcolocale. Therefore, Raney-spatialization of subcolocales is a coframe surjection*

$$sp : \mathsf{RS}(L, C) \to \mathsf{RS}_{sp}(L, C),$$

and this is the Raney-spatialization of the coframe $\mathsf{RS}(L, C)$.

Proof. Closure under joins is clear. Consider a Raney-spatial subcolocale S and an arbitrary subcolocale T. We have $S \backslash T = \bigvee \{\mathfrak{rb}(x) : x \in \mathsf{pt}_R(S)\} \backslash T = \bigvee_{x \in \mathsf{pt}_R(S)} (\mathfrak{rb}(x) \backslash T) = \bigvee \{\mathfrak{rb}(x) : x \in \mathsf{pt}_R(S), x \notin \mathsf{pt}_R(T)\}$. The rest of the claim follows by Lemma 6.4. □

Lemma 6.6. *For a Raney extension* (L, C) *and for* $X \subseteq \mathsf{pt}_R(L)$, *we have that*

$$\mathsf{pt}_R\left(\bigvee \{\mathfrak{rb}(x) : x \in X\}\right) = X.$$

Proof. Suppose that (L, C) is a Raney extension and that $X \subseteq \mathsf{pt}_R(C)$. We prove the nontrivial inclusion $\mathsf{pt}_R(\bigvee \{\mathfrak{rb}(x) : x \in X\}) \subseteq X$. We have $\bigvee \{\mathfrak{rb}(x) : x \in X\} = \mathcal{J}(X)$, where recall that $\mathcal{J}(-)$ maps a subset of a complete lattice to its closure under arbitrary joins. Because all elements in X are completely join-prime, every element which is in $\mathcal{J}(X)$ must be in X, already. □

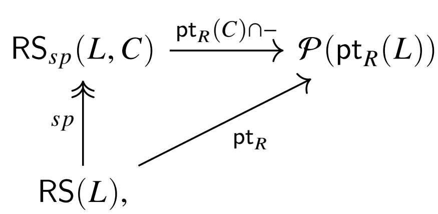
Proposition 6.7. *For a Raney extension* (L, C), *we have an isomorphism*

$$\begin{gathered} \mathsf{RS}_{sp}(L, C) \to \mathcal{P}(\mathsf{pt}(L, C)) \\ S \mapsto \mathsf{pt}_R(C) \cap S. \end{gathered}$$

Proof. We claim that the inverse of the assignment is the map $X \mapsto \bigvee \{\mathfrak{rb}(x) : x \in X\}$. Consider a Raney-spatial subcolocale $S \subseteq C$. We have $\bigvee \{\mathfrak{rb}(x) : x \in \mathsf{pt}_R(C) \cap S\} = S$. Consider a subspace $X \subseteq \mathsf{pt}_R(L, C)$. We have $\mathsf{pt}_R(C) \cap \bigvee \{\mathfrak{rb}(x) : x \in X\} = X$, by Lemma 6.6. □

We come to the desired adaptation of our diagram.

Theorem 6.8. *For every Raney extension* (L, C), *we have a diagram as follows in the category of coframes:*



where $\mathsf{pt}_R(C) \cap -$ *is an isomorphism.*

Proof. By Proposition 6.7, the horizontal arrow is an isomorphism. By Lemma 6.5, the vertical arrow is a coframe surjection. Finally, to see that the diagram commutes, notice that for a subcolocale S its Raney-spatialization is $\bigvee\{\mathfrak{rb}(x) : x \in \mathsf{pt}_R(S)\}$, and that by Lemma 6.6 its points are, indeed, the elements of $\mathsf{pt}_R(S)$. □

Next, we will prove versions of Niefield and Rosenthal's famous results on total spatiality of frames (see [37]). We observe that, because of Proposition 6.1, for Raney extension total spatiality suffices for the subcolocales of (L, C) to represent the subspaces of $\mathsf{pt}_R(L, C)$ perfectly. We shall soon see that in the case of Raney extensions it suffices for the frame component to be fit for total spatiality to collapses to spatiality.

Remark 6.4. *Let us sketch right away the reason why we have this lack of symmetry with frame theory. An* essential *prime for an element a of a frame L is a prime $p \in L$ with $a \leq p$ and such that $p \in S$ whenever $a \in S$ for any sublocale $S \subseteq L$. Total spatiality, in [37], amounts to every element being a meet of essential primes. In our case, because the points of a Raney extension are* completely *join-prime elements, it suffices for a completely join-prime element to be maximal for it to satisfy the analogue of essentiality.*

In the following, for a coframe C and $c \in C$ we denote $\{x \in \mathsf{pt}_R(C) : x \leq c\}$ simply as $\mathsf{pt}_R(c)$. In [37] the authors observe that for a frame L if we assume Zorn's Lemma every element a which is a meet of primes it is a meet of primes which are minimal in $\uparrow a \cap L$. The analogue of this does not hold in our case. It is not true in general that for an element $c \in C$ the collection $\mathsf{pt}_R(c)$ is such that all chains have an upper bound[3]. Let us call $\mathsf{maxpt}_R(c)$ the collection of maximal elements of $\mathsf{pt}_R(c)$. We say that a completely join-prime element $x \in \mathsf{pt}_R(c)$ is *essential* for c, or simply *essential*, should c be clear from the context, if $\bigvee \mathsf{pt}_R(c) \setminus \{x\} \neq c$.

Lemma 6.9. *For a coframe C and for $c \in C$ such that $c = \bigvee \mathsf{pt}_R(c)$, an element $x \in \mathsf{pt}_R(c)$ is essential if and only if it is maximal in $\mathsf{pt}_R(c)$.*

Proof. We observe that for an element $x \in \mathsf{pt}_R(c)$ the condition $c \nleq \bigvee \mathsf{pt}_R(c) \setminus \{x\}$ is equivalent to $x \nleq \bigvee \mathsf{pt}_R(c) \setminus \{x\}$. In turn, by complete join-primality of x, this condition is equivalent to $x \nleq y$ for all $y \in \mathsf{pt}_R(c)$ with $x \neq y$, that is, maximality of x in $\mathsf{pt}_R(c)$. □

Lemma 6.10. *For a coframe C and for $c \in C$ such that $c = \bigvee \mathsf{pt}_R(c)$, an element $x \in \mathsf{pt}_R(c)$ is essential if and only if for every subcolocale $S \subseteq C$ we have $c \in S$ implies $x \in S$.*

[3]To see this, it suffices to consider the element ω in the coframe $\omega + 1$: this is the join of the collection of completely join-prime elements of natural numbers, which does not have an upper bound.

Proof. Suppose that $x \in \mathsf{pt}_R(c)$ is essential, and that $S \subseteq C$ is a subcolocale with $c \in S$. We claim that $d := \bigvee \mathsf{pt}_R(c) \backslash \{x\}$ is such that $x = c \backslash d$. Computing this expression:

$$c \backslash d = \bigvee \{y \backslash d : y \in \mathsf{pt}_R(c)\} = \bigvee \{y \in \mathsf{pt}_R(c) : y \nleq d\}.$$

We observe that $\{y \in \mathsf{pt}_R(c) : y \nleq d\} = \{x\}$, by essentiality of x. To conclude the proof of the first part of the claim, we observe that $x = c \backslash d$ implies that $x \in \mathfrak{rb}(c)$, the smallest subcolocale containing c. For the converse, suppose that $x \in \mathfrak{rb}(c)$, that is, $x = c \backslash (c \backslash x)$. Observe that d, defined as above, is $c \backslash x$. Therefore, we have $x = \bigvee \{y \in \mathsf{pt}_R(c) : y \nleq d\}$. Observe that this means that $x \nleq d$, and this means (by definition of d) that $x \nleq y$ whenever $y \in \mathsf{pt}_R(c)$ is such that $y \neq x$. So, x must be maximal, hence essential by Lemma 6.9. □

We say that a Raney extension (L, C) is *totally spatial* if all its subcolocales are Raney-spatial.

Proposition 6.11. *A Raney extension is totally spatial if and only if for every element $c \in C$ we have $c = \bigvee \mathsf{maxpt}_R(c)$.*

Proof. Suppose that (L, C) is spatial, and that $S \subseteq C$ is a subcolocale. We show that $S \subseteq \bigvee \{\mathfrak{rb}(x) : x \in \mathsf{pt}_R(C) \cap S\}$. By spatiality $a = \bigvee \mathsf{pt}_R(a)$. If $a = \bigvee \mathsf{maxpt}_R(a)$, by Lemma 6.10, we must then have that $\mathsf{maxpt}_R(a) \subseteq S$. In particular, $\mathsf{maxpt}_R(a) \subseteq \mathsf{pt}_R(C) \cap S$, and so $\mathcal{J}(\mathsf{pt}_R(a)) \subseteq \bigvee \{\mathfrak{rb}(x) : x \in \mathsf{pt}_R(C) \cap S\}$, as subcolocales are closed under arbitrary joins. In particular, $a \in \bigvee \{\mathfrak{rb}(x) : x \in \mathsf{pt}_R(C) \cap S\}$. Conversely, suppose that (L, C) is such that for each subcolocale $S \subseteq C$ we have $S = \bigvee \{\mathfrak{rb}(x) : x \in \mathsf{pt}_R(C) \cap S\}$. In particular, for each $c \in C$, the subcolocale $\mathfrak{rb}(c)$ is Raney-spatial, and thus by Lemma 6.10 we have that for each $x \in \mathsf{pt}_R(c)$ the condition $x \in \mathfrak{rb}(c)$ is equivalent to x being essential. By Raney-spatiality of $\mathfrak{rb}(c)$ we have $c = \bigvee \mathsf{pt}_R(C) \cap S \cap \mathsf{pt}_R(c) = \bigvee \mathsf{maxpt}_R(c)$. The required result follows from Lemma 6.9. □

Theorem 6.12. *For a Raney extension (L, C), the following are equivalent.*

1. *(L, C) is totally spatial.*
2. *$c = \bigvee \mathsf{maxpt}_R(c)$ for all $c \in C$.*
3. *The map $\mathsf{pt}_R : \mathsf{RS}(L, C) \to \mathcal{P}(\mathsf{pt}_R(L, C))$ is an isomorphism.*
4. *The coframe $\mathsf{RS}(L, C)$ is Raney-spatial.*

Proof. That (1) is equivalent to (2) is the content of Proposition 6.11. That (2) implies (3) follows from the diagram in Theorem 6.8. It is clear that (3) implies (4), as powersets are atomic Boolean algebras and these are spatial as coframes. Finally, suppose that $\mathsf{RS}(L, C)$ is spatial. By Lemma 6.5, (1) holds. □

For a coframe C, we say that the collection $\mathsf{pt}_R(L)$ is *discretely ordered* as a short-hand for saying that the collection $\mathsf{pt}_R(C)$ with the order inherited from C is such that for each $x, y \in \mathsf{pt}_R(C)$ we have $x \leq y$ implies $x = y$. We note that for a Raney extension (L, C) the collection $\mathsf{pt}_R(L, C)$ is discretely ordered if and only if the space $\mathsf{pt}_R(L, C)$ is a T_1 space.

Corollary 6.13. *If a Raney extension (L, C) is such that $\mathsf{pt}_R(L, C)$ is discretely ordered, then (L, C) is totally spatial whenever it is spatial.*

Proof. Suppose that (L, C) is such that the collection $\mathsf{pt}_R(L, C)$ is discretely ordered. Then, for each $c \in C$, we have $\mathsf{pt}_R(a) = \mathsf{maxpt}_R(a)$. The required conclusion follows from Proposition 6.11. □

Corollary 6.14. *If a frame L is such that all primes of L are maximal, every spatial Raney extension (L, C) is totally spatial. In particular, this holds for L fit.*

Proof. Suppose that L is such that $\mathsf{pt}(L) = \mathsf{maxpt}(L)$. Then, in the collection of completely prime filters of L, too, we have $P \subseteq Q$ implies $P = Q$. In particular, then, for completely join-prime elements $x, y \in C$ we have that $x \leq y$ implies that $\uparrow^L y \subseteq \uparrow^L x$, and as these are completely prime filters, we have $\uparrow^L x = \uparrow^L y$, that is, $x = y$. Thus, $\mathsf{pt}_R(L, C)$ is discretely ordered, and we have the required result by Corollary 6.13. □

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