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Abstract

Inflation is currently among the most debated topics at both a political and economical level, given its direct consequences on people's lives. Therefore, both estimating its current and estimating its future values is a task of fundamental importance to set a correct monetary policy and achieve the desired targets. In particular, multiple measures are used to forecast inflation, but none seems to outperform all others over different time periods, as described in Meyer and Pasaogullari (2010).
In order to provide an estimate of future expected inflation, in this thesis we will use the Jarrow-Yildirim (2003) model, which we will first analyze from a theoretic point of view. Despite this model was proposed to price TIPS, a security issued by the US Treasury and linked to an inflation index to return a constant real amount to its holders, it can be adopted to measure the difference between market's expected future real and nominal rates, thus providing an estimate of market's expected inflation by Fisher law.

Introduction

After the end of the Covid-19 pandemic, inflation has been in all the headlines and one of major concerns for policymakers at every level, causing the fastest interest rate increase the Western world has seen since the 1970s. Everybody has already experienced its effects on purchasing power, causing almost anywhere tensions between different social and economic classes.

In order to take educated decisions, it is important that policymakers and investors alike have effective ways to estimate inflation and make guesses on how it is going to evolve in the future, on a short-, mid- and long-term basis. In order to do so, using only macro-economic variables and past relations among them may be misleading in light of Lucas critique (Lucas, 1976). This thesis therefore will use market data from inflation-linked bonds and nominal bonds to avoid this problem and have a direct insight into market participants' expectations of future inflation. In particular, as explained in Wojtowicz (2023), inflation-linked bonds, known also as linkers or ILBs, have the same credit quality of nominal bonds issued by the same country, but, differently, from them, provide an hedge against inflation by indexing the cash flows to investors to an inflation index. Threrefore, if inflation is positive, the amounts received by investors will be higher.

Indeed, when the first inflation-linked bonds were issued during the American Revolutionary War (Shiller, 2005), they were seen as an extraordinary measure and were rarely issued after the end of the war for 200 years, despite an abundant stream of economic literature. It was in the 1980s that the UK and other countries started to issue regularly inflation-linked bonds out of a deliberate political choice, thus contributing to the creation of a flourishing market (Garcia and van Rixtel, 2007).

In particular, inflation-linked bonds have been, at this point, issued by many countries for a couple of decades. We will focus on the US market for two main reasons:

1. It has the biggest and most liquid market for inflation linked bonds, thus providing more data for the estimation process. Indeed, the size of the market for inflation-linked bonds has been increasing steadily to a value of USD 2.82 trillion compared to USD 1.72 trillion of one decade ago, measured by the value of bonds outstanding as of April 2023, with the US counting for USD 1.29 trillion (Wojtowicz, 2023);

2. The FED has been increasing the rates more quickly and has moved earlier than the ECB.

The inflation-linked bonds that we will use in our estimation process are TIPS - Treasury Inflation

Protected Securities - which the US Treasury has been issuing since January 1997. In order to estimate the expected inflation, we will adopt the method outlined in Jarrow and Yildirim (2022):

1. Select the period to be studied, which in our case will be from January 3rd, 2022 to August 31st, 2023;

2. Estimate either the forward rate curve or the yield curve by finding the curve that minimizes the squared difference between the bonds' theoretical prices calculated by discounting each payoff and the actual market prices;

3. Assume an evolution for the term structure, which in our case will be given by the Hull-White model, and estimate the relative parameters. In particular, the Hull-White model is a continuous-time stochastic and mean reverting model, with Gaussian dynamics and exponentially affine formulas for bond prices;

4. Adjust for possible liquidity premium, given the difference liquidity between TIPS and Treasuries.

Repeating this process both for TIPS and Treasuries will allow us to exploit the Fisher equation to estimate the expected inflation, by subtracting the estimated real forward rates from the estimated nominal forward rates. This process shows clearly the implementation of the Heath-Jarrow-Merton (HJM) foreign currency analogy (Jarrow and Turnbull, 1998) and adopts the modeling technology of Amin and Jarrow (1991).

Even though new models have been presented in the literature to estimate the variables of interest, such as the ones described in Chen, Liu and Cheng (2010), Brace, Gatarek, and Musiela (2010), and Ho, Huang, and Yildirim (2014), the original model by Jarrow and Yildirim in still used in the industry, and it is the one we will adopt given its relative simplicity. Furthermore, for a more detailed review of the current literature, the reader can be referred to Kupfer (2018), which presents an analysis of different term structure models and a discussion of regression based approaches.

The work is organized as follows. Chapter 1 presents some theoretical preliminaries that are fundamental for the following chapters and focuses mainly on the definition of interest rates and of incomplete markets. Chapter 2 defines the probabilistic models used in the estimation process, the Hull-White model and the Jarrow-Yildirim model. Chapter 3 describes the data and the methodology used in the estimation process. Chapter 4 presents the results of the analysis and the estimates for the variables involved. Chapter 5 concludes.

1 Theoretical preliminaries

In this section we will focus on defining the fundamental concepts we will need in the following chapters. In particular, we will focus on defining different types of interest rates and we will introduce the concept of incomplete markets.

1.1 Interest rates

There are two basic types of bonds: zero coupon bonds (ZCB) and coupon bonds (CB). The difference between these two types of bonds is that, while CB have intermediate payments between the date of issuance and the maturity date, ZCB have just one payment at the maturity date. Both these instruments are known as fixed income instruments, since the lender theoretically knows in advance when and how much the borrower will repay its debt (differently for example from equity, where the firm does not have any obligation of issuing dividends). In this section, we will focus on ZCB and we will simply provide the pricing formula for CB.

As mentioned previously, the repayment of ZCB occurs in only one moment: at the maturity time T, when the principal is paid back to the lender. According to the interest rate of the bond and the main interest rates, the price at time t of the ZCB $p(t, T)$ can be lower or higher than the face value of the bond itself during the lifetime of the bond, given the stochastic nature of interest rates.

1.1.1 Definitions

For the following discussion and the modelling framework, we will follow Björk (2020, Chapter 19). Further references can be Musiela and Rutkowski (1995, Chapters 9.1 and 9.2), Brigo and Mercurio (2007, Chapter 1), Filipović (2009), and Hull (2018, Chapter 4).

In order to guarantee the existence of a sufficiently rich and regular bond market, we will introduce the following assumptions:

- There exists a (frictionless) market for T-bonds for every $T > 0$;
- The relation $p(t, t) = 1$ holds for all t[1];
- For each fixed t, the bond price $p(t, T)$ is differentiable w.r.t. time of maturity T.

[1] This relation is necessary in order to avoid arbitrage

Multiple interest rates can be defined on this market and of particular interest is the concept of forward rate, which allows us to lock in advance a given interest rate for an investment that will start in the future for a given period. In particular, if we consider the three points in time $t < S < T$, we are able to have a deterministic rate of return at the contract time t over the interval period $[S, T]$. In the following definitions, we define spot rates as forward rate with $t = S$ and instantaneous rate the limit of the continuously compounded forward rate when $S \to T$. Although our theoretical discussion will focus on continuous time, we provide the definition for forward and spot rates in discrete time, since we will need them in our empirical analysis.

Definition 1.1.

1. *The simple forward rate for* $[S, T]$ *is defined as*

$$L(t; S, T) = -\frac{p(t,T) - p(t,S)}{(T-S)p(t,T)}; \tag{1.1}$$

2. *The simple spot rate for* $[S, T]$ *is defined as*

$$L(S, T) = -\frac{p(S,T) - 1}{(T-S)p(S,T)}; \tag{1.2}$$

3. *The continuously compounded forward rate for the period* $[S, T]$ *is defined as*

$$R(t; S, T) = -\frac{\log p(t,T) - \log p(t,S)}{T-S}; \tag{1.3}$$

4. *The continuously compounded spot rate for the period* $[S, T]$ *is defined as*

$$R(S, T) = -\frac{\log p(S,T)}{T-S}; \tag{1.4}$$

5. *The instantaneous forward rate with maturity* T *is defined as*

$$f(t, T) = -\frac{\partial \log p(t,T)}{\partial T}; \tag{1.5}$$

6. *The instantaneous short rate at time* t *is defined as*

$$r_t = f(t, t). \tag{1.6}$$

Another definition is necessary in this introductory setting, the one for the bank account process B.

Definition 1.2.

The bank account process is defined by

$$B_t = e^{\int_0^t r_s ds}, \tag{1.7}$$

or, equivalently,

$$\begin{cases} dB_t = r_t \cdot B_t dt, \\ B_t = 1. \end{cases} \tag{1.8}$$

From the previous definitions, we can derive the following lemma.

Lemma 1.3.

For $t \leq s \leq T$ we have

$$p(t,T) = p(t,s) \cdot \exp\left\{-\int_s^T f(t,u)du\right\}, \tag{1.9}$$

and for $t = s$

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}. \tag{1.10}$$

In order to make a model for the bond market, we can proceed in multiple ways by considering dynamics of different forms:

- **Short rate dynamics**

$$dr_t = a(t)dt + b(t)dW_t. \tag{1.11}$$

- **Bond price dynamics**

$$dp(t,T) = p(t,T)m(t,T)dt + p(t,T)v(t,T)dW(t). \tag{1.12}$$

- **Forward rate dynamics**

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t). \tag{1.13}$$

All these different dynamics are related, but before getting to know how they are connected, we need to introduce some technical assumptions.

Assumption 1.4.

1. *For each fixed* ω, t *all the objects* $m(t,T), v(t,T), \alpha(t,T) and \sigma(t,T)$ *are assumed to be continuously differentiable in the T-variable. This partial T-derivative will be denoted by a* T *subscript.*

2. *All processes are assumed to be regular enough to allow us to differentiate under the integral sign as well as to interchange the order of integration.*

From this, the next proposition follows, which holds without assuming that markets are free of arbitrage.

Proposition 1.5.
If $f(t,T)$ *satisfies equation (1.13) then* $p(t,T)$ *satisfies*

$$dp(t,T) = p(t,T)\left\{r_t + A(t,T) + \frac{1}{2}||S(t,T)||^2\right\}dt + p(t,T)S(t,T)dW_t,$$

where $||\cdot||$ *denotes the Euclidean norm, and*

$$\begin{cases} A(t,T) = -\int_t^T \alpha(t,s)ds, \\ S(t,T) = -\int_t^T \sigma(t,s)ds. \end{cases} \tag{1.14}$$

Proof For a proof see Björk (2020, Chapter 19)

1.1.2 Coupon bonds

As mentioned in the previous section, it is possible to price CB in terms of ZCB, independently of whether the coupon is fixed or floating. Since the fixed coupon bond is the simplest CB, we will first describe this type and we will then move to the floating one.
Indeed, it is easy to show that it is possible to replicate the payoff of a fixed coupon bond through a portfolio of ZCB, and that therefore the price $p(t)$ at a time $t < T_1$, of the CB is

$$p(t) = K \cdot p(t,T_n) + \sum_{i=1}^{n} c_i \cdot p(t,T_i). \tag{1.15}$$

where K is the face value, c_i is the deterministic coupon and T_n is the maturity.

Regarding floating rate bonds, it is possible to show that the cash flows they generate are possible to replicate using a self-financing bond strategy, to the initial cost $p(t, T_{i-1})$ and that the valuation formula for the floating rate bond is

$$p(t) = p(t, T_n) + \sum_{i=1}^{n} [p(t, T_{i-1} - p(t, T_i)] = p(t, T_0). \quad (1.16)$$

For the proof of this equation, we refer to Björk (2020, Chapter 19)

1.2 Incomplete markets

Differently from the Black-Scholes (1973) case for stock markets, it is not possible to find a unique martingale measure for the pricing of bonds. In this section, we will simply provide a definition of complete markets and state a meta-theorem to identify if a market is complete, free or arbitrage or both. In this discussion, we will closely follow (Björk, 2020, Chapter 9).

Definition 1.6.

We say that a contingent T-claim X can be replicated, alternatively that it is reachable or hedgeable, if there exists a self-financing portfolio h such that its value V at time T V_T^h *respects the equality*

$$V_T^h = X, \; P - a.s. \quad (1.17)$$

In this case we say that h is a hedge against X. Alternatively, h is called a replicating or hedging portfolio. If every contingent claim is reachable, we say that the market is complete.

Meta-theorem 1.7.

Let N denote the number of underlying ***traded*** *assets in the model* ***excluding*** *the risk free asset, and let R denote the number of sources of randomness. Generically we then have the following relations:*

1. *The model is arbitrage free if and only if* $N \leq R$.
2. *The model is complete if and only if* $N \geq R$.
3. *The model is complete and arbitrage free if and only if* $N = R$.

It is therefore clear that the standard Black-Scholes model is complete and arbitrage free since there is one source of randomness and one traded asset (excluding the risk-free asset). On the other hand, the bond markets are not complete, since there is one source of randomness, but no traded assets. Indeed, we cannot consider the interest rate as a traded asset and it would be nonsensical to say that we are buying or selling it. It has to be stated that in some models ZCBs are assumed to be primitive assets, instead of derivatives written on the interest rate, and in this case even the bond market would be complete, given that we would have a source of randomness and a traded asset. However, going forward, we will not consider these models and we will keep assuming the bond market to be incomplete.

2 Models

In this chapter we will use the tools developed in the previous one to analyse the different models we will need in our empirical analysis. In particular, the short rate models and the HJM framework (Heath, Jarrow and Morton, 1992) were introduced to analyse bond prices, while the Jarrow-Yildirim model (2003) uses a HJM framework to study the pricing of TIPS.

2.1 Short rate models

In this section, we will analyze the problem of modelling an arbitrage free family of zero coupon bond price processes $\{p(\cdot, T); T \geq 0\}$. In particular, we will focus on the Hull-White model after a general introduction. We will closely follow the analysis made in (Björk 2020, Chapters 20 and 21).

Since the price $p(t, T)$ of a bond depends on the behavior of the short rate r over the interval $[t, T]$, we will start the discussion by modeling the short rate, under the objective probability measure P, as the solution of an SDE of the form

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t. \tag{2.1}$$

The short rate r is the only object given *a priori*, so the only exogenously given asset is the money account, whose price process B is defined like in the previous chapter as

$$dB_t = r_t B_t dt. \tag{2.2}$$

As we mentioned in the previous chapter, this market is incomplete since, as described by meta-theorem 1.7 the number of random sources is bigger than the number of underlying traded assets. Indeed, it can be noted that there is one random source in the short rate SDE and no underlying traded asset (since we do not count the risk-free rate). However, since there must still be some internal consistency rules to avoid arbitrage, we can achieve the following result, where λ indicates the market price of risk, defined as the ratio between an asset's expected excess return and its volatility.

Proposition 2.1.

In an arbitrage free bond market, there will exist a process $\lambda(t, r_t)$ such that, for every maturity T, *the bond pricing function F^T will satisfy the term structure equation*

$$\begin{cases} F_t^T + \{\mu - \lambda\sigma\}F_r + \dfrac{1}{2}\sigma^2 F_{rr} - rF^T = 0, \\ \qquad\qquad\qquad\qquad\qquad\quad F^T(T, r) = 1. \end{cases} \tag{2.3}$$

where F_t indicates the first partial derivative of F w.r.t t, F_r the the first partial derivative of F w.r.t t and F_{rr} the function F derived twice w.r.t r.

Proof For a proof see Björk (2020, Chapters 9 and 20)

Furthermore, we can enunciate the following stochastic representation formula.

Proposition 2.2.

Bond prices are given by the formula $p(t, T) = F(r, r_t; T)$ where

$$F(t, r; T) = E_{t,r}^Q \left[e^{-\int_t^T r_s ds} \right]. \tag{2.4}$$

The Q-dynamics of r are given by

$$dr_t = \{\mu(t, r_t) - \lambda(t, r_t)\sigma(t, r_t)\}ds + \sigma(t, r_t)dW_t^Q, \tag{2.5}$$

where W^Q is a Q-Wiener.

Proof For a proof see Björk (2020, Chapter 20)

It can be pointed out that equation (2.4) has the standard economic interpretation: the bond price is given as the expected value of the final payoff discounted. Furthermore, it has to be noticed that the expectation is to be taken under the martingale measure Q. Since the market is not complete, there is not a unique martingale measure, but we have to infer it directly through market data. The way to do it will be explained at a later stage, while for now we will limit ourselves to state that for analytical tractability and computational efficiency, the models must have an affine term structure to allow an easy analytical analysis.

Before moving to it, we will present a list with some of the most popular short rate models. If a parameter is time dependent, this is written out explicitly, otherwise all parameters are positive constants:

1. Vasiček (1977)

$$dr_t = (b - ar_t)dt + \sigma dW_t, \tag{2.6}$$

2. Cox-Ingersoll-Ross (CIR) (1985)

$$dr_t = (b - ar_t)dt + \sigma\sqrt{r_t}dW_t, \tag{2.7}$$

3. Dothan (1978)

$$dr_t = ardt + \sigma r_t dW_t, \tag{2.8}$$

4. Black-Derman-Toy (1990)

$$dr_t = \theta(t)r_t dt + \sigma(t)r_t dW_t, \tag{2.9}$$

5. Ho-Lee (1986)

$$dr_t = \theta(t)dt + \sigma dW_t, \tag{2.10}$$

6. Hull-White (extended Vasiček) (1990)

$$dr_t = [\theta(t) - ar_t]dt + \sigma dW_t, \tag{2.11}$$

7. Hull-White (extended CIR) (1990)

$$dr_t = [\theta(t) - ar_t]dt + \sigma\sqrt{r_t}dW_t. \tag{2.12}$$

In the following chapter, we will use the Hull-White model for our estimation process and we will see more in detail how this is achieved. However, we can already note that, once we have chosen the model we will work with, we will have to choose the parameter vector in a way that the theoretical curve, the curve produced as output by the model, fits the empirical curve, the curve actually observed in the market, or the bond prices we will calculate will not be correct.

2.1.1 Affine Term Structures (ATS)

Definition 2.3.

If the term structure $\{p(t,T);\ 0 \leq t \leq T,\ T > 0\}$ *has the form*

$$p(t,T) = F(t, r_t; T), \tag{2.13}$$

where F *has the form*

$$F(t,r;T) = e^{A(t,T)-B(t,T)r}, \tag{2.14}$$

and where A *and* B *are deterministic functions, then the model is said to possess an ATS.*

Since the existence of an ATS facilitates importantly the analytical and computational task, it is important to understand for which choices of μ and σ in the Q-dynamics for r we get an ATS. We will proceed formulating the following proposition.

Proposition 2.4.

Assume that μ *and* σ *are of the form*

$$\begin{cases} \mu(t,r) = \alpha(t)r + \beta(t), \\ \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}. \end{cases} \tag{2.15}$$

Then the model admits an ATS of the form (2.14) where A and B satisfy the system

$$\begin{cases} B_t(t,T) + \alpha(t)B(t,T) - \dfrac{1}{2}\gamma(t)B^2(t,T) = -1, \\ \qquad\qquad\qquad\qquad\qquad\qquad B(T,T) = 0. \end{cases} \tag{2.16}$$

$$\begin{cases} A_t(t,T) = \beta(t)B(t,T) - \dfrac{1}{2}\delta(t)B^2(t,T), \\ A(T,T) = 0. \end{cases} \tag{2.17}$$

Given the high importance of this proposition, we will provide the proof below.

Proof

We will assume that we have the Q-dynamics

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t, \tag{2.18}$$

and that the bond prices have the form (2.14). This allows us to compute easily the partial derivatives of F, and since F must solve the term structure equation (2.3), we obtain

$$A_t(t,T) - \{1 + B_t(t,T)\}r - \mu(t,r)B(t,T) + \frac{1}{2}\sigma^2(t,r)B^2(t,T) = 0 \tag{2.19}$$

The boundary value $F(T, r; T) \equiv 1$ implies

$$\begin{cases} A(T,T) = 0, \\ B(T,T) = 0. \end{cases} \tag{2.20}$$

We can notice that if μ and σ^2 are both affine functions of r, with possibly time-dependent coefficients, then equation (2.19) become a separable differential for the unknown functions A and B. Assume that μ and σ have the form

$$\begin{cases} \mu(t,r) = \alpha(t)r + \beta(t), \\ \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}. \end{cases} \tag{2.21}$$

After collecting terms, (2.19) transforms into

$$\begin{aligned} &A_t(t,T) - \beta(t)B(t,T) + \frac{1}{2}\gamma(t)B^2(t,T) \\ &- \left\{1 + B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T)\right\} r = 0. \end{aligned} \tag{2.22}$$

Since the equation holds for all values of r, the coefficient of r must be equal to zero. Thus we have the equation

$$B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = -1. \tag{2.23}$$

This implies that also the other term in (2.22) must be equal to zero and therefore we obtain

$$A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T). \tag{2.24}$$

We note that equation (2.16) is a Riccati equation for the determination of B which does not involve A. Having solved equation (2.16), we may then insert the solution B into equation (2.17) and simply integrate in order to obtain A. □

2.1.2 Hull-White model

As mentioned before, it is important to have a perfect fit between the theoretical and the observed bond prices. In order to obtain a perfect fit, we have to use a model with an infinite dimensional parameter vector by letting some or all parameters be time dependent. This will allow us to find an unique solution to the infinite dimensional system of equations (one equation for each T). In this section we will analyze the Hull-White (extended Vasiček) model, which uses that method to solve that issue, while in the following one we will discuss the HJM framework.
Since the model has an ATS, as we can see from the stochastic differential equation defining the short rate

$$dr_t = [\theta(t) - ar_t]dt + \sigma dW_t,$$

we can use the previous discussion to find the formula for bond pricing. In particular, A and B solve

$$\begin{cases} B_t(t,T) = aB(t,T) - 1, \\ B(T,T) = 0. \end{cases}$$

$$\begin{cases} A_t(t,T) = \theta(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T), \\ A(T,T) = 0. \end{cases}$$

The solutions to these equations are given by

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\},$$

$$A(t,T) = \int_t^T \left\{ \frac{1}{2}\sigma^2 B^2(s,T) - \theta(s)B(s,T) \right\} ds.$$

In order to fit the theoretical prices to the observed ones, we will use the forward rate, given the one-to-one correspondence. In any affine model the forward rates are given by

$$f(0,T) = B_T(0,T)r_0 - A_T(0,T),$$

which, after inserting the previous two equations into it, becomes

$$f(0,T) = e^{-aT}r_0 + \int_0^T e^{-a(T-s)}\theta(s)ds - \frac{\sigma^2}{2a^2}(1-e^{-aT})^2$$

Given an observed forward rate structure $f^\star$ our problem is to find a function θ that solves the previous equation $\forall T > 0$. This can be done by writing it as

$$f^\star(0,T) = x(T) - g(T)$$

where x and g are defined by

$$\begin{cases} \dot{x} = -ax(t) + \theta(t), \\ x(0) = r(0), \end{cases}$$

$$g(t) = \frac{\sigma^2}{2a^2}(1-e^{-at})^2 = \frac{\sigma^2}{2}B^2(0,t).$$

We now have

$$\begin{aligned} \theta(t) &= \dot{x}(T) + ax(T) = f_T^\star(0,T) + \dot{g}(T) + ax(T) \\ &= f_T^\star(0,T) + \dot{g}(T) + a\{f^\star(0,T) + g(t)\}, \end{aligned} \quad (2.25)$$

and have therefore proved the following result.

Lemma 2.5.

Fix an arbitrary bond curve $\{p^\star(0,T); T > 0\}$*, subject only to the condition that* $p^\star(0,T)$ *is twice differentiable w.r.t T. Choosing* θ *according to (2.25) will then produce a term structure* $\{p^\star(0,T); T > 0\}$ *such that* $p(0,T) = p^\star(0,T) \forall T > 0$.

By choosing θ according to (2.25) we have determined our martingale measure for a fixed choice of a and σ and it is now possible to state the following.

Proposition 2.6.

Consider the Hull-White model with a and σ fixed. Having inverted the yield curve by choosing θ according to (2.25) we obtain the bond prices as

$$p(t,T) = \frac{p^\star(0,T)}{p^\star(0,t)} exp\left\{ B(t,T)f^\star(0,t) - \frac{\sigma^2}{4a}B^2(t,T)(1-e^{-2at}) - B(t,T)r_t \right\}, \qquad (2.26)$$

where B is given in the previous page.

Proof

If we substitute the equations for B and θ in the one for A, we can express it as

$$A(t,T) = \frac{\sigma^2}{2a^2}\int_t^T (1-e^{-a(T-s)})^2 ds - \int_t^T \frac{1}{a}(1-e^{-a(T-s)})^2 (f^\star_T(0,s) + \dot{g}(s) + a\{f^\star(0,s)+g(s)\})ds.$$

If we integrate by parts and solve the first integral, we obtain

$$\begin{aligned} &= \frac{\sigma^2}{2a^2}\left\{ T - t - \frac{2}{a} + \frac{2e^{-a(T-t)}}{a} + \frac{1}{2a} - \frac{e^{-2a(T-t)}}{2a} \right\} + f^\star(0,t)B(t,T) - \\ &- \int_t^T f^\star(0,s)e^{-a(T-s)}ds + g(t)B(t,T) - \int_t^T g(s)e^{-a(T-s)}ds - \log P^\star(0,t)(1-e^{-a(T-t)}) + \\ &+ \int_t^T a \log P^\star(0,s)e^{-a(T-s)}ds - \int_t^T g(s)ds + \int_t^T g(s)e^{-a(T-s)}ds. \end{aligned}$$

After some arithmetic simplifications, we get to the following form

$$= f^\star(0,t)B(t,T) - \frac{\sigma^2}{4a}B^2(t,T)(1-e^{-2at}) + \log\frac{P^\star(0,T)}{P^\star(0,t)},$$

and after substituting the value for A in (2.14), we have proved the proposition. □

2.2 HJM framework

As mentioned before, the HJM framework is a popular method to fit the observed prices with the theoretical ones. Differently from the short rate models that we saw before, the HJM framework does not use a single explanatory variable, but instead uses the entire forward curve as the infinite dimensional framework. It is important to remark that the HJM method does not offer a specific model, but rather a framework. It is indeed possible to formulate any short rate model in forward rate terms (and we will indeed show below how this can be done for the Hull-White model analyzed before). We will follow the discussion made in (Björk, 2020, Chapter 22) and we will start stating the following assumption.

Assumption 2.7.

We assume that, for every fixed $T > 0$, the forward rate $f(\cdot, T)$ has a stochastic differential which under the objective measure P is given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t^P, \tag{2.27}$$

$$f(0,T) = f^{\star}(0,T), \tag{2.28}$$

where W^P is a d-dimensional P-Wiener process whereas $\alpha(\cdot, T)$ and $\sigma(\cdot, T)$ are adapted processes.

The following has to be noted:

- Once we have specified α, σ and $\{f^{\star}(0,T);\, T \geq 0\}$, we have specified the entire term structure $\{p(t,T);\, T > 0,\, 0 \leq t \leq T\}$;
- Since we have d sources of randomness and an infinite number of traded assets, we need to understand how the processes α and σ are related to avoid arbitrage possibilities.

Before stating the HJM drift condition, which solves the problem raised in the second point, we have to state two theorems, the First Fundamental Theorem of Asset Pricing (FTAP) and the Girsanov Theorem, which we will need in the proof of the HJM drift condition.

Theorem 2.8 (The First Fundamental Theorem of Asset Pricing)**.** *The market model is free of arbitrage if and only if there exists a martingale measure, i.e. a measure Q ∼ P such that the processes*

$$\frac{S_t^0}{S_t^0}, \frac{S_t^1}{S_t^0}, \dots, \frac{S_t^N}{S_t^0}$$

are (local) martingales measures under Q.

Proof For a proof see Björk (2020, Chapter 11)

Theorem 2.9 (The Girsanov Theorem)**.**

Let W be a d-dimensional standard P-Wiener process on (Ω, $\mathcal{F}$, P, **F***) and let φ be any d-dimensional* *adapted column vector process. Choose a fixed T and definie the process L on [0,T]*

$$dL_t = \varphi_t^\star L_t dW_t,$$
$$L_0 = 1,$$

i.e.

$$L_t = e^{\int_0^t \varphi_s^\star dW_s - \frac{1}{2}\int_0^t ||\varphi_s||^2 ds}.$$

Assume that

$$E^P[L_T] = 1,$$

and define the new probability measure Q on $\mathcal{F}_T$ by

$$L_T = \frac{dQ}{dP}, \qquad \text{on } \mathcal{F}_T.$$

Then

$$dW_t = \varphi_t dt + dW_t^Q,$$

where W^Q is a Q-Wiener process.

Proof For a proof see Björk (2020, Chapter 12)

The process φ is referred to as the Girsanov kernel of the measure transformation and it is given by $\varphi = -\lambda$, where λ is the market price of risk.

It is now possible to state and prove the following HJM drift condition.

Theorem 2.10.

Assume that the family of forward rates is given by (2.27) and that the induced bond market is arbitrage free. Then there exists a d-dimensional column-vector process

$$\lambda(t) = [\lambda_1(t), \dots, \lambda_d(t)]'$$

with the property that for all $T \geq 0$ and for all $t \leq T$, we have

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)'ds - \sigma(t,T)\lambda(t). \tag{2.29}$$

In these formulas $'$ denotes transpose.

Proof

From Proposition (1.5) we have the bond dynamics

$$dp(t,T) = p(t,T)\left\{r_t + A(t,T) + \frac{1}{2}||S(t,T)||^2\right\}dt + p(t,T)S(t,T)dW_t,$$

where

$$\begin{cases} A(t,T) = -\int_t^T \alpha(t,s)ds, \\ S(t,T) = -\int_t^T \sigma(t,s)ds. \end{cases}$$

Based on the First Fundamental Theorem, there must be a martingale measure Q to obtain an arbitrage free model. We have therefore to convert the P-dynamics to Q-dynamics and this is achieved through the Girsanov Theorem. Indeed we can write that

$$r_t + A(t,T) + \frac{1}{2}||S(t,T)||^2 + \sum_{i=1}^{d} S_i(t,T)\varphi_i(t) = r_t,$$
$$A(t,T) + \frac{1}{2}||S(t,T)||^2 - \sum_{i=1}^{d} S_i(t,T)\lambda_i(t) = 0.$$

Eventually, taking the T-derivative of the last equation gives us equation (2.29). □

This result shows us that once we have specified the forward rate dynamics under Q and the volatility structure, the drift parameter will be uniquely determined and we are able to compute bond prices.

We will show now how this can be achieved by taking into consideration a volatility structure, that will eventually return the Hull-White model that we have studied in the previous section. We will specify σ as follows

$$\sigma(t,T) = \sigma e^{-a(T-t)}$$

and, by the HJM drift condition, we have that

$$\alpha(t,T) = \frac{\sigma^2}{a} e^{-a(T-t)}(1 - e^{-a(T-t)})$$

We can restate assumption (1.5) in integral form and we have

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dW_s$$

We notice that, considering equation 1.5 and equation 2.14, we can write that

$$\begin{aligned} f(0,T) &= \int_0^T \theta(s)\frac{\partial B(s,T)}{\partial T} ds + \frac{\sigma^2}{2a^2}\left(\frac{\partial B(0,T)}{\partial T} - 1\right) + \frac{\sigma^2}{2a} B(0,T)\frac{\partial B(0,T)}{\partial T} + \frac{\partial B(0,T)}{\partial T} r_0 \\ &= \int_0^T \theta(s)e^{-a(t-s)}ds + \frac{\sigma^2}{2a^2}\left(e^{-aT} - 1\right) + \frac{\sigma^2}{2a^2}\left(e^{-aT} - e^{-2aT}\right) + e^{-aT} r_0 \\ &= \int_0^T \theta(s)e^{-a(t-s)}ds - \frac{\sigma^2}{2a^2}\left(e^{-aT} - 1\right)^2 + e^{-aT} r_0. \end{aligned}$$

Furthermore we notice that

$$\begin{aligned} \int_0^t \alpha(s,T)ds &= \frac{\sigma^2}{a}\int_0^t e^{-a(T-s)}(1 - e^{-a(T-s)})ds \\ &= \frac{\sigma^2}{2a^2}\left[\left(e^{-a(T-t)} - 1\right)^2 - \left(e^{-aT} - 1\right)^2\right] \end{aligned}$$

Combining the three pieces together and recalling that $r_t = f(t,t)$, we have

$$r_t = e^{-at}\int_0^t \theta(s)e^{as}ds + e^{-at} r_0 + e^{-at}\int_0^t \sigma e^{as} dW_s$$

and, taking the derivative of it, we get

$$\begin{aligned} dr_t &= -ar_t dt + \theta(t)dt + \sigma dW_t \\ &= [\theta(t) - ar_t]dt + \sigma dW_t, \end{aligned}$$

which corresponds to the short rate Hull-White model as we have defined it in equation (2.11).

2.3 Jarrow-Yildirim model

The Jarrow-Yildirim model (2003) was among the first models developed to price TIPS. In particular, it uses an HJM model to consistently price TIPS and the Hull-White model as specification for the term structure during the estimation process. In this section, we will focus on the theoretical framework, while leaving the estimation process to a later chapter. Furthermore, before beginning with the analysis, it is important to acknowledge that the HJM foreign currency analogy (as developed in Jarrow and Turnbull, 1998) is used to implement the methodology, where the two currencies are the nominal and the real rates (with the real rate being the foreign currency), and the index for measuring the inflation rate is the spot exchange rate. As indicated in Brigo and Mercurio (2007), this analogy is perfectly motivated because, denoting by $I(t)$ the CPI value at time t, it is possible to switch from real to nominal values simply by multiplying the price of a basket in real terms by $I(t)/I(0)$ to obtain the price in nominal terms. Since $I(0)$ is constant, this is quite similar to converting amounts from one currency to another.

Notation

- r: real;
- n: nominal;
- $P_n(t,T)$: time t price of a nominal ZCB maturing at time T in dollars;
- $I(t)$: time t inflation index;
- $P_r(t,T)$: time t price of a real ZCB maturing at time T in inflation index units;
- $f_k(t,T)$: time t forward rates for date T, where $k \in \{r,n\}$;
- $r_k(t) = f_k(t,t)$: the time t spot rate where $k \in \{r,n\}$;
- $B_k(t) = exp\{\int_0^t r_k(s)ds\}$: time t money market account value for $k \in \{r,n\}$;
- $P_{TIPS}(t,T) = I(t)P_r(t,T)$: time t price of a real ZCB maturing at time T in dollars;

Uncertainty is introduced in the economy through three P-Wiener processes $(W_n(t), W_r(t), W_I(t))$. These Wiener processes have correlations given by $dW_n(t)dW_r(t) = \rho_{nr}dt$, $dW_n(t)dW_I(t) = \rho_{nI}dt$ and $dW_r(t)dW_I(t) = \rho_{rI}dt$, thus generating a three-factor model.

Assumption 2.11.

We assume that the initial nominal forward rate curve $f_n(0,T)$, the initial real forward rate curve $f_r(0,T)$ and the inflation index $I(t)$ evolve according to the following processes

$$df_n(t,T) = \alpha_n(t,T)dt + \sigma_n(t,T)dW_n(t), \tag{2.30}$$

$$df_r(t,T) = \alpha_r(t,T)dt + \sigma_r(t,T)dW_r(t), \tag{2.31}$$

$$\frac{dI(t)}{I(t)} = \mu_I(t)dt + \sigma_I(t)dW_I(t). \tag{2.32}$$

These processes are arbitrage free if and only if the following process are Q-martingales (Amin and Jarrow, 1991)

$$\eta(t) = \frac{P_n(t,T)}{B_n(t)},\ \zeta(t) = \frac{I(t)P_r(t,T)}{B_n(t)} \text{ and } \xi(t) = \frac{I(t)B_r(t)}{B_n(t)},$$

and this implies that the following conditions must hold

$$\alpha_n(t,T) = \sigma_n(t,T)\int_t^T \sigma_n(t,s)ds, \tag{2.33}$$

$$\alpha_r(t,T) = \sigma_r(t,T)\left(\int_t^T \sigma_t(t,s)ds - \sigma_I(t)\rho_{rI}\right), \tag{2.34}$$

$$\mu_I(t) = r_n(t) - r_r(t). \tag{2.35}$$

Proof

In the following proofs, we will use Itô's product and quotient rules and the approximations $dtdW_t \approx 0, dt^2 \approx 0$ and $dW^2(t) \approx dt$, but it has to be noted that the same results can be achieved in a completely rigorous mathematical method, with a similar proof to the one provided in Amin and Jarrow (1991).

1. $d\eta(t) = \frac{dP_n(t,T)}{B_n(t)} - \frac{P_n(t,T)}{B_n^2(t)}dB_n(t).$

   By substituting in the values for the two price processes and using Proposition (1.5) we get

$$\begin{aligned}\frac{d\eta(t)}{\eta(t)} &= [r_n(t) + A_n(t,T) + \frac{1}{2}||S_n(t,T)||^2]dt + S_n(t,T)dW_n(t) - r_n(t)dt \\ &= [A_n(t,T) + \frac{1}{2}||S_n(t,T)||^2]dt + S_n(t,T)dW_n(t).\end{aligned}$$

In order to be a martingale, η must have null drift, so we set the dt term to zero and derive the result w.r.t T:

$$A_n(t,T) = -\frac{1}{2}||S_n(t,T)||^2$$
$$\alpha_n(t,T) = \sigma_n(t,T)\int_t^T \sigma_n(t,u)du$$

2. $d\zeta = \frac{d(I(t)P_r(t,T))}{B_n(t)} - \frac{I(t)P_r(t,T)}{B_n^2(t)}dB_n(t)$

   By applying Itô product rule and using Proposition (1.5) we get

$$\begin{aligned}\frac{d\zeta(t)}{\zeta(t)} &= (r_r(t) + A_r(t,T) + \frac{1}{2}||S_r(t,T)||^2)dt + S_r(t,T)dW_r(t) + \mu_I(t)dt + \sigma_I(t)dW_I(t)+ \\ &+ \sigma_I S_r(t,T)\rho_{rI}dt - r_n(t)dt \\ &= (r_r(t) + A_r(t,T) + \frac{1}{2}||S_r(t,T)||^2 + \mu_I(t) + \sigma_I S_r(t,T)\rho_{rI} - r_n(t))dt+ \\ &+ S_r(t,T)dW_r(t) + \sigma_I(t)dW_I(t).\end{aligned}$$

   As before, in order to be a martingale, ζ must have null drift. Therefore, we set the dt term to zero and derive w.r.t. T to obtain

$$-\alpha_r(t,T) + \sigma_r(t,T)\int_t^T \sigma_r(t,s)ds - \sigma_r(t,T)\sigma_I\rho_{rI} = 0$$

   that we can easily see corresponding to the second condition.

3. $d\xi = \frac{d(I(t)B_r(t)}{B_n(t)} - \frac{I(t)B_r(t)}{B_n^2(t)}dB_n(t)$.

   By applying Itô product rule and after some arithmetical semplifications we get

$$\begin{aligned}d\xi(t) &= \frac{dI(t)B_r(t) + I(t)dB_r(t)}{B_n(t)} - r_n(t)dt\xi(t) \\ &= \frac{(I(t)\mu_I(t)dt + I(t)\sigma_I dW_I(t))B_r(t) + I(t)r_r(t)B_r(t)dt}{B_n(t)} - r_n(t)dt\xi(t) \\ &= \xi(t)[\mu_I d(t) + \sigma_I dW_I(t) + r_r d(t) - r_n(t)] \\ \frac{d\xi(t)}{\xi(t)} &= (\mu_I(t) + r_r(t) - r_n(t))dt + \sigma_I dW_I(t).\end{aligned}$$

   By setting the dt term equal to zero in order for ξ to be a martingale, we have proved the third and last condition. □

From equations (2.33) and (2.34), together with Proposition (1.5), we can easily see that nominal and real ZCB must have the following price processes:

$$\frac{dP_n(t,T)}{P_n(t,T)} = r_n(t)dt - \int_t^T \sigma_n(t,s)dsdW_n(t) \tag{2.36}$$

$$\frac{dP_r(t,T)}{P_r(t,T)} = \left[r_r(t) + \rho_{rI}\sigma_I \int_t^T \sigma_r(t,s)ds\right]dt - \int_t^T \sigma_r(s)dsdW_r(t) \tag{2.37}$$

Finally, we can show that

$$\frac{dP_{TIPS(t,T)}}{P_{TIPS(t,T)}} = r_n(t)dt + \sigma_I dW_I(t) - \int_t^T \sigma_r(t,s)dsdW_r(t) \tag{2.38}$$

Proof Equation (2.38)

We recall that $P_{TIPS}(t,T) = I(t)P_r(t,T)$. Therefore, by Itô product rule, we have

$$\begin{aligned}
dP_{TIPS(t,T)} &= I(t)dP_r(t,T) + P_r(t,T)dI(t) + dI(t)dP_r(t,T) \\
\frac{dP_{TIPS(t,T)}}{P_{TIPS(t,T)}} &= \left[r_r(t) + \rho_{rI}\sigma_I \int_t^T \sigma_r(t,s)ds\right]dt - \int_t^T \sigma_r(s)dsdW_r(t) + \\
&\quad + [r_n(t) - r_r(t)]dt + \sigma_I dW_I(t) - \sigma_I \rho_{rI} \int_t^T \sigma_r(t,s)dsdt \\
&= r_n(t)dt + \sigma_I dW_I(t) - \int_t^T \sigma_r(t,s)dsdW_r(t)
\end{aligned}$$

which completes the proof. □

3 Empirical analysis

In this chapter, we will use the Hull-White and Jarrow-Yildirim models described in the previous chapter to analyze TIPS and Treasuries data in order to deduce the market expectation on future inflation implied in market data. We will first briefly describe the different types of securities issued by the US Treasury, then present the data, describe the methodology adopted and finally analyze the results. The Python code used can be found in Appendix A.

3.1 US Treasury Debt Survey

The US Treasury issues six different securities, but only five of which are marketable. The only non-marketable security, US Saving Bonds[2], cannot indeed be sold or transferred to someone else since each is registered to one person's social security number. The five marketable securities are:

1. **Treasury Bills** have maturities ranging from 4 week to 52 weeks. They are sold at a discount or at par and when it matures, the face value is paid;

2. **Treasury Notes** have maturities of 2, 3, 5, 7, or 10 years and pay a fixed rate of interest every six months until they mature;

3. **Treasury Bonds** have maturities of either 20 or 30 years and pay a fixed rate of interest every six months until they mature;

4. **Treasury Inflation-Protected Securities (TIPS)** have maturities of 5, 10, or 30 years. They are indexed to the CPI-U. Coupons and principal are adjusted on the basis of the the indexation coefficient for the relevant date and cannot be less than their nominal value.

5. **Floating Rate Notes (FRNs)** mature in two years and pay interest four times each year. The interest rate is the sum of an index rate and a spread. The index rate is tied to the highest accepted discount rate of the most recent 13-week Treasury Bill and therefore is reset every week, while the spread is constant and is determined at the auction when the FRN is first offered.

It is possible to view the debt outstanding per marketable security in Figure 1.

[2] More information on US Saving Bonds can be found on the US Treasury's website at the link `https://treasurydirect.gov/savings-bonds/`

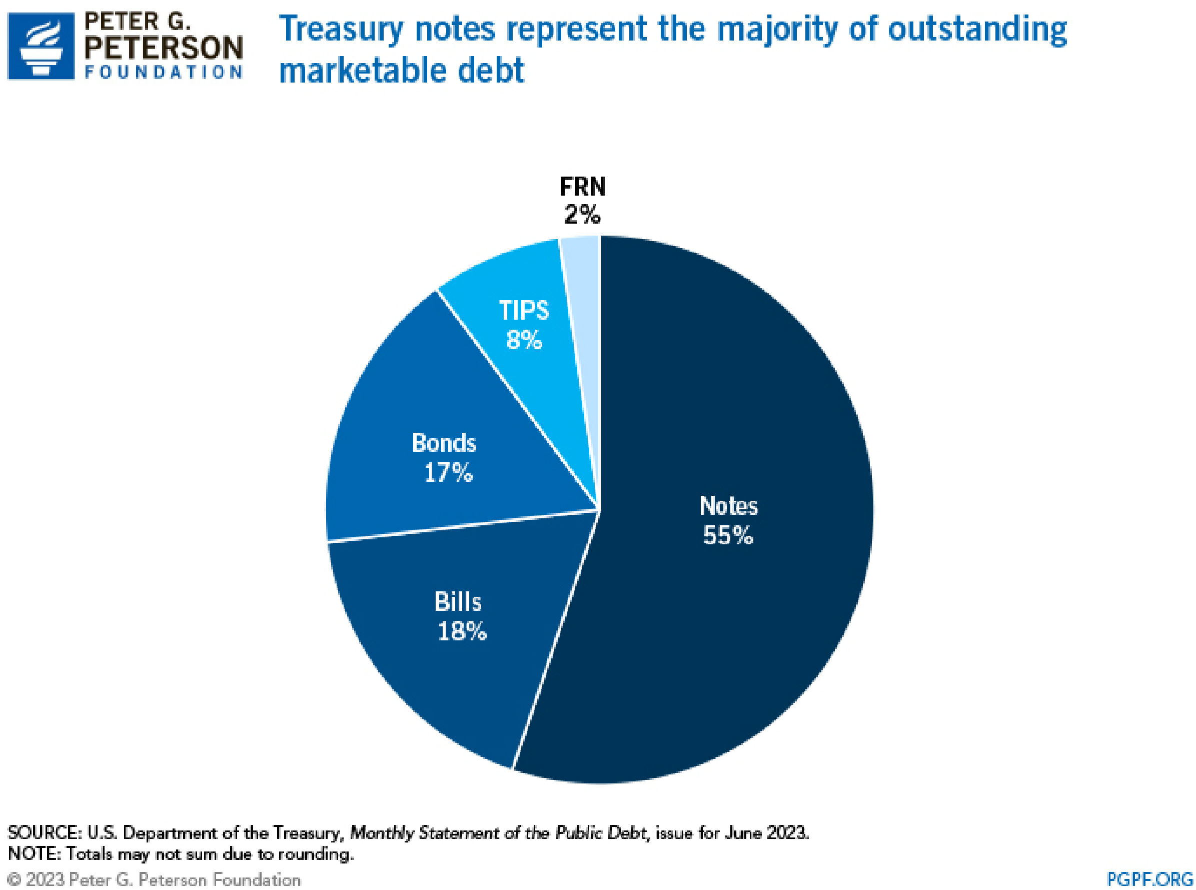

Figure 1: Outstanding Debt by Marketable Security as of June 2023.
Source Peter G. Peterson Foundation

3.2 Data

Our analysis employs daily prices of Treasury Notes, Treasury Bonds and TIPS. Securities' prices and characteristics were retrieved from CapIQ on September 2$^{\text{nd}}$, 2023 for a 20-months window, spanning from January 3$^{\text{rd}}$, 2022 to August 31$^{\text{st}}$, 2023. The time window consists of 414 daily observations. At the day of retrieval 51 TIPS and 1474 nominal securities were outstanding.

To minimize the possible misspecification that the frictionless market assumption may have on the estimation, we exclude from the analysis the securities that mature either in 2023 or in 2024 given the marginal trader's tax treatment for coupons and capital gains income that may differ for these bonds as compared to the remaining ones. Furthermore, we remove all securities that have been issued after the begin of the time window. Finally, as Kupfer (2018) notes, when calculating the difference between nominal and real return to estimate the break-even inflation rate (BEIR), there is the possibility of biased estimates due to difference in liquidity between Treasuries and TIPS.

Therefore, we remove all the securities issued after before January 1st, 2020, and we are left with 13 TIPS and 91 Treasuries. In Table 1 it is possible to see the list of TIPS used in the analysis.

Table 1: TIPS Data

CUSIP	Coupon %	Issue Date	Maturity Date
91282CDC2	0.125	10/29/2021	10/15/2026
91282CCM1	0.125	07/30/2021	07/15/2031
91282CCA7	0.125	04/30/2021	04/15/2026
912810SV1	0.125	02/26/2021	02/15/2051
91282CBF7	0.125	01/29/2021	01/15/2031
91282CAQ4	0.125	10/30/2020	10/15/2025
912828ZZ6	0.125	07/31/2020	07/15/2030
912828ZJ2	0.125	04/30/2020	04/15/2025
912810SM1	0.125	02/28/2020	02/15/2050
912828Z37	0.125	01/31/2020	01/15/2030
9128287D6	0.250	07/31/2019	07/15/2029
912810SG4	1.000	02/28/2019	02/15/2049
9128285W6	0.875	01/31/2019	01/15/2029

3.2.1 Treasuries

Because of the selection rules described above, our analysis deals only with Treasury Notes and Treasury Bonds, since Treasury Bills do not have a maturity of at least 20 months. Treasury Notes are issued with maturities of 2, 3, 5, 7, or 10 years, while Treasury Bonds are issued with maturities of either 20 or 30 years. Both securities are auctioned and bids can be presented both in a non-competitive or competitive way[3]. Coupons are paid twice a year and their rate is never lower than 0.125%. Treasuries are then redeemed at maturity with a single payment. Treasuries are a fundamental in global finance and are generally considered the risk-free investment. In Figure 2 we have plotted the clean price of three representative Treasuries for the time window period, therefore not including accrued interest to the quoted price, because those are the prices which will be later used in the analysis.

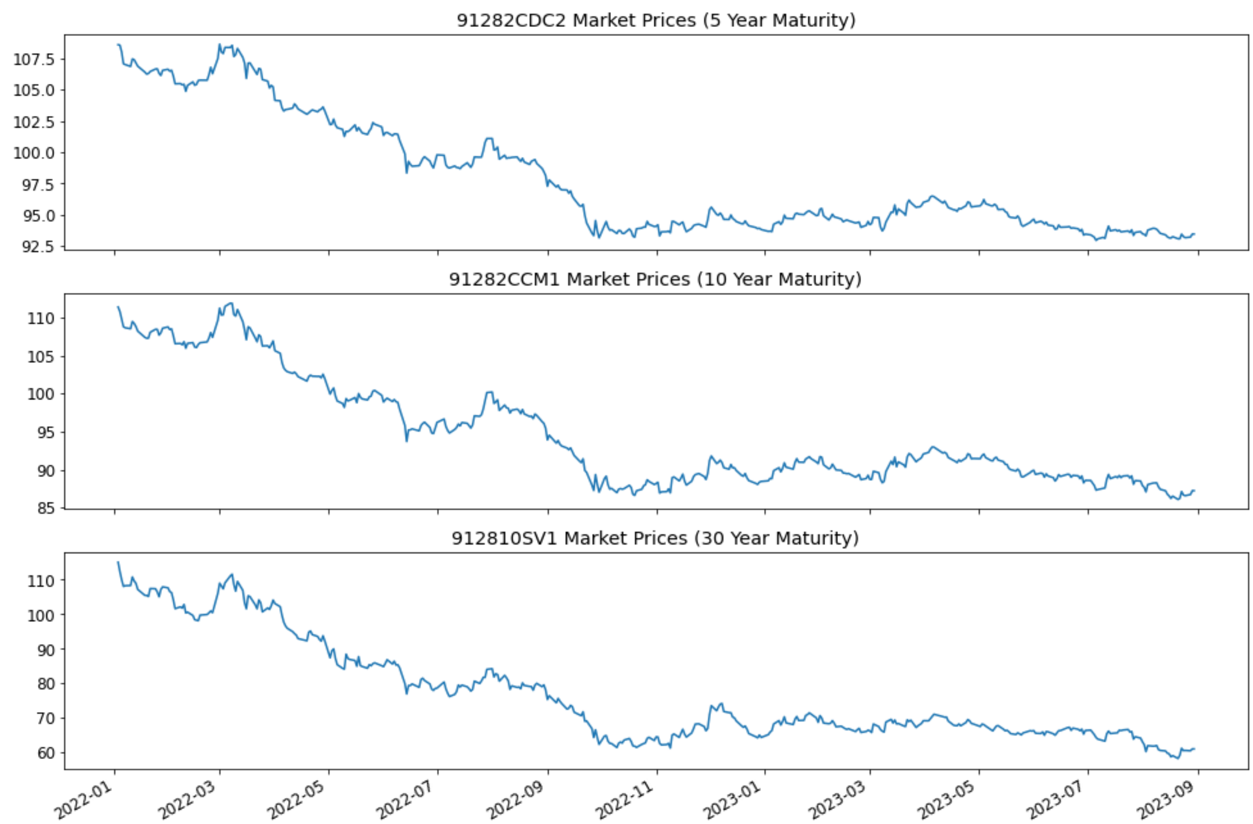

Figure 2: Clean Daily Prices of Three Representative Treasuries,
January 3rd, 2022 to August 31st, 2023

3.2.2 TIPS

TIPS are issued with maturities of 5, 10, or 30 years. Similarly to Treasuries, TIPS are auctioned, bids can be presented both in a non-competitive or competitive way, coupons are paid twice a year, their rate is never lower than 0.125%, and are then redeemed at maturity with a single payment.

[3]For a description of the difference between non-competitive and competitive bids, we refer to Treasury website at `https://www.treasurydirect.gov/auctions/how-auctions-work/`

However, differently from Treasuries, TIPS coupon payments are not constant throughout the life of the security, but are indexed to the CPI-U, an inflation index discussed below. In particular, the Treasury publishes on a monthly basis daily indexation coefficient $\text{IC}_{\text{y,m,d}}$ calculated as

$$\text{IC}_{\text{y,m,d}} = \frac{\text{RI}_{\text{y,m,d}}}{\text{IB}}$$

where IB is the inflation base, which is set at the dated date and represents the inflation at the time of issuance, and $\text{RI}_{\text{y,m,d}}$ is the reference inflation for the day d, month m and year y[4]. As indicated both in Barone and Castagna (1998) and in Gürkaynak, Sack and Wright (2010), $\text{RI}_{\text{y,m,d}}$ is an interpolation of the CPI-U values three and two months previous to the date of interest. The formula to calculate it is

$$\text{RI}_{\text{y,m,d}} = \Omega_M + \frac{g-1}{n} \cdot (\Omega_{M+1} - \Omega_M)$$

where g is the calendar day of the day of interest, n is the number of days in the month in which the day of interest falls and Ω_M is the CPI-U reported as of the third month previous to that of the day of interest.

Another important TIPS characteristic to be aware of is that there is an embedded option on the coupon payments. Indeed TIPS never pay less than the original principal, but the greater between the inflation-adjusted price or the original principal. Therefore each coupon payments is calculated as

$$\max\ \left(\frac{\text{c}}{2} \cdot \text{RI}_{\text{y,m,d}}\ ,\ \frac{\text{c}}{2} \right)$$

Finally, in Figure 3 the prices of three TIPS have been plotted. We have used clean prices in this case as well. We can see that, similarly to Treasuries prices plotted in Figure 2, the increase in inflation and consequent increase in interest rates brings to a sharp reduction in securities prices.

[4]Both the IB and $\text{RI}_{\text{y,m,d}}$ are publicly available at the page `https://www.treasurydirect.gov/auctions/announcements-data-results/tips-cpi-data/` by selecting any TIPS

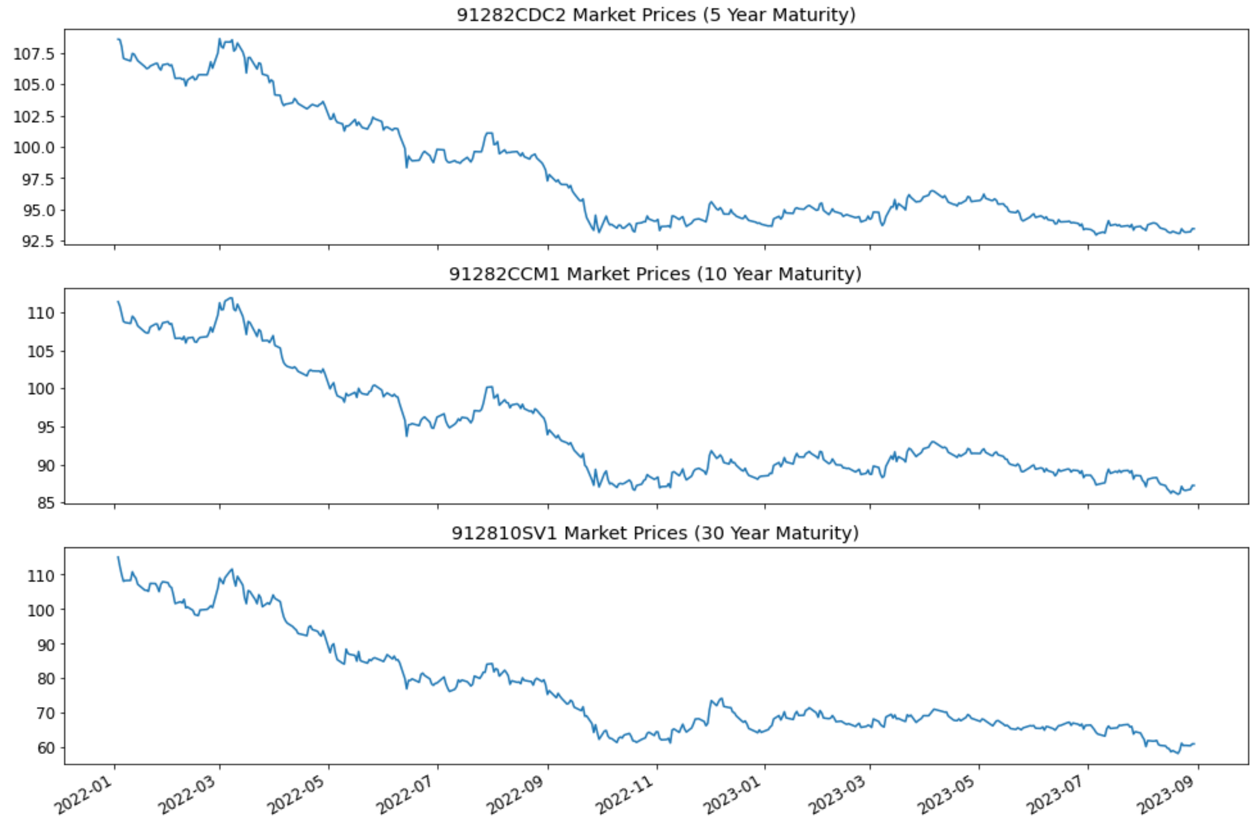

Figure 3: Clean Daily Prices of Three Representative TIPS,
January 3rd, 2022 to August 31st, 2023

3.2.3 CPI-U

There are multiple price indexes available for the US, and TIPS are indexed to the Consumer Price Index for All Urban Consumers (CPI-U), which covers approximately 93% of the total population. The CPI represents changes in prices of all goods and services purchased for consumption by urban households, including user fees and sales and excise taxes. The CPI-U includes expenditures by urban wage earners and clerical workers, professional, managerial, and technical workers, the self-employed, short-term workers, the unemployed, retirees and others not in the labor force.

Because of data collection and computation issues, the index is always reported with a two-month lag, which causes TIPS not to provide an exact real return, but only an approximate one.

CPI-U is available monthly from January 1913 to September 2023[5]. In order to use the index for daily securities prices, a daily linear interpolation is calculated as described previously. Finally,

[5]The time series can be downloaded from the Bureau of Labor Statistics' website at `https://data.bls.gov/pdq/SurveyOutputServlet`

it can be noted from Figure 4 that the only moment in the last 20 years when the CPI-U index has decreased is during the 2008 financial crisis, therefore giving little value to the aforementioned TIPS embedded option, which can safely not be calculated during the securities' pricing process.

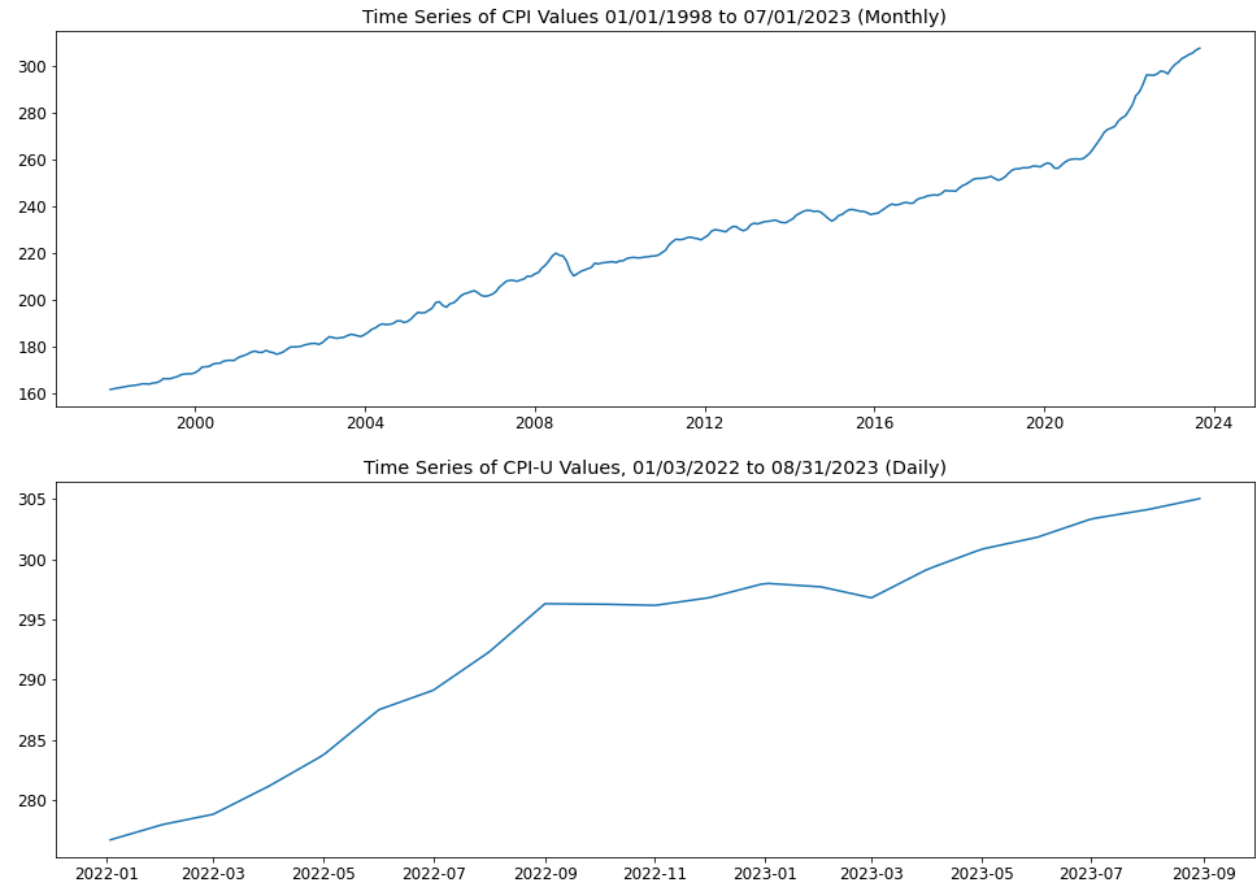

Figure 4: Daily and Monthly CPI-U Index Levels

3.3 Methodology

As we have seen with formula (1.15), the price of a nominal coupon bond is equal to the sum of the present value of its cash flows. In order to estimate the market's discount factors for Treasuries by stripping nominal and real zero-coupon bond prices from the observed market prices of the coupon-bearing securities, we use the quadratic programming estimation method described in Jarrow (2020) to minimize the sum of squared error differences between market and model prices. This can be done by defining the following problem

$$\min_{p(0,t)} \sum_{j=1}^{m} \left[p_j^\star(0) - \left(K \cdot p(0,T_j) + \sum_{i=1}^{j} c_j \cdot p(0,T_i) \right) \right]^2 \tag{3.1}$$

where $p_j^\star(0)$ denotes the market price of the j-Treasury at time 0, m the number of Treasuries in our data set, K the face value of each Treasury (assumed to be constant at 100 for all Treasury), c_j the coupon payment of the j-Treasury and $p(0,T_i)$ the discount factor for maturity T_i at time 0.

A similar formula can be used for TIPS, taking into account the indexation coefficient, and is defined as

$$\min_{p(0,t)} \sum_{j=1}^{m} \left[p_j^\star(0) - \frac{I(0)}{I(t_{0,j})} \left(K \cdot p(0,T_j) + \sum_{i=1}^{j} c_j \cdot p(0,T_i) \right) \right]^2 \tag{3.2}$$

where $p_j^\star(0)$ denotes the market price of the j-TIPS, $I(t_{0,j})$ the REF CPI on the issue date and m the number of TIPS in our data set.

This procedures generates a vector of zero-coupon bond prices that have maturities with discrete spacing. It is therefore impossible to determine unequivocally the continuously compounded forward rates of all maturities, which is a continuous curve. The simplest approach to solve this issue is assuming constant forward rates over the missing maturities, therefore enabling us to parameterize a continuous curve with a finite number of parameters. Furthermore, Bliss (1996) provides evidence that piecewise constant forms work well. Therefore, as seen in equation 1.5, it is possible to calculate the forward rate with the formula

$$\frac{p(t,T)}{p(t,T+\Delta)} = \exp\left(\int_{T}^{T+\Delta} f(t,s)ds \right) \tag{3.3}$$

which, given the piecewise constant forward rate hypothesis, simplifies to

$$\frac{p(t,T)}{p(t,T+\Delta)} = \exp\left(f(t,T)\Delta \right) \tag{3.4}$$

It has to be noted that this process cannot be applied directly to the securities we have described previously. Indeed, in order to estimate the discount factors through expression (3.1) we need a zero-coupon bond price to be estimated for each coupon payment date for each of the coupon bonds under consideration. Given that there Treasuries with still approximately 30 years left in their life, it means that for each of them about 60 zero-coupon bond prices will need to be estimated. This would be possible for Treasuries, given the high number of them available, but it would not be the case for TIPS, which would not have a unique solution to the minimization problem.

In order to overcome this difficulty and obtain a unique solution, the zero-coupon bond price stripping procedure and the forward rate estimation procedure can be combined into one operation by substituting the forward rate expression 3.4 into the minimization problems 3.1 and 3.2, thus generating the new minimization problems

$$\begin{aligned} \min_{f_n(0,t)} \sum_{j=0}^{m} \Bigg[p_j^\star(0) - \Bigg(K \cdot \exp\Bigg(- \sum_{k=1}^{T_j} f_n(0,k) \Bigg) \\ + \sum_{i=1}^{T_j} c_j \cdot \exp\Bigg(- \sum_{k=1}^{i} f_n(0,k) \Bigg) \Bigg) \Bigg]^2 \end{aligned} \tag{3.5}$$

for Treasuries and

$$\begin{aligned} \min_{f_r(0,t)} \sum_{j=0}^{m} \Bigg[p_j^\star(0) - \frac{I(0)}{I(t_{0,j})} \Bigg(K \cdot \exp\Bigg(- \sum_{k=1}^{T_j} f_r(0,k) \Bigg) \\ + \sum_{i=1}^{n} c_j \cdot \exp\Bigg(- \sum_{k=1}^{i} f_r(0,k) \Bigg) \Bigg) \Bigg]^2 \end{aligned} \tag{3.6}$$

for TIPS.

As mentioned before, it would not have been strictly necessary to implement the same piecewise constant forward rate curve procedure for both types of securities, but we decided to do so for comparative purposes. In order to highlight short-, medium-, and long-term expectations, we decided to assume forward rates are constant over four different intervals, specifically zero-three years, three-five years, five-ten years and ten-thirty years, as it was done in Jarrow-Yildirim (2003) as well.

This procedure is repeated for every day of the time window and generates a daily series for each maturity of real and nominal rates. This allows us to estimate the volatility functions used in the three-factor HJM model described in the previous chapter. In particular, two ways to estimate the volatility functions in an HJM model could be implemented:

1. Principal Component Analysis, a method to reduce the dimension of the dataset by preserving as much variability as possible to focus on the most important underlying factors (Rencher, 2003);

2. Functional form.

Here we will implement the second method and we consider a one-factor model with an exponentially declining volatility, the Hull-White model, which has the form

$$\sigma_i(t,T) = \sigma_i \exp^{-a_i(T-t)} \tag{3.7}$$

with i $\in$ {n, r}, where r denotes the real factors and n the nominal ones, and where σ_i and a_i are constants.
Using equation 2.37 and this volatility function, it can be easily shown that bond returns evolve according to the following normal distribution

$$\begin{aligned} \frac{dP_r(t,T)}{P_r(t,T)} - \left[r_r(t) + \rho_{rI}\sigma_I \int_t^T \sigma_r(t,s)ds \right] \Delta t \\ \sim N\left[0, \left(\int_t^T \sigma_r(t,s)ds \right)^2 \Delta t \right] \end{aligned} \tag{3.8}$$

with Δt being equal to $1/360$ since we are using daily observations. Thanks to this, the expected return on the bond $(r_r(t) + \rho_{rI}\sigma_I \int_t^T \sigma_r(t,s)ds)\Delta t$ is small relative to its standard deviation $(\int_t^T \sigma_r(t,s)ds)\sqrt{\Delta t}$ and can be neglected in estimation procedure. Although that being an approximation, it allows us to estimate the sample variance of the real spot rate without the need of initially estimating either the correlation of the index with the real spot rate or the volatility of the inflation index.
Therefore, we can conclude that, based on equations 3.7 and 3.8, the variance of the real zero-coupon bond prices satisfies the equation

$$\text{var}\left(\frac{\Delta P_r(t+\Delta,T)}{P_r(t,T)} \right) = \frac{\sigma^2(e^{-a_r(T-t)} - 1)^2 \Delta}{a_r^2} \tag{3.9}$$

Using equation 3.4 we can compute the left side of 3.9 and then run a cross-sectional nonlinear regression across the different maturity zero-coupon bond prices to estimate the parameters σ_r and a_r. An analogous process can be used to estimate the nominal forward rate parameters.

Finally, from equations 2.37 and 2.38, we see that the volatility of the inflation rate σ_I, and the correlations between the inflation index and the real spot rate ρ_{rI}, the inflation index and the nominal spot rate ρ_{nI}, and the real and nominal spot rate ρ_{rn} are crucial parameters to estimate the price dynamics. Using the sample moments, we can derive the following formulas to be used for computing estimates of these parameters

$$\sigma_I = \left\{ \frac{1}{\Delta} \text{var} \left(\frac{\Delta I(t)}{I(t)} \right) \right\}^{1/2} \tag{3.10}$$

$$\rho_{rI} = \text{cor} \left(\Delta r_r(t), \frac{\Delta I(t)}{I(t)} \right) \tag{3.11}$$

$$\rho_{nI} = \text{cor} \left(\Delta r_n(t), \frac{\Delta I(t)}{I(t)} \right) \tag{3.12}$$

$$\rho_{rn} = \text{cor} \left(\Delta r_r(t), \Delta r_n(t) \right), \tag{3.13}$$

using the historical CPI-U data, and the real and nominal interest rates calculated through the stripping procedure.

For this last part, we have to use monthly data and therefore have $\Delta = 1/12$ because we cannot use the linearly interpolated daily CPI-U values. Indeed, if we did, we would misspecify an estimate of a daily inflation rate's volatility because the linear interpolation procedure for creating daily index values is deterministic. We are therefore left with only 21 monthly observations.

3.4 Results

In this section we can finally present the results of our analysis.

In Figure 5 the forward rates for the 4 different maturities have been plotted. It can be immediately noted that, although the nominal rates behave as expected and remain into reasonable ranges, this is not the case for the real ones. In particular, the minimization process for the 0-3 year spot rates returns value that steadily increase until they reach values higher than 8%, which is clearly unrealistic. Multiple steps have been taken to solve this issue, but since no outliers were detected, increasing the number of estimated rates would not have been feasible for computing reasons, and

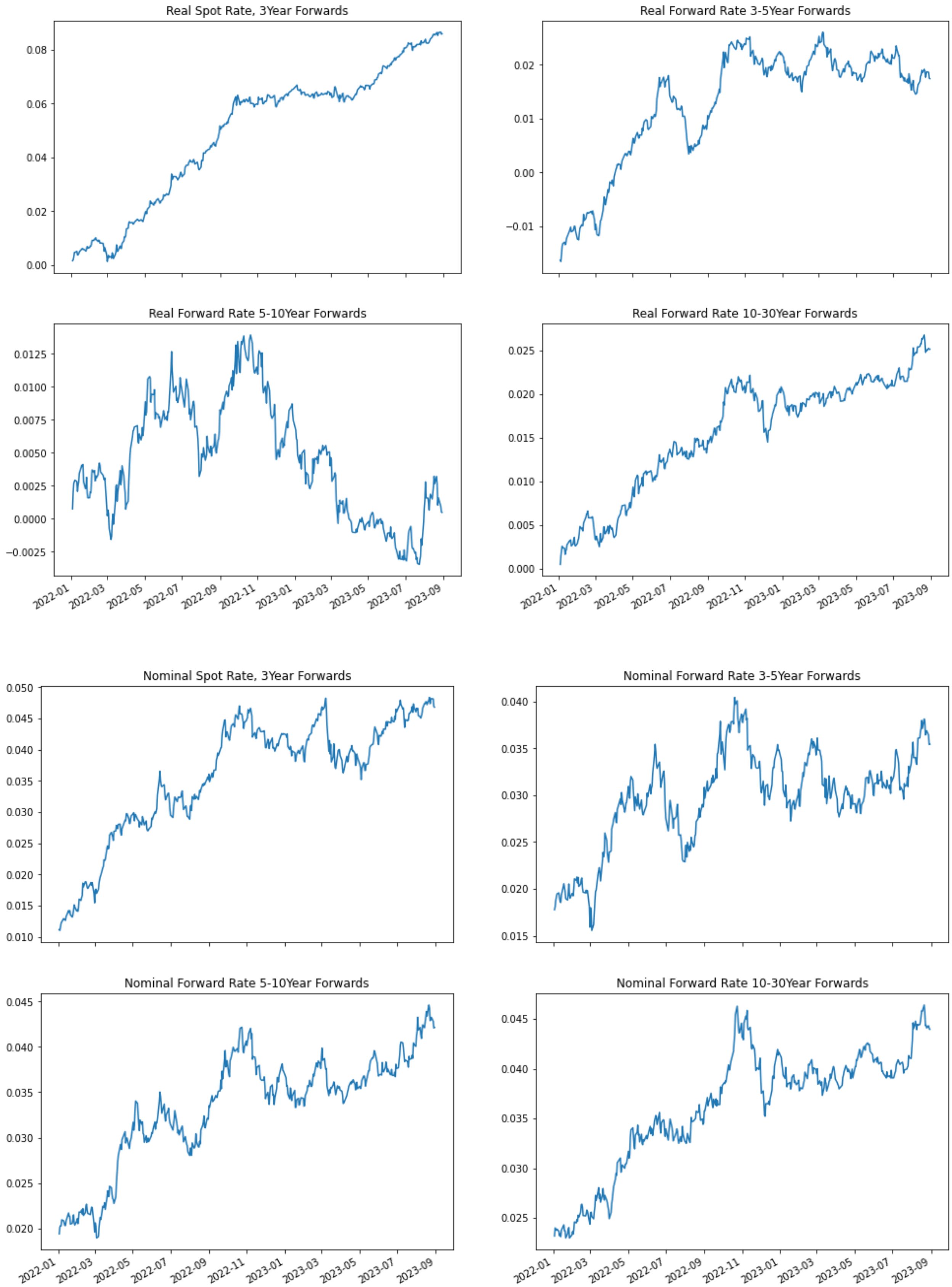

Figure 5: 3-, 5-, 10-, and 30-Year Real and Nominal Forward Rates
(January 3, 2022 - August 31, 2023)

the algorithm written to test that the code was actually returning the minimum values was working as expected, we have decided to present the current results, and to leave for future research the understanding of the reasons for these results.

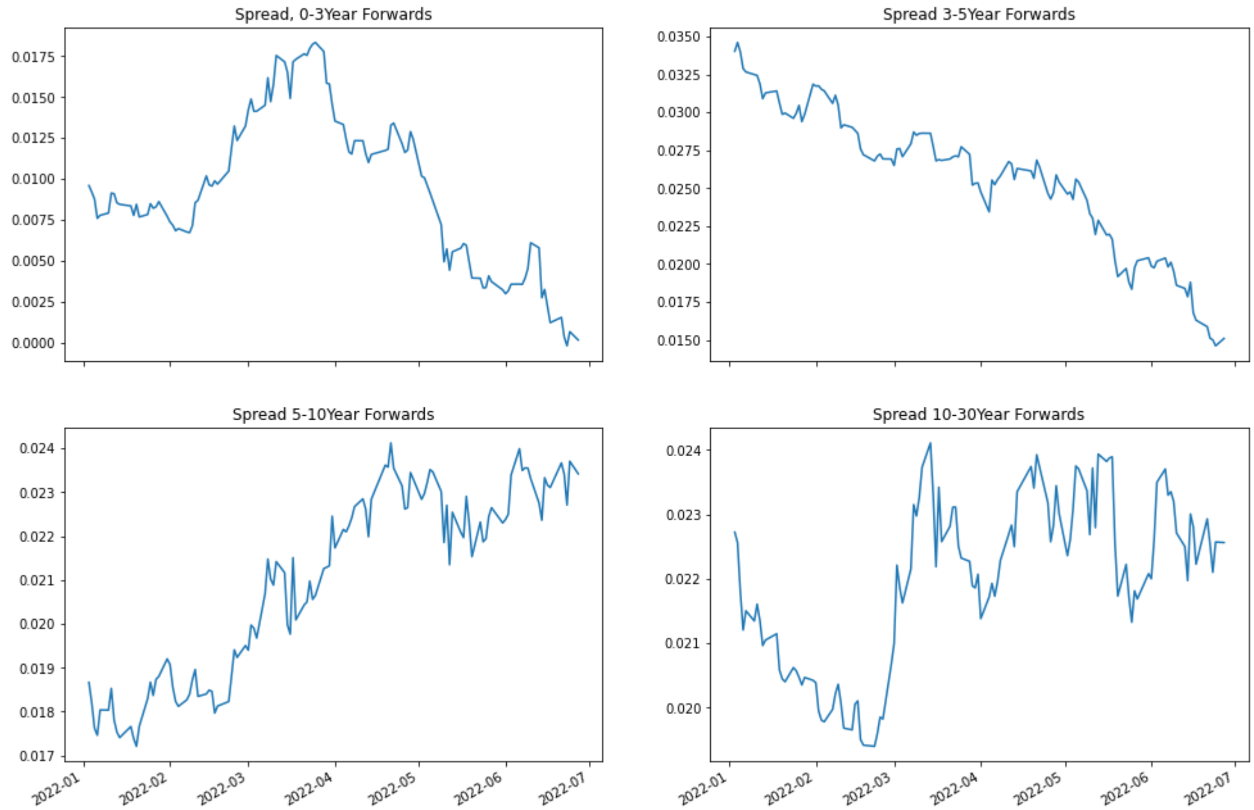

Figure 6: Nominal vs. Real Forward Spreads
(January 3, 2022 - June 28, 2022)

For the calculation of the expected inflation we have therefore used only the first 120 observations, reducing the time window to the period January 3rd, 2022 to June 21st, 2022. As explained in Barro (1998) and Blanchard (2016), the Fisher equation provides an approximation for inflation by the formula

$$\pi(t) = n(t) - r(t)$$

where $\pi(t)$ is the forward inflation rate and can be interpreted as the rate of inflation expected over a given period which begins at some future date. Similarly, $n(t)$ indicates the nominal forward rate and $r(t)$ the real forward rate.

In Figures 7 and 6 the spread for the reduced time-window has been plotted. It can be seen that the

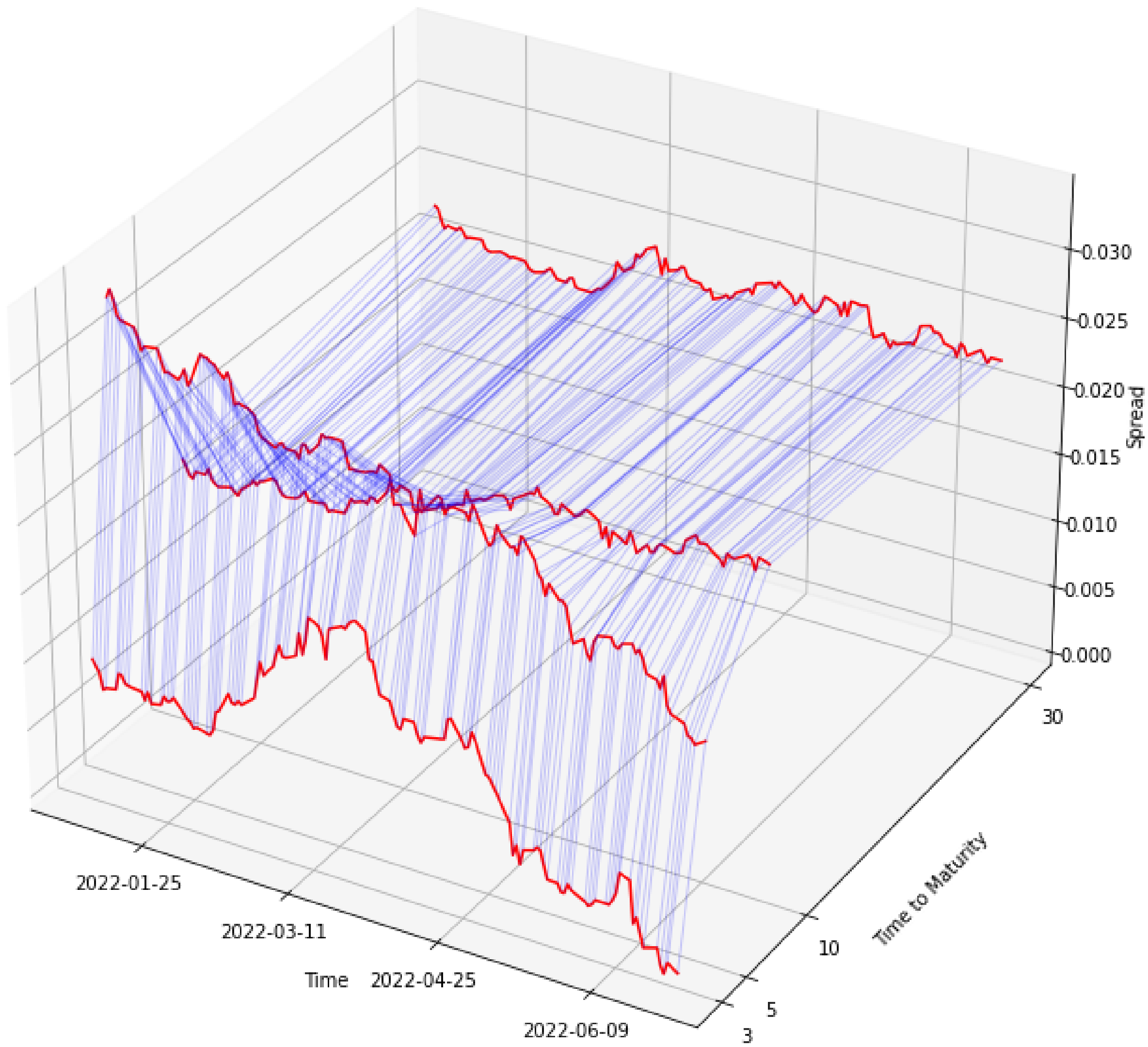

Figure 7: Nominal vs. Real Forward Spreads
(January 3, 2022 - June 28, 2022)

5-, 10- and 30-Year spread are either constant at 2% or move towards that value, thus indicating market confidence in the FED ability to bring inflation back to the target level. Instead, the 3-Year spread starts from extremely low values, as result of the extremely low interest environment that characterized the economy after the Covid-19 pandemic, and then increases until the beginning of April 2022, before moving back to low values. Since the FED has been raising rates since April 2022, this behavior is quite unexpected and probably relates to the minimization puzzle described earlier.

Despite these results, we have proceeded with the calibration of the model. The results are displayed in Table 2.

Table 2: Calibration parameters

Parameters	Estimate
σ_n	0.01656
a_n	0.02293
σ_r	0.01652
a_r	0.02830
σ_I	0.01303
ρ_{rI}	0.03558
ρ_{nI}	-0.00347
ρ_{rn}	0.73530

Finally, in Figure 8 we have plotted histograms with the daily absolute change in real forward rates and the monthly relative change in inflation. The daily absolute change in nominal rates is similar to the one for real ones and has therefore not been displayed. Furthermore, it has to be noted that, given the low number of monthly CPI-U observations both in the full and in the reduced time window, we have used all the monthly values from January 1998 to produce the inflation plot.

It can be noted that the real and nominal rates and the inflation volatility are quite similar, the nominal and real rates are barely correlated with the inflation index, and the correlation between the real and nominal rates is extremely strong, as can be seen from the above plots as well.

Finally, in Figure 9 the plot of the two volatility functions has been displayed for the 4 different maturities. Given the similarity between the real and nominal parameters, the two graphs looks really similar to each other and, as expected, decline exponentially in the forward rate's maturity.

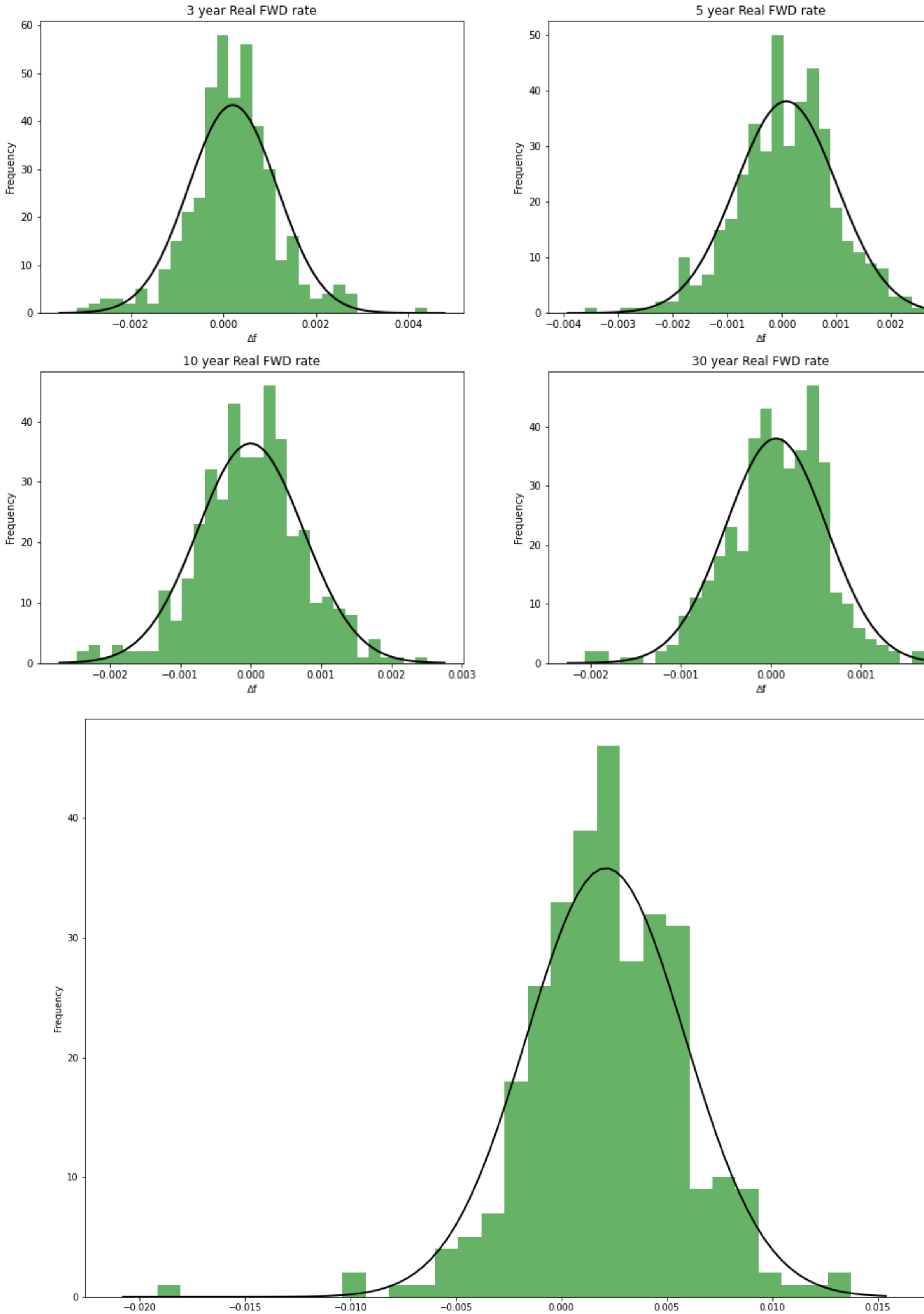

Figure 8: Normality Assumption on Real Rates and Inflation

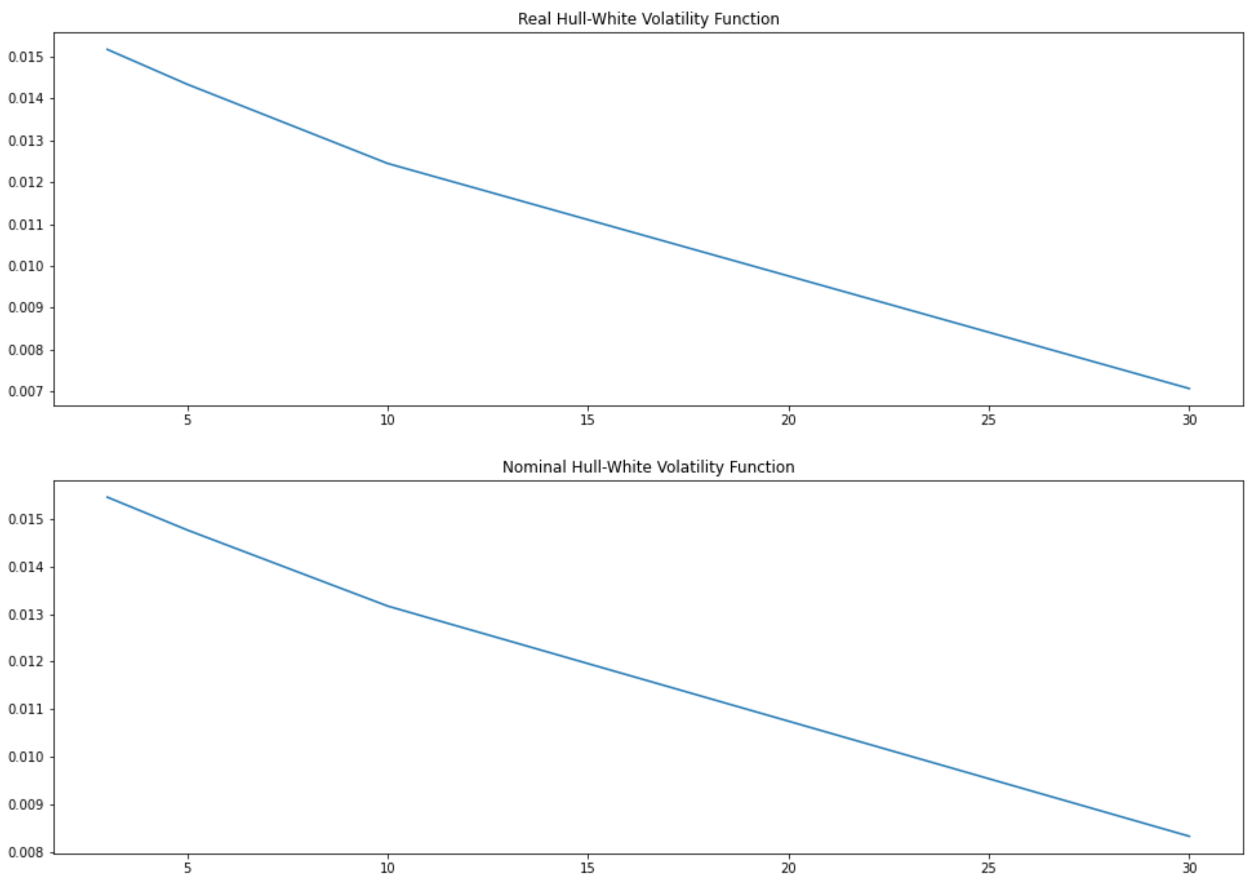

Figure 9: Real and Nominal Hull-White Volatility Functions

4 Conclusion

This thesis has been built on three deeply interrelated pillars, where each has constituted a chapter of the present work. The first one is the introductory and definitional part, which provided the basis for the remaining parts by introducing the most relevant concepts of different interest rates and incomplete market. The second one is the modeling part, where both the Hull-White model and the Jarrow-Yildirim model were introduced and discussed from a theoretical standpoint. Finally the last part is the empirical one and applies the concepts developed in the previous chapter to securities' data.

When planning the work at the beginning of this project, the goal was to estimate the market's expected inflation through 2022 and 2023, two years that have been marked by public interest in the topic and an interest environment that cut clearly from the one experienced after the 2008 financial crisis and during the Covid-19 pandemics, as the coupon rates of the different securities can show. Unfortunately, this project has been hampered by the difficulties that arose during the estimation process of the real forward interest rates and that, at this point, are still missing a solution. This caused the analysis to be limited to a 6-months time window, where the results still seemed reasonable. Indeed, if all the initial time-window had been plotted, it would have resulted in an increasing expected deflation in the short-term, which obviously is not compatible with the current market situation. This issue aside, all the other parts of the empirical analysis produce the expected results including, among others, the normality hypothesis for the change in real and nominal interest rates.

After 2003, other models for the estimation of TIPS prices, and derivatives written on them, have been published, such as Mercurio (2005), Jacoby and Shiller (2008), Hinnerich (2008), Chen, Liu and Cheng (2010), Singor, Grzelak,van Bragt, and Oosterlee (2013), Ho, Huang, and Yildirim (2014), D'Amico, Kim and Wei (2018), and Dam, Macrina, Skovmand and Sloth (2020), whose estimates could be more precise and provide a solution to the estimation issue. Furthermore, different interpolation methods, different from the piecewise constant one, could be implemented. A list of the methods available and currently used in the industry can be found in Rebonato (1998).

Another direction this thesis can be expanded is by estimating the real forward rates from securities different from TIPS. As Jarrow and Yildirim (2022) show, the traded notional volume of inflation swaps in 2021 hit a record $1.67 trillion, steadily increasing from the previous years, with inflation cap and floors following a similar trend, as indicated in Chipeniuk and Walker (2021).

To conclude, it is important to highlight that the estimating precisely market's inflation expectation is fundamental to governmental institutions, such as central banks, to plan effectively the monetary policy and the effects that it has on the economy more in general.

Appendix A: Python Code

A.1 TIPS Stripping

```python
import pandas as pd
import numpy as np
import calendar
import os
from scipy.optimize import minimize
import matplotlib.pyplot as plt

os.chdir("C:/Users/FreePC/Desktop/Universit /Facolt /Magistrale/Tesi/TIPS/Parte empirica/Database/CapIQ")


# CPI_U inflation data
inflation_df = pd.read_excel("Data thesis.xlsx", sheet_name="CPI - U")
CPI_U = inflation_df.iloc[:, 1]
DateINF = inflation_df.iloc[:, 0]
DateINF = pd.to_datetime(DateINF, format="%d-%m-%Y")
CPI_U = pd.DataFrame({"CPI_U": CPI_U})
CPI_U["Time"] = DateINF
CPI_U = CPI_U[["Time", "CPI_U"]]


# TIPS traded
Price = pd.read_excel("Data thesis.xlsx", sheet_name="TIPS - Daily Prices", header=None)
CUSIP_TIPS = Price.iloc[0, 1:]
Coupon_TIPS = pd.to_numeric(Price.iloc[1, 1:])
Issue_TIPS = pd.to_datetime(Price.iloc[3, 1:])
Maturity_TIPS = pd.to_datetime(Price.iloc[4, 1:])
TIPS = Price.iloc[5:, 1:]
Time = pd.to_datetime(Price.iloc[5:, 0])
TIPS.columns = CUSIP_TIPS


fig, (ax1, ax2, ax3) = plt.subplots(3, 1)
```

```python
fig.set_size_inches(420/25.4, 297/25.4)
ax1.plot(Time, TIPS.iloc[:,0])
ax2.plot(Time, TIPS.iloc[:,1])
ax3.plot(Time, TIPS.iloc[:,3])
ax1.title.set_text('91282CDC2 Market Prices (5 Year Maturity)')
ax2.title.set_text('91282CCM1 Market Prices (10 Year Maturity)')
ax3.title.set_text('912810SV1 Market Prices (30 Year Maturity)')
fig.autofmt_xdate()


# Indexation Coefficient
A = CPI_U["CPI_U"].values
IB = np.zeros(len(TIPS.columns))
IE2 = np.zeros(len(Time))
IE3 = np.zeros(len(Time))
IR = np.zeros(len(Time))
CR = np.zeros((len(Time), len(TIPS.columns)))
yy = np.zeros(len(TIPS.columns))
mm = np.zeros(len(TIPS.columns))
dd = np.zeros(len(TIPS.columns))
y = np.zeros(len(TIPS.index))
m = np.zeros(len(TIPS.index))
d = np.zeros(len(TIPS.index))
R = np.zeros(len(TIPS.index))

for i in range(len(TIPS.columns)):
    for j in range(len(Time)):
        yy[i] = Issue_TIPS.iloc[i].year
        mm[i] = Issue_TIPS.iloc[i].month
        dd[i] = Issue_TIPS.iloc[i].day

        if mm[i] < 4:
            if mm[i] == 3:
                IB[i] = A[np.where((CPI_U["Time"].dt.month == 9 + mm[i]) & (CPI_U["Time"].dt.year == yy[i] - 1))] + \
                    14 / (calendar.monthrange(Issue_TIPS.iloc[i].year, Issue_TIPS.iloc[i].month))[1] * (
                            A[np.where((CPI_U["Time"].dt.month == mm[i] - 2)
```

```
& (CPI_U["Time"].dt.year == yy[i]))] - \
                        A[np.where((CPI_U["Time"].dt.month == 9 + mm[i]) & (CPI_U["Time"].dt.year == yy[i] - 1))])
        else:
            IB[i] = A[np.where((CPI_U["Time"].dt.month == 9 + mm[i]) & (CPI_U["Time"].dt.year == yy[i] - 1))] + \
                14 / (calendar.monthrange(Issue_TIPS.iloc[i].year, Issue_TIPS.iloc[i].month))[1] * (
                        A[np.where((CPI_U["Time"].dt.month == 10 + mm[i]) & (CPI_U["Time"].dt.year == yy[i] - 1))] - \
                        A[np.where((CPI_U["Time"].dt.month == 9 + mm[i]) & (CPI_U["Time"].dt.year == yy[i] - 1))])



    else:
        IB[i] = A[np.where((CPI_U["Time"].dt.month == mm[i] - 3) & (CPI_U["Time"].dt.year == yy[i]))] + \
                14 / (calendar.monthrange(Issue_TIPS.iloc[i].year, Issue_TIPS.iloc[i].month))[1] * (
                        A[np.where((CPI_U["Time"].dt.month == mm[i] - 2) & (CPI_U["Time"].dt.year == yy[i]))] - \
                        A[np.where((CPI_U["Time"].dt.month == mm[i] - 3) & (CPI_U["Time"].dt.year == yy[i]))])


    y[j] = Time.iloc[j].year
    m[j] = Time.iloc[j].month
    d[j] = Time.iloc[j].day
    R[j] = (d[j] - 1) / (calendar.monthrange(Time.iloc[j].year, Time.iloc[j].month)[1])

    if m[j] < 3:
        IE2[j] = A[np.where((CPI_U["Time"].dt.month == 10 + m[j]) & (CPI_U["Time"].dt.year == y[j] - 1))]
    else:
        IE2[j] = A[np.where((CPI_U["Time"].dt.month == m[j] - 2) & (CPI_U["Time"].dt.year == y[j]))]
```

```
        if m[j] < 4:
            IE3[j] = A[np.where((CPI_U["Time"].dt.month == 9 + m[j]) & (CPI_U["Time"].dt.year == y[j] - 1))]
        else:
            IE3[j] = A[np.where((CPI_U["Time"].dt.month == m[j] - 3) & (CPI_U["Time"].dt.year == y[j]))]

        IR[j] = IE3[j] + R[j] * (IE2[j] - IE3[j])
        CR[j, i] = IR[j] / IB[i]

# Lower Bound IC = 1
CR[CR >= 1] = CR[CR >= 1]
CR[CR < 1] = 1



fig, (ax1, ax2) = plt.subplots(2, 1)
#fig.suptitle('Time Series Graphs of the 3-, 5-, 10-, and 30-Year Real Forward Rates \n(January 3, 2022 - August 31, 2023)')
fig.set_size_inches(420/25.4, 297/25.4)
ax1.plot(DateINF, CPI_U.iloc[:, 1])
ax2.plot(Time, IR)
ax1.title.set_text('Time Series of CPI Values 01/01/1998 to 07/01/2023 (Monthly)')
ax2.title.set_text('Time Series of CPI-U Values, 01/03/2022 to 08/31/2023 (Daily)')


# Coupon Date
CD_TIPS = []
for i in range(len(TIPS.columns)):
    cd = pd.date_range(start=Issue_TIPS.iloc[i], end=Maturity_TIPS.iloc[i], freq='6M')
    cd = cd[cd <= Maturity_TIPS.iloc[i]]
    CD_TIPS.append(cd)

maximum = [len(cd) for cd in CD_TIPS]
```

```

Dist_C_TIPS = np.zeros((len(Time), max(maximum), len(TIPS.columns)))

for i in range(len(TIPS.columns)):
    for z in range(len(Time)):
        for j in range(len(CD_TIPS[i])):
            Dist_C_TIPS[z, j, i] = (CD_TIPS[i][j] - Time.iloc[z]).days / 365
            if Dist_C_TIPS[z, j, i] < 0 or pd.isna(Dist_C_TIPS[z, j, i]):
                Dist_C_TIPS[z, j, i] = 0


# Distance day from Maturity payment
Dist_M_TIPS = np.zeros((len(Time), len(TIPS.columns)))

for i in range(len(TIPS.columns)):
    for z in range(len(Time)):
        Dist_M_TIPS[z, i] = (Maturity_TIPS.iloc[i] - Time.iloc[z]).days / 365




#Function with 4 intervals

x0 = np.random.uniform(low=0.02, high=0.06, size=(4))
TIPS_ifull = TIPS.values
RealFWD = np.zeros((len(Time),4))


for z in range(150):

    def RealStripfun(f):

        # Initialize variables
        PV_C_TIPS = np.zeros((Dist_C_TIPS.shape[1], TIPS.shape[1]))
        PV_M_TIPS = np.zeros((len(Time), TIPS.shape[1]))
        TheoricPrice = np.zeros((len(Time), TIPS.shape[1]))
        Scarto = np.zeros((len(Time), TIPS.shape[1]))

```

```python

        for i in range(len(TIPS.columns)):
            for j in range(PV_C_TIPS.shape[0]):
                PV_C_TIPS[j,i] =  Coupon_TIPS.iloc[i]/2 * (0 < Dist_C_TIPS[z, j, i] <=3) * np.exp(-(Dist_C_TIPS[z, j, i]*f[0])) + \
                    Coupon_TIPS.iloc[i]/2 * (3 < Dist_C_TIPS[z, j, i] <= 5) * np.exp(-(f[0]*3 + (Dist_C_TIPS[z, j, i] - 3)*f[1])) + \
                    Coupon_TIPS.iloc[i]/2 * (5 < Dist_C_TIPS[z, j, i] <= 10) * np.exp(-(f[0]*3 + f[1]*2 + (Dist_C_TIPS[z, j, i] - 5)*f[2])) + \
                    Coupon_TIPS.iloc[i]/2 * (Dist_C_TIPS[z, j, i] > 10) * np.exp(-(f[0]*3 + f[1]*2 + f[2]*5 + f[3] * (Dist_C_TIPS[z, j, i] - 10)))


        # Face Value Present Value
        for i in range(len(TIPS.columns)):
            PV_M_TIPS[z, i] = (0 < Dist_M_TIPS[z, i] <= 3) * np.exp(-(f[0] * Dist_M_TIPS[z, i])) + \
                (3 < Dist_M_TIPS[z, i] <= 5) * np.exp(-(f[0]*3 + f[1] * (Dist_M_TIPS[z, i] - 3))) + \
                (5 < Dist_M_TIPS[z, i] <= 10) * np.exp(-(f[0]*3 + f[1]*2 + f[2]* (Dist_M_TIPS[z, i] - 5))) + \
                (Dist_M_TIPS[z, i] > 10) * np.exp(-(f[0]*3 + f[1]*2 + f[2]*5 + f[3]* (Dist_M_TIPS[z, i] - 10)))

        # Theoretical Price
        for i in range(len(TIPS.columns)):
            TheoricPrice[z,i] = CR[z,i] * 100 * (np.sum(PV_C_TIPS[:,i])+PV_M_TIPS[z,i])

        for i in range(len(TIPS.columns)):
            Scarto[z,i] = (TIPS_ifull[z,i] - TheoricPrice[z,i]) ** 2

        return np.sum(Scarto)

    result = minimize(RealStripfun, x0)
    RealFWD[z, :] = result.x
```

```
fig, ((ax1, ax2), (ax3, ax4)) = plt.subplots(2, 2, sharex = True)
#fig.suptitle('Time Series Graphs of the 3-, 5-, 10-, and 30-Year Real
   Forward Rates \n(January 3, 2022 - August 31, 2023)')
fig.set_size_inches(420/25.4, 297/25.4)
ax1.plot(Time, RealFWD[:,0])
ax2.plot(Time, RealFWD[:,1])
ax3.plot(Time, RealFWD[:,2])
ax4.plot(Time, RealFWD[:,3])
ax1.title.set_text('Real Spot Rate, 3Year Forwards')
ax2.title.set_text('Real Forward Rate 3-5Year Forwards')
ax3.title.set_text('Real Forward Rate 5-10Year Forwards')
ax4.title.set_text('Real Forward Rate 10-30Year Forwards')
fig.autofmt_xdate()

for ax in fig.get_axes():
    ax.label_outer()
    ax.axis['left'].set_visible(True)
```

A.2 Treasuries Stripping

```

import pandas as pd
import numpy as np
import os
from scipy.optimize import minimize
import matplotlib.pyplot as plt

os.chdir("C:/Users/FreePC/Desktop/Universit /Facolt /Magistrale/Tesi/TIPS/Parte empirica/Database/CapIQ")


# Treasuries traded
Price = pd.read_excel("Data Thesis.xlsx", sheet_name="Notes and Bonds - Daily Prices", header=None)
CUSIP_Treasuries = Price.iloc[0, 1:]
Coupon_Treasuries = pd.to_numeric(Price.iloc[1, 1:])
Issue_Treasuries = pd.to_datetime(Price.iloc[3, 1:])
Maturity_Treasuries = pd.to_datetime(Price.iloc[4, 1:])
Treasuries = Price.iloc[5:, 1:]
Time = pd.to_datetime(Price.iloc[5:, 0])
Treasuries.columns = CUSIP_Treasuries

fig, (ax1, ax2, ax3) = plt.subplots(3, 1)
fig.set_size_inches(420/25.4, 297/25.4)
ax1.plot(Time, Treasuries.iloc[:,1])
ax2.plot(Time, Treasuries.iloc[:,6])
ax3.plot(Time, Treasuries.iloc[:,5])
ax1.title.set_text('91282CDQ1 Market Prices (5 Year Maturity)')
ax2.title.set_text('91282CDJ7 Market Prices (10 Year Maturity)')
ax3.title.set_text('912810TB4 Market Prices (30 Year Maturity)')
fig.autofmt_xdate()


# Coupon Date
CD_Treasuries = []
for i in range(len(Treasuries.columns)):
```

```
    cd = pd.date_range(start=Issue_Treasuries.iloc[i], end=
   Maturity_Treasuries.iloc[i], freq='6M')
    cd = cd[cd <= Maturity_Treasuries.iloc[i]]
    CD_Treasuries.append(cd)

maximum = [len(cd) for cd in CD_Treasuries]

Dist_C_Treasuries = np.zeros((len(Time), max(maximum), len(Treasuries.columns
   )))

for i in range(len(Treasuries.columns)):
    for z in range(len(Time)):
        for j in range(len(CD_Treasuries[i])):
            Dist_C_Treasuries[z, j, i] = (CD_Treasuries[i][j] - Time.iloc[z])
   .days / 365
            if Dist_C_Treasuries[z, j, i] < 0 or pd.isna(Dist_C_Treasuries[z,
    j, i]):
                Dist_C_Treasuries[z, j, i] = 0

# Distance day from Maturity payment
Dist_M_Treasuries = np.zeros((len(Time), len(Treasuries.columns)))

for i in range(len(Treasuries.columns)):
    for z in range(len(Time)):
        Dist_M_Treasuries[z, i] = (Maturity_Treasuries.iloc[i] - Time.iloc[z
   ]).days / 365


#4 rates

x0 = np.random.uniform(low=0.03, high=0.05, size=(4))
NomFWD = np.zeros((len(Time), 4))
Treasuries_ifull = Treasuries.values

for z in range(len(Treasuries)):

    def NomStripfun(f):
        # Initialize variables
```

```
    PV_C_Treasuries = np.zeros((Dist_C_Treasuries.shape[1], Treasuries.shape[1]))
    PV_M_Treasuries = np.zeros((len(Time), Treasuries.shape[1]))
    TheoricPrice = np.zeros((len(Time), Treasuries.shape[1]))
    Scarto = np.zeros((len(Time), Treasuries.shape[1]))

    for i in range(len(Treasuries.columns)):
        for j in range(PV_C_Treasuries.shape[0]):
            PV_C_Treasuries[j,i] =  100 * Coupon_Treasuries.iloc[i]/2 * (0 < Dist_C_Treasuries[z, j, i] <=3) * np.exp(-(Dist_C_Treasuries[z, j, i]*f[0])) + \
                100 * Coupon_Treasuries.iloc[i]/2 * (3 < Dist_C_Treasuries[z, j, i] <= 5) * np.exp(-(f[0]*3 + (Dist_C_Treasuries[z, j, i] - 3)*f[1])) + \
                100 * Coupon_Treasuries.iloc[i]/2 * (5 < Dist_C_Treasuries[z, j, i] <= 10) * np.exp(-(f[0]*3 + f[1]*2 + (Dist_C_Treasuries[z, j, i] - 5)*f[2])) + \
                100 * Coupon_Treasuries.iloc[i]/2 * (10 < Dist_C_Treasuries[z, j, i]) * np.exp(-(f[0]*3 + f[1]*2 + f[2]*5 + f[3] * (Dist_C_Treasuries[z, j, i] - 10)))


    # Face Value Present Value
    for i in range(len(Treasuries.columns)):
        PV_M_Treasuries[z, i] = 100 * (0 < Dist_M_Treasuries[z, i] <= 3) * np.exp(-(f[0] * Dist_M_Treasuries[z, i])) + \
            100 * (3 < Dist_M_Treasuries[z, i] <= 5) * np.exp(-(f[0]*3 + f[1] * (Dist_M_Treasuries[z, i] - 3))) + \
            100 * (5 < Dist_M_Treasuries[z, i] <= 10) * np.exp(-(f[0]*3 + f[1]*2 + f[2]* (Dist_M_Treasuries[z, i] - 5))) + \
            100 * (10 < Dist_M_Treasuries[z, i]) * np.exp(-(f[0]*3 + f[1]*2 + f[2]*5 + f[3]* (Dist_M_Treasuries[z, i] - 10)))


    # Theoretical Price
    for i in range(len(Treasuries.columns)):
        TheoricPrice[z,i] = np.sum(PV_C_Treasuries[:,i])+PV_M_Treasuries[z,i]
```

```

        for i in range(len(Treasuries.columns)):
            Scarto[z,i] = (Treasuries_ifull[z,i] - TheoricPrice[z,i]) ** 2

        return np.sum(Scarto)

    result = minimize(NomStripfun, x0)
    NomFWD[z, :] = result.x



fig, ((ax1, ax2), (ax3, ax4)) = plt.subplots(2, 2, sharex = True)
#fig.suptitle('Time Series Graphs of the 3-, 5-, 10-, and 30-Year Nominal Forward Rates \n(January 3, 2022 - August 31, 2023)')
fig.set_size_inches(420/25.4, 297/25.4)
ax1.plot(Time, NomFWD[:,0])
ax2.plot(Time, NomFWD[:,1])
ax3.plot(Time, NomFWD[:,2])
ax4.plot(Time, NomFWD[:,3])
ax1.title.set_text('Nominal Spot Rate, 3Year Forwards')
ax2.title.set_text('Nominal Forward Rate 3-5Year Forwards')
ax3.title.set_text('Nominal Forward Rate 5-10Year Forwards')
ax4.title.set_text('Nominal Forward Rate 10-30Year Forwards')
fig.autofmt_xdate()

for ax in fig.get_axes():
    ax.label_outer()
    ax.axis['left'].set_visible(True)
```

A.3 Calibration

```python

import numpy as np
import pandas as pd
import scipy.stats as stats
from scipy.optimize import curve_fit
import matplotlib.pyplot as plt
import os


os.chdir("C:/Users/FreePC/Desktop/Universit /Facolt /Magistrale/Tesi/TIPS/Parte empirica/Database/CapIQ")

Time = Time.reset_index()
Time.drop(columns = ['index'], inplace = True)

# Real Calibration
RealFWD = np.load('RealFWD_definitive.npy')
MaturityZCB = np.array([3, 5, 10, 30])

ZCBprice = np.zeros((len(Time), 4))

for z in range(len(Time)) :
     ZCBprice[z,0] = np.exp(-RealFWD[z,0] * MaturityZCB[0])
     ZCBprice[z,1] = np.exp(-(RealFWD[z,0] * MaturityZCB[0] + RealFWD[z,1] * (MaturityZCB[1] - MaturityZCB[0])))
     ZCBprice[z,2] = np.exp(-(RealFWD[z,0] * MaturityZCB[0] + RealFWD[z,1] * (MaturityZCB[1] - MaturityZCB[0]) + RealFWD[z,2] * (MaturityZCB[2] - MaturityZCB[1])))
     ZCBprice[z,3] = np.exp(-(RealFWD[z,0] * MaturityZCB[0] + RealFWD[z,1] * (MaturityZCB[1] - MaturityZCB[0]) + RealFWD[z,2] * (MaturityZCB[2] - MaturityZCB[1]) + RealFWD[z,3] * (MaturityZCB[3] - MaturityZCB[2])))


Delta_r = np.diff(RealFWD, axis=0)

# Sigma_r and a_r
```

```
ZCBchange = np.diff(ZCBprice, axis=0) / ZCBprice[:-1, :]
VarZCBchange = np.var(ZCBchange, axis=0)


# Define the function to fit
def Realfunc(x, a, sigma):
    return (sigma**2 * (np.exp(-a * x) - 1)**2 * (1/365)) / (a**2)

# Perform the curve fitting

params, covariance = curve_fit(Realfunc, MaturityZCB, VarZCBchange, p0=np.
    random.uniform(low=0, high=0.05, size=(2)))

# Extract the parameters
aR, SigmaR = params

Delta_r_df = pd.DataFrame(Delta_r)
Delta_r_df['Time'] = Time

filtered_Delta_r = Delta_r_df.groupby(Delta_r_df['Time'].dt.to_period('M')).
    first().reset_index(drop=True)


# Sigma_I

#All data in the dataset
DeltaI = np.diff(CPI_U.CPI_U, axis=0) / CPI_U.iloc[:-1, 1]

#Data after 01/01/2022
filtered_CPI_U = CPI_U.CPI_U[CPI_U["Time"] >= np.datetime64('2022-01-01')]
filtered_DeltaI = np.diff(filtered_CPI_U) / filtered_CPI_U[:-1]

VarI = np.var(DeltaI)
SigmaI = np.sqrt(VarI * 12)



# Nominal Calibration
```

```
NomFWD = np.load('NomFWD.npy')

for z in range(len(Time)) :
    ZCBprice[z,0] = np.exp(-NomFWD[z,0] * MaturityZCB[0])
    ZCBprice[z,1] = np.exp(-(NomFWD[z,0] * MaturityZCB[0] + NomFWD[z,1] * (
   MaturityZCB[1] - MaturityZCB[0])))
    ZCBprice[z,2] = np.exp(-(NomFWD[z,0] * MaturityZCB[0] + NomFWD[z,1] * (
   MaturityZCB[1] - MaturityZCB[0]) + NomFWD[z,2] * (MaturityZCB[2] -
   MaturityZCB[1])))
    ZCBprice[z,3] = np.exp(-(NomFWD[z,0] * MaturityZCB[0] + NomFWD[z,1] * (
   MaturityZCB[1] - MaturityZCB[0]) + NomFWD[z,2] * (MaturityZCB[2] -
   MaturityZCB[1]) + NomFWD[z,3] * (MaturityZCB[3] - MaturityZCB[2])))


Delta_n = np.diff(NomFWD, axis=0)

# Sigma_n and a_n
ZCBchange = np.diff(ZCBprice, axis=0) / ZCBprice[:-1, :]
VarZCBchange = np.var(ZCBchange, axis=0)


# Define the function to fit
def Nomfunc(x, a, sigma):
    return (sigma**2 * (np.exp(-a * x) - 1)**2 * (1/365)) / (a**2)

# Perform the curve fitting

params, covariance = curve_fit(Nomfunc, MaturityZCB, VarZCBchange, p0=np.
   random.uniform(low=0, high=0.05, size=(2)))

# Extract the parameters
aN, SigmaN = params


Delta_n_df = pd.DataFrame(Delta_n)
Delta_n_df['Time'] = Time

filtered_Delta_n = Delta_n_df.groupby(Delta_n_df['Time'].dt.to_period('M')).
```

```
    first().reset_index(drop=True)



# Correlation r-I
Rho_rI = stats.pearsonr(filtered_Delta_r.iloc[:,0], filtered_DeltaI)

# Correlation n-I
Rho_nI = stats.pearsonr(filtered_Delta_n.iloc[:,0], filtered_DeltaI)

# Correlation r-n

Rho_rn = stats.pearsonr(Delta_r[:,0], Delta_n[:,0])




# Normality Plot real
# Create a 2x2 grid of subplots
fig, axes = plt.subplots(nrows=2, ncols=2, figsize=(10, 8))
fig.set_size_inches(420/25.4, 297/25.4)

# Iterate over the subplots and plot your data
for i, ax in enumerate(axes.flat):
    row = i // 2
    col = i % 2

    ax.hist(Delta_r[:, i], bins=30, density=False, alpha=0.6, color='g')
    ax.set_title(f'{MaturityZCB[i]} year Real FWD rate')
    ax.set_xlabel('  f  ')
    ax.set_ylabel('Frequency')
    mu, std = stats.norm.fit(Delta_r[:, i])
    xmin, xmax = ax.get_xlim()
    x = np.linspace(xmin, xmax, 100)
    hist, bins = np.histogram(Delta_r[:,i], bins=30, density=False)
    bin_width = bins[1] - bins[0]
    scaling_factor = len(Delta_r[:,i]) * bin_width
    p = stats.norm.pdf(x, mu, std) * scaling_factor
```

```
    ax.plot(x, p, 'k', linewidth=2)



# Normality Plot nominal
# Create a 2x2 grid of subplots
fig, axes = plt.subplots(nrows=2, ncols=2, figsize=(10, 8))
fig.set_size_inches(420/25.4, 297/25.4)

# Iterate over the subplots and plot your data
for i, ax in enumerate(axes.flat):
    row = i // 2
    col = i % 2

    ax.hist(Delta_n[:, i], bins=30, density=False, alpha=0.6, color='g')
    ax.set_title(f'{MaturityZCB[i]} year Nominal FWD rate')
    ax.set_xlabel('  f  ')
    ax.set_ylabel('Frequency')
    mu, std = stats.norm.fit(Delta_n[:, i])
    xmin, xmax = ax.get_xlim()
    x = np.linspace(xmin, xmax, 100)
    hist, bins = np.histogram(Delta_n[:,i], bins=30, density=False)
    bin_width = bins[1] - bins[0]
    scaling_factor = len(Delta_n[:,i]) * bin_width
    p = stats.norm.pdf(x, mu, std) * scaling_factor
    ax.plot(x, p, 'k', linewidth=2)


# Normality Plot inflation index
plt.subplot()
fig = plt.figure()
fig.set_size_inches(420/25.4, 297/25.4)
plt.hist(DeltaI, bins=30, density=False, alpha=0.6, color='g')
#plt.title('Inflation Rate of Change')
plt.xlabel('  I  /I')
plt.ylabel('Frequency')
mu, std = stats.norm.fit(DeltaI)
xmin, xmax = plt.xlim()
```

```python
x = np.linspace(xmin, xmax, 100)
hist, bins = np.histogram(DeltaI, bins=30, density=False)
bin_width = bins[1] - bins[0]
scaling_factor = len(DeltaI) * bin_width
p = stats.norm.pdf(x, mu, std) * scaling_factor
plt.plot(x, p, 'k', linewidth=2)
plt.show()

Volatility_HW_r = np.zeros(4)
Volatility_HW_n = np.zeros(4)

def HWfunc(x, a, sigma):
    return (sigma * (np.exp(-a * x)))

for i in range(len(MaturityZCB)):
    Volatility_HW_r[i] = HWfunc(MaturityZCB[i], aR, SigmaR)


for i in range(len(MaturityZCB)):
    Volatility_HW_n[i] = HWfunc(MaturityZCB[i], aN, SigmaN)


fig, (ax1, ax2) = plt.subplots(2, 1)
fig.set_size_inches(420/25.4, 297/25.4)
ax1.plot(MaturityZCB, Volatility_HW_r)
ax2.plot(MaturityZCB, Volatility_HW_n)
ax1.title.set_text('Real Hull-White Volatility Function')
ax2.title.set_text('Nominal Hull-White Volatility Function')
```

A.4 Inflation Graph

```python

import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d.art3d import Poly3DCollection
import numpy as np
import matplotlib.dates as mdates


# Real Calibration
RealFWD = np.load('RealFWD_definitive.npy')
Maturity = np.array([3, 5, 10, 30])

RealFWD = np.load('RealFWD_definitive.npy')
NomFWD = np.load('NomFWD.npy')


Time = pd.to_datetime(Price.iloc[5:, 0])
Inflation = NomFWD[0:120] - RealFWD[0:120]
Time = Time[0:120]



# Convert datetime values to numeric values for the y-axis
numeric_time = mdates.date2num(Time)

# Create a 3D plot
fig = plt.figure()
fig.set_size_inches(420/25.4, 297/25.4)
ax = fig.add_subplot(111, projection='3d')

# Plot the series with fixed y-coordinate after converting Time to numeric values
ax.plot(numeric_time, np.full_like(Inflation[:, 0], Maturity[0]), zs=Inflation[:, 0], label='Series 1', zdir='z', color = 'red')
ax.plot(numeric_time, np.full_like(Inflation[:, 1], Maturity[1]), zs=Inflation[:, 1], label='Series 2', zdir='z', color = 'red')
ax.plot(numeric_time, np.full_like(Inflation[:, 2], Maturity[2]), zs=
```

```
    Inflation[:, 2], label='Series 3', zdir='z', color = 'red')
ax.plot(numeric_time, np.full_like(Inflation[:, 3], Maturity[3]), zs=
    Inflation[:, 3], label='Series 4', zdir='z', color = 'red')

# Create polygons between the lines to fill the space
polygon1_points = []
polygon2_points = []
polygon3_points = []
for i in range(len(numeric_time)):
    x = numeric_time[i]
    y_bottom1 = Maturity[0]
    y_top1 = Maturity[1]
    z_bottom1 = Inflation[i, 0]
    z_top1 = Inflation[i, 1]
    polygon1_points.append([(x, y_bottom1, z_bottom1), (x, y_top1, z_top1)])

    y_bottom2 = Maturity[1]
    y_top2 = Maturity[2]
    z_bottom2 = Inflation[i, 1]
    z_top2 = Inflation[i, 2]
    polygon2_points.append([(x, y_bottom2, z_bottom2), (x, y_top2, z_top2)])

    y_bottom3 = Maturity[2]
    y_top3 = Maturity[3]
    z_bottom3 = Inflation[i, 2]
    z_top3 = Inflation[i, 3]
    polygon3_points.append([(x, y_bottom3, z_bottom3), (x, y_top3, z_top3)])

# Create Poly3DCollections to fill the space between lines
poly3d1 = Poly3DCollection(polygon1_points, alpha=0.2, color='blue')
poly3d2 = Poly3DCollection(polygon2_points, alpha=0.2, color='blue')
poly3d3 = Poly3DCollection(polygon3_points, alpha=0.2, color='blue')
ax.add_collection3d(poly3d1)
ax.add_collection3d(poly3d2)
ax.add_collection3d(poly3d3)

# Set labels for axes
ax.set_xlabel('Time')
```

```
ax.set_ylabel('Time to Maturity')
ax.set_zlabel('Spread')
plt.title('Nominal vs. Real Forward Spreads \n(January 3, 2022 - June 28, 2022)')

ax.xaxis.set_major_locator(mdates.DayLocator(interval=45))
ax.xaxis.set_major_formatter(mdates.DateFormatter('%Y-%m-%d'))

# Fix the y-axis position
ax.set_yticks(Maturity)


# Show the 3D plot
plt.show()


#4 subplots

fig, ((ax1, ax2), (ax3, ax4)) = plt.subplots(2, 2, sharex = True)
#fig.suptitle('Time Series Graphs of the 3-, 5-, 10-, and 30-Year Real Forward Rates \n(January 3, 2022 - August 31, 2023)')
fig.set_size_inches(420/25.4, 297/25.4)
ax1.plot(Time, Inflation[:,0])
ax2.plot(Time, Inflation[:,1])
ax3.plot(Time, Inflation[:,2])
ax4.plot(Time, Inflation[:,3])
ax1.title.set_text('Spread, 0-3Year Forwards')
ax2.title.set_text('Spread 3-5Year Forwards')
ax3.title.set_text('Spread 5-10Year Forwards')
ax4.title.set_text('Spread 10-30Year Forwards')
fig.autofmt_xdate()

for ax in fig.get_axes():
    ax.label_outer()
    ax.axis['left'].set_visible(True)
```

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