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New black hole solutions in $\mathcal{N} = 2$ U(1) gauged supergravity

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"Humanity's deepest desire for knowledge is justification enough for our continuing quest. And our goal is nothing less than a complete description of the universe we live in."

- A Brief History of Time, Stephen Hawking

Abstract

The study of black hole physics is fundamental in the understanding of strong gravity effects and, in general, serves as a crucial test for a quantum theory of gravity. In this context, string theory provides an elegant framework for conducting explicit calculations, achieving striking results such as the counting of black hole microstates that reproduce the Bekenstein-Hawking entropy formula. In order to find solutions in string theory, one possible approach is to study its low energy limit: supergravity. A well-known aspect of supergravity black holes is the “attractor mechanism”, namely the fact that the area of the horizon of extremal solutions - and consequently, the entropy of black holes - does not depend on the asymptotic values of the scalars but only on the charges. Over the past decades, different configurations have been studied, starting from the simplest case of ungauged theories. Progress was then made by studying black hole solutions with an Anti-de Sitter (AdS) vacuum, which have important implications for gauge/gravity correspondence. A significant class of solutions yet to be explored includes gauged solutions where the cosmological constant has not been fixed to be negative. A better understanding of these solutions may have numerous consequences, including a deeper comprehension of the attractor mechanism itself.

In this thesis, we analyse $D = 4$, $\mathcal{N} = 2$ gauged supergravity theories that include both vector multiplets and hypermultiplets, without fixing a priori a gauging choice that leads to AdS vacua. In this context, a first-order description for the most general black hole solution in terms of the gradient flow of a real superpotential was obtained, generalizing the previous results obtained for AdS black holes. Moreover, the explicit construction of some simplified models with one vector multiplet and one hypermultiplet was carried out, underlying their limitations and suggesting possible extensions to be considered in future works.

Contents

Introduction

Black holes are a fundamental prediction of the theory of General Relativity formulated by Albert Einstein in 1915. After over a century of intense theoretical work, the first direct detection of gravitational waves from the merger of two black holes by the LIGO experiment [1], and the images of accretion disks from the Event Horizon Telescope [2], provided striking experimental evidence for the existence of these fascinating objects, displaying a stunning agreement with predictions from General Relativity. Alongside all the experimental verifications the theory has gathered over the years, from the first groundbreaking result on the precession of Mercury perihelion to the precise estimate of the energy emission via gravitational waves for binary pulsars (see [3] for a review), these achievements reinforce the theory's foundational role in our understanding of gravity.
Despite its undeniable success, it is nowadays universally recognized that General Relativity cannot represent the final description of the gravitational interaction. In particular, as fundamental as they proved to be, black holes carry the seeds of the theory's own failure. From singularities sitting behind the event horizon to inconsistencies in their evaporation process arising from the contact of General Relativity with Quantum Field Theories, black holes testify the need to go beyond General Relativity and serve as the quintessential theoretical laboratory for a *quantum theory of gravity*.

The first cracks in the General Relativity description emerge already at the semiclassical level: one of the most profound discoveries is that black holes are not entirely black; instead, they emit particles with a thermal spectrum [4], a phenomenon known as Hawking radiation. This thermodynamic analogy could be further formalized, getting to the "black holes laws of thermodynamics" and leading to the identification of an entropy, S, proportional to the area A of their event horizon (measured in Planck units), as expressed by the renowned Bekenstein–Hawking formula:

$$S = \frac{k_B}{l_P^2}\frac{A}{4}. \tag{1}$$

In analogy with known thermodynamic systems, this entropy is expected to have a statistical interpretation in terms of microscopic configurations. Just as the properties of gases could be derived from the collective behaviour of individual molecules, we would like to identify a set of *microstate geometries* that could let black holes macroscopic properties emerge. However, due to a series of *no-hair theorems*, such a task is beyond the reach of General Relativity. In particular, the classical theory associates a unique geometry with a black hole for given properties like mass and charges, leaving no room for the vast ensemble of microstates that the Bekenstein–Hawking entropy suggests. This mismatch clearly exposes the inadequacy of General Relativity in possessing the necessary degrees of freedom to be considered the ultimate theory for describing our Universe already at the classical level, pointing to the need for a more fundamental, UV-complete theory that could resolve the contradictions and, eventually, reconcile gravity with the quantum world.

Over the last decades, *string theory* has emerged as the most promising candidate for a quantum theory of gravity. In string theory, the fundamental entities are not point-like particles, but extended objects that exist in a ten-dimensional spacetime. Additionally, it is the only theoretical framework that intrinsically predicts the existence of the gravitational interaction, and therefore, it must address the unresolved issues inherited from General Relativity to ensure its consistency. Notably, starting from the groundbreaking work by Strominger and Vafa [5], there are compelling examples for which string theory is able to provide a microscopic explanation of the Bekenstein–Hawking entropy, shedding light on the nature of microstates accounting for black hole entropy. However, as it often happens in theoretical physics, these successes frequently rely on strong simplifying assumptions, rendering the configurations studied somewhat idealized and making it imperative to intensify searches for more complex and yet realistic models.

In this grand quest, a pivotal role is played by supergravity, the low-energy effective field theory that emerges from string theory. Supergravity allows us to find explicit black hole solutions within the four-dimensional spacetime we observe, effectively bridging the high-energy, ten-dimensional world of string theory with our familiar four-dimensional Universe. In this thesis, we will focus specifically on four-dimensional $\mathcal{N} = 2$ supergravity. While being distant from any potential phenomenological application, our choice is inspired by the fact that $\mathcal{N} = 2$ supergravity strikes a balance between the extremely rigid maximally supersymmetric $\mathcal{N} = 8$ theory and the poorly governable $\mathcal{N} = 1$ models, thus allowing for results that could shed some light also onto more realistic situations, particularly as we strive to comprehend scenarios in which supersymmetry is broken.

Being a direct extension of General Relativity, supergravity naturally encompasses all its solutions, making it a valuable framework for the study of black hole physics. However, supergravity introduces additional complexity due to new particles arising from supersymmetry. Notably, it includes the gravitino, the spin-3/2 supersymmetric partner of the graviton, as well as a multitude of scalars. The first results obtained in these theories heavily relied on the inputs from supersymmetry, which allowed for greater control over the equations governing the solutions, reducing them from second-order to first-order differential equations. Through a careful analysis, however, it was eventually realized that the possibility of having a first-order description does not depend on the supersymmetry content of the theory but rather on an intrinsic property of black holes themselves: *extremality*.
This realization led to several consequences, including the possibility of studying more intricate black hole configurations, such as *multi-centre solutions*. These solutions are highly non-trivial due to the intrinsic non-linear nature of General Relativity and they represent the fundamental building blocks for the construction of black holes microstate geometries.
Another intriguing aspect of extremal black holes is their behaviour at the event horizon. Regardless of the influence of scalar fields, the characteristics of the horizon are determined solely by the black hole's electric and magnetic charges. Specifically, scalar fields flow to fixed values dictated by the

ratio of these charges, making the horizon an "attractor" point. This feature, firstly explored in [6], is usually referred to as "*attractor mechanism*" and has been crucial for understanding how black holes entropy in supergravity depends only on the charges, as it should to explain its microscopic origin.

Building upon these insights, after a thorough review of black hole physics within supergravity theories, this work aims to extend the existing theoretical framework to encompass a broader class of black hole configurations.
Specifically, we will analyse black holes solutions in four-dimensional $\mathcal{N} = 2$ supergravity with Abelian gaugings. As we will present, these theories can describe a wide variety of physical scenarios, including the presence of a non-vanishing cosmological constant and the possibility of partial supersymmetry breaking. Our goal is precisely to include this richness, extending previous analyses that have been limited to asymptotically flat spacetimes without gaugings or Anti-de Sitter backgrounds.
The work presented in this thesis represent a significant advancement, as we were able to study the supersymmetry equations of the theory coupled to the most general allowed matter content. In particular, we showed that the supersymmetry variations can be expressed as first-order flow equations for the bosonic content of the theory in terms of a real function, W, which is directly related to the ADM mass of the black hole. The obtained results successfully include previous analyses and extend their range of applications.
This work represents a crucial and necessary first step towards a deeper understanding of black hole configurations. Future investigations should focus on finding explicit solutions fulfilling the equation we've derived, verifying the equations of motion by rewriting the action in a BPS squared form, and exploring the near-horizon geometry of this new class of black holes.

This thesis will be structured as follows:

- in chapter 1, we will provide a broad overview of black hole physics. We begin our discussion with black hole solutions in General Relativity, introducing the concept of extremality and outlining their thermodynamic properties. Next, we will explore how string theory addresses questions that remain unresolved from a classical perspective, examining the work of Strominger and Vafa on microstate counting, the explicit construction of black hole microstates and the Fuzzball programme, and the original interpretative framework provided by the AdS/CFT holographic correspondence. Finally, we will examine how black hole solutions are realized within Supergravity theories, introducing the attractor mechanism.
- Chapter 2 is dedicated to a comprehensive introduction to four-dimensional $\mathcal{N} = 2$ gauged Supergravity. After an overview of all the basic ingredients of the theory, we will describe how electromagnetic duality and R-symmetry constrain the scalar sector of the theory. We will therefore proceed to define both Special and Quaternionic Kähler manifolds, summarizing their key features and properties. Next, we will review the general *gauging procedure*, the role of symplectic invariance, and introduce holomorphic and triholomorphic *prepotentials* as essential components of a gauged theory. With these foundations in place, we will present the bosonic Lagrangian of the theory and the supersymmetry transformation rules for the fermionic fields. Finally, we will properly define the concept of a *vacuum* and introduce the topic of spontaneous

supersymmetry breaking.

- In chapter 3, we will delve into the study of black hole physics within $D = 4$, $\mathcal{N} = 2$ supergravity theories. We will start our discussion from single-centre solutions in asymptotically flat spacetimes, discussing the discovery of the attractor mechanism, the first-order description for BPS solutions and the subsequent extension of the formalism to non-supersymmetric cases. Then, we will explore the field of multi-centre black holes in Minkowski spacetime and their string theory origin, focusing on both BPS and non-BPS states. To conclude, we will examine BPS black holes in Anti-de Sitter spacetimes within gauged supergravity theories and their related attractor mechanism.

- In chapter 4, we will develop a comprehensive framework for describing black hole solutions in $\mathcal{N} = 2$ supergravity with Abelian gaugings, focusing on cases where the theory is coupled to both vector multiplets and hypermultiplets, a scenario not previously explored in existing solutions. After specializing the general theory to the specific gauging choices we made, we will derive a set of first-order differential equations governing the solutions directly from the analysis of fermionic supersymmetry variations. Moreover, we will verify that the whole solution could be described in terms of a real *superpotential*. Finally, we will attempt to construct explicit realizations for some simplified models, discussing the general procedure for finding the vacua of the theory.

Chapter 1

Black Holes, String Theory and Supergravity

In this chapter, we will delve into black hole physics, starting with a review of classical solutions and their thermodynamics [7–10]. Next, we will examine how black holes are addressed in *string theory*, the most promising quantum theory of gravity to date [11,12]. In particular, we will summarize how string theory provides a microscopic interpretation of black hole entropy, the construction of black hole geometries from stringy components and the Fuzzball programme [13–17], and the thoughtful insights provided by the AdS/CFT correspondence [18–20]. Finally, we will review some basic aspects of black holes in *supergravity*, the low-energy effective realization of string theory. Specifically, we will understand how supersymmetry allows us to describe black holes solutions via first-order *flow equations* and review the *attractor mechanism* [9,10,20].

1.1 Black Holes: Classical Solutions and Thermodynamic Properties

1.1.1 Charged Black Holes and Extremality Condition

One of the simplest and yet most instructive examples of black hole solutions we can consider arises in Einstein–Maxwell theory, which describes Einstein gravity coupled to an Abelian gauge field in 4 dimensions and whose Lagrangian density is given by

$$e^{-1}\mathscr{L} = R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1.1}$$

where e denotes the determinant of the metric.
In this framework, we can consider the Reissner–Nördstrom solution, whose metric describes a static, spherically symmetric black hole of mass M, electric charge Q and magnetic charge P[1]:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{P^2+Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{P^2+Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2, \tag{1.2}$$

where $d\Omega^2$ represents the metric on the unit 2-sphere.
For the specific choice of the parameters $Q = P = 0$, this metric reduces to the one for the well-known Schwarzschild solution.

[1]Here and in the rest of the chapter we will employ natural units, i.e. $c = \hbar = k_B = G_N = 1$.

By looking directly at the metric (1.2), the Reissner–Nördstrom metric exhibits a singularity for $r = 0$. Additionally, depending on the values of the charges, there might be additional singular points, identified by setting $g^{rr} = 0$:

$$r_{\pm} = M \pm \sqrt{M^2 - (P^2 + Q^2)}. \tag{1.3}$$

As it becomes clear by computing the curvature invariant in terms of the Ricci tensor

$$R_{\mu\nu}R^{\mu\nu} = 4\frac{\left(Q^2 + P^2\right)^2}{r^8}, \tag{1.4}$$

only $r = 0$ is a true curvature singularity, while $r_{\pm}$ are coordinate singularities and represent the horizons of this specific gravitational configuration.
If $M^2 < P^2+Q^2$, the two horizons disappear and the black hole is characterized by a *naked singularity*. According to the cosmic censorship conjecture, this situation is believed to be non-physical, therefore, for this specific class of solutions, the conjecture directly implies the bound $M^2 \geq P^2 + Q^2$.
When the bound is exactly saturated, i.e. for $M^2 = P^2 + Q^2$, the two horizons coincide and the black hole is said to be *extremal*.
To better understand the differences between these two classes of solutions, it is interesting to analyse the near-horizon geometry as we vary the charges. Specifically, we can focus on the $g^{rr} = -g_{tt}$ components, which approach

$$1 - \frac{2M}{r} + \frac{P^2 + Q^2}{r^2} = \frac{(r - r_+)(r - r_-)}{r^2} \quad \xrightarrow{r \to r_+} \quad \frac{r_+ - r_-}{r_+^2}\rho, \tag{1.5}$$

where we defined $\rho = r - r_+$ so as to measure the distance from the outer horizon.
In the non-extremal case, the resulting near-horizon geometry can be interpreted as the product of a 2-dimensional Rindler spacetime with a 2-sphere. This becomes clear by rewriting the metric as:

$$ds^2_{\text{Non-Extremal}} \quad \xrightarrow{r \to r_+} \quad e^{2\alpha\xi}(-d\tau^2 + d\xi^2) + r_+^2 d\Omega^2, \tag{1.6}$$

where we introduced a new set of coordinates $\rho = e^{2\alpha\xi}$, $t = \frac{\tau}{4\alpha^2}$ and the constant $\alpha = \frac{\sqrt{r_+ - r_-}}{2r_+}$.
On the other hand, in the extremal case, since the two horizons coincide, the near-horizon behaviour is described by a quadratic function of ρ; in particular, introducing $z = -\frac{M^2}{\rho}$, the metric becomes

$$ds^2_{\text{Extremal}} \quad \xrightarrow{r \to r_+} \quad M^2\left(\frac{-dt^2 + dz^2}{z^2}\right) + M^2 d\Omega^2. \tag{1.7}$$

Therefore, extremal black holes may be regarded as solitons of classical general relativity, interpolating between two vacua of the theory: the flat Minkowski space-time, which is recovered at spatial infinity $r \to \infty$, and the *Bertotti–Robinson metric*, describing the conformally flat geometry $AdS_2 \times S^2$ in the near horizon limit.
Another insightful perspective on the extremal case can be obtained upon rewriting the metric (1.2) in an isotropic form:

$$ds^2 = -H^{-2}(\vec{x})dt^2 + H^2(\vec{x})d\vec{x}_3^2. \tag{1.8}$$

By imposing the equation of motions, $H(\vec{x})$ turns out to be a generic harmonic function which, in principle, may have multiple centres:

$$H = 1 + \sum_i \frac{M_i}{|\vec{x} - \vec{x}_i|}, \qquad M_i^2 = P_i^2 + Q_i^2. \tag{1.9}$$

Given the intrinsic non-linear nature of General Relativity, this is a remarkable result and it relates to the fact that extremality can be interpreted as a perfect balance between gravitational attraction and electromagnetic repulsion.
As will become clear later on, extremal black holes represent a particularly interesting class of solutions, especially as we move from classical general relativity toward a quantum theory of gravity.

1.1.2 Black Holes and Thermodynamics

Black holes are thermal systems that obey the *laws of black holes thermodynamics*.
The interplay between two seemingly distinct branches of physics originates from Hawking's "Area law" [21]. This result of classical general relativity states that the black hole horizon area cannot decrease in any process. Moreover, when two black holes merge, the area of the resulting black hole cannot be smaller than the sum of initial areas.
The compelling analogy between the area of the horizon and the entropy of a thermal system inspired Bekenstein [22] to first attempt to unify black hole physics with thermodynamics. In his seminal work, he concluded that black holes must indeed be characterized by some form of entropy so as not to violate the second law of thermodynamic, and that this entropy should be related to the horizon area via some universal coefficient.
A fundamental step forward towards the complete understanding of the matter was then taken through the groundbreaking work of Hawking and Unruh [4, 23], showing that black holes actually emit radiation (Hawking radiation) with a perfect black body spectrum at a temperature $T = \frac{\kappa}{2\pi}$, where κ is the *surface gravity*. It's worth noticing that for extremal black hole configurations, the Hawking temperature is actually vanishing, which implies their thermodynamic stability.
Dealing with a thermodynamic system for which we defined the energy (given by the mass of the black hole) and a temperature, it is natural to define an entropy, such that

$$\frac{dS_{BH}}{dM} = \frac{1}{T}. \tag{1.10}$$

This leads to the formulation of the *Bekenstein–Hawking entropy*:

$$S_{BH} = \frac{A}{4}, \tag{1.11}$$

where A is the area of the black hole horizon expressed in Plank units. This relation not only ties thermodynamics to the geometric properties of a black hole, but also highlights a peculiar aspect of General Relativity. In ordinary Quantum Field Theories, most physical quantities (like energy or entropy) typically scale with the volume of the system. The fact that the black hole entropy depends on the horizon area rather than the volume thus signals a fundamental distinction in the nature of the gravitational interaction, suggesting that its description as a quantum theory should be profoundly different from other known physical theories.
Moreover, this observation raises a more fundamental question: what are the internal, microscopic degrees of freedom that the Bekenstein–Hawking entropy is counting?
In most physical systems, thermodynamic entropy has a statistical interpretation, keeping track of the microscopic degrees of freedom via Boltzmann's relation

$$S = \log \Omega(M, Q, P), \tag{1.12}$$

where Ω is the total number of microstates of the system for a given energy and fixed charges.
In General Relativity, as a consequence of the *no-hair theorem*, the black hole geometry is completely specified by the charges measured at infinity, thus $\Omega = 1$. On the other hand, for instance, based on its horizon area, the black hole at the centre of the Milky Way (Sgr A*) should have about $\Omega \approx e^{10^{90}}$ microstates, leading to one of the largest discrepancies in modern theoretical physics.
Shedding light on formula (1.12) at a more fundamental level and resolving this discrepancy are essential duties any quantum theory of gravity is called to fulfil.

1.2 A String Theory perspective on Black Holes

String theory is a theoretical framework for quantum gravity, where the fundamental constituents of the theory are not point particles, but rather one-dimensional vibrating strings whose vibration frequencies give rise to various particles. In addition to strings, the theory includes extended, non-perturbative objects known as *D-branes*.
Since the study of black holes involves strong coupling, the need to go beyond simple string perturbation theory renders D-branes fundamental ingredients for the study of black holes. In particular, black hole solutions are interpreted as bound states of D-branes in a space-time compactified to four or five dimensions.

1.2.1 Microstates counting

In 1996, Strominger and Vafa [5] provided the first striking theoretical evidence that the Bekenstein–Hawking entropy formula could be matched with a microscopic counting of degrees of freedom. The core idea behind their result is to compare the entropy related to the horizon area, as determined from the low-energy effective theory (i.e. *supergravity*), with the counting of stringy-like states degeneracy. To understand how this matching works, it is important to recall that, as it will be further described later on, supergravity offers an effective description of superstring theory that is valid at the lowest order in the string loop expansion and when the space-time curvature is much smaller than the typical string scale l_s. Therefore, as long as charged black holes are concerned, the supergravity description is reliable when the horizon radius is much larger than the string scale, corresponding to the limit of large charges. Schematically, by introducing the string coupling g_s and the number of D-branes N [2], this regime is identified by

$$g_s N \gg 1 \qquad \text{and} \qquad g_s \to 0. \tag{1.13}$$

On the string theory side, the microstate counting can be effectively performed in the weakly coupled open string picture. In particular, as long as $g_s \to 0$, D-branes do not source gravitons (i.e. closed strings), therefore we can consider open strings on D-branes in flat spacetime[3]. Therefore, a reliable calculation in string theory terms can be carried out in the regime

$$g_s N \ll 1 \qquad \text{and} \qquad g_s \to 0. \tag{1.14}$$

To actually perform the matching, one should extrapolate this result and compare it to the one obtained from supergravity within the same range of validity. The problem is that, in general, as the

[2]The supergravity charges are related to the number of D-branes, which act like electric/magnetic sources.
[3]Closed strings perturbation theory goes with powers of g_s while open strings expansion parameter gets enhanced to $g_s N$.

coupling increases, non-perturbative effects are expected to arise, potentially modifying the counting. This problem could be avoided by examining *BPS states* of string theory, i.e. solutions that preserve supersymmetry, at least partially. In this particular scenario, supersymmetry protects the degeneracy counting, extending the validity of the result to arbitrary values of the coupling.
In the specific case studied by Strominger and Vafa, five-dimensional extremal black hole solutions carrying an electric charge Q_F and an axion charge Q_H were studied. The Bekenstein–Hawking entropy, as determined from the low-energy effective action, is

$$S_{BH} = 2\pi\sqrt{\frac{Q_H Q_F^2}{2}}. \tag{1.15}$$

On the other hand, the leading order result for the logarithm of the bound-state degeneracy for large Q_H and fixed Q_F as obtained in Type IIB string theory compactified on $K3 \times S^1$ reads

$$S_{stat} = 2\pi\sqrt{Q_H\left(\frac{1}{2}Q_F^2 + 1\right)}, \tag{1.16}$$

which leads to $S_{BH} = S_{stat}$ for large charges.
This is actually a highly non-trivial test of string theory, testifying that, in principle, it has the right microscopic degrees of freedom required for a consistent quantum gravity theory.

1.2.2 The Information Paradox and the Fuzzball proposal

As we have discussed, string theory successfully reproduces the Bekenstein–Hawking entropy formula by counting microscopic degrees of freedom. However, for it to claim its primacy as the final theory of quantum gravity, more details are needed.
Specifically, not only do we want to count black holes microstates but also understand what these microstates actually *look* like so as to address another fundamental problem arising from the contact of General Relativity and Quantum Mechanics: the *Information Paradox*.
This paradox arises from the fact that, since Hawking radiation originates just above the horizon, the uniqueness of black holes in GR implies that the radiation is universal, thermal and featureless. Consequently, it is impossible to reconstruct the interior state of a black hole from the final state of the Hawking radiation. Therefore, the evaporation process cannot be described as a unitary transformation of states in a Hilbert space, which is inconsistent with foundational postulates of quantum mechanics.
In 2009, Mathur [24] showed that, under some general assumptions, the information paradox cannot be solved by higher-order corrections to either GR or quantum field theory. Before this result, the key to the resolution of the paradox was thought to reside in small corrections to the Hawking result that, together with the extremely large evaporating time, could incrementally resolve the problem. As a consequence of Mathur's result, to solve the information paradox there should be some "order 1" modification to our current description of black holes. In particular, we should relax at least one of these two hypothesis: locality of the interactions or absence of additional structures at the horizon scale.
The Fuzzball and Microstate Geometry programs are promising attempts within string theory working with the latter hypothesis. In this context, traditional black holes are replaced with *fuzzballs*, a new phase that emerges when matter is compressed to black-hole densities, consisting of branes and

other stringy ingredients; this new phase prevents the formation of a horizon or singularity, which only arise when gravity is described using a theory that has too few degrees of freedom.

Ground states and sufficiently coherent excitations of these new configurations can be described by supergravity and are referred to as *microstate geometries*. They have the huge advantage that detailed computations can be done and one might reasonably hope that simple, semi-classical quantization of the supergravity phase space could provide further details of the quantum fuzzball, similar to the way kinetic theory describes gases. Additionally, if one can find a smooth geometry with no horizon that corresponds to every microstate, then a black hole would be nothing more than a classical effective description of the statistical ensamble of microstate geometries.

The first success of this identification was obtained for the D1-D5 two-charge system [25], for which the smooth supergravity solutions dual to microstates have been classified and shown to precisely account for the entropy of this setup. As interesting as this result is, the two-charge system does not give rise to a true black hole with non-zero horizon area.

The simplest generalization displaying a finite horizon area is given by the three-charge system. In this framework, Bena and Warner [26, 27] exploited supersymmetry to simplify non-linear supergravity equations to linear, first-order differential relations (i.e. BPS equations) which make it possible to construct huge classes of solutions with the same supersymmetries and charges of the three-charge black holes. Actually, the obtained solutions are generally BPS black rings, which are five-dimensional black holes with an event horizon of topology $S^1 \times S^2$. For the purpose of finding three-charge geometries dual to black hole microstates, one is actually interested in their zero-entropy limit [4]; this limit would actually lead to singularities but, via a peculiar geometric transition, one actually gets a huge moduli space of smooth, horizonless "bubbled" geometries. As a result, the actual black hole is replaced with multi-centre configurations and the position of the centers are constrained in terms of the charges through "bubble equations".

Therefore, despite many questions remain open, the fuzzball and microstate geometry approaches have the potential to shed light on the physics of black-holes microstructure.

1.2.3 AdS/CFT correspondence

Both microstate counting and the interpretation of black holes as statistical ensembles of different stringy configurations sit perfectly inside the picture of the *AdS/CFT correspondence* conjecture, one of the greatest achievements of string theory. Broadly speaking, AdS/CFT correspondence is a specific example of gauge/gravity duality, that is an equality between two theories: a quantum field theory in d space-time dimensions and a gravity theory on a $d+1$ dimensional spacetime, which has an asymptotic boundary which is d dimensional.

The basic prescription for the correspondence is that fields on the gravity side (the *bulk*) act as sources for the CFT fields on the boundary; schematically, to each operator O of the CFT we associate a source $h(x^\mu)$ which is considered the boundary value of an *on-shell* bulk field $\hat{h}(x^\mu, x^{d+1})$ (i.e., $\hat{h}$ solves the equations of motions of the gravity theory). With this construction, the foundational statement of AdS/CFT is that:

$$e^{W(h)} = \left\langle e^{\int hO} \right\rangle_{QFT} = e^{S_{Bulk}(\hat{h})}, \tag{1.17}$$

[4]As it is typical in statistical mechanics, a single microstate is expected to have vanishing entropy.

for $W(h)$ the generating functional for connected correlation functions and S_{Bulk} the action for the gravity theory.

The pivotal example of such a correspondence in the context of string theory was brought up by Maldacena in 1997 [28], exploring the duality between $\mathcal{N} = 4$ Super-Yang-Mills theory and the type IIB string background $AdS_5 \times S^5$.

To understand the motivation for such a correspondence, one must first recall that quantizing open strings ending on a D-brane leads to massless excitations corresponding to a supersymmetry vector multiplet. Moreover, a set of N D-branes carries a Yang-Mills theory with gauge group U(N) on its world-volume.

The original Maldacena conjecture stems from the observation that the two theories can be obtained through the same decoupling limit, $\alpha' \to 0$, performed on the world-volume theory and the back-reacted metric curved by the presence of D-branes. Specifically, $\mathcal{N} = 4$ SYM theory can be realized on N parallel D3-branes in Type IIB; in general, the world-volume theory interacts with the bulk fields which live in 10 dimensions, but in the limit $\alpha' \to 0$ the theory decouples from the bulk.

On the other hand, looking at the metric obtained by deformations of the background due to the presence of D3-branes and taking the same limit, one ends up precisely with the product of $AdS_5 \times S^5$.

By a comparison of the parameters of the two theories, one finds that

$$4\pi g_s = \frac{x}{N}, \tag{1.18}$$

$$\frac{R^2}{\alpha'} = \sqrt{x}, \tag{1.19}$$

where we introduced the t'Hooft coupling $x = g_{YM}^2 N$. This reveals that the weak coupling regime of the gravity theory ($g_s \to 0$) corresponds to the strong coupling regime of the CFT and vice-versa, making this correspondence extremely useful and powerful.

In the context of black holes physics, the AdS/CFT connects the entropy of a black hole with the ordinary thermal entropy of a field theory, giving a statistical foundation for black hole entropy and reinforcing the interpretation of black holes as ordinary thermal states in a unitary quantum field theory.

From the practical point of view, the entropy for extremal black holes should be recovered by enumerating supersymmetric states in the dual theory, that basically is equivalent to computing the partition function for a suitable statistical ensemble. As it is common in statistical mechanics, getting an explicit expression for the partition function may result in too hard a task, therefore one usually relies on some supersymmetric indices or consider some limiting case [20] (e.g. the high temperature limit allows to estimate the entropy for a two dimensional CFT via Cardy's formula).

1.3 Supergravity

As already mentioned several times, supergravity theories arise as the low-energy effective actions of superstring theories. Even as an infrared limit, its highly non-linear interactions enable supergravity to capture certain non-perturbative properties of string theory, such as D-branes configurations.

Historically, inspired by the striking success of local gauge invariances in Standard Model physics, supergravity was initially conceived as the *gauge theory of supersymmetry*. In particular, since the supersymmetry parameter is a spinorial quantity, it was soon realized that the theory internal

consistency naturally led to the inclusion of spin-2 particles, i.e. *gravitons*.
As a consequence, general relativity and its solutions are automatically included in supergravity, leading to a natural embedding of black hole configurations. The essential new ingredients are provided by supersymmetry, which requires the presence of additional vector and scalar fields.

1.3.1 Black Holes in Supergravity: an overview

At the two-derivative level, a generic Lagrangian describing the bosonic degrees of freedom for a supergravity theory has the form

$$e^{-1}\mathscr{L} = \frac{R}{2} - \frac{1}{2}g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(\phi)F^\Lambda_{\mu\nu}F^{\Sigma\,\mu\nu} + \frac{1}{4}\mathcal{R}_{\Lambda\Sigma}(\phi)\frac{\varepsilon^{\mu\nu\rho\sigma}}{2\sqrt{-g}}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}, \tag{1.20}$$

where $g_{ij}(\phi)$ is the metric for the scalar σ-model, $\mathcal{I}$ is definite negative and describes non-minimal gauge kinetic couplings, and $\mathcal{R}$ is the generalization of the θ-terms.
Looking for single centre, static, spherically symmetric, charged and asymptotically flat black hole solutions, we can introduce magnetic and electric charges, defined respectively as:

$$\frac{1}{4\pi}\int_{S^2} F^\Lambda = p^\Lambda\ , \qquad \frac{1}{4\pi}\int_{S^2} G_\Lambda = q_\Lambda, \tag{1.21}$$

where we introduced the dual field strengths $G_\Lambda \equiv -\frac{\delta\mathcal{L}}{\delta F^\Lambda}$. These two-forms are related by the usual *electric-magnetic duality*; as will be thoroughly reviewed in the following chapters, in a general supergravity theory, this duality can be extended to a larger group of duality transformations that leave both the Bianchi identities and the equations of motion invariant, playing a crucial role in the construction of the theory itself.
To find explicit solutions for the theory, one can specify suitable ansatze for the metric and the vector fields and obtain the equations of motion, resulting in a set of second-order differential equations. However, if we focus on supersymmetry-preserving states, i.e. BPS solutions, one can actually describe the fields dynamics through *first-order differential equations*.
Specifically, BPS states preserve a fraction, 1/2 or 1/4 or 1/8, of the original supersymmetries. This implies the existence of a suitable projection operator $S^2 = S$ acting on the supersymmetry charge Q_{SUSY}, such that

$$(S\cdot Q_{SUSY})\,|\text{BPS state}\rangle = 0. \tag{1.22}$$

Since the supersymmetry transformation rules of any supersymmetric field theory are linear in the first derivatives of the fields, eq. (1.22) is actually a system of first-order differential equations. In particular, since, as we will present in the following, fermions vanish on the solutions we typically take into account, the supersymmetric variations of bosonic fields are automatically satisfied. On the other hand, the request of vanishing fermionic variations furnishes a set of equations for the bosonic fields content.
While the resulting BPS equations typically do not solve the full set of equations of motion for the theory, they can be easily supplemented with Bianchi Identities or some conditions for the form-fields to restore full equivalence.
Although BPS configurations will be the main focus of our work, it is important to highlight that it is actually possible to obtain a first order-description also for solutions that do not preserve any supersymmetry [29].

1.3.2 Attractor mechanism

As we discussed previously, in the weak-coupling, low-energy limit, we can compute the entropy for a black hole from the Bekenstein–Hawking formula. The area of the horizon can be explicitly extracted from the relevant supergravity solution that, in general, will depend on many scalar fields. As a consequence, in principle, the area may depend on many parameters, including asymptotic values for the moduli. On the other hand, the microscopic entropy of extremal black holes is a function of the conserved charges only, therefore the horizon should loose all the information about the scalar fields.

This feature is explicitly realized through the *attractor mechanism* [6, 30]: for extremal black holes, scalar fields, independently of their value at spatial infinity, flow to a fixed point given in terms of the charges of the solution at the horizon. Therefore, we conclude that the entropy of extremal black holes does not depend on continuous parameters and it is given in terms of quantized charges only.

To get the physical intuition behind this result, we recall that for extremal black holes the horizon is at an infinite proper distance from any observer; in particular, starting from the metric (1.2) adapted to the extremal case and introducing the radius $r_H^2 = M^2 = (P^2 + Q^2)$, one immediately sees that the length L of any radial curve (for fixed t, θ and φ) is indeed log-divergent:

$$L = \int_{r_*}^{r_H} \frac{dr}{1 - \frac{r_H}{r}} = \infty. \tag{1.23}$$

Therefore, as they descend down the AdS_2 infinite throat, scalar fields lose memory of their initial conditions as a direct consequence of the request of having regular solutions. Specifically, their derivative with respect to the radial coordinate should vanish while approaching the horizon, so as to prevent their values from growing indefinitely.

By looking at the equation of motions for the scalars obtained from (1.20), this condition on the derivatives actually implies that, at the horizon, the moduli reach a critical point of the *black hole potential* V_{BH}[5]:

$$\partial_i V_{BH}(\phi^i_{\text{horizon}}, q, p) = 0. \tag{1.24}$$

Since the only parameters appearing in the minimization condition are the black hole charges, the attractor values of the scalar fields are going to be given in terms of the charges

$$\phi^i_{\text{horizon}} = \phi^i_{\text{horizon}}(p, q). \tag{1.25}$$

As a consequence, at the horizon, the black hole charges are the only parameters specifying the solution and all the dependence on the asymptotic value of the moduli fields is lost.

[5]The black hole potential emerges once we integrate out formally t, θ and φ from (1.20) and we describe our system via an effective 1-dimensional action. It is completely specified once the scalar σ-model and the charges of the solutions are fixed.

Chapter 2

$\mathcal{N}=2,\ D=4$ Gauged Supergravity

Besides its string theory origin, supergravity represents an intrinsically rich and fascinating theory, laying its foundations on deep geometrical principles and providing profound insights on black hole physics. Since our main interest in this thesis is the study of black hole solutions in the context of $\mathcal{N}=2$ supergravity in four spacetime dimensions, we provide in this chapter a concise and yet comprehensive introduction to the theory, mostly following the presentation in [31][1]. After an overview of the matter content and its general structure, we will thoroughly review the scalar sector of the theory. In particular, we will describe how generalized electro-magnetic duality and R-symmetry force scalar dynamics to be described via sigma models on Special Kähler and quaternionic-Kähler manifolds [32]. These peculiar geometrical objects will be properly defined and the most important geometric relations will be presented [31–33].

Next, we will describe the *gauging* of the isometries of the scalar manifold, an essential ingredient in extended supergravity theories to obtain a non-vanishing scalar potential. After a general outline of the required steps to gauge an isometry [32], we introduce *momentum maps* or *prepotentials* as essential geometric constructions for the gauging, following our main reference.

The full bosonic Lagrangian of the theory and fermionic supergravity variations will then be presented [31], thus collecting all the relevant definitions that will serve as a starting point for our subsequent work.

Finally, we will properly define the vacua of a supergravity theory and present the supersymmetry breaking mechanism in Minkowski backgrounds, mainly following the presentation in [33].

2.1 Matter Content and Geometrical Structure

As intricate as it may appear at first glance, the theory for $\mathcal{N}=2$ supergravity in four dimensions can be successfully understood in terms of a few geometrical inputs.

To begin our examination, we first introduce the field content of the theory, which includes:

- a *gravitational multiplet*, described by the vielbein 1-form V^a, $(a=0,1,2,3)$, the spin-connection 1-form ω^{ab}, the SU(2) doublet of gravitino 1-forms $\psi^A,\ \psi_A$ (respectively left and right chirality), and the graviphoton 1-form A^0;

[1] We will mainly follow the reference conventions, with some minor changes that will be pointed out throughout the chapter.

- n_V *vector multiplets*, each containing a gauge boson 1-form A^I ($I=1,...,n_V$), a doublet of spinors (gauginos) $\lambda^{iA}, \lambda^{i*}_A$, and a complex scalar field z^i ($i=1,...,n_V$). The scalar fields z^i can be thought as coordinates on a manifold $\mathcal{SM}$ of complex dimension n_V that supersymmetry dictates to be Special Kähler;

- n_H *hypermultiplets*, each carrying a doublet of spinors (hyperinos) ζ^α ($\alpha=1,\ldots,2n_H$, with lower/upper index denoting left/right chirality) and four scalar fields q^u, ($u=1,\ldots,4n_H$), which parametrize a quaternionic-Kähler manifold $\mathcal{QM}$ of real dimension $4n_H$. Moreover, as will be presented later in the discussion, any quaternionic manifold has a holonomy group

$$\mathcal{H}ol(\mathcal{QM}) \subset \mathrm{SU}(2) \otimes \mathrm{Sp}(2n_H, \mathbb{R}),$$

  and the index α of the hyperinos sits in the fundamental representation of $Sp(2n_H, \mathbb{R})$.

Given the rich matter content of the theory, the supersymmetric Lagrangian and the supersymmetry transformation rules are indeed quite involved. However, a closer examination reveals that all the couplings, the mass matrices and the vacuum energy are completely fixed by supersymmetry once three geometrical quantities are specified:

1. a special Kähler manifold $\mathcal{SM}$ describing the vector multiplets σ-model;
2. a quaternionic-Kähler manifold $\mathcal{QM}$ describing the hypermultiplets σ-model;
3. the choice of a *gauge group* $\mathcal{G}$, that in general must be a subgroup of the isometry group of the complete scalar manifold $\mathcal{M}_{scalar} \equiv \mathcal{SM} \otimes \mathcal{QM}$, and its immersion in the symplectic group $Sp(2n_V+2, \mathbb{R})$ of electric-magnetic duality rotations.

In the following we will therefore illustrate the main features and properties of the scalar geometry and the gauging procedure.

2.2 Scalar Geometry

2.2.1 Vector Multiplets

Electromagnetic Duality

The field equations for a general theory of Abelian vector fields in four spacetime dimensions reveal the presence of an underlying electric–magnetic duality.
In the context of extended supergravity theories (namely, for $\mathcal{N}>1$), since multiplets with vectors contain scalar fields as well, this additional duality structure will directly constrain the other matter content of the theory. In particular, it will force the scalar fields sitting in the vector multiplets to parametrize a *Special Kähler* manifold.
Let's consider a 2-derivative Lagrangian containing n_V Abelian vectors, A^Λ (for $\Lambda=1,...,n_V$), appearing through their field strengths $F^\Lambda = \mathrm{d}A^\Lambda$. Additionally, we will consider arbitrary couplings with other scalar fields[2] ϕ^i, so that the Lagrangian takes the general form:

$$e^{-1}\mathscr{L} = \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(\phi) F^\Lambda_{\mu\nu} F^{\Sigma\,\mu\nu} + \frac{1}{4}\mathcal{R}_{\Lambda\Sigma}(\phi) F^\Lambda_{\mu\nu} \tilde{F}^{\Sigma\,\mu\nu} + \frac{1}{2}\mathcal{O}^{\mu\nu}_\Lambda F^\Lambda_{\mu\nu} + e^{-1}\mathscr{L}_{\text{rest}}, \tag{2.1}$$

[2]The discussion follows similar steps also for other bosonic or fermionic fields.

where we introduced two field-dependent, symmetric matrices $\mathcal{I}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$ (the former being negative definite to ensure unitarity), a generic tensor function, $\mathcal{O}_{\Lambda}^{\mu\nu}$, of the other fields containing at most a single derivative, and $\mathcal{L}_{\text{rest}}$ contains all the terms that do not depend on the vector field strengths. By definition, the n_V vector fields satisfy the Bianchi identities:

$$\mathrm{d}F^{\Lambda}=0. \tag{2.2}$$

Additionally, following the usual variational method, their equations of motion can be read from

$$\nabla^{\mu}\frac{\partial\mathscr{L}}{\partial F^{\mu\nu\,\Lambda}}=0. \tag{2.3}$$

Introducing the dual field variables

$$\tilde{G}_{\Lambda\,\mu\nu}=2\frac{\partial\mathscr{L}}{\partial F^{\mu\nu\,\Lambda}}, \tag{2.4}$$

for $\tilde{G}_{\Lambda\,\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}G_{\Lambda}^{\rho\sigma}$, the equations of motion (2.3) can be actually recast in terms of Bianchi identities for the dual fields, i.e.

$$\mathrm{d}G_{\Lambda}=0. \tag{2.5}$$

Naïvely, the system of Bianchi identities and equations of motion seems invariant under generic, constant $\mathrm{GL}(2n_V,\mathbb{R})$ transformations,

$$\mathbb{F}'=S\mathbb{F},\qquad \mathbb{F}=\mathbb{F}\equiv\begin{pmatrix}F^{\Lambda}\\ G_{\Sigma}\end{pmatrix}. \tag{2.6}$$

However, in order to preserve the very same definition (2.4) for G_{Λ}, S is actually constrained to be an element of the subgroup $\mathrm{Sp}(2n_V,\mathbb{R})$, as was first realized in the seminal work by Gaillard and Zumino [34]; a modern, expanded presentation can be also found in [32].
Apart from the technicalities of the derivation, it is important to underline that the invariance of the system $\mathrm{d}\mathbb{F}=0$ *does not* imply an invariance of the Lagrangian. In particular, introducing the complex kinetic matrix

$$\mathcal{N}_{\Lambda\Sigma}=\mathcal{R}_{\Lambda\Sigma}+i\mathcal{I}_{\Lambda\Sigma} \tag{2.7}$$

and the self-dual combination

$$\mathcal{O}^{+}=\frac{1}{2}\left(\mathcal{O}-i\tilde{\mathcal{O}}\right), \tag{2.8}$$

one finds that they must transform under duality transformation as:

$$\mathcal{N}'=\left(C+D\mathcal{N}\right)\left(A+B\mathcal{N}\right)^{-1}, \tag{2.9}$$

$$\mathcal{O}^{+'}=\mathcal{O}^{+}\left(A+B\mathcal{N}\right)^{-1}, \tag{2.10}$$

for

$$S=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathrm{Sp}(2n_V,\mathbb{R}). \tag{2.11}$$

Since, in general, both $\mathcal{N}$ and $\mathcal{O}$ depend on the scalar fields, these transformations can be interpreted also as a non-trivial action of the duality group on the scalars. Explicitly, this implies that, given the scalar manifold $\mathcal{M}_{\text{scalar}}$, there exist a homomorphism

$$\iota_{\delta}:\mathrm{Diff}(\mathcal{M}_{\text{scalar}})\longrightarrow\mathrm{Sp}(2n_V,\mathbb{R}), \tag{2.12}$$

such that

$$\forall\, \xi \in \mathrm{Diff}(\mathcal{M}_{\mathrm{scalar}}) \quad \exists\, \iota_\delta(\xi) = S_\xi \in \mathrm{Sp}(2n_V, \mathbb{R}). \tag{2.13}$$

Moreover, for consistency, $\mathcal{N}(\xi(\phi))$ should transform as prescribed by (2.9).
Given the non-trivial action on the scalars, in order to ensure invariance of the *full* set of equations of motion, the duality group should be further reduced, identifying the *U-duality group* G_U,

$$G_U \subset \mathrm{Sp}(2n_V, \mathbb{R}). \tag{2.14}$$

Interpreting scalar fields as coordinates on suitable manifolds, the U-duality group will coincide with the symmetry group under which the reparametrization of the scalar fields leaves the Lagrangian invariant, i.e., the isometry group of the scalar manifold $\mathrm{Iso}(\mathcal{M}_{\mathrm{scalars}})$.
To summarize, in general, the global symmetry group leaving invariant the set of Bianchi identities and equations of motion is $G_{\mathrm{global}} = G_U \times G_{\mathrm{inert}}$, where G_{inert} is the global symmetry group of those fields that do not have direct couplings to vector fields (e.g., hypermultiplets in $\mathcal{N}=2$ supergravity).

Special Kähler Geometry

As already pointed out, the scalar fields z^i and their complex conjugates can be thought as coordinate on a Special Kähler manifold; in view of the preceding discussion, we expect this result to emerge from the interplay between supersymmetry and symplectic duality covariance of the Bianchi identities and equations of motion for the n_V+1 vector fields[3].
Let's start by considering a complex (n_V+1)-dimensional Kähler manifold, namely a complex manifold with an Hermitian metric $g_{i\bar{\jmath}}(z,\, \bar{z})$ and closed fundamental two-form

$$\mathcal{K} = i g_{i\bar{\jmath}} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}}, \qquad \mathrm{d}\mathcal{K} = 0. \tag{2.15}$$

This last relation implies the existence of a real function $K(z^i, \bar{z}^{\bar{\jmath}})$ dubbed *Kähler potential*, such that

$$g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K. \tag{2.16}$$

In general, this function is not globally defined but differs on different coordinate patches U_α and U_β by a *Kähler transformation*:

$$K_\alpha = K_\beta + f_{\alpha\beta} + f^*_{\alpha\beta}\ , \tag{2.17}$$

where $f_{\alpha\beta}$ is a holomorphic function on the overlap $U_\alpha \cap U_\beta$.
A Kähler manifold $\mathcal{M}$ admitting a line bundle $\mathcal{L} \longrightarrow \mathcal{M}$ such that its first Chern class equals the de Rham cohomology class $[K]$ of the Kähler form is said to be *Hodge–Kähler*. This implies that, introducing the composite connection $\mathcal{Q}$ on the U(1)-bundle associated to the line bundle $\mathcal{L}$,

$$\mathcal{Q} = \frac{i}{2}\left[(\partial_{\bar{\jmath}} K)\mathrm{d}z^{\bar{\jmath}} - (\partial_i K)\mathrm{d}z^i\right], \tag{2.18}$$

its curvature will be related to the Kähler form

$$\mathrm{d}\mathcal{Q} = \mathcal{K}, \tag{2.19}$$

[3]In $\mathcal{N}=2$ supergravity coupled to n_V vector multiplets we have a total of n_V+1 Abelian vectors, the additional one being the graviphoton sitting in the gravitational multiplet.

up to an exact two-form.
Hodge–Kähler manifolds are the scalar geometry of $\mathcal{N}=1$ supergravity, as required by local supersymmetry and invariance under Kähler transformations [32]. Moving to $\mathcal{N}=2$, the symplectic duality group will further constrain this kind of geometries, leading to the definition of *Special Kähler manifolds.*
To characterize Special geometry, let's start by introducing a new holomorphic, flat vector bundle $\mathcal{SV} \longrightarrow \mathcal{M}$ of rank $2n_V+2$ and structural group $\mathrm{Sp}(2n_V+2,\mathbb{R})$. Consider now a tensor bundle $\mathcal{H}=\mathcal{SV}\otimes\mathcal{L}$, whose typical holomorphic sections will be of the form:

$$\mathscr{V} = \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix}, \quad \Lambda,\Sigma = 0,\ 1,\ \ldots,\ n_V. \tag{2.20}$$

As a consequence, the transition functions between two different local trivializations of $\mathcal{H}$ on U_α and U_β have the form

$$\mathscr{V}_\alpha = e^{-f_{\alpha\beta}} S_{\alpha\beta} \mathscr{V}_\beta, \tag{2.21}$$

for $S_{\alpha\beta}$ a constant element in $\mathrm{Sp}(2n_V+2,\mathbb{R})$.
Moreover, to ensure a consistent definition, the transition functions are subject to the cocycle condition on a triple overlap:

$$e^{f_{\alpha\beta}+f_{\beta\gamma}+f_{\gamma\alpha}} = 1 = S_{\alpha\beta}S_{\beta\gamma}S_{\gamma\alpha}. \tag{2.22}$$

Definition. A (local) Special Kähler manifold is a Hodge–Kähler manifold equipped with a tensor bundle $\mathcal{H}=\mathcal{SV}\otimes\mathcal{L}$ such that for some holomorphic section $\mathscr{V}$ the Kähler potential is given by:

$$K = -\log\left(i\left\langle \mathscr{V},\overline{\mathscr{V}}\right\rangle\right) = -\log\left(i\mathscr{V}^T\Omega\overline{\mathscr{V}}\right) = -\log\left[i\left(\overline{X}^\Lambda F_\Lambda - \overline{F}_\Sigma X^\Sigma\right)\right], \tag{2.23}$$

where Ω denotes the symplectic invariant matrix and $\langle\cdot,\cdot\rangle$ is a Hermitian and symplectic metric on $\mathcal{H}$.
Additionally, the sections satisfy:

$$\langle \nabla_i \mathscr{V}, \nabla_j \mathscr{V}\rangle = 0, \tag{2.24}$$

for $\nabla_i\mathscr{V} \equiv (\partial_i + \partial_i K)\mathscr{V}$ the Kähler covariant derivative of $\mathscr{V}$.

For $n_V > 1$, the last identity implies the existence of a symplectic frame with a *holomorphic prepotential* $F(X^\Lambda)$, i.e., there exist a symplectic transformation $S\in\mathrm{Sp}(2(n_V+1),\mathbb{R})$ such that $\tilde{\mathscr{V}} = S\cdot\mathscr{V} = \begin{pmatrix}\tilde{X}^\Lambda \\ \tilde{F}_\Sigma\end{pmatrix}$ and $\tilde{F}_J = \partial F(\tilde{X})/\partial\tilde{X}^J$, with $F(X^\Lambda)$ a homogeneous function of degree two.
Starting from $\mathscr{V}$, we can also introduce a covariantly holomorphic section

$$V = \begin{pmatrix} L^\Lambda \\ M_\Sigma \end{pmatrix} \equiv e^{K/2}\mathscr{V}, \tag{2.25}$$

so that (2.23) can be rewritten as

$$1 = i\left\langle V,\overline{V}\right\rangle = i\left(\overline{L}^\Lambda M_\Lambda - \overline{M}_\Sigma L^\Sigma\right). \tag{2.26}$$

By definition, it immediately follows that

$$\nabla_{\bar{i}} V = \left(\partial_{\bar{i}} - \frac{1}{2}\partial_{\bar{i}} K\right) V = 0. \tag{2.27}$$

Additionally, defining

$$U_i = \begin{pmatrix} f_i^{\Lambda} \\ h_{\Sigma i} \end{pmatrix} \equiv \nabla_i V = \left(\partial_i + \frac{1}{2}\partial_i K\right) V, \tag{2.28}$$

one can introduce the *period matrix* or *gauge kinetic matrix* $\mathcal{N}_{\Lambda\Sigma}$ via the relations:

$$\overline{M}_{\Lambda} = \overline{\mathcal{N}}_{\Lambda\Sigma}\overline{L}^{\Sigma}, \qquad h_{\Sigma i} = \overline{\mathcal{N}}_{\Lambda\Sigma} f_i^{\Sigma}. \tag{2.29}$$

They can be explicitly solved introducing two $(n_V+1)\times(n_V+1)$ vectors

$$f_I^{\Lambda} = \begin{pmatrix} f_i^{\Lambda} \\ \overline{L}^{\Lambda} \end{pmatrix}, \qquad h_{\Lambda I} = \begin{pmatrix} h_{\Lambda i} \\ M_{\Lambda} \end{pmatrix} \tag{2.30}$$

and setting:

$$\overline{\mathcal{N}}_{\Lambda\Sigma} = h_{\Lambda|I} \circ \left(f^{-1}\right)^{I}_{\Sigma}. \tag{2.31}$$

Starting from its definition, it is possible to verify that the matrix $\mathcal{N}$ transforms exactly as prescribed by (2.9). Moreover, the additional condition in eq. (2.24) ensures the symmetry and uniqueness of $\mathcal{N}_{\Lambda\Sigma}$.
Given all the ingredients introduced so far, it is possible to derive a set of useful relations, some of which we will report in the following [35]:

$$\mathrm{Im}\mathcal{N}_{\Lambda\Sigma} L^{\Lambda}\bar{L}^{\Sigma} = -\frac{1}{2}, \tag{2.32}$$

$$\langle V, U_i\rangle = \langle V, U_{\bar{\imath}}\rangle = 0, \tag{2.33}$$

$$U^{\Lambda\Sigma} \equiv f_i^{\Lambda} f_{\bar{\jmath}}^{\Sigma} g^{i\bar{\jmath}} = -\frac{1}{2}(\mathrm{Im}\mathcal{N})^{-1|\Lambda\Sigma} - \bar{L}^{\Lambda}L^{\Sigma}, \tag{2.34}$$

$$g_{i\bar{\jmath}} = -\mathrm{i}\left\langle U_i \mid \bar{U}_{\bar{\jmath}}\right\rangle = -2 f_i^{\Lambda}\, \mathrm{Im}\mathcal{N}_{\Lambda\Sigma} f_{\bar{\jmath}}^{\Sigma}. \tag{2.35}$$

As a final remark, we notice that, as opposed to what happens for $\mathcal{N}=2$ supersymmetry, we cannot locally identify the z^i with the $X^I(z)$ (since we have one X^I more due to the presence of the graviphoton). Instead, one can interpret the X^I as homogeneous coordinates of a projective space and $z^i \equiv X^i/X^0$ as the corresponding inhomogeneous coordinates (only for symplectic frames in which a prepotential exists). For this reason, Special Kähler geometry is also referred to as *projective Special Kähler* and this specific coordinate system is usually dubbed as *special coordinates*.

2.2.2 Hypermultiplets

Since the $\mathcal{N}=2$ hypermultiplets do not contain any fields with spin greater than 1/2, the scalar geometry is directly constrained only by the supersymmetry algebra.
In particular, from the structure of the R-symmetry group (i.e., $\mathrm{U}(2)_R$), one finds that the linearized supersymmetry transformations for n_H hypermultiplets are invariant under $\mathrm{Sp}(2n_H,\mathbb{R})\times\mathrm{SU}(2)$. Additionally, since the SU(2) doublet of supersymmetry parameters is not constant, the SU(2) part of the curvature has to be non-trivial [32].
Therefore, scalars q^u sitting in hypermultiplets should span a manifold of real dimension $4n_H$ with holonomy contained in $\mathrm{Sp}(2n_H,\mathbb{R})\times\mathrm{SU}(2)$ and non-trivial SU(2) curvature. As we will discuss, these properties precisely define a *quaternionic-Kähler manifold*.
Consider a real manifold $\mathcal{QM}$ of dimension $4n_H$ equipped with a metric

$$ds^2 = h_{uv}(q)\,\mathrm{d}q^u\otimes\mathrm{d}q^v, \qquad (u,v\ =\ 1,...,\ 4n_H), \tag{2.36}$$

and three complex structures J^x $(x = 1,2,3)$ satisfying the quaternionic algebra

$$J^x J^y = -\delta^{xy}\mathbb{I} + \varepsilon^{xyz} J^z. \qquad (2.37)$$

Generalizing the usual construction for Kähler manifolds, the metric is hermitian with respect to the complex structures:

$$\forall X, Y \in T(\mathcal{QM}) \quad : \quad h(J^x X, J^x Y) = h(X,Y) \quad (x = 1,2,3), \qquad (2.38)$$

hence we can introduce a triplet of two-forms

$$K^x = K^x_{uv} \mathrm{d}q^u \wedge \mathrm{d}q^v = h_{uw} \left(J^x\right)^w_v \, \mathrm{d}q^u \wedge \mathrm{d}q^v. \qquad (2.39)$$

This SU(2) Lie-algebra valued two-form is often referred to as *HyperKähler form*, as it provides a generalization of the usual Kähler form (which is U(1) Lie-algebra valued). Unlike the Kähler form, however, the HyperKähler form is not generically closed.
Let us now define a principal SU(2)-bundle $\mathcal{SU} \longrightarrow \mathcal{QM}$ and an associated connection ω^x.

> **Definition.** $\mathcal{QM}$ is said to be *quaternionic-Kähler* if the associated HyperKähler two-form is covariantly closed with respect to the connection ω^x:
>
> $$\nabla K^x \equiv \mathrm{d}K^x + \varepsilon^{xyz}\omega^y \wedge K^z = 0. \qquad (2.40)$$
>
> Additionally, the curvature of the $\mathcal{SU}$-bundle is proportional to the HyperKähler form, that is:
>
> $$\Omega^x \equiv \mathrm{d}\omega^x + \frac{1}{2}\varepsilon^{xyz}\omega^y \wedge \omega^z = -K^x. \qquad (2.41)$$

By construction, the manifold $\mathcal{QM}$ has precisely the holonomy group required by supersymmetry:

$$\mathrm{Hol}(\mathcal{QM}) = \mathrm{SU}(2) \otimes \mathrm{H}, \quad \mathrm{H} \subset \mathrm{Sp}(2n_H, \mathbb{R}), \qquad (2.42)$$

as it will be also clear by looking at the structure of the Riemann tensor.
We can now introduce flat indices running in the fundamental representation of SU(2) and $\mathrm{Sp}(2n_H,\mathbb{R})$ and the vielbein one-form

$$\mathcal{U}^{A\alpha}_u = \mathcal{U}^{A\alpha}_u \mathrm{d}q^u, \qquad (A = 1,2 \text{ and } \alpha = 1,...,2n_H), \qquad (2.43)$$

such that

$$h_{uv} = \mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v \mathbb{C}_{\alpha\beta}\varepsilon_{AB}, \qquad (2.44)$$

for $\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}$ and $\varepsilon_{AB} = -\varepsilon_{BA}$ the flat $\mathrm{Sp}(2n_H,\mathbb{R})$ and SU(2) metrics, respectively.
Introducing also a $\mathrm{Sp}(2n_H,\mathbb{R})$-Lie Algebra valued connection $\Delta^{\alpha\beta} = \Delta^{\beta\alpha}$, the vielbein is covariantly closed:

$$\nabla\mathcal{U}^{A\alpha} \equiv d\mathcal{U}^{A\alpha} + \frac{i}{2}\omega^x \left(\varepsilon\sigma_x\varepsilon^{-1}\right)^A{}_B \wedge \mathcal{U}^{B\alpha} + \Delta^{\alpha\beta} \wedge \mathcal{U}^{A\gamma}\mathbb{C}_{\beta\gamma} = 0, \qquad (2.45)$$

where $\left(\sigma^x\right)_A{}^B$ denotes the standard Pauli matrices.
Moreover, $\mathcal{U}^{A\alpha}$ satisfies the reality condition:

$$\mathcal{U}_{A\alpha} \equiv \left(\mathcal{U}^{A\alpha}\right)^* = \varepsilon_{AB}\mathbb{C}_{\alpha\beta}\mathcal{U}^{B\beta}. \qquad (2.46)$$

Finally, we can introduce the inverse vielbein $\mathcal{U}^u_{A\alpha}$, such that:

$$\mathcal{U}^u_{A\alpha}\mathcal{U}^{A\alpha}_v = \delta^u_v. \qquad (2.47)$$

As already mentioned, it is actually interesting to flatten a pair of indices of the Riemann tensor ${\mathcal{R}^{uv}}_{ts}$ and get

$$ {\mathcal{R}^{uv}}_{ts}\mathcal{U}_u^{A\alpha}\mathcal{U}_v^{B\beta} = -\frac{1}{2}\Omega^{x\ AC}_{ts}\left(\sigma_x\right)_C{}^B\mathbb{C}^{\alpha\beta} + \mathbb{R}^{\alpha\beta}_{ts}\varepsilon^{AB}, \tag{2.48}$$

for $\mathbb{R}^{\alpha\beta}_{ts}$ the field-strength of the $\mathrm{Sp}(2n_H,\mathbb{R})$ connection. As anticipated, the structure of the Riemann tensor explicitly realises the statement that the Levi–Civita connection associated with the metric h has the correct holonomy group. Moreover, the previous equations actually imply that $\mathcal{QM}$ is an Einstein space [36] with Ricci tensor given by

$$ \mathcal{R}_{uv} = -(2+n_H)h_{uv}. \tag{2.49}$$

At this point, it is useful to introduce some quaternionic identities that will be largely employed in the following. First of all, let's notice that one can actually write down a stronger version of (2.44):

$$ \left(\mathcal{U}_u^{A\alpha}\mathcal{U}_v^{B\beta} + \mathcal{U}_v^{A\alpha}\mathcal{U}_u^{B\beta}\right) = h_{uv}\varepsilon^{AB}. \tag{2.50}$$

Additionally, as a consequence of the quaternionic nature of the complex structures J^x and the relation between the SU(2) curvature and the HyperKähler two-form, one can actually write:

$$ h^{st}K^x_{us}K^y_{tv} = -\delta^{xy}h_{uv} + \varepsilon^{xyz}K^z_{uv}, \tag{2.51}$$

which can be equivalently recast also in terms of Ω^x

$$ h^{st}\Omega^x_{us}\Omega^y_{tv} = -\delta^{xy}h_{uv} - \varepsilon^{xyz}\Omega^z_{uv}. \tag{2.52}$$

In particular, this last relation implies that the intrinsic components of the curvature two-form Ω^x yield a representation of the quaternionic algebra. As a consequence, one can see that

$$ \frac{i}{2}\Omega^x\left(\sigma_x\right)_A{}^B = -\mathcal{U}_{A\alpha}\wedge\mathcal{U}^{B\alpha}. \tag{2.53}$$

Finally, exploiting eqs. (2.50, 2.53) we can express the product of two vielbeins as:

$$ \mathbb{C}_{\alpha\beta}\mathcal{U}_u^{A\alpha}\mathcal{U}_v^{B\beta} = \frac{1}{2}h_{uv}\varepsilon^{AB} + \frac{i}{2}\Omega^x_{uv}\left(\sigma_x\right)_C{}^B\varepsilon^{AC}. \tag{2.54}$$

2.3 The Gauging

After having thoroughly reviewed the underlying geometrical features associated to the scalars, we are now ready to review the *gauging procedure*. This step is actually crucial in the context of extended supergravity theories, as gauging provides the only mechanism that can generate a *scalar potential*, a crucial ingredient for moduli stabilization and supersymmetry breaking.
Before delving into technical details, we will shortly review the basic gauging structure and the role of duality transformations.

2.3.1 Gauging and Symplectic Frames

An overview

By *gauging* we usually refer to the procedure of making local a global symmetry of a given Lagrangian. In general, these symmetries coincide with the isometries of the scalar manifold $\mathcal{M}_{\text{scalar}}$ and are described in terms of *Killing vectors* k^i_Λ,

$$ \delta\phi^i = \alpha^\Lambda k^i_\Lambda, \qquad \text{for} \qquad \nabla_{(i}k_{\Lambda j)} = 0, \tag{2.55}$$

where α^Λ denote infinitesimal symmetry parameters.
The first step towards a gauged theory is therefore the choice of a suitable gauge group $\mathcal{G} \subset \mathrm{Iso}(\mathcal{M}_{\text{scalar}})$, whose dimension should not exceed the number of vectors in the theory.
Allowing for non-constant symmetry parameters, $\alpha^\Lambda(x)$, requires the introduction of a number of modifications of the theory. In particular, covariant derivatives for the scalar fields should be introduced

$$\mathcal{D}_\mu \phi^i \equiv \partial_\mu \phi^i - A_\mu^\Lambda k_\Lambda^i, \tag{2.56}$$

and the Maxwell-type transformations for the vector fields should be replaced by

$$\delta A_\mu^\Lambda = \partial_\mu \alpha^\Lambda + f^\Lambda{}_{\Sigma\Delta} A_\mu^\Sigma \alpha^\Delta, \tag{2.57}$$

for f the structure constants associated to $\mathcal{G}$.
As long as the kinetic terms of the vector fields are concerned, gauge invariance demands the replacement of Abelian field strengths with their non-Abelian counterpart

$$F_{\mu\nu}^\Lambda = 2\partial_{[\mu} A_{\nu]}^\Lambda + f^\Lambda{}_{\Sigma\Delta} A_\mu^\Sigma A_\mu^\Delta. \tag{2.58}$$

Rescaling the gauge vectors and making the gauge coupling explicit $A \to gA$, one immediately realizes that the adjustments introduced so far correspond to $\mathcal{O}(g^2)$ modifications, which inevitably break explicitly supersymmetry. After properly covariantizing also the kinetic terms for the fermions and all the derivatives appearing in the supersymmetry transformation rules, additional adjustments are required for the theory to completely restore its supersymmetry content:

1. the introduction of a "fermionic shift" involving scalar field-dependent terms at order $\mathcal{O}(g)$ in the fermionic supersymmetry variations;
2. the addition of a fermion-fermion, Yukawa-like term of order $\mathcal{O}(g)$ in the Lagrangian;
3. a contribution to the scalar potential at order $\mathcal{O}(g^2)$.

Even if we will not deal with the detailed calculations behind these steps, we point out that this procedure is entirely general and will help us understand the origin of the various terms present in the general Lagrangian and in the supersymmetry variations.

The role of Symplectic Frames

Up to this point, we have introduced two different relevant groups: the group of global symmetries of a given Lagrangian, $G_{\mathcal{L}}$, and the symmetry group of Bianchi identities and equations of motion, i.e., the U-duality group G_U.
Taking a closer look, this actually implies that, for a fixed set of multiplets, a given theory may have different Lagrangian realizations with different $G_{\mathcal{L}}$. The set of Lagrangians that cannot be mapped to each other by local field redefinitions is identified with the double quotient space

$$\mathrm{GL}(n_V+1,\mathbb{R}) \setminus \mathrm{Sp}(2(n_V+1),\mathbb{R}) \;/\; G_U \; . \tag{2.59}$$

This result follows from the simple observation that we can reabsorb a duality transformation through a local field redefinition of the vector fields, which is accounted for by the $\mathrm{GL}(n_V+1,\mathbb{R})$ quotient, but also via redefinitions of the other fields contained in the U-duality group G_U.

Each different Lagrangian defines a distinct *symplectic frame* and is invariant under a particular *electric subgroup*[4] of the G_U duality group. The choice of the symplectic frame is therefore important to find a purely electric realization of the gauge group $\mathcal{G}$ (i.e., $\mathcal{G} \subset G_{\mathcal{L}}$).
In any case, before the gauging, the resulting equations of motion and Bianchi identities for any Lagrangian defined through (2.59) are still equivalent, being related by symplectic field redefinitions.

2.3.2 Killing Prepotentials

All the modifications of the Lagrangian and of the supersymmetry transformation rules required upon gauging a subgroup of isometries of the scalar manifold can be actually described in terms of a generic geometric construction associated with the action of Lie-Groups on manifolds admitting a symplectic structure: the *prepotentials* or *momentum maps*.
In the following, we will construct explicitly such objects on both Special Kähler and quaternionic-Kähler manifolds.

Holomorphic prepotentials on Special Kähler manifolds

Let's consider a set of Killing vectors $k^i_\Lambda(z)$ associated with the isometries of the metric $g_{i\bar{\jmath}}$ of a Special Kähler manifold $\mathcal{SM}$.
Given the complex nature of the manifold, we expect k^i_Λ to be holomorphic, i.e. $\partial_{\bar{\jmath}} k^i_\Lambda(z) = 0$, so as not to mix z^i and $\bar{z}^{\bar{\imath}}$. Moreover, they satisfy the usual Killing equation that, in holomorphic indices, reads

$$\nabla_i k_{j,\Lambda} + \nabla_j k_{i,\Lambda} = 0 \quad \text{and} \quad \nabla_{\bar{\imath}}\, k_{j,\Lambda} + \nabla_j\, k_{\bar{\imath},\Lambda} = 0. \tag{2.60}$$

The request for the isometry group to have an embedding into the symplectic group can be formulated by writing that

$$\mathcal{L}_\Lambda V \equiv k^i_\Lambda \partial_i V + k^{\bar{\imath}}_\Lambda \partial_{\bar{\imath}} V = T_\Lambda V + V f_\Lambda(z), \tag{2.61}$$

where $\mathcal{L}_\Lambda$ denotes the Lie derivative along k^i_Λ, V is the covariantly holomorphic section introduced in (2.25), T_Λ is an element of the real symplectic Lie algebra and f_Λ corresponds to an infinitesimal Kähler transformation.
Since the holomorphic Killing vectors preserve both the metric on $\mathcal{SM}$ and its complex structure, they will also preserve the Kähler form:

$$\mathcal{L}_\Lambda \mathcal{K} = i_\Lambda \mathrm{d}\mathcal{K} + \mathrm{d} i_\Lambda \mathcal{K} = \mathrm{d} i_\Lambda \mathcal{K} = 0, \tag{2.62}$$

for i_Λ denoting the interior product along k_Λ and where we exploited the Cartan formula for the Lie derivative and the fact that $\mathrm{d}\mathcal{K} = 0$ for a Kähler manifold.
This implies that, at least locally, there must exist functions $\mathcal{P}^0_\Lambda$ such that

$$i_\Lambda \mathcal{K} = \mathrm{d}\mathcal{P}^0_\Lambda. \tag{2.63}$$

These functions are known as *Killing prepotentials* or *momentum maps* and they can be expressed explicitly in terms of the Killing vectors and the Kähler potential as

$$i\mathcal{P}^0_\Lambda = \frac{1}{2}\left(k^i_\Lambda \partial_i K + k^{\bar{\imath}}_\Lambda \partial_{\bar{\imath}} K\right) = k^i_\Lambda \partial_i K = -k^{\bar{\imath}}_\Lambda \partial_{\bar{\imath}} K. \tag{2.64}$$

[4] Although $\delta\mathcal{L}_{\text{rest}} = 0$, under a generic transformation in G_U the vectorial sector of the Lagrangian gets modified. The full Lagrangian is invariant, up to a total derivative, only under "electric" transformations, obtained by setting $B = 0$ in (2.11).

Exploiting Special geometry identities, one can actually recast the prepotential in terms of symplectic invariants. In particular, focusing on electric gaugings, the symplectic image of the generators is block-diagonal and coincides with the adjoint representation in each block, that is

$$T_\Lambda = \begin{pmatrix} f^\Sigma{}_{\Lambda\Delta} & 0 \\ 0 & -f^\Sigma{}_{\Lambda\Delta} \end{pmatrix}. \tag{2.65}$$

For this specific case we can write

$$\mathcal{P}^0_\Lambda = \langle V | T_\Lambda \overline{V} \rangle = e^K \left(F_\Delta f^\Delta{}_{\Lambda\Sigma} \overline{X}^\Sigma + \overline{F}_\Delta f^\Delta{}_{\Lambda\Sigma} X^\Sigma \right). \tag{2.66}$$

As a final remark, we underline that to every generator of the abstract Lie algebra of the group $\mathcal{G}$ we have associated a function $\mathcal{P}^0_\Lambda$. From a geometrical point of view, the prepotential is the Hamiltonian function providing the Poissonian realization of the Lie algebra on the Special Kähler manifold. In particular, it holds[5]:

$$\left\{ \mathcal{P}^0_\Lambda, \mathcal{P}^0_\Sigma \right\} \equiv 4\pi \mathcal{K}(\Lambda, \Sigma) = f_{\Lambda\Sigma}{}^\Gamma \mathcal{P}^0_\Lambda, \tag{2.67}$$

where $\left\{ \mathcal{P}^0_\Lambda, \mathcal{P}^0_\Sigma \right\}$ are the Poisson bracket of $\mathcal{P}^0_\Lambda$ with $\mathcal{P}^0_\Sigma$ and $\mathcal{K}(\Lambda,\Sigma) \equiv \mathcal{K}(k_\Lambda, k_\Sigma)$ denotes the value of the Kähler form along the pair of Killing vectors.
Equation (2.67) is usually referred to as *equivariance condition* and in components it reads

$$\frac{i}{2} g_{i\bar{\jmath}} \left(k^i_\Lambda k^{\bar{\jmath}}_\Sigma - k^i_\Sigma k^{\bar{\jmath}}_\Lambda \right) = \frac{1}{2} f_{\Lambda\Sigma}{}^\Gamma \mathcal{P}^0_\Gamma. \tag{2.68}$$

The triholomorphic prepotentials on Quaternionic manifolds

To construct suitable prepotentials on quaternionic-Kähler manifolds, we have to generalize the construction we carried out in the previous section. Specifically, in this context, the Kähler form is replaced by the SU(2) Lie-algebra valued HyperKähler two-form and, as a consequence, holomorphic Killing vectors ought to be generalized to *triholomorphic* ones.
Triholomorphicity implies that the Killing vector fields leave the HyperKähler form invariant up to SU(2) rotations, that is

$$\mathcal{L}_\Lambda K^x = \varepsilon^{xyz} K^y W^z_\Lambda \quad \text{and} \quad \mathcal{L}_\Lambda \omega^x = \nabla W^x_\Lambda, \tag{2.69}$$

for W^x_Λ an SU(2) compensator associated to the Killing vector k^u_Λ. The previous relation can be obviously expressed also in terms of the SU(2) curvature by virtue of its proportionality to the HyperKähler form.
In full analogy with the Special Kähler case, to each Killing vector we can associate a triplet of 0-form prepotentials $\mathcal{P}^x_\Lambda(q)$ via the following relation:

$$i_\Lambda \Omega^x = -\nabla \mathcal{P}^x \equiv -(d\mathcal{P}^x + \varepsilon^{xyz} \omega^y \mathcal{P}^z), \tag{2.70}$$

where ∇ denotes the SU(2)-covariant derivative. In components, the relation reads

$$2 k^u_\Lambda \Omega^x_{uv} = -\nabla_v \mathcal{P}^x_\Lambda. \tag{2.71}$$

To find an explicit expression for the prepotentials in terms of the Killing vectors, starting from the Killing equation and using the fact that the Ricci tensor on quaternionic-Kähler manifolds takes the

[5] This relation actually holds only if the Lie algebra has a trivial second cohomology group, which is always the case for semi-simple Lie algebras.

form in eq. (2.49), one can notice that the prepotentials are actually eigenfunction of the covariant Laplacian [36], that is

$$\nabla_v \nabla^v \mathcal{P}^x_\Lambda - 4 n_H \mathcal{P}^x_\Lambda = 0. \tag{2.72}$$

This last relation, together with the defining equation in (2.70), leads to

$$\mathcal{P}^x_\Lambda = \frac{1}{2n_H} \nabla^u k^v_\Lambda \Omega^x{}_{uv} \tag{2.73}$$

Finally, we point out that Killing prepotentials provide once again a Poissonian realization of the gauge group Lie algebra; in particular, introducing the triholomorphic Poisson bracket

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\}^x \equiv i_\Lambda i_\Sigma K^x + \varepsilon^{xyz} \mathcal{P}^y_\Lambda \mathcal{P}^z_\Sigma \quad \text{for} \quad \frac{1}{2} i_\Lambda i_\Sigma K^x \equiv -\Omega^x_{uv} k^u_\Lambda k^v_\Sigma, \tag{2.74}$$

the equivariance condition now reads

$$\{\mathcal{P}_\Lambda, \mathcal{P}_\Sigma\}^x = f^\Delta{}_{\Lambda\Sigma} \mathcal{P}^x_\Delta, \tag{2.75}$$

or, in components

$$\frac{1}{2} \varepsilon^{xyz} \mathcal{P}^y_\Lambda \mathcal{P}^z_\Sigma - \Omega^x_{uv} k^u_\Lambda k^v_\Sigma = \frac{1}{2} f^\Delta{}_{\Lambda\Sigma} \mathcal{P}^x_\Delta. \tag{2.76}$$

2.4 Lagrangian and Supersymmetry Variations

Having introduced the main concepts and geometric structures for the ungauged theory, and described the key ingredients in the gauging procedure, we are now ready to present the Lagrangian and the supersymmetry variations for $\mathcal{N}=2$ gauged supergravity. Given that our following analysis will focus on solutions with vanishing fermionic fields, we will concentrate on the bosonic part of the Lagrangian and on the supersymmetry variations for the fermions. Our presentation will primarily follow [31]; however, since we will employ the mostly plus signature, there will be a sign difference whenever there is an upper spacetime index.

Schematically, the bosonic $N=2$ supergravity Lagrangian can be split as

$$\mathscr{L}_{\text{bosonic}} = \mathscr{L}_k - V(z, \bar{z}, q), \tag{2.77}$$

where $\mathcal{L}_k$ consists of the properly covariantized kinetic terms and $V(z,\bar{z},q)$ denotes the $\mathcal{O}(g^2)$ contributions to the *scalar potential* arising from the gauging[6]. Explicitly, each term reads:

$$\begin{aligned} \mathscr{L}_k &= \frac{1}{2}R - g_{i\bar{\jmath}} \nabla^\mu z^i \nabla_\mu \bar{z}^{\bar{\jmath}} - h_{uv} \nabla_\mu q^u \nabla^\mu q^v + i\left(\overline{\mathcal{N}}_{\Lambda\Sigma} F^{-\Lambda}_{\mu\nu} F^{-\Sigma\mu\nu} - \mathcal{N}_{\Lambda\Sigma} F^{+\Lambda}_{\mu\nu} F^{+\Sigma\mu\nu}\right), && (2.78) \\ V(z,\bar{z},q) &= g^2\left[\left(g_{i\bar{\jmath}} k^i_\Lambda k^{\bar{\jmath}}_\Sigma + 4 h_{uv} k^u_\Lambda k^v_\Sigma\right) \bar{L}^\Lambda L^\Sigma + \left(g^{i\bar{\jmath}} f^\Lambda_i f^\Sigma_{\bar{\jmath}} - 3\bar{L}^\Lambda L^\Sigma\right) \mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma\right], && (2.79) \end{aligned}$$

where we introduced explicitly the coupling constant g so as to clearly identify the terms due to gauging.

The covariant derivatives appearing in the Lagrangian are

$$\nabla z^i = \mathrm{d}z^i + gA^\Lambda k^i_\Lambda(z), \quad \nabla \bar{z}^{\bar{\imath}} = \mathrm{d}\bar{z}^{\bar{\imath}} + gA^\Lambda k^{\bar{\imath}}_\Lambda(z), \tag{2.80}$$

$$\nabla q^u = \mathrm{d}q^u + gA^\Lambda k^u_\Lambda(q). \tag{2.81}$$

[6] As we discussed, in principle we also expect order $\mathcal{O}(g)$ Yukawa-like contributions to the Lagrangian but they vanish in the absence of fermions.

Moreover, the vector field strengths appear as $F^{\pm\Lambda}_{\mu\nu} = \frac{1}{2}\left(F^{\Lambda}_{\mu\nu} \pm \frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}F^{\Lambda}_{\rho\sigma}\right)$, i.e. through their self-dual and anti self-dual combinations, respectively.
As long as the supergravity transformation rules of the fermionic fields are concerned, we have:

$$\delta\psi_{A\mu} = \mathcal{D}_\mu\epsilon_A - T^{-}_{\mu\nu}\gamma^\nu\varepsilon_{AB}\epsilon^B + ig\eta_{\mu\nu}\gamma^\nu S_{AB}\epsilon^B; \tag{2.82}$$

$$\delta\lambda^{iA} = -i\nabla_\mu z^i\gamma^\mu\epsilon^A + G^{-i}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^{AB}\epsilon_B + gW^{iAB}\epsilon_B; \tag{2.83}$$

$$\delta\zeta_\alpha = -i\mathcal{U}^{B\beta}_u\nabla_\mu q^u\gamma^\mu\varepsilon_{AB}\mathbb{C}_{\alpha\beta}\epsilon^A + gN^A_\alpha\epsilon_A. \tag{2.84}$$

In the above supersymmetry variations the field strengths appear in their anti self-dual combinations, dressed by the scalar fields as follow:

$$T^{-}_{\mu\nu} = 2i\mathcal{I}_{\Lambda\Sigma}L^\Sigma F^{-\Lambda}_{\mu\nu}; \quad G^{-i}_{\mu\nu} = -\bar{D}^i\bar{L}^\Gamma\mathcal{I}_{\Gamma\Lambda}F^{-\Lambda}_{\mu\nu}. \tag{2.85}$$

Moreover, as expected, contributions at order $\mathcal{O}(g)$ appears in the supersymmetry variations through the *mass matrices*, given by

$$S_{AB} = \frac{i}{2}(\sigma_x)_A{}^C\varepsilon_{BC}\mathcal{P}^x_\Lambda L^\Lambda, \tag{2.86}$$

$$W^{i\,AB} = \varepsilon^{AB}k^i_\Lambda\bar{L}^\Lambda + i(\sigma_x)_C{}^B\varepsilon^{CA}\mathcal{P}^x_\Lambda g^{i\bar{\jmath}}\bar{f}^\Lambda_{\bar{\jmath}}, \tag{2.87}$$

$$N^A_\alpha = 2\mathcal{U}^A_{\alpha u}k^u_\Lambda\bar{L}^\Lambda. \tag{2.88}$$

The covariant derivative for the supersymmetry parameters reads explicitly

$$\mathcal{D}_\mu\epsilon_A = \partial_\mu\epsilon_A + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}\epsilon_A + \frac{i}{2}\hat{\mathcal{Q}}_\mu\epsilon_A + (\hat{\omega}_\mu)_A{}^B\epsilon_B, \tag{2.89}$$

for the U(1) and SU(2) gauged connection defined respectively as

$$\hat{\mathcal{Q}} = \mathcal{Q} + gA^\Lambda\mathcal{P}^0_\Lambda \tag{2.90}$$

and

$$(\hat{\omega})_A{}^B = \frac{i}{2}\hat{\omega}^x(\sigma_x)_A{}^B, \quad \text{with} \quad \hat{\omega}^x = \omega^x_u\mathrm{d}q^u + gA^\Lambda\mathcal{P}^x_\Lambda. \tag{2.91}$$

Finally, as outlined when discussing the general gauging procedure, we would like to stress that the scalar potential of the theory is indeed related to the fermionic shifts appearing in the supersymmetry variation. In particular, it holds [33]

$$g^{-2}\delta^A_B\,V(z,\bar{z},q) = g_{i\bar{\jmath}}W^{iAC}W^{\bar{\jmath}}_{BC} + 2N^A_\alpha N^\alpha_B - 12S^{AC}S_{BC}. \tag{2.92}$$

This relation is usually referred to as *Ward Identity* and can be generalized also to other supergravity theories.

2.5 Vacua and Supersymmetry Breaking

A Lorentz preserving *vacuum* of a supergravity theory is a *maximally symmetric solution*, i.e., a solution exhibiting Minkowski, de Sitter, or anti-de Sitter spacetime geometry, depending on the value of a cosmological constant Λ.
As a consequence of the combined request of Lorentz invariance and maximal spacetime symmetry,

only spin-0 fields can develop a non-vanishing, uniform vacuum expectation value (v.e.v.), that is, specifying our treatment to $\mathcal{N}=2$ supergravity,

$$\left\langle z^i(x)\right\rangle = z_0^i, \qquad \langle q^u(x)\rangle = q_0^u \qquad \text{and} \qquad \langle\psi_{\mu A}\rangle = \left\langle\lambda_A^i\right\rangle = \langle\zeta^\alpha\rangle = \left\langle A_\mu^\Lambda\right\rangle = 0, \tag{2.93}$$

for $\langle\ \rangle$ denoting quantities evaluated on the vacuum.
Denoting collectively $\phi_0 = (z_0, \bar{z}_0, q_0)$, such a value for the scalar fields identifies a point in the moduli space which extremize the scalar potential $V(z,\bar{z},q)$:

$$\left.\frac{\partial V}{\partial z^i}\right|_{\phi_0} = \left.\frac{\partial V}{\partial \bar{z}^{\bar{\imath}}}\right|_{\phi_0} = \left.\frac{\partial V}{\partial q^u}\right|_{\phi_0} = 0, \qquad \forall\, i \in \{1,...,n_V\}\,,\ u \in \{1,...,4n_H\} \tag{2.94}$$

and the value $V(z_0,\bar{z}_0,q_0)$ provides the effective cosmological constant describing the background geometry:

$$\Lambda = V(z_0,\bar{z}_0,q_0). \tag{2.95}$$

A given vacuum, identified by the critical values for the scalars, can preserve a certain amount of supersymmetry of the original theory. In such cases, this implies that there should exist a local supersymmetry parameter $\epsilon_A(x)$ along which the supersymmetry variations of the fermionic fields vanish on the solution. The analogous condition on the supersymmetry variations of the bosonic field content of the theory would be trivially satisfied since these are expressed in terms of the fermionic fields which go to zero when evaluated on the vacuum.
By looking at the explicit expressions for the fermionic transformation rules given in eqs. (2.82)-(2.84), on the vacuum one obtains

$$\begin{aligned} \delta\psi_{A\mu} &= \mathcal{D}_\mu\epsilon_A + ig\eta_{\mu\nu}\gamma^\nu S_{AB}\epsilon^B = 0, \\ \delta\lambda^{iA} &= gW^{iAB}\epsilon_B = 0, \\ \delta\zeta_\alpha &= gN_\alpha^A\epsilon_A = 0, \end{aligned} \tag{2.96}$$

for $\mathcal{D}_\mu$ reducing to the Lorentz-covariant derivative.
These equation are usually referred to as *Killing spinor equations* and the background preserves a number $\mathcal{N}' \leq \mathcal{N}$ of the original $\mathcal{N}$ supersymmetries of the theory if they admit $\mathcal{N}'$ distinct solutions. As it becomes clear from eq. (2.96), the presence of residual supersymmetries translates to conditions on the mass matrices S_{AB}, $W^{i\,AB}$ and N_α^A, that correspond, via their definitions, to geometrical constraints on the scalar manifold and its gauged isometries.
In our work, we'll be interested in solutions that preserve supersymmetry, at least partially. In the context of $\mathcal{N}=2$ supergravity, this implies that solutions either preserve the whole $\mathcal{N}=2$ or spontaneously break the supersymmetry content down to $\mathcal{N}=1$.
In the first case, it immediately follows from (2.96) that

$$W^{iAB} = 0 \qquad \text{and} \qquad N_\alpha^A = 0. \tag{2.97}$$

Moreover, restricting our treatment to Minkowski vacua, from the Ward identity in (2.92) it is clear that it should also hold

$$S_{AB} = 0, \tag{2.98}$$

which effectively implies that both gravitini should remain massless on the solution.
On the other hand, in the case of spontaneous supersymmetry breaking $\mathcal{N}=2 \to \mathcal{N}=1$, the Killing

spinor equations admit only one solution, that we identify with ϵ_1 without any loss of generality. This implies that

$$W^{iA1} = 0, \qquad N_\alpha^1 = 0, \tag{2.99}$$

so that the Ward identity now becomes

$$g^{-2}\delta_B^1\, V(z, \bar{z}, q) = -12 S^{1C} S_{BC}. \tag{2.100}$$

The last equation is easily solved by writing

$$g^2 S_{AC} S^{BC} = \begin{pmatrix} -\frac{1}{12}V & 0 \\ 0 & S_{2C}S^{2C} \end{pmatrix}. \tag{2.101}$$

For the Minkowski case, this means that S_{AB} eigenvalues should be non-degenerate and one of them has to vanish, i.e., one of the two gravitini has to become massive while the second remains massless.

Chapter 3

Black hole solutions in $\mathcal{N} = 2$, $D = 4$ Supergravity

The aim of this chapter is to give a comprehensive overview of the state of the art in black hole physics within $D = 4$, $\mathcal{N} = 2$ supergravity theories. In addition to tracing the historical developments in the field, with this review we aim to describe the essential ideas and tools that form the foundational background for the new solutions that we will present in the following chapter.

We will begin by reviewing black hole solutions in asymptotically flat spacetimes. In particular, we will start from the original work on supersymmetric, single centre black holes [37], and the subsequent discovery of the attractor mechanism. Then, we will explain how certain properties of these solutions, initially thought to be exclusively related to supersymmetry, are in fact shared by all extremal black holes [38]. Moreover, we will introduce the role of the *superpotential* as the object effectively describing the solution by a gradient flow, laying the groundwork for extending the formalism, mutatis mutandis, to non-supersymmetric configurations through the introduction of a "fake-superpotential", as developed in [29].

We then shift our focus to the physics of multi-centre black hole solutions. Firstly, we will present Denef's pioneering work on BPS configurations [39], explaining how these solutions can be fully characterized by harmonic functions. Secondly, we explore the challenges emerging in the attempt of generalizing this framework to non-BPS solutions. Specifically, we will discuss how multi-centre configurations can be uplifted to 11-dimensional supergravity. Then, by carefully choosing a compactification scheme, we present how Denef's results can be recovered and extended to the non-supersymmetric case, introducing the idea of the "floating brane" ansatz and the role of U-duality transformations [40, 41].

Finally, we turn our attention to BPS black holes in Anti-de Sitter spacetimes within gauged supergravity theories [42]. In particular, we will examine how the attractor mechanism emerges in these AdS configurations and explore how the Killing spinor equations lead to the first-order flow equations that describe BPS solutions.

3.1 Asymptotically flat Black Holes

We will start our review by focusing on theories without gauging; as we described in the previous chapter, this implies the absence of a scalar potential and, as a consequence, we will be able to study exclusively black holes configurations that are asymptotically flat.

3.1.1 Single Centre Solutions

Supersymmetric black holes... and beyond?

The efforts towards an understanding of black hole physics in the context of $D=4$, $\mathcal{N}=2$ supergravity theories went initially exclusively into supersymmetric (BPS) configurations because, as we presented in the first chapter, they open up a string theory perspective on the puzzle of black holes entropy. Moreover, enforcing the vanishing of the supersymmetry transformations for fermions (as sketched in (1.22)), BPS configurations are expected to admit a simplified description in terms of first-order differential equations.

In this direction, the first results were obtained for static, spherically symmetric and magnetically charged black holes by Ferrara, Kallosh and Strominger [37] in $\mathcal{N}=2$ ungauged supergravity coupled to n_V vector multiplets.

As a first step, a suitable metric ansatz was chosen

$$\mathrm{d}s^2 = -e^{2U(r)}\mathrm{d}t^2 + e^{-2U(r)}\mathrm{d}\vec{x}^2, \tag{3.1}$$

for U an arbitrary function of the radial coordinate ($r=\sqrt{\vec{x}^2}$) only, and, via Bianchi identities ($\mathrm{d}\hat{F}=0$), the radial component of the magnetic field strengths were fixed to their values

$$\hat{F}^\Lambda_r = \frac{p^\Lambda}{r^2}e^{U(r)}, \quad \text{for} \quad \hat{F}^\Lambda_r \equiv 2\varepsilon_r{}^{\theta\phi}\hat{F}^\Lambda_{\theta\phi}. \tag{3.2}$$

At this point, inserting the obtained ansatze in the variations for the gravitino and the gaugino fields and demanding that they vanish for some choice of the supersymmetry parameter, one can derive the first-order differential equations:

$$4U' = -\sqrt{\frac{(\bar{Z}\mathcal{N}p)(Z\mathcal{N}p)(\bar{Z}\mathcal{N}Z)}{(\bar{Z}\mathcal{N}Z)(\bar{Z}\mathcal{N}\bar{Z})}}e^U, \tag{3.3}$$

$$(Z^\Lambda)' = -\frac{e^U}{4}\sqrt{\frac{(Z\mathcal{N}Z)(\bar{Z}\mathcal{N}p)(\bar{Z}\mathcal{N}Z)}{(\bar{Z}\mathcal{N}\bar{Z})(Z\mathcal{N}p)}}(Z^\Lambda p^0 - p^\Lambda), \tag{3.4}$$

where the primes denotes the derivative with respect to $\rho \equiv 1/r$, $Z^\Lambda(z^i)$ are the inhomogeneous special coordinates on the Special Kähler manifold (i.e., $Z^0=1$ and $Z^i=z^i$), p^Λ represent the magnetic charges and where we dropped contracted indices (e.g., the $Z\mathcal{N}p \equiv Z^\Lambda\mathcal{N}_{\Lambda\Sigma}p^\Sigma$).

By looking at (3.4), it is immediately clear that, once initial conditions are specified at infinity ($\rho=0$), Z^Λ will evolve until it reaches a fixed point. In particular, imposing that $(Z^\Lambda)'=0$, we find

$$Z^\Lambda_{\text{fixed}} = \frac{p^\Lambda}{p^0}, \tag{3.5}$$

so that the fixed points are completely specified in terms of the charges of the black hole. This was indeed the first example of the *attractor mechanism* mentioned in (1.3.2).

Additionally, upon integration of eq. (3.3), one finds that, at such fixed point, in the near-horizon limit, the solution corresponds to the maximally symmetric charged Bertotti–Robinson universe.

These results were soon generalised to arbitrary electric-magnetic charges by Strominger [43].

Shortly after, Ferrara, Gibbons and Kallosh [38] realized that regularity requirements on the solutions for scalar fields were sufficient to explain the attractor mechanism, without employing supersymmetry whatsoever.

To understand their result, let's start by considering the general Lagrangian described in (1.20), together with the ansatz for extremal black holes in eq. (3.1) expressed in terms of a new coordinate $\tau \equiv (r - r_H)^{-1}$

$$\mathrm{d}s^2 = -e^{2U}\mathrm{d}t^2 + e^{-2U}\left[\frac{\mathrm{d}\tau^2}{\tau^4} + \frac{1}{\tau^2}\mathrm{d}\Omega^2\right]. \tag{3.6}$$

As it is clear from the definition, in these coordinates the horizon sits at $\tau \to -\infty$.

Since we are interested in static, spherically symmetric solutions, one can actually reduce the system by integrating out formally the (t, θ, φ) coordinates, ending up with a one-dimensional effective Lagrangian [38]

$$\mathscr{L}_{1D} = \left(\frac{dU}{d\tau}\right)^2 + g_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau} + e^{2U}V_{BH}(\phi, p, q), \tag{3.7}$$

where we introduced the black hole potential V_{BH} arising in the integration procedure. It is defined as

$$V_{BH}(p, q, \phi^i) = -\frac{1}{2}Q^T\mathcal{M}Q \equiv -\frac{1}{2}Q^T\begin{pmatrix} \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix}Q, \quad \text{for} \quad Q \equiv \begin{pmatrix} p \\ q \end{pmatrix}. \tag{3.8}$$

For the solutions of the reduced system to actually solve the equations of motion of the complete theory, we need to add an additional constraint (usually referred to as the Hamiltonian constraint)

$$\left(\frac{dU}{d\tau}\right)^2 + g_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau} - e^{2U}V_{BH}(\phi, p, q) = 0. \tag{3.9}$$

By looking at the metric ansatz, in order to obtain a finite area solution, it is clear that it must hold:

$$e^{-2U} \to \left(\frac{A}{4\pi}\right)\tau^2 \quad \text{as} \quad \tau \to -\infty. \tag{3.10}$$

Moreover, we demand that the scalars kinetic term in the original Lagrangian remains finite as we approach the horizon, i.e.,

$$g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j = g_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau}e^{2U}\tau^4 < \infty, \tag{3.11}$$

which, together with (3.10) implies that

$$g_{ij}\frac{d\phi^i}{d\tau}\frac{d\phi^j}{d\tau}\left(\frac{4\pi}{A}\right)\tau^2 \to X^2 \quad \text{as} \quad \tau \to -\infty, \tag{3.12}$$

for X a real, finite constant. To better understand the physical consequences of such a requirement, it is useful to introduce a "proper-distance" coordinate $\omega \equiv -\log(-\tau)$[1], so that we obtain

$$g_{ij}\frac{d\phi^i}{d\omega}\frac{d\phi^j}{d\omega}\left(\frac{4\pi}{A}\right) \to X^2 \quad \text{as} \quad \omega \to -\infty. \tag{3.13}$$

It is therefore clear that the only compatible choice with the request of finite moduli near the horizon is $X^2 = 0$ and, as a consequence,

$$\left.\frac{d\phi^i}{d\omega}\right|_{\text{horizon}} = 0. \tag{3.14}$$

[1]In these coordinates, for extremal black holes the horizon sits at $\omega_H = -\infty$, at infinite proper distance from any observer.

Under this condition, solving the equations of motion for the scalars in the proximity of the horizon one gets

$$\phi^i = \left(\frac{2\pi}{A}\right)\frac{\partial V_{BH}}{\partial \phi^i}\log\tau + \phi^i_{\text{horizon}}. \tag{3.15}$$

Enforcing again a regular value of the scalars in the near horizon region, it follows that the fixed value of the scalars must be an extremum of the black-hole potential

$$\frac{\partial V_{BH}(p,q,\phi^i_{\text{horizon}})}{\partial \phi^i} = 0. \tag{3.16}$$

This last relation implicitly ties the values of the moduli at the horizon to the charges of the black holes.

Therefore, with no use of supersymmetry, we were able to prove that scalar fields must reach a fixed value on the black hole horizon and that this value is related to the charges of the black hole via (3.16). We stress how the whole construction heavily relies on the extremal nature of the black hole. The horizon of non-extremal black holes, in fact, sits at finite proper distance from an arbitrary observer, thus the previous intuition does not apply.

Specializing now the discussion to $\mathcal{N}=2$ supergravity, we interpret $\phi^i = z^i$ as the scalar fields sitting in the vector multiplets parametrizing a Special Kähler manifold, whose metric is denoted by $g_{i\bar{\jmath}}$. In this context, the black hole potential actually corresponds to the symplectic invariant I_1 of Special geometry [44]:

$$V_{BH} = I_1 \equiv |\mathcal{Z}(z,p,q)|^2 + |D_i\mathcal{Z}(z,p,q)|^2, \tag{3.17}$$

for $\mathcal{Z} \equiv \langle Q, V\rangle = (L^\Lambda q_\Lambda - M_\Lambda p^\Lambda)$ the $\mathcal{N}=2$ central charge. The one-dimensional Lagrangian and the Hamiltonian constraint now read

$$\mathscr{L}_{1D} = \left(\frac{dU}{d\tau}\right)^2 + g_{i\bar{\jmath}}\frac{dz^i}{d\tau}\frac{d\bar{z}^{\bar{\jmath}}}{d\tau} + e^{2U}\left(|\mathcal{Z}(z,p,q)|^2 + |D_i\mathcal{Z}(z,p,q)|^2\right), \tag{3.18}$$

$$\left(\frac{dU}{d\tau}\right)^2 + g_{i\bar{\jmath}}\frac{dz^i}{d\tau}\frac{d\bar{z}^{\bar{\jmath}}}{d\tau} - e^{2U}\left(|\mathcal{Z}(z,p,q)|^2 + |D_i\mathcal{Z}(z,p,q)|^2\right) = 0. \tag{3.19}$$

Making use of some properties of Special geometry, one can actually rewrite the Lagrangian as a sum of squares

$$\mathscr{L}_{1D} = \left(\frac{dU}{d\tau} \pm e^U|\mathcal{Z}|\right)^2 + \left|\frac{dz^i}{d\tau} \pm e^U g^{i\bar{k}}\bar{D}_{\bar{k}}\bar{\mathcal{Z}}\right|^2 \pm \frac{d}{d\tau}\left(e^U|\mathcal{Z}|\right), \tag{3.20}$$

from which we immediately read the first-order flow equations

$$\frac{dU}{d\tau} = \mp e^U|\mathcal{Z}| \qquad \text{and} \qquad \frac{dz^i}{d\tau} = \mp e^U g^{i\bar{k}}\bar{D}_{\bar{k}}\bar{\mathcal{Z}}. \tag{3.21}$$

Although both sign choices are in principle admitted, only the lower sign leads to meaningful black hole solutions. In particular, the flow equation for the warp factor should increase along the flow, as it approaches one at infinity and becomes proportional to $|\tau|$ near the horizon.

It can be shown that these flow equations derived without any input from supersymmetry precisely reproduce the ones reported in eqs. (3.3)-(3.4).

As a final remark, it is important to notice that the critical points of the black hole potential coincide with the critical point of the central charge $\mathcal{Z}$. In particular, extremal points such that

$$D_i\mathcal{Z} = 0 \quad \text{and} \quad \mathcal{Z} \neq 0 \tag{3.22}$$

give rise to supersymmetric black holes.

Non-BPS black holes and "Fake Superpotentials"

As we reviewed in the previous section, the attractor mechanism and the existence of first-order flow equations seem to be related to the extremal nature of black hole solutions rather than to their supersymmetry content. This realization, along with advancement in the study of non-BPS domain-wall solutions, inspired the quest for a first-order formalism also for non-supersymmetric extremal black holes.

Ceresole and Dall'Agata [29] found out that non-BPS configurations could indeed be described by first-order differential equations by replacing the central charge $\mathcal{Z}$ by another function $W(z)$, dubbed "*fake-superpotential*".

To see this, we can start by generalizing the results obtained before. In particular, we notice that the Hamiltonian constraint (3.9) actually relates the black hole potential, the derivatives of the scalar fields and the warp factor U. As a consequence, finding a real "superpotential" $W(z, \bar{z})$ such that

$$V_{BH} = W^2 + 4g^{i\bar{j}}\partial_i W \partial_{\bar{j}} W, \tag{3.23}$$

the Hamiltonian constraint is identically satisfied and the equations of motion take the form

$$\frac{dU}{d\tau} = \pm e^U W \qquad \text{and} \qquad \frac{dz^i}{d\tau} = \pm 2e^U g^{i\bar{k}}\partial_{\bar{k}} W. \tag{3.24}$$

This is evident upon rewriting the one-dimensional Lagrangian as

$$\mathscr{L}_{1D} = \left(U' \pm e^U W\right)^2 + \left|z^{i'} \pm 2e^U g^{i\bar{k}}\partial_{\bar{k}} W\right|^2 \mp \frac{d}{d\tau}\left(e^U W\right). \tag{3.25}$$

By identifying the superpotential W with $|\mathcal{Z}|$, this generalization immediately reduces to the BPS case that we analysed previously.

Going beyond the supersymmetric case, we are interested in theories for which the Hamiltonian constraint admits multiple solutions. In particular, the black hole potential might not univocally identify a superpotential, allowing for different equivalent choices of W, only one of which correspond to the central charge $\mathcal{Z}$. When such a "fake superpotential" exists (which is not simply proportional to $\mathcal{Z}$), the first-order equations would not imply anymore preserved supersymmetries since they would differ from the Killing spinor equations. Therefore, critical points of the form $\partial_i W = 0$ would give rise to stable non-BPS black holes.

Specializing our discussion for $\mathcal{N} = 2$ supergravity, we present one possible strategy for the construction of W [29]. In particular, starting from the matrix $\mathcal{M}$ appearing in the definition of V_{BH} (3.8), we can define an additional symplectic matrix M via the relation

$$\mathcal{M} = \Omega M = \Omega \begin{pmatrix} D & C \\ B & A \end{pmatrix}, \quad \text{for} \quad \Omega = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \tag{3.26}$$

and

$$A = -D^T = \mathcal{R}\mathcal{I}^{-1}, \quad C = \mathcal{I}^{-1}, \quad B = -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R}. \tag{3.27}$$

Notice that $M^2 = -\mathbb{I}$.

By looking again at the definition for V_{BH}, we see that we can perform a symplectic rotation of the charge vector $Q \to SQ$ without changing the value of the potential, provided that

$$S^T \mathcal{M} S = \mathcal{M} \implies S^T \Omega M S = \Omega M \implies [S, M] = 0, \tag{3.28}$$

where we made use of the fact that $S^T\Omega S=\Omega$.
Therefore, by looking at the relation among V_{BH} and the central charge (eq. (3.17)), we deduce that, anytime we find a constant symplectic matrix S that commutes with M, we can define a new "fake superpotential"

$$\mathcal{W}=Q^TS^T\Omega V \quad (\text{with } W=|\mathcal{W}|) \tag{3.29}$$

giving rise to the same potential as $\mathcal{Z}$ and such that its critical points describe non-supersymmetric black holes.

3.1.2 Multicentre solutions

After the study of single centre black holes in asymptotically flat spacetimes, it is natural to look for solutions that allow for multiple black holes sitting at different centres. These configurations are particularly interesting because they seem to play a significant role in the study of black hole physics within the String Theory framework, the aforementioned *microstate geometries* programme (chapter 1.2.2) being one representative example of application. Therefore, it is of compelling importance to study and classify both supersymmetric and non-supersymmetric multicentre solutions.

Supersymmetric case

Pivotal results in this direction were obtained by Denef [39], which succeeded in completely classifying supersymmetric, multicentre configurations. To summarize the main findings contained in its work, let's start by considering a general metric ansatz for stationary solutions

$$\mathrm{d}s^2=-e^{2U}\left(\mathrm{d}t+\omega_i\mathrm{d}x^i\right)^2+e^{-2U}\left(\mathrm{d}x^i\right)^2, \tag{3.30}$$

where both U and ω are arbitrary functions of $\vec{x}$, with the only requirement that U, $\omega\to 0$ as $r\to\infty$ to recover asymptotically Minkowski. Notice that setting $\omega=0$ and $r^2=\delta_{ij}x^ix^i$, we recover the single centre ansatz employed before.
By rewriting the action in the usual BPS squared form (just like it was done in eq. (3.20)), one finds that the entire gravitational solution for N sources sitting at positions $\vec{x}_I$, each with charges $(p^\Lambda_I, q_{\Lambda\,I})$, is described in terms of a symplectic vector of harmonic functions $\mathcal{H}$ of the form

$$\mathcal{H}=-\sum_{I=1}^{N}\frac{Q_I}{|\vec{x}-\vec{x}_I|}+2\mathrm{Im}\left(e^{-i\alpha}V\right)\big|_{r\to\infty}, \tag{3.31}$$

for V the usual covariantly holomorphic section of Special geometry, Q_i the vector of electric-magnetic charges of the i-th centre and α a real-valued function.
In particular, defining the 1-form

$$\zeta\equiv-\langle\mathrm{d}\mathcal{H},V\rangle=\sum_{I=1}^{N}\langle Q_I,V\rangle\ \mathrm{d}\tau_I=\sum_{I=1}^{N}\mathcal{Z}_I\ \mathrm{d}\tau_I, \quad \text{for} \quad \tau_I=\frac{1}{|\vec{x}-\vec{x}_I|}, \tag{3.32}$$

we can get the flow equations:

$$\mathrm{d}U = -e^U\mathrm{Re}\left(e^{-i\alpha}\zeta\right), \tag{3.33}$$

$$\mathrm{d}z^i = -e^Ug^{i\bar{j}}e^{i\alpha}\bar{D}_{\bar{j}}\zeta, \tag{3.34}$$

which precisely generalize the ones obtained in eq. (3.21) for the single centre case, together with the phase constraint

$$\mathcal{Q} + \mathrm{d}\alpha = e^{U}\mathrm{Im}\left(e^{i\alpha}\zeta\right) = -\frac{1}{2}e^{2U}\left\langle \mathrm{d}\mathcal{H}, \mathcal{H}\right\rangle, \tag{3.35}$$

for $\mathcal{Q}$ the U(1) connection on the Kähler manifold.
Explicitly, for $N=1$, identifying $\alpha = \arg \mathcal{Z}$, one gets

$$\zeta = \mathcal{Z}\mathrm{d}\tau, \tag{3.36}$$

and

$$\begin{aligned} \mathrm{d}U &= -e^{U}\left|\mathcal{Z}\right|\mathrm{d}\tau, \\ \mathrm{d}z^{i} &= -e^{U}g^{i\bar{\jmath}}e^{i\alpha}\bar{D}_{\bar{\jmath}}\bar{\mathcal{Z}}\mathrm{d}\tau = -2e^{U}g^{i\bar{\jmath}}\partial_{\bar{\jmath}}\left|\mathcal{Z}\right|\mathrm{d}\tau, \end{aligned} \tag{3.37}$$

where in the last equality we used the fact that $e^{2i\alpha} = \mathcal{Z}/\bar{\mathcal{Z}}$. Written in this form, the obtained equations exactly reproduce the ones for single centre configurations, as written in eq. (3.24).
Moreover, the additional non-static contribution to the metric is also determined starting from $\mathcal{H}$, via

$$\star_0\, \mathrm{d}\omega = \left\langle \mathrm{d}\mathcal{H}, \mathcal{H}\right\rangle, \tag{3.38}$$

for $\star_0$ the 3D Hodge dual with respect to the flat metric.
As a final remark, just like standard General Relativity, it is interesting to define the angular momentum vector $\vec{J}$ from the asymptotic form of the metric

$$\omega_i = 2\varepsilon_{ijk}J^{i}\frac{x^k}{r^3} + \mathcal{O}\left(\frac{1}{r^3}\right), \quad \text{as} \quad r\to\infty. \tag{3.39}$$

Employing the flow equation for ω, one finds that

$$\vec{J} = \frac{1}{2}\sum_{I<J}\left\langle Q_I, Q_J\right\rangle\,\frac{\vec{r}_{IJ}}{|\vec{r}_{IJ}|}, \tag{3.40}$$

for $\vec{r}_{IJ} \equiv \vec{x}_I - \vec{x}_J$. This implies that multicentre configurations can be characterized by *intrinsic* angular momentum, even when centres are at rest. This is actually a more general phenomenon, arising for example also in the context of ordinary electrodynamics.

Non-supersymmetric configurations: to String Theory and Back

Just like we did for the single centre solutions, we would like to understand how the results obtained for BPS multicentre configurations extend to the non-BPS case. Unfortunately, this problem has proven to be significantly more challenging then other studied scenarios since, even for the simplest solutions involving two centres, solving the non-linear Einstein's equations is too difficult of a task without any input coming from supersymmetry.
The first results were obtained by Goldstein and Katmadas [45] by noticing that the very same equations governing five and four dimensional supersymmetric multicentre black holes could also describe non-BPS solutions by opportunely changing a few signs. Actually, these type of solutions are usually called "almost-BPS" since, except for global constraints, they locally preserve supersymmetry.
A supergravity approach for constructing more general non-BPS solutions, encompassing also the advancements introduced in [46], was provided by Bena, Giusto, Ruef and Warner [40], who employed

a "floating brane" ansatz to simplify the problem. Specifically, they realized that an insightful perspective on non-supersymmetric configurations could be obtained by studying five dimensional $\mathrm{U}(1)^3$ supergravity solutions which could be uplifted to solutions of $D=11$ supergravity and that, when further compactified to four-dimensions, correspond to solutions for the STU model in $\mathcal{N}=2$ supergravity. By imposing that a probe M2 brane in the supergravity ansatz experiences no net force (hence the term "floating brane"), the equations of motion factorize, enabling the derivation of more general solutions. Finally, working in this framework, Dall'Agata, Giusto and Ruef [41] were able to generate new solutions via the action of U-duality, getting to the most general class of extremal non-BPS multicentre under-rotating[2] solutions of the STU model.

In the following, we will therefore review how these solutions are obtained, starting from the higher-dimensional supergravity theory and showing how the multicentre solutions are recovered by successive compactifications, first to five and then to four dimensions. Our presentation will mainly follow [41] and [10].

Consider eleven-dimensional supergravity[3] carrying M2 and M5 branes, together with KK6 monopoles and momentum charges. We will focus on compactifications on a Calabi-Yau and a circle, $CY_6 \times S^1$, for the specific choice

$$CY_6 = \frac{T^6}{\mathbb{Z}_2 \times \mathbb{Z}_2} \simeq (T^3)^2 \tag{3.41}$$

since this configuration leads to the STU model in four-dimensions.

The ansatze for both the metric and the 3-form potential from the M-theory perspective reads:

$$\mathrm{d}s^2_{11} = -Z^{-2}(\mathrm{d}t+k)^2 + Z\,\mathrm{d}s^2_4(x) + \sum_{I=1}^{3} \frac{Z}{Z^I}\,\mathrm{d}s^2_I, \qquad A^{(3)} = \sum_{I=1}^{2} \left(-\frac{\mathrm{d}t+k}{Z_I} + a_I \right) \wedge \mathrm{d}T_I, \tag{3.42}$$

for $(t,\vec{x})$ the coordinates of 4-dimensional spacetime, Z_I the warp factors which define $Z \equiv (Z_1 Z_2 Z_3)^{1/3}$, $\mathrm{d}s^2_4$ denotes the metric on a Ricci-flat 4-dimensional Euclidean space and $\mathrm{d}s^2_I$ and $\mathrm{d}T_I$ are respectively the metric and the volume form on the I-th 2-torus.

Subsequently, we choose to specialize $\mathrm{d}s^2_4$ to the one for a Gibbons–Hawking space, i.e.

$$\mathrm{d}s^2_4 = \frac{1}{V}\left(\mathrm{d}\psi + \vec{A}\right)^2 + V\,\mathrm{d}s^2_3(\vec{x}), \quad \text{where} \quad \star\,\mathrm{d}A = \pm \mathrm{d}V, \tag{3.43}$$

$\mathrm{d}s^2_3(\vec{x})$ denotes the 3-dimensional flat space, ψ ia a U(1) isometry of the metric and V is the harmonic function generating the NUT charge. Each sign choice corresponds to a different orientation of the spacetime manifold and, consequently, to different solutions. In particular, the plus sign corresponds to BPS configurations while the minus sign is related to non-supersymmetric solutions.

Finally, having specified the four-dimensional base space, we can also specialize the one-forms a_I and k appearing in the metric ansatz

$$a_I = P_I\,(\mathrm{d}\psi + A) + w^I\ , \quad k = \mu\,(\mathrm{d}\psi + A) + \omega\ , \tag{3.44}$$

for w^I and ω 1-forms on $\mathrm{d}s^2_3(\vec{x})$.

At this point, one can explicitly write down the equations of motion that regulate the solutions of

[2]Geometries of the under-rotating type are the ones that, like the extremal Reisner–Nordström black hole, have a conformally flat three-dimensional base.

[3]We briefly recall that $D=11$ supergravity describes a graviton, a gravitino and a 3-form gauge field.

the eleven-dimensional supergravity theory for the BPS case, which read

$$\begin{aligned} \mathrm{d}\star\mathrm{d}Z_I &= \frac{C_{IJK}}{2}\mathrm{d}\star\mathrm{d}\left(VP_JP_K\right), &(3.45)\\ \star\mathrm{d}w^I &= -\mathrm{d}\left(VP_I\right), &(3.46)\\ \star\mathrm{d}\omega &= V\mathrm{d}\mu-\mu\mathrm{d}V-VZ_I\mathrm{d}P_I, &(3.47)\end{aligned}$$

and the non-BPS case, for which we have

$$\begin{aligned} \mathrm{d}\star\mathrm{d}Z_I &= \frac{C_{IJK}}{2}V\mathrm{d}\star\mathrm{d}\left(P_JP_K\right), &(3.48)\\ \star\mathrm{d}w^I &= P_I\mathrm{d}V-V\mathrm{d}P_I, &(3.49)\\ \star\mathrm{d}\omega &= \mathrm{d}(\mu V)-VZ_I\mathrm{d}P_I. &(3.50)\end{aligned}$$

As anticipated before, the resulting theory obtained from the reduction on the circle identified by the ψ direction and on the CY_6 could be described via the $\mathcal{N}=2$ STU supergravity model, i.e., a theory coupled to three vector multiplets whose scalar manifold is given by

$$\mathcal{M}_{\text{scalar}}=\left[\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right]^3. \tag{3.51}$$

The 11-dimensional ansatze employed before imply a constrained form for both the 4-dimensional metric and the three scalar fields, which will explicitly read

$$\mathrm{d}s^2_{4d}=-e^{2U}\left(\mathrm{d}t+\omega\right)^2+e^{-2U}\mathrm{d}s^2_3(\vec{x})\,,\quad\text{for}\quad e^{-2U}=\sqrt{VZ^3-V^2\mu^2} \tag{3.52}$$

and

$$z^I=\frac{\left(VZ_IP_I-V\mu\right)-ie^{-2U}}{VZ_I}, \tag{3.53}$$

where Z_IP_I is not summed over I. Similar expressions could be also found for the vector fields A^Λ. As long as the BPS case is concerned, one can explicitly verify that the equations of motion (3.45)-(3.47) admit solutions in terms of 8 harmonic functions $\left\{H^\Lambda,H_\Lambda\right\}$, exactly reproducing the results obtained by Denef [39].

For the non-supersymmetric case, on the other hand, the full solution (which can be found in [46] and we do not report here for conciseness) cannot be fully expressed in terms of harmonic functions. Moreover, it can be shown that, in this case, regularity conditions of the solutions translate into equations for the distance between the centres, implying that such configurations describe bound states of black holes.

To conclude, we would like to analyse the role of U-duality transformations on the obtained solutions. Specifically, as shown in [41], duality transformations acting on BPS solutions simply rotate the harmonic functions among themselves. More interestingly, the same transformations act non-trivially on non-supersymmetric solutions. This implies that, in principle, one could choose a specific "seed" solution and, via U-duality, construct new non-BPS configurations.

The construction we have presented here for a compactification on $\left(T^3\right)^2$ which leads to the STU model can be clearly generalized by following the standard procedures for obtaining $\mathcal{N}=2$ four-dimensional supergravity theories from Calabi–Yau compactifications.

3.2 Anti-de Sitter Black Holes

Up to this point we have focused our attention on ungauged supergravity theories and black hole solutions in asymptotically flat space-times.
In this section, mainly following [42], we will be interested instead in BPS black holes in four-dimensional anti-de Sitter space (AdS), where the presence of a non-trivial cosmological constant is made possible by suitable gauging choices.
Motivations for studying supersymmetric AdS black holes arises in different context. Firstly, they provide an additional benchmark for the universality of the attractor mechanism. Secondly, in the light of $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence, controlling the gravitational solution is fundamental for the computation of the microscopic entropy and for a comparison with Bekenstein–Hawking formula. Lastly, within the framework of flux compactifications in String theory, it is important to determine whether the attractor mechanism is still at work in gauged supergravity theories, so as not to destabilize the vacuum.
The first results for supersymmetric, static black hole solutions in AdS backgrounds were obtained in [47,48], although both works are significantly limited by the assumption of constant scalars profiles, which led to naked singularities and irregular geometries. A major breakthrough was obtained by Cacciatori and Klemm [49], who focussed on $\mathcal{N}=2$ gauged supergravity with vector multiplets only and U(1) gauging provided by Fayet–Iliopoulos terms. Their results were later fully generalized by Dall'Agata and Gnecchi [42], providing a completely covariant solution that allows for both electric and magnetic gauging, and describing the entire black hole solutions by means of first-order flow equations governed by a superpotential W, echoing the approach adopted for the asymptotically flat case.
Therefore, in the final part of this chapter, we will review the results of [42] to gain a deeper understanding of the physical content of this class of solutions, and to establish the starting point for the results we will present in the next chapter.
As anticipated, let's start by considering dyonic black hole solutions of $\mathcal{N}=2$ U(1) gauged supergravity coupled to n_V vector multiplets, a linear combination of which will gauge a U(1) factor via Fayet–Iliopoulos (FI) terms. Starting from the general Lagrangian in eq. (2.77), for this class of models we can write

$$\mathscr{L} \quad = \quad \frac{1}{2}R - g_{i\bar{\jmath}}\partial^{\mu} z^i \partial_{\mu}\bar{z}^{\bar{\jmath}} + \frac{1}{4}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma\,\mu\nu} + \frac{1}{4}\mathrm{Re}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}\tilde{F}^{\Sigma\,\mu\nu} - V_g, \tag{3.54}$$

where the scalar potential can be written in terms of the superpotential $\mathcal{L}$ as

$$V_g = g^{i\bar{\jmath}}D_i\mathcal{L}\bar{D}_{\bar{\jmath}}\bar{\mathcal{L}} - 3|\mathcal{L}|^2, \quad \text{for} \quad \mathcal{L} \equiv \langle \mathcal{G}, \mathcal{V}\rangle\,, \tag{3.55}$$

where $\mathcal{G} \equiv \left(g^{\Lambda}, g_{\Lambda}\right)$ denotes the FI terms and $\mathcal{V}$ is the usual covariantly holomorphic symplectic section of Special geometry.
At this point, we specify the metric ansatz

$$\mathrm{d}s^2 = -e^{2U}\mathrm{d}t^2 + e^{-2U}\left(\mathrm{d}r^2 + e^{2\psi}\mathrm{d}\Omega^2\right) \tag{3.56}$$

and the appropriate expressions for the vector fields, so that

$$\int_{S^2} F^{\Lambda} = 4\pi p^{\Lambda}, \qquad \int_{S^2} G_{\Lambda} = 4\pi q_{\Lambda} \quad (\text{where } G_{\Lambda} \equiv \frac{\delta\mathscr{L}}{\delta F^{\Lambda}}), \tag{3.57}$$

for $Q \equiv \left(p^\Lambda, q_\Lambda\right)$ the black hole magnetic and electric charges. As always, we will assume spherical symmetry of the solution, so that the scalars $z^i(r)$ and the warp factors $U(r)$ and $\psi(r)$ are functions of the radial coordinate only.

Just like we did for the asymptotically flat case, by employing the anstaze introduced before, we can obtain a 1-dimensional effective action for the theory

$$\begin{aligned} S_{1d} &= \int dr \left\{ e^{2\psi}\left[U'^2 - \psi'^2 + g_{i\bar{j}} z^{i\prime} \bar{z}^{\bar{j}\prime} + e^{2U-4\psi} V_{BH} + e^{-2U} V_g\right] - 1\right\} \\ &\quad + \int dr \frac{d}{dr}\left[e^{2\psi}\left(2\psi' - U'\right)\right], \end{aligned} \tag{3.58}$$

where the primes denote derivatives with respect to the radial coordinate and the black hole potential is the same as eq. (3.17).

To extract the first-order flow equations describing the solutions, one could rewrite the action as a sum of BPS squares by using Special geometry identities. The result of such an approach gives

$$\begin{aligned} S_{1d} = \int dr \Big\{ & -\frac{1}{2} e^{2(U-\psi)} \mathcal{E}^T \mathcal{M} \mathcal{E} - e^{2\psi}\left[\left(\alpha' + \mathcal{Q}_r\right) + 2e^{-U} \operatorname{Re}\left(e^{-i\alpha}\mathcal{L}\right)\right]^2 \\ & - e^{2\psi}\left[\psi' - 2e^{-U} \operatorname{Im}\left(e^{-i\alpha}\mathcal{L}\right)\right]^2 - \left(1 + \langle \mathcal{G}, Q\rangle\right) \\ & -2\frac{d}{dr}\left[e^{2\psi-U} \operatorname{Im}\left(e^{-i\alpha}\mathcal{L}\right) + e^{U} \operatorname{Re}\left(e^{-i\alpha}\mathcal{Z}\right)\right]\Big\}, \end{aligned} \tag{3.59}$$

where we introduced

$$\mathcal{E}^T \equiv 2e^{2\psi}\left(e^{-U}\operatorname{Im}\left(e^{-i\alpha}\mathcal{V}\right)\right)'^T - e^{2(\psi-U)}\mathcal{G}^T\Omega\mathcal{M}^{-1} + 4e^{-U}\left(\alpha' + \mathcal{Q}_r\right)\operatorname{Re}\left(e^{-i\alpha}\mathcal{V}\right)^T + Q^T, \tag{3.60}$$

for $\mathcal{Q}$ the U(1) composite connection of Special geometry.

Therefore, the action has been rewritten as a sum of squares, provided that the charges fulfill the constraint

$$\langle \mathcal{G}, Q\rangle = -1. \tag{3.61}$$

The resulting flow equations are

$$\begin{aligned} U' &= -e^{U-2\psi}\operatorname{Re}\left(e^{-i\alpha}\mathcal{Z}\right) + e^{-U}\operatorname{Im}\left(e^{-i\alpha}\mathcal{L}\right), \\ \psi' &= 2e^{-U}\operatorname{Im}\left(e^{-i\alpha}\mathcal{L}\right), \\ z^{i\prime} &= -e^{i\alpha} g^{i\bar{j}}\left[e^{U-2\psi} D_{\bar{j}}\bar{\mathcal{Z}} + i e^{-U} D_{\bar{j}}\bar{\mathcal{L}}\right], \end{aligned} \tag{3.62}$$

together with the phase constraint

$$e^{2i\alpha} = \frac{\mathcal{Z} - i e^{2(\psi-U)}\mathcal{L}}{\overline{\mathcal{Z}} + i e^{2(\psi-U)}\overline{\mathcal{L}}}. \tag{3.63}$$

An important outcome of this analysis is that, just like we saw for the ungauged case, we can introduce a superpotential

$$W = e^U \left|\mathcal{Z} - i e^{2(\psi-U)}\mathcal{L}\right| \tag{3.64}$$

so that the flow equations become:

$$\begin{aligned} U' &= -g^{UU}\partial_U W, \\ \psi' &= -g^{\psi\psi}\partial_\psi W, \\ z^{i\prime} &= -2\tilde{g}^{i\bar{j}}\partial_{\bar{j}} W, \end{aligned} \tag{3.65}$$

for $g_{UU}=-g_{\psi\psi}=e^{2\psi}$, $\tilde{g}_{i\bar{\jmath}}=e^{2\psi}g_{i\bar{\jmath}}$. Notice that, for $\mathcal{G}=0$, i.e. turning off the gauging, we recover exactly the ungauged superpotential.

As we already mentioned, the same flow equations could be actually obtained also by imposing that the supersymmetry variations for the fermionic fields vanish. In particular, by finding suitable projectors relating spinorial components of the supersymmetry parameters, it is possible to translate supergravity variations to first-order differential equations for the bosonic content of the theory. Moreover, each independent projector condition will halve the number of preserved supersymmetry. In this particular case, the relevant supersymmetry variations could be obtained by adapting the general ones in eqs. (2.82)-(2.83):

$$\begin{aligned}\delta\psi_{\mu A} &= D_\mu\epsilon_A-\varepsilon_{AB}T^-_{\mu\nu}\gamma^\nu\epsilon^B-\frac{i}{2}\mathcal{L}\delta_{AB}\gamma^\nu\eta_{\mu\nu}\epsilon^B,\\ \delta\lambda^{iA} &= -i\partial_\mu z^i\gamma^\mu\epsilon^A+G^{-i}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^{AB}\epsilon_B+\bar{D}^i\overline{\mathcal{L}}\delta^{AB}\epsilon_B.\end{aligned} \tag{3.66}$$

Employing the metric ansatz in (3.56) and taking

$$\begin{aligned}F^\Lambda_{tr} &= \frac{e^{2U-2\psi}}{2}(\mathcal{I}^{-1})^{\Lambda\Sigma}\left(\mathcal{R}_{\Sigma\Gamma}p^\Gamma-q_\Sigma\right),\\ F^\Lambda_{\theta\phi} &= -\frac{1}{2}p^\Lambda\sin\theta\end{aligned} \tag{3.67}$$

as ansatz for the field strengths, we can explicitly write down the variations.

Let's start by considering, for example, the time component of the gravitino $\delta\psi_{tA}=0$:

$$\frac{1}{2}e^{2U}U'\gamma^{01}\epsilon_A+\frac{1}{2}A^\Lambda_t g_\Lambda\delta_{AC}\varepsilon^{CB}\epsilon_B+\frac{i}{2}e^{3U-2\psi}\mathcal{Z}\gamma^1\varepsilon_{AB}\epsilon^B-\frac{i}{2}e^U\mathcal{L}\delta_{AB}\gamma^0\epsilon^B=0. \tag{3.68}$$

To actually identify the right projectors it is convenient to rewrite the equation in the following fashion:

$$U'\epsilon_A=e^{-2U}A^\Lambda_t g_\Lambda\delta_{AC}\gamma^1\gamma^0\varepsilon^{CB}\epsilon_B+ie^{U-2\psi}\mathcal{Z}\gamma^0\varepsilon_{AB}\epsilon^B-ie^{-U}\mathcal{L}\delta_{AB}\gamma^1\epsilon^B. \tag{3.69}$$

Written in this form, it is evident that by imposing the two projector conditions

$$\gamma^0\epsilon_A=ie^{i\alpha}\varepsilon_{AB}\epsilon^B \quad \text{and} \quad \gamma^1\epsilon_A=e^{i\alpha}\delta_{AB}\epsilon^B, \tag{3.70}$$

we can rewrite $\delta\psi_{tA}=0$ as a single differential equation multiplying the same spinor ϵ_A. In particular, we obtain

$$\left(-U'+ie^{-2U}A^\Lambda_t g_\Lambda-e^{U-2\psi}\mathcal{Z}e^{-i\alpha}-ie^{-U}\mathcal{L}e^{-i\alpha}\right)\epsilon_A=0, \tag{3.71}$$

whose real part matches precisely the first equation in (3.62). By similar arguments applied to the other variations, employing again the same projectors in eq. (3.70), one is able to reproduce all the flow equations (as described thoroughly in Appendix A, [42]).

Chapter 4

Black Holes in $\mathcal{N} = 2$ U(1) Gauged Supergravity with Hypermultiplets

In the previous chapter, our review highlighted a significant gap in the current understanding of black hole physics within supergravity theories: the absence of a systematic study of the equations governing solutions that include *hypermultiplets*.
Hypermultiplets are expected to arise naturally in generic string compactifications, making their inclusion essential for a complete description of black hole solutions. Unlike ungauged supergravity, in which their dynamics decouples from the rest of the system, in gauged theories, quaternionic scalars may be charged and thus actively participate in the solutions. Despite some progress in this direction, such as the construction of new solutions with non-trivial hypermultiplets in [50], a comprehensive treatment using a superpotential has yet to be developed.
In this chapter, we aim at filling this gap by extending the work of [42], developing a general framework that could describe black hole solutions in supergravity theories with Abelian U(1) gaugings coupled to both vector multiplets and hypermultiplets.

We begin by illustrating how the general $\mathcal{N} = 2$ theory adapts to the specific gauging choices of our interests. Then, we outline the key properties of the solutions we seek to describe. With these foundations in place, we then proceed to write down the most general supersymmetry variations, specifying the ansatze for the solutions. Subsequently, we will identify the necessary projectors required to extract first-order BPS equations that govern the solution. These equations will naturally reduce to those found in [42] when the hypermultiplets are turned off. Finally, we present some attempts to construct explicit solutions for simple cases involving one hypermultiplet and one vector multiplet, describing the general procedure for finding vacua whenever moduli parametrize coset manifolds and the limitations that the simple models considered present.

4.1 $\mathcal{N} = 2$, $D = 4$ supergravity with Abelian gaugings

In this chapter we consider 4-dimensional $\mathcal{N} = 2$ gauged supergravity theories coupled to n_V vector multiplets and n_H hypermultiplets, specifically focusing on U(1) Abelian gaugings of the quaternionic-Kähler manifold isometry group. Therefore, we begin our examination by detailing how the general theory presented in chapter 2 is adapted to this specific setup.

One direct simplification arises at the Lagrangian level and in the supersymmetry variations for the gaugini. In particular, the choice to only consider Abelian gaugings of the quaternionic isometries effectively implies the cancellation of all the terms proportional to the holomorphic Killing vectors k^i_Λ. Therefore, recalling the general expressions reported in eqs. (2.78), (2.79) and (2.83), and removing the book-keeping gauge parameter g (that is, setting it to one), we obtain

$$\mathscr{L}_k = \frac{1}{2}R - g_{i\bar{\jmath}}\partial^\mu z^i\partial_\mu \bar{z}^{\bar{\jmath}} - h_{uv}\nabla_\mu q^u \nabla^\mu q^v + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\Sigma\,\mu\nu} + \frac{1}{4}\mathcal{R}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}\tilde{F}^{\Sigma\,\mu\nu}, \tag{4.1}$$

$$V(z,\bar{z},q) = 4h_{uv}k^u_\Lambda k^v_\Sigma \bar{L}^\Lambda L^\Sigma + \left(g^{i\bar{\jmath}}f^\Lambda_i f^\Sigma_{\bar{\jmath}} - 3\bar{L}^\Lambda L^\Sigma\right)\mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma \tag{4.2}$$

and

$$\delta\lambda^{iA} = -i\partial_\mu z^i\gamma^\mu\epsilon^A + G^{-i}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^{AB}\epsilon_B + W^{iAB}\epsilon_B, \tag{4.3}$$

for W^{iAB} that reduces to

$$W^{i\,AB} = i(\sigma_x)_C{}^B\varepsilon^{CA}\mathcal{P}^x_\Lambda g^{i\bar{\jmath}}\bar{f}^\Lambda_{\bar{\jmath}}. \tag{4.4}$$

The other variations, instead, maintain their general forms presented in (2.82) and (2.84).
Now, let's consider the equivariance condition for the triholomorphic prepotentials, which in components is given by

$$\frac{1}{2}\varepsilon^{xyz}\mathcal{P}^y_\Lambda\mathcal{P}^z_\Sigma - \Omega^x_{uv}k^u_\Lambda k^v_\Sigma = \frac{1}{2}f^\Delta{}_{\Lambda\Sigma}\mathcal{P}^x_\Delta. \tag{4.5}$$

The request of having an Abelian gauge group immediately translates into the need for vanishing structure constants, i.e., $f^\Delta{}_{\Lambda\Sigma}=0$.
Additionally, the expression can be further simplified as we will require that at least one of the gauged isometries remains unbroken on the vacuum, i.e., $k^u_\Lambda = 0$ for some choice of Λ. The need for such a condition can be understood by examining the kinetic term for the quaternionic scalars:

$$\mathscr{L}_k \supset h_{uv}\nabla_\mu q^u\nabla^\mu q^v = ... + h_{uv}\,k^u_\Lambda\,k^v_\Sigma\;A^\Lambda_\mu A^{\Sigma\,\mu}, \tag{4.6}$$

where we extracted the contribution to the vector field masses arising in the gauging procedure.
It is therefore clear that the request of an unbroken isometry on the vacuum ensures that at least one vector field A^μ_Λ continues to be massless, allowing for a consistent definition of the black holes charges.
As a consequence, assuming without any loss of generality that $k^u_1 = 0$ along the solution, since we will focus on U(1)×U(1) gaugings, the only non-trivial contribution proportional to the SU(2) curvature will be of the form $\Omega^x_{uv}k^u_2 k^v_2$, which vanishes due to antisymmetry.
It is important to note that this condition poses stronger constraints than simply requiring the Killing vectors to commute.
Combining these two requests, eq.(4.5) becomes

$$\varepsilon^{xyz}\mathcal{P}^y_\Lambda\mathcal{P}^z_\Sigma = 0 \tag{4.7}$$

or, equivalently,

$$\vec{\mathcal{P}}_\Lambda \times \vec{\mathcal{P}}_\Sigma = 0. \tag{4.8}$$

The prepotentials are therefore *parallel* in SU(2) space and they admit a decomposition of the type

$$\mathcal{P}^x_\Lambda = g_\Lambda(q)Q^x(q), \tag{4.9}$$

for Q^x identifying a direction in SU(2) and satisfying $Q^x Q^x = 1$.
Introducing Pauli matrices, we can also define

$$Q_A{}^B = i(Q^x \sigma_x)_A{}^B, \tag{4.10}$$

so that

$$Q_A{}^B Q_B{}^C = -\delta_A^C \quad \text{and} \quad Q_{AB} Q^{BC} = \delta_A^C, \tag{4.11}$$

as it follows from $(\sigma_x)_A{}^B (\sigma_y)_B{}^C = \delta_{xy}\delta_A^C + i\varepsilon_{xyz}(\sigma_z)_A{}^C$.
Additionally, we can define $\mathcal{L}^x \equiv \mathcal{P}^x_\Lambda L^\Lambda$ and its complex conjugate $\overline{\mathcal{L}}^x \equiv \mathcal{P}^x_\Lambda \bar{L}^\Lambda$. Exploiting the decomposition in eq. (4.9), these quantities can be also rewritten as

$$\mathcal{L}^x = g_\Lambda\, L^\Lambda\, Q^x \equiv\, \mathcal{L}\, Q^x \qquad \text{and} \qquad \overline{\mathcal{L}}^x = g_\Lambda\, \bar{L}^\Lambda\, Q^x \equiv\, \bar{\mathcal{L}}\, Q^x. \tag{4.12}$$

Finally, it is useful for our purposes to derive a simple equality for $\mathcal{L}$. Consider the SU(2)-covariant derivative acting on $\mathcal{L}^x$

$$\begin{aligned} \nabla_u \mathcal{L}^x &= (\partial_u \mathcal{L})\, Q^x + \mathcal{L}\nabla_u Q^x \\ &= L^\Lambda \nabla_u \mathcal{P}^x_\Lambda, \end{aligned} \tag{4.13}$$

where in the first line we have used the decomposition introduced in (4.12) while the second follows from the $\mathcal{L}^x$ definition itself.
Then, since $Q^x \nabla_u Q^x = 0$ as a direct consequence of $Q^x Q^x = 1$, contracting both sides of the previous relation with Q^x we end up with

$$\partial_u \mathcal{L} = L^\Lambda Q^x \nabla_u \mathcal{P}^x_\Lambda. \tag{4.14}$$

Finally, recalling the defining relation for the prepotentials given in eq. (2.71), we get

$$\partial_u \mathcal{L} = -2 L^\Lambda Q^x k^v_\Lambda \Omega^x_{vu}. \tag{4.15}$$

Similar calculations lead to

$$\partial_u \bar{\mathcal{L}} = -2 \bar{L}^\Lambda Q^x k^v_\Lambda \Omega^x_{vu}. \tag{4.16}$$

4.2 Analysis of Supersymmetry Variations

Working within the context of four-dimensional $\mathcal{N} = 2$ U(1) gauged supergravity coupled to both vector multiplets and hypermultiplets, we now present the general analysis of supergravity variations of the fermionic fields and how these lead to a set of differential equations describing supersymmetric black hole configurations.
Specifically, after outlining the properties of the solutions we aim to describe, we will choose appropriate ansatze for the metric and the vector fields, specify the supergravity variations and identify the projection conditions that leads to BPS equations. To conclude, we will show how the whole solution can be described in terms of a suitable superpotential W.

4.2.1 Expected properties of the solutions and Ansatze

We are interested in the analysis of static, spherically symmetric, single centre black holes in an asymptotically flat spacetime. Notice that, working in the context of gauged supergravity, the last assumption actually poses a constraint on the scalar potential given in eq. (4.2). In particular, following the general treatment presented in chapter 2, $V(z,\bar{z},q)$ should exhibit a minimum $(z_0^i, \bar{z}_0^{\bar{\imath}}, q_0^u)$ such that

$$V(z_0,\bar{z}_0,q_0)=0. \tag{4.17}$$

Additionally, as mentioned in the derivation of eq. (4.7), we require that at least one of the gauged isometries (or, equivalently, a linear combination of them) remains unbroken on the vacuum, i.e., $k_\Lambda^u=0$ for some choice of Λ.
As a last request, we would like our solution to preserve supersymmetry, at least partially.
We remark that the combined request of having a vanishing scalar potential and some residual supersymmetry on the solution could be only fulfilled in the presence of hypermultiplets. In particular, setting $n_H=0$, we can still obtain a $\mathrm{U}(1)^{n_V+1}$ gauging via Fayet-Iliopoulos terms

$$\mathcal{P}_\Lambda^x=\xi_\Lambda^x, \tag{4.18}$$

which correspond to constant gauge prepotentials. The scalar potential would then read

$$V(z,\bar{z}) \quad = \quad \left(g^{i\bar{\jmath}}f_i^\Lambda f_{\bar{\jmath}}^\Sigma-3\bar{L}^\Lambda L^\Sigma\right)\xi_\Lambda^x\xi_\Sigma^x. \tag{4.19}$$

Although a particular choice of the prepotential $F(X)$ on the Special manifold could possible result in $V=0$, it has been proven [51] that this class of models completely breaks supersymmetry.

Ansatze for the solution

Consider the generic ansatz for the metric

$$\mathrm{d}s^2=-e^{2U(r)}\mathrm{d}t^2+e^{-2U(r)}\left(\mathrm{d}r^2+e^{2\psi(r)}\mathrm{d}\Omega^2\right), \tag{4.20}$$

valid for static, spherically symmetric, single centre black hole solutions. Working in the framework of gauged supergravity, the additional warp factor ψ is required to compensate for the curvature contributions expected to arise from the varying non-trivial cosmological constant.
A suitable vielbein choice for this metric is given by

$$e^0=e^U\mathrm{d}t, \qquad e^1=e^{-U}\mathrm{d}r, \qquad e^2=e^{\psi-U}\mathrm{d}\theta, \qquad e^3=e^{\psi-U}\sin\theta\,\mathrm{d}\varphi, \tag{4.21}$$

so that $g_{\mu\nu}=e_\mu^a e_\nu^b\eta_{ab}$, for η_{ab} the Minkowski metric.
Following the standard procedure, we can derive the spin-connection from the torsion-free condition

$$De^a=\mathrm{d}e^a+\omega^a{}_b\wedge e^b=0\,, \tag{4.22}$$

from which we get

$$\begin{gathered}\omega^0{}_1=U'\,e^{2U}\mathrm{d}t, \qquad \omega^0{}_2=\omega^0{}_3=0,\\ \omega^1{}_2=\left(U'-\psi'\right)e^\psi\mathrm{d}\theta, \qquad \omega^1{}_3=\left(U'-\psi'\right)e^\psi\sin\theta\mathrm{d}\varphi,\\ \omega^2{}_3=-\cos\theta\mathrm{d}\varphi,\end{gathered} \tag{4.23}$$

where the primes denote derivatives with respect to the radial coordinate.
As long as the vector fields are concerned, for spherically symmetric solutions we can introduce electric potentials χ^Λ and write

$$A^\Lambda=\chi^\Lambda(r)\,\mathrm{d}t+p^\Lambda\cos\theta\,\mathrm{d}\varphi, \tag{4.24}$$

so that

$$F^\Lambda = \mathrm{d}A^\Lambda = \left(\chi^\Lambda\right)' \mathrm{d}r \wedge \mathrm{d}t - p^\Lambda \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\varphi \quad \text{and} \quad \int_{S^2} F^\Lambda = 4\pi p^\Lambda. \tag{4.25}$$

An analogous construction could be also carried out for the dual fields A_Λ, introducing magnetic potentials ϕ_Λ and

$$A_\Lambda = \phi_\Lambda(r)\, \mathrm{d}t + q^\Lambda \cos\theta\, \mathrm{d}\varphi, \tag{4.26}$$

from which

$$G_\Lambda \equiv \frac{\delta \mathscr{L}}{\delta F^\Lambda} = \mathrm{d}A_\Lambda \quad \text{and} \quad \int_{S^2} G_\Lambda = 4\pi q_\Lambda. \tag{4.27}$$

By employing the equations of motion, one could actually integrate out the electric potentials and rewrite them in terms of the charges and the real/imaginary part of the gauge kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$. Specifically, imposing that

$$\mathrm{d}G_\Lambda = \mathrm{d}\left(\mathcal{R}_{\Lambda\Sigma} F^\Sigma - \mathcal{I}_{\Lambda\Sigma} \tilde{F}^\Sigma\right) = 0 \tag{4.28}$$

and enforcing the duality relation between F^Λ and G_Λ, one gets

$$\left(\chi^\Lambda\right)' = e^{2U-2\psi} \left(\mathcal{I}^{-1}\right)^{\Lambda\Sigma} \left(q_\Sigma - \mathcal{R}_{\Sigma\Gamma} p^\Gamma\right). \tag{4.29}$$

Therefore, in components

$$\begin{aligned} F^\Lambda_{tr} &= \frac{1}{2} e^{2U-2\psi} \left(\mathcal{I}^{-1}\right)^{\Lambda\Sigma} \left(\mathcal{R}_{\Sigma\Gamma} p^\Gamma - q_\Sigma\right), \\ F^\Lambda_{\theta\varphi} &= -\frac{1}{2} p^\Lambda \sin\theta. \end{aligned} \tag{4.30}$$

As we have seen in chapter 2, in the supersymmetry transformation rules the field strengths actually appear in their dressed versions, as defined in (2.85).
Thus, as a first step, we introduce the antiself-dual combination

$$F^{-\Lambda}_{\mu\nu} = \frac{1}{2}\left(F^\Lambda_{\mu\nu} - \frac{i}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\Lambda\rho\sigma}\right). \tag{4.31}$$

Starting from the field strengths components in eq. (4.30) and the metric ansatz, we can write

$$\begin{aligned} F^{-\Lambda}_{tr} &= \frac{1}{4} e^{2U-2\psi} \left[\left(\mathcal{I}^{-1}\right)^{\Lambda\Sigma} \left(\mathcal{R}_{\Sigma\Gamma} p^\Gamma - q_\Sigma\right) + i p^\Lambda\right], \\ F^{-\Lambda}_{\theta\varphi} &= \frac{i}{4} \sin\theta \left[\left(\mathcal{I}^{-1}\right)^{\Lambda\Sigma} \left(\mathcal{R}_{\Sigma\Gamma} p^\Gamma - q_\Sigma\right) + i p^\Lambda\right]. \end{aligned} \tag{4.32}$$

At this point, we can explicitly compute

$$\begin{aligned} T^-_{tr} &= 2i \mathcal{I}_{\Lambda\Sigma} L^\Sigma F^{-\Lambda}_{tr} \\ &= \frac{i}{2} e^{2U-2\psi} \left[L^\Sigma (\mathcal{R}_{\Sigma\Gamma} + i \mathcal{I}_{\Sigma\Gamma}) p^\Gamma - L^\Sigma q_\Sigma\right] \\ &= \frac{i}{2} e^{2U-2\psi} \left[L^\Sigma \mathcal{N}_{\Sigma\Gamma} p^\Gamma - L^\Sigma q_\Sigma\right] \\ &= -\frac{i}{2} e^{2U-2\psi} \mathcal{Z}, \end{aligned} \tag{4.33}$$

where in the last equality we have used the fact that $M_\Lambda = \mathcal{N}_{\Lambda\Sigma} L^\Sigma$ and recalled the definition for the central charge $\mathcal{Z} = \langle Q, V\rangle = L^\Sigma q_\Sigma - M_\Sigma p^\Gamma$.
Identical calculations lead to

$$T^-_{\theta\varphi} = \frac{1}{2} \sin\theta\, \mathcal{Z}. \tag{4.34}$$

As long as $G^{-i}_{\mu\nu}$ is concerned, through similar calculations

$$\begin{aligned} G^{-i}_{tr} &= -\bar{D}^i\bar{L}^\Gamma\mathcal{I}_{\Gamma\Lambda}F^{-\Lambda}_{tr} \\ &= -\frac{1}{4}e^{2U-2\psi}\left[\bar{D}^i\bar{L}^\Gamma\mathcal{N}_{\Gamma\Lambda}p^\Lambda-\bar{D}^i\bar{L}^\Gamma q_\Gamma\right] \\ &= \frac{1}{4}e^{2U-2\psi}\bar{D}^i\overline{\mathcal{Z}}, \end{aligned} \tag{4.35}$$

where we employed the fact that $\bar{D}^iM_\Lambda=\mathcal{N}_{\Lambda\Sigma}\bar{D}^iL^\Sigma$. Similarly,

$$G^{-i}_{\theta\varphi} = \frac{i}{4}\sin\theta\bar{D}^i\overline{\mathcal{Z}}. \tag{4.36}$$

As a final remark, we emphasise how, given the spherical symmetry of the solutions, we will assume all the scalar fields to depend on the radial coordinate only, i.e., $z^i=z^i(r)$ and $q^u=q^u(r)$.
We now have all the ingredients to write down explicitly the fermionic supersymmetry variations.

4.2.2 Fermionic Supersymmetry Variations

We can now begin our analysis by specializing the general supersymmetry transformations presented in eqs. (2.82)-(2.84). In this way, the general structure of the Killing spinor equations will emerge, allowing us to determine the appropriate projectors to extract the BPS equations.

Gravitini

Consider the transformation rule for the gravitini:

$$\delta\psi_{A\mu} = \mathcal{D}_\mu\epsilon_A-T^-_{\mu\nu}\gamma^\nu\varepsilon_{AB}\epsilon^B+i\eta_{\mu\nu}\gamma^\nu S_{AB}\epsilon^B=0. \tag{4.37}$$

The mass matrix S_{AB}, recalling the decomposition in eq. (4.9) and the definition for $Q_A{}^B$, can be rewritten as

$$S_{AB} = \frac{i}{2}(\sigma_x)_A{}^C\varepsilon_{BC}\mathcal{P}^x_\Lambda L^\Lambda=\frac{1}{2}(iQ^x\sigma_x)_A{}^C\varepsilon_{BC}\mathcal{L}=\frac{1}{2}Q_{AB}\mathcal{L}. \tag{4.38}$$

Let's start by looking at the time component of the variation; recalling that $\gamma^\mu\equiv e^\mu_a\gamma^a$, we can write

$$\begin{aligned} \delta\psi_{At} &= \mathcal{D}_t\epsilon_A-T^-_{tr}\gamma^r\varepsilon_{AB}\epsilon^B+i\eta_{tt}\gamma^tS_{AB}\epsilon^B \\ &= \partial_t\epsilon_A+\frac{1}{4}\eta_{ab}\left(\omega^a{}_c\right)_t\gamma^{bc}\epsilon_A+\frac{i}{2}\mathcal{Q}_t\epsilon_A+\frac{i}{2}\left(\omega^x_t+A^\Lambda_t\mathcal{P}^x_\Lambda\right)(\sigma_x)_A{}^B\epsilon_B+ \\ &\quad -T^-_{tr}e^r_a\gamma^a\varepsilon_{AB}\epsilon^B-\frac{i}{2}\mathcal{L}\ e^t_a\gamma^aQ_{AB}\epsilon^B \\ &= \frac{1}{4}\eta_{ab}\left(\omega^a{}_c\right)_t\gamma^{bc}\epsilon_A+\frac{1}{2}A^\Lambda_tg_\Lambda Q_A{}^B\epsilon_B+\frac{i}{2}e^{3U-2\psi}\mathcal{Z}\varepsilon_{AB}\gamma^1\epsilon^B-\frac{i}{2}e^U\mathcal{L}\gamma^0Q_{AB}\epsilon^B=0, \end{aligned} \tag{4.39}$$

where in the last equation we assumed the supersymmetry parameters to be time independent, and exploited the fact that $\mathcal{Q}_t=\omega^x_t=0$, as a consequence of $\partial_tz^i=\partial_tq^u=0$.
Writing down the only non vanishing spin-connection component, we end up with

$$\delta\psi_{At}=-\frac{1}{2}U'e^{2U}\gamma^{01}\epsilon_A+\frac{1}{2}A^\Lambda_tg_\Lambda Q_A{}^B\epsilon_B+\frac{i}{2}e^{3U-2\psi}\mathcal{Z}\varepsilon_{AB}\gamma^1\epsilon^B-\frac{i}{2}e^U\mathcal{L}\gamma^0Q_{AB}\epsilon^B=0. \tag{4.40}$$

Since $\gamma^{01}=\gamma^0\gamma^1$ and $\left(\gamma^1\right)^2=-\left(\gamma^0\right)^2=1$, the last equation can be rewritten as

$$U'\epsilon_A=ie^{U-2\psi}\mathcal{Z}\varepsilon_{AB}\gamma^0\epsilon^B-i\mathcal{L}e^{-U}Q_{AB}\gamma^1\epsilon^B-e^{-2U}A^\Lambda_tg_\Lambda Q_A{}^B\gamma^1\gamma^0\epsilon_B. \tag{4.41}$$

Let's now turn to the θ-component, that is

$$\delta\psi_{A\theta} \quad = \quad \mathcal{D}_\theta\epsilon_A - T^-_{\theta\varphi}\gamma^\varphi\varepsilon_{AB}\epsilon^B + i\eta_{\theta\theta}\gamma^\theta S_{AB}\epsilon^B = 0. \tag{4.42}$$

Using again the fact that $\mathcal{Q}_\theta = \omega^x_\theta = 0$, and recalling from our ansatz (4.24) that $A^\Lambda_\theta = 0$, we obtain

$$\begin{aligned}\delta\psi_{A\theta} &= \partial_\theta\epsilon_A + \frac{1}{2}\omega_\theta^{12}\gamma_{12}\epsilon_A - \frac{1}{2}e^{U-\psi}\mathcal{Z}\varepsilon_{AB}\gamma^3\epsilon^B + \frac{i}{2}e^{\psi-U}\mathcal{L}Q_{AB}\gamma^2\epsilon^B \\ &= \partial_\theta\epsilon_A + \frac{1}{2}\left(U'-\psi'\right)e^\psi\gamma^{12}\epsilon_A - \frac{1}{2}e^{U-\psi}\mathcal{Z}\;\varepsilon_{AB}\gamma^3\epsilon^B + \frac{i}{2}e^{\psi-U}\mathcal{L}\;Q_{AB}\gamma^2\epsilon^B = 0.\end{aligned} \tag{4.43}$$

In a completely similar fashion, we find for the φ-component

$$\begin{aligned}\delta\psi_{A\varphi} \quad = \quad & \partial_\varphi\epsilon_A + \frac{1}{2}\left(U'-\psi'\right)\sin\theta\; e^\psi\gamma^{13}\epsilon_A + \frac{1}{2}\sin\theta\; e^{U-\psi}\mathcal{Z}\;\varepsilon_{AB}\gamma^2\epsilon^B + \\ & + \frac{i}{2}\sin\theta\; e^{\psi-U}\mathcal{L}\;Q_{AB}\gamma^3\epsilon^B + \frac{1}{2}A^\Lambda_\varphi g_\Lambda Q_A{}^B\epsilon_B - \frac{1}{2}\cos\theta\;\gamma^{23}\epsilon_A = 0.\end{aligned} \tag{4.44}$$

It is worth noticing that, setting aside the last two terms, the equation for the φ-component is basically proportional to the θ-one, as expected by spherical symmetry. As we will see, the additional terms will provide a constraint on the black hole charges.
As long as the gravitini are concerned, the only missing variation is the radial one. In this case $\omega^{ab}_r = 0 = A^\Lambda_r$ and we will have non-vanishing contributions from both the U(1) and the SU(2) connections. Explicitly

$$\begin{aligned}\delta\psi_{Ar} &= \mathcal{D}_r\epsilon_A - T^-_{rt}\gamma^t\varepsilon_{AB}\epsilon^B + i\eta_{rr}\gamma^r S_{AB}\epsilon^B \\ &= \partial_r\epsilon_A + \frac{i}{2}\mathcal{Q}_r\epsilon_A + \frac{i}{2}(\sigma_x)_A{}^B\omega^x_r\epsilon_B + \frac{i}{2}e^{-U}\mathcal{L}Q_{AB}\gamma^1\epsilon^B - \frac{i}{2}e^{U-2\psi}\mathcal{Z}\varepsilon_{AB}\gamma^0\epsilon^B,\end{aligned} \tag{4.45}$$

so that, all in all,

$$\delta\psi_{Ar} = \partial_r\epsilon_A + \frac{i}{2}\mathcal{Q}_r\epsilon_A + \frac{i}{2}(q^u)'(\omega^x_u\sigma_x)_A{}^B\epsilon_B + \frac{i}{2}e^{-U}\mathcal{L}Q_{AB}\gamma^1\epsilon^B - \frac{i}{2}e^{U-2\psi}\mathcal{Z}\varepsilon_{AB}\gamma^0\epsilon^B = 0. \tag{4.46}$$

Gaugini

Moving on onto the gaugini variation, we recall that

$$\delta\lambda^{iA} \quad = \quad -i\partial_\mu z^i\gamma^\mu\epsilon^A + G^{-i}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^{AB}\epsilon_B + W^{iAB}\epsilon_B, \tag{4.47}$$

and the mass matrix W^{iAB} now reads

$$W^{iAB} \quad = \quad i(\sigma_x)_C{}^B\varepsilon^{CA}\mathcal{P}^x_\Lambda g^{i\bar{\jmath}}\bar{D}_{\bar{\jmath}}\bar{L}^\Lambda = Q^{AB}g^{i\bar{\jmath}}\bar{D}_{\bar{\jmath}}\bar{\mathcal{L}}. \tag{4.48}$$

Therefore

$$\begin{aligned}\delta\lambda^{iA} &= -i\partial_r z^i\gamma^r\epsilon^A + 2G^{-i}_{tr}\gamma^{tr}\varepsilon^{AB}\epsilon_B + 2G^{-i}_{\theta\varphi}\gamma^{\theta\varphi}\varepsilon^{AB}\epsilon_B + g^{i\bar{\jmath}}\bar{D}_{\bar{\jmath}}\bar{\mathcal{L}}Q^{AB}\epsilon_B \\ &= -ie^U\left(z^i\right)'\gamma^1\epsilon^A + \frac{1}{2}e^{2U-2\psi}\bar{D}^i\overline{\mathcal{Z}}\left(\gamma^{01}+i\gamma^{23}\right)\varepsilon^{AB}\epsilon_B + g^{i\bar{\jmath}}\bar{D}_{\bar{\jmath}}\bar{\mathcal{L}}Q^{AB}\epsilon_B = 0.\end{aligned} \tag{4.49}$$

Hyperini

Finally, looking at the hyperini

$$\begin{aligned}\delta\zeta_\alpha &= -i\mathcal{U}^{B\beta}_u\nabla_\mu q^u\gamma^\mu\varepsilon_{AB}\mathbb{C}_{\alpha\beta}\epsilon^A + N^A_\alpha\epsilon_A \\ &= -i\mathcal{U}^{B\beta}_u\left[(q^u)'\gamma^r + A^\Lambda_\mu k^u_\Lambda\gamma^\mu\right]\varepsilon_{AB}\mathbb{C}_{\alpha\beta}\epsilon^A + 2\mathcal{U}^A_{\alpha v}k^v_\Lambda\bar{L}^\Lambda\epsilon_A \\ &= -i\mathcal{U}^{B\beta}_u\left(q^u\right)'\gamma^r\varepsilon_{AB}\mathbb{C}_{\alpha\beta}\epsilon^A + 2\mathcal{U}^A_{\alpha v}k^v_\Lambda\bar{L}^\Lambda\epsilon_A = 0.\end{aligned} \tag{4.50}$$

In the last equality we used the fact that, under our working assumptions, the combination $A^\Lambda_\mu k^u_\Lambda$ is always vanishing. Specifically, whenever k^u_Λ is related to a preserved isometry, $k^u_\Lambda = 0$; on the other hand, for broken isometries the corresponding vector fields A^Λ_μ become massive and we set them to zero on the solution.

This equation can be actually recast in a much more instructive form by contracting it with another vielbein $\mathcal{U}^{A\alpha}_u$. In particular, we can write

$$\mathcal{U}^{A\alpha}_u \left(i\mathcal{U}^{B\beta}_v \left(q^v\right)' \gamma^r \varepsilon_{AB} \mathbb{C}_{\alpha\beta} \epsilon^A - 2\mathcal{U}^{B\beta}_v \mathbb{C}_{\alpha\beta} k^v_\Lambda \bar{L}^\Lambda \epsilon_B \right) = 0, \tag{4.51}$$

so that, recalling the quaternionic identity

$$\mathbb{C}_{\alpha\beta} \mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v = \frac{1}{2} h_{uv} \varepsilon^{AB} + \frac{i}{2} \Omega^x_{uv} (\sigma_x)_C{}^B \varepsilon^{AC}, \tag{4.52}$$

we end up with

$$\left[h_{uv} \epsilon^{AB} + i\Omega^x_{uv} (\sigma_x)_C{}^B \epsilon^{AC} \right] \left[i \left(q^v\right)' e^U \varepsilon_{DB} \gamma^1 \epsilon^D - 2\bar{k}^v \epsilon_B \right] = 0, \tag{4.53}$$

for $\bar{k}^u \equiv k^u_\Lambda \bar{L}^\Lambda$.

4.2.3 Projectors and BPS equations

Having written down explicitly all the relevant equations, the next crucial step is to identify the appropriate projectors for the supersymmetry parameters. Specifically, as already mentioned in the context of AdS black holes, due to the non-trivial spinorial structure of the equations arising from the supersymmetry variations, it is necessary to identify suitable projectors that can reduce the number of independent ϵ^A components, allowing for a rewriting in terms of first-order differential equations for the bosonic fields alone.

To establish their correct form, let's start by looking at the time-component for the gravitini variation

$$\delta\psi_{At} = 0 \quad : \quad U' \epsilon_A = i e^{U-2\psi} \mathcal{Z} \varepsilon_{AB} \gamma^0 \epsilon^B - i\mathcal{L} e^{-U} Q_{AB} \gamma^1 \epsilon^B - e^{-2U} A^\Lambda_t g_\Lambda Q_A{}^B \gamma^1 \gamma^0 \epsilon_B. \tag{4.54}$$

Written in this form, recalling that both ε_{AB} and Q_{AB} square to unity (modulo a minus sign), it is clear that imposing

$$\gamma^0 \epsilon_A \propto \varepsilon_{AB} \epsilon^B \qquad \text{and} \qquad \gamma^1 \epsilon_A \propto Q_{AB} \epsilon^B \tag{4.55}$$

we should be able to rewrite the supersymmetry variation as a first-order differential equation for the bosonic content of the theory multiplying the same spinor ϵ_A.

Specifically, we introduce two independent projector conditions relating the spinor components as

$$\gamma^0 \epsilon_A = i e^{i\alpha} \varepsilon_{AB} \epsilon^B, \tag{4.56}$$

and

$$\gamma^1 \epsilon_A = e^{i\alpha} Q_{AB} \epsilon^B. \tag{4.57}$$

For consistency, we should also impose that

$$\gamma^0 \epsilon^A = -i e^{-i\alpha} \varepsilon^{AB} \epsilon_B \qquad \text{and} \qquad \gamma^1 \epsilon^A = e^{-i\alpha} Q^{AB} \epsilon_B. \tag{4.58}$$

These requirement can be actually recast in terms of two projectors Π^0 and Π^1, defined on the four-dimensional spinor (ϵ_A, ϵ^A) as

$$\Pi^0\begin{pmatrix}\epsilon_A\\ \epsilon^A\end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix}\epsilon_A + ie^{i\alpha}\gamma^0\varepsilon_{AB}\epsilon^B\\ \epsilon^A - ie^{-i\alpha}\gamma^0\varepsilon^{AB}\epsilon_B\end{pmatrix} \tag{4.59}$$

and

$$\Pi^1\begin{pmatrix}\epsilon_A\\ \epsilon^A\end{pmatrix} \equiv \frac{1}{2}\begin{pmatrix}\epsilon_A - e^{i\alpha}\gamma^1 Q_{AB}\epsilon^B\\ \epsilon^A - e^{-i\alpha}\gamma^1 Q^{AB}\epsilon_B\end{pmatrix}. \tag{4.60}$$

Conditions (4.56)-(4.58) thus correspond to

$$\Pi^0\begin{pmatrix}\epsilon_A\\ \epsilon^A\end{pmatrix} = 0 \qquad \text{and} \qquad \Pi^1\begin{pmatrix}\epsilon_A\\ \epsilon^A\end{pmatrix} = 0. \tag{4.61}$$

The only necessary non-trivial checks are to confirm that Π^0 and Π^1 truly define projectors and that the two projections are mutually independent.
The first check comes from explicit calculations:

$$\begin{aligned}\left[\left(\Pi^0\right)^2\epsilon\right]^A &= \frac{1}{2}\Pi^0\left[\epsilon^A - ie^{-i\alpha}\gamma^0\varepsilon^{AB}\epsilon_B\right]\\ &= \frac{1}{4}\left[\epsilon^A - ie^{-i\alpha}\gamma^0\varepsilon^{AB}\epsilon_B - 2ie^{-i\alpha}\gamma^0\varepsilon^{AB}\left(\Pi^0\epsilon\right)_B\right]\\ &= \frac{1}{4}\left[\epsilon^A - ie^{-i\alpha}\gamma^0\varepsilon^{AB}\epsilon_B - ie^{-i\alpha}\gamma^0\varepsilon^{AB}\left(\epsilon_B + ie^{i\alpha}\gamma^0\varepsilon_{BC}\epsilon^C\right)\right]\\ &= \frac{1}{2}\left[\epsilon^A - ie^{-i\alpha}\gamma^0\varepsilon^{AB}\epsilon_B\right]\\ &= \left(\Pi^0\epsilon\right)^A,\end{aligned} \tag{4.62}$$

where we exploited the fact that $\left(\gamma^0\right)^2 = -1$.
Analogously, for Π^1:

$$\begin{aligned}\left[\left(\Pi^1\right)^2\epsilon\right]^A &= \frac{1}{2}\Pi^1\left[\epsilon^A - e^{-i\alpha}\gamma^1 Q^{AB}\epsilon_B\right]\\ &= \frac{1}{4}\left[\epsilon^A - e^{-i\alpha}\gamma^1 Q^{AB}\epsilon_B - 2e^{-i\alpha}\gamma^1 Q^{AB}\left(\Pi^1\epsilon\right)_B\right]\\ &= \frac{1}{4}\left[\epsilon^A - e^{-i\alpha}\gamma^1 Q^{AB}\epsilon_B - e^{-i\alpha}\gamma^1 Q^{AB}\left(\epsilon_B - e^{i\alpha}\gamma^1 Q_{BC}\epsilon^C\right)\right]\\ &= \frac{1}{2}\left[\epsilon^A - e^{-i\alpha}\gamma^1 Q^{AB}\epsilon_B\right]\\ &= \left(\Pi^1\epsilon\right)^A,\end{aligned} \tag{4.63}$$

as $\left(\gamma^1\right)^2 = 1$ and $Q^{AB}Q_{BC} = \delta^A_C$. Similar calculations hold for the lower components of both $\left(\Pi^0\epsilon\right)_A$ and $\left(\Pi^1\epsilon\right)_A$.
To verify instead the independence of the two conditions, we can compute the commutator of the projectors and check that it vanishes. In particular, focusing for simplicity on ϵ_A:

$$\begin{aligned}\left[\Pi^0,\ \Pi^1\right]\epsilon_A &= \frac{1}{2}\Pi^0\left(\epsilon_A - e^{i\alpha}\gamma^1 Q_{AB}\epsilon^B\right) - \frac{1}{2}\Pi^1\left(\epsilon_A + ie^{i\alpha}\gamma^0\varepsilon_{AB}\epsilon^B\right)\\ &= \frac{i}{4}Q_{AB}\varepsilon^{BC}\gamma^1\gamma^0\epsilon_C + \frac{i}{4}\varepsilon_{AB}Q^{BC}\gamma^0\gamma^1\epsilon_C\\ &= \frac{i}{4}\left[Q_{AB}\varepsilon^{BC} - \varepsilon_{AB}Q^{BC}\right]\gamma^1\gamma^0\epsilon_C\\ &= 0.\end{aligned} \tag{4.64}$$

One last crucial information is the dimension of the space Π^0 and Π^1 project on. Specifically, this can be determined by computing

$$\mathrm{Tr}\left(\Pi^0\right)=4 \qquad \text{and} \qquad \mathrm{Tr}\left(\Pi^1\right)=4, \tag{4.65}$$

where the trace is taken on both the spinor and the SU(2) space and the fact that $\mathrm{Tr}\left(\gamma^\mu\right)=0$ was explicitly employed.
Since we start from 8 independent supercharges, these results imply that each projector will halve the number of independent degrees of freedom. Therefore, imposing two compatible projectors, the resulting solutions will preserve only 1/4 of the original supersymmetry (in this case, the solutions are said to be 1/4-BPS). This can be also explicitly verified by computing

$$\mathrm{Tr}\left(\Pi^1\Pi^0\right)=2. \tag{4.66}$$

Finally, turning off the contribution from the hypermultiplets, i.e., $Q^x=\delta^{x2}$, the condition on $\gamma^1\epsilon_A$ becomes

$$\gamma^1\epsilon_A=e^{i\alpha}Q_{AB}\epsilon^B=e^{i\alpha}\left(i\delta^{x2}\sigma_x\right)_A{}^C\varepsilon_{BC}\epsilon^B=e^{i\alpha}\delta_{AB}\epsilon^B, \tag{4.67}$$

precisely recovering the projector in eq. (3.70) employed in [42] for the study of AdS black holes.

Gravitini

Starting once again from the time-component of the gravitini variation, we have

$$\begin{aligned}
U'\epsilon_A &= ie^{U-2\psi}\mathcal{Z}\varepsilon_{AB}\gamma^0\epsilon^B-i\mathcal{L}e^{-U}Q_{AB}\gamma^1\epsilon^B-e^{-2U}A_t^\Lambda g_\Lambda Q_A{}^B\gamma^1\gamma^0\epsilon_B\\
&= ie^{U-2\psi}\mathcal{Z}\varepsilon_{AB}(-ie^{-i\alpha}\varepsilon^{BC}\epsilon_C)-i\mathcal{L}e^{-U}Q_{AB}\left(e^{-i\alpha}Q^{BC}\epsilon_C\right)+\\
&\quad -e^{-2U}A_t^\Lambda g_\Lambda Q_A{}^B\gamma^1\left(ie^{i\alpha}\varepsilon_{BC}\epsilon^C\right)\\
&= -e^{U-2\psi}e^{-i\alpha}\mathcal{Z}\epsilon_A-ie^{-i\alpha}\mathcal{L}e^{-U}\epsilon_A-e^{-2U}A_t^\Lambda g_\Lambda Q_A{}^B\left(ie^{i\alpha}\varepsilon_{BC}\right)\left(e^{-i\alpha}Q^{CD}\epsilon_D\right)\\
&= -e^{U-2\psi}e^{-i\alpha}\mathcal{Z}\epsilon_A-ie^{-i\alpha}\mathcal{L}e^{-U}\epsilon_A+ie^{-2U}A_t^\Lambda g_\Lambda\epsilon_A.
\end{aligned} \tag{4.68}$$

Therefore, we can write

$$\left(U'+e^{U-2\psi}e^{-i\alpha}\mathcal{Z}+ie^{-U}e^{-i\alpha}\mathcal{L}-ie^{-2U}A_t^\Lambda g_\Lambda\right)\epsilon_A=0 \tag{4.69}$$

and, separating the real and imaginary part, we obtain equations for the warp factor $U(r)$ and the vectors time component:

$$\boxed{U' = -e^{U-2\psi}\mathrm{Re}\left(e^{-i\alpha}\mathcal{Z}\right)+e^{-U}\mathrm{Im}\left(e^{-i\alpha}\mathcal{L}\right)} \tag{4.70}$$

and

$$\boxed{A_t^\Lambda g_\Lambda(q) = e^{3U-2\psi}\mathrm{Im}\left(e^{-i\alpha}\mathcal{Z}\right)+e^{U}\mathrm{Re}\left(e^{-i\alpha}\mathcal{L}\right)}. \tag{4.71}$$

Let's now apply the same projectors to the θ-component. As we will see, this will lead to the flow equation for the warp factor ψ.
In particular, recalling that

$$\delta\psi_{A\theta}=0 \quad : \quad \partial_\theta\epsilon_A+\frac{1}{2}\left(U'-\psi'\right)e^\psi\gamma^{12}\epsilon_A\underbrace{-\frac{1}{2}e^{U-\psi}\mathcal{Z}\,\varepsilon_{AB}\gamma^3\epsilon^B}_{(i)}+\underbrace{\frac{i}{2}e^{\psi-U}\mathcal{L}\,Q_{AB}\gamma^2\epsilon^B}_{(ii)}=0, \tag{4.72}$$

we study each term separately:

(i) looking at the first term,

$$-\frac{1}{2}e^{U-\psi}\mathcal{Z}\,\varepsilon_{AB}\gamma^3\epsilon^B = \frac{i}{2}e^{U-\psi}e^{-i\alpha}\mathcal{Z}\gamma^3\left(ie^{i\alpha}\varepsilon_{AB}\epsilon^B\right) = \frac{i}{2}e^{U-\psi}e^{-i\alpha}\mathcal{Z}\gamma^3\gamma^0\epsilon_A. \tag{4.73}$$

Then, since in our conventions $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, we can rewrite $i\gamma^3\gamma^0 = -\gamma^2\gamma^1\gamma_5$, so that

$$\frac{i}{2}e^{U-\psi}e^{-i\alpha}\mathcal{Z}\gamma^3\gamma^0\epsilon_A = -\frac{1}{2}e^{U-\psi}e^{-i\alpha}\mathcal{Z}\gamma^{21}\gamma_5\epsilon_A = -\frac{1}{2}e^{U-\psi}e^{-i\alpha}\mathcal{Z}\gamma^{21}\epsilon_A, \tag{4.74}$$

given that $\gamma_5\epsilon_A = \epsilon_A$;

(ii) in this case, we straightforwardly obtain

$$\frac{i}{2}e^{\psi-U}\mathcal{L}\,Q_{AB}\gamma^2\epsilon^B = \frac{i}{2}e^{\psi-U}e^{-i\alpha}\mathcal{L}\gamma^2\left(e^{i\alpha}Q_{AB}\epsilon^B\right) = \frac{i}{2}e^{\psi-U}e^{-i\alpha}\mathcal{L}\gamma^{21}\epsilon_A. \tag{4.75}$$

Therefore, equation (4.72) becomes

$$\partial_\theta\epsilon_A + \frac{e^\psi}{2}\left[\psi' - U' - e^{U-2\psi}e^{-i\alpha}\mathcal{Z} + ie^{-U}e^{-i\alpha}\mathcal{L}\right]\gamma^{21}\epsilon_A = 0. \tag{4.76}$$

At this point, by making use of the equation for U' in (4.70) and decomposing both $e^{-i\alpha}\mathcal{Z}$ and $e^{-i\alpha}\mathcal{L}$ into real and imaginary part, we get

$$\partial_\theta\epsilon_A + \frac{e^\psi}{2}\left[\psi' - 2e^{-U}\mathrm{Im}\left(e^{-i\alpha}\mathcal{L}\right) + i\left(e^{-U}\mathrm{Re}\left(e^{-i\alpha}\mathcal{L}\right) - e^{U-2\psi}\mathrm{Im}\left(e^{-i\alpha}\mathcal{Z}\right)\right)\right]\gamma^{21}\epsilon_A = 0, \tag{4.77}$$

from which we can extract

$$\boxed{\psi' = 2e^{-U}\mathrm{Im}\left(e^{-i\alpha}\mathcal{L}\right)}, \tag{4.78}$$

together with the constraint

$$\boxed{e^{-U}\mathrm{Re}\left(e^{-i\alpha}\mathcal{L}\right) = e^{U-2\psi}\mathrm{Im}\left(e^{-i\alpha}\mathcal{Z}\right)} \tag{4.79}$$

and the condition

$$\boxed{\partial_\theta\epsilon_A = 0}. \tag{4.80}$$

We can now turn to the φ-component:

$$\begin{aligned}\delta\psi_{A\varphi} = 0 \quad : \quad & \partial_\varphi\epsilon_A + \left[\frac{1}{2}\left(U' - \psi'\right)\sin\theta\, e^\psi\gamma^{13}\epsilon_A + \frac{1}{2}\sin\theta\, e^{U-\psi}\mathcal{Z}\,\varepsilon_{AB}\gamma^2\epsilon^B + \right.\\ & \left. + \frac{i}{2}\sin\theta\, e^{\psi-U}\mathcal{L}\,Q_{AB}\gamma^3\epsilon^B\right] + \frac{1}{2}A^\Lambda_\varphi g_\Lambda Q_A{}^B\epsilon_B - \frac{1}{2}\cos\theta\,\gamma^{23}\epsilon_A = 0.\end{aligned} \tag{4.81}$$

Consider the expression included in square brackets:

$$\begin{aligned}&\frac{1}{2}\left(U' - \psi'\right)\sin\theta\, e^\psi\gamma^{13}\epsilon_A + \frac{1}{2}\sin\theta\, e^{U-\psi}\mathcal{Z}\,\varepsilon_{AB}\gamma^2\epsilon^B + \frac{i}{2}\sin\theta\, e^{\psi-U}\mathcal{L}\,Q_{AB}\gamma^3\epsilon^B\\ &\quad = \frac{e^\psi}{2}\sin\theta\left[\left(U' - \psi'\right)\gamma^{13}\epsilon_A - ie^{U-2\psi}e^{-i\alpha}\mathcal{Z}\gamma^2\left(ie^{i\alpha}\varepsilon_{AB}\epsilon^B\right) + i\,e^{-U}e^{-i\alpha}\mathcal{L}\gamma^3\left(e^{i\alpha}Q_{AB}\epsilon^B\right)\right]\\ &\quad = \frac{e^\psi}{2}\sin\theta\left[\left(U' - \psi'\right)\gamma^{13}\epsilon_A - ie^{U-2\psi}e^{-i\alpha}\mathcal{Z}\gamma^{20}\epsilon_A + i\,e^{-U}e^{-i\alpha}\mathcal{L}\gamma^{31}\epsilon_A\right]\\ &\quad = \frac{e^\psi}{2}\sin\theta\left[\psi' - U' - e^{U-2\psi}e^{-i\alpha}\mathcal{Z} + ie^{-U}e^{-i\alpha}\mathcal{L}\right]\gamma^{31}\epsilon_A,\end{aligned} \tag{4.82}$$

where in the last equation we used $i\gamma^2\gamma^0 = \gamma^3\gamma^1\gamma_5$.

From a direct comparison with eq. (4.76), it is evident that the whole expression vanishes once we

consider the supersymmetry equations (4.78)-(4.79). As anticipated, this is a direct consequence of the spherical symmetry of the solution.
The analysis of the φ-component thus reduces to

$$\partial_\varphi \epsilon_A + \frac{1}{2} A^\Lambda_\varphi g_\Lambda Q_A{}^B \epsilon_B - \frac{1}{2}\cos\theta\, \gamma^{23}\epsilon_A = 0. \tag{4.83}$$

Recalling that $\epsilon_{BA}\epsilon^{BC} = \delta^C_A$, we can rewrite the second term as

$$\begin{aligned} \frac{1}{2} A^\Lambda_\varphi g_\Lambda Q_A{}^B \epsilon_B &= \frac{1}{2} A^\Lambda_\varphi g_\Lambda Q_A{}^C \delta^B_C \epsilon_B \\ &= \frac{1}{2} A^\Lambda_\varphi g_\Lambda Q_A{}^C \left(\varepsilon_{DC}\varepsilon^{DB}\right)\epsilon_B \\ &= \frac{i}{2} A^\Lambda_\varphi g_\Lambda \left[e^{i\alpha} Q_A{}^C \varepsilon_{DC}\left(-ie^{-i\alpha}\varepsilon^{DB}\epsilon_B\right)\right] \\ &= \frac{i}{2} A^\Lambda_\varphi g_\Lambda (q)\gamma^{01}\epsilon_A = \frac{1}{2} A^\Lambda_\varphi g_\Lambda(q)\gamma^{32}\gamma_5\epsilon_A. \end{aligned} \tag{4.84}$$

So, given the ansatz for the vector field (4.24), we are left with

$$\partial_\varphi \epsilon_A + \frac{1}{2}\cos\theta\, \left[1 + p^\Lambda g_\Lambda(q)\right]\gamma^{32}\epsilon_A = 0, \tag{4.85}$$

resulting in

$$\boxed{\partial_\varphi \epsilon_A = 0}, \tag{4.86}$$

together with the constraint on the charges

$$\boxed{p^\Lambda g_\Lambda(q) = -1}. \tag{4.87}$$

The only variation yet to be studied is the radial one. Specifically,

$$\delta\psi_{Ar} = 0 \quad : \quad \partial_r \epsilon_A + \frac{i}{2}\mathcal{Q}_r \epsilon_A + \frac{i}{2}(q^u)'(\omega^x_u \sigma_x)_A{}^B\epsilon_B \; + \underbrace{\frac{i}{2}e^{-U}\mathcal{L}Q_{AB}\gamma^1\epsilon^B}_{(i)} \underbrace{-\frac{i}{2}e^{U-2\psi}\mathcal{Z}\varepsilon_{AB}\gamma^0\epsilon^B}_{(ii)} = 0. \tag{4.88}$$

Also in this case, we can employ the projectors to rewrite each single term:

(i)

$$\frac{i}{2}e^{-U}\mathcal{L}Q_{AB}\gamma^1\epsilon^B = \frac{i}{2}e^{-U}\mathcal{L}Q_{AB}\left(e^{-i\alpha}Q^{BC}\epsilon_C\right) = \frac{i}{2}e^{-U}e^{-i\alpha}\mathcal{L}\,\epsilon_A, \tag{4.89}$$

(ii)

$$-\frac{i}{2}e^{U-2\psi}\mathcal{Z}\varepsilon_{AB}\gamma^0\epsilon^B = -\frac{i}{2}e^{U-2\psi}\mathcal{Z}\varepsilon_{AB}\left(-ie^{-i\alpha}\varepsilon^{BC}\epsilon_C\right) = \frac{1}{2}e^{U-2\psi}e^{-i\alpha}\mathcal{Z}\epsilon_A. \tag{4.90}$$

The resulting expression

$$\partial_r \epsilon_A + \frac{i}{2}\mathcal{Q}_r\epsilon_A + \frac{i}{2}(q^u)'(\omega^x_u\sigma_x)_A{}^B\epsilon_B + \frac{i}{2}e^{-U}e^{-i\alpha}\mathcal{L}\epsilon_A + \frac{1}{2}e^{U-2\psi}e^{-i\alpha}\mathcal{Z}\epsilon_A = 0 \tag{4.91}$$

can be further simplified by employing the usual trick of decomposing $e^{-i\alpha}\mathcal{Z}$ and $e^{-i\alpha}\mathcal{L}$ into real and imaginary part, and recalling equation (4.70) for the warp factor U, getting to

$$\partial_r\epsilon_A + \frac{i}{2}\mathcal{Q}_r\epsilon_A + \frac{i}{2}(q^u)'(\omega^x_u\sigma_x)_A{}^B\epsilon_B - \frac{1}{2}U'\epsilon_A + \frac{i}{2}e^{-U}\mathrm{Re}\left(e^{-i\alpha}\mathcal{L}\right)\epsilon_A + \frac{i}{2}e^{U-2\psi}\mathrm{Im}\left(e^{-i\alpha}\mathcal{Z}\right)\epsilon_A = 0. \tag{4.92}$$

To conclude, we can define

$$\begin{aligned} \tilde{\mathcal{Q}} \;&\equiv\; \mathcal{Q}_r + e^{-U}\mathrm{Re}\left(e^{-i\alpha}\mathcal{L}\right) + e^{U-2\psi}\mathrm{Im}\left(e^{-i\alpha}\mathcal{Z}\right) \\ &=\; \mathcal{Q}_r + 2e^{-U}\mathrm{Re}\left(e^{-i\alpha}\mathcal{L}\right), \end{aligned} \tag{4.93}$$

where we used the constraint in eq. (4.79), so that the equation for the radial component of the supersymmetry parameters can be written in the compact form

$$\boxed{\partial_r \epsilon_A - \frac{1}{2}\left(U' - i\tilde{\mathcal{Q}}\right)\epsilon_A + \frac{i}{2}(\partial_r q^u)\left(\omega_u^x \sigma_x\right)_A{}^B \epsilon_B = 0}. \tag{4.94}$$

If the combination $(q^u)'\left(\omega_u^x\sigma_x\right)_A{}^B$ vanishes, we exactly recover the result obtained for AdS black holes [42].

Gaugini

As long as the gaugini are concerned, we can extract a differential equation for the scalar fields z^i through properly employing the projectors defined in eqs. (4.56)-(4.57).
In particular, $\delta\lambda^{iA} = 0$ implies

$$-ie^U\left(z^i\right)'\gamma^1\epsilon^A + \frac{1}{2}e^{2U-2\psi}\bar{D}^i\overline{\mathcal{Z}}\left(\gamma^{01} + i\gamma^{23}\right)\varepsilon^{AB}\epsilon_B + g^{i\bar{\jmath}}\bar{D}_{\bar{\jmath}}\bar{\mathcal{L}}Q^{AB}\epsilon_B = 0. \tag{4.95}$$

The second term of the equation can be rewritten as

$$\begin{aligned}
\frac{1}{2}e^{2U-2\psi}\bar{D}^i\overline{\mathcal{Z}}\left(\gamma^{01}+i\gamma^{23}\right)\varepsilon^{AB}\epsilon_B &= \frac{i}{2}e^{2U-2\psi}e^{i\alpha}\bar{D}^i\overline{\mathcal{Z}}\left(\gamma^{01}+i\gamma^{23}\right)\left(-ie^{-i\alpha}\varepsilon^{AB}\epsilon_B\right)\\
&= \frac{i}{2}e^{2U-2\psi}e^{i\alpha}\bar{D}^i\overline{\mathcal{Z}}\left(\gamma^{01}+i\gamma^{23}\right)\gamma^0\epsilon^A\\
&= \frac{i}{2}e^{2U-2\psi}e^{i\alpha}\bar{D}^i\overline{\mathcal{Z}}\left(\gamma^{01}+\gamma^{01}\gamma_5\right)\gamma^0\epsilon^A,
\end{aligned} \tag{4.96}$$

where the last step follows from the definition of γ_5.
Recalling now that $\{\gamma_5, \gamma^\mu\} = 0$ and $\gamma_5\epsilon^A = -\epsilon^A$, we can further simplify the last equation and write

$$\frac{i}{2}e^{2U-2\psi}e^{i\alpha}\bar{D}^i\overline{\mathcal{Z}}\left(\gamma^{01}+\gamma^{01}\gamma_5\right)\gamma^0\epsilon^A = ie^{2U-2\psi}e^{i\alpha}\bar{D}^i\overline{\mathcal{Z}}\gamma^{01}\gamma^0\epsilon^A = ie^{2U-2\psi}e^{i\alpha}\bar{D}^i\overline{\mathcal{Z}}\gamma^1\epsilon^A. \tag{4.97}$$

The term proportional to $\bar{D}_{\bar{\jmath}}\bar{\mathcal{L}}$ in eq. (4.95) can be recast as

$$g^{i\bar{\jmath}}\bar{D}_{\bar{\jmath}}\bar{\mathcal{L}}Q^{AB}\epsilon_B = g^{i\bar{\jmath}}e^{i\alpha}\bar{D}_{\bar{\jmath}}\bar{\mathcal{L}}\left(e^{-i\alpha}Q^{AB}\epsilon_B\right) = g^{i\bar{\jmath}}e^{i\alpha}\bar{D}_{\bar{\jmath}}\bar{\mathcal{L}}\gamma^1\epsilon^A, \tag{4.98}$$

so that the whole variation becomes

$$\left[-ie^U\left(z^i\right)' + ie^{2U-2\psi}e^{i\alpha}\bar{D}^i\overline{\mathcal{Z}} + g^{i\bar{\jmath}}e^{i\alpha}\bar{D}_{\bar{\jmath}}\bar{\mathcal{L}}\right]\gamma^1\epsilon^A = 0, \tag{4.99}$$

from which we conclude

$$\boxed{\left(z^i\right)' = e^{i\alpha}e^{-2\psi}g^{i\bar{\jmath}}\bar{D}_{\bar{\jmath}}\left[e^U\overline{\mathcal{Z}} + ie^{2\psi-U}\bar{\mathcal{L}}\right]}. \tag{4.100}$$

Hyperini

Let's finally consider the hyperini variation:

$$\delta\zeta^\alpha = 0 \quad : \quad \left[h_{uv}\epsilon^{AB} + i\Omega^x_{uv}(\sigma_x)_C{}^B\epsilon^{AC}\right]\left[i\left(q^v\right)'e^U\varepsilon_{DB}\gamma^1\epsilon^D - 2\bar{k}^v\epsilon_B\right] = 0. \tag{4.101}$$

Projecting the spinor components of the supersymmetry parameters

$$\begin{aligned}
&\left[h_{uv}\epsilon^{AB} + i\Omega^x_{uv}(\sigma_x)_C{}^B\epsilon^{AC}\right]\left[i\left(q^v\right)'e^U\varepsilon_{DB}\left(e^{-i\alpha}Q^{DN}\epsilon_N\right) - 2\bar{k}^v\epsilon_B\right]\\
&\quad = \left[h_{uv}\epsilon^{AB} + i\Omega^x_{uv}(\sigma_x)_C{}^B\epsilon^{AC}\right]\left[\left(q^v\right)'e^U e^{-i\alpha}\left(\sigma_y Q^y\right)_B{}^D - 2\bar{k}^v\delta_B^D\right]\epsilon_D = 0,
\end{aligned} \tag{4.102}$$

where in the last equality we've used the explicit definition for Q_{AB} and renamed some dummy indices.
We will now show that this seemingly involved expression could be actually recast in a matrix equation of the form $\mathcal{B}_u^{AD}\epsilon_D=0$. Specifically, multiplying the quantities in the square brackets and expanding the product of two Pauli matrices $(\sigma_x)_A{}^B(\sigma_y)_B{}^C=\delta_{xy}\delta_A^C+i\varepsilon_{xyz}(\sigma_z)_A{}^C$, we end up with

$$\left[\mathcal{B}_u^0\varepsilon^{AB}\delta_B^D+\mathcal{B}_u^z(i\sigma_z)_B{}^D\varepsilon^{AB}\right]\epsilon_D=0, \tag{4.103}$$

where we defined the following quantities:

$$\mathcal{B}_u^0 \equiv ie^Ue^{-i\alpha}\,\Omega^x_{uv}Q^x\,(q^v)'-2\bar{k}_u, \tag{4.104}$$
$$\mathcal{B}_u^z \equiv -ie^Ue^{-i\alpha}\,Q^z\,h_{uv}\,(q^v)'+ie^Ue^{-i\alpha}\,\varepsilon_{xyz}\Omega^x_{uv}Q^y\,(q^v)'-2\Omega^z_{uv}\bar{k}^v. \tag{4.105}$$

In principle one should impose $\mathcal{B}_u^0=\mathcal{B}_u^z=0$ separately but, employing the identities of quaternionic geometry, one can show that actually they are not independent conditions. In particular, the first equation is solved by writing

$$\bar{k}_u=\frac{i}{2}e^Ue^{-i\alpha}\,\Omega^x_{uv}Q^x\,(q^v)'\,. \tag{4.106}$$

Then, plugging this expression in the second equation we get

$$h_{uv}Q^z\,\,(q^v)'-\varepsilon_{xyz}\Omega^x_{uv}Q^y\,(q^v)'+h^{vt}\Omega^z_{uv}\Omega^x_{tm}Q^x\,(q^m)'=0. \tag{4.107}$$

Finally, recalling the relation (2.52) for the contraction of two SU(2) curvatures

$$h^{st}\Omega^x_{us}\Omega^y_{tv}=-\delta^{xy}h_{uv}-\varepsilon^{xyz}\Omega^z_{uv}, \tag{4.108}$$

equation (4.107) identically vanishes.
As a side comment, it is interesting to notice that this pattern is not exclusive of the physical setting we are analysing. In particular, the same formal structure arises also in the study of domain wall solutions in five dimensional $\mathcal{N}=2$ supergravity [52].
To extract a flow equation for the quaternionic scalars, we can plug the expression (4.106) in the relation obtained in (4.16) and get

$$\begin{aligned}\partial_u\bar{\mathcal{L}} &= -2Q^x\left[\frac{i}{2}e^Ue^{-i\alpha}\,h^{vs}\Omega^y_{st}Q^y\left(q^t\right)'\right]\Omega^x_{vu}\\ &= ie^Ue^{-i\alpha}\,\left(q^t\right)'\left[h^{vs}\Omega^x_{uv}\Omega^y_{st}\right]Q^xQ^y\\ &= ie^Ue^{-i\alpha}\,\left(q^t\right)'\left[-h_{ut}\delta^{xy}-\varepsilon^{xyz}\Omega^z_{ut}\right]Q^xQ^y\\ &= -ie^Ue^{-i\alpha}\,h_{ut}\left(q^t\right)',\end{aligned} \tag{4.109}$$

where we used the fact that $Q^xQ^x=1$ and the additional term proportional to the antisymmetric ε^{xyz} vanishes in the contraction with the symmetric combination Q^xQ^y.
Finally, we can invert the obtained relation and write

$$\boxed{(q^u)'=ie^Ue^{i\alpha}h^{uv}\partial_v\bar{\mathcal{L}}}\,. \tag{4.110}$$

As we will present shortly, we can describe the whole solution by introducing a suitable superpotential W. As a consequence, equation (4.110) will further split, yielding also a constraint on the phase α.

4.2.4 Superpotential description

From our review on the previously known black hole solutions in supergravity carried out in chapter 3, we have realized that every scenario admits a description in terms of a properly defined superpotential W. It is therefore interesting to verify that the equations we have derived in the previous section fall into the same pattern.

Let's start by considering the constraint (4.79) obtained from the variation of the gravitini θ-component, i.e.,

$$e^{-U}\mathrm{Re}\left(e^{-i\alpha}\mathcal{L}\right) = e^{U-2\psi}\mathrm{Im}\left(e^{-i\alpha}\mathcal{Z}\right). \tag{4.111}$$

This relation can be actually recast into an expression that identifies the phase α as:

$$e^{2i\alpha} \quad = \quad \frac{\mathcal{Z} - ie^{2(\psi-U)}\mathcal{L}}{\overline{\mathcal{Z}} + ie^{2(\psi-U)}\overline{\mathcal{L}}}. \tag{4.112}$$

Written in this form, we realize that we can interpret $e^{i\alpha}$ as the phase of a complex quantity $\mathcal{W}$ whose norm is given in terms of the superpotential W defined as

$$\mathcal{W} \equiv e^{i\alpha}W, \qquad \text{for} \qquad W \equiv e^{U}\left|\mathcal{Z} - ie^{2(\psi-U)}\mathcal{L}\right| \tag{4.113}$$

or, equivalently,

$$W = e^{U}\mathrm{Re}\left(e^{-i\alpha}\mathcal{Z}\right) + e^{-U+2\psi}\mathrm{Im}\left(e^{-i\alpha}\mathcal{L}\right). \tag{4.114}$$

For convenience we also define the complex conjugate quantity $\overline{\mathcal{W}} = e^{-i\alpha}W$.

The whole set of differential equations for the warp factors and the scalars can be actually rewritten in terms of W.

In particular, let's start by rewriting the equation for $U(r)$ as

$$\begin{aligned} U' &= -e^{U-2\psi}\mathrm{Re}\left(e^{-i\alpha}\mathcal{Z}\right) + e^{-U}\mathrm{Im}\left(e^{-i\alpha}\mathcal{L}\right) \\ &= -e^{-2\psi}\left[e^{U}\mathrm{Re}\left(e^{-i\alpha}\mathcal{Z}\right) - e^{2\psi-U}\mathrm{Im}\left(e^{-i\alpha}\mathcal{L}\right)\right] \\ &= -e^{-2\psi}\partial_U\left[e^{U}\mathrm{Re}\left(e^{-i\alpha}\mathcal{Z}\right) + e^{2\psi-U}\mathrm{Im}\left(e^{-i\alpha}\mathcal{L}\right)\right], \end{aligned} \tag{4.115}$$

so that we end up with

$$\boxed{U' \quad = \quad -e^{-2\psi}\partial_U W}. \tag{4.116}$$

In a similar fashion, we can recast the condition on $\psi(r)$ in the form

$$\begin{aligned} \psi' &= 2e^{-U}\mathrm{Im}\left(e^{-i\alpha}\mathcal{L}\right) \\ &= e^{-2\psi}\left[2e^{2\psi-U}\mathrm{Im}\left(e^{-i\alpha}\mathcal{L}\right)\right] \\ &= e^{-2\psi}\partial_\psi\left[e^{2\psi-U}\mathrm{Im}\left(e^{-i\alpha}\mathcal{L}\right)\right] \\ &= e^{-2\psi}\partial_\psi\left[e^{U}\mathrm{Re}\left(e^{-i\alpha}\mathcal{Z}\right) + e^{2\psi-U}\mathrm{Im}\left(e^{-i\alpha}\mathcal{L}\right)\right], \end{aligned} \tag{4.117}$$

where the fact that $\partial_\psi\left[e^{U}\mathrm{Re}\left(e^{-i\alpha}\mathcal{Z}\right)\right] = 0$ was explicitly used.

Therefore, we immediately read

$$\boxed{\psi' \quad = \quad e^{-2\psi}\partial_\psi W}. \tag{4.118}$$

As long as the scalars are concerned, consider the equation for z^i :

$$(z^i)' = -e^{-2\psi}e^{i\alpha}g^{i\bar{\jmath}}\,\overline{D}_{\bar{\jmath}}\left[e^{U}\overline{\mathcal{Z}} + ie^{2\psi-U}\overline{\mathcal{L}}\right] = -e^{-2\psi}e^{i\alpha}g^{i\bar{\jmath}}\,\overline{D}_{\bar{\jmath}}\overline{\mathcal{W}}. \tag{4.119}$$

To rewrite this last expression in terms of W, we can employ the phase constraint in eq. (4.112). Specifically, by taking derivatives of both sides with respect to the scalars, one can obtain

$$\partial_{\bar{\jmath}} e^{2i\alpha} = \frac{1}{\overline{\mathcal{W}}}\partial_{\bar{\jmath}}\mathcal{W} - \frac{\mathcal{W}}{\overline{\mathcal{W}}^2}\partial_{\bar{\jmath}}\overline{\mathcal{W}} = e^{2i\alpha}\left[\frac{1}{2}\left(\partial_{\bar{\jmath}}K\right)\overline{\mathcal{W}} - \frac{1}{\overline{\mathcal{W}}}\partial_{\bar{\jmath}}\overline{\mathcal{W}}\right], \tag{4.120}$$

where in the second equality we employed the fact that $D_{\bar{\imath}}\mathcal{W} = \left[\partial_{\bar{\imath}} - \frac{1}{2}\left(\partial_{\bar{\imath}}K\right)\right]\mathcal{W} = 0$, as it follows directly from the definitions of $\mathcal{Z}$ and $\mathcal{L}$.
After some trivial calculations, the previous relation can be rewritten as

$$2i\partial_{\bar{\jmath}}\alpha \quad = \quad -\frac{1}{\overline{\mathcal{W}}}\left[\partial_{\bar{\jmath}} - \frac{1}{2}\left(\partial_{\bar{\imath}}K\right)\right]\overline{\mathcal{W}}. \tag{4.121}$$

As a consequence, we immediately obtain

$$e^{i\alpha}g^{i\bar{\jmath}}\,\overline{D_{\bar{\jmath}}\mathcal{W}} = 2g^{i\bar{\jmath}}\partial_{\bar{\jmath}}W, \tag{4.122}$$

so that

$$\boxed{(z^i)' \quad = \quad -2e^{-2\psi}g^{i\bar{\jmath}}\partial_{\bar{\jmath}}W}. \tag{4.123}$$

Finally, since $\partial_u\mathcal{Z} = \partial_u U(r) = \partial_u\psi(r) = 0$, we can rewrite the equation for the quaternionic scalars as

$$\begin{aligned}(q^u)' &= ie^U e^{i\alpha}h^{uv}\partial_v\bar{\mathcal{L}}\\ &= ie^{-2\psi}e^{i\alpha}h^{uv}\partial_v\left[e^U\overline{\mathcal{Z}} + ie^{2\psi-U}\overline{\mathcal{L}}\right]\\ &= e^{-2\psi}e^{i\alpha}h^{uv}\partial_v\left[e^{-i\alpha}W\right]\\ &= e^{-2\psi}h^{uv}\partial_v W - ie^{-2\psi}h^{uv}\partial_v\alpha.\end{aligned} \tag{4.124}$$

Separating real and imaginary part, we end up with

$$\boxed{(q^u)' \quad = \quad e^{-2\psi}h^{uv}\partial_v W} \qquad \text{and} \qquad \boxed{\partial_u\alpha \quad = \quad 0}. \tag{4.125}$$

As anticipated, together with the flow equation for the scalars q^u, we learn that the phase α associated to the projectors *does not* depend on the quaternions.
Therefore, to summarize, considering Abelian U(1) gaugings of the isometries of the hypermultiplets scalar geometry, solutions for spherically symmetric, static and charged BPS black holes are described by the following first-order differential equations:

$$\begin{aligned}U' &= -e^{-2\psi}\partial_U W,\\ \psi' &= e^{-2\psi}\partial_\psi W,\\ (z^i)' &= -2e^{-2\psi}g^{i\bar{\jmath}}\partial_{\bar{\jmath}}W,\\ (q^u)' &= e^{-2\psi}h^{uv}\partial_v W,\end{aligned} \tag{4.126}$$

together with the constraints

$$\begin{aligned}A_t^\Lambda g_\Lambda(q) &= e^{3U-2\psi}\mathrm{Im}\left(e^{-i\alpha}\mathcal{Z}\right) + e^U\mathrm{Re}\left(e^{-i\alpha}\mathcal{L}\right),\\ e^{2i\alpha} &= \frac{\mathcal{Z} - ie^{2(\psi-U)}\mathcal{L}}{\overline{\mathcal{Z}} + ie^{2(\psi-U)}\overline{\mathcal{L}}},\\ p^\Lambda g_\Lambda(q) &= -1,\\ \partial_u\alpha &= 0.\end{aligned} \tag{4.127}$$

4.3 Searches for Explicit Solutions

In this last section, we'd like to present our attempts to build explicit models as benchmarks for the equations we have obtained. In particular, we will focus on supergravity models coupled to one vector multiplet and one hypermultiplet, restricting ourselves to scalar geometries that could be described as *coset spaces*.

Before entering into the details of such models, we point out that for these simple setups we will not be able to study solutions that partially break supersymmetry. Specifically, as discussed in chapter 2, the spontaneous $\mathcal{N}=2 \to \mathcal{N}=1$ pattern implies that one gravitino has to become massive. The residual $\mathcal{N}=1$ supersymmetry, as observed for the first time in [53], then forces the massive gravitino to sit into a massive $\mathcal{N}=1$ representation for spin-3/2 fields[1] and, consequently, also two vectors have to become massive. Therefore, since models with $n_V = n_H = 1$ are coupled to two vectors only, this scenario is clearly in contrast with our request of having at least one preserved isometry (i.e., one massless vector).

As a result, we will look for $\mathcal{N}=2$ preserving solutions employing the constraints derived in eqs. (2.97)-(2.98), together with the general assumptions under which we have derived the flow equations. In particular, by looking at the explicit expressions for the mass matrices, for Abelian gaugings one finds that the request of $\mathcal{N}=2$ supersymmetry implies the following constraints on both the Killing vectors and the prepotentials:

$$\mathcal{P}^x_\Lambda L^\Lambda = 0, \qquad \mathcal{P}^x_\Lambda D^i L^\Lambda = 0, \qquad k^u_\Lambda \bar{L}^\Lambda = k^u_\Lambda L^\Lambda = 0. \tag{4.128}$$

Actually, recalling the definiton in eq. (2.30), the first two equations could be recast as

$$\mathcal{P}^x_\Lambda \begin{pmatrix} f_i^\Lambda \\ \bar{L}^\Lambda \end{pmatrix} = \mathcal{P}^x_\Lambda f_I^\Lambda = 0 \tag{4.129}$$

and, being f_I^Λ invertible, this implies

$$\mathcal{P}^x_\Lambda = 0. \tag{4.130}$$

We remark that these relations characterize the asymptotic vacuum. Then, in general, the black hole solution will lead to the partial or complete breaking of supersymmetry, which could then be enhanced at the horizon.

4.3.1 Vacua on Coset Manifolds

Before specializing our discussion to a specific choice for the scalar manifold, we would like to outline some basic properties of coset spaces (for a full review on the topic, see [32]) and describe how the search for the vacua of the theory can be simplified thanks to their structure.

Let's start by recalling that a *homogeneous space* is a manifold with a metric whose isometry group, G, has a transitive action on the space, i.e., any point on the space can be reached from any other by the group action. The subgroup H of G that leaves a point of the manifold fixed is called the *isotropy group*. Any homogeneous space can be therefore described as the *coset space* G/H, that is the set of equivalence classes of elements of G with respect to the right action of H elements:

$$g \sim g', \quad \text{if} \quad g = g'h \quad \text{for} \quad g, g' \in G, h \in H. \tag{4.131}$$

[1]We recall that, for $\mathcal{N}=1$, the matter content of such a multiplet is given by: one spin-1/2 fermion, two vectors and one spin-3/2 field [9].

The dimension of such a space is therefore given by $d = \dim(G) - \dim(H)$.
As we previously described, vacua of supergravity theories are identified by extremizing the scalar potential, i.e. by finding the values of the moduli that minimize the potential obtained for a given gauge choice. In general, this implies solving a set of coupled equations involving both the moduli and the gauging parameters and solving them analytically usually results in too hard of a task.
For scalars parametrizing a coset manifold[2] a different approach can be followed, simplifying the standard procedure and allowing for a more general scan of the theory vacua [54].
In particular, following the presentation given in [33], let's start from the observation that the potential actually depends on both the scalars and the choice of a gauge group $\mathcal{G} \subset G$, so that $V = V(\phi, \mathcal{G})$. Additionally, V is a *singlet* with respect to the group action, that is

$$\forall L \in G, \quad V(L\phi, L\mathcal{G}) = V(\phi, \mathcal{G}), \tag{4.132}$$

for L a coset representative, and $L\phi$ and $L\mathcal{G}$ denoting the action of a group transformation on the scalars (as described via the Killing vectors) and on the gauging (as dictated by the Lie Algebra associated to G), respectively.
This implies that, if $V(\phi, \mathcal{G})$ has an extremum for ϕ_0,

$$\left.\frac{\partial V(\phi, \mathcal{G})}{\partial \phi}\right|_{\phi_0} = 0, \tag{4.133}$$

then $V(\phi, L\mathcal{G})$ will display a critical point at $L\phi_0$,

$$\left.\frac{\partial V(\phi, L\mathcal{G})}{\partial \phi}\right|_{L\phi_0} = 0, \tag{4.134}$$

with the same value of the potential.
As a consequence, when working with coset manifolds, we can map any generic vacuum ϕ_0 of a given theory defined by $\mathcal{G}$ to a chosen *base point* of the moduli space via a transformation $L(\phi_0) \in G$. Then, the base point will now be a vacuum for the theory defined by $L(\phi_0)\mathcal{G}$. This implies that, when searching for vacua with given properties (residual supersymmetry, cosmological constant, mass spectrum, etc.), we can compute all the relevant quantities at the base point and vary the gauging parameters only, thus systematically exploring all the possible gauging choices.

4.3.2 Explicit Examples

We will now describe our attempts to find explicit realizations of the solutions for the equations we derived from the supersymmetry variations. As we anticipated, we will consider supergravity models coupled to one vector multiplet and one hypermultiplet.
As long as the Special geometry is concerned, we focused our attention on

$$\mathcal{M}_{\text{vector}} = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1,1)}. \tag{4.135}$$

Following [55], we can describe the manifolds by choosing the holomorphic sections as

$$X^0(z) = -\frac{1}{2}, \quad X^1(z) = \frac{i}{2}, \quad F_0 = iz, \quad F_1 = z. \tag{4.136}$$

[2]This is always the case for $\mathcal{N} \geq 3$ supergravity [32].

Employing the relations introduced in chapter 2, we can therefore derive the Kähler potential

$$K = -\log\left[i\left(\overline{X}^{\Lambda}F_{\Lambda} - \overline{F}_{\Sigma}X^{\Sigma}\right)\right] = -\log(z+\bar{z}), \tag{4.137}$$

and the metric

$$g_{z\bar{z}} = \frac{1}{(z+\bar{z})^2} \tag{4.138}$$

on the Special-Kähler manifold.

Ferrara - Girardello - Porrati Model

One possible choice for the hypermultiplet geometry is given by the "Euclidean anti-de Sitter" space (EAdS), defined as

$$\mathcal{M}_{\text{EAdS}} \quad = \quad \frac{\text{SO}(4,1)}{\text{SO}(4)}. \tag{4.139}$$

Together with the Special-Kähler manifold introduced before, this is precisely the model studied by Ferrara, Girardello and Porrati (FGP) [55]. The quaternionic geometry is determined from the SU(2) connection $\omega^x = \omega^x_u \mathrm{d}q^u$ and their field strength $\Omega^x = \Omega^x_{uv}\mathrm{d}q^u \wedge \mathrm{d}q^v$, which read

$$\omega^x_u = \frac{1}{q^0}\delta^x_u, \quad \Omega^x_{0u} = -\frac{1}{2(q^0)^2}\delta^x_u, \quad \Omega^x_{yz} = \frac{1}{2(q^0)^2}\varepsilon^{xyz}. \tag{4.140}$$

Following the quaternionic identities, one finds that the metric is given by

$$h_{uv} = \frac{1}{2(q^0)^2}\delta_{uv} \tag{4.141}$$

and a consistent choice for the vielbein $\mathcal{U}^{\alpha A}_u$ is

$$\mathcal{U}^{\alpha A} = \frac{1}{2q^0}\varepsilon^{\alpha\beta}\left(\mathrm{d}q^0 - i\sigma^x \mathrm{d}q^x\right)_\beta^{\ A}. \tag{4.142}$$

As long as the isometries of the quaternionic manifold are concerned, it is useful to exploit the isomorphism that ties the *conformal algebra* to the one for SO(4,1). Employing the parametrization of the former, we introduce the generators:

$$\begin{aligned}
P^i &\qquad \text{(translation)},\\
M^{ij} &\qquad \text{(rotations)},\\
D &\qquad \text{(scale transformations)},\\
K^i &\qquad \text{(special conformal transformations)},
\end{aligned}$$

with $i,j = 1,2,3$, for a total of ten isometries.
The corresponding Killing vectors could be written explicitly as

$$k_{Pi} = \delta^u_i \partial_u, \tag{4.143}$$

$$k_{M_{32}} \equiv k_{R_1} = q^3\partial_2 - q^2\partial_3, \qquad k_{M_{13}} \equiv k_{R_2} = q^1\partial_3 - q^3\partial_1, \qquad k_{M_{21}} \equiv k_{R_3} = q^2\partial_1 - q^1\partial_2, \tag{4.144}$$

$$k_D = q^u\partial_u, \tag{4.145}$$

$$k^u_{K_i} = 2q^u q^i - \delta^{ui}\delta_{mn}q^m q^n. \tag{4.146}$$

Finally, the SO(4, 1) algebra is recovered by identifying

$$M^{ij} = T^{ij}, \quad P^i = T^{i4} + T^{i5}, \quad K^i = T^{i4} - T^{i5}, \quad D = T^{45}, \tag{4.147}$$

or, inverting these relations,

$$T^{ij} = M^{ij}, \quad T^{i4} = \frac{1}{2}(P^i + K^i), \quad T^{i5} = \frac{1}{2}(P^i - K^i), \quad T^{45} = D. \tag{4.148}$$

It is therefore useful to introduce the following Killing vectors combinations

$$k_i^{\pm} = \frac{1}{2}\left[k_{Pi} \pm k_{K_i}\right]. \tag{4.149}$$

In the gauging procedure, we can consider arbitrary linear combinations of Killing vectors (as long as they commute). In the basis we have just introduced, the most general Killing vectors read

$$k_\Lambda = \sum_{i=1}^{3}\left[a_{\Lambda,i}k_i^+ + b_{\Lambda,i}k_i^-\right] + c_\Lambda k_D + \sum_{i=1}^{3} r_{\Lambda,i}k_{R_i} \quad (\Lambda = 1,2), \tag{4.150}$$

for $a_{\Lambda,i},\ b_{\Lambda,i},\ c_\Lambda, r_{\Lambda,j}$ real numbers.
We choose as base point for our scalar manifold $q^0 = 1 = \mathrm{Re}z$ and $q^1 = q^2 = q^3 = \mathrm{Im}z = 0$.
With this choice, one can explicitly verify that:

$$k^u_{+,i} = k^u_{R_i} = 0, \qquad k^u_{-,i} = \delta^{iu}, \qquad k_D = \delta^{0u} \quad (i = 1,2,3), \tag{4.151}$$

and the associated prepotentials (computed employing eq. (2.73)):

$$\mathcal{P}^x_{-,i} = \mathcal{P}^x_D = 0, \qquad \mathcal{P}^x_{+,i} = \delta^{ix}, \qquad \mathcal{P}^x_{R_i} = -\delta^{ix} \quad (i = 1,2,3). \tag{4.152}$$

We notice that, for the chosen origin of the moduli space, the only non-vanishing Killing vectors are the one associated to non-compact isometries. As long as the prepotentials are concerned, instead, the opposite is true, with the compact isometries yielding the only non-trivial contributions.
For this model, the conditions for a maximally supersymmetric Minkowski vacuum introduced in eqs. (4.128) - (4.130) read:

$$a_{\Lambda,i} = r_{\Lambda,i}, \qquad c_\Lambda L^\Lambda = 0, \qquad b_{\Lambda,i}L^\Lambda = 0. \tag{4.153}$$

As long as the vector masses are concerned, since at the base point the metric on the quaternionic manifold becomes $h_{uv} = \frac{1}{2}\delta_{uv}$, the request of having at least one massless vector can be written as

$$\det(h_{uv}k^u_\Lambda k^v_\Sigma) = 0 \quad \longrightarrow \quad (k_1)^2(k_2)^2 - (k_1^u k_{2,u})^2 = 0, \tag{4.154}$$

which takes the form of a saturated Cauchy–Schwarz inequality.
At the base point, this translates into a condition on the non-compact components of the two Killing vectors; in particular

$$k^u_{2,non-compact} = \lambda k^u_{1,non-compact},$$

that is

$$c_2 = \lambda c_1, \quad b_{2,i} = \lambda b_{1,i} \quad (\lambda \in \mathbb{R}). \tag{4.155}$$

Having obtained the general constraints our gauging choice should fulfil, we can now try to build some explicit realizations.

As a first step, we notice that thanks to the quotient with SO(4) we can partially simplify the expression for one of the two Killing vectors. In particular, looking at eq. (4.148) we realize that T^{i5} and T^{45} are actually related via an SO(4) rotation, so we can choose the latter as the only relevant non-compact direction and turn off all the others, i.e., $b_{1,i} = 0$. At this point, we realize that we still have an SO(3) redundancy in our description among T^{i4} and T^{ij} that can be further eliminated. This is also confirmed by the fact that in the previous step we only used 3 out of the 6 degrees of freedom coming from SO(4).
Choosing $i = 3$ as the only relevant direction ($a_{1,1} = a_{1,2} = r_{1,3} = 0$), we end up with

$$k_1 = a_{1,3}k_{+,3} + c_1 k_D + r_{1,1}k_{R_1} + r_{1,2}k_{R_2},$$

while the other one should be kept generic.
At this point, imposing the first condition in eqs. (4.153) and (4.155), we end up with:

$$\begin{aligned} k_1 &= c_1 k_D, \\ k_2 &= \lambda c_1 k_D + a_{2,1}(k_{+,1} + k_{R_1}) + a_{2,2}(k_{+,2} + k_{R_2}) + a_{2,3}(k_{+,3} + k_{R_3}). \end{aligned}$$

Before imposing the second condition in (4.153), we observe that the commutator between these two vectors vanishes if and only if $c_1 = 0$ or $a_{2,i} = 0$, resulting in parallel Killing vectors. Thus, we conclude that this model cannot provide an explicit realization of the solutions of our interest.

Universal Hypermultiplet

Another possible choice for the quaternionic geometry is given by the *universal hypermultiplet*, whose manifold is defined as

$$\mathcal{M}_{\text{UH}} = \frac{\text{SU}(2,1)}{\text{U}(2)}. \tag{4.156}$$

In our discussion we will mainly follow the description of the geometry given in [52], adapting all the definitions to our conventions.
Explicitly, denoting with $(V, \sigma, \theta, \tau)$ the scalar fields on the manifold, the SU(2) curvatures are given by

$$\begin{aligned} \Omega^1 &= \frac{1}{V^{3/2}}[(\mathrm{d}\sigma + 2\theta \mathrm{d}\tau) \wedge \mathrm{d}\theta + \mathrm{d}\tau \wedge \mathrm{d}V], \\ \Omega^2 &= \frac{1}{V^{3/2}}[(\mathrm{d}\sigma - 2\tau \mathrm{d}\theta) \wedge \mathrm{d}\tau + \mathrm{d}\theta \wedge \mathrm{d}V], \\ \Omega^3 &= \frac{1}{V}\mathrm{d}\theta \wedge \mathrm{d}\tau - \frac{1}{2V^2}[(\mathrm{d}\sigma - 2\tau \mathrm{d}\theta + 2\theta \mathrm{d}\tau) \wedge \mathrm{d}V], \end{aligned} \tag{4.157}$$

while the metric reads

$$\mathrm{d}s^2 = \frac{\mathrm{d}V^2}{4V^2} + \frac{1}{4V^2}\left[\mathrm{d}\sigma + 2\theta \mathrm{d}\tau - 2\tau \mathrm{d}\theta\right]^2 + \frac{1}{V}\left[\mathrm{d}\tau^2 + \mathrm{d}\theta^2\right]. \tag{4.158}$$

Tbe SU(2,1) isometry group of the metric is generated by the following Killing vectors:

$$\vec{k}_1=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \vec{k}_2=\begin{pmatrix}0\\2\theta\\0\\1\end{pmatrix},\quad \vec{k}_3=\begin{pmatrix}0\\-2\tau\\1\\0\end{pmatrix},\quad \vec{k}_4=\begin{pmatrix}0\\0\\-\tau\\\theta\end{pmatrix},$$
$$\vec{k}_5=\begin{pmatrix}2V\sigma\\\sigma\\\theta/2\\\tau/2\end{pmatrix},\quad \vec{k}_6=\begin{pmatrix}\sigma^2-\left(V+\theta^2+\tau^2\right)^2\\\sigma\theta-\tau\left(V+\theta^2+\tau^2\right)\\\sigma\tau+\theta\left(V+\theta^2+\tau^2\right)\end{pmatrix},\tag{4.159}$$
$$\vec{k}_7=\begin{pmatrix}-2V\theta\\-\sigma\theta+V\tau+\tau\left(\theta^2+\tau^2\right)\\\frac{1}{2}\left(V-\theta^2+3\tau^2\right)\\-2\theta\tau-\sigma/2\end{pmatrix},\quad \vec{k}_8=\begin{pmatrix}-\sigma\tau-V\theta-\theta\left(\theta^2+\tau^2\right)\\-2\theta\tau+\sigma/2\\\frac{1}{2}\left(V+3\theta^2-\tau^2\right)\end{pmatrix}.$$

The first three vectors correspond to some constant shift of the coordinates, the fourth is the generator of the rotation symmetry between the θ and τ coordinates and the fifth generates dilatations. The remaining three Killing vectors correspond to some highly non-trivial isometries of the metric.
To let the underlying coset structure emerge more clearly, it is actually convenient to define the following combinations

$$\mathrm{SU}(2)\begin{cases}T_1=\frac{1}{4}(k_2-2k_8),\\T_2=\frac{1}{4}(k_3-2k_7),\\T_3=\frac{1}{4}(k_1+k_6-3k_4),\end{cases}\qquad \mathrm{U}(1)\quad\begin{cases}T_8=\frac{\sqrt{3}}{4}\left(k_4+k_1+k_6\right),\end{cases}\tag{4.160}$$

$$\frac{\mathrm{SU}(2,1)}{\mathrm{U}(2)}\begin{cases}T_4=\mathrm{i}k_5,\\T_5=-\mathrm{i}\frac{1}{2}\left(k_1-k_6\right),\\T_6=-\mathrm{i}\frac{1}{4}\left(k_3+2k_7\right),\\T_7=-\mathrm{i}\frac{1}{4}\left(k_2+2k_8\right).\end{cases}\tag{4.161}$$

that realize explicitly the SU(3) algebra. Notice that the generators T_4, T_5, T_6 and T_7 are imaginary, so that the corresponding real algebra is SU(2,1).
Dropping the imaginary units, we will employ this base to build the prepotentials. In particular, choosing as a base point $(V,\,\sigma,\,\theta,\,\tau)=(1,0,0,0))$, we find that:

$$T_1^u=T_2^u=T_3^u=T_8^u=0,\qquad \begin{cases}T_4^u=\delta^{u0},\\T_5^u=-\delta^{u1},\\T_6^u=-\frac{1}{2}\delta^{u2},\\T_7^u=-\frac{1}{2}\delta^{u3},\end{cases}\tag{4.162}$$

and

$$\mathcal{P}_1^x=-\delta^{x1},\quad \mathcal{P}_2^x=\delta^{x2},\quad \mathcal{P}_3^x=-\delta^{x3},\qquad \mathcal{P}_i^x=0\quad(i=4,5,6,7,8).\tag{4.163}$$

For a generic Killing vector $k_\Lambda=\sum_{i=1}^{8}a_\Lambda^i T_i^u$, the condition in (4.130) implies that

$$a_\Lambda^i=0,\quad \text{for i = 1, 2, 3.}\tag{4.164}$$

At this point, we can consider the request for the presence of a massless vector and, following the same reasoning adopted in the FGP model, we get that, at the base point,

$$T^u_{2,non-compact}=\lambda T^u_{1,non-compact}\quad\longrightarrow\quad a_2^i=\lambda a_1^i,\quad \text{for i = 4, 5, 6, 7 and }\lambda\in\mathbb{R}.\tag{4.165}$$

Imposing all the conditions established so far, the two Killing vectors take the forms

$$\begin{aligned} k_1 &= \sum_{i=4}^{7} a_1^i T_i + a_1^8 T_8, \\ k_2 &= \lambda \sum_{i=4}^{7} a_1^i T_i + a_2^8 T_8. \end{aligned} \tag{4.166}$$

Again, before even imposing the other conditions for a $\mathcal{N} = 2$ supersymmetric vacuum, requiring the Killing vectors commutator to vanish we realize that the only possibility is that the two Killing vectors are parallel.

Therefore, we conclude that both models considered are too simple to capture the physics we would like to describe, and further analyses should be conducted allowing for more complex models.

Chapter 5

Summary and Outlook

In this last chapter, we would like to briefly summarize the content of this thesis and explore possible extensions of the results we have presented.

The aim of this work was to describe the most general black hole solution in the context of $\mathcal{N}=2$ U(1) gauged supergravity coupled to both vector multiplets and hypermultiplets, extending the results presented in [42] for vectors only. Such scenarios naturally emerge in the context of flux compactifications in string theory, and their comprehension is therefore crucial for obtaining a comprehensive view of black hole physics in a theory of quantum gravity.

With this goal in mind, chapter 1 was devoted to a wide introduction to black holes, introducing some basic notions coming from the classical theory of General Relativity and presenting how string theory effectively describes them as a collection of D-branes, allowing, with some caveats, to correctly identify the microscopic geometries that could account for the Bekenstein–Hawking entropy. Additionally, we briefly discussed the embedding of black hole solutions within supergravity theories, giving a first general introduction to the concepts of BPS equations and the attractor mechanism.

In chapter 2, we specialized the discussion on the theory we chose to focus on. In particular, we provided an overview of four-dimensional $\mathcal{N}=2$ gauged supergravity, properly introducing Special and Quaternionic Kähler geometries as essential geometrical ingredients of the theory and describing the general gauging procedure for the isometries of the scalar manifold. The complete bosonic Lagrangian and the fermionic supersymmetry variations were then presented, together with a proper definition of the theory *vacua* and an introduction to the topic of spontaneous supersymmetry breaking.

With the theoretical framework in place, we reviewed in chapter 3 the current understanding of black holes in both ungauged and gauged $\mathcal{N}=2$ theories. Starting from the seminal work in [6], we presented how both BPS and non-BPS black holes allow for a first-order description in terms of a real superpotential W. The same also applies to multicentre black holes, though the non-BPS case is much more involved and requires some insights from the string theory point of view. Finally, we briefly described the work in [42] for AdS black holes, obtained in the context of $\mathcal{N}=2$ U(1) gauged supergravity coupled to vector multiplets only.

Finally, in chapter 4 we addressed the question that inspired this work. The first necessary step was to show how the general theory gets simplified under the joint assumptions of dealing with $U(1)\times U(1)$ gaugings and having one preserved isometry or, equivalently, one massless vector on the solution. In particular, we realized that the equivariance condition forces the triholomorphic prepotentials to be parallel in SU(2) space, thus allowing for a rewriting of the form

$$\mathcal{P}^x_\Lambda = g_\Lambda(q) Q^x(q) \qquad (\text{for } Q^x Q^x = 1) \tag{5.1}$$

together with the definition of

$$\mathcal{L}^x = \mathcal{P}^x_\Lambda L^\Lambda = \mathcal{L} Q^x. \tag{5.2}$$

Starting from the general fermionic supersymmetry variations, we were then able to derive a set of first-order differential equations describing static, spherically symmetric black holes in an asymptotically flat background. Subsequently we showed that, just like all the other known scenarios, the whole solution encompassing both vector and quaternionic scalars could be described in terms of a real superpotential

$$W \equiv e^U \left| \mathcal{Z} - i e^{2(\psi - U)} \mathcal{L} \right|. \tag{5.3}$$

The resulting equations take the form of gradient flows

$$\begin{aligned} U' &= -e^{-2\psi} \partial_U W, \\ \psi' &= e^{-2\psi} \partial_\psi W, \\ (z^i)' &= -2 e^{-2\psi} g^{i\bar{\jmath}} \partial_{\bar{\jmath}} W, \\ (q^u)' &= e^{-2\psi} h^{uv} \partial_v W, \end{aligned} \tag{5.4}$$

or algebraic constraints

$$\begin{aligned} A_t^\Lambda g_\Lambda(q) &= e^{3U - 2\psi} \mathrm{Im}\left(e^{-i\alpha} \mathcal{Z}\right) + e^U \mathrm{Re}\left(e^{-i\alpha} \mathcal{L}\right), \\ e^{2i\alpha} &= \frac{\mathcal{Z} - i e^{2(\psi - U)} \mathcal{L}}{\overline{\mathcal{Z}} + i e^{2(\psi - U)} \overline{\mathcal{L}}}, \\ p^\Lambda g_\Lambda(q) &= -1, \\ \partial_u \alpha &= 0. \end{aligned} \tag{5.5}$$

Since the equations were obtained by imposing two independent projectors on the supersymmetry parameters ϵ_A, the resulting solutions will preserve only 1/4 of the original supersymmetry content, though they may have enhanced supersymmetry at the horizon and at infinity.

Despite the progress presented in this thesis, several important directions for future research remain open. First, there is a need to refine our search for explicit realizations of the class of black holes that we discussed. As demonstrated in chapter 4, models involving one hypermultiplet and one vector multiplet, together with our request of one preserved isometry, do not allow for any supersymmetry breaking patterns. Additionally, the requirement for full $\mathcal{N} = 2$ supersymmetry to be preserved at infinity imposes too stringent a constraint, forcing the Killing vectors chosen in the gauging procedure to be parallel. Therefore, a crucial step forward would be to consider larger models with at least three vector fields that could potentially fulfil our requests.
Another possible direction is to rewrite the bosonic action in a “BPS squared form” to show that the flow equations we’ve derived directly imply the equations of motion. As explored in chapter 3, flow equations can be directly derived from the action after formally integrating out the (t, θ, φ) components and rewriting the resulting effective action as a sum of squares. Applying a similar procedure to our setup does not present any conceptual difficulties, so we expect such a rewriting to be feasible.

Finally, it would be particularly insightful to perform a near-horizon analysis of the solutions, so as to generalize the one conducted in [42] (which did not consider hypermultiplets). Specifically, given the flow information on the warp factors and scalar fields, we would expect the attractor mechanism to be at work also in this scenario, for scalars reaching a critical point at the horizon so as not to diverge and spoil the regularity of the solution. Such an analysis should eventually shed some light onto the details of the mechanism, extracting the explicit conditions at the horizon and determining their general properties.

To conclude, over the last century, black holes have proven themselves to represent profoundly rich and intricate objects within the broader and highly ambitious project of formulating a final theory of quantum gravity, one of the most compelling challenges contemporary theoretical physics has to face. In this context, the work presented in this thesis represents a small contribution in the pursuit of more realistic black hole solutions. And as vast as the complexities involved in the quest might appear, so too are the potential rewards, with each step bringing us even closer to a complete and unified understanding of the Universe we live in.

Appendix A

Normalizations and conventions

- Minkowski metric:

$$\eta_{ab} = (-,+,+,+). \tag{A.1}$$

- Decomposition of tensors in self-dual and antiself-dual parts ($\varepsilon_{0123} = 1$):

$$T^{\mp}_{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} \mp \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}T^{\rho\sigma}\right). \tag{A.2}$$

- Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad [\gamma^\mu, \gamma^\nu] = 2\gamma^{\mu\nu}, \tag{A.3}$$

$$\gamma_5 \equiv -i\gamma_0\gamma_1\gamma_2\gamma_3 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{A.4}$$

- Decomposition of fermions in chiral and antichiral parts:

$$\begin{aligned} \gamma_5 \begin{pmatrix} \lambda^{iA} \\ \zeta_\alpha \\ \psi_A \end{pmatrix} &= \begin{pmatrix} \lambda^{iA} \\ \zeta_\alpha \\ \psi_A \end{pmatrix}, \\ \gamma_5 \begin{pmatrix} \lambda^{i\star}_A \\ \zeta^\alpha \\ \psi^A \end{pmatrix} &= -\begin{pmatrix} \lambda^{i\star}_A \\ \zeta^\alpha \\ \psi^A \end{pmatrix}. \end{aligned} \tag{A.5}$$

- SU(2) and Sp($2n$) metrics:

$$\epsilon^{AB}\epsilon_{BC} = -\delta^A_C, \qquad \epsilon^{AB} = -\epsilon^{BA}; \tag{A.6}$$

$$\mathbb{C}^{\alpha\beta}\mathbb{C}_{\beta\gamma} = -\delta^\alpha_\gamma, \qquad \mathbb{C}^{\alpha\beta} = -\mathbb{C}^{\beta\alpha}. \tag{A.7}$$

For any SU(2) vector P_A we have

$$\epsilon_{AB}P^B = P_A, \qquad \epsilon^{AB}P_B = -P^A. \tag{A.8}$$

The same conventions apply to Sp($2n$) vectors.

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