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Mathematics and social sciences in high schools

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Introduction

This master thesis work aims to understand if the presentation of applications of mathematics of a humanistic/social type in high schools can interest and bring mathematics closer even to those students who do not feel inclined to the subject and who believe it will not be necessary or useful for them in life.

Mathematics is a subject with great potential. Unlike what is believed, it is not used purely in scientific fields. This belief has meant that many students, convinced of the fact that they do not have a scientific mind, judged it as far from their needs and their interests.
It is important to undermine this idea; mathematics is truly present in every area of life, both daily and at work. It has applications in every field, including the humanistic and social fields, considered the furthest from it. When young minds who approach this subject from a school point of view for the first time will have the opportunity to come into contact with the most varied applications of mathematics, ranging between the various possible contexts, their interest in this subject will increase, aware of its usefulness in every field.

The first objective of this work is to analyze the current school context, to understand if applications of this type are present in Italian high school classes, if they are starting to take hold or if they are still mostly absent.
To make this analysis concrete, the research was carried out in three main stages:

- the administration of a questionnaire to mathematics and physics teachers of various Italian high schools, in which the applications of mathematics proposed by them were explored, also trying to understand the reasons for their choices in order to have a more detailed view of this context;
- the analysis of the national guidelines for high schools, looking for references to applications of mathematics that detached themselves from the common physical or chemical applications, and which instead went to investigate more the usefulness of mathematics in the

humanistic/social field;

- the analysis of two cycles of mathematics textbooks commonly used in Italian high schools to verify the presence or absence of insights, examples or exercises that could present humanistic/social applications of the subject to students.

Another objective was to verify the effectiveness of presenting humanistic/social applications to students in order to bring them closer to the subject. To do this we have chosen a possible application of this type and we have designed and then carried out an activity in a class of a mathematical scientific high school.
The humanistic/social application that has been chosen is political power. The activity, on the basis of the necessary prerequisites, was then proposed to a fourth grade of the high school who joined. Political power as an application of mathematics, in addition to being a very interesting application, also gives the possibility of introducing the concept of model, a concept considered very important, increasingly present in the everyday life of each of us. Furthermore, the choice of methodologies to be used in proposing the activity to the class was dictated by recent educational studies, and in particular in mathematics education; we used the theory of the four teaching situations, considered one of the best to introduce a new topic, making students feel directly involved in the definition of new concepts. This makes it easier for students to learn and consolidate the topic.
A verification factor of our hypothesis, in addition to the analysis of the results of the task carried out at the end of the activity, was a satisfaction questionnaire. In the questionnaire there were questions relating to both the satisfaction of the activity and the interest and predisposition to mathematics of each respondents, in this way it was possible to analyze in particular the answers of those who defined themselves as less akin to the subject.

This master thesis consists of five chapters. The first chapter briefly introduces some key concepts of mathematics education, to give a general context on the studies carried out in these first few decades of research, given that the interest in how to teach certain concepts was born recently compared to the interest in pure concepts.
The second chapter instead contains all three fundamental moments, previously described, of the analysis of the Italian context as regards the importance of presenting humanistic/social applications of mathematics to students and as regards the presence of these applications in Italian high schools.
The third chapter has the role of presenting the application in an in-depth manner which then, in an adequate way with respect to the listeners, was presented through the activity mentioned above in a class of a mathematical scientific high school. The chapter therefore explores as much as possible

the issue of calculating an individual's political power in a yes-no voting system through the four most established indexes. It will also present concrete examples of existing yes-no voting systems, with the relative calculation of the four indexes for each member of the system.
The fourth chapter describes the motivation for choosing the activity, the class in which it was carried out, the methodologies used and the structure of the activity. The satisfaction questionnaire is then analyzed and there is also an interesting further possible study of the subject in the IT field, aimed above all at scientific high schools with an applied science option.
The last chapter is dedicated to the conclusions, in which the various information derived from the analyzes conducted intertwine to give an affirmative answer to the question that motivates the development of the thesis.

Chapter 1

Mathematics education

The mathematical sciences are an integral part of our daily lives and are constantly growing to deal with new areas of science and technology, thus being crucial for the economic development and competitiveness of the production system. For this reason they find themselves forced to continually change their scope, leaving a narrow and academic vision of their disciplinary role.

> The truth is that we meet mathematicians everywhere, every day, but we hardly notice it [...] I often think that the best way to change the way people look at mathematics would be to put a red label on everything that use mathematics [...] Our whole life floats like a small boat on a vast ocean of mathematics, but hardly anyone notices it.

Words of Ian Nicholas Stewart (Folkestone, 24 September 1945), British mathematician and writer, emeritus professor of mathematics at the University of Warwick in England, famous as a science fiction writer and science popularizer.
To materialize what Stewart says, if we imagine the life of an average adult in a Western country, it depends on algorithms and mathematical ideas deeply connected to each other. These ideas can be based both on classic mathematical topics and on more recent results, often derived from fields other than those in which the application is then used. Here are some examples, many of them adapted from Committee on the Mathematical Sciences in 2025.

- The radio, or television, from which we hear the news when we wake up, receives signals which are then cleaned of noise and disturbances through some signal processing algorithm. Furthermore, images and sounds will be compressed and converted from similar analog signals.
- Special effects in films use advanced mathematical techniques for their

realization, latest generation algorithms are used in solving equations that describe the different physical processes involved.

- Weather forecasts are based on numerical solutions of systems of non-linear equations, then connected with the statistical analysis of historical series and with data on the behavior of the atmosphere.
- When we use a search engine to browse the web we use a sophisticated mathematical algorithm. Over the years it has been greatly improved, to this day dozens of algorithms also allow you to take into account people's tastes.
- When we buy a plane ticket, or a ticket for a high-speed train, the search site we rely on will use an algorithm to optimize offers and prices.
- Medical investigations such as magnetic resonance or CT are applications of some transforms of functions, Radon and Fourier.
- In the preliminary study phase of a drug, computational models are used to evaluate the interactions of the various molecules with each other and with our organs. Furthermore, the use of a drug on the market depends on statistical techniques that allow you to evaluate the effectiveness.
- Finally, although there would be many other examples, multi-user video games, which allow you to play against opponents from all over the world, are based on languages that are a transcription of pure function calculus developed by Church, and other mathematicians, almost a hundred years ago.

The reasons why the influence of mathematics in science and technology is so powerful lie in several factors: the mathematization of science has not stopped at physics, it has actually reached much more advanced objectives, large amounts of data can be managed today by mathematical technologies, the creation of increasingly efficient algorithms and the considerably increased calculation capacity interact effectively, the abstraction capacity of mathematics allows progress to be translated from one application sector to another quickly identifying the common features.

1.1 What is Mathematics Education?

Mathematics education is a young discipline, and has therefore seen its research interests and methodologies evolve in just a few years.
Initially this discipline was focused exclusively on the mathematical content, due to the belief that it was enough to present the contents well to solve the problems of the learning process.

Mathematics education began its actual development when, in addition to the "mathematical knowledge" variable, the "student" variable and the "teacher" variable were considered. Quoting Cornu-Vergnioux, 1992,

> Addressing the issues of teaching and learning in terms of education means that the transmission of knowledge is a complex phenomenon, which requires numerous mediations, and that it is always necessary to keep together the three poles, of the teacher, of knowledge and of the pupil, but without reducing the analysis to one of the three.

One of the most important aspects of mathematics education is research into mathematics education, which arises as an external reflection on the teaching activity.
It is important to note how different mathematics research and mathematics education research are. The results of the latter are almost never definitive and the evidence is not obtained through demonstrations but is obtained by accumulating multiple sources in support of the thesis which lead to conclusions that can be considered true.
Research in mathematics education essentially presents two classes of results: conclusions, therefore adequate models with respect to a broad field of action within the context being discussed, and experiences which challenge existing models.
Progress in this field clearly goes hand in hand with the development and general pedagogical reflection on learning models, in fact every educational research must have a specific learning model as its starting point.

A fundamental moment in mathematics education is the transition from knowledge to be taught to knowledge taught. An important notion is the notion of **didactic contract** ([3]). Every social interaction is governed by some type of contract which defines the "rights and duties" of those who belong to a certain community, in the form of a social pact, a belief introduced by Rousseau. The didactic contract is the contract that governs the school or class community, it is the set of rules, implicit or explicit, which attributes to each one their own responsibilities with respect to a particular mathematical knowledge taught.
The clauses of the didactic contract that seem to guide the behavior of the students have been studied in depth, one of these is the belief of the students that the problems posed by the teacher always have solutions. It is also noted that children tackle problems by putting into practice pre-established series activated by impulses that are independent from the mathematical meaning of the problem.
These beliefs that are created in students regarding what are the "expected" responses to certain tasks and the activation of procedures based on these beliefs have an important consequence: the error is outlined with respect to

a code of conduct which establishes which is the answer that is expected, which will also be the answer that is mathematically correct. This means that a wrong answer corresponds to an "unexpected" answer but that it is not necessarily mathematically wrong. Conversely, an "expected" answer that turns out to be mathematically incorrect puts the student in crisis, problems like these are said to "break" the didactic contract. An example can be problems where the solution cannot be found, students feel "betrayed" by the teacher who has not complied with the implicit laws of the contract. Very often, however, breaking the didactic contract can be used as a didactic strategy. An example can also be the exchange of roles between students and teachers, this can lead to awareness of unknown aspects and can give the possibility to renegotiate inadequate meanings.

1.2 Didactic action

According to some scholars of the theory of didactic situations (see e.g. [3], [12], [1], [13]), a theory born as a response to various critical issues of the teaching-learning process, to arrive at a new knowledge, the didactic action must be divided into four phases which represent four types of didactic situations:

1. **Action situation**, the teacher's job is to choose and formulate a problem and then exit the scene. The problem must be such as to interest and intrigue students and students must have the means to independently reach the solution, inventing a new procedure or using one they already know. Knowledge at this stage is a means of solving the given problem.

2. **Formulation situation**, students exchange and share the results obtained from an activity previously carried out in class. The language may not be rigorous, so at this stage we also try to create an adequate common vocabulary. This time the teacher has the task of directing the exchanges and underlining some of the students' formulations by repeating them aloud so that everyone gets in touch with the observations of the others. Knowledge this time appears as the result of a personal experience, which in order to be understood by others must be shared, therefore it must be depersonalized and decontextualized.

3. **Validation situation**, the teacher becomes a scholar who evaluates the papers of other scholars, a role assumed by the students. The students' goal is to verify a certain conjecture, the teacher is a sort of speaker of the debate. Students therefore feel active participants in an academic debate, knowledge therefore represents more of a theory that is being constructed than a finished and institutionalized theory.

4. **Institutionalization situation**, the teacher becomes the representative of official mathematics, official culture and introduces students to the officially accepted definitions and terminology and theorems deemed important from the point of view of the institution. Knowledge therefore now has the characteristics of a law, validated by the authority of the institution.

The situation of institutionalization is the most frequent in teaching practice, other situations hardly appear, if not in a degenerate way, i.e. the teacher does not carry out his task correctly but intervenes to ensure that the students give the correct answer.
Those presented are just some of the didactic situations presented by this theory, which, however fine and well-constructed, still deserves to be known and studied.

1.3 Error handling

A very important aspect of mathematics teaching, which unfortunately still today is not valued enough in Italian schools, is error management. Raffaella Borasi, an Italian scholar who has been in America for a long time, has produced revolutionary writings with respect to the approach to error (see e.g. [10], [9], [3]), considering it a true teaching resource.

The error, in the traditional educational setting, is considered a failure, the cause of a low grade. The need to complete a rigid pre-established program forces students and teachers to do mathematics under the pressure of the variable time, thus creating the will to avoid mistakes, which are therefore linked to negative emotions.
Borasi therefore argues that there is a need for time that students can dedicate to "get lost", because getting lost and making mistakes are excellent opportunities for new learning opportunities to be exploited. In one of his written productions, Borasi brings examples of questions that can stimulate the use of error, so as to use them as a starting point for new mathematical explorations.

The teacher's ability to manage errors is therefore important, with "managing" we mean many different actions: noticing the errors, understanding where they come from, finding significant insights starting from the errors and creating paths that lead to thinking about the causes of these errors.
In 2013, Mellone, Ribeiro and Jakobsen introduced a new construct of **interpretative knowledge** ([18]), which deals with the knowledge necessary to be able to interpret the mathematical productions of students. Requiring great attention to the processes, it is strongly linked to the work of argumentation by the students, which is both fundamental from an educational point of view for the student and an essential means for the teacher.

The fact that behind a student's answer there is not a single possible interpretation is truly fundamental in mathematics teaching, it is what leads to differentiating observation and interpretation, making the latter become a working hypothesis and not the absolute truth, being a completely subjective activity. The difference between observation and interpretation is crucial and the fact that correct answers can hide difficulties, and conversely that some errors can develop significant explorations help us to become aware of this difference.
This awareness should then also lead us to take an interest in educational research and its results, key means for increasing one's interpretative knowledge.

1.4 Justification problem and mathematical competence

The justification problem is identified with the need to justify the meaning of mathematics education for everyone. The fundamental role that mathematics plays in today's society is not questioned, but this is not enough to justify the fact that everyone must study mathematics. Some scholars, including Mogens Niss ([21]), a Danish mathematics education researcher, argue that the answer to the justification problem must cover two contexts: on the one hand, it must justify why society has to invest resources to ensure that everyone is taught mathematics, on the other, the most important according to many, it is necessary to justify to the individual the reasons why he must study mathematics throughout his school career, regardless of the path he wants to take in the future.

To find an answer that justifies teaching mathematics to the individual, we want to find the objective of the individual's mathematics education and then show how the achievement of this objective is significant for the individual regardless of future choices.
By addressing this new issue, Niss and his team start from an affinity found between mastery of mathematics and mastery of language, often called literacy. Several studies already exist on the latter, which therefore provide a good starting point for defining mathematical literacy, evocative of the various degrees of mathematical competence.
According to Niss, literacy has three fundamental characteristics:

- mastering a language implies knowing how to use different linguistic registers based on the context in which one finds oneself;
- the main components of language proficiency are the same for a literature teacher and for a primary school student;
- knowledge and skills related to spelling, grammar and vocabulary are

certainly necessary to master the language but they are not sufficient.

These three characteristics, according to an assumption by Niss and his group, must also characterize mathematical literacy. Indeed we note that these components are didactically important:

- the need to present mathematical tools and to challenge students in different contexts, thus having the ability, for example, to notice a situation that can be modeled mathematically;
- the need to work transversely between the various school levels;
- knowledge and skills are certainly necessary to master mathematics but they are not sufficient.

Mathematical competence, that is the goal we are looking for, cannot be reduced to knowledge and skills alone, but, starting from these, it is something more. Starting from these assumptions, Niss proposes a definition of mathematical competence, which is updated and evolved by OECD (The Organization for Economic Cooperation and Development) in the theoretical framework of the famous international survey PISA 2003 and again in PISA 2012 ([29]) and has become the following:

> **Mathematical literacy** is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain, and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens.

Another important result present in the theoretical framework of PISA is the **modeling cycle** (1.1), which explains what is practically meant by mathematical literacy that each individual should acquire from mathematics education. A criticism frequently leveled at mathematics education, especially

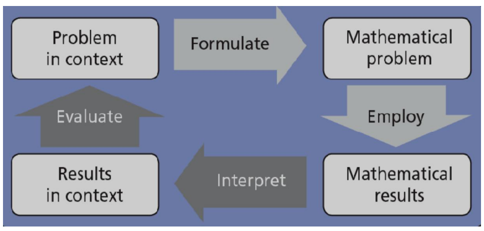


Figure 1.1: Modeling cycle

in the Italian context, is that it focuses mainly, if not only, on the right side of the cycle.

Niss highlights the fact that, despite having given an answer to the justification problem, there are still major difficulties in teaching mathematics. The main critical issues are the following four (it should be noted that each of these is linked to the transversal aspect of school levels):

- the implementation problem, related to the problem of training and extraction of mathematics teachers;
- the problem of transition, linked to the irregularity in the presentation of mathematics going from one school level to another;
- the problem of identity and coherence, the problem of transition then leads to greater difficulties in students who try to give an identity to mathematics and who try to understand what is required in mathematics;
- the problem of assessment, in a context of discontinuity between the various levels, it becomes difficult for students to identify, pursue, characterize and measure a progression in the mastery of mathematics.

In Italy, through various projects, an attempt has been made to find a solution to the critical issues presented by Niss. The projects that contributed the most are the **Mathematics for Citizens** Project and the **M@t.abel** Project.
The first represents a concrete response to the problems of transition, identity and coherence and evaluation. The second, still inspired by the first, has as its central topic the problem of implementation, which remained in the background of the first project.

1.4.1 Problem solving

Problem solving can be considered as an educational goal (see e.g. [3], [16]). It is clear that it is a cross-disciplinary skill, but it is certainly an activity that characterizes doing mathematics, a fact supported by many prominent mathematicians. The reasons why this practice is generally shared are developed on three main levels: that of mathematical education, that linked to the taste and enjoyment of trying to solve problems and that linked to the transversality of the activity which highlights the potential of mathematics education outside the mathematical context.

The first motivation also depends on the certainty that problem solving is able to stimulate intellectual challenges which then lead to a mathematical development of the interested parties.

The second motivation starts from the belief that solving challenging problems is much more interesting than doing mathematics. This may be true in a non-evaluative context, where one challenges oneself and others to solve a problem without having a shadow of evaluation following it. For these reasons it is important to find moments in which, in the classroom, students are confronted with problems with the aim of developing skills and not evaluating them. This method can enhance the pleasure of doing mathematics, whether or not you succeed in solving the given problems.

The third motivation is associated with the training of the adult citizen, the contribution of mathematics education to this training is linked to the possibility of transferring what has been learned to contexts other than the mathematical one. This contribution is therefore related to the evolution of a *forma mentis* that can be applied to different situations and to the realization of productive rather than reproductive processes.

1.5 Conclusion

We conclude by saying that mathematics is, therefore, essential to fully understand reality. It is also important to underline that the mathematization of sciences no longer stops at the classical sciences, but it is expanding and it is also involving the social and biological sciences as well. Unfortunately, however, this central and fundamental role of mathematics is not yet recognized by those who are not specialists in the field, it is therefore a difficult but fundamental task to start changing this altered perception, and there is no more suitable place than school to start with this revolution.

Chapter 2

Mathematics and human/social sciences in Italy

The previous chapter ended by recalling the fact that nowadays the mathematization of the sciences is increasingly involving areas such as the social sciences. In this chapter we want to try to understand if in the Italian context we begin to notice this change in mathematics education, no longer aimed only at pure mathematics, or at most willing to link up with sciences such as physics and chemistry, but also interested in involving in its paths the humanistic/social sciences.

2.1 The survey

The following survey is aimed at investigating how often, in Italian high schools, mathematics teachers consider applications of mathematics not only to subjects in the scientific field, but also to subjects in the humanistic/social field.
Below are the questions of the survey and the related answers obtained. The survey was compiled by 105 Italian high school teachers.

- **Question 1.** Do you usually give space, during the school year, to applications of mathematics in other areas than mathematics itself?
  Possible answers.

  1. Yes, both scientific and humanistic/social applications
  2. Yes, only scientific applications
  3. Yes, only humanistic/social applications

4. No, I don't have the time, if I had I would like to do it

5. No, I don't think it's necessary

Percentage for each response detected:

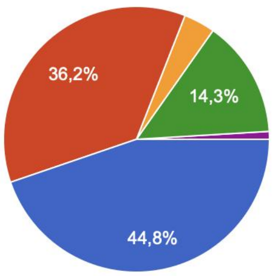
1. $44,8\%$

2. $36,2\%$

3. $3,8\%$

4. $14,3\%$

5. $0,9\%$



If the answer to **Question 1** is 1 or 3, then next questions are:

- **Question 2.** With regard to humanistic/social applications, do you usually collaborate with teachers whose subject is affected by these applications?
  Possible answers.

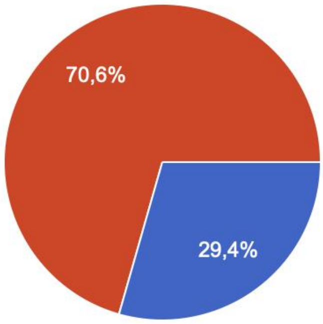
  1. I usually do

  2. I usually don't

  Percentage for each response detected:

  1. $70,6\%$

  2. $29,4\%$



– **Question 3.** What applications did you present to your students in particular?
Possible answers. (Multiple choice)

1. Social choice theory
2. Different voting systems
3. The functioning of the auction
4. Political power of an individual or a group of individuals
5. Simple game theory models applied to historical conflict situations
6. Malthus model
7. Zeno's paradox
8. Aristotelian syllogisms
9. Others

Percentage for each response detected:

1. $9,8\%$
2. $27,5\%$
3. $7,8\%$
4. $13,7\%$
5. $23,5\%$
6. $19,6\%$
7. $54,9\%$
8. $39,2\%$

Other answers detected:

* The phenomenon of recidivism, in the ambit of those admitted to alternative measures to prison detention
* Analysis of graphs to explain phenomena, in particular social phenomena
* Choice problems in general
* Articles relating to statistical surveys on environmental or social issues
* Gambling and mathematical models to describe the trend of an epidemic
* Probability applied to the game, riddles concerning reality, number theory (example: decomposition into prime factors for banking systems), burning mirrors of the Greeks (parables and comics)
* Astronomy and literature, transformers and solar panels, the fiber
* Application to art, drawing and music
* Simple and compound capitalization
* Achilles and the tortoise, games of chance, the golden section in Architecture and in Art
* Text analysis
* Use of probability distributions in the context of choices

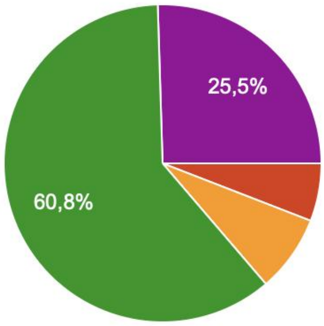
– **Question 4.** Did you find the applications presented to your students useful?
Possible answers.

1. Not at all
2. Not very
3. indifferent
4. Quite
5. Very

Percentage for each response detected:

1. $0,0\%$
2. $5,9\%$
3. $7,8\%$

4. $60,8\%$
5. $25,5\%$



– **Question 5.** Some positive and/or negative aspects of bringing these applications to class.
Open answer. The most common answers to this question are the following, divided into positive and negative aspects.
Positive aspects:

  * Make students perceive that subjects are not closed drawers but that there is global knowledge, a general knowledge of which we can become aware.
  * By experimenting with different applications of the subject, they re-evaluate the very concept of mathematics.
  * By recognizing concepts learned in other subjects, students feel less afraid and see the subject a little closer.
  * Sometimes these concrete applications help those who often wonder what math is for in life.
  * This type of lessons allows you to get away from the classic lesson centered on exercises/problems/theorems.

Negative aspects:

  * There is not enough time to devote to these applications, risking not developing them at their best.
  * Many teachers do not feel prepared enough to present these applications.

Finally there is an aspect that would be good for student learning, but sometimes it is difficult to achieve it due to lack of collaboration between teachers:

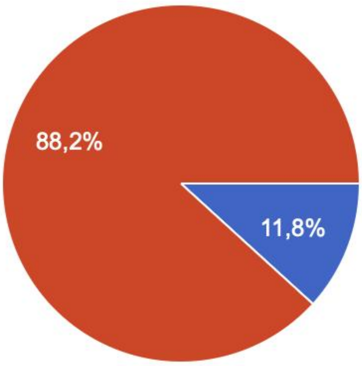
  * The need for greater sharing between different disciplines, especially to integrate the activity with the specific skills of each.

- **Question 6.** Were you aware of all the applications listed in the previous question?
  Possible answers.

  1. Yes, all of them
  2. No, only some of them
  3. No, none of them

  Percentage for each response detected:

  1. $11,8\%$
  2. $88,2\%$
  3. $0,0\%$



- **Question 7.** Now that you have read some of the possible applications of mathematics in the humanistic/social field, are you interested in exploring new applications and trying to present them to your students?
  If you like, you can give your reasons on the "Other" box.
  Possible answers. (Multiple choice)

  1. Yes
  2. No
  3. Other

  Percentage for each response detected:

  1. $86,3\%$
  2. $11,8\%$

The most common answers collected under the "other" option are:

* Yes, in this way one can give transversal information of mathematics.

* Having more time it would be nice to present them, but often the time is not even enough to carry out the standard program.

* New possible applications to be presented are closely linked to finding colleagues who are open and willing to collaborate.

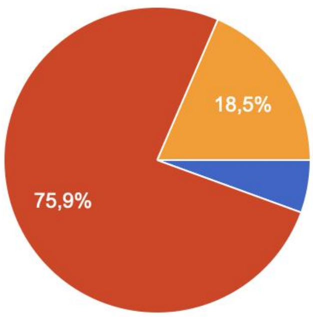
If the answer to **Question 1** is 2, 4 or 5, then next questions are:

- **Question 8.** Some of the possible applications of mathematics in the humanistic social field can be: Aristotle's syllogisms, Zeno's paradoxes, the famous Malthus model, the functioning of the auction, simple game theory models applied to historical conflict situations , the different voting systems, the political power of an individual or a group of individuals, the theory of social choice.
  Were you aware of all these applications? **Possible answers.**

  1. Yes, all of them
  2. No, only some of them
  3. No, none of them

  Percentage for each response detected:

  1. $5,6\%$
  2. $75,9\%$
  3. $18,5\%$



If the answer to **Question 8** is 1 or 2, then next question is:

* **Question 9.** Why, despite knowing some of these applications, have you never presented them to your students?
  Open answer. The most common answers to this question are the following:
  · Lack of time due to programs that are too dense to complete.
  · Lack of in-depth knowledge on the topics on the part of the teachers.
  · Difficulty in searching and creating new content that involves linking with social applications.
  · There has not yet been an opportunity to address these topics.
  · Because those in the scientific field are more congenial.
  · Difficult to link these themes to the program.

If the answer to **Question 8** is 3, then next question is:

* **Question 10.** Now that you have read some of the possible applications of mathematics in the humanistic/social field, are you interested in exploring new applications and trying to present them to your students?
  If you like, you can give your reasons on the "Other" box.
  Possible answers. (Multiple choice)
  1. Yes
  2. No
  3. Other

  Percentage for each response detected:
  1. 80%
  2. 20%

  No answers have been collected under the "other" option.

- **Question 11.** Do you think that applications of this type can bring students closer to mathematics? If you like, you can give your reasons on the "Other" box
  Possible answers. (Multiple choice)
  1. Yes
  2. No

3. Other

Percentage for each response detected:

1. $82,9\%$

2. $17,1\%$

The most common answers collected under the "other" option are:

- Students see mathematics in action: they see this citizen's mathematics as a necessary premise to be able to credibly support opinions, based on data and not sensations: a true bulwark against populism.

- The applications allow you to fully grasp the meaning of mathematical concepts and to grasp their potential.

- Don't know yet.

- Students often ask what is the use of what is done in mathematics, these would be real and useful applications that can arouse interest and bring students closer to the subject.

- **Question 12.** As a teacher, do you think you have been sufficiently trained to deal with applications of this type?
Possible answers.

1. Not at all

2. Not very

3. Quite

4. Very

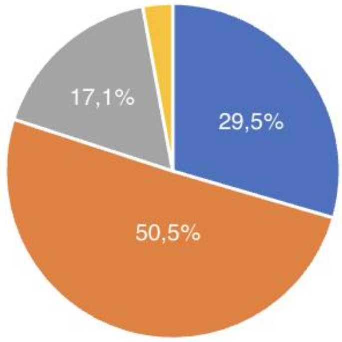
Percentage for each response detected:

1. $29,5\%$

2. $50,5\%$

3. $17,1\%$

4. $2,9\%$



From the answers obtained we can isolate some interesting aspects.
First of all, unlike what we expected, the percentage of teachers who carry out both scientific and humanistic/social applications is higher than the percentage of teachers who carry out only scientific applications. Despite this, as pointed out by some open answers and by Section 2.3, the textbooks most used in high schools do not help teachers. To realize this, it is sufficient to see the scarcity of proposals for applications of a humanistic/social nature present in these textbooks.
After that, we can see that teachers presenting humanistic/social applications, in most cases, do not collaborate with colleagues who deal with the subjects that intersect mathematics in this type of applications. Collaboration between teachers of such different subjects, on the other hand, could make students understand even better the presence of mathematics in all areas, and it is also known that any interdisciplinary project benefits students thanks to the vision of different points of view of the same subject.
Furthermore, despite the percentage approaching 50% of teachers presenting humanistic/social applications, it is interesting to note that 80% of those who filled out the questionnaire do not consider themselves trained in this field, and these teachers often fear presenting applications of this type to the class due to poor preparation. To be more precise, among the teachers presenting applications of a humanistic/social type, 72.6% do not feel adequately prepared to deal with applications of this type. Among the teachers who, on the other hand, do not present this type of application, 87% do not feel adequately prepared; the percentage is much higher as expected.
Finally, it can be concluded by highlighting the fact that almost all teachers believe that this type of humanistic/social applications are to be considered useful for students, both for those who are already interested in the subject and for those who feel less akin to scientific subjects but thanks to these insights they can feel closer to mathematics.

2.2 National Guidelines for high schools

The National Guidelines for high schools ([19]) are intended to set the general objectives, learning objectives and related goals for the development of children's and young people's skills for each discipline or field of experience. To investigate the Italian Context it is necessary to analyse this document, searching for references to applications of mathematics in the humanistic/social field.
References in the document are presented below.
The National Guidelines are divided into sections, every section is dedicated to a different type of high school: artistic high school visual arts address, artistic high school architecture and environment address, artistic high school design address, artistic high school audiovisual and multimedia address, artistic high school graphics address, artistic high school set design address, classical high school, linguistic high school, musical and dance high school, scientific high school, scientific high school applied sciences address, human sciences high school, human sciences high school social economic address. There are some sentences that recall applications of mathematics in the humanistic/social field in common for every type of high school listed before in the subsection *"Mathematics"*:

- *"The student will have acquired a historical-critical view of the relationship between the main themes of mathematical thought and the philosophical, scientific and technological context. In particular, he will have acquired the meaning and scope of the three main moments that characterize the formation of mathematical thought: mathematics in Greek civilization, the infinitesimal calculus that was born with the scientific revolution of the seventeenth century and which leads to the mathematization of the physical world, the which takes its cue from Enlightenment rationalism and which leads to the formation of modern mathematics and to a new process of mathematization which invests new fields (technology, **social**, economic, biological sciences) and which has changed the face of scientific knowledge."*

- *"This articulation of themes and approaches will form the basis for establishing links and conceptual and methodological comparisons with other disciplines such as physics, natural sciences, **philosophy** and **history**."*

Then there are some sentences that are specific for every high school. In the subsection *"Mathematics"* of all different addresses of linguistic high schools it is written: *"In the artistic high school, particular attention will be paid to all those mathematical concepts and techniques that have particular relevance in the **graphic**, **pictorial** and **architectural arts** and which pertain in particular to analytical, descriptive and projective geometry."*

In the subsection *"Mathematics"* of the classical high school, all different addresses of linguist high school and musical and dance high school it is written: *"In the classical high school, particular attention will be paid to the relationship between mathematical thought and **philosophical thought**; in the linguistic high school, to the role of linguistic expression in mathematical reasoning; in the musical and dance high school, to the role of mathematical structures in **musical language**; in the high school of human sciences, to a critical view of the role of mathematical modeling in the **analysis of social processes**."*
Finally, in the paragraph *"Data and Forecasts"* of the second two-year period in the section *"Mathematics"* of the human science high school social economic address there are some additional interesting sentences:

- *"The multidisciplinary teaching of the human sciences, to be foreseen in close contact with the economy and legal disciplines, mathematics, geography, philosophy, history, literature, provides the student with the useful skills:*

    1. *to understand the dynamics of social reality, with particular attention to the world of work, personal services, intercultural phenomena and the contexts of coexistence and the construction of citizenship;*

    2. *to understand the sociopolitical and economic transformations induced by the phenomenon of globalization, the issues relating to the management of multiculturalism and the sociopolitical and economic significance of the so-called "third sector";*

    3. *to develop an adequate cultural awareness of psycho-social dynamics;*

    4. *to master the principles, methods and techniques of research in the economic-social field."*

- *"The use of mathematics in social and economic disciplines will be explored according to a modeling approach."*

It is clear that applications of mathematics to humanistic/social field are rarely mentioned, except for human science high school social economic address, where it is underlined the usefulness of studying the relations between human science and mathematics.

2.3 High school textbooks

We analyzed two complete cycles of math high school books. What we were searching for were some references, in any form, to humanistic/social application of mathematics. The cycles of book analyzed are *Matematica* *blu* for the first ([4]), second ([5]) grade of high school, and *Matematica blu 2.0* for the third ([6]), fourth ([7]) and fifth ([8]) grade of high school, both editions by Bergamini, Barozzi and Trifone and *I colori della Matematica*, for all the five grades of high school ([24], [25], [26], [27], [28]), by Sasso and Zanone.

Matematica blu. Each chapter of this book is divided into a first theoretical part, inside which there are two particular paragraphs called "Exploration" and "Problems, reasoning, deductions", which propose insights of different types. At the beginning of each chapter there is a question relating to different areas, the answer to which is then found at the end of the theoretical part, we will call this part "Question&Answer".
After this first theoretical part there is a part with a considerable number of exercises followed by two other particular paragraphs, called "Laboratory of mathematics" and "Mathematics for the citizen". The first aims to give some connection points between the topics covered and the computer programs that can be used in support, the second, instead, wants to take a look at the applications of mathematics in the life of each of us, as the name itself suggests.
Each chapter concludes with "end of chapter tests".

Matematica blu 2.0. Each chapter of this book is divided into a first theoretical part, inside which there is, as in *Matematica blu*, the paragraph "Exploration" and "Question&Answer".
Between the theoretical part and the part of exercises there is the paragraph "Laboratory of mathematics", the same one present in the previous edition we talked about above.
After the part of exercises, always numerous, there is a new paragraph, which replaces the one called "Mathematics for the citizen", which is called "Reality and models". This paragraph consists of a maximum of two pages in which there are exercises based on real situations.
Each chapter concludes with "Towards the state exam", a section in which there are exercises which, as the name implies, have the aim of preparing the student for the state exam.

I colori della matematica. This book is divided into two main parts, the physical book and the online book. Each chapter of the physical book is divided into a first theoretical part, and a second part in which there are many exercises. In this last part there are different types of exercises: the classical ones that do not have a particular denomination, the ones under the denomination of "Reality and models", that consist on exercises based on real situations, the ones under the denomination of "From the newspaper", that are exercises based on newspaper articles, and finally the ones under the denomination of "Mathematics and physics/chemistry/history/economics/electronics/...", that are exercises that link mathematics to other subjects.

Each chapter of the online book can contain several sections: insights, mathematics in reality, mathematics in history, use of different tools and problem solving. Not all of the sections listed are in every chapter.

After reading all the textbooks looking for humanistic/social applications of mathematics between the chapters, we created a list containing everything we found, specifying in which specific section of the chapter it was present. We propose the list below:

- **Matematica blu 1:**
  - page 91, Exploration: *Numbers and music*;
  - page 176, Question&Answer: *Blood groups*;
  - page 247, Exploration: *World population model*;
  - page 283, Mathematics for the citizen: *Genealogical tree*;
  - page α10, Exploration: *Is smoking bad?*;
  - page G23, Question&Answer: *Without compass*;
  - page G54, Exploration: *Triangles on doors, mathematics and architecture.*
- **Matematica blu 2:**
  - page 1130, Exploration: *Canon, music and geometric transformations*;
  - page β18, Exploration: *Lottery game.*
- **Matematica blu 2.0 3:**
  - page 88, Exploration: *Cryptography*;
  - page 258, Question&Answer: *Measurement of the diameter of a tree trunk with three rods*;
  - page 393, Exploration: *The focus of the president (Meeting between architecture and politics)*;
  - page 430, Reality and Models: *It is whispered but it is heard*;
  - page 628, Reality and Models: *A bacterial population*;
  - page β44, Exploration: *Statistics and the labor market*;
  - page β45, Question&Answer: *Reliability of survey results.*
- **Matematica blu 2.0 4:**
  - page 1099, Exploration: *Debt, deficit and GDP*;

  - page α17, Question&Answer: *Always around (trip optimization of a salesman).*

- **Matematica blu 2.0 5:**
  - page 1366, Question&Answer: *The right price (supply and demand curve)*;
  - page 1398, Reality and Models: *Flu*;
  - page 1562, Reality and Models: *Personal income tax (Irpef)*;
  - page 1585, Exploration: *Zeno's paradoxes*;
  - page 1649, Question&Answer: *Inflation*;
  - page 1725, Exploration: *Fines, sicve and speed cameras compared*;
  - page 1732, Question&Answer: *Survival boards*;
  - page 2032, Question&Answer: *Shape of the Eiffel Tower*;
  - page 2092, Exploration: *Prey and predators*;
  - page 2118, Reality and Models: *Force interest*;
  - page σ31, Question&Answer: *Math and law.*

- **I colori della matematica 1:**
  - page 44, Reality and Models: *Archaeologist*;
  - unit 2 online, Mathematics in reality: *Proportional electoral law*;
  - page 120, From the Newspaper: *Apple's revenue decline*;
  - page 243, Reality and Models: *Construction of formulas for car tax*;
  - page 264, From the Newspaper: *Unemployment rate*;
  - page 299, Mathematics and Economics: *Investments (interest rate, fixed installment, compound capitalization regime, amount, capital,...)*;
  - unit 7 online, Mathematics in reality: *Cryptography*;
  - page 362, Mathematics and Economics: *Wages, production of a good, increase in prices*;
  - page 407, From the Newspaper: *Easter holidays 2016*;
  - page 457, From the Newspaper: *2020 without cocoa*;

  - page 567, Mathematics and Economics: *Investments (interest rate, fixed installment, compound capitalization regime, amount, capital,...)*;
  - page 603, Mathematics and Economics: *Investments (interest rate, fixed installment, compound capitalization regime, amount, capital,...)*;
  - unit 15 online, Mathematics in reality: *Weighted average and city taxis*;
  - page 664 and 665, Reading graphs: *Diffusion of social media in the period 2005-2010 by age group. Percentage incidence on GDP of pension expenditure in Italy in the years 1981-2013. Comparison of causes of mortality in the years 1931, 1971, 2012*;
  - page 665, From the Newspaper: *Benetton group sales in 2015*;
  - unit 17 online, Mathematics in reality: *Tetris.*

- **I colori della matematica 2:**
  - page 184 and 185, Choice problems: *More convenient options for companies, banks, telephone rates, rentals,...*;
  - page 307, From the Newspaper: *China: new promised land for Italians*;
  - page 573, From the Newspaper: *"Scratch and win": to make ends meet the state gambles.*

- **I colori della matematica 3:**
  - page 53, Reality and Models: *Evolution of an epidemic*;
  - page 66, Mathematics and Economics: *Investments (interest rate, fixed installment, compound capitalization regime, amount, capital,...)*;
  - unit 2 online, Mathematics in reality: *Cryptography* (exactly the same resource used in volume 1);
  - page 163, Reality and Models: *Growth of an insect population*;
  - page 175, Reality and Models: *Types of contracts*;
  - unit 5 online, Insights: *Linear programming problems: maximizing profit*;
  - page from 272 to 274, Choice problems: *More convenient options for companies, banks, telephone rates, rentals,...*;
  - page 493, Reality and Models: *Internet*;

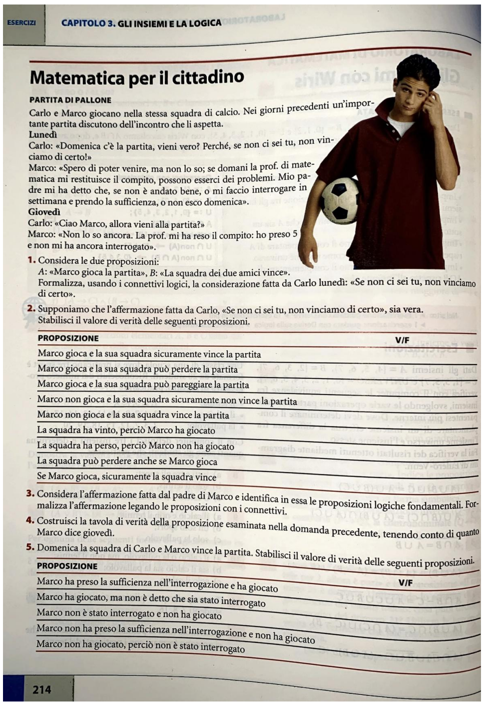
  - page 495, Reality and Models: *Flu epidemic*;
  - page 570, Reality and Models: *Spread of a fire*;
  - page 738, Reading graphs: *Diffusion of social media in the period 2005-2010 by age group. Percentage incidence on GDP of pension expenditure in Italy in the years 1981-2013. Comparison of causes of mortality in the years 1931, 1971, 2012* (exactly the same exercises as in volume 1);
  - page 738, From the Newspaper: *Benetton group sales in 2015* (exactly the same exercise as in volume 1).

- **I colori della matematica 4:**
  - page 26, Mathematics and Economics: *Investments (interest rate, fixed installment, compound capitalization regime, amount, capital,...)*;
  - page 26, Reality and Models: *Fish population. Half-life of a drug. Bacteria. Population of a city*;
  - page 27, Mathematics and Economics: *Investments (interest rate, fixed installment, compound capitalization regime, amount, capital,...)*;
  - page 35, Reality and Models: *Bacteria. Parasites. Absorption of a drug*;
  - page 45, Reality and Models: *Colony of bacteria. Concentration of a drug*;
  - page 84, Mathematics and Economics: *Upright. Inflation rate. Compound capitalisation. Price of a good*;
  - page 84 and 85, Reality and Models: *Half-life of a drug. Concentration of a drug in the blood*;
  - page 87, Reality and Models: *Alcohol and accident risk. Archaeology*;
  - page 100, Reality and Models: *Organ donors. Influenza incidence rate*;
  - page 548, Reality and Models: *Queues at the tollbooth. Evolution of a fish population*;
  - page 557, Problems from reality: *Bacteria. Defective parts. Evolution of a population. An investment*;
  - page 570 and 571, Problems from reality: *SIS model. Gompertz model*;

- page 619 and 620, Reality and Models: *Absorption of a drug. Evolution of a population.*

- **I colori della matematica 5:**
  - page 46, Reality and Models: *Spread of an epidemic. Average cost*;
  - page 78, Reality and Models: *Spread of an epidemic*
  - page 128, Reality and Models: *Bacterial reproduction*;
  - page 149, Reality and Models: *Estimate of an animal population*;
  - page 156, Problems from reality: *Growth of a population. Growth of a tumor*;
  - page 312, Reality and Models: *Spread of the flu*;
  - page 334, Reality and Models: *Evolution of a population of lady-bugs*;
  - page 391, Reality and Models: *Growth of an insect population. Concentration of a drug in the blood*;
  - page 454, Problems from reality: *Growth of a colony of bacteria. Dissemination of a product. Evolution of a population*;
  - page 510, Reality and Models: *Lotto game*;
  - page 511, Reality and Models: *Booking an airline ticket.*

We can notice that in the cycle of textbooks "Matematica blu" and "Matematica blu 2.0" most of the humanistic/social applications found are under the section called "Explorations". The part of exercises called "Mathematics for citizens" in volumes 1 and 2, which becomes "Reality and Models" in the following three volumes, for the most part does not present situations that are actually useful for the citizen or that represent reality and introduce models.
We report two examples of exercises which, despite being realistic in nature, are not really useful. The first example is taken from "Mathematics for citizens":



ESERCIZI CAPITOLO 3. GLI INSIEMI E LA LOGICA

Matematica per il cittadino

PARTITA DI PALLONE

Carlo e Marco giocano nella stessa squadra di calcio. Nei giorni precedenti un'importante partita discutono dell'incontro che li aspetta.

Lunedì

Carlo: «Domenica c'è la partita, vieni vero? Perché, se non ci sei tu, non vinciamo di certo!»

Marco: «Spero di poter venire, ma non lo so; se domani la prof. di matematica mi restituisce il compito, possono esserci dei problemi. Mio padre mi ha detto che, se non è andato bene, o mi faccio interrogare in settimana e prendo la sufficienza, o non esco domenica».

Giovedì

Carlo: «Ciao Marco, allora vieni alla partita?»

Marco: «Non lo so ancora. La prof. mi ha reso il compito: ho preso 5 e non mi ha ancora interrogato».

1. Considera le due proposizioni:
A: «Marco gioca la partita», *B*: «La squadra dei due amici vince».
Formalizza, usando i connettivi logici, la considerazione fatta da Carlo lunedì: «Se non ci sei tu, non vinciamo di certo».

2. Supponiamo che l'affermazione fatta da Carlo, «Se non ci sei tu, non vinciamo di certo», sia vera. Stabilisci il valore di verità delle seguenti proposizioni.

PROPOSIZIONE	V/F
Marco gioca e la sua squadra sicuramente vince la partita	
Marco gioca e la sua squadra può perdere la partita	
Marco gioca e la sua squadra può pareggiare la partita	
Marco non gioca e la sua squadra sicuramente non vince la partita	
Marco non gioca e la sua squadra vince la partita	
La squadra ha vinto, perciò Marco ha giocato	
La squadra ha perso, perciò Marco non ha giocato	
La squadra può perdere anche se Marco gioca	
Se Marco gioca, sicuramente la squadra vince	

3. Considera l'affermazione fatta dal padre di Marco e identifica in essa le proposizioni logiche fondamentali. Formalizza l'affermazione legando le proposizioni con i connettivi.

4. Costruisci la tavola di verità della proposizione esaminata nella domanda precedente, tenendo conto di quanto Marco dice giovedì.

5. Domenica la squadra di Carlo e Marco vince la partita. Stabilisci il valore di verità delle seguenti proposizioni.

PROPOSIZIONE	V/F
Marco ha preso la sufficienza nell'interrogazione e ha giocato	
Marco ha giocato, ma non è detto che sia stato interrogato	
Marco non è stato interrogato e non ha giocato	
Marco non ha preso la sufficienza nell'interrogazione e non ha giocato	
Marco non ha giocato, perciò non è stato interrogato	

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The proposed exercise is interesting as a test of the student's understanding, but it does not help the citizen in everyday life in any way. It is a classic exercise, contextualized in a situation that we can define as real.
The second example is taken from "Reality and Models":

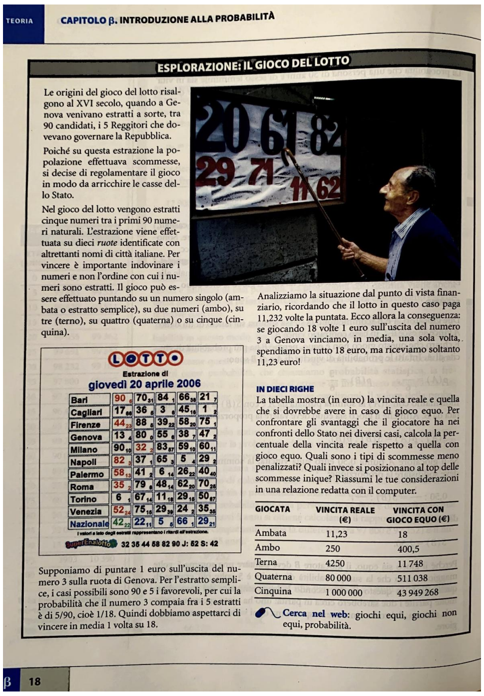
3 **Il planisfero**

Un bambino vuole colorare ogni continente di un planisfero con un colore diverso e per fare questo ha a disposizione 10 colori.

- In quanti modi può colorare i continenti?
- Se pittura subito l'Europa di verde, in quanti modi può poi colorare gli altri continenti? Qual è la relazione con il caso precedente?
- Quanti colori dovrebbe avere a disposizione per poter colorare Asia e Africa in più di dieci possibili modi?

Also in this case, as in the previous example, we are faced with a classic exercise, in a real context, which however does not in any way help the student to discover new sides of mathematics in reality and does not introduce the concept of model.
Unlike the humanistic/social applications we are looking for, scientific applications, of a physical/chemical type, are much more present in this cycle of books, especially in the "Reality and Models" exercises.

It is also interesting to note that among the humanistic/social applications proposed in the questionnaire and those collected from the answers, seen previously, in this cycle of textbooks we find only a few references to the lottery game, some applications of mathematics to architecture, some references to statistical surveys and a paragraph dedicated to Zeno's paradoxes. We report below some examples of the references we found. The first example is the Exploration *Lottery game*:



TEORIA CAPITOLO β. INTRODUZIONE ALLA PROBABILITÀ

ESPLORAZIONE: IL GIOCO DEL LOTTO

Le origini del gioco del lotto risalgono al XVI secolo, quando a Genova venivano estratti a sorte, tra 90 candidati, i 5 Reggitori che dovevano governare la Repubblica.

Poiché su questa estrazione la popolazione effettuava scommesse, si decise di regolamentare il gioco in modo da arricchire le casse dello Stato.

Nel gioco del lotto vengono estratti cinque numeri tra i primi 90 numeri naturali. L'estrazione viene effettuata su dieci *ruote* identificate con altrettanti nomi di città italiane. Per vincere è importante indovinare i numeri e non l'ordine con cui i numeri sono estratti. Il gioco può essere effettuato puntando su un numero singolo (ambata o estratto semplice), su due numeri (ambo), su tre (terno), su quattro (quaterna) o su cinque (cinquina).

LOTTO

Estrazione di

giovedì 20 aprile 2006

Bari	90_{6}	70_{31}	84_{1}	66_{38}	21_{7}
Cagliari	17_{66}	36_{6}	3_{6}	45_{16}	1_{2}
Firenze	44_{23}	88_{8}	39_{22}	58_{20}	75_{1}
Genova	13_{4}	80_{0}	55_{6}	38_{7}	47_{2}
Milano	90_{10}	32_{2}	83_{87}	59_{10}	60_{11}
Napoli	82_{3}	37_{1}	65_{3}	5_{4}	29_{2}
Palermo	58_{13}	41_{2}	6_{14}	26_{22}	40_{40}
Roma	35_{2}	79_{8}	48_{14}	62_{20}	70_{26}
Torino	6_{1}	67_{14}	11_{18}	29_{16}	50_{87}
Venezia	52_{24}	75_{16}	29_{39}	24_{2}	35_{36}
Nazionale	42_{22}	22_{11}	5_{8}	66_{1}	29_{21}

I valori a lato degli estratti rappresentano i ritardi all'estrazione.

SuperEnalotto 32 35 44 58 82 90 J: 52 S: 42

Supponiamo di puntare 1 euro sull'uscita del numero 3 sulla ruota di Genova. Per l'estratto semplice, i casi possibili sono 90 e 5 i favorevoli, per cui la probabilità che il numero 3 compaia fra i 5 estratti è di 5/90, cioè 1/18. Quindi dobbiamo aspettarci di vincere in media 1 volta su 18.

Analizziamo la situazione dal punto di vista finanziario, ricordando che il lotto in questo caso paga 11,232 volte la puntata. Ecco allora la conseguenza: se giocando 18 volte 1 euro sull'uscita del numero 3 a Genova vinciamo, in media, una sola volta, spendiamo in tutto 18 euro, ma riceviamo soltanto 11,23 euro!

IN DIECI RIGHE

La tabella mostra (in euro) la vincita reale e quella che si dovrebbe avere in caso di gioco equo. Per confrontare gli svantaggi che il giocatore ha nei confronti dello Stato nei diversi casi, calcola la percentuale della vincita reale rispetto a quella con gioco equo. Quali sono i tipi di scommesse meno penalizzati? Quali invece si posizionano al top delle scommesse inique? Riassumi le tue considerazioni in una relazione redatta con il computer.

GIOCATA	VINCITA REALE (€)	VINCITA CON GIOCO EQUO (€)
Ambata	11,23	18
Ambo	250	400,5
Terna	4250	11 748
Quaterna	80 000	511 038
Cinquina	1 000 000	43 949 268

Cerca nel web: giochi equi, giochi non equi, probabilità.

β 18

The second example is the Exploration *The focus of the president*:

ESPLORAZIONE L'ELLISSE IN ARCHITETTURA

ESPLORAZIONE

L'ellisse in architettura

Il fuoco del presidente

La National Statuary Hall, situata all'interno del Campidoglio di Washington (sede dei due rami del Congresso degli Stati Uniti), ospita una collezione di cento statue donate dagli stati dell'Unione, che rappresentano importanti personalità della storia del Paese.
In origine era la sede della Camera dei rappresentanti (l'equivalente della nostra Camera dei deputati), ma nel 1857, per i problemi acustici dovuti alla sua forma ellittica, i parlamentari si trasferirono in un'altra sala del Campidoglio.
Si racconta che John Quincy Adams, sesto presidente degli Stati Uniti (1825-1829), quando tornò a frequentare la Hall come membro del Congresso, scelse una posizione che corrispondeva a uno dei fuochi dell'ellisse che costituiva la pianta della sala. In questo modo poteva sfruttare la sua acustica per origliare i discorsi dei deputati che avevano i propri banchi nei pressi dell'altro fuoco.
Un metodo ingegnoso e lecito di spionaggio politico, molto prima del Watergate.

L'ellisse nel Barocco

Nell'epoca del Barocco l'ellisse venne usata dagli architetti sia nella pianta delle chiese sia in quella delle piazze. L'esempio più famoso è piazza San Pietro a Roma, opera del Bernini. Nel centro di simmetria della sua parte ellittica c'è un obelisco e poco distante, nella pavimentazione, c'è una pietra circolare che segna uno dei due fuochi. Stando su quella pietra, il colonnato, che è composto da quattro file di colonne, per un gioco prospettico appare come se fosse formato da una sola fila.

The third example is the Question&Answer *Reliability of survey results*:

POSSIAMO FIDARCI?
Quanto sono attendibili i risultati dei sondaggi?

Il quesito completo a pag. β1

In Italia vi sono sessanta milioni di abitanti. I sondaggi, al massimo, interpellano poche migliaia di individui. Eppure i risultati, per lo più, non si discostano molto dai valori reali e riescono a presumere gli orientamenti della popolazione con un accettabile grado di approssimazione. Com'è possibile?

Per capire il metodo della cosiddetta «inferenza statistica», su cui si basano i sondaggi, si può ricorrere a un modello molto semplice: il lancio della moneta. Dal punto di vista del calcolo delle probabilità, infatti, intervistare i cittadini equivale a contare il numero di esiti testa o croce. Si tratta solo di capire quante monete si devono lanciare (ovvero quante persone interpellare) perché la percentuale di esiti che osserviamo sia vicina, con un piccolo margine di errore, alla vera probabilità dell'evento stesso.
Per la legge dei grandi numeri, maggiore è il numero dei lanci, più alta è la probabilità che la percentuale degli esiti (per esempio, testa) si approssimi al 50%. Analogamente, maggiore è il numero di intervistati in un sondaggio, tanto più il risultato ottenuto sarà vicino alla realtà.

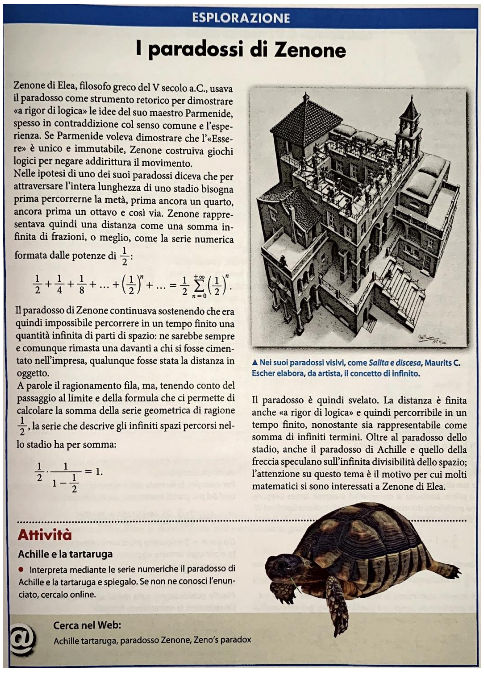
Per un grande numero di lanci, la distribuzione di probabilità del lancio delle monete può essere ricondotta alla gaussiana standard. Utilizzando la curva a campana, si può determinare un criterio per stabilire il margine di errore, valido anche per i sondaggi. Si tratta di determinare l'intervallo in cui l'area al di sotto della curva a campana equivale a una probabilità totale del 95% (ovvero la probabilità che i risultati siano esatti diciannove volte su venti).
Una delle proprietà della gaussiana è che il 95,45% dei valori tende a essere incluso nell'intervallo di estremi $M - 2\sigma$ e $M + 2\sigma$. Nel caso della curva normale standardizzata, la deviazione standard è pari a 1 e il valore medio a 0. Pertanto, si può dimostrare che lanciando N monete la stima che i risultati siano esatti al 95% è affetta da un margine di errore pari a $\frac{98}{\sqrt{N}}$.

In altre parole significa che, su 20 serie di lanci, per 19 volte la frazione che dà il numero di teste uscite differirà dalla vera probabilità di una percentuale non superiore al margine di errore calcolato.
Per fare qualche esempio, per $N = 10$, il margine di errore è pari al 31%; per $N = 1000$, il margine di errore che si ottiene è pari a circa il 3,1%.

La stessa formula viene utilizzata nei sondaggi: vale a dire che, se intervistiamo 1000 persone, nel 95% dei casi otterremo una percentuale di risposte che si discosta al massimo del 3,1% rispetto alla media della popolazione generale. Ecco quindi la formula per determinare il margine di errore di un sondaggio: 98% diviso la radice quadrata del numero delle persone intervistate.

I sondaggi, comunque, hanno i loro limiti anche quando il margine di errore è molto piccolo. Per questo a volte sbagliano, come è effettivamente successo prima del voto elettorale o durante gli exit poll. Nel primo caso, il risultato può essere sbagliato perché i sondaggi non sono in grado di rendere conto dei cambiamenti futuri e fino al giorno prima del voto qualcosa può cambiare; nel secondo, i risultati possono essere inficiati dalle false risposte dei cittadini. Altre volte, inoltre, il vantaggio di una parte politica sull'altra è così piccolo da sfidare qualunque previsione.

The last example is the Exploration *Zeno's paradoxes*:



ESPLORAZIONE

I paradossi di Zenone

Zenone di Elea, filosofo greco del V secolo a.C., usava il paradosso come strumento retorico per dimostrare «a rigor di logica» le idee del suo maestro Parmenide, spesso in contraddizione col senso comune e l'esperienza. Se Parmenide voleva dimostrare che l'«Essere» è unico e immutabile, Zenone costruiva giochi logici per negare addirittura il movimento.
Nelle ipotesi di uno dei suoi paradossi diceva che per attraversare l'intera lunghezza di uno stadio bisogna prima percorrerne la metà, prima ancora un quarto, ancora prima un ottavo e così via. Zenone rappresentava quindi una distanza come una somma infinita di frazioni, o meglio, come la serie numerica formata dalle potenze di $\frac{1}{2}$:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \left(\frac{1}{2}\right)^n + \ldots = \frac{1}{2}\sum_{n=0}^{+\infty}\left(\frac{1}{2}\right)^n.$$

Il paradosso di Zenone continuava sostenendo che era quindi impossibile percorrere in un tempo finito una quantità infinita di parti di spazio: ne sarebbe sempre e comunque rimasta una davanti a chi si fosse cimentato nell'impresa, qualunque fosse stata la distanza in oggetto.
A parole il ragionamento fila, ma, tenendo conto del passaggio al limite e della formula che ci permette di calcolare la somma della serie geometrica di ragione $\frac{1}{2}$, la serie che descrive gli infiniti spazi percorsi nello stadio ha per somma:

$$\frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2}} = 1.$$

▲ Nei suoi paradossi visivi, come *Salita e discesa*, Maurits C. Escher elabora, da artista, il concetto di infinito.

Il paradosso è quindi svelato. La distanza è finita anche «a rigor di logica» e quindi percorribile in un tempo finito, nonostante sia rappresentabile come somma di infiniti termini. Oltre al paradosso dello stadio, anche il paradosso di Achille e quello della freccia speculano sull'infinita divisibilità dello spazio; l'attenzione su questo tema è il motivo per cui molti matematici si sono interessati a Zenone di Elea.

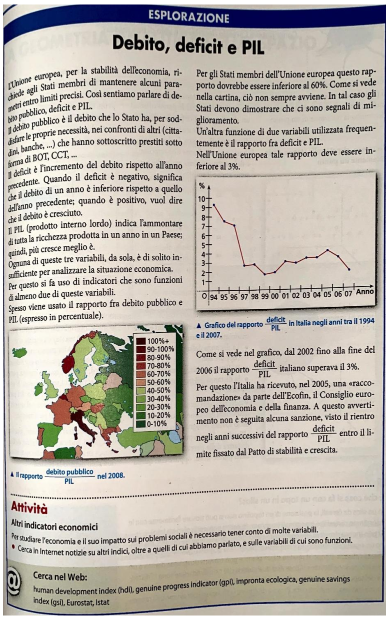
Attività

Achille e la tartaruga

- Interpreta mediante le serie numeriche il paradosso di Achille e la tartaruga e spiegalo. Se non ne conosci l'enunciato, cercalo online.

Cerca nel Web:
Achille tartaruga, paradosso Zenone, Zeno's paradox

Certainly there are some other interesting ideas in these textbooks, such as for example the exploration dedicated to *Dept, deficit and GDP* or the question&answer dedicated to *Inflation*, which we report below:



ESPLORAZIONE

Debito, deficit e PIL

L'Unione europea, per la stabilità dell'economia, richiede agli Stati membri di mantenere alcuni parametri entro limiti precisi. Così sentiamo parlare di debito pubblico, deficit e PIL.
Il debito pubblico è il debito che lo Stato ha, per soddisfare le proprie necessità, nei confronti di altri (cittadini, banche, ...) che hanno sottoscritto prestiti sotto forma di BOT, CCT, ...
Il deficit è l'incremento del debito rispetto all'anno precedente. Quando il deficit è negativo, significa che il debito di un anno è inferiore rispetto a quello dell'anno precedente; quando è positivo, vuol dire che il debito è cresciuto.
Il PIL (prodotto interno lordo) indica l'ammontare di tutta la ricchezza prodotta in un anno in un Paese; quindi, più cresce meglio è.
Ognuna di queste tre variabili, da sola, è di solito insufficiente per analizzare la situazione economica.
Per questo si fa uso di indicatori che sono funzioni di almeno due di queste variabili.
Spesso viene usato il rapporto fra debito pubblico e PIL (espresso in percentuale).

100%+
90-100%
80-90%
70-80%
60-70%
50-60%
40-50%
30-40%
20-30%
10-20%
0-10%

▲ Il rapporto $\frac{\text{debito pubblico}}{\text{PIL}}$ nel 2008.

Per gli Stati membri dell'Unione europea questo rapporto dovrebbe essere inferiore al 60%. Come si vede nella cartina, ciò non sempre avviene. In tal caso gli Stati devono dimostrare che ci sono segnali di miglioramento.
Un'altra funzione di due variabili utilizzata frequentemente è il rapporto fra deficit e PIL.
Nell'Unione europea tale rapporto deve essere inferiore al 3%.

%
10 9 8 7 6 5 4 3 2 1
O 94 95 96 97 98 99 00 01 02 03 04 05 06 07 Anno

▲ Grafico del rapporto $\frac{\text{deficit}}{\text{PIL}}$ in Italia negli anni tra il 1994 e il 2007.

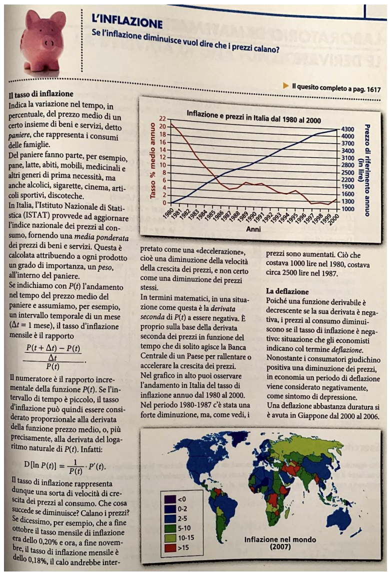
Come si vede nel grafico, dal 2002 fino alla fine del 2006 il rapporto $\frac{\text{deficit}}{\text{PIL}}$ italiano superava il 3%.
Per questo l'Italia ha ricevuto, nel 2005, una «raccomandazione» da parte dell'Ecofin, il Consiglio europeo dell'economia e della finanza. A questo avvertimento non è seguita alcuna sanzione, visto il rientro negli anni successivi del rapporto $\frac{\text{deficit}}{\text{PIL}}$ entro il limite fissato dal Patto di stabilità e crescita.

Attività

Altri indicatori economici

Per studiare l'economia e il suo impatto sui problemi sociali è necessario tener conto di molte variabili.
- Cerca in Internet notizie su altri indici, oltre a quelli di cui abbiamo parlato, e sulle variabili di cui sono funzioni.

@ **Cerca nel Web:**
human development index (hdi), genuine progress indicator (gpi), impronta ecologica, genuine savings index (gsi), Eurostat, Istat



L'INFLAZIONE

Se l'inflazione diminuisce vuol dire che i prezzi calano?

▶ Il quesito completo a pag. 1617

Il tasso di inflazione
Indica la variazione nel tempo, in percentuale, del prezzo medio di un certo insieme di beni e servizi, detto *paniere*, che rappresenta i consumi delle famiglie.
Del paniere fanno parte, per esempio, pane, latte, abiti, mobili, medicinali e altri generi di prima necessità, ma anche alcolici, sigarette, cinema, articoli sportivi, discoteche.
In Italia, l'Istituto Nazionale di Statistica (ISTAT) provvede ad aggiornare l'indice nazionale dei prezzi al consumo, fornendo una *media ponderata* dei prezzi di beni e servizi. Questa è calcolata attribuendo a ogni prodotto un grado di importanza, un *peso*, all'interno del paniere.
Se indichiamo con $P(t)$ l'andamento nel tempo del prezzo medio del paniere e assumiamo, per esempio, un intervallo temporale di un mese ($\Delta t = 1$ mese), il tasso d'inflazione mensile è il rapporto

$$\frac{\frac{P(t+\Delta t)-P(t)}{\Delta t}}{P(t)}.$$

Il numeratore è il rapporto incrementale della funzione $P(t)$. Se l'intervallo di tempo è piccolo, il tasso d'inflazione può quindi essere considerato proporzionale alla derivata della funzione prezzo medio, o, più precisamente, alla derivata del logaritmo naturale di $P(t)$. Infatti:

$$D[\ln P(t)] = \frac{1}{P(t)} \cdot P'(t).$$

Il tasso di inflazione rappresenta dunque una sorta di velocità di crescita dei prezzi al consumo. Che cosa succede se diminuisce? Calano i prezzi? Se dicessimo, per esempio, che a fine ottobre il tasso mensile di inflazione era dello 0,20% e ora, a fine novembre, il tasso di inflazione mensile è dello 0,18%, il calo andrebbe interpretato come una «decelerazione», cioè una diminuzione della velocità della crescita dei prezzi, e non certo come una diminuzione dei prezzi stessi.
In termini matematici, in una situazione come questa è la *derivata seconda* di $P(t)$ a essere negativa. È proprio sulla base della derivata seconda dei prezzi in funzione del tempo che di solito agisce la Banca Centrale di un Paese per rallentare o accelerare la crescita dei prezzi.
Nel grafico in alto puoi osservare l'andamento in Italia del tasso di inflazione annuo dal 1980 al 2000.
Nel periodo 1980-1987 c'è stata una forte diminuzione, ma, come vedi, i prezzi sono aumentati. Ciò che costava 1000 lire nel 1980, costava circa 2500 lire nel 1987.

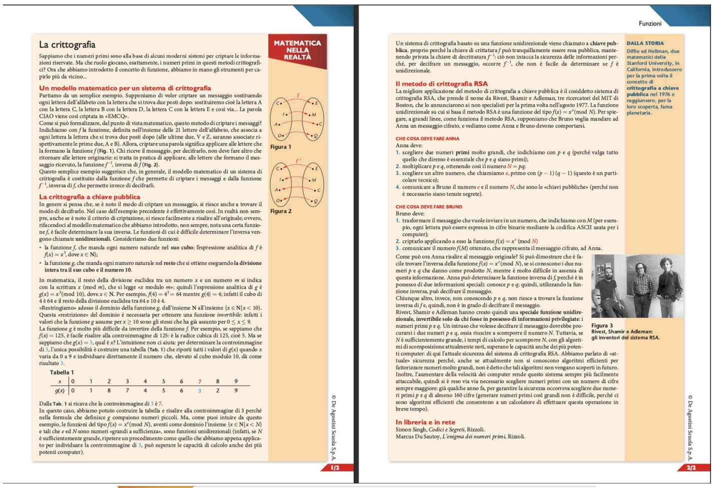
La deflazione
Poiché una funzione derivabile è decrescente se la sua derivata è negativa, i prezzi al consumo diminuiscono se il tasso di inflazione è negativo: situazione che gli economisti indicano col termine *deflazione*.
Nonostante i consumatori giudichino positiva una diminuzione dei prezzi, in economia un periodo di deflazione viene considerato negativamente, come sintomo di depressione.
Una deflazione abbastanza duratura si è avuta in Giappone dal 2000 al 2006.

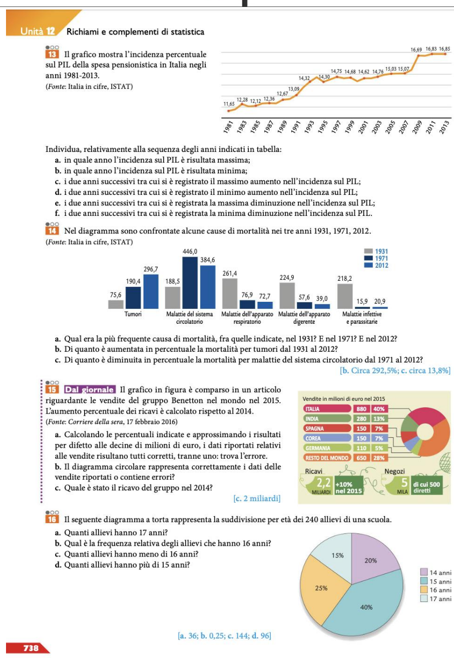
Inflazione e prezzi in Italia dal 1980 al 2000
Tasso % medio annuo
22 20 18 16 14 12 10 8 6 4 2 0
Prezzo di riferimento annuo (in lire)
4300 4000 3700 3400 3100 2800 2500 2200 1900 1600 1300 1000
1980 1981 1982 1983 1984 1985 1986 1987 1988 1989 1990 1991 1992 1993 1994 1995 1996 1997 1998 1999 2000
Anni

<0
0-2
2-5
5-10
10-15
>15
Inflazione nel mondo (2007)

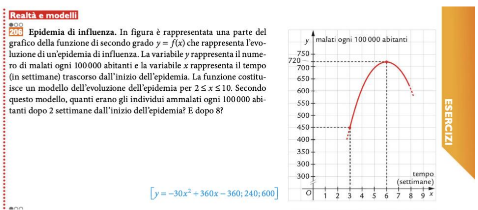
Although these examples and exercises are interesting, it would have been useful to go deeper into these topics, also introducing a certain continuity with the subsequent volumes, for example.

In the cycle of textbooks "I colori della matematica" we find many more references to humanistic/social applications of mathematics, as we can see from the list above. It is interesting to note, however, that the majority of applications found are among the exercises in the physical book and instead the interesting insights taken from the "Mathematics in reality" section of the online book are few and are not connected to the exercises found. Perhaps it would have been more constructive to have resources in the online book that deepen humanistic, social and economic concepts that are then presented without a solid background in the exercises under the paragraphs "Reality and Models", "From the Newspaper" and "Mathematics and Economics".
On the other hand, a positive note is deserved by the fact that many of the proposed humanistic/social exercises, such as for example those concerning the growth of a population, the spread of an epidemic and others, are very often taken up so as to consolidate the idea that mathematics is connected to many fields, not only purely scientific. The only downside to this is that sometimes some insights or exercises are simply repeated the same, such as the cryptography tab or the graph reading exercises, that we report respectively below, that are present both in volume 1 and 3.





However, the exercises and online resources are very interesting. Below there is an example by every type of exercise and in-depth information sheets. Example of exercise in "Reality and Models":



Example of exercise in "From the Newspaper":

161 **Dal giornale** **"Gratta e vinci": per far tornare i conti lo Stato gioca d'azzardo.**

Siamo i primi al mondo: stampiamo un quinto dei biglietti di tutto il pianeta. [...] Quel che è certo è che il 73,6% del denaro raccolto viene restituito ai giocatori (pay out) in forma di premi, il resto (più di 2 miliardi l'anno) entra nelle casse dell'Erario [...]. Ma un pay out così alto non deve trarre in inganno [...] Per fare degli esempi concreti: quante possibilità ci sono di grattare il premio più ghiotto (500 mila euro) di un "Nuovo Miliardario"? Una su 5 milioni e 280 mila biglietti, lo 0,000019% (circa, NdR). *E quante probabilità esistono di mollare tutto e diventare "Turista per sempre"? Una su 2 milioni e 880 mila, lo 0,000035%* (circa, NdR).
(*Fonte:* l'Espresso, 8 gennaio 2016)

a. In base alle informazioni espresse nell'articolo, che cosa si può affermare circa l'ammontare del denaro che, nell'anno preso in considerazione dall'articolo, è stato restituito ai giocatori?
b. Se si compra un biglietto del "Nuovo Miliardario" e uno di "Turista per Sempre", qual è la probabilità di vincere il premio più alto, in entrambi i casi? E in almeno uno dei due casi? Esprimi i risultati in notazione scientifica.
c. Le probabilità considerate al punto precedente sono maggiori o minori della probabilità di puntare un numero alla roulette francese ogni giorno, per una settimana, e vincere tutti i giorni?

[**a.** è superiore a 5,57 miliardi l'anno; **b.** circa $6{,}65 \cdot 10^{-14}$ e $5{,}4 \cdot 10^{-7}$ (utilizzando i valori approssimati forniti nell'articolo) **c.** la prima è inferiore, mentre la seconda è superiore]

Example of exercise in "Mathematics and Economics":

Matematica e... economia

223 Il sig. Bianchi impiega 10 000 euro in una forma di investimento che prevede un tasso d'interesse annuo dell'x%. Gli interessi maturati dopo un anno vengono versati nell'investimento e contribuiscono a generare nuovi interessi l'anno successivo. Dopo il primo anno il tasso d'interesse aumenta dell'1%. Esprimi tramite un polinomio ridotto il capitale complessivo (comprensivo di tutti gli interessi maturati) a disposizione del sig. Bianchi dopo 2 anni. [$x^2 + 201x + 10\,100$]

224 Il sig. Rossi impiega 10 000 euro in una forma di investimento che prevede un tasso d'interesse annuo dell'x%. Gli interessi maturati dopo un anno vengono versati nell'investimento e contribuiscono a generare nuovi interessi l'anno successivo. Dopo il primo anno il tasso d'interesse diminuisce dell'1%. Esprimi tramite un polinomio ridotto il capitale complessivo (comprensivo di tutti gli interessi maturati) a disposizione del sig. Rossi dopo 2 anni. [$x^2 + 199x + 9900$]

Example of exercise of "Choice problems":

Problemi di scelta

206 Per il noleggio di un'auto, due diverse compagnie offrono le seguenti condizioni.
a. La compagnia *A* applica 20 euro di costo fisso più 50 euro per ogni giorno di noleggio.
b. La compagnia *B* non applica nessun costo fisso e richiede 60 euro per ogni giorno di noleggio.

Stabilisci, in dipendenza del numero di giorni per cui si vuole noleggiare l'auto, qual è la scelta più conveniente.
[Per un solo giorno di noleggio conviene la compagnia *B*, per più di 2 giorni conviene la compagnia *A*, per 2 giorni è indifferente]

Example of exercise of "Problem from reality":

503 **Crescita di una popolazione.** La numerosità di una popolazione di insetti è ben modellizzata dalla funzione:

$$P(t) = \frac{8000}{1 + 40e^{-0{,}2t}}$$

dove $P(t)$ rappresenta il numero di insetti della popolazione e t è il tempo, misurato in mesi. Stabilisci dopo quanto tempo dall'inizio dell'osservazione ($t = 0$) la velocità di crescita della popolazione inizia a diminuire. Esprimi il risultato in mesi e giorni, arrotondato ai giorni (assumi che un mese abbia 30 giorni).

[18 mesi e 13 giorni]

504 **Crescita di un tumore.** La crescita di un tumore è ben modellizzata da una funzione del tipo:

$$N(t) = e^{a(1-e^{-bt})}$$

dove $N(t)$ rappresenta il numero di cellule tumorali dopo t giorni dall'inizio dell'osservazione, mentre a e b sono due costanti positive. Determina il punto di flesso della funzione $N(t)$ e illustrane il significato in relazione alla crescita del tumore. $\left[t = \frac{\ln a}{b}\right]$

Example of "Mathematics in reality" online tab:

Leggi elettorali proporzionali

MATEMATICA NELLA REALTÀ

Nella politica di un paese democratico esistono numerose circostanze in cui entra in gioco la matematica: sistemi elettorali, scelte sociali, ricerca del consenso, alleanze tra partiti...

Analizziamo qui il problema della ripartizione dei seggi secondo una legge elettorale di tipo proporzionale puro. Si tratta di una situazione in cui le approssimazioni giocano un ruolo determinante. Sai già che il calcolo approssimato è importante nelle discipline tecnico-scientifiche, dove è necessario ridurre al minimo i rischi legati al fenomeno della propagazione degli errori. Vedremo che anche in questo contesto le approssimazioni hanno un ruolo critico, sebbene di natura diversa.
Supponiamo, per semplicità, che un parlamento abbia 100 seggi e che essi vadano suddivisi tra i partiti A, B, C secondo una legge elettorale proporzionale. Supponiamo anche che gli elettori che eleggono i loro rappresentati siano 1000.
La legge elettorale proporzionale impone che il numero di seggi assegnati al partito A sia proporzionale al numero di voti ottenuti, e così per gli altri partiti.
Ipotizziamo che si siano ottenuti i risultati in **Tab. 1**.

Tabella 1

	Voti	Percentuale	Seggi
Partito A	140	14%	14
Partito B	430	43%	43
Partito C	430	43%	43
Totale	1000	100%	100

In questo caso è semplice attribuire i seggi ai tre partiti.
Il partito A ha avuto 140 preferenze, perciò ottiene 14 seggi; analogamente, i partiti B e C hanno avuto 430 voti ciascuno e perciò ottengono entrambi 43 seggi.
Ma considera ora la situazione (molto più realistica) presentata in **Tab. 2**.

Tabella 2

	Voti	Percentuale	Seggi (valore decimale teorico)
Partito A	134	13,4%	13,4
Partito B	433	43,3%	43,3
Partito C	433	43,3%	43,3
Totale	1000	100%	100

I partiti A, B, C hanno ottenuto rispettivamente 134, 433, 433 voti, a cui corrisponderebbero 13,4 eletti per il partito A e 43,3 eletti sia per il partito B sia per il partito C. Ma tale ripartizione è impossibile, perché ogni parlamentare è ovviamente indivisibile. Il problema sta quindi nell'approssimare i valori decimali in maniera giusta.

app_leggi_ele_prop © De Agostini Scuola S.p.A. 1/2

MATEMATICA NELLA REALTÀ

Si potrebbe pensare di adottare la seguente procedura: in prima istanza si assegnano a ogni partito tanti seggi quanti risultano approssimando all'unità per difetto la rispettiva quota; successivamente, si assegnano parlamentari "extra" ai partiti maggiormente penalizzati nella prima fase, fino al raggiungimento del totale dei seggi.
In **Tab. 3** è stato applicato questo metodo.

Tabella 3

	Voti	Percentuale	Seggi (valore decimale teorico)	Seggi (valore intero)
Partito A	134	13,4%	13,4	13 + 1 = 14
Partito B	433	43,3%	43,3	43
Partito C	433	43,3%	43,3	43
Totale	1000	100%	100	99 + 1 = 100

La procedura appena descritta ha però dei problemi. Osserva ora la **Tab. 4**: noti qualcosa?

Tabella 4

	Voti	Percentuale	Seggi (valore decimale teorico)	Seggi (valore intero)
Partito A	136	13,6%	13,6	13
Partito B	437	43,7%	43,7	43 + 1 = 44
Partito C	427	42,7%	42,7	42 + 1 = 43
Totale	1000	100%	100	98 + 2 = 100

Il partito A ha incrementato il numero dei voti da 134 a 136, eppure ha perso un seggio. Come è possibile che, in un sistema elettorale proporzionale, *aumentando* la percentuale dei propri elettori *si riduca* il numero dei propri eletti? La causa per cui il partito A ha perso un seggio è la stessa per cui prima ne aveva acquisito uno. Tutto dipende dalla distribuzione dei voti assegnati ai vari partiti in competizione, e di conseguenza dalle approssimazioni per passare dai valori decimali ai valori interi.
Ma non è tutto: c'è ancora una stranezza dovuta alle (inevitabili) approssimazioni intere. Confronta la **Tab. 2**, già analizzata, con la **Tab. 5**, ottenuta assumendo che il numero dei parlamentari eletti sia 101 invece di 100 (si ha dunque *un seggio in più* rispetto alla situazione precedente).

Tabella 5

	Voti	Percentuale	Seggi (valore decimale teorico)	Seggi (valore intero)
Partito A	134	13,4%	13,534	13
Partito B	433	43,3%	43,733	43 + 1 = 44
Partito C	433	43,3%	43,733	43 + 1 = 44
Totale	1000	100%	101	99 + 2 = 101

Anche qui si registra un'apparente stranezza: *a parità di voti ottenuti da tutti e tre i partiti*, il partito A perde un seggio, nonostante i seggi in parlamento siano aumentati. Di nuovo, tale fenomeno si spiega con la distribuzione dei voti assegnati ai vari partiti in competizione, che anche stavolta influenza le approssimazioni per passare dai valori decimali ai valori interi.
L'apparente anomalia che abbiamo descritto è nota in letteratura come "paradosso dell'Alabama", perché fu osservata per la prima volta a seguito del censimento statunitense del 1880: ipotizzando 299 seggi complessivi, allo stato dell'Alabama sarebbero spettati 8 seggi, ipotizzandone invece 300, i seggi assegnati al medesimo stato sarebbero stati solo 7.

In libreria
George G. Szpiro, *Matematica per la democrazia*, Bollati Boringhieri.

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Chapter 3

Political Power: an application of Mathematics

Political Power is one of the central concepts of political science, that is one of the reasons because it has been chosen as central topic of the activity that will be held in class. Power in general is many-faceted with aspects like influence, intimidation, authority and others. Our analysis concerns the narrower sphere involving power, which is reflected in formal voting situations related to specific yes-no issues.
In literature there are different definitions of index of power, in this chapter we will introduce **Shapley-Shubik index of power**, **Banzhaf index of power**, **Johnston index of power** and **Deegan-Packel index of power** (see e.g. [30], [14]).
The principle subjects of this topic we are going to analyze are voting systems. From now on we will always consider systems with a finite number of members.
First of all we need some definitions and then we need to add a convention, with which we can turn to our discussion of power:

Definition 3.0.1. A **yes-no voting system** is a set of rules that specifies which collections of "yes" votes yield passage of the issue at hand.
Any collection of voters is called a **coalition**.
A coalition is said to be **winning** if passage is guaranteed by "yes" votes from exactly the voters in that coalition.
Coalitions that are not winning are called **losing**.

Definition 3.0.2. A **monotone voting system** is a voting system such that adding extra voters to a winning coalition again yields to a winning coalition. For monotone systems, one can focus on the **minimal winning coalitions**, namely those winning coalitions with the property that the deletion of one of any of the voters yields a losing coalition.

Convention. Whenever we say **voting system** we mean monotone voting system in which the grand coalition, i.e. the one to which all the voters belong, is winning, and the empty coalition, i.e. the one to which none of the voters belong, is losing.

3.1 Shapley-Shubik index of power

We need a first definition that we will use to define Shapley-Shubik index of power:

Definition 3.1.1. Let X be the finite set of all voters of a voting system (yes-no voting system in our case), let σ be an ordering of the elements of X. A voter is said to be **pivotal** for σ if the coalition formed by the elements of σ that appear before him is a non-winning coalition, but the coalition formed by the addition of this voter to the the previous coalition is a winning coalition.

Example 3.1.2. Suppose $X = \{p_1, ..., p_8\}$ is the set of voters of a voting system, let each voter has one vote except for p_2 and p_8 who have two each. Suppose six votes are needed for passage of a certain proposal. Consider the ordering $(p_2, p_4, p_8, p_1, p_5, p_6, p_3, p_7)$. The coalition $\{p_2, p_4, p_8\}$ is a non-winning coalition, but if p_1 joins it, it becomes $\{p_2, p_4, p_8, p_1\}$ that is a winning coalition, so p_1 is pivotal for this ordering.

We are now ready to define the first index of power we are going to see.

Definition 3.1.3. Suppose p is a voter in a yes-no voting system and let X be the finite set of all voters. The **Shapley-Shubik index** of p, denoted by $\mathbf{SSI}(p)$ is the ratio of the number of orderings of X for which p is pivotal to the total number of possible orderings of the set X.

Observation 3.1.4.

- The denominator in $SSI(p)$ is just $n!$ if there are n voters. Indeed the number of total possible orderings of n people is given by $n!$.
- $0 \leq SSI(p) \leq 1$ for all p. Indeed the orderings of X for which p is pivotal is a subset of all the possible orderings. Hence the number at the numerator in calculating the Shapley-Shubik index of p is less than or (at most) equal to the denominator, it means that $SSI(p) \leq 1$. Meantime the numerator is either a positive integer or zero and the denominator is a positive integer, then $SSI(p) \geq 0$.
- $\sum_{p \in X} SSI(p) = 1$. Indeed every ordering σ has one and only one element that is pivotal for that ordering. Hence the disjoint union of all orderings with a pivotal element is precisely the set of all possible orderings. Thanks to this fact: $\sum_{p \in X}$ number of orderings of X for

which p is pivotal= total number of possible orderings of the set X, and $\sum_{p \in X} SSI(p) = \frac{TotOrderings}{TotOrderings} = 1$.

Example 3.1.5. Let us consider the previous Example 3.1.2, and calculate $SSI(p_1)$. Notice that p_1 is pivotal for an ordering precisely when the number of votes held by the voters to the left of it is five, cause p_1 has only one vote. We have different cases:

1. p_1 is preceded by both p_2 and p_8 and any other voter;

2. p_1 is preceded by only one between p_2 and p_8 and any three other voters;

3. p_1 is preceded by all remaining voters with one vote each.

Case 1. We have five different possibilities (because there are five voters we can choose between). For every possibility we have to calculate the number of possible orderings: for the first three voters we have 3! orderings and for the last four voters we have 4! orderings. So for this case we will have $5 \cdot 3! \cdot 4! = 720$ orderings.
Case 2. We have $2 \cdot \binom{5}{3} = 20$ different possibilities (because we have two choices between p_2 and p_8 and we have to choose three voters out of five and so there are $\binom{5}{3}$ choices for the other three elements). For every possibility we have to calculate the number of possible orderings: for the first four voters we have 4! orderings and for the last three voters we have 3! orderings. So for this case we will have $20 \cdot 4! \cdot 3! = 2880$ orderings.
Case 3. We have only one possibility because there are no voters to choose between (we need all five remaining voters with one vote each before p_1). We have to calculate the number of possible orderings: for the first five voters we have 5! orderings and for the last two voters we have 2! orderings. So for this case we will have $5! \cdot 2! = 240$ orderings.
Hence, as numerator of the $SSI(p_1)$ we have $720 + 2880 + 240 = 3840$ orderings for which p_1 is pivotal, as denominator we have $8! = 40320$, then $SSI(p_1) = \frac{3840}{40320} = \frac{2}{21} \simeq 0.095$.
We can say p_1 has about 9.5% as percentage of power in this voting system, the same would be true for all other voters with only one vote.
For p_2 and p_8 we have a different result. Let us find the result for p_2, then it would be the same for p_8. Notice that p_2 is pivotal for an ordering precisely when the number of votes held by the voters to the left of it is either four or five, cause p_2 has two votes. We have four different cases:

1. p_2 is preceded by p_8 and any two others voters;

2. p_2 is preceded by p_8 and any three others voters;

3. p_2 is preceded by any four voters excluding p_8;

4. p_2 is preceded by any five voters excluding p_8.

Case 1. We have $\binom{6}{2} = 15$ different possibilities (because we have to choose two voters out of six). For every possibility we have to calculate the number of possible orderings: for the first three voters we have 3! orderings and for the last four voters we have 4! orderings. So for this case we will have $15 \cdot 3! \cdot 4! = 2160$ orderings.
Case 2. We have $\binom{6}{3} = 20$ different possibilities (because we have to choose three voters out of six). For every possibility we have to calculate the number of possible orderings: for the first four voters we have 4! orderings and for the last three voters we have 3! orderings. So for this case we will have $20 \cdot 4! \cdot 3! = 2880$ orderings.
Case 3. We have $\binom{6}{4} = 15$ different possibilities (because we have to choose four voters out of six). For every possibility we have to calculate the number of possible orderings: for the first four voters we have 4! orderings and for the last three voters we have 3! orderings. So for this case we will have $15 \cdot 4! \cdot 3! = 2160$ orderings.
Case 4. We have $\binom{6}{5} = 6$ different possibilities (because we have to choose five voters out of six). For every possibility we have to calculate the number of possible orderings: for the first five voters we have 5! orderings and for the last three voters we have 2! orderings. So for this case we will have $6 \cdot 5! \cdot 2! = 1440$ orderings.
Hence, as numerator of the $SSI(p_1)$ we have $2160+2880+2160+1440 = 8640$ orderings for which p_2 is pivotal, as denominator we have $8! = 40320$, then $SSI(p_2) = \frac{8640}{40320} = \frac{3}{14} \simeq 0.214$.
We can say p_2 and p_8 have about 21.4% as percentage of power in this voting system. As we saw in the Observation 3.1.4 $0 \leq SSI(p_i) \leq 1$ for all $i = 1, ..., 8$ and $\sum\limits_{p \in X} SSI(p) = 6 \cdot \frac{2}{21} + 2 \cdot \frac{3}{14} = 1$.

It is clear that with a more complex voting system it would not be that easy to calculate Shapley-Shubik index of power of a voter. In the particular case of a yes-no weighted voting system where every voter has one vote, there is a theorem that makes finding the Shapley-Shubik index of power of a voter much easier.
First of all we need to define what a yes-no weighted voting system is.

Definition 3.1.6. A yes-no voting system is said to be **weighted** if it can be described by specifying real number weights for the voters and a real number quota, with no provisos or mention of veto power, such that a coalition is winning precisely when the sum of the weights of the voters in the coalition meets or exceeds the quota.

Notation. Suppose we have a weighted voting system with n voters $p_1, ..., p_n$ with, respectively, weights $w_1, ..., w_n$. Suppose that q is the quota. Then this system can be denoted in this way: $[q : w_1, ..., w_n]$.

The Example 3.1.2 we made before is an example of a weighted voting system and it's notation would be $[6 : 1, 2, 1, 1, 1, 1, 1, 2]$.

If some voters decide to unite and vote together, becoming a so-called voting bloc, then these voters are considered as a single voter with a number of votes equal to the sum of the votes of each participant in the voting bloc.
For example, if we take again the system of Example 3.1.2, and suppose p_1, p_3, p_8 became a voting bloc, the notation will be $[6 : 4, 2, 1, 1, 1, 1]$.

We are ready now to present this theorem:

Theorem 3.1.7. *Suppose we have n voters and that a single bloc of size c forms. Consider the resulting weighted voting system: $[q : c, 1, ..., 1]$, with n-c 1's.*
Assume $c - 1 \leq q - 1 \leq n - c$. Then the Shapley-Shubik index of power of the bloc is given by:

$$SSI(bloc) = \frac{c}{n-c+1}.$$

Proof. Notice that $n-c+1$ is just the number of distinct orderings, because we have $n-c$ 1's and the number of places where c can be inserted is just one more than this. We are not distinguishing between two orderings in which the 1's have been rearranged because, if we break this collection of $(n-c+1)!$ orderings into $n - c + 1$ equal size classes determined by the place occupied by the c bloc in the string of $n - c$ 1's, then the different orderings within any class are arrived at simply by permuting the 1's involved. In particular, the c-vote bloc is pivotal for one ordering in the class only if it is pivotal for every ordering in the class. Thus, to calculate the Shapley-Shubik index of the voting bloc, we must simply determine how many of the $n - c + 1$ distinct orderings have the c bloc in a pivotal position. The c bloc will be pivotal when the initial sequence of 1's is of length at least $q - c$ and not more than $q - 1$. Hence, the c bloc is pivotal when the initial sequence of 1's is any of the following lengths: $q - 1$, $q - 2$,..., $q - c$.
Since $q-1 \leq n-c$ and $n-c$ is the number of 1's available, we can construct sequences with all of these lengths.
There are exactly c numbers in the list of different lengths.
The Shapley-Shubik index of the bloc of size c is hence given by:

$$SSI(bloc) = \frac{c}{n-c+1}.$$

□

Example 3.1.8. Suppose we have a yes-no voting system with 7 voters with one vote each, the quota is 5. The notation for this voting system is $[5 : 1, 1, 1, 1, 1, 1, 1]$. Suppose that 3 voters decide to vote together, so they become a voting bloc. Now the system can be represented by this notation:

$[5 : 3, 1, 1, 1, 1]$. In this system we can use Theorem 3.1.7. This theorem implies that $SSI(bloc) = \frac{3}{4+1} = \frac{3}{5}$.
We now want to calculate $SSI(bloc)$ using the definition given above, assuming that the three voters who make up the bloc are a single voter with three votes at his disposal. In this system we have 5 voters, $p_1, ..., p_5$, assume p_1 to be the one with three votes.

$$SSI(p_1) = \frac{\binom{4}{2} \cdot 2! \cdot 2! + \binom{4}{3} \cdot 3! + 4!}{5!} = \frac{72}{120} = \frac{3}{5}.$$

The two results are actually the same.

Remark. Suppose we have a voting system where Suppose that each member of a voting bloc has equal power, namely the power of each member is the ratio of the power of the voting bloc to the number of members belonging to the voting bloc. Than the Shapley-Shubik index of power is a **superadditive function**.
A function f is **superadditive** if $f(x \cup y) \geq f(x) + f(y)$.
Theorem 3.1.7 shows that in a voting system with n members, where each member has equal power, SSI is a superadditive function. Indeed, let p be a single voter, $SSI(p) = \frac{1}{n}$. If p is a member of a voting bloc with c members, then $SSI(c) = \frac{c}{n-c+1}$ and $SSI(p_c) = \frac{c}{n-c+1} \cdot \frac{1}{c} = \frac{1}{n-c+1}$ where p_c is the voter p that has joined the voting bloc. We can easily observe that $SSI(p) = \frac{1}{n} \leq \frac{1}{n-c+1} = SSI(p_c)$ and then, of course, $SSI(c) = \sum\limits_{p \in c} SSI(p_c) = \frac{c}{n-c+1} \geq \sum\limits_{p \in c} SSI(p) = \frac{c}{n}$, that is the definition of superadditive function.

3.1.1 Paradoxical aspects

In general, power indexes could happen to have paradoxical aspects. Here there is an example regarding Shapley-Shubik index of power:

Example 3.1.9. Suppose we have a bicameral yes-no voting system wherein an issue, in order to pass, must win in both the House and the Senate. Considering the House as a yes-no voting system in its own right, if a member of the House has n (natural number) times as much power as another member of the House have, then everyone expects that the first member still have n times as much power as the other member when we are considering the bicameral yes-no voting system.
Suppose that the House consists of three people, called A, B and C, and there are two minimal winning coalition: A alone and B and C together. The Senate consists of two people, called D and E, and there are two minimal winning coalitions: D alone and E alone. The Shapley-Shubik indexes of House voters, when we are considering the House as yes-no voting system in its own right, are $\frac{4}{6}$, $\frac{1}{6}$ and $\frac{1}{6}$, then A has four times as much power as B and C has. Now, considering the bicameral yes-no voting system, the Shapley-

Shubik indexes of House voters are $\frac{44}{120}$, $\frac{14}{120}$ and $\frac{14}{120}$, then A no longer has four times as much power as B and C has.
In 3.2.6 we will see that this paradoxical aspect does not happen using Banzhaf index of power, that is the second index that we will introduce in the next section.

3.2 Banzhaf index of power

We now introduce a different, even if similar, measure of power, the Banzhaf index of power. It is interesting to know that this power index was introduced by the attorney John F. Banzhaf III in connection with a lawsuit involving the county board of Nassau County, New York in the 1960s.
To define this new index we first need the notion of **total Banzhaf power** of a voter:

Definition 3.2.1. Suppose that p is a voter in a yes-no voting system. Then the **total Banzhaf power** of p, denoted by $\mathbf{TBP}(p)$, is the number of coalitions C satisfying the following conditions:

1. p is a member of C;
2. C is a winning coalition;
3. if p is deleted from C, the resulting coalition is not a winning one. (Usually we say that p's defection from C is critical).

Now we are ready to define the new index of power.

Definition 3.2.2. Let X be the set of all voters in a yes-no voting system, suppose that p_1 is one of these voters. Then the **Banzhaf index** of p_1, denoted by $\mathbf{BI}(p_1)$, is the number given by

$$BI(p_1) = \frac{TBP(p_1)}{\sum\limits_{p \in X} TBP(p)}.$$

Observation 3.2.3.
- $0 \leq BI(p) \leq 1$ for all p.
- $\sum\limits_{p \in X} BI(p) = 1$.

Here are two examples that compute and compare Banzhaf and Shapley-Shubik indexes of power.

Example 3.2.4. Let's take again the system of the Example 3.1.2, we now calculate the total Banzhaf power for every voter:

- For every p voter with one vote $TBP(p)$ is the same and it is

$$TBP(p) = 5 + 2 \cdot \binom{5}{3} + 1 = 5 + 20 + 1 = 26;$$

- For every p voter with two votes $TBP(p)$ is the same and it is

$$TBP(p) = \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} = 15 + 20 + 15 + 6 = 56;$$

Hence we have $TBP(p) = 26$ for every $p \in X$ that has one vote. We will have the following Banzhaf index of power:

$$BI(p) = \frac{26}{26 \cdot 6 + 56 \cdot 2} = \frac{26}{268} = \frac{13}{134} \simeq 0,097.$$

Then every voter with one vote of the system has the 9.7% as percentage of power with respect to the Banzhaf definition.
Moreover we have $TBP(p) = 56$ for every $p \in X$ that has two votes. We will have the following Banzhaf index of power:

$$BI(p) = \frac{56}{26 \cdot 6 + 56 \cdot 2} = \frac{56}{268} = \frac{14}{67} \simeq 0,209.$$

Then every voter with two votes of the system has the 20.9% as percentage of power with respect to the Banzhaf definition.
Recall that for the same example we had $SSI(p_1) \simeq 0,095$ (as for any other voter with one vote) and $SSI(p_2) = SSI(p_8) \simeq 0,214$, these results are similar but different from the last results.

Example 3.2.5. Suppose we have a three-person weighted voting system in which p_1 has fifty votes, p_2 has forty-nine votes and p_3 has one vote. The quota is fixed at fifty-one.
The winning coalitions are: $C_1 = \{p_1, p_2, p_3\}$, $C_2 = \{p_1, p_2\}$, $C_3 = \{p_1, p_3\}$.
Shapley-Shubik index of power:

$$\begin{array}{l} SSI(p_1) = \frac{4}{3!} = \frac{2}{3} \\ SSI(p_2) = \frac{1}{3!} = \frac{1}{6} \\ SSI(p_3) = \frac{1}{3!} = \frac{1}{6} \end{array}$$

Total Banzhaf power:

$$TBP(p_1) = 3,\ TBP(p_2) = 1,\ TBP(p_3) = 1 \Rightarrow \sum_{p \in X} TBP(p) = 5$$

Banzhaf index of power:

$$\begin{array}{l} BI(p_1) = \frac{3}{5} \\ BI(p_2) = \frac{1}{5} \\ BI(p_3) = \frac{1}{5} \end{array}$$

We can observe that even in this example the two indexes are consistently different.

3.2.1 Procedures

There are two different (but equivalent as we will see) procedures to compute Total Banzhaf Power. Both procedures begin with a simple chart that has the winning coalitions enumerated in a vertical list down to the left side of the page, and the individual voters enumerated in a horizontal list across the top.

Procedure 1. Assign each voter a +1 (plus one) for each winning coalition of which it is a member, and assign it a -1 (minus one) for each winning coalition of which it is not a member. The sum of these +1 and -1 turns out to be the Total Banzhaf Power of the voter.

Procedure 2. Assign each voter a +2 for each winning coalition of which it is a member, and assign nothing for each winning coalition for which it is not a member. The result of subtracting the total number of winning coalitions from the sum of these +2 turns out to be the Total Banzhaf Power of the voter.

It is easy to see that these two procedures are equivalent: going from Procedure 1 to Procedure 2, all the -1 became 0 and all the +1 became +2. Hence, the sum for each voter increased by one for each winning coalition and when we subtracted off the number of winning coalitions the result from Procedure 2 became the same as the result from Procedure 1.
Now we have to explain why Procedure 1 works. Let us fix a voter p of the voting system, the list of winning coalitions can be divided into 3 blocks of coalitions:

1. Those winning coalitions for which p is not a member;

2. The coalition of **Block 1** with p added to them;

3. All others winning coalitions.

We need to notice several thing about these blocks:

- the coalitions in **Block 2** are all winning coalitions because we simply add a new voter to coalitions that were already winning (remember that we are considering only monotone voting systems);

- the number of coalitions in **Block 2** is the same as in **Block 1**;

- every coalition in **Block 3** contains p;

- p's defection from a winning coalition is critical precisely for the coalitions in **Block 3**.

The Procedure 1 is now clear because the -1s in **Block 1** are exactly offset by the +1s in **Block 2**, this way only the +1s for coalitions of **Block 3** are contributing to the sum, and these are precisely the ones for which p's defection is critical.

3.2.2 Paradoxical aspects

Example 3.2.6. Now we can conclude the Example 3.1.9, since we introduced the Banzhaf index of power.
Remember the system we are talking about: we have a bicameral yes-no voting system wherein an issue, in order to pass, must win in both the House and the Senate. Considering the House as a yes-no voting system in its own right, if a member of the House has n (natural number) times as much power as another member of the House have, then everyone expects that the first member still have n times as much power as the other member when we are considering the bicameral yes-no voting system.
indices Suppose, as before, that the House consists of three people, called A, B and C, and there are two minimal winning coalition: A alone and B and C together. The Senate consists of two people, called D and E, and there are two minimal winning coalitions: D alone and E alone. The Banzhaf indexes of House voters, when we are considering the House as yes-no voting system in its own right, are $\frac{3}{5}$, $\frac{1}{5}$ and $\frac{1}{5}$, then A has three times as much power as B and C has. Now, considering the bicameral yes-no voting system, the Banzhaf indexes of House voters are $\frac{9}{25}$, $\frac{3}{25}$ and $\frac{3}{25}$, then A still has three times as much power as B and C has.

Banzhaf index of power is not free from these paradoxical aspects, let us see an example now.

Example 3.2.7. Consider the following weighted system: [8:5,3,1,1,1]. Using the Procedure 1 it is easy to see that the Banzhaf indexes of the voters are $\frac{9}{19}$, $\frac{7}{19}$, $\frac{1}{19}$, $\frac{1}{19}$ and $\frac{1}{19}$.
Now suppose that the first voter, the one with five votes, gives one of his votes to the second voter, the one with three votes. The system that arises from this exchange is the following: [8:4,4,1,1,1]. If we calculate Banzhaf indices of this new system we find the paradoxical aspect we were talking about. The new indices are $\frac{1}{2}$, $\frac{1}{2}$, 0, 0, 0. It means that, although the first voter has four votes instead of five, he has $\frac{1}{2}$ of total power, as measured by the Banzhaf index, that is strictly larger than $\frac{9}{19}$. This situation happens because the least three voters lose all of their power and the first two voters are both pivotal in every winning coalition, so they split in equal part the power of the voting system, as measured by the Banzhaf index.
This paradoxical aspect does not arise if the power is measured as in Shapley-Shubik index of power. Indeed, considering the first weighted system [8:5,3,1,1,1], Shapley-Shubik indices of the voters are $\frac{11}{20}$, $\frac{6}{20}$, $\frac{1}{20}$, $\frac{1}{20}$, $\frac{1}{20}$. Considering the second weighted system [8:4,4,1,1,1], Shapley-Shubik indices of the voters are $\frac{1}{2}$, $\frac{1}{2}$, 0, 0, 0, so the power of the first voter is strictly smaller that before, as everyone expected.

3.3 Johnston index of power

The idea underlying the Johnston index of power is that the Banzhaf index of power does not take into consideration the total number of voters whose defection from a given coalition is critical. To explain better: if a voter p is the only one whose defection from C is critical, then this is a stronger indication of power than if every voter in C has a critical defection.
Let us now define the Johnston index of power with the following two definitions as for the Banzhaf index of power.

Definition 3.3.1. Suppose that p is one of the voters in a yes-no voting system. The **total Johnston power** of p, denoted by $\mathbf{TJP}(p)$, is the number arrived at as follows: suppose $C_1, ..., C_j$ are the winning coalitions for which p's defection is critical, suppose $n_1, ..., n_j$ are respectively the numbers of voters whose defection in $C_1, ..., C_j$ is critical. Then

$$TJP(p) = \frac{1}{n_1} + ... + \frac{1}{n_j}.$$

Definition 3.3.2. Let X be the set of all voters of a yes-no voting system and suppose p_1 is one of these voters. The **Johnston index** of p_1, denoted by $\mathbf{JI}(p_1)$, is the number given by:

$$JI(p_1) = \frac{TJP(p_1)}{\sum\limits_{p \in X} TJP(p)}.$$

Example 3.3.3. Suppose we are in the system of Example 3.2.5. Let us compute the Johnston index of power of the three voters and compare it with the two others indexes we already know.
Recall that the winning coalitions are: $C_1 = \{p_1, p_2, p_3\}$, $C_2 = \{p_1, p_2\}$, $C_3 = \{p_1, p_3\}$.
We have:

$$\begin{array}{l} TJP(p_1) = \frac{1}{1} + \frac{1}{2} + \frac{1}{2} = 2 \\ TJP(p_2) = 0 + \frac{1}{2} + 0 = \frac{1}{2} \\ TJP(p_3) = 0 + 0 + \frac{1}{2} = \frac{1}{2} \end{array}$$

and then we obtain:

$$\begin{array}{l} JI(p_1) = \frac{2}{2+\frac{1}{2}+\frac{1}{2}} = \frac{2}{3} \\ JI(p_1) = \frac{\frac{1}{2}}{2+\frac{1}{2}+\frac{1}{2}} = \frac{1}{6} \\ JI(p_1) = \frac{\frac{1}{2}}{2+\frac{1}{2}+\frac{1}{2}} = \frac{1}{6}. \end{array}$$

Notice that these results are the same results we obtained for Shapley-Shubik index.

3.4 Deegan-Packel index of power

Deegan and Packel in 1978 introduced a new power index, that is uniquely determined by three assumptions:

1. the only coalitions that should be considered in determining the relative power of voters are the minimal winning coalitions;
2. all minimal coalitions form with equal probability;
3. the amount of power a voter derives from belonging to some minimal winning coalition is the same for all voters in that particular coalition.

Let us now define the Deegan-Packel index of power with the following two definitions as for the Banzhaf and Johnston indexes of power.

Definition 3.4.1. Suppose that p is one of the voters in a yes-no voting system. The **total Deegan-Packel power** of p, denoted by **TDPP**(p), is the number arrived at as follows: suppose $C_1, ..., C_j$ are the minimal winning coalitions to which p belongs, suppose $n_1, ..., n_j$ are respectively the numbers of voters in $C_1, ..., C_j$. Then

$$TDPP(p) = \frac{1}{n_1} + ... + \frac{1}{n_j}.$$

Definition 3.4.2. Let X be the set of all voters of a yes-no voting system, suppose p_1 is one of these voters. The **Deegan-Packel index** of p_1, denoted by **DPI**(p_1), is the number given by:

$$DPI(p_1) = \frac{TDPP(p_1)}{\sum\limits_{p \in X} TDPP(p)}.$$

Example 3.4.3. Suppose we are in the system of Example 3.2.5. Let us compute the Deegan-Packel index of power of the three voters and compare it with the three other indexes we already know.
The minimal winning coalitions are: $C_2 = \{p_1, p_2\}$ and $C_3 = \{p_1, p_3\}$. We have:

$$\begin{aligned} TDPP(p_1) &= \tfrac{1}{2} + \tfrac{1}{2} = 1 \\ TDPP(p_2) &= \tfrac{1}{2} + 0 = \tfrac{1}{2} \\ TDPP(p_3) &= 0 + \tfrac{1}{2} = \tfrac{1}{2} \end{aligned}$$

and then we obtain:

$$\begin{aligned} DPI(p_1) &= \frac{1}{1+\frac{1}{2}+\frac{1}{2}} = \frac{1}{2} \\ DPI(p_2) &= \frac{\frac{1}{2}}{1+\frac{1}{2}+\frac{1}{2}} = \frac{1}{4} \\ DPI(p_3) &= \frac{\frac{1}{2}}{1+\frac{1}{2}+\frac{1}{2}} = \frac{1}{4}. \end{aligned}$$

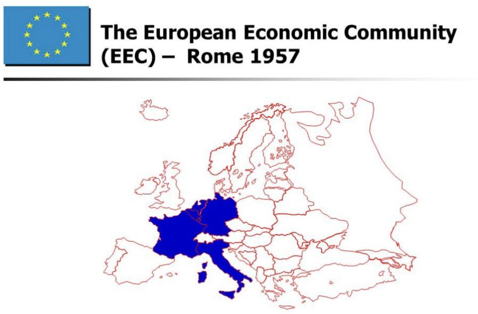
As we can see these results are different from the results relative to the other indexes.

3.5 Real-world examples of yes-no voting systems

We now introduce five real-world examples of yes-no voting systems that will be used in the activity and some of them will be used as examples to underlying the fact that the four indexes give really different results.

3.5.1 The European Economic Community of 1958

The European Economic Community was born in 1958 as one of the consequences of the Treaty of Rome. This Community is a yes-no voting system and the voters in 1958 were the following countries: Italy, France, Germany, Belgium, the Netherlands and Luxembourg.



Italy, France and Germany were given four votes each, Belgium and the Netherlands were given two votes each and Luxembourg was given one vote. The Passage requires at least twelve votes.
In 1973 new countries were added to the European Economic Community and the votes were redistributed.

Indexes of power of the six voters

Shapley-Shubik index of power. We need to calculate $SSI(Italy)$, that is the same for France and Germany, $SSI(Belgium)$, that is the same for the Netherlands, and $SSI(Luxembourg)$.

$$SSI(Italy) = \frac{2 \cdot 3! + 2 \cdot 3! \cdot 2 + 3! \cdot 2 + 2 \cdot 4! + 2 \cdot 3! \cdot 2 + 2 \cdot 4!}{6!} = \frac{168}{720} = \frac{14}{60}$$

This is because, to be pivotal, Italy may have from 8 to 11 votes before it.
To have 8 votes before the pivot, we can have France and Germany before Italy. This case explains the addendum $2 \cdot 3!$ because we can permute the two countries before the pivot and the three countries after the pivot. Another possibility is to have one between France and Germany and the countries with two votes each, namely Belgium and the Netherlands. This case would explain the addendum $2 \cdot 3! \cdot 2$.
To have 9 votes before the pivot, we can have both cases exposed above, to which we add Luxembourg that has only one vote. This case explains the

addenda $3! \cdot 2 + 2 \cdot 4!$.
To have 10 votes before the pivot, we can have France, Germany and one between Belgium and the Netherlands before Italy. This case explains the addendum $2 \cdot 3! \cdot 2$.
Finally, to have 11 votes before the pivot, we can take the "10 votes before"-case and add Luxembourg. This case explains the last addendum $2 \cdot 4!$.
The denominator will always be 6! in this system because we have six countries that can be exchanged between them.

$$SSI(Belgium) = \frac{\binom{3}{2}\cdot 3! \cdot 2 + \binom{3}{2}\cdot 4!}{6!} = \frac{108}{720} = \frac{9}{60}$$

This is because, to be pivotal, Belgium may have 10 or 11 votes before it.
To have 10 votes before the pivot, we can have two countries between Italy, France and Germany and the Netherlands before Belgium. This case explains the addendum $\binom{3}{2} \cdot 3! \cdot 2$.
To have 11 votes before the pivot, we can take the "10 votes before"-case and add Luxembourg. This case explains the second addendum $\binom{3}{2} \cdot 4!$.

$$SSI(Luxembourg) = \frac{0}{6!} = 0$$

This is because, to be pivotal, Luxembourg may have 11 votes before it, but this is impossible.
Hence, with respect to the Shapley-Shubik index of power, Italy, France and Germany have approximately 23.3% of the power each, Belgium and the Netherlands have 15.0% of the power each and Luxembourg has 0.0% of the power.

Banzhaf index of power. Let's use Procedure 1:

	I	**F**	**G**	**B**	**N**	**L**
IFG	1	1	1	-1	-1	-1
IFBN	1	1	-1	1	1	-1
IGBN	1	-1	1	1	1	-1
FGBN	-1	1	1	1	1	-1
IFGL	1	1	1	-1	-1	1
IFBNL	1	1	-1	1	1	1
IGBNL	1	-1	1	1	1	1
FGBNL	-1	1	1	1	1	1
IFGB	1	1	1	1	-1	-1
IFGN	1	1	1	-1	1	-1
IFGBL	1	1	1	1	-1	1
IFGNL	1	1	1	-1	1	1
IFGBN	1	1	1	1	1	-1
IFGBNL	1	1	1	1	1	1
TBP	10	10	10	6	6	0

Now we can calculate the Banzhaf index for every voter:

$$BI(I) = BI(F) = BI(G) = \frac{10}{42} = \frac{5}{21}$$
$$BI(B) = BI(N) = \frac{6}{42} = \frac{3}{21}$$
$$BI(L) = 0$$

Hence, with respect to the Banzhaf index of power, Italy, France and Germany have approximately 23.8% of the power each, Belgium and the Netherlands have approximately 14.3% of the power each and Luxembourg has 0.0% of the power.

Johnston index of power. Let us use a chart as we did for Total Banzhaf Power with the difference that in this case one needs to identify which voters have critical defections from which winning coalitions. Here's the procedure:

	I	**F**	**G**	**B**	**N**	**L**
IFG	1/3	1/3	1/3			
IFBN	1/4	1/4		1/4	1/4	
IGBN	1/4		1/4	1/4	1/4	
FGBN		1/4	1/4	1/4	1/4	
IFGL	1/3	1/3	1/3			
IFBNL	1/4	1/4		1/4	1/4	
IGBNL	1/4		1/4	1/4	1/4	
FGBNL		1/4	1/4	1/4	1/4	
IFGB	1/3	1/3	1/3			
IFGN	1/3	1/3	1/3			
IFGBL	1/3	1/3	1/3			
IFGNL	1/3	1/3	1/3			
IFGBN						
IFGBNL						
TBP	3	3	3	3/2	3/2	0

Now we can calculate the Johnston index for every voter:

$$JI(I) = JI(F) = JI(G) = \frac{3}{12} = \frac{1}{4}$$
$$JI(B) = JI(N) = \frac{\frac{3}{2}}{12} = \frac{1}{8}$$
$$JI(L) = 0$$

Hence, with respect to the Johnston index of power, Italy, France and Germany have 25.0% of the power each, Belgium and the Netherlands have 12.5% of the power each and Luxembourg has 0.0% of the power.

Deegan-Packel index of power. Let us use a chart as we did for Total Johnston Power with the difference that includes only the minimal winning coalitions. Here it is the procedure:

	I	**F**	**G**	**B**	**N**	**L**

IFG	1/3	1/3	1/3			
IFBN	1/4	1/4		1/4	1/4	
IGBN	1/4		1/4	1/4	1/4	
FGBN		1/4	1/4	1/4	1/4	
TDPP	5/6	5/6	5/6	3/4	3/4	0

Now we can calculate the Deegan-Packel index for every voter:

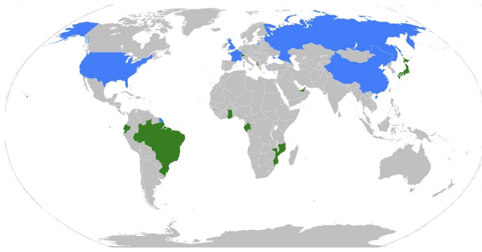
$$DPI(I) = DPI(F) = DPI(G) = \frac{\frac{5}{6}}{4} = \frac{5}{24}$$
$$DPI(B) = DPI(N) = \frac{\frac{3}{4}}{4} = \frac{3}{16}$$
$$DPI(L) = 0$$

Hence, with respect to the Deegan-Packel index of power, Italy, France and Germany have approximately 20.8% of the power each, Belgium and the Netherlands have approximately 18.8% of the power each and Luxembourg has 0.0% of the power.

As we can see we have four different results with respect to four different indexes of power.

3.5.2 The United Nations Security Council

In this system there are five countries that are called permanent members, that are China, England, France, Russia and the United States, and then there are ten other countries that are called non-permanent members, for a total of fifteen countries. Passage requires a quota of nine of the fifteen possible votes, subject to a veto from any one of the five permanent members.



Indexes of power of the members

Shapley-Shubik index of power. Permanent members (P) will all have the same index of power, as will non-permanent members (NP) with each other.
The denominator in the calculation of the Shapley-Shubik index of power of every member is 15!, because there are 15 members in total. We consider a permanent member **P**. In order for him to be the pivotal element of the order, the possibilities are: before him there must be the other four per-

manent members and from four to ten non-permanent members, given that for the coalition to be successful there must be are all permanent members thanks to their right of veto. So:
4P+4NP+pivot+remaining members. The number of possible sorts of this type is $\binom{10}{4} \cdot 8! \cdot 6!$. $\binom{10}{4}$ because we can choose four NP countries out of the ten we have, 8! because we can permute the eight countries we have before the pivot and 6! because we can trade-in the six countries we have after the pivot.
Same procedure for the other possibilities.
Therefore, the Shapley-Shubik index of power of a permanent member is:

$$SSI(P) = \frac{\binom{10}{4} \cdot 8! \cdot 6! + \binom{10}{5} \cdot 9! \cdot 5! + \ldots + \binom{10}{10} \cdot 14!}{15!} = \frac{421}{2145} \simeq 0.196$$

We consider a non-permanent member **NP**. To make it the pivotal element of the order, the only possibility is that before it there are the five permanent members and any three among the remaining non-permanent members. So the sorting will be like **5P+3NP+pivot+remaining members**. The number of possible sorts of this type is $\binom{9}{3} \cdot 8! \cdot 6!$. $\binom{9}{3}$ because I can choose three NP members among the remaining nine, 8! because we can permute the eight countries we have before the pivot and 6! because we can trade-in the six countries we have after the pivot.
Therefore, the Shapley-Shubik index of power of a non-permanent member is:

$$SSI(NP) = \frac{\binom{9}{3} \cdot 8! \cdot 6!}{15!} = \frac{4}{2145} \simeq 1.86 \times 10^{-3}$$

Hence, with respect to the Shapley-Shubik index of power, permanent members have approximately 19.6% of the power each, non-permanent members have approximately 0.186% of the power each.

Banzhaf index of power. The winning coalitions are all those containing 5 permanent members and containing 4 to 10 non-permanent members.
The 5 permanent members are critical in any coalition, so their total Banzhaf power is given by the total number of winning coalitions:

$$TBP(P) = \binom{10}{4} + \binom{10}{5} + \ldots + \binom{10}{10} = 848$$

This is because $\binom{10}{4}$ is the number of possible coalitions containing five permanent members and four non-permanent members chosen from the 10 available; the same reasoning applies to the other possibilities of having winning coalitions.
Non-permanent members are critical only in coalitions that contain five permanent members and exactly four non-permanent members. Therefore, given a non-permanent member **NP** whose Banzhaf power we want to calculate, the number of these coalitions is given by $\binom{9}{3} = 84$, because it is sufficient to choose three non-permanent members out of the remaining nine

available.
Therefore, the Banzhaf indexes of power of the members are:

$$BI(P) = \frac{TBP(P)}{5 \cdot TBP(P) + 10 \cdot TBP(NP)} = \frac{848}{5 \cdot 848 + 10 \cdot 84} = \frac{106}{635} \simeq 0.167$$
$$BI(NP) = \frac{TBP(NP)}{5 \cdot TBP(P) + 10 \cdot TBP(NP)} = \frac{84}{5 \cdot 848 + 10 \cdot 84} = \frac{21}{1270} \simeq 16.5 \times 10^{-3}$$

Hence, with respect to the Banzhaf index of power, permanent members have approximately 16.7% of the power each, non-permanent members have approximately 1.65% of the power each.

Johnston index of power. The Johnston index of power is calculated in a similar way to the Banzhaf index of power, the difference is that for each winning coalition considered in the Banzhaf, the number of critical members of the coalition must be taken into account.
For the permanent members we have that in coalitions with five permanent members and exactly four non-permanent members, the critical members are all nine, whereas in the other coalitions where there are five to ten non-permanent members, the critical members are only the five permanent.
Therefore the total Johnston power of the permanent members is given by:

$$TJP(P) = \binom{10}{4} \cdot \frac{1}{9} + \left(\binom{10}{5} + ... + \binom{10}{10}\right) \cdot \frac{1}{5} = \frac{2264}{15}$$

Non-permanent members are critical only in coalitions containing nine members, including the five permanent ones. In these coalitions, the critical members, as seen above, are all nine.
Therefore the total Johnston power for non-permanent members is given by:

$$TJP(NP) = \binom{9}{3} \cdot \frac{1}{9} = \frac{28}{3}$$

Therefore, the Johnston indexes of power of the members are:

$$JI(P) = \frac{TJP(P)}{5 \cdot TJP(P) + 10 \cdot TJP(NP)} = \frac{283}{1590} \simeq 0.178$$
$$JI(NP) = \frac{TJP(NP)}{5 \cdot TJP(P) + 10 \cdot TJP(NP)} = \frac{7}{636} \simeq 11.0 \times 10^{-3}$$

Hence, with respect to the Johnston index of power, permanent members have 17.8% of the power each, non-permanent members have 1.10% of the power each.

Deegan-Packel index of power. The Deegan-Packel index of power is calculated in a similar way to Johnston's, the difference is that only minimal winning coalitions are considered, in which every member is critical.
Minimal coalitions are only those that contain exactly nine members, including 5 permanent ones. Total Deegan-Packel power for non-permanent members remains equal to Johnston's as only minimal winning coalitions were considered. For permanent members it is simplified because we have fewer winning coalitions to consider. Therefore, the total Deegan-Packel power of permanent members is given by:

$$TDPP(P) = \binom{10}{4} \cdot \frac{1}{9} = \frac{70}{3}$$

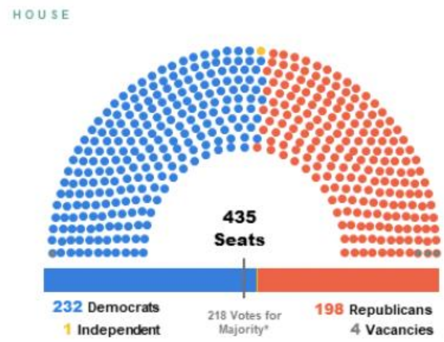
Therefore, the Deegan-Packel indexes of power of the members are:

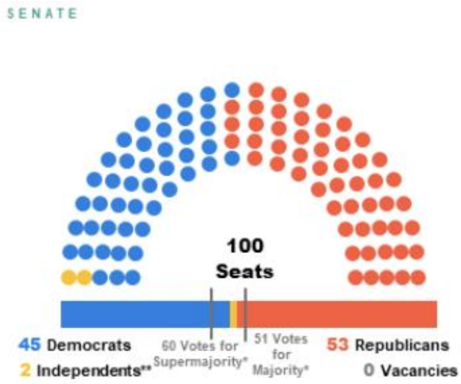
$$BI(P) = \frac{TDPP(P)}{5 \cdot TDPP(P) + 10 \cdot TDPP(NP)} = \frac{1}{9} \simeq 0.111$$
$$BI(NP) = \frac{TDPP(NP)}{5 \cdot TDPP(P) + 10 \cdot TDPP(NP)} = \frac{2}{45} \simeq 44.4 \times 10^{-3}$$

Hence, with respect to the Deegan-Packel index of power, permanent members have approximately 11.1% of the power each, non-permanent members have approximately 4.44% of the power each.

3.5.3 The United States Federal System

This yes-no voting system is composed by the House of Representatives, with 435 members, the Senate, with 100 members, the Vice President and the President, for a total of 537 members.





The Vice President has the role of tie-breaker in the Senate, the President has veto power that can be overridden by a two-thirds vote of both the House and the Senate.

For a bill to pass, therefore, it must be supported by either:

1. at least 218 representatives and at least 51 senators and the president;

2. at least 218 representatives, 50 senators, the vice president and the president;

3. at least 290 representatives and at least 67 senators.

President's power

Something really interesting is comparing the fraction of power held by the president with respect to the four different indexes that we have defined. We will consider a version of the U.S. federal system in which the tie-breaking role of the vice president is ignored.

The Shapley-Shubik index of the President. We recall that passage in this system requires either two-thirds of both houses or half of each house and the president.
The numerator of the $SSI(president)$ is given by the following formula, the denominator is 536!. Since it is really too long to be calculated by hand we will also present an expression that, inserted in some computer programs, like $Mathematica$ or $Python$ for example, will give the result.

$$\begin{aligned} SSI(president) = &\binom{435}{218}[\binom{100}{51}(218+51)!(535-218-51)! + ... \\ &+ \binom{100}{100}(218+100)!(535-218-100)!] \\ &+ ... \\ &+ \binom{435}{289}[\binom{100}{51}(289+51)!(535-289-51)! + ... \\ &+ \binom{100}{100}(289+100)!(535-289-100)!] \\ &+ \binom{435}{290}[\binom{100}{51}(290+51)!(535-290-51)! + ... \\ &+ \binom{100}{66}(290+66)!(535-290-66)!] \\ &+ ... \\ &+ \binom{435}{435}[\binom{100}{51}(435+51)!(535-435-51)! + ... \\ &+ \binom{100}{66}(435+66)!(535-435-66)!] \end{aligned} \tag{3.1}$$

Input for *Mathematica*:

$$\begin{aligned} &Sum[Binomial[435,h]Binomial[100,s](s+h)!(535-s-h)!,\\ &\qquad \{h,218,289\},\{s,51,100\}]/536!+\\ &Sum[Binomial[435,h]Binomial[100,s](s+h)!(535-s-h)!,\\ &\qquad \{h,290,435\},\{s,51,66\}]/536! \end{aligned} \tag{3.2}$$

The output is $SSI(president) \simeq 0.16047$, then the president has about the **16%** of the power in the U.S. federal system, according to the Shapley-Shubik index.

The Banzhaf index of the President. We need to determine the number of winning coalitions and to how many a House member, a senator and the president belong.

Notation. S denotes the number of coalitions within the Senate that contain at least two-thirds of the members of the Senate.
s denotes the number of coalitions within the Senate that contain at least one-half of the members of the Senate.
H denotes the number of coalitions within the House that contain at least two-thirds of the members of the House.
h denotes the number of coalitions within the House that contain at least one-half of the members of the House.

Thanks to this notation we have:

$$\begin{aligned} S &= \tbinom{100}{67} + \dots + \tbinom{100}{100} \\ s &= \tbinom{100}{51} + \dots + \tbinom{100}{100} \\ H &= \tbinom{435}{290} + \dots + \tbinom{435}{435} \\ h &= \tbinom{435}{218} + \dots + \tbinom{435}{435} \end{aligned}$$

To calculate the total Banzhaf power of the president we need four numbers:

1. the number of winning coalitions of the system, it can be seen as the sum of those which contain the president and those which do not. Hence, the total number is

$$(h \cdot s) + (H \cdot S)$$

2. the number of winning coalitions to which the president belongs. Thanks to the previous point this is

$$h \cdot s$$

3. the number of winning coalitions to which a fixed member of the Senate belongs. We can separate these into those that contain the president and those that do not. We need to know two numbers: fixed a senator p, how many coalitions there are within the Senate that contain p and

also contain at least two-thirds of the members of the Senate (including p) and how many coalitions there are within the Senate that contain p and also contain at least one-half the members of the Senate (including p) . Hence, the numbers of these coalitions are, respectively,

$$\tbinom{99}{66} + ... + \tbinom{99}{99}$$

and, in the same way,

$$\tbinom{99}{50} + ... + \tbinom{99}{99}.$$

Thus, the number we are searching for is

$$\begin{aligned} TBP(senator) = [\binom{99}{50} + ... + \binom{99}{99}] \cdot h \\ + [\binom{99}{66} + ... + \binom{99}{99}] \cdot H \end{aligned} \tag{3.3}$$

4. the number of winning coalitions to which a fixed member of the House belongs. As we did in the previous point, the result is

$$\begin{aligned} TBP(deputy) = [\binom{434}{217} + ... + \binom{434}{434}] \cdot s \\ + [\binom{434}{289} + ... + \binom{434}{434}] \cdot S \end{aligned} \tag{3.4}$$

Using now Procedure 2 to calculate the TBP of the voters we easily obtain the result.
As for Shapley-Shubik index the result is given by a computer program with an appropriate input. The output of the program is $BI(president) \simeq 0.038$, then the president has about the **4%** of the power in the U.S. federal system, according to the Banzhaf index.

The Johnston index of the President. In this situation we also worry about how many voters in each coalition have critical defection, we have to divide the different cases, and we do it through the following chart:

Type of winning coalition	Number of critical defection	Whose defection is critical		
T_{11}:67 S, 290 H	357		S	H
T_{12}:67 S, 291-435 H	67		S	
T_{13}:68-100 S, 290 H	290			H
T_{21}:P, 51 S, 218 H	270	P	S	H
T_{22}:P, 51 S, 219-289 H	52	P	S	
T_{23}:P, 52-66 S, 218 H	219	P		H
T_{24}:P, 52-66 S, 219-289 H	1	P		
T_{31}:P, 67-100 S, 218 H	219	P		H
T_{32}:P, 67-100 S, 219-289 H	1	P		
T_{41}:P, 51 S, 290-435 H	52	P	S	
T_{42}:P, 52-66 S, 290-435 H	1	P		

The winning coalitions involved in calculating the total Johnston power of the president are $T_{21}, T_{22}, T_{23}, T_{24}, T_{31}, T_{32}, T_{41}, T_{42}$. Then,

$$TJP(president) = \frac{|T_{21}|}{270} + \frac{|T_{22}|}{52} + \frac{|T_{23}|}{219} + |T_{24}| + \frac{|T_{31}|}{219} + |T_{32}| + \frac{|T_{41}|}{52} + |T_{32}| \quad (3.5)$$

With $|T_{ij}|$ we denote the number of coalitions of that type, these numbers are calculated in a similar way to what was done in the calculation of Banzhaf power. We now consider a fixed member s of the Senate, we have to consider the following types of winning coalitions: $T_{11}, T_{12}, T_{21}, T_{22}, T_{41}$.
We have to be more careful when we compute $TJP(s)$ because we can not use, for example, $|T_{11}|$ since our fixed senator doesn't belong to many of these coalitions. The computation to do it is the following:

$$\begin{aligned} TJP(s) = \frac{1}{357}\binom{99}{66}\binom{435}{290} + \frac{1}{67}\binom{99}{66}\left[\binom{435}{291} + ... + \binom{435}{435}\right] \\ + \frac{1}{270}\binom{99}{50}\binom{435}{218} + \frac{1}{52}\binom{99}{50}\left[\binom{435}{219} + ... + \binom{435}{289}\right] \\ + \frac{1}{52}\binom{99}{50}\left[\binom{435}{290} + ... + \binom{435}{435}\right]. \end{aligned} \quad (3.6)$$

Finally we consider a fixed member h of the House, we have to consider the following types of winning coalitions: $T_{11}, T_{13}, T_{21}, T_{23}, T_{331}$.
Similarly to what we did for a fixed senator, the computation to do is the

following:

$$TJP(h) = \frac{1}{357}\binom{100}{67}\binom{434}{289} + \frac{1}{290}\left[\binom{100}{68} + ... + \binom{100}{100}\right]\binom{434}{289}$$
$$+ \frac{1}{270}\binom{100}{51}\binom{434}{217} + \frac{1}{219}\left[\binom{100}{52} + ... + \binom{100}{66}\right]\binom{434}{217}$$
$$+ \frac{1}{219}\left[\binom{100}{67} + ... + \binom{100}{100}\right]\binom{434}{217}. \quad (3.7)$$

We can now calculate the Johnston index of the president that turn out to be $JI(president) \simeq 0.77$, then the president has about the **77%** of the power in the U.S. federal system, according to the Johnston index.

The Deegan-Packel index of the President. In this situation we need minimal winning coalitions, we can calculate their total number summing those that contain the president and those that do not:

$$|\text{Minimal winning coalitions}| = \tbinom{100}{51}\tbinom{435}{218} + \tbinom{100}{67}\tbinom{435}{290}.$$

Every minimal winning coalition that contains the president contains 270 voters, every minimal winning coalition that does not contain the president contains 357 voters.
Let us now calculate the total Deegan-Packel power for each type of voter:

$$TDPP(president) = \frac{\binom{100}{51}\binom{435}{218}}{270},$$

consider now a fixed member s of the Senate and h of the House,

$$TDPP(s) = \tfrac{1}{270}\tbinom{99}{50}\tbinom{435}{218} + \tfrac{1}{357}\tbinom{99}{66}\tbinom{435}{290}$$
$$TDPP(h) = \tfrac{1}{270}\tbinom{100}{51}\tbinom{434}{217} + \tfrac{1}{357}\tbinom{100}{67}\tbinom{434}{289}.$$

Thanks to these computations, dividing each for the number of minimal winning coalitions, we obtain that the $DPI(president) \simeq 0.0037$, then the president has about the **0.4%** of the power in the U.S. federal system, according to the Deegan-Packel index.

Recap. In conclusion we obtained that the president has these different percentages of power according to these different indexes:

- **Shapley-Shubik**: 16%;
- **Banzhaf**: 4%;
- **Johnston**: 77%;
- **Deegan-Packel**: 0.4%.

Although every definition of these indexes seems reasonable to us, the results are very different.

3.5.4 The System to Amend the Canadian Constitution

An amendment to the Canadian Constitution becomes law only if it is approved by seven or more Canadian provinces, provided that at least half of Canada's population is among the approving provinces. We now report the population percentage taken from the 1961 census for the ten Canadian provinces:

- Alberta 7%
- British Columbia 9%
- Manitoba 5%
- New Brunswick 3%
- Newfoundland 3%
- Nova Scotia 4%
- Ontario 34%
- Prince Edward Island 1%
- Quebec 29%
- Saskatchewan 5%

Indexes of power of provinces

Shapley-Shubik index of power. We can see that the Canadian provinces can be divided into two groups, one containing Ontario and Quebec, the two major provinces, and the other with all the remaining minor provinces. We note that the eight minor provinces together do not reach 50% of the population, therefore to be pivotal they must always be preceded by at least one of the major provinces. However, we also note that adding the percentages of population of the 6 smallest provinces $(1\%, 3\%, 3\%, 4\%, 5\%, 5\%)$ with the smallest province among the two largest (Quebec 29%) we get exactly 50%

of the population, so any choice of 6 minor provinces and one major one assures us the majority.
Having said this, let us start calculating the Shapley-Shubik index of one of the minor provinces (without choosing which one in particular, let us see if it will be necessary), we will call it m. For m to be pivotal we have two possibilities:

1. The first possibility is to have both major provinces and any four of the remaining minor provinces before m; the major provinces together already exceed half of the population so the pivotal element will be the seventh province, m in our case. The number of sorts of this type is $\binom{7}{4} \cdot 6! \cdot 3!$, this is because $\binom{7}{4}$ indicates the fact that we have to choose four minor provinces among the seven left, 6! because we can permute the six provinces we have before the pivot and 3! because we can permute the three provinces we have after the pivot.

2. The second possibility is to have only one of the two major provinces and any five of the remaining minor provinces before m; as we have seen, with one of the major provinces and six of the minor provinces, we always reach half of the population, so we don't have to make further evaluations. The number of sorts of this type is $\binom{2}{1}\binom{7}{5} \cdot 6! \cdot 3!$, this is because $\binom{2}{1}$ indicates the fact that we have to choose one of the two major provinces, $\binom{7}{4}$ indicates the fact that we have to choose four minor provinces among the seven left, 6! because we can permute the six provinces we have before the pivot and 3! because we can permute the three provinces we have after the pivot.

Therefore, the Shapley-Shubik index of power of any of the minor provinces is:

$$SSI(m) = \frac{\binom{7}{4}\cdot 6!\cdot 3! + \binom{2}{1}\binom{7}{5}\cdot 6!\cdot 3!}{10!} = \frac{11}{120} \simeq 91.7 \times 10^{-3}$$

The denominator is 10! since we can permute ten provinces. Now let us consider one of the two major provinces (without choosing which one in particular, let us see if it will be necessary), we will call it M. For M to be pivotal, there are three main possibilities:

1. The first possibility is that before M there are any six of all provinces (both major and minor), since even if we take the six smallest provinces, adding M would reach at least half of the population. The number of sorts of this type is $\binom{9}{6} \cdot 6! \cdot 3!$, this is because $\binom{9}{6}$ indicates the fact that we have to choose six provinces among the nine left, 6! because we can permute the six provinces we have before the pivot and 3! because we can permute the 3 provinces we have after the pivot.

2. The second possibility is to have seven minor provinces before M, since we know that, alone, they do not reach half of the population. The

number of sorts of this type is $\binom{8}{7} \cdot 7! \cdot 2!$, this is because $\binom{8}{7}$ indicates the fact that we have to choose seven minor provinces among the eight we have, 7! because we can permute the seven provinces we have before the pivot and 2! because we can permute the two provinces we have after the pivot.

3. The third and final possibility is to have eight minor provinces before M, for the same reason as in the previous point. The number of sorts of this type is $\binom{8}{8} \cdot 8!$, this is because $\binom{8}{8}$ indicates the fact that we have to choose eight minor provinces among the eight we have, so all of them, 8! because we can permute the eight provinces we have before the pivot.

Therefore, the Shapley-Shubik index of power of any of the major provinces is:

$$SSI(M) = \frac{\binom{9}{6} \cdot 6! \cdot 3! + \binom{8}{7} \cdot 7! \cdot 2! + \binom{8}{8} \cdot 8!}{10!} = \frac{2}{15} \simeq 0.133$$

Hence, with respect to the Shapley-Shubik index of power, major provinces have approximately 13.3% of the power each, minor provinces have approximately 9.17% of the power each.

Banzhaf index of power. Winning coalitions can be divided into 2 large groups: those containing both major provinces and those containing only one.
Let us consider one of the major provinces M.
The winning coalitions in which it is critical in the first group are those in which we have only five minor provinces, since from the sixth onwards it is no longer necessary to have both major provinces to reach half the population. The number of coalitions of this type is $\binom{8}{5}$, since I have to choose five among the eight minor provinces.
The winning coalitions in which it is critical in the second group are all those that contain, in addition to it, from six to eight minor provinces. The number of coalitions of this type is $\binom{8}{6} + \binom{8}{7} + \binom{8}{8}$, since I have to choose six, seven or eight among the eight minor provinces.
Therefore the total Banzhaf power of the major provinces is:

$$TBP(M) = \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8} = 93$$

Let us consider one of the minor provinces m.
The winning coalitions in which it is critical in the first group are those that contain exactly seven provinces, since reaching half of the population is ensured by the two largest provinces. The number of coalitions of this type is $\binom{7}{4}$, since I have to choose four out of the seven remaining minor provinces.
The winning coalitions in which it is critical in the second group are, as before, those that contain exactly seven provinces: one of the two largest,

five of the remaining minor ones, and m. The number of coalitions of this type is $\binom{2}{1} \cdot \binom{7}{5}$, since I have to choose one of the two major provinces and five of the seven remaining minor provinces. Therefore the total Banzhaf power of the minor provinces is:

$$TBP(m) = \binom{7}{4} + \binom{2}{1} \cdot \binom{7}{5} = 77$$

The Banzhaf index of power of any major province (M) and any minor province (m) is given by:

$$BI(M) = \frac{TBP(M)}{2 \cdot TBP(M) + 8 \cdot TBP(m)} = \frac{93}{802} \simeq 0.116$$
$$BI(m) = \frac{TBP(m)}{2 \cdot TBP(M) + 8 \cdot TBP(m)} = \frac{77}{802} \simeq 96.0 \times 10^{-3}$$

Hence, with respect to the Banzhaf index of power, major provinces have approximately 11.6% of the power each, minor provinces have approximately 9.60% of the power each.

Johnston index of power. Johnston index of power is calculated in a similar way to that of Banzhaf, the difference is that for each winning coalition considered in Banzhaf the number of critical members of the coalition must be taken into account.
Let us consider one of the major provinces M.
The winning coalitions in which it is critical are of different types, as we have seen previously.
The first possibility is that the coalitions contain the two major provinces and only five minor provinces. In this type of coalition the critical elements are all, i.e. seven. The number of coalitions of this type is $\binom{8}{5}$, as we have seen.
Another possibility is that in addition to the major province considered, there are six to eight minor provinces. When there are only six minor provinces, all the elements are critical. The number of coalitions of this type is $\binom{8}{6}$.
When there are seven or eight minor provinces, the only critical element is the major province we are considering. The number of coalitions of this type is $\binom{8}{7} + \binom{8}{8}$, as we have seen.
Therefore the total Johnston power of the major provinces is:

$$TJP(M) = (\binom{8}{5} + \binom{8}{6}) \cdot \frac{1}{7} + (\binom{8}{7} + \binom{8}{8}) \cdot \frac{1}{1} = 21$$

Let us consider one of the minor provinces m.
The winning coalitions in which it is critical are of different types, as we have seen previously.
The first possibility is that the coalitions contain the two major provinces and only five minor provinces. In this type of coalition the critical elements are all, i.e. seven. The number of coalitions of this type is $\binom{7}{4}$, as we have seen.
The other possibility is that there are one of the two major provinces and

exactly six minor provinces, five beyond the one we are considering. In this type of coalition the critical elements are all, i.e. 7. The number of coalitions of this type is $\binom{2}{1} \cdot \binom{7}{5}$, as we have seen.
Therefore the total Johnston power of the minor provinces is:

$$TJP(m) = (\tbinom{7}{4} + \tbinom{2}{1} \cdot \tbinom{7}{5}) \cdot \tfrac{1}{7} = 11$$

The Johnston index of power of any major province (M) and any minor province (m) is given by:

$$\begin{aligned} JI(M) &= \tfrac{TJP(M)}{2 \cdot TJP(M) + 8 \cdot TJP(m)} = \tfrac{21}{130} \simeq 0.162 \\ JI(m) &= \tfrac{TJP(m)}{2 \cdot TJP(M) + 8 \cdot TJP(m)} \tfrac{11}{130} \simeq 84.6 \times 10^{-3} \end{aligned}$$

Hence, with respect to the Johnston index of power, major provinces have 16.2% of the power each, minor provinces have 8.46% of the power each.

Deegan-Packel index of power. The Deegan-Packel index of power is calculated in a similar way to Johnston's, the difference is that only minimal winning coalitions are considered, in which every member is critical.
Considering one of the major provinces M, it suffices to consider two types of coalitions: coalitions with both major provinces and only five of the minor ones, and coalitions with the considered province M and six other minor provinces.
Therefore the total Deegan-Packel power of the major provinces is:

$$TDPP(M) = (\tbinom{8}{5} + \tbinom{8}{6}) \cdot \tfrac{1}{7} = 12$$

Taking into account one of the minor provinces m, the total Johnston power of m already takes into account only minimal winning coalitions so it corresponds to the total Deegan-Packel power:

$$TDPP(m) = (\tbinom{7}{4} + \tbinom{2}{1} \cdot \tbinom{7}{5}) \cdot \tfrac{1}{7} = 11$$

The Deegan-Packel index of power of any major province (M) and any minor province (m) is given by:

$$\begin{aligned} DPI(M) &= \tfrac{TDPP(M)}{2 \cdot TDPP(M) + 8 \cdot TDPP(m)} = \tfrac{3}{28} \simeq 0.107 \\ DPI(m) &= \tfrac{TDPP(m)}{2 \cdot TDPP(M) + 8 \cdot TDPP(m)} = \tfrac{11}{112} \simeq 98.2 \times 10^{-3} \end{aligned}$$

Hence, with respect to the Deegan-Packel index of power, major provinces have approximately 10.7% of the power each, minor provinces have approximately 9.82% of the power each.

It is interesting to note that, although the percentages of population within the provinces are almost all different from each other, regardless of the index we are considering, the smaller provinces always have the same index between them, as well as the two major provinces.

3.5.5 Passing a constitutional law in the Italian parliament

To understand better Italian situation we report below the division of the Parliament, namely of the chamber of deputies and the senate after the Italian political elections of last 25 September ([22], [15]).
The chamber of deputies has 400 members. This is the breakdown of members by political party:

- Majority:
  - 118 members belong to Fratelli d'Italia;
  - 66 members to Lega;
  - 44 members to Forza Italia;
  - 9 members to Noi Moderati and Maie (Movimento associativo italiani all'estero) together.
- Opposition:
  - 69 members belong to Partito Democratico;
  - 52 members to Movimento 5 Stelle;
  - 21 members to Azione-Italia Viva;
  - 21 members of the mixed group in which 12 members are from Verdi-Sinistra italiana, 3 members are from +Europa, 3 members of linguistic minorities and 3 members not enrolled in any component.

The senate has 200 members in addition to senators for life that are 6 people. This is the breakdown of members by political party:

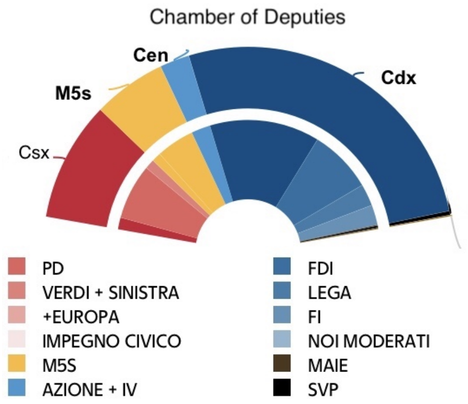
- Majority:
  - 63 members belong to Fratelli d'Italia;
  - 29 members to Lega;
  - 18 members to Forza Italia;
  - 6 members to Civici d'Italia-Noi Moderati-Maie.
- Opposition:
  - 38 members belong to Partito Democratico;
  - 28 members to Movimento 5 Stelle;
  - 9 members to Azione-Italia Viva;

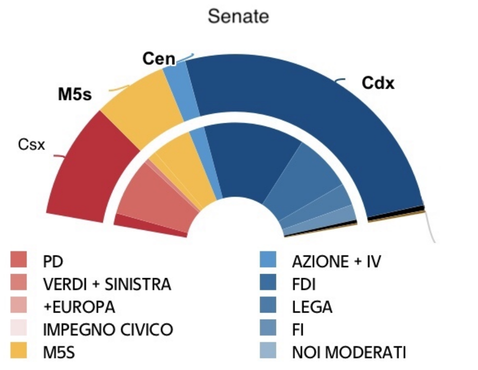
– 7 members of the mixed group in which 4 members are from Verdi-Sinistra italiana and 3 senators for life that are Mario Monti, Renzo Piano and Liliana Segre;

– 7 members of the autonomy group composed by local political forces as Svp-Patt, Campobase and Sud chiama Nord, and 2 senators for life that are Elena Cattaneo and Giorgio Napolitano;

– last senator for life Carlo Rubbia that did not join any group.

We can visualize this subdivision in the following figures:





Passing a constitutional review law or a constitutional law. For a constitutional review law or a constitutional law to pass ([2]) the scenarios are:

- two-thirds of both Chamber of Deputies and Senate approve the law, then the promulgation by the President of the Republic and finally the publication in the Official Gazette;
- absolute majority of both Chamber of Deputies and Senate approve the law, and the possibility of requesting a constitutional referendum is envisaged. The request can be presented by one fifth of the members

of a Chamber, by 500,000 electors or by 5 regional councils within three months of publication in the Official Gazette.

Let us now calculate the indexes of power of deputies and senators considering only the need to reach the $\frac{2}{3}$ of voters in both chambers.

Indexes of power of deputies and senators

Shapley-Shubik index of power. First of all, we note that the $\frac{2}{3}$ of the House and the Senate correspond, respectively, to 267 members and 138 members.
The denominator in the calculation of Shapley-Shubik indexes of power is always 606! since this is the number of possible permutations of the 400 members of the Chamber of Deputies and the 206 members of the Senate.
Let us consider a deputy, let's call him D. In order for him to be pivotal in an ordering, it is necessary that before him there are at least 138 members of the Senate, so that the victory in the Senate is assured, and exactly 266 members of the Chamber of Deputies. The number of sorts of this type is given by:

$$\begin{aligned} SSI(D) = & \frac{\binom{206}{138}\binom{399}{266} \cdot (138+266)! \cdot (606-138-267)!}{606!} + \\ & + \frac{\binom{206}{139}\binom{399}{266} \cdot (139+266)! \cdot (606-139-267)!}{606!} + \ldots \\ & + \frac{\binom{206}{206}\binom{399}{266} \cdot (206+266)! \cdot (606-206-267)!}{606!} \end{aligned} \tag{3.8}$$

Let us analyze the reason for the first addend: $\binom{206}{138}$ gives us all the possible choices of 138 senators among the 206 available, $\binom{399}{266}$ gives us all the possible choices of 266 deputies among the remaining 399, (138+266)! gives us all possible permutations of the members of parliament before D, (606-138-267)! gives us all permutations of the members of parliament after D.
The other addends always have one more senator than the previous one up to having the entire Senate before D.
Using the computer program *Mathematica*, we type the following as input:

```
Sum[Binomial[206,s]Binomial[399,266](266+s)!(339-s)!,{s,138,206}]/606!
```

Then we type "N[%]" to obtain the answer as a nice decimal.
The output is

$$SSI(D) \simeq 1.22 \times 10^{-3}$$

Let us consider a member of the Senate, let's call him S. In order for him to be pivotal in an ordering, it is necessary that before him there are at least 267 members of the Chamber, so that the victory in the Chamber of deputies is assured, and exactly 137 members of the Senate. The number of sorts of this type is given by:

$$\begin{aligned} SSI(S) = & \frac{\binom{400}{267}\binom{205}{137}\cdot(137+267)!\cdot(606-267-138)!}{606!}+ \\ & + \frac{\binom{400}{268}\binom{205}{137}\cdot(138+268)!\cdot(606-268-138)!}{606!}+\dots \\ & + \frac{\binom{400}{400}\binom{205}{137}\cdot(400+137)!\cdot(606-400-138)!}{606!} \end{aligned} \tag{3.9}$$

Let us analyze the reason for the first addendum: $\binom{400}{267}$ gives us all the possible choices of 267 deputies among the 400 available, $\binom{205}{137}$ gives us all the possible choices of 137 senators among the 205 remaining, (137+267)! gives us all possible permutations of members of parliament before S, (606-267-138)! gives us all permutations of members of parliament after S.
The other addends always have one more deputy than the previous one up to having all the Chamber of Deputies before S.
Using the computer program *Mathematica*, we type the following as input:

```
Sum[Binomial[400,s]Binomial[205,137](137+s)!(468-s)!,{s,267,400}]/606!
```

Then we type "N[%]" to obtain the answer as a nice decimal.
The output is

$$SSI(S) \simeq 2.48 \times 10^{-3}$$

Hence we have that every deputy has, due to the Shapley-Shubik index, approximately 0.12% of the power, and every senator has approximately 0.25% of the power. Therefore we have that the Chamber of Deputies holds approximately 48% of the power, and the Senate holds approximately 52% of the power.

It interesting to note the fact that, as we can see in [14], Shapley and Shubik in 1954 stated that:

> In pure bicameral systems using simple majority votes, each chamber gets 50% of the power (as it turns out), regardless of the relative sizes.

This only turns out to be true if both chambers are the same size or if both chambers have an odd number of members, proof of this claim was made by Professor John Duncan.
Since, in the Italian Parliament, both chambers have an even number of members, the Chamber of Deputies and the Senate, as announced, do not have the same amount of power. In this particular case the Senate is more powerful than the Chamber of Deputies.

Banzhaf index of power. The winning coalitions of this system are all those involving at least 138 senators and at least 267 deputies.
Considering a deputy D, the winning coalitions to which he belongs and in which his defection is critical are all those with only 267 deputies including him.
Therefore the total Banzhaf power of a fixed deputy is:

$$TBP(D) = (\binom{206}{138} + ... + \binom{206}{206}) \cdot \binom{399}{266}$$

This is because in any such coalition I have to choose 266 deputies from the remaining 399, and then I can choose from 138 to 206 senators.
Taking a Senator S into consideration, the winning coalitions to which he belongs and in which his defection is critical are all those with only 138 Senators including him.
Therefore the total Banzhaf power of a fixed senator is:

$$TBP(S) = (\binom{400}{267} + ... + \binom{400}{400}) \cdot \binom{205}{137}$$

This is because in any coalition of this type I have to choose 137 out of 205 senators, and then I can choose from 267 to 400 deputies from the ones I have. Therefore Banzhaf indexes of power of a deputy and a senator are:

$$BI(D) = \frac{TBP(D)}{400 \cdot TBP(D) + 206 \cdot TBP(S)}$$
$$BI(S) = \frac{TBP(S)}{400 \cdot TBP(D) + 206 \cdot TBP(S)}$$

Again using *Mathematica* we obtained

$$BI(D) \simeq 1.64 \times 10^{-3}$$
$$BI(S) \simeq 1.68 \times 10^{-3}$$

Hence we have that every deputy has, due to the Banzhaf index, approximately 0.16% of the power, and every senator has approximately 0.17% of the power. Therefore we have that the Chamber of Deputies holds approximately 64% of the power, and the Senate holds approximately 36% of the power.

Johnston index of power. Johnston index of power is calculated in a similar way to the Banzhaf index, the difference is that for each winning coalition considered for the Banzhaf index, the number of critical members of the coalition must be taken into account.
Taking into consideration a deputy D, the number of critical members in the coalitions considered to find the total Banzhaf power of D is 138+267 when there are exactly 138 senators and 267 deputies, instead it is 267 when there are from 139 to 206 senators and 267 deputies.
Therefore the total Johnston power of a fixed deputy is:

$$TJP(D) = \binom{206}{138} \cdot \binom{399}{266} \cdot \frac{1}{138+267} + (\binom{206}{139} + ... + \binom{206}{206}) \cdot \binom{399}{266} \cdot \frac{1}{267}$$

Taking into consideration a senator S, the number of critical members in the coalitions considered to find the total Banzhaf power of S is 138+267 when there are exactly 138 Senators and 267 Deputies, instead it is 138 when there are from 268 to 400 deputies and 138 senators.
Therefore the total Johnston power of a fixed senator is:

$$TJP(S) = \binom{400}{267} \cdot \binom{205}{137} \cdot \frac{1}{138+267} + (\binom{400}{268} + ... + \binom{400}{400}) \cdot \binom{205}{137} \cdot \frac{1}{138}$$

Therefore Johnston indexes of power of a deputy and a senator are:

$$JI(D) = \frac{TJP(D)}{400 \cdot TJP(D) + 206 \cdot TJP(S)}$$
$$JI(S) = \frac{TJP(S)}{400 \cdot TJP(D) + 206 \cdot TJP(S)}$$

Again using *Mathematica* we obtained

$$JI(D) \simeq 1.37 \times 10^{-3}$$
$$JI(S) \simeq 2.20 \times 10^{-3}$$

Hence we have that every deputy has, due to the Johnston index, approximately 0.14% of the power, and every senator has approximately 0.22% of the power. Therefore we have that the Chamber of Deputies holds approximately 56% of the power, and the Senate holds approximately 44% of the power.

Deegan-Packel index of power. Deegan-Packel index of power is calculated in a similar way to the Johnston index, the difference is that only minimal winning coalitions are considered, in which every member is critical.
Taking into consideration a deputy D, the minimal winning coalitions are those in which there are exactly 138 senators and 267 deputies.
Therefore the total Deegan-Packel power of a fixed deputy is:

$$TDPP(D) = \binom{206}{138} \cdot \binom{399}{266} \cdot \frac{1}{138+267}$$

Taking into consideration a senator S, the minimal winning coalitions are the same as before.
Therefore the total Deegan-Packel power of a fixed senator is:

$$TDPP(S) = \binom{400}{267} \cdot \binom{205}{137} \cdot \frac{1}{138+267}$$

Therefore Deegan-Packel indexes of power of a deputy and a senator are:

$$JDP(D) = \frac{TDPP(D)}{400 \cdot TDPP(D) + 206 \cdot TDPP(S)}$$
$$JDP(S) = \frac{TDPP(S)}{400 \cdot TDPP(D) + 206 \cdot TDPP(S)}$$

$$DPI(D) \simeq 1.65 \times 10^{-3}$$
$$DPI(S) \simeq 1.65 \times 10^{-3}$$

Hence we have that every deputy has, due to the Johnston index, approximately 0.17% of the power, and every senator has approximately 0.17% of the power. Therefore we have that the Chamber of Deputies holds approximately 68% of the power, and the Senate holds approximately 32% of the power.

Recap. We can observe that the Chamber of Deputies and the Senate does not have the same power with respect to most of the indexes. And besides, the chamber with the most power isn't always the same either. Indeed we obtained (approximately):

- **Shapley-Shubik index**: CoD 48%, S 52%;
- **Banzhaf index**: CoD 64%, S 36%;
- **Johnston index**: CoD 56%, S 44%;
- **Deegan-Packel index**: CoD 68%, S 32%.

Chapter 4

Classroom activity

4.1 Teaching ideas

Political power, or rather, the calculation of its indices for individuals in a yes-no voting system, is an excellent activity to present to a fourth or fifth grade in a high school. It is not an activity suitable for previous grades due to the need for a base of combinatorics to calculate the indexes defined in the previous chapter.
There are several reasons for presenting this study; first of all it is an excellent example of the connection between mathematics and social sciences. The presentation of links between mathematics and other subjects, not only scientific but also humanistic/social, is one of the objectives underlined by the previously analyzed National Guidelines for High Schools.
It is also a great application to demonstrate the fact that mathematics is really used in every field and it is present even where you least expect it. It can therefore help answer the question "What is mathematics for?", which was discussed earlier.
Probably the most useful and important motivation for students is the fact that it is an application of mathematics that lends itself perfectly to further study or to the introduction of mathematical models, which now have a fundamental role in the National Guidelines for bringing students the acquisition of the required skills.
First of all it is important to understand why mathematical models are so important (see e.g. [20], [11], [23]).
A **mathematical model** is an artificial construction to represent some properties of real objects using quantitative techniques and mathematical tools. Initially, mathematical models were used in the physical sciences to represent natural phenomena. Subsequently, the modeling approach was also adopted by the social and economic sciences.
Mathematical models are useful both in understanding phenomena and in

finding solutions to problems. The main advantages of mathematical modeling are the following:

- **Understanding of reality.** Mathematical models are a structured and simplified representation of reality or of a particular problem. This representation allows us to better understand the properties of reality that would otherwise not be evident. A mathematical model can also allow other properties of the problem or of reality not yet known to be analytically deduced (analytic deduction).

- **Rational strategy.** Mathematical models make it possible to identify a rational strategy to achieve a specific final objective. In addition to indicating the most rational strategy to achieve a goal, mathematical models also provide detailed quantitative information on the choices to be made.

- **Simulation.** Mathematical models allow for simulations of reality. Simulation allows you to study the effects of a decision without necessarily having to adopt it in reality. An effective mathematical model performs an important forecasting function which considerably reduces the risks of the decision-making choices of a decision maker.

- **Qualitative aspects.** A mathematical model represents reality through quantitative quantities (measurable quantities). However, not all aspects of reality can be measured objectively and unequivocally. There are qualitative aspects that assume great importance in the explanation of reality. In order not to be excluded from the mathematical model, these qualitative aspects must necessarily be translated in a quantitative way. However, the transformation of qualitative quantities into quantitative quantities is notoriously a subjective process, strongly influenced by the judgments of the researcher. In conclusion, it is not always true that a mathematical model reflects the reality of phenomena. In many cases, mathematical models only reflect the researcher's point of view of reality. This happens, in particular, in the formulation of mathematical models in the social sciences (this is our case).

We also report the fact that one of the most common complaints from teachers is the fact that students fail to retain mathematical knowledge and are unable to apply a previously learned skill to a new type of problem. Students who are taught exclusively from traditional textbook problems fail to grasp the importance of what they are learning. They form a subjective perception of what a subject is about based on the tasks they are assigned. This explains why most students view mathematics as a set of rules or procedures on how to move symbols around.
The current method of instruction encourages students to divide concepts

and procedures into different compartments and to approach mathematics as a series of topics to be memorized and quickly forgotten.
Using mathematical procedures in the context of authentic activities allows students to see procedures as tools rather than the end result of their knowledge, making them more likely to be able to adapt and use procedures in other situations. While it is valuable to repeat procedures until they become automatic, students should be required to achieve a higher level of understanding of concepts. The use of high-quality assignments in the classroom can serve as a bridge between process and concept. Mathematical modeling encourages a deeper understanding of mathematical ideas and trains students to think, interpret, and formulate a plan when presented with a non-traditional problem. When used correctly, mathematical modeling encourages students to stop seeing mathematics as techniques and procedures and to start seeing it as a tool for solving problems.
Zbiek and Conner, in their article "Beyond Motivation: Exploring Math Modeling as a Context for Deepening Students' Understandings of Curricular Mathematics" ([31]), say that modeling tasks lead students to a deeper conceptual understanding of mathematical entities by requiring them to combine multiple mathematical objects, properties, and parameters into a single mathematical entity and that modeling tasks improve students' procedural understanding by requiring them to select the appropriate procedure and perform mathematical manipulations.
The National Guidelines, as we have seen before, state that students must learn to apply the mathematical content and procedures learned in class to different situations. Teaching students to think mathematically is beneficial since the purpose of teaching mathematics is for students to develop reasoning skills so they are able to analyze and solve a problem.
Until now, teachers have focused on teaching content to students by expecting students to develop math habits and thinking skills on their own. However, this teaching style only works for high-achieving students who are interested in math. It will be necessary for teachers to incorporate mathematical modeling problems that require students to use thinking and reasoning skills in conjunction with content knowledge.

4.2 Activity and teaching methodologies used

When designing this activity, the aim was to create lessons that are as interactive as possible, where the teacher's part of the lesson is minimized and group work is encouraged. In this way, students are given more responsibility and they have more motivation to look for a solution to the problem posed.

4.2.1 Students involved

The school in which I carried out the activity entitled *Power as an application of mathematics* is the "Liceo Scientifico Eugenio Curiel" in Padua. The participants in this project are the students of the fourth year of the "Liceo Matematico" of the Institute.
The "Liceo Matematico" is an additional option to the scientific and applied sciences courses. It is aimed at all students who want to improve their skills in mathematical modeling, it is a path that aims to cultivate excellence in this sector. It provides additional hours to the curricular ones, at least 33 hours of enhancement for each year.

The principles and values of this project are explained in depth on the official site of the "Liceo Matematico" (`https://www.liceomatematico.it`).
The general idea of the new structure is to give more space to mathematics and the sciences, not to introduce a greater number of notions, but to reflect on foundations and ideas, broaden cultural horizons, deepen, better understand, and in particular underline connections with other disciplines, including the humanistic ones. Therefore, a strongly interdisciplinary approach of the initiative is highlighted. In this order of ideas, the project is not reserved only for scientific high schools, but for all high schools that intend to offer their students a diversified and expanded cultural challenge; the activities will then be adapted to the various addresses.
It is good to underline the all-round cultural and social importance of the initiative, and its medium and long-term impact also on the production and employment level, in a society that requires an ever greater capacity for scientific data analysis and an approach to complex situations. Mathematics is in fact increasingly present in our daily life with its multiple applications, as well as having a decisive cultural impact on the development of our civilization.

The "Liceo Matematico" project represents an opportunity to profitably reflect and discuss the contents and teaching methodologies in high school classes. From this point of view, the project can have a significant impact on the revision process of the National Guidelines. As far as the methodology is concerned, laboratory-type practices are systematically used in the project, also bearing in mind the experiences gained in the Scientific Degree Plan.
The "Liceo Matematico" is characterized by a strong collaboration between university teachers and school teachers, in both cases not only in mathematics. The collaboration takes place through the organization of periodic meetings aimed at designing and discussing laboratory courses to be tested and implemented in the classrooms.

Speaking of projects with university teachers, during the first year of high

school, the class that carried out the activity had taken part in a project with professor Samuele Maschio, again within the "Liceo Matematico", which dealt with the voting procedures, constituency voting and gerrymandering. As we can read in [17], the project is divided into three activities.
The first two focus on a critical analysis of voting procedures:

1. the first activity, in particular, deals with the vote between two alternatives, presenting the **May's theorem**;

2. the second activity deals with the vote between more than two alternatives, presenting the **Arrow's Theorem**.

The third activity instead focuses on the issues of voting by constituencies and gerrymandering. These concepts are introduced thanks also to the analysis of the elections for the American president, which lend themselves well to the explanation of the topics just mentioned.

It is therefore interesting to note how the activity on political power can be seen as a continuation of the activity in which these students participated in the first year of high school.
Let us now analyze the four hours of lessons in which this new project took place.

4.2.2 First two-hour lesson

The didactic methodology used in the first two hours of this activity was that of the four didactic situations, presented previously.
The activity started with a discussion on some preliminary knowledge necessary for the continuation of the activity. Definitions of yes-no voting system, coalition, monotone voting system and minimum winning coalition have been given. Also in this preliminary discussion, the presentation of the official definitions was preceded by a short discussion in which the students themselves looked for a suitable definition, getting very close to the official definition which was then made explicit.

After having laid the necessary foundations to continue with the activity, the concept of the power of an individual in a voting system was introduced to the students. The blackboard was used, in a collective discussion mode, to identify and collect the variables that can affect the amount of power an individual enjoys in a yes-no voting system.
After this collective discussion the students were divided into groups of four/five and they were given time to find together a model to calculate the power of an individual. It is therefore the action situation, the first of the four phases of the method that is being used. In this phase, the commitment by the members of the various groups to find a solution was considerable, the problem was rather complicated and did not have a single

solution. The objective of this first phase was for the students to become aware of the fact that the solution was not unique and to try to use the variables previously collected on the blackboard in the resolution.

When the time available to the students was over, the second phase began, i.e. the formulation situation, in which each group presented its idea of resolution to the others. An exponent from each group had the task of outlining their idea on the blackboard, so that it was always visible to everyone. During the exposition there was a constructive discussion among the students, moderated by me, in which the different solutions were analysed.

The validation situation was carried out immediately afterwards, in this phase the critical points and strengths of the various proposed theories were evaluated (by the teacher). The students also played an active role in this phase, trying to resolve the critical issues that were brought to their attention, with the intention of formulating a valid theory from all points of view. Given the complexity of the subject, it was not possible to arrive at a precise definition of any of the four power indices seen previously, but the understanding of the critical issues and having come very close to the possible solutions was nonetheless remarkable.

Finally it was the moment of the institutionalization situation, in which I presented the students with the official definitions of the four indices of power. The definitions of the Shapley-Shubik and Banzhaf indices have been well received, due to being rather intuitive; the Johnston and Deegan-Packel indices instead required a more in-depth discussion to better understand their meaning. For each index of power, after defining and discussing it together, we formulated an example and calculated the values of the four indexes for it.
The example was formulated by the students, under the condition of obtaining a realistic yes-no voting system. The result is a hypothetical system in which the head teacher **h**, the class coordinator **c**, another teacher **t** of the class in question and one of the two class representatives **r** have to decide whether or not to join a certain project. The students gave the head teacher 4 votes, the coordinator 3 votes, the other teacher 1 vote and the representative 2 votes. The quota they have set is 3 votes, arguing that the class coordinator and the head teacher can choose to have a class join an activity even without the consent of students or of other teachers; instead, if students are interested in a certain activity, they have no option to join except with the approval of one of their teachers.
We can denote the weighted system in this way: $[3 : 4, 3, 2, 1]$.
Shapley-Shubik index of power. The orderings in which **h** (respectively **c**) is pivotal are the following: when **h** (respectively **c**) is the first of the ordering, when **h** (respectively **c**) is preceded by **t** or **r**. Then

$$SSI(h) = SSI(c) = \tfrac{3!+2\cdot 2}{4!} = \tfrac{10}{24}.$$

The ordering in which **r** (respectively **t**) is pivotal is when **r** (respectively **t**) is preceded by **t** (respectively **r**). Then

$$SSI(r) = SSI(t) = \frac{2}{4!} = \frac{2}{24}.$$

Banzhaf index of power. The winning coalitions that contain **h** (respectively **c**) and in which **h** (respectively **c**) is critical are **h** (respectively **c**), **hr** (respectively **cr**) and **ht** (respectively **ct**). Then $TBP(h) = TBP(c) = 3$.
The winning coalition that contain **r** (respectively **t**) and in which **r** (respectively **t**) is critical is **tr**. Then $TBP(t) = TBP(r) = 1$.
So we obtain

$$BI(h) = BI(c) = \frac{3}{3+3+1+1} = \frac{3}{8}$$
$$BI(r) = BI(t) = \frac{1}{3+3+1+1} = \frac{1}{8}$$

Johnston index of power. The winning coalitions that contain **h** (respectively **c**) and in which **h** (respectively **c**) is critical are **h** (respectively **c**) that has only one critical member, **hr** (respectively **cr**) that has only one critical member and **ht** (respectively **ct**) that, again, has only one critical member. Then $TBJ(h) = TBJ(c) = \frac{1}{1} + \frac{1}{1} + \frac{1}{1} = 3$.
The winning coalition that contain **r** (respectively **t**) and in which **r** (respectively **t**) is critical is **tr** that has two critical members. Then $TBJ(t) = TBJ(r) = \frac{1}{2}$.
So we obtain

$$JI(h) = JI(c) = \frac{3}{3+3+\frac{1}{2}+\frac{1}{2}} = \frac{3}{7} = \frac{6}{14}$$
$$JI(r) = JI(t) = \frac{\frac{1}{2}}{3+3+\frac{1}{2}+\frac{1}{2}} = \frac{1}{14}$$

Deegan-Packel index of power. The minimal winning coalition that contain **h** (respectively **c**) is **h** (respectively **c**). Then $TDPP(h) = TDPP(c) = 1$.
The minimal winning coalition that contain **r** (respectively **t**) is **tr**. Then $TDPP(t) = TDPP(r) = \frac{1}{2}$.
So we obtain

$$DPI(h) = DPI(c) = \frac{1}{1+1+\frac{1}{2}+\frac{1}{2}} = \frac{1}{3} = \frac{2}{6}$$
$$DPI(r) = DPI(t) = \frac{\frac{1}{2}}{1+1+\frac{1}{2}+\frac{1}{2}} = \frac{1}{6}$$

Thanks to this example it was possible to notice the fact that the results, even if they have common traits such as the fact that **h** and **c** always have the same power, as well as **r** and **t**, give all different results.
This observation facilitated the introduction of the concept of mathematical model, arousing the interest of the students, also demonstrated by the answers to the questionnaire proposed at the end of the activity which will be discussed later.

The lesson ended with the introduction of Example 3.1.9 presented previously, which clearly illustrates a situation in which the Shapley-Shubik power index is not the right model, unlike the Banzhaf power which is a better model. This example is useful to give a real idea of what it means to choose a good model for a certain situation.

4.2.3 Second two-hour lesson

The second, and last, two hours of lessons, on the day following the first two hours, had two main focuses: the first was to complete the necessary calculations to show that the example introduced the previous day excluded the Shapley-Shubik power index from being chosen as a good role model in that given situation; the second is to challenge students to calculate the four defined indices of power of members of a certain yes-no voting system taken from reality.

After finishing the example related to the paradoxical aspect of the Shapley-Shubik index and checking that in that given example the Banzahf index works correctly, Procedure 1 was presented to calculate the total Banzhaf power with an alternative, often facilitating method. The students, intrigued by this new method, explicitly requested the demonstration that this method was equivalent to the classical method for calculating the total Banzhaf power. So, gladly, some time was spent on this demonstration.

The following hour was devoted to the final work, the students split into the same groups as in the previous lesson and randomly extracted one of the, previously prepared, real situations of yes-no voting systems. The voting systems were chosen from those present in the previous chapter and were the following:

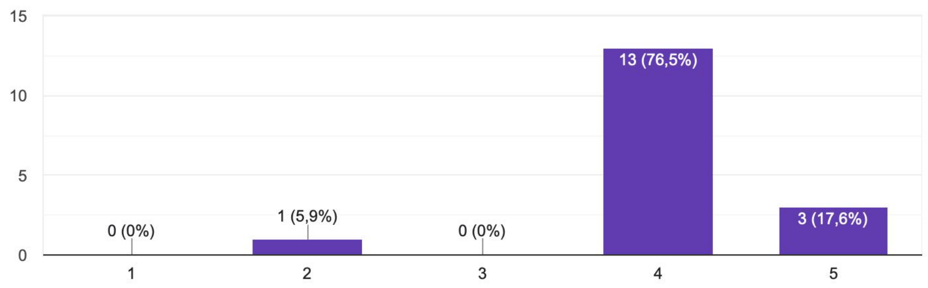
- The European Economic Community of 1958;
- The United Nations Security Council;
- The System to Amend the Canadian Constitution;
- Passing a constitutional law in the Italian parliament, considering only the need to reach the $\frac{2}{3}$ of voters in both chambers.

During the hour at their disposal, the students had the task of calculating the four distinct indices of power for each member of the voting system they had taken charge of.
The task was quite difficult so if necessary the teacher (i.e. myself) was allowed to give suggestions to help the groups.
The students worked with interest and enthusiasm and tried to complete the exercise within the allotted time. Although it wasn't easy, most of the groups managed it, even if with some sporadic mistakes due to haste.

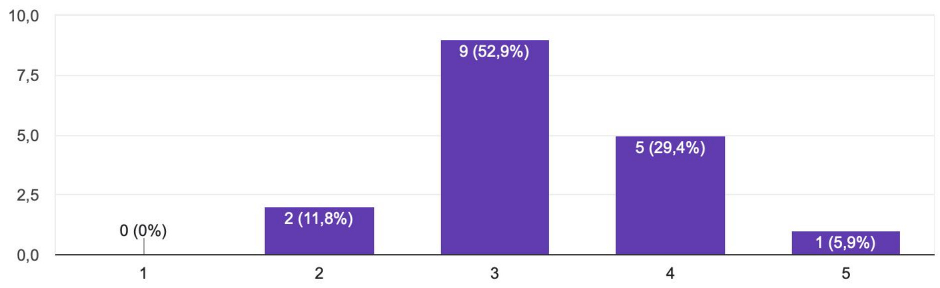
Before the end of the lesson, a quarter of an hour was spent allowing the students to fill out a satisfaction questionnaire on the activity and on the study of mathematics in general.

4.3 Satisfaction questionnaire

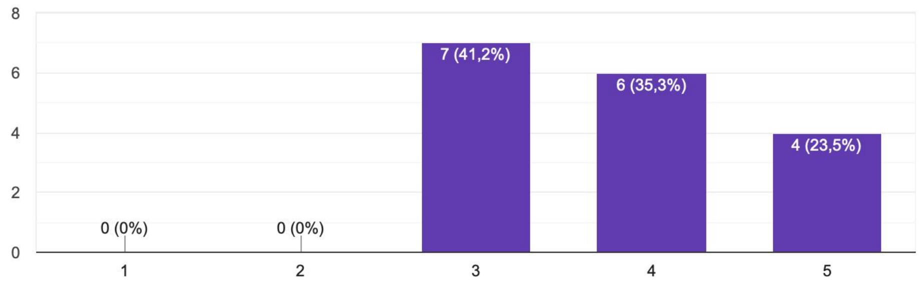
The following is the satisfaction questionnaire of the activity proposed to the students, every question presented is followed by the answers given by the 17 students who participated to the activity:
Question 1. How much do you like studying math on a scale of 1 to 5? Where 1 is "Not at all, it's my least favorite subject" and 5 is "Very much, it's my favorite subject".



Question 2. How good do you think you are at math on a scale of 1 to 5? Where 1 is "A little, I have to study a lot to get results" and 5 is "A lot, I don't have to study a lot to get results".



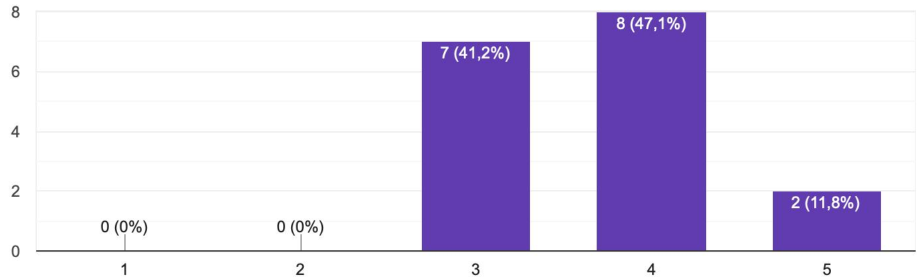
Question 3. How interesting do you find the activity carried out on a scale of 1 to 5? Where 1 is "Not at all interesting" and 5 is "Very interesting".



Question 4. Did you find the activity easier or more difficult to understand than the classic topics usually covered in class? Why?

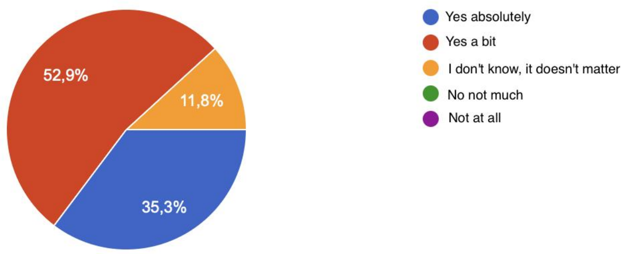
14 students found the activity more difficult in general but more stimulating and interesting. They were particularly pleased with the fact that it was an activity based on real examples. They attribute the fact that it was more difficult to having introduced a completely new topic that united several previously seen topics (positive aspect of the activity), and to the fact that there were only a few hours to tackle the activity.
The remaining 3 students found it easier because the topics covered were real and tangible in reality.

Question 5. How useful do you consider the activity carried out on a scale of 1 to 5? Where 1 is "Not at all helpful" and 5 is "Very helpful".



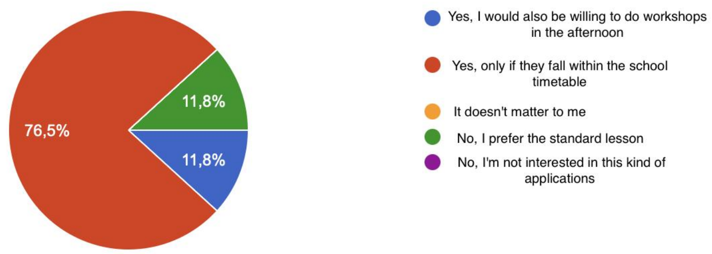
Question 6. Do you think that seeing these humanistic/social applications of mathematics can bring you closer to the subject?
Possible answers:

- Yes absolutely
- Yes a bit
- I don't know, it doesn't matter
- No not much
- Not at all



Question 7. Would you like to see more applications of this type?
Possible answers:

- Yes, I would also be willing to do workshops in the afternoon
- Yes, only if they fall within the school timetable
- It doesn't matter to me
- No, I prefer the standard lessons
- No, I'm not interested in this kind of applications



Question 8. Space for further comments on the activity, advice, strengths, weaknesses,...

I report the most common answers given by students.
8 students expressed their willingness to devote more hours to the activity, to better understand the topic on the one hand and to deepen other aspects on the other.
6 students appreciated the teaching methodology used, in particular the work in groups, reducing the teacher's time for frontal lessons.
5 students found the activity very interesting, also thanks to the fact that it was a practical application of mathematics.

Therefore, basing ourselves on the answers to the questionnaire, we can state that the activity was appreciated by the students and that seeing more humanistic/social applications of mathematics also helps the approach to mathematics of even less interested students. In fact, those who expressed little interest in mathematics in the first questions of the questionnaire later stated that seeing this type of application brings them closer to mathematics

and increases their interest in this area.
We also received positive feedback from the teachers of this high school, who were enthusiastic and expressed their desire to repeat the activity in the next years.

4.4 Possible insights

Another interesting possibility to develop this activity in scientific high school, applied science option, is to collaborate with the computer science teacher.
This activity sees as natural developments the creation of programs that speed up the calculation of the winning coalitions of a voting system or the calculation of the power indexes analyzed.
An example of MATLAB code to calculate all the winning coalition of a voting system is the following:

```
   %Ask as input voters and their weights
n = input('Enter the number of voters: ');
voters = cell(n,1);
weights = zeros(n,1);
for i = 1:n
    voters{i} = input('Enter a name for the voter: ','s');
    pesi(i) = input('Enter the weight associate to this voter: ');
end
%Ask as input the quota
quota = input('Enter the quota: ');
%Generate all possible coalitions of voters
coalitions = cell(2^n-1,1);
k = 1;
for i = 1:n
    for j = 1:k
        coalitions{k+j} = [coalitions{j} i];
    end
    k=k*2;
end
coalitions = coalitions(2:end);
%Filter coalitions that meet the quota
results = {};
for i=1:length(coalitions)
    coalition = coalitions{i};
    sum = sum(weights(coalition));
    if sum >= quota
        results{end+1}=coalition;
    end
```

```
end
%Print the results
fprintf('Coalitions where the sum of weights meets or exceeds the quota are:\n');
for i = 1:length(risultati)
    coalition = results{i};
    stringa = voters{coalition(1)};
    for j = 2:length(coalition)
        stringa = strcat(stringa,',',voters{coalition(j)});
    end
    fprintf('%s\n',stringa);
end
```

The created program can be easily adapted to other commonly used programming languages and has a rather simple code. It does not contain advanced commands and is therefore suitable for a fourth year of a scientific high school, applied sciences option, who should have a good basic knowledge of at least one programming language.
There are many possible programs to be implemented, and they can be designed with increasing difficulty to challenge students and make lessons more interesting.

Chapter 5

Conclusions

This study aimed to answer the following question: can humanistic/social applications of mathematics bring more students closer to mathematics?
The search for an answer to this question involved several phases:

- the administration of a questionnaire concerning the teaching of this type of applications to mathematics and physics teachers of various Italian high schools;
- the analysis of the National Guidelines for High Schools, looking for references to the applications in question;
- the analysis of two cycles of mathematics textbooks, often adopted by high schools, seeking humanistic/social applications or insights;
- the activity carried out in a class of a mathematical scientific high school on political power, seen as an application of mathematics;
- the administration of a satisfaction questionnaire of the activity carried out to the class in question.

The cross-analysis of the results obtained from the first three phases brings out a certain coherence between the activity of the teachers, the references of the guidelines and the applications present in the books. By coherence we mean the fact that teachers, in choosing the topics to present, are conditioned by the textbooks provided to them, which in turn are conditioned by the national guidelines.
From the guidelines it emerged that an interdisciplinary path has become very important for students, but interdisciplinarity often indicates the link between scientific subjects or between humanities subjects, the intersection of these two macro areas is rarely mentioned and when mentioned it appears ambiguous. As a result, we find few noteworthy examples in textbooks where some humanistic/social applications are presented. It certainly also depends

on which textbook one takes into consideration, in any case, although there are few really well-structured examples, it is fair to point out the fact that over the years the situation has improved and the quantity of references to humanistic/social applications is grown. This also suggests that their usefulness is probably starting to be increasingly recognized on several social levels.
Finally, thanks to the answers to the questionnaire, it was learned that a good percentage of teachers has also begun to propose applications other than the usual scientific applications, however they require more time to implement them (therefore a revision of the guidelines is necessary, which must keep new needs of teachers in order to create the best path for their students) and require more examples and insights from textbooks.

Despite the current context that we have investigated, thanks to various researches in the field of mathematics education, the importance of presenting humanistic/social examples and insights to students has emerged. Thanks to this it will be possible to make students aware of the potential of mathematics and its presence in all fields of study, giving concrete examples and not motivational sentences that make no differences. Furthermore, the importance of clarifying the concept of model, through concrete examples and activities, is often underlined by scholars and is also present in the guidelines, even if without a description of the modalities with which to present it. This ambiguity consequently also makes the task of textbooks vague. In these it is rare to find examples of mathematical models that are really useful for fully understanding their concept and importance.

To give a concrete proof of the importance and interest that applications of mathematics of this type can arouse, we proposed an activity in a fourth grade of a mathematical scientific high school. The created activity focused on the theme of political power as an application of mathematics. Before thinking about the activity, we analyzed and deepened the foundations of this application through the study of the four internationally recognized indexes of power, also through the analysis of the paradoxical aspects of some of them and, finally, through concrete examples in which to calculate them. We then built the activity on the basis of the knowledge acquired, both regarding the topic itself and regarding the most recommended teaching methodologies in terms of research in mathematics education.
The four hours of lessons provided to the students can be considered a success, based on the results obtained from the final activity, the answers to the final satisfaction questionnaire and the feedback from the teacher who participated.
In particular, the last hour of the activity was dedicated to the calculation of the four indexes of power related to different really existing voting systems. Although the assignment was very challenging and complex, the students, with the help of some suggestions, managed to complete most of

the work. Furthermore, from the satisfaction questionnaire it emerged that the students evaluated the activity very positively, even those students who consider themselves less inclined and less passionate about mathematics. They also expressed the desire to deepen it by dedicating more time to it and to do other similar activities to see the usefulness of this subject materialize in other situations as well. The teacher who participated in the activity then reported the students' satisfaction with the activity carried out and expressed the will to do the activity again in the next years.

To conclude, we can say that humanistic/social applications have the power to bring even less interested students closer to mathematics, and have the possibility of introducing new and above all useful concepts for the citizens of tomorrow.
It is clear that there is work to be done to keep national guidelines, textbooks and teacher constantly updated. This will allow the two fields of education and of research to align, in order to provide an educational program that is increasingly more sensitive about the cultural, psychological, physical and social well-being of students. The students are the final users of the entire process described, therefore the value that is given to each individual activity must be attributed thinking exclusively of the benefit they can derive from it.
I conclude with a quote from Taiichi Ohno, the father of Japanese philosophy known as *Lean Thinking*. The Lean philosophy is already known and used in the corporate world, but it would bring great benefits if adapted to the world of education.

> Something is wrong if workers do not look around each day, find things that are tedious or boring, and then rewrite the procedures. Even last month's manual should be out of date.

Bibliography

[1] M. Artigue, M. Haspekian, and A. Corblin-Lenfant. Introduction to the theory of didactical situations (TDS). *Networking of theories as a research practice in mathematics education*, pages 47–65, 2014.

[2] Assemblea Costituente. La costituzione, parte II: Ordinamento della repubblica, titolo VI: Garanzie costituzionali, sezione II: Revisione della costituzione. leggi costituzionali., 1946.

[3] A. Baccaglini Frank, P. Di Martino, R. Natalini, and G. Rosolini. *Didattica della Matematica.* Mondadori Università, 2018.

[4] M. Bergamini, G. Barozzi, and A. Trifone. *Matematica Blu*, volume 1. Zanichelli, second edition, 2016.

[5] M. Bergamini, G. Barozzi, and A. Trifone. *Matematica Blu*, volume 2. Zanichelli, second edition, 2016.

[6] M. Bergamini, G. Barozzi, and A. Trifone. *Matematica Blu 2.0*, volume 3. Zanichelli, second edition, 2017.

[7] M. Bergamini, G. Barozzi, and A. Trifone. *Matematica Blu 2.0*, volume 4. Zanichelli, second edition, 2017.

[8] M. Bergamini, G. Barozzi, and A. Trifone. *Matematica Blu 2.0*, volume 5. Zanichelli, second edition, 2017.

[9] R. Borasi. Exploring mathematics through the analysis of errors. *For the learning of Mathematics*, 7(3):2–8, 1987.

[10] R. Borasi. *Reconceiving mathematics instruction: A focus on errors.* Greenwood Publishing Group, 1996.

[11] P. Brandi and A. Salvadori. I modelli matematici entrano a scuola. percorsi di sperimentazione didattica: C— dai modelli lineari ai primi modelli non lineari. *Matematica e Realtà*, 2015.

[12] G. Brousseau and V. Warfield. Didactic situations in mathematics education. *Encyclopedia of mathematics education*, pages 206–213, 2020.

[13] C. Kieran. Theory of didactical situations in mathematics. *Journal for Research in Mathematics Education*, 29(5):595, 1998.

[14] A. King. Thesis digest: Mathematical interpretation of political power and the arkansas state government. *Inquiry: The University of Arkansas Undergraduate Research Journal*, 4:7, 2003.

[15] La Repubblica. Elezioni politiche 2022. https://elezioni.repubblica.it/2022/elezioni-politiche.

[16] T. Y. Madu. Effective methodology in teaching mathematics: The way forward. *Journal of Resourcefulness and Distinction*, 8:10, 2014.

[17] S. Maschio. Scelta sociale a scuola. percorsi di matematica, cittadinanza e costituzione. *L'insegnamento della matematica e delle scienze integrate*, 45:22, 2022.

[18] M. Mellone, C. M. Ribeiro, and A. Jakobsen. Characterizing prospective teachers' knowledge in/for interpreting students' solutions. In *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education*, volume 4, pages 89–96. PME Kiel, 2013.

[19] Ministero dell'Istruzione e dell'Università e della Ricerca. *Indicazioni nazionali riguardanti gli obiettivi specifici di apprendimento concernenti le attività e gli insegnamenti compresi nei piani degli studi previsti per i percorsi liceali.*, 2010.

[20] A. Moser. Introduzione alla modellizzazione matematica nella scuola secondaria di secondo grado. *Didattica della matematica. Dalla ricerca alle pratiche d'aula*, 2021.

[21] M. Niss. Mathematics in society. *Didactics of mathematics as a scientific discipline*, 13:367–378, 1994.

[22] Openpolis. La XIX legislatura e gli equilibri del nuovo parlamento. https://www.openpolis.it/esercizi/la-composizione-delle-nuove-aule-parlamentari, 2022.

[23] S. Oswalt. Mathematical modeling in the high school classroom. Master's thesis, Louisiana State University and Agricultural and Mechanical College, 2012.

[24] L. Sasso and C. Zanone. *Colori della Matematica*, volume 1. Petrini, blu edition, 2017.

[25] L. Sasso and C. Zanone. *Colori della Matematica*, volume 2. Petrini, blu edition, 2017.

[26] L. Sasso and C. Zanone. *Colori della Matematica*, volume 3. Petrini, blu edition, 2017.

[27] L. Sasso and C. Zanone. *Colori della Matematica*, volume 4. Petrini, blu edition, 2017.

[28] L. Sasso and C. Zanone. *Colori della Matematica*, volume 5. Petrini, blu edition, 2017.

[29] K. Stacey. The international assessment of mathematical literacy: PISA 2012 framework and items. In *Selected regular lectures from the 12th International Congress on Mathematical Education*, pages 771–790. Springer, 2015.

[30] A. D. Taylor and A. M. Pacelli. *Mathematics and politics*. Springer, New York, second edition, 2008. Strategy, voting, power and proof.

[31] R. M. Zbiek and A. Conner. Beyond motivation: Exploring mathematical modeling as a context for deepening students' understandings of curricular mathematics. *Educational Studies in mathematics*, 63:89–112, 2006.