Università degli Studi di Padova

Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"

Corso di Laurea Triennale in Matematica

BESICOVITCH SETS AND REGULARITY

Relatore:
Prof. Davide Vittone

Laureando: Francesco Salmaso
Matricola: 2016099

Anno Accademico 2022/2023

21/07/2023

Contents

Introduction

In 1917 Abram Besicovitch was studying a problem in Riemann integration.

He wanted to know if, given $f : \mathbb{R}^2 \to \mathbb{R}$ Riemann integrable there exists a system of coordinates such that the Riemann integral $\int_{\mathbb{R}} f(x, y)dx$ exists for all y and $\int_{\mathbb{R}} f(x, y)dx$ is Riemann integrable.

The answer is negative because we can create a counterexample using the fact that there exists a set of measure zero in the plane that contains a segment of length one parallel to every direction.

He managed to construct it in 1919. Moreover he understood that with little effort he could do the same to find a Kakeya set which is a set of as small area as one wants where we can continuously move a unit segment from a certain direction to the same one with reversed orientation.

In our work we will build the Besicovitch set adding another result about its existence and showing that in higher dimension there is more regularity so the analogue set can not exist.

We will follow the book from Stein and Shakarchi [5] for the regularity and the book from Falconer [2] for the construction of the Besicovitch set.

The information comes also from a paper of Falconer himself written in 1979 [6].

Moreover in the part about regularity we will introduce the Radon transform defined as in the paper from Oberlin and Stein done in 1982 [3].

In our work we will start with regularity in dimension greater or equal to three. The main theorem states that for a set of finite n-measure the function that gives the $n-1$-measure of the intersection of the plane orthogonal to a certain direction and at a certain distance from a given point, is continuous in the distance for almost every direction.

To obtain this we will need some lemmas about Radon transform. In particular in one of them it will be fundamental that the dimension is greater or equal to three which is the reason why the same arguments do not work in the plane.

Finally we will prove a stronger version of the theorem where the function is much more general and it is not necessarily the one that measures the slices described.

Moreover we will also show that a Besicovitch set must have Hausdorff dimension two using a tool similar to the Radon transform. We will also slightly change the results used in the previous part.

About this topic there is also the Kakeya conjecture which is an important open problem in mathematics. It states that a set which contains a segment parallel to every direction in $\mathbb{R}^n$(the analogue of Besicovitch set in higher dimension) must have Hausdorff dimension n.

Then we will introduce the Besicovitch set as the natural consequence of the fact that in the plane the previous result does not hold.

Before constructing the set itself we will show a stronger result where the Baire's lemma will be fundamental.

In a square of side one we take the set of the subsets of the square made of segments from one side of the square to the opposite one and that contain a segment of length one parallel to every direction. The subset of this set made of the Besicovitch sets is dense with the Hausdorff distance.

In the end we will give the construction of a Besicovitch set with the Perron tree method.

More precisely we will divide a triangle and we will move its parts onto each other infinitely many times and then with a converging argument we will obtain our Besicovitch set.

Chapter 1

Regularity

1.1 Theorems

We want to show that, intersecting a measurable set of finite measure with all the hyperplanes perpendicular to a given direction, we obtain a continuous function for almost all the directions. We have a system of coordinates in $\mathbb{R}^n$ and we define

Definition 1. *$P_{t,\gamma} = \{x \in \mathbb{R}^n : x \cdot \gamma = t\}$ is the plane at signed distance t from the origin perpendicular to $\gamma \in \mathbb{S}^{n-1}$.*

Now let E be a measurable set of finite measure, then we define

Definition 2. *The section of E cut by the plane $P_{t,\gamma}$ is defined as*

$$E_{t,\gamma} = E \cap P_{t,\gamma}$$

Definition 3. *$f_\gamma : \mathbb{R} \to \mathbb{R}^+$ by $f_\gamma(t) = m_{n-1}(E_{t,\gamma})$ where m_{n-1} is the $n-1$-dimensional Lebesgue measure naturally carried by the plane $P_{t,\gamma}$.*

We claim that

Theorem 4. *For all $n \geq 3$ for almost every direction $\gamma \in \mathbb{S}^{n-1}$ we have:*
$E_{t,\gamma}$ is measurable for all $t \in \mathbb{R}$
$f_\gamma(t)$ is Hölder continuous with exponent α for all $\alpha \in]0, \frac{1}{2}[$.

When we say for almost every direction, intuitively, we intend with respect to the standard surface measure on $\mathbb{S}^{n-1}$.

Definition 5. *The standard surface measure is defined as*

$$\sigma(A) = n \cdot m_n \left(\{x \in \mathbb{R}^n : \frac{x}{|x|} \in A, 0 \leq |x| < 1\} \right)$$

We have an obvious corollary which is not true for $n = 2$ and the counter-example proves that Theorem 1 is false for $n = 2$.

Corollary 6. *For all $n \geq 3$ if $E \subseteq \mathbb{R}^n$ has measure 0 then for almost every direction γ we have that $m_{n-1}(E_{t,\gamma}) = 0$ for all $t \in \mathbb{R}$*

The fact that this is false for $n = 2$ is due to the existence of Besicovitch sets.

Definition 7. *A Besicovitch set is a compact set in* $\mathbb{R}^2$ *of measure* 0 *which contains a segment of length(*1*-dimensional Lebesgue measure)* 1 *for every direction in the plane.*

If Theorem 4 were true for $n = 2$ then we would have $f_\gamma(t) = 1$ for some t and if f_γ were continuous then close to t we would have $f_\gamma(t) > \frac{1}{2}$ a contradiction since $f_\gamma(t) = 0$ for all $t \in \mathbb{R}$ for almost every γ.

Moreover we have that

Theorem 8. *A compact set in the plane which contains a segment of length* 1 *parallel to every direction has Hausdorff dimension* 2*.*

Where Hausdorff dimension is defined as

Definition 9. *The Hausdorff dimension of a set* A *is defined as* $\inf\{\alpha, \mathcal{H}^\alpha(A) = 0\} = \sup\{\alpha, \mathcal{H}^\alpha(A) = +\infty\}$*.*

We have that Theorem 8 for $n \geq 3$ is an open problem.

Theorem 10 (Conjecture)**.** *For all* $n \geq 3$ *a compact set in* $\mathbb{R}^n$ *which contains a segment of length* 1 *parallel to every direction has Hausdorff dimension* n*.*

In order to prove Theorem 4 and Theorem 8 we have to introduce the Radon transform.

1.2 The Radon transform

Definition 11. *Let* f *be a function on* $\mathbb{R}^n$*. Then the Radon transform* $R(f) : \mathbb{R} \times \mathbb{S}^{n-1} \to \mathbb{R}$ *is defined by*

$$R(f)(t,\gamma) = \int_{P_{t,\gamma}} f$$

We see that $R(f)(t,\gamma)$ could be defined almost everywhere for f measurable. Moreover if E is a measurable set we have $R(\chi_E)(t,\gamma) = m_{n-1}(E_{t,\gamma})$ for all γ for a.e. t.

Definition 12. *We want also the superior in* t *of the Radon transform*

$$R^*(f)(\gamma) = \sup_{t \in \mathbb{R}^n} R(f)(t,\gamma)$$

We state this theorem:

Theorem 13. *Let* f *be a continuous function with compact support on* $\mathbb{R}^n$*,* $n \geq 3$*. Then there exists* c *such that*

$$\int_{\mathbb{S}^{n-1}} R^*(f)(\gamma)\, d\sigma(\gamma) \ \leq c(\|f\|_{L^1(\mathbb{R}^n)} + \|f\|_{L^2(\mathbb{R}^n)})$$

Moreover for all $\alpha \in]0, \frac{1}{2}[$ *we have*

$$\int_{\mathbb{S}^{n-1}} \sup_{t_1 \neq t_2} \frac{|R(f)(t_2,\gamma) - R(f)(t_1,\gamma)|}{|t_2 - t_1|^\alpha}\, d\sigma(\gamma) \ \leq c(\|f\|_{L^1(\mathbb{R}^n)} + \|f\|_{L^2(\mathbb{R}^n)})$$

To show this theorem we need some lemmas.

We introduce the Fourier transform of $f(x)$ by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2i\pi x\cdot\xi}\, dx$.

Lemma 14. *If f is continuous and has compact support, then we have $\hat{R}(f)(s,\gamma) = \hat{f}(s\gamma)$ $\forall\gamma \in \mathbb{S}^{n-1}$ where $\hat{R}(f)(s,\gamma)$ is the Fourier transform with respect to s.*

Proof. We take a coordinate system with $\gamma = e_n$.

We have $\int_{P_{t,\gamma}} f \ = \int_{\mathbb{R}^{n-1}} f(x',t)\, dx'$

We have by Fubini-Tonelli and because $x \cdot \gamma = t$

$$\hat{f}(s\gamma) = \int_{\mathbb{R}^n} f(x) e^{-2i\pi x\cdot s\gamma}\, dx \ = \int_{-\infty}^{+\infty} \int_{\mathbb{R}^{n-1}} f(x',t) e^{-2i\pi (x',t)\cdot s\gamma}\, dx'\, dt \ =$$
$$= \int_{-\infty}^{+\infty} \left(\int_{P_{t,\gamma}} f(x) \right) e^{-2i\pi st}\, dt \ = \hat{R}(f)(s,\gamma).$$

□

In the following lemma we will use the Plancherel formula from Fourier theory and the formula to change to polar coordinates in $\mathbb{R}^n$ (see [1, page 182] and [2, page 279] for more details).

Lemma 15. *As in the previous lemma let f be a continuous function with compact support, we have*

$$\int_{\mathbb{S}^{n-1}} \left(\int_{-\infty}^{+\infty} |\hat{R}(f)(s,\gamma)|^2 |s|^{n-1}\, ds \right) d\sigma(\gamma) \ = 2 \int_{\mathbb{R}^n} |f(x)|^2\, dx$$

Proof. By the Plancherel formula we have that $\int_{\mathbb{R}^n} |f(x)|^2\, dx \ = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\, d\xi$

Now we change the coordinates from ξ to $(s,\gamma) \in \mathbb{R} \times \mathbb{S}^{n-1}$ with $\xi = s\gamma$

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\, d\xi \ = \int_{\mathbb{S}^{n-1}} \left(\int_0^{+\infty} |\hat{f}(s\gamma)|^2 s^{n-1}\, ds \right) d\sigma(\gamma)$$

We change the variable $s \to -s$ and we obtain, since $\mathbb{R}^n \to \mathbb{R}^n$ and $-\mathbb{S}^{n-1} = \mathbb{S}^{n-1}$

$$\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\, d\xi \ = \int_{\mathbb{S}^{n-1}} \left(\int_0^{-\infty} |\hat{f}(s(-\gamma))|^2 (-s)^{n-1} (-1)\, ds \right) d\sigma(\gamma) \ =$$
$$= \int_{\mathbb{S}^{n-1}} \left(\int_{-\infty}^{0} |\hat{f}(s(\gamma))|^2 (-s)^{n-1}\, ds \right) d\sigma(\gamma)\ .$$

Summing up the last two equations we obtain

$$2 \int_{\mathbb{R}^n} |f(x)|^2\, dx \ = 2 \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\, d\xi \ = \int_{\mathbb{S}^n} \left(\int_{-\infty}^{+\infty} |\hat{R}(f)(s,\gamma)|^2 |s|^{n-1}\, ds \right) d\sigma(\gamma)\ .$$

□

Lemma 16. *Let $F:\mathbb{R}\to\mathbb{R}$ be a function that satisfies the Fourier inversion formula $F(t)=\int_{-\infty}^{+\infty}\hat{F}(s)e^{2i\pi ts}\,ds$. If there exist $A,B\in\mathbb{R}$ such that*

$$\sup_{s\in\mathbb{R}}|\hat{F}(s)|=A$$

and

$$\int_{-\infty}^{+\infty}|\hat{F}(s)|^2|s|^{n-1}\,ds\ =B^2$$

then there exist $c,c_\alpha\in\mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}}|F(t)|\le c(A+B)$$

and

$$|F(t_2)-F(t_1)|\le c_\alpha|t_2-t_1|^\alpha(A+B)$$

for all $t_1,t_2\in\mathbb{R},\alpha\in]0,\frac{1}{2}[$

Proof. First we show $\sup_{t\in\mathbb{R}}|F(t)|\le c(a+b)$

$$\begin{aligned}|F(t)|&\le\int_{|s|\le1}|\hat{F}(s)e^{2\pi ist}|\,ds\ +\int_{|s|>1}|\hat{F}(s)e^{2\pi ist}|\,ds\ \le\\ &\le A\int_{|s|\le1}ds\ +\left(\int_{|s|>1}|\hat{F}(s)|^2|s|^{n-1}\,ds\right)^{\frac{1}{2}}\left(\int_{|s|>1}|s|^{1-n}\,ds\right)^{\frac{1}{2}}\end{aligned}$$

where we have used the Cauchy-Schwarz inequality on $|\hat{F}(s)||s|^{\frac{n-1}{2}}$ and $|s|^{\frac{1-n}{2}}$ and the fact that $|e^{2\pi ist}|=1$.

We see that $\int_{|s|>1}|s|^{1-n}\,ds\ <+\infty\Leftrightarrow1-n<1\Leftrightarrow n>2$ as we are supposing, so we have

$$|F(t)|\le c(A+B)$$

For the second inequality

$$|F(t_2)-F(t_1)|\le\int_{-\infty}^{+\infty}|\hat{F}(s)||e^{2\pi ist_2}-e^{2\pi ist_1}|\,ds\ \le$$

We see that

$$\begin{aligned}&|e^{2\pi ist_2}-e^{2\pi ist_1}|\le|e^{2\pi ist_1}||e^{2\pi is(t_2-t_1)}-1|\le1\cdot2\pi\{|s(t_2-t_1)|\}\le\\ &\le2\pi\{|s(t_2-t_1)|\}^\alpha\le2\pi|s|^\alpha|t_2-t_1|^\alpha\end{aligned}$$

where $\frac{1}{2}\le\alpha<1$ and $\{x\}$ is the fractional part of x. We see that $|e^{2\pi ix}-1|\le2\pi|x|$ for $|x|\le1$ because the length of an arc is greater than the length of the correspondent segment between the same two points. Resuming the chain of inequalities

$$\leq c\int_{|s|\leq 1}|\hat{F}(s)||s|^{\alpha}|t_2-t_1|^{\alpha}\,ds\ +c\int_{|s|>1}|\hat{F}(s)|s|^{\alpha}|t_2-t_1|^{\alpha}\,ds\ \leq$$

$$\leq c_{\alpha}A|t_2-t_1|^{\alpha}+c\left(\int_{|s|>1}|\hat{F}(s)|^2|s|^{n-1}\,ds\right)^{\frac{1}{2}}\left(\int_{|s|>1}|s|^{1-n+2\alpha}\,ds\right)^{\frac{1}{2}}|t_2-t_1|^{\alpha}\leq$$

$$\leq c_{\alpha}(A+B)$$

using Cauchy-Schwarz on $|\hat{F}(s)||s|^{\frac{n-1}{2}}$ and $|s|^{\frac{1-n}{2}+\alpha}$ and because
$\int_{|s|>1}|s|^{1-n+2\alpha}\,ds\ \leq +\infty \Leftrightarrow 1-n+2\alpha<-1 \Leftrightarrow n>2+2\alpha.$

□

Proof of Theorem 13. By Lemma 16 we have, choosing
$F(t)=R(f)(t,\gamma),\ A(\gamma)=\sup_{s\in\mathbb{R}}\hat{R}(f)(s,\gamma)$ and $B(\gamma)=\left(\int_{-\infty}^{+\infty}|\hat{R}(s,\gamma)|^2|s|^{n-1}\,ds\right)^{\frac{1}{2}}$,
that $R^*(f)(\gamma)\leq c(A(\gamma)+B(\gamma))$ or $\frac{R(f)(t_2,\gamma)-R(f)(t_1,\gamma)}{|t_2-t_1|^{\alpha}}\leq c_{\alpha}(A(\gamma)+B(\gamma))$

In fact $R(f)(t,\gamma)$ satisfies the Fourier inversion formula because $f\in L^1(\mathbb{R}^n)\Rightarrow R(f)(\cdot,\gamma)\in L^1(\mathbb{R})$ for a.e. $\gamma\in\mathbb{S}^{n-1}$

We have, by lemma 14

$$\hat{R}(f)(s,\gamma)=\hat{f}(s\gamma)\Rightarrow A(\gamma)=\sup_{s\in\mathbb{R}}\hat{R}(f)(s,\gamma)=\sup_{s\in\mathbb{R}}\hat{f}(s\gamma)=$$
$$=\sup_{s\in\mathbb{R}}\int_{\mathbb{R}^n}f(x)e^{-2\pi i s\gamma\cdot x}\,dx\ \leq\sup_{s\in\mathbb{R}}\int_{\mathbb{R}^n}|f(x)|\,dx\ =\|f\|_{L^1(\mathbb{R}^n)}.$$

Moreover we have, since $\sqrt{x}$ is a concave function, by Jensen inequality and by Lemma 2, that there exists c such that

$$\int_{\mathbb{S}^{n-1}}B(\gamma)\,d\sigma(\gamma)\ =\int_{\mathbb{S}^{n-1}}\sqrt{B(\gamma)^2}\,d\sigma(\gamma)\ \leq c\sqrt{\int_{\mathbb{S}^{n-1}}B(\gamma)^2\,d\sigma(\gamma)}\ =$$
$$=c\sqrt{\int_{\mathbb{S}^{n-1}}\int_{-\infty}^{+\infty}|\hat{R}(f)(s,\gamma)^2|s|^{n-1}\,ds\ \ d\sigma(\gamma)}\ =c\sqrt{2\int_{\mathbb{R}^n}|f(x)|^2\,dx}\ =\sqrt{2}c\|f\|_{L^2(\mathbb{R}^n)}$$

where c is greater than $\sqrt{\sigma(\mathbb{S}^{n-1})}$.

So we have

$$\int_{\mathbb{S}^{n-1}}R^*(f)(\gamma)\,d\sigma(\gamma)\ \leq c\int_{\mathbb{S}^{n-1}}A(\gamma)\,d\sigma(\gamma)\ +c\int_{\mathbb{S}^{n-1}}B(\gamma)\,d\sigma(\gamma)\ \leq$$
$$\leq c\left(\|f\|_{L^1(\mathbb{R}^n)}+\|f\|_{L^2(\mathbb{R}^n)}\right).$$

We can perform the same argument with $\sup_{t_1\neq t_2}\frac{|R(f)(t_2,\gamma)-R(f)(t_1,\gamma)|}{|t_2-t_1|^{\alpha}}$ instead of $R^*(f)(\gamma)$ and we obtain the other part of the theorem.

□

All we have said holds for $n \geq 3$ but for $n = 2$ we can obtain similar results using a function similar to the Radon transform.

Definition 17. *Let $R_\delta(f)(t,\gamma)$ be the average of $R(f)(t,\gamma)$ in the strip of width 2δ around the line $P_{t,\gamma}$*

$$R_\delta(f)(t,\gamma) = \frac{1}{2\delta}\int_{t-\delta}^{t+\delta} R(f)(s,\gamma)\,ds \;= \frac{1}{2\delta}\int_{t-\delta}^{t+\delta}\left(\int_{P_{s,\gamma}} f\right)ds \;= \frac{1}{2\delta}\int_{t-\delta\leq x\cdot\gamma\leq t+\delta} f(x)\,dx\;.$$

Definition 18. *We want also the analogue of $R^*(f)(\gamma)$*

$$R^*_\delta(f)(\gamma) = \sup_{t\in\mathbb{R}} |R_\delta(f)(t,\gamma)|$$

We will show the analogue of theorem 13.

Theorem 19. *If f is continuous function with compact support on $\mathbb{R}^2$, then there exists c such that for all $\delta \in]0,\frac{1}{2}]$*

$$\int_{\mathbb{S}^{n-1}} R^*_\delta(f)(\gamma)\,d\sigma(\gamma) \;\leq c\left(\log\frac{1}{\delta}\right)^{\frac{1}{2}}(\|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)})$$

Lemma 20. *Let $F_\delta : \mathbb{R}\to\mathbb{R}$ be the function*

$$F_\delta(t) = \int_{-\infty}^{+\infty}\hat{F}(s)\left(\frac{e^{2\pi i(t+\delta)s} - e^{2\pi i(t-\delta)s}}{4\pi i s\delta}\right)ds$$

If we have that there exist $A, B\in\mathbb{R}$ such that

$$\sup_{s\in\mathbb{R}}|\hat{F}(s)| = A$$

and

$$\int_{-\infty}^{+\infty}|\hat{F}(s)|^2|s|\,ds \;= B^2$$

then there exist $c, c_\alpha\in\mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}}|F_\delta(t)| \leq c\left(\log\frac{1}{\delta}\right)^{\frac{1}{2}}(A+B)$$

Proof. Let $2\pi ts = m$ and $2\pi s\delta = n$ then we have

$$\begin{aligned}
&e^{2\pi i(t+\delta)s} - e^{2\pi i(t-\delta)s} = \cos m\cos n - \sin m\sin n + i(\sin m\cos n + \cos m\sin n) - \\
&- \cos m\cos n - \sin m\sin n + i(-\sin m\cos n + \cos m\sin n) = \\
&= -2\sin m\sin n + 2i\cos m\sin n = 2i\sin n(\cos m + i\sin m) = 2i\sin n e^{im} = \\
&= 2i\sin(2\pi s\delta)e^{2\pi its}
\end{aligned}$$

We can say, since $|\frac{\sin x}{x}| \le \min\{1, |\frac{1}{x}|\}$, and using Cauchy-Schwarz as in Lemma 3, that

$$\int_{-\infty}^{+\infty} \hat{F}(s) \left(\frac{e^{2\pi i(t+\delta)s} - e^{2\pi i(t-\delta)s}}{4\pi i s\delta} \right) ds \ \le \int_{-\infty}^{+\infty} \hat{F}(s) \left| \frac{\sin(2\pi\delta s)}{2\pi s\delta} e^{2\pi i t s} \right| ds \ \le$$

$$\le \int_{|s|\le 1} |\hat{F}(s)|\, ds \ + \int_{1<|s|\le\frac{1}{\delta}} |\hat{F}(s)|\, ds \ + \frac{c'}{\delta} \int_{|s|>\frac{1}{\delta}} |\hat{F}(s)||s|^{-1}\, ds \ \le$$

$$\le cA + \left(\int_{1<|s|\le\frac{1}{\delta}} |\hat{F}(s)|^2 |s|\, ds \right)^{\frac{1}{2}} \left(\int_{1<|s|\le\frac{1}{\delta}} |s|^{-1}\, ds \right)^{\frac{1}{2}} +$$

$$+ \frac{c'}{\delta} \left(\int_{|s|>\frac{1}{\delta}} |\hat{F}(s)|^2 |s|\, ds \right)^{\frac{1}{2}} \left(\int_{|s|>\frac{1}{\delta}} |s|^{-3}\, ds \right)^{\frac{1}{2}} \le c \left(\log \frac{1}{\delta} \right)^{\frac{1}{2}} (A+B)$$

□

Proof of Theorem 19. We use the same arguments of Theorem 13 but instead of Lemma 16 we use Lemma 20. In fact

$$\frac{1}{2\delta} \int_{t-\delta}^{t+\delta} R(f)(s,\gamma)\, ds \ = \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} \int_{-\infty}^{+\infty} \hat{R}(f)(\xi,\gamma) e^{2\pi i \xi s}\, d\xi \ \ ds \ =$$

$$= \frac{1}{2\delta} \int_{-\infty}^{+\infty} \int_{t-\delta}^{t+\delta} \hat{R}(f)(\xi,\gamma) e^{2\pi i \xi s}\, ds \ \ d\xi \ = \int_{-\infty}^{+\infty} \hat{R}(f)(\xi,\gamma) \left(\frac{e^{2\pi i(t+\delta)\xi} - e^{2\pi i(t-\delta)\xi}}{4\pi i s\delta} \right) d\xi \ .$$

So we can choose $F_\delta(t) = R_\delta(f)(t,\gamma)$ and we know

$$R_\delta(f)(t,\gamma) = \int_{-\infty}^{+\infty} \hat{R}(f)(\xi,\gamma) \left(\frac{e^{2\pi i(t+\delta)\xi} - e^{2\pi i(t-\delta)\xi}}{4\pi i s\delta} \right) d\xi$$

so we obtain

$$\int_{\mathbb{S}^{n-1}} R^*_\delta(f)(\gamma)\, d\sigma(\gamma) \ \le c \left(\log \frac{1}{\delta} \right)^{\frac{1}{2}} (\|f\|_{L^1(\mathbb{R}^2)} + \|f\|_{L^2(\mathbb{R}^2)})$$

□

1.3 Proofs

The proof of Theorem 4 will follow as a particular case of the following

Theorem 21. *For $n \ge 3$ and f bounded and vanishing outside a set of finite measure, for a.e. $\gamma \in \mathbb{S}^{n-1}$ we have that*

1.f is integrable on the plane $P_{t,\gamma}$ $\forall t \in \mathbb{R}$

2.$R(f)(t,\gamma)$ is continuous in t and is Hölder continuous with exponent α, $\forall \alpha \in]0, \frac{1}{2}[$

We see that, with $f = \chi_E$ we have that χ_E is bounded and vanishing outside a set of finite measure, since E is measurable of finite measure and therefore $E_{t,\gamma}$ is measurable for all $t \in \mathbb{R}$ and $m_n(E_{t,\gamma})$ is continuous and Hölder continuous with exponent α, for all $\alpha \in]0, \frac{1}{2}[$.

Proof. Step 1. Let A be an open and bounded set and assume $f = \chi_A$. Then $A \cap P_{t,\gamma}$ is open and bounded in $P_{t,\gamma}$ so it is measurable.

Let f_d be a sequence of non-negative continuous functions with compact support such that $f_d(x)$ is increasing and $f_d(x) \to f(x)$ for a.e. x.[1]

By Beppo-Levi theorem we have, since $R(f)(t,\gamma) = \int_{P_{t,\gamma}} f$, that $R(f_d)(t,\gamma) \to R(f)(t,\gamma)$ for all t, γ.

We find t such that $R^*(f)(\gamma) - R(f)(t,\gamma) < \epsilon$ and d such that $R(f)(t,\gamma) - R(f_d)(t,\gamma) < \epsilon$ so $R^*(f)(\gamma) - R(f_d)(t,\gamma) < 2\epsilon$. So $R^*(f)(\gamma) \leq \lim_{d \to +\infty} R^*(f_d)(\gamma)$

Moreover we have that $R(f_d)(t,\gamma) \leq R(f)(t,\gamma) \Rightarrow R(f_d)(t,\gamma) \leq R^*(f)(\gamma) \Rightarrow R^*(f_d)(\gamma) \leq R^*(f)(\gamma)$ so we have $R^*(f_d)(\gamma) \to R^*(f)(\gamma)$ for all t, γ and the first inequality of Theorem 4 holds for f.

Step 2.Now we take E bounded of measure 0. We can take a sequence $(A_d)_{d \in \mathbb{N}}$ of open sets such that $E \subseteq A_d \ \forall d$ and $m_n(A_d) \to 0$.

If $E' = \bigcap_{d \in \mathbb{N}} A_d$ we have $m_n(E') = 0$ and $E' \cap P_{t,\gamma}$ is measurable since it is the intersection of measurable sets$(A_n \cap P_{t,\gamma})$. So we can say that $R(\chi_{E'})(t,\gamma)$ and $R^*(\chi_{E'})(\gamma)$ are well defined.

$$R^*(\chi_{E'})(\gamma) \leq R^*(\chi_{A_d})(\gamma) \Rightarrow \int_{\mathbb{S}^{n-1}} R^*(\chi_{E'})(\gamma)\, d\sigma(\gamma) \ \leq \int_{\mathbb{S}^{n-1}} R^*(\chi_{A_d})(\gamma)\, d\sigma(\gamma) \ \leq$$
$$\leq c\|\chi_{A_d}\|_{L^1(\mathbb{R}^n)} + c\|\chi_{A_d}\|_{L^2(\mathbb{R}^n)} \to 0$$

so $R^*(\chi_{E'})(\gamma) = 0$ for a.e. $\gamma \Rightarrow m_{n-1}(E \cap P_{t,\gamma}) = 0 \ \forall t \in \mathbb{R}$ for a.e. $\gamma \in \mathbb{S}^{n-1}$ since $E \subseteq E'$.

When E has measure 0 but is not bounded it can be seen as the countable union of bounded sets yielding to the same result.

This also proves Corollary 6.

Step 3. Now let f be a function as in the statement of the theorem. Then we find a sequence $(f_d)_{d \in \mathbb{N}}$ of continuous functions with compact support uniformly bounded (the same bound M of f) and non-vanishing only in a set where f is non-vanishing (this set will be called H), that converges to f a.e.

We have that for a.e. $\gamma \ \forall t \ f_d(x) \to f(x)$ except in a set of measure $0(E \cap P_{t,\gamma}$ in the step 2) in $P_{t,\gamma}$ with respect to m_{n-1} since $f_d(x) \to f(x)$ except in a set of measure $0(E$ in the step 2) by what we have just proved in step 2.

Since we can take $M\chi_H$ as a dominant, by the dominated convergence theorem we have $R(f_d)(t,\gamma) \to R(f)(t,\gamma)$ for a.e. $\gamma \ \forall t$.

We have also that $\|f_d - f\|_{L^1(\mathbb{R}^n)}, \|f_d - f\|_{L^2(\mathbb{R}^n)} \to 0$ so we find a subsequence $(d_k)_{k \in \mathbb{N}}$ such that $\|f_{d_k} - f\|_{L^1(\mathbb{R}^n)} + \|f_{d_k} - f\|_{L^2(\mathbb{R}^n)} \leq 2^{-k}$.

[1] We know that we can find a sequence of compact sets $(K_d)_{d \in \mathbb{N}}$ such that $m_n(A \setminus K_d) < \frac{1}{2^d}$ and $K_d \subseteq K_{d+1} \subseteq A$ (if this doesn't hold we take $K_{d+1} \cup K_d$) so by Urysohn's lemma we can find a continuous function f_d with support contained in A and equal to 1 in K_d. We see that we can choose it increasing because if $f_d(x) < f_{d-1}(x)$ for some x we can take $f_d^1(x) = \max\{f_d(x), f_{d-1}(x)\}$ which is continuous.

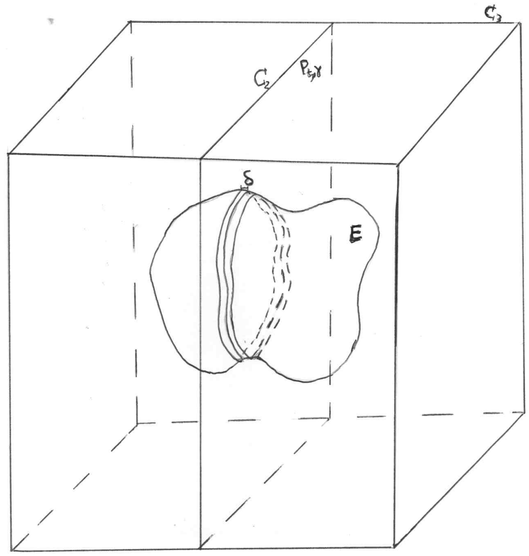

Figure 1.1

By Theorem 13 and from what we know about integration on series we have

$$\int_{\mathbb{S}^{n-1}} \sum_{d=1}^{+\infty} R^*(f_{d_k} - f_{d_{k-1}})(\gamma)\, d\sigma(\gamma) \ = \sum_{d=1}^{+\infty} \int_{\mathbb{S}^{n-1}} R^*(f_{d_k} - f_{d_{k-1}})(\gamma)\, d\sigma(\gamma) \ \leq$$
$$c\sum_{d=1}^{+\infty} 2^{-k} < +\infty \Rightarrow \sup_{t\in\mathbb{R}} \sum_{d=1}^{+\infty} |R(f_{d_k})(t,\gamma) - R(f_{d_{k-1}})(t,\gamma)| \leq$$
$$\leq \sum_{d=1}^{+\infty} \sup_{t\in\mathbb{R}} |R(f_{d_k})(t,\gamma) - R(f_{d_{k-1}})(t,\gamma)| < +\infty$$

for a.e. $\gamma \in \mathbb{S}^{n-1}$.

For such γ we have therefore that $R(f_{d_k})(t,\gamma)$ converges uniformly to $R(f)(t,\gamma)$. Since $R(f_{d_k})(t,\gamma)$ is continuous in t because f_d is continuous, then $R(f)(t,\gamma)$ is continuous.

In fact $f_d(x)$ has compact support so also its intersection with $P_{t,\gamma}$ is compact and for all ε there exist δ such that for $|t-s|<\delta$ we have

$$|f_d(t\gamma) - f_d(s\gamma)| < \varepsilon \Rightarrow |R(f_d)(t,\gamma) - R(f_d)(s,\gamma)| = \left| \int_{P_{t,\gamma}} f_d - \int_{P_{s,\gamma}} f_d \right| < \varepsilon m_{n-1}(C_{n-1})$$

where, since the support of f_d can be put in a n-cube C_n with a side orthogonal to γ, its intersection with $P_{t,\gamma}$ gives a $n-1$ cube C_{n-1} with finite measure(Figure 1.1).

We can repeat the same argument with $\frac{|R(f)(t_2,\gamma)-R(f)(t_1,\gamma)|}{|t_2-t_1|^{\alpha}}$ in place of $R^*(f)(\gamma)$ and we obtain the Hölder condition of the theorem. □

Proof of Theorem 8. The inequality of theorem 19 can be extended to functions of $L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ since we can find a sequence $(f_d)_{d\in\mathbb{N}}$ of continuous function with compact support that converges to f in the L^1 norm so their integral in the subset of $\mathbb{R}^n$ given by $\bigcup_{s\in]t-\delta,t+\delta[} P_{s,\gamma}$ must converge to the integral of f in the same set so $R_\delta(f_d)(t,\gamma)\to R_\delta(f)(t,\gamma)\Rightarrow R^*_\delta(f_d)(t,\gamma)\to R^*_\delta(f)(t,\gamma)$.

Let us take F a Besicovitch set. We will show that for all $\alpha\in]0,2[$ every covering of F with balls $(B_i)_{i\in\mathbb{N}}$ satisfies $\sum_{i=0}^{+\infty}\omega_\alpha\left(\frac{diam(B_i)}{2}\right)^\alpha\geq c_\alpha>0$ so the Hausdorff dimension of F is necessarily greater than or equal to 2 and so it is 2.

We take B_i such that $diam(B_i)<1$. Then we call N_k the number of balls B_i such that $2^{-k}\leq diam(B_i)\leq 2^{-k+1}$ which is finite (otherwise $\sum_{i=0}^{+\infty}(diam(B_i))^\alpha=+\infty$ and our job is done).

We will show that there exists k_0 such that $N_{k_0}2^{-k_0\alpha}\geq c_\alpha$ so $\sum_{i=0}^{+\infty}(diam(B_i))^\alpha\geq c_\alpha$. We define

$$F_k=F\cap\left(\bigcup_{2^{-k}\leq diam B_i\leq 2^{-k+1}} B_i\right)$$

and

$$G_k=\bigcup_{2^{-k}\leq diam B_i\leq 2^{-k+1}}\hat{B}_i$$

where $\hat{B}_i$ is the ball with same center as B_i and whose diameter is twice the diameter of B_i.

We have $m_2(G_k)\leq\omega_2 N_k\left(\frac{2^{-k+2}}{2}\right)^2=\omega_2N_k2^{-2k+2}$.

We call s_γ one of the segments of length 1 which is in F. Then we chose a sequence $(a_k)_{k\in\mathbb{N}}$ such that $a_k=(1-2^{-\varepsilon})2^{-\varepsilon k}$. We see $a_k\geq 0$ and $\sum_{k=0}^{\infty}a_k=(1-2^{-\varepsilon})\frac{1}{1-2^{-\varepsilon}}=1$. Since $s_\gamma\in F$ there exists k such that $m_1(s_\gamma\cap F_k)\geq a_k$ otherwise $1=m(s_\gamma\cap F)\leq\sum_{k=1}^{+\infty}m_1(s_\gamma\cap F_k)<\sum_{k=1}^{+\infty}a_k=1$.

We have that for this k there exists $t_0\in\mathbb{R}$ such that $s_\gamma\subseteq P_{t_0,\gamma}$ so

$$R^*_{2^{-k}}(\chi_{G_k})(\gamma)\geq\frac{1}{2^{-k+1}}\int_{]t_0-2^{-k},t_0+2^{-k}[}\left(\int_{P_{t,\gamma}}\chi_{G_k}\right)dt\ \geq\frac{1}{2^{-k+1}}a_k2^{-k+1}=a_k$$

since $m_1(s_\gamma\cap F_k)\geq a_k$ and every point with distance less than 2^{-k} from F_k is in G_k because G_k is a union of balls centered in the same point of balls centered in a point of F_k. These balls cover completely F_k itself and the balls of G_k are of diameter of length at the least 2^{-k+1} (Figure 2).

Now we take $E_k=\{\gamma:R^*_{2^{-k}}(\chi_{G_k})(\gamma)\geq a_k\}$.

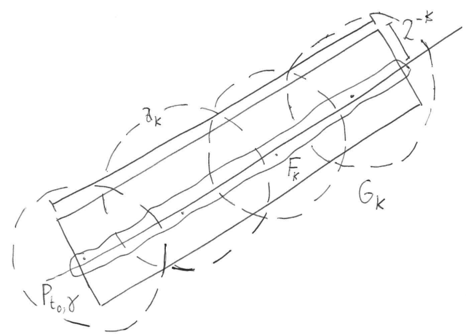

Figure 1.2

Necessarily $\mathbb{S}^1 = \bigcup_{k=1}^{+\infty} E_k$.

Since $\sigma(\mathbb{S}^1) = 2\pi$ we have that there exists k_0 such that $\sigma(E_{k_0}) \geq 2\pi a_{k_0}$ otherwise $\sigma(\mathbb{S}^1) \leq \leq \sum_{k=0}^{+\infty} \sigma(E_k) < \sum_{k=0}^{+\infty} 2\pi a_k = 2\pi$.

Now $2\pi a_{k_0}^2 \leq \sigma(E_{k_0}) a_{k_0} = \int_{E_{k_0}} a_{k_0}\, d\sigma(\gamma) \ \leq \int_{\mathbb{S}^1} R^*_{2^{-k_0}}(\chi_{G_{k_0}})(\gamma)\, d\sigma(\gamma)$.

By theorem 19 we obtain

$$a_k^2 \leq c\left(\log \frac{1}{2^{-k_0}}\right)^{\frac{1}{2}} \left(\|\chi_{G_{k_0}}\|_{L^1(\mathbb{R}^n)} + \|\chi_{G_{k_0}}\|_{L^2(\mathbb{R}^n)}\right)$$

We see $\|\chi_{G_{k_0}}\|_{L^2(\mathbb{R}^n)} \leq (cN_{k_0} 2^{-2k_0+2})^{\frac{1}{2}} = c' N_{k_0}^{\frac{1}{2}} 2^{-k_0+1}$.

Since for the characteristic function of a set A of finite measure we have that $\|\chi_A\|_{L^1(\mathbb{R}^n)} = m_n(A)$ and $\|\chi_A\|_{L^2(\mathbb{R}^n)} = \sqrt{m_n(A)}$, since for all k the distance of a point in G_k from F is less than 1 since G_k is a union of balls centered in a point of F of diameter less than 1 and F is compact, $\|\chi_{G_{k_0}}\|_{L^2(\mathbb{R}^n)}$ is bounded and so we have

$$\|\chi_{G_{k_0}}\|_{L^1(\mathbb{R}^n)} = \|\chi_{G_{k_0}}\|_{L^2(\mathbb{R}^n)} \|\chi_{G_{k_0}}\|_{L^2(\mathbb{R}^n)} \leq c'' N_{k_0}^{\frac{1}{2}} 2^{-k_0+1}$$

Since $a_k = (1-2^{-\varepsilon})2^{-\varepsilon k}$ we have

$$(1-2^{-\varepsilon})^2 2^{-2\varepsilon k_0} \leq c'''(\log(2^{k_0}))^{\frac{1}{2}} N_{k_0}^{\frac{1}{2}} 2^{-k_0+1} \Rightarrow$$
$$\Rightarrow 2^{-\alpha k_0} N_{k_0} \geq \frac{2^{-2}(1-2^{-\varepsilon})^4 2^{-4\varepsilon k_0 + 2k_0}}{c'''^2 k_0 \log 2} 2^{-\alpha k_0} \geq c_\alpha$$

as long as $2-\alpha-4\varepsilon > 0 \Leftrightarrow \alpha < 2-4\varepsilon$ because we have that $\frac{2^{-2}(1-2^{-\varepsilon})^4 2^{-4\varepsilon k_0+2k_0}}{c'''^2 k_0 \log 2} 2^{-\alpha k_0}$ tends to $+\infty$ for $k_0 \to +\infty$ so it has a positive minimum.

□

Chapter 2

Besicovitch sets

2.1 Density of the sets in $\mathbb{R}^2$

Let us take a square of side 1 in the plane $\mathbb{R}^2$. We define L the set of the sets made of segments that begin in a point of a side of the square and end in a point of the opposite side, that contains a segment of length 1 parallel to every direction We claim that the subset $T \subseteq L$ of sets of measure 0 is dense in L with the Hausdorff distance.

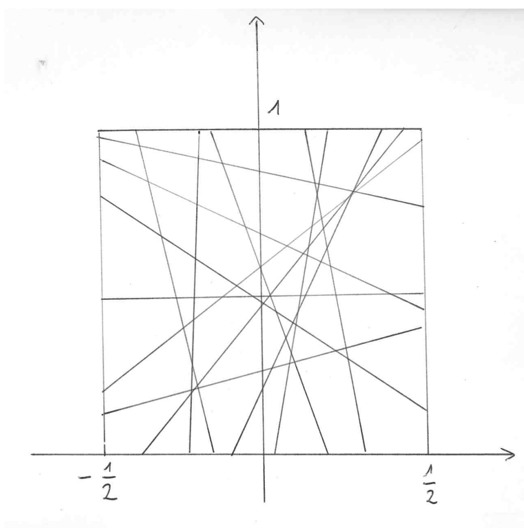

Figure 2.1

First we define the Hausdorff distance.
In order to do this we define the δ-neighborhood

Definition 22. *Let $A \subseteq \mathbb{R}^2$, we have for all $\delta \geq 0$*

$$A^\delta = \{x : \exists y \in A : |x - y| < \delta\} = \{x : d(x, A) < \delta\}$$

Definition 23. *The Hausdorff distance d of two compact sets A and B is defined as*

$$d(A, B) = \min\{\delta : B \subseteq A^\delta, A \subseteq B^\delta\}$$

We note that the distance is well defined and it cannot be $+\infty$ since A and B are compact.

The Hausdorff distance satisfies the properties of a distance.

1.$d(A, B) = 0 \Leftrightarrow B \subseteq A,\ A \subseteq B \Leftrightarrow A = B$.

2.$d(A, B) = d(B, A)$. It is obvious by the symmetry of the definition.

3.We want to show that $d(A, B) = \delta$, $d(B, C) = \delta' \Rightarrow d(A, C) \leq \delta + \delta'$. We have

$$A \subseteq B^{\delta}, B \subseteq C^{\delta'} \Rightarrow \{x \in B^{\delta} \Rightarrow \delta \geq d(x, B) \geq d(x, C^{\delta'})\} \Rightarrow B^{\delta} \subseteq C^{\delta+\delta'} \Rightarrow A \subseteq C^{\delta+\delta'}$$

In the same way we can show that $C \subseteq A^{\delta+\delta'}$.

Moreover this theorem holds.

Theorem 24. *The set of the compact subsets of $\mathbb{R}^2$ with the Hausdorff distance is a complete metric space.*

Proof. We take a Cauchy sequence of compact sets $(K_i)_{i\in\mathbb{N}}$.

We want to show that the sequence converges to

$$K = \bigcap_{n=1}^{+\infty} \overline{\bigcup_{i=n}^{\infty} K_i}$$

We see that K is closed. Moreover K is bounded because there exists n_0 such that for all $n \geq n_0$

$$d(K_n, K_{n_0}) \leq \varepsilon \Rightarrow K_n \subseteq K_{n_0}^{\varepsilon} \Rightarrow \overline{\bigcup_{i=n}^{\infty} K_i} \subseteq K_{n_0}^{2\varepsilon}$$

and so K is compact and since K is the intersection of decreasing sets and we can take a sequence $x_n \in \overline{\bigcup_{i=n}^{\infty} K_i}$ $\forall n \geq n_0$ that converges up to a subsequence to a point in K, K itself is non-empty.

Let $\varepsilon > 0$ be fixed. We want to prove that there exists n_1 such that for $n \geq n_1$ we have $d(K_n, K) < 2\varepsilon$. In order to do that we will show the two inclusions in Definition 23.

We see that for all $n \geq n_1$ we have that for all $i \geq n$

$$K_i \subseteq K_n^{\varepsilon} \Rightarrow \bigcup_{i=n}^{\infty} K_i \subseteq K_n^{\varepsilon} \Rightarrow \overline{\bigcup_{i=n}^{\infty} K_i} \subseteq K_n^{2\varepsilon} \Rightarrow K \subseteq K_n^{2\varepsilon}$$

We define $T_n = \overline{\bigcup_{i=n}^{\infty} K_i}$. We want to show that $d(T_n, K) \to 0$. In fact in that case there exists n_1 such that for all $n \geq n_1$ $d(T_n, K) < \varepsilon$ and for all $i \geq n$

$$K_n \subseteq K_i^{\varepsilon} \Rightarrow K_n \subseteq \bigcup_{i=n}^{+\infty} K_i^{\varepsilon} = \left(\bigcup_{i=n}^{+\infty} K_i\right)^{\varepsilon} \Rightarrow K_n \subseteq \left(\overline{\bigcup_{i=n}^{+\infty} K_i}\right)^{\varepsilon} = T_n^{\varepsilon} \Rightarrow K_n \subseteq T_n^{\varepsilon} \subseteq K^{2\varepsilon}$$

In fact

$$x \in K_i^{\varepsilon} \Rightarrow d(x, K_i) < \varepsilon \Rightarrow d(x, \bigcup_{i=n}^{+\infty} K_i) < \varepsilon \Rightarrow x \in \left(\bigcup_{i=n}^{+\infty} K_i\right)^{\varepsilon}$$

and

$$x \in \left(\bigcup_{i=n}^{+\infty} K_i\right)^{\varepsilon} \Rightarrow \exists i: \ d(x, K_i) < \varepsilon \Rightarrow x \in K_i^{\varepsilon} \Rightarrow x \in \bigcup_{i=n}^{+\infty} K_i^{\varepsilon}$$

If we assume by contradiction that $d(T_n, K)$ does not tend to 0 there exists $\varepsilon' > 0$ and $(T_{n_k})_{k\in\mathbb{N}}$ such that $d(T_{n_k}, K) > \varepsilon'$. So we can take $x_{n_k} \in T_{n_k}$ such that $d(x_{n_k}, K) > \varepsilon'$.

Since $T_n \subseteq T_m$ for all $m \leq n$ we have that $x_{n_k} \in T_1$ for all k and because T_1 is compact we can assume $x_{n_k} \to x \in T_1$ (otherwise we take a subsequence).

However for all k_0 such that $n_{k_0} \geq n_0$, $\forall k \geq k_0$ we have that $x_{n_k} \in T_{n_{k_0}}$ so $x \in T_{n_{k_0}}$ since $T_{n_{k_0}}$ is compact.

Since this holds for all k_0 large enough and $T_{n_{k_0}} \subseteq T_n$ for all $n \leq n_{k_0}$ we have that $x \in T_n$ for all n so we can say that $x \in K$, a contradiction because $d(x, K) \geq \varepsilon'$.

□

Now we define L as

Definition 25. *Let L be the set made by closed subsets K of $[-\frac{1}{2}, \frac{1}{2}] \times [0, 1]$ with the following properties:*

1.K is made of segments from a point in $[-\frac{1}{2}, \frac{1}{2}] \times \{0\}$ to a point in $[-\frac{1}{2}, \frac{1}{2}] \times \{1\}$.

2.For all $\alpha \in [-\frac{\pi}{4}, \frac{\pi}{4}]$ K contains a segment that makes an angle α with the y-axis.

Now we are ready to enunciate the theorem.

Theorem 26. *The subset of L of closed sets of measure 0 is dense in L with the Hausdorff distance.*

We see that adding the set made of the rotation of an angle $\frac{\pi}{2}$ of all the sets in L and calling L' this new set we have that $\dot{\bigcup}_{K\in L, K'\in L'} \{K \cup K'\}$ contains a dense subset of $L \cup L'$ of sets of measure 0 that contains a segment parallel to every direction.

In fact we note that $\dot{\bigcup}_{K\in L, K'\in L'} \{K \cup K'\}$ is the set of subsets of $[\frac{1}{2}, \frac{1}{2}] \times [0, 1]$ which contains a segment parallel to every direction from one side to the opposite one.

In order to give the proof of the theorem we are going to prove a weaker version and then we can use Baire's Lemma.

Definition 27. *Take $y_0 \in [0, 1]$ and let $L(y_0, \varepsilon)$ be the subset of L of the elements K in L such that for all $y \in [y_0 - \varepsilon, y_0 + \varepsilon]$ we have that there exists η such that*

$$m_1(\{x : (x, y) \in K^{\eta}\}) < 3\varepsilon$$

Then we have

Theorem 28. *We have that for all $y_0 \in [0, 1]$, for all $\varepsilon > 0$ $L(y_0, \varepsilon)$ is open and dense in L.*

Lemma 29 (Baire's Lemma)**.** *The intersection of countably many open and dense subsets of a complete metric space A is dense in A.*

Proof. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of open and dense sets in A(not necessarily different each one from all the others). We take $G = \bigcap_{n=1}^{+\infty} A_n$.

If $B(x,r)$ is an open ball, we want to show that $G \cap B(x,r) \neq \varnothing$.

We have that $A_1 \cap B(x,r) \neq \varnothing$ since A_1 is dense.

Let x_1 be a point in this intersection. Since A_1 and $B(x,r)$ are open we have that there exists r_1 such that $\overline{B(x_1,r_1)} \subseteq B(x,r) \cap A_1$. We ask also $r_1 < \frac{r}{2}$ (we can take r_1 as small as we want).

In the same way we build a sequence $(x_n)_{n\in\mathbb{N}}$ where we take $x_n \in B(x_{n-1},r_{n-1}) \cap A_n$ (A_n dense) and r_n such that $\overline{B(x_n,r_n)} \subseteq B(x_{n-1},r_{n-1}) \cap A_n$, $r_n < \frac{r_{n-1}}{2}$.

We have that $r_n < \frac{r}{2^n} \Rightarrow d(x_m,x_n) < d(x_m,x_{m+1}) + ... + d(x_{n-1},x_n) < \frac{r}{2^{m-1}}$ for all $m < n \Rightarrow (x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. So $x_n \to z \in X$ (X complete) for $n \to +\infty$.

We see that for all n_0 the sequence $(x_n)_{n\geq n_0}$ is in $\overline{B(x_{n_0},r_{n_0})} \subseteq A_{n_0}$ because of the way we have built the sequence. In fact we have $x_n \in B(x_{n_0},r_{n_0})$ for all $n \geq n_0$. So $z \in \overline{B(x_{n_0},r_{n_0})}$ because the set is closed and $z \in B(x,r)$.

Therefore we have just shown that $z \in A_n$ $\forall n \Rightarrow z \in G$. So for every ball we find z such that $z \in G \Rightarrow G$ is dense in X.

□

Proof of Theorem 26. We take

$$T = \bigcap_{M=1}^{+\infty} \bigcap_{m=1}^{M} L\left(\frac{m}{M}, \frac{1}{M}\right)$$

By Theorem 28 we know that $L\left(\frac{m}{M}, \frac{1}{M}\right)$ is open and dense and by Baire's Lemma then T is dense in L.

We must show that the elements in T have measure 0.

We define K_y as

$$K_y = \{x : (x,y) \in K\}$$

We see by the definition of $L\left(\frac{m}{M}, \frac{1}{M}\right)$ that

$$m_1(K_y) \leq \frac{3}{M}$$

for all $y \in \left[\frac{m-1}{M}, \frac{m+1}{M}\right]$ for all $K \in L\left(\frac{m}{M}, \frac{1}{M}\right)$.

Moreover

$$m_1(K_y) \leq \frac{3}{M}$$

for all $y \in [0,1]$ for all $K \in \bigcap_{m=1}^{M} L\left(\frac{m}{M}, \frac{1}{M}\right)$.

Since this holds for every M we have that

$$m_1(K_y) \leq \frac{3}{M}$$

for all M, for all $y \Rightarrow m_1(K_y) = 0$ so we obtain by Fubini's theorem that $m_2(K) = 0$ for all $K \in T$.

□

Proof of Theorem 28. Step 1. We first show that the set is open.

We will prove that, given $K \in L(y_o, \varepsilon)$, there exists $\varepsilon' > 0$ such that for every $K' \in L$ such that $d(K, K') < \varepsilon'$ we have that $K' \in L(y_o, \varepsilon)$.

We see immediately that, since there exists η such that

$$m_1(\{x : (x, y) \in K^{\eta}\}) < 3\varepsilon$$

for all $y \in [y_0 - \varepsilon, y_0 + \varepsilon]$, if $\varepsilon' < \frac{\eta}{2}$ we have that $K'^{\frac{\eta}{2}} \subseteq K^{\eta}$ by the triangular inequality $(d(K'^{\frac{\eta}{2}}, K) \leq d(K'^{\frac{\eta}{2}}, K') + d(K', K) \leq \eta)$.

So we have that

$$m_1(\{x : (x, y) \in K'^{\frac{\eta}{2}}\}) < 3\varepsilon$$

for all $y \in [y_0 - \varepsilon, y_0 + \varepsilon]$.

So we have that $K' \in L(y_0, \varepsilon)$.

Step 2. We want to show that $L(y_0, \varepsilon)$ is dense in L.

Given $K \in L$ we want to build a set K' which is in $L(y_0, \varepsilon)$ and whose distance from K is at most some fixed δ.

We take the square of Definition 25. From now on by lines we will mean the segments from a side to the opposite one.

First we take $N \in \mathbb{N}$ and we consider all the intervals

$$\left[\frac{n}{N}\frac{\pi}{4}, \frac{n+1}{N}\frac{\pi}{4}\right]$$

$n = -N, ..., N - 1$.

For each interval there is a line $l_n \subseteq K$ whose angle with the y-axis is $\frac{2n+1}{2N}\frac{\pi}{4}$ (we call B the set of these lines). For all n we add to K' this line l_n and all the lines that pass trough the point $\{(x, y_0) : x \in \left[-\frac{1}{2}, \frac{1}{2}\right]\} \cap l_n$ and whose angle with the y-axis is in $\left[\frac{n}{N}\frac{\pi}{4}, \frac{n+1}{N}\frac{\pi}{4}\right]$.

For all n the set of these lines is the union of two triangles. We see that it could be possible for these triangles not to be in $[-\frac{1}{2}, \frac{1}{2}] \times [0, 1]$ but we can chose N such that all their points are closer than $\frac{\delta}{2}$ from the square so we can translate the triangles of $\frac{\delta}{2}$ in order to put them inside the square.

Then we want to add to K' some lines that are close to the points of K (so K' is close enough to K).

We see that taking $\bigcup_{l \subseteq K} l^{\delta}$, it is an open covering of K compact (K is closed and bounded) so we can take a finite subcovering of K. We add to K' each line l such that l^{δ} has been taken by the subcovering (and we call A the set of these lines).

We see that K' contains the lines with angle α for all $\alpha \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ and that $K \subseteq K'^{\delta}$ because $K'^{\delta} \supseteq \bigcup_{l \in A} l^{\delta} \supseteq K$.

We must show that $K' \subseteq K^{\delta}$.

We see that $l \in B \Rightarrow l \subseteq K$. Moreover the triangles associated to $l \in B$ have sides adjacent to $\{(x, y_0) : x \in \left[-\frac{1}{2}, \frac{1}{2}\right]\} \cap l$ whose angle with the y-axis is less than $\frac{\pi}{4} + \frac{1}{N}\frac{\pi}{8}$ and more than $-\frac{\pi}{4} - \frac{1}{N}\frac{\pi}{8}$. So the third side can be small how much we want choosing N properly.

So the triangle can be in l^{δ} and therefore $K' \subseteq K^{\delta}$.

Now we want to show that $K' \in L(y_0, \varepsilon)$, that is there exists η such that $m_1(\{x : (x, y) \in K^\eta\}) < 3\varepsilon$ for all $y \in [y_0 - \varepsilon, y_0 + \varepsilon]$.

We see that $m_1(\{x : (x, y) \in (\bigcup_{l \in A} l)^\eta\}) \leq 2\sqrt{2} d\eta$ for all y where $d = |A|$ (because the lines form an angle in $[\frac{\pi}{4}, \frac{3\pi}{4}]$ with the x-axis).

We must find $m_1(\{x : (x, y) \in K'^\eta\})$, $y \in [y_0 - \varepsilon, y_0 + \varepsilon]$. We see that for each double triangle T, $m_1(\{x : (x, y) \in T^\eta\})$ is less than the measure of the union of the projections on the x-axis of the two basis of the triangles cut at $y = y_0 + \varepsilon$ and $y = y_0 - \varepsilon$ (similar to the original ones) adding a segment of measure η on each side (Figure 2).

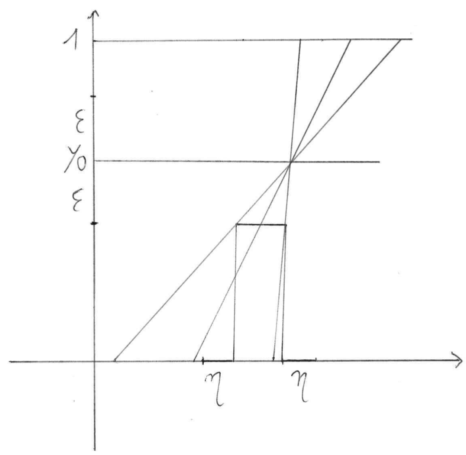

Figure 2.2

We see that these bases have length less than $2\varepsilon \sin(\frac{\pi}{4N}) \leq 2\varepsilon \frac{\pi}{4N}$. In fact building the circumscribed circle of the triangle we have that it has diameter less than 2ε since the angle with the y-axis of the sides is in $[-\frac{\pi}{4}, +\frac{\pi}{4}]$.[1]

So we have that

$$m_1(\{x : (x, y) \in K'^\eta\}) \leq (2\varepsilon \frac{\pi}{4N} + 2\eta)N = \frac{\varepsilon\pi}{2} + 2\eta N$$

In the end we can chose η such that

$$\frac{\varepsilon\pi}{2} + 2N\eta + 2\sqrt{2} d\eta < 3\varepsilon$$

So we have that

$$m_1(\{x : (x, y) \in K'^\eta\} < 3\varepsilon$$

for all $y \in [y_0 - \varepsilon, y_0 + \varepsilon]$.

□

[1]Let the triangle be BCD in the figure 3. Then we build the diameter AB and we intersect $y = y_0$ with AD obtaining H. Since $m_1(HD) = \varepsilon$ and $A\hat{D}B \leq \frac{\pi}{4}$ we have $m_1(BD) \leq \sqrt{2}\varepsilon$. Similarly $m_1(CB) \leq \sqrt{2}\varepsilon$. Since $A\hat{C}B \leq \frac{\pi}{4}$, we have $m_1(AB) \leq \sqrt{2}\varepsilon$ and so $A\hat{B}C = \frac{\pi}{2} \Rightarrow m_1(AC) \leq 2\varepsilon$

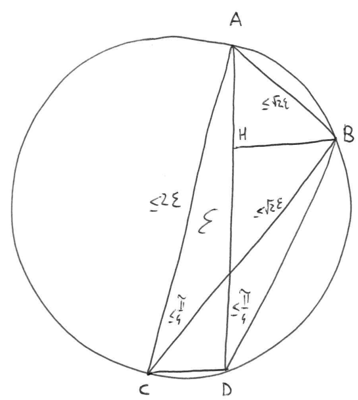

Figure 2.3

2.2 Example of a Besicovitch set

We will first show that moving parts of a triangle we can obtain a figure as small as we want (small 2-measure) containing a segment parallel to all the directions. Then we will intersect open sets containing these figures and we will show that its 2-measure is 0.

(1) Given a triangle T_1 we divide its base in two parts of equal length l obtaining two new triangles (V_1 and V_2) with bases of equal length. Then we move one of the triangles (we suppose the right one in the figure) partially over the other one, of a vector parallel to the base and of length $2(1-t)l$ with $\frac{1}{2} < t < 1$.

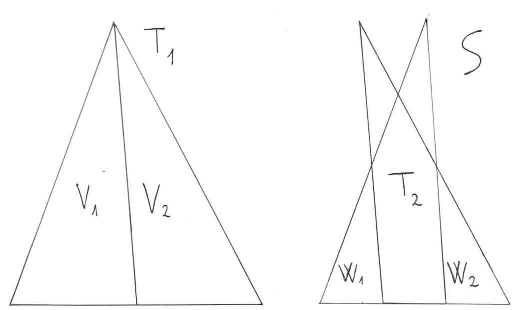

Figure 2.4

We can do this so that the new figure is made of a triangle T_2, homothetic to T_1 and two other triangles with, taking t large enough, very small area.

In particular

$$m_2(T_2) = t^2 m_2(T_1)$$

and, calling S the new figure we have that

$$m_2(T_1) - m_2(S) = m_2(T_1)(3t-1)(1-t)$$

In fact we have that T_2 is homothetic to T_1 and $\frac{2l-2(1-t)l}{2l} = t$ is the ratio between the bases and

$$m_2(S) = m_2(T_1) - m_2(V_3) = m_2(T_1) - m_2(T_2) + m_2(W_1) + m_2(W_2)$$

where W_1 and W_2 are the two small triangles homothetic to V_1 and V_2 respectively that we can see in the figure. Moreover V_3 is the intersection of V_1 and V_2 after the translation.

$$m_2(S) = m_2(T_1)\left(1 - t^2 + \left(\frac{l - 2(1-t)l}{l}\right)^2 \cdot 2 \cdot \frac{1}{2}\right) \Rightarrow$$
$$\Rightarrow m_2(T_1) - m_2(S) = m_2(T_1)(t^2 - 4t^2 + 4t - 1) = m_2(T_1)(1-t)(3t-1)$$

(2)Now we take an open set A containing T and whose area is as near to the area of T as we want.

We divide the base of T in segments whose length is less than ε where ε is the distance between A^C (closed) and T (compact). We know that this distance exists.

Now we work only on one of the triangles built on one of the segments taken previously(the opposite vertex is the same of T) and for the others it will be the same. We call T_1 this triangle.

We can choose t and k such that, dividing the base of T_1 in 2^k segments of equal length, we can build a figure with small area.

First we call T_1^i the triangles created by the segments, with $i = 1, 2, ..., 2^k$.

By induction we take two consecutive figures S_j^{2i-1}, S_j^{2i} (at the first step they are the triangles) with bases of length l (a in the figure) and we move the right one over the left one of a vector parallel to the base and of length $2l(1-t)$(b).

Then we move each figure just created of a vector parallel to the base so there is no space between the bases of two consecutive new figures (S_{j+1}^{i-1}, S_{j+1}^{i}), i.e. their bases have exactly one point in common.

In the end we have created the figures S_{j+1}^i, $i = 1, 2, ..., 2^{k-j}$(c).

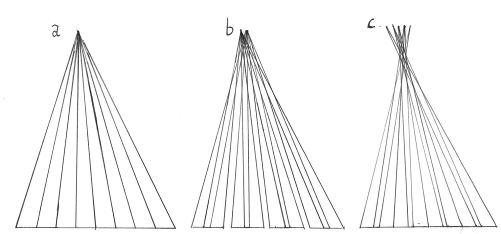

Figure 2.5

We see that at every step the number of the figures we consider is half the number of the figures in the previous step.

Now we focus on the triangle contained in this figures. As we have seen before after the first step the figure S_2^i (from $S_1^{2i-1} = T_1^{2i-1}$ and $S_1^{2i} = T_1^{2i}$) is made of a triangle T_2^i homothetic to the previous one and two other triangles.

T_2^i has its sides (the left one and the right one) parallel respectively to the right side of T_2^{i-1} and the left side of T_2^{i+1} since the four triangles T_1^{2i-2}, T_1^{2i-1}, T_1^{2i} and T_1^{2i+1} are consecutive.

By induction we see that this is true at every step since the figure S_j^i is made of a triangle T_j^i and another part we are not interested in. In fact T_j^i is created due to the overlapping of T_{j-1}^{2i-1} and T_{j-1}^{2i} as seen before.

Let us give and esteem of the area.

At the first step the area of all the figures is such that

$$m_2(T_1) - m_2\left(\bigcup_{i=1}^{2^{k-1}} S_2^i\right) = m_2(T_1)(1-t)(3t-1)$$

and this is the reduction of the area.

On the following steps we see that at the least the area reduces of the area that reduces from the overlapping of T_j^{2i-1} and T_j^{2i} for all i so at the least of

$$\sum_{i=1}^{2^{k-j}} m_2(T_j^{2i-1} \cup T_j^{2i})(1-t)(3t-1) = \sum_{i=1}^{2^{k-j}} m_2(T_1^i)t^{2j-2}(1-t)(3t-1) = m_2(T_1)t^{2j-2}(1-t)(3t-1)$$

So we have that the area at the last step (k-th step) is at the most

$$m_2(T_1) - (1-t)(3t-1)(1+t^2+...+t^{2k-2})m_2(T_1) = m_2(T_1)\left(1 - \frac{(3t-1)(1-t^{2k})}{t+1}\right)$$

It is clear that, choosing t close enough to 1 and then k great enough, the area is as small as we want.

Moreover all the segments from the bases to the opposite vertex have not been divided in any step so we can find them in the final figure.

The figure is also contained in A since every point has been moved of less than ε.

We see also that the final figure is a union of 2^k triangles overlapping because each of the triangles in the initial figure have only been translated and not 'broken' in any way.

(3)Finally we want to create a set of measure 0 which contains a segment parallel to every directions from $-\frac{2\pi}{3}$ to $-\frac{\pi}{3}$ from an equilateral triangle. Adding five copies of it rotated in the obvious way we obtain our Besicovitch set.

Let T_1 be an equilateral triangle whose side has length 1.

We take an open set A_1 such that $T_1 \subseteq A_1$, $m_2(A_1) \leq 2m_2(T_1)$ and $m_2(A_1) < 2m_2(\overline{A_1})$ (we can because A_1 contains T_1 and we can choose A_1 itself so that it and its closure are contained in a ball of radius small enough). From this triangle we create a figure T_2 of area less than 2^{-2} contained in A_1 as we have seen in step (2).

This figure can be seen as an union of triangles with the same height of T_1.

Now we take an open set A_2 such that $T_2 \subseteq A_2 \subseteq A_1$, $m_2(A_2) \leq 2m_2(T_2)$ and $m_2(A_2) < 2m_2(\overline{A_2})$.

We can do this due to the properties of the Lebesgue measure and because if A_2 is not a subset of A_1 we can intersect A_2 with A_1 obtaining an open set with the desired properties.

We proceed like this and at every step n we will apply step (2) to the triangles of the figure T_{n-1} and we will obtain a figure T_n, with area less than 2^{-n}, and which is the union of triangles and an open set $A_n \supseteq T_n$ such that $A_n \subseteq A_{n-1}$, $m_2(A_n) \leq 2m_2(T_n)$ and $m_2(A_n) < 2m_2(\overline{A_n})$.

We have that $A_1 \supseteq A_2 \supseteq A_3 \supseteq ....$

We take $C = \bigcap_{i=1}^{+\infty} \overline{A_i}$. This is the intersection of closed sets so it is closed. Moreover

$$m_2(C) \leq 2m_2(A_i) \leq 4m_2(T_i) \forall i \Rightarrow m_2(C) = 0$$

We take a segment parallel to every direction in every figure T_i (they exist due to our construction in (2)).

So, for every direction, we can create a sequence of segments $(l_i)_{i \in \mathbb{N}}$. Taking the distance between two of these segments as the distance between the points of these segments which are in the bases of the correspondent figure we have that we can take a subsequence of segments $(l_{k_i})_{i \in \mathbb{N}}$ such that the distance between two of them converges to 0. For every line parallel to the base of T_1 (and of all the figures) we can intersect it with the segments so we have a convergent sequence of points.

Since C is closed the points converge to a point in C. This is true for every line and we see that the points of convergence form a segment parallel to all the segments in the sequence.

So this segment is in C and therefore C contains a segment of length 1 parallel to every direction.

So we have our Besicovitch set.

Bibliography

[1] K.J. Falconer. *Continuity properties of k-plane integrals and Besicovitch sets.* Cambridge University Press, 1979.

[2] K.J. Falconer. *The geometry of fractal sets.* Cambridge University Press, 1985.

[3] D.M. Oberlin & E.M. Stein. *Mapping properties of the Radon transform.* Indiana University Mathematical Journal, 1982.

[4] Elias M. Stein & Rami Shakarchi. *Fourier Analysis: an introduction.* Princeton lectures in analysis I, 2003.

[5] Elias M. Stein & Rami Shakarchi. *Real Analysis: measure theory, integration & Hilbert spaces.* Princeton lectures in analysis III, 2005.

[6] Elias M. Stein & Rami Shakarchi. *Functional Analysis: introduction to further topics in analysis.* Princeton lectures in analysis IV, 2011.