Università degli Studi di Padova

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DIPARTIMENTO DI MATEMATICA “TULLIO LEVI-CIVITA”

Master’s Degree in Mathematics

Two-term Silting Complexes over Gentle Algebras

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Introduction

Gentle algebras constitute an important and rather large subclass of finite-dimensional path algebras, characterised by particularly nice bound quivers. Their study is the main topic of this Master's thesis. Gentle algebras are of great interest since they enjoy a favourable representation theory: they are indeed tame algebras and their finite-dimensional representations are completely classified. Thanks to their tractable nature, gentle algebras represent an optimal class of examples for the testing of theoretical conjectures or for inspiring new ideas. For this reason, the study of gentle algebras is a highly active area of research, with links between representation theory and different areas of mathematics. Their wide importance in the theory of cluster algebras is noteworthy, where they occur as surface algebras. In representation theory, an important class of modules are the support τ-tilting modules, introduced by Adachi, Iyama and Reiten in [AIR14], as a generalisation of tilting modules, where τ stands for the Auslander-Reiten translation. As the name may intuitively suggest, τ-tilting theory connects two very important branches in representation theory: tilting and Auslander-Reiten theory. The authors also showed that support τ-tilting modules are in one-to-one correspondence with two-term silting complexes in $K^b(\operatorname{proj}\Lambda)$. This implies that two-term silting complexes control homological properties of Λ-mod and that they offer significant insights into $K^b(\operatorname{proj}\Lambda)$.

Therefore the aim of this thesis is not only to introduce and to study the properties of modules over gentle algebras, but we will also give a complete classification of their two-term silting complexes.

In Chapter 1, we fix our notations and we describe the setting where we will be working throughout this thesis, presenting some basic concepts of general representation theory. We always consider a basic finite dimensional algebra over an algebraically closed field. Bearing in mind that our goal is to provide a complete classification of the two-term silting complexes, in this chapter, we also study the association between modules and their minimal projective presentations. This is aimed at understanding the structure of Λ-mod by looking at $K^{[-1,0]}(\operatorname{proj}\Lambda)$, since this is the environment where the two-term silting complexes live and where we may naturally study τ-tilting theory.

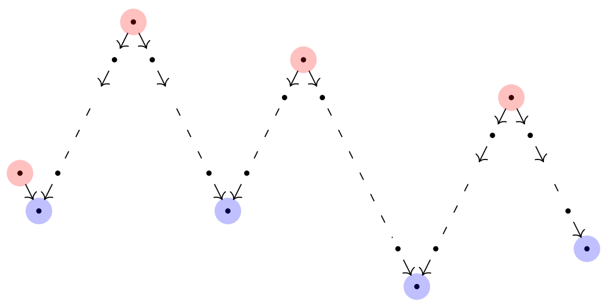
The second chapter of this project lays the foundations of the theory regarding gentle algebras. In this introduction we give a brief and informal summary of these concepts.

Definition 2.3.1. A *gentle quiver* $\mathrm{Q} := ((\mathrm{Q}_0, \mathrm{Q}_1), I)$ is a bound quiver, i.e. a directed graph with relations, such that:

- each vertex $v \in Q_0$ has at most two incoming and two outgoing arrows,
- I is generated by paths of length exactly two,
- for any arrow $\beta \in Q_1$, there is at most one arrow $\alpha \in Q_1$ such that $t(\alpha) = s(\beta)$ and $\beta\alpha \notin I$, and there is at most one arrow $\alpha' \in Q_1$, such that $t(\alpha') = s(\beta)$ and $\beta\alpha' \in I$,
- for any arrow $\beta \in Q_1$, there is at most one arrow $\gamma \in Q_1$ such that $t(\beta) = s(\gamma)$ and $\gamma\beta \notin I$ and there is at most one arrow $\gamma' \in Q_1$ such that $t(\beta) = s(\gamma')$ and $\gamma'\beta \in I$.

The algebra $\Lambda = \frac{k\,Q}{I}$ associated to a gentle bound quiver is called a *gentle algebra.*

We call a composition of direct and inverse paths of the quiver a *string*, and to every string ω, we associate a *string module* $M(\omega)$. We can visualise every string as a diagram of the form:



where the direct (resp. inverse) path are the one pointing to the right (resp. to the left). Intuitively the *peak* vertices are the one colored in red, while the *deep* are the blue ones.

Revisiting a more general proof made by Crawley-Boevey in [Cra98], we will show that each homomorphism between string modules corresponds to a linear combination of partial maps between their respective strings. In this way we associate a fact about modules to the structure of the strings that generate them, making it also easier to visualise. In general, being able to visualise objects makes them easier to understand, hence we will follow the same idea throughout the project: whenever possible, we will link our statements to a combinatorial phenomenon occurring in the string.

Moreover, we also prove that string modules are indecomposable. This result is important because the categories Λ-mod and $K^b(\operatorname{proj}\Lambda)$ both satisfy the Krull-Schmidt property, namely each object can be uniquely decomposed into a finite direct sum of indecomposable objects. Thus, understanding the indecomposables is fundamental, which is why the study of indecomposable objects will be a recurring theme across this thesis.

We conclude Chapter 2 by proving how, given a string ω, one can compute the minimal projective presentation $P(\omega)$ of the respective string module. To reach this result, it is necessary to examine the indecomposable projective modules in $\operatorname{proj}\Lambda$ and the homomorphisms between them. These modules correspond bijectively to the vertices of the gentle quiver and they turn out to be string modules. Namely, for a vertex $a \in Q_0$, we show that the string generating the indecomposable projective module P_a consists of the two maximal-length paths starting from a, with a as the unique peak vertex of the string.

The indecomposable objects of $K^b(\operatorname{proj}\Lambda)$ of a gentle algebra Λ have been classified by Bekkert and Merklen in [BM03]. Using the computation, that we have just determined, and their classification, we will show that every minimal projective presentation of a string module corresponds to one of the indecomposable objects of $K^b(\operatorname{proj}\Lambda)$.

For the last two chapter of this thesis we will focus on the two-term silting complexes.

Definition 3.0.1. Let P in $K^b(\operatorname{proj}\Lambda)$. We call P *silting* if

- $\operatorname{Hom}_{K^b(\operatorname{proj}\Lambda)}(P, P[i]) = 0$ for any $i > 0$ and
- $\operatorname{thick}(P) = K^b(\operatorname{proj}\Lambda)$,

where $\operatorname{thick}(P)$ is the smallest full subcategory of $K^b(\operatorname{proj}\Lambda)$ which contains P and is closed under cones, shifts, direct summands and isomorphisms. If only the first condition is satisfied, then we call P *presilting*. If P is a silting object in $K^{[-1,0]}(\operatorname{proj}\Lambda)$, then it is a *two-term silting* object.

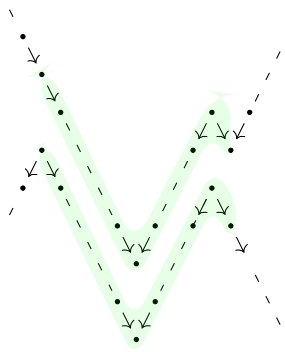
In Chapter 3, in a general context, we present characterisations of these complexes and describe several of their properties. Following [AIR14], we show all the necessary steps to establish the previously mentioned bijection between these complexes and the support τ-tilting modules.

In Chapter 4, we show how, by enlarging our gentle algebras, we can more naturally describe the minimal projective presentation $P(\omega) = P^1 \to P^0$ of a string module $M(\omega)$. We introduce the so-called *blossoming quiver* $Q^{❀}$, which is constructed by adding specific arrows to the original quiver Q. This construction is not limited to the quiver alone, we also expand the set of strings. The *blossoming string* $\omega^{❀}$ is created by adding a blossoming cohook at both the beginning and the end of the original string ω. Then P^0 (resp. P^1) corresponds to the direct sum of the projective indecomposable modules of the peak vertices (resp. the deep) of the blossoming string. Therefore, the combinatorial information given by $\omega^{❀}$ is sufficient to compute $P(\omega)$.

We can now characterize presilting minimal projective presentation of string modules over a gentle algebra, with the following original result.

Proposition 4.2.3. *A minimal projective presentation of a string module $M(\omega)$ is a two-term* presilting *complex for Λ if and only if the relative blossoming string $\omega^{❀}$ does not kiss itself.*

We say that two strings are kissing if there exists a common substring following certain rules. Thus, with this proposition, we successfully link the combinatorial data arising from the string to an algebraic property. We present, below, an example of two kissing strings, where the "kiss" is highlighted in green:



We will conclude this Master's thesis by providing a complete classification of the two-term silting complexes over gentle algebras. This classification is made possible by the results achieved in the previous chapters, the bijection between support τ-tilting modules and two-term silting complexes, as discussed in Chapter 3, and the classification of support τ-tilting modules, made in [PPP21] and [Brü+20].

Chapter 1

Preliminaries

Before entering into the more important topics discussed in this thesis, it is essential to recall basic preliminary concepts. Therefore, following consistently the first three chapters of [ASS06], we will provide an overview of the basic definitions, results, and techniques related to the study of the theory of modules over finite-dimensional algebra. Since the preliminaries in the first sections of this chapter may already have been studied during a Master's course in homological algebra, we have chosen to omit certain details or even entire proofs. This decision allows us to focus more on the general construction of the projective resolution, of which we will present an important application in Chapter 2, that will be the starting point of the main result stated and proved in Chapter 4. In Sections 1.3 and 1.4 we introduce the concept of triangulated categories and prove some of their properties. We will return to them in Chapter 3. Despite recalling very elementary concepts of finite-dimensional algebra, we have chosen not to include a review of category theory. This is on the assumption that the reader already has sufficient knowledge of these concepts, and also because category theory is not the main focus of our work. For a full background, the readers, who may not be familiar with this theory, can refer to [Mac98]. This chapter also serves to establish the notations that will be employed throughout the project.

1.1 Finite-dimensional algebras

A **ring** is a triple $(R, +, \cdot)$ consisting of a set R and two binary operations: addition and multiplication, such that $(R, +)$ is an abelian group, with zero element $0 \in R$, and such that multiplication is associative and left and right distributive over the addition. We will consider only rings with identity. A ring homomorphism $f : R \to R'$, is a map which preserves addition and multiplication.

Throughout this thesis $\mathbb{K}$ denotes an algebraically closed field, i.e. a commutative division ring, where any nonconstant polynomial $p(x)$ in one indeterminate x with coefficients in $\mathbb{K}$ has a root in $\mathbb{K}$.

Definition 1.1.1. A **$\mathbb{K}$-algebra** is a ring Λ with an identity element such that Λ has a $\mathbb{K}$-vector space structure compatible with the multiplication of the ring.

A $\mathbb{K}$-algebra Λ is said to be **finite-dimensional**, if the dimension of the $\mathbb{K}$-vector space is finite. An algebra Λ is **connected** if it is not a direct product of two algebras. A $\mathbb{K}$-**subalgebra** of Λ is a $\mathbb{K}$-vector subspace Λ' having the same identity of Λ and closed under multiplication. A $\mathbb{K}$-vector subspace I of the $\mathbb{K}$-algebra Λ is a right **ideal** (or left ideal) if $xa \in I$ (or $ax \in I$, respectively) for all $x \in I$ and $a \in \Lambda$. A two-sided ideal of Λ (or simply an ideal of A) is both a left and a right ideal.

A $\mathbb{K}$-algebra homomorphisms is a ring homomorphism, which is also a $\mathbb{K}$-linear map.

Definition 1.1.2. A **left Λ-module** consists of an abelian group M and an operation $\cdot : \Lambda \times M \to M$, called the **action** of Λ over M, such that for all $a, b \in \Lambda$ and $m, m' \in M$, we have :

- $a \cdot (m + m') = a \cdot m + a \cdot m'$;
- $(a + a') \cdot m = a \cdot m + a' \cdot m$;
- $(aa') \cdot m = a \cdot (a' \cdot m)$;
- $1 \cdot m = m$.

We will often omit the symbol $\cdot$. Right modules are defined similarly, but since we primarily focus on left modules, we may sometimes omit the term 'left' when the context is clear. Note that the reference we follow, [ASS06], works with right modules instead, so all results are dual to the ones in the book.

A **submodule** M' of M is a $\mathbb{K}$-vector subspace closed under the action of Λ, namely am belongs to M', for each $a \in \Lambda$ and $m \in M'$. A submodule M' of M is **maximal** if it is different than M and for each N submodule of M such that $M' \subseteq N$, then N is isomorphic to M.

A Λ-module M is said to be **indecomposable**, if M is non-zero and has no direct sum decomposition $M \simeq M' \oplus M''$, where both M' and M'' are non-zero Λ-modules. A Λ-module M is said to be **finite-dimensional** if the dimension of the underlying $\mathbb{K}$-vector space is finite. Unless otherwise specified, we will mostly deal with finite-dimensional modules. A Λ-module M is said to be **simple**, if M is non-zero and any submodule is either zero or M itself.

For our purposes, the most important submodule of a left Λ-module M is the **Jacobson radical**: denoted with $\operatorname{rad} M$. It is defined as the intersection of all the maximal submodules of M. The Jacobson radical is a crucial concept that we will frequently utilize. Observe that the radical of a finite-dimensional module M can be computed as $\operatorname{rad} M = \operatorname{rad} \Lambda \cdot M$. The quotient $\operatorname{top} M = \frac{M}{\operatorname{rad} M}$ is called the **top** of M, and is a left $\frac{\Lambda}{\operatorname{rad} \Lambda}$-module.

Definition 1.1.3. Let M, N be two left Λ-module. **A Λ-module homomorphism**

$$f : M \to N$$

is a $\mathbb{K}$-linear map such that, for all $m \in M$ and $a \in \Lambda$, we have $f(a \cdot m) = a \cdot f(m)$.

With Λ-Mod we denote the abelian category of all left Λ-modules, whose morphisms are Λ-module homomorphisms. We denote with Λ-mod, the full subcategory of Λ-Mod whose object are finite-dimensional left Λ-modules. To simplify, we call $\mathrm{Hom}_{\Lambda\text{-}\mathrm{Mod}}(M, N) = \mathrm{Hom}_\Lambda(M, N)$ the set of Λ-module homomorphism from M to N. Observe that it is a $\mathbb{K}$-vector space, which is moreover finite-dimensional if both M and N are finite-dimensional.

A surjective Λ-module homomorphisms is also called **epimorphism**, while an injective Λ-module homomorphisms will be called **monomorphism**.

The following theorem, see [ASS06, Theorem I.4.10], is of vital importance for the representation theory of finite-dimensional algebras and moreover it shows the reasons why we will focus on indecomposable modules.

Theorem 1.1.1 (Unique decomposition theorem)**.** *Let Λ be a finite-dimensional $\mathbb{K}$-algebra. Every module M in* Λ-mod *has a decomposition*

$$M \simeq M_1 \oplus \cdots \oplus M_n,$$

where $M_1, \ldots, M_n$ are indecomposable modules. This decomposition is "unique" in the sense that if $M \simeq M_1 \oplus \cdots \oplus M_n \simeq N_1 \oplus \ldots N_m$, where M_i and N_j are indecomposable then $m = n$ and there exists a permutation σ of $\{1, \ldots, n\}$ such that $M_{\sigma(i)} \simeq N_i$ for each $i = 1, \ldots, n$.

We also refer to this results saying that the category Λ-mod satisfies the Krull-Schmidt property.

1.1.1 Idempotents

To fully describe the theory of finite-dimensional algebras, we need to introduce particular elements of Λ, known as idempotents. An element $e \in \Lambda$ is called an **idempotent** if $e^2 = e$. Two idempotents e_1 and e_2 are called **orthogonal** if $e_2e_1 = e_1e_2 = 0$. An idempotent e is said to be **primitive** if it cannot be expressed as a sum $e = e_1 + e_2$, where e_1 and e_2 are non-zero orthogonal idempotents of Λ.

Every algebra Λ has two trivial idempotents, 0 and 1, and if e is an idempotent, then $1 - e$ is also an idempotent of Λ.

Observe that, given a primitive idempotent e, $e\Lambda e$ is a ring with identity e. For any left Λ-module M, eM is a left $e\Lambda e$-module. The action of $e\Lambda e$ over eM, is defined as $eae \cdot em = eaem$, for each $m \in M$ and $a \in \Lambda$. Similarly, one can prove that Λe is a right $e\Lambda e$-module. Hence, we have that $\mathrm{Hom}_\Lambda(\Lambda e, M)$ is a left $e\Lambda e$-module. The action is defined as $eae \cdot f(be) = f(beae)$, for each $a, b \in \Lambda$ and $f \in \mathrm{Hom}_\Lambda(\Lambda e, M)$.

We can now state the following useful result.

Lemma 1.1.2. *Let Λ be a $\mathbb{K}$-algebra, e a primitive idempotent and M a left Λ-module. We have an isomorphism of left $e\Lambda e$-modules:*

$$\mathrm{Hom}_\Lambda(\Lambda e, M) \simeq eM.$$

Moreover, it induces a natural equivalence between functors $\mathrm{Hom}_\Lambda(\Lambda e, -) : \Lambda\text{-}\mathrm{Mod} \to e\Lambda e\text{-}\mathrm{Mod}$ *and* $e(-) : \Lambda\text{-}\mathrm{Mod} \to e\Lambda e\text{-}\mathrm{Mod}$.

Proof. We prove that the $\mathbb{K}$-linear map $\phi : \operatorname{Hom}_\Lambda(\Lambda e, M) \to eM$, which associates to $f \in \operatorname{Hom}_\Lambda(\Lambda e, M)$, $ef(e) \in eM$, is an isomorphism of left $e\Lambda e$-modules. Let $a, b \in \Lambda$ and $m \in M$. Firstly, this is a well-defined homomorphism between left $e\Lambda e$-modules, since $eae \cdot \phi(f) = eae \cdot ef(e) = eaef(e)$, while $\phi(eae \cdot f) = f(e \cdot eae) = f(eeae) = f(eae)$. Because f is a homomorphism of left Λ-module $eaef(e) = f(eaee) = f(eae)$. We prove that ϕ is an isomorphism by proving that there exists an inverse. Consider the $\mathbb{K}$-linear map

$$\phi' : eM \to \operatorname{Hom}_\Lambda(\Lambda e, M)$$

defined by the formula $em \to f_{em}$, with $f_{em} : \Lambda e \to M$ such that $f_{em}(ae) = aem$. Then f_{em} is a well-defined homomorphism of left Λ-modules, since $bf_{em}(ae) = baem = f_{em}(bae)$. While ϕ' is an homomorphism between left $e\Lambda e$-modules since $(eae \cdot \phi'(em))(be) = eae \cdot f_{em}(be) = f_{em}(beae) = beaem$ and $\phi'(eae \cdot em)(be) = \phi'(eaem)(be) = f_{eaem}(be) = beaem$. Lastly, $\phi\phi'(em) = \phi(f_{em}) = ef_{em}(e) = eeem = em$, and $\phi'\phi(f)(ae) = \phi'(ef(e))(ae) = f_{ef(e)}(ae) = aeef(e) = af(e) = f(ae)$, where the last equality is due to the fact that f is a left Λ-module homomorphism. Thus, they are inverse to each other, and with this we can conclude that ϕ is an isomorphism of left $e\Lambda e$-module.

To prove that it is also functorial on the variable M, we just need to prove that, given $f \in \operatorname{Hom}_\Lambda(M, N)$ the following commutes

$$\begin{array}{ccc} \operatorname{Hom}_\Lambda(\Lambda e, M) & \xrightarrow{\phi} & eM \\ {\scriptstyle \operatorname{Hom}(\Lambda e, f)}\downarrow & & \downarrow{\scriptstyle ef} \\ \operatorname{Hom}_\Lambda(\Lambda e, N) & \xrightarrow[\phi]{} & eN \end{array} .$$

Let $h \in \operatorname{Hom}_\Lambda(\Lambda e, M)$, then $\phi\operatorname{Hom}(\Lambda e, f)(h) = efh(e)$, while $ef(\phi(h)) = ef(eh(e)) = efh(e)$, thanks to the definition of ef, which sends an element $em \in eM$ to $ef(m) \in eN$. □

Consider a set $\{e_1, \dots, e_n\}$ of primitive pairwise orthogonal idempotents of Λ, if moreover $1 = e_1 + \cdots + e_n$, the set is called **complete**, and it induces a decomposition $\Lambda = \Lambda e_1 \oplus \cdots \oplus \Lambda e_n$ with indecomposable left modules $\Lambda e_1, \dots, \Lambda e_n$.

We list some properties, see [ASS06, Proposition I.4.4 and Proposition I.4.5] which show the interconnection between the notion of idempotents and the one of Jacobson radical:

Proposition 1.1.3. *Let Λ be a finite-dimensional algebra. Let $B = \frac{\Lambda}{\operatorname{rad}\Lambda}$.*

- *Every right ideal I of B is a direct sum of simple right ideals of the form Be, where e is a primitive idempotent of B.*
- *Any module N in B-mod is isomorphic to a direct sum of simple right ideals of the form Be, where e is a primitive idempotent of B.*
- *If $e \in \Lambda$ is a primitive idempotent of Λ, then $\operatorname{top}\Lambda e$ is simple and $\operatorname{rad}(\Lambda e) = (\operatorname{rad}\Lambda)e \subset \Lambda e$ is the unique maximal proper submodule of Λe.*

This proposition implies that, given $\Lambda = \Lambda e_1 \oplus \cdots \Lambda e_n$ a decomposition of Λ into indecomposable submodules, every simple left Λ-submodule is isomorphic to one of the modules

$$S(1) = \operatorname{top} \Lambda e_1, \cdots, S(n) = \operatorname{top} \Lambda e_n$$

Thus there exists a bijection between a complete set of primitive orthogonal idempotents and a complete set of pairwise non-isomorphic simple Λ-modules given by

$$e_j \to \operatorname{top} \Lambda e_j.$$

1.1.2 Projective modules and projective resolutions

Now that we have covered some basic concepts, we can start to look at the so-called projective modules and construct the projective resolution. These objects and their relative properties will be used often throughout our thesis project.

A left Λ-module P is **projective** if, for any epimorphism $g : M \to N$ and any $f \in \operatorname{Hom}_\Lambda(P, N)$, there is $\tilde{f} \in \operatorname{Hom}_\Lambda(P, M)$ such that the following diagram is commutative:

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{\tilde{f}} & \downarrow{\scriptstyle f} & & \\ M & \xrightarrow[g]{} & N & \longrightarrow & 0. \end{array}$$

It is known that the category Λ-mod **has enough projectives**, namely for every left Λ-module there is an epimorphism $f : P \to M$ with P projective. Moreover, we have another equivalent characterization of projective module, indeed a module is projective if and only if is a direct summand of Λ^n, for some natural number $n \geq 0$. This implies that Λe is a projective left Λ-module for any idempotent e.

We define a **projective resolution** of a left Λ-module M to be a chain complex

$$\cdots \longrightarrow P^n \xrightarrow{d_n} P^{n-1} \to \cdots \to P^1 \xrightarrow{d_1} P^0 \longrightarrow 0 \longrightarrow \ldots$$

of projective Λ-modules together with an epimorphism $P^0 \stackrel{d_0}{\to} M$ of left Λ-modules such that the sequence is exact, namely $\operatorname{Ker} d_n = \operatorname{Im} d_{n-1}$ for any $n > 0$. Since Λ-mod has enough projectives, any module M has a projective resolution.

A Λ-submodule M' of M is **superfluous** if for every submodule X of M the equality $L + X = M$ implies $X = M$. The Jacobson radical $\operatorname{rad} M$ of a module M, for instance, is a superfluous submodule.

A Λ-epimorphism $h : M \to N$ in Λ-mod is **minimal** if $\operatorname{Ker} h$ is superfluous as a submodule of M. An epimorphism $h : P \to M$ in Λ-mod is called a **projective cover** of M, if P is a projective module and h is a minimal epimorphism.

Lemma 1.1.4. *An epimorphism $h : P \to M$ is a projective cover of an Λ-module M if and only if P is projective and for any Λ-homomorphism $g : N \to P$ the surjectivity of hg implies the surjectivity of g.*

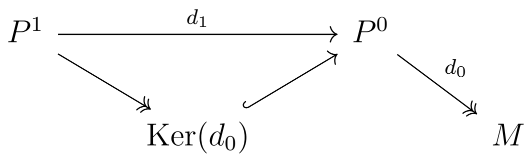
Proof. Let $h : P \to M$ be a projective cover of M and let $g : N \to P$ be an homomorphism such that hg is surjective. We want to show that $P = \operatorname{Ker} h + \operatorname{Im} g$. Let $x \in P$. Since $\operatorname{Im}(hg) = \operatorname{Im}(h) = M$, we have that $h(x)$ belongs to $\operatorname{Im}(hg)$, so there exists $y \in N$ such that $h(x) = hg(y)$. This implies that $h(x - g(y)) = 0$. Then $x = x - g(y) + g(y)$, where $x - g(y)$ belongs to the kernel of h, while $g(y)$ belongs to the image of g. The other inclusion is clear. Since $\operatorname{Ker} h$ is superfluous, then g is surjective.

Conversely, we need now to prove that if h has the stated property, then it is a projective cover. Since hg is surjective, then h is surjective. Let N be a submodule of P such that $N + \operatorname{Ker} h = P$. Let $\epsilon : N \to P$ be the natural inclusion, then $h\epsilon : N \to M$ is surjective, so also ϵ must be surjective. This implies that ϵ is an isomorphism and so $\operatorname{Ker} h$ is superfluous. □

A sequence in Λ-mod

$$P^1 \xrightarrow{d_1} P^0,$$

together with an epimorphism $P^0 \xrightarrow{d_0} M$ is called a **minimal projective presentation** of a Λ-module M, if the Λ-module homomorphism d_0 is a projective cover, while d_1 is the composition of the projective cover of $\operatorname{Ker}(d_0)$ and its natural inclusion into P^0.



A projective resolution of M is called **minimal** if $d_j : P^j \to \operatorname{Im} d_j$ is a projective cover for all $j \geq 0$.

The next result shows that any module M in Λ-mod admits a minimal projective presentation and a minimal projective resolution in Λ-mod.

Theorem 1.1.5. *Let Λ be a finite-dimensional $\mathbb{K}$-algebra and let $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents of Λ. For any Λ-module, M there exists a projective cover $h : P(M) \to M \to 0$ where*

$$P(M) \simeq (\Lambda e_1)^{s_1} \oplus \cdots \oplus (\Lambda e_n)^{s_n}$$

and $s_1 \geq 0, \dots, s_n \geq 0$. The homomorphism h induces an isomorphism $\frac{P(M)}{\operatorname{rad} P(M)} \simeq \frac{M}{\operatorname{rad} M}$.

Proof. Set $B = \frac{\Lambda}{\operatorname{rad} \Lambda}$ and let $\rho : \Lambda \to B$ the natural projection and $\overline{e_j} = \rho(e_j)$. Because $\{e_1, \dots, e_n\}$ is a complete set of primitive orthogonal idempotents of Λ and ρ is a Λ-module epimorphism, $\{\overline{e}_1, \dots, \overline{e}_n\}$ is a complete set of primitive orthogonal idempotents of B and $B = B\overline{e}_1 \oplus \cdots \oplus B\overline{e}_n$ is the induced indecomposable decomposition. We have that, for each j, the epimorphism $\rho_j : \Lambda e_j \to \operatorname{top} \Lambda e_j$ is a projective cover of $\operatorname{top} \Lambda e_j$, since its kernel $\operatorname{rad} \Lambda e_j$ is superfluous and Λe_j is projective.

Let M a Λ-module. Then, $\operatorname{top} M = \frac{M}{\operatorname{rad} M}$ is a module over B. By Proposition 1.1.3, any object in B-mod is isomorphic to a direct sum of left B-modules of the type Be_j. Namely,

$$\operatorname{top} M \simeq (B\overline{e_1})^{s_1} \oplus \cdots \oplus (B\overline{e_n})^{s_n} \simeq (\operatorname{top} \Lambda e_1)^{s_1} \oplus \cdots \oplus (\operatorname{top} \Lambda e_n)^{s_n},$$

for some $s_1 \geq 0, \dots, s_n \geq 0$.

Set $P(M) = (\Lambda e_1)^{s_1} \oplus \cdots \oplus (\Lambda e_n)^{s_n}$. This is a projective Λ-module. We choose this because $\operatorname{top} P(M) = \operatorname{top}((\Lambda e_1)^{s_1} \oplus \cdots \oplus (\Lambda e_n)^{s_n}) \simeq (\operatorname{top} \Lambda e_1)^{s_1} \oplus \cdots \oplus (\operatorname{top} \Lambda e_n)^{s_n} \simeq \operatorname{top} M$.

So, if we consider the canonical epimorphisms $\pi : P(M) \to \operatorname{top} P(M)$ and $\pi' : M \to \operatorname{top} M$, by the projectivity of the module $P(M)$, there exists a Λ-module homomorphism $h : P(M) \to M$, making the following commute:

$$\begin{array}{ccccc} & & P(M) & & \\ & \overset{h}{\swarrow} & \downarrow{\scriptstyle \phi\pi} & & \\ M & \xrightarrow[\pi']{} & \operatorname{top} M & \longrightarrow & 0 \end{array}$$

It follows that h is surjective. Moreover $\operatorname{Ker} h \subseteq \operatorname{Ker} \pi = \operatorname{rad} P(M)$.

Since $\operatorname{rad} P(M)$ is superfluous in $P(M)$, $\operatorname{Ker} h$ is also superfluous in $P(M)$. Therefore, the epimorphism h is a projective cover of M. □

Note that the proof is constructive, and later in the thesis, we will apply the technique provided by the theorem to construct projective covers specifically in the context of gentle algebras

Corollary 1.1.6. *Let Λ be a finite-dimensional $\mathbb{K}$-algebra. Any module M in* Λ-mod *admits a minimal projective presentation and a minimal projective resolution in* Λ-mod.

Proof. Let M be in Λ-mod and let $P^0 := P(M) \overset{d_0}{\to} M$ its projective cover. Then $\operatorname{Ker} d_0 \subseteq P^0$ is a finite-dimensional submodule, hence it exists its projective cover $P^1 := P(\operatorname{Ker} d_0) \overset{d_0'}{\to} \operatorname{Ker} d_0$. Considering the natural inclusion $\epsilon_0 : \operatorname{Ker} d_0 \to P^0$, set $d_1 := \epsilon_0 d_0'$ we have that $P^1 \overset{d_1}{\to} P^0 \overset{d_0}{\to} M \to 0$ is a minimal projective presentation of M. By induction, we can continue with the same reasoning and obtain a minimal projective resolution of M in Λ-mod.

$$\begin{array}{ccccccccc} \cdots \longrightarrow P_2 & & \xrightarrow{\quad d_2 \quad} & & P^1 & & \xrightarrow{\quad d_1 \quad} & & P^0 \xrightarrow{\ d_0\ } M \\ & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & \operatorname{Ker} d_1 & & & & \operatorname{Ker} d_0 & & \end{array}$$

□

Given its crucial role in constructing the minimal resolution, and thereby the minimal presentation, we refer to the kernel of the projective cover $d_0 : P^0 \to M$ as the **syzygy** of M, denoted with $\Omega(M)$.

Corollary 1.1.7. *If P is a projective module in* Λ-mod, *then the canonical epimorphism $\pi : P \to \operatorname{top} P$ is a projective cover of* $\operatorname{top} P$ *and there exists an isomorphism*

$$P \simeq (\Lambda e_1)^{s_1} \oplus \cdots \oplus (\Lambda e_n)^{s_n}$$

for some $s_1 \geq 0, \dots, s_n \geq 0$.

This result implies that, given a decomposition of Λ into indecomposable submodules $\Lambda = \Lambda e_1 \oplus \cdots \Lambda e_n$, a complete list of the indecomposable projective finite-dimensional Λ-module is given by

$$P(1) = \Lambda e_1, P(2) = \Lambda e_2, \ldots, P(n) = \Lambda e_n.$$

The dual notion of projective modules are the so called **injective** modules. The injective cover and the injective resolution are then constructed similarly.

1.2 Quivers and path algebras

A quiver is just a directed graph, a very simple mathematical object. Nevertheless, as we will show in this section, there is a strong connection between quivers and the representation theory of finite-dimensional algebras. Moreover, working with quivers offers the important advantage of dealing with objects that can be visualised. These are just some of the reasons why the main application of our work, which will be presented in the next chapter, focuses on a particular type of algebra derived from quivers. Therefore, we need to outline some basic definitions and properties of quivers, their algebras and their representations, highlighting how they can be seen as an application of the result shown in the previous section. For instance, we show how the projective and simple modules are computed.

Definition 1.2.1. A **quiver** $\mathrm{Q} = (\mathrm{Q}_0, \mathrm{Q}_1, \mathscr{s}, \mathscr{t})$ is a quadruple consisting of two sets: Q_0 (whose elements are called **vertices**) and Q_1 (whose elements are called **arrows**), and two maps $\mathscr{s}, \mathscr{t} : \mathrm{Q}_1 \to \mathrm{Q}_0$ which associate to each arrow $\alpha \in \mathrm{Q}_1$, its source $\mathscr{s}(\alpha) \in \mathrm{Q}_0$ and its target $\mathscr{t}(\alpha) \in \mathrm{Q}_0$, respectively.

An arrow $\alpha \in \mathrm{Q}_1$ of source $a = \mathscr{s}(\alpha)$ and target $b = \mathscr{t}(\alpha)$ is usually denoted by $\alpha : a \to b$. A quiver $\mathrm{Q} = (\mathrm{Q}_0, \mathrm{Q}_1, \mathscr{s}, \mathscr{t})$ is usually denoted briefly by $\mathrm{Q} = (\mathrm{Q}_0, \mathrm{Q}_1)$ or even simply by Q.

Thus, a quiver is nothing but an oriented graph without any restriction as to the number of arrows between two points, to the existence of loops or oriented cycles.

A quiver Q is said to be **finite** if Q_0 and Q_1 are finite sets. The underlying graph Q of a quiver Q is obtained from Q by forgetting the orientation of the arrows. The quiver Q is said to be **connected** if Q is a connected graph.

From now on, unless otherwise specified, when we talk about a quiver, we always imply that it is connected and finite.

When drawing a quiver, we agree to represent each vertex by a dot, and each arrow will be pointing towards its target, as showed below.

Let $\mathrm{Q} = (\mathrm{Q}_0, \mathrm{Q}_1, \mathscr{s}, \mathscr{t})$ be a quiver and $a, b \in \mathrm{Q}_0$. A **path** of length l greater or equal than 1 with source a and target b (or, more briefly, from a to b) is a sequence $\alpha_1, \alpha_2, \ldots, \alpha_l$, where $\alpha_k \in \mathrm{Q}_1$ for all $1 \leq k \leq l$, and we have $\mathscr{s}(\alpha_1) = a$, $\mathscr{t}(\alpha_k) = \mathscr{s}(\alpha_{k+1})$ for each $1 \leq k < l$, and finally $\mathscr{t}(\alpha) = b$. Such a path is denoted briefly by $\alpha_1 \ldots \alpha_l$ and may be visualized as follows:

$$a = a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} a_2 \to \cdots \xrightarrow{\alpha_l} a_l = b.$$

Example 1.2.1. In Figure 1.1, we list some examples of finite connected quivers. Then $\delta\beta\alpha$ is a path of Q_A of length 3, $\delta\mu\alpha$ is a path of Q_B of length 3, $\mu\mu\beta\alpha\nu\nu\nu$ is a path of Q_C of length 7,$\beta\alpha'$ is a path of Q_D of length 2.

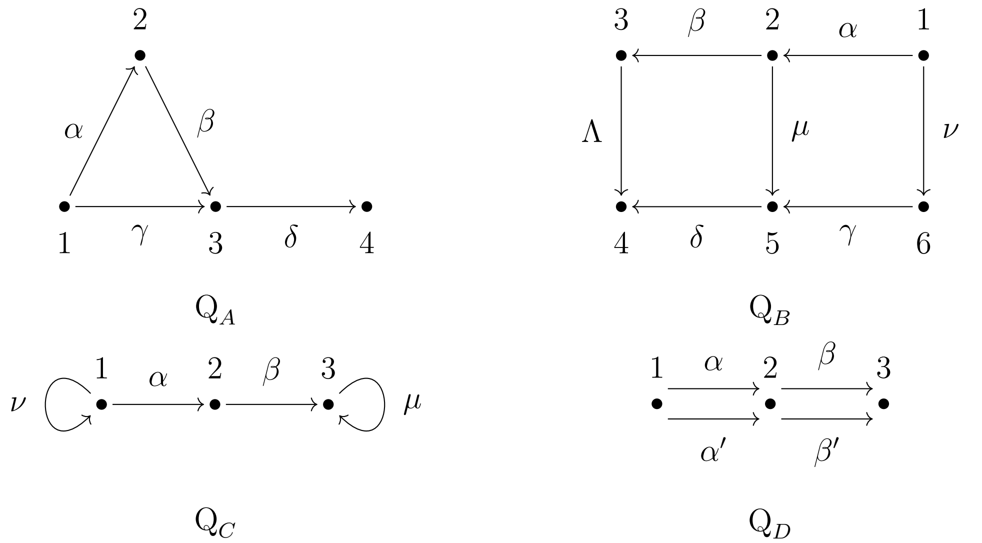
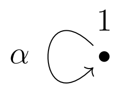


Figure 1.1

We denote by Q_l the set of all paths in Q of length l . We also agree to associate with each point $a \in Q_0$ a path of length 0, called the **trivial path** and denoted by e_a Thus, the paths of lengths 0 and 1 are in bijective correspondence with the elements of Q_0 and Q_1, respectively. A path of length $l \geq 1$ is called a **cycle** whenever its source and target coincide. A cycle of length 1 is called a **loop**. A quiver is called **acyclic** if it contains no cycles. If there exists in Q a path from a to b, then a is said to be a **predecessor** of b, and b is said to be a **successor** of a.

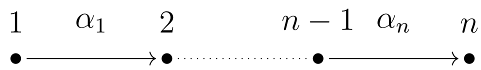
Definition 1.2.2. The **path algebra** $\mathbb{K}\,Q$ of Q is the $\mathbb{K}$-algebra whose underlying $\mathbb{K}$-vector space has as its basis the set of all paths of length $l \geq 0$ in Q and such that the product of two paths $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_k$ is equal to zero if $t(\alpha) \neq s(\beta_1)$ and is equal to the composed path $\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_k$, if $t(\alpha) = s(\beta_1)$. The product of basis elements is then extended to arbitrary elements of $\mathbb{K}\,Q$ by distributivity.

Example 1.2.2. • If Q is the Jordan quiver



then $\mathbb{K}\,Q \simeq \mathbb{K}[x]$, the polynomials in one variable.

- If Q is the quiver of type A_n, with $n-1$ arrows and n vertices:

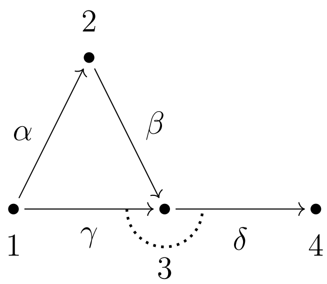


  then $\mathbb{K}\,\mathrm{Q}$ is isomorphic to the lower triangular matrices of $M(\mathbb{K})_n$.

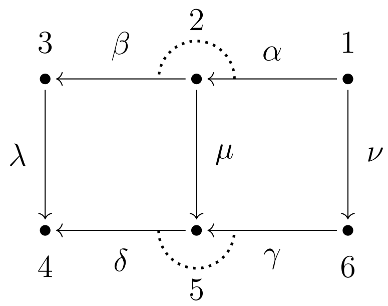
The **arrow ideal** of $\mathbb{K}\,\mathrm{Q}$ is the ideal Arr generated by the arrows of Q. A two-sided ideal I of $\mathbb{K}\,\mathrm{Q}$ is **admissible** if there exists $m \geq 2$ such that $\mathrm{Arr}^m \subseteq I \subseteq \mathrm{Arr}^2$.

The pair $\mathrm{Q} := (\mathrm{Q}, I)$ is then called a bound quiver and the quotient $\frac{\mathbb{K}\,\mathrm{Q}}{I}$ is a bound quiver algebra. We will call the generators of the set I, relations and if two arrows are in a relation, we will then visualise it, with a dotted lined, as in the following example.

Example 1.2.3. We add some relations in the quivers Q_A and Q_B of example 1.2.1:



$I_A =< \delta\gamma >$



$I_B =< \beta\alpha, \delta\gamma >$

The following result show the profound interconnection between finite-dimensional algebra and quivers. For the proof one can refer to [ASS06, Corollary II.2.12, Theorem II.3.7].

Theorem 1.2.1. *Given* Q *a finite connected quiver and* I *an admissible ideal, the bound quiver algebra* $\frac{\mathbb{K}\,\mathrm{Q}}{I}$ *is a basic and connected finite-dimensional algebra with an identity, having* $\frac{\mathrm{Arr}}{I}$ *as radical and* $\{\overline{e_a} \mid a \in \mathrm{Q}_0\}$ *as a complete set of pairwise orthogonal primitive idempotents, where* $\overline{e_a}$ *is the residual class of the trivial path* e_a *in* $\mathbb{K}\,\mathrm{Q}$ *modulo* I.

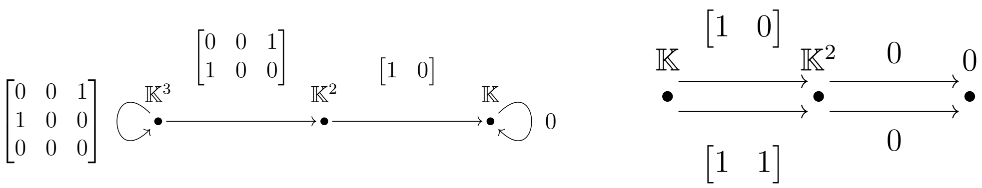
Conversely, let Λ *be a basic connected finite-dimensional* $\mathbb{K}$*-algebra. There exists a unique quiver* Q_Λ *and an admissible ideal* I *of* $\mathbb{K}\,\mathrm{Q}_\Lambda$ *such that* $A \simeq \frac{\mathbb{K}\,\mathrm{Q}_\Lambda}{I}$.

Definition 1.2.3. Let Q be a finite quiver. A $\mathbb{K}$-linear representation or, more briefly, a **representation** M of Q is defined by the following data:

- To each vertex a in Q_0 is associated a $\mathbb{K}$-vector space M_a,
- To each arrow $\alpha : a \to b$ in Q_1 is associated a $\mathbb{K}$-linear map $\phi_\alpha : M_a \to M_b$.

Such a representation is denoted as $M = (M_a, \phi_\alpha)_{a \in \mathrm{Q}_0, \alpha \in \mathrm{Q}_1}$ or simply $M = (M_a, \phi_\alpha)$.

Example 1.2.4. We list examples of representation of the quivers Q_C and Q_D, of Example 1.2.1.



A representation is **finite-dimensional** if each M_a is finite-dimensional. The above representations are all finite-dimensional.

Let $M = (M_a, \phi_\alpha)$ and $M' = (M'_a, \phi'_\alpha)$ be two representations of Q. A **morphism (of representations)** $f : M \to M'$ is a family $f = (f_a)_{a\in \mathrm{Q}_0}$ of $\mathbb{K}$-linear maps $(f_a : M_a \to M'_a)_{a\in\mathrm{Q}_0}$ that are compatible with the structure maps ϕ_α, that is, for each arrow $\alpha : a \to b$, we have $\phi_\alpha f_a = f_b\phi_\alpha$ or, equivalently, the following square is commutative:

$$\begin{array}{ccc} M_a & \xrightarrow{\phi_\alpha} & M_b \\ \big\downarrow{\scriptstyle f_a} & & \big\downarrow{\scriptstyle f_b} \\ M'_a & \xrightarrow[\phi'_\alpha]{} & M'_b. \end{array}$$

Let $f : M \to M'$ and $g : M' \to M''$ be two morphisms of representations of Q, where $f = (f_a)_{a\in\mathrm{Q}_0}$ and $g = (g_a)_{a\in\mathrm{Q}_0}$. Their composition is defined to be the family $gf = (g_a f_a)_{a\in\mathrm{Q}_0}$. Then gf is easily seen to be a morphism from M to M''. We have thus defined a category Rep(Q) of $\mathbb{K}$-linear representations of Q. We denote by rep(Q) the full subcategory of Rep(Q) consisting of the finite-dimensional representations.

Let Q be a finite quiver and $M = (M_a, \phi_\alpha)$ be a representation of Q. For any non-trivial path $p = \alpha_1\alpha_2\ldots\alpha_l$ from a to b in Q, we define the **evaluation** of M on the path P to be the $\mathbb{K}$-linear map from M_a to M_b defined by $\phi_p = \phi_{\alpha_1}\phi_{\alpha_2}\ldots\phi_{\alpha_{l-1}}\phi_{\alpha_l}$. The definition of evaluation extends to $\mathbb{K}$-linear combinations of paths with a common source and a common target; thus let

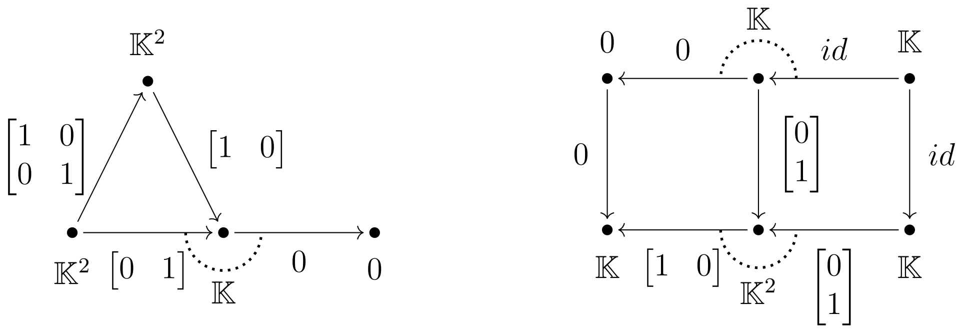
$$q = \sum_{i=1}^{m} \lambda_i p_i$$

be such a combination, where λ_i belongs to $\mathbb{K}$ and p_i is a path in Q, for each i, then

$$\phi_q = \sum_{i=1}^{m} \lambda_i \phi_{p_i}.$$

We are now able to define a notion of **representation of a bound quiver.** Let thus Q be a finite quiver and I be an admissible ideal of $\mathbb{K}\,\mathrm{Q}$. A representation $M = (M_a, \phi_\alpha)$ of Q is said to be bound by I, or to satisfy the relations in I, if we have $\phi_q = 0$, for all relations $q \in I$. If I is generated by the finite set of relations $\{q_1, \ldots, q_m\}$, the representation M is bound by I if and only if $\phi p_j = 0$, for all j such that $1 \leq j \leq m$. We denote by $\mathrm{Rep}_{\mathbb{K}}(\mathrm{Q}, I)$ (or by $\mathrm{rep}_{\mathbb{K}}(\mathrm{Q}, I)$ the full subcategory of $\mathrm{Rep}_{\mathbb{K}}(\mathrm{Q})$ (or of $\mathrm{rep}_{\mathbb{K}}(\mathrm{Q})$, respectively) consisting of the representations of Q bound by I.

Example 1.2.5. Here are presented some bound representations of the bound quivers (Q_A, I_A) and (Q_B, I_B) of Example 1.2.3:



Theorem 1.2.2. *Let* $\Gamma = \frac{\mathbb{K}\,\mathrm{Q}}{I}$, *where* Q *is a finite, connected quiver and* I *is an admissible ideal of* $\mathbb{K}\,\mathrm{Q}$. *There exists an equivalence of categories between the category of* Λ-mod *of finite-dimensional left* Λ*-modules and the category of finite-dimensional bound quiver representations:*

$$\Lambda\text{-}\operatorname{mod} \simeq \operatorname{rep}_{\mathbb{K}}(\mathrm{Q}, I).$$

Even if we are not giving a complete proof of this theorem, for it one can refer to [ASS06, Theorem III.1.6], we want to highlight the main idea behind, hence we show how, from M a left Λ-module, we associate its equivalent representation. We set for each $a \in \mathrm{Q}_0$, $M_a = e_a M$, which is the vector space consisting of all $e_a m$, with $m \in M$, and, for any $\alpha : a \to b$ arrow in Q_1, let $\overline{\alpha}$, be its class modulo I. Then for any $x \in M_a$, the map $\phi_\alpha : M_a \to M_b$ is given by

$$\phi_\alpha(x) = \overline{\alpha}x.$$

As a consequence, $\operatorname{rep}_{\mathbb{K}}(\mathrm{Q}, I)$ is abelian, has enough projective and injective objects, and thus every object of $\operatorname{rep}_{\mathbb{K}}(\mathrm{Q}, I)$ admits a projective and an injective resolution. In view of the above, we will often use the words “module” and “representation” to mean the same object.

Let $a \in \mathrm{Q}_0$; we denote by $S(a)$ the representation $(S(a)_b, \phi_\alpha)$ of Q defined as follows

$$S(a)_b = \begin{cases} 0 \text{ if } b \neq a, \\ K \text{ if } b = a, \\ \phi_\alpha = 0 \text{ for all } \alpha \in \mathrm{Q}_1 . \end{cases}$$

Clearly, $S(a)$ is a bound representation of (Q, I) (for any admissible I), and we have the following lemma.

Lemma 1.2.3. *Let* $\Lambda = \frac{\mathbb{K}\,\mathrm{Q}}{I}$ *be the bound quiver algebra of* (Q, I).

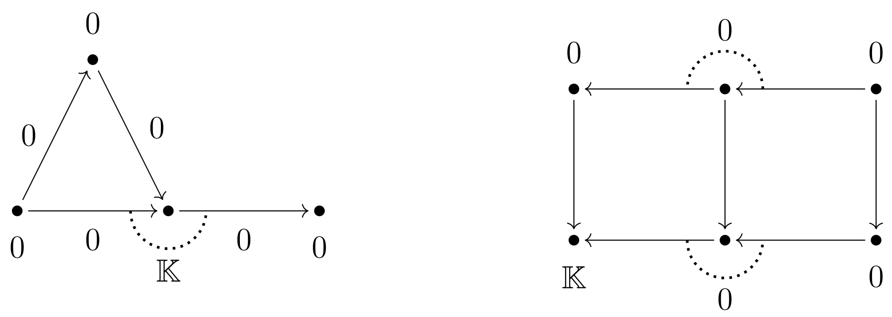
- *For any* $a \in \mathrm{Q}_0$, $S(a)$ *viewed as a* Λ*-module is isomorphic to the top of the indecomposable projective* Λ*-module* Λe_a.

- *The set* $\{S(a) \mid a \in \mathrm{Q}_0\}$ *is a complete set of representatives of the isomorphism classes of the simple* Λ*-modules.*

Proof. By definition, given $a \in \mathrm{Q}_0$, $S(a)$ is one-dimensional, hence it is a simple representation and so, thanks to the equivalence in Theorem 1.2.2, is a simple Λ-module. Thus, it must be isomorphic to one of the modules $\operatorname{top} \Lambda e_j$, which are simple by Proposition 1.1.3. We have that $\operatorname{Hom}_\Lambda(\Lambda e_a, S(a)) \neq 0$. Indeed, by Lemma 1.1.2, $\operatorname{Hom}_\Lambda(\Lambda e_a, S(a)) \simeq e_a S(a)$, and, by Theorem 1.2.2, $e_a S(a)$ is isomorphic to $S(a)_a$, which is different from zero. So we have at least one homomorphism different from zero between Λe_a and $S(a)$, which must be an epimorphism, since $S(a)$ is simple. This implies that $S(a) \simeq \operatorname{top} \Lambda e_a$.

If $a \neq b$ are two different vertices of Q_0, it is clear that $\operatorname{Hom}_\Lambda(S(a), S(b)) = 0$ and in particular $S(a) \simeq S(b)$. So, the simple modules $S(a)$, $a \in \mathrm{Q}_0$, are pairwise non-isomorphic. The last statement then follows, since we proved that each simple indecomposable Λ-module is of the type $\operatorname{top} \Lambda e_a$ for some $a \in \mathrm{Q}_0$. □

Example 1.2.6.



The simple module $S(2)$ of $\Lambda_A = \frac{\mathbb{K}\,\mathrm{Q}_A}{I_A}$. The simple module $S(4)$ of $\Lambda_B = \frac{\mathbb{K}\,\mathrm{Q}_B}{I_B}$.

The radical of a bound representation M of a quiver (Q, I) is described by the following lemma:

Lemma 1.2.4. . *Let* $M = (M_a, \phi_\alpha)$ *be a bound representation of* (Q, I)*. Then* $\operatorname{rad} M = (R_a, \rho_\alpha)$ *with*

$$R_a = \sum_{\alpha : b \to a} \operatorname{Im}(\phi_\alpha : M_b \to M_a)$$

and

$$\rho_\alpha = \phi_\alpha\big|_{R_a}$$

for every arrow α *of source* a*.*

Proof. Let Arr be the arrow ideal of $\mathbb{K}\,\mathrm{Q}$. For each α in Arr denote with $\overline{\alpha} \in \frac{\mathrm{Arr}}{I}$ its residual class modulo I. Then $\operatorname{rad} M = \operatorname{rad} \Lambda \cdot M = \frac{\mathrm{Arr}}{I} \cdot M$. So $\operatorname{rad} M$ is generated by the action of $\overline{\alpha}$ over M for each $\alpha \in \mathrm{Q}_1$, namely $\operatorname{rad} M = \sum_{\alpha \in \mathrm{Q}_1} \overline{\alpha} M$. Hence, by the construction showed in Theorem 1.2.2, $(\operatorname{rad} M)_a = e_a \operatorname{rad} M$, which is equal to $e_a \sum_{\alpha \in \mathrm{Q}_1} \overline{\alpha} M = \sum_{\alpha \in \mathrm{Q}_1 \mid t(\alpha) = a} \overline{\alpha} M$. Given an arrow $\alpha : b \to a$ of target a, by the same construction, we have that ϕ_α is given by the right multiplication by $\overline{\alpha}$, thus $\overline{\alpha}M = \overline{\alpha}e_bM = \overline{\alpha}M_b = \phi_\alpha(M_b) = \operatorname{Im}\phi_\alpha$. We can now conclude that

$$(\operatorname{rad} M)_a = \sum_{\alpha\in \mathrm{Q}_1 \ \mid t(\alpha)=a} \operatorname{Im}\phi_\alpha.$$

Since $\operatorname{rad} M$ is a submodule of M, the linear map ρ_α relative to the arrow $\alpha : a \to b$ is just equal to the restriction of the map of M, namely $\rho_\alpha = \phi_\alpha|_{(\operatorname{rad} M)_a}$ for each $\alpha \in \mathrm{Q}_1$. □

We now show how to compute the indecomposable projective Λ-modules. Since Λ is basic and $\{e_a \mid a \in \mathrm{Q}_0\}$ is a complete set of primitive orthogonal idempotents of Λ, the decomposition $\Lambda = \oplus_{a\in \mathrm{Q}_0}\Lambda e_a$ is a decomposition of Λ as a direct sum of pairwise non-isomorphic indecomposable projective Λ-modules. We wish to describe the modules $P(a) = \Lambda e_a$, with $a \in \mathrm{Q}_0$.

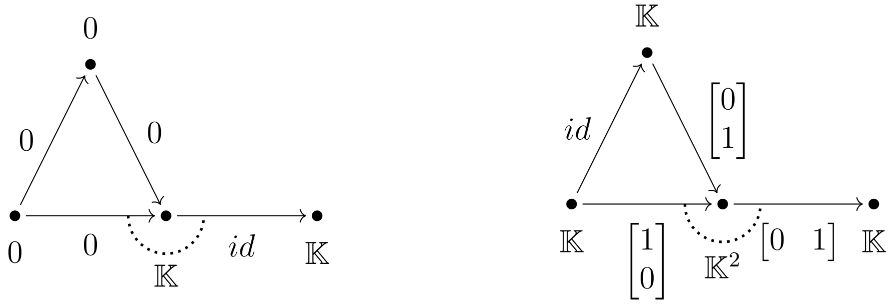
Lemma 1.2.5. *Let (Q, I) be a bound quiver, $\Lambda = \frac{\mathbb{K}\mathrm{Q}}{I}$, and $P(a) = \Lambda e_a$, where $a \in \mathrm{Q}_0$. If $P(a) = (P(a)_b, f_\alpha)_{b\in \mathrm{Q}_0,\alpha\in \mathrm{Q}_1}$, then $P(a)_b$ is the $\mathbb{K}$-vector space with basis the set of all the p path, not in a relation, from a to b. For an arrow $\alpha : b \to c$, the $\mathbb{K}$-linear map $f_\alpha : P(a)_b \to P(a)_c$ is given by the left multiplication by α.*

Proof. From the construction in Theorem 1.2.2, the representation corresponding to Λe_a is such that for each $b \in \mathrm{Q}_0$,

$$P(a)_b = e_bP(a) = e_b\Lambda e_a = e_b\frac{\mathbb{K}\,\mathrm{Q}}{I}e_a = \frac{e_b\mathbb{K}\,\mathrm{Q}\,e_a}{e_bIe_a}.$$

Note that $e_b\mathbb{K}\,\mathrm{Q}\,e_a$ corresponds exactly to the path algebra which has as vectors of a basis the paths from a to b. Hence, $\frac{e_b\mathbb{K}\,\mathrm{Q}\,e_a}{e_bIe_a}$ is generated by the paths $p : a \to b$ which are not in a relation. Moreover, if $\beta : b \to c$ is an arrow of Q, then $\phi_\beta : e_b\Lambda e_a = P(a)_b \to e_c\Lambda e_a = P(a)_c$ is such that if w is the residual class of a path w from a to b, then $f_\beta(w) = \beta w$, namely it corresponds to the left multiplication, as we were looking for. □

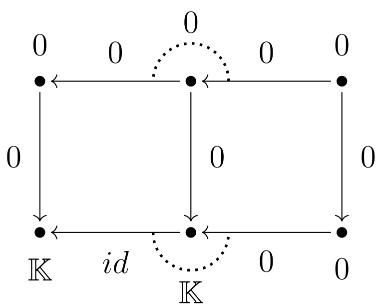
Example 1.2.7. In Figure 1.3, we present some examples of projective indecomposable modules over the path algebras defined in the previous examples.



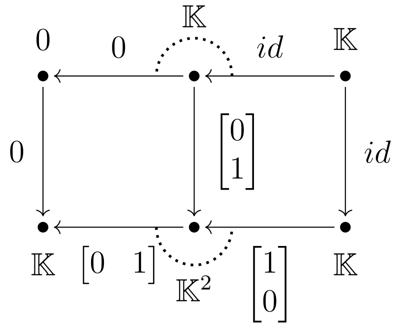
The indecomposable projective Λ_A-module $P(3)$.

The indecomposable projective Λ_A-module $P(1)$.

Figure 1.2



The indecomposable projective Λ_B-module $P(5)$



The indecomposable projective Λ_B-module $P(1)$

Figure 1.3

1.3 Standard functors

Let Λ be a finite-dimensional $\mathbb{K}$-algebra. We denote with Λ^{op} the opposite algebra of Λ, namely the K-algebra whose underlying set and vector space structure are just those of Λ, but the multiplication "$*$" in Λ^{op} is defined as $a * b = b \cdot a$, where "$\cdot$" is the multiplication of Λ, as defined in Section 1.1. The left module over Λ^{op} can be viewed as right Λ-module. Therefore, to indicate right Λ-module, we will also use Λ^{op}-module.

For instance, $(\frac{\mathbb{K}\,\mathrm{Q}}{I})^{\mathrm{op}} = \frac{\mathbb{K}\,\mathrm{Q}^{\mathrm{op}}}{I^{\mathrm{op}}}$, where Q^{op} is the quiver obtained from Q by reversing each arrow and I^{op} is the admissible ideal obtained from I by reversing all paths, i.e if $p = \alpha_1 \dots \alpha_l$ belongs to I, then $p^{\mathrm{op}} = \alpha_l \dots \alpha_1$ belongs to I^{op}.

Definition 1.3.1. We define the contravariant functor

$$D : \Lambda\text{-}\mathrm{mod} \to \Lambda^{\mathrm{op}}\text{-}\mathrm{mod},$$

by assigning, to each left module M in Λ-mod, its dual $\mathbb{K}$-vector space

$$D(M) = \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K}).$$

This is well-defined, indeed $D(M)$ is a right Λ-module, with the right action of Λ over $D(M)$ defined as $(\phi \cdot a)(m) = \phi(am)$, for any $\phi \in \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$, $m \in M$, $a \in \Lambda$. Moreover, given M, N in Λ-mod, we have that $\mathrm{Hom}_\Lambda(M, N) \simeq \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(DN, DM)$, i.e. this action is compatible with functoriality of $\mathrm{Hom}_{\mathbb{K}}(-, \mathbb{K})$ which means that it takes Λ-homomorphisms to Λ^{op}-homomorphisms.

Similarly, one can define the quasi-inverse of D which associates to a right Λ-module its dual $\mathbb{K}$-vector space $\mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$, which has now a left Λ-module structure. It is denoted with the same symbol of D: $D : \Lambda^{\mathrm{op}}\text{-}\mathrm{mod} \to \Lambda\text{-}\mathrm{mod}$, and one can prove that there exists a natural equivalence such that

$$1_{\Lambda\text{-}\mathrm{mod}} \simeq D \circ D \quad \text{and} \quad 1_{\Lambda^{\mathrm{op}}\text{-}\mathrm{mod}} \simeq D \circ D. \tag{1.1}$$

We will refer to the functor $D(-)$ as the **standard duality**. One can prove that every indecomposable injective right Λ-module is isomorphic to one of the modules

$$D(Ae_1), \dots, D(Ae_n).$$

Proposition 1.3.1. *Let Λ be a $\mathbb{K}$-algebra, e a primitive idempotent and M a left Λ-module. We have an isomorphism of right $e\Lambda e$-modules:*

$$D(eM) \simeq (DM)e$$

It is, moreover, functorial on M.

Proof. Let M be a left Λ-module and e a primitive idempotent. Then e gives a decomposition of M into $eM \oplus (1-e)M$ and so, by additivity of the Hom functor, $DM \simeq D(eM) \oplus D((1-e)M)$. Since DM is a right Λ-module, we also get that $DM \simeq DMe \oplus DM(1-e)$.

Now we have:

$$(DM)e = \{f \in DM \mid f(m) = 0 \text{ for all } m \in (1-e)M\}.$$

Indeed, if $f \in (DM)e$, by definition of the action of Λ over DM, there exists $f' \in DM$, such that $f(m) = f' \cdot e(m) = f'(em)$ for all $m \in M$. Thus if $x \in (1-e)M$, i.e. exists $m' \in M$ such that $x = (1-e)m'$, we get that $f(x) = f'(e(1-e)m') = f'(0) = 0$. If f in $\{f \in DM \mid f(m) = 0 \text{ for all } m \in (1-e)M\}$, then for each $m \in M$, we have that $f(m) = f(em) + f((1-e)m) = f(em)$, hence f could be written as $f = f' \cdot e$, with f' in DM, i.e. it belongs to $(DM)e$. This proves that, as sets, $D(eM)$ is the same as DMe.

This observation gives us the idea on how to construct an isomorphism ϕ of right $e\Lambda e$-modules between $D(eM)$ and $(DM)e$. We define ϕ such that, given $f \in (DM)e$, then $\phi(f) = f'$. This is, by what said above, well-defined, injective and surjective. It preserves moreover the $e\Lambda e$ action, indeed $\phi(f) \cdot eae(em) = f' \cdot eae(em) = f'(eaem)$. While $f \cdot eae = f' \cdot e \cdot eae = (f' \cdot eae) = (f' \cdot ea) \cdot e$, so $\phi(f \cdot eae)(em) = (f' \cdot ea)(em) = f'(eaem)$

The isomorphism ϕ induces also an equivalence of functors since it does not depend on the variable m, making, given a Λ-module homomorphism $f : M \to N$, the following commute

$$\begin{array}{ccc} (DN)e & \xrightarrow{\phi} & D(eN) \\ {\scriptstyle (Df)e}\downarrow & & \downarrow{\scriptstyle D(ef)} \\ (DM)e & \xrightarrow[\phi]{} & D(eM) \end{array} .$$

In fact, consider a map $h' : eN \to k$ in $D(eN)$ By the definitions of the functors $D(-)$, $e(-)$, and $(-)e$, for any $m \in M$, we have

$$D(ef)(h)(em) = h(ef(em)) = h(ef(m)) = h(f(em)),$$

where the last equality holds because f is a Λ-module homomorphism. Now consider $h = h' \cdot e \in (DN)e$, we get that

$$(Df)e(h)(m) = h'f(em),$$

where h' is the homomorphism in DN associated with h. Since $\phi(h) = h'$, we conclude that the map ϕ establishes a natural equivalence $D(eM) \simeq (DM)e$. $\square$

If M is a left Λ-module, then the vector space $\mathrm{Hom}_\Lambda(M,\Lambda)$ becomes a right Λ-module via $(f\cdot a)(m) = f(a\cdot m)$. So we can consider the dual functor

$$\mathrm{Hom}_\Lambda(-,\Lambda) : \Lambda\text{-}\mathrm{mod} \to \Lambda^{\mathrm{op}}\text{-}\mathrm{mod}\,.$$

We observe that given Λe a projective left Λ-module, with e a primitive idempotent, then, by Lemma 1.1.2, $\mathrm{Hom}_\Lambda(\Lambda e,\Lambda) \simeq e\Lambda$ is a projective right Λ-module. The additivity of the Hom functor, implies that $\mathrm{Hom}_\Lambda(-,\Lambda)$ induces a duality between the category $\mathrm{proj}\,\Lambda$ of projective left Λ-modules, and the category $\mathrm{proj}\,\Lambda^{\mathrm{op}}$ of projective right Λ-modules.

We can finally define the so called **Nakayama functor** as the composition of the standard duality $D(-)$ and $\mathrm{Hom}_\Lambda(-,\Lambda)$. Formally:

Definition 1.3.2. The Nakayama functor ν is defined to be the endofunctor

$$\nu = D\,\mathrm{Hom}_\Lambda(-,\Lambda) : \Lambda\text{-}\mathrm{mod} \to \Lambda\text{-}\mathrm{mod}\,.$$

Lemma 1.3.2. *Given P a projective left Λ-module, there is an equivalence of functors $\Lambda\text{-}\mathrm{mod} \to \Lambda\text{-}\mathrm{mod}$:*

$$D\,\mathrm{Hom}_\Lambda(-,\nu P) \simeq \mathrm{Hom}_\Lambda(P,-).$$

Proof. Start by considering a projective indecomposable left Λ-module $P = \Lambda e$, with $e \in \Lambda$ primitive idempotent. We proved in Lemma 1.1.2 that there exists a functorial isomorphism $\mathrm{Hom}_\Lambda(\Lambda e, M) \simeq eM$. Dually, we have that $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(e\Lambda, M) \simeq Me$. Moreover, thanks to the same isomorphism, we get $\nu\Lambda e = D\,\mathrm{Hom}_\Lambda(\Lambda e,\Lambda) = D(e\Lambda)$. Being $D(-)$ a duality functor, if we have M in Λ-mod and N in Λ^{op}-mod, we get $\mathrm{Hom}_\Lambda(M, DN) = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(N, DM)$. By using this and Proposition 1.3.1, we obtain that

$$\begin{aligned} eM &\simeq e(D(DM)) \simeq D((DM)e) \simeq D(\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(e\Lambda, DM)) \\ &\simeq D(\mathrm{Hom}_\Lambda(M, D(e\Lambda)) \simeq D(\mathrm{Hom}_\Lambda(M,\nu\Lambda e)), \end{aligned}$$

where each isomorphism is functorial. We conclude thanks to the additivity of the functors Hom and $D(-)$. □

This result also implies that $\mathrm{Hom}_\Lambda(-,\nu P) \simeq D\,\mathrm{Hom}_\Lambda(P,-)$.

1.4 Homotopy category of chain complexes

Let $\mathrm{proj}\,\Lambda$, be the full subcategory of Λ-mod, whose objects are the finite-dimensional projective modules. A **chain complex** of $\mathrm{proj}\,\Lambda$, also referred with just *complex*, consists of a sequence of finite-dimensional left projective Λ-modules and a sequence of Λ-module homomorphisms between consecutive modules such that the image of each homomorphism is included in the kernel of the next. Formally:

Definition 1.4.1. A chain complex $C^\bullet = (C^i, d^i)_{i\in\mathbb{Z}}$ is a sequence of left projective Λ-modules $C^i \in \mathrm{proj}\,\Lambda$ connected by Λ-homomorphisms $d^i : A^i \to A^{i+1}$ such that $d^{i+1}\circ d^i = 0$.

We often write a complex as

$$\cdots \longrightarrow C^i \xrightarrow{d^i} C^{i+1} \longrightarrow \cdots \longrightarrow C^{-1} \xrightarrow{d^{-1}} C^0 \xrightarrow{d^0} C^1 \longrightarrow \cdots$$

Note that we sometimes omit the bullet at the top if it does not cause more confusion and if it is clear that we are talking about complexes. The object C^i will be referred to as the term of C in degree i.

A **chain map** $f^\bullet$ between chain complexes $A^\bullet = (A^i, a^i)_{i\in\mathbb{Z}}$ and $B^\bullet = (B^i, b^i)_{i\in\mathbb{Z}}$ is a sequence of left Λ-module homomorphism $f^i : A^i \to B^i$ for each $i \in \mathbb{Z}$ such that $b^i \circ f^i = f^{i-1} \circ a^i$. Namely, in the diagram:

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & A^i & \xrightarrow{a^i} & A^{i+1} & \longrightarrow \cdots \longrightarrow & A^{-1} & \xrightarrow{a^{-1}} & A^0 & \xrightarrow{a^0} & A^1 & \longrightarrow \cdots \\
& \downarrow{\scriptstyle f^i} & & \downarrow{\scriptstyle f^{i+1}} & & \downarrow{\scriptstyle f^{-1}} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & \\
\cdots \longrightarrow & B^i & \xrightarrow{b^i} & B^{i+1} & \longrightarrow \cdots \longrightarrow & B^{-1} & \xrightarrow{b^{-1}} & B^0 & \xrightarrow{b^0} & B^1 & \longrightarrow \cdots
\end{array}$$

all square commutes.

The category of chain complexes $\mathcal{C}(\operatorname{proj}\Lambda)$ is the category whose objects are chain complexes and whose morphisms are chain map.

We introduce an equivalence relation between chain maps. Let $f^\bullet, g^\bullet$ be chain maps between chain complexes $A^\bullet = (A^i, a^i)_{i\in\mathbb{Z}}$ and $B^\bullet = (B^i, b^i)_{i\in\mathbb{Z}}$, f and g are **chain homotopic** (or simply homotopic), written $f \sim g$, if there exists a homotopy h, i.e. a sequence of Λ-homomorphism $h^i : A^i \to B^{i-1}$ such that $h^{i+1}a^i + b^{i-1}h^i = f^i - g^i$.

The **homotopy category of chain complexes** $\mathcal{K}(\operatorname{proj}\Lambda)$ is then defined as follows: its objects are the same as the objects of $\mathcal{C}(\operatorname{proj}\Lambda)$, namely, chain complexes. Its morphisms are "maps of complexes modulo homotopy", namely

$$\operatorname{Hom}_{\mathcal{K}(\operatorname{proj}\Lambda)}(A^\bullet, B^\bullet) = \frac{\operatorname{Hom}_{\mathcal{C}(\operatorname{proj}\Lambda)}(A, B)}{\sim}$$

We denote with $\mathcal{K}^b(\operatorname{proj}\Lambda)$ the full subcategory of $\mathcal{K}(\operatorname{proj}\Lambda)$ whose objects are the **bounded complexes**, namely $A^n = 0$ for $|n| >> 0$. Define $K^{[-1,0]}(\operatorname{proj}\Lambda)$ as the full subcategory of $\mathcal{K}^b(\operatorname{proj}\Lambda)$, consisting of chain complexes with at most two non-zero objects, which appear in degrees 0 and -1.

Given a left Λ-module M, its minimal projective resolution is defined as a complex

$$P^\bullet_M = \cdots \longrightarrow P^n \xrightarrow{d_n} P^{n-1} \longrightarrow \cdots \to P^1 \xrightarrow{d_1} P^0 \longrightarrow 0 \longrightarrow \ldots,$$

with an epimorphism $d_0 : P^0 \to M$. This complex is an object in $K(\operatorname{proj}\Lambda)$, with the convention that in degree $-i$ there is the term P^i. Note that, when we will refer to the projective resolution of a module, we will mean the complex, but the map d_0 is always implicitly understood. Equivalently, when discussing the minimal projective presentation $P^1 \to P^0$, we treat it as an object in $K^{[-1,0]}(\operatorname{proj}\Lambda)$, with the existence of the projective cover $d_0 : P^0 \to M$ implied.

This association turns out to be functorial; namely, there exists a well-defined functor $k : \Lambda\text{-}\operatorname{mod} \to K^b(\operatorname{proj}\Lambda)$, such that $k(M) = P^\bullet_M$. This came from the fact that, given two different projective resolutions of the same module, there exist a homotopy between them.

Due to this correspondence, the homotopy category serves as a powerful tool in the study of module theory over finite-dimensional algebras. Moreover, $K^{[-1,0]}(\operatorname{proj}\Lambda)$ is precisely the category where the two-term silting complexes, mentioned in the introduction, live. These are some of the reasons why, we have introduced it.

Throughout this thesis, we denote $K^b(\operatorname{proj}\Lambda)$ by $\mathcal{K}$, and $K^{[-1,0]}(\operatorname{proj}\Lambda)$ by $\mathcal{K}^{[-1,0]}$.

1.4.1 Triangulated structure and its properties

Contrary to what one might have initially thought, myself included, the homotopy category is not abelian. Instead, it has a triangulated structure, which generalizes the concept of exact sequences, a fundamental aspect of the theory discussed so far. In this section, we follow the notations and definitions from [Hap88]. This reference also provides a helpful background on the general theory of triangulated structures for anyone interested.

We define a triangle T in $\mathcal{K}$ as a sextuple of elements $(A, B, C, \alpha, \beta, \gamma)$, where A, B, C are in $\mathcal{K}$, $\alpha : A \to B$, $\beta : B \to C$ and $\gamma : C \to A[1]$. The **shift** $A[1] = (A[1]^i, d^i_{A[1]})$ of $A = (A^i, d^i_\Lambda)$ is the complex in $\mathcal{K}$ defined as $A[1]^i = A^{i+1}$ and $d^i_{A[1]} = -d^{i+1}_\Lambda$.

We will denote a triangle as

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1].$$

$\mathcal{K}$ is a **triangulated category**, namely the set of triangles satisfy the following axioms:

(TR1) Every sextuple isomorphic to a triangle is a triangle. Every morphism $f : A \to B$ in $\mathcal{K}$ can be embedded into a triangle $A \xrightarrow{f} B \to C \to A[1]$ and $A \xrightarrow{id} A \to 0 \to A[1]$ is a triangle.

(TR2) If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ is a triangle, then the two rotated triangle $B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1] \xrightarrow{-\alpha[1]} B[1]$ and $C[-1] \xrightarrow{-\gamma[1]} A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ are also triangles.

(TR3) Given two triangles $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$ and $A' \xrightarrow{\alpha'} B' \xrightarrow{\beta'} C' \xrightarrow{\gamma'} A'[1]$, and morphisms $f : A \to A'$, $g : B \to B'$, such that the following commutes

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A' & \xrightarrow{\alpha'} & B'. \end{array}$$

Then it exists $h : C \to C'$, making (f, g, h) a morphism between triangles, that is the following commute:

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & A[1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f[1]} \\ A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \xrightarrow{\gamma'} & A'[1]. \end{array}$$

(TR4) Given triangles:

$$A \xrightarrow{u} B \xrightarrow{j} C' \xrightarrow{k} A[1],$$
$$B \xrightarrow{v} C \xrightarrow{l} A' \xrightarrow{i} B[1],$$
$$A \xrightarrow{vu} C \xrightarrow{m} B' \xrightarrow{n} A[1],$$

there exists a triangle $C' \xrightarrow{f} B' \xrightarrow{g} A' \xrightarrow{h} C'[1]$ such that $l = gm,\ k = nf,\ h = j[1]i,\ i[-1]g[-1] = un[-1],\ fj = mv$. Namely the following diagram has commuting squares and the third row is a triangle:

$$\begin{array}{ccccccccc}
B'[-1] & \xrightarrow{n[-1]} & A & \xrightarrow{id} & A & & & & \\
\downarrow{\scriptstyle g[-1]} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle vu} & & & & \\
A'[-1] & \xrightarrow{i[-1]} & B & \xrightarrow{v} & C & \xrightarrow{l} & A' & \xrightarrow{i} & B[1] \\
 & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle j[1]} \\
 & & C' & \xrightarrow{f} & B' & \xrightarrow{g} & A' & \xrightarrow{h} & C'[1] \\
 & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle n} & & & & \\
 & & A[1] & \xrightarrow{id} & A[1]. & & & &
\end{array}$$

In $\mathcal{K}$, we can embed any morphism between complexes $A = (A^i, d^i_\Lambda),\ B = (B^i, d^i_B)$ $f : A \to B$ into a triangle, thanks to the following construction: define the complex $C_f := (A^{i+1} \oplus B^i, d^i_{C_f})$ in $\mathcal{K}$, with differentials

$$d^i_{C_f} = \begin{bmatrix} -d^{i+1}_A & f^{i+1} \\ 0 & d^i_B \end{bmatrix} : A^{i+1} \oplus B^i \to A^{i+2} \oplus B^{i+1}.$$

C_f is called the **mapping cone** of f. Then $A \xrightarrow{f} B \xrightarrow{\epsilon_B} C_f \xrightarrow{\pi_{A[1]}} A[1]$ is a triangle, where ϵ_B is the natural injection and $\pi_{A[1]}$ is the natural projection.

Not only is $\mathcal{K}$ a triangulated category, but also it satisfies the **Krull-Schmidt** theorem, namely every object decomposes into a finite direct sum of objects having local endomorphism rings, which are called **indecomposable**. This result comes from a more general one, combination of Theorem 6.1 of [Sha23] and Theorem 3.4 of [Sch11].

We now present and demonstrate some implications of the axioms of triangulated categories.

Proposition 1.4.1. *Let $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ be a triangle. For any Q in $\mathcal{K}$, there are two long exact sequences:*

$$\cdots \longrightarrow \mathrm{Hom}(Q, A[n]) \xrightarrow{\mathrm{Hom}(Q,u[n])} \mathrm{Hom}(Q, B[n]) \xrightarrow{\mathrm{Hom}(Q,v[n])} \mathrm{Hom}(Q, C[n]) \xrightarrow{\mathrm{Hom}(Q,w[n])} \mathrm{Hom}(Q, A[n+1]) \longrightarrow \cdots,$$

$$\cdots \longrightarrow \mathrm{Hom}(C[n], Q) \xrightarrow{\mathrm{Hom}(v[n],Q)} \mathrm{Hom}(B[n], Q) \xrightarrow{\mathrm{Hom}(u[n],Q)} \mathrm{Hom}(A[n], Q) \xrightarrow{\mathrm{Hom}(w[n],Q)} \mathrm{Hom}(C[n+1], Q) \longrightarrow \cdots.$$

Proof. Fix an integer n and take f in $\mathrm{Hom}_{\mathcal{K}}(Q, A[n])$. The commutative diagram,

$$\begin{array}{ccccccc} Q & \xrightarrow{id} & Q & \xrightarrow{0} & 0 & \xrightarrow{0} & Q[1] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle u[n]f} & & & & \\ A[n] & \xrightarrow{u[n]} & B[n] & \xrightarrow{v[n]} & C[n] & \xrightarrow{w[n]} & A[n+1], \end{array}$$

can be completed into a morphism between triangles, thanks to the third axiom (TR3). The construction is shown below:

$$\begin{array}{ccccccc} Q & \xrightarrow{id} & Q & \xrightarrow{0} & 0 & \xrightarrow{0} & Q[1] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle u[n]f} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f} \\ A[n] & \xrightarrow{u[n]} & B[n] & \xrightarrow{v[n]} & C[n] & \xrightarrow{w[n]} & A[n+1]. \end{array}$$

This implies that $vuf = 0$, namely $0 = \mathrm{Hom}(Q, v)((\mathrm{Hom}(Q, u)(f)))$ for any $f \in \mathrm{Hom}(Q, A[n])$, hence the image of $\mathrm{Hom}(Q, u)$ is contained in the kernel of $\mathrm{Hom}(Q, v)$. To prove the other inclusion, we follow the same reasoning. Let g be in the kernel of $\mathrm{Hom}(Q, v)$, i.e. $vg = 0$. This implies that the following is a commutative square:

$$\begin{array}{ccccccc} Q & \xrightarrow{id} & Q & \xrightarrow{0} & 0 & \xrightarrow{0} & Q[1] \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 0} & & \\ A[n] & \xrightarrow{u[n]} & B[n] & \xrightarrow{v[n]} & C[n] & \xrightarrow{w[n]} & A[n+1] \end{array}$$

and so, by the third axiom (TR3) combined with the second (TR2), the diagram can be completed to a morphism between triangles:

$$\begin{array}{ccccccc} Q & \xrightarrow{id} & Q & \xrightarrow{0} & 0 & \xrightarrow{0} & Q[1] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f[1]} \\ A[n] & \xrightarrow{u[n]} & B[n] & \xrightarrow{v[n]} & C[n] & \xrightarrow{w[n]} & A[n+1]. \end{array}$$

This implies that $g = uf$ for some f in $\mathrm{Hom}(Q, A[n])$, i.e. g belongs to the image of $\mathrm{Hom}(Q, u)$. This concludes the proof of the exactness in $\mathrm{Hom}(Q, B[n])$. The exactness in the other degree of the sequence is given by the second axiom (TR2).

The exactness of the second long sequence can be proved similarly. □

Proposition 1.4.2.

(1) Let

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[1] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & A'[1] \end{array}$$

be a morphism of triangles. If f and g are isomorphism, then the same is true for h.

(2) If one of the vertices of a triangle is 0*, then the map between the other vertices is an isomorphism.*

(3) Given $A \to B \to C \to A[1]$ *and* $A' \to B' \to C' \to A'[1]$ *two triangles, then* $A \oplus A' \to B \oplus B' \to C \oplus C' \to A[1] \oplus A'[1]$ *is a triangle.*

(4) One of the side of a triangle is the 0*-morphism,* $A \xrightarrow{0} B$*, if and only if the third vertex is isomorphic to* $B \oplus A[1]$*.*

Proof. (1) Thanks to Proposition 1.4.1, there are long exact sequences,

$$\begin{array}{ccccccccc}
\mathrm{Hom}(Q,A) & \xrightarrow{u\circ} & \mathrm{Hom}(Q,B) & \xrightarrow{v\circ} & \mathrm{Hom}(Q,C) & \xrightarrow{w\circ} & \mathrm{Hom}(Q,A[1]) & \xrightarrow{u[1]\circ} & \mathrm{Hom}(Q,B[1]) \\
\downarrow{\scriptstyle f\circ} & & \downarrow{\scriptstyle g\circ} & & \downarrow{\scriptstyle h\circ} & & \downarrow{\scriptstyle f[1]\circ} & & \downarrow{\scriptstyle g[1]\circ} \\
\mathrm{Hom}(Q,A') & \xrightarrow{u'\circ} & \mathrm{Hom}(Q,B') & \xrightarrow{v'\circ} & \mathrm{Hom}(Q,C') & \xrightarrow{w'\circ} & \mathrm{Hom}(Q,A'[1]) & \xrightarrow{u'[1]\circ} & \mathrm{Hom}(Q,B'[1]),
\end{array}$$

where the morphisms between the sequences are induced by the ones between the triangles. We had denoted $f\circ = \mathrm{Hom}(Q,f)$, we will use this notation when the object Q we refer to is obvious and can be omitted.

Since f and g are isomorphisms, also $f[i]\circ$ and $g[i]\circ$ are isomorphisms for any $i \geq 0$. Thus, due to the Five lemma, $\mathrm{Hom}(Q,h)$ is an isomorphism for each Q in $\mathcal{K}$. In particular, if we choose $Q = C$ and consider $id_{C'}$ in $\mathrm{Hom}(C',C')$, exists $h' \in \mathrm{Hom}(C',C)$ such that $hh' = id_{C'}$, namely, we found a right inverse. The left inverse can be found by looking at the sequence $\mathrm{Hom}(-,C')$. The same reasoning applies.

(2) Thanks to the second axiom (TR2), we can assume, without loss of generality, that we have a triangle of the type $A \xrightarrow{f} B \to 0 \to A[1]$. We have obvious commutative squares of the type

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle id} & & & & \\
0 & \longrightarrow & A & \xrightarrow{id} & A & \longrightarrow & 0
\end{array}
\quad \text{and} \quad
\begin{array}{ccccccc}
0 & \longrightarrow & B & \xrightarrow{id} & B & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle id} & & \downarrow \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & 0.
\end{array}$$

Thanks to the second and third axioms (TR2), (TR3), there exist h and g, which complete the following commutative diagram:

$$\begin{array}{ccccccc}
B & \xrightarrow{id} & B & \longrightarrow & 0 & \longrightarrow & B[1] \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow \\
A & \xrightarrow{f} & B & \longrightarrow & 0 & \longrightarrow & A[1] \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle h} & & \downarrow & & \downarrow \\
A & \xrightarrow{id} & A & \longrightarrow & 0 & \longrightarrow & A[1].
\end{array}$$

So $fg = id_B$ and $hf = id_A$, namely g is the right inverse, while h is the left one.

(3) Let $A \xrightarrow{f} B \to C \to A[1]$ and $A' \xrightarrow{f'} B' \to C' \to A'[1]$ be two triangles. The map

$$\begin{bmatrix} f & 0 \\ 0 & f' \end{bmatrix} : A \oplus A' \to B \oplus B'$$

can be fitted into a triangle

$$A \oplus A' \to B \oplus B' \to Q \to (A \oplus A')[1] \tag{1.2}$$

by the first axiom (TR1). Note that $(A \oplus A')[1] = A[1] \oplus A'[1]$. Consider the natural projections $\pi_A, \pi_{A'}, \pi_B, \pi_{B'}$. We can complete the commutative squares

$$\begin{CD} A \oplus A' @>>> B \oplus B' @>>> Q @>>> (A \oplus A')[1] \\ @V{\pi_A}VV @V{\pi_B}VV @. @. \\ A @>>> B @>>> C @>>> A[1] \end{CD} \quad \text{and} \quad \begin{CD} A \oplus A' @>>> B \oplus B' @>>> Q @>>> (A \oplus A)[1] \\ @V{\pi_{A'}}VV @V{\pi_{B'}}VV @. @. \\ A' @>>> B' @>>> C' @>>> A'[1] \end{CD}$$

into morphisms of triangles:

$$\begin{CD} A \oplus A' @>>> B \oplus B' @>>> Q @>>> (A \oplus A')[1] \\ @V{\pi_A}VV @V{\pi_B}VV @VV{\phi}V @VVV \\ A @>>> B @>>> C @>>> A[1] \end{CD} \quad \text{and} \quad \begin{CD} A \oplus A' @>>> B \oplus B' @>>> Q @>>> (A \oplus A)[1] \\ @V{\pi_{A'}}VV @V{\pi_{B'}}VV @VV{\phi'}V @VVV \\ A' @>>> B' @>>> C' @>>> A'[1]. \end{CD}$$

The induced map ϕ and ϕ' give rise to a morphism $\phi'' : Q \to C \oplus C''$, thanks to the universal property of the direct sum, such that the following is a commutative diagram:

$$\begin{CD} A \oplus A' @>>> B \oplus B' @>>> Q @>>> (A \oplus A')[1] \\ @V{id}VV @V{id}VV @V{\phi''}VV @V{id}VV \\ A \oplus A' @>>> B \oplus B' @>>> C \oplus C' @>>> A[1] \oplus A'[1]. \end{CD} \tag{1.3}$$

Following quite the same reasoning as in (1), we prove that ϕ'' must be an isomorphism. Observe that we can not use (1) directly, since we do not have two triangles, as in the hypothesis of (1). We have, by Proposition 1.4.1, two long exact sequences for any Q in $\mathcal{K}$:

$$\mathrm{Hom}(Q, A) \longrightarrow \mathrm{Hom}(Q, B) \longrightarrow \mathrm{Hom}(Q, C) \longrightarrow \mathrm{Hom}(Q, A[1]) \longrightarrow \mathrm{Hom}(Q, B[1]),$$

$$\mathrm{Hom}(Q, A') \longrightarrow \mathrm{Hom}(Q, B') \longrightarrow \mathrm{Hom}(Q, C') \longrightarrow \mathrm{Hom}(Q, A'[1]) \longrightarrow \mathrm{Hom}(Q, B'[1]),$$

and due to the additivity of the functor Hom, we get the long exact sequence:

$$\mathrm{Hom}(Q, A \oplus A') \longrightarrow \mathrm{Hom}(Q, B \oplus B') \longrightarrow \mathrm{Hom}(Q, C \oplus C') \longrightarrow$$

$$\longrightarrow \mathrm{Hom}(Q, A[1] \oplus A'[1]) \longrightarrow \mathrm{Hom}(Q, B[1] \oplus B'[1]).$$

The commutative diagram 1.3 gives rise to morphisms between this latter long exact sequence and the one arising from the triangle 1.2. Then, by using the Five Lemma, we obtain that ϕ'' must be an isomorphism. Hence, $A \oplus A' \to B \oplus B' \to C \oplus C' \to A[1] \oplus A'[1]$ is a triangle since it is isomorphic to one.

(4) Let $A \xrightarrow{0} B \xrightarrow{u} C \xrightarrow{v} A[1]$ be a triangle associated to the 0-morphism. Consider the trivial triangles $A \to 0 \to A[1] \to A[1]$ and $0 \to B \to B \to 0$. Using (3), we get that

$$A \xrightarrow{0} B \to B \oplus A[1] \to A[1]$$

is a triangle.

Then we have a commutative square,

$$\begin{array}{ccccccc} A & \xrightarrow{0} & B & \xrightarrow{u} & C & \xrightarrow{w} & A[1] \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & & & \\ A & \xrightarrow{0} & B & \longrightarrow & B \oplus A[1] & \longrightarrow & A[1], \end{array}$$

which can be completed in a morphism of triangles by the third axiom (TR3):

$$\begin{array}{ccccccc} A & \xrightarrow{0} & B & \xrightarrow{u} & C & \xrightarrow{w} & A[1] \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle id} \\ A & \xrightarrow{0} & B & \longrightarrow & B \oplus A[1] & \longrightarrow & A[1]. \end{array}$$

Since the first two map are isomorphisms, then also ϕ is an isomorphism by (1).

□

1.5 Minimal right approximations

During the first section, we encountered the concept of minimal projective covers. In some sense, projective covers constitute the best way to approximate a left Λ-module as a projective one. Now, we will try to generalize this idea.

Let $P = (P_1 \xrightarrow{d_P} P_0)$ be in $\mathcal{K}^{[-1,0]}$ and call $\operatorname{add} P$ the full subcategory of $\mathcal{K}^{[-1,0]}$ whose object are direct summands of direct sum of P, namely it is the smallest full subcategory of $\mathcal{K}$ closed under isomorphisms, direct sums and direct summands containing P. Then $\operatorname{add} P$ is **covariantly finite**, i.e. for any $N = (N_1 \xrightarrow{d_N} N_0)$ in $\mathcal{K}^{[-1,0]}$, there exists a right $\operatorname{add} P$-approximation of N, namely there exists a complex morphism $f : P' \to N$ where P' is in $\operatorname{add} P$, such that the map $\operatorname{Hom}_{\mathcal{K}^{[-1,0]}}(Q, f) : \operatorname{Hom}_{\mathcal{K}^{[-1,0]}}(Q, P') \to \operatorname{Hom}_{\mathcal{K}^{[-1,0]}}(Q, N)$ is surjective for any Q in $\operatorname{add} P$.

Consider $\{f_i = (f_i^1, f_i^0)\}_{i=1}^n$ a K-basis of the K-vector space $\operatorname{Hom}_{\mathcal{K}^{[-1,0]}}(P, N)$, so, for any i, $f_i^0 \circ d_P = d_N \circ f_i^1$. In order to prove that $\operatorname{add} P$ is covariantly finite, we will show that the map

$$f = \begin{bmatrix} f_1 & f_2 & \dots & f_n \end{bmatrix} : P^n \to N$$

is a right $\operatorname{add} P$-approximation of N.

Let g in $\mathrm{Hom}_{\mathcal{K}^{[-1,0]}}(Q, N)$. Since Q is in add P, there exists Q' in $\mathcal{K}$ such that $Q \oplus Q' = P^m$ for some m natural number m. Call π_q the natural projection $P^m \to Q$, and ϵ_q the natural injection $Q \to P^m$.

Then $g \circ \pi_q$ is in $\mathrm{Hom}_{\mathcal{K}^{[-1,0]}}(P^m, N)$ and $g \circ \pi_q = (g \circ \pi_q)_{i=1}^m$, such that, for every i, $(g \circ \pi_q)_i$ is in $\mathrm{Hom}_{\mathcal{K}^{[-1,0]}}(P, N)$. Then $(g \circ \pi_q)_i = \sum_{j=1}^n \lambda_j^i f_j$. So for every $p = (p_i^1, p_i^0)_{i=1}^m$ in P^m, $g \circ \pi_q(p) = \sum_{i=1}^m \sum_{j=1}^n \lambda_j^i f_j(p_i^1, p_i^0) = \sum_{i=1}^m \sum_{j=1}^n \lambda_j^i (f_j^1(p_i^1), f_j^0(p_i^0))) = \sum_{i=1}^m \sum_{j=1}^n (f_j^1(\lambda_j^i p_i^1), f_j^0(\lambda_j^i p_i^0)) = \sum_{j=1}^n (f_j^1(\sum_{i=1}^m \lambda_j^i p_i^1), f_j^0(\sum_{i=1}^m \lambda_j^i p_i^0)) = f(L \cdot p^1, L \cdot p^0) = f(L \cdot p)$, where

$$L = \begin{bmatrix} \lambda_1^1 & \dots & \lambda_1^m \\ \\ \lambda_n^1 & \dots & \lambda_n^m \end{bmatrix}$$

.

Then $g \circ \pi_q \circ \epsilon_q(q) = g(q) = f(L \cdot \epsilon_q(q))$, where $L \cdot \epsilon_q$ is in $\mathrm{Hom}_{\mathcal{K}^{[-1,0]}}(Q, P^n)$. This is the module homomorphism, preimage of g through $\mathrm{Hom}_{\mathcal{K}^{[-1,0]}}(Q, f)$, that we were looking for.

Note that the same reasoning could be repeated if P was in $\mathcal{K}$ or in Λ-mod, but also if we consider subcategories closed under isomorphisms, direct summands and direct sums.

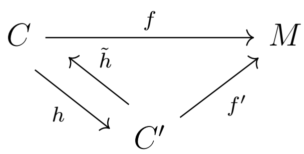
We call a morphism $f \in \mathrm{Hom}_{\mathcal{K}}(A, B)$ **right minimal**, if for any $g \in \mathrm{End}_{\mathcal{K}}(A)$ such that $fg = f$, then g is an isomorphism.

In [KS98], as an application of the dual statement of Proposition 1.2, the authors prove that any morphism in a Krull-Schmidt category has a minimal version and that, given a morphism $f : A \to B$ in this category, there exists a decomposition $f = (f', f'') : A = A' \oplus A'' \to B$ such that f' is right minimal and $f'' = 0$.

So not only does there always exist and add P-approximation, but also a minimal one.

Note that, as introduced at the start of this section, projective covers are minimal right approximations from the category Λ-Mod of the full subcategory proj Λ.

Let $\mathcal{C}$ be a subcategories of $\mathcal{K}$ closed under isomorphisms, direct summands and direct sums, then minimal right $\mathcal{C}$-approximation are unique up to isomorphism. Indeed let $f : C \to M, f' : C' \to M$ be two minimal right $\mathcal{C}$-approximations of M in $\mathcal{K}$. Then, since both are right $\mathcal{C}$-approximations, we have that there exists $h : C \to C'$, $\tilde{h} : C' \to C$, making the following diagram commutative:



.

Then $f\tilde{h}h = f'h = f$, so since f is minimal, $\tilde{h}h = id_C$, and, equivalently, $h\tilde{h} = id_{C'}$. Namely, h is the inverse of $\tilde{h}$. So h is an isomorphism.

We denote with $A * B$ the collection of objects C of $\mathcal{K}$ such that exists a triangle of the type $A \to C \to B \to A[1]$. Observe that, due to the octahedral axiom (TR4), $A*(B*C) = (A*B)*C$. Indeed, let X in $(A*B)*C$, then we have triangle $A \xrightarrow{a} Y \to B \to A[1]$ and $Y \xrightarrow{b} X \to C \to Y[1]$. We can then create a triangle $A \xrightarrow{ba} X \to C_{ba} \to A[1]$, by putting

C_{ba} as the cone of the composition ba. Then we have a triangle $B \to C_{ba} \to C \to B[1]$, i.e. X also belongs to $A * (B * C)$. To prove vice versa, one follows the same reasoning.

In general this collection is not closed under direct summands, but we have the following sufficient condition:

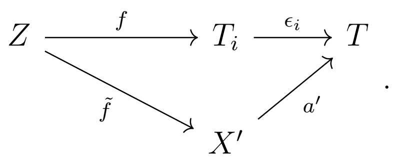
Lemma 1.5.1 ([IY08]). *Given two subcategories of $\mathcal{K}$, $\mathcal{X}$, $\mathcal{Y}$ closed under summands, directs sums and isomorphisms, such that* $\mathrm{Hom}_{\mathcal{K}}(X, Y) = 0$ *for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, then $\mathcal{X} * \mathcal{Y}$ is closed under summands.*

Proof. Let $X \xrightarrow{a} T \xrightarrow{b} Y \xrightarrow{c} X[1]$ be a triangle, with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, and let $T = T_1 \oplus T_2$ be a complex in $\mathcal{K}$. Then a is a right $\mathcal{X}$-approximation, i.e. $\mathrm{Hom}_{\mathcal{K}}(Z, a) : \mathrm{Hom}_{\mathcal{K}}(Z, X) \to \mathrm{Hom}_{\mathcal{K}}(Z, T)$ is an epimorphism for each Z in $\mathcal{X}$. Indeed, thanks to Proposition 1.4.1, we get the following long exact sequence,

$$\mathrm{Hom}_{\mathcal{K}}(Z, X) \xrightarrow{\mathrm{Hom}_{\mathcal{K}}(Z,a)} \mathrm{Hom}_{\mathcal{K}}(Z, T) \to \mathrm{Hom}_{\mathcal{K}}(Z, Y),$$

and the last item of the sequence is zero if Z belongs to $\mathcal{X}$, by hypothesis.

Let a' be the minimal right $\mathcal{X}$-approximation induced by a, namely $a = (a' \ 0) : X = X' \oplus X'' \to T$. Consider the natural projection $\pi_i : T \to T_i$ for $i = 1, 2$, then $\pi_i a' : X' \to T_i$ is a right $\mathcal{X}$- approximation. Indeed, let $f \in \mathrm{Hom}_{\mathcal{K}}(Z, T_i)$, we get that $\epsilon_i f$, with $\epsilon_i : T_i \to T$ the natural inclusion, belongs to $\mathrm{Hom}_{\mathcal{K}}(Z, T)$. Thus exists $\tilde{f}$ making the following diagram commute:



So $a'\tilde{f} = \epsilon_i f$, and $\pi_i a' \tilde{f} = \pi_i \epsilon_i f = f$. Hence, $\pi_i a'$ is a right $\mathcal{X}$-approximation for $i = 1, 2$.

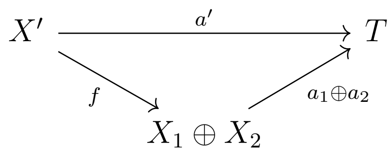
Now we can decompose $\pi_i a'$ into $(a_i \ 0) : X' = X_i \oplus X_i' \to T_i$, such that $a_i : X_i \to T_i$ is a minimal right $\mathcal{X}$-approximation.

The map

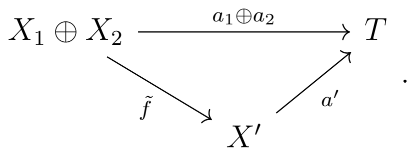
$$a_1 \oplus a_2 = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} : X_1 \oplus X_2 \to T_1 \oplus T_2$$

is still a minimal right $\mathcal{X}$-approximation, thanks to the additivity of the functor Hom. Indeed, let Z in $\mathcal{X}$, $\mathrm{Hom}(Z, X_1) \oplus \mathrm{Hom}(Z, X_2) = \mathrm{Hom}(Z, X_1 \oplus X_2)$ and $\mathrm{Hom}(Z, T_1) \oplus \mathrm{Hom}(Z, T_2) = \mathrm{Hom}(Z, T_1 \oplus T_2)$, so if $\mathrm{Hom}(Z, a_i)$ is an epimorphism between $\mathrm{Hom}(Z, X_i) \to \mathrm{Hom}(Z, T_i)$, then $\mathrm{Hom}(Z, a_1 \oplus a_2) = \mathrm{Hom}(Z, a_1) \oplus \mathrm{Hom}(Z, a_2)$ is an epimorphism between $\mathrm{Hom}(Z, X_1 \oplus X_2)$ and $\mathrm{Hom}(Z, T_1 \oplus T_2)$.

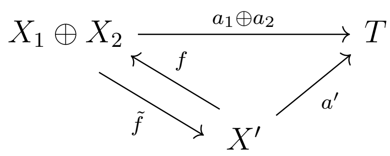
We now show that a' is isomorphic to $\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ and so $X' \simeq X_1 \oplus X_2$. Since they are both right $\mathcal{X}$-approximation, there exist f and $\tilde{f}$ making the following diagrams commute:



and



The commutativity of these diagrams implies that $a'\tilde{f}f = (a_1 \oplus a_2)f = a'$.



Since a' is minimal, $\tilde{f}f \simeq id_{X'}$. Equivalently $f\tilde{f} \simeq id_{X_1 \oplus X_2}$.

Call X_3 the minimal decomposition of a, then $X \simeq X_1 \oplus X_2 \oplus X_3$. We have triangles:

$$X_1 \xrightarrow{a_1} T_1 \to C_1 \to X_1[1],$$
$$X_2 \xrightarrow{a_2} T_2 \to C_2 \to X_2[1],$$
$$X_3 \to 0 \to X_3[1] \xrightarrow{id} X_3[1],$$

where C_i is the mapping cone of a_i, for $i = 1, 2$.

Using Proposition 1.4.2, the direct sum of these triangles is a triangle and is isomorphic to $X \xrightarrow{a} T \xrightarrow{b} Y \xrightarrow{c} X[1]$. So $Y \simeq C_1 \oplus C_2 \oplus X_3[1]$ and thus T_i belongs to $\mathcal{X} * \mathcal{Y}$.

□

Chapter 2

String and gentle algebras

A gentle algebra is a path algebra arising from a quiver Q, as defined in Chapter 1, together with a set of relations so that the algebra has particularly nice properties. Gentle algebras and string algebras are particularly interesting in representation theory because, thanks to how the set of relations is chosen, they have a well-understood module category; indeed, the representations of gentle algebras can often be described in terms of combinatorial data associated with the quiver and its relations. We mostly follow the notation of [BR87], the only difference being that we are not dealing with algebras arising from the opposite quiver as the author does, but from the original one.

The first sections introduces the notions of string algebras and outlines the initial properties of their modules. The final section is, instead, dedicated to detailing all the steps necessary to achieve a complete description of the construction of the projective presentation of a string module.

2.1 First definitions

Definition 2.1.1. A **string quiver** $\mathrm{Q} := (\mathrm{Q}, I)$ is a bound quiver such that:

(S1) each vertex $v \in \mathrm{Q}_0$ has at most two incoming and two outgoing arrows,

(S2) for any arrow $\beta \in \mathrm{Q}_1$, there is at most one arrow $\alpha \in \mathrm{Q}_1$ such that $t(\alpha) = s(\beta)$ and $\alpha\beta \notin I$, and there is at most one arrow $\gamma \in \mathrm{Q}_1$ such that $t(\beta) = s(\gamma)$ and $\beta\gamma \notin I$,

(S3) for any arrow $\beta \in \mathrm{Q}_1$, there is some bound $n(\beta)$ such that any path $\alpha_1, \cdots, \alpha_{n(\beta)}$ with $\alpha_1 = \beta$ contains a subpath in I and there is some bound $n'(\beta)$ such that any path $\alpha_1, \cdots, \alpha_{n'(\beta)}$ with $\alpha_{n'(\beta)} = \beta$ contains a subpath in I.

The algebra $\Lambda = \frac{\mathbb{K}\mathrm{Q}}{I}$ associated to a string bound quiver is called a **string algebra.**

Example 2.1.1. The path algebras arising from the bound quivers defined in 1.2.3 are string algebras.

By what was said in Theorem 1.2.1, Λ is a basic and connected finite-dimensional algebra, having $\{\overline{e_a} \mid a \in Q_0\}$ as a complete set of pairwise orthogonal primitive idempotents. where $\overline{e_a}$ is the residual class of the trivial path e_a in $\mathbb{K}Q$ modulo I. If the context is clear, we will omit the line over e_a and just refer to the trivial path as e_a.

Given an arrow $\beta \in Q_1$, we define its **formal inverse** as, β^{-1}, such that $s(\beta^{-1}) = t(\beta)$ and $t(\beta^{-1}) = s(\beta)$. Then $\beta^{-1^{-1}} = \beta$, thus $(-)^{-1}$ is an involution between the set of arrows and the set of formal inverses of arrows of Q. We extend this involution to the set of all paths by further setting $e_a^{-1} = e_a$ for all vertices $a \in Q_0$. Moreover $\alpha^{-1}\alpha = e_{t(\alpha)}$ and $\alpha\alpha^{-1} = e_{s(\alpha)}$.

Using the convention of concatenating paths from right to left, a **string** of length $l \geq 1$ is a sequence of arrows $\omega = \alpha_1^{\epsilon_1}, \cdots, \alpha_l^{\epsilon_l}$, such that

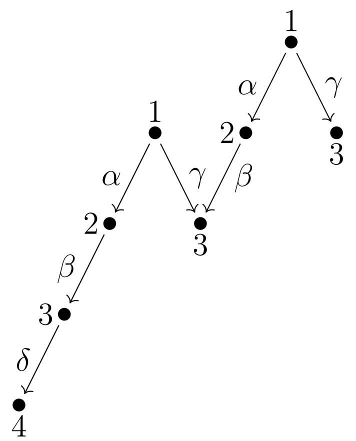
- α_i belong to Q_1 and $\epsilon_i \in \{1, -1\}$, for each $1 \leq i \leq l$;
- $\alpha_i^{\epsilon_i} \neq (\alpha_{i+1})^{-\epsilon_{i+1}}$, for each $1 \leq i < l$;
- $s(\alpha_{i+1}^{\epsilon_{i+1}}) = t(\alpha_i^{\epsilon_i})$ for each $1 \leq i \leq l-1$;
- there is no subpath $\alpha_i^{\epsilon_i}\alpha_{i+1}^{\epsilon_{i+1}} \ldots \alpha_{i+k}^{\epsilon_{i+k}}$ such that neither it nor its inverse $\alpha_{i+k}^{-\epsilon_{i+k}} \ldots \alpha_i^{-\epsilon_i}$ belongs to I;
- ω is reduced, in the sense that no factor $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$ appears for any $\alpha \in Q_1$.

Note also that we could have, by convention, strings of length 0. Indeed, for each vertex a in Q_0 we have an arrow of length zero e_a and so we include the trivial strings $\omega = e_a$ and $\omega = e_a^{-1}$. Moreover, the string could also have infinite length.

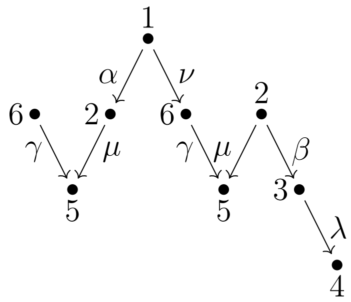
The following notation will be useful for dealing with strings. For a given string $\omega = \alpha_1^{\epsilon_1}, \cdots, \alpha_l^{\epsilon_l}$, we draw ω as follows:

- draw all arrows $\alpha_1^{\epsilon_1}, \cdots, \alpha_l^{\epsilon_l}$ from left to right,
- draw all arrows pointing downwards.

Example 2.1.2. Consider the string bound quiver (Q_A, I_A), an example of a string of length 7 is $\omega_A = \delta\beta\alpha\gamma^{-1}\beta\alpha\gamma^{-1}$, which is visualized as:



If, instead, we consider the string bound quiver (Q_B, I_B), $\omega_B = \gamma^{-1}\mu\alpha\nu^{-1}\gamma^{-1}\mu\beta^{-1}\lambda^{-1}$ is a string of length 8, visualized as:



The strings e_a of length zero are depicted simply as one vertex. Be aware that even if the string is depicted linearly from left to right, it might have cycles since some substrings can be repeated along the string.

2.2 String modules

Definition 2.2.1. Let $\omega = \alpha_1^{\epsilon_1}, \cdots, \alpha_l^{\epsilon_l}$ be a string for a string algebra $\Lambda = \frac{\mathbb{K}\mathrm{Q}}{I}$. The **string module** $M(\omega)$ is the left Λ-module defined as a representation of Q as follows:

- let, for each vertex a in Q_0, $I_a = \{i \mid s(\alpha_i^{\epsilon_i}) = a\} \cup \{0 \text{ if } t(\alpha_1^{\epsilon_1}) = a\}$
- For each vertex $a \in \mathrm{Q}_0$, let $M(\omega)_a$ be the vector space with basis given by $\{z_i \mid i \in I_a\}$
- For each arrow $\beta : a \to b$ of Q_1, the $\mathbb{K}$-linear map $\phi_\beta(\omega) : M(\omega)_a \to M(\omega)_b$ is defined on the basis of $M(\omega)_a$ by

$$\phi_\beta(\omega)(z_i) = \begin{cases} z_{i-1} \text{ if } \alpha_i = \beta \text{ and } \epsilon_i = 1, \\ z_{i+1} \text{ if } \alpha_{i+1} = \beta \text{ and } \epsilon_{i+1} = -1, \\ 0 \text{ otherwise.} \end{cases}$$

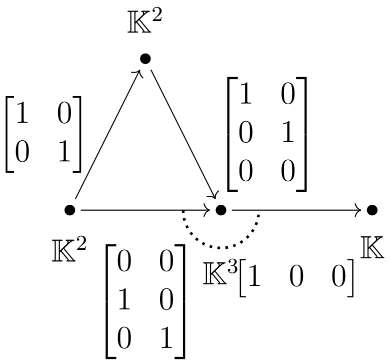
The action of Λ on $M(\omega)$ is then defined for a non trivial path $p = \beta_1 \dots \beta_l$, from a to b and $z = (z_a)_{a \in \mathrm{Q}_0} \in \oplus_{a \in \mathrm{Q}_0} M_a$ as $\phi_{\beta_1}(\omega) \circ \cdots \circ \phi_{\beta_l}(\omega)(z_a)$. The action is then extended by $\mathbb{K}$-linearity.

It follows from the definition, that, for any string ω, the string modules $M(\omega)$ and $M(\omega^{-1})$ are isomorphic. This suggests that we include a relation on the set of strings, such that $\omega \sim \omega^{-1}$. Hence, from now on in this thesis, strings had to be considered up to their inverses.

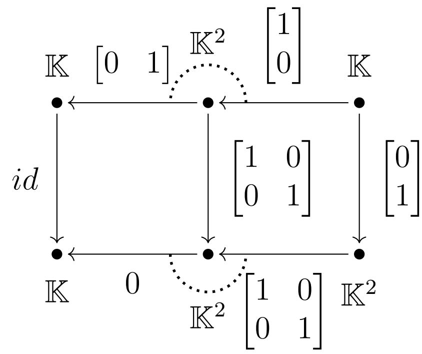
It is easy to see, from the definition, that the simple module $S(a)$, as described in Lemma 1.2.3, associated to a vertex a in Q_0, is generated by the string of length zero e_a.

If the string has finite length, then the module associated will be finite-dimensional.

Example 2.2.1. We compute the modules arising from the strings in Example 2.1.2. From these examples one could see that, in some sense, the string module maintains the structure of the string.



The Λ_A-module $M(\omega_A)$



The Λ_B-module $M(\omega_B)$

2.2.1 Module homomorphisms between string modules

We have described what a string algebra is and introduced an important class of modules over it, those generated from a string. Not only are they easy to visualise, but their importance lies in the fact that they are indecomposable. As seen in Theorem 1.1.1, Λ- Mod is a Krull-Schmidt category, so it is essential to understand the indecomposable modules. To prove this important result, we adapt a more general proof of [Cra98], involving not only finite strings, but also modules generated by possibly infinite strings.

We start by looking at the homomorphisms between string modules. It turns out that they can be computed just by looking at the maps between the relative strings. To see this, we need to formally define what we mean by a *map between the strings*.

Given a string $\omega = \alpha_1^{\epsilon_1} \dots \alpha_k^{\epsilon_k} = \omega_1 \dots \omega_k$, the quiver $\Gamma_\omega = (\Gamma_{\omega,0}, \Gamma_{\omega,1})$, which has $k+1$ vertices, $\Gamma_{\omega,0} = \{0, \dots, k\}$ and k arrows

$$\Gamma_{\omega,1} = \{\gamma_i : i \to i-1 \text{ if } \omega_i \text{ is direct}, \gamma_i : i-1 \to i \text{ if } \omega_i \text{ is inverse} \mid i = 1 \dots k\},$$

the quiver **underlying** the string.

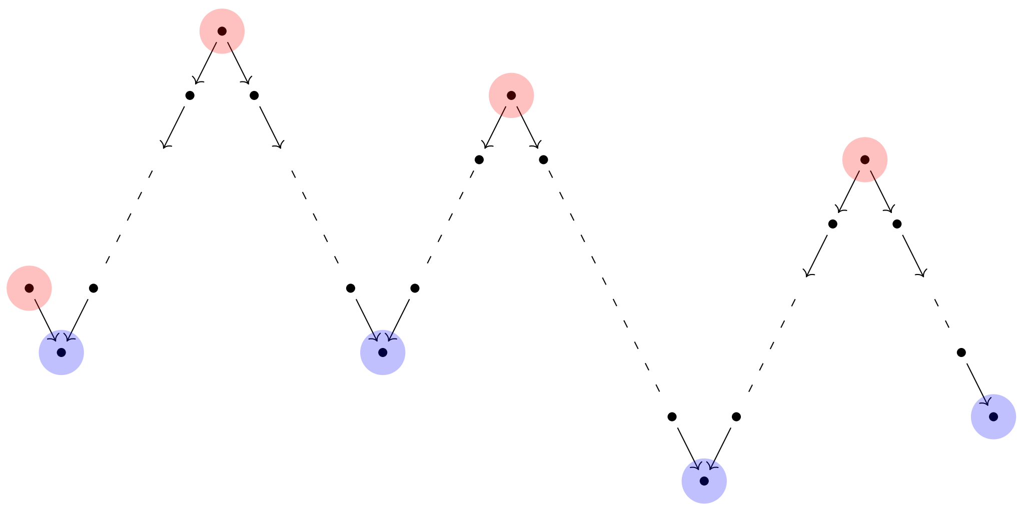
Observe that this construction can be computed also for strings of infinite length. Moreover, note that Γ_ω, by definition, is a connected quiver, has no cycles, even unoriented, and it is uniquely defined.

We call a vertex i of a finite string s of length l a **peak** if it is a source in the underlying quiver, i.e. one of these hold

- w_i is direct and w_{i+1} is inverse;
- $i = 0$ and w_1 is inverse;
- $i = l$ and w_l is direct.

With the same idea, we call a vertex z_i of s a **deep** if it is a sink in the underlying quiver.

Example 2.2.2. In the following string we have highlighted in red the peak vertices and in blue the deep ones.



We define a map between quivers: $F_\omega : \Gamma_\omega \to \mathrm{Q}$ such that

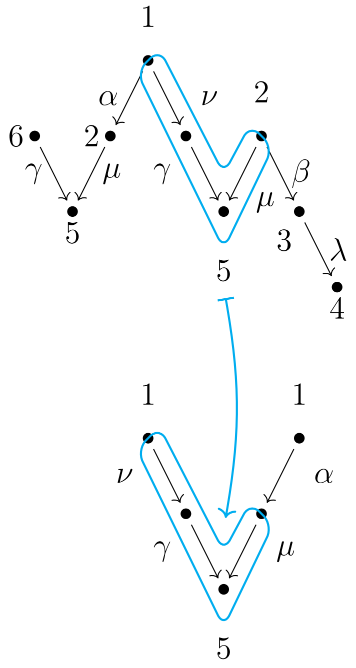
$$F_\omega(i) = \begin{cases} s(\omega_i) \text{ if } i \neq 0, \\ t(\omega_1) \text{ if } i = 0 \end{cases} \quad \text{and} \quad F_\omega(\gamma_i) = \alpha_i.$$

Note that for a vertex $a \in \mathrm{Q}_0$, I_a, as defined above, is equal to $F_\omega^{-1}(a)$, indeed z_i is a basis vector of $M(\omega)_a$ if and only if i belongs to I_a, i.e. $s(\omega_i) = a$ or, if $i = 0$, $t(\omega_1) = a$, but this actually means that $F_\omega(i) = a$. We will often indicate a basis element of $M_{F_\omega(i)}$ as z_i.

Then F_ω is unramified, meaning that for each vertex $i \in \Gamma_\omega$ and arrow $\alpha \in \mathrm{Q}$ with head, resp. tail, at $F_\omega(i) = a$, there is at most one arrow $\gamma \in F_\omega^{-1}(\alpha)$ with head, resp. tail, at i. Observe also that no path in Γ_ω is sent to a path occurring (with non-zero coefficient) in any element of I. The map F_ω can be represented by labeling each vertex and arrow of Γ with its corresponding image in Q, doing so we essentially recover the original string. Thus F_ω also provides a formal description of the passage from the string to its graphical representation. Consequently, F_ω serves as a tool to distinguish between the elements of the string, important because they are part of the string, and the actual arrows they represent in the quiver. This distinction is crucial for avoiding confusion and clarifying the discussion, especially when dealing with maps between string modules and the corresponding partial maps.

Definition 2.2.2. By a **partial map between two strings** $\Theta : \omega \rightsquigarrow \omega'$, we mean an isomorphism $\Theta : D_\Theta \to R_\Theta$ satisfying $F_{\omega'} \circ \Theta = F_\omega|_{D_\Theta}$, where $D_\Theta \subseteq \Gamma_\omega$ is a non-empty full connected subquiver of Γ_ω which is closed under predecessors, and $R_\Theta \subseteq \Gamma_{\omega'}$ is a non-empty full connected subquiver of $\Gamma_{\omega'}$ which is closed under successors.

Example 2.2.3. We show a partial map Θ colored in cyan, between the string $\omega_B = \gamma^{-1}\mu\alpha\nu^{-1}\gamma^{-1}\mu\beta^{-1}\lambda^{-1}$ and the string $\omega'_A = \nu^{-1}\gamma^{-1}\mu\alpha$.



Given vertices $r \in \Gamma_\omega$ and $s \in \Gamma_{\omega'}$, we write $r \rightsquigarrow s$ to mean that there is a partial map $\Theta : \omega \rightsquigarrow \omega'$, and such that $r \in D_\Theta$ and $s \in R_\Theta$ and $s = \Theta(r)$. We list some properties of this relation:

Lemma 2.2.1 (First properties of partial maps)**.** *Let $\omega, \omega', \omega''$ be three strings with vertices $r \in \Gamma_\omega, s \in \Gamma_{\omega'}, t \in \Gamma_{\omega''}$.*

(i) There is at most one partial map $\Theta : \omega \to \omega'$ inducing $r \rightsquigarrow s$.

(ii) If $r \rightsquigarrow s \rightsquigarrow t$, then $r \rightsquigarrow t$.

(iii) If $r \rightsquigarrow s$ and $s \rightsquigarrow r$, then the corresponding partial maps are inverse isomorphisms between Γ_ω and $\Gamma_{\omega'}$. In particular $\Gamma_\omega \simeq \Gamma_{\omega'}$.

Proof. (i) Given $r \in \Gamma_\omega$ and $s \in \Gamma_{\omega'}$, let $\Theta : D_\Theta \to R_\Theta$ and $\Theta' : D'_\Theta \to R'_\Theta$ be two partial maps such that both $\Theta(r) = s$ and $\Theta'(r) = s$.

Let p be in D_Θ. By construction of the underlying quiver and the map related, there exists a substring of ω, $\tilde{\omega}$, such that either $F_\omega(p)$ is the head of this substring and $F_\omega(r)$ its tail, or p is the tail and r its head. Without loss of generality, we can assume $F_\omega(p) = \mathfrak{s}(\tilde{\omega})$ and $F_\omega(r) = \mathfrak{t}(\tilde{\omega})$. So $\tilde{\omega} = \omega_{r+1} \dots \omega_p$.

If ω_{r+1} is inverse, then the arrow of Γ_ω, γ_{r+1} is between $r+1 \to r$, and both D_Θ and $D_{\Theta'}$ are closed under predecessors, so $r+1$ belongs to them. Moreover, by definition of partial map, $F_\omega(\mathfrak{s}(\gamma_{r+1})) = F_\omega(r+1) = F_{\omega'}(\Theta'(r+1)) = F_{\omega'}(\Theta(r+1))$. Since $F_{\omega'}$ is unramified, there exists at most one arrow γ' in $\Gamma_{\omega'}$ with tail at $\Theta(r) = s = \Theta'(r)$. This implies that $\Theta(r+1) = \Theta'(r+1)$, since they are quiver isomorphism.

If ω_{r+1} is direct, then $s = s(\Theta(\gamma_{r+1})) = s(\Theta'(\gamma_{r+1})) \in R_\Theta$. Since both $R_\Theta, R_{\Theta'} \subseteq \Gamma_{\omega'}$ are closed under successors, $t(\Theta(\gamma_{r+1}))$ belongs to R_Θ and $t(\Theta'(\gamma_{r+1}))$ belongs to $R_{\Theta'}$. Then $\Theta^{-1}(t(\Theta(\gamma_{r+1}))) = t(\gamma_{r+1}) = r+1 = \Theta'^{-1}(t(\Theta'(\gamma_{r+1})))$ belongs to also D_Θ and $D_{\Theta'}$. In any case, $r+1$ belongs to D'_Θ. By definition of partial map, $F_\omega(t(\gamma_{r+1})) = F_\omega(r+1) = F_{\omega'}(\Theta'(r+1)) = F_{\omega'}(\Theta(r+1))$. Since $F_{\omega'}$ is unramified exists at most one arrow γ' in $\Gamma_{\omega'}$ with head at $\Theta(r) = s = \Theta'(r)$. This implies that, again, $\Theta(r+1) = \Theta'(r+1)$.

By induction, repeating the same reasoning, $r+l$ belongs to $D_{\Theta'}$ and $\Theta(r-l) = \Theta'(r-l)$, for each $1 \leq l \leq p-r$.

Equivalently, one can prove the converse, namely that if p' is in $D_{\Theta'}$, then it is also in D_Θ and $\Theta'(p') = \Theta(p')$. Hence, the two partial maps are the same.

(ii) Call $\Theta : \Gamma_\omega \to \Gamma_{\omega'}$, resp. $\Theta' : \Gamma_{\omega'} \to \Gamma_{\omega''}$, the partial map inducing $r \rightsquigarrow s$, resp. $s \rightsquigarrow t$. Then $\Phi = \Theta' \circ \Theta|_{D_\Theta \cap \Theta^{-1}(D'_\Theta)}$ is the partial map inducing $r \rightsquigarrow t$. Indeed, D'_Θ is closed under predecessors and Θ is a quiver isomorphism, then $\Theta^{-1}(D'_\Theta)$ is closed under predecessors. Since s belongs to D'_Θ and $\Theta^{-1}(s) = r$, then r belongs to $\Theta^{-1}(D'_\Theta)$. So r is in $D_\Phi = D_\Theta \cap \Theta^{-1}(D'_\Theta)$. Intersection of connected quivers closed under predecessors, resp. successors, is closed under predecessors, resp. successors, moreover also the image of a quiver closed under predecessors, resp. successors, is closed under predecessors, resp. successors. So the domain of $\Phi = D_\Phi$ is closed under predecessor and the codomain $R_\Phi = \Theta' \circ \Theta(D_\Theta) \cap \Theta'(D'_\Theta) = \Theta'(R_\Theta) \cap R_{\Theta'}$ is closed under successors. This implies that Φ is a partial map. Obviously $\Phi(r) = \Theta'(\Theta(r)) = t$ and $F'_\omega \circ \Phi = F_\omega$. Then we proved the existence of the partial map inducing $r \rightsquigarrow t$.

(iii) Let $r \in \Gamma_\omega$ and $s \in \Gamma_{\omega'}$, such that exists $\Theta : \Gamma_\omega \to \Gamma_{\omega'}$ inducing $r \rightsquigarrow s$, and exists $s \rightsquigarrow r$ induced by $\Theta' : \Gamma_{\omega'} \to \Gamma_\omega$. Then the construction of (ii), gives a partial map, $\Phi = \Theta' \circ \Theta|_{D_\Theta \cap \Theta^{-1}(D'_\Theta)}$, inducing $r \rightsquigarrow r$. The identity is another partial map which induce $r \rightsquigarrow r$, then, by (i), $\Phi = id_{\Gamma_\omega}$. Equivalently $\Theta \circ \Theta'|_{D'_\Theta \cap \Theta'^{-1}(D_\Theta)} = id_{\Gamma_{\omega'}}$. Thus, the partial maps Θ and Θ' are inverse to each other.

□

A partial map $\Theta : \omega \rightsquigarrow \omega'$ induces a linear map $f_\Theta : M(\omega) \to M(\omega')$, between their respective strings modules defined as the map which sends z_i to $z_{\Theta(i)}$, if $i \in D_\Theta$, to zero otherwise. The construction of f_Θ is then extended by $\mathbb{K}$-linearity on the other elements of $M(\omega)$.

To prove that is a Λ-module homomorphism, we need to show that $f_\Theta(\beta m) = \beta f_\Theta(m)$, for m in $M(\omega)$ and β in Λ. Without loss of generality, we can assume that $m = z_i$ for some basis vector of $M_{F_\omega(i)}$ and $\beta : a \to b$ to be an arrow in Q_1.

By definition of the action of Λ over $M(\omega)$, If β is not trivial, we have that

$$\beta z_i = \phi_\beta(\omega)(z_i) = \begin{cases} z_{i-1} & \text{if } \alpha_i = \beta \text{ and } \epsilon_i = 1, \\ z_{i+1} & \text{if } \alpha_{i+1} = \beta \text{ and } \epsilon_{i+1} = -1, \\ 0 & \text{otherwise.} \end{cases}$$

. We can rephrase this, in terms of the underlying quiver:

$$\beta z_i = \begin{cases} z_j & \text{if there exists } \gamma : i \to j \mid F_\omega(\gamma) = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

In particular it implies that $F_\omega(i) = a$ and $F_\omega(j) = b$.

Then

$$f_\Theta(\beta z_i) = \begin{cases} z_{\Theta(j)} & \text{if there exists } \gamma : i \to j \in \Gamma_\omega \mid F_\omega(\gamma) = \beta \text{ and } j \in D_\Theta, \\ 0 & \text{otherwise.} \end{cases}$$

While

$$\beta f_\Theta(z_i) = \beta \cdot \begin{cases} z_{\Theta(i)} & \text{if } i \in D_\Theta, \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} z_{j'} & \text{if there exists } \gamma' : \Theta(i) \to j' \in \Gamma_{\omega'} \mid F_{\omega'}(\gamma') = \beta \text{ and } i \in D_\Theta, \\ 0 & \text{otherwise.} \end{cases}$$

If there exists $\gamma : i \to j$ such that $F_\omega(\gamma) = \beta$ and $j \in D_\Theta$, then since D_Θ is closed under predecessors, i also belongs to D_Θ. Then, by definition of partial map, we have that $\Theta(\gamma) : \Theta(i) \to \Theta(j) \in \Gamma_{\omega'}$ and $F_{\omega'}(\Theta(\gamma)) = \beta$. Since $F_{\omega'}$ is unramified, $\Theta(\gamma)$ is the unique arrow in $\Gamma_{\omega'}$, with head in $\Theta(i)$ and such that its image through $F_{\omega'}$ is equal to β.

Conversely if there exists $\gamma' : \Theta(i) \to j' \mid F_{\omega'}(\gamma') = \beta$ and $i \in D_\Theta$, then $\Theta(i)$ is in R_Θ, which is closed under successors. Hence j' belongs to R_Θ, namely there exists j'' such that $\Theta(j'') = j'$ and the same holds for γ', i.e. there exists $\gamma'' \in \Gamma_\omega$ such that $\Theta(\gamma'') = \gamma'$. Moreover, since Θ is a quiver isomorphism $\gamma'' : i \to j''$ and, by definition of a partial map, $F_\omega(\gamma'') = F_{\omega'}(\gamma') = \beta$. Since F_ω is unramified, γ'' is the unique arrow with head in i and with image through F_ω equal to β.

This implies that $f_\Theta(\beta z_i) \neq 0$ if and only if $\beta f_\Theta(z_i) \neq 0$ and, in this case, $\gamma' = \theta(\gamma)$ and $j' = \Theta(j)$. Hence, $f_\Theta(\beta z_i) = \beta f_\Theta(z_i)$, so we showed that f_Θ is a left Λ-module homomorphism.

Let m be in $M(\omega)$, we denote with $c_s(m)$ the coefficient of z_s in m. Now we are ready to give a characterisation of the module homomorphism between string modules.

Lemma 2.2.2. *Let ω, ω' be two strings and $M(\omega)$, $M(\omega')$ be their relative string module. Then any Λ-module homomorphism $f : M(\omega) \to M(\omega')$ can be written uniquely as a (possibly infinite) linear combination*

$$f = \sum_{\Theta:\omega \rightsquigarrow \omega'} \lambda_\Theta f_\Theta$$

with $\lambda_\Theta \in \mathbb{K}$, such that for each vertex $r \in \Gamma_\omega$, there are only finitely many non-zero λ_Θ with $r \in D_\Theta$. In particular, if $c_s(f(z_r)) \neq 0$ then $r \rightsquigarrow s$.

Proof. Let r be a vertex of Γ_ω and s be a vertex of $\Gamma_{\omega'}$, such that $c_s(f(z_r)) \neq 0$. We aim to construct a partial map Θ that induces $r \rightsquigarrow s$. To do this, we must define a domain D_Θ, closed under predecessors, and a codomain R_Θ, closed under successors. For each $p \in D_\Theta$, we must then define $\Theta(p) \in R_\Theta$.

Since $c_s(f(z_r)) \neq 0$, it implies that $F_{\omega'}(s)) = F_\omega(r)$, indeed let $a := F_{\omega'}(s)$ and consider the trivial path e_a. Since f is a Λ-module homomorphism, $f(e_a z_r) = e_a f(z_r)$. Hence $c_s(f(e_a z_r)) = c_s(f(z_r)) \neq 0$. In particular it implies that $e_a z_r = z_r$. Thus $F_\omega(r) = a$.

We say that a vertex v of Γ_ω and a vertex v' of $\Gamma_{\omega'}$ are *matching* if

$$c_{v'}(f(z_v)) = c_s(f(z_r)) \neq 0 \quad \text{and} \quad F_{\omega'}(v') = F_\omega(v).$$

For what proved above, s and r are matching.
Claim: If there exists an arrow $\gamma : a \to b$ in Γ_ω and b is matching with a vertex b' in $\Gamma_{\omega'}$, then there exists a unique arrow $\gamma' : a' \to b'$ in $\Gamma_{\omega'}$ such that a is matching with a'. Moreover $F_{\omega'}(\gamma') = F_\omega(\gamma)$.

Proof of Claim 1. Since b is matching with b', we have that $c_{b'}(f(z_b)) = c_s(f(z_r)) \neq 0$. Call $\alpha := F_\omega(\gamma)$, then, by definition, $\alpha z_a = z_b$. Since f is a Λ-module homomorphism, $f(\alpha z_a) = f(z_b)$, then $c_{b'}(f(\alpha z_a)) \neq 0$. This implies that there exists a vertex a' in $\Gamma_{\omega'}$, such that $\alpha z_{a'} = z_{b'}$ and $c_{a'}(f(z_a)) = c_s(f(z_r)) \neq 0$. Then, since $\alpha z_{a'} = z_{b'}$, there exist an arrow in $\Gamma_{\omega'}$, $\gamma : a' \to b'$, such that the image, through $F_{\omega'}$, is equal to α. Because $F_{\omega'}$ is unramified, we have that γ is unique with those properties. This implies that a and a' are matching. □

Claim: If there exists an arrow $\gamma' : a' \to b'$ in $\Gamma_{\omega'}$ and a' is matching with a vertex a in Γ_ω, then there exists a unique arrow $\gamma : a \to b$ in Γ_ω such that b is matching with b'. Moreover $F_{\omega'}(\gamma') = F_\omega(\gamma)$.

Proof of Claim 2. Since a is matching with a', we have that $c_{a'}(f(z_a)) = c_s(f(z_r)) \neq 0$. Call $\alpha := F_{\omega'}(\gamma')$, then, by definition, $\alpha z_{a'} = z_{b'}$. Since f is a Λ-module homomorphism, $f(\alpha z_{a'}) = f(z_{b'})$, then $c_{b'}(f(\alpha z_{a'})) \neq 0$. This implies that $\alpha z_{a'} \neq 0$. Thus, there exist an arrow in Γ_ω, $\gamma : a \to b$, such that the image, through F_ω, is equal to α. Because F_ω is unramified, we have that γ is unique with those properties. This implies that b and b' are matching. □

Thanks to these claims, it is now possible to define R_Θ and D_Θ. To construct such sets, we will describe a recursive process involving two different types of steps: steps (a) and (b). In the following, we will outline the ideas behind this recursion

Process (a) We begin this step with a deep matching vertex x, along with a direct predecessor y in Γ_ω. We then consider all the predecessors of y, that are in a finite number by (S3). Let z be the predecessor of x that has no other predecessors. By the first claim, there exists a set of matching vertices in $\Gamma_{\omega'}$. Now, consider the matching vertex x' corresponding to x.

If x' has no other successors, we stop. Otherwise, if x' has another successor, say x'_1, we proceed to Step (b) with the pair of vertices (x', x'_1).

Process (b) In this step, we start with a peak matching vertex y', along with a direct successor x' in $\Gamma_{\omega'}$. We consider all the successors of x', that are in finite number by (S3). Let z' be the successor of x' that has no other successors. By the first claim, there exists a set of matching vertices in Γ_ω. Now, consider the matching vertex x corresponding to x'.

If x has no other predecessors, we stop. Otherwise, if x has another predecessor, say x_1, we proceed to Step (a) with the pair of vertices (x, x_1).

We need to start this process with the matching vertices s and r. However, we do not know whether s or r are peak or deep vertices. So, in general, we cannot directly start with one of the two steps.

- If s has no successors, set $f = 0$ and $g = 0$.

- If s has at least one direct successor, choose one and call it s_1 and consider the set of vertices in $\Gamma_{\omega'}$, $\{s_1, \ldots, s_{i_0}\}$, where s_{j+1} is a direct successor of s_j, for $1 \leq j \leq i_0$, and s_{i_0} has no successors. Observe that i_0 must be a natural number different from infinity, due to condition (S3). By the first claim, we have matching vertices $\{r_1, \ldots, r_{i_0}\}$ in Γ_ω, where r_j matches with s_j for $1 \leq j \leq i_0$. If r_{i_0} has no direct predecessor other than r_{i_0-1}, we set $f := i_0$ and stop. If r_{i_0} has a direct predecessor different from r_{i_0-1}, we continue. Observe that we have just described how to perform Step(b), following the mechanism outlined before, and starting with the pair (s, s_1), even if s is not a peak vertex.

  Now, denote by $\{r_{i_0+1}, \ldots, r_{i_0+i_1}\}$, the set of vertices of Γ_ω, such that r_{i_0+j+1} is a direct predecessor of r_{i_0+j} for $1 \leq j \leq i_1$, and $r_{i_0+i_1}$ has no predecessors. Again, due to condition (S3), we have $i_1 \neq \infty$. By the second claim, we have matching vertices $\{s_{i_0+1}, \ldots, s_{i_0+i_1}\}$ in $\Gamma_{\omega'}$, such that s_{i_0+j} matches with r_{i_0+j} for $1 \leq j \leq i_1$. If $s_{i_0+i_1}$ has no direct successor other than $s_{i_0+i_1-1}$, set $f := i_0 + i_1$. If $s_{i_0+i_1}$ has other predecessor, then we continue. We have just described how to compute Step (a), following Process (a).

  We can now repeat this process, alternating between Step (a) and Step (b). We will stop after defining $f = \sum_j i_j$, when if, after a step of type (a), s_f has no direct successor other than s_{f-1}, or if, after a step of type (b), r_f has no other predecessors direct other than r_{f-1}. Note that this process could theoretically continue indefinitely, as strings can have infinite lengths. In that case we can set $f = \infty$.

- If s has no direct successors different from s_1, set $g = 0$

- If s has two direct successors, then it has a direct successor different from s_1. Denote the successors by $\{s_{-1}, \ldots, s_{-l_0}\}$, where s_{-j-1} is a direct predecessor of s_j for $-1 \geq j \geq -l_0$, and s_{-l_0} has no successors. Observe that l_0 must also be a natural number different from infinity due to condition (S3). By second claim, matching vertices exist in Γ_ω. As before, we can continue alternating between Step (b), starting with (s, s_{-1}), and Step (a). We will stop after defining: $-g = \sum_j -l_j$, if, after a step of type (a), s_{-g} has no direct successor other than s_{-g+1}, or if , after a step of type (b), r_{-g} has no other predecessors direct other than r_{-g+1}. Notice that, again this process could theoretically continue indefinitely, as strings can have infinite lengths. In this case $g = \infty$.

We will follow the same steps, but starting with vertex r. In particular, if it exists a predecessors $\tilde{r}_1$, we start with Step (a) with the pair $(r, \tilde{r}_1)$. If another predecessor $\tilde{r}_{-1}$ exists, we restart Step (a) with the pair $(r, \tilde{r}_{-1})$.

By alternating between Step (a) and Step (b), as described before, we will define a set of vertices $\{\tilde{r}_j \mid -\tilde{g} \geq \tilde{f}\}$, closed under predecessors, with matching vertices in $\Gamma_{\omega'}$. If r has no predecessors, set $\tilde{f} = 0$ and $\tilde{g} = 0$, and if it has only one predecessor, set $\tilde{g} = 0$.

Thus, we can set

$$D_\Theta = \{r_j \mid -g \leq j \leq f\} \cup \{\tilde{r}_j \mid -\tilde{g} \leq j \leq \tilde{f}\}$$

and

$$R_\Theta = \{s_j \mid -g \leq j \leq f\} \cup \{\tilde{s}_j \mid -\tilde{g} \leq j \leq \tilde{f}\}.$$

By construction, D_Θ is closed under predecessors, and R_Θ is closed under successors. Each element $v \in D_\Theta$ has a matching element $v' \in R_\Theta$, such that

$$c_{v'}(f(z_v)) = c_s(f(z_r)) \neq 0 \quad \text{and} \quad F_{\omega'}(v') = F_\omega(v).$$

Therefore, we define $\Theta(v) = v'$. Then Θ is a partial map inducing $r \rightsquigarrow s$.

So we proved that if $r \in \Gamma_\omega$ and $s \in \Gamma_{\omega'}$ are vertices with $c_s(f(z_r)) \neq 0$, then there is a partial map Θ inducing $r \rightsquigarrow s$ and with $\lambda_\Theta = c_{\Theta(t)}(f(z_t)) = c_s(f(z_r))$ for each $t \in D_\Theta$. Observe that by Lemma 2.2.1, there exists a unique partial map inducing $r \rightsquigarrow s$.

Now, for each r in Γ_ω, we have

$$f(z_r) = \sum_{s \in \Gamma_{\omega'} \mid c_s(f(z_r)) \neq 0} c_s(f(z_r)) z_s = \sum_{s \mid r \rightsquigarrow s} c_s(f(z_r)) z_s.$$

And this is equal to say:

$$f(z_r) = \sum_{\Theta : \omega \rightsquigarrow \omega' \mid r \in D_\Theta} \lambda_\Theta z_{\Theta(r)}$$

Since this is true fron any r in Γ_ω, we have that:

$$f = \sum_{\Theta : \omega \rightsquigarrow \omega'} \lambda_\Theta f_\Theta$$

Now we prove uniqueness. By contradiction, let f have two different decomposition:

$$f = \sum_{\Theta : \omega \rightsquigarrow \omega'} \lambda_\Theta f_\Theta = \sum_{\Theta' : \omega \rightsquigarrow \omega'} \lambda_{\Theta'} f_{\Theta'}.$$

Let r a vertex in Γ_ω. Then, since we are summing over the same group and the partial map f_Θ depends only on Θ, we get

$$f(z_r) = \sum_{\Theta : \omega \rightsquigarrow \omega',\, r \in D_\Theta} \lambda_\Theta f_\Theta(z_r) = \sum_{\Theta : \omega \rightsquigarrow \omega',\, r \in D_\Theta} \lambda_{\Theta'} f_\Theta(z_r),$$

so

$$\sum_{\Theta : \omega \rightsquigarrow \omega',\, r \in D_\Theta} \lambda_\Theta z_{\Theta(r)} = \sum_{\Theta : \omega \rightsquigarrow \omega',\, r \in D_\Theta} \lambda'_\Theta z_{\Theta(r)}.$$

By Lemma 2.2.1, there exists a unique partial map Θ inducing $r \rightsquigarrow \Theta(r)$, so $c_{\Theta(r)}(f(x_r)) = \lambda_\Theta = \lambda'_\Theta$. By arbitrariness of Θ, we conclude. □

The importance of this lemma extends beyond just providing a basis for the $\mathbb{K}$-vector space of module homomorphisms between string modules; it also offers a way to visualize these morphisms. Indeed, by simply looking at the string, we can determine the computations needed to construct such a morphism. This will prove to be very useful throughout this project, allowing us to create figures that will serve as a guide and as an example to follow in order to better understand the proofs and constructions, which will come later.

2.2.2 Indecomposability of the string modules

Finally, to conclude this section on string modules, we will prove that they are indeed indecomposable, as was mentioned at the beginning. However, to prove this result, we need to first demonstrate the validity of the following statement, as it plays a relevant role in the proof of the theorem.

Lemma 2.2.3. *The ring $\mathbb{K}[x, x^{-1}]$ has no non-trivial idempotents.*

Proof. This follows from the fact that there exists a natural inclusion from $\mathbb{K}[x, x^{-1}]$ into the field of rational functions $\mathbb{K}(x)$. This field consists of all rational functions in the variable x with coefficients in $\mathbb{K}$, where each rational function is of the form $\frac{p(x)}{q(x)}$, with $p(x), q(x) \in \mathbb{K}[x]$ and $q(x) \neq 0$.

Since $\mathbb{K}[x, x^{-1}]$ is included in the field $\mathbb{K}(x)$, it is an integral domain. Consequently, the ring of Laurent polynomials has no non-trivial idempotents, since idempotents in a domain are either 0 or 1. □

During the development of this thesis, we proved this fact in an alternative way. We showed that $\mathbb{K}\big[[x]\big]$ has non-trivial idempotents and that there exists an injective ring-homomorphism from $\mathbb{K}[x, x^{-1}]$ into $\mathbb{K}\big[[x]\big]$. Thanks to this we arrive at the same conclusions. For further details, see Appendix A.0.1.

We can now state and prove the indecomposability of the string modules.

Theorem 2.2.4. *[Cra98] Given a string ω, possibly of infinite length, then $M(\omega)$ is an indecomposable Λ-module. If ω' is another string, then $M(\omega) \simeq M(\omega')$ if and only if there is a quiver isomorphism Θ between Γ_ω and $\Gamma_{\omega'}$ such that $F_{\omega'} \circ \Theta = F_\omega$.*

Proof. Let G be the group of automorphisms of Γ_ω over Q, consisting of those quiver automorphisms g of Γ with $F_\omega \circ g = F_\omega$, i.e.

$$G = \{g \in \mathrm{Aut}(\Gamma_\omega) \mid F_\omega \circ g = F_\omega\}.$$

This is a group since composition of automorphisms $g \circ h$ is still an automorphism, and $F_\omega \circ g \circ h = F_\omega$. Note that each elements of G is a partial map $\Theta : \omega \rightsquigarrow \omega$, but not every partial map is an element of G. We prove that if ω is finite or aperiodic, the only element of G is the identity, while if ω is infinite and periodic, then G is isomorphic to $\mathbb{Z}$.

- If ω is finite, let $\Theta : \omega \rightsquigarrow \omega$ be an isomorphism between Γ_ω and itself, i.e. an element of G. If it sends the vertex $0 \in \Gamma_{\omega,0}$ to itself, then, for 2.2.1, Θ is the identity. Observe that, since we are assuming that the string $\omega = \omega_1 \dots \omega_k$ is finite, there are only two vertices, 0 and k, that have just one arrow, respectively γ_0 and γ_k, linked to them. Since Θ is a quiver isomorphism, we have $s \circ \Theta = \Theta \circ s$ and $t \circ \Theta = \Theta \circ t$, then $\Theta(0)$ must be equal to k and either both vertices are the start of the respective arrow, or the tails. For the same reason, $\Theta(\gamma_0) = \gamma_k$ and $\Theta(1) = k-1$. Using induction we get that k must be even, and $\Theta(\gamma_l) = \gamma_{k-l}$ and $\Theta(l) = k - l$ for $\frac{k}{2} \geq l \geq 1$. By definition of a partial map, $F_\omega \circ \Theta = F_\omega$, so $F_\omega(\gamma_{k-l}) = F_\omega(\gamma_l)$ and $F_\omega(k-l) = F_\omega(l)$ for $\frac{k}{2} \geq l \geq 1$. This means that Γ_ω is symmetric with respect to the vertex $\frac{k}{2}$. Hence, by construction of the underlying quiver, the string is equal to

$$\omega = \omega_1 \dots \omega_k = \alpha_1^{\epsilon_1} \dots \alpha_{\frac{k}{2}}^{\epsilon_{\frac{k}{2}}} \alpha_{\frac{k}{2}}^{-\epsilon_{\frac{k}{2}}} \alpha_1^{-\epsilon_1}.$$

Note that this implies that $\omega = \omega^{-1}$. Since we are considering strings up to inverses, without loss of generality we can replace ω with its inverse, Θ became the identity.

- If the string, ω, is infinite and aperiodic, let p, s different vertices of Γ_ω and assume $\Theta(s) = p$. Denote $n = |p - s|$. The vertex s, resp. p, is linked with two arrows, γ_s and γ_{s+1}, resp. γ_p and γ_{p+1}, if they exist. Then we could only have

  (a) either $\Theta(\gamma_s) = \gamma_p$,

  (b) or $\Theta(\gamma_s) = \gamma_{p+1}$.

  Indeed $\Theta(\gamma_s)$ is the unique arrow, γ_x, belonging to $F_\omega^{-1}(F_\omega(\gamma_p))$, where x could be either p or $p+1$, and $\Theta(\gamma_{s+1})$ is the other one. If (a) happens, by induction and by looking at the structure of the quiver Γ_ω, we get that

  $$\Theta(\gamma_{s-l}) = \Theta(\gamma_{p-l}) \text{ and } \Theta(\gamma_{s+l}) = \Theta(\gamma_{p+l}), \text{ if } l \geq 1.$$

  However this also implies that $F_\omega(\gamma_{s-l}) = F_\omega(\gamma_{p-l})$ and $F_\omega(\gamma_{s+l}) = F_\omega(\gamma_{p+l})$, if $l \geq 1$, namely $\alpha_{s-l} = \alpha_{p-l}$ and $\alpha_{s+l} = \alpha_{p+l}$. Hence the string w is equal to

  $$\begin{array}{ll} \dots \alpha_{s+n}^{\epsilon_{s+n}} \alpha_{s+1}^{\epsilon_{s+1}} \dots \alpha_{s+n}^{\epsilon_{s+n}} \alpha_{s+1}^{\epsilon_{s+1}} \dots & \text{if } p > s, \\ \dots \alpha_{p+n}^{\epsilon_{p+n}} \alpha_{p+1}^{\epsilon_{p+1}} \dots \alpha_{p+n}^{\epsilon_{p+n}} \alpha_{p+1}^{\epsilon_{p+1}} \dots & \text{if } s > p. \end{array}$$

  Namely the string is periodic and this is a contradiction. So $p = s$ and $\Theta = id$.

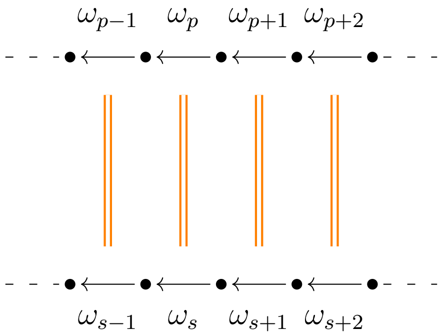
  If (b) happens, following quite the same reasoning, we get that n must be even and

  $$\Theta(\gamma_{s-l}) = \Theta(\gamma_{p+1+l}) \text{ and } \Theta(\gamma_{s+l}) = \Theta(\gamma_{p+l-1}), \text{ if } l \geq 1,$$

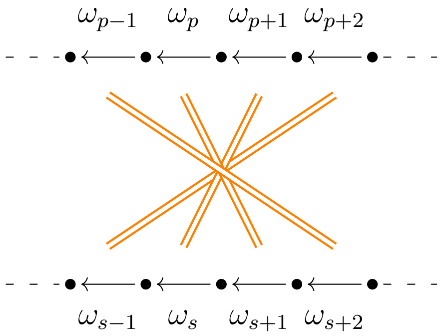
  and so the string w is equal to:

  $$\begin{array}{ll} \dots \alpha_{s-1}^{\epsilon_{s-1}} \alpha_s^{\epsilon_s} \alpha_{s+1}^{\epsilon_{s+1}} \dots \alpha_{s+n/2}^{\epsilon_{s+n/2}} \alpha_{s+n/2}^{-\epsilon_{s+n/2}} \dots \alpha_{s+1}^{-\epsilon_{s+1}} \alpha_s^{-\epsilon_s} \alpha_{s-1}^{-\epsilon_{s-1}} \dots & \text{if } p > s, \\ \dots \alpha_{p-1}^{\epsilon_{p-1}} \alpha_p^{\epsilon_p} \alpha_{p+1}^{\epsilon_{p+1}} \dots \alpha_{p++n/2}^{\epsilon_{p+n/2}} \alpha_{p+n/2}^{-\epsilon_{p+n/2}} \dots \alpha_{p+1}^{-\epsilon_{p+1}} \alpha_p^{-\epsilon_p} \alpha_{p-1}^{-\epsilon_{p-1}} \dots & \text{if } s > p, \end{array}$$

  This implies that $\omega = \omega^{-1}$. Again, up to replace the string ω with its inverse, Θ is the identity.



Case (a)



Case (b)

- If ω is infinite and periodic, then ω is the infinite composition of the same finite substring $\tilde{\omega}$, i.e $\omega = \dots\tilde{\omega}\tilde{\omega}\tilde{\omega}\dots = \prod_{i\in\mathbb{Z}}\tilde{\omega}_i$. We can assume that $\tilde{\omega} = \tilde{\omega}_1 \dots \tilde{\omega}_k$ is the period of ω, i.e. is the substring of minimal length which is aperiodic. We, firstly, rename the vertices and arrows of Γ_ω such that they match the period of the string. Thus, the vertices of $\Gamma_{\tilde{\omega}_i}$ are renamed as $i_0, \dots, i_k$ for each $i \in \mathbb{Z}$ and also the names of the arrows follow the same idea: they will be $\gamma_{i_1}, \dots, \gamma_{i_k}$. Let p, s be two different vertices of Γ_ω and Θ a quiver isomorphism, different than the identity, such that $\Theta(p) = s$ and $F_\omega \circ \Theta = F_\omega$. We showed that the only isomorphism between finite string is the identity, so if $s = i_a$ belongs to $\Gamma_{\tilde{\omega}_i}$ with $i, a \in \mathbb{Z}$ and $0 \le a \le k$, then $p = j_b$ must belong to $\Gamma_{\tilde{\omega}_j}$ with $i \neq j$ and $j, b \in \mathbb{Z}$ and $0 \le b \le k$, and $\Theta|_{\Gamma_{\tilde{\omega}_i}} = \Gamma_{\tilde{\omega}_j} = id_{\Gamma_{\tilde{\omega}}}$, i.e. $a = b$. Moreover, since Θ is a quiver isomorphism and the string is connected, $\Theta(\gamma_{i_k}) = \gamma_{j_k}$, then $\Theta(i + 1_0) = j + 1_0$. Hence, call $n = i - j$, $\Theta(x_a) = (x + n)_a$ for each $x \in \mathbb{Z}$. This gives us the idea of how to construct the group homomorphism ϕ between G and $\mathbb{Z}$: consider the vertex 0_a in $\Gamma_{\tilde{\omega}_0}$. and its image through $\Theta \in G$, $\Theta(0_a) = n_a$, with $n \in \mathbb{Z}$. We then define $\phi(\Theta) = n$. Let Θ' be in G, such that $\Theta'(0_a) = n'_a$, then $(\Theta \circ \Theta')(0_a) = (n + n')_a$, this proves that ϕ is a well-defined group homomorphism. It is then obviously surjective and injective, namely it is a group isomorphism.

We now consider its group algebra $\mathbb{K}G$. If $G \simeq \{id\}$, then $\mathbb{K}G \simeq \mathbb{K}$, since any element $a \in \mathbb{K}G$ can be written as $a = \sum_{g\in G} \lambda_g g = \lambda id$, with λ in $\mathbb{K}$. Being isomorphic to a field, $\mathbb{K}G$ has no non-trivial idempotents.

If $G \simeq \mathbb{Z}$, then $\mathbb{K}G \simeq \mathbb{K}[x, x^{-1}]$, since any element $a \in \mathbb{K}G$ can be written as $a = \sum_{g\in G} \lambda_g g = \sum_{i\in\mathbb{Z}} \lambda_i i$, that could be seen as a polynomial in one variable x: $\sum_{i\in\mathbb{Z}} \lambda_i x^i$, with λ in $\mathbb{K}$. Namely, the group algebra is isomorphic to the Laurent polynomials. Due to Lemma 2.2.3, also in this case, $\mathbb{K}G$ has no non-trivial idempotents.

Each element in G is, by definition, a partial map $\Theta : \omega \rightsquigarrow \omega$, and we can link it with its corresponding linear map $f_\Theta : M(\omega) \to M(\omega')$. By Lemma 2.2.2, any module map $f : M(\omega) \to M(\omega)$, i.e. $f \in \mathrm{Aut}(M(\omega))$ can be written uniquely as $f = \sum_{\Theta:\omega\rightsquigarrow\omega} \lambda_\Theta f_\Theta$, with $\lambda_\Theta \in \mathbb{K}$. This sum is equal to say

$$f = \sum_{\substack{\Theta:\omega\rightsquigarrow\omega \\ \Theta\notin G}} \lambda_\Theta f_\Theta + \sum_{\Theta\in G} \lambda_\Theta f_\Theta.$$

Namely we can say that $\mathrm{End}(M(\omega)) = S \oplus J$, where

$$S = \{f \in \mathrm{End}(M(\omega)) \text{ such that } f \text{ can be written as } \sum_{\Theta \in G} \lambda_\Theta f_\Theta\}$$

and

$$J = \{f \in \mathrm{End}(M(\omega)) \text{ such that } f \text{ can be written as } \sum_{\substack{\Theta:\omega \rightsquigarrow \omega \\ \Theta \notin G}} \lambda_\Theta f_\Theta\}.$$

We can give a different, but equivalent, description of this decomposition which will be very convenient later. Let s, r two vertices of Γ_ω, if there exists a partial map Θ inducing $s \rightsquigarrow r$ and if $c_s(f_\Theta(z_r)) \neq 0$, then $r \rightsquigarrow s$, by Lemma 2.2.1, then Θ is an isomorphism, i.e. Θ belongs to G. Instead, if there exists a partial map Θ inducing $s \rightsquigarrow r$ and such that $\Theta \notin G$, then $\Theta(r) \neq s$, so $c_s(f_\Theta(z_r)) = 0$. So

$$J = \{f \in \mathrm{End}(M(\omega)) \mid c_s(f(z_r)) = 0, \ \forall\, r, s, \ s \rightsquigarrow r\}.$$

Consider S, it is by definition isomorphic to $\mathbb{K}G$, and is a subalgebra of $\mathrm{End}(M(\omega))$, since given $f, g \in S$, then $f \circ g = f(\sum_{\Theta \in G} \lambda^g_\Theta f_\Theta) = (\sum_{\Phi \in G} \lambda_\Phi f_\Phi) \circ (\sum_{\Theta \in G} \lambda_\Theta f_\Theta) = \sum_{\Phi, \Theta \in G} \lambda_\Phi \lambda_\Theta f_\Phi \circ f_\Theta = \sum_{\Psi \in G} \lambda_\Psi f_\Phi$, where $\Psi = \Phi \circ \Theta$ is still an element of G and $f_\Phi \circ f_\Theta$ correspond exactly to f_Ψ, while $\lambda_\Psi = \lambda_\Theta \cdot \lambda_\Phi$.

Meanwhile, J is an ideal of $\mathrm{End}(M(\omega))$, since given f in $\mathrm{End}(M(\omega))$ and g in J, both $f \circ g$ and $g \circ f$ belong to J. Indeed, let

$$f = \sum_{\substack{\Theta:\omega \rightsquigarrow \omega \\ \Theta \notin G}} \lambda_\Theta f_\Theta + \sum_{\Theta \in G} \lambda_\Theta f_\Theta \quad \text{and} \quad g = \sum_{\substack{\Theta:\omega \rightsquigarrow \omega \\ \Theta \notin G}} \lambda'_\Theta f_\Theta,$$

with $\lambda_\Theta, \lambda'_\Theta \in \mathbb{K}$. Then we have

$$f \circ g = \sum_{\substack{\Theta, \Theta':\omega \rightsquigarrow \omega \\ \Theta, \Theta' \notin G}} \lambda_\Theta \lambda'_{\Theta'} f_{\Theta \circ \Theta'} + \sum_{\substack{\Theta:\omega \rightsquigarrow \omega \\ \Theta \notin G, \Theta' \in G}} \lambda_\Theta \lambda'_{\Theta'} f_{\Theta \circ \Theta'}$$

If $\Theta, \Theta' : \omega \rightsquigarrow \omega$ and $\Theta, \Theta' \notin G$, then $\Theta \circ \Theta'$ is a partial map which does not belong to G, since $D_{\Theta \circ \Theta'} = D_\Theta \cap \Theta^{-1}(D_{\Theta'})$ which is different from all Γ_ω. The same reasoning applies if $\Theta, \Theta' : \omega \rightsquigarrow \omega$ such that $\Theta \notin G$ and $\Theta' \in G$. Then $D_{\Theta \circ \Theta'} = D_\Theta \cap \Theta^{-1}(D_{\Theta'}) \neq \Gamma_\omega$, hence $\Theta \circ \Theta'$ is a partial map $\omega \rightsquigarrow \omega$ not in G. This means that $f \circ g$ belongs to J. Similarly also $g \circ f$ belongs to J.

Let now $f \neq 0$ in $\mathrm{End}(M(\omega))$, such that $f^2 = f$. Let $\overline{f}$ be the projection of f into the quotient group $\frac{\mathrm{End}\, M(\omega)}{J} \simeq S \simeq \mathbb{K}G$. Since the projection is a group morphism $\overline{f} = \overline{f^2} = \overline{f}^2$, i.e. $\overline{f}$ is still an idempotent element of $\mathbb{K}G$, but $\mathbb{K}G$ has no non-trivial idempotents, so $\overline{f} = 0$ or $\overline{f} = 1$. Without loss of generality, we can assume $\overline{f} = 0$, if not we consider $1 - \overline{f}$ which is still an idempotent. Then f belongs to J and is different from zero, so $\exists\, r \in \Gamma_\omega$ such that $f(z_r) \neq 0$, then $f(z_r) = \sum_{i=1}^m \lambda_i z_{r_i}$ for some r_i vertices of Γ_ω and $c_{r_i}(f(z_r)) = \lambda_i \neq 0 \in \mathbb{K}$, then $r \rightsquigarrow r_i$ and $r_i \not\rightsquigarrow r$, this is due to Lemma 2.2.2 and by the fact that f belong to J.

Assume that $r_j \rightsquigarrow r_1$. Let $\Theta_1 : D_1 \to R_1$ be the partial map inducing $r \rightsquigarrow r_1$, $\Theta_j : D_j \to R_j$ be the partial map inducing $r \rightsquigarrow r_j$, and $\Theta : D_\Theta \to R_\Theta$ be the partial map inducing $r_j \rightsquigarrow r_1$.

Since there exists a unique partial map inducing $r \rightsquigarrow r_1$, it follows that

$$\Theta \circ \Theta_j\big|_{\Theta_j^{-1}(D_\Theta)\cap D_j} = \Theta_1 \neq 0.$$

In particular, $D_1 = \Theta_j^{-1}(D_\Theta) \cap D_j$. This implies that if $r_j \rightsquigarrow r_1$, then D_1 is a subquiver of D_j. Now, consider the collection of domains $\{D_i \mid \Theta_i : D_i \to R_i\}$, partially ordered by inclusion, and take the *maximal* one. Without loss of generality, assume that D_1 is maximal.

Hence, from the above reasoning, if $r_j \rightsquigarrow r_1$, the only possibility is that $D_j = D_1$, and therefore $r_1 \rightsquigarrow r_j$.

Since $f^2 = f$, then $f(z_r) = f(f(z_r)) = \sum_i \lambda_i f(z_{r_i})$. So, for some i, $c_{r_{-1}}(f(z_{r_i})) \neq 0$, then $r_i \rightsquigarrow r_1$, but for our choice of r_1, $r_1 \rightsquigarrow r_i$. However, since f belongs to J, $c_{r_{-1}}(f(z_{r_i})) = 0$, and this is a contradiction. Then $M(\omega)$ has no non-trivial idempotent endomorphisms, but this means that $M(\omega)$ is indecomposable. Indeed, if $M(\omega) = M_1 \oplus M_2$, then there exist the natural projection π_1 and injection ι_1. Set $f = \iota_1\pi_1$, this is an idempotent endomorphism different than zero.

Now we need to prove the second part of the theorem. Let $M(\omega) \overset{\psi}{\simeq} M(\omega')$. If r is a vertex of Γ_ω, then, by Lemma 2.2.2, $\psi(z_r) = \sum_i \lambda_i z_{r_i}$, with $c_{r_i}(\psi(z_r)) = \lambda_i \in \mathbb{K}$ and r_i vertices of Γ_ω such that $r \rightsquigarrow r_i$. Then $z_r = \psi^{-1}(\sum_{i=1}^m \lambda_i z_{r_i}) = \sum_{i=1}^m \lambda_i \psi^{-1}(z_{r_i})$ and $\psi^{-1}(z_{r_i}) = \sum_s \lambda_s z_s$, where $c_s(\psi^{-1}(z_{r_i})) = \lambda_s \in \mathbb{K}$ and s vertices of Γ_ω such that $r_i \rightsquigarrow s$. Then $z_r = \sum_i \lambda_i \sum_s \lambda_s z_s$. Since z_r is an element of a basis of a vector space, it must exist at least one i, such that $r \rightsquigarrow r_i \rightsquigarrow r$. By Lemma 2.2.1, it exists a quiver isomorphism $\Theta : \omega \rightsquigarrow \omega'$.

□

2.3 Projective presentations of a string modules

Gentle algebras are a particular class of string algebras where we need to put more rules on the relations of the quiver such that, for instance, if an arrow has two arrows with which it can be composed, one of them will be in a relation, while the other not. Formally we have:

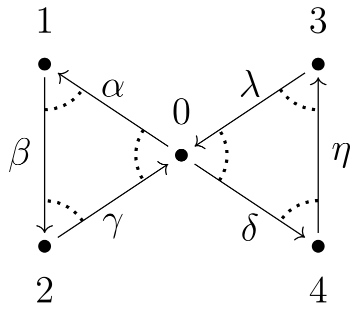
Definition 2.3.1. A **gentle quiver** $\mathrm{Q} := ((\mathrm{Q}_0, \mathrm{Q}_1), I)$ is a bound quiver, i.e. a directed graph with relations, such that:

- each vertex $v \in \mathrm{Q}_0$ has at most two incoming and two outgoing arrows,
- I is generated by paths of length exactly two,
- for any arrow $\beta \in \mathrm{Q}_1$, there is at most one arrow $\alpha \in \mathrm{Q}_1$ such that $t(\alpha) = s(\beta)$ and $\alpha\beta \notin I$, and there is at most one arrow $\alpha' \in \mathrm{Q}_1$, such that $t(\alpha') = s(\beta)$ and $\alpha\beta \in I$,

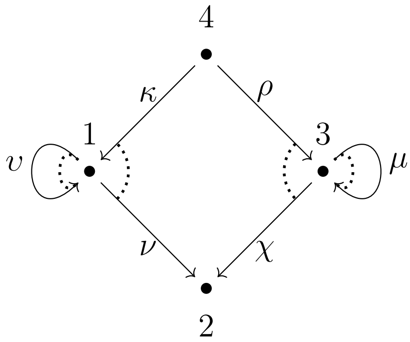
- for any arrow $\beta \in \mathrm{Q}_1$, there is at most one arrow $\gamma \in \mathrm{Q}_1$ such that $t(\beta) = s(\gamma)$ and $\beta\gamma \notin I$ and there is at most one arrow $\gamma' \in \mathrm{Q}_1$ such that $t(\beta) = s(\gamma')$ and $\beta\gamma' \in I$.

The algebra $\frac{k\mathrm{Q}}{I}$ associated to a gentle bound quiver is called a **gentle algebra.**

Example 2.3.1. The following are examples of gentle quivers, we have also indicated the generators of the admissible ideal.



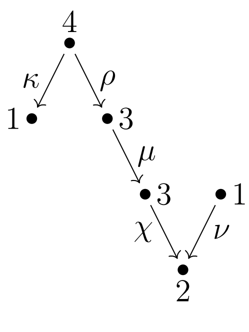
$\Lambda_1 = \frac{\mathbb{K}Q_1}{I_1}$, with I_1 generated by $\beta\alpha,\ \gamma\beta,\ \alpha\gamma,\ \delta\eta,\ \lambda\eta,$ and $\eta\delta$.



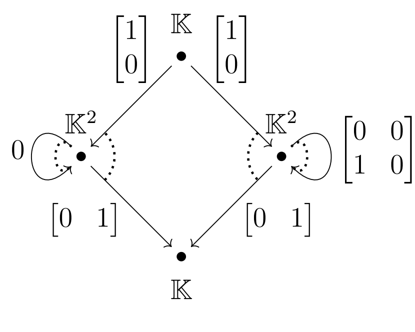
$\Lambda_2 = \frac{\mathbb{K}Q_2}{I_2}$, with I_2 generated by $\upsilon^2,\ \nu\kappa,\ \chi\rho,\ \mu^2$.

Since string algebras are a generalization of gentle algebras, everything we have proved so far is still true for gentle algebras. For instance the string modules are indecomposable. From now on, when we talk about strings, we will always consider strings to be gentle, finite and up to inverses. Also the quivers will be always gentle, so Λ will corresponds to a gentle algebra, i.e. is basic, connected and finite-dimensional.

Example 2.3.2. Referring to the gentle algebra Λ_2 of the example above, we present a gentle string and its string module associated.



The string
$\omega = \kappa\rho^{-1}\mu^{-1}\chi^{-1}\nu$



The left Λ_2-module associated
$M(\omega)$

The aim of this section is to construct the projective presentation of a string gentle module, and, in doing so, we will also see how to compute its projective resolution. We will show that, given a string ω, the combinatorial information provided by the string is all we need to know in order to give a complete description of the relative projective presentation.

2.3.1 Projective indecomposable modules

Let a be a vertex of Q. We start by proving that $P(a)$, the projective indecomposable module corresponding to the vertex a is a string module. Firstly, we construct the string $p(a)$, which, as we will prove, generates $P(a)$.

Construction of $p(a)$.

By (S1), any vertex is the starting point of at most two arrows. If there is none starting from a, then $p(a)$ is the string of length zero $p(a) = e_a$.

Consider the set S of all the possible path starting from a. If a is the starting point of two arrows, call them α_1 and α_{-1}, then $S = S_1 \dot\cup S_{-1}$, where $S_{\pm 1}$ is the subset of S containing all the path starting from $\alpha_{\pm 1}$. If there is only one arrow starting from a, then we can define, without loss of generality, $S = S_1$ and $S_{-1} = \emptyset$.

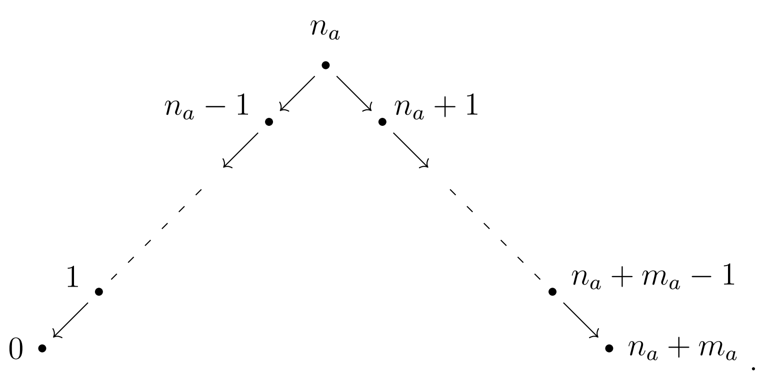
We can order the element of $S_{\pm 1}$ by their length and, since there exists only a finite number of paths starting with a, due to (S3), there exists a maximal path in S_1, resp. S_{-1} (if it is not the empty set), call it M_1, resp. M_{-1}. Any path with length greater than zero and starting with a is either a subpath of M_1 or a subpath of M_{-1}, depending on the starting arrow. In particular, this means that the maximal paths, if they exist, are respectively unique.

Let n_a be the length of M_1 and m_a the length of M_{-1}, condition (S3) ensures that n_a and m_a are natural number not infinite. Denote the maximal paths $M_1 = \alpha_{n_a}\alpha_{n_a-1}\dots\alpha_1$ and $M_{-1} = \alpha_{-m_a}\alpha_{-(m_a-1)}\dots\alpha_{-1}$, with α_i in Q_1 for each $-m_a \le i \le -1, 1 \le i \le n_a$.

Finally, we set:

$$\begin{aligned} p(a) &= M_1(M_{-1})^{-1} = \\ &= \alpha_{n_a}\alpha_{n_a-1}\dots\alpha_1(\alpha_{-m_a}\alpha_{-(m_a-1)}\dots\alpha_{-1})^{-1} = \alpha_{n_a}\alpha_{n_a-1}\dots\alpha_1\alpha_{-1}^{-1}\dots\alpha_{-m_a}^{-1} \\ &= p(a)_1 p(a)_2 \dots p(a)_{n_a} p(a)_{n_a+1} \dots p(a)_{n_a+m_a}. \end{aligned}$$

To summarize in a less formal and detailed way, $p(a)$ consists of the two paths of maximum length starting from a, which will become the unique peak vertex of the string. From now on, the quiver underlying $p(a)$ will be denoted with Γ_a and the relative map by F_a.



Γ_a, the quiver underlying $p(a)$

Proposition 2.3.1. *For each vertex a in Q_0, the projective indecomposable module $P(a)$ associated to that vertex is isomorphic to the string module $M(p(a))$ generated by the string $p(a)$.*

Proof. We need to construct an isomorphism

$$\phi : M(p(a)) \to P(a),$$

where $M(p(a)) = (M(p(a))_b, f_\alpha)_{b\in Q_0, \alpha\in Q_1}$ is the module generated by the string $p(a)$, and $P(a) = (P(a)_b, \cdot\alpha)_{b\in Q_0,\alpha\in Q_1}$ is the indecomposable projective module corresponding to the vertex a as described in Lemma 1.2.5.

Let b be a vertex of Q, by the same lemma, $P(a)_b = e_a A e_b$ is the vector space with, as a basis, the paths starting from a and arriving in b. Let q be an element of this basis, i.e. is a path such that $a = s(q)$ and $b = t(q)$ of length $j \geq 0$. If it has length greater than zero, it must be either a subpath of M_1 or of M_{-1}, starting with the same arrow respectively i.e. $q = \alpha_j \ldots \alpha_1$ or $q = \alpha_{-j} \ldots \alpha_{-1}$. Then q corresponds also to a substring of $p(a)$ or $p(a)^{-1}$, indeed $q = p(a)_{n_a-j+1} \ldots p(a)_{n_a}$ or $q = \big(p(a)_{n_a+1} \ldots p(a)_{n_a+j}\big)^{-1}$.

Define

$$\phi_b(q) = \begin{cases} z_{n_a-j} & \text{if } q = \alpha_j \ldots \alpha_1, \\ z_{n_a+j} & \text{if } q = \alpha_{-j} \ldots \alpha_{-1}, \\ z_{n_a} & \text{if } j = 0, \end{cases}$$

with z_i basis vector of $M(p(a))_{F_a(i)}$. Observe that $\phi_b(q)$ is well-defined, indeed $F_a(n_a - j) = b$ if and only if $s(p(a)_{n_a-j}) = b$, and if $q = \alpha_j \ldots \alpha_1$, then $t(p(a)_{n_a-j+1}) = s(p(a)_{n_a-j}) = b$. Equivalently also the other cases work. The construction of ϕ is extended by $\mathbb{K}$-linearity on the other elements of $P(a)_b$.

Note that ϕ_b is an isomorphism of $\mathbb{K}$-vector spaces for each vertex $b \in Q_0$. Indeed, by construction ϕ_b is a $\mathbb{K}$-linear map and, just by looking at the definition, one can see that is also injective, since we assign to each element of the basis of $P(a)_b$ an element of the basis of $M(p(a))_b$. We just need to prove that this correspondence works also in the other sense, i.e. that ϕ_b is surjective. Let z_k be a basis vector of $M(p(a))_b$, then, by definition, $F_a(k) = b$, namely $b = s(p(a)_k)$ if $k \neq 0$, or $b = t(p(a)_k)$ if $k = 0$. So, if $k < n_a$, then $\alpha_{n_a-k} \ldots \alpha_1$ is a path in Q from a to b and $\phi_b(\alpha_{n_a-k} \ldots \alpha_1) = z_k$. While, if $k > n_a$, then $\alpha_{n_a+k} \ldots \alpha_{-1}$ corresponds to the preimage of z_k and if $k = n_a$ $\phi_b(e_a) = z_k$. Thus, we showed a one to one correspondence between elements of the basis, proving injectivity and surjectivity of ϕ_b.

Now we just have to prove that $\phi = (\phi_b)_{b\in Q_0}$ is a morphism between representations, i.e. given an arrow $\alpha : b \to c$ in Q_1 the following commutes:

$$\begin{array}{ccc} P(a)_b & \xrightarrow{\cdot\alpha} & P(a)_c \\ \downarrow{\scriptstyle \phi_b} & & \downarrow{\scriptstyle \phi_c} \\ M(p(a))_b & \xrightarrow{f_\alpha} & M(p(a))_b. \end{array}$$

By definition, given q, different from zero, basis element of $P(a)_b$ of length j, we have

$$f_\alpha(\phi_b(q)) = \begin{cases} z_{n_a-j-1} & \text{if } \ p(a)_{n_a-j} = \alpha_{j+1} = \alpha, \\ z_{n_a+j+1} & \text{if } \ p(a)_{n_a+j+1} = \alpha^{-1}_{-(j+1)} = \alpha^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

While

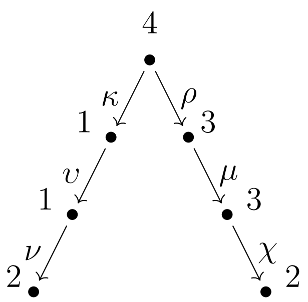
$$\phi_b(q \cdot \alpha) = \begin{cases} z_{n_a-j-1} & \text{if } \ q \cdot \alpha = \alpha_{j+1} \dots \alpha_1, \\ z_{n_a+j+1} & \text{if } \ q \cdot \alpha = \alpha_{-(j+1)} \dots \alpha_{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

and we can conclude, because if $q \cdot \alpha$ is different from zero, then $q \cdot \alpha$ is a path of length $j + 1$ which starts with the same arrow as q.

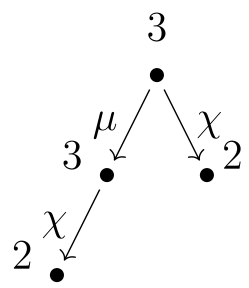
□

From now on, the string $p(a)$ that generates $P(a)$ will be referred to as the projective string associated with the vertex a. When the vertex is clear from context, we will simply call it the projective string.

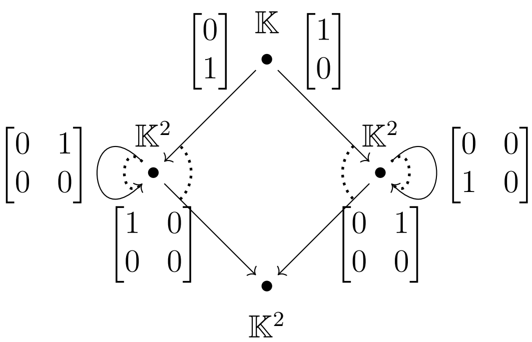
Example 2.3.3. We present here some projective modules over Λ_2, as defined in 2.3.1, and the respective projective strings.



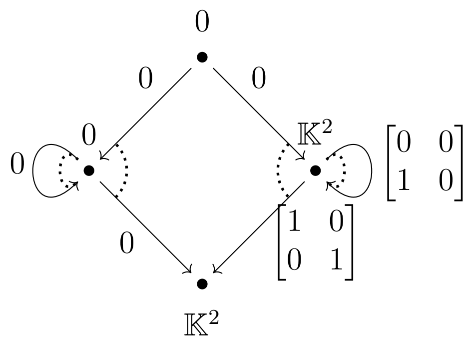
String generating $P(4)$ in Λ_2



String generating $P(3)$ in Λ_2



$P(4)$



$P(3)$

Partial maps between projective strings

Let a, b be vertices in Q_0 and let $p(a) = p(a)_1 \dots p(a)_{n_a} p(a)_{n_a+1} \dots p(a)_{n_a+m_a}$, and $p(b) = p(b)_1 \dots p(b)_{n_b} p(b)_{n_b+1} \dots p(b)_{n_b+m_b}$ be the projective strings related. Let Γ_a, Γ_b, be the respective underlying quivers and F_a, F_b the relative maps.

It is well-known that there is a one-to-one correspondence between

$$\begin{array}{ccc} \{\text{Basis elements of } \operatorname{Hom}_\Lambda(P(a), P(b))\} & \underset{1:1}{\longleftrightarrow} & \{\text{path in } Q \text{ from } b \text{ to } a\} \\ u \to p \cdot u & \longleftrightarrow & p. \end{array}$$

Thanks to Lemma 2.2.2, we also have the following bijection:

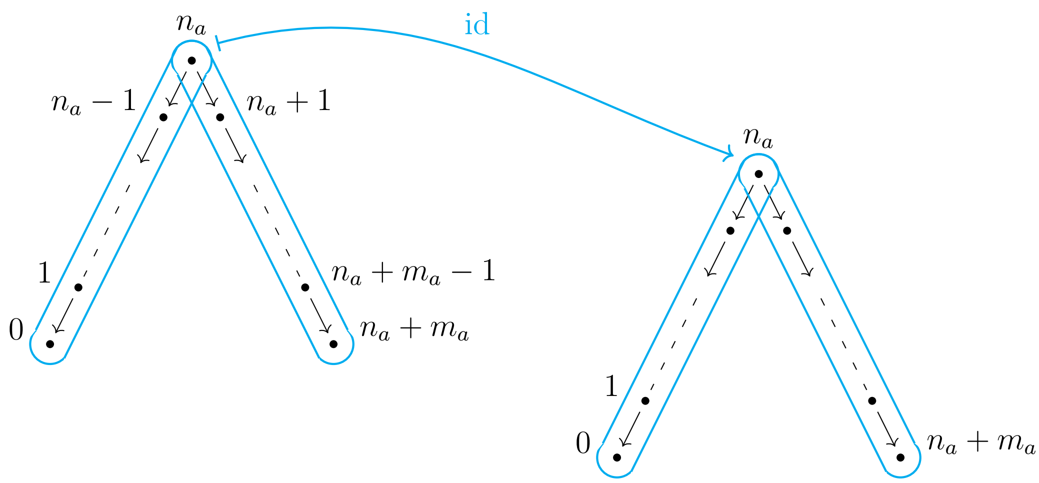
$$\begin{array}{ccc} \{\text{Basis elements of } \operatorname{Hom}_\Lambda(P(a), P(b))\} & \underset{1:1}{\longleftrightarrow} & \{\text{partial map between } p(a) \text{ and } p(b)\} \\ f_\Theta & \longleftrightarrow & \Theta. \end{array}$$

We now aim to explicitly demonstrate the correspondence between partial maps $\Theta : p(a) \rightsquigarrow p(b)$ and paths $p : b \to a$. To begin, we will describe in detail the possible partial maps between $p(a)$ and $p(b)$.

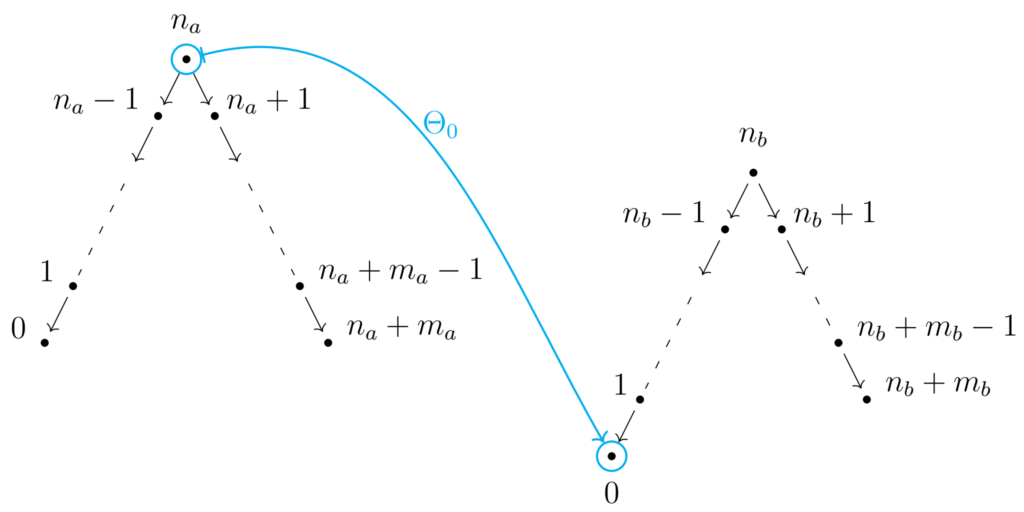
Recall that a partial map Θ between $p(a)$ and $p(b)$ is a quiver isomorphism from $D_\Theta \subseteq \Gamma_a$, closed under predecessors, to $R_\Theta \subseteq \Gamma_b$, closed under successor. So, by construction of $p(a)$, D_Θ must contain the vertex n_a of Γ_a for all Θ.

Let Θ be a partial map such that $x \in \Gamma_{b,0} = \{0, \dots, n_b + m_b\}$ is the vertex image of n_a. By Lemma 2.2.1, given the image of a vertex, a partial map is defined univocally, i.e. if it exists Θ partial map s.t. $\Theta(n_a) = x$, this is unique. We denote this map Θ_x. By definition of partial map $F_b \circ \Theta_x = F_a$, then $F_b(x) = a$. Up to replacing $p(b)$ with its inverse, we can assume that x is less or equal than n_b, without loss of generality.

If $x = n_b$, then $b = F_b(n_b) = a$, so $p(a) = p(b)$ and $\Theta \in \operatorname{Aut}\Gamma_a$. By Lemma 2.2.1, then Θ_{n_b} is the identity, and its module homomorphism associated corresponds to the trivial path. The partial map Θ_{n_b} is visualized below:



Now, we consider the case when $x \neq n_b$. If $x = 0$, we observe that zero is a sink vertex of Γ_b and n_a is a source vertex of Γ_a, so, since Θ is a quiver isomorphism, D_{Θ_0} is made of just the vertex n_a and R_{Θ_0} is made of just the vertex zero. Namely, Θ_0 sends the vertex n_a to the vertex 0 of Γ_b and nothing else, as shown in the picture below.



If $x \neq 0$, x is not a source vertex in Γ_b (indeed, by construction, n_b is the only source vertex), so x is the head of an arrow and the tail of another, while n_a is a source, i.e. is the head of two arrows. Each vertex $x-i$ for $1 \leq i \leq x$ is a successor of x in Γ_b, then $x-i$ must belong to R_Θ for $1 \leq i \leq x$. This implies, being Θ a quiver isomorphism, that either n_a+i or n_a-i must belong to D_Θ for $1 \leq i \leq x$ and moreover $\Theta(n_a \pm i) = x-i$. Since $F_b \circ \Theta = F_a$, $p(b)_1 \dots p(b)_x = \beta_1 \dots \beta_x$ is equal to either $(p(a)_{n_a+1} \dots p(a)_{n_a+x})^{-1} = \alpha_{n_a+x} \dots \alpha_{n_a+1}$ or to $p(a)_{n_a-x+1} \dots p(a)_{n_a} = \alpha_{n_a-x+1}, \dots \alpha_{n_a}$.

Observe that, by construction of $p(b)$, $\beta_1 \dots \beta_x$ is a maximal path of length x in Λ starting from $F_b(\jmath(p(b)_1)) = F_b(x) = a$. However, as showed during the construction of the string $p(a)$, the maximal path starting from a are at most two and are respectively unique. They correspond to $\alpha_1 \dots \alpha_{n_a}$ and $\alpha_{n_a+m_a} \dots \alpha_{n_a+1}$. This means that x could be either equal to n_a or to m_a. Refer to Figure 2.1.

Summarizing, let $\Theta : p(a) \rightsquigarrow p(b)$ be a partial map inducing $n_a \rightsquigarrow x$, then, up to replacing $p(b)$ with its inverse, x could only be equal to $0, n_a, m_a$; or n_b if $a = b$, we have no other possibilities.

Now, our goal is to link the Λ-module homomorphism $f_x = f_{\Theta_x} : P(a) \to P(b)$ associated to one of the above possible partial map Θ_x to the multiplication on the left by a path in Λ with head in b and tail equal to a.

- If $x = n_b$, we've already seen that $a = b$ and the partial Θ_{n_b} map correspond to the identity between the underlying quivers, then its associated module homomorphism is the identity, which is linked with the trivial path e_a.

In the other cases, consider the substring

$$p := p(b)_{x+1} \dots p(b)_{n_b-1} p(b)_{n_b} = \beta_{x+1} \dots \beta_{n_b-1} \beta_{n_b},$$

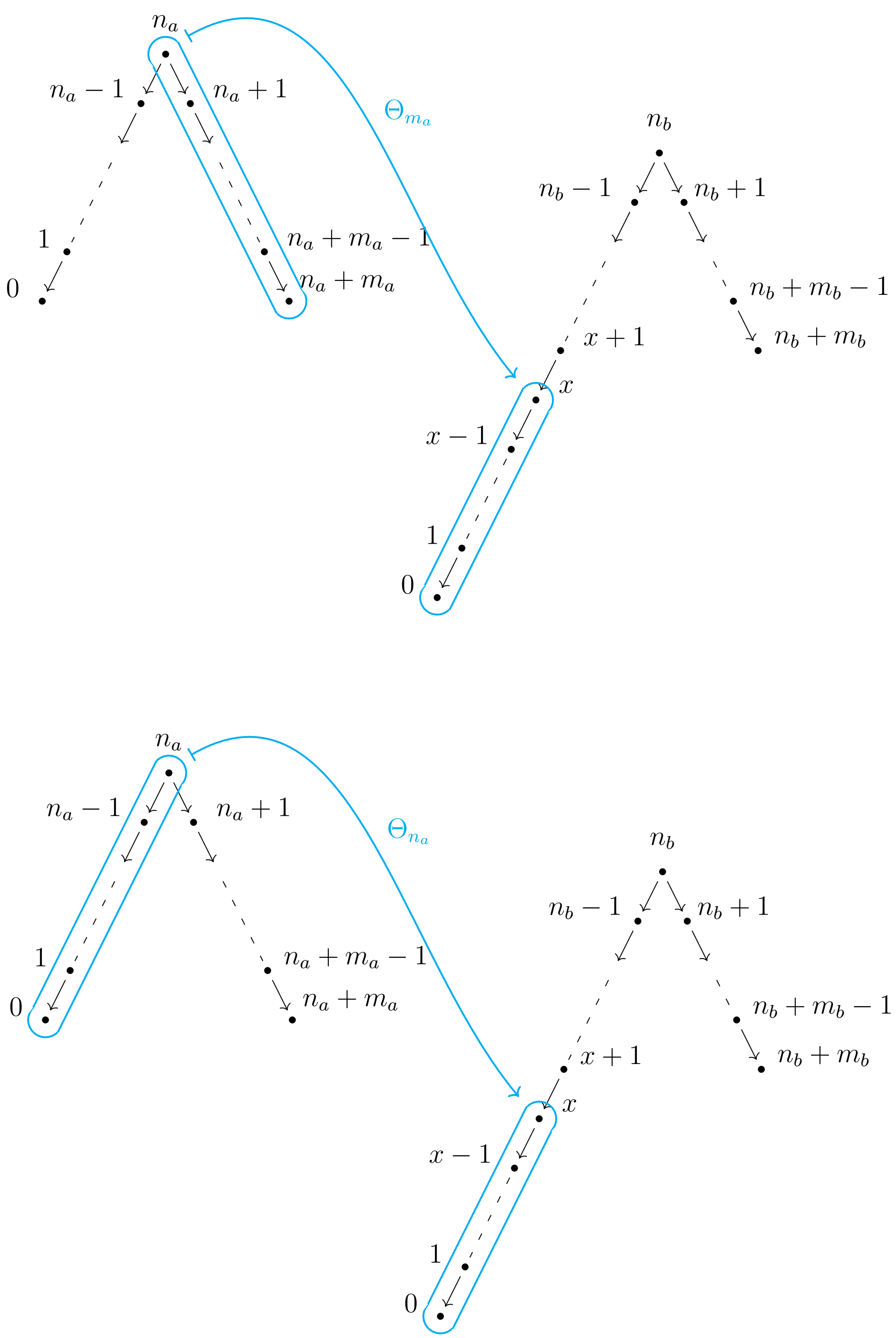

n_a
$n_a - 1$
$n_a + 1$
Θ_{m_a}
n_b
$n_b - 1$
$n_b + 1$
1
$n_a + m_a - 1$
0
$n_a + m_a$
$n_b + m_b - 1$
$x + 1$
$n_b + m_b$
x
$x - 1$
1
0
n_a
$n_a - 1$
$n_a + 1$
Θ_{n_a}
n_b
$n_b - 1$
$n_b + 1$
1
$n_a + m_a - 1$
0
$n_a + m_a$
$n_b + m_b - 1$
$x + 1$
$n_b + m_b$
x
$x - 1$
1
0

Figure 2.1

which is a path in Q from b to a, since $a = F_a(n_a) = F_b(\Theta_x(n_a)) = F_b(x) = F_b(t(p(b)_{x+1}))$, while $F_b(n_b) = F_b(s(\beta_{n_b})) = b$.

Now we prove that the module homomorphism f_x works the same as the multiplication on the right by p.

Let c be another vertex of Q and $q \in P(a)_c = e_c A e_a$ a path from a to c of length $j \geq 0$, if $j \neq 0$ then $q = p(a)_{n_a-j+1} \dots p(a)_{n_a} = \alpha_{n_a-j+1} \dots \alpha_{n_a}$ or $q = (p(a)_{n_a+1} \dots p(a)_{n_a+j})^{-1} = \alpha_{n_a+j} \dots \alpha_{n_a+1}$, if $j = 0$, q is the trivial path e_a.

- If $x = 0$, p is a path of maximal length starting from b and ending in a. So the multiplication on the right by p correspond to the map:

$$q \cdot p = \begin{cases} p & \text{if } \text{ j=0}, \\ 0 & \text{otherwise}. \end{cases}$$

  This corresponds to the module homomorphism f_0 which sends z_{n_a} in z_0 and is zero otherwise.

- If $x \neq 0$ and $x \neq n_b$:

$$q \cdot p = \begin{cases} p & \text{if } j = 0, \\ qp & \text{if } q = p(b)_{x-j+1} \dots p(b)_x, \\ 0 & \text{otherwise}, \end{cases}$$

  indeed, the multiplication is not zero if and only if q starts with $p(b)_x$.

Since f_x is defined with vector coordinates, we may use the isomorphism described in Proposition 2.3.1, for which

$$\phi_c(q) = \begin{cases} z_{n_a-j} & \text{if } q = p(a)_{n_a-j+1} \dots p(a)_{n_a}, \\ z_{n_a+j} & \text{if } q = (p(a)_{n_a+1} \dots p(a)_{n_a+j})^{-1}, \\ z_{n_a} & \text{if } j = 0, \end{cases}$$

and

$$\phi_c(p(q)) = \begin{cases} z_x & \text{if } j = 0, \\ z_{x-j} & \text{if } q = p(b)_{x-j-1} \dots p(b)_x, \\ 0 & \text{otherwise}. \end{cases}$$

Then we divide the last two cases remaining

- If $x = n_a$ then $p(a)_1 \dots p(a)_{n_a} = p(b)_1 \dots p(b)_x$ so

$$\Theta_{n_a}(\phi_c(q)) = \begin{cases} z_{n_a} & \text{if } j = 0, \\ z_{n_a-j} & \text{if } q = p(a)_{n_a-j+1} \dots p(a)_{n_a}, \\ 0 & \text{otherwise}, \end{cases}$$

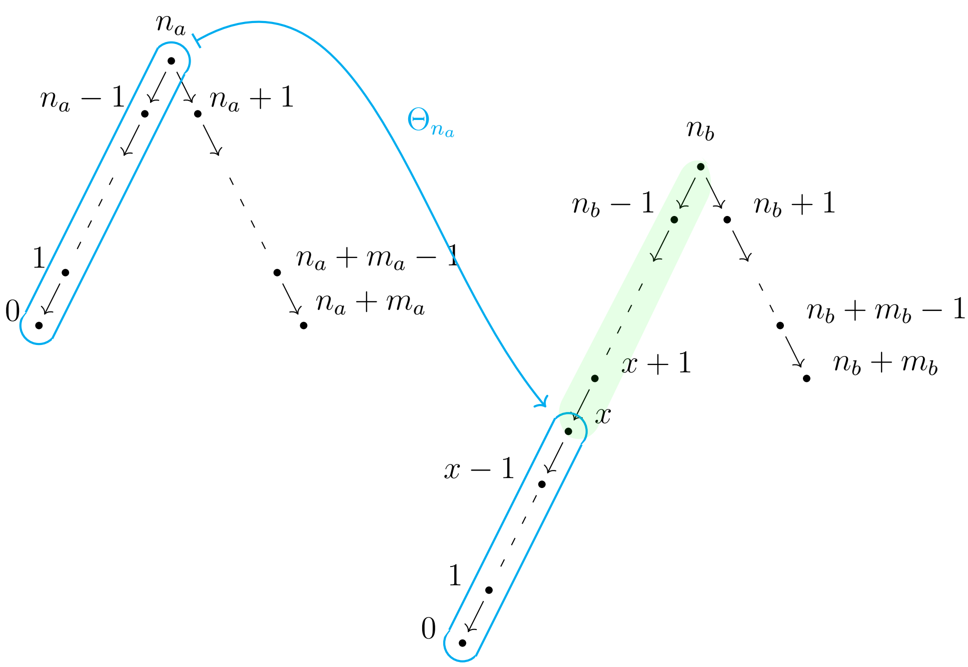
$$= \begin{cases} z_x & \text{if } j = 0, \\ z_{x-j} & \text{if } q = p(b)_{x-j+1} \dots p(b)_x, \\ 0 & \text{otherwise}. \end{cases}$$

- If $x = m_a$, then $(p(a)_{n_a+1} \dots p(a)_{n_a+m_a})^{-1} = p(b)_1 \dots p(b)_x$ and so

$$\Theta_{m_a}(\phi_c(\beta)) = \begin{cases} z_{m_a} & \text{if } j = 0, \\ z_{m_a-j} & \text{if } q = (p(a)_{n_a+1} \dots p(a)_{n_a+j})^{-1}, \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} z_x & \text{if } j = 0, \\ z_{x-j} & \text{if } q = p(b)_{x-j+1} \dots p(b)_x, \end{cases}$$

With this, we showed how to link a partial map between indecomposable projective string modules to a path.

The path p associated with the map Θ_x can be easily visualized by looking at the strings, as shown below, where the partial map is colored in cyan and the path is highlighted in green:



We need to show that the correspondence we have just created also works in the other sense. So, consider p a path in Q from b to a of length $j > 0$. It must start with $p(b)_{n_b}$ or with $p(b)^{-1}_{n_b+1}$. Up to replacing $p(b)$ with its inverse, we can assume, without loss of generality, that it starts with $p(b)_{n_b}$. Then p is a subpath of $p(b)_1 \dots p(b)_{n_b}$, in particular, it must be equal to $p(b)_{n_b-j+1} \dots p(b)_{n_b}$. If $j \neq n_b$, by construction of the string $p(b)$, $\beta_1 \dots \beta_{n_b+j}$ is a maximal path starting with a, then it must be of length either

m_a or n_a. Thus, if $j = n_b - n_a$, we have $p = p(b)_{n_a+1} \dots p(b)_{n_b}$, if $j = n_b - m_a$, we get $p = p(b)_{m_a+1} \dots p(b)_{n_b}$. So, all paths, with length other than zero or n_b, from b to a are of the type $p_{x+1} \dots p_{n_b}$, with $x = n_a, m_a$. With this last observation, we can conclude.

Due to this one-to-one correspondence, we will use the term *path* to refer to both the partial map and the module homomorphism associated when dealing with maps between projective indecomposable modules.

2.3.2 Projective covers of string modules

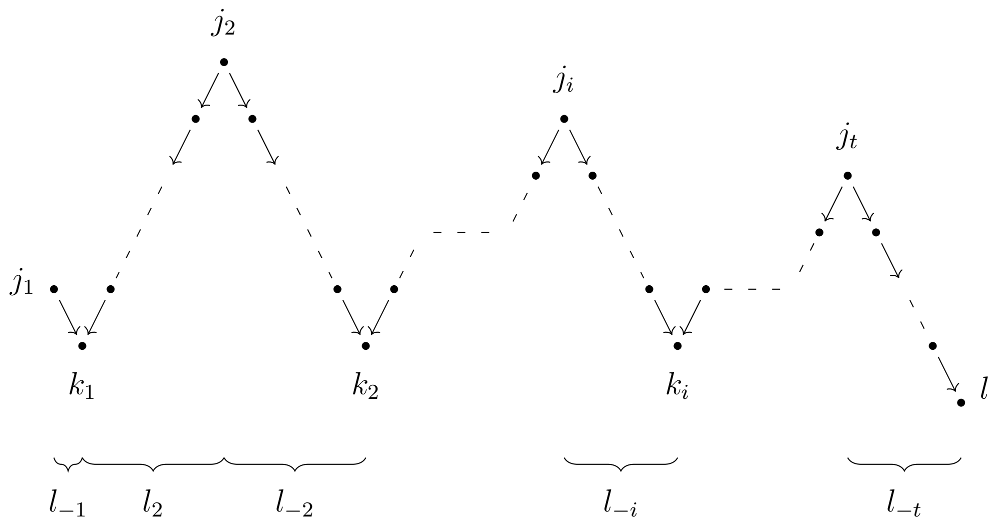
Now that we understand how to compute projective indecomposables and the morphisms between them, we will conclude this chapter by finally demonstrating the computations necessary to fully describe the projective presentation of a string module. Be aware that the following construction will be very detailed, often making it difficult and annoying to follow. We therefore encourage the reader to refer to the accompanying figures for guidance.

Proposition 2.3.2. *Given ω a finite string, the projective cover $P(\omega)$ of the string module $M(\omega)$ is:*

$$P(\omega) = \bigoplus_{i \text{ peak vertices of } \omega} P(F_\omega(i))$$

Proof. Let $\omega = \omega_1 \dots \omega_l$ a string and $M(\omega)$ its associated module. The idea behind this proof is to construct the projective cover following the recipe given in the proof of Theorem 1.1.5.

The following picture presents the notations we will use for the rest of this section, we will after define it formally:



Given a vertex a in Q_0, set $J_a := \{j \in \{0, \dots, l\}$ such that $F_\omega(j) = a$ and j is a peak$\}$, which is a subset of $\Gamma_{\omega,0}$. Let $J := \dot{\bigcup}_{a \in Q_0} J_a$. Since the string is finite, J is a finite set, so we can order its element and rename it such that $J = \{j_1, \dots, j_t\}$, where $t = |J| = \sum_{J_a \neq \emptyset} |J_a|$.

With the same reasoning, define a subset of $\Gamma_{\omega,0}$ as $K_a := \{k \in \{0,\dots,l\} \mid F_\omega(k) = a$ and k is a deep$\}$. Let $K := \dot\bigcup_{a\in Q_0} K_a$. Set $\tilde{K} = K - \{0,l\}$. If K does not contain the indices 0 and l, $K = \tilde{K}$. Note that each deep vertex of the string, which belongs to $\tilde{K}$, lies between two peak vertices. Hence we can order the elements of $\tilde{K}$ and rename them such that $\tilde{K} = \{k_1,\dots,k_{t-1}\}$, where $j_i < k_i < j_{i+1}$ for $i = 1,\dots,t-1$. Then K, by definition, could be either $\{0,k_1,\dots,k_{t-1}\}$, or $\{0,k_1,\dots,k_{t-1},l\}$, or $\{k_1,\dots,k_{t-1}\}$ or $\{k_1,\dots,k_{t-1},l\}$.

For $i = 1,\dots,t-1$ we call $l_{-i} = k_i - j_i$ and for $i = 2,\dots,t$ $l_i = j_i - k_{i-1}$, while $l_1 = j_1$ and $l_{-t} = l - j_t$. Note that l_1 and l_t could also be zero if neither zero nor l are not deep vertices, i.e if $K = \tilde{K}$.

Now our goal is to prove that $T := \operatorname{top}(M(\omega))$ is equal to $\bigoplus_{i=1}^t S(F_\omega(j_i))$. In order to do this, we start by looking at the radical of $M(\omega)$. By Lemma 1.2.4, we know the explicit description of radicals of any quiver representation, M. We recall it briefly: $R = \operatorname{rad}(M) = (R_a,\nu_\alpha)_{a\in Q_O,\alpha\in Q_1}$, where $R_a = \sum_{\alpha:b\to a}\operatorname{Im}(f_\alpha : M_b \to M_a)$ and $\nu_\alpha = f_\alpha|_{R_a}$ for every a in Q_0 and α in Q_1. Set $R(\omega) = R(M(\omega))$.

Since we are dealing with string quivers, by (S1), for any vertex there are at most two arrows with it as a target. Let a be a vertex of Q and let, if they exist, α and β be the two arrows for which $t(\alpha) = t(\beta) = a$. Then $R(\omega)_a = \operatorname{Im}(f_\alpha) + \operatorname{Im}(f_\beta)$. If one of them (or both) does not exist, we could just set $\operatorname{Im}(f_\alpha)$, resp. $\operatorname{Im}(f_\beta)$, to be zero.

We want to prove that

$$R(\omega)_a = \bigoplus_{i \,\mid\, F_\omega(i)=a \text{ and } i\notin J_a} \mathbb{K}z_i.$$

If $a = F_\omega(j_i)$ for some j_i in J_a, then z_{j_i} is a base vector of $M(\omega)_a$, but not of $R(\omega)_a$. Indeed, by definition, z_{j_i} belongs to the image of f_α if and only if w_i is inverse and α is equal to $(w_i)^{-1}$ or if w_{i+1} is direct and α is equal to w_{i+1}. But since j_i is a *peak*, this is not possible. Equivalently, it is not in the image of f_β. Conversely, if $F_\omega(h) = a$ and $h \notin J_a$, then, not only z_h is a base vector of $M(\omega)_a$, but also of $R(\omega)_a$. Indeed, by definition, $h = t(\omega_h) = s(\omega_{h+1})$ and since h is not a *peak* in Γ_h, either both ω_h and ω_{h+1} have the same *direction* or ω_h is inverse and ω_{h+1} is direct, i.e. h is a deep vertex. If h is a deep or if they are both direct, implies that $a = t(F_\omega(\gamma_h)) = t(\omega_h)$, so ω_h is equal to α or β, while if they are both inverse ω_h is the inverse of an arrow ending in a, namely α or β. And so z_h is a basis vector of $R(\omega)_a$.

By definition $T = \frac{M}{\operatorname{rad}(M)} = (T_a,\tau_\alpha)_{a\in Q_0,\alpha\in Q_1}$, then for any vertex

$$T_a = \frac{\bigoplus_{i \,\mid\, F_\omega(i)=a}\mathbb{K}z_i}{\bigoplus_{i \,\mid\, F_\omega(i)=a \text{ and } i\notin J_a}\mathbb{K}z_i} = \bigoplus_{j_i\in J_a}\mathbb{K}z_i$$

and $\tau_\alpha = 0$ for every arrow, since we are factoring through the image. This show that

$$T = S(F_\omega(j_1)) \oplus S(F_\omega(j_2)) \oplus \cdots \oplus S(F_\omega(j_t))$$

and for the construction presented in the proof of Theorem 1.1.5, we can conclude, obtaining

$$P(\omega) = \bigoplus_{j_i\in J} P(F_\omega(j_i)).$$

□

To simplify notation, from now on, we denote for $i = 1, \dots, m$, $P(j_i) = P(F_\omega(j_i))$ generated by the projective string $p^i = p^i_1, \dots p^i_{n_i} p^i_{n_i+1} \dots p^i_{m_i+n_i}$. The vectors of the basis of $P(j_i)_{F_\omega(h)}$ will be called x^i_h for $h = 1, \dots, n_i + m_i$. The simple module $S(j_i) = S(F_\omega(j_i))$ is generated by the string of length zero $s^i = e_{F_\omega(j_i)}$, which will be also as we denote the only vector of the basis of $S(j_i)_{F_\omega(j_i)}$.

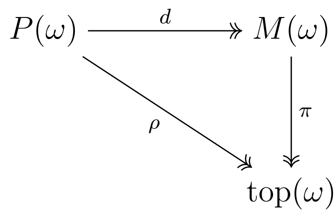
Proposition 2.3.3. *Given ω a finite string, with the notations introduced before, the epimorphism between the projective cover $P(\omega)$ of the string module and the module itself $M(\omega)$ is:*

$$d_\omega = (f_{\Theta_i})_{i=1,\dots,t}$$

where Θ_i is the partial map $P(j_i) \rightsquigarrow M(\omega)$, inducing $n_i \rightsquigarrow j_i$.

Proof. We want to describe the module homomorphism $d = d_\omega$ between $M(\omega)$ and $P(\omega)$.

By construction in Theorem 1.1.5, ϕ is the surjective module homomorphism which makes the following diagram commute



where π and ρ are just the quotient projection. Since $P(\omega) = \oplus_{j_i \in J} P(j_i)$, d could be seen as a vector of homomorphism $(d_i)_i$, with $d_i = d|_{P(j_i)} : P(j_i) \to M(\omega)$. Since d_i is a module homomorphism between string modules, by Lemma 2.2.2, d_i could be written in a unique way as a linear combination of module homomorphisms associated to a partial map $\Theta : p^i \rightsquigarrow \omega$, namely $d_i = \sum_{\Theta : p^i \rightsquigarrow \omega} \lambda_\Theta f_\Theta$.

Moreover, we have that

$$\pi_a : M_a = \bigoplus_{i=0,\dots,l \mid F_\omega(i)=a} \mathbb{K} z_i \to T_a = \bigoplus_{j_i \in J_a} S(j_i)$$

sends the element of the basis z_{j_i} in s^i and is zero otherwise. Since $\pi d = \rho$, $\pi^{-1}(s^i) = \{z_{j_i}\}$ and $d(x^i_{n_i}) = s^i$, then $d(x_{n_i}) = d_i(x_{n_i}) = z_{j_i}$.

By Lemma 2.2.1, there is a unique partial map Θ_i from the underlying quiver of $P(j_i)$ to the underlying quiver of ω, which sends n_i in j_i. We represent Θ_i (for $i \neq 0, m$) in Figure 2.2.

Note that even if there are other partial maps ν between p^i and ω, then λ_ν, the coefficient of ν in $d_i = \sum_{\Theta : p^i \rightsquigarrow \omega} \lambda_\Theta f_\Theta$ must be zero. Indeed, for any partial map $\nu : p^i \rightsquigarrow \omega$, we have, since $F_\omega \circ \nu = F_{p^i}$ and ν is a quiver isomorphism, $\nu(n_i) \in J_a$. Call the image of n_i through ν, j_ν. This means that $d_i(x_{n_i}) = z_{j_i} + \sum_\nu \lambda_\nu z_{j_\nu}$, with $\lambda_\nu \in \mathbb{K}$ for any ν partial map. But then we get $\sum_\nu \lambda_\nu z_{j_\nu} = 0$ and, since this is a linear combination of elements of a basis, $\lambda_\nu = 0$.

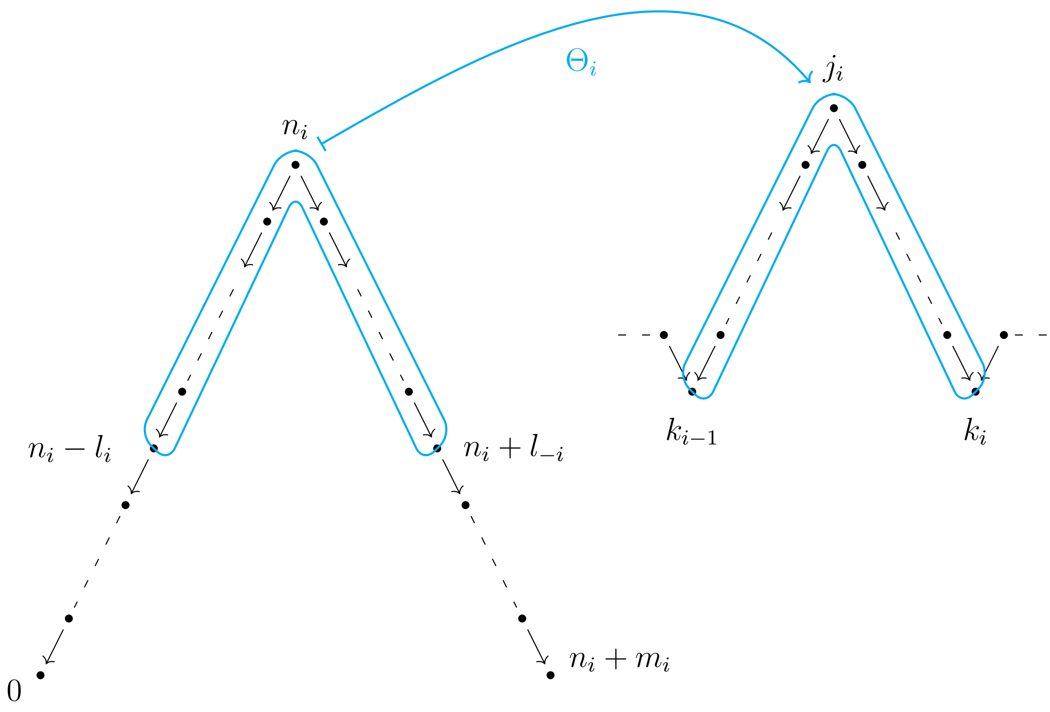


Figure 2.2

Hence $d_i = f_{\Theta_i}$, for each $i = 1, \ldots, t$, in particular, we showed that

$$d(x^i_h) = z_{\Theta_i(h)} = \begin{cases} z_{h-(n_i-l_i)+k_{i-1}} & \text{if } n_i - l_i \leq h \leq n_i, \\ z_{h-n_i+j_i} & \text{if } n_i \leq h \leq n_i + l_{i-1}, \\ 0 & \text{otherwise,} \end{cases}$$

for each $i = 1, \ldots, t$. □

Syzygies of string modules

Now call $(\Omega(\omega), \iota)$ the kernel of $(P(\omega), d)$, this exists since we are in an abelian category. Set $\Omega(\omega) = (\Omega(\omega)_a, n_\alpha)_{a \in \mathrm{Q}_0, \alpha \in \mathrm{Q}_1}$ and $\iota = (\iota_a)_{a \in \mathrm{Q}_0}$, by definition $\iota \circ d = 0$ and each ι_a is injective.

Let $\alpha : a \to b$ be an arrow, since ι is a morphism between representations, the following must commute:

$$\begin{array}{ccc} \Omega(\omega)_a & \xrightarrow{\iota_a} & P(\omega)_a \\ \downarrow n_\alpha & & \downarrow f_\alpha \\ \Omega(\omega)_b & \xrightarrow{\iota_b} & P(\omega)_b \end{array} .$$

Call, for $i = 1, \dots, t-1$, $u_i = n_{i+1} - l_{i+1}$ and $r_i = n_i + l_{-i}$. If $u_i \neq 0$, observe that the substring $p_1^{i+1} \dots p_{n_{i+1}-l_{i+1}}^{i+1}$ of p^{i+1}, by construction, correspond to a maximal path with head in $F_{p^{i+1}}(x_{n_{i+1}-l_{i+1}}^{i+1}) = F_\omega(\Theta_{i+1}(x_{n_{i+1}-l_{i+1}}^{i+1})) = F_\omega(k_i)$. If $r_i \neq n_i + m_i$, the same is true for the substring $p_{n_i+l_{-i}+1}^i \dots p_{n_i+m_i}^i$ of p^i, which correspond to an inverse of a maximal path of Q with head in $F_{p^i}(x_{n_i-l_{-i}}^i) = F_\omega(\Theta_i(x_{n_i-l_{-i}}^i)) = F_\omega(k_i)$. They are of length u_i, respectively r_i.

This implies that if we define

$$q^i = \begin{cases} q_1^i \dots q_{u_i}^i q_{u_i+1}^i \dots q_{u_i+r_i}^i := p_1^{i+1} \dots p_{n_{i+1}-l_{i+1}}^{i+1} p_{n_i+l_{-i}+1}^i \dots p_{n_i+m_i}^i & \text{if } u_i \neq 0 \text{ and } r_i \neq n_i + m_i, \\ q_1^i \dots q_{u_i}^i := p_1^{i+1} \dots p_{n_{i+1}-l_{i+1}}^{i+1} & \text{if } r_i = n_i + m_i, \\ q_1^i \dots q_{r_i}^i := p_{n_i+l_{-i}+1}^i \dots p_{n_i+m_i}^i & \text{if } u_i = 0, \\ e_{F_\omega(k_i)} & \text{if } u_i = 0 \text{ and } r_i = n_i + m_i, \end{cases}$$

then q^i corresponds to the string generating $P(k_i)$.

We showed that, for $i = 1, \dots, t-1$, $d(x_h^i) = 0$ and so x_h^i belongs to the image of ι for $0 \le h \le n_i - l_i - 1$, if $n_i - l_i \neq 0$, and $n_i + l_{-i} + 1 \le h \le n_i + m_i$, if $n_i + l_{-i} \neq n_i + m_i$. Moreover, we have $\Theta^i(n_i + l_{-i}) = \Theta^{i+1}(n_{i+1} + l_{i+1})$, or equivalently $d(x_{n_i+l_{-i}}^i) = d(x_{n_{i+1}-l_{i+1}}^{i+1}) = z_{k_i}$, so also $x_{n_i+l_{-i}}^i - x_{n_{i+1}-l_{i+1}}^{i+1}$ belongs to the image of ι.

We will address the case when both $r_i \neq n_i + m_i$ and $u_i \neq 0$, as it is the most general scenario. The other cases can be derived in a similar manner, requiring only adjustments to the indices.

Thus, denote,

$$\begin{aligned} y_{u_i}^i &= \iota^{-1}(x_{n_{i+1}-l_{i+1}}^{i+1} - x_{n_i+l_{-i}}^i), \\ y_h^i &= \iota^{-1}(x_h^{i+1}) && \text{if } 0 \le h \le u_i - 1, \\ -y_h^i &= \iota^{-1}(x_{n_i+l_{-i}+h-u_i}^i) && \text{if } u_i + 1 \le h \le u_i + r_i, \end{aligned}$$

where, by the definition of the kernel of a morphism between representations, each y_h^i is a vector basis element of $\Omega(\omega)_{F_{p^{i+1}}(h)}$ if $0 \le h \le u_i$, or of $\Omega(\omega)_{F_{p^i}(n_i+l_{-i}+h-u_i)}$ if $u_i + 1 \le h \le u_i + r_i$.

Let $\alpha : a \to b$ be an arrow, then

$$f_\alpha(\iota(y_h^i)) = \begin{cases} f_\alpha(x_h^{i+1}) & \text{if } 0 \le h \le u_i - 1, \\ f_\alpha(-x^i_{n_i+l_{-i}+h-u_i}) & \text{if } u_i + 1 \le h \le u_i + r_i - 1, \\ f_\alpha(x^{i+1}_{n_{i+1}-l_{i+1}} - x^i_{n_i+l_{-i}}) & \text{if } h = u_i, \end{cases}$$
$$= \begin{cases} x^{i+1}_{h-1} & \text{if } 1 \le h \le u_i - 1 \text{ and } \alpha = q^i_h, \\ -x^i_{n_i+l_{-i}+h-u_i+1} & \text{if } u_i + 1 \le h \le u_i + r_i - 1 \text{ and } \alpha^{-1} = q^i_{h+1}, \\ -x^i_{n_i+l_{-i}+1} & \text{if } h = u_i \text{ and } \alpha^{-1} = q^i_{u_i+1}, \\ x^{i+1}_{n_{i+1}-l_{i+1}-1} & \text{if } h = u_i \text{ and } \alpha = q^i_{u_i}, \\ 0 & \text{otherwise.} \end{cases}$$

By the commutativity of the diagram above, we can describe how n_α works, indeed:

$$\begin{array}{ccc} y^i_h & \xrightarrow{\iota_a} & x^{i+1}_h \\ {\scriptstyle n_\alpha}\downarrow & & \downarrow{\scriptstyle f_\alpha} \\ y^i_{h-1} & \xrightarrow[\iota_b]{} & x^{i+1}_{h-1} \end{array} \qquad \begin{array}{ccc} y^i_h & \xrightarrow{\iota_a} & -x^i_{n_i+l_{-i}+h-u_i} \\ {\scriptstyle n_\alpha}\downarrow & & \downarrow{\scriptstyle f_\alpha} \\ y^i_{h+1} & \xrightarrow[\iota_b]{} & -x^i_{n_i+l_{-i}+h-u_i+1} \end{array}$$

i.e.

$$n_\alpha(y^i_h) = \begin{cases} y^i_{h-1} & \text{if } 1 \le h \le u_i \text{ and } \alpha = q^i_h, \\ y^i_{h+1} & \text{if } u_i \le h \le u_i + r_i - 1 \text{ and } \alpha^{-1} = q^i_{h+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that we have just described the module $P(j_i)$, namely, $P(j_i)$ is a direct summand of $\Omega(\omega)$ for $i = 1, \dots, t-1$. Given that the proof may be challenging to follow due to the various indices and names, refer to Figure 2.3 for clarity, in order to understand how ι works.

The remaining description of $\Omega(\omega)$ depends on what happens at the start and, symmetrically, at the end, of the string ω. Denote, if it exists, by ω_0, the unique direct arrow which makes $\omega_0\omega_1 \dots \omega_l$ a string.

We will consider later the case when ω_0 does not exist. Denote $k_0 = t(\omega_0)$.

Consider the maximal path q_0 starting in k_0 such that $q_0\omega_0$ is different than zero. Observe that q_0 could also have length zero. Then p^1, the projective string generating $P(j_1)$, up to replacing it with its inverse, must start with q_0. The length of q_0 is $n_1 - l_1 - 1$, denote this number u_0. Set

$$q_0^+ = p^1_1 \dots p^1_{u_0} = q^0_1 \dots q^0_{u_0}.$$

If q_0 has length zero, it means that k_0 is a sink vertex in Q_0, then $\Omega(\omega)_0 := S(k_0)$.

Recall that $d(x^1_h) = 0$ for $0 \le h \le n_1 - l_1$ and so we can denote

$$y^0_h = \iota^{-1}(x^1_h) \quad \text{if } 0 \le h \le u_0,$$

with y_h^0 is an element of the basis of the $\mathbb{K}$-vector space $\Omega(\omega)_{F_{p^1}(h)}$ Repeating the same reasoning as above

$$n_\alpha(y_h^0) = \begin{cases} y_{h-1}^0 & \text{if } 1 \leq h \leq u_0 \text{ and } \alpha = q_h^0, \\ 0 & \text{otherwise.} \end{cases}$$

In this case $\Omega(\omega)_0 := M(q_0^+)$

Symmetrically consider the end of the string. Denote by ω_{l+1}, if it exists, the direct arrow which makes $\omega_1 \dots \omega_l \omega_{l+1}$ a string. Again, we will deal later with the case when ω_{l+1} does not exist. Denote $k_t = \mathfrak{s}(\omega_{l+1})$.

Consider q^t the inverse of the maximal path starting in k_t such that $w_{l+1}^{-1} q^t$ is different than zero. Call u_t the length of q_t. If $u_t = 0$, set $\Omega(\omega)_t := S(k_t)$. Since p^t must end with q^t, we get that $u_t = m_t - (l - j_t) - 1$. So set the string

$$q_t^- = p_{n_t+l_{-t}+2}^t \dots p_{n_t+m_t}^t = q_1^t \dots q_{u_t}^t.$$

Again $d(x_h^t) = 0$ for $n_t + l_{-t} + 2 \leq h \leq n_t + m_t$ and denote

$$y_h^0 = \iota^{-1}(x_h^t) \quad \text{if } n_t + l_{-t} + 2 \leq h \leq n_t + m_t,$$

the vertex in $\Omega(\omega)_{F_{p^t}(h)}$ and, so,

$$n_\alpha(y_h^t) = \begin{cases} y_{h+1}^t & \text{if } n_t + l_{-t} + 2 \leq h \leq u_t - 1 \text{ and } \alpha = q_h^t \\ 0 & \text{otherwise} \end{cases}.$$

In this case $\Omega(\omega)_t := M(q_t^-)$.

Summarizing, we have just described the module generated by the strings q^0, and, symmetrically, the one generated by q^t. If ω_0, resp ω_{l+1}, does not exists set $\Omega(\omega)_0 = 0$, resp. $\Omega(\omega)_t = 0$. Then $\Omega(\omega)$ is the direct summand of the modules generated by the strings q^i, namely

$$\Omega(\omega) = \Omega(\omega)_0 \oplus \bigoplus_{i=1}^{t-1} P(k_i) \oplus \Omega(\omega)_t = \bigoplus_{i=0}^{t} \Omega(\omega)_i$$

.

We can describe the module homomorphism ι as a matrix, $(\iota_{h,i})_{h=1,\dots,t;\, i=0,\dots,t}$ where each component $\iota_{h,i}$ is a Λ-module homomorphism from $\Omega(\omega)_i \to P(j_h)$. Observe that for $i \neq 0$ and $i \neq t$, $\iota_{h,i}$ corresponds to a module homomorphism between indecomposable projectives $P(k_i) \to P(j_h)$.

For $i \neq 0$, and $i \neq t$, $\iota_{i+1,i} : P(k_i) \to P(j_{i+1})$ is the module homomorphism corresponding to the path

$$q_i^+ = \omega_{n_{i+1}-l_{i+1}} \dots \omega_{n_{i+1}} = \omega_{k_i+1} \dots \omega_{j_{i+1}}.$$

While, $\iota_{i,i}$ is minus the homomorphism corresponding to the inverse of

$$w_{n_i+l_{-i}} \dots w_{n_i+m_i} = w_{j_i+1} \dots w_{k_i},$$

call the inverse of this substring, which is a path in Q, q_i^-. Moreover for $h \neq i$ and $h \neq i+1$, $\iota_{h,i} = 0$.

In other words, $\iota_{i,i} = f_{\Theta_{q_i^+}}$ and $\iota_{i+1,i} = -f_{\Theta_{q_i^-}}$, where $\Theta_{q_i^+} : q^i \rightsquigarrow p^{i+1}$ is the unique partial map sending u_i in $n_{i+1} - l_{i+1}$, while $\Theta_{q_i^-} : q^i \rightsquigarrow p^i$ is the partial map inducing $u_i \rightsquigarrow n_i + l_{-i}$. Below are pictured the partial map, and the path associated for each $i \neq 0$ and $i \neq t$.

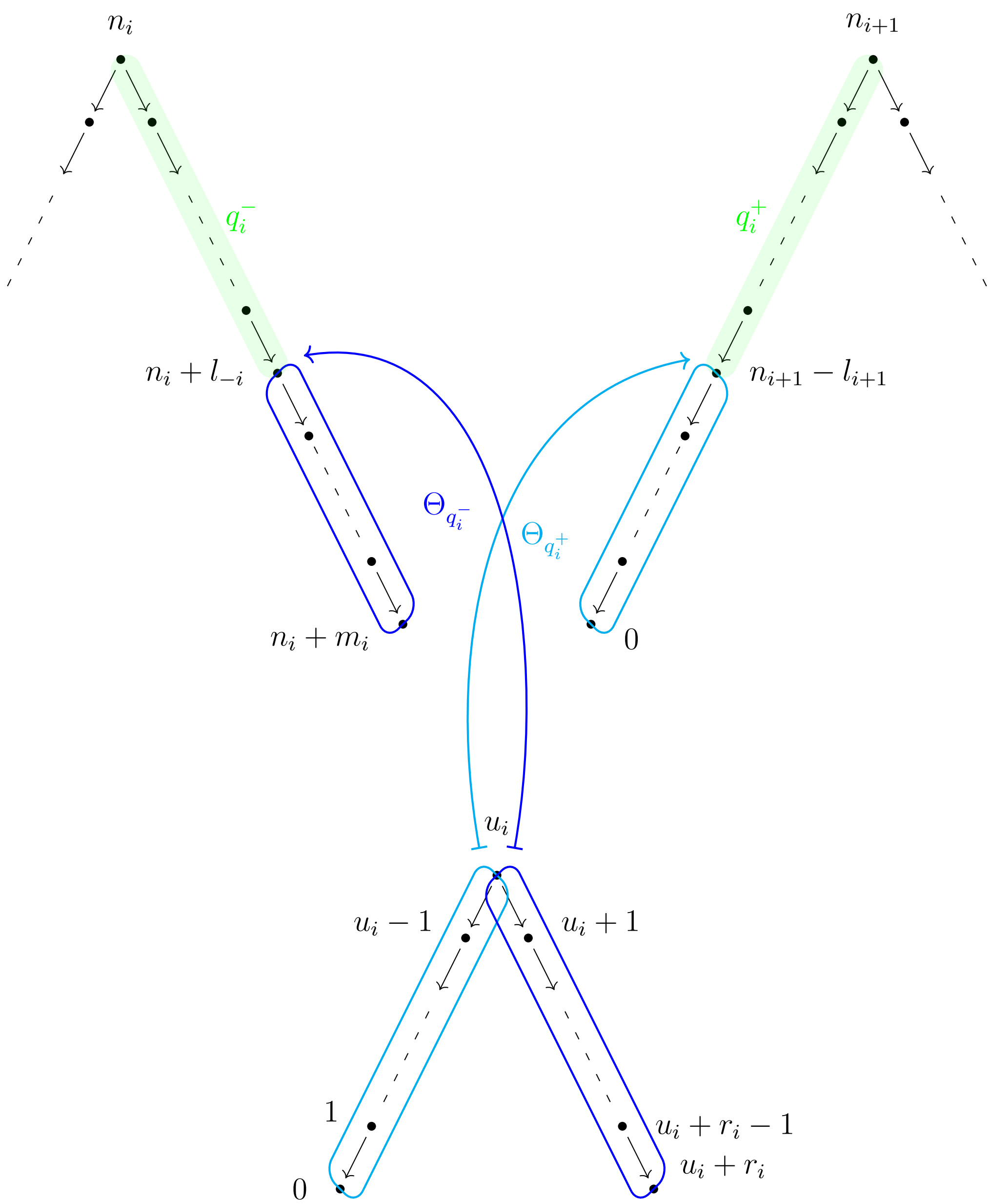
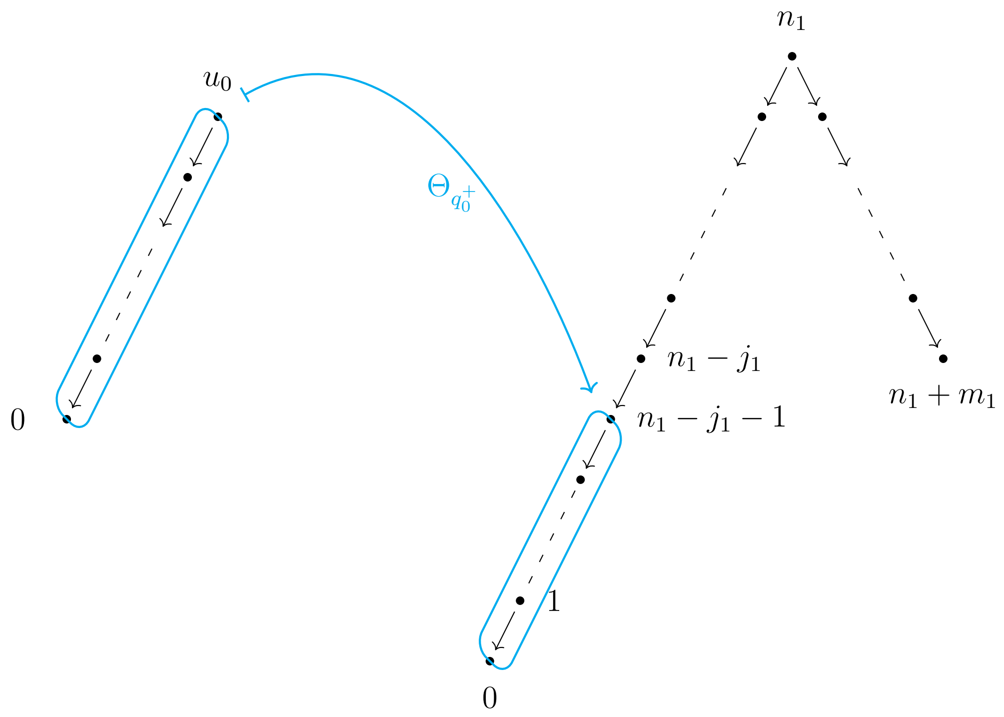
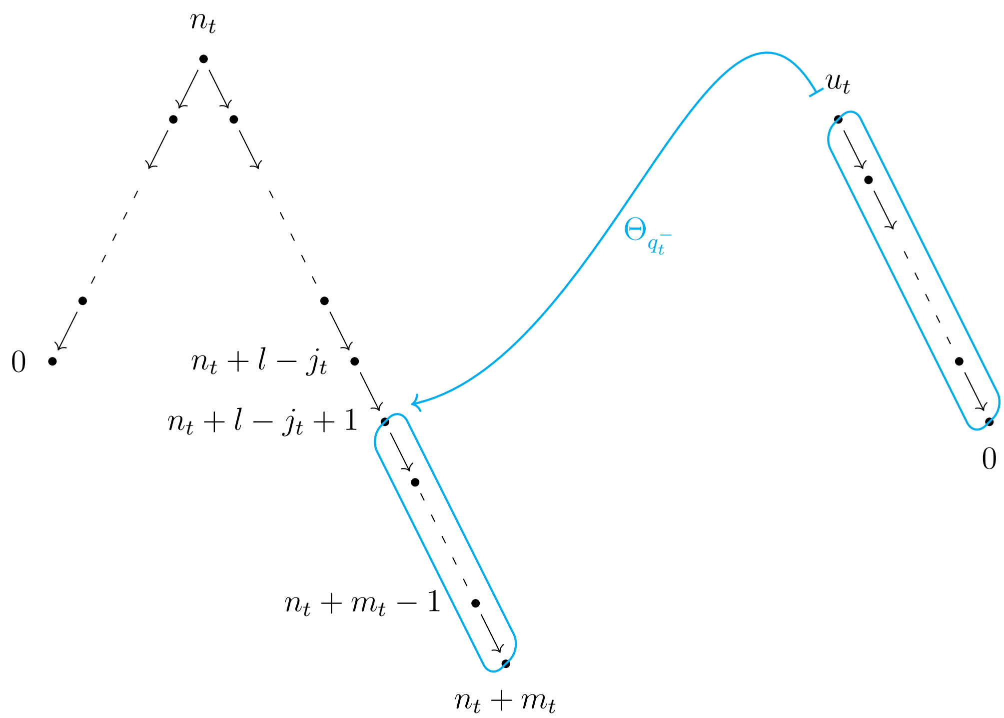


Figure 2.3: $\iota_{i,i}$ and $\iota_{i+1,i}$

Now consider $i = 0$, then $\iota_{1,0}$ is equal to $f_{\Theta_{q_0^+}}$, where $\Theta_{q_0^+} : q^0 \rightsquigarrow p^1$ is the unique partial map which sends u_0 in $n_1 - j_1 - 1$, as pictured below:



Symmetrically, we have $\iota_{t,t} = -f_{\Theta_{q_t^-}}$, where $\Theta_{q_t^-} : q^t \rightsquigarrow p^t$ is the partial map inducing $u_t \rightsquigarrow n_t + l - j_t + 1$.



Putting all together, we get

$$\iota_{h,i} = \begin{cases} -q_i^- & \text{if } h = i \text{ and } i \neq 0, t, \\ q_i^+ & \text{if } h = i+1 \text{ and } i \neq 0, t, \\ f_{\Theta_{q_0^+}} & \text{if } i = 0 \text{ and } h = 1, \\ -f_{\Theta_{q_t^-}} & \text{if } i = t \text{ and } h = t, \end{cases}$$

namely

$$\iota = \begin{bmatrix} f_{\Theta_{q_0^+}} & -q_1^- & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & q_1^+ & -q_2^- & 0 & & & & & \vdots \\ \vdots & 0 & q_2^+ & -q_3^- & \ddots & & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & & & \vdots \\ \vdots & & & & & & \ddots & \ddots & \vdots \\ \vdots & & & & & \ddots & \ddots & -q_{t-1}^- & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & & 0 & q_{t-1}^+ & -f_{\Theta_{q_t^-}} \end{bmatrix}.$$

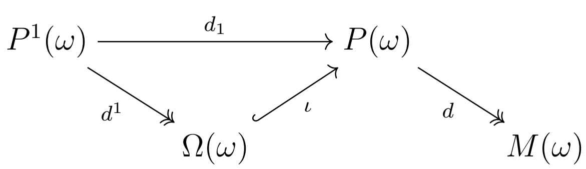
2.3.3 Projective presentations and resolutions of string modules

Given a string $\omega = \omega_1 \dots \omega_l$, to compute the projective presentation $P^1(\omega) \xrightarrow{d_1} P(\omega)$ of the string module $M(\omega)$, as described in 1.1.6, we need only to use the results obtained thus far, i.e. it is necessary just to know how to construct the projective cover of the syzygy and the associated epimorphism. We will continue to use the notations and terms introduced in this chapter.

Observe that the projective cover of a direct sum of modules is the direct sum of the projective cover of each summand. Hence $P^1(\omega) = P(\Omega(\omega)) = \bigoplus_{i=0}^{t} P(\Omega(\omega)_i)$. Referring to Proposition 2.3.2, the projective cover of the syzygy is the direct sum of the projectives correspondent to the peak vertices of the string. In particular, $\Omega(\omega)_i = P(k_i)$ for $1 \leq i \leq t-1$, then $P(\Omega(\omega)_i) = \Omega(\omega)_i = P(k_i)$, namely correspond to the projective module of the deep vertices k_i of the string ω. The peak vertex of the string q^0, resp. q^t generating $\Omega(\omega)_0$, resp. $\Omega(\omega)_t$, is k_0, resp. k_t, hence, if they exist, $P(\Omega(\omega)_0) = P(k_0)$, resp. $P(\Omega(\omega)_t) = P(k_t)$. Summarizing

$$P^1(\omega) = \bigoplus_{i=0}^{t} P(k_i).$$

The map $d_1 : P^1(\omega) \to P(\omega)$ is the composition of the surjective map $d^1 : P^1(\omega) \to \Omega(\omega)$ and the monomorphism $\iota : \Omega(\omega) \to P(\omega)$.



We can rewrite d^1 as a matrix as $d^1 = (d^1_{i,h})_{h=0,\ldots,t, i=0,\ldots,t}$. In particular for any $i = 0, \ldots, t$, $d^1_{i,i} = f_{\Theta_i} : P(k_i) \to \Omega(\omega)_i$, where Θ_i is the partial map sending the vertex k_i in u_i. It corresponds to the identity for $i \neq 0$, and $i \neq t$. Moreover, $d^1_{h,i}$ is equal to zero if $h \neq i$.

Then

$$d_1 = \iota \circ d^1 =$$

$$= \begin{bmatrix} f_{\Theta_{q_0^+}} & -q_1^- & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & q_1^+ & -q_2^- & 0 & & & \vdots \\ \vdots & 0 & q_2^+ & -q_3^- & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & -q_{t-1}^- & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & q_{t-1}^+ & -f_{\Theta_{q_t^-}} \end{bmatrix} \cdot \begin{bmatrix} f_{\Theta_0} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & id & 0 & & & & \vdots \\ \vdots & 0 & id & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & id & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & f_{\Theta_t} \end{bmatrix}$$

So d_1 is equal to

$$\begin{bmatrix} f_{\Theta_{q_0^+}} \circ f_{\Theta_0} & -q_1^- & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & q_1^+ & -q_2^- & 0 & & & \vdots \\ \vdots & 0 & q_2^+ & -q_3^- & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & -q_{t-1}^- & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & q_{t-1}^+ & -f_{\Theta_{q_t^-}} \circ f_{\Theta_t} \end{bmatrix},$$

where

- $d_{1_{0,0}} = f_{\Theta_{q_0^+}} \circ f_{\Theta_0} : P(k_0) \to \Omega(\omega)_0 \to P(j_0)$, being the composition of two module homomorphism related to two partial maps, which induces respectively $k_0 \rightsquigarrow u_0$ and $u_0 \rightsquigarrow n_1 - j_1 - 1$, then it corresponds to the module homomorphism of the composition of the partial maps. Thus $d_{1_{0,0}} = f_{\Theta_{q_0^+} \circ \Theta_0}$, where $\Theta_{q_0^+} \circ \Theta_0$ induces $k_0 \rightsquigarrow n_1 - j_1 - 1$, i.e. $d_{1_{0,0}}$ corresponds to the multiplication by the path q_0^+.
- Symmetrically, $d_{1_{t,t}} = -f_{\Theta_{q_t^-}} \circ f_{\Theta_t} = -f_{\Theta_{q_t^-} \circ \Theta_t} : P(k_t) \to P(j_t)$, where $\Theta_{q_t^-} \circ \Theta_t$ corresponds to the partial map $p(k_t) \rightsquigarrow p(j_t)$ inducing $k_t \rightsquigarrow n_t - j_t + 1 - l$, i.e. $d_{1_{t,t}}$ corresponds at minus the multiplication by the path q_t^-.

We refer to Figures 2.3 and 2.4 to better visualize these maps.

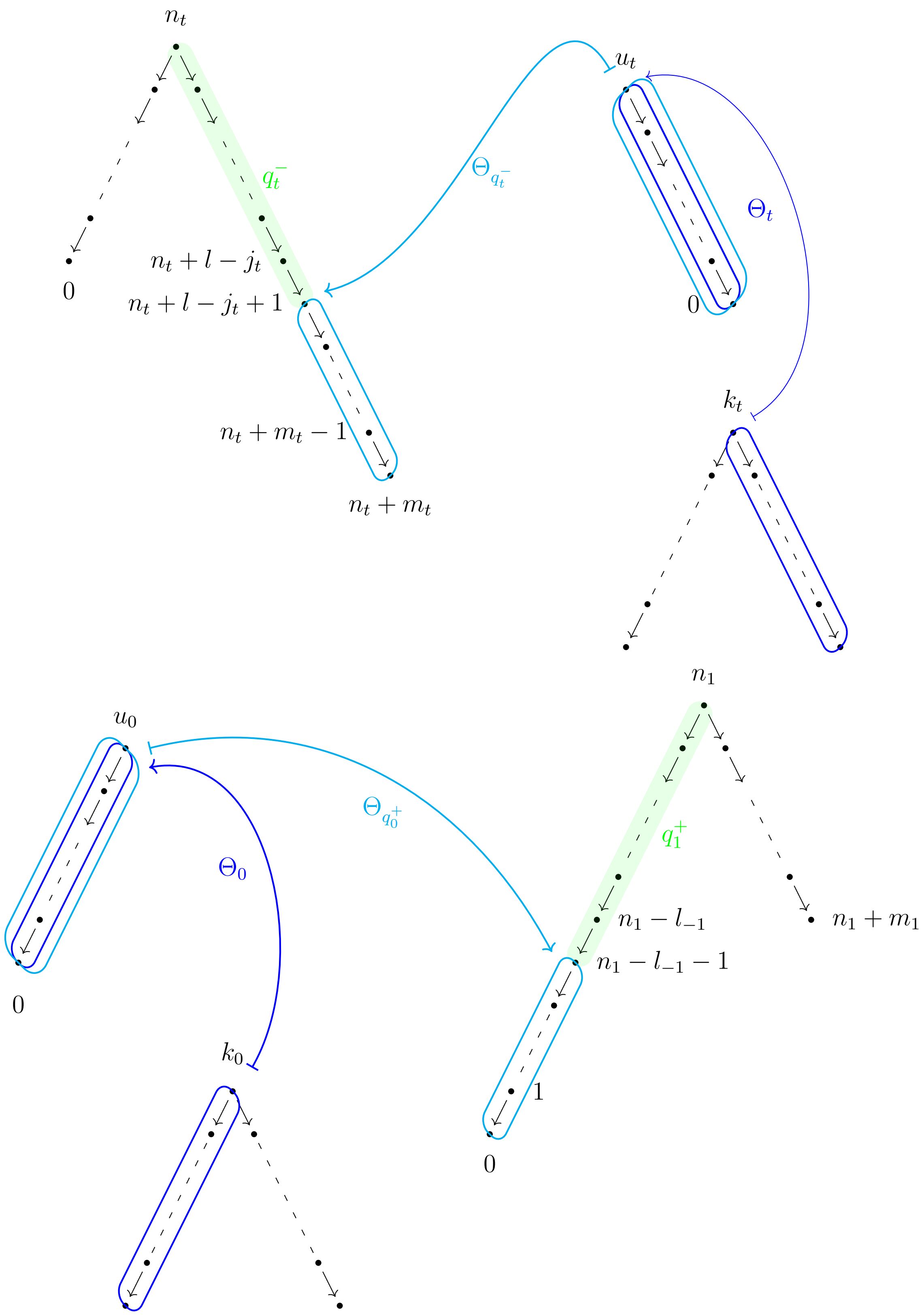


Figure 2.4: Visualization of $\Theta_{q_0^+} \circ \Theta_0$ and $\Theta_{q_t^-} \circ \Theta_t$

Concluding, we get that:

$$d_1 = \begin{bmatrix} q_0^+ & -q_1^- & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & q_1^+ & -q_2^- & 0 & & & \vdots \\ \vdots & 0 & q_2^+ & -q_3^- & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & -q_{t-1}^- & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & q_{t-1}^+ & -q_t^- \end{bmatrix}.$$

Example 2.3.4. Consider the string $\omega = \kappa\rho{-}1\mu^{-1}\chi^{-1}\nu$ of Example 2.3.2. Its minimal projective presentation $P(\omega) = P^1(\omega) \xrightarrow{d_1} P^0(\omega)$ is:

$$P^1(\omega) = P(1) \oplus P(2) \oplus P(1) \xrightarrow{d_1 = \begin{bmatrix} \upsilon\kappa & -\chi\mu\rho & 0 \\ 0 & \nu & -\mu \end{bmatrix}} P^0(\omega) = P(4) \oplus P(1)$$

We conclude this chapter with a brief overview of how to construct the projective resolution of a string module. Having demonstrated how to compute of the minimal projective presentation, the process now becomes relatively straightforward: one simply needs to replicate the steps outlined so far, following Corollary 1.1.6.

However, we want to just emphasize an important observation. Since $d^1_{i,i}$ corresponds to the identity for $i \neq 0$ and $i \neq t$, the only contributions to the kernel of this map come from Ω_0 and Ω_t. If these do not exist, then the minimal projective presentation is identical to the minimal projective resolution.

Instead, if they do exist, we must consider whether there is another arrow α_0 having as source k_0, resp. α_t with source k_t, distinct from q_1^0, resp. q_1^t. If such arrows exist, then P^2_ω will be equal to the direct sum of $P(\mathit{t}(\alpha_0))$ and $P(\mathit{t}(\alpha_t))$. Therefore, for $i \geq 2$, P^i_ω will have at most two summands.

Chapter 3

Two-term silting complexes

This chapter aims to prove the existence of a bijection between two-term silting complexes and support τ-tilting modules. This bijection was established by Adachi, Iyama, and Reiten in [AIR14], here we outline the steps leading to this bijection. In their paper, the authors demonstrated that support τ-tilting modules parametrize torsion pairs in module categories. This result implies that two-term silting complexes play a crucial role in controlling many homological properties of the homotopy category, while also representing a generalization of tilting module. Thus silting theory may be seen as a completion of tilting theory and this is one of the reasons why we are so interested in it.

Chapter 3 is structured as follows: first, we define a two-term silting complex and provide its characterizations. Next, we introduce the Auslander-Reiten translation τ and explore the initial connections between τ- rigid modules and two-term complexes. Finally, we arrive at the proof of the aforementioned bijection.

Definition 3.0.1. Let P in $\mathcal{K}$. We call P **silting** if

- $\mathrm{Hom}_{\mathcal{K}}(P, P[i]) = 0$ for any $i > 0$ and
- $\mathrm{thick}(P) = \mathcal{K}$,

where $\mathrm{thick}(P)$ is the smallest full subcategory of $\mathcal{K}$ which contains P and is closed under cones, $[\pm 1]$ shifts, direct summands and isomorphisms. If only the first condition is satisfied, we call P **presilting**. If P is a silting object in $\mathcal{K}^{[-1,0]}$, then it is a **two-term silting** object. We call silt-Λ the set of isomorphism classes of silting complexes for Λ. 2-silt Λ will denote the set of isomorphism classes of two-term silting complexes.

We note that Λ, viewed as the complex $\cdots \to 0 \to \Lambda \to 0 \to \dots$, is a silting complex in any degree.

3.1 Completion of silting complexes

Whenever two complexes M, N in $\mathcal{K}$ verify $\mathrm{Hom}_{\mathcal{K}}(M, N[i]) = 0$ for any $i > 0$, we denote it as $M \geq N$. We observe that, for any two-term complex P in $\mathcal{K}^{[-1,0]}$, we have $\Lambda \geq P \geq \Lambda[1]$.

The following proposition shows how a two-term presilting complex can be completed to a silting one.

Proposition 3.1.1 ([Aih13]). *Every two term presilting complex P for Λ is a direct summand of a two-term silting complex for Λ.*

Proof. Since $\operatorname{add} P$ is covariantly finite, see Section 1.5, there exists $f : P' \to \Lambda$ a right-$\operatorname{add} P$ approximation of Λ, with P' in $\operatorname{add} P$, i.e. there exists P'' in $\mathcal{K}$ and n natural number such that $P' \oplus P'' = P^n$. We can then create the triangle

$$T := V \to P' \xrightarrow{f} \Lambda \to V[1] \tag{3.1}$$

, by setting $C_f = V[1]$, thanks to the first axiom (TR1).

We want to show that $W := P \oplus V$ belongs to $2\text{-}\operatorname{silt} \Lambda$.

Firstly, we need to prove that $\operatorname{Hom}_{\mathcal{K}}(W, W[i])$ is equal to zero for every $i > 0$. Observe that

$$\begin{aligned}\operatorname{Hom}_{\mathcal{K}}(W, W[i]) \simeq \operatorname{Hom}_{\mathcal{K}}(P, P[i]) \oplus \operatorname{Hom}_{\mathcal{K}}(V, V[i]) \oplus \\ \oplus \operatorname{Hom}_{\mathcal{K}}(P, V[i]) \oplus \operatorname{Hom}_{\mathcal{K}}(V, P[i]),\end{aligned}$$

where $\operatorname{Hom}_{\mathcal{K}}(P, P[i]) = 0$, since P is presilting.

For any $i > 0$, we have a long exact sequence, thanks to Proposition 1.4.1:

$$\operatorname{Hom}_{\mathcal{K}}(P, U'[i-1]) \xrightarrow{\operatorname{Hom}_{\mathcal{K}}(P, f[i-1])} \operatorname{Hom}_{\mathcal{K}}(P, \Lambda[i-1]) \to \operatorname{Hom}_{\mathcal{K}}(P, V[i]) \to \operatorname{Hom}_{\mathcal{K}}(P, P'[i]).$$

Since $\operatorname{Hom}_{\mathcal{K}}(P, P[i]) = 0$, then $0 = \operatorname{Hom}_{\mathcal{K}}(P, P[i])^n \simeq \operatorname{Hom}_{\mathcal{K}}(P, P^n[i]) \simeq \operatorname{Hom}_{\mathcal{K}}(P, P'[i]) \oplus \operatorname{Hom}_{\mathcal{K}}(P, P''[i])$, so $\operatorname{Hom}_{\mathcal{K}}(P, P'[i]) = 0$.

Since f is a right $\operatorname{add} P$- approximation $\operatorname{Hom}_{\mathcal{K}}(P, f[i-1])$ is surjective. So the map $\operatorname{Hom}_{\mathcal{K}}(P, \Lambda[i-1]) \to \operatorname{Hom}_{\mathcal{K}}(P, V[i])$ is zero. This implies, by exactness of the sequence, that

$$\operatorname{Hom}_{\mathcal{K}}(P, V[i]) = 0 \text{ for } i > 0.$$

Consider now the sequence which arises from the triangle:

$$\operatorname{Hom}_{\mathcal{K}}(P', W[i]) \to \operatorname{Hom}_{\mathcal{K}}(V, W[i]) \to \operatorname{Hom}_{\mathcal{K}}(\Lambda[-1], W[i]) \to \operatorname{Hom}_{\mathcal{K}}(P'[-1], W[i]).$$

For any $i > 0$, $\operatorname{Hom}_{\mathcal{K}}(P', W[i]) \simeq \operatorname{Hom}_{\mathcal{K}}(P', P[i]) \oplus \operatorname{Hom}_{\mathcal{K}}(P', V[i])$. We just showed that $\operatorname{Hom}_{\mathcal{K}}(P, V[i])$ is equal to zero, then $0 = \operatorname{Hom}_{\mathcal{K}}(P, V[i])^n \simeq \operatorname{Hom}_{\mathcal{K}}(P', V[i]) \oplus \operatorname{Hom}_{\mathcal{K}}(P'', V[i])$, so $\operatorname{Hom}_{\mathcal{K}}(P', V[i]) = 0$. Equivalently, since P is presilting, $0 = \operatorname{Hom}_{\mathcal{K}}(P, P[i])^n \simeq \operatorname{Hom}_{\mathcal{K}}(P', P[i]) \oplus \operatorname{Hom}_{\mathcal{K}}(P'', P[i])$, then $\operatorname{Hom}_{\mathcal{K}}(P', P[i]) = 0$. Moreover, $\operatorname{Hom}_{\mathcal{K}}(P'[-1], W[i]) \simeq \operatorname{Hom}_{\mathcal{K}}(P', W[i+1])$ which is equal to zero for any $i > 0$. Then by exactness of the sequence

$$\operatorname{Hom}_{\mathcal{K}}(V, W[i]) \simeq \operatorname{Hom}_{\mathcal{K}}(\Lambda[-1], W[i]).$$

Consider the long exact sequence arising from the triangle 3.1, by applying the functor $\operatorname{Hom}_{\mathcal{K}}(\Lambda, -)$:

$$\mathrm{Hom}_{\mathcal{K}}(\Lambda, \Lambda[i]) \to \mathrm{Hom}_{\mathcal{K}}(\Lambda, V[i+1]) \to \mathrm{Hom}_{\mathcal{K}}(\Lambda, P'[i+1]).$$

Since Λ is silting, $\mathrm{Hom}_{\mathcal{K}}(\Lambda, \Lambda[i]) = 0$. Since $\Lambda \geq P$, also $\Lambda \geq P[1]$, i.e $0 = \mathrm{Hom}_{\mathcal{K}}(\Lambda, P[i+1])^n \simeq \mathrm{Hom}_{\mathcal{K}}(\Lambda, P''[i+1]) \oplus \mathrm{Hom}_{\mathcal{K}}(\Lambda, P'[i+1])$. So $\mathrm{Hom}_{\mathcal{K}}(\Lambda, P'[i+1])$ is equal to zero for $i > 0$. By exactness of the sequence

$$\mathrm{Hom}_{\mathcal{K}}(\Lambda, V[i+1]) = 0 \text{ for } i \geq 0.$$

Then $\mathrm{Hom}_{\mathcal{K}}(V, W[i]) \simeq \mathrm{Hom}_{\mathcal{K}}(\Lambda[-1], W[i]) \simeq \mathrm{Hom}_{\mathcal{K}}(\Lambda, W[i+1]) \simeq \mathrm{Hom}_{\mathcal{K}}(\Lambda, V[i+1]) \oplus \mathrm{Hom}_{\mathcal{K}}(\Lambda, P[i+1])$, which is zero for any $i > 0$. This show that W is presilting.

By definition, $\mathrm{thick}\, W$ is closed under direct summands and $[\pm 1]$ shifts, so it contains both P' and $V[1]$. Moreover, it is closed under extension, so Λ also belongs to $\mathrm{thick}\, W$. Since Λ is silting, $\mathrm{thick}\, A = \mathcal{K}$. Then $\mathrm{thick}\, W = \mathcal{K}$, namely, we proved that W is silting. □

3.2 Characterization of silting complexes

Now we give some results which relate silting complexes to the triangulated structure of $\mathcal{K}$, in order to prove the following characterization of silting objects:

Theorem 3.2.1. *Any two-term presilting complex P in* $\mathrm{silt}\,\Lambda$ *is silting if and only if has the same number of pairwise non-isomorphic direct summands of Λ, i.e. $|P| = |\Lambda|$.*

We note that $\mathrm{thick}\, P = \mathrm{thick}\,\mathrm{add}\, P$. Indeed, $\mathrm{thick}\, P$ is closed under extensions, isomorphisms, direct summands and $[\pm 1]$ shifts, while $\mathrm{thick}\,\mathrm{add}\, P$ contains also direct sums of P. Thanks to Proposition 1.4.2, direct sum of triangles is a triangle. So, let $X, Y \in \mathrm{thick}\, P$, then $X \oplus Y$ is in a triangle of the type $X \to X \oplus Y \to Y \to X[1]$, namely $X \oplus Y$ is an extension of X and Y, then $X \oplus Y$ belongs to $\mathrm{thick}\, P$. Thus $\mathrm{thick}\, P$ is closed under direct sums, in particular it contains direct sums of P, i.e $\mathrm{thick}\, P \supseteq \mathrm{add}\, P$, so $\mathrm{thick}\, P \supseteq \mathrm{thick}\,\mathrm{add}\, P$. The other inclusion is due to the fact that P is contained in $\mathrm{add}\, P$.

We note also that, if $\mathrm{Hom}_{\mathcal{K}}(P, P[i]) = 0$ for some $i > 0$, then $\mathrm{Hom}_{\mathcal{K}}(X, Y[i]) = 0$ for X, Y in $\mathrm{add}\, P$. Indeed, there exists X' and Y' such that $X \oplus X' \simeq P$ and $Y \oplus Y' \simeq P$, then

$$\begin{aligned} 0 = \mathrm{Hom}_{\mathcal{K}}(P, P[i]) \simeq & \mathrm{Hom}_{\mathcal{K}}(X \oplus X', Y[i] \oplus Y'[i]) \simeq \\ & \simeq \mathrm{Hom}_{\mathcal{K}}(X, Y[i]) \oplus \mathrm{Hom}_{\mathcal{K}}(X, Y'[i]) \\ & \oplus \mathrm{Hom}_{\mathcal{K}}(X', Y'[i]) \oplus \mathrm{Hom}_{\mathcal{K}}(X', Y[i]). \end{aligned}$$

So $\mathrm{Hom}_{\mathcal{K}}(X, Y[i]) = 0$. Moreover $\mathrm{Hom}_{\mathcal{K}}(Q, N[i]) = 0$ for any $i > 0$ and any $Q \in \mathrm{add}\, P$, if and only if $\mathrm{Hom}_{\mathcal{K}}(P, N[i]) = 0$ for any $i > 0$. Let $Q \in \mathrm{add}\, P$, then it exists $P' \in \Lambda\text{-}\mathrm{mod}$ such that $Q \oplus P' \simeq P^n$, so by additivity of the Hom functor, $0 = (\mathrm{Hom}_{\mathcal{K}}(P, N[i])^n \simeq \mathrm{Hom}_{\mathcal{K}}(Q, N[i]) \oplus \mathrm{Hom}_{\mathcal{K}}(P', N[i])$. The other implication is obvious.

From now on, we denote the subcategory $\mathrm{add}\, P$, with $\mathcal{P}$.

Proposition 3.2.1 ([AI12])**.** *Let P be a silting complex, then*

$$\mathcal{K} = \operatorname{thick} P = \bigcup_{l \geq 0} \mathcal{P}[-l] * \mathcal{P}[1-l] * \cdots * \mathcal{P}[l-1] * \mathcal{P}[l].$$

Proof. Firstly observe that $\mathcal{P}[n]$ is closed under extension, namely

$$\mathcal{P}[n] * \mathcal{P}[n] = \mathcal{P}[n]. \tag{3.2}$$

Indeed, any triangle $A \to X \to B \to A[1]$, with both A, B in $\mathcal{P}[n]$, due to the fact that P is silting and by Proposition 1.4.2, is such that $X \simeq A \oplus B$ and $\mathcal{P}[n]$ is closed under direct summand.

Consider $n \geq m$ and a triangle $A \to X \to B \xrightarrow{f} A[1]$, where A belongs to $\mathcal{P}[n]$ and B belongs to $\mathcal{P}[m]$. The fact that P is silting implies that $f = 0$, so, by Proposition 1.4.2, $X \simeq A \oplus B$, i.e. X belongs to $\mathcal{P}[m] * \mathcal{P}[n]$. This means that

$$\mathcal{P}[n] * \mathcal{P}[m] \subseteq \mathcal{P}[m] * \mathcal{P}[n], \quad \text{if } n \geq m.$$

We now prove that

$$\operatorname{thick} P = \bigcup_{l \geq 0,\, n_1 \geq n_2 \geq \cdots \geq n_l \in \mathbb{Z}} \mathcal{P}[n_1] * \mathcal{P}[n_2] * \cdots * \mathcal{P}[n_{l-1}] * \mathcal{P}[n_l].$$

The right-hand side contains P and is a thick subcategory:

- is closed under isomorphisms. Indeed, let $X \simeq Y$ isomorphic complexes, with X in $\mathcal{P}[n_1] * \mathcal{P}[n_2] * \cdots * \mathcal{P}[n_{l-1}] * \mathcal{P}[n_l]$ for some $l \geq 0$, $n_1 \geq n_2 \geq \cdots \geq n_l \in \mathbb{Z}$. So there exists a triangle $A \to X \to B \to A[1]$, with A in $\mathcal{P}[n_1]$ and B in $\mathcal{P}[n_2] * \cdots * \mathcal{P}[n_{l-1}] * \mathcal{P}[n_l]$. We get the commutative diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & X & \xrightarrow{g} & B & \longrightarrow & A[1] \\ \Big\| {\scriptstyle id} & & \Big| \simeq & & \Big\downarrow {\scriptstyle \phi} & & \Big| {\scriptstyle id} \\ A & \xrightarrow{f} & Y & \longrightarrow & C_f & \longrightarrow & A[1], \end{array}$$

  where ϕ exists by the second axiom of the triangulated category (TR2) and is an isomorphism due to Proposition 1.4.2. Thus Y belongs to $\mathcal{P}[n_1] * \mathcal{P}[n_2] * \cdots * \mathcal{P}[n_{l-1}] * \mathcal{P}[n_l]$.

- Is closed under direct summands. Indeed, due to the fact that P is silting and to Lemma 1.5.1, we get $\mathcal{P}[n_1] * \mathcal{P}[n_2] * \cdots * \mathcal{P}[n_{l-1}] * \mathcal{P}[n_l] = \operatorname{smd} \mathcal{P}[n_1] * \mathcal{P}[n_2] * \cdots * \mathcal{P}[n_{l-1}] * \mathcal{P}[n_l]$. Given a category $\mathcal{C}$, with $\operatorname{smd} \mathcal{C}$ we denote the subcategory whose objects are the summands of $\mathcal{C}$.

- Is closed under $[\pm 1]$-shifts. Given X in $\mathcal{P}[n_1] * \mathcal{P}[n_2] * \cdots * \mathcal{P}[n_{l-1}] * \mathcal{P}[n_l]$ for some $l \geq 0$, $n_1, \ldots, n_l \in \mathbb{Z}$, we want to show that $X[\pm 1]$ belongs to $\mathcal{P}[n_1 \pm 1] * \mathcal{P}[n_2 \pm 1] * \cdots * \mathcal{P}[n_{l-1} \pm 1] * \mathcal{P}[n_l \pm 1]$. We use induction on l. If $l = 1$, the statement is obviously true. If $l = 2$, there exists a triangle $A \to X \to B \to A[1]$, where

A belongs to $\mathcal{P}[n_1]$ and B to $\mathcal{P}[n_2]$ for some n_1, n_2 in $\mathbb{Z}$. Then there are triangles $A[1] \to X[1] \to B[1] \to A[2]$ and $A[-1] \to X[-1] \to B[-1] \to A$. So $X[1]$ belongs to $\mathcal{P}[n_1+1] * \mathcal{P}[n_2+1]$ and $X[-1]$ belongs to $\mathcal{P}[n_1-1] * \mathcal{P}[n_2-1]$. Now let the statement be true for l and take X in $\mathcal{P}[n_1] * \cdots * \mathcal{P}[n_l] * \mathcal{P}[n_{l+1}]$. There exists a triangle $A \to X \to B \to A[1]$, with $A \in \mathcal{P}[n_1] * \cdots * \mathcal{P}[n_l]$ and $B \in \mathcal{P}[n_{l+1}]$, then it exists a triangle $A[\pm 1] \to X[\pm 1] \to B[\pm 1] \to A[\pm 1 + 1]$ and $A[\pm 1]$, by induction, belongs to $\mathcal{P}[n_1 \pm 1] * \cdots * \mathcal{P}[n_l \pm 1]$ and $B[\pm 1]$ is in $\mathcal{P}[n_{l+1}]$. Then $X[\pm 1]$ belongs to $\mathcal{P}[n_1 \pm 1] * \cdots * \mathcal{P}[n_l \pm 1] * \mathcal{P}[n_{l+1} \pm 1]$ as we wanted.

- Is closed under cones, indeed is closed under extensions. Consider A in $\mathcal{P}[n_1] * \mathcal{P}[n_2] * \cdots * \mathcal{P}[n_{l-1}] * \mathcal{P}[n_l]$ and B in $\mathcal{P}[m_1] * \mathcal{P}[m_2] * \cdots * \mathcal{P}[m_{t-1}] * \mathcal{P}[m_t]$ for some $l \geq 0$, $n_1, \ldots, n_l \in \mathbb{Z}$ and $t \geq 0$, $m_1, \ldots, m_t \in \mathbb{Z}$. If exists a triangle $A \to X \to B \to A[1]$, then X belongs to $(\mathcal{P}[n_1] * \cdots * \mathcal{P}[n_l]) * (\mathcal{P}[m_1] * \cdots * \mathcal{P}[m_t])$. Since any mapping cone C_f of $A \xrightarrow{f} B$ is an extension of B and $A[1]$, we conclude.

By definition, thick P is the smallest thick subcategory containing P, so we have shown that the right-hand side includes the left one.

Conversely, it is easy to see that each object in $\mathcal{P}[n_1] * \mathcal{P}[n_2] * \cdots * \mathcal{P}[n_{l-1}] * \mathcal{P}[n_l]$ belongs to thick P. Thus we proved that:

$$\operatorname{thick} P = \bigcup_{l \geq 0,\ n_1 \geq n_2 \geq \cdots \geq n_l \in \mathbb{Z}} \mathcal{P}[n_1] * \mathcal{P}[n_2] * \cdots * \mathcal{P}[n_{l-1}] * \mathcal{P}[n_l].$$

Now we want to prove that

$$\bigcup_{l \geq 0,\ n_1, \ldots, n_l \in \mathbb{Z}} \mathcal{P}[n_1] * \mathcal{P}[n_2] * \cdots * \mathcal{P}[n_{l-1}] * \mathcal{P}[n_l] = \bigcup_{l \geq 0} \mathcal{P}[-l] * \mathcal{P}[1-l] * \cdots * \mathcal{P}[l-1] * \mathcal{P}[l].$$

Note that

$$\mathcal{P}[m] * \mathcal{P}[n] \subseteq \mathcal{P}[m] * \mathcal{P}[m+1] * \cdots * \mathcal{P}[n-1] * \mathcal{P}[n] \subseteq \mathcal{P}[-n] * \mathcal{P}[1-n] * \cdots * \mathcal{P}[n-1] * \mathcal{P}[n].$$

Indeed, consider a triangle $A \to X \to B \to A[1]$, where A belongs to $\mathcal{P}[m]$ and B belongs to $\mathcal{P}[n]$, then we also have triangles $X \to X \to 0 \to X[1]$ and $B[-1] \to 0 \to B \to B$. Refer to the following diagram for clearance:

$$\begin{array}{ccccccc}
A[1] & & & & B & & \\
\uparrow & & & & \uparrow & & \\
B & & & & B & & \\
\uparrow & & & & \uparrow & & \\
X & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & X[1] \\
\uparrow & & & & \uparrow & & \\
A & & & & B[-1] & &
\end{array} .$$

This implies, using 3.2, that X belongs to

$$(\mathcal{P}[m] * \mathcal{P}[n]) * (\mathcal{P}[n-1] * \mathcal{P}[n]) = \mathcal{P}[m] * \mathcal{P}[n-1] * \mathcal{P}[n] * \mathcal{P}[n] = \mathcal{P}[m] * \mathcal{P}[n-1] * \mathcal{P}[n].$$

By repeating the same reasoning, we conclude.

□

Proposition 3.2.2. *If M in $\mathcal{K}$ is such that $\mathrm{Hom}_{\mathcal{K}}(P, M[i]) = 0$ for any $i > 0$, i.e. $P \geq M$, then M belongs to $\mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l-1] * \mathcal{P}[l]$ for some $l \geq 0$. Moreover, we have triangles:*

$$M_1 \longrightarrow P_0 \xrightarrow{f_0} M \longrightarrow M_1[1],$$

$$\cdots \qquad \cdots$$

$$M_l \longrightarrow P_{l-1} \xrightarrow{f_{l-1}} M_{l-1} \longrightarrow M_l[1],$$

$$0 \longrightarrow P_l \xrightarrow{f_l} M_l \longrightarrow 0,$$

such that $P_i \in \mathcal{P}$ and f_i is a minimal right $\mathcal{P}$-approximation for $0 \leq i \leq l$. These triangles are unique up to isomorphisms.

Proof. Let M in $\mathcal{K}$ be such that $\mathrm{Hom}_{\mathcal{K}}(\mathcal{P}, M[i]) = 0$ for any $i > 0$ and take the biggest n such that M belongs to $\mathcal{P}[n] * \mathcal{P}[n+1] * \cdots * \mathcal{P}[l-1] * \mathcal{P}[l]$, for some $l \geq 0$. So there exists a triangle $A \xrightarrow{f} M \to B \to A[1]$, with A in $\mathcal{P}[n]$ and $B \in \mathcal{P}[n+1] * \cdots * \mathcal{P}[l]$. If $f \in \mathrm{Hom}_{\mathcal{K}}(\mathcal{P}[n], M)$ is zero, then, by Proposition 1.4.2, $B = M \oplus A[1]$. Hence M belongs to $\mathcal{P}[n+1] * \cdots * \mathcal{P}[l]$, since this is closed under summands, however this contradicts the maximality of n. Since $\mathrm{Hom}_{\mathcal{K}}(\mathcal{P}, M[i]) = 0$ for any $i > 0$ and $\mathrm{Hom}_{\mathcal{K}}(\mathcal{P}[n], M) = \mathrm{Hom}_{\mathcal{K}}(\mathcal{P}, M[-n]) \neq 0$, then $-n$ must be less than zero, i.e. $n \geq 0$. So M belongs to $\mathcal{P}[n] * \mathcal{P}[n+1] * \cdots * \mathcal{P}[l-1] * \mathcal{P}[l] \subseteq \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l-1] * \mathcal{P}[l]$.

Now we prove the second part of the proposition. If $l = 0$, M belongs to $\mathcal{P}$, then the triangle correspondent is just the trivial one : $0 \to M \to M \to 0$.

So we can assume $l > 0$. We have a triangle

$$P_1' \to P_0' \xrightarrow{f} M \to P_1'[1]$$

with P_0' in $\mathcal{P}$ and P_1' in $\mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l-1]$. We have, by Proposition 1.4.1, a long exact sequence for every $Z \in \mathcal{P}$:

$$\mathrm{Hom}_{\mathcal{K}}(Z, P_1') \to \mathrm{Hom}_{\mathcal{K}}(Z, P_0') \to \mathrm{Hom}_{\mathcal{K}}(Z, M) \to \mathrm{Hom}_{\mathcal{K}}(Z, P_1'[1]).$$

Since P_1' is in $\mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l-1]$, then $\mathrm{Hom}_{\mathcal{K}}(Z, P_1'[1]) = 0$ for every $Z \in \mathcal{P}$. Then f is a right $\mathcal{P}$-approximation. We can decompose f, $f = (f_0\ 0) : P_0' = P_0 \oplus P_0'' \to M$, such that f_0 is right minimal and, thanks to the first axiom (TR1), we can construct the triangle

$$M_1 \to P_0 \xrightarrow{f_0} M \to P_1[1].$$

Consider the trivial triangle $P_0'' \to P_0'' \to 0 \to P_0''[1]$, then the direct sum of these two triangles is again a triangle and is isomorphic to $P_1' \to P_0' \xrightarrow{f} M \to P_1'[1]$. In particular $P_1' \simeq M_1 \oplus P_0''$, so M_1 is a summand of P_1', then it belongs to $\mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l-1]$ by Lemma 1.5.1. We can repeat the same reasoning with M_1 and so on, until M_l belongs to $\mathcal{P}$.

Starting the construction of the triangles related to M with the same triangle $P_1' \to P_0' \xrightarrow{f} M \to P_1'[1]$, will always get the same triangles up to isomorphisms, thanks to the second and third axioms, (TR3) and (TR2), and Proposition 1.4.2.

Let $Q_1' \to Q_0' \xrightarrow{g} M \to Q_1'[1]$ be a different triangle than $P_1' \to P_0' \xrightarrow{f} M \to P_1'[1]$, with Q_0 in $\mathcal{P}$ and $Q_1' \in \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l-1]$.Then we can consider $g_0 : Q_0 \to M$ a right minimal $\mathcal{P}$-approximation. By what proved in Section 1.5, f_0 and g_0 are unique up to isomorphism, namely there exists $h : Q_0 \to P_0$ Λ-module isomorphism such that, $f_0 h = g_0$. Thanks to the second and third axioms, (TR3) and (TR2), we get the following commutative diagram of triangles:

$$\begin{array}{ccccccc} Q_1 & \longrightarrow & Q_0 & \xrightarrow{g_0} & M & \longrightarrow & Q_1[1] \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle id} & & \downarrow \\ M_1 & \longrightarrow & P_0 & \xrightarrow{f_0} & M & \longrightarrow & P_1[1], \end{array}$$

where ϕ is an isomorphism, thanks to Proposition 1.4.2.

We have shown that even if we start with a different triangle, we get one that is isomorphic, and so, following the same construction, we get triangles that are equal up to isomorphism.

□

We are now ready to prove the first implication of Theorem 3.2.1 aforementioned.

Theorem 3.2.3 ([AI12])**.** *Any silting complex P in* Λ-silt *has the same number of pairwise non-isomorphic direct summands.*

Proof. Let P be a silting complex. The idea is to prove that the non-isomorphic indecomposable summands of P, denoted by $\operatorname{ind} P$, form a basis of the Grothendieck group $G_0(\mathcal{K})$. Let $n = |\operatorname{ind} P|$. Note that $\operatorname{ind} P = \operatorname{ind} \mathcal{P}$, indeed let $\operatorname{ind} P = \{Q_1, \dots Q_n\}$ and let $P' \in \mathcal{P}$, then there exists $P'' \in \mathcal{P}$ such that $P' \oplus P'' \simeq P^m$ for some $m \in \mathbb{N}$. So, $P^m \simeq \oplus_i^n Q_i^{s_i} \simeq P' \oplus P''$, for $s_i > 0$, since $\mathcal{K}$ is a Krull-Schmidt category, this decomposition is unique, hence $P' \simeq \oplus_i^n Q_i^{r_i}$ for $r_i \geq 0$.

The $\mathrm{G}_0(\mathcal{K})$ is defined as the abelian group generated by isomorphism classes of objects of the category $\mathcal{K}$ and the relation:

$$[L] - [M] + [N] = 0,$$

for each triangle

$$L \to M \to N \to L[1].$$

We shall define a group homomorphism $\gamma : \mathrm{G}_0(\mathcal{K}) \to \mathbb{Z}^{|\operatorname{ind} P|}$. We divide the construction of this map into different steps. Firstly, we describe a map $\gamma : \mathcal{K} \to \mathbb{Z}^{|\operatorname{ind} P|}$ and we prove that is well-defined group homomorphism. Then we show that it can be restricted to $\mathrm{G}_0(\mathcal{K})$.

Construction of the map

We need to show how $\gamma(M)$ works for each M in $\mathcal{K}$.

- If M belongs to $\mathcal{P}$, then $M = \oplus_{i=1,\dots n} Q_i^{n_i}$, for some $n_i \geq 0 \in Z$, then $\gamma(M) = (n_i)_i$.
- Consider M in $\mathcal{K}$ such that $P \geq M$. Then by Proposition 3.2.2 we have triangles, and we can set
$$\gamma(M) := \sum_{i=0}^{l} (-1)^i \gamma(P_i).$$
- For a general M object of $\mathcal{K}$, there must exist a sufficiently large k such that $M[k]$ satisfies $\mathrm{Hom}_{\mathcal{K}}(P, M[k+i]) = 0$ for $i > 0$. Then
$$\gamma(M) := (-1)^k \gamma(M[k]).$$

Is well-defined?

- If M belongs to $\mathcal{P}$, then, since we are in a Krull-Schmidt category, its decomposition is unique up to isomorphism, namely if $M = \oplus_{i=1,\dots,n} P_i^{n_i} = \oplus_{i=1,\dots,m} P_i^{m_i}$, for some $n_i \geq 0 \in Z, m_i \geq 0 \in Z$, then $n_i = m_i$ and $n = m$.
- If M in $\mathcal{K}$ is such that $\mathcal{P} \geq M$, then $\gamma(M) := \sum_{i=0}^{l} (-1)^i \gamma(P_i)$. The triangles of Proposition 3.2.2 are constructed up to isomorphisms. So even if we have a different sequence of triangles related to M, i.e $\gamma(M) = \sum_{i=0}^{l} (-1)^i \gamma(Q_i)$, then $Q_i \simeq P_i \in \mathcal{P}$ for every $0 \leq i \leq l$. For what said above $\gamma(Q_i) = \gamma(P_i)$. Equivalently if $M \simeq N$, then $\gamma(M) = \gamma(N)$, since, again, the triangles related to them are isomorphic.
- For a general M object of $\mathcal{K}$, given the minimum s such that $M[s]$ satisfies $P \geq M[s]$, we want to prove that $(-1)^s \gamma(M[s]) = (-1)^k \gamma(M[k])$ for any $k \geq s$, for which $P \geq M[k]$.

  Hence, let $M[k]$ be such that $P \geq M[k]$, i.e. $\mathrm{Hom}_{\mathcal{K}}(P, M[k+i]) = 0$ for $i \geq 0$, thus it belongs to $\mathcal{P} * \dots * \mathcal{P}[l]$, for some $l \geq 0$. So there exist $l+1$ triangles
$$M_1 \longrightarrow P_0 \xrightarrow{f_0} M[k] \longrightarrow M_1[1],$$
$$\dots \qquad \dots,$$
$$M_l \longrightarrow P_{l-1} \xrightarrow{f_{l-1}} M_{l-1} \longrightarrow M_l[1],$$
$$0 \longrightarrow P_l \xrightarrow{f_l} M_l \longrightarrow 0,$$
  and, by definition, $\gamma(M) = (-1)^k \gamma(M[k]) = (-1)^k \sum_{i=0}^{l} (-1)^i \gamma(P_i)$. Suppose that also $P \geq M[k-1]$, we want to show that $\gamma(M[k]) = (-1)^{k-1} \gamma(M[k-1])$. Again

we have l triangles:

$$M_1' \longrightarrow P_0' \xrightarrow{f_0'} M[k-1] \longrightarrow M_1'[1],$$

$$\dots \qquad \dots,$$

$$M_{l-1}' \longrightarrow P_{l-2}' \xrightarrow{f_{l-2}'} M_{l-2}' \longrightarrow M_{l-1}'[1],$$

$$0 \longrightarrow P_{l-1}' \xrightarrow{f_{l-1}'} M_{l-1}' \longrightarrow 0,$$

Since $\mathrm{Hom}_{\mathcal{K}}(P, M[k-1+i]) = 0$ for $i \geq 0$, in particular $\mathrm{Hom}_{\mathcal{K}}(P, M[k]) = 0$, namely f_0 is zero. Then, by Proposition 1.4.2, the triangle

$$M_1 \to P_0 \xrightarrow{f_0} M[k] \to M_1[1]$$

must be isomorphic to the trivial triangle:

$$M[k-1] \to 0 \to M[k] \to M[k].$$

This implies that, by minimality of f_{j+1} and f_j',

$$M_{j+2} \to P_{j+1} \xrightarrow{f_{j+1}} M_{j+1} \to M_{j+2}[1]$$

is isomorphic to

$$M_{j+1}' \to P_j' \xrightarrow{f_j} M_j' \to M_{j+1}'[1],$$

for each $l-1 \geq j \geq 0$.

Namely

$$(-1)^{k-1}\gamma(M[k-1]) = (-1)^{k-1}\sum_{i=0}^{l-1}(-1)^i\gamma(P_i') = (-1)^{k-1}\sum_{i=0}^{l-1}(-1)^i\gamma(P_i) = -\gamma(M[k]).$$

By repeating the same reasoning, i times until $k-i=s$, we prove our statement, this also implies that the map is well-defined for any objects in $\mathcal{K}$.

We proved that $\gamma(M) = \gamma(N)$, if $M \simeq N$ for any M in $\mathcal{K}$. The Krull-Schmidt property also implies that γ behaves well with direct sums, i.e. if $X \simeq M \oplus N$, then $\gamma(X) = \gamma(M) + \gamma(N)$. Indeed, let M, N in $\mathcal{P}$, $N = \oplus_i Q_i^{n_i}, M = \oplus_i Q_i^{m_i}$. Then $\gamma(M) = (m_i)_i$ and $\gamma(N) = (n_i)_i$. While $X = \oplus_i Q_i^{t_i}$, so $\gamma(X) = (t_i)_i$. However, $t_i = m_i + n_i$ for every i. So we proved that $\gamma(X) = \gamma(M) + \gamma(N)$ for X in $\mathcal{P}$. By construction of the map γ, this is also true for every object of $\mathcal{K}$.

Restriction to $\mathrm{G}_0(\mathcal{K})$

We now prove that the relations given by the Grothendieck group are maintained by γ. Namely, we are proving that we can restrict γ to $\mathrm{G}_0(\mathcal{K})$. Let $X \to Y \to Z \to X[1]$ be a triangle. Then we will prove that $\gamma(X) - \gamma(Y) + \gamma(Z) = 0$.

If X, Y, Z are general objects of $\mathcal{K}$ and there exists a triangle $X \to Y \to Z \to X[1]$, then $\gamma(X) - \gamma(Y) + \gamma(Z) = (-1)^k \gamma(X[k]) - (-1)^k \gamma(Y[k]) + (-1)^k \gamma(Z[k])$, with k sufficiently large such that $\mathcal{P} \geq X[k]$, $\mathcal{P} \geq Y[k]$, $\mathcal{P} \geq Z[k]$. Then there exists a triangle $X[k] \to Y[k] \to Z[k] \to X[k+1]$, so without loss of generality, we can assume X, Y, Z be such that $\mathcal{P} \geq X$, $\mathcal{P} \geq Y$, $\mathcal{P} \geq Z$.

By Proposition 3.2.2 we can take the biggest l, such that all three X, Y, Z belong to $\mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l]$. We want to use induction. Consider the following assertions for $l \geq 0$ and a triangle $X \to Y \to Z \to X[1]$:

(i_l) If $X, Y, Z \in \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l]$, then $\gamma(X) - \gamma(Y) + \gamma(Z) = 0$.

(ii_l) If $X, Y \in \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l]$ and $Z \in \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l] * \mathcal{P}[l+1]$, then $\gamma(X) - \gamma(Y) + \gamma(Z) = 0$.

(iii_l) If $X \in \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l]$ and $Y, Z \in \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l] * \mathcal{P}[l+1]$, then $\gamma(X) - \gamma(Y) + \gamma(Z) = 0$.

We prove first that (i_0) is true, and then the following chain of assertions: (i_l) $\Rightarrow$ (ii_l) $\Rightarrow$ (iii_l) $\Rightarrow$ (i_{l+1}) for any $l \geq 0$.

(i_0) Given a triangle $X \to Y \to Z \to X[1]$, with $X, Y, Z \in \mathcal{P}$, it splits, since the map $Z \to X[1]$ correspond to the zero map due to the fact that P is silting. Then by Proposition 1.4.2, $Y \simeq X \oplus Z$. By construction of the map, $\gamma(X) - \gamma(Y) + \gamma(Z) = 0$.

(i_l) $\Rightarrow$ (ii_l) Let $X \to Y \to Z \overset{w}{\to} X[1]$ be a triangle, with $X, Y \in \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l]$ and $Z \in \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l] * \mathcal{P}[l+1]$. Then it exists a triangle $Z_1 \to A \overset{u}{\to} Z \to Z_1[1]$, with A in $\mathcal{P} \subseteq \mathcal{P} * \cdots * \mathcal{P}[l]$ and $Z_1[1]$ in $\mathcal{P}[1] * \mathcal{P}[2] * \cdots * \mathcal{P}[l] * \mathcal{P}[l+1]$, i.e. $Z_1[1]$ in $\mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l-1] * \mathcal{P}[l]$. We can choose this triangle such that u is a minimal right $\mathcal{P}$-approximation. Then, by definition of $\gamma(Z)$, we have:

$$\gamma(A) = \gamma(Z_1) + \gamma(Z).$$

Then we have, $wu : A \to X[1]$ and using the mapping cone we can construct the triangle $X \to C_{wu} \to A \overset{wu}{\to} X[1]$. In particular C_{wu} belongs to $\mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l] * \mathcal{P} = \mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l]$ by what had been proved in Proposition 3.2.1. Then we get the triangles:

$$\begin{aligned} &Z[-1] \to X \to Y \to Z, \\ &A[-1] \to Z[-1] \to Z_1 \to A, \\ &A[-1] \to X \to C_{wu} \to A. \end{aligned}$$

By the octadrehal axiom (TR4), there exists a triangle $Z_1 \to C_{wu} \to Y \to Z_1[1]$. But then all three, Z_1, C_{wu}, Y, belong to $\mathcal{P} * \mathcal{P}[1] * \cdots * \mathcal{P}[l]$. By $(i)_l$,

$$\gamma(C_{wu}) = \gamma(Z_1) + \gamma(Y).$$

Moreover, since wu belongs to $\mathrm{Hom}_{\mathcal{K}}(A, X[1]) \subseteq \mathrm{Hom}_{\mathcal{K}}(\mathcal{P}, X[1]) = 0$, using Proposition 1.4.2 related to the triangle $X \to C_{wu} \to A \overset{wu}{\to} X[1]$, we get that $C_{wu} \simeq A \oplus X$, thus

$$\gamma(C_{wu}) = \gamma(A) + \gamma(X).$$

Then

$$\gamma(X) - \gamma(Y) + \gamma(Z) = \gamma(X) - \gamma(Y) + \gamma(A) - \gamma(Z_1) = \gamma(C_{wu}) - \gamma(Y) - \gamma(Z_1) = 0.$$

So $(ii)_l$ holds.

By using exactly the same reasoning and same commutative diagram,

$$\begin{array}{ccccccc}
 & & Z_1[1] & = & Z_1[1] & & \\
 & & \uparrow & & \uparrow & & \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\| & & \uparrow & & \uparrow & & \| \\
X & \longrightarrow & C_{wu} & \longrightarrow & A & \longrightarrow & X[1] \\
 & & \uparrow & & \uparrow & & \\
 & & Z_1 & = & Z_1, & &
\end{array}$$

one can show the other implications.

Conclusion

Since $\operatorname{thick} P = \mathcal{K}$, then $\operatorname{ind} P$ generate $\mathrm{G}_0(\mathcal{K})$. Indeed, let $[X]$ be an iso-classes of $\mathcal{K}$. Then X and all the complexes isomorphic to it belong to $\mathcal{P}[-l] * \cdots * \mathcal{P}[l]$ for some $l \geq 0$. But then, thanks to the relation in $\mathrm{G}_0(\mathcal{K})$, $[X] = \sum_{i=-l}^{l} [P_i]$, with P_i in $\mathcal{P}[i]$.

As we observed at the start, $\operatorname{ind} P = \operatorname{ind} \mathcal{P} = \operatorname{ind} \mathcal{P}[i]$, in particular this means that the set $\operatorname{ind} P$ generates $\mathrm{G}_0(\mathcal{K})$.

Since $\gamma(\operatorname{ind} P))$ is a basis of $\mathbb{Z}^{|\operatorname{ind} P|}$, then the set $\operatorname{ind} P$ must be linearly independent in $\mathrm{G}_0(\mathcal{K})$. Thus, it forms a basis of $\mathrm{G}_0(\mathcal{K})$.

This also implies that if we take another silting complex Q, then $\operatorname{ind} Q$ forms a basis of $\mathrm{G}_0(\mathcal{K})$. But $\mathrm{G}_0(\mathcal{K})$ depends only on the category, then the cardinality of these two sets must be the same, i.e. $|\operatorname{ind} P| = |\operatorname{ind} Q|$ □

We will denote the cardinality of the set $\operatorname{ind} P$, which represents the number of pairwise non-isomorphic indecomposable summands of P, with $|P|$, for any object $P \in \mathcal{K}$.

Now we are ready to prove the main theorem of this section, 3.2.1, we recall it:

Theorem 3.2.1. *Any two-term presilting complex P in* $\operatorname{silt} \Lambda$ *is silting if and only if has the same number of pairwise non-isomorphic direct summands of Λ, i.e. $|P| = |\Lambda|$.*

Proof. $(\Rightarrow)$ The "only if" part follows from Theorem 3.2.3.

$(\Leftarrow)$ Given a two-term presilting complex P with $|P| = |A|$, by Theorem 3.1.1, there exists a complex Q, such that $P \oplus Q$ is silting. Then, by Theorem 3.2.3, $|P \oplus Q| = |A| = |P|$. Since the category is Krull-Schmidt, Q belongs to $\mathcal{P}$, thus P is silting. □

3.3 Support τ-tilting modules

3.3.1 Auslander-Reiten translations

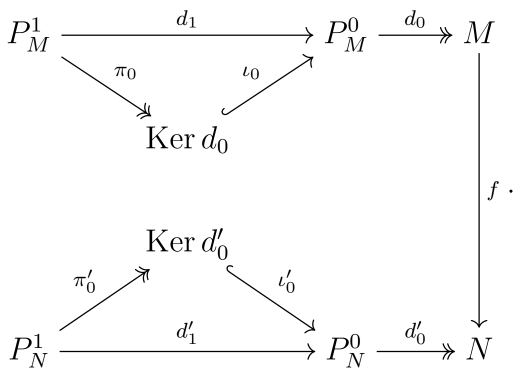
Let M be a left Λ-module, and consider its projective presentation $(P_M^1 \xrightarrow{d_1} P_M^0)$. Apply the contravariant functor $\mathrm{Hom}_\Lambda(-,\Lambda)$, as defined in 1.3, obtaining:

$$\mathrm{Hom}_\Lambda(M,\Lambda) \xrightarrow{\mathrm{Hom}_\Lambda(d_0,\Lambda)} \mathrm{Hom}_\Lambda(P_M^0,\Lambda) \xrightarrow{\mathrm{Hom}_\Lambda(d_1,\Lambda)} \mathrm{Hom}_\Lambda(P_M^1,\Lambda)$$

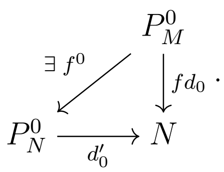
. Let $\mathrm{Tr}\, M := \mathrm{Coker}\,(\mathrm{Hom}_\Lambda(d_1,\Lambda))$, this is called the **transpose** of M.

Since projective covers are unique up to isomorphism, as proved in Section 1.5, so are minimal projective presentations, hence $\mathrm{Tr}\, M$ is unique up to isomorphism. We can prove that the correspondence given by the transposition define a contravariant functor $\Lambda\text{-mod} \to \Lambda^{\mathrm{op}}\text{-mod}$, We need to describe how is defined $\mathrm{Tr}\, f : \mathrm{Tr}\, N \to TrM$, given a left Λ-module homomorphism $f : M \to N$.

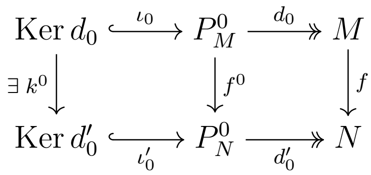
Let $(P_M^1 \xrightarrow{d_1} P_M^0)$, resp. $(P_N^1 \xrightarrow{d_1'} P_N^0)$ be the projective presentation of M, resp. N. Consider the following diagram:



Due to the projectivity of P_M^0, there exists a Λ-module homomorphism $f^0 : P_M^0 \to P_N^0$ making the following diagram commute:



Since $d_0' f^0 \iota_0 = f d_0 \iota_0 = 0$, the universal property of the kernel ensures the existence of $k_0 : \mathrm{Ker}\, d_0 \to \mathrm{Ker}\, d_0'$ such that

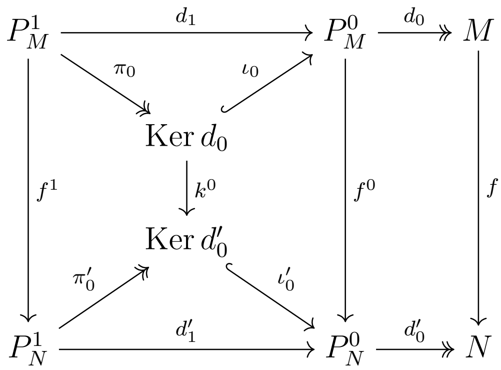


commutes.

Furthermore, since P_M^1 is projective, we have a Λ- module homomorphism, $f^1 : P_M^1 \to P_N^1$, satisfying:

$$\begin{array}{ccc} & & P_M^1 \\ & \overset{\exists f^1}{\swarrow} & \big\downarrow{\scriptstyle k_0\pi_0} \\ P_N^1 & \xrightarrow[\pi_0']{} & \operatorname{Ker} d_0' \end{array} .$$

Thus $d_1' f^1 = \iota_0' \pi_0' f^{-1} 1 = \iota_0' k_0 \pi_0 = f^0 \iota_0 \pi_0 = f^0 d_1$, as shown in the diagram below.



Hence we proved the existence of the following commutative diagram with exact rows, induced by f:

$$\begin{array}{ccccc} P_M^1 & \xrightarrow{d_1} & P_M^0 & \xrightarrow{d_0} & M \\ \big\downarrow{\scriptstyle f^1} & & \big\downarrow{\scriptstyle f^0} & & \big\downarrow{\scriptstyle f} \\ P_N^1 & \xrightarrow{d_1'} & P_N^0 & \xrightarrow{d_0'} & N \end{array} .$$

Applying the functor $\operatorname{Hom}_\Lambda(-, \Lambda)$, yields a commutative diagram with exact rows

$$\begin{array}{ccccc} \operatorname{Hom}_\Lambda(P_M^1, \Lambda) & \xrightarrow{\operatorname{Hom}_\Lambda(d_1,\Lambda)} & \operatorname{Hom}_\Lambda(P_M^0, \Lambda) & \longrightarrow & \operatorname{Tr} M \\ \big\uparrow{\scriptstyle \operatorname{Hom}_\Lambda(f^1,\Lambda)} & & \big\uparrow{\scriptstyle \operatorname{Hom}_\Lambda(f^0,\Lambda)} & & \big\uparrow{\scriptstyle \operatorname{Tr} f} \\ \operatorname{Hom}_\Lambda(P_N^1, \Lambda) & \xrightarrow{\operatorname{Hom}_\Lambda(d_1',\Lambda)} & \operatorname{Hom}_\Lambda(P_N^0, \Lambda) & \longrightarrow & \operatorname{Tr} N \end{array} ,$$

where $\operatorname{Tr} f$ exists by the universal property of the cokernels.

Note that $\operatorname{Tr} M = 0$ if and only if M is projective. Indeed, if M is projective, P_M^1 is zero, then also $\operatorname{Hom}_\Lambda(d_1, \Lambda) = 0$ and it implies that $\operatorname{Tr} M = 0$. Conversely, if $\operatorname{Tr} M = 0$, then $\operatorname{Hom}_\Lambda(d_1, \Lambda)$ is surjective,namely d_1 is injective. This implies that $P_M^1 \overset{d_1}{\to} P_M^0 \overset{d_0}{\to} M$ is a splitting short exact sequence, indeed, it would mean that $P_M^1 \simeq \operatorname{Ker} d_1$. Hence M is a direct summand of a projective, i.e. is a projective module.

The minimal projective presentation of $\operatorname{Tr} M$ is $\operatorname{Hom}_\Lambda(P_M^0, \Lambda) \overset{\operatorname{Hom}_\Lambda(d_1,\Lambda)}{\to} \operatorname{Hom}_\Lambda(P_M^1, \Lambda)$. If not, call the minimal projective presentation of $\operatorname{Tr} M$, $(Q_0 \to Q_1)$. Then $\operatorname{Hom}_\Lambda(P_M^0, \Lambda) \overset{\operatorname{Hom}_\Lambda(d_1,\Lambda)}{\to} \operatorname{Hom}_\Lambda(P_M^1, \Lambda)$ must be isomorphic to a direct sum, where one of the summand is $(Q_0 \to Q_1)$. This implies that, by additivity of the functor Hom, $\operatorname{Hom}_\Lambda(Q_1, \Lambda) \to \operatorname{Hom}_\Lambda(Q_0, \Lambda)$, would be a summand of $P_M^1 \overset{d_1}{\to} P_M^0$ and this is obviously a contradiction, since the latter is minimal.

Observe also that if M is not projective then $\operatorname{Tr}\operatorname{Tr} M \simeq M$. Indeed, by what just proved,

$$\operatorname{Hom}_\Lambda(\operatorname{Hom}_\Lambda(P^1_M,\Lambda),\Lambda)) \to \operatorname{Hom}_\Lambda(\operatorname{Hom}_\Lambda(P^0_M,\Lambda),\Lambda))$$

is the projective presentation of $\operatorname{Tr}\operatorname{Tr} M$.

As shown in Section 1.3, $\operatorname{Hom}_\Lambda(-,\Lambda)$ induces a duality if restricted to the subcategory of projective modules. This implies that we have the following diagram:

$$\begin{array}{ccccc} P^1 & \longrightarrow & P^0 & \longrightarrow & M \\ \downarrow \simeq & & \downarrow \simeq & & \\ \operatorname{Hom}_\Lambda(\operatorname{Hom}_\Lambda(P^1_M,\Lambda),\Lambda)) & \longrightarrow & \operatorname{Hom}_\Lambda(\operatorname{Hom}_\Lambda(P^0_M,\Lambda),\Lambda)) & \longrightarrow & \operatorname{Tr}\operatorname{Tr} M \end{array}.$$

Then, by universal property of the cokernel and using the five Lemma, we get that there exists an isomorphism of left modules, completing the diagram, as shown below.

$$\begin{array}{ccccc} P^1 & \longrightarrow & P^0 & \longrightarrow & M \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \operatorname{Hom}_\Lambda(\operatorname{Hom}_\Lambda(P^1_M,\Lambda),\Lambda)) & \longrightarrow & \operatorname{Hom}_\Lambda(\operatorname{Hom}_\Lambda(P^0_M,\Lambda),\Lambda)) & \longrightarrow & \operatorname{Tr}\operatorname{Tr} M \end{array}.$$

These observations shows that the correspondence given by the transposition define a functor $\Lambda\text{-mod} \to \Lambda^{\text{op}}\text{-mod}$, but can not define a duality, even if we would like so, since it annihilates the projectives. This justify the need of the following construction.

Let $\Lambda\text{-}\underline{\text{mod}}$ be the category whose object are the same of $\Lambda\text{-mod}$, while

$$\underline{\operatorname{Hom}}_\Lambda(M,N) := \operatorname{Hom}_{\Lambda\text{-}\underline{\text{mod}}}(M,N) = \frac{\operatorname{Hom}_\Lambda(M,N)}{P(M,N)},$$

where $P(M,N)$ is the $\mathbb{K}$-vector subspace of the homomorphism which factors through a projective. In some sense $\Lambda\text{-}\underline{\text{mod}}$ is the module category where projectives are *factored out*.

Dually, one can define $\Lambda\text{-}\overline{\text{mod}}$, the module category whose injective are *factored out*.

It turns out that if $f \in \operatorname{Hom}_\Lambda(M,N)$ does not factor through projectives, neither does $\operatorname{Tr} f$, and this implies that

Proposition 3.3.1. *[ASS06, Proposition IV.2.2] The correspondence*

$$\begin{gathered} \Lambda\text{-}\underline{\text{mod}} \to \Lambda^{\text{op}}\text{-}\underline{\text{mod}} \\ M \to \operatorname{Tr} M \end{gathered}$$

defines a $\mathbb{K}$-linear duality functor.

Definition 3.3.1. The **Auslander-Reiten translation** is the endofunctor defined as

$$\tau(-) = D\operatorname{Tr}(-) : \Lambda\text{-}\overline{\text{mod}} \to \Lambda\text{-}\overline{\text{mod}}\,.$$

Given a left Λ-module M, its translation τM can be constructed starting with its projective presentation $(P_M^1 \xrightarrow{d_1} P_M^0)$. By first applying the functor $\mathrm{Hom}_\Lambda(-,\Lambda)$ to this sequence, computing the cokernel of $\mathrm{Hom}_\Lambda(d_0,\Lambda)$, and finally applying the duality functor D, we obtain the exact sequence:

$$0 \to D\,\mathrm{Tr}\,M \to D(\mathrm{Hom}_\Lambda(P_M^1,\Lambda)) \to D(\mathrm{Hom}_\Lambda(P_M^0,\Lambda)) \to D(\mathrm{Hom}_\Lambda(M,\Lambda))$$

Recalling the definition of the Nakayama functor ν, as Definition 1.3.2, this sequence is equal to :

$$0 \to \tau M \to \nu(P_M^1) \to \nu(P_M^0) \to \nu(M),$$

This implies that $\tau M = \ker \nu(d_1)$.

Theorem 3.3.2 (A-R formulas)**.** *Let Λ be a $\mathbb{K}$-algebra and M, N be two Λ-modules in* Λ-mod*. Then there exists an isomorphism:*

$$\mathrm{Ext}^1_\Lambda(M,N) \simeq D\underline{\mathrm{mod}}_\Lambda(\tau^{-1}N, M) \simeq D\overline{\mathrm{mod}}_\Lambda(N,\tau M)$$

which is functorial in both variables.

See [ASS06, Theorem IV.2.13].

Definition 3.3.2. A left Λ-module is said to be **τ-rigid** if $\mathrm{Hom}_\Lambda(M,\tau M) = 0$.

Definition 3.3.3. A left Λ-module is said to be **τ-tilting** if is τ-rigid and if it has the same number of isomorphism classes of indecomposable direct summands as Λ, namely $|M| = n$.

Proposition 3.3.3 ([AS81])**.** *Let M be a left Λ-module. The following are equivalent:*

(i) M is τ- rigid,

(ii) $\mathrm{Ext}^1(M,M') = 0$ *for all factor modules M' of M, i.e.* $\mathrm{Ext}^1(M, \mathrm{Fac}\,M) = 0$.

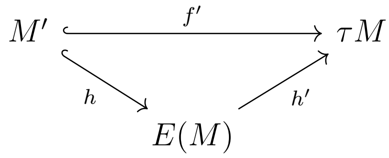
Proof. Let M' be a factor module of M. Saying that $\mathrm{Ext}^1(M,M') = 0$ is equivalent, thanks to the AR-formulas, to saying $D\overline{\mathrm{Hom}}_\Lambda(M',\tau M) = 0$. This happens if and only if $\overline{\mathrm{Hom}}_\Lambda(M',\tau M) = 0$.

Without loss of generality, we can assume that M is a non-projective indecomposable module, since both τ and $\overline{\mathrm{Hom}}$ are additive functors and τM is zero if M is projective indecomposable.

$(i \Rightarrow ii)$ This implication is now obvious, since if $\mathrm{Hom}_\Lambda(M,\tau M) = 0$ then $\overline{\mathrm{Hom}}_\Lambda(M', \tau M) = 0$ for all M' factors of M. Thus, we conclude.

$(ii \Rightarrow i)$ We need to prove that if $\overline{\mathrm{Hom}}_\Lambda(M',\tau M) = 0$ for any M' factor module of M then $\mathrm{Hom}_\Lambda(M,\tau M) = 0$. By contradiction, let $f : M \to \tau M$ be non zero and let $M' = \mathrm{Im}\, f$. Then M' is a factor of M. and we have a natural injection $f' : M' \to \tau M$. Since $f \neq 0$, then $f' \neq 0$. So f' must factor through injectives, i.e. there exist $h : M' \to E$ and $h' : E \to \tau M$ with E injective such that $f' = h' \circ h$. Since f is a monomorphism, h

must be a monomorphism, so we can assume that $E = E(M)$ is the injective envelope of M'.



Since, by Proposition 3.3.1, τM is indecomposable non-injective, then h' is not a monomorphism, and so $\ker h' \neq 0$. Indeed, if h were a monomorphism, we would have a splitting short exact sequence: $E(M) \xrightarrow{h'} \tau M \to \operatorname{Coker} h'$. Moreover, by the construction of the injective envelope, $E(M)$ is the maximal essential extension of M and so $\operatorname{Im} h$ is essential in $E(M)$. Then $\operatorname{Ker} h' \cap \operatorname{Im} h$ must be non-zero. So $h' \circ h$ can not be a monomorphism, thus f' can not factor through injectives. This is the contradiction we were looking for. □

We introduce support τ-tilting module and τ-tilting pair.

Definition 3.3.4. A left τ-rigid Λ-module M is said to be **support τ-tilting** if there exists an idempotent $e \in \mathcal{A}$ such that M is a τ-tilting $\frac{\Lambda}{\Lambda e \Lambda}$-module, where $\Lambda e \Lambda =< e >$ is the two-sided ideal generated by e in Λ.

We denote with $s\tau\text{-}\operatorname{tilt}\Lambda$ the set of iso-classes of support τ-tilting modules.

Definition 3.3.5. A left τ-rigid Λ-module M and a left projective Λ-module P constitute a **τ-rigid pair** (M, P) if $\operatorname{Hom}_\Lambda(P, M) = 0$. A τ-rigid pair (M, P) is said to be **τ-tilting pair** if $|M| + |P| = |\Lambda|$.

Note that Λ has a complete set of orthogonal primitive idempotent $\{e_1, \ldots, e_n\}$, such that $\Lambda = \oplus_{i=1}^n \Lambda e_i$. So $|\Lambda| = n = |\frac{\Lambda}{<e>}| + |e\Lambda|$. Moreover, $\operatorname{Hom}_\Lambda(e\Lambda, M)$ is isomorphic to Me, [ASS06]. Due to these facts, we can show that these two definitions are equivalent.

Let M be a support τ-tilting module with associated idempotent e, so is τ-rigid and $|M| = |\frac{\Lambda}{<e>}|$. Since it is a submodule of $\frac{\Lambda}{<e>}$, then Me is zero. Conversely, if (M, P) is a τ-tilting pair, then we have that $P = e\Lambda$ for some idempotent $e \in \mathcal{A}$. Then $|M| = |\frac{\Lambda}{<e>}|$. In order to say that M is a τ-tilting $\frac{\Lambda}{\Lambda e \Lambda}$-module, we also need to prove that if $\operatorname{Hom}_\Lambda(M, \tau M) = 0$, then $\operatorname{Hom}_{\Lambda/<e>}(M, \tau M) = 0$. This follows from Proposition 3.3.3, since M is τ-rigid then $0 = \operatorname{Ext}^1_\Lambda(M, \operatorname{Fac} M)$. This contains $\operatorname{Ext}^1_{\Lambda/<e>}(M, \operatorname{Fac} M)$, which is thus equal to zero. Using again Proposition 3.3.3, we get that

$$\operatorname{Hom}_{\Lambda/<e>}(M, \tau M) = 0.$$

Since being a τ-tilting pair is equivalent to being a support τ-tilting module, we are often going to interchange the definitions and use the one that is most convenient depending on each case.

3.4 Connection between two term silting complexes and τ-rigidity

With the following two propositions, we start to show the connection between being a two-term silting complex and being τ-rigid. The key idea is the correspondence between Λ-modules M and their projective presentations $P_M^1 \xrightarrow{d_1} P_M^0 \xrightarrow{d_0} M \to 0$, done already in Chapter 1. Specifically, for a Λ-module M, we can uniquely associate a two-term complex in degrees -1 and 0 within $\mathcal{K}^{[-1,0]}$, which is given by $P_M = \cdots \to 0 \to P_M^1 \to P_M^0 \to 0$, also denoted simpler with just $P_M = (P_M^1 \to P_M^0)$, with the convention that the minimal epimorphism $P_M^0 \xrightarrow{d_0} M$ is always implied, though it is not part of the complex.

Lemma 3.4.1. *Let M, N in Λ-mod. Let $P^1 \xrightarrow{p_1} P^0 \xrightarrow{p_0} M$ and $Q^1 \xrightarrow{q_1} Q^0 \xrightarrow{q_0} M$ be their minimal projective presentation. Denote with $P = (\cdots \to 0 \to P^1 \xrightarrow{p_1} P^0 \to 0 \to \cdots) = (P^1 \to P^0)$ and $Q = (\cdots \to 0 \to \mathrm{Q}^1 \xrightarrow{q_1} Q^0 \to 0 \to \cdots) = (Q^1 \to Q^0)$, the two-term complexes associated, objects of $\mathcal{K}^{[-1,0]}$. Then*

$$\mathrm{Hom}_\Lambda(N, \tau M) = 0 \quad \overset{i}{\Longleftrightarrow} \quad \mathrm{Hom}_\Lambda(p_1, N) \textit{ is surjective} \quad \overset{ii}{\Longleftrightarrow} \quad \mathrm{Hom}_\mathcal{K}(P, Q[1]) = 0$$

If we consider the particular case when $M = N$, this lemma affirms that M is a τ-rigid module if and only if P is a two-term presilting complex for Λ, since by degree reasons, also $\mathrm{Hom}_\mathcal{K}(P, P[i]) = 0$, if $i \geq 0$.

Proof. By what said in the introduction of the functor τ, we have an exact sequence $O \to \tau M \to \nu P^1 \xrightarrow{\nu p_1} \nu P^0$. Applying $\mathrm{Hom}_\Lambda(N, -)$ we get:

$$0 \to \mathrm{Hom}_\Lambda(N, \tau M) \to \mathrm{Hom}_\Lambda(N, \tau P^1) \xrightarrow{\nu\,\mathrm{Hom}_\Lambda(p_1, N)} \mathrm{Hom}_\Lambda(N, \nu P^0).$$

While applying $D\,\mathrm{Hom}_\Lambda(-, N)$ to the exact sequence $P^1 \xrightarrow{p_1} P^0 \xrightarrow{p_0} M$ we obtain:

$$D\,\mathrm{Hom}_\Lambda(P^1, N) \xrightarrow{D\,\mathrm{Hom}_\Lambda(p_1, N)} D\,\mathrm{Hom}_\Lambda(P^0, N) \to D\,\mathrm{Hom}_\Lambda(M, N) \to 0.$$

Thanks to Lemma 1.3.2, we get the following diagram with exact rows:

$$\begin{array}{ccccccc} 0 \longrightarrow \mathrm{Hom}_\Lambda(N, \tau M) \longrightarrow & \mathrm{Hom}_\Lambda(N, \nu P^1) & \longrightarrow & \mathrm{Hom}_\Lambda(N, \nu P^0) & & \\ & \downarrow \simeq & & \downarrow \simeq & & \\ & D\,\mathrm{Hom}_\Lambda(P^1, N) & \longrightarrow & D\,\mathrm{Hom}_\Lambda(P^0, N) & \longrightarrow D\,\mathrm{Hom}_\Lambda(M, N) \longrightarrow 0. \end{array}$$

Then $\mathrm{Hom}_\Lambda(N, \tau M)$ is zero if and only if $D\,\mathrm{Hom}_A(p_1, N)$ is injective. The first "if and only if" ($\overset{i}{\Longleftrightarrow}$) follows.

($\overset{ii}{\Rightarrow}$) A morphism f between complexes in $\mathrm{Hom}_\mathcal{K}(P, Q[1])$, is a sequence of maps, but, in this case, is just determined by the left Λ-module homomorphism in degree -1: $f_1 : P^1 \to Q^0$. Indeed, in other degree, we just have the null-map. With the same reasoning, a homotopy h between the complexes P and $Q[1]$ has just the terms in degree zero and -1 different than zero. One can see the reasons why just by looking at the diagram below, which represent a morphism between these complexes.

$$\begin{array}{ccccccccccc} & & \cdots & \longrightarrow & 0 & \longrightarrow & P^1 & \xrightarrow{p_1} & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & Q^1 & \xrightarrow{q_1} & Q^0 & \longrightarrow & 0 & \longrightarrow & \cdots & & \end{array}$$

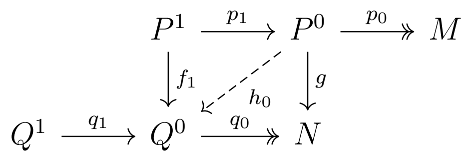
We want to show that if $\mathrm{Hom}_\Lambda(p_1, N)$ is surjective, then f_1 is homotopic to zero.

$$\begin{array}{ccccc} & & P^1 & \xrightarrow{p_1} & P^0 \xrightarrow{p_0} M \\ & & \downarrow{\scriptstyle f_1} & & \\ Q^1 & \xrightarrow{q_1} & Q^0 & \xrightarrow{q_0} & N \end{array}$$

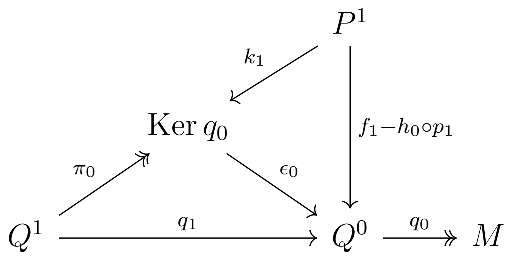
We have that $q_0 \circ f_1$ belongs to $\mathrm{Hom}_\Lambda(P^1, N)$ and since $\mathrm{Hom}_\Lambda(p_1, N)$ is surjective, there exists g in $\mathrm{Hom}_\Lambda(P^0, N)$ such that $q_o \circ f_1 = \mathrm{Hom}_\Lambda(p_1, N)(g) = g \circ p_1$.

$$\begin{array}{ccccccc} & & P^1 & \xrightarrow{p_1} & P^0 & \xrightarrow{p_0} & M \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\ Q^1 & \xrightarrow{q_1} & Q^0 & \xrightarrow{q_0} & N & & \end{array}$$

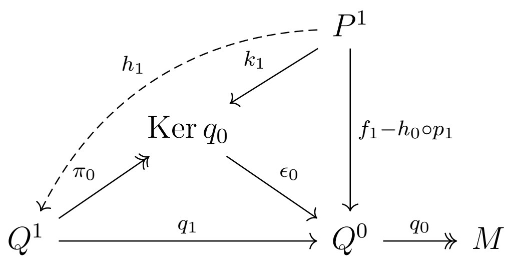
But also q_0 is surjective, so by projectiveness of P^0, there exists a map h_0 in $\mathrm{Hom}_\Lambda(P^0, Q^0)$ such that $q_0 \circ h_0 = g$.



Then $q_o \circ f_1 = q_0 \circ h_0 \circ p_1$, i.e. $q_0 \circ (f_1 - h_0 \circ p_1) = 0$. By universal property of the kernel of Q^0, there exists a module homomorphism $k_1 : P^1 \to \mathrm{Ker}\, q_0$ such that $\epsilon_0 \circ k_1 = f_1 - h_0 \circ p_1$.



Then, by projectiveness of P^1, there exists $h_1 : P^1 \to Q^1$ such that $\pi_0 \circ h_1 = k_1$.



Thus $q_1 \circ h_1 = \epsilon_0 \circ \pi_0 \circ h_1 = \epsilon_0 \circ k_1 = f_1 - h_0 \circ p_1$, i.e. $f_1 = h_0 \circ p_1 + q_1 \circ h_1$. So $h = (h_1, h_0)$ is the homotopy that we were looking for.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & P^1 & \xrightarrow{p_1} & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & \swarrow^{h_1} & \downarrow f_1 & \swarrow^{h_0} & & & & & \\ \cdots \longrightarrow 0 & \longrightarrow & Q^1 & \xrightarrow{q_1} & Q^0 & \longrightarrow & 0 & \longrightarrow & \cdots & & \end{array}$$

($\overset{ii}{\Leftarrow}$) Conversely, we want to show that the morphism $\operatorname{Hom}_\Lambda(p_1, N) : \operatorname{Hom}_\Lambda(P^0, N) \to \operatorname{Hom}_\Lambda(P^1, N)$ is surjective. So take any f in $\operatorname{Hom}_\Lambda(P^1, N)$. Since P^1 is projective and q_0 surjective, there exists g in $\operatorname{Hom}_\Lambda(P^1, Q^0)$ such that $f = q_0 \circ g$.

$$\begin{array}{ccccccc} & & P^1 & \xrightarrow{p_1} & P^0 & \xrightarrow{p_0} & M \\ & & \downarrow g & \searrow^{f} & & & \\ Q^1 & \xrightarrow{q_1} & Q^0 & \xrightarrow{q_0} & N & & \end{array}$$

We can consider g as an element of $\operatorname{Hom}_{\mathcal{K}}(P, Q[1])$, by setting the maps in the other degree as zero. By hypothesis this is equal to zero, so it exists a homotopy $h = (h_1, h_0)$, where $h_1 : P^1 \to Q^1$ and $h_0 : P^0 \to Q^0$ such that $g = h_0 \circ p_1 + q_1 \circ h_1$.

$$\begin{array}{ccccccccccc} & \longrightarrow & 0 & \longrightarrow & P^1 & \xrightarrow{p_1} & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & \swarrow^{h_1} & \downarrow g & \swarrow^{h_0} & & & & & \\ \longrightarrow 0 & \longrightarrow & Q^1 & \xrightarrow{q_1} & Q^0 & \longrightarrow & 0 & \longrightarrow & \cdots & & \end{array}$$

Then, $f = q_0 \circ (h_0 \circ p_1 + q_1 \circ h_1)$ which is equal to $q_0 \circ h_0 \circ p_1$ since the sequence is exact, i.e. $f = \operatorname{Hom}_\Lambda(p_1, N)(q_0 \circ h_0)$. And we can conclude. ☐

Lemma 3.4.2. *Let M be a left A-module and $P^1 \overset{p_1}{\to} P^0 \overset{p_0}{\to} 0$ be its projective presentation. Denote with $P := (P^1 \overset{p_1}{\to} P^0)$ the two-term complex in $\mathcal{K}^{[-1,0]}$ arising from the presentation. For any Q projective left A-module, the following are equivalent:*

(i) $\operatorname{Hom}_\Lambda(Q, M) = 0$,

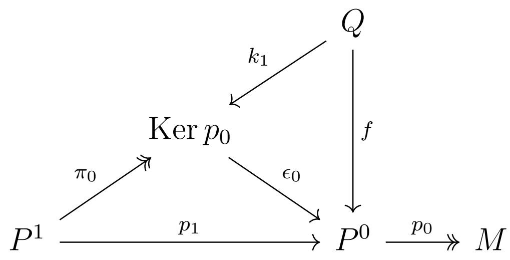
(ii) $\operatorname{Hom}_{\mathcal{K}}(Q, P) = 0$,

where Q is considered as a complex of $\mathcal{K}$, with the only non-zero term equal to Q in degree 0.

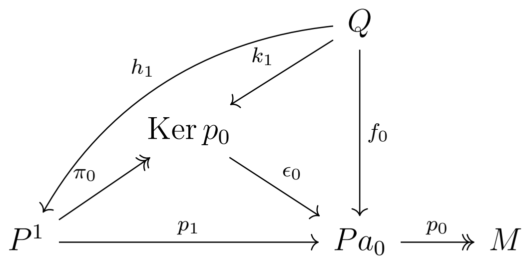
Proof. ($i \Rightarrow ii$) Let f be in $\operatorname{Hom}_{\mathcal{K}}(Q, P)$. By degree reasons, this is determined by a non-zero homomorphism in degree zero $f = f_0 : Q \to P^0$.

$$\begin{array}{ccccccccccc} & & \cdots & \longrightarrow & 0 & \longrightarrow & Q & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & & & \downarrow f_0 & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & P^1 & \xrightarrow{p_1} & P^0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

Then $p_0 \circ f_0$ belongs to $\mathrm{Hom}_\Lambda(Q, M)$ and so is equal to zero. By universal property of the kernel of p_0, there exists $k_1 : Q \to \operatorname{Ker} p_0$ such that $f_0 = \epsilon_0 \circ k_1$.

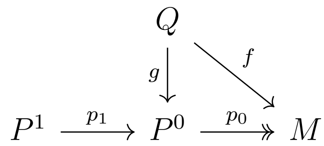


By projectiveness of Q, there exists $h_1 : Q \to P^1$ such that $\pi_0 \circ h_1 = k_1$.

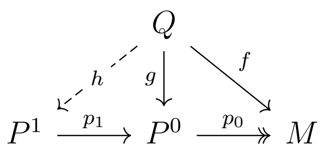


Then $f_0 = \epsilon_0 \circ k_1 = \epsilon_0 \circ \pi_0 \circ h_1 = p_1 \circ h_1$. So $h = (h_1, 0)$ is the homotopy that we were looking for and $f_0 \sim 0$, i.e $\mathrm{Hom}_{\mathcal{K}}(Q, P) = 0$.

($ii \Rightarrow i$) Conversely, let f be in $\mathrm{Hom}_\Lambda(Q, M)$. Since p_0 is surjective and by projectiveness of Q, there exists g in $\mathrm{Hom}_\Lambda(Q, P^0)$ such that $f = p_0 \circ g$.



Then g gives a map in $\mathrm{Hom}_{\mathcal{K}}(Q, P)$. By hypothesis g must be homotopic to zero, i.e. it must exist $h : Q \to P^1$ module homomorphism, such that $g = p_1 \circ h$.



But then $f = p_0 \circ p_1 \circ h = 0$. □

3.5 Bijection between two-term silting complexes and support τ-tilting modules

Now we are ready to prove the main theorem of this chapter:

Theorem 3.5.1. *Let Λ be a finite-dimensional $\mathbb{K}$-algebra. Then there exists a bijection*

$$2\text{-}\operatorname{silt}\Lambda \longleftrightarrow s\tau\text{-}\operatorname{tilt}\Lambda,$$

given by

$$P \mapsto H^0(P),$$
$$(P_M^1 \oplus Q \xrightarrow{(f,0)} P_M^0) \leftarrow\!\shortmid (M, Q),$$

where $P = (P^1 \xrightarrow{d} P^0)$ and $H^0(P) = coker(d)$, while $f : P_M^1 \to P_M^0$ is a minimal projective presentation of M.

Proof. Firstly, we prove that two-term silting complexes for Λ give support τ-tilting modules.

Let $P := (P^1 \xrightarrow{d} P^0)$ be a two-term silting complex and $M := \operatorname{Coker} d$. Then P is isomorphic to ${P^1}_M \oplus P'' \xrightarrow{(d',0)} P_M^0$, where $P_M := P_M^1 \xrightarrow{d'} P_M^0 \to M$ is the minimal projective presentation of M. Then $P = P_M \oplus P''[1]$. We want to prove that (M, P'') is a support τ-tilting pair for A. Since P is silting, then M is τ-rigid, by Lemma 3.4.1.

Moreover since $\operatorname{Hom}_{\mathcal{K}}(P, P[i]) = 0$ for every $i > 0$, then $\operatorname{Hom}_{\mathcal{K}}(P'', P) = 0$. Indeed, by additivity of the functor Hom,

$$\begin{aligned} 0 = \operatorname{Hom}_{\mathcal{K}}(P, P[i]) &\simeq \operatorname{Hom}_{\mathcal{K}}(P_M \oplus P''[1], (P_M \oplus P''[1])[i]) \\ &\simeq \operatorname{Hom}_{\mathcal{K}}(P_M, P_M[i]) \oplus \operatorname{Hom}_{\mathcal{K}}(P_M \oplus P''[i+1]) \oplus \\ &\oplus \operatorname{Hom}_{\mathcal{K}}(P''[1], P_M) \oplus \operatorname{Hom}_{\mathcal{K}}(P''[1], P''[1+i]) \end{aligned}$$

By hypothesis, we have that P is a two-term complex and P'' is projective, then by Lemma 3.4.2, $\operatorname{Hom}_{\mathcal{K}}(P'', M) = 0$. Thus (M, P'') is a τ-rigid pair for Λ.

Moreover, since $P_M^1 \xrightarrow{(d',0)} P_M^0$ is the minimal projective presentation of M, we have $|M| = |P_M^1 \xrightarrow{d'} P_M^0|$. Indeed, if M' is a direct summand of M, then its minimal projective presentation is a direct summand of $P_M^1 \xrightarrow{d'} P_M^0$. This is due to the Horseshoe lemma.

Thus, we have:

$$|M| + |P''| = |P_M^1 \xrightarrow{d'} P_M^0| + |P''| = |P| = |A|,$$

where the last equality holds because P is silting by Theorem 3.2.1. Hence, (M, P'') is a support τ-tilting pair for Λ.

Now we show that support τ-tilting Λ-modules give silting complexes for Λ.

Let (M, Q) be a τ-tilting pair for Λ and $P_M = (P_M^1 \xrightarrow{f} P_M^0)$ its minimal projective presentation. Since M is τ-rigid, then P_M is a presilting complex for A, as showed by Lemma 3.4.1. Set $P = (P_M^1 \oplus Q \xrightarrow{(f,0)} P_M^0) = P_M \oplus Q[1]$ and observe that since Q is projective, its minimal projective presentation is just $(0 \to Q)$. Consider $\operatorname{Hom}_{\mathcal{K}}(P, P[i]) = \operatorname{Hom}_{\mathcal{K}}(P_M \oplus Q[1], P_M[i] \oplus Q[1+i])$ for any $i > 0$ then

$$\begin{aligned} \operatorname{Hom}_{\mathcal{K}}(P, P[i]) &\simeq \operatorname{Hom}_{\mathcal{K}}(P_M, P_M[i]) \oplus \operatorname{Hom}_{\mathcal{K}}(Q[1], P_M[i]) \\ &\operatorname{Hom}_{\mathcal{K}}(Q[1], Q[i+1]) \oplus \operatorname{Hom}_{\mathcal{K}}(P_M, Q[1+i]). \end{aligned}$$

We have that $\mathrm{Hom}_{\mathcal{K}}(P_M, P_M[i]) = 0$ because P_M is presilting. Since $\mathrm{Hom}_{\Lambda}(Q, M) = 0$, by Lemma 3.4.2, $\mathrm{Hom}_{\mathcal{K}}(Q[1], P_M[1]) = 0$. Then $\mathrm{Hom}_{\mathcal{K}}(Q[1], P_M[i]) = 0$ for every $i > 1$ by degree reasons. While $\mathrm{Hom}_{\mathcal{K}}(Q[1], Q[i+1])$ and $\mathrm{Hom}_{\mathcal{K}}(P_M, Q[1+i])$ are zero by definition.

This show how P is a presilting complex for Λ. We have again that $|P_M| = |M|$. And since (M, Q) is a τ-tilting pair for Λ, we have $|M| + |Q| = |\Lambda|$.

Thus,

$$|P| = |P_M^1 \to P_M^0| + |Q| = |M| + |Q| = |\Lambda|.$$

So we showed that P is a silting complex for Λ by Theorem 3.2.1. This concludes the proof of the theorem.

□

Chapter 4

Two-terms silting complexes over gentle algebras

In the previous chapter, we examined the characteristics and properties of two-term silting complexes and established the bijection between these complexes and support τ-tilting modules. This bijection is fundamental for achieving the main goal of this thesis: to provide a complete classification of two-term silting complexes over gentle algebras. We believe that giving this classification will be a satisfying conclusion to this thesis project, as it will allow one to produce a wide range of useful and practical examples of these powerful algebraic structures.

The key tool for achieving this classification is the result from [Brü+20], where the authors classified support τ-tilting modules over gentle algebras. It is worth noting that this classification was independently obtained by another group of authors in [PPP21]. While both sets of authors reached the same conclusion, they use slightly different terminology: [Brü+20] refers to what [PPP21] calls a "blossom algebra" as a "fringed algebra." We will adopt the notation of [PPP21]; however, we find the approach taken by [Brü+20] to be more aligned with our project, which is why we primarily utilize their work.

In the paper by [Brü+20], support τ-tilting modules are correlated with certain non-kissing collections of strings. This non-kissing criterion will be analyzed in terms of two-term complexes in the second section of this chapter. Indeed, we will establish a correspondence between the two-term presilting minimal projective presentation of a string module and non-kissing blossoming strings. In particular, we will prove that the property of being presilting can be determined only by looking at the combinatorial information provided by the blossoming string. This result will then be used to refine the final classification.

4.1 Blossoming quivers and blossoming strings

To provide the most accurate description of a minimal projective presentation of a string module, we discovered that the optimal approach is to enlarge our quiver. We extend it so that each original vertex has now four neighbors. This enlargement is employed solely for the combinatorial insights it offers; we are not concerned with the geometric or algebraic properties of the algebra that may arise from such an extension. In this first section, we provide the details of this construction, which will also enlarge the strings, and demonstrate its utility.

Definition 4.1.1 ([PPP21]). The **blossoming quiver** of a gentle bound quiver (Q, I) is the gentle bound quiver $(\mathrm{Q}^{\text{❀}}, I^{\text{❀}})$, also denoted simply with $\mathrm{Q}^{\text{❀}}$, obtained by adding at each vertex $v \in Q_0$:

- $2 - |\{\alpha \in \mathrm{Q}_1 \mid t(\alpha) = v\}|$ incoming arrows and $2 - |\{\alpha \in \mathrm{Q}_1 \mid s(\alpha) = v\}|$ outgoing arrows,
- relations such that vertex v fulfills the gentle bound quiver conditions of Definition 2.3.1.

Then $\frac{\mathbb{K}\mathrm{Q}^{\text{❀}}}{I^{\text{❀}}}$ is a gentle algebra and is called the blossom algebra.

Note that each vertex of Q_0 has precisely two incoming and two outgoing arrows in $\mathrm{Q}^{\text{❀}}$. The vertices of $\mathrm{Q}_0^{\text{❀}} - \mathrm{Q}_0$ are called **blossom vertices**, and the arrows $\mathrm{Q}_1^{\text{❀}} - \mathrm{Q}_1$ are called **blossom arrows**. Note that in a blossoming quiver, among the set of paths starting, resp. ending, at a vertex of $\mathrm{Q}^{\text{❀}}$, the longest path starts, resp. ends, at a blossom vertex.

To distinguish blossom vertices and blossom arrows from the original ones, these will be depicted in green.

Example 4.1.1. In Figure 4.1, we illustrate the construction of blossoming quivers starting from the gentle quivers presented in Example 2.3.1.

Definition 4.1.2. Let v a vertex in Q. A **left blossom cohook** of v is a string $\sigma = \sigma_1\sigma_2 \dots \sigma_l$ of $\mathrm{Q}^{\text{❀}}$, such that $\sigma_1 = \alpha^{-1}$, with α arrow of $\mathrm{Q}_1^{\text{❀}}$ with head v, while $\sigma_2 \dots \sigma_l$ corresponds to a path of maximal length in $\mathrm{Q}^{\text{❀}}$ ending in v. Symmetrically, a **right blossom cohook** of v is the inverse string of a left cohook of v.

A left blossom cohook of v is not unique, but since we are in a gentle quiver, we have that for each vertex there exists at most two left different cohook.

By construction of the blossoming quiver, each original vertex has two incoming arrows and two outgoing arrows, so if we consider a vertex of Q in the blossoming quiver, has always two different non-trivial left blossom cohooks. However, we also authorize the situation where the blossom cohook could also be constituted of just an arrow, this happens if one of the direct successors of v is a blossom vertex, and so there are no other arrow ending in that vertex.

Note that in a left blossom cohook $\sigma = \sigma_1\sigma_2 \dots \sigma_l$ of a vertex v in Q_0, σ_l is always a blossom arrow.

Definition 4.1.3. Let $\omega = \omega_1 \dots \omega_l$ be a string of Q, we call **blossoming string** and denote it with $\omega^{\text{❀}}$, the gentle string obtained by adding a right blossom cohook σ a the right of ω and a left blossom cohook $\tilde{\sigma}$ at its left ending. Namely $\omega^{\text{❀}} = \sigma\omega\tilde{\sigma}$.

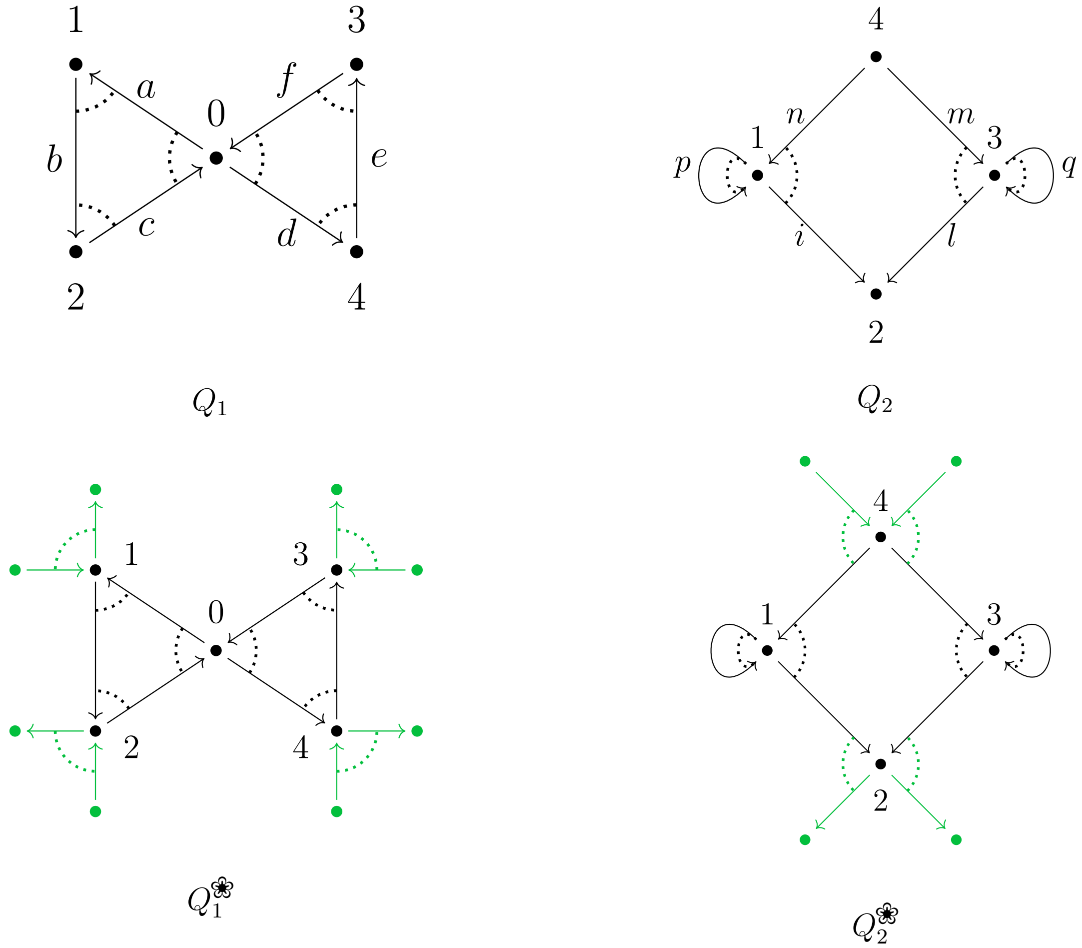


Figure 4.1

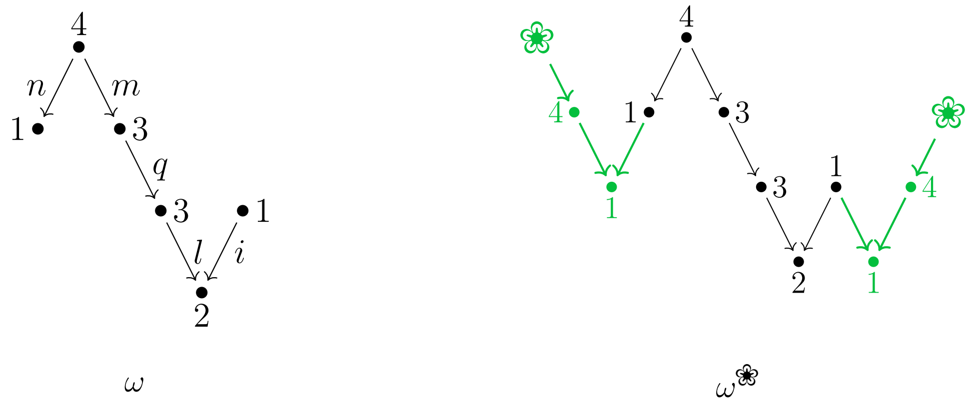


Figure 4.2

Notice that a blossoming strings always starts and ends with a blossom vertex. Moreover blossoming strings $\omega^{❀}$ are uniquely determined, when ω is not trivial. We will depict the blossom cohook attached at a string in green, and depict its blossoming vertices with "❀".

Example 4.1.2. Figure 4.2 depicts the construction of a blossoming string in $Q_2^{❀}$, which is the blossoming quiver of Example 4.1.1. Note that the original string ω of Q_1 was previously presented in Example 2.3.2.

Recall the description provided in Chapter 2 of the minimal projective presentation of a string module $M(\omega)$ and assume that ω_{l+1} exists. We have defined ω_{l+1} as the unique inverse arrows which make $\omega_l\omega_{l+1}$ a string. Now, considering the blossoming string $\omega^{❀} = \sigma\omega\tilde{\sigma}$, it follows easily that ω_{l+1} must coincide with the first arrow of $\tilde{\sigma}$. Therefore, $k_t = \mathfrak{s}(\omega_{l+1})$ is its only deep vertex. If ω_{l+1} does not exist, then $\tilde{\sigma}$ consists solely of a single blossom arrow. The same reasoning applies to ω_0.

This observation yields a more natural description of the minimal projective presentation of a string module and demonstrates that the combinatorial information encoded in the blossoming string is sufficient for its computation.

Proposition 4.1.1. *Let $\omega = \omega_1 \dots \omega_l$ a string of Q and let $P = P^1 \xrightarrow{p_1} P^0$ be the minimal projective presentation of the string module $M(\omega)$. Let $\omega^{❀}$ be the associated respective blossoming string, created by added a left cohook $\sigma = \sigma_0 \dots \sigma_r$ at the left and a right coohok $\tilde{\sigma} = \tilde{\sigma}_0 \dots \tilde{\sigma}_s$ at the right. Set*

$$\omega_0 = \begin{cases} \sigma_0 & \text{if } \sigma_1 \text{is not trivial,} \\ 0 & \text{otherwise.} \end{cases} \quad \text{and } \omega_{l+1} = \begin{cases} \tilde{\sigma}_0 & \text{if } \tilde{\sigma}_1 \text{is not trivial,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$P^0 = \bigoplus_{\substack{j_i \text{ is a peak} \\ \text{non-blossom vertex of } \omega^{❀}}} P(F_{\omega^{❀}}(j_i)), \quad P^1 = \bigoplus_{\substack{k_i \text{ is a deep} \\ \text{non-blossom vertex of } \omega^{❀}}} P(F_{\omega^{❀}}(k_i)),$$

and

$$p_1 = \begin{bmatrix} p_0^+ & -p_1^- & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & p_1^+ & -p_2^- & 0 & & & \vdots \\ \vdots & 0 & p_2^+ & -p_3^- & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & -p_{t-1}^- & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & p_{t-1}^+ & -p_t^- \end{bmatrix}.$$

where

$$p_i^+ = \omega_{k_i+1} \dots \omega_{j_{i+1}}, \qquad p_i^- = (\omega_{j_i+1} \dots \omega_{k_i})^{-1}, \qquad \text{for } t-1 \geq i \geq 1,$$

and

$$p_0^+ = \omega_0\omega_1 \dots \omega_{j_1}, \qquad p_t^- = (\omega_{j_t+1} \dots \omega_l\tilde{\sigma}_0)^{-1}.$$

Note that since ω_0 may be zero, p_0^+ could also be null; similarly, the same holds for p_t^-.

To better visualise a minimal projective presentation of a string module we notice that this two-term complex 'unfolds' as

$$\cdots \overset{P(k_{r-1})}{\bullet} \xrightarrow{p^+_{r-1}} \overset{P(j_r)}{\bullet} \xleftarrow{-q^-_r} \overset{P(k_r)}{\bullet} \xrightarrow{q^+_r} \overset{P(j_{r+1})}{\bullet} \xleftarrow{-q^-_{r+1}} \overset{P(k_{r+1})}{\bullet} \cdots$$

where the projective modules $P(j_i)$ for $t \geq i \geq 1$ has homological degree zero, while the projective modules $P(k_i)$ for $t \geq i \geq 0$ has homological degree -1.

We will indicate this unfolding of the minimal projective presentation simply as:

$$\cdots \overset{k_{r-1}}{\bullet} \xrightarrow{p^+_{r-1}} \overset{j_r}{\bullet} \xleftarrow{-p^-_r} \overset{k_r}{\bullet} \xrightarrow{p^+_r} \overset{j_{r+1}}{\bullet} \xleftarrow{-p^-_{r+1}} \overset{k_{r+1}}{\bullet} \cdots$$

This method of visualising the minimal projective presentation will be particularly useful when describing homomorphisms between complexes. Therefore, from now on we will use this to represent $P(\omega)$.

Note that it corresponds to the description of a string complex arising from an homotopy string given by [ALP16]. In [BM03], the authors proved that an element of this type is an indecomposable element of $\mathcal{K}^{[-1,0]}$.

4.2 Two-term presilting complexes and non-kissing strings

Having established the basic terminology and construction relative to the blossoming algebra, we are now ready to define what a "kiss" is. We will then relate this combinatorial concept of the string to the algebraic property of being presilting.

Definition 4.2.1. Let ω, ω' be two blossoming strings. We say that ω **kisses** ω' if there exist a **kiss**, i.e a maximal common substring σ of ω and ω' such that the arrows of ω incident to σ are both incoming while the arrows of ω' incident to σ are both outgoing.

Note that we authorize the situation where σ is reduced to a vertex v, meaning that v is a peak of ω and a deep of ω'. Observe also that ω can kiss ω' several times, that ω and ω' can mutually kiss, and that ω can kiss itself.

We will say that two strings ω and ω' are non kissing, if neither ω kisses ω', nor ω' kisses ω.

Example 4.2.1. In Figure 4.3, the string above kisses the string below, the common substring is depicted in blue.

Proposition 4.2.1. *Let* Q *be a gentle quiver. Let $Q = Q^1 \xrightarrow{q_1} Q^0$ be the minimal projective presentation of a string module $M(\omega)$, and let $P = P^1 \xrightarrow{p_1} P^0$ be the minimal projective presentation of another string module $M(\omega')$. Consider the blossoming strings $\omega^{❀}$ and $\omega'^{❀}$. If there exists a map not homotopic to zero in $\mathrm{Hom}_{\mathcal{K}}(Q, P[1])$, then $\omega^{❀}$ kisses $\omega'^{❀}$.*

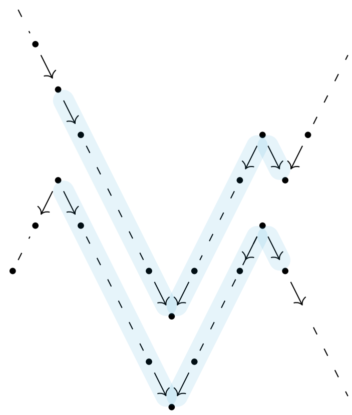


Figure 4.3

Proof. As showed in Proposition 4.1.1, implying the map $F_{\omega^{❀}}$, We have:

$$Q^0 = \bigoplus_{j_i \text{ peak non blossom vertices of } \omega^{❀}} P(j_i), \quad Q^1 = \bigoplus_{k_i \text{ deep non blossom vertices of } \omega^{❀}} P(k_i),$$

$$P^0 = \bigoplus_{j'_i \text{ peak non blossom vertices of } (\omega')^{❀}} P(j'_i), \quad P^1 = \bigoplus_{k'_i \text{ deep non blossom vertices of } \omega'^{❀}} P(k'_i).$$

The maps p_1 and q_1 corresponds to

$$q_1 = \begin{bmatrix} q_0^+ & -q_1^- & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & q_1^+ & -q_2^- & 0 & & & \vdots \\ \vdots & 0 & q_2^+ & -q_3^- & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & -q_{t-1}^- & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & q_{t-1}^+ & -q_t^- \end{bmatrix}.$$

and

$$p_1 = \begin{bmatrix} p_0^+ & -p_1^- & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & p_1^+ & -p_2^- & 0 & & & \vdots \\ \vdots & 0 & p_2^+ & -p_3^- & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & -p_{t-1}^- & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & p_{t-1}^+ & -p_t^- \end{bmatrix}.$$

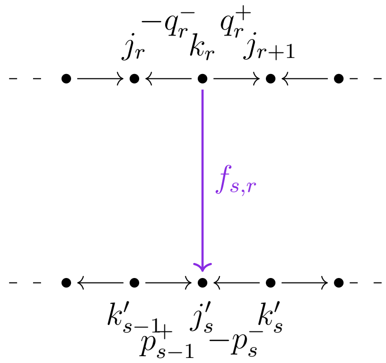
Notice that the two matrices could have different dimension, since t could be different from t'.

A map in $\text{Hom}_{\mathcal{K}}(Q, P[1])$ is determined by a Λ-module homomorphism from Q^1 to P^0, while a homotopy consists of a pair of maps $h^0 : Q^0 \to P^0$ and $h^1 : Q^1 \to P^1$. Any homomorphism f in $\text{Hom}_\Lambda(Q^1, P^0)$ can be expressed as a matrix $f = (f_{s,r})_{s,r}$, where each $f_{s,r}$ represents a homomorphism $P(k_r) \to P(j'_s)$ for $t \geq s \geq 1$ and $t' \geq r \geq 0$.Assume that $f_{s,r} \neq 0$ for some s, r.

As shown in Chapter 2, a basis vector for $\text{Hom}_\Lambda(P(k_r), P(j'_s))$ is given by multiplication by a path with head in j'_s and tail in k_r. Without loss of generality, since being null-homotopic is an additive property, we can assume that $f_{s,r}$ corresponds to the multiplication by a path starting at j_s and ending at k_r. We will often denote the path and the map corresponding to the multiplication by the path with the same name.

The homomorphism h^0 is represented by a matrix with entries $h^0_{s,r} : P(j_r) \to P(j'_s)$, while h^1 corresponds to a matrix with entries $h^1_{s,r} : P(k_r) \to P(k'_s)$.

Since $f_{s,r}$ is a path from j'_s to k_r, it must share its initial arrow with either q^+_r or q^-_r, and its final arrow with either p^-_s or p^+_{s-1}. We start by assuming that $f_{s,r}$ has as a subpath only one of these four. This assumption leads to the first step in our proof, which is divided into five cases based on which one of these paths is a subpath of $f_{s,r}$. The fifth case occurs when none of the paths are subpaths.



We label the five cases,

- (a) if q^+_r is a subpath of $f_{s,r}$,
- (b) if q^-_r is a subpath of $f_{s,r}$,
- (c) if $-p^-_s h^1_{s,r}$ is a subpath of $f_{s,r}$,
- (d) if p^+_{s-1} is a subpath of $f_{s,r}$,
- (e) Otherwise .

Although these four cases may initially appear different, we will analyse only one of them in detail. In fact, if we are in case (b), we can assume that we are in case (a) since we are considering strings up to their inverses. Indeed, by replacing ω with ω^{-1}, we observe that this replacing results in an index shift, where j_r becomes j_{r+1}, q^-_r becomes q^+_r, and so on. This implies that if we are in case (b), we can substitute ω with ω^{-1} and assume, without loss of generality, that we are in case (a). The same reasoning applies to cases (c) and (d).

This reduces our task to analysing cases (a), (c), and (e). However, we will not discuss case (c) in detail, as it follows the same pattern as case (a). In fact, the diagram shown in Figure 4.5 is identical. For the sake of completeness, we will depict how $\omega^{\text{❀}}$ kisses $\omega'^{\text{❀}}$ in some subcases of this in Figure 4.6.

Case (e)

We begin with the last case, (e), as it is the quickest to address. In this scenario, $f_{s,r}$ neither starts with q_r^+ nor q_r^-, nor does it end with p_s^- or p_{s-1}^+. This means that we can not write $f_{s,r}$ as

$$f_{s,r} = h^0_{s,r+1}q_r^+ - h^0_{s,r}q_r^- - p_s^- h^1_{s,r} + p^+_{s-1}h^1_{s-1,r}. \tag{4.1}$$

This implies that $f_{s,r}$ is homotopic only to itself. Since we have started with assuming that $f_{s,r}$ is different from zero, we need to show that $\omega^{\text{❀}}$ kisses $\omega'^{\text{❀}}$. As already said, since $f_{s,r}$ is a path from j'_s to k_r, it must share its starting arrow with either q_r^+ or q_r^-, and its ending arrow with either p_s^- or p_{s-1}^+. Then one of the pairs (q_r^+, p_s^-), (q_r^+, p_{s-1}^+), (q_r^-, p_s^-), or (q_r^-, p_{s-1}^+) shares a common subpath, with the first element in the pair matching with it at the start and the second at the end. We can visualize these four cases as shown in Figure 4.4.

As one can easily deduce from the figures, it means that $\omega^{\text{❀}}$ kisses $\omega'^{\text{❀}}$, as we wanted. In this case, $f_{s,r}$ could also be the identity, corresponding to the trivial path. If this occurs, we still have a kiss, as we allow for the possibility of a kiss being reduced to a single vertex.

Case (a)

We provide a brief summary of the structure of the proof, the detailed explanation of the reasons will be given in the course of the discussion. This part of the demonstration follows a tree diagram, consisting of various cases, as illustrated in Figure 4.5. We invite the reader to follow the diagram for clarity and guidance.

In the tree, each case generally divides into two branches: one stops, while the other continues. It is only at the end of the diagram, which corresponds to the end of one of the strings, that the cases become more specific and are distinguished into different types. Each stop alternatively corresponds to a case where $f_{s,r}$ is not homotopic to zero, there we will show that the two strings kiss, and one where $f_{s,r}$ is homotopic to zero, which is a contradiction.

At the beginning, the process appears different, as it starts with a split into three cases. However, one of these, the case (a.2'), can be excluded without losing generality, as we shall see.

The split is driven by two main motives, which alternate between the cases and also apply to the final cases. This aspect is highlighted in the figure, where equal colours indicate splits based on the same motives.

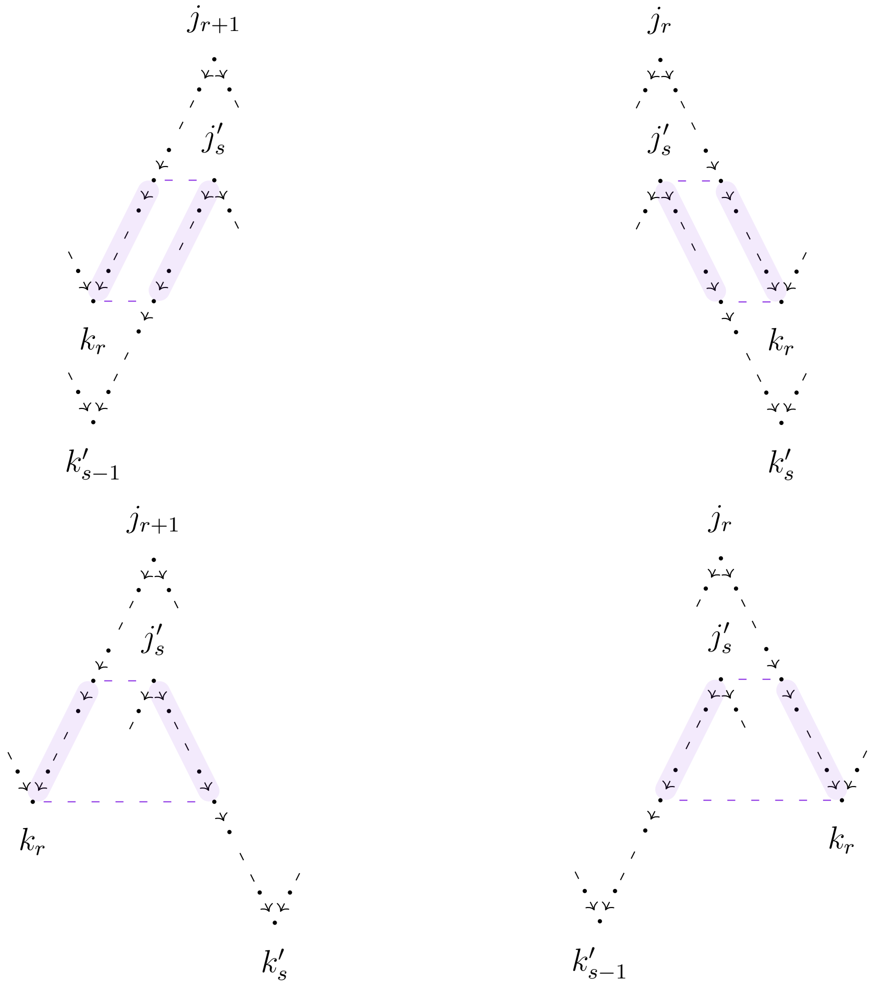


Figure 4.4

As said at the start, we have that $f_{s,r}$ is a path from j'_s to k_r, having as a ending subpath q_r^+. Since q_r^+ is a path with tail in k_r and head in j_{r+1}, there exists a path α_1 in Q starting at j'_s and ending at j_{r+1} such that $f_{s,r} = q_r^+\alpha_1$. Call $h^0_{s,r+1}$ the homomorphism corresponding to the multiplication by α_1, so it is different from zero. Then $f_{s,r} = h^0_{s,r+1}q_r^+$.

We observe that if there exists a homotopy that makes f homotopically trivial, then we also have that

$$f_{s,r+1} = h^0_{s,r+2}q^+_{r+1} - h^0_{s,r+1}q^-_{r+1} - p^-_s h^1_{s,r+1} + p^+_{s-1}h^1_{s-1,r+1}.$$

However, by assumption, $f_{s,r+1} = 0$.

Therefore, for $f_{s,r}$ to be homotopic to zero, we must have $h^0_{s,r+2}q^+_{r+1} - h^0_{s,r+1}q^-_{r+1} - p^-_s h^1_{s,r+1} + p^+_{s-1}h^1_{s-1,r+1} = 0$. This occurs, for instance, if $h^0_{s,r+1} \neq id$ and all other entries of the matrices corresponding to h^0 and h^1 are set to zero. This is the reason for the subsequent division of case (a) into two branches. As mentioned at the beginning of this case, these considerations will apply to all subsequent discussions.

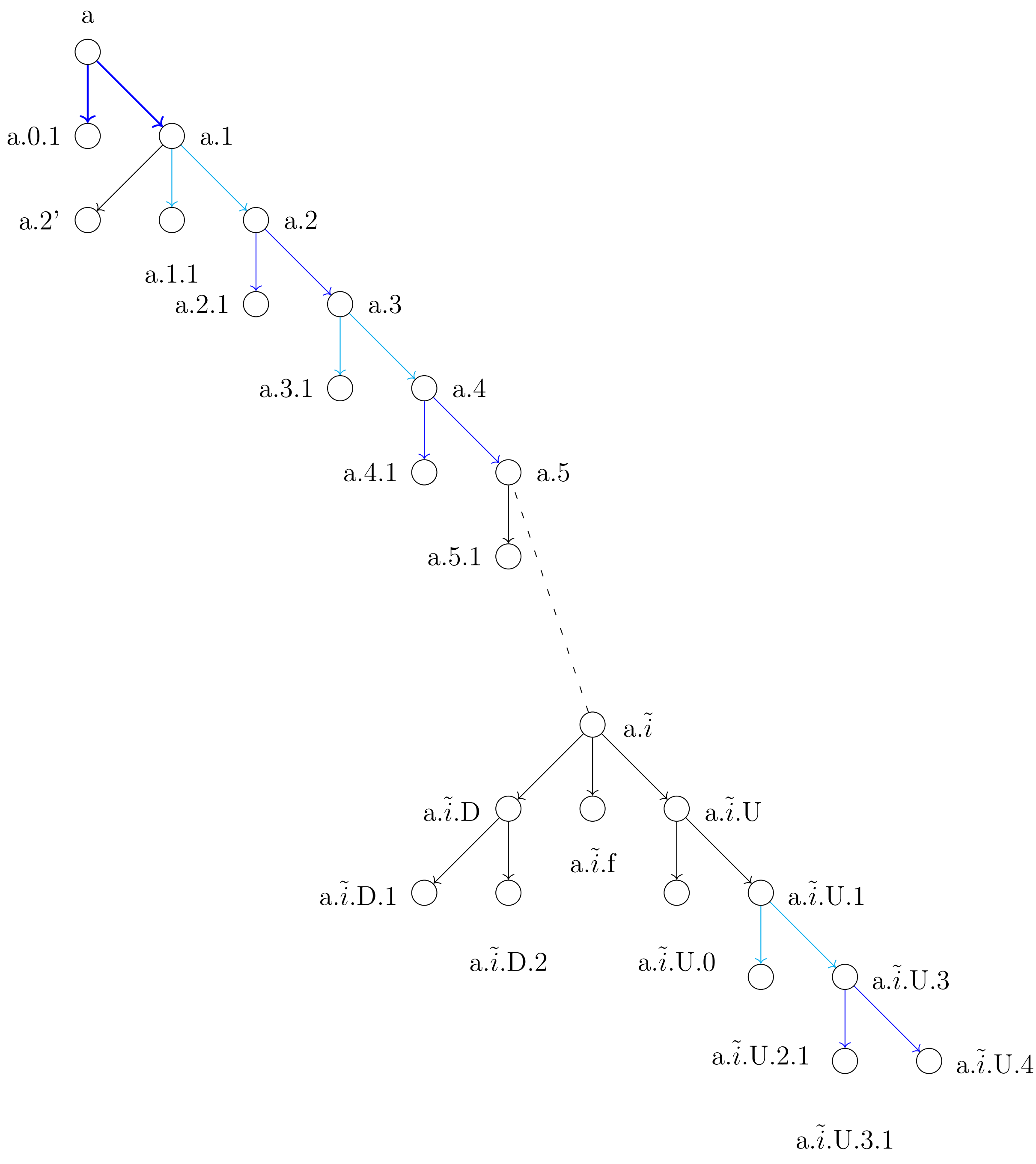
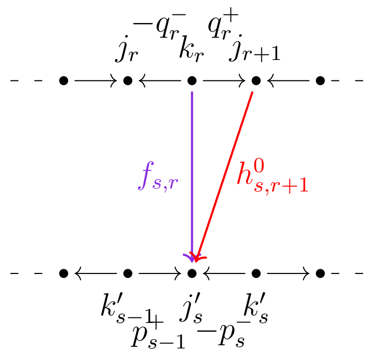


Figure 4.5: Scheme of case (a)

- (a.0.1) If $h^0_{s,r+1} \neq id$, then $f_{s,r}$ is homotopic to zero. Indeed, the homotopy is given by setting $h^1 = 0$ and, obviously, h^0 has the only entry different from zero equal to $h^0_{s,r+1}$, in position $(s, r+1)$.

  By the definition of the gentle quiver Q, there cannot be two paths that, when postcomposed with $h^0_{s,r+1}$, are different from zero, in fact one of the two must be in a relation. Therefore, $h^0_{s,r+1}q^-_{r+1} = 0$ and so, as already mentioned, $h^0_{s,r+2}q^+_{r+1} - h^0_{s,r+1}q^-_{r+1} - p^-_s h^1_{s,r+1} + p^+_{s-1}h^1_{s-1,r+1} = 0$. However this implies that $f_{s,r}$ is homotopic to zero and so it contradicts our hypothesis.



a.0.1

- (a.1) If $h^0_{s,r+1} = id$, then $j_{r+1} = j'_s$, and α_1 is the trivial path, so $f_{s,r}$ is homotopic to $-q^-_{r+1}$. Indeed, by applying the same settings to h^0 and h^1 as in case (a.0.1), we obtain the equation

$$h^0_{s,r+2}q^+_{r+1} - h^0_{s,r+1}q^-_{r+1} - p^-_s h^1_{s,r+1} + p^+_{s-1}h^1_{s-1,r+1} = -q^-_{r+1}.$$

  This does not conclude our discussion, however, because we still do not know whether $-q^-_{r+1} : P(k_{r+1}) \to P(j'_s)$ is homotopic to zero, and if it is not, whether there is a kiss between the strings.

Notice that we have shifted our focus to the path that follows the position where $f_{s,r}$ is located.

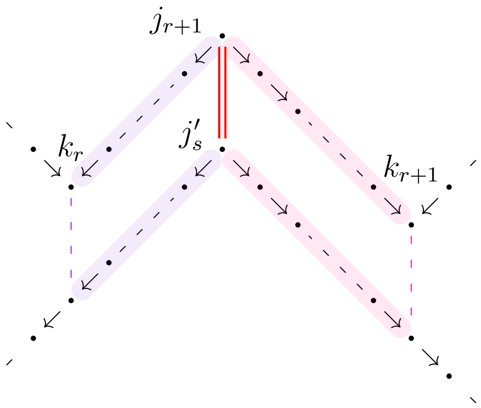
We have that q^-_{r+1} is a path from j_{r+1} to k_{r+1}. Moreover, both p^+_s and p^-_{s-1} are paths in Q that have their heads at $j_{r+1} = j'_s$. By the definition of a gentle quiver, since there are at most two arrows starting from each vertex, either p^+_s or p^-_{s-1} must share the same starting arrow with q^-_{r+1}. Observe that, by construction of the string, q^-_{r+1} has not the starting arrow in common with q^+_r. So we obtain three cases:

- (a.1.1) Neither p^-_s nor p^+_{s-1} is a starting subpath of q^-_{r+1}. This means that we cannot construct a non-null homomorphism from $P(k_{r+1})$ to either $P(k'_{s-1})$ or $P(k'_s)$ such that postcomposing it with p^+_s or p^+_{s-1} yields q^-_{r+1}. Namely, we can not find homomorphisms $h^1_{s,r+1}$ and $h^1_{s,r+1}$ such that

$$q^+_{r+1} + -p^-_s h^1_{s,r+1} + p^+_{s-1}h^1_{s,r+1} \neq 0.$$

Hence the homomorphism corresponding to the negative multiplication by q^-_{r+1} cannot be homotopic to zero, which subsequently implies that $f_{s,r}$ is also not homotopic to zero.

Moreover, this implies that either p^-_s or p^+_{s-1} must start with q^-_{r+1}. The strings in these two cases look respectively like:



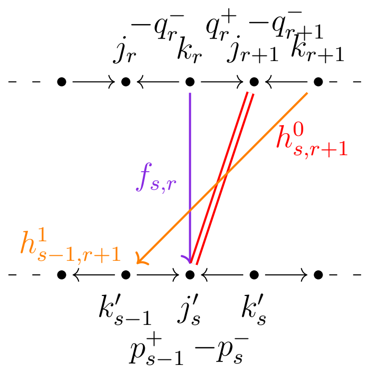
One can easily observe that $\omega^{❀}$ kisses $\omega'^{❀}$, where the highlighted parts of the strings in the figure above are identical.

- (a.2) If q^-_{r+1} is a path starting with p^-_s, there exists a path β_1 in Q starting at k'_s and ending at k_{r+1} such that $q^-_{r+1} = \beta_1 p^-_s$. By setting $h^1_{s,r+1}$ to be the homomorphism corresponding to the multiplication by β_1, we get that

$$-q^-_{r+1} = -p^-_s h^1_{s,r+1}$$

  We do not have any conclusive information yet, so we need again to continue under this hypothesis.

- (a.2') If q^-_{r+1} is a path starting with p^+_{s-1}, there exists a homomorphism $h^1_{s-1.r+1}$ such that $-q^-_{r+1} = -p^+_{s-1} h^1_{s-1,r+1}$.

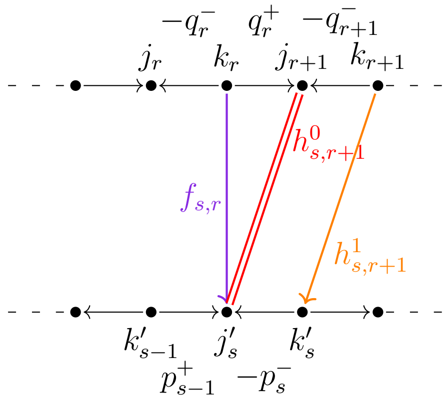


(a.2')

Since we are working with strings up to inverses, we can interchange ω' with ω'^{-1}. If we consider the latter, we notice that the index k'_{s-1} changes to k'_s, p^+_{s-1} becomes $-p^-_s$, and so on. In particular, this means that we can simply replace ω' with its inverse and continue, without loss of generality, under the hypothesis of case $(a.2)$. This justifies the name of this vertex in Figure 4.5.

We proceed under the assumption of case $(a.2)$. Similarly, the reasoning behind the bifurcation of case (a) applies here, splitting case (a.2) into two distinct scenarios.

- (a.2.1) If $h^1_{s,r+1} \neq id$, then, $f_{s,r}$ is homotopic to zero. The homotopy (h^1, h^0) is constructed by setting h^1 to be the matrix with the only non-zero entry being $h^1_{r+1,s}$. Meanwhile, h^0 is defined to be the identity at entry $(s, r+1)$ with all other entries equal to zero. Indeed, by the same reasoning as in case (a.0.1), this construction works due to the definition of the gentle quiver. However, this represents a contradiction of our assumption.



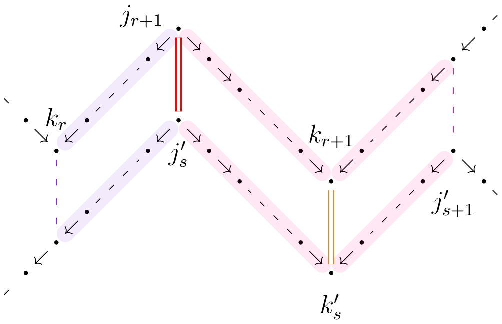
(a.2.1)

- (a.3) If $h^1_{s,r+1} = id$, then $k_{r+1} = k'_s$ and $-q^-_{r+1}$ is equal to p^+_s, so, $f_{s,r}$ is homotopic to p^+_s. Again, we do not have any conclusive information; instead, we now ask whether p^+_s is homotopic to zero.

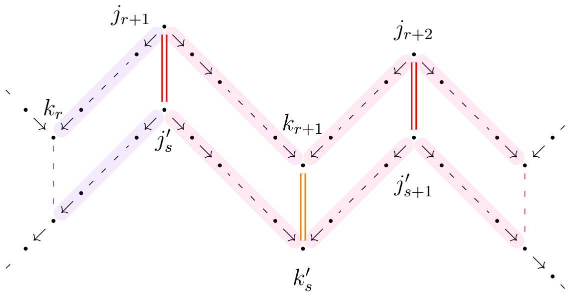
Once more, we have shifted from talking about a path to talk about the one in the subsequent position in the string complex.

We will again encounter a bifurcation depending on whether this path can be viewed as an ending subpath of q^+_{r+1} or not. The reasons for this bifurcation are analogous to those behind the splitting of case (a.1). If the path cannot be seen as an ending subpath of q^+_{r+1}, it implies that $f_{s,r}$ is not homotopic to zero, and, by the reasoning already discussed, $\omega^{❀}$ kisses $\omega'^{❀}$. On the other hand, if the path can be seen as an ending subpath, we need to examine whether p^+_s and q^+_{r+1} are equal or not, which brings us back to the same scenario as in case (a.3). These alternating splitting continue to occur following the same idea. The scheme is illustrated in Figure 4.5.

For completeness, the figures below illustrate the configurations of the strings in cases (a.3.1) and (a.5.1). In case (a.3.1), we have $h^0_{s,r+i} = h^1_{s,r+1} = id$ and p_s^+ is an ending subpath of q^+_{r+1}. In case (a.5.1), the conditions are $h^0_{s,r+i} = h^1_{s,r+1} = h^0_{s+1,r+2} = id$, and q^-_{r+2} is an ending subpath of q^-_{s+1}.



(a.3.1)



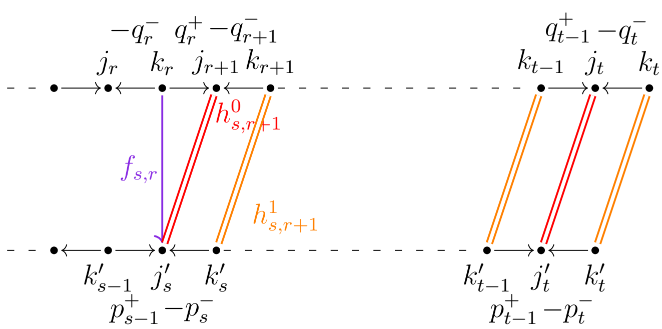
(a.5.1)

However, since the strings are finite, this process cannot continue indefinitely. Eventually, we must reach the end of one of the strings. Specifically, the final steps of case (a) depend on whether we reach the end of the string ω or the string ω' first.

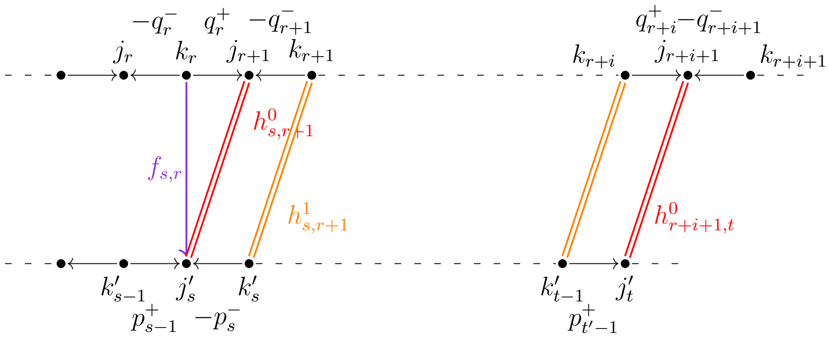
Set $i = \min\{t' - s, t - r - 1\}$. After $\tilde{i} = 4i + 1$ steps from case (a), we will have constructed $h^0_{s+i,r+i+1} = id$ and $h^1_{s+i-1,r+i+1} = id$. We could have:

- (a.$\tilde{i}$.D) $i = t' - s$, so it ends before the string ω', the one we have always depicted down in the figures;
- (a.$\tilde{i}$.U) $i = t - r - 1$, so it ends before the string ω, that we have always depicted up in the figure.
- (a.$\tilde{i}$.f) $t' = t$ and either neither k_t nor k'_t exists, or both exist and are equal. These are the only cases not covered in the previous scenarios. For instance, if k'_t exists but k_t does not, we can refer to case (a.$\tilde{i}$.U.1).

  In these last cases, $f_{s,r}$ turns out to be homotopic to zero. The homotopy consists of the pair h^0 and h^1, with entries equal to the identity for each position: $s + n, r + n + 1$ and $s + n - 1, r + n + 1$ respectively for $t - 1 \geq n \geq 0$. Additionally, we include the entry $h^1_{t,t'} = id$, if both k_t and k'_t exist.



Case (a.$\tilde{i}$.D).



Our proof bifurcates into two distinct cases, depending on the existence of the vertex $k'_{t'}$. We aim to determine whether a contradiction can be reached by constructing a null-homotopy, and if not, to understand how the strings are connected to each other.

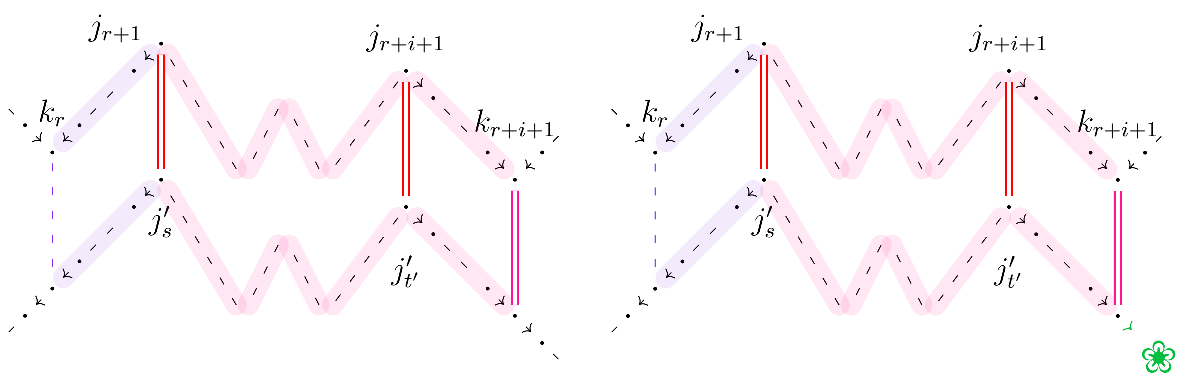
- (a.$\tilde{i}$.D.1) If $k'_{t'}$ does not exist, or it exists, but q_{r+i+1}^- is a subpath of $p_{t'}$, different from it, then $f_{s,r}$ is not homotopic to zero. Specifically, $f_{s,r}$ is homotopic to $-q_{r+i+1}^-$. Indeed, it is impossible to construct a map $h^1_{t,r+i+1}$ from $P(k_{r+i+1})$ that can be precomposed with $-p_{t'}^-$ such that the composition equals q_{r+i+1}^-, i.e.,

  $$q_{r+i+1}^- \neq -p_t^- h^1_{t',r+i+1}.$$

  This confirms that $f_{s,r}$ cannot be homotopic to anything else.

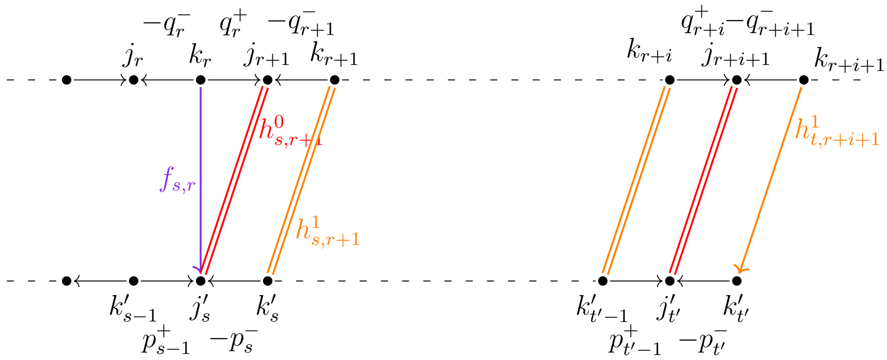
  Moreover, if $k'_{t'}$ does not exist, this implies that the subpath $\tilde{\omega'} = \omega'_{j_{t'}+1} \dots \omega'_l$ is a maximal path starting from $j'_{t'}$. Given that $h_{t',r+i+1} = id$, we have $j'_{t'} = j_{r+i+1}$.

  So it follows that q_{r+i+1}^- must either be a substring of $\tilde{\omega'}$ or equal to it. In either case, $\omega^{❀}$ kisses $\omega'^{❀}$, as depicted in the figure below:

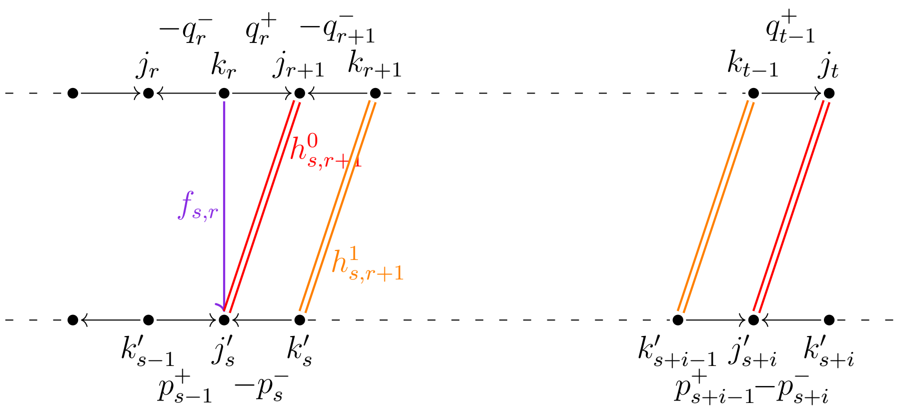
  Note that the left figure also represents the case when $k'_{t'}$ exists, but q_{r+i+1}^- is a subpath of $p_{t'}^-$.

- (a.$\tilde{i}$.D.2) If there exists $k'_{t'}$ and $p_{t'}^-$ is a subpath of q_{r+i+1}^-, or they are equal, then there exists a path β_{i+1} in Q such that $q_{r+i+1}^- = \beta_{i+1} p_{t'}^-$. Note that β_{i+1} could also be the trivial path if $p^-_{t'}$ and q^-_{r+i+1} are equal. By setting $h^1_{t',r+i+1}$ as the homomorphism corresponding to multiplication by the path β_{i+1}, we obtain the homotopy that makes $f_{s,r}$ homotopically trivial. But this is impossible under our starting assumption.



Case (a.$\tilde{i}$.U). We found ourselves in this case if $r+i = t-1$ and $h^0_{s+i,t} = id$, namely, if the string above ends before the string below. This case ramifies into other two branches, whether k_t exists or not.

- (a.$\tilde{i}$.U.0) If k_t does not exists, then $f_{s,r}$ is homotopic to zero and the homotopy is represented below:



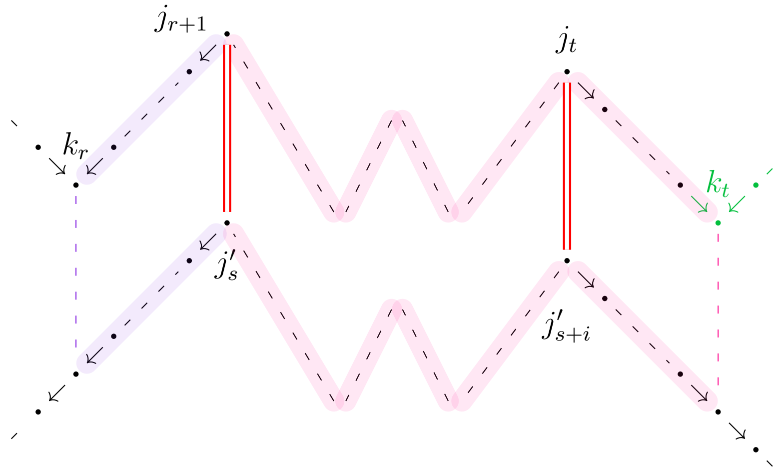
This contradicts the hypothesis under which we are working, so this case is not possible.

- (a.$\tilde{i}$.U.1) If there k_t, then, with the same homotopy just defined in the case above, $f_{s,r}$ is homotopic to $-q^-_t$, since $h^0_{s+i,t+1}q^+_t - h^0_{s+i,t}q^-_t - p^-_{s+i}h^1_{s+i,t} + p^+_{s+i}h^1_{s+i,t} = -q^-_t$

The path q^-_t starts in j_t and ends in k_t, since $h^0_{s+i,t} = id$ also p^+_{s+i-1} and p^-_{s+i} correspond to paths with head in j_t. However, since p^+_{s+i-1} is equal to q^+_{t-1}, the only possibility is that p^-_{s+i} and q^-_t share the same starting arrow. Then the subsequent splitting follows the same reasoning used, for instance, in case (a.2).

- (a.$\tilde{i}$.U.1.1) If q_t^- is a starting subpath of p_{s+i}^-, then q_t^- is not homotopic to zero. Consequently, $f_{s,r}$ is also not homotopic to zero, since it is impossible to construct a module homomorphism from $P(k_t)$ to $P(k'_{s+i})$ such that precomposing it with the multiplication by p_{s+i}^- yields q_t^-.

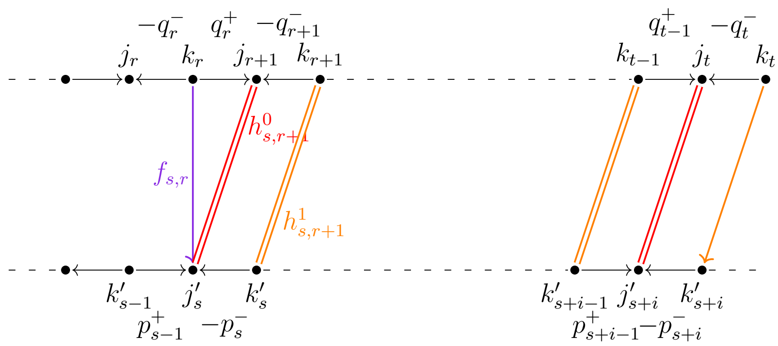
  In the following figure, the parts of the strings which are in common are highlighted, and it is easy to see that also in this case there is a kiss.



- (a.$\tilde{i}$.U.2) If p_{s+i}^- is a starting subpath of q_t^-, then there exists β_{i+1}, path in Q starting in k'_{s+i} and ending in k_t such that $q_t^- = \beta_{i+1}p_{s+i}^-$. Thus we set $h^1_{s+i,t}$ as the multiplication by β_{i+1}.

The splitting now follows the same reasoning as case (a), (a.2) or (a.4) and so on, given by the fact that if $h^1_{s+i,t} \neq id$ is a contradiction, if not we need to continue our examination.

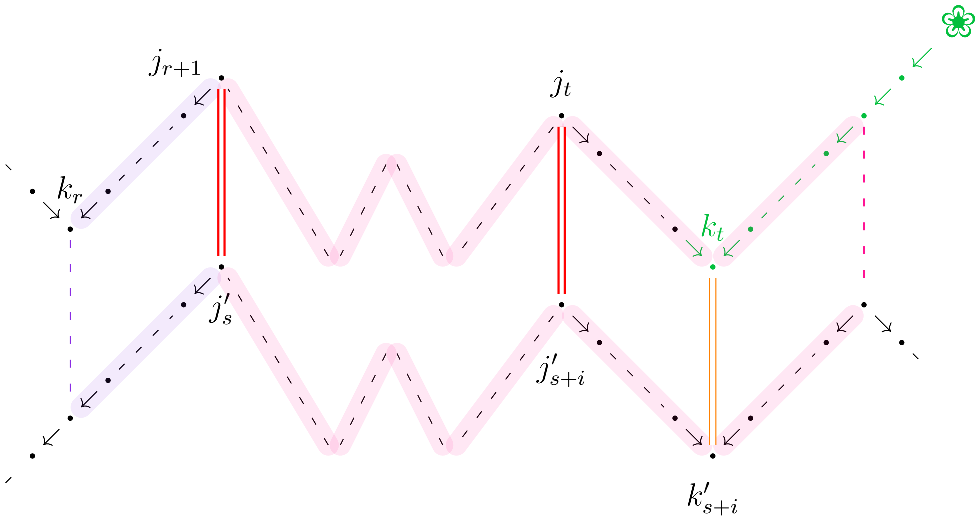
- (a.$\tilde{i}$.U.2.1) If $h^1_{s+i,t} \neq id$, then, as previously discussed, $f_{s,r}$ is homotopic to zero and the homotopy is represented in the following



  This case is a contradiction of our hypothesis, so it is impossible.

- (a.$\tilde{i}$.U.3) If $h^1_{s+i,t} = id$, then $k_t = k'_{s+i}$ and $q_t^- = p_{s+i}^-$. This implies that $f_{s,r}$ is homotopic to p_{s+i}^-, which is not homotopic to zero.

Note that the path p^-_{s+i} ends with $k_t = k'_{s+i}$, and so it must have in common with the cohook $\tilde{\sigma}'$, at least $\tilde{\sigma}'_1$. However, by definition of the cohook, $\tilde{\sigma}'_1 \dots \tilde{\sigma}'$ is a maximal path in Q with head in k_t. This means that p^+_{s+i} is a subpath of it, so the strings look like:



and it is obvious that $\omega^{❀}$ kisses $\omega'^{❀}$.

Recall that at the beginning, we have assumed that only one among $h^0_{s,r+1}q^+_r - h^0_{s,r}q^-_r$, $-p^-_s h^1_{s,r}$, and $p^+_{s-1}h^1_{s-1,r}$ is equal to $f_{s,r}$. We showed that with this assumption everything turns out as desired. However, to conclude the proof, we must not forget to look back at what happens if more than one among $h^0_{s,r+1}q^+_r - h^0_{s,r}q^-_r$, $-p^-_s h^1_{s,r}$, and $p^+_{s-1}h^1_{s-1,r}$ are equal to $f_{s,r}$.

Since, by construction, $q^+_r \neq q^-_r$ and $p^-_s \neq p^+_{s-1}$, $f_{s,r}$ can be equal to at most two of these expressions: one involving a "q" path and the other a "p" path. Given that $f_{s,r}$ is assumed to be non-homotopic to zero, the only significant case is when $f_{s,r}$ equals two of these four terms, $h^0_{s,r+1}q^+_r - h^0_{s,r}q^-_r$, $-p^-_s h^1_{s,r}$, and $p^+_{s-1}h^1_{s-1,r}$, with the corresponding "h" map being the identity. Indeed, it is not important to look at the case when for at least one of the element of the pair, the "h" map related is not the identity. If this happens we have that $f_{s,r}$ is homotopic to zero, as discussed earlier (see case (a.0.1)), which contradicts the hypothesis.

Consider the case where $f_{s,r} = q^+_r = p^+_{s-1}$; similar reasoning applies to other cases. This assumption implies that q^-_r and p^-_{s-1} share the same starting arrow, as do q^-_{r+1} and p^-_s. If $q^-_r \neq p^-_{s-1}$, one must be a subpath of the other. Suppose p^-_{s-1} is the subpath; then $f_{s,r}$ is homotopic to p^-_{s-1}, with the homotopy given by setting $h^0_{s,r} = h^1_{s,r} = id$.

For the path $p^-_{s-1} : j'_{s-1} \to k_r$, only one of the following can equal it: $h^0_{s-1,r+1}q^+_r$, $-h^0_{s-1,r}q^-_r$, $-p^-_{s-1}h^1_{s,r}$, or $p^+_s h^1_{s-1,r}$, which obviously corresponds to set $h^1_{s,r} = id$. Thus, without loss of generality, we can replace $f_{s,r}$ with p^-_{s-1} and return to the beginning of this proof, since now the division in five cases is mandatory. The same logic applies if $q^-_{r+1} \neq p^-_s$.

If both $q^-_r = p^-_{s-1}$ and $q^-_{r+1} = p^-_s$, we examine whether the pairs (q^+_{r-1}, p^+_{s-2}) and (q^+_{r+1}, p^+_s) are made of equal terms. If they are not, $f_{s,r}$ is homotopic to a path for which there is only one possible way to construct a homotopy , as discussed above, allowing us to replace $f_{s,r}$ with it and restart the discussion.

If the pairs are equal, we continue by examining the pairs (q^-_{r-1}, p^-_{s-2}) and (q^-_{r+2}, p^-_{s+1}) and so on. If we never find a pair $(q^{\pm}_{r-i}, p^{\pm}_{s-i-1})$ for $\min\{s-1, r\} \geq i \geq 0$ or $(q^{\pm}_{r+i+1}, p^{\pm}_{s+i})$ for $\min\{t'-1-r, t-s\} \geq i \geq 0$ that has unequal terms, it means that ω and ω' are identical, and $f_{s,r}$ is homotopic to the identity.

□

Proposition 4.2.2. *Let ω, ω' be two gentle string of* Q. *Consider the blossoming strings $\omega^{❀}$ and $\tilde{\omega}^{❀}$ and let $Q = Q^1 \xrightarrow{q_1} Q^0$, resp. $P = P^1 \xrightarrow{p_1} P^0$, be the minimal projective presentation of $M(\omega)$, resp. $M(\tilde{\omega})$. If $\omega^{❀}$ kisses $\tilde{\omega}^{❀}$, there exists a map not homotopic to zero in* $\mathrm{Hom}_{\mathcal{K}}(Q, P[1])$.

Proof. By assumption there exists a common substring σ of $\omega^{❀}$ and $\tilde{\omega}^{❀}$ such that the arrows of $\omega^{❀}$ incident to σ are both incoming while the arrows of $\tilde{\omega}^{❀}$ incident to σ are both outgoing.

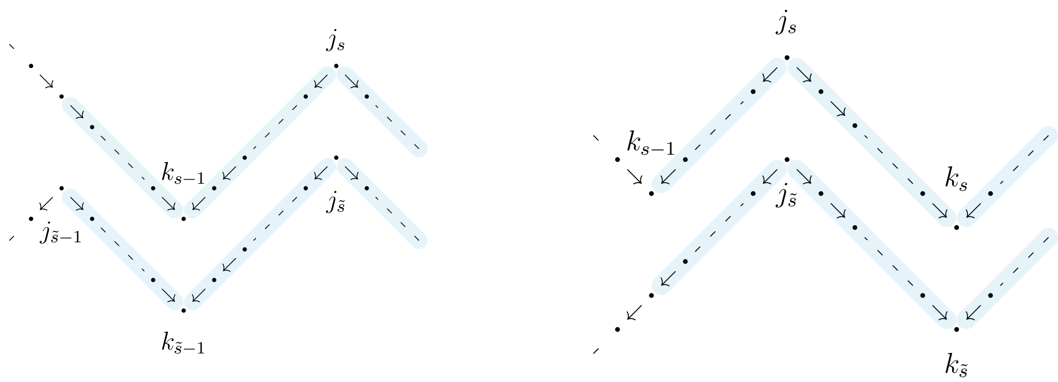
Call $j_s, \dots j_{s+n}$ the peak vertices of $\omega^{❀}$, which are in common with the peak vertices $j_{\tilde{s}}, \dots j_{\tilde{s}+n}$ of $\tilde{\omega}^{❀}$ and call $k_r, \dots k_{r+m}$ the deep vertices of $\omega^{❀}$, which are in common with the deep vertices $k_{\tilde{r}}, \dots k_{\tilde{r}+m}$ of $\tilde{\omega}^{❀}$.

Consider the first vertex that the two strings have in common, the one that links σ with $\omega^{❀}$ and $\tilde{\omega}^{❀}$. We denote this vertex as v for $\omega^{❀}$ and $\tilde{v}$ for $\tilde{\omega}^{❀}$.

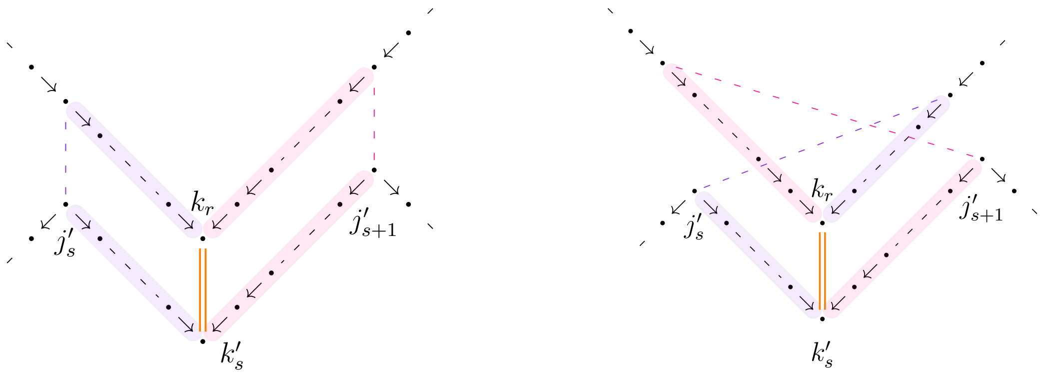
By construction, only one between v and $\tilde{v}$ the must be a vertex in which the respective string changes direction, in particular either

- (a) $\tilde{v}$ is a peak vertex of $\tilde{\omega}^{❀}$ or
- (b) v is a deep vertex of $\omega^{❀}$.

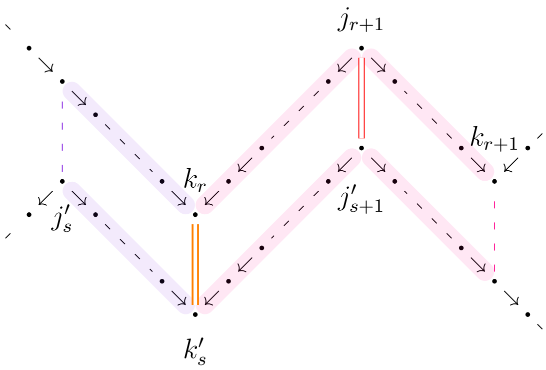
Notice that if we are in case (a), then $r = s-1$ and $\tilde{r} = \tilde{s}-1$ and $\tilde{v}$ is equal to $j_{\tilde{s}-1}$, while in case (b) $r = s$ and $\tilde{r} = \tilde{s}$ and v corresponds to k_{s-1}.



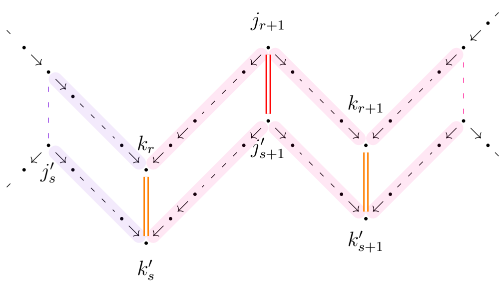
(a) (b)



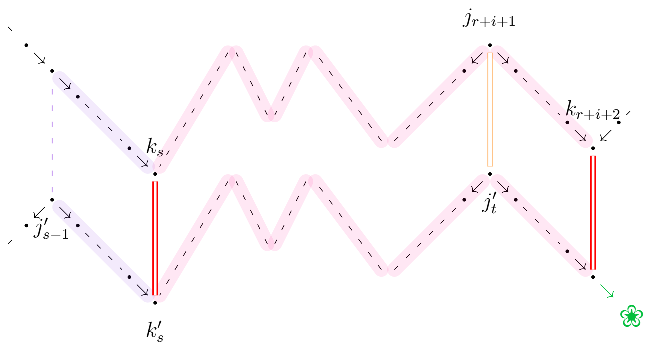
(c.1.1)



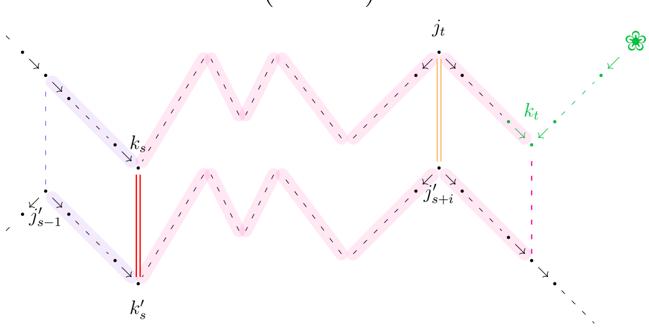
(c.3.1)



(c.5.1)



(c.$\tilde{i}$.D.1)



(c.$\tilde{i}$.U.1.1)

Figure 4.6: Case (c)

We define a map

$$f : \bigoplus_i P(k_i) = P^1 \to \bigoplus_i P(j_{\tilde{i}}) = Q^0$$

as a matrix where all entries are zero except for specific positions. In case (a), we set

$$f_{\tilde{s}-1,s-1} := p^-_{\tilde{s}-1} : P(k_{s-1}) = P(k_{\tilde{s}-1}) \to P(j_{\tilde{s}-1})$$

in position $(s-1, \tilde{s}-1)$, while in case (b), we set

$$f_{\tilde{s},s-1} := q^+_{s-1} : P(k_{s-1}) \to P(j_s) = P(j_{\tilde{s}})$$

in position $(\tilde{s}, s-1)$.

Our goal is to prove that these maps are not null-homotopic. We will focus to show the details of case (a), as the proof for case (b) follows similarly.

Assume, by contradiction, that f is null-homotopic. This implies that for each pair $(i, \tilde{i})$, the following equation holds:

$$f_{\tilde{i},i} = h^0_{\tilde{i},i+1} q^+_i - h^0_{\tilde{i},i} q^-_i - p^-_{\tilde{i}} h^1_{\tilde{i},i} + p^+_{\tilde{i}-1} h^1_{\tilde{i}-1,i},$$

for Λ-module homomorphisms $h^0 : \bigoplus_i P(j_i) \to \bigoplus_i P(j_{\tilde{i}})$ and $h^1 : \bigoplus_i P(k_i) \to \bigoplus_i P(k_{\tilde{i}})$.

Since $f_{\tilde{s}-1,s-1} := p^-_{\tilde{s}-1}$, it follows that $h^1_{\tilde{s}-1,s-1} = id$, while $h^0_{\tilde{s}-1,s}$, $-h^0_{\tilde{s}-1,s-1}$ and $h^1_{\tilde{s}-2,s-1}$ must be equal to zero, since $p^+_{\tilde{s}-2}$ is not a subpath of neither q^+_{s+n} nor q^-_{s+n}.

We have:

$$h^0_{\tilde{s}+i,s+i+1} q^+_{s+i} - h^0_{\tilde{s}+i,s+i} q^-_{s+i} - p^-_{\tilde{s}+i} h^1_{\tilde{s}+i,s+i} + p^+_{\tilde{s}+i-1} h^1_{\tilde{s}+i-1,s+i} = 0 \quad \text{for } i \neq -1.$$

By construction, we know:

$$q^+_{s+i} = p^+_{\tilde{s}+i} \quad \text{for } n-1 \geq i \geq -1, \quad \text{and} \quad q^-_{s+i} = p^-_{\tilde{s}+i} \quad \text{for } n-1 \geq i \geq 0.$$

Putting these two equations together, it implies that $h^1_{\tilde{s}+i,s+i} = id$, for $n-1 \geq i \geq -1$ and $h^0_{\tilde{s}+i,s+i} = id$, for $n \geq i \geq 0$.

Now we have two possibilities, either $q^-_{s+n} = p^-_{\tilde{s}+n}$ or not. This depends on whether σ ends with a peak vertex or not.

If $q^-_{s+n} \neq p^-_{\tilde{s}+n}$, by the definition of the kiss, q^-_{s+n} is a subpath of $p^-_{\tilde{s}+n}$. This means that there exists a homomorphism $h^1_{\tilde{s}+n,\tilde{s}+n}$ that is different from the identity such that $-p^-_{\tilde{s}+n} = -q^-_{s+n} h^1_{\tilde{s}+n,\tilde{s}+n}$. However, this leads to:

$$0 \neq h^0_{\tilde{s}+n,s+n+1} q^+_{s+n} - h^0_{\tilde{s}+n,s+n} q^-_{s+n} - p^-_{\tilde{s}+n} h^1_{\tilde{s}+n,s+n} + p^+_{\tilde{s}+n-1} h^1_{\tilde{s}+n-1,s+n},$$

which contradicts our assumption.

If $q^-_{s+n} = p^-_{\tilde{s}+n}$, then, again, by definition of the kiss, $p^+_{\tilde{s}+n}$ is a subpath of q^+_{s+n}. Following an equivalent reasoning as above, we reach a contradiction.

Since both cases lead to contradictions, we conclude that f, as defined, is not null-homotopic.

□

Putting together these two last propositions, we have proved then the following

Proposition 4.2.3. *A minimal projective presentation of a string module $M(\omega)$ is a two-term* presilting *complex for Λ if and only if the relative blossoming string $\omega^{❀}$ does not kiss itself.*

4.3 Classification of two-term silting complexes over a gentle algebra

In this concluding section, we complete our work by providing a complete classification of the two-term silting complexes over a gentle algebra. We will illustrate how these algebraic objects can be described using only combinatorial information, and how each of the results obtained so far contributes to the classification.

Let Q be a gentle quiver. We consider the set of τ-rigid indecomposable modules, along with the shifted projective modules $P_v[1]$ for each vertex v of the quiver Q. We say that two Λ-modules M and N are *compatible* if $\mathrm{Hom}(M, \tau N) = 0 = \mathrm{Hom}(N, \tau M)$. A Λ-module M and $P_v[1]$ are *compatible* if $\mathrm{Hom}(P_v, M) = 0$.

Let C be a maximal compatible collection, and consider the pair (M, P), where $M = \bigoplus_{M_i \in C} M_i$ and $P = \bigoplus_{v | P_v[1] \in C} P_v$. Then (M, P) is a τ-rigid pair. Indeed, $\mathrm{Hom}_\Lambda(P, M) = 0$ because $\mathrm{Hom}_\Lambda(P_v, M_i) = 0$ for all v and i, and $\mathrm{Hom}_\Lambda(M, \tau M) = 0$, since $\mathrm{Hom}_\Lambda(M_j, M_i)$ is equal to zero for all i and j.

Because C is *maximal*, the following lemma proves that the relation established above yields a one-to-one correspondence between maximal compatible collections and support τ-tilting pairs:

Lemma 4.3.1. *Let (T, P) be a τ-rigid pair. The following are equivalent:*

- *If $(T \oplus X, P)$ is τ-rigid pair for some X Λ-module, then $X \in \mathrm{add}\, T$.*
- *(T, P) is a support τ-tilting pair.*

See [AIR14, Theorem 2.12 and Corollary 2.13].

In [Brü+20, Theorem 5.1], Brüstle, Douville, Mousavand, Thomas, and Yıldırım, proved that there is a bijective correspondence from maximal compatible collections of Λ-modules to maximal non-kissing collections of *long strings*.

With the term *long strings*, they denote the set of blossoming strings, along with the so-called *injective blossoming strings*. An injective blossoming string I_v is obtained by adding to the lazy path at a vertex v both maximal sequences of arrows oriented towards v in the blossoming quiver $Q^{❀}$.

The correspondence is then created by associating to each blossoming string ω its string module $M(\omega)$ and to each injective blossoming string I_v the module $P_v[1]$.

$$\left\{ \begin{array}{c} \text{Maximal collection of} \\ \text{non-kissing long strings} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Maximal compatible collection} \\ \text{of } \Lambda\text{-modules} \end{array} \right\}$$
$$\omega \longleftrightarrow M(\omega)$$
$$I_v \longleftrightarrow P_v[1]$$

We now want to translate this results in terms of two-term silting complexes.

By Theorem 3.5.1, we have a bijection between support τ-tilting pairs and two-term silting complexes given by associating to each support τ-tilting pair (M, P), the two-term silting complex $P^1 \oplus P \to P^0$, where $(P^1 \to P^0)$ is the minimal projective presentation of M.

This implies that, by combining the previous results, we have established a one-to-one correspondence between maximal collections of non-kissing long strings S and two-term silting complexes. Specifically, this correspondence is given by

$$\begin{Bmatrix} \text{Maximal collection of} \\ \text{non-kissing long strings} \end{Bmatrix} \longleftrightarrow \{\text{Two-term silting complexes}\}$$
$$S \longleftrightarrow \bigoplus_{\omega \in S} P(\omega) \oplus \bigoplus_{v | I_v \in S} P_v[1],$$

where $P(\omega)$ denotes the minimal projective presentation of the string module $M(\omega)$.

Using the results achieved so far in our thesis project, we can describe this correspondence in greater detail: let M, N be Λ modules, P be a projective Λ-module, and $P_M := (P_M^1 \to P_M^0)$, resp. $P_N := (P_N^1 \to P_N^0)$, be the minimal projective presentation of M, resp. N. Thanks to Lemma 3.4.1, we have $\mathrm{Hom}_\Lambda(M, \tau N) = 0$ if and only if $\mathrm{Hom}_\mathcal{K}(P_M, P_N[1]) = 0$. Moreover, due to Lemma 3.4.2, we know that $\mathrm{Hom}_\Lambda(P, M) = 0$ if and only if $\mathrm{Hom}_\mathcal{K}(P, P_M) = 0$.

So if we denote a subset $\mathcal{S}$ of the set

$$\left\{P = (P^1 \to P^0) \mid P \text{ is presilting}\right\} \cup \{P_v[1] \mid v \in \mathrm{Q}_0\}$$

as *coherent* when $\mathrm{Hom}_\mathcal{K}(P, Q[1]) = 0$ and $\mathrm{Hom}_\mathcal{K}(P_v, P) = 0$ for each P, Q, P_v in $\mathcal{S}$, it follows that a maximal compatible collection of Λ-modules defined above is in bijective correspondence with a maximal coherent collection of two-term complexes. Indeed, we associate to each module M its minimal projective presentation, while preserving each $P_v[1]$ that appears.

Thanks to Proposition 4.2.3, we know that a minimal projective presentation of a string module $P(\omega)$ is a two-term presilting complex if and only if the blossoming string $\omega^{❀}$ does not kiss itself.

With the following lemma, we demonstrate the connection between the fact that $\mathrm{Hom}_\mathcal{K}(P_v, P(\omega)) = 0$ and the non-kissing criterion.

Lemma 4.3.2. *Given a vertex v of a gentle quiver* Q*, ω a gentle string, let P_v be the indecomposable projective module related to vertex v and $P(\omega)$, the minimal projective presentation of string module $M(\omega)$. Then* $\mathrm{Hom}_\mathcal{K}(P_v, P(\omega)) = 0$ *if and only if I_v does not kiss $\omega^{❀}$.*

Proof. Let f be in $\mathrm{Hom}_\mathcal{K}(P_v, P(\omega)) = 0$, then f can be seen as a vector $[f_1, \dots f_t]$, where f_i is a Λ-module homomorphism from P_v to P_{j_i}. If f is not zero, then at least there exists some i, such that $f_i \neq 0$.

Since f_i is a Λ-module homomorphism between two indecomposable projective modules, we can assume that f_i corresponds to a multiplication by a path $j_i \to v$ in Q. We denote the path with f_i. Call p_v^+ and p_v^- the two maximal paths ending in v of $\mathrm{Q}^{❀}$, namely $I_v = (p_v^-)^{-1} p_v^+$. Then f_i is a subpath of either p_v^- or p_v^+.

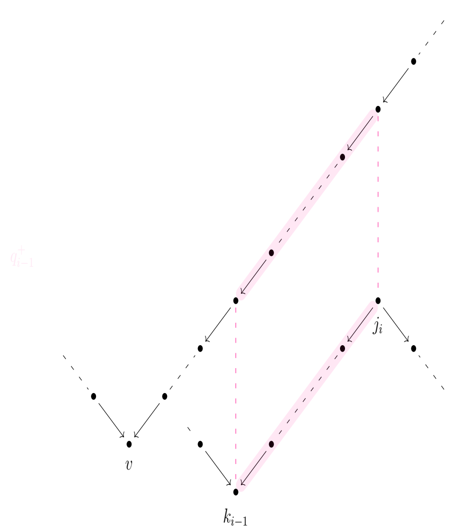


Figure 4.7: q_{i-1}^+ is a subpath of f_i

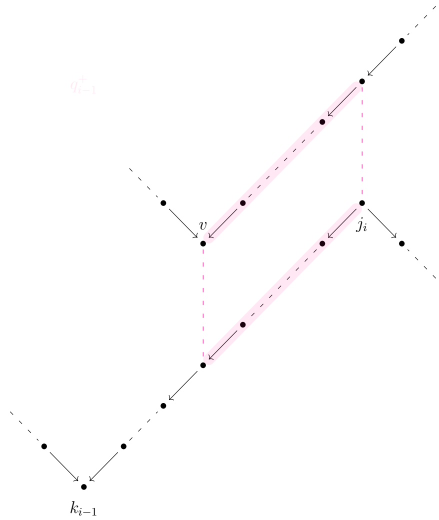


Figure 4.8: f_i is a subpath of q_{i-1}^+

Moreover, let $q_{i-1}^+ = \omega^{\text{❀}}_{k_{i-1}+1} \dots \omega^{\text{❀}}_{j_i}$, and $q_i^- = (\omega^{\text{❀}}_{j_i+1} \dots \omega^{\text{❀}}_{k_i})^{-1}$. These are two different path of Q starting with j_i . This implies that f_i must share the same starting arrow as one of them.

Since we are working with strings up to inverses, we can assume that f_i is a subpath of p_v^+ and has in common the starting arrow with q_{i-1}^+. The other combinations arise from simply replacing $\omega^{\text{❀}}$ with $(\omega^{\text{❀}})^{-1}$ or I_v with I_v^{-1}.

The homomorphism f_i is null-homotopic if and only if f_i can be written as $f_i = q_{i-1}^+ h_{i-1}$, where h_{i-1} is a Λ-module homomorphism from P_v to $P_{k_{i-1}}$. Since both f_i and q_{i-1}^+ are path , h_{i-1} must correspond to a multiplication by a path from k_{i-1} to v. Hence f_i is null-homotopic if and only if q_{i-1}^+ is a starting subpath of f_i.

If q_{i-1}^+ is a starting subpath of f_i or if they are equal , we have that the two strings are not kissing. Indeed q_{i-1}^+ is a common subpath, but the arrow $\omega^{\text{❀}}_{k_{i-1}}$ and $\omega^{\text{❀}}_{j_i+1}$, incident to it, have the same direction, as one can see in Figure 4.7.

If q_{i-1}^+ is not a starting subpath of f_i. it means that, since they share the same starting arrow, f_i is a starting subpath of q_{i-1}^+. The the common substring f_i represents a kiss, since the arrow incident to it of I_v are both incoming and the one of $\omega^{\text{❀}}$ are both outgoing. Refer to Figure 4.8.

So we proved that f_i is null-homotopic if and only if I_v does not kiss $\omega^{\text{❀}}$. Due to the fact that being null-homotopic is an additive property, we conclude. □

The result of [Brü+20] may then be rephrased, in light of the work presented here and in the preceding chapters.

Definition 4.3.1. Let Q be a gentle quiver. Define the set

$$\mathrm{Str}_{\mathrm{nk}} := \left\{ P(\omega) \mid \omega^{\text{❀}} \text{ is a string not kissing itself} \right\} \cup \{P_v[1] \mid v \in \mathrm{Q}_0\}.$$

where we denote by $P(\omega) = (P^1(\omega) \to P^0(\omega))$ the minimal projective presentation of the string module $M(\omega)$ and P_v are the indecomposable projective modules related to the vertices of Q.

Thanks to Propositions 4.2.1, 4.2.2, and the above Lemma, a subset S of $\mathrm{Str}_{\mathrm{nk}}$ is *coherent* if and only if for each $P(\omega)$ and $P(\tilde{\omega})$ in S, the strings $\omega^{❀}$ and $\tilde{\omega}^{❀}$ are non-kissing, and for each $P_v[1]$ and $P(\omega)$ in S, the strings I_v and $\omega^{❀}$ are also non-kissing.

We can then state:

Proposition 4.3.3. *Let Λ be a gentle algebra. There is a one-to-one correspondence between two-term silting complex over Λ and maximal coherent subsets of Str_{nk}, where the correspondence is given by associating to each maximal compatible subset the direct sum of its elements. In particular each two-term silting complex corresponds to a maximal collection of non-kissing long strings.*

Appendix A

An alternative proof: the ring of Laurent polynomials has no non-trivial idempotents

Lemma A.0.1. $\mathbb{K}[x, x^{-1}]$ *has no non-trivial idempotents.*

Proof. Firstly, we note that the commutative ring of formal series $\mathbb{K}\big[[x]\big]$ has no non-trivial idempotents. Indeed, let p an idempotent element of $\mathbb{K}\big[[x]\big]$, then $p = \sum_i^{+\infty} p_i x^i$, with p_i in $\mathbb{K}$ and $p^2 = \sum_i (\sum_{j+k=i} p_j p_k) x^i$. Note that the coefficient $q_i = \sum_{j+k=i} p_j p_k$ of x^i in p^2 depends only on the coefficients with index less or equal than it. If $p^2 = p$, then $p_0 = q_0 = p_0^2$, since $\mathbb{K}$ is a field, then $p_0 = 0$. Equivalently the equation $q_1 = 2p_0p_1$ must holds, thus p_1 must be equal than zero. Continuing with the same reasoning, we get $p = 0$.

To prove that $\mathbb{K}[x, x^{-1}]$ has no non-trivial idempotents, we would like to find an injective ring homomorphism between $\mathbb{K}[x, x^{-1}]$ and $\mathbb{K}\big[[x]\big]$, such that, if restricted to $\mathbb{K}$, is the identity. This will implies that each idempotent of $\mathbb{K}[x, x^{-1}]$ is sent to an idempotent of $\mathbb{K}\big[[x]\big]$, which has only trivial ones. Hence, $\mathbb{K}[x, x^{-1}]$ has no non-trivial idempotents.

So now we need to find such an injective ring morphism.

We start by looking at how the ring $\mathbb{K}[x, x^{-1}]$ is constructed. Let $R = \mathbb{K}[x]$ be the polynomial's ring given by the set of polynomials in one variable with coefficient on a field $\mathbb{K}$. This is a commutative ring. We can consider the multiplicative closed set $S = \{1, x, x^2, \dots\}$, then the localization $S^{-1}R$ is canonically isomorphic to $R[x^{-1}] = \mathbb{K}[x, x^{-1}]$, which is the ring of Laurent polynomials.

Since S does not contain any zero divisors, we have an injective ring homomorphism

$$\iota : R \hookrightarrow S^{-1}R$$
$$\sum_{i=o}^{n} a_i x^i \hookrightarrow \sum_{i=0}^{n} a_i x^i$$

which satisfies the following universal property of the localization:

if $f : R \to T$ is a ring homomorphism that maps every element of S to an invertible element in T, then there exist a unique ring homomorphism $g : S^{-1}R \to T$ such that $f = g \circ \iota$.

Claim:

$$\begin{aligned} \phi : \mathbb{K}[x, x^{-1}] &\to \mathbb{K}\big[[x]\big] \\ 1 &\mapsto 1 \\ x &\mapsto 1 - x \\ x^{-1} &\mapsto (1-x)^{-1} \end{aligned}$$

is an injective ring homomorphism.

If we show that this claim is true, we can conclude the proof of Lemma A.0.1.

We first prove that is a well-defined ring homomorphism, using the universal property of the localization. We define the map ψ from $\mathbb{K}[x]$ to $\mathbb{K}\big[[x]\big]$ on the generator of $\mathbb{K}[x]$ and then extend it by $\mathbb{K}-$linearity: $\psi(x) = 1 - x$, $\psi(1) = 1$.

Let $a = \sum_{i=0}^{n} a_i x^i$, $b = \sum_{j=0}^{n} b_j x^j \in \mathbb{K}[x]$. We can assume without loss of generalization $n \geq m$.

Then

$$\psi(a+b) = \psi\Big(\sum_{i=0}^{m}(a_i + b_i)x^i + \sum_{i=m+1}^{n} a_i x^i\Big) = \sum_{i=0}^{m}(a_i+b_i)(1-x)^i + \sum_{i=m+1}^{n} a_i(1-x)^i.$$

While

$$\begin{aligned} \psi(a) + \psi(b) = \psi(\sum_{i=0}^{n} a_i x^i) + \psi(\sum_{i=0}^{m} b_i x^i) = \sum_{i=0}^{n} a_i(1-x)^i + \sum_{i=0}^{m} b_i(1-x)^i = \\ = \sum_{i=0}^{m}(a_i+b_i)(1-x)^i + \sum_{i=m+1}^{n} a_i(1-x)^i. \end{aligned}$$

So ψ preserves the sum. Hence

$$\psi(ab) = \psi\Big(\sum_{i=0}^{m+n} c_i x^i\Big) = \sum_{i=0}^{m+n} c_i(1-x)^i.$$

where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$, defining $b_j = 0$ if $j > m$. While

$$\begin{aligned} \psi(a)\psi(b) = \psi(\sum_{i=0}^{n} a_i x^i)\psi(\sum_{i=0}^{m} b_i x^i) = \sum_{i=0}^{n} a_i(1-x)^i \sum_{i=0}^{m} b_i(1-x)^i = \\ = \sum_{i=0}^{m+n} c_i(1-x)^i. \end{aligned}$$

where c_i is as above. So ψ preserves also the product. Since, by construction, the image of the identity is the identity, we showed that ψ is a ring homomorphism.

Moreover $\psi(S) = \{1, (1-x), (1-x)^2, \dots\}$, which are invertible elements in $\mathbb{K}[[x]]$. Then, by universal property of the localization, there exists a unique ring homomorphism ϕ from $\mathbb{K}[x, x^{-1}]$ to $\mathbb{K}[[x]]$ such that $\phi \circ \iota = \psi$. Thus $\phi(\iota(x)) = \phi(x) = 1 - x$ and so $\phi(x^{-1}) = \phi(x)^{-1} = (1-x)^{-1}$. Thus actually we proved the first part of our claim, i.e. ϕ is a ring homomorphism.

Now, we prove the injectivity of ϕ. Let $p \in \mathbb{K}[x, x^{-1}]$, then $p = \sum_{i=-n}^{m} a_i x^i$ with $a_i \in \mathbb{K}$ and $\phi(p) = \sum_{i=0}^{m} a_i(1-x)^i + \sum_{i=1}^{n} a_{-i}(1-x)^{-i}$.

Recalling Newton's formula, $(1-x)^i = \sum_{k=0}^{i} \binom{i}{k}(-1)^k x^k$, we get

$$\sum_{i=0}^{m} a_i(1-x)^i = \sum_{i=0}^{m} a_i\Big(\sum_{k=0}^{i}\binom{i}{k}(-1)^k x^k\Big) = \sum_{k=0}^{m}(-1)^k\Big(\sum_{i=k}^{m} a_i\binom{i}{k}\Big)x^k.$$

Then $(1-x)^{-1} = \sum_{i=0}^{\infty} x^i \in \mathbb{K}[[x]]$, and we could identify this with the infinite vector: $(1, 1, 1, 1, \dots\dots)$.

We have

$$(1-x)^{-2} = \Big(\sum_{i=0}^{\infty} x^i\Big)\cdot\Big(\sum_{j=0}^{\infty} x^j\Big) = \sum_{h=0}^{\infty} p_h x^h$$

where $p_h = \sum_{i=0}^{h} 1^i 1^{h-i} = \sum_{i=0}^{h} 1^h = (h+1) = \binom{h+1}{1}$.

While

$$(1-x)^{-3} = \sum_{h=0}^{\infty} q_h x^h$$

where $q_h = \sum_{i=0}^{h} 1^{h-i} p_i = \sum_{i=0}^{h}(i+1) = \sum_{i=0}^{h}(i+1) = \frac{(h+1)(h+2)}{2} = \binom{h+2}{2}$.

Using the same identification as above $(1-x)^{-2} = (1, 2, 3, 4, 5 \dots\dots)$ and $(1-x)^{-3} = (1, 3, 6, 10, \dots\dots)$.

By induction, since

$$(1-x)^{-i} = \sum_{h=0}^{\infty} q_h^i x^h$$

where $q_h^i = \sum_{j=0}^{h} 1^{h-j}\binom{i-2+j}{j} = \binom{i-1+h}{h}$, the infinite vector representing the series $(1-x)^{-i}$ is the $(i-1)-$th diagonal of the Pascal's triangle, i.e. $(t_{(i-1,i-1)}, t_{(i-1+1,i-1)}, t_{(i-1+2,i-1)}, \dots)$.

$$\begin{array}{ccccccc} & & & t_{(0,0)} & & & \\ & & t_{(1,0)} & & t_{(1,1)} & & \\ & t_{(2,0)} & & t_{(2,1)} & & t_{(2,2)} & \\ t_{(3,0)} & & t_{(3,1)} & & t_{(3,2)} & & t_{(3,3)} \\ \iddots & & & \vdots & & & \ddots \end{array}$$

which corresponds to

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ \iddots & & & \vdots & & & \ddots \end{array}$$

So $(1-x)^{-i+1} = \sum_{j=0}^{\infty} t_{(i+j,i)} x^j$ and this implies that

$$\sum_{i=1}^{n} a_{-i}(1-x)^{-i} = \sum_{j=0}^{\infty} \Big(\sum_{i=1}^{n} a_{-i} \binom{j+i-1}{j} \Big) x^j.$$

We just showed that

$$\phi(p) = \sum_{k=0}^{m} (-1)^k \Big(\sum_{i=k}^{m} a_i \binom{i}{k} \Big) x^k + \sum_{j=0}^{\infty} \Big(\sum_{i=1}^{n} a_{-i} \binom{j+i-1}{j} \Big) x^j.$$

Now, we consider the coefficient of x^k with $k > m$, that is only $c_k = \sum_{i=1}^{n} a_{-i} \binom{k+i-1}{k}$. If $p \in \ker(\phi)$ then $c_k = 0$ for all k. We want to prove that $a = (a_{-i})_i$ is the null vector. In order to show this, we need to find n natural numbers, call them $k_1, \dots, k_n$ such that the following matrix

$$M = \begin{bmatrix} \binom{k_1}{k_1} & \binom{k_1+1}{k_1} & \binom{k_1+2}{k_1} & \dots & \dots & \binom{k_1+n-1}{k_1} \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ \binom{k_n}{k_n} & \binom{k_n+1}{k_n} & \binom{k_n+2}{k_n} & \dots & \dots & \binom{k_n+n-1}{k_n} \end{bmatrix}$$

has maximal rank, i.e. $\mathrm{rank}(M) = n$.

Given $a = (a_{-1}, \dots, a_{-n})$, if we can find such a set $\{k_1, \dots k_n\}$, where $k_i > m$ for i between 1 and n, then the homogeneous linear system with variables a:

$$Ma = 0$$

has, by Rouche Capelli, a unique solution, which is the null vector, then $a = 0$.

We will work by induction on n, the dimension of the vector a.

If $n = 1$, $M = \binom{k}{k} = 1$, has rank 1, for all k.

Assume that we have a family of n natural numbers $\{k_1, \dots, k_n\}$ all greater than m, such that M has maximal rank. We want to find another natural number k_{n+1} greater than m, such that, by adding a column $c = (\binom{k_1+n}{k_1}, \binom{k_2+n}{k_2}, \dots, \binom{k_n+n}{k_n})^T$ and a row $r = (\binom{k_{n+1}}{k_{n+1}}, \binom{k_{n+1}+1}{k_{n+1}}, \dots, \binom{k_{n+1}+n}{k_{n+1}}) = (1, r_1, \dots, r_{n+1})$ to M, the matrix

$$\tilde{M} = \begin{bmatrix} \begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix} & \begin{bmatrix} \binom{k_1+n}{k_1} \\ \binom{k_2+n}{k_2} \\ \vdots \\ \binom{k_n+n}{k_n} \end{bmatrix} \\ \binom{k_{n+1}}{k_{n+1}} \quad \binom{k_{n+1}+1}{k_{n+1}} \quad \dots \quad \dots & \binom{k_{n+1}+n}{k_{n+1}} \end{bmatrix}$$

has maximal rank.

Since M has maximal rank, we can triangularize it, in order to have every pivot equal to 1 and each $p_{j,i}$ is a linear combination of elements of the same column. We get:

$$T_{\tilde{M}} = \begin{bmatrix} 1 & p_{1,2} & p_{1,3} & \cdots & \\ 0 & 1 & p_{2,3} & \cdots & \\ \vdots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

By doing the same operation on the column c, we obtain $\tilde{c} = (p_{1,n+1}, p_{2,n+1}, \ldots, p_{n,n+1})^T$.

We want to obtain a matrix of the type :

$$\tilde{\tilde{M}} = \begin{bmatrix} \begin{bmatrix} 1 & p_{1,2} & p_{1,3} & \cdots & \\ 0 & 1 & p_{2,3} & \cdots & \\ \vdots & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} & \begin{bmatrix} p_{1,n+1} \\ p_{2,n+1} \\ \vdots \\ \vdots \\ p_{n,n+1} \end{bmatrix} \\ 0 & p_{n+1,n+1} \end{bmatrix}.$$

In order to get this, we need to subtract from the last row, $(1, r_1, \ldots, r_n)$, $q_1 = 1$-times the first row, obtaining $(0, r_1 - p_{1,2}, \ldots, r_n - p_{1,n+1})$, then we need to subtract $q_2 = (r_1 - p_{1,2})$-times the second row, obtaining

$$(0, 0, r_3 - p_{1,3} - q_2 \cdot p_{2.3}, \ldots, r_{n+1} - p_{1,n+1} - q_2 \cdot p_{2,n+1}),$$

and so on. Until we obtain $(0, \cdots, 0, p_{n+1,n+1})$, where the last pivotal element became equal to :

$$p_{n+1,n+1} = r_n - (\sum_{i=1}^{n} q_i p_{i,n+1}),$$

with q_i defined for recurrence as $q_i = r_{i-1} - (\sum_{j=1}^{i-1} q_j p_{j,i})$ and initial value $q_1 = 1$.

By definition, $r_j = \binom{k_{n+1}+j}{k_{n+1}}$ is a polynomial in the variable k_{n+1} of grade j, for all j. In particular, the coefficient of $(k_{n+1})^n$ in r_n is equal to one, since $r_n = (k_{n+1}+1)(k_{n+1}+2)\cdots(k_{n+1}+n)$. So $p_{n+1,n+1}$ is a monic polynomial of degree n, giving at most n possible choice of k_{n+1} for which $p_{n+1,n+1}$ is equal to zero.

However, if $k_{n+1} = k_i$ for $1 \leq i \leq n$, then $\det(\tilde{M}) = 0$ and this implies that $p_{n+1,n+1} = 0$. Thus, the n-roots of $p_{n+1,n+1}$ are actually $\{k_1, \ldots, k_n\}$. So we just need to take k_{n+1} different from k_i for all i and with this choice $\text{rank}(\tilde{M}) = n+1$.

So we showed that if $p = \sum_{i=-n}^{m} a_i x^i$ belongs to $\ker(\phi)$, then $a_{-i} = 0$ for $1 \leq i \leq n$.

Now, consider the coefficient of x^k, for $0 \leq k \leq m$, in $\phi(p)$, this is equal to

$$c_k = (-1)^k \Big(\sum_{i=k}^{m} a_i \binom{i}{k} \Big),$$

and, since $p \in \ker(\phi)$, we have $c_k = 0$ for all k. But, if $k = m$,

$$c_m = (-1)^m a_m \binom{m}{m} = 0 \iff a_m = 0.$$

If $k = m - 1$,

$$c_{m-1} = (-1)^{m-1} \left(a_{m-1} \binom{m-1}{m} + a_m \binom{m}{m} \right) = 0 \iff a_{m-1} = 0.$$

By induction we get $a_i = 0$ for all $i = 0, \ldots, m$. Thus, if $p \in \ker(\phi)$, we get $p = 0$, hence ϕ is injective, and so we proved our claim.

□

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