Università degli Studi di Padova

Dipartimento di Matematica "Tullio Levi-Civita"

Corso di Laurea Magistrale in Matematica

Logarithmic construction of the moduli space of admissible covers

Candidato

SURAJ DASH

Matricola- 2004302

Supervisore

PROF. ORSOLA TOMMASI

Dipartimento di Matematica

Anno Accademico 2020/2021

data ufficiale di laurea: 21 luglio 2021

Abstract

For the thesis project, we are interested in learning about the Harris-Mumford modular compactification of the classical Hurwitz stack using *log admissible covers*. The classical Hurwitz stack parametrizes d-sheeted, simple branched coverings of $\mathbb{P}^1$ with b branched points. Harris and Mumford first introduced the notion of admissible covers in [16] as a tool for compactifying the classical Hurwitz space. In this thesis, we present a complete proof of the fact that the stack of *log admissible covers* is a *proper Deligne-Mumford logarithmic stack*. The use of logarithmic structures is necessary for obtaining a full fledged modular interpretation of the space of admissible covers. In particular, the logarithmic structures allow the admissibility condition on covers of curves to enjoy scheme-like properties. Following Mochizuki ([21]), this is achieved by showing the existence of a minimal logarithmic structure on the family of admissible covers.

Acknowledgements

First and foremost, I wish to express my heartfelt gratitude to my advisor Prof. Orsola Tommasi for countless things. I am grateful to her for accepting me as her student and introducing me to the beautiful world of logarithmic moduli problems. Needless to say, it would not have been possible for me to write this thesis without her invaluable guidance, unrelenting support and endless patience through out. Prof. Tommasi helped me resolve every mathematical hurdle I encountered, however silly they were, painstakingly corrected every nook and cranny of the thesis many a time and oft and helped me improve my mathematical writing skill. I thank her deeply for always being there to listen to my talks, mathematical or otherwise. Not in my wildest dreams I would have thought to find such a wonderful person to learn from. I owe her a huge debt of gratitude for her unconditional support while applying to graduate schools and guiding me through each minute detail of the process. I keenly look forward to keep learning from Prof. Tommasi in the future.

I would like to thank the ALGANT Master program for this great opportunity of learning mathematics in two different countries. It is a pleasure for me to thank all the professors I had the privilege to learn from at Essen and Padova; especially, Prof. Marc Levine and Prof. Jochen Heinloth. I am grateful to the ALGANT coordinators Prof. Vytautas Paškūnas and Prof. Remke Kloosterman for their support. Special thanks to Mrs. Elisa Zambon and Mr. Udo Rasum for helping me steer through the bureaucratic hurdles during the trying times of the pandemic. I thank all my friends I came across in the last two years, especially Sambit for his crucial advice at every juncture.

Last but not the least, I would like to express my deepest gratitude to my parents for their incessant love and support throughout my life.

SURAJ DASH

Contents

Introduction

Moduli problems deal with one of the most fundamental problems in mathematics, namely classifying a family of geometric objects modulo an appropriate notion of isomorphism of the objects. Being interested in classifying families of geometric objects, we would like a moduli space to be naturally endowed with a geometric structure such that its points correspond bijectively to isomorphism classes of objects. Thus, one of the striking properties of a moduli space is that it provides us ample insights into the properties of the objects we want to classify. For instance, the projective space $\mathbb{P}^n_{\mathbb{R}}$ is a moduli space classifying lines passing through the origin in $\mathbb{R}^{n+1}$.

If we have a notion of family of objects over a base and a notion of isomorphism of families; then for any base S, we would like the moduli space $\mathcal{M}$ to satisfy the property that there is a bijection:

$$\{\text{Families over } S\}/\sim \;\longleftrightarrow\; \{\text{Morphisms } \ S \longrightarrow \mathcal{M}\}$$

For instance, if one defines the geometric family over S to be families of smooth algebraic curves of arithmetic genus g, then this leads to one of the important objects in algebraic geometry, $\mathcal{M}_g$, *the moduli space of smooth curves of genus g.*

The solution $\mathcal{M}$ to the moduli problem satisfying the property above is called the *fine moduli space* for the moduli problem. One can rephrase this by saying that the fine moduli space $\mathcal{M}$ is a space that admits a family $\mathcal{U}$ over it such that every other family over S is determined uniquely up to isomorphism by pulling back the family $\mathcal{U} \longrightarrow \mathcal{M}$ via a unique map $S \longrightarrow \mathcal{M}$. The family $\mathcal{U} \longrightarrow \mathcal{M}$ is called a universal family for the moduli problem. In some sense, the universal family carries all information about the geometric objects we want to classify.

It is not at all obvious that there exists a universal family for a moduli problem.

The main problem one faces when trying to construct a universal family is the presence of non-trivial automorphisms of objects that one tries to classify. The solution proposed by Deligne and Mumford to study moduli problems was to expand the notion of geometric objects from schemes and introduce the notion of stacks. Morally, a stack includes information about the automorphisms of objects and hence one expects to find a universal family for the concerned moduli problem.

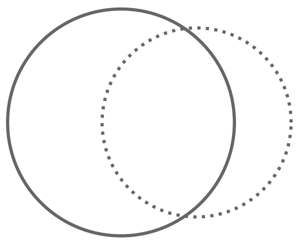
A preliminary example of moduli space arises from the moduli problem of parametrizing circles in $\mathbb{R}^2$. To define a circle in $\mathbb{R}^2$ it is enough to specify its center (x_0, y_0) and its radius r. Thus the moduli space of circles in $\mathbb{R}^2$ is given by

$$\mathcal{M} = \{(x_0, y_0, r) \subset \mathbb{R}^3 \mid r > 0\}$$

The universal family for the moduli problem is given by

$$\mathcal{U} = \{(x, y, x_0, y_0, r) \mid (x - x_0)^2 + (y - y_0)^2 = r^2,\ r > 0\} \longrightarrow \mathcal{M}$$

$$(x, y, x_0, y_0, r) \longmapsto (x_0, y_0, r)$$



Note that this moduli space $\mathcal{M}$ is not compact.

Formally speaking, a functor that maps a scheme to the set of isomorphism classes of families over it is called a moduli functor. By the Yoneda's lemma, the existence of a universal family for a moduli functor is equivalent to saying that the moduli functor originates from a scheme, i.e. the moduli functor is *representable by a scheme.*

The Hurwitz moduli space $\mathcal{H}^{d,b}$ parametrizing isomorphism classes of simple branched coverings of $\mathbb{P}^1$ of a fixed degree d with b branched points plays a central role in this thesis. Recall that a connected branched covering $f : C \longrightarrow \mathbb{P}^1$ is simple if for every branch point $p \in \mathbb{P}^1$, there is exactly one point in $f^{-1}(p)$ with ramification index two and all other points in the fiber are unramified. The discrete invariants, i.e. the genus of the curve, number of branched points and the degree of the covering are related by the Riemann-Hurwitz formula $b = 2d+2g-2$. It was proven by Fulton in [12] that the moduli functor $\mathcal{H}^{d,b}$ is represented by a Noetherian scheme $H^{d,b}$ and admits a locally finite étale covering

$$\delta : H^{d,b} \longrightarrow \mathbb{P}^b_S \setminus \Delta_b$$

where Δ_b is the discriminant hypersurface and δ is called the discriminant map. δ sends the class of a simple branched cover $f : C \longrightarrow \mathbb{P}^1$ to its branched locus. One of the striking properties of the Hurwitz space is that it admits a natural map to the moduli space $\mathcal{M}_{g,b}$ of smooth curves of genus g with b distinct marked points by forgetting the map to $\mathbb{P}^1$. For $d \geq g+1$, this map is dominant, hence the canonical morphism $\mathcal{H}^{d,b} \longrightarrow \mathcal{M}_g$ serves as an important bridge between the geometry of the Hurwitz space and the moduli space of curves. For instance, it was proved in [12] that the Hurwitz space $\mathcal{H}^{d,b}$ is irreducible by showing that the fundamental group $\pi_1(\mathbb{P}^b_S \setminus \Delta_b)$ acts transitively on the fibers of δ. This in turn gives an alternative proof of the irreducibility of $\mathcal{M}_g$. For a detailed survey on the various constructions of the Hurwitz space and its applications, we shall refer to Matthieu Romagny and Stefan Wewers' survey [27].

It may so happen that a moduli space does not contain limits of some families of objects, hence need not be compact. For instance, the moduli space of circles in $\mathbb{R}^3$ is not compact. The moduli space $\mathcal{M}_{0,4}$ of smooth curves of genus zero with four distinct marked points over a field k is the open subset $\mathbb{P}^1_k \setminus \{0, 1, \infty\}$. The moduli space $\mathcal{M}_{0,5}$ of smooth curves of genus zero with five distinct marked points over a field k is the open subset $\mathbb{P}^1_k \times_k \mathbb{P}^1_k \setminus \{\text{diagonal}\}$ of $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ (See Figure 1 below).

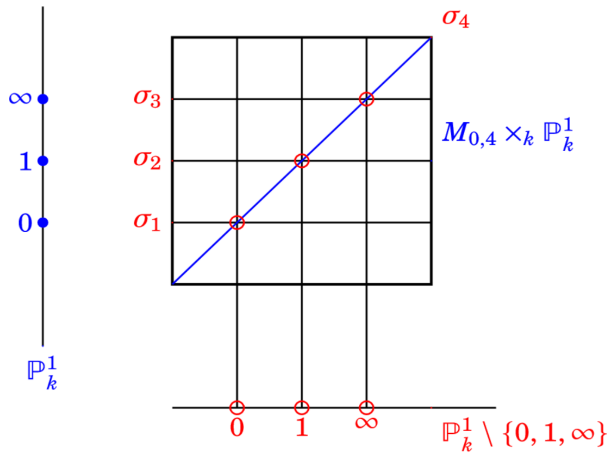


Figure 1: Moduli space of genus zero smooth curves with five marked points

Non-compactness of a family of geometric objects can be observed more clearly in the following example. Consider the family of elliptic curves given by

$$y^2 z = x(x-z)(x-tz)$$

where t is the parameter of the family.

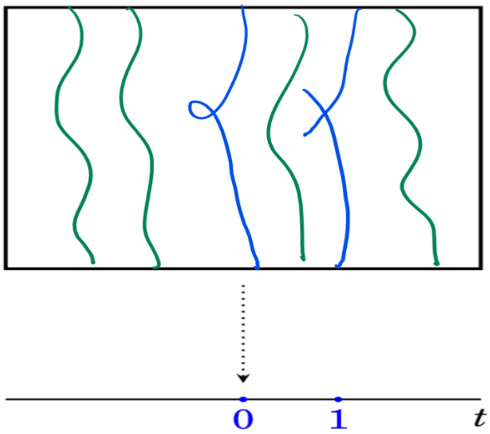


Figure 2: $y^2 z = x(x-z)(x-tz)$

This family degenerates into non-singular curves with nodal singularities at $t = 0, 1$. In this example, the family of elliptic curves does not have sufficiently many objects to be compact. With the exception of $\mathcal{M}_{0,3}$, which is just a point, $\mathcal{M}_{g,n}$ is not compact since a family of smooth curves can degenerate to a curve with singularities.

Thus a natural question in the study of moduli problems is how to construct a compactification of a moduli space $\mathcal{M} \subset \overline{\mathcal{M}}$ that satisfies some *nice* properties.

Namely,

- The compactification $\overline{\mathcal{M}}$ should have a nice interpretation as a moduli space.
- The boundary points $\partial\overline{\mathcal{M}} := \overline{\mathcal{M}} \setminus \mathcal{M}$ of the compactified space should correspond to degenerate objects of the family we are trying to classify.
- The singularities of the boundary divisors of the compactified space decide the validity of the modular compactification. Hence, the singularities of the boundary divisor should be reasonable, like normal crossings.

One of the most important examples is the Deligne-Mumford-Knudsen modular compactification of the moduli space $\mathcal{M}_{g,n}$ of smooth curves of arithmetic genus g with n distinct marked points. The philosophy behind the Deligne-Mumford-Knudsen modular compactification is *not to allow the marked points to coincide*. For this reason, the limit of the family is obtained by considering curves with nodes. For the compactification of moduli space of curves to have a modular interpretation, one has to restrict to *stable curves*. Stable curves are basically nodal curves with finitely many automorphisms. This philosophy can be illustrated in the Deligne-Mumford-Knudsen compactification of $\mathcal{M}_{0,5}$ which is given by blowing up $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$ at the three diagonal points. This blow up gives rise to the moduli space of stable curves of genus zero with five marked points. It was proved by Deligne-Mumford-Knudsen in [10] that the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable curves of genus g with n distinct marked points is a modular compactification for $\mathcal{M}_{g,n}$ such that $\partial\overline{\mathcal{M}}_{g,n} := \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ is an effective Cartier normal crossing divisor. Note that there may exist mutiple modular compactifications of a moduli space, depending on what kind of degenerate objects we want to include in the boundary.

The notion of compactness in algebraic geometry is replaced by the notion of *properness* which is usually checked by the *valuative criterion of properness*. This criterion roughly states that families of geometric objects over a punctured disc extend uniquely possibly after a base change to the entire disc. Thus the properness of $\overline{\mathcal{M}}_{g,n}$ is equivalent to being able to extend a family of stable curves to a stable curve over the special point of the family. For families of stable curves, such an extension is achieved by the *stable reduction theorem* [30, Tag 0E8C].

In principle, one generally follows the following standard approach to obtain a full fledged modular compactification of a moduli space $\mathcal{M}$:

- Define a larger family of geometric objects $\mathcal{M} \subset \overline{\mathcal{M}}$ we want to parametrize by including possible degenerate objects.

- Study the local structure of $\overline{\mathcal{M}}$. In other words, study the properties of the diagonal $\overline{\mathcal{M}} \longrightarrow \overline{\mathcal{M}} \times_{\mathbb{Z}} \overline{\mathcal{M}}$ and construct a smooth surjective atlas $U \longrightarrow \overline{\mathcal{M}}$. The presence of a smooth atlas demands the degenerate objects to behave very much like schemes.

- Study the coarse moduli space of $\overline{\mathcal{M}}$, which acts as a best approximation to the moduli space, at the cost of losing the universality.

- Verify the valuative criterion for the moduli space $\overline{\mathcal{M}}$ and conclude by standard openness properties that $\mathcal{M} \subset \overline{\mathcal{M}}$ is an open substack.

In this thesis we are interested in understanding a full fledged modular interpretation of the compactification of the Hurwitz moduli space $\mathcal{H}^{d,b}$. We follow Harris and Mumford's idea which is very similar to the Deligne-Mumford-Knudsen compactification of the moduli space of smooth curves, namely, *do not allow the branched points to coincide, i.e. include coverings of stable families with an admissibility condition*. The admissibility condition captures the idea that in order to compactify the Hurwitz moduli space, one allows both the source and target curves to become singular. Moreover, we would like this compactification to behave well with respect to the canonical map $\mathcal{H}^{d,b} \longrightarrow \mathcal{M}_{g,b}$, i.e. there exists an extension $\overline{\mathcal{H}}^{d,b} \longrightarrow \overline{\mathcal{M}}_{g,b}$ of $\mathcal{H}^{d,b} \longrightarrow \mathcal{M}_{g,b}$ such that the following map of moduli stacks commutes:

$$\begin{array}{ccc} \mathcal{H}^{d,b} & \longrightarrow & \mathcal{M}_{g,b} \\ \downarrow & & \downarrow \\ \overline{\mathcal{H}}^{d,b} & \longrightarrow & \overline{\mathcal{M}}_{g,b} \end{array}$$

where $\overline{\mathcal{H}}^{d,b}$ is the compactification of Hurwitz space we are seeking for.

As observed by Mochizuki in [21], the *admissibility condition* for coverings of curves, which we define in Definition 3.1.1, does not satisfy the scheme-like properties that are necessary to obtain a full fledged modular interpretation. The solution proposed by Mochizuki in [21] is to study moduli problems with logarithmic structures. As we will see, the study of logarithmic structures boils down to studying certain line bundles. The moduli space of line bundles are classically well studied, hence, they are easier to deal with. Hence, we shall study in the thesis how logarithmic structures serve as a natural tool to obtain a modular interpretation.

Logarithmic structures can be studied from numerous perspectives, namely:

- They make the degenerate/boundary objects behave as if they are smooth. As mentioned in [17], the motivating philosophy for logarithmic moduli problems is that *log smoothness includes degenerating objects like semistable reductions, hence one expects moduli spaces of log smooth objects to already be compact*.

  Hence, logarithmic structures serve as an important tool for modular compactification.

- Logarithmic structures encode schemes with boundary in a natural way.

- They serve as a bridge between algebraic geometry and tropical geometry.

Thus, the primary goal of this thesis is to understand logarithmic moduli problems, in particular in the case of curves and admissible coverings of curves. In understanding logarithmic moduli problems, an important step is the hands on construction of special families, called minimal objects, which capture the geometry of all other families. Following [21], we give a proof of the following main result in this thesis in Theorem 3.3.1:

Theorem. *Fix non-negative integers g, r, q, s, d such that $2g-2+r = d(2q-2+s)$. Let us consider log admissible covers where the source logarithmic curve is stable of genus g with n distinct marked points, the target logarithmic curve is stable of genus q with s distinct marked points and the map from the source to the target curve is of degree d. Then the moduli space of such log admissible covers*

$$\mathcal{LA}dm^{g,r}_{q,s,d} \longrightarrow \boldsymbol{LogSch}^{fs}_{st,\mathbb{Z}[1/d!]}$$

is a logarithmic DM stack, proper of finite type with a separated diagonal. The open substack $\mathcal{LA}dm^{g,r,min}_{q,s,d} \longrightarrow Sch_{\mathbb{Z}[1/d!]}$ of minimal log admissible covers admits a finite log étale morphism

$$\mathcal{LA}dm^{g,r,min}_{q,s,d} \longrightarrow \mathcal{LM}^{min}_{g,r} \cong \overline{\mathcal{M}}_{q,s}$$

Moreover, $\mathcal{LA}dm^{g,r,min}_{q,s,d} \longrightarrow Sch_{\mathbb{Z}[1/d!]}$ *admits a projective coarse moduli space* $LAdm^{g,r,min}_{q,s,d}$*, which is a finite étale scheme over the coarse moduli scheme* $\overline{M}_{q,s}$ *associated to* $\overline{\mathcal{M}}_{q,s}$*.*

Notations and conventions

For a scheme X, we won't distinguish between the scheme, its functor of points $h_X := Hom(\cdot, X)$ and the comma category (Sch/X). The objects in the comma category (Sch/X) are given by pairs (y, Y), where $y : Y \longrightarrow X$ is a morphism of schemes. An arrow $(y, Y) \longrightarrow (z, Z)$ is given by a morphism $f : Y \longrightarrow Z$ such that $z \circ f = y$.

All schemes are assumed to be noetherian of finite type. Since we are interested in noetherian schemes of finite type, we will not distinguish between the notions of formally smooth morphisms and smooth morphisms of schemes and algebraic stacks (see [30, Tag 0DNV]).

CFG Category fibered in groupoid

$\mathfrak{X}$ Letters in fraktur font such as $\mathfrak{X}$ will be used to denote the two-categorical **CFG**, stacks, algebraic spaces and so on

$\mathcal{X}$ Calligraphy letters will be used to denote a log scheme. If $\mathcal{X} = (X, \mathcal{M}_X)$ is a log scheme, then $\underline{\mathcal{X}}$ denotes the underlying scheme. For a morphism $f : \mathcal{X} \longrightarrow \mathcal{Y}$ of log schemes, $\underline{f}$ denotes the morphism of underlying schemes. Schemes upon which we do not want to keep track of log structures will be denoted by simple letters

$\mathcal{M}on$ Category of monoids (see Section 1.1)

$P\,\mathrm{LOG_X}$ Category of pre-log structures on a scheme X (see Section 1.1)

$\mathrm{LOG_X}$ Category of log structures on a scheme X (see Section 1.1)

$\overline{\mathcal{M}}_X$ Characteristics sheaf of monoids $\overline{\mathcal{M}}_X := \mathcal{M}_X / \mathcal{O}_X^*$ (see Section 1.1)

$\overline{\mathcal{M}}_{X/Y}$ For a morphism of logarithmic schemes $f : \mathcal{X} \longrightarrow \mathcal{Y}$, the relative characteristic sheaf of monoids $\overline{\mathcal{M}}_{X/Y} := \mathcal{M}_X/\mathrm{im}(f^*\mathcal{M}_Y \longrightarrow \mathcal{M}_X)$ (see Section 1.1)

$\Omega^1_{\mathcal{X}/\mathcal{Y}}$ The logarithmic sheaf of relative differentials (see Section 1.5)

$(Sch)_{\text{ét}}$ The small étale site on the category of schemes. (see Section A.1)

$\mathbf{LogSch}^{fs}_{st}$ The category of fine saturated log schemes with strict morphisms (see Section 2.2)

$\mathbf{LogSch}^{fs}_{st,\text{ét}}$ The category of fine saturated log schemes with strict morphisms endowed with the strict étale topology (see Section 2.2)

LOG 2-Cat$/(Sch)_{\text{ét}}$ The category of 2-categories equipped with a log structure defined over $(Sch)_{\text{ét}}$ (see Section 2.2)

LOG $\mathbf{CFG}/(Sch)_{\text{ét}}$ The category of categories fibered in groupoids equipped with a log structure defined over $(Sch)_{\text{ét}}$ (see Section 2.2)

LOG Stacks$/(Sch)_{\text{ét}}$ The category of stacks equipped with a log structure defined over $(Sch)_{\text{ét}}$ (see Section 2.2)

$2-\mathrm{Cat}/\mathbf{LogSch}^{fs}_{st,\text{ét}}$ The category of 2-categories defined over $\mathbf{LogSch}^{fs}_{st,\text{ét}}$ (see Section 2.2)

$\mathbf{CFG}/\mathbf{LogSch}^{fs}_{st,\text{ét}}$ The category of categories fibered in groupids defined over $\mathbf{LogSch}^{fs}_{st,\text{ét}}$ (see Section 2.2)

Stacks$/\mathbf{LogSch}^{fs}_{st,\text{ét}}$ The category of stacks defined over $\mathbf{LogSch}^{fs}_{st,\text{ét}}$ (see Section 2.2)

$\mathcal{H}^{d,b}$ The Hurwitz moduli space that parametrizes isomorphism classes of simple branched coverings of P^1 with b-branched points of a fixed degree d (see Section 3.4)

$\mathcal{LM}_{g,n}$ The moduli stack of logarithmic curves of type (g,n) (see Section 2.1)

$\mathcal{LA}dm^{g,r}_{q,s,d}$ The moduli stack of admissible covers (see Section 3.1)

Geometric point of a scheme: Let x be a schematic point of a separated scheme X. Then a *geometric point* $\overline{x}$ is a separable closure $k(\overline{x})$ of the residue field $k(x)$ of x. In other words, we have a morphism $\operatorname{Spec} k(\overline{x}) \longrightarrow Spec\ k(x) \longrightarrow X$. An étale neighbourhood of a geometric point $\overline{x}$ is an étale morphism of schemes $U \longrightarrow X$ with a geometric point $k(u) \hookrightarrow k(\overline{u})$ such that the following diagram commutes:

$$\begin{array}{ccccc} \operatorname{Spec} k(\overline{u}) & \longrightarrow & \operatorname{Spec} k(u) & \longrightarrow & U \\ \downarrow & & \downarrow & & \downarrow \text{étale} \\ \operatorname{Spec} k(\overline{x}) & \longrightarrow & \operatorname{Spec} k(x) & \longrightarrow & X \end{array}$$

In the notes, we will suppress the geometric point $\overline{u}$ lying over the geometric point $\overline{x}$ while referring to an étale neighbourhood of $\overline{x}$.

The stalk at $\overline{x}$ of the structure sheaf O_X with X endowed with the étale topology is defined as

$$O_{X,\overline{x}} := \varinjlim_{(\overline{u},U)} \Gamma(U, O_U)$$

where the directed limit is considered over all connected étale neighbourhoods $(\overline{u}, U)$ of $\overline{x}$. Indeed, the connected étale neighbourhoods $(\overline{u}, U)$ of $\overline{x}$ form a directed set defined by:

$$(\overline{u}, U) \leq (\overline{v}, V) \text{ if } \exists \text{ a map } (\overline{u}, U) \longrightarrow (\overline{v}, V)$$

Since we are considering étale morphism of schemes, there exists at most one morphism $U \longrightarrow V$ that maps the geometric point $\overline{u}$ to the geometric point $\overline{v}$.

In the notes, we will use that $O_{X,\overline{x}}$ is a noetherian local ring of dimension equal to that of X and

$$O_{X,\overline{x}} \cong O^{sh}_{X,x}$$

where $O^{sh}_{X,x}$ denotes the strict Henselisation of the local ring $O_{X,x}$. We can analogously generalize the above definitions for any étale sheaf of monoids over X. For more details, see [20, I.4].

Family of smooth algebraic curves A family of smooth algebraic curves of genus g over a scheme S is a smooth proper morphism of schemes $C \longrightarrow S$ such that every geometric fiber is an irreducible curve of genus g. Unless specified, by genus of a curve we will always refer to the arithmetic genus of the curve, i.e. $g := \dim_k H^1(C, \mathcal{O}_C)$.

Nodal curves An algebraic curve C over an algebraically closed field k has a nodal singularity at P if there exists an isomorphism $\mathcal{O}^{\wedge}_{C,P} \cong k[[x,y]]/(xy)$. For any arbitrary field k, P is a nodal point if P is a nodal point in $C_{\overline{k}}$ in the above sense. Moreover, after a separable extension $k \hookrightarrow k'$, we have an isomorphism $\mathcal{O}^{\wedge}_{C_{k'},P} \cong k'[[x,y]]/(xy)$.

Stable curve A connected algebraic curve C over a field k with n ordered distinct marked points $(p_1, \ldots, p_n)$ in the smooth locus in C with at worst nodal singularities is said to be stable if it satisfies either of the following equivalent conditions:

A Every smooth rational curve $\mathbb{P}^1 \subset C$ contains at least three special points, i.e. either a marked point or a nodal singularity.

B $\omega_C(p_1 + \cdots + p_n)$ is an ample invertible sheaf.

C The automorphism group $\mathrm{Aut}(C; p_1, \ldots, p_n)$ is finite.

For every n-pointed stable curve of genus g, we have $2g - 2 + n > 0$. A family of curves $C \longrightarrow S$ with sections $\sigma_1, \ldots, \sigma_n : C \longrightarrow S$ is said to be stable if $(C_s; \sigma_1(s), \ldots, \sigma_n(s))$ is a stable curve with n marked points for all $s \in S$.

The Deligne-Mumford-Knudsen stack of moduli space of n-pointed stable curves is defined as the **CFG**

$$\overline{\mathcal{M}}_{g,n} \longrightarrow (Sch/\mathbb{Z})_{\text{ét}}$$

with objects and arrows defined as

$$Obj(\overline{\mathcal{M}}_{g,n}) := \left\{ \begin{array}{c} X \\ {\scriptstyle \{s_i\}_{i=1}^n} \uparrow\downarrow \\ S \end{array} \text{ family of stable curves of genus } g \text{ with } n \text{ marked points} \right\}$$

$$Arr(\overline{\mathcal{M}}_{g,n}) := \left\{ \begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle \{s_i'\}_{i=1}^n} \uparrow \downarrow {\scriptstyle \pi'} & & {\scriptstyle \pi} \downarrow \uparrow {\scriptstyle \{s_i\}_{i=1}^n} \\ S' & \xrightarrow{f} & S \end{array} \text{ cartesian diagram with } f' \circ s_i' = s_i \circ f \ \forall i \right\}$$

CHAPTER 1

Basic logarithmic geometry

In this chapter we will define the basic setting for the study of logarithmic geometry, which will be employed throughout the rest of this thesis to study logarithmic moduli problems. Starting from the prototype examples of normal crossing varieties and the semistable reduction model, we study logarithmic structures, which should be thought of as the study of functions without poles along a divisor. As we will see, the study of such functions keeps track of properties of the scheme and the normal crossing divisor simultaneously. As a result, families with normal crossing singularities in the fiber turn out to be smooth in the logarithmic world. Vital to the study of functions without poles along a divisor is the study of the geometry of monoids. We introduce the basic notions in the geometry of monoids necessary for the subsequent chapters of this thesis and refer to [22] for a detailed analysis.

1.1 Logarithmic schemes

Definition 1.1.1. A *monoid* $(M, \cdot, 1)$ is a commutative semigroup with a unit[1]. A morphism of commutative monoids preserves the unit element. We denote the category of monoids as $\mathcal{M}on$.

Example 1.1.2.

1. $\{1\}$ is the final and initial object in the category of monoids. Moreover, this is an abelian group.
2. $(\mathbb{N}, +, 0), (\mathbb{Z}, \cdot, 1)$ are commutative monoids.

Definition 1.1.3. Let X be a scheme. A *pre-logarithmic* structure on X is a pair

[1] We will interchangably use the additive and multiplicative notation based on convenience.

of datum $(\mathcal{M}_X, \alpha_X : \mathcal{M}_X \longrightarrow \mathcal{O}_{X_{\text{ét}}})$ where $\mathcal{M}_X$ is a sheaf of monoids on the ètale site $X_{\text{ét}}$,[2] i.e.

$$\mathcal{M}_X : (Sch/X)^{op}_{\text{ét}} \longrightarrow \mathcal{M}on$$

is a sheaf and α_X is a morphism of sheaves of monoids with $\mathcal{O}_X$ considered as a sheaf of monoids with respect to the multiplicative structure.

A pre-logarithmic structure $(\mathcal{M}_X, \alpha_X)$ is called *logarithmic* if α_X identifies the units, i.e.$\alpha_X^{-1}(\mathcal{O}^*_{X_{\text{ét}}}) \cong \mathcal{O}^*_{X_{\text{ét}}}$. A scheme X admitting a logarithmic structure is called a *log scheme*. A morphism of log structures $(\mathcal{M}_X, \alpha_X) \longrightarrow (\mathcal{N}_X, \beta_X)$ is a commutative diagram of sheaf of monoids :

$$\begin{array}{ccc} & & \mathcal{N}_X \\ & \nearrow & \downarrow{\scriptstyle \beta_X} \\ \mathcal{M}_X & \xrightarrow{\alpha_X} & \\ & \searrow & \mathcal{O}_X \end{array}$$

A *log scheme* $(X, \mathcal{M}_X)$ is a scheme X equipped with a log structure $(\mathcal{M}, \alpha_X)$ as above. We denote the category of log schemes by **LogSch** (Morphisms of log schemes will be defined in the next section). Calligraphy letters will be used to denote a log scheme. If $\mathcal{X} = (X, \mathcal{M}_X)$ is a log scheme, then $\underline{\mathcal{X}}$ denotes the underlying scheme and $\underline{f}$ denotes the morphism of underlying schemes.

For a log structure $(\mathcal{M}_X, \alpha_X)$ on X, we define the *characteristics sheaf of monoids* $\overline{\mathcal{M}}_X := \mathcal{M}_X/\mathcal{O}^*_X$ (where, $\mathcal{O}^*_X$ is considered as a sub-sheaf of monoids of $\mathcal{M}_X$ via the identification of units, by an abuse of notation). In colloquial language, the characteristic sheaf takes care of the 'non-trivial' part of the log structure.

We denote the category of pre-log and log structures on the scheme X as $P\,\mathrm{LOG_X}$ and $\mathrm{LOG_X}$ respectively. Moreover, we have a canonical inclusion functor :

$$i : \{\mathrm{LOG_X}\} \hookrightarrow \{P\,\mathrm{LOG_X}\}$$

Remark 1.1.4. 1. It is not hard to see that the conditions $\alpha_X^{-1}(\mathcal{O}^*_{X_{\text{ét}}}) \cong \mathcal{O}^*_{X_{\text{ét}}}$ is equivalent to the condition $\mathcal{M}^*_X \cong \mathcal{O}^*_{X_{\text{ét}}}$ and $\alpha_X^{-1}(\mathcal{O}^*_{X_{\text{ét}}}) = \mathcal{M}^*_X$. The last equality is the analogue of the definition of a *local morphism* of locally ringed

[2] In the following chapters we often omit the topology in the notation and use the étale site unless otherwise specified.

space. This makes sense by defining ideals, prime ideals and irreducibility in the category $\mathcal{M}on$. For a monoid M, M^* is the unique maximal ideal of the monoid, hence we have an analogy:

$$\{\text{Geometry on monoids}\} \longleftrightarrow \{\text{Geometry on local rings}\}$$

Example 1.1.5. 1. Every scheme X admits a *trivial* log structure which is an initial object in the category $\mathrm{LOG_X}$ by considering the pair $(\mathcal{O}_X^*, \mathcal{O}_X^* \hookrightarrow \mathcal{O}_X)$. This gives a fully-faithful embedding of the category of schemes in the category of log schemes

$$(Sch) \hookrightarrow \mathbf{LogSch}$$

For any scheme X we have the *identity log structure* which is the final object in the category $\mathrm{LOG_X}$, namely, $(\mathcal{M}_X, \alpha_X) := (\mathcal{O}_X, \mathcal{O}_X = \mathcal{O}_X)$

2. Let X be any scheme and P be a monoid. Consider $\mathcal{M}_X := \mathcal{O}_X^* \oplus P_X$, where P_X is the constant sheaf on X defined by the monoid P and the structure morphism is defined on local sections by

$$\alpha_X : \mathcal{O}_X^* \oplus P_X \longrightarrow \mathcal{O}_X$$

$$\alpha_X(s, p) = \begin{cases} s & \text{for } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

This defines a log structure on X. In particular, if $X :=$ Spec k, where k is a field, then the log scheme as defined above is called a *log point*.

3. Let X be a regular scheme and $D \subset X$ a normal crossing divisor, then we have the *divisorial log structure* on X with respect to the divisor D, étale locally given by :

$$\mathcal{M}_X^D(V) := \{f \in \mathcal{O}_X(V) \mid f_{|V-D} \in \mathcal{O}_X^*(V-D)\} \subset \mathcal{O}_X(V)$$

In other words, if $j : U = X - D \hookrightarrow X$ is the canonical inclusion, then

$$\mathcal{M}_X^D = j_*\mathcal{O}_U^* \cap \mathcal{O}_X$$

For instance, take $X = \mathbb{A}^2_k$ and $D = Z(x) \cup Z(y)$. Then we have a morphism of sheaves of monoids

$$\begin{aligned} \mathbb{N}^2_X &\longrightarrow \mathcal{M}^D_X \\ (n_1, n_2) &\longmapsto x^{n_1} y^{n_2} \end{aligned}$$

The morphism above serves as a prototype for a 'local chart/ local model' for the log scheme. Moreover, the stalks of the sheaf of monoid are given by

$$\mathcal{M}^D_{X,p} = \begin{cases} \phi & \text{if } p \in U \\ \mathbb{N} & \text{if } p \in Z(x) \text{ or } p \in Z(y) \\ \mathbb{N}^2 & \text{if } p = (0,0) \end{cases}$$

Note that we have the canonical inclusions $\mathcal{O}^*_X \subset \mathcal{M}^D_X \subset \mathcal{O}_X$. The second inclusion need not hold in general, as in example (2). As discussed in the motivation, $\mathcal{M}^D_X$ takes care of invertible elements away from the divisor D, in other words, 'functions' with zeroes only on D.

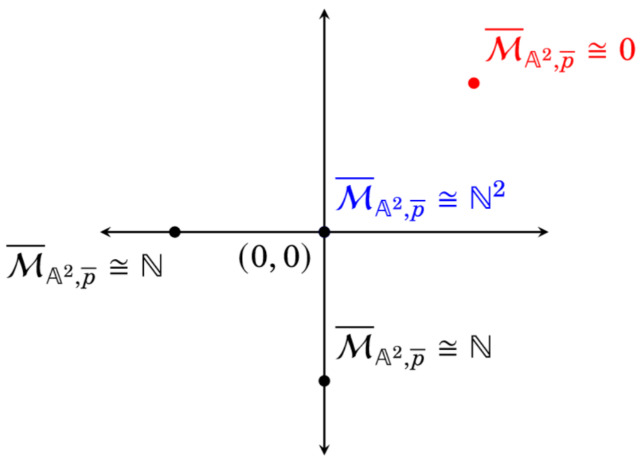


Figure 1.1: Divisorial log structure on $\{xy = 0\}$

Thus, $\mathbb{A}^2_k = \bigsqcup_i U_i$ admits a stratification such that the divisorial log structure is constructible, i.e. the log structure $\overline{\mathcal{M}}^D_X$ is given by a single monoid on each stratum U_i.

4. Consider a family of schemes $X \longrightarrow S$ over a disc, i.e.$S = \operatorname{Spec} A$, where (A, m) is a DVR with unique closed point $m = (\pi)$. Let $\mathcal{M}^m_S$ be the normal

crossing log structure on S with respect to the divisor m. Then we have a divisorial log structure $\mathcal{M}_X^{X_m}$ with respect to the fiber X_m.

We will be interested in the case where $X \longrightarrow S$ étale locally has a factorisation

$$X \underset{\text{ét}}{\longrightarrow} \operatorname{Spec} A[x_1, ..., x_n](x_1 \cdots x_d - \pi) \longrightarrow S; \ d \leq n,$$

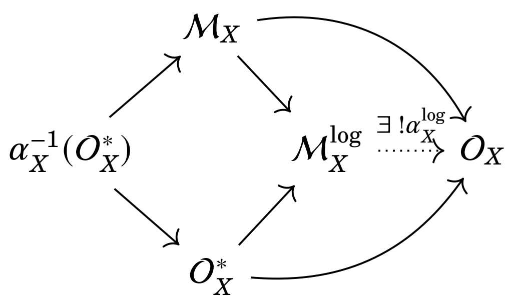
called the *semistable reduction* model. Vaguely speaking, we are interested in ‘classifying’ all log structures on a semistable model. See chapter 2 for more details.

1.1.1 Log-structure associated to a pre-log structure

Analogously to the sheafification functor, we can define the logarithmification functor. Recall that we have the canonical inclusion functor

$$i : \{\mathrm{LOG_X}\} \hookrightarrow \{P\,\mathrm{LOG_X}\}$$

We want to associate a log structure $\alpha_X^{\log} : \mathcal{M}_X^{\log} \longrightarrow \mathcal{O}_X$ with a scheme X with a pre-log structure $\alpha_X : \mathcal{M}_X \longrightarrow \mathcal{O}_X$. It will be defined by the following co-cartesian diagram in the category of étale sheaf of monoids :



In other words, the log structure $\mathcal{M}_X^{\log}$ associated to the pre-log structure $\mathcal{M}_X$ is obtained by the étale sheafification of the presheaf .

$$U \longmapsto (\mathcal{M}_X \oplus_{\alpha_X^{-1}(\mathcal{O}_X^*)} \mathcal{O}_X^*)(U)$$

for every étale cover $U \longrightarrow X$.

Remark 1.1.6. In general, the log structures associated with $\mathcal{M}_X$ with respect to the Zariski and étale site need not be isomorphic. In case $\mathcal{M}_X$ is a fine sheaf of monoid (see Definition 1.2.8), then the sheafification carried out in the Zariski and étale topologies are isomorphic. See [22, Section III.1.4] for details about the comparision of log structures in the Zariski and étale topologies.

Theorem 1.1.7. *The logarithmification functor*

$$\log : \{P\,\mathrm{LOG_X}\} \longrightarrow \{\mathrm{LOG_X}\}$$

$$\{\alpha_X : \mathcal{M}_X \longrightarrow \mathcal{O}_X\} \longmapsto \{\alpha_X^{\log} : \mathcal{M}_X^{\log} \longrightarrow \mathcal{O}_X\}$$

is a left adjoint to the inclusion functor

$$i : \{\mathrm{LOG_X}\} \hookrightarrow \{P\,\mathrm{LOG_X}\}$$

The main part of the proof of the theorem above is constructing a push-out in the category of monoids, which we recall in the next section.

1.1.2 The geometry of monoids-I

Arbitrary *projective limits* exist in the category $\mathcal{M}on$ and the projective limit functor commutes with the forgetful functor $\mathcal{M}on \longrightarrow (Sets)$. This functor has a right adjoint and hence commutes with all projective limits. In particular we have (arbitrary) products and fiber products in the category of monoids.

Remark 1.1.8. For a morphism $f : P \longrightarrow Q$ of monoids, $ker\, f$ makes sense, being a projective limit, i.e.the equaliser of the morphisms f and the constant morphism 0. The notion of monomorphism in $\mathcal{M}on$ coincides with the notion of monomorphism in $(Sets)$. Hence, a monomorphism has a trivial kernel, whereas a map with a trivial kernel is not necessarily a monomorphism. For example, consider $f : \mathbb{N} \oplus \mathbb{N} \longrightarrow \mathbb{N}$ given by $(a, b) \longmapsto a + b$. We see that $ker\, f = \{(0, 0)\}$ but f is not a monomorphism.

Definition 1.1.9. 1. If P is a monoid, then a *congruence relation* on P is a subset $E \subset P \oplus P$ which is both a submonoid and a set-theoretic equivalence relation.

2. A subset $S \subset E$ generates the congruence relation E if E is the smallest congruence relation on P containing S.

Remark 1.1.10. For any equivalence relation E on P, the surjection $P \longrightarrow P/E$ induces a structure of a monoid on P/E if and only if E is a congruence relation. Therefore, there is a natural bijection between the isomorphism classes of surjective maps of monoids $P \longrightarrow Q$ and the set of congruence relations on P. See [22, I.1.1.1] for more details.

Definition 1.1.11 (Push-outs in the category of monoids)**.** Let $f : P \longrightarrow M$ and $g : P \longrightarrow N$ be morphism of monoids. Then, we have a co-cartesian square in $\mathcal{M}on$

$$\begin{array}{ccc} P & \xrightarrow{f} & M \\ \downarrow{\scriptstyle g} & & \downarrow \\ N & \longrightarrow & M \oplus_P N \end{array}$$

where $M \oplus_P N := (M \oplus N)/\sim$ with '$\sim$' being the smallest equivalence relation stable under the monoid operation such that $(f(p), 0) \sim (0, g(p))$ $\forall p \in P$. More explicitly, the congruence relation '$\sim$' can be described as follows:

Let S be the set of pairs $((m_1, n_1), (m_2, n_2)) \in (M \oplus N) \times (M \oplus N)$ such that there exists a $p \in P$ such that $m_2 = m_1 + f(p)$ and $n_1 = n_2 + g(p)$. Set $-S := \{(a, b) | (b, a) \in S\}$. Then '$\sim$' is given by the set of pairs $(a, b) \in (M \oplus N) \times (M \oplus N)$ such that there exists a sequence $(r_0, \ldots, r_n) \in (M \oplus N)^{n+1}$ such that $a = r_0, b = r_n$ and satisfying $(r_i, r_{i+1}) \in S$ if i is even and $(r_i, r_{i+1}) \in -S$ if i is odd.

Remark 1.1.12.

1. In case in which at least one of M and N is a group, the congruence relation is generated by: $(m_1, n_1) \sim (m_2, n_2)$ if and only if $\exists\, c, d \in P$ such that $m_1 + f(c) = m_2 + f(d),\ n_1 + g(d) = n_2 + g(c)$. We will see in Definition 1.2.16 that an integral morphism of monoids is characterised by the above congruence relation.

2. In particular, taking $N = 0$, we obtain the cokernel of the morphism $f : P \longrightarrow M$. An epimorphism in $\mathcal{M}on$ has a trivial cokernel, while a map with a trivial cokernel is not necessarily an epimorphism. For example, $f : \mathbb{N} \oplus \mathbb{N} \longrightarrow \mathbb{N} \oplus \mathbb{N}$ given by $(a, b) \longmapsto (a, a + b)$ has trivial cokernel but is not a surjective map of monoids.

1.1.3 Functoriality of log structures

Definition 1.1.13. 1. Let $f : X \longrightarrow Y$ be a morphism of schemes. The pull-back functor

$$f^{*\log} : Log_Y \longrightarrow Log_X$$
$$\mathcal{M}_Y \longmapsto f^{*\log}(\mathcal{M}_Y)$$

is defined as the log structure associated to the canonical morphism $f^{-1}\mathcal{M}_Y \longrightarrow f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$.

2. A morphism of log schemes $(f, f^{@}) : (X, \mathcal{M}_X) \longrightarrow (Y, \mathcal{M}_Y)$ is given by a morphism $f : X \longrightarrow Y$ of the underlying schemes and together with morphism of log structures on X :

$$f^{@} : f^{*\log}\mathcal{M}_Y \longrightarrow \mathcal{M}_X$$

such that the morphism $f^{@}$ is compatible with the structure morphism, i.e.the following diagram commutes :

$$\begin{array}{ccc} f^{*\log}(\mathcal{M}_Y) & \xrightarrow{f^{@}} & \mathcal{M}_X \\ \downarrow{\scriptstyle f^*(\alpha_X)} & & \downarrow{\scriptstyle \alpha_X} \\ f^*\mathcal{O}_Y & \longrightarrow & \mathcal{O}_X \end{array}$$

Thus, this defines morphisms in **LogSch**. The *relative characteristic sheaf* of f is defined by $\overline{\mathcal{M}}_{X/Y} := \mathcal{M}_X/\mathrm{im}(f^{*\log}\mathcal{M}_Y \longrightarrow \mathcal{M}_X)$.

3. A morphism of log schemes $(f, f^{@}) : (X, \mathcal{M}_X) \longrightarrow (Y, \mathcal{M}_Y)$ is called *strict* if the morphism $f^{@} : f^{*\log}\mathcal{M}_Y \longrightarrow \mathcal{M}_X$ is an isomorphism of sheaves of monoids. For instance, the pull-back of the trivial log structure $\mathcal{O}_Y^*$ on Y is isomorphic to the trivial log structure $\mathcal{O}_X^*$ on X, hence is a strict morphism of log schemes.

   Furthermore, for any log scheme $(X, \mathcal{M}_X)$, we have a canonical morphism of log schemes $p_X : (X, \mathcal{M}_X) \longrightarrow (X, \mathcal{O}_X^*)$.

4. The cartesian product of log schemes is given by :

$$\begin{array}{ccc} (X\times_Z Y, \mathcal{M}_{X\times_Z Y}) & \xrightarrow{\pi_X} & (X,\mathcal{M}_X) \\ \downarrow{\scriptstyle \pi_Y} & \searrow^{\pi_Z} & \downarrow \\ (Y,\mathcal{M}_Y) & \longrightarrow & (Z,\mathcal{M}_Z) \end{array}$$

where the log structure $\mathcal{M}_{X\times_Z Y} \longrightarrow \mathcal{O}_{X\times_Z Y}$ is given by the logarithmification of the push out under π_2 of the following diagram :

$$\begin{array}{ccc} \pi_Z^*\mathcal{M}_Z & \longrightarrow & \pi_X^*\mathcal{M}_X \\ \downarrow & & \downarrow \\ \pi_Y^*\mathcal{M}_Y & \longrightarrow & \pi_X^*\mathcal{M}_X \oplus_{\pi_Z^*\mathcal{M}_Z} \pi_Y^*\mathcal{M}_Y \end{array}$$

Thus, the formation of fiber products in the category **LogSch** commutes with the forgetful functor:

$$\mathbf{LogSch} \longrightarrow Sch$$
$$\mathcal{X} \longrightarrow \underline{\mathcal{X}}$$

5. Let $f : X \longrightarrow Y$ be a morphism of schemes. For a log structure $\alpha_X : \mathcal{M}_X \longrightarrow \mathcal{O}_X$ on X, the pushforward functor

$$f_*^{\log} : Log_X \longrightarrow Log_Y$$

is defined as the fiber product of the below diagram in the category of étale sheaves of monoids :

$$\begin{array}{ccc} f_*^{\log}\mathcal{M}_X & \xrightarrow{f_*^{\log}(\alpha_X)} & \mathcal{O}_Y \\ \downarrow & & \downarrow \\ f_*\mathcal{M}_X & \xrightarrow{f_*(\alpha_X)} & f_*\mathcal{O}_X \end{array}$$

Remark 1.1.14. 1. The pushforward functor $f_*^{\log}$ is right adjoint to the pull back functor $f^{*\log}$.

2. Any morphism of log schemes $(f, f^@) : (X,\mathcal{M}_X) \longrightarrow (Y,\mathcal{M}_Y)$ can be uniquely factored as $(X,\mathcal{M}_X) \xrightarrow{(Id, f^@)} (X, f^{*\log}(\mathcal{M}_Y)) \xrightarrow{f^{\text{strict}}} (Y,\mathcal{M}_Y)$, where

f^{strict} is a strict morphism of log schemes. Equivalently by adjointness, we also have a unique factorization of $(f, f^{@}) : (X, \mathcal{M}_X) \to (Y, f_*^{\log}(\mathcal{M}_X)) \to (Y, \mathcal{M}_Y)$ such that the following diagram commutes :

$$\begin{array}{ccc} (X, \mathcal{M}_X) & \longrightarrow & (X, f^{*\log}(\mathcal{M}_Y)) \\ \downarrow & & \downarrow f^{\text{strict}} \\ f_*^{\log}(\mathcal{M}_X) & \longrightarrow & (Y, \mathcal{M}_Y) \end{array}$$

3. A morphism of log schemes $(f, f^{@}) : (X, \mathcal{M}_X) \longrightarrow (Y, \mathcal{M}_Y)$ is strict if and only if the diagram

$$\begin{array}{ccc} (X, \mathcal{M}_X) & \xrightarrow{f} & (Y, \mathcal{M}_Y) \\ \downarrow \pi_X & & \downarrow \pi_Y \\ (X, \mathcal{O}_X^*) & \xrightarrow{\underline{f}} & (Y, \mathcal{O}_Y^*) \end{array}$$

is cartesian in the category of log schemes.

4. The characteristics sheaves are stable under pull-backs, i.e.the canonical morphism $f^{-1}(\overline{\mathcal{M}_Y}) \longrightarrow \overline{f^{*\log}(\mathcal{M}_Y)}$ is an isomorphism for every morphism f of log schemes.

5. For the sake of simplicity, from now on we will omit the *log* in the superscripts of the notation of the pullback and pushforward functors.

1.1.4 Charts of log schemes

Just as affine schemes give a local picture of schemes and assist in the local geometry, affine log schemes serve as a similar analogue in the category of log schemes.

Example 1.1.15. **Affine log scheme**:

Let A be a commutative ring. Let $\mathcal{M}on_A$ be the category where the objects are given by pairs (y, P), where P is a monoid and $y : P \longrightarrow A$ is a morphism of monoids. Here A is considered as a monoid with respect to its multiplicative structure. An arrow $(y, P) \longrightarrow (z, Q)$ is given a morphism $f : P \longrightarrow Q$ such that $z \circ f = y$. An object $P \longrightarrow A$ in $\mathcal{M}on_A$ induces a morphism of monoidal algebras

$$P \longrightarrow A[P]$$

where $A[P]$ is the free A-module generated by the elements $\{e_p \mid p \in P\}$ with multiplication defined as $e_p \cdot e_q := e_{p+q}$ extended linearly over A. Thus,the forgetful functor from the category of A-algebras to the category of monoids

$$\mathcal{A}lg_A \longrightarrow \mathcal{M}on$$
$$(R, +, \cdot) \longmapsto (R, +)$$

admits a left adjoint (in particular, commutes with colimits) given by

$$\mathcal{M}on_A \longrightarrow \mathcal{A}lg_A$$
$$P \longrightarrow A[P]$$

This induces a unique morphism of sheaves of monoids

$$P_{\operatorname{Spec} A[P]} \longrightarrow \mathcal{O}_{\operatorname{Spec} A[P]}$$

where $P_{\operatorname{Spec} A[P]}$ is the constant sheaf on $\operatorname{Spec} A[P]$. Let $P^{\log}_{\operatorname{Spec} A[P]} \longrightarrow \mathcal{O}_{\operatorname{Spec} A[P]}$ be the morphism of associated sheaves of monoids. Thus,we have a log scheme, called the *affine log scheme* $\mathbb{A}^1_P := (\operatorname{Spec} A[P], P^{\log}_{\operatorname{Spec} A[P]})$. Moreover, for every morphism of A-monoids $P \longrightarrow Q$, there exists a commutative diagram :

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow & & \downarrow \\ A[P] & \xrightarrow{\exists !} & A[Q] \end{array}$$

This in turn induces a morphism between the affine log schemes

$$(\operatorname{Spec} A[Q], Q^{\log}_{\operatorname{Spec} A[Q]}) \longrightarrow (\operatorname{Spec} A[P], P^{\log}_{\operatorname{Spec} A[P]})$$

Thus, we have a contravariant functor :

$$\operatorname{LogSpec} : \mathcal{M}on^{op} \longrightarrow \mathbf{AffLogSch} \hookrightarrow \mathbf{LogSch}$$

Lemma 1.1.16. *The canonical association*

$$Hom_{\boldsymbol{LogSch}}((X, \mathcal{M}_X), (\operatorname{Spec} \mathbb{Z}[P], P^{\log}_{\operatorname{Spec} \mathbb{Z}[P]})) \longrightarrow Hom_{\mathcal{M}on}(P, \Gamma(X, \mathcal{M}_X))$$

is bijective.

Proof. Given a morphism $f : (X, \mathcal{M}_X) \longrightarrow (\operatorname{Spec} \mathbb{Z}[P], P_X^{\log})$, we have a morphism $f^{-1}P_X^{\log} \longrightarrow \mathcal{M}_X$ of sheaves of monoids. Taking global sections of the sheaves, we obtain $P \longrightarrow \Gamma(X, \mathcal{M}_X))$.

Conversely, a morphism $P \longrightarrow \Gamma(X, \mathcal{M}_X)) \longrightarrow \Gamma(X, \mathcal{O}_X))$ uniquely induces a morphism of A-algebras $\mathbb{Z}[P] \longrightarrow \Gamma(X, \mathcal{O}_X))$, which induces a morphism of schemes $X \longrightarrow \operatorname{Spec} \mathbb{Z}[P]$. The morphism of log structures is induced by the logarithmification of the morphism $P_X \longrightarrow \mathcal{M}_X$.

The fact that both the constructions are inverse to one another follows as in the case of schemes. See [22] for more details.

■

Definition 1.1.17 (Charts for log schemes)**.** Let $(X, \mathcal{M}_X)$ be a log scheme, let P be a monoid and consider the constant sheaf P_X on X. A *global chart* for $(X, \mathcal{M}_X)$ over the monoid P is a strict morphism of log schemes

$$c : (X, \mathcal{M}_X) \longrightarrow (\operatorname{Spec} \mathbb{Z}[P], P^{\log}_{\operatorname{Spec} \mathbb{Z}[P]}) = \mathbb{A}^1_P$$

Equivalently, a morphism of monoids $P \longrightarrow \Gamma(X, \mathcal{M}_X)$, i.e. a morphism of sheaves of monoids $P_X \longrightarrow \mathcal{M}_X$ such that the associated morphism of log structures $P_X^{\log} \longrightarrow \mathcal{M}_X$ is an isomorphism, gives a *global chart* for $(X, \mathcal{M}_X)$ over the monoid P.

Example 1.1.18 (Divisorial log structure). Take $X = \mathbb{A}^2_k$ and $D = Z(x) \cup Z(y)$. Then we have a morphism of sheaves of monoids

$$\mathbb{N}^2_X \longrightarrow \mathcal{M}^D_X$$
$$(n_1, n_2) \longmapsto x^{n_1} y^{n_2}$$

The morphism above is a local chart for the log scheme.

Definition 1.1.19. Charts for morphisms of log schemes

Let $f : \mathcal{X} = (X, \mathcal{M}_X) \longrightarrow (Y, \mathcal{M}_Y) = \mathcal{Y}$ be a morphism of log schemes. A chart for the morphism f is a morphism of monoids $\theta : Q \longrightarrow P$ such that there exists twp charts $c_P : \mathcal{X} \longrightarrow \mathbb{A}^1_P$ and $c_Q : \mathcal{Y} \longrightarrow \mathbb{A}^1_Q$, making the following diagram commute.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \\ \downarrow{\scriptstyle c_P} & & \downarrow{\scriptstyle c_Q} \\ \mathbb{A}^1_P & \xrightarrow{\mathbb{A}^1_\theta} & \mathbb{A}^1_Q \end{array}$$

In the above setup, we have a commuatative diagram of log schemes

$$\begin{array}{ccccc} \mathcal{X} & & \overset{c_P}{\curvearrowright} & & \\ & \searrow^{b_\theta} & & & \\ & & \mathcal{Y}_\theta & \xrightarrow{b} & \mathbb{A}^1_P \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f_\theta} & & \downarrow{\scriptstyle \mathbb{A}^1_\theta} \\ & & \mathcal{Y} & \xrightarrow{c_Q} & \mathbb{A}^1_Q \end{array}$$

where the underlying scheme of $\mathcal{Y}_\theta$ is given by $Y \times_{\underline{\mathbb{A}^1_Q}} \underline{\mathbb{A}^1_P}$ and the log structure is given by the pullback of the log structure on the affine log scheme $\mathbb{A}^1_P$ via $\underline{b}$. The morphisms c_Q, b and c_P are strict morphisms of log schemes. The commutative square is cartesian in the category of log schemes. Hence, there exists a unique morphism of log schemes $b_\theta : \mathcal{X} \longrightarrow \mathcal{Y}_\theta$ which is strict by construction.

Equivalently, it is sufficient to give two morphisms $P_X \longrightarrow \mathcal{M}_X$ and $Q_Y \longrightarrow \mathcal{M}_Y$ with isomorphisms of associated log structures such that the following diagram commutes:

$$\begin{array}{ccc} Q_X & \longrightarrow & P_X \\ \downarrow & & \downarrow \\ f^*\mathcal{M}_Y & \longrightarrow & \mathcal{M}_X \end{array}$$

Remark 1.1.20. 1. Since we are dealing with log structures, the morphism of monoids $\theta : Q \longrightarrow P$ is automatically *local*, i.e. $\theta^{-1}(Q^*) = P^*$.

Example 1.1.21 (Semistable reduction). Let $X \longrightarrow S$ be a semistable reduction model with divisorial log structures on the base and the total space as introduced in example 3 of Example 1.1.5, with a factorisation

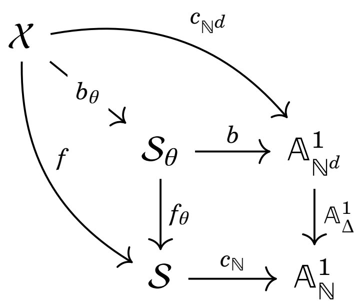
$$X \underset{\text{ét}}{\longrightarrow} \operatorname{Spec} A[x_1, ..., x_n](x_1 \cdots x_d - \pi) \longrightarrow S;\ d \leq n$$

Thus, we have local charts

$$\mathbb{N} \longrightarrow \mathcal{O}_S;\ 1 \longmapsto \pi$$

$$\mathbb{N}^d \longrightarrow \mathcal{O}_X; (n_1, \dots, n_d) \longmapsto \prod_{i=1}^{d} x_i^{n_i}$$

$$\Delta : \mathbb{N} \longrightarrow \mathbb{N}^d; 1 \longmapsto (1, \dots, 1)$$



1.2 The geometry of monoids-II

In this section we review some properties of morphisms of monoids, for instance exact (Definition 1.2.12), integral (Definition 1.2.16) and saturated (Definition 1.2.20) morphisms. These properties serve as important tools to translate the classical geometric notions of flat morphisms of schemes and morphisms of schemes with reduced fibers to the logarithmic world.

A monoid P has an associated group defined by $P^{\mathrm{gp}} := (P \times P)/\sim$ where $(a, b) \sim (c, d)$ if and only if $\exists s \in P$ such that $s+a+d = s+b+c$. Thus, we have a canonical morphism $P \longrightarrow P^{\mathrm{gp}}$. For instance, $\mathbb{N}^{\mathrm{gp}} = \mathbb{Z}$ and if P is a group, then $P^{\mathrm{gp}} \cong P$.

Definition 1.2.1 (Integral monoid)**.** A monoid P is *integral* if the canonical map $P \longrightarrow P^{\mathrm{gp}}$ is injective, i.e. for any $a, b, c \in P$ we have $a + b = a + c \implies b = c$. Define $P^{\mathrm{int}} := \mathrm{img}(P \longrightarrow P^{\mathrm{gp}})$ which is an integral monoid. In other words, P is integral if and only if $P = P^{\mathrm{int}}$. Thus, we have a factorisation $P \longrightarrow P^{\mathrm{int}} \longrightarrow P^{\mathrm{gp}}$. Moreover, by definition the canonical morphism $P^{\mathrm{gp}} \longrightarrow (P^{\mathrm{gp}})^{\mathrm{int}}$ is an isomorphism. The association $P \longrightarrow P^{\mathrm{int}}$ is a left adjoint to the inclusion functor from the subcategory of integral monoids $\mathcal{M}on^{\mathrm{int}}$ to the category monoids $\mathcal{M}on$.

$$\mathcal{M}on^{\mathrm{int}} \hookrightarrow \mathcal{M}on \longrightarrow \mathcal{M}on^{\mathrm{int}}$$

$$P \longmapsto P \longmapsto P^{\mathrm{int}}$$

For instance, $\mathbb{N} \hookrightarrow \mathbb{Z}$ is an integral monoid.

Definition 1.2.2 (Saturated monoid)**.** An integral monoid P is said to be *saturated* if for any $x \in P^{\mathrm{gp}}$ with $nx \in P$ for some $n \geq 1$, then $x \in P$. Define $P^{\mathrm{sat}} := \{x \in P^{\mathrm{gp}} \mid \exists n \geq \ \text{such that } nx \in P\}$ which is a saturated monoid. In other words, P is saturated if and only if $P = P^{\mathrm{sat}}$. The association $P \longrightarrow P^{\mathrm{sat}}$ is a left adjoint to the inclusion functor from the subcategory of saturated monoids $\mathcal{M}on^{\text{sat}}$ to the category integral monoids.

$$\mathcal{M}on^{\text{sat}} \hookrightarrow \mathcal{M}on^{\text{int}} \longrightarrow \mathcal{M}on^{\text{sat}}$$

$$P \longmapsto P \longmapsto P^{\text{sat}}$$

Definition 1.2.3 (Sharp monoid)**.** Let P be a monoid. We denote the group of units of P by P^*. We denote the quotient monoid P/P^* by $\overline{P}$. A monoid P is said to be *sharp* if $P^* = \{0\}$.

Remark 1.2.4. If P is saturated, then $\overline{P}^{\text{gp}} \cong P^{\text{gp}}/P^*$ is torsion free. In fact, as we will see later, torsion freeness of cokernel of morphisms characterizes smooth integral morphisms of log schemes with reduced fibers.

Example 1.2.5. $P = \mathbb{N} \setminus \{1\}$ is not saturated.

Definition 1.2.6 (Coherent monoid)**.** A monoid P is said to be *coherent* if it admits a finite family of generators, i.e. if and only if there exists a surjective morphism $\mathbb{N}^m \longrightarrow P$.

Lemma 1.2.7. *A monoid P is coherent if and only if P^* is finitely generated as a group and $\overline{P}$ is coherent.*

Proof. If P is coherent then clearly P^* is finitely generated as a group and $\overline{P}$ is coherent. Conversely, let $\{s_i\}$ and $\{t_j\}$ be finite set of generators for P^* and $\overline{P}$ respectively. Then the finite set $\{s_i, -s_i, t_j\}$ generates the monoid P.

■

Definition 1.2.8 (Fine saturated (fs) monoids)**.** A monoid P is *fine* if it is coherent and integral; *fs* (fine and saturated) if it is both fine and saturated; *toric* if it is fs and torsion-free.

Remark 1.2.9. If P is toric, then $P^{\text{gp}} \cong \mathbb{Z}^d$.

Definition 1.2.10 (Irreducible elements)**.** If P is a sharp monoid, then an element $p \in P$ is *irreducible* if for any equality $p_1 + p_2 = p$ in P we have $p_1 = 0$ or $p_2 = 0$.

We will use the following result about irreducible elements of a fs monoid in the upcoming sections. It is analogous to the unique factorisation of elements into irreducibles in a unique factorisation domain.

Theorem 1.2.11. *If P is a fine sharp monoid, then the set* $\mathrm{Irr}(P)$ *of irreducible elements in P is a finite set which generates P as a monoid.*

Proof. See [22, Section I.2.1].

■

Definition 1.2.12 (Exact morphism)**.** A morphism $f : P \longrightarrow Q$ of monoids is *exact* if the following commutative diagram is cartesian int the category of monoids[3]:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \downarrow & & \downarrow \\ P^{\mathrm{gp}} & \xrightarrow{f^{\mathrm{gp}}} & Q^{\mathrm{gp}} \end{array}$$

Remark 1.2.13.

1. If P is integral, then $f : P \longrightarrow Q$ is exact if and only if $(f^{\mathrm{gp}})^{-1}(Q) = P$. Intuitively, exactness can be thought of as the equivalence $f(a)|f(b) \Leftrightarrow a|b$ for any $a, b \in P$.
2. Since the diagram is cartesian, the canonical map $\ker f \longrightarrow \ker f^{\mathrm{gp}}$ is an isomorphism. Thus, f is a monomorphism (i.e. injective) if and only if $\ker(f)$ is trivial. In the subsequent chapters, we will be dealing with morphisms which are exact, hence we don't have to deal with the fallacy mentioned in Remark 1.1.8.
3. $f : P \longrightarrow Q$ is exact if and only if $\overline{f} : \overline{P} \longrightarrow \overline{Q}$ is an exact morphism.
4. The family of exact morphisms is stable under composition, pullback and pushout.

Example 1.2.14. The diagonal morphism $\Delta : \mathbb{N} \longrightarrow \mathbb{N}^n$ is exact, hence by definition the semistable reduction $\mathcal{X} \longrightarrow \mathcal{S}$ in Example 1.5.9 is exact.

Remark 1.2.15. Pushouts in the category of integral and saturated monoids are subtler compared to the category of monoids as the family of integral and saturated morphisms need not be closed under pushouts. This motivates the following definition, which also has important geometric consequences.

Definition 1.2.16 (Integral morphism)**.** A morphism $f : P \longrightarrow Q$ of integral monoids is *integral* if the following equivalent conditions hold[4]:

[3] See [22, section I.4.2] for more details on exact morphisms

[4] See [22, sections I.4.5 and I.4.6] for the equivalence of $(1) - (3)$.

1. For any morphism $g : P \longrightarrow R$ of integral monoids, the push-out $Q \oplus_P R$ in the category of monoids is integral.

2. For any morphism $P \longrightarrow R$ of integral monoids, the push-out $Q \oplus_P R$ in the category of monoids is described as the quotient of $Q \oplus R$ by the equivalence relation. $(m_1, n_1) \sim (m_2, n_2)$ if and only if $\exists\, c, d \in P$ such that $m_1 + f(c) = m_2 + f(d),\ \ n_1 + g(d) = n_2 + g(c)$. Note that this is the same equivalence relation used to define the push-out in the category of groups.

3. The transporter category $\mathcal{T}_P Q$ is cofiltering, i.e. whenerver $q_1, q_2 \in Q$ and $p_1, p_2 \in P$ satisfy $f(p_1) + q_1 = f(p_2) + q_2$, then there exists $q' \in Q$ and $p_i^{'} \in P$ such that $q_i = f(p_i{}') + q'$ and $p_1 + p_1' = p_2 + p_2'$. In other words, the following diagram is completed by the dotted arrows

$$\begin{array}{ccc} q' & \overset{p_1'}{\dashrightarrow} & q_1 \\ {\scriptstyle p_2'}\downarrow & & \downarrow{\scriptstyle p_1} \\ s_2 & \overset{p_2}{\longrightarrow} & q \end{array}$$

Recall that the objects of the transporter category $\mathcal{T}_P Q$ are the elements of Q and for which the morphisms from an object q_1 to an object q_2 are the elements p of P such that $f(p) + q_1 = q_2$.

If P is a sharp monoid (which implies f is injective), then we have the following important geometric consequence.

Theorem 1.2.17. *Let $f : P \longrightarrow Q$ be a morphism of integral monoids with P a sharp monoid. Then the following conditions are equivalent.*

1. *f is integral.*

2. *The homomorphism $\mathbb{Z}[f] : \mathbb{Z}[P] \longrightarrow \mathbb{Z}[P]$ is a flat morphism of $\mathbb{Z}$-algebras.*

3. *For any field k, the homomorphism $k[f] : k[P] \longrightarrow k[P]$ is a flat morphism.*

Proof. See [22, I.4.6.7]. ■

Remark 1.2.18. 1. A morphism of monoids $f : P \longrightarrow Q$ is integral if and only if $\overline{f} : \overline{P} \longrightarrow \overline{Q}$ is an integral morphism.

2. The family of integral morphisms is stable under compositions, pullbacks and pushouts.

3. An integral morphism is exact if and only if it is local (i.e. $f^{-1}(Q^*) = P^*$). All our morphisms in the subsequent chapters will fall under this category.

Example 1.2.19. The diagonal morphism $\Delta : \mathbb{N} \longrightarrow \mathbb{N}^n$ is integral, hence by definition the semistable reduction $\mathcal{X} \longrightarrow \mathcal{S}$ in Example 1.5.9 is integral. Moreover, the map $f : \mathbb{N}^m \longrightarrow \mathbb{N}^{m+n}$ defined by

$$e_i \longmapsto \begin{cases} e_i \ i \neq m \\ e_m + \cdots + e_{m+n} \ i = m \end{cases}$$

is integral since it is a base change of the diagonal $\Delta : \mathbb{N} \longrightarrow \mathbb{N}^{n+1}$.

Definition 1.2.20 (Saturated morphism)**.** An integral morphism $f : P \longrightarrow Q$ of saturated monoids is called *saturated* if for any morphism $P \longrightarrow R$ with R a saturated monoid, the push-out $Q \oplus_P R$ in the category of integral monoids is a saturated monoid.

Example 1.2.21. The diagonal morphism $\Delta : \mathbb{N} \longrightarrow \mathbb{N}^n$ is saturated, hence by definition the semistable reduction $\mathcal{X} \longrightarrow \mathcal{S}$ in Example 1.5.9 is saturated.

Remark 1.2.22. 1. In addition, if P and Q are sharp monoids, then $f : P \longrightarrow Q$ is saturated if and only if Coker (f^{gp}) is torsion free. See [22, Section I.4.8] for more details. All our morphisms in the subsequent chapters will fall under this category.

2. $f : P \longrightarrow Q$ is saturated if and only if $\overline{f} : \overline{P} \longrightarrow \overline{Q}$ is an integral saturated morphism.

3. The family of saturated morphisms is stable under composition, pullback and pushout.

1.3 Properties of morphisms of log schemes

Definition 1.3.1. Let $\mathbf{Q}$ be one of the properties of monoids defined in Section 1.2. For example, fine, integral, saturated, etc. Then a log structure $\mathcal{M}_X$ on X is said to satisfy property if X admits an open covering $\{U_\alpha\}_\alpha$ in the étale topology such that the restriction of $\mathcal{M}_X$ to each $\{U_\alpha\}$ admits a chart $P^\alpha \longrightarrow \mathcal{M}_{X|U_\alpha}$ where the monoid P_α satisfies the property **Q**.

Remark 1.3.2. $\mathcal{M}_X$ satisfies property **Q** if and only if the monoid $\mathcal{M}_{X,\overline{x}}$ has property **Q** for all geometrc points $\overline{x} \in X$. See [22, II.1.1.3] for more details.

Definition 1.3.3. Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of *fine* log schemes in the sense of Definition 1.3.1. Let **P** be one of the properties (*integral* or *saturated* morphism) of monoids as defined in Section 1.2. Let $\overline{x}$ be a geometric point lying over a point $x \in X$. Then, f is said to satisfy the property **P** if for every geometric point $\overline{x}$ lying over x and every point $x \in X$, the morphism of monoids $f^{@}_{\overline{x}} : \mathcal{M}_{Y,f(\overline{x})} \longrightarrow \mathcal{M}_{X,\overline{x}}$ satisfies **P**.

The properties **P** of morphisms of log schemes above obey the usual functoriality and openness conditions. Namely,

Theorem 1.3.4. *Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ and $g : \mathcal{Y} \longrightarrow \mathcal{Z}$ be morphisms of fine log schemes. Let $\in X, y := \underline{f}(x), z := \underline{g}(y)$.*

1. *Openness of **P**: If f has property **P** at x, then the property holds in an étale neighbourhood of x.*

2. *Stable under composition: If f has property **P** at x and g has property **P** at y, then $g \circ f$ has property **P** at x. If $g \circ f$ has property **P** at x and f has property **P** at x then g has property **P** at y.*

3. *Stable under base change: The family of morphisms satisfying **P** is stable under base change in the category of log schemes. Moreover, if $f : \mathcal{X} \longrightarrow \mathcal{Y}$ and $g : \mathcal{W} \longrightarrow \mathcal{Y}$ are morphisms of fine (resp. fine saturated) log schemes and f is integral (resp. saturated) morphism of log schemes, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{W}$ is an integral (resp. saturated) log scheme with the parallel transport $\mathcal{X} \times_{\mathcal{Y}} \mathcal{W} \longrightarrow \mathcal{W}$ an integral (resp. saturated) morphism.*

Proof. The proof follows from the corresponding statements for monoids satisfying **P**. See [22, III.2.5.3]. ■

1.4 More about charts of log schemes

In this section we recall some facts about the existence of *neat charts* for fs log schemes that we will be using in the subsequent chapters.

Remark 1.4.1. We first observe that for a log scheme $(X, \mathcal{M}_X)$, the morphism $f : (X, \mathcal{M}_X) \longrightarrow \mathbb{A}^1_P$ is a chart if $\overline{f}^{@}_x : \overline{\mathcal{M}}_{\mathbb{A}^1_P, \overline{y}} \longrightarrow \overline{\mathcal{M}}_{X,\overline{x}}$ is an isomorphism for every geometric point $\overline{x} \in X$ and $y := \underline{f}(\overline{x})$. Conversely, if $\mathcal{M}_{X,\overline{x}}$ (equivalently $\overline{\mathcal{M}}_{X,\overline{x}}$) is an integral monoid and $\overline{f}^{@}_x : \overline{\mathcal{M}}_{\mathbb{A}^1_P, \underline{f}(\overline{x})} \longrightarrow \overline{\mathcal{M}}_{X,\overline{x}}$ is an isomorphism for every geometric point $\overline{x} \in X$, then $f : (X, \mathcal{M}_X) \longrightarrow \mathbb{A}^1_P$ is a chart. The converse basically follows from the fact that in the commutative diagram below, θ is an isomorphism if and only if $\overline{\theta}$ is an isomorphism whenever Q is an integral monoid.

$$\begin{array}{ccc} P & \xrightarrow{\theta} & Q \\ \downarrow & & \downarrow \\ \overline{P} & \xrightarrow{\overline{\theta}} & \overline{Q} \end{array}$$

Definition 1.4.2. Let $(X, \mathcal{M}_X)$ be an integral log scheme and let $\theta : P_X \longrightarrow \mathcal{M}_X$ be a chart. By the above remark P is also an integral monoid. Let $\overline{x} \longrightarrow X$ be a geometric point of X. Then, $\theta(X) : P \longrightarrow \mathcal{M}_X(X)$ is a *neat* chart at $\overline{x}$ if it satisfies one of the following equivalent conditions:

1. The canonical map $P \longrightarrow \overline{\mathcal{M}}_{X,\overline{x}}$ is an isomorphism.
2. The canonical map $P^{\text{gp}} \longrightarrow \overline{\mathcal{M}}^{\text{gp}}_{X,\overline{x}}$ is an isomorphism.

Remark 1.4.3.
1. By the above conditions, P is in fact a sharp monoid.
2. Since $P \longrightarrow \mathcal{M}_X$ is a chart and $P \cong \overline{\mathcal{M}}^{\text{gp}}_{X,\overline{x}}$, the morphism $\mathcal{M}^{\text{gp}}_{X,\overline{x}} \longrightarrow \overline{\mathcal{M}}^{\text{gp}}_{X,\overline{x}}$ admits a section.
3. Moreover, any chart of a fs log scheme étale locally factors through a neat chart.

Theorem 1.4.4 (Neat charts for fine log schemes)**.** *Let $(X, \mathcal{M}_X)$ be a fine log scheme and let $x \in X$ be a point. Then*

1. *$(X, \mathcal{M}_X)$ admits a local neat chart at a geometric point $\overline{x}$ lying over x if and only if there is a split short exact sequence*

$$0 \longrightarrow \mathcal{M}^*_{X,\overline{x}} \cong \mathcal{O}^*_{X,\overline{x}} \xrightarrow{\alpha} \mathcal{M}^{\text{gp}}_{X,\overline{x}} \underset{s}{\rightleftarrows} \overline{\mathcal{M}}^{\text{gp}}_{X,\overline{x}} \longrightarrow 0 \qquad (1.4.1)$$

2. *There exists an fppf neighbourhood $Y \longrightarrow X$ of x such that the short exact sequence 1.4.1 splits.*

3. *In case $\overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}} \otimes k(x)$ is torsion free (for instance if $\mathcal{M}_{X,\bar{x}}$ is saturated), then the sequence 1.4.1 always splits in an étale neighbourhood $Y \longrightarrow X$ of x.*

Sketch of proof. 1. For a local neat chart at x, the spliting is clear from Remark 1.4.3. Conversely, if the sequence splits, by Remark 1.4.1 it is enough to prove $\overline{\mathcal{M}}^{\log}_{X,\bar{x}} \cong \mathcal{M}_{X,\bar{x}}$. Then by a standard openness argument, we can extend the chart to a neighbourhood of $\bar{x}$. We have a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M}^*_{X,\bar{x}} \cong \mathcal{O}^*_{X,\bar{x}} & \longrightarrow & \mathcal{M}_{X,\bar{x}} \cong \mathcal{M}^{\mathrm{gp}}_{X,\bar{x}} \times_{\overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}}} \overline{\mathcal{M}}_{X,\bar{x}} & \longrightarrow & \overline{\mathcal{M}}_{X,\bar{x}} & \longrightarrow & 0 \\
 & & \| & & \downarrow & \overset{\exists !}{\curvearrowleft} & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}^*_{X,\bar{x}} \cong \mathcal{O}^*_{X,\bar{x}} & \longrightarrow & \mathcal{M}^{\mathrm{gp}}_{X,\bar{x}} & \longrightarrow & \overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}} & \longrightarrow & 0 \\
 & & & & & \underset{s}{\curvearrowleft} & & &
\end{array}$$

such that the right hand sidesquare is cartesian since $\mathcal{M}_{X,\bar{x}} \longrightarrow \overline{\mathcal{M}}_{X,\bar{x}}$ is an exact morphism. Hence, there exists a unique morphism $\overline{\mathcal{M}}_{X,\bar{x}} \longrightarrow \mathcal{M}_{X,\bar{x}}$. Now by a similar argument as in Remark 1.4.1 and the construction of logarithmification, we have $\overline{\mathcal{M}}^{\log}_{X,\bar{x}} \cong \mathcal{M}_{X,\bar{x}}$.

2. Since we are working over coherent log structures, there exists an isomorphism $\overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}} \cong \mathbb{Z}^r \bigoplus_{i=1}^m C_i$, where for $i \in \{1, \dots, m\}$, C_i is a cyclic group of order d_i with generator g_i. For the free part $\mathbb{Z}^r$, there always exists a section $\mathbb{Z}^r \longrightarrow \mathcal{M}^{\mathrm{gp}}_{X,\bar{x}}$. For the torsion part C_i, consider the lifts $\overline{g}_i$ of the generators g_i. Then by the exactness in Equation 1.4.1, there exists $u_i \in \mathcal{O}^*_{X,x}$ such that $d_i g_i = \alpha(u_i)$ for each i. Replacing x by an fppf neighbourhood with the new local ring isomorphic to $\frac{\mathcal{O}_{X,x}[T_1,\dots,T_m]}{(T_1^{d_1}-u_1 \dots T_m^{d_m}-u_m)}$, then the morphisms $g_i \longmapsto \overline{g}_i - \alpha(T_i)$ serve as sections since T_i's are units in $\frac{\mathcal{O}_{X,x}[T_1,\dots,T_m]}{(T_1^{d_1}-u_1 \dots T_m^{d_m}-u_m)}$.

3. Since $\overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}} \otimes k(x)$ is torsion free and finitely generated, it is free, hence the short exact sequence splits. Moreover, $\frac{\mathcal{O}_{X,x}[T_1,\dots,T_m]}{(T_1^{d_1}-u_1 \dots T_m^{d_m}-u_m)}$ is an étale $\mathcal{O}_{X,x}$-

algebra if and only if d_i's are invertible in $k(x)$, i.e.the torsion part of $\overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}} \otimes k(x)$ vanishes.

■

In the proof of (1) above, we *extended a chart to a neighbourhood of x*. This can be done precisely by the following result.

Lemma 1.4.5. *Let $(X, \mathcal{M}_X)$ be a fs log scheme. Let $x \in X$ be a point and G a finitely generated abelian group. Let $h^{\mathrm{gp}} : G \longrightarrow \mathcal{M}^{\mathrm{gp}}_{X,\bar{x}}$ be a homomorphism such that $G \longrightarrow \overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}}$ is surjective. Then $P := (h^{\mathrm{gp}})^{-1}(\mathcal{M}_{X,\bar{x}}) \longrightarrow \mathcal{M}_{X,\bar{x}}$ can be extended to a chart $P_Y \longrightarrow \mathcal{M}_{X|_Y}$ in an étale neighbourhood Y of $\bar{x}$.*

Proof. See [18, Lemma 2.10]. ■

The next corollary shows the existence of a local neat chart for morphisms of fine saturated log schemes.

Corollary 1.4.6 (Neat charts for fs log morphisms). Let $\mathcal{X} \longrightarrow \mathcal{Y}$ be a mophism of fine log schemes. Let $\bar{x} \longrightarrow X$ be a geometric point with $\underline{f}(\bar{x}) = \bar{y}$. Let $P \longrightarrow \mathcal{M}_Y$ be a fine chart. Then fppf locally around $\bar{x}$, f admits a chart $P \longrightarrow Q$ such that $Q \longrightarrow \mathcal{M}_X$ is neat at $\bar{x}$. If $\mathcal{M}^{\mathrm{gp}}_{X/Y,\bar{x}}$ is torsion free (for instance if f is a saturated morphism), then we can replace the chart in an étale neighbourhood of $\bar{x}$.

Moreover, étale locally, the morphism $P \longrightarrow Q$ can be always chosen to be injective, which implies that $P \longrightarrow \mathcal{M}_Y$ is also neat at $\bar{y}$.

Proof. Follows by applying Theorem 1.4.4. ■

Remark 1.4.7. The proof of the openness of property **P** in the first part of Theorem 1.3.4 follows now by choosing a neat chart in an fppf neighbourhood.

Example 1.4.8 (Log point). Recall example 2 in Example 1.1.5. Let $X = \operatorname{Spec} k$ where k is an algebraically closed field. Let $\mathcal{M}_X := k^* \oplus P_X$, where P_X is the constant sheaf on X defined by a coherent monoid P such that P is sharp. The structure morphism is defined on local sections by

$$\alpha_X : \mathcal{O}_X^* \oplus P_X \longrightarrow \mathcal{O}_X$$

$$\alpha_X(s,p) = \begin{cases} s & \text{for } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

Note that $P \longrightarrow \mathcal{M}_X$ is a chart for the log point. Since P is sharp, we have an isomorphism $\overline{\mathcal{M}}_X \cong P$. Hence, $P \longrightarrow \mathcal{M}_X$ is a neat chart. Equivalently, the following short exact sequence of splits.

$$0 \longrightarrow \mathcal{M}_X^* \cong k^* \xrightarrow{\alpha} \mathcal{M}_X^{gp} \underset{s}{\rightleftarrows} \overline{\mathcal{M}}_X^{gp} \longrightarrow 0 \tag{1.4.2}$$

Corollary 1.4.9. Let $(X, \mathcal{M}_X)$ be a coherent log scheme. Let $n \in \mathbb{N}$. Then $X_n := \{x \in X \mid \mathrm{rank}(\overline{\mathcal{M}}^{gp}_{X,\bar{x}}) \leq n\}$ is an open subscheme of X. In particular, $X_1 := \{x \in X \mid \mathcal{M}^*_{X,\bar{x}} = \mathcal{M}_{X,\bar{x}}\}$ is an open subscheme of X. Note that X_1 need not be non-empty.

Moreover, we have a stratification of a log scheme by constructible subschemes $X = C_0 \supseteq C_1 \supseteq \cdots \supseteq C_m \supseteq \dots$.

An alternative way to think about the characteristic sheaf of fs log structures is by considering the log structure on each strata as observed in example 2 of Example 1.1.5. Precisely, for a finitely generated monoid P, the space $\mathbb{A}^1_P$ is a Kolmogoroff space, i.e. for any $x \neq y \in \mathbb{A}^1_P$, either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. Thus, every point is locally closed and hence every sheaf of monoids on $\mathbb{A}^1_P$ is constructible. The following result gives a converse to the above discussion.

Theorem 1.4.10. *Let $\mathcal{M}_X$ be an integral saturated log structure on a scheme X. Then $\mathcal{M}_X$ is coherent if and only if it satisfies the following conditions*

1. *X admits an open covering on which $\overline{\mathcal{M}}_X$ is constructible.*
2. *$\overline{\mathcal{M}}_{X,\bar{x}}$ is coherent for every geometric point $\bar{x} \in X$.*

Proof. The converse uses Theorem 1.4.4. See [22, II.2.5.4] for more details.

∎

Remark 1.4.11. The advantage of working with constructible sheaves on the étale site is that they are representable by étale algebraic spaces (see Definition A.1.3 for the definition) as we shall see later.

1.5 Log derivations and differentials

Definition 1.5.1. Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of log schemes and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. A *log derivation* of $\mathcal{X}/\mathcal{Y}$ with values in $\mathcal{F}$ is a pair (D, δ), where $D : \mathcal{O}_X \longrightarrow \mathcal{F}$ is a morphism of abelian sheaves and $\delta : \mathcal{M}_X \longrightarrow \mathcal{F}$ is a morphism of sheaves of monoids such that the following conditions are satisfied:

1. $D(\alpha_X(m)) = \alpha_X(m)\delta(m)$ for every local section m of $\mathcal{M}_X$, where $\alpha_X : \mathcal{M}_X \longrightarrow \mathcal{O}_X$.
2. $\delta(f^{@}(n)) = 0$ for every local section n of $f^{-1}(\mathcal{M}_Y)$.
3. $D(ab) = aD(b) + bD(a)$ for all local sections a, b of $\mathcal{O}_X$.
4. $D(f^*(c)) = 0$ for every local section c of $f^{-1}\mathcal{O}_Y$.

We define $\mathrm{Der}_{\mathcal{X}/\mathcal{Y}}(\mathcal{F})$ to be the set of all such log derivations. Then the functor

$$\mathrm{Der}_{\mathcal{X}/\mathcal{Y}} : \mathcal{O}_X - \mathrm{Mod} \longrightarrow (Sets)$$

$$\mathcal{F} \longmapsto \mathrm{Der}_{\mathcal{X}/\mathcal{Y}}(\mathcal{F})$$

is representable, with a universal object

$$\mathcal{O}_X \xrightarrow{d} \Omega^1_{\mathcal{X}/\mathcal{Y}} \quad \mathcal{M}_X \xrightarrow{dlog} \Omega^1_{\mathcal{X}/\mathcal{Y}}$$

where $\Omega^1_{\mathcal{X}/\mathcal{Y}} := (\Omega^1_{X/Y} \bigoplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{\mathrm{gp}}))/\sim$ and $\sim$ is generated by the relations:

i $(d\alpha_X(m), 0) - (0, \alpha_X(m) \otimes m)$ for all $m \in \mathcal{M}_X$.

ii $(0, 1 \otimes f^{@}(n))$ for all local section n of $f^{-1}(\mathcal{M}_Y)$.

Thus, we have the canonical morphisms

$$\mathcal{O}_X \xrightarrow{d} \Omega^1_{\mathcal{X}/\mathcal{Y}}$$

$$b \longmapsto db$$

$$\mathcal{M}_X \xrightarrow{dlog} \Omega^1_{\mathcal{X}/\mathcal{Y}}$$

$$m \longmapsto [(0, 1 \otimes m)]$$

Moreover, the first relation above ensures that $\alpha_X(m)dlog(m) = d\alpha_X(m)$. See [22, IV.1.1 and IV.1.2] for more details on log derivations and differentials.

Remark 1.5.2. The logarithmic sheaf of differentials $\Omega^1_{\mathcal{X}/\mathcal{Y}}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules. In fact, let $\mathbb{A}^1_P \longrightarrow \mathbb{A}^1_Q$ be a (local) chart for $f : \mathcal{X} \longrightarrow \mathcal{Y}$. Then $\Omega^1_{\mathcal{X}/\mathcal{Y}|\mathbb{A}^1_P} = \widetilde{\Omega^1_{\mathbb{A}^1_P/\mathbb{A}^1_Q}}$. We denote $\Omega^1_{P/Q} := \Omega^1_{\mathbb{A}^1_P/\mathbb{A}^1_Q}$. Moreover, the morphism

$$1 \otimes dlog : \mathbb{Z}[P] \otimes_{\mathbb{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}}) \xrightarrow{\sim} \Omega^1_{P/Q}$$

is an isomorphism of $\mathbb{Z}[P]$-modules.

Example 1.5.3. Consider the morphism $\mathcal{X} := \mathrm{Spec}(k[x_1, \dots, x_n]/(x_1 \cdots x_r)) \longrightarrow \mathrm{Spec} k$ endowed with the divisorial logarithmic structure. Then using the above remark, $\Omega^1_{\mathcal{X}/k}$ is a free $\mathcal{O}_X$ module generated by $\frac{dx_1}{x_1}, \dots, \frac{dx_r}{x_r}, dx_{r+1}, \dots, dx_n$ such that

$$\Sigma^r_{1=1} \frac{dx_i}{x_i} = 0$$

1.5.1 Log smooth and log étale morphisms

In this section we recall various local properties of morphisms of log schemes. All the definitions and results below hold true for arbitrary morphisms of log schemes but to stick to our purpose, we restrict to working over fine saturated log schemes.

Definition 1.5.4 (Log thickening)**.** A *n-th order log thickening* is a strict closed immersion (i.e. a closed immersion of underlying schemes and a strict morphism of log structures) $i : \mathcal{S} \longrightarrow \mathcal{T}$ of log schemes such that sheaf of ideals $\mathcal{I}$ defining $\underline{i} : S \longrightarrow T$ is nilpotent, i.e. $\mathcal{I}^{n+1} = 0$ for some $n \in \mathbb{N}$.

Remark 1.5.5.

1. The action of $1 + \mathcal{I} \subseteq \mathcal{O}^*_T \cong \mathcal{O}^*_T$ on $\mathcal{M}_T$ (resp. $\mathcal{M}^{\mathrm{gp}}_T$) makes it a torsor over $\mathcal{M}_S$ (resp. $\mathcal{M}^{\mathrm{gp}}_S$), i.e.the canonical morphisms

$$(1 + \mathcal{I}) \times \mathcal{M}_T \xrightarrow{\sim} \mathcal{M}_T \times_{\mathcal{M}_S} \mathcal{M}^{\mathrm{gp}}_T$$

$$(1 + \mathcal{I}) \times \mathcal{M}^{\mathrm{gp}}_T \xrightarrow{\sim} \mathcal{M}^{\mathrm{gp}}_T \times_{\mathcal{M}^{\mathrm{gp}}_S} \mathcal{M}^{\mathrm{gp}}_T$$

$$(u, m) \longmapsto (m, um)$$

$$\mathcal{M}_T \xrightarrow{\sim} \mathcal{M}_S \times_{\mathcal{M}^{\mathrm{gp}}_S} \mathcal{M}^{\mathrm{gp}}_T$$

are isomorphisms.

2. $\ker(\mathcal{O}^*_T \longrightarrow \mathcal{O}^*_S) = \ker(\mathcal{M}^*_T \longrightarrow \mathcal{M}^*_S) = 1 + \mathcal{I}$

3. By standard reductions as in the classical setup, it is enough to study only first order thickenings, i.e. $\mathcal{I}^2 = 0$.

As in the classical setup, we study *infinitesimal liftings* aka *deformations* of a morphism $f : \mathcal{X} \longrightarrow \mathcal{Y}$ of log schemes. By a *log infinitesimal lifting* we refer to the existence of a lifting $\tilde{g} : \mathcal{T} \longrightarrow \mathcal{X}$ in the following commutative diagram of log schemes

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{g} & \mathcal{X} \\ \downarrow{\scriptstyle i} & \overset{\tilde{g}}{\nearrow} & \downarrow{\scriptstyle f} \\ \mathcal{T} & \xrightarrow{h} & \mathcal{Y} \end{array} \tag{1.5.1}$$

where $i : \mathcal{S} \longrightarrow \mathcal{T}$ is a log thickening.

Moreover, let $\mathrm{Lif}_{\mathcal{X}/\mathcal{Y}}(g, \mathcal{T})$ denote the set of liftings $\tilde{g}$ in the above diagram. Then there is a natural action

$$\mathrm{Der}_{\mathcal{X}/\mathcal{Y}}(g_*\mathcal{I}) \times g_*\mathrm{Lif}_{\mathcal{X}/\mathcal{Y}}(g, \mathcal{T}) \longrightarrow g_*\mathrm{Lif}_{\mathcal{X}/\mathcal{Y}}(g, \mathcal{T}) \tag{1.5.2}$$

Definition 1.5.6 (Log smooth, log unramified and log étale morphisms)**.** A morphism of log schemes $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is said to be *log smooth* (resp. *log unramified*, resp. *log étale*) if for every n-th order log thickening $i : \mathcal{S} \longrightarrow \mathcal{T}$, there exists at least one (resp. at most one, resp. a unique) local lifting for the Diagram 1.5.1.

Remark 1.5.7.

1. The family of smooth (resp. unramified, resp. étale) morphisms is stable under composition and base change and satisfies the standard openness properties in the category of fs log schemes.

2. Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a log smooth morphism of fs log schemes. For any geometric point $\overline{x} \in X$, f factors (not necessarily uniquely) as

$$\mathcal{X} \longrightarrow \mathcal{Y} \times \mathbb{A}^1_{\mathbb{N}^r} \longrightarrow \mathcal{Y}$$

in a strict étale neighbourhood of $\overline{x}$ such that $\mathcal{X} \longrightarrow \mathcal{Y} \times \mathbb{A}^1_{\mathbb{N}^r}$ is log étale and $r := \mathrm{rank}\, \Omega^1_{\mathcal{X}/\mathcal{Y},\overline{x}}$.

3. A morphism of log schemes $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is log smooth if and only if $\Omega^1_{\mathcal{X}/\mathcal{Y}}$ is a locally free $\mathcal{O}_X$-module of finite rank. See [22, IV.3.2.1].

This implies that the morphism from $\mathrm{Spec}(k[x_1,\ldots,x_n]/(x_1\cdots x_r))$ to $\mathrm{Spec}\,k$ endowed with the divisorial logarithmic structure is log smooth.

4. A morphism of log schemes $f:\mathcal{X}\longrightarrow\mathcal{Y}$ is log unramified if and only if $\Omega^1_{\mathcal{X}/\mathcal{Y}}=0$. See [22, IV.3.1.3].

5. A strict morphism of log schemes is log smooth (resp. log unramified, resp. log étale) if and only if the underlying morphism of schemes is smooth (unramified, resp. étale). See [22, IV.3.1.6].

6. A morphism of log schemes $f:\mathcal{X}\longrightarrow\mathcal{Y}$ is log smooth (resp. log étale) if the action in Equation 1.5.2 is pseudo-torsorial (torsorial).

1.5.2 Chart criterion

In this section we recall K. Kato's *chart criterion* (also called *toroidal criterion*) for log smooth (resp. log étale) morphisms and illustrate that to show that our prototype example of semistable reduction is log smooth.

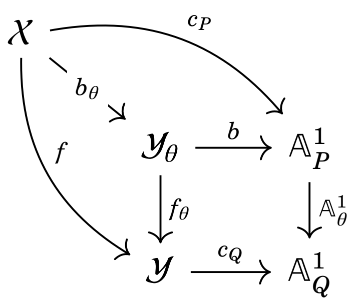
Theorem 1.5.8 (Chart criterion)**.** *Let $f:\mathcal{X}\longrightarrow\mathcal{Y}$ be a morphism of log schemes and let $\mathcal{Y}\longrightarrow\mathbb{A}^1_P$ be a chart of $\mathcal{Y}$. Then the following are equivalent:*

1. *f is log smooth (resp. log étale).*

2. *Étale locally there exists a chart $\theta:P\longrightarrow Q$ for f with the following properties:*

   (a) *The morphism $\theta:P\longrightarrow Q$ is injective and the torsion part of $\mathrm{coker}(\theta)^{\mathrm{gp}}$ has order invertible in $\mathcal{O}_X$ (resp. is of finite order in $\mathcal{O}_X$).*

   (b) *The morphism b_θ in the following diagram is log étale and strict (In other words, $\underline{b_\theta}:X\longrightarrow Y\times_{\mathbb{A}^1_Q}\mathbb{A}^1_P$ is étale in the classical sense).*



Moreover, if either of the two conditions hold, then for every x in the local neighbourhood, the chart b is exact at $\overline{x}$. If f is log étale, the chart can be chosen to be neat at $\overline{x}$. If f is log smooth and the order of the torsion subgroup of $\mathcal{M}^{\mathrm{gp}}_{X/Y,\overline{x}}$ is invertible in $k(x)$, the chart can be chosen to be neat at $\overline{x}$, provided b_θ is allowed to be log smooth (but not necessarily étale).

Proof. See [22, IV.3.1.13 and IV.3.3.1]. ■

Example 1.5.9 (Semistable reduction). Let $X \longrightarrow S$ be a semistable reduction model with divisorial log structures on the base and the total space as introduced earlier, with a factorisation

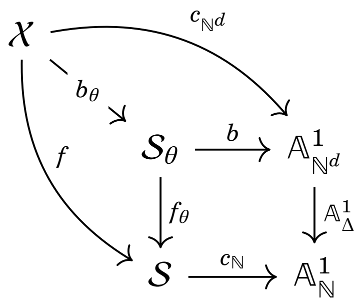
$$X \underset{\text{ét}}{\longrightarrow} \operatorname{Spec} A[x_1, ..., x_n](x_1 \cdots x_d - \pi) \longrightarrow S;\ d \leq n$$

Thus, we have local charts

$$\mathbb{N} \longrightarrow \mathcal{O}_S;\ 1 \longmapsto \pi$$

$$\mathbb{N}^d \longrightarrow \mathcal{O}_X; (n_1, \dots, n_d) \longmapsto \prod_{i=1}^{d} x_i$$

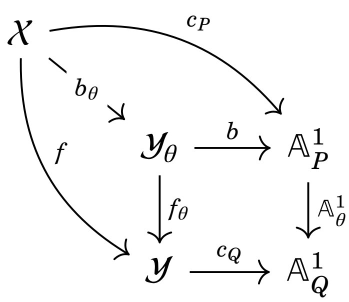
$$\Delta : \mathbb{N} \longrightarrow \mathbb{N}^d; 1 \longmapsto (1, \dots, 1)$$



Clearly, $\Delta : \mathbb{N} \longrightarrow \mathbb{N}^d$ is injective and $\operatorname{coker}(\Delta^{\mathrm{gp}})_{\mathrm{tor}} = 0$ and thus has an invertible order. Moreover, locally $\underline{b_\theta} = Id$, hence, by Theorem 1.5.8 the semistable model above is log étale.

1.5.3 Log flatness

Definition 1.5.10. A morphism of log schemes $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is *log flat* if fppf locally on X and Y, there exists a chart $\theta : P \longrightarrow Q$ for f with θ injective such that the morphism $\underline{b_\theta} : X \longrightarrow Y \times_{\mathbb{A}^1_Q} \mathbb{A}^1_P$ is a flat morphism of schemes.



Remark 1.5.11. 1. Log smooth morphisms are log flat. This follows directly from the chart criterion Theorem 1.5.8.

2. A strict morphism of log schemes is log flat if and only if the underlying morphism of schemes is flat. See [22, IV.4.1.2].

3. Log flat morphisms are stable under composition and base change in the category of log schemes.

Next, we state two results used in the subsequent chapters which states that for a log smooth integral morphism of fine log schemes, the underlying morphism of schemes is flat in the classical sense. Moreover, a log smooth integral morphism of fine log schemes is saturated if and only if the fibers of the underlying morphisms of schemes are reduced.

Theorem 1.5.12. *Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of locally noetherian fine log schemes, where f is locally of finite presentation.*

1. *If f is log flat and integral, then $\underline{f}$ is also flat.*

2. *If f is log smooth and integral, then the morphisms f and $\underline{f}$ are flat, and if in addition $\mathcal{X}$ and $\mathcal{Y}$ are saturated, then the fibers of $\underline{f}$ are Cohen–Macaulay.*

Proof. See [22, IV.3.5]. ■

Theorem 1.5.13. *Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a log smooth and integral morphism of fine saturated log schemes. Then f is a saturated morphism if and only if $\underline{f}$ has reduced fibers.*

Proof. See [22] IV.4.3.6.

■

1.6 An altenative viewpoint: DF log structures

As motivated earlier, log structures keep track of a divisor and its inclusion in the ambient scheme. This viewpoint is made concrete by the following definition first introduced in [18].

Definition 1.6.1. A *Deligne-Faltings (DF) log structure of rank r* on a scheme X is given by a pair $(\mathcal{M}_X, l)$ where $\mathcal{M}_X$ is a fine log structure on X and $l : \mathbb{N}^r \longrightarrow \overline{\mathcal{M}}_X$ is a morphism which étale locally lifts to a chart for $\mathcal{M}_X$.

A morphism $(\mathcal{M}_X, l) \longrightarrow (\mathcal{M}'_X, l')$ of DF log structures of rank r on X is given by a pair of morphisms $\mathcal{M}_X \longrightarrow \mathcal{M}'_X$ and $\mathbb{N}^r \longrightarrow \mathbb{N}^{r'}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{N}^r & \longrightarrow & \mathcal{M}_X \\ \downarrow & & \downarrow \\ \mathbb{N}^{r'} & \longrightarrow & \mathcal{M}'_X \end{array}$$

Theorem 1.6.2. *A DF log structure of rank r on a scheme X is equivalent to the data of a finite sequence of line bundles $\{\mathcal{L}_i\}_{i=1}^r$ and a finite sequence of sections $\gamma_i : \mathcal{L}_i \longrightarrow O_X, \forall 1 \leq i \leq r$.*

Sketch proof. Let $\pi : \mathcal{M}_X \longrightarrow \overline{\mathcal{M}}_X$ be the canonical map. Since, locally there exists a lifting of $l : \mathbb{N}^r \longrightarrow \overline{\mathcal{M}}_X$ to a chart $\bar{l} : \mathbb{N}^r \longrightarrow \mathcal{M}_X$, then $\pi^{-1}(l(\epsilon_i))$ is a O_X^* torsor, for every generator ϵ_i of $\mathbb{N}^r$. Hence, $\pi^{-1}(l(\epsilon_i))$ defines a line bundle $\mathcal{L}_i$ with sections given by $\pi^{-1}(l(\epsilon_i)) \hookrightarrow \mathcal{M}_X \longrightarrow O_X$.

Conversely, consider a sheaf $\mathcal{F}$ whose sections on a connected open set U consists of pairs (a, I), where $I := (I_1, \ldots, I_n) \in \mathbb{N}^n$ and a is a local generator of $\mathcal{L}^I := \mathcal{L}_1^{I_1} \otimes \cdots \otimes \mathcal{L}_n^{I_n}$. The sections $\{\gamma\}_i$ define a morphism $\gamma^I := \gamma^{I_1} \otimes \cdots \otimes \gamma^{I_n} : \mathcal{L}^I \longrightarrow O_X$. Let

$$\gamma : \mathcal{F} \longrightarrow O_X$$

be defined as

$$(a, I) \longmapsto \gamma^I(a)$$

Then define $\mathcal{M}_X$ to be the log structure associated to the morphism above. Moreover, we have a natural homomorphism $\mathcal{F} \longrightarrow \mathbb{N}^r$ mapping $(a, I) \longmapsto I$. Conversely, on a common trivialising neighbourhood of the line bundles $\mathcal{L}_i$, we have a morphism $\mathbb{N}^r \longrightarrow \mathcal{F}$ defined by $I \longmapsto (a_1^{I_1} \ldots a_n^{I_n}, I)$ where $a_i \in \mathcal{L}_i^{I_i}$ is a local generator in a common trivialising neighborhood of all the line bundles for every i. Thus, we have a morphism $\mathbb{N}^r \longrightarrow \overline{\mathcal{M}}_X$ which one verifies lifts to a local chart. See [22, II.1.7.3] for more details.

■

Remark 1.6.3. If $f : X \longrightarrow Y$ is a morphism of schemes, then a DF log structure on Y pulls back naturally to a DF log structure on X via f.

Example 1.6.4. Let D be an effective Cartier divisor on a scheme X equipped with the divisorial log structure $\mathcal{M}_X^D$ as in example 2 of Example 1.1.5. Then the inclusion $\mathcal{O}_X(-D) \hookrightarrow \mathcal{O}_X$ defines a DF log structure on X.

CHAPTER 2

Log stacks and moduli space of log curves

In this chapter, we use logarithmic structures to understand moduli problems, in particular the moduli stack of stable curves. Following Kato's philosophy, logarithmic structures serve as natural tools to compactify moduli spaces since they already include degenerate objects. After defining logarithmic (stable) curves in the first section, we proceed to define the various notions of logarithmic stacks. An important notion to obtain a full fledged logarithmic moduli problem, in particular studying algebraicity of the moduli stack, is defining 'minimal logarithmic objects' in the moduli space of interest. In fact, the minimal logarithmic objects capture the geometry of all objects parameterized by the moduli problem. Using the general theory of minimal logarithmic objects as in [13], we give a concrete construction of minimal logarithmic stable curves, following [17] in the last section.

2.1 Log curves: Definition and examples

Definition 2.1.1. A log curve over a fine saturated log scheme $\mathcal{Y}$ is a log-smooth, integral morphism of fs log schemes $f : \mathcal{X} \longrightarrow \mathcal{Y}$ such that the underlying morphism of schemes $\underline{f}$ is a proper family of curves with the geometric fibers connected and reduced.

Remark 2.1.2.

1. The first simple example of a log curve is given by a smooth proper family of curves $f : X \longrightarrow Y$ such that all the geometric fibers are connected and reduced, endowed with the trivial log structures.

2. The assumption of log smoothness and integrality in Definition 2.1.1 ensures that the underlying morphism of schemes is flat (see [22, IV.4.3.5]).

3. The assumption of log smoothness and reduced fibers in Definition 2.1.1 is equivalent to the fact that the morphism of log schemes $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is saturated (see [22, IV.4.3.6]).

   Counterexample: Consider a morphism of affine log schemes over k

$$\mathbb{A}^1_P \xrightarrow{f} \mathbb{A}^1_{\mathbb{N}}$$

   where P is the monoid generated by a, b, c satisfying the relation $2a + b = c$ and f is induced by the morphism

$$\mathbb{N} \longrightarrow P$$
$$1 \longmapsto c$$

   Then the morphism of k-algebras

$$k[x, y, z] \longrightarrow k[P]$$
$$x \longmapsto e_a$$
$$y \longmapsto e_b$$
$$z \longmapsto e_c$$

   where $e : P \longrightarrow k[P]$ is the canonical morphism from the monoid P to its associated monoidal k-algebra, gives the following underlying morphism of schemes

$$\underline{\mathbb{A}^1_P} \cong Spec\ k[x, y, z]/(x^2y - z) \longrightarrow \operatorname{Spec} k[z]$$

   which has a reduced fiber $\operatorname{Spec} k[x, y]/(x^2y)$ over the point $z = 0$. This happens since the morphism of monoids $\mathbb{N} \longrightarrow P$ given by $1 \longmapsto c$ is not saturated. Indeed the pushout $P \oplus_{\mathbb{N}} \mathbb{N}$ along the morphism $\mathbb{N} \longrightarrow \mathbb{N}$ given by $n \longmapsto 2n$ is generated by a, b, c' satisfying the relation $2a + b = 2c'$ in $\mathbb{N}$ and it is not saturated (see [22, I.4.8.10] for equivalent characterisations of saturated morphism of monoids).

4. The assumption of saturation in Definition 2.1.1 ensures that we have at worst nodal singularities in the underlying morphism of schemes.

   Counterexample: Consider the affine log scheme $\mathbb{A}^1_P$ over a field k with the trivial log structure, where P is the monoid generated by a, b satisfying the relation $2a = 3b$. Note that P is not saturated. The morphism of k-algebras

$$k[x,y] \longrightarrow k[P]$$
$$x \longmapsto e_a$$
$$y \longmapsto e_b$$

   where $e : P \longrightarrow k[P]$ is the canonical morphism from the monoid P to its associated monoidal k-algebra, gives the following underlying morphism of schemes

$$\underline{\mathbb{A}^1_P} \cong Spec\ k[x,y]/(x^2 - y^3) \longrightarrow \operatorname{Spec} k$$

   which has cuspidal singularities.

   Thus, the definition of log curve itself gives a good hold on the singularities of the underlying morphism of schemes.

5. Let A be a henselian local ring with uniformiser π. Endow $Y := \operatorname{Spec} A$ with a log structure $\mathcal{M}_Y$ determined by the chart $\mathbb{N} \longrightarrow A$ given by $1 \longmapsto \pi$. Endow the nodal curve $X := \operatorname{Spec} A[x,y]/(xy)$ with a chart $\mathbb{N}^2 \longrightarrow A[x,y]/(xy)$ given by $(m,n) \longmapsto x^m y^n$, i.e. the log structure $\mathcal{M}^D_X$ associated to the normal crosing divisor $D := \{xy = 0\}$. Then the diagonal morphism $\Delta : \mathbb{N} \longrightarrow \mathbb{N}^2$ gives a chart for the morphism of log schemes $f : (X, \mathcal{M}_X) \longrightarrow (Y, \mathcal{M}_Y)$. Hence, f is a log curve since Δ is an integral, saturated morphism of fs log schemes and it is log smooth by the chart criterion (see [22, IV.3.1.18 and IV.3.1.19] for the more general example of a morphism arising from a semi-stable reduction).

   This example serves as a prototype for many of the constructions below.

In view of Kato's philosophy that logarithmic objects contain degenerate objects, we will prove the following structure theorem of log curves in Theorem 2.3.1.

Theorem 2.1.3. *Let k be a separably closed field and consider $\mathcal{Y} = Spec\ k$ with the canonical log structure. Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a log curve such that $r_1, \ldots, r_l$ are the nodes of $\underline{\mathcal{X}}$, then there exist smooth points $s_1, \ldots, s_n$ of $\underline{\mathcal{X}}$ such that:*

$$\overline{\mathcal{M}}_{X/Y} \cong \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_l} \oplus \mathbb{N}_{s_1} \oplus \cdots \oplus \mathbb{N}_{s_n}$$

where $\mathbb{Z}_{r_i}$ and $\mathbb{N}_{s_i}$ are the skyscraper sheaves of monoids supported on the nodal points and the smooth points, respectively.

Remark 2.1.4. Thus, the smooth points $s_1, \ldots, s_n$ in the theorem above can be thought of as the n disjoint sections (or marked points) in the classical case of (pre)-stable curves.

In the classical set up, stable curves are characterized among pre-stable curves by the absence of infinitesimal deformations. Using the above Theorem 2.1.3, we can define *log stable curves of type (g,n)* as follows:

Definition 2.1.5. A log curve $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is called a *stable log curve of type* (g, n) if for any geometric point t of $\underline{\mathcal{Y}}$ and a characterisation of the characteristic sheaves of monoids as in the theorem:

$$\overline{\mathcal{M}}_{X/\mathrm{Spec}\, k(t)} \cong \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_{l(t)}} \oplus \mathbb{N}_{s_1} \oplus \cdots \oplus \mathbb{N}_{s_{n(t)}}$$

we have

1. $n(t) = n$ for every geometric point $t \in \underline{\mathcal{Y}}$,
2. the underlying family of curves has genus g,
3. The stability condition is captured by the vanishing of the global sections of the tangent bundle of all the geometric fibers, i.e:

$$H^0(\underline{\mathcal{X}}_t, T_{\underline{\mathcal{X}}_t}) = 0$$

   for every geometric point t in the base scheme.

Remark 2.1.6. In the section on log curves and its characterisations, we will indeed see a proof of the fact that the underlying scheme of a log stable curve of type (g, n) is indeed an $n-$pointed stable curve of genus g in the classical sense.

Thus, in view of the above remark, we can define the stack of log stable curves of type (g,n) with the forgetful morphism mapping to the base fine saturated log scheme:

$$\mathcal{LM}_{g,n} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}}$$

Precisely, we have

$$Obj(\mathcal{LM}_{g,n}) := \left\{ \begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathcal{Y} \end{array} \text{ stable log curves of type } (g,n) \right\}$$

$$Arr(\mathcal{LM}_{g,n}) := \left\{ \begin{array}{ccc} \mathcal{X}' & \xrightarrow{f'} & \mathcal{X} \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ \mathcal{Y}' & \xrightarrow{f} & \mathcal{Y} \end{array} \text{ cartesian diagram of stable log curves of type } (g,n) \right\}$$

Remark 2.1.7.

1. The fibered product above is considered in the category of fs log schemes. Since by assumption the log curves $\mathcal{X} \longrightarrow \mathcal{Y}$ are *integral* and *saturated*, the fibered product is indeed isomorphic to the one considered in the category of log schemes. Hence, in what follows, a commutative diagram of log curves is cartesian if and only if the underlying diagram of schemes is cartesian and the diagram of sheaves of monoids is co-cartesian (See [22, III.2.1] for more about fibered products in $\mathbf{LogSch}^{fs}$).

2. The standard pull back properties of fs log schemes, integral, saturated, log smooth morphisms, see Remark 2.1.6 and Corollary 2.3.3, imply that log stable curves of type (g,n) are stable under pull-back.

3. The sections of the underlying morphisms of log stable curves are automatically compatible. This follows from the last assertion in Corollary 2.3.3.

4. To conclude that $\mathcal{LM}_{g,n} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}}$ is a stack (see Definition A.0.1 for the definition of a stack), we assume étale descent in the category of fine saturated sheaves of monoids.

Moreover, we have the classical Deligne–Mumford–Knudsen moduli stack

$$\overline{\mathcal{M}}_{g,n} \longrightarrow (Sch)_{\text{ét}}$$

of stable curves. Thus, our next major goal would be the following:

Goal 2.1.8. Compare the stacks $\overline{\mathcal{M}}_{g,n} \longrightarrow (Sch)_{\text{ét}}$ and $\mathcal{LM}_{g,n} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}} \longrightarrow (Sch)_{\text{ét}}$, where the latter arrow is the forgetful functor, forgetting the log structure on the fs log schemes.

To this end, we equip the stack $\overline{\mathcal{M}}_{g,n} \longrightarrow (Sch)_{\text{ét}}$ with a *log structure* as defined in the next section and consider what *nice* log structures on the moduli stack of pointed stable curves help us compare it with the logarithmic stack.

As we will see below, this is equivalent to answering following the question: *what log structures can we equip a classical pointed stable curve with so that the geometric properties of every stable log curve can be read off from these special objects?*

2.2 Stacks equipped with a log structure

Definition 2.2.1 (Log stacks)**.** A stack (resp. algebraic space) $p : \mathfrak{X} \longrightarrow (Sch)_{\text{ét}}$ is said to be equipped with a log structure if there exists a morphism of 2-categories $\mathcal{M}_{\mathfrak{X}} : \mathfrak{X} \longrightarrow \mathbf{LogSch}^{fs}_{st,\ et}$ such that $\underline{\mathcal{M}_{\mathfrak{X}}} = p$, i.e. such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\mathcal{M}_{\mathfrak{X}}} & \mathbf{LogSch}^{fs}_{st,\text{ét}} \\ & \searrow^{p} & \downarrow \\ & & (Sch)_{\text{ét}} \end{array} \tag{2.2.1}$$

where the vertical morphism is the forgetful functor. Stacks (resp. algebraic spaces) equipped with a log structure are also called *log stacks*.

Moreover, if the stack $p : \mathfrak{X} \longrightarrow (Sch)_{\text{ét}}$ is algebraic (resp. DM), we call the log stack $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ a *log algebraic (resp. DM) stack*. (See Definition A.1.3, Definition A.1.6 and Definition A.1.7 for the definitions of an algebraic space, a DM stack and an algebraic stack respectively.

Remark 2.2.2. 1. The morphism $\mathbf{LogSch}^{fs}_{st,\text{ét}} \longrightarrow (Sch)_{\text{ét}}$ is a **CFG**. The distinction between the fibered categories $\mathcal{M}_X : \mathfrak{X} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}}$ and $p : \mathfrak{X} \longrightarrow$

$(Sch)_{\text{ét}}$ is that morphisms in the former are required to be cartesian over the category of schemes, while in the latter they are only required to be cartesian over the category of logarithmic schemes.

2. The definition of a log stack is compatible with the definition of log structure on a scheme when the stack represents an honest scheme.

   Indeed, let us suppose we have a commutative diagram with X a scheme

$$\begin{array}{ccc} h_X & \xrightarrow{\mathcal{M}_X} & \mathbf{LogSch}^{fs}_{st,\text{ét}} \\ & \searrow & \downarrow \\ & & (Sch)_{\text{ét}} \end{array}$$

   Then $\mathcal{M}_X(id_X)$ defines a log structure on X.

   Conversely, suppose we have a log scheme $\mathcal{X} = (X, \mathcal{M}_X)$, then a commutative diagram

$$\begin{array}{ccc} h_X & \xrightarrow{\mathcal{M}_X} & \mathbf{LogSch}^{fs}_{st,\text{ét}} \\ & \searrow & \downarrow \\ & & (Sch)_{\text{ét}} \end{array}$$

   can be defined by sending $f : Y \longrightarrow X$ in $h_X(Y)$ to the pullback of the log structure $\mathcal{M}_X$.

3. Analogously to the definition of a log structure on a scheme in the fppf (resp. Zariski) topology, we can also define a log structure on an algebraic (DM) stack in the fppf (resp. Zariski) topology.

Morphisms of log stacks

A 1-morphism of log stacks $(f, f') : (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \longrightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ is given by a morphism of stacks $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ and a natural transformation of functors $f' : \mathcal{M}_{\mathfrak{Y}} \longrightarrow f^*\mathcal{M}_{\mathfrak{X}}$ where $f^* = \mathcal{M}_{\mathfrak{Y}} \circ f$ such that the underlying morphism of schemes satisfies $\underline{f'}(x) = Id_x, \ \forall\ x \in Obj(\mathfrak{X})$. This definition of morphism of log stacks is compatible with the definition of morphism of log schemes. The 2-morphisms are defined in the usual manner. A 2-morphism $(f, f') \longrightarrow (g, g')$ is given by a natural transformation

$$\eta : f \longrightarrow g$$

with $\underline{\eta}(x) = Id_x, \ \forall\, x \in Obj(\mathfrak{X})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_X & \xrightarrow{f'} & \mathcal{M}_Y \circ g \\ & \searrow^{g'} & \downarrow \mathcal{M}_Y(x) \\ & & \mathcal{M}_Y \circ f \end{array}$$

Thus, we can consider the following 2-categories equipped with a log structure defined over $(Sch)_{\text{ét}}$:

$$\text{LOG 2-Cat}/(Sch)_{\text{ét}} \subset \text{LOG } \mathbf{CFG}/(Sch)_{\text{ét}} \subset \text{LOG stack}/(Sch)_{\text{ét}}$$

Analogously to the definition of strict morphisms of log schemes, we can define strict morphisms of log stacks. Precisely,

Definition 2.2.3. A 1-morphism of log stacks (resp. log algebraic spaces) $(f, f') : (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \longrightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ is is said to be *strict* if the natural transformation of functors

$$f' : \mathcal{M}_{\mathfrak{Y}} \longrightarrow f^* \mathcal{M}_{\mathfrak{X}}$$

is an isomorphism, where $f^* = \mathcal{M}_{\mathfrak{Y}} \circ f$ such that the underlying morphism of schemes satisfies $\underline{f'}(x) = Id_x, \ \forall\, x \in Obj(\mathfrak{X})$.

Another definition of log (algebraic/ DM) stack

Definition 2.2.4. Let $\mathfrak{X} \longrightarrow (Sch)_{\text{ét}}$ be an algebraic (resp. DM) stack. A log structure $\mathcal{M}_{\mathfrak{X}}$ on $\mathfrak{X}$ is given by a morphism of sheaves of monoids on the fppf (resp. étale) site of the stack $\alpha : \mathcal{M}_{\mathfrak{X}} \longrightarrow \mathcal{O}_{\mathfrak{X}}$ such that α preserves units. (The structure sheaf $\mathcal{O}_{\mathfrak{X}}$ is defined on the stack using atlasses in the standard manner.)

A 1-morphism of log stacks $(f, f') : (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \longrightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ is given by a morphism of stacks $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ and a natural transformation of functors $f' : \mathcal{M}_{\mathfrak{Y}} \longrightarrow f^* \mathcal{M}_{\mathfrak{X}}$ where $f^* = \mathcal{M}_{\mathfrak{Y}} \circ f$. The 2- morphisms are defined in the usual manner.

Similarly, we can define a log structure on an algebraic space using this alternative Definition 2.2.4.

Remark 2.2.5. The category of fine log structures on an algebraic stack (resp. DM) $\mathfrak{X}$ as in Definition 2.2.4 is equivalent to the datum of smooth (étale) charts $c_i : U_i \longrightarrow \mathfrak{X}$ together with pairs $(\mathcal{M}_i, \sigma_i^j)_{i,j}$, where $\mathcal{M}_i$ is a fs log structure on the schemes U_i, the σ_i^j are isomorphisms

$$\sigma_i^j : pr_{ij}^* \mathcal{M}_i \cong pr_{ji}^* \mathcal{M}_j$$

where $pr_{ij} : U_i \times_{\mathfrak{X}} U_j \longrightarrow U_j$ are the canonical projections such that the pairs $(\mathcal{M}_i, \sigma_i^j)_{i,j}$ also agree on triple intersections (i.e. they satisfy a co-cycle condition).

We have the following natural functors:

$$\begin{array}{ccc} & & \{(\mathcal{M}_i, \sigma_i^j)_{i,j}\} \\ & \overset{F}{\nearrow} & \downarrow G \\ \{\text{fs log str. on } \mathfrak{X} \text{ as in Definition 2.2.4}\} & \underset{H}{\searrow} & \\ & & \{(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \mid \text{as in Definition 2.2.1}\} \end{array}$$

The fact that the arrow F defines an equivalence of categories follows from [24, Proposition 5.6]. The arrow G defines an equivalence of categories in view of the 2-Yoneda's lemma. Thus, this rough sketch argues the equivalence of Definition 2.2.1 and Definition 2.2.4 of log algebraic (resp. DM) stacks.

Since we are interested in working with fine saturated sheaves of monoids, we define the following notion:

Definition 2.2.6. A log structure $\mathcal{M}_{\mathfrak{X}}$ on an algebraic stack $\mathfrak{X} \longrightarrow (Sch)_{\text{ét}}$ is said to be fine saturated (fs) if for each smooth atlas $U \longrightarrow \mathfrak{X}$, the pullback of the sheaf of monoids $\mathcal{M}_{\mathfrak{X}|U}$ is a fine saturated sheaf of monoids on the scheme U.

We now define log étale (resp. log unramified, log smooth) morphism of log algebraic stacks, analogously to the corresponding definition for morphisms of log schemes (see Definition 1.5.6).

Definition 2.2.7 (Log smooth, log unramified and log étale morphisms of log stacks)**.** A morphism of log algebraic stacks $f : (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \longrightarrow (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}})$ is said to be *log smooth* (resp. *log unramified*, resp. *log étale*) if for every n-th order log thickening $i : \mathcal{S} \longrightarrow \mathcal{T}$ (see Definition 1.5.4), there exists at least one (resp. at most one, resp. a unique) local lifting for the following diagram:

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{g} & (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \\ \downarrow{\scriptstyle i} & \overset{\tilde{g}}{\nearrow} & \downarrow{\scriptstyle f} \\ \mathcal{T} & \xrightarrow{h} & (\mathfrak{Y}, \mathcal{M}_{\mathfrak{Y}}) \end{array}$$

Analogously, we can define log étale (resp. log unramified, log smooth) morphisms of log algebraic spaces.

Recall that according to Goal 2.1.8, we want to equip $\overline{\mathcal{M}}_{g,n}$ with a nice log structure and compare its geometry with that of the stack $\mathcal{LM}_{g,n} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}}$ of log stable curves of type (g, n) over the category of fs log schemes in the strict étale topology. In order to compare these two stacks, we define the following morphism between 2-categories:

$$\Phi_{Cat} : \text{LOG 2-Cat}/(Sch)_{\text{ét}} \longrightarrow 2 - \text{Cat}/\mathbf{LogSch}^{fs}_{st,\text{ét}}$$

$$(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \longmapsto (\mathfrak{X}^{\log} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}})$$

where for each $\mathcal{S} = (S, \mathcal{M}_S) \in \mathbf{LogSch}^{fs}_{st,\text{ét}}$, the objects in the fiber category $\mathfrak{X}^{\log}_S$ are given by pairs

$$\{(x, f) \mid x \in Obj(\mathfrak{X}_S),\ f : \mathcal{S} \longrightarrow \mathcal{M}_{\mathfrak{X}}(x) \text{ is a strict morphism of log schemes, } \underline{f} = Id_{\underline{S}}\}$$

In other words, the log structure $\mathcal{M}_S$ on S and the log structure induced on S by the composition $\underline{S} \xrightarrow{x} \mathfrak{X} \xrightarrow{\mathcal{M}_{\mathfrak{X}}} \mathbf{LogSch}^{fs}_{st,\text{ét}}$ are isomorphic. A morphism between the objects of $\mathfrak{X}^{\log}$

$$((x, f) \mid f : \mathcal{S} \longrightarrow \mathcal{M}_{\mathfrak{X}}(x)\ ) \longrightarrow ((y, g) \mid g : \mathcal{T} \longrightarrow \mathcal{M}_{\mathfrak{X}}(y))$$

is given by $a : x \longrightarrow y$ and $b : \mathcal{S} \longrightarrow \mathcal{T}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{f} & \mathcal{M}_{\mathfrak{X}}(x) \\ \downarrow b & & \downarrow \mathcal{M}(a) \\ \mathcal{T} & \xrightarrow{g} & \mathcal{M}_{\mathfrak{X}}(y) \end{array}$$

Remark 2.2.8. It is not hard to verify that:

1. The association in Φ_{Cat} is functorial.
2. Φ_{Cat} maps to 2-morphisms in LOG 2-Cat$/(Sch)_{\text{ét}}$ to 2-morphisms in $2-$ Cat$/\mathbf{LogSch}^{fs}_{st,\text{ét}}$.
3. Φ_{Cat} restricts to morphisms of 2-categories

$$\Phi_{\mathbf{CFG}} : \text{LOG } \mathbf{CFG}/(Sch)_{\text{ét}} \longrightarrow \mathbf{CFG}/\mathbf{LogSch}^{fs}_{st,\text{ét}}$$

$$\Phi_{Stack} : \text{LOG Stack}/(Sch)_{\text{ét}} \longrightarrow \text{Stack}/\mathbf{LogSch}^{fs}_{st,\text{ét}}$$

   In other words, if $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ is a stack over $(Sch)_{\text{ét}}$ in the étale topology, then $\mathfrak{X}^{\log}$ is a stack over $\mathbf{LogSch}^{fs}_{st,\text{ét}}$ in the strict étale topology.

4. There is nothing sacred about étale or smooth topology here, the notations are used till now just in view of the examples we have in mind. But one needs to be careful while in the fppf topology where we need to use Olsson's results that there is a bijective correspondence of log structures on a scheme in either étale, smooth or fppf topology.

We are interested in studying our moduli problems in the logarithmic world, i.e. we are interested in the category Stack$/\mathbf{LogSch}^{fs}_{st,\text{ét}}$ of stacks over the category of fs log schemes in the strict étale topology. A priori it is not clear how to define the geometric phenomena of algebraicity and so on on stacks over $\mathbf{LogSch}^{fs}_{st,\text{ét}}$. One way out is to resort to geometric properties of objects in the category LOG Stack$/(Sch)_{\text{ét}}$. Thus, this boils down to answering the following question.

Question 2.2.9. *Under what conditions is a stack $\mathfrak{X} \longrightarrow \boldsymbol{LogSch}^{fs}_{st,ét}$ obtained from a stack $\mathfrak{Y} \longrightarrow (Sch)_{ét}$ equipped with a log structure? In other words, the question boils down to studying the essential image of the morphism $\Phi_{\boldsymbol{CFG}}$.*

Suppose $\mathfrak{X} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}}$ is a stack, we can canonically obtain a stack $\mathfrak{X}' \longrightarrow (Sxh)$ by composing $\mathfrak{X}$ with the forgetful functor $\mathbf{LogSch}^{fs}_{st,\text{ét}} \to (Sch)_{\text{ét}}$. More explicitly, the objects of the fiber category $\mathfrak{X}'_S$, $S \in (Sch)_{\text{ét}}$ are given by

$$\{(\mathcal{M}_S, a) \mid \mathcal{M}_S \in Log_S,\ a \in Obj(\mathfrak{X}_S)\}$$

In other words, we have a functorial association

$$\Psi_{\mathbf{CFG}} : \mathbf{CFG}/\mathbf{LogSch}^{fs}_{st,\text{ét}} \longrightarrow \text{LOG } \mathbf{CFG}/(Sch)_{\text{ét}}$$
$$\mathfrak{X} \longrightarrow \mathfrak{X}'$$

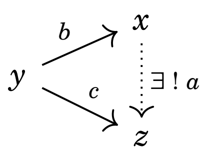
Thus, following Gillam's idea, one defines a full substack $\mathfrak{X}^{\min}$ of $\mathfrak{X}$ consisting of *minimal* objects such that equipping $\mathfrak{X}^{\min}$ with the canonical log structure obtained from $\Psi_{\mathbf{CFG}}$ gives an answer to Question 2.2.9 above. Thus, studying the essential image of $\Phi_{\mathbf{CFG}}$ is essentially the same as defining and studying the *minimal objects* in the stack $\mathfrak{X} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}}$.

As we will see in the following sections, defining the minimal objects of the stack $\mathcal{LM}_{g,n}$ gives a comparison with the classical stack $\overline{\mathcal{M}}_{g,n}$, as we needed in Goal 2.1.8.

2.2.1 Minimal objects in a stack over log schemes

As mentioned in the previous section, we want to study the essential image of $\Phi_{\mathbf{CFG}}$, which is encoded in the following result of Gillam:

Definition 2.2.10. An object z in the **CFG** $F : \mathfrak{X} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}}$ is said to be *minimal* if for any object x, y in $\mathfrak{X}$, the following diagram in $\mathfrak{X}$



with $\underline{F}(b) = \underline{F}(c) = Id$ has a unique completion/retract $a : x \longrightarrow z$

Theorem 2.2.11 (Gillam's main theorem). *A **CFG** $F : \mathfrak{X} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}}$ lies in the essential image of $\Phi_{\mathbf{CFG}}$ if and only if it satisfies the following conditions:*

1. *$\mathfrak{X}$ has enough minimal objects, i.e. for every $w \in \mathfrak{X}$ there exists a minimal object $z \in \mathfrak{X}$ and a morphism $i : w \longrightarrow z$ such that $\underline{F}(i) = Id$.*

2. *For any morphism $i : w \longrightarrow z$ in $\mathfrak{X}$ with z a minimal object, then $F(i)$ is a strict morphism of fs log schemes if and only if w is a minimal object.*

   If the above two conditions are satisfied, then we have an isomorphism of stacks over log schemes

$$(\mathfrak{X}^{\min,'})^{\log} \cong \mathfrak{X}$$

   where $\mathfrak{X}^{\min} \longrightarrow \boldsymbol{LogSch}^{fs}_{st,ét}$ is the full sub-category of $\mathfrak{X} \longrightarrow \boldsymbol{LogSch}^{fs}_{st,ét}$ consisting of the minimal objects and $\mathfrak{X}^{\min,'} \longrightarrow (Sch)_{ét}$ is the CFG obtained by forgetting the log structure on the base.

   In other words, the minimal objects of $\mathfrak{X}$ equipped with the log structure obtained from the canonical forgetful functor gives a comparison with the stack $\mathfrak{X} \longrightarrow \boldsymbol{LogSch}^{fs}_{st,ét}$.

Remark 2.2.12. 1. Condition (2) in the above theorem guarantees that if $\mathfrak{X} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}} \longrightarrow (Sch)_{\text{ét}}$ is a groupoid fibration, then so is $(\mathfrak{X}^{\min,'})^{\log}$. Similarly, if $F : \mathfrak{X} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}}$ is a stack, then $\mathfrak{X}^{\min} \hookrightarrow \mathfrak{X} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}} \longrightarrow (Sch)_{\text{ét}}$ is also a stack. In particular, $\mathcal{LM}^{\min}_{g,n} \longrightarrow (Sch)_{\text{ét}}$ is a stack.

2. For the stack $\mathcal{LM}_{g,n}$, the statement (1) above says that every $n-$pointed stable curve of genus g in the classical sense is the underlying morphism of schemes of a unique minimal stable log curve of type (g,n). In other words, the canonical morphism of stacks considered over $(Sch)_{\text{ét}}$ forgetting the log structures on the log curves

$$\mathcal{LM}^{\min}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

   is an isomorphism of stacks. Thus, $\overline{\mathcal{M}}^{\log}_{g,n} \cong \mathcal{LM}_{g,n}$ is the comparison we were seeking for and hence $\mathcal{LM}_{g,n}$ is a log DM stack.

3. Thus, an important step in logarithmic moduli problems is the explicit construction of the minimal objects.

Before proving Theorem 2.2.11, we prove the following lemma which asserts that *minimality is an open condition* for families of log objects obtained from a stack with a log structure. Precisely,

Lemma 2.2.13. *Let* $\mathfrak{X}^{\min} \longrightarrow \boldsymbol{LogSch}^{fs}_{st,ét} \longrightarrow (Sch)_{ét}$ *be the full sub-stack of a stack* $F : \mathfrak{X} \longrightarrow \boldsymbol{LogSch}^{fs}_{st,ét} \longrightarrow (Sch)_{ét}$ *consisting of the minimal objects. Further assume that* $\mathfrak{X}$ *satisfies the conditions* (1) *and* (2) *in Theorem 2.2.11. Then* $\mathfrak{X}^{\min}$ *is an open sub-stack of* $\mathfrak{X}$*.*

Proof. Consider a morphism $f : S \longrightarrow \mathfrak{X}$ with $S \in (Sch)_{\text{ét}}$ corresponding to an object x^f lying over S, i.e. $x^f \in \mathfrak{X}_S$. We need to show that $\mathfrak{X}^{\min} \times_{\mathfrak{X}} S \longrightarrow S$ is an open immersion of schemes (in the étale topology). Equivalently, we need to show that the locus $\{s \in S \mid x^f_s \text{ is minimal}\}$ is an open subscheme of S (in the étale topology); here x^f_s is the pull back (chosen up to isomorphism) of x via the morphism $\{s\} \hookrightarrow S$ in the category fibered in groupoids $\mathfrak{X} \longrightarrow (Sch)_{\text{ét}}$. Using condition (1) of Theorem 2.2.11, there exists a unique minimal object $z \in \mathfrak{X}^{\min}$ and a morphism $i : x^f \longrightarrow w$ such that $\underline{F}(i) = Id$. Let $\mathcal{M}^{x^f}$ and $\mathcal{M}^w$ be the log structures associated with the schemes $\underline{F}(x^f) = \underline{F}(w) = S$ respectively via F. Using condition (2) of Theorem 2.2.11 and the fact that $F : \mathfrak{X} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}}$ is a category fibered in groupoids, we have $\{s \in S \mid x^f_s \text{ is minimal}\} = \{s \in S \mid x^f_s \cong w_s\} = \{s \in S \mid \mathcal{M}^{x^f}|_s \cong \mathcal{M}^w|_s\}$.

Thus, if $\bar{s} \in S$ is a geometric point, then we need to show that the isomorphism $\mathcal{M}^{x^f}|_s \cong \mathcal{M}^w|_s$ can be extended over an étale neighbourhood of $\bar{s}$. Using [22, III.1.2.7], in an étale neighbourhood $S_0 \xrightarrow{\text{étale}} S$ of $\bar{s}$, we can find neat charts, i.e. $P \longrightarrow \mathcal{M}^{x^f}_{S_0}$ and $Q \longrightarrow \mathcal{M}^w_{S_0}$ are charts such that $P \cong \overline{\mathcal{M}^{x^f}|_{\bar{s}}}$ and $Q \cong \overline{\mathcal{M}^w|_{\bar{s}}}$. Hence, in the étale neighbourhood S_0, we have the isomorphism $\mathcal{M}^{x^f}_{S_0} \cong \mathcal{M}^w_{S_0}$ as required.

■

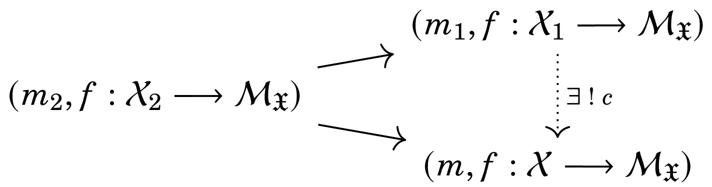
2.2.2 Proof of Gillam's minimality criterion

In this section, we give a sketch of the proof of Gillams's Theorem 2.2.11. We shall refer to [13, Section 2.5] for a detailed proof of the theorem.

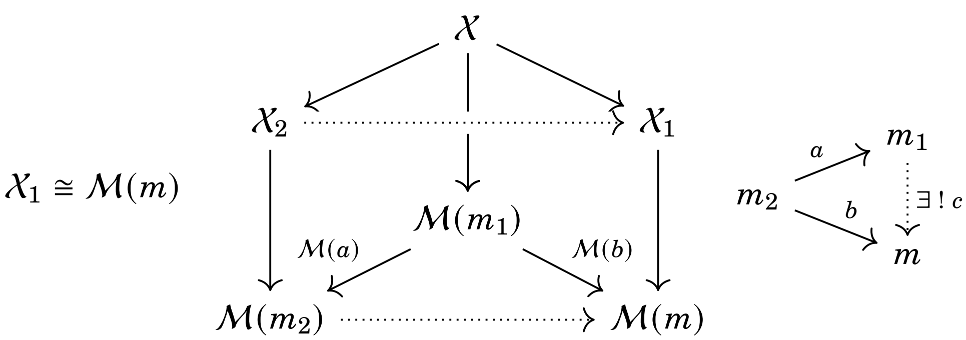
Lemma 2.2.14. *An object* $(m, f : \mathcal{X} \longrightarrow \mathcal{M}_{\mathfrak{X}})$ *of* $\mathfrak{X}^{\log}$ *is minimal if and only if* f *is an isomorphism. It follows from this that the essential image of* Φ_{CFG} :

LOG CFG$/(Sch)_{ét} \longrightarrow$ CFG/$\boldsymbol{LogSch}^{fs}_{st,ét}$ *satisfies the hypothesis* (1) − (2) *in Theorem 2.2.11.*

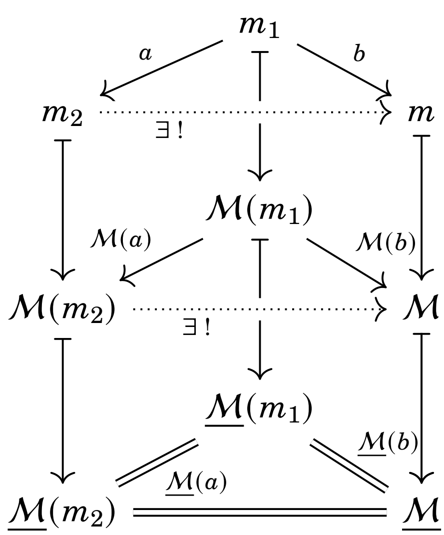
Proof: Assume that the morphism $f : \mathcal{X} \longrightarrow \mathcal{M}_{\mathfrak{X}}$ is an isomorphism. Suppose we have a diagram as in Definition 2.2.10, we need to find a completion c in the diagram below,



Thus, by definition we have commutative diagrams:



such that the underlying morphism of schemes is the identity. Consider the tower of maps in the fibered categories:



Since $\mathcal{X} \longrightarrow (Sch)$ is a **CFG** and $\underline{\mathcal{M}}(a) = Id$, the retraction maps are obtained from the above diagram.

Conversely, let $(m, f : \mathcal{X} \longrightarrow \mathcal{M}_{\mathfrak{X}})$ be a minimal object. Now consider the morphism $(m, id : \mathcal{M}(m) \longrightarrow \mathcal{M}(m)) \longrightarrow (m, f : \mathcal{X} \longrightarrow \mathcal{M}_{\mathfrak{X}})$. Since the object $(m, id : \mathcal{M}(m) \longrightarrow \mathcal{M}(m))$ is minimal by definition, we are done.

Thus, the required direction of Gillam's minimality result follows using the element $(m, id : \mathcal{M}(m) \longrightarrow \mathcal{M}(m))$ as the minimal element above.

Now we show the other direction of the theorem, namely,

Lemma 2.2.15. *If the stack $\mathfrak{X} \longrightarrow \textbf{LogSch}^{fs}_{st,ét}$ satisfies the conditions $(1) - (2)$, then it lies in the essential image of the functor Φ_{CFG}. In other words, it suffices to prove that we have an isomorphism of stacks $\mathfrak{X}^{\min,\log} \cong \mathfrak{X}$*

Proof: We first define a 1-morphism $\Psi : \mathfrak{X}^{\min,\log} \longrightarrow \mathfrak{X}$ as stacks over $\textbf{LogSch}^{fs}_{st,ét}$ by mapping $(x, f : \mathcal{X}$ to $\mathcal{M}_{\mathfrak{X}}) \longmapsto z_x$, where z_x is the unique element obtained from the following diagram:

$$\begin{array}{ccccc} \exists\,!\, z_x & \longmapsto & \mathfrak{X} & \longmapsto & \underline{\mathfrak{X}} \\ \downarrow & & \downarrow & & \downarrow \\ \underline{x} & \longmapsto & \mathcal{M}(x) & \longmapsto & \underline{\mathcal{M}}(x) \end{array} \qquad \text{such that } \underline{f} = Id$$

The action of morphisms $(x, f : \mathcal{X} \longrightarrow \mathcal{M}(x) \longrightarrow (y, g : \mathcal{Y} \longrightarrow \mathcal{M}(y))$ is the unique morphism obtained from the following towers in fibered categories

The 1-morphism $\phi : \mathfrak{X} \longrightarrow \mathfrak{X}^{\min,\log}$ in the other direction is defined as follows: For any $z \in \mathfrak{X}$, there exists a unique x_z and a morphism $f_z : z \longrightarrow x_z$, Thus, we map z to the object $(x_z, \mathcal{M}(f_z)) : \mathcal{M}(z) \longrightarrow \mathcal{M}(x_z)$. It is easy to verify that that both the morphisms are inverses. See [13, Section 2.5] for more details.

2.2.3 An equivalent definition of logarithmic algebraic stacks

In view of Theorem 2.2.11, we now give an equivalent definition of a logarithmic algebraic stack which is more consistent with the classical definition of an algebraic stack (see Definition A.1.7).

In this section, a log algebraic space refers to an algebraic space equipped with a log structure as defined in Definition 2.2.1.

Definition 2.2.16. Let $\mathfrak{X} \longrightarrow \mathbf{LogSch}^{fs}_{st,ét}$ be a stack that satisfies conditions (1) and (2) in Theorem 2.2.11. Then $\mathfrak{X} \longrightarrow \mathbf{LogSch}^{fs}_{st,ét}$ is said to be a *log algebraic stack* if it satisfies the following conditions:

1. The diagonal $\mathfrak{X} \longrightarrow \mathfrak{X} \times_{\mathbf{LogSch}^{fs}_{st,ét}} \mathfrak{X}$ is representable by a log algebraic space, i.e. for any morphism $\mathcal{U} \longrightarrow \mathfrak{X} \times_{\mathbf{LogSch}^{fs}_{st}} \mathfrak{X}$ with $\mathcal{U} \in \mathbf{LogSch}^{fs}_{st,ét}$ considered as a stack equipped with a log structure (see Remark 2.2.2), the fibre product $\mathcal{U} \times_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X}$ in the category of stacks over $\mathbf{LogSch}^{fs}_{st,ét}$ is representable by a log algebraic space and the morphism $\mathcal{U} \times_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X} \longrightarrow \mathcal{U}$ in the cartesian diagram

$$\begin{array}{ccc} \mathcal{U} \times_{\mathfrak{X}\times\mathfrak{X}} \mathfrak{X} & \longrightarrow & \mathcal{U} \\ \downarrow & & \downarrow \\ \mathfrak{X} & \longrightarrow & \mathfrak{X} \times \mathfrak{X} \end{array}$$

   is a strict morphism of log algebraic spaces (see Definition 2.2.3).

2. The stack $\mathfrak{X} \longrightarrow \mathbf{LogSch}^{fs}_{st,ét}$ admits a smooth strict cover, i.e. there exists $\mathcal{U} \in \mathbf{LogSch}^{fs}_{st,ét}$ and a morphism $\mathcal{U} \longrightarrow \mathfrak{X}$ such that for all $\mathcal{V} \in \mathbf{LogSch}^{fs}_{st,ét}$ and for all morphisms $\mathcal{V} \longrightarrow \mathfrak{X}$, the fibre product $\mathcal{U} \times_{\mathfrak{X}} \mathcal{V}$ in the category of stacks over $\mathbf{LogSch}^{fs}_{st,ét}$ is representable by a log algebraic space and the morphism $\mathcal{U} \times_{\mathfrak{X}} \mathcal{V} \longrightarrow \mathcal{V}$ in the cartesian diagram

$$\begin{array}{ccc} \mathcal{U} \times_{\mathfrak{X}} \mathcal{V} & \longrightarrow & \mathcal{V} \\ \downarrow & & \downarrow \\ \mathcal{U} & \longrightarrow & \mathfrak{X} \end{array}$$

is a smooth (see Definition 2.2.7), strict and surjective morphism of log algebraic spaces.

Similarly, one can define $\mathfrak{X} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}}$ to be a *log DM stack* by requiring that it admits an étale strict cover $\mathcal{U} \longrightarrow \mathfrak{X}$.

Remark 2.2.17. The equivalence of Definition 2.2.1 and Definition 2.2.16 of log algebraic stacks follows from the construction of $\mathfrak{X}^{\log}$ in Section 2.2 and Theorem 2.2.11.

Thus, in order to study a logarithmic moduli problem $\mathfrak{X} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}}$, we identify the minimal objects in the category $\mathfrak{X}$ and work with either of the definitions based on our convenience.

2.3 Characterization of log curves

In this section we will present the local structure theorem for log curves following [17] and [13].

Theorem 2.3.1 (Local structure of log curves)**.** *Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a log curve. Let x be a schematic point of X and $\overline{x}$ a geometric point corresponding to x. Let $y := \underline{f}(x) \in Y$ a schematic point with $\overline{y}$ a geometric point corresponding to y. Then, $\overline{\mathcal{M}}_{X/Y,\overline{x}}$ is isomorphic to either 0, $\mathbb{N}$ or $\mathbb{Z}$. Thus, we have the following three cases:*

Case 1: ***(Smooth points in the fiber)*** *If $\overline{\mathcal{M}}_{X/Y,\overline{x}} \cong 0$, then f is strict in an étale neighbourhood of $\overline{x}$ and $\underline{f}$ is smooth near x.*

Case 2: ***(Sections defining the marked points in the fiber)*** *If $\overline{\mathcal{M}}_{X/Y,\overline{x}} \cong \mathbb{N}$, then there exists a unique $p \in \overline{\mathcal{M}}_{X,\overline{x}}$ such that*

$$(\overline{f}^{@}_{x}, p) : \overline{\mathcal{M}}_{Y,\overline{y}} \oplus \mathbb{N} \cong \overline{\mathcal{M}}_{X,\overline{x}}$$

is an isomorphism of fine saturated monoids, where $\overline{f}^{@}_{x}$ is the induced morphism

$$\overline{f}^{@}_{x} : \overline{\mathcal{M}}_{Y,\overline{y}} \longrightarrow \overline{\mathcal{M}}_{X,\overline{x}}$$

Thus, we have the following short exact sequence:

$$0 \longrightarrow \overline{\mathcal{M}}^{\mathrm{gp}}_{Y,\overline{y}} \xrightarrow{\overline{f}^{@,\mathrm{gp}}_{x}} \overline{\mathcal{M}}^{\mathrm{gp}}_{X,\overline{x}} \underset{p}{\leftrightarrows} \mathbb{Z} \longrightarrow 0$$

Furthermore, there exists a lift $\overline{p} \in \mathcal{M}_U(U)$ of p in an étale neighbourhood $U \xrightarrow{\text{étale}} X$ of $\overline{x}$ and an étale neighbourhood $V \xrightarrow{\text{étale}} Y$ of $\overline{y}$ such that for the morphism

$$(f^{@}, \overline{p}) : f^{-1}\mathcal{M}_V \oplus \mathbb{N} \longrightarrow \mathcal{M}_U$$

induces an isomorphism of the the morphism of associated log structures. Moreover, the underlying morphism of schemes

$$U \longrightarrow V \times \mathbb{A}^1$$

is an étale morphism determined by

$$\mathcal{O}_V[t] \longrightarrow \mathcal{O}_U$$
$$t \longmapsto \alpha_U(\overline{p})$$

where $\alpha_U : \mathcal{M}_U \longrightarrow \mathcal{O}_U$ *is the log structure. In particular,* $\underline{f}$ *is smooth in an étale neighbourhood of* x *and* f *is strict away from the zero locus of* $\alpha_U(\overline{p}) \in \mathcal{O}_U(U)$.

Furthermore, there exists a strict étale covering $\{\mathcal{Y}_i \longrightarrow \mathcal{Y}\}_{i \in I}$ *such that for each* $i \in I$, *there exist sections* $\sigma_{i,j} : Y_i \longrightarrow X_i := X \times_Y Y_i$ *such that for each* $x \in X_i$ *with* $\overline{\mathcal{M}}_{X/Y,\overline{x}} \cong \mathbb{N}$, *the log structure of an étale neighbourhood of* $x \in X_i$ *is determined by the divisor defined by the sections* $\sigma_{i,j}$.

Thus, using the last claim, the points $x \in X$ *with* $\overline{\mathcal{M}}_{X/Y,\overline{x}} \cong \mathbb{N}$ *can be though of as 'smooth marked points'.*

Case 3: ***(Nodes in the fiber)*** *If* $\overline{\mathcal{M}}_{X/Y,\overline{x}} \cong \mathbb{Z}$, *then there is a unique morphism* $q_x : \mathbb{N} \longrightarrow \overline{\mathcal{M}}_{Y,\overline{y}}$ *(called the smoothing parameter of the node x) and a pair* $(p_1, p_2) \in \overline{\mathcal{M}}_{X,\overline{x}}$ *unique up to transposition such that the following diagram is co-cartesian in the category of fine saturated monoids:*

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{q_x} & \overline{\mathcal{M}}_{Y,\overline{y}} \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \overline{f}^{@}_x} \\ \mathbb{N}^2 & \xrightarrow{(p_1,p_2)} & \overline{\mathcal{M}}_{X,\overline{x}} \end{array} \tag{2.3.1}$$

The element $q_x(1)$ *is irreducible in the monoid* $\overline{\mathcal{M}}_{Y,\overline{y}}$. *Thus, we have a morphism of sets*

$$\{\textit{Nodal points over } \overline{y}\} \longrightarrow \overline{\mathcal{M}}_{Y,\overline{y}} = \mathrm{Irr}(\overline{\mathcal{M}}_{Y,\overline{y}})$$
$$x \longmapsto q_x(1)$$

Furthermore, there exist étale neighbourhoods U and V of x and y respectively such that there exist lifts $\overline{q} \in \mathcal{M}_V(V)$ of $q_x(1)$ and $(\overline{p}_1, \overline{p}_2) \in (\mathcal{M}_U(U))^2$ satisfying $\overline{p}_1+\overline{p}_2 = f^@ q$ and such that the morphism of log structures induced by

$$(f^@, \overline{p}_1, \overline{p}_2) : f^*\mathcal{M}_V \oplus_{\mathbb{N}} \mathbb{N}^2 \longrightarrow \mathcal{M}_U$$

is an isomorphism. Moreover, the underlying morphism of schemes

$$U \longrightarrow V \times_{\mathbb{A}^1} \mathbb{A}^2 = \operatorname{Spec} \mathcal{O}_V(V)[u,v]/(uv - \alpha_V(\overline{q}))$$

is an étale morphism determined by

$$\mathcal{O}_V(V)[u,v]/(uv - \alpha_V(\overline{q})) \longrightarrow \mathcal{O}_U(U)$$
$$u \longmapsto \alpha_U(\overline{p}_1)$$
$$v \longmapsto \alpha_U(\overline{p}_2)$$

where $\alpha_V : \mathcal{M}_V \longrightarrow \mathcal{O}_V$ is the log structure. In particular, x is a nodal singularity in the fiber over y and f is smooth away from the common zero locus of $\alpha_U(\overline{p}_1), \alpha_U(\overline{p}_2) \in \mathcal{O}_U(U)$.

Thus, this case justifies the motivation that log smooth objects contain degenerations like semi-stable reduction.

Remark 2.3.2 (See [14]). If $\overline{\mathcal{M}}_{X/Y,\overline{x}} \cong \mathbb{Z}$, then $\overline{\mathcal{M}}_{X,\overline{x}} \cong \overline{\mathcal{M}}_{Y,\overline{y}} \oplus_{\mathbb{N}} \mathbb{N}^2$ can be considered as a submonoid of $\overline{\mathcal{M}}_{Y,\overline{y}} \oplus \overline{\mathcal{M}}_{Y,\overline{y}}$ via the morphism:

$$i : \overline{\mathcal{M}}_{Y,\overline{y}} \oplus_{\mathbb{N}} \mathbb{N}^2 \longrightarrow \overline{\mathcal{M}}_{Y,\overline{y}} \oplus \overline{\mathcal{M}}_{Y,\overline{y}}$$
$$[(m,(a,b))] \longmapsto (m + a \cdot q_x(1), m + b \cdot q_x(1))$$

where q_x is as above in Case 3 of Theorem 2.3.1. Since $\overline{\mathcal{M}}_{Y,\overline{y}} \longrightarrow \overline{\mathcal{M}}_{X,\overline{x}}$ is an integral morphism of integral monoids, the pushout above is defined by the relations

$$(x_1, x_2) \sim (y_1, y_2)$$

if and only if there exists $c, d \in \mathbb{N}$ such that

$$x_1 + dq_x(1) = y_1 + cq_x(1)$$

$$x_2 + \Delta(d) = y_2 + \Delta(c)$$

Thus, using the above relations i is well defined. If $i[(m,(a,b))] = 0$, then $m + a.q_x(1) = m + b.q_x(1) = 0$. Since $q_x(1) \neq 0$, we have $a = b$. Moreover, $[(m,(a,a))] = [(m + aq_x(1),(0,0))] = [(0_{\overline{\mathcal{M}}_{Y,\bar{y}}},(0,0)]$ using the above relations. Hence, i is injective.

Moreover, the image of i can be explicitly described as

$$\overline{\mathcal{M}}_{X,\bar{x}} \cong \overline{\mathcal{M}}_{Y,\bar{y}} \oplus_{\mathbb{N}} \mathbb{N}^2 \cong \{(m_1,m_2) \in \overline{\mathcal{M}}_{Y,\bar{y}} \oplus \overline{\mathcal{M}}_{Y,\bar{y}} \mid m_1 - m_2 \in \mathbb{Z}q_x(1) \text{ in } \overline{\mathcal{M}}^{\mathrm{gp}}_{Y,\bar{y}}\}$$

An important consequence of the local structure theorem of log curves is the following corollary, which justifies that log curves behave nicely under pull back (see Remark 2.1.7).

Corollary 2.3.3. Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ and $f' : \mathcal{X}' \longrightarrow \mathcal{Y}'$ be log curves. Then a commutative diagram of log curves

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{h} & \mathcal{X}' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \mathcal{Y} & \xrightarrow{g} & \mathcal{Y}' \end{array}$$

is cartesian if the underlying diagram of underlying schemes is cartesian. In particular, we have an isomorphism of fine saturated sheaves of monoids

$$\overline{\mathcal{M}}_{X/Y,\bar{x}} \cong \overline{\mathcal{M}}_{X'/Y',\bar{x}'} \quad \text{for each } x \in X, x' \in X' \text{ such that } \underline{h}(x) = x'$$

The last assertion shows that the sections defining the marked points in the fibre of f (see Case 2 of Theorem 2.3.1) are mapped to the sections defining the marked points in the fibre of f', i.e. it guarantees that the sections of the underlying stable curves are compatible.

Proof. Since $f : \mathcal{X} \longrightarrow \mathcal{Y}$ and $f' : \mathcal{X}' \longrightarrow \mathcal{Y}'$ are saturated (in particular integral) morphisms of fine saturated log schemes, it suffices to show that the corresponding commutative diagram of log structures is co-cartesian. It is enough to show this at the level of stalks. Hence, without loss of generality, let $\underline{\mathcal{Y}'} = \underline{\mathcal{Y}} = \operatorname{Spec} k = \overline{y}$, for some separably closed field k be a geometric point in the base and $\underline{f'} = \underline{f}$. Let $\overline{x}$ be a geometric point lying over $\operatorname{Spec} k = \overline{y}$. Let us check the property case by case.

Case 1: $\overline{\mathcal{M}}_{X/Y,\overline{x}} \cong 0$. In this case we have a commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{Y',\overline{y}} & \longrightarrow & \overline{\mathcal{M}}_{Y,\overline{y}} \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ \overline{\mathcal{M}}_{X',\overline{x}} & \longrightarrow & \overline{\mathcal{M}}_{X,\overline{x}} \end{array}$$

As in Theorem 2.3.1, the log curve $\mathcal{X} \longrightarrow \mathcal{Y}$ is strict near x. Hence, our required diagram of monoids is co-cartesian.

Case 2: $\overline{\mathcal{M}}_{X/Y,\overline{x}} \cong \mathbb{N}$. In this case we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{\mathcal{M}}_{Y',\overline{y}} & \longrightarrow & \overline{\mathcal{M}}_{Y,\overline{y}} & \rightleftarrows & \mathbb{N} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{f'^{@}_x}} & & \downarrow{\scriptstyle \overline{f^{@}_x}} & & \| & & \\ 0 & \longrightarrow & \overline{\mathcal{M}}_{X',\overline{x}} & \longrightarrow & \overline{\mathcal{M}}_{X,\overline{x}} & \rightleftarrows & \mathbb{N} & \longrightarrow & 0 \end{array}$$

and it follows from Theorem 2.3.1 that the left hand side square in the diagram is co-cartesian.

Case 3: $\overline{\mathcal{M}}_{X/Y,\overline{x}} \cong \mathbb{Z}$. In this case we have a commutative diagram

$$\begin{array}{ccccc} \mathbb{N} & \xrightarrow{q_x} & \overline{\mathcal{M}}_{Y',\overline{y}} & \longrightarrow & \overline{\mathcal{M}}_{Y,\overline{y}} \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \overline{f'^{@}_x}} & & \downarrow{\scriptstyle \overline{f^{@}_x}} \\ \mathbb{N}^2 & \xrightarrow{(p_1,p_2)} & \overline{\mathcal{M}}_{X',\overline{x}} & \longrightarrow & \overline{\mathcal{M}}_{X,\overline{x}} \end{array}$$

and using Theorem 2.3.1, the left hand side square and the big rectangle are co-cartesian. Hence, the right hand side square is co-cartesian as required.

The co-cartesian diagrams in the above three cases induce an isomorphism of the cokernels of the vertical arrows:

$$\operatorname{coker} \overline{f^{@}_x} = \overline{\mathcal{M}}_{X/Y,\overline{x}} \cong \overline{\mathcal{M}}_{X'/Y',\overline{x}} = \operatorname{coker} \overline{f'^{@}_x} \quad \text{for each } x \in X.$$

Remark 2.3.4. The cokernels of the vertical (resp. horizontal) arrows in a cocartesian diagram of monoids are always isomorphic (i.e. we did not use that the morphisms are integral or saturated for concluding the last assertion).

■

Before proving the main structure theorem for log curves Theorem 2.3.1, we prove some facts in the category of fine saturated monoids.

Definition 2.3.5. Let $h : Q \hookrightarrow P$ be a monomorphism of integral, saturated and sharp (i.e. with trivial group of units) monoids.

1. An element $p \in P$ is said to be *Q-primitive* if $p = p_1 + q$ for some $q \in Q$, $p_1 \in P$ implies $q = 0$.
2. We set $I_Q \subseteq P$ to be the ideal $(Q \setminus \{0\}) + P = \{ q + p \mid q \in Q \setminus \{0\}, p \in P\}$.
3. An element $p \in P$ is said to be nilpotent with respect to the morphism h if $p \notin I_Q$ but $np \in I_Q$ for some $n \in \mathbb{Z}_{\geq 1}$.

Lemma 2.3.6 (Integral splitting lemma). *Let $h : Q \hookrightarrow P$ be a monomorphism of fine, saturated and sharp monoids. Then every $p \in P$ can be uniquely written as $p = p_1 + q$ for a Q-primitive element $p_1 \in P$, $q \in Q$. In other words, every element of P/Q has a unique Q-primitive representative in P.*

Proof. Let $p_1, p_2 \in P$. Set $p_1 \leq p_2$ if and only if there exists a $p \in P$ such that $p_1 + p = p_2$. Since P is finitely generated, there is no infinitely strictly decreasing chain in P. So, for every $p \in P$, there exists $p' \in P$ with the same image as p in P/Q such that p' is minimal with respect to this order. Hence, by definition p' is Q-primitive. Thus, $p = p'' + q, p' = p'' + q'$ for $p'' \in P, q, q' \in Q$. By Q-primitivity of p', we have $q' = 0$, so $p' = p''$ and hence $p = p' + q$ is the required unique decomposition.

■

Lemma 2.3.7. *Let $h : Q \hookrightarrow P$ be an integral injective morphism*[1] *of fine, sharp monoids without nilpotents. Then:*

[1]Any integral morphism $h : Q \longrightarrow P$ of fine and sharp monoids is automatically injective.

1. *If P is saturated then P^{gp}/Q^{gp} is a toric monoid, i.e. it is torsion free.*[2] *If $P^{gp}/Q^{gp} \cong \mathbb{Z}$ then P/Q is saturated. Thus, P/Q is isomorphic to either $0, \mathbb{N}$ or $\mathbb{Z}$.*

2. *If $P/Q \cong \mathbb{N}$, then there is a unique $p \in P$ such that*

$$(h,p) : Q \oplus \mathbb{N} \longrightarrow P$$

$$(q,n) \longmapsto h(q) + np$$

is an isomorphism of monoids.

3. *If $P/Q \cong \mathbb{Z}$, then there is a unique $q \in Q$ and $p_1, p_2 \in P$ unique up to transposition such that the following diagram is co-cartesian in the category of fine saturated monoids*

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{q} & Q \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle h} \\ \mathbb{N}^2 & \xrightarrow{(p_1,p_2)} & P \end{array}$$

Moreover, $q(1)$ is an irreducible (see Definition 1.2.10) element in Q.

Proof. (1) Let $p_1, p_2 \in P$ such that $p_1 - p_2 \notin Q^{gp}$ but $np_1 - np_2 \in Q^{gp}$ for some positive integer n, i.e. P^{gp}/Q^{gp} has a torsion element. Then there exist $q_1, q_2 \in Q$ such that $np_1 + q_1 = np_2 + q_2$. By Lemma 2.3.6, we can without loss of generality assume p_1, p_2 are Q-primitive and write $np_1 = a_1 + b, np_2 = a_2 + b$, where $b \in P$ is Q-primitive. Note that either a_1 or a_2 is non-zero. Indeed, if $a_1 = a_2 = 0$, then $n(p_1 - p_2) = 0$ and P is saturated, hence $p_1 - p_2 \in P$ and by symmetry, $p_2 - p_1 \in P$. Thus, $p_1 - p_2$ is a unit in P but P is a sharp monoid, which implies $p_1 = p_2$. But this contradicts $0 = p_1 - p_2 \notin Q^{gp}$. So, without loss of generality let $a_1 \neq 0$. Then $np_1 = a_1 + b \in I_Q$ (see Definition 2.3.5). Hence, we have $p_1 \in I_Q$, otherwise h would be nilpotent and by Q-primitivity, $p_1 \in Q$. This implies $b = 0$ so that $np_2 = a_2 \in Q$ and in particular $np_2 \in I_Q$, in contradiction with $p_1 - p_2 \notin Q^{gp}$.

If $P^{gp}/Q^{gp} \cong \mathbb{Z}$, then we need to show that P/Q is saturated, so that we can conclude that $P/Q \cong 0, \mathbb{Z}$ or $\mathbb{N}$ by [22, I.2.4.2] .

[2]If h is also assumed to be saturated (without finitely generated and sharpness assumptions), then P^{gp}/Q^{gp} being torsion free also holds true. See [22, I.4.8.11].

Since $P^{gp}/Q^{gp} \cong \mathbb{Z}$ and we are working over integral monoids, we can consider $P/Q \hookrightarrow \mathbb{Z}$. So, there exist $m, n \in P/Q \subseteq \mathbb{Z}$ with $(m, n) = 1$ such that $am - bn = \pm 1$ for some $a, b \in \mathbb{N}$. For $m \in P/Q$, let $p_m \in P$ be its unique Q-primitive represenatative. Now consider $ap_m - bp_n \in P^{gp}$.

We claim that $ap_m - bp_n \in P$ is an element of P. Then since $ap_m - bp_n$ maps to $\pm 1 \in P/Q \subseteq \mathbb{Z}$ and any submonoid containing ± 1 is saturated. Note that for any $t \in \mathbb{N}$, we have $tp_m = p_{mt}$ by the nilpotent free assumption. Thus, $m(ap_m - bp_n) = p_{amn} - p_{bmn}$. By Definition 2.3.5, we have

$$p_{amn} + q = p_{bmn} + p_m = bnp_m + p_m = (bn + 1)p_m$$

for some $q \in Q$.

Since h is nilpotent free, $q = 0$ and by saturatedness of P, we have $ap_m - bp_n \in P$ as required.

(2) The splitting $(h, p) : Q \oplus \mathbb{N} \longrightarrow P$ is easily obtained from the existence of Q-primitive representative as in Definition 2.3.5.

(3) Since $P/Q \cong \mathbb{Z}$ for each $n \in \mathbb{Z}$, there is a unique Q-primitive $p_n \in P$ mapping to $n \in \mathbb{Z}$. Set $q_0 := p_1 + p_{-1}$. Now we can define the map of monoids $\mathbb{N} \longrightarrow Q$ by $n \longmapsto nq_0$ and $\mathbb{N}^2 \longrightarrow P$ by $(m, n) \longmapsto p_m + p_{-n}$. As in the proof of (1), we have $p_n = np_1$. Thus, by the above definition the diagram in the claim commutes. Hence, it is enough to check that the canonical map

$$Q \oplus_{\mathbb{N}} (\mathbb{N} \oplus \mathbb{N}) \longrightarrow P$$

$$[q, (m, n)] \longmapsto q + p_m + p_{-n}$$

is an isomorphism. Surjectivity is clear and injectivity follows from the fact that the pushout of an integral morphism is integral. (See [22, I.4.6.2]).

Remark 2.3.8. Since $q = p_1 + p_2$ in the co-cartesian diagram in Lemma 2.3.7, q is not an irreducible element when considered as an element of P via the injective map $h : Q \longrightarrow P$.

■

With the above results at our disposal, we are now ready to present the **proof of the local structure theorem of log curves (Theorem 2.3.1)**.

Proof. For a log curve $f : \mathcal{X} \longrightarrow \mathcal{Y}$, let x be a schematic point of X and $\overline{x}$ a geometric point corresponding to x. Let $y := \underline{f}(x) \in Y$ a schematic point with $\overline{y}$ a geometric point corresponding to y.

Step 1: We first verify that the integral (and saturated) morphism of fine saturated monoids $\overline{\mathcal{M}}_{Y,\overline{y}} \xrightarrow{\overline{f_x^@}} \overline{\mathcal{M}}_{X,\overline{x}}$ satisfies the hypothesis of Lemma 2.3.7 with $h = \overline{f_x^@}$, $Q = \overline{\mathcal{M}}_{Y,\overline{y}}$ and $P = \overline{\mathcal{M}}_{X,\overline{x}}$.

1. Since $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is an integral morphism, $\mathcal{M}_{Y,\overline{y}} \xrightarrow{f_x^@} \mathcal{M}_{X,\overline{x}}$ is an integral morphism of monoids by definition. The canonical morphisms $\mathcal{M}_{Y,\overline{y}} \longrightarrow \overline{\mathcal{M}}_{Y,\overline{y}}$ and $\mathcal{M}_{X,\overline{x}} \longrightarrow \overline{\mathcal{M}}_{X,\overline{x}}$ are always integral. This implies that $\overline{\mathcal{M}}_{Y,\overline{y}} \xrightarrow{\overline{f_x^@}} \overline{\mathcal{M}}_{X,\overline{x}}$ is also integral. By the same same argument $\overline{\mathcal{M}}_{Y,\overline{y}} \xrightarrow{\overline{f_x^@}} \overline{\mathcal{M}}_{X,\overline{x}}$ is also saturated.

2. Since $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a saturated morphism of monoids, $\overline{\mathcal{M}}^{\text{gp}}_{X/Y,\overline{x}}$ is torsion free (see [22, III.2.5.5]). In particular, one does not need the monoids to be sharp as in Lemma 2.3.7.

3. Let us prove that $Q = \overline{\mathcal{M}}_{Y,\overline{y}} \xrightarrow{h=\overline{f_x^@}} \overline{\mathcal{M}}_{X,\overline{x}} = P$ is an exact injective morphism of integral monoids:

   Let $q_1 - q_2 \in Q^{\text{gp}}$ such that $p := h(q_1 - q_2) \in P$. Since h is an integral morphism, $h(q_1) = h(q_2) + p$ in P and there exists $p' \in P$ and $q'_i \in Q$ such that $h(q'_1) + p' = 0$, $p = h(q'_2) + p'$ and $q'_1 + q_1 = q'_2 + q_2$. Then p' is a unit in P and q'_1 is an unit in Q. Thus, $q_1 - q_2 = q'_2 - q'_1 \in Q$. Thus, h is an exact morphism.

   To show injectivity, let $a, b \in Q$ be such that $h(a) = h(b)$ holds. Then $h^{\text{gp}}(a - b) = 0 \in P$ implies $a - b \in Q$ by exactness of h, where h^{gp} is the induced morphism $Q^{\text{gp}} \longrightarrow P^{\text{gp}}$. Similarly $b - a \in Q$. Thus, $a - b \in Q^* = \{0\}$ (since Q is sharp). Hence, $a = b$.

4. Now we verify that the morphism $Q = \overline{\mathcal{M}}_{Y,\overline{y}} \xrightarrow{h=\overline{f_x^@}} \overline{\mathcal{M}}_{X,\overline{x}} = P$ has no nilpotents.

   Without loss of generality, let us assume $Y = \operatorname{Spec} k$ for some separably closed field k and replace X by an etale neighbourhood $(\operatorname{Spec} A, u) \xrightarrow{\text{étale}}$

(X, x) of x. Since f is log smooth, by the chart criterion for log smoothness (see [22, III.3.3.1]), there exist fine charts which fit into the following commutative diagram of fine saturated monoids

$$\begin{array}{ccccc} Q & \longrightarrow & \mathcal{M}_Y(Y) & \xrightarrow{\alpha_Y(Y)} & k \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ P_1 & \longrightarrow & \mathcal{M}_X(X) & \xrightarrow{\alpha_X(X)} & A \end{array}$$

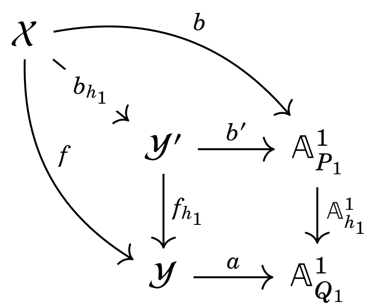
such that the morphism $k \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_1] \longrightarrow A$ is étale. Here, $k \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_1] \cong k[P_1]/I^k_{P_1}$, where $I^k_{P_1} := \{\Sigma_i a_i p_i \mid p_i \in (Q_1 \setminus \{0\}) + P_1\}$. Moreover, $k[P_1]/I^k_{P_1}$ is reduced since A is reduced ($Q \longrightarrow P$ is saturated) and $k \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_1] \longrightarrow A$ is étale. Since $P \longrightarrow \mathcal{M}_X(X)$ is a chart,the morphism $P_1 \longrightarrow \overline{\mathcal{M}}_{X,\bar{x}} = P$ is surjective and we have the follwing commutative diagram of monoids

$$\begin{array}{ccc} Q & \xhookrightarrow{h} & P_1 \\ & \searrow & \big\downarrow \\ & & P = \overline{\mathcal{M}}_{X,\bar{x}} \end{array}$$

Let $p \in P$ such that $p \notin Q \setminus \{0\} + P$ and there exists $n \in \mathbb{Z}_{\geq 1}$ such that $np \in Q - \{0\} + P$. Let $\bar{p}$ be a lift of $p \in P$ to P_1. Clearly, $\bar{p} \notin Q - \{0\} + P_1$ and $n\bar{p} \in Q - \{0\} + P_1$. Thus, $k \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_1] \cong k[P_1]/I^k_{P_1}$ is non-reduced, which is a contradiction. Hence, $h : Q \hookrightarrow P$ is without nilpotents.

5. Finally it remains to verify that $\overline{\mathcal{M}}^{\mathrm{gp}}_{X/Y,\bar{x}} \cong \mathbb{Z}$, for all $\bar{x} \in \mathrm{Supp}(\overline{\mathcal{M}}_{X/Y})$. Then by Lemma 2.3.7, $\overline{\mathcal{M}}_{X/Y,\bar{x}} \cong 0, \mathbb{N}$ or $\mathbb{Z}$:

   Without loss of generality, let us assume $Y = \operatorname{Spec} k$ for some separably closed field k. Since f is log smooth, by the chart criterion for log smoothness (see [22, III.3.3.1]), étale locally around $x \in X$ and $y \in Y$ (by an abuse of notation, we again denote the étale neighbourhoods by X and Y), there exist fine charts $b : \mathcal{X} \longrightarrow \mathbb{A}^1_{P_1}$ and $a : \mathcal{Y} \longrightarrow \mathbb{A}^1_{Q_1}$ with $h_1 : Q_1 \longrightarrow P_1$ injective such that they fits into a diagram:



where the square is cartesian with $\underline{\mathcal{Y}} = \operatorname{Spec} k$, $\underline{\mathcal{Y}'} = Spec(k \otimes_{\mathbb{Z}[Q_1]} \mathbb{Z}[P_1]) \cong Spec(k[P_1]/I^k_{P_1})$, $I^k_{P_1} := \{\Sigma_i a_i p_i \mid p_i \in (Q_1 \setminus \{0\}) + P_1\}$ and b' is a strict morphism and b_{h_1} is a log étale morphism. Moreover the chart $b : \mathcal{X} \longrightarrow \mathbb{A}^1_{P_1}$ can be chosen to be neat at $\bar{x}$, i.e. $P_1^{\mathrm{gp}} \cong \overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}}$ (see [22, II.2.3] for more details). Thus, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_1^{\mathrm{gp}} & \hookrightarrow & P_1^{\mathrm{gp}} & \longrightarrow & P_1^{\mathrm{gp}}/Q_1^{\mathrm{gp}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \cong & & \twoheaddownarrow & & \\ 0 & \longrightarrow & \overline{\mathcal{M}}^{\mathrm{gp}}_{Y,\bar{y}} & \hookrightarrow & \overline{\mathcal{M}}^{\mathrm{gp}}_{X,\bar{x}} & \longrightarrow & \overline{\mathcal{M}}^{\mathrm{gp}}_{X/Y,\bar{x}} & \longrightarrow & 0 \end{array}$$

Hence,we conclude that:

$$1 = \dim\ (X/Y) = \dim\ Spec(k[P_1]/I^k_{P_1}) = \operatorname{Rank}_{\mathbb{Z}}(P_1^{\mathrm{gp}}/Q_1^{\mathrm{gp}}) \geq \operatorname{Rank}_{\mathbb{Z}} \overline{\mathcal{M}}^{\mathrm{gp}}_{X/Y,\bar{x}} > 0$$

where the first inequality follows from the discussion in [22, Cor. I.2.3.8]. Thus, we have the assertion $\overline{\mathcal{M}}^{\mathrm{gp}}_{X/Y,\bar{x}} \cong \mathbb{Z}$, for all $\bar{x} \in \operatorname{Supp}(\overline{\mathcal{M}}_{X/Y})$.

Thus, the above verification together with Lemma 2.3.7 proves the assertions at the level of stalks in Case 2 and Case 3 of Theorem 2.3.1.

Proof of Case 1: The injective integral morphism $\overline{\mathcal{M}}_{Y,\bar{y}} \xrightarrow{\overline{f^@_x}} \overline{\mathcal{M}}_{X,\bar{x}}$ of fine, sharp and saturated monoids is an isomorphism since $0 = \overline{\mathcal{M}}_{X/Y,\bar{x}} = \operatorname{coker}\ \overline{f^@_x}$. Then $f^@_x : \mathcal{M}_{Y,\bar{y}} \longrightarrow \mathcal{M}_{X,\bar{x}}$ is surjective by the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{M}_{Y,\bar{y}} & \xrightarrow{f_x^@} & \mathcal{M}_{X,\bar{x}} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{Y,\bar{y}} & \xrightarrow{\cong} & \overline{\mathcal{M}}_{X,\bar{x}} \end{array}$$

If $f_x^@(p_1) = f_x^@(p_2)$ for some $p_1, p_2 \in \mathcal{M}_{Y,\bar{y}}$, then $\overline{f_x^@}$ is an isomorphism implies there exists $u \in \overline{\mathcal{M}}^*_{Y,\bar{y}}$ such that $p_2 = p_1 + u$. Then $f_x^@(p_2) = f_x^@(p_1) + f_x^@(u) = f_x^@(p_2) + f_x^@(u)$. $\mathcal{M}_{X,\bar{x}}$ is integral implies that $f_x^@(u) = 0$ and $\overline{f_x^@}$ is an isomorphism implies that $u = 0$. Hence, $p_1 = p_2$. Thus, $f_x^@$ is an isomorphism $\forall x \in X$ such that $\overline{\mathcal{M}}_{X/Y,\bar{x}} = 0$. Hence, f is strict in an étale neighbourhood of x (see [22, II.2.1.6] for more details) and hence the underlying morphism of schemes is smooth by [22, IV.3.1.6].

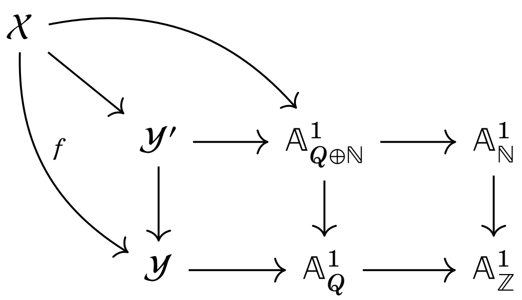
Proof of Case 2: We have shown that there exists a unique $p \in \overline{\mathcal{M}}_{X,\bar{x}}$ such that

$$(\overline{f}_x^@, p) : \overline{\mathcal{M}}_{Y,\bar{y}} \oplus \mathbb{N} \cong \overline{\mathcal{M}}_{X,\bar{x}}$$

is an isomorphism of fine saturated monoids. In an étale neighbourhood of x (which we by an abuse of notation denote by X), we can lift p to $\overline{p} \in \mathcal{M}_X(X)$ such that the morphism

$$(f^@, \overline{p}) : f^*\mathcal{M}_Y \oplus \mathbb{N} \longrightarrow \mathcal{M}_X$$

is an isomorphism. Let $\mathcal{Y} \longrightarrow \mathbb{A}^1_Q$ be a local chart in an étale neighbourhood (which we again denote by Y) of $y = \underline{f}(x)$. Since $Q \oplus \mathbb{N} \cong \overline{\mathcal{M}}_{Y,\bar{y}} \oplus \mathbb{N} \cong \overline{\mathcal{M}}_{X,\bar{x}}$, we can indeed consider $\mathcal{X} \longrightarrow \mathbb{A}^1_{Q+\mathbb{N}}$ to be a local chart in an étale neighbourhood (which we again call as X) of x. Thus, we have the following commutative diagram of log schemes



where the horizontal arrows in the left hand side square and $\mathcal{X} \longrightarrow \mathbb{A}^1_{Q\oplus\mathbb{N}}$ are strict. Hence, $\mathcal{X} \longrightarrow \mathcal{Y}'$ is a strict morphism of log schemes. Since f is log smooth and $\Omega^1_{\mathcal{X}/\mathcal{Y}}$ and $\Omega^1_{Q\oplus\mathbb{N}/Q}$ are free of rank one, $\mathcal{X} \longrightarrow \mathcal{Y}'$ is log étale and in particular, the underlying morphism of schemes

$$X \longrightarrow Y \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[Q\oplus\mathbb{N}] \cong Y \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[\mathbb{N}] \cong Y \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[t]$$

is étale.

Now we need to show the existence of local sections which define the marked points in the fiber. Set $X_{\mathbb{N}} := \{x \in X \mid \overline{\mathcal{M}}_{X/Y,\bar{x}} \cong \mathbb{N}\}$. Note that $X_{\mathbb{N}}$ is closed by [22, II.2.1.6]. We have the morphism $Q = \overline{\mathcal{M}}_{Y,\bar{y}} \xrightarrow{h=\overline{f_x^@}} \overline{\mathcal{M}}_{X,\bar{x}} = P$ as before and by the above arguments we have that $P \cong Q \oplus \mathbb{N}$ and we can indeed consider $\mathcal{X} \longrightarrow \mathbb{A}^1_P$ to be a fine chart in an étale neighbourhood of x which we denote again by X. Hence,we have the morphisms

$$\mathbb{N} \longrightarrow P \longrightarrow \mathcal{M}_X(X) \longrightarrow \mathcal{O}_{X_{\text{ét}}}(X) \longrightarrow \mathcal{O}_{X,\bar{x}}$$

The image of $1 \in \mathbb{N}$ in $\mathcal{O}_{X,\bar{x}}$ defines the closed subscheme $X_{\mathbb{N}}$. Thus, $X_{\mathbb{N}} \longrightarrow Y$ defines the necessary sections étale locally. One only needs to verify that the choice of étale local charts above is compatible with the construction of $X_{\mathbb{N}}$, which we omit here[3].

Proof of Case 3: We have shown that there is a unique $q_x : \mathbb{N} \longrightarrow \overline{\mathcal{M}}_{Y,\bar{y}}$ and elements $(p_1, p_2) \in \overline{\mathcal{M}}_{X,\bar{x}}$ unique up to order such that the following diagram is co-cartesian in the category of fine saturated monoids:

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{q_x} & \overline{\mathcal{M}}_{Y,\bar{y}} \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\overline{f_x^@}} \\ \mathbb{N}^2 & \xrightarrow{(p_1,p_2)} & \overline{\mathcal{M}}_{X,\bar{x}} \end{array}$$

In an étale neighbourhood of y (which by an abuse of notation we again denote by Y), we can lift $q_x(1)$ to $\overline{q} \in \mathcal{M}_Y(Y)$ and $p_1(1,0)$ and $p_2(0,1)$ to $\overline{p}_1$ and $\overline{p}_2$ in an

[3]See [22, III.1.7.3] for a similar discussion using Deligne-Faltings structures determined by closed subschemes

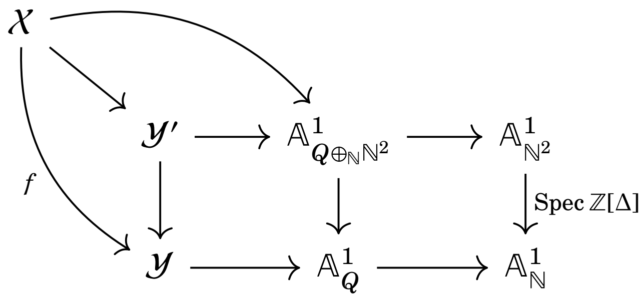
étale neighbourhood of x (denoted again by X for the sake of simplicity) satisfying

$$\overline{p}_1 + \overline{p}_2 = f^*\overline{q}$$

and such that the morphism of log structure induced by

$$(f^@, \overline{p}_1, \overline{p}_2) : f^*\mathcal{M}_Y \oplus_{\mathbb{N}} \mathbb{N}^2 \longrightarrow \mathcal{M}_X$$

is an isomorphism. Let $\mathcal{Y} \longrightarrow \mathbb{A}^1_Q$ be a local chart in an étale neighbourhood (which we again denote by Y) of $y = \underline{f}(x)$. Since $Q \oplus_{\mathbb{N}} \mathbb{N}^2 \cong \overline{\mathcal{M}}_{Y,\overline{y}} \oplus_{\mathbb{N}} \mathbb{N}^2 \cong \overline{\mathcal{M}}_{X,\overline{x}}$, we can indeed consider $\mathcal{X} \longrightarrow \mathbb{A}^1_{Q\oplus_{\mathbb{N}}\mathbb{N}^2}$ to be a local chart in an étale neighbourhood (which we again denote by X) of x. Thus, we have the following commutative diagram of log schemes



where both the horizontal arrows in the left hand side cartesian square and $\mathcal{X} \longrightarrow \mathbb{A}^1_{Q\oplus_{\mathbb{N}}\mathbb{N}^2}$ are strict.

Note that the right hand side square in the diagram above is cartesian since the forgetful functor from the category of A-algebras to the category of monoids $\mathcal{A}lg_A \longrightarrow \mathcal{M}on$ admits a left adjoint given by $\mathcal{M}on \longrightarrow \mathcal{A}lg_A$, $P \longrightarrow A[P]$.

Hence, $\mathcal{X} \longrightarrow \mathcal{Y}'$ is a strict morphism of log schemes. Since f is log smooth and $\Omega^1_{\mathcal{X}/\mathcal{Y}}$ and $\Omega^1_{Q\oplus_{\mathbb{N}}\mathbb{N}^2/Q}$ are free of rank one, $\mathcal{X} \longrightarrow \mathcal{Y}'$ is log étale and in particular, the underlying morphism of schemes

$$X \longrightarrow Y \times_{\mathrm{Spec}\,\mathbb{Z}[Q]} \mathrm{Spec}\,\mathbb{Z}[Q \oplus_{\mathbb{N}} \mathbb{N}^2] \cong Y \times_{\mathrm{Spec}\,\mathbb{Z}[N]} \mathrm{Spec}\,\mathbb{Z}[\mathbb{N}^2] \cong Y \times_{\mathbb{A}^1} \mathbb{A}^2$$

is étale, where $\mathbb{A}^1 \longrightarrow \mathbb{A}^2$ is induced by the diagonal morphism of monoids $\Delta : \mathbb{N} \longrightarrow \mathbb{N}^2$. The fact that $\Omega^1_{Q\oplus_{\mathbb{N}}\mathbb{N}^2/Q}$ are free of rank one follows from the fact that for a morphism of monoids $Q \longrightarrow P$, we have a natural isomorphism of $\mathcal{O}_{\operatorname{Spec}\mathbb{Z}[P]}$-modules $\Omega^1_{\mathbb{A}^1_P/\mathbb{A}^1_Q} \cong \mathbb{Z}[P] \widetilde{\otimes_{\mathbb{Z}} (P^{\mathrm{gp}}}/Q^{\mathrm{gp}})$. See [22, IV.1.1.4] for a proof of this fact.

■

2.4 Minimal logarithmic curves

Following [13], we will present the construction of minimal log objects in the category of log (stable) curves.

Consider a log curve:

$$\begin{array}{c} \mathcal{X} \\ \downarrow f \\ \mathcal{Y} \end{array}$$

Let $y \in Y$ be a scheme theoretic point in the underlying scheme Y.

Denote by $X_{\mathbb{Z}}^y := \{x \in \underline{f}^{-1}(y) \mid \overline{\mathcal{M}}_{X/Y,\bar{x}} \cong \mathbb{Z}\}$, the set of nodal points in the fiber over y. We recall from Theorem 2.3.1 that $X_{\mathbb{Z}}^y$ is a finite set. Further we have a unique element $q_x(1) \in \overline{\mathcal{M}}_{Y,\bar{y}}$ and unique elements (up to transposition) $p_1, p_{-1} \in \overline{\mathcal{M}}_{X,\bar{x}}$ such that the diagram

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{q_x} & \overline{\mathcal{M}}_{Y,\bar{y}} \\ \downarrow \Delta & & \downarrow f_x^{@} \\ \mathbb{N}^2 & \xrightarrow{(p_1,\, p_2)} & \overline{\mathcal{M}}_{X,\bar{x}} \end{array}$$

is co-cartesian in the category of monoids.

We have a map of sets

$$q : X_{\mathbb{Z}}^y \longrightarrow \overline{\mathcal{M}}_{Y,\bar{y}} = \operatorname{Irr}(\overline{\mathcal{M}}_{Y,\bar{y}})$$

$$x \longmapsto q_x(1)$$

Let P_y be the free monoid $\mathbb{N}^{X_{\mathbb{Z}}^y}$ over the finite set of nodal points over y, which is unique up to an automorphism of $\mathbb{N}^{X_{\mathbb{Z}}^y}$. The morphism

$$X_{\mathbb{Z}}^y \longrightarrow \mathbb{N}^{X_{\mathbb{Z}}^y} \cong P_y$$

$$x \longmapsto e_x$$

gives an adjunction

$$\mathrm{Hom}_{\mathcal{M}on}(P_y, \overline{\mathcal{M}}_{Y,\overline{y}}) \cong \mathrm{Hom}_{\mathcal{S}ets}(X^y_{\mathbb{Z}}, \overline{\mathcal{M}}_{Y,\overline{y}}) \cong \mathrm{Hom}_{\mathcal{S}ets}(X^y_{\mathbb{Z}}, \mathrm{Irr}(\overline{\mathcal{M}}_{Y,\overline{y}}))$$

Thus, by an abuse of notation, the morphism q above induces a morphism of monoids

$$q : P_y \longrightarrow \overline{\mathcal{M}}_{Y,\overline{y}}$$

We shall see in the next results (Theorem 2.4.1 and Theorem 2.4.4) that minimal log curves are characterized by the isomorphism of $q : P_y \longrightarrow \overline{\mathcal{M}}_{Y,\overline{y}}$, i.e. the log structure on the base is étale locally given by a neat chart whose generators are parametrized by the nodes in the fiber.

More specifically, the log structure on a minimal log curve would look as in the following schematic diagram.

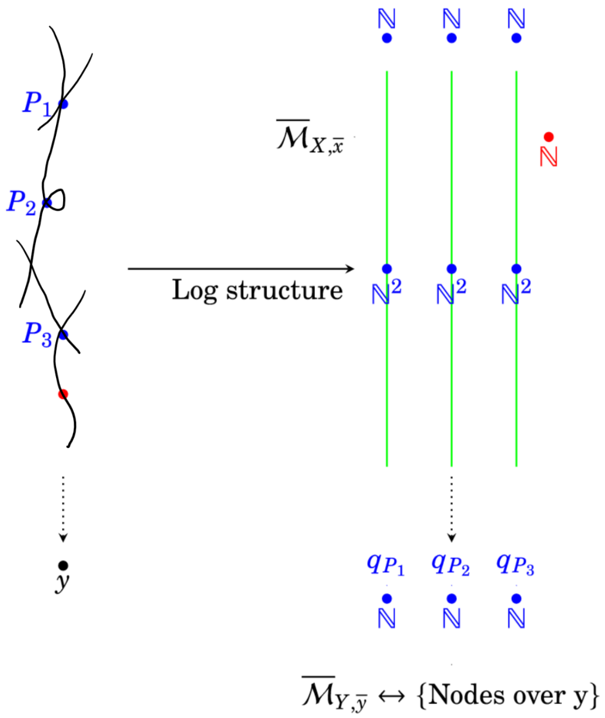
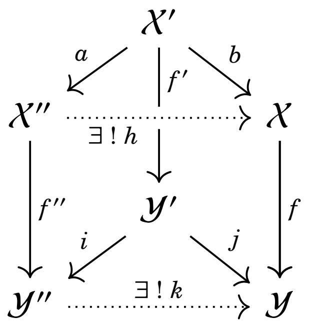


Figure 2.1: The left hand side picture is a nodal curve with three nodal points P_1, P_2, P_3 and one marked point (red). The right hand side is a schematic picture of the minimal log structure on the same nodal curve. The log structure on the base y corresponds to the free monoid generated by the nodal points over y. Each nodal point in the total space corresponds to $\mathbb{N}^2$ obtained from the diagonal $\Delta : \mathbb{N} \longrightarrow \mathbb{N}^2$. The marked point corresponding to $\mathbb{N}$ is labelled in red.

The minimal log structure obtained over the point y can be extended to an étale neighbourhood and we then have to check the gluing of these sheaves of monoids.

Theorem 2.4.1. *Let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a log curve such that the morphism of monoids $q : P_y \longrightarrow \overline{\mathcal{M}}_{Y,\bar{y}}$ is an isomorphism for all $y \in Y$. In other words, the smoothing parameters q_x of the nodes x lying over y generate $\overline{\mathcal{M}}_{Y,\bar{y}}$ as a free monoid. Then $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is a minimal log curve, i.e. it is a minimal object in the category of log curves $\mathcal{LM}_g$ as per Definition 2.2.10.*

Proof. Given any commutative triangle of log curves f, f' and f'' as below, we need to find unique morphisms $h : \mathcal{X}'' \longrightarrow \mathcal{X}$ and $k : \mathcal{Y}'' \longrightarrow \mathcal{Y}$ completing the diagram with $\underline{k} = Id$.



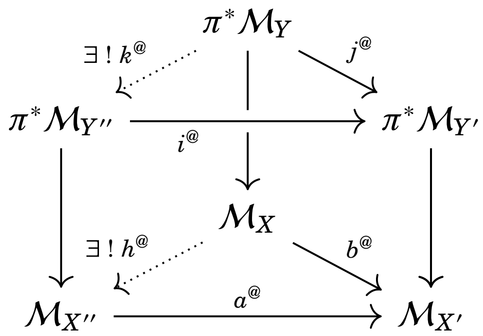
$\underline{i} = \underline{j} = Id$

All the squares in the above diagram are cartesian by definition of morphism of log curves in the category $\mathcal{LM}_g$. (Both the solid squares are cartesian automatically guarantee that the dotted square is cartesian if h and k exist.) In particular, the corresponding squares of underlying morphism of schemes is cartesian. Since $\underline{i} = \underline{j} = Id$, we can without loss of generality assume $\underline{a} = \underline{b} = Id$. Further we can assume that the underlying morphism of schemes $\underline{f} = \underline{f'} = \underline{f''} = \pi$.

Hence, we can now restrict our attention to only the log structures involved, i.e. it is enough to prove there exist unique morphisms of log structures

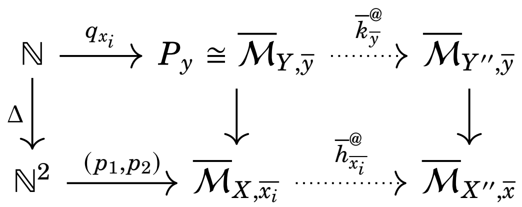
$$k^{@} : \mathcal{M}_Y \longrightarrow \mathcal{M}_{Y''},\ h^{@} : \mathcal{M}_X \longrightarrow \mathcal{M}_{X''}$$

completing the following co-cartesian diagram:



Step 1: Proving the uniqueness of the extension pair $(k^@, h^@)$.

Let $y \in Y$ be a schematic point and let $X^y_{\mathbb{Z}} = \{x_1, ..., x_n\}$ be the nodal points in the fiber over y. Assuming the existence of the pair $(k^@, h^@)$, we have the following commutative diagram with every square co-cartesian in the category of fine saturated sheaves of monoids for every $x_i \in X^y_{\mathbb{Z}}$ (the left hand side square is co-cartesian using Theorem 2.3.1 and the right hand square is co-cartesian by definition of morphism of log curves in $\mathcal{LM}_g$.



$\bar{y}, \bar{x_i}$ are geometric points

The morphism of fine saturated monoids $\overline{k}^@_{\bar{y}} : \overline{\mathcal{M}}_{Y,\bar{y}} \longrightarrow \overline{\mathcal{M}}_{Y'',\bar{y}}$ is determined uniquely by the images of $q_x(1)$ under $\overline{k}^@_{\bar{y}}$ for x varying over $X^y_{\mathbb{Z}}$ since $q : P_y \cong \overline{\mathcal{M}}_{Y,\bar{y}}$, where $P_y = \mathbb{N}^{X^y_{\mathbb{Z}}}$. If $\overline{k}^{@,1}_{\bar{y}}$ is another morphism completing the top dotted arrow, then using the structure Theorem 2.3.1 for the log curve $\mathcal{X}'' \longrightarrow \mathcal{Y}''$ we have $\overline{k}^{@,1}_{\bar{y}}(q_{x_i}(1)) = \overline{k}^@_{\bar{y}}(q_{x_i}(1)), \quad$ for all $x_i \in X^y_{\mathbb{Z}}$. (In view of the uniqueness of the morphism q_{x_i} for each commutative diagram as above corresponding to the nodal points x_i). Since the morphisms are determined by $q_{x_i}(1)$'s, we have $\overline{k}^@_{\bar{y}} = \overline{k}^{@,1}_{\bar{y}} : \overline{\mathcal{M}}_{Y,\bar{y}} \longrightarrow \overline{\mathcal{M}}_{Y'',\bar{y}}$. Since the morphisms $\overline{k}^@, \overline{k}^{@,1}$ agree on the stalk of the characteristic sheaves of monoids of the log structure on Y (i.e. the units of the monoids can be identified) for every $y \in Y$, the morphism $\overline{k}^@$ exists and is unique. More precisely, we are in the following situation:

If $F : P \longrightarrow Q$ is a morphism of monoids such that under F, $P^* \cong Q^*$, $F^{-1}(Q^*) =$

P^*, then if $\overline{F} = 0$, then $F = 0$

Let $p \in P$ such that $F(p) = q$ s.t $q \neq 0$. Since $\overline{F}(p) = 0$, we have $q \in Q^*$. Hence, by the identification of units, $q = F(p')$, for some $p' \in P^*$. Since we are assuming F is logarithmic, i,e satisfies the two identification of units, we have $Ker(F) = 0$. Hence, $p = p'$. Thus, we have the factorisation of F as

$$P^* \hookrightarrow P \xrightarrow{F} Q^* \cong P^* \cong \mathrm{img}(F) \hookrightarrow Q$$

The claim now follows from the integrality of Q.

Now, we need to show the uniqueness of

$$h^@ : \mathcal{M}_X \longrightarrow \mathcal{M}_{X''}$$

For any schematic point $x \in X, y := \pi(x) \in Y$, consider the following three cases as in Theorem 2.3.1.

Case 1: $\overline{\mathcal{M}}_{X/Y,\overline{x}} \cong 0$

As in Theorem 2.3.1, the log curve $\mathcal{X} \longrightarrow \mathcal{Y}$ is strict near x. Hence, the uniqueness of $k^@$ and the co-cartesian diagram gives the uniqueness of $\overline{h}^@_{\overline{x}}$

$$\begin{array}{ccc} \overline{\mathcal{M}}_{Y,\overline{y}} & \xrightarrow{\overline{k}^@_{\overline{y}}} & \overline{\mathcal{M}}_{Y'',\overline{y}} \\ \downarrow \cong & & \downarrow \\ \overline{\mathcal{M}}_{X,\overline{x}} & \overset{\overline{h}^@_{\overline{x}}}{\dashrightarrow} & \overline{\mathcal{M}}_{X'',\overline{x}} \end{array}$$

Case 2: $\overline{\mathcal{M}}_{X/Y,\overline{x}} \cong \mathbb{N}$

As in Theorem 2.3.1, the isomorphism $q : \mathbb{N}^{X^y_{\mathbb{Z}}} \longrightarrow \overline{\mathcal{M}}_{Y,\overline{y}}$ gives us:

$$\overline{\mathcal{M}}_{Y,\overline{y}} \oplus \mathbb{N} \cong \mathbb{N}^{X^y_{\mathbb{Z}}} \oplus \mathbb{N} \cong \overline{\mathcal{M}}_{X,\overline{x}}$$

The morphism $\overline{h}^@_{\overline{x}} : \overline{\mathcal{M}}_{X,\overline{x}} \cong \mathbb{N}^{X^y_{\mathbb{Z}}} \oplus \mathbb{N} \longrightarrow \overline{\mathcal{M}}_{X'',\overline{x}}$ is uniquely determined on the generators. Hence, the uniqueness of $\overline{k}^@_{\overline{y}}$ and the co-cartesian diagram below gives the uniqueness of $\overline{h}^@_{\overline{x}}$.

$$\begin{array}{ccc} \overline{\mathcal{M}}_{Y,\bar{y}} \cong \mathbb{N}^{X^y_{\mathbb{Z}}} & \xrightarrow{\overline{k}^{@}_{\bar{y}}} & \overline{\mathcal{M}}_{Y'',\bar{y}} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{X,\bar{x}} \cong \mathbb{N}^{X^y_{\mathbb{Z}}} \oplus \mathbb{N} & \overset{\overline{h}^{@}_{\bar{x}}}{\dashrightarrow} & \overline{\mathcal{M}}_{X'',\bar{x}} \end{array}$$

Case 3: $\overline{\mathcal{M}}_{X/Y,\bar{x}} \cong \mathbb{Z}$.

We have the following co-cartesian diagram as before:

$$\begin{array}{ccccc} \mathbb{N} & \xrightarrow{q_x} & P_y \cong \overline{\mathcal{M}}_{Y,\bar{y}} & \xrightarrow{\exists !\, \overline{k}^{@}_{\bar{y}}} & \overline{\mathcal{M}}_{Y'',\bar{y}} \\ \Delta\downarrow & & \downarrow & & \downarrow \\ \mathbb{N}^2 & \xrightarrow{(p_1,p_2)} & \overline{\mathcal{M}}_{X,\bar{x}} & \overset{\overline{h}^{@}_{\bar{x}}}{\dashrightarrow} & \overline{\mathcal{M}}_{X'',\bar{x}} \end{array}$$

Thus, $\overline{h}^{@}_{\bar{x}}$ is unique by the uniqueness of $\overline{k}^{@}_{\bar{y}}$ and Theorem 2.3.1. Hence, the uniqueness of $h^{@}$ follows as in the case of $h^{@}$.

Step 2: Constructing $k^{@} : \mathcal{M}_Y \longrightarrow \mathcal{M}''_Y$ étale locally around a point $y \in Y$.

By the assumption that the canonical morphism $q : P_y \longrightarrow \overline{\mathcal{M}}_{Y,\bar{y}}$ is an isomorphism, there exists an étale neighbourhood of y: $Y_1 \xrightarrow{\text{étale}} Y$ such that q extends to a local chart $q_1 : P_y \longrightarrow \mathcal{M}_Y(Y_1) = \mathcal{M}_{Y_1}(Y_1)$, i.e. $P_y^{\log} \cong \mathcal{M}_{Y_1}$. Similarly the canonical map $P_y \longrightarrow \overline{\mathcal{M}}_{Y'',\bar{y}}$ extends to a morphism of sheaves of monoids on an étale neighbourhood $Y_2 \xrightarrow{\text{étale}} Y$ of y: $q_2 : P_y \longrightarrow \mathcal{M}_{Y''}(Y_2) = \mathcal{M}_{Y_2}(Y_2)$. (This need not be a local chart for $\mathcal{Y}''$). Set $Y_3 := Y_1 \times_Y Y_2 \xrightarrow{\text{étale}} Y$. Moreover, we have the equality of morphisms of monoids

$$i^{@}_y \circ q_{2,y} = j^{@}_y \circ q_{1,y} : P_y \longrightarrow \overline{\mathcal{M}}_{Y',\bar{y}}$$

which extends to an equality on an étale neighbourhood $Y_4 \xrightarrow{\text{étale}} Y_3 \longrightarrow Y$. Hence, the logarithmification of q_2 pulled back to Y_4 gives us the necessary local extension $\mathcal{M}_{Y_4} \longrightarrow \mathcal{M}''_{Y_4}$. In other words, the following diagram is commutative:

$$\begin{array}{ccc} & \overset{q_2^{\log}}{\nearrow} & \mathcal{M}''_{Y_4} \\ \mathcal{M}_{Y_4} \cong P_y^{\log} & & \downarrow \\ & \searrow & \mathcal{M}'_{Y_4} \end{array}$$

By the construction of $q_2^{\log}$, it is not hard to see that these morphism of sheaves of monoids glue under étale pull-backs.

Step 3: Construct étale local morphisms $h^{@} : \mathcal{M}_X \longrightarrow \mathcal{M}_{X''}$ locally around each x lying in the fiber over $y = \pi(x)$, for every schematic point $y \in Y$ (Recall that $\pi : X \longrightarrow Y$ is the underlying morphism of schemes of the log curve $\mathcal{X} \longrightarrow \mathcal{Y}$).

As observed in Corollary 2.3.3, since we have cartesian diagrams of log curves, the relative characteristic sheaves of monoids are isomorphic, i.e.

$$\overline{\mathcal{M}}_{X/Y,\overline{x}} \;\cong\; \overline{\mathcal{M}}_{X'/Y',\overline{x}} \;\cong\; \overline{\mathcal{M}}_{X''/Y'',\overline{x}} \quad \text{for each } x$$

Thus, we need to complete the diagram in the three cases of Theorem 2.3.1:

Case 1: $\overline{\mathcal{M}}_{X/Y,\overline{x}} \;\cong\; 0$

By Theorem 2.3.1, the log curve $\mathcal{X} \longrightarrow \mathcal{Y}$ is strict in an étale neighbourhood of x. Thus, we have the following co-cartesian diagram:

$$\begin{array}{ccc} \mathbb{N}^{X^y_{\mathbb{Z}}} \cong \overline{\mathcal{M}}_{Y,\overline{y}} & \xrightarrow{\overline{k}^{@}_{\overline{y}}} & \overline{\mathcal{M}}_{Y'',\overline{y}} \\ \downarrow \cong & & \downarrow \\ \overline{\mathcal{M}}_{X,\overline{x}} & \xrightarrow{\overline{h}^{@}_{\overline{x}}} & \overline{\mathcal{M}}_{X'',\overline{x}} \end{array} \qquad \overline{h}^{@}_{\overline{x}} \text{ is as constructed in Step 2}$$

And, $\overline{k}^{@}_{\overline{y}}$ is the natural morphism $\mathbb{N}^{X^y_{\mathbb{Z}}} \longrightarrow \overline{\mathcal{M}}_{Y,\overline{y}}$. Since $\overline{\mathcal{M}}_{Y,\overline{y}} \;\cong\; \overline{\mathcal{M}}_{X,\overline{x}}$ in an étale neighbourhood of x, we can extend $\overline{k}^{@}_{\overline{y}}$ to the same étale neighbourhood.

Case 2: $\overline{\mathcal{M}}_{X/Y,\overline{x}} \cong \mathbb{Z}$

We have the following co-cartesian diagram as before:

$$\begin{array}{ccccc}
\mathbb{N} & \xrightarrow{q_x} & P_y \cong \overline{\mathcal{M}}_{Y,\overline{y}} & \xrightarrow{\exists !\, \overline{k}^{@}_{\overline{y}}} & \overline{\mathcal{M}}_{Y'',\overline{y}} \\
\Delta\downarrow & & \downarrow & & \downarrow \\
\mathbb{N}^2 & \xrightarrow{(p_1,p_2)} & \overline{\mathcal{M}}_{X,\overline{x}} & \xrightarrow{\overline{h}^{@}_{\overline{x}}} & \overline{\mathcal{M}}_{X'',\overline{x}}
\end{array}$$

Since $\overline{\mathcal{M}}_{X/Y,\overline{x}} \cong \overline{\mathcal{M}}_{X''/Y'',\overline{x}}$ for each x, Theorem 2.3.1 says that the outer commutative rectangle above is co-cartesian, hence the right hand side commutative square is co-cartesian and $\overline{h}^{@}_{\overline{x}}$ is the unique map to the push-out. Now we need to extend $\overline{h}^{@}_{\overline{x}}$ to an étale neighbourhood of x.

There exists a lift $h : P_y \longrightarrow \mathcal{M}_U(U)$ of the isomorphism $q : P_y \longrightarrow \overline{\mathcal{M}}_{Y,\overline{y}}$ to an étale neighbourhood $U \xrightarrow{\text{étale}} Y$ of $y = \pi(x)$ such that h is a local chart for $\mathcal{M}_Y$. Set $V := U \times_Y X \xrightarrow{\text{étale}} X$. Let $\overline{h} : P_y \xrightarrow{h} \mathcal{M}_U(U) \to \mathcal{M}_V(V)$. After intersecting with another étale neighbourhood of x, which by an abuse of notation we denote again by V, there exist liftings $\overline{p}_1, \overline{p}_2$ of the unique elements $p_1, p_2 \in \overline{\mathcal{M}}_{X,\overline{x}}$ that satisfies

$$\overline{p}_1 + \overline{p}_2 = \overline{h}(q^{-1}(q_x(1)))$$

where

$$q : P_y \longrightarrow \overline{\mathcal{M}}_{Y,\overline{y}}$$

and $q_x(1)$ is the smoothing parameter as in Theorem 2.3.1. The universal property for the co-cartesian diagram above gives a morpism

$$s := (\overline{h}, \overline{p}_1, \overline{p}_2) : \overline{\mathcal{M}}_{V,\overline{x}} \cong \overline{\mathcal{M}}_{X,\overline{x}} \cong P_y \oplus_{\mathbb{N}} \mathbb{N}^2 \longrightarrow \mathcal{M}_V(V)$$

Moreover, $s : \overline{\mathcal{M}}_{V,\overline{x}} \longrightarrow \mathcal{M}_V(V)$ can be considered as a chart since $\mathcal{M}_X$ is a saturated sheaf of monoid (see Theorem 1.4.4).

Consider the composition

$$\overline{k} : P_y \xrightarrow{h} \mathcal{M}_U(U) \xrightarrow{k^{@}_U(U)} \mathcal{M}_{U''}(U) \to \mathcal{M}_{V''}(V)$$

After intersecting with another étale neighbourhood of x, which by an abuse of notation we denote again by V, there exist liftings $\overline{t}_1, \overline{t}_2$ of the unique elements

$p_1, p_2 \in \overline{\mathcal{M}}_{X,\bar{x}}$ that satisfies

$$\bar{t}_1 + \overline{pt}_2 = \bar{h}(q^{-1}(q_x(1)))$$

Thus, we have a morphism

$$t := (\bar{k}, \bar{t}_1, \bar{t}_2) : \overline{\mathcal{M}}_{V'',\bar{x}} \cong \overline{\mathcal{M}}_{X'',\bar{x}} \cong P_y \oplus_{\mathbb{N}} \mathbb{N}^2 \longrightarrow \mathcal{M}_{V''}(V)$$

Moreover, we have the equality as in Case 1

$$a^{@}(V)t = b^{@}(V)s$$

which agrees on the stalks at x, hence in another étale neighbourhood which by an abuse of notation is $V \xrightarrow{\text{étale}} X$. Thus, we have a commuatative diagram:

$$\begin{array}{ccc} & \overset{s^{\log}}{\nearrow} & \mathcal{M}_{V''} \\ \mathcal{M}_V \cong (P_y \oplus_{\mathbb{N}} \mathbb{N}^2)^{\log} & & \downarrow \\ & \searrow & \mathcal{M}'_{Y_4} \end{array}$$

Thus, $s^{\log}$ induces the local extension of $h^{@} : \mathcal{M}_X \longrightarrow \mathcal{M}_{X''}$

Case 3: $\overline{\mathcal{M}}_{X/Y,\bar{x}} \cong \mathbb{N}$

As in Theorem 2.3.1, the isomorphism $q : \mathbb{N}^{X^y_{\mathbb{Z}}} \longrightarrow \overline{\mathcal{M}}_{Y,\bar{y}}$ gives us:

$$\overline{\mathcal{M}}_{Y,\bar{y}} \oplus \mathbb{N} \cong \mathbb{N}^{X^y_{\mathbb{Z}}} \oplus \mathbb{N} \cong \overline{\mathcal{M}}_{X,\bar{x}}$$

We have the following co-cartesian diagram as before:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{Y,\bar{y}} \cong \mathbb{N}^{X^y_{\mathbb{Z}}} & \xrightarrow{\bar{k}^{@}_{\bar{y}}} & \overline{\mathcal{M}}_{Y'',\bar{y}} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{X,\bar{x}} \cong \mathbb{N}^{X^y_{\mathbb{Z}}} \oplus \mathbb{N} & \overset{\bar{h}^{@}_{\bar{x}}}{\dashrightarrow} & \overline{\mathcal{M}}_{X'',\bar{x}} \end{array}$$

$\bar{k}^{@}_{\bar{y}}$ is as obtained in Step 2

And $\bar{h}^{@}_{\bar{x}}$ is the unique morphism defined on the generators. As in the previous case, we will extend this to an étale neighbourhood of x.

There exists a lift $h : P_y \longrightarrow \mathcal{M}_U(U)$ of the isomorphism $q : P_y \longrightarrow \overline{\mathcal{M}}_{Y,\bar{y}}$ to an étale neighbourhood $U \xrightarrow{\text{étale}} Y$ of $y = \pi(x)$ such that h is a local chart for $\mathcal{M}_Y$. Set $V := U \times_Y X \xrightarrow{\text{étale}} X$. Let $\bar{h} : P_y \xrightarrow{h} \mathcal{M}_U(U) \to \mathcal{M}_V(V)$. After intersecting with another étale neighbourhood of x, which by an abuse of notation we denote again by V, there exist liftings $\overline{p}$ of the unique elements $p \in \overline{\mathcal{M}}_{X,\bar{x}}$ The universal property for the co-cartesian diagram above gives a morpism

$$s := (\bar{h}, \overline{p}) : \overline{\mathcal{M}}_{V,\bar{x}} \cong \overline{\mathcal{M}}_{X,\bar{x}} \cong P_y \oplus \mathbb{N} \longrightarrow \mathcal{M}_V(V)$$

Moreover, $s : \overline{\mathcal{M}}_{U,\bar{x}} \longrightarrow \mathcal{M}_U(U)$ can be considered as a chart since $\mathcal{M}_X$ is a saturated sheaves of monoids (see Theorem 1.4.4).

Consider the composition

$$\bar{k} : P_y \xrightarrow{h} \mathcal{M}_U(U) \xrightarrow{k_U^@(U)} \mathcal{M}_{U''}(U) \to \mathcal{M}_{V''}(V)$$

After intersecting with another étale neighbourhood of x, which by an abuse of notation we denote again by V, there exist liftings $\bar{t}$ of the unique elements $p \in \overline{\mathcal{M}}_{X,\bar{x}}$.

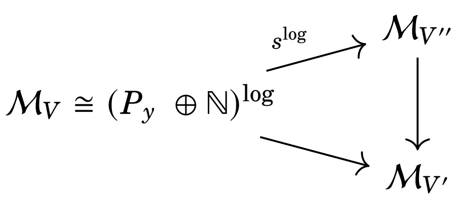
Thus, we have a morphism

$$t := (\bar{k}, \bar{t}) : \overline{\mathcal{M}}_{V'',\bar{x}} \cong \overline{\mathcal{M}}_{X'',\bar{x}} \cong P_y \oplus \mathbb{N} \longrightarrow \mathcal{M}_{V''}(V)$$

Moreover, we have the equality as in Case 2

$$a^@(V)t = b^@(V)s$$

which agrees on the stalks at x, hence in another étale neighbourhood which by an abuse of notation is $V \xrightarrow{\text{étale}} X$. Thus, we have a commuatative diagram:



Thus, $s^{\log}$ induces the local extension of $h^@ : \mathcal{M}_X \longrightarrow \mathcal{M}_{X''}$

Step 4: All the morphisms in Steps 2 and 3 glue together: This follows from the following lemma in [21, §3.7 and §3.8]. ■

Lemma 2.4.2. *Let* (A, m_A) *be a noetherian local henselian ring with* $s \in m_A$. *Let us denote by* $(R, m_R) := \frac{A[X,Y]}{(XY-s)}^{sh}$ *the strict henselization with respect to the ideal* (X, Y, m_A). *Under the canonical faithfully flat morphism, let*

$$\frac{A[X,Y]}{(XY-s)} \longrightarrow R$$
$$X \longmapsto x;\ Y \longmapsto y$$

1. *Suppose there exists* $x', y' \in R;\ s' \in A$ *such that* $x'y' = s'$ *and* $(x, y, m_A) = (x', y', m_A)$. *Then the pairs* (x, y) *and* (x', y') *dif and only ifer by units in* R.
2. *Suppose there exist units* $u_x, u_y \in R^*$ *such that* $x^l u_x = x^l,\ y^l u_y = y^l$ *for some* $l \in \mathbb{N}$ *and* $u_x u_y \in A^*$. *Then* $u_x = u_y = 1$.

Now we will check that the fibered category of log curves $\mathcal{LM}_g$ and hence the fibered category of stable log curves of type (g, n): $\mathcal{LM}_{g,n}$ satisfy the two conditions in Gillam's main result in Theorem 2.2.11.

First we shall show that every log curve is a pull-back of a unique minimal log curve (Uniqueness basically follows from the definition of minimality). The idea of the proof is to locally construct a log curve $\mathcal{X}^{\min} \longrightarrow \mathcal{Y}^{\min}$ and verify that the morphism $q : P_y \longrightarrow \overline{\mathcal{M}}_{Y,\bar{y}}$ is an isomorphism for every y and then use Theorem 2.4.1 to claim minimality.

Remark 2.4.3.

1. In Theorem 2.4.1, we do not get to see how the log structure on the total space looks like. This will be described in the next result.
2. The next result also gives a converse to Theorem 2.4.1, i.e. every minimal log curve satisfies the property that the morphism $q : P_y \longrightarrow \overline{\mathcal{M}}_{Y,\bar{y}}$ is an isomorphism for every y. Thus, we will get a complete classification of all minimal log curves (hence minimal stable log curves) characterized by the log structure on the base.

Theorem 2.4.4. *For any log curve $f : \mathcal{X} \longrightarrow \mathcal{Y}$, there exists a unique minimal log curve $f^{\min} : \mathcal{X}^{\min} \longrightarrow \mathcal{Y}^{\min}$ and a morphism of log curves*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{h} & \mathcal{X}^{\min} \\ \downarrow f & & \downarrow f^{\min} \\ \mathcal{Y} & \xrightarrow{g} & \mathcal{Y}^{\min} \end{array}$$

such that $\underline{g} = Id_{\underline{\mathcal{Y}}}$.

Proof. The uniqueness statement in the theorem follows once we verify that the constructed log curve $f^{\min} : \mathcal{X}^{\min} \longrightarrow \mathcal{Y}^{\min}$ is minimal.

The morphism g will be constructed étale locally on Y. Using the gluing Lemma 2.4.2 in the previous theorem, the minimal log structures will glue together. Let $y \in Y$ be any schematic point. Let $X^y_{\mathbb{Z}} = \{x_1, ..., x_n\}$ be the finite set of nodal points in the fiber over y as before. Similarly, set $X^y_{\mathbb{N}} := \{x \in f^{-1}(y) \mid \overline{\mathcal{M}}_{X/Y,\bar{x}} \cong \mathbb{N}\} = \{x_{n+1}, ..., x_{n+m}\}$.

Let $U \xrightarrow{\text{étale}} Y$ be an étale neighbourhood of y to which we extend the unique elements $q_{x_i} \in \overline{\mathcal{M}}_{Y,y}$ from Theorem 2.3.1 to $\overline{q}_i$. Let $U_i \xrightarrow{\text{étale}} X$ be an étale neighbourhood of x_i to which we extend the unique pair of elements $(p_{1,i}, p_{2,i})$ to $(\overline{p}_{1,i}, \overline{p}_{2,i})$ $\forall\, i \in \{1, ..., m\}$ and similarly lift the unique elements $p_i \in \overline{\mathcal{M}}_{X,\overline{x_i}}$ $\forall\, i \in \{m+1, ..., m+n\}$. Moreover, f is a strict morphism of log schemes away from the union (in the sense of Grothendieck's topology) of the U_i's.

Let $\mathcal{M}_{U^{\min}}$ be the log structure associated to the morphism of sheaves of monoids $q := (\overline{q}_1, ..., \overline{q}_n) : \mathbb{N}^n \longrightarrow \mathcal{M}_U(U) \longrightarrow \mathcal{O}_U(U)$. In other words, $\mathbb{N}^n \longrightarrow \mathcal{M}_{U^{\min}}(U)$ is a chart. Note that, the associated log structure $\mathcal{M}_{U^{\min}}$ is a fine saturated sheaves of monoids on U. Since the associated log structure is defined by the pushout out diagram

$$\begin{array}{ccc} q^{-1}(\mathcal{O}_U^*) & \longrightarrow & \mathcal{O}_U^* \\ \downarrow & & \downarrow \\ \mathbb{N}^n & \longrightarrow & (\mathbb{N}^n)^{\log} \end{array}$$

Hence, by the universal property of the co-cartesian diagram there exists a unique morphism

$$\mathcal{M}_{U^{\min}} \longrightarrow \mathcal{M}_U$$

For $i \in \{1, ..., n\}$, set $\mathcal{M}_{U_i^{\min}}$ to be the log structure associated to

$$h_i := (f^@ q, \overline{p}_{i,1}, \overline{p}_{i,2}) : \mathbb{N}^n \oplus_{\mathbb{N}} \mathbb{N}^2 \longrightarrow \mathcal{M}_{U_i}(U_i) \longrightarrow \mathcal{O}_{U_i}(U_i)$$

where $f^@ q : \mathbb{N}^n \xrightarrow{\underline{f}^* q} \underline{f}^* \mathcal{M}_U(U_i) \xrightarrow{f^@} \mathcal{M}_{U_i}(U_i) \to \mathcal{O}_{U_i}(U_i)$

By similar arguments as above, there exists a unique morphism

$$\mathcal{M}_{U_i^{\min}} \longrightarrow \mathcal{M}_{U_i}$$

The logarithmification of the morphism $\mathbb{N}^n \longrightarrow \mathbb{N}^n \oplus_{\mathbb{N}} \mathbb{N}^2$ determines the morphism $(\underline{f}^* \mathcal{M}_{U^{\min}})_{|U_i} \longrightarrow \mathcal{M}_{U_i^{\min}}$. In other words, the morphism of monoids $\mathbb{N}^n \longrightarrow \mathbb{N}^n \oplus_{\mathbb{N}} \mathbb{N}^2$ serves as a chart for $(\underline{f}^* \mathcal{M}_{U^{\min}})_{|U_i} \longrightarrow \mathcal{M}_{U_i^{\min}}$.

$$\begin{array}{ccc} \mathbb{N}^n & \longrightarrow & \mathbb{N}^n \oplus_{\mathbb{N}} \mathbb{N}^2 \\ \downarrow & & \downarrow \\ (\underline{f}^* \mathcal{M}_{U^{\min}})_{|U_i}(U_i) & \longrightarrow & \mathcal{M}_{U_i^{\min}}(U_i) \end{array}$$

For $i \in \{n+1, ..., m+n\}$ set $\mathcal{M}_{U_i^{\min}}$ to be the log structure associated to the morphism

$$h_i := (f^@ q, \overline{p}_i) : \mathbb{N}^n \oplus \mathbb{N} \longrightarrow \mathcal{M}_{U_i}(U_i) \longrightarrow \mathcal{O}_{U_i}(U_i)$$

By similar arguments as above, there exists a unique morphism

$$\mathcal{M}_{U_i^{\min}} \longrightarrow \mathcal{M}_{U_i}$$

The logarithmification of the morphism $\mathbb{N}^n \longrightarrow \mathbb{N}^n \oplus \mathbb{N}$ determines the morphism $(\underline{f}^* \mathcal{M}_{U^{\min}})_{|U_i} \longrightarrow \mathcal{M}_{U_i^{\min}}$. In other words, the morphism of monoids $\mathbb{N}^n \longrightarrow \mathbb{N}^n \oplus \mathbb{N}$ serves as a chart for $(\underline{f}^* \mathcal{M}_{U^{\min}})_{|U_i} \longrightarrow \mathcal{M}_{U_i^{\min}}$.

$$
\begin{array}{ccc}
\mathbb{N}^n & \longrightarrow & \mathbb{N}^n \oplus \mathbb{N} \\
\downarrow & & \downarrow \\
(\underline{f}^* \mathcal{M}_{U^{\min}})_{|U_i}(U_i) & \longrightarrow & \mathcal{M}_{U_i^{\min}}(U_i)
\end{array}
$$

Thus, we have a morphism of log schemes

$$
f_i^{\min} : (U_i, \mathcal{M}_{U_i^{\min}}) \longrightarrow (U, \mathcal{M}_{U^{\min}}) \;\; \forall\, i \in \{1, \ldots, m+n\}
$$

This is log étale by the torroidal chart criterion (see [22, IV.3.1.13]) and hence defines a log curve. The morphism f_i' is an integral morphism of log schemes since the underlying schemes in the construction remained same as in the original log curve.

Further, we have a commutative diagram of log schemes:

$$
\begin{array}{ccccccc}
 & & (U_i, \mathcal{M}_{U_i}) & & \longrightarrow & & (U_i, \mathcal{M}_{U_i^{\min}}) \\
 & \swarrow & \Big\downarrow {\scriptstyle f_i} & & & \swarrow & \Big\downarrow {\scriptstyle f_i^{\min}} \\
(X, \mathcal{M}_X) & & \longrightarrow & & (X, \mathcal{M}_{X^{\min}}) & & \\
\Big\downarrow & & (U, \mathcal{M}_U) & & \longrightarrow & & (U, \mathcal{M}_{U^{\min}}) \\
 & \swarrow & & & \Big\downarrow & \swarrow & \\
(Y, \mathcal{M}_Y) & & \longrightarrow & & (Y, \mathcal{M}_{Y^{\min}}) & &
\end{array}
$$

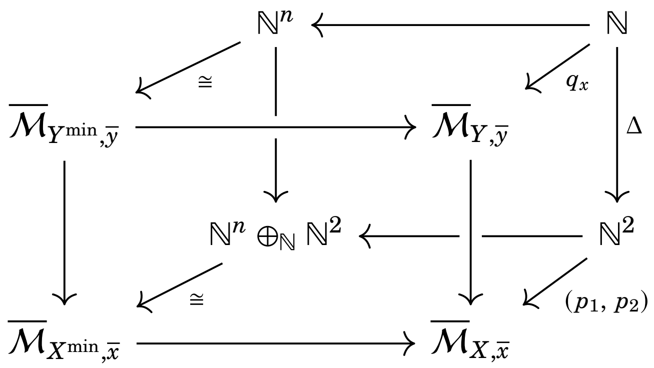
The commutativity of the square behind follows from the construction. More precisely, we need to check that the following diagram is co-cartesian in the category of monoids:

$$
\begin{array}{ccc}
\overline{\mathcal{M}}_{Y^{\min},\overline{y}} & \longrightarrow & \overline{\mathcal{M}}_{Y,\overline{y}} \\
\downarrow & & \downarrow \\
\overline{\mathcal{M}}_{X^{\min},\overline{x}} & \longrightarrow & \overline{\mathcal{M}}_{X,\overline{x}}
\end{array}
\qquad \text{for any } x \in X, y = f(x)
$$

We check this for x in each of the following cases:

Case 1: x does not lie in any of the U_i constructed above. In this case, f is a strict morphism, hence the diagram above is co-cartesian.

Case 2: $x \in U_i,\ i \in \{1, ..., m\}$. Then we have the following commutative diagram of monoids (using the explicit chart constructed for the minimal log curves):



Hence, the claim follows.

Case 3: $x \in U_i,\ i \in \{n+1, ..., m+n\}$. Then we have the following commutative diagram of monoids:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{Y^{\min},\overline{y}} \cong \mathbb{N}^n & \longrightarrow & \overline{\mathcal{M}}_{Y,\overline{y}} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{X^{\min},\overline{x}} \cong \mathbb{N}^n \oplus \mathbb{N} & \longrightarrow & \overline{\mathcal{M}}_{X,\overline{x}} \cong \overline{\mathcal{M}}_{Y,\overline{y}} \oplus \mathbb{N} \end{array}$$

Hence, the claim follows.

Remark 2.4.5. The co-cartesian diagrams in **Case 2** and **Case 3** indeed justify the schematic diagram in the beginning of this section. More elaborately, in **Case 2** each $\mathbb{N}$ in the base corresponding to each nodal point in the fibre is embedded diagonally in $\mathbb{N}^2$ whereas in **Case 3**, the base $\mathbb{N}$ has *no relations* with the log structure in the total space corresponding to the *marked points*.

The commutativity of the front diagram follows after checking that the minimal log structures constructed on the total space can be glued together. This holds true away from the union (in the sense of Grothendieck's topology) of the étale neighbourhoods U_i.

We need to show that we have an isomorphism

$$\mathcal{M}_{U_i^{\min}|U_i \times_U U_j} \cong \mathcal{M}_{U_j^{\min}|U_i \times_U U_j} \quad \forall i \neq j$$

If $(i, j) \in \{1, ..., m\}$ or $(i, j) \in \{m + 1, ..., m + n\}$, then the above isomorphism holds by using Theorem 2.3.1. We need to verify the isomorphism in the case $i \in \{1, ..., m\}$ and $j \in \{m + 1, ..., m + n\}$ (or vice versa).

Lastly we verify that the log curve constructed above satisfies that the morphism $q : P_y \longrightarrow \overline{\mathcal{M}}_{Y^{\min},\bar{y}}$ is an isomorphism for every y. Thus, using Theorem 2.4.1 we conclude that the log curve $f^{\min} : \mathcal{X}^{\min} \longrightarrow \mathcal{Y}^{\min}$ is in minimal. This would also justify Remark 2.4.3 that every minimal log curve satisfies that the morphism $q : P_y \longrightarrow \overline{\mathcal{M}}_{Y,\bar{y}}$ is an isomorphism for every y.

Here we claim that the morphism of monoids is given by

$$q : P_y \cong \mathbb{N}^{X^y_{\mathbb{Z}}} \longrightarrow \overline{\mathcal{M}}_{Y^{\min},\bar{y}}$$

$$e_{x_i} \longmapsto \bar{q}_{i,y}$$

where x_i are the schematic points in $X^{\min}$ lying over the fibers over $y \in Y^{\min}$ such that $X^y_{\mathbb{Z}} = \{x_1, ..., x_n\}$ be the finite set of nodal points in the fiber over y as before. Similarly, set $X^y_{\mathbb{N}} := \{x \in f^{-1}(y) \mid \overline{\mathcal{M}}_{X/Y,\bar{x}} \cong \mathbb{N}\} = \{x_{n+1}, ..., x_{n+m}\}$. By the explicit chart description of $f^{\min} : \mathcal{X}^{\min} \longrightarrow \mathcal{Y}^{\min}$, we have the following co-cartesian diagram

$$\begin{array}{ccccc}
\mathbb{N} & \xrightarrow{\bar{q}_{i,y}} & \overline{\mathcal{M}}_{Y^{\min},\bar{y}} & \longrightarrow & \overline{\mathcal{M}}_{Y,\bar{y}} \\
{\scriptstyle\Delta}\downarrow & & \downarrow & & \downarrow \\
\mathbb{N}^2 & \longrightarrow & \overline{\mathcal{M}}_{X^{\min},\bar{x}} & \longrightarrow & \overline{\mathcal{M}}_{X,\bar{x}}
\end{array}$$

Again by the structure theorem for log curves Theorem 2.3.1, the morphism q as defined above serves our purpose.

■

Remark 2.4.6.

1. The proof of the above theorem now fully justifies the schematic picture of minimal log structures.
2. It is evident from the proof of the above theorem that if $f : X \longrightarrow Y$ is a family of *smooth* curves and Y is the underlying scheme of a fine saturated log scheme $\mathcal{Y} = (Y, \mathcal{M}_Y)$, then the log structure on X is uniquely determined by the log structure $\mathcal{M}_Y$.

Now we verify that a minimal log curve satisfies condition 2 in Gillam's Theorem 2.2.11. Precisely,

Theorem 2.4.7. *Consider a morphism of log curves*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{h} & \mathcal{X}^{\min} \\ \downarrow f & & \downarrow f^{\min} \\ \mathcal{Y} & \xrightarrow{g} & \mathcal{Y}^{\min} \end{array}$$

with $\mathcal{X}^{\min} \longrightarrow \mathcal{Y}^{\min}$ minimal log curve, then $\mathcal{X} \longrightarrow \mathcal{Y}$ is minimal if and only if g is a strict morphism of log schemes.

Proof. Since $\mathcal{X}^{\min} \longrightarrow \mathcal{Y}^{\min}$ is a minimal log curve,

$$\begin{array}{ccc} \overline{\mathcal{M}}_{Y^{\min},\overline{y}} \cong P_y & \longrightarrow & \overline{\mathcal{M}}_{Y,\overline{g(y)}} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{X^{\min},\overline{x}} & \longrightarrow & \overline{\mathcal{M}}_{X,\overline{h(x)}} \end{array} \qquad \text{for any } x \in X, y = f(x)$$

If g is strict then clearly $\mathcal{X} \longrightarrow \mathcal{Y}$ is minimal. Conversely, $\mathcal{X} \longrightarrow \mathcal{Y}$ minimal says that $\overline{\mathcal{M}}_{Y,\overline{g(y)}} \cong P_{g(y)}$. Since we are working with integral monoids, isomorphism of characteristic monoids lifts to an isomorphism of monoids (See [22, I.4.1.2] for details).

∎

Remark 2.4.8. *An example of a log curve which is not minimal:*

Consider $f : X \longrightarrow Y$ where $Y = \operatorname{Spec} A$ is such that A is a Henselian local ring with uniformiser π. Endow Y with a log structure $\mathcal{M}_Y$ determined by the chart $\mathbb{N} \longrightarrow A$ given by $1 \longmapsto \pi$. Let the generic fiber X_0 of f be a smooth curve of aritheoremetic genus $g > 0$ and the special fiber X_π be the gluing up of two rational curves at two nodes. Endow X with the log structure associated to the normal crossing divisor X_π. Then $\mathcal{X} \longrightarrow \mathcal{Y}$ is a log curve with $\overline{\mathcal{M}}_{Y,\overline{y}} \cong \mathbb{N}$ while there are two nodes in the special fiber, hence not minimal by Theorem 2.4.1 and Theorem 2.4.4.

Thus, for the stack $\mathcal{LM}_{g,n}$, Gillam's Theorem 2.2.11 yields that every $n-$pointed stable curve of genus g in the classical sense is the underlying morphism of schemes of a unique minimal stable log curve of type (g,n). In other words, the canonical morphism of stacks considered over $(Sch)_{\text{ét}}$ forgetting the log structures on the log curves

$$\mathcal{LM}^{\min}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

is an isomorphism of stacks. Thus, inducing $\overline{\mathcal{M}}_{g,n}$, the classical moduli stack of $n-$pointed stable curves of genus g with the log stack structure obtained from the obvious log stack structure on $\mathcal{LM}^{\min}_{g,n} \longrightarrow (Sch)_{\text{ét}}$ gives us the isomorphism:

$$\begin{array}{ccc} & & \mathcal{LM}_{g,n} \\ & \overset{\cong}{\nearrow} & \downarrow \\ \overline{\mathcal{M}}^{\log}_{g,n} & \searrow & \\ & & \mathbf{LogSch}^{fs}_{st,\text{ét}} \end{array}$$

This is the comparison we were seeking for. Hence, $\mathcal{LM}_{g,n} \longrightarrow \mathbf{LogSch}^{fs}_{st,\text{ét}}$ is a logarithmic Deligne-Mumford stack.

Remark 2.4.9. The construction of minimal logarithmic structures has been generalised to the case of semi-stable varieties using similar techniques as in the construction of minimal logarithmic curves in [24, Section 4.3]. Generalising the definition of a log curve (see Definition 2.1.1), M. Olsson defines the notion of an *essentially semi-stable* variety. Precisely,

A log smooth proper morphism of fs log schemes $f : \mathcal{X} \longrightarrow \mathcal{Y}$ is said to be *essentially semi-stable* if for every geometric point $\overline{x} \in X$ and $\overline{y} = \underline{f}(\overline{x})$, there exist isomorphisms $\overline{\mathcal{M}}_{X,\overline{x}} \cong \mathbb{N}^{r+s}$ and $\overline{\mathcal{M}}_{S,\overline{s}} \cong \mathbb{N}^{r}$ for some positive integers r and s. Moreover, the morphism

$$\overline{\mathcal{M}}_{S,\overline{y}} \cong \mathbb{N}^{r} \longrightarrow \overline{\mathcal{M}}_{X,\overline{x}} \cong \mathbb{N}^{r+s}$$

is determined by

$$\epsilon_i \longmapsto \begin{cases} \epsilon_i \; i \neq r \\ \epsilon_m + \cdots + \epsilon_{m+n} \; i = r \end{cases}$$

where ϵ_i are the standard basis vectors. Recall from Example 1.2.19 that the map above is integral and saturated.

Analogous to the structure Theorem 2.3.1, M. Olsson proves that the singularities of an essentially semi-stable variety are at worst normal crossings in the classical sense. For a singular point x lying over y, we have a co-cartesian diagram of fs monoids as in the case of Theorem 2.3.1.

$$\begin{array}{ccc} \mathbb{N}^r & \xrightarrow{q_x} & \overline{\mathcal{M}}_{Y,\overline{y}} \\ \downarrow & & \downarrow \overline{f}^{@}_x \\ \mathbb{N}^{r+s} & \xrightarrow[(p_1,\dots,p_{r+s})]{} & \overline{\mathcal{M}}_{X,\overline{x}} \end{array}$$

Furthermore, there exist étale neighbourhoods U and V of x and y respectively such that there exist lifts $\overline{q} \in \mathcal{M}_V(V)$ of $q_x(1)$ and $(\overline{p}_1, \dots \overline{p}_{r+s}) \in (\mathcal{M}_U(U))^{r+s}$ satisfying $\overline{p}_1 + \cdots + \overline{p}_{r+s} = f^{@}q$. Moreover, the underlying morphism of schemes

$$U \longrightarrow V \times_{\mathbb{A}^r} \mathbb{A}^{r+s} = \operatorname{Spec} \mathcal{O}_V(V)[u_r, \dots, u_{r+s}]/(u_r \cdots u_{r+s} - \alpha_V(\overline{q}))$$

is an étale morphism, where $\alpha_V : \mathcal{M}_V \longrightarrow \mathcal{O}_V$ is the log structure and $u_1, \dots, u_{r+s}$ are the affine co-ordinates.

Moreover, we have a morphism of sets

$$q : \{\text{Nodal points over } \overline{y}\} \longrightarrow \overline{\mathcal{M}}_{Y,\overline{y}} = \operatorname{Irr}(\overline{\mathcal{M}}_{Y,\overline{y}})$$
$$x \longmapsto q_x(1)$$

Analogous to Theorem 2.4.1, M. Olsson proves that an essentially semi-stable variety is minimal in the sense of Definition 2.2.11 if and only if the morphism q above is an isomorphism and also verifies that the category of essentially semi-stable varieties has enough minimal objects in the sense of Theorem 2.2.11.

Thus, one obtains a complete classification of log structures on semi-stable varieties, similar to the case of log curves.

CHAPTER 3

The moduli space of admissible covers

In this chapter we first define logarithmic admissible covers following [21] and give an explicit construction of minimal log admissible covers by identifying the smoothing deformation parameters of the minimal log structures on the source and target log curves up to the local ramification indices. Then we give the full fledged modular interpretation of the space of log admissible covers in Theorem 3.3.1. Finally, we close the chapter by comparing the moduli space of log admissible covers with the compactification of the classical Hurwitz space as introduced by Harris and Mumford in [16].

3.1 Logarithmic admissible covers

Definition 3.1.1. Fix non-negative integers g, r, q, s, d such that $2g - 2 + r = d(2q - 2 + s)$. Let $C \longrightarrow \mathcal{S}$ and $\mathcal{X} \longrightarrow \mathcal{S}$ be log curves of type (g, r) and (q, s) respectively. A *log admissible cover of type* (g, r, q, s, d) is a commutative diagram of fine saturated log schemes

$$\zeta^{C \longrightarrow X} : \quad \begin{array}{ccc} C & \xrightarrow{\pi} & \mathcal{X} \\ & \searrow^{h} & \downarrow f \\ & & \mathcal{S} \end{array}$$

with an underlying morphism of schemes

$$\underline{\zeta}^{C\longrightarrow X} : \quad \begin{array}{ccc} C & \xrightarrow{\underline{\pi}} & X \\ \uparrow \{s'_i\}_{i=1}^{r} & \searrow^{\underline{h}} & \underline{f}\downarrow \uparrow \{s_i\}_{i=1}^{s} \\ & & S \end{array}$$

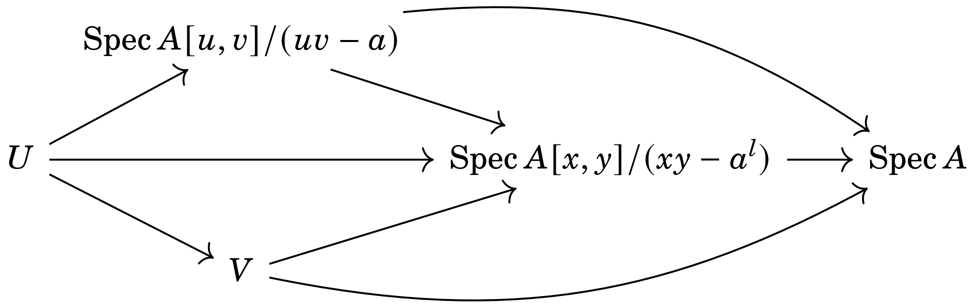
such that:

1. π is log-étale.
2. $\underline{\pi}^{-1}(X_{smooth}) = C_{smooth}$.
3. $\underline{\pi}$ is a finite morphism of degree $d > 0$ over the smooth locus of the fiber.
4. Sections defining the marked points are compatible, i.e.

$$O_C(\Sigma s'_i) \subseteq \underline{\pi}^*(O_X(\Sigma s_i)) \subseteq d.O_C(\Sigma s'_i)$$

   Note that we are considering unordered set of marked points. Hence, to be precise, we will be considering the quotient of the stack of (log) stable curves by the appropriate symmetric group.

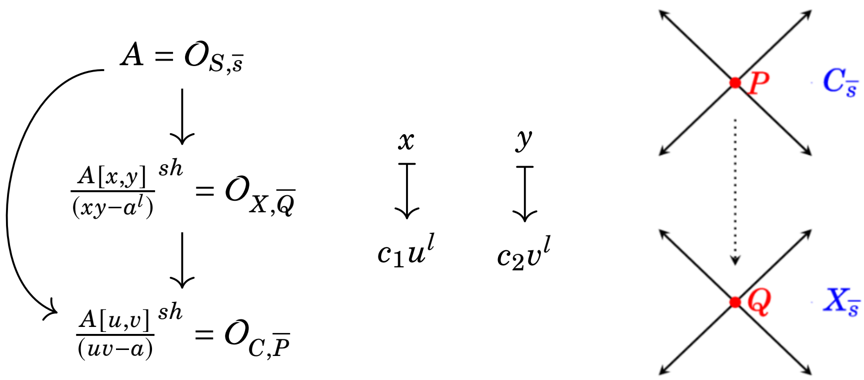
5. $\underline{\pi} : C \longrightarrow X$ has at worst nodal singularities and $\underline{\pi}^{-1}(X_{sing}) = C_{sing}$, i.e. the set of nodes of C is precisely the preimage under $\underline{\pi}$ of the set of nodes of X.
6. $\underline{\pi}$ is étale over the smooth locus of the morphism except over the closed subscheme defined by the sections, where it exhibits tame ramification.
7. *Admissibility condition over the nodal points*: For every nodal point P of C lying over a nodal point Q of X, $\exists\ l := l^{Q,P} \leq d \in \mathbb{Z}$ and formally étale neighbourhoods $U \longrightarrow C$ of P, $V \longrightarrow X$ of Q and $\operatorname{Spec} A \longrightarrow S$ of $s := \underline{f}(Q) = \underline{h}(P)$ such that $U \longrightarrow V \longrightarrow \operatorname{Spec} A$ factorises as in the commutative diagram:



for some $a \in m_A$ in the local ring A, where

$$A[x,y]/(xy-a^l) \longrightarrow A[u,v]/(uv-a)$$
$$x \longmapsto u^l$$
$$y \longmapsto v^l$$

Remark 3.1.2. The admissibility condition above can be restated as: For every nodal point P of C lying over a nodal point Q of X, $\exists\ l := l^{Q,P} \leq d \in \mathbb{Z}$ and $a \in m_{S,\overline{s}} = m^{sh}_{S,s} \subset \mathcal{O}_{S,\overline{s}} =: A,\ u,v \in m_{C,\overline{P}} = m^{sh}_{C,P} \subset \mathcal{O}_{C,\overline{P}},\ x,y \in m_{X,\overline{Q}} = m^{sh}_{X,Q} \subset \mathcal{O}_{X,\overline{Q}}$ such that we have a factorisation given by:



where $c_1, c_2 \in \mathcal{O}^*_{X,\overline{Q}}$, $c_1 \cdot c_2 \in A^*$ and the strict henselizations are taken with respect to the ideals (x,y,m_A) and (u,v,m_A) respectively. Moreover, choosing a smaller étale neighbourhood, we can arrange $a = \alpha_C(\overline{q}_P)$, where $\overline{q}_P$ is a lift of the smoothing parameter q_P of the node P (see Theorem 2.3.1) to the chosen étale neighbourhood and α_C is the log structure on the curve C. The equivalence of both the definitions follows from Lemma 2.4.2.

Thus, we have the fibered category of log admissible covers of type (g,r,q,s,d) defined by

$$\mathcal{LAdm}^{g,r}_{q,s,d} \longrightarrow \mathbf{LogSch}^{fs}_{st,\mathbb{Z}[1/d!]}$$

$$Obj(\mathcal{LAdm}^{g,r}_{q,s,d}) := \left\{ \begin{array}{ccc} C & \xrightarrow{\pi} & \mathcal{X} \\ & \underset{h}{\searrow} & \downarrow f \\ & & \mathcal{S} \end{array} \text{ log admissible covers} \right\}$$

$$Arr(\mathcal{LAdm}^{g,r}_{q,s,d}) := \left\{ \begin{array}{ccccccc} & & C & \longrightarrow & & & C' \\ & \swarrow & & & & \swarrow & \\ \mathcal{X} & & \longrightarrow & & \mathcal{X}' & & \downarrow \\ & & \downarrow & & & & \\ \downarrow & & \mathcal{S} & \longrightarrow & \downarrow & & \mathcal{S}' \\ & // & & & & // & \\ \mathcal{S} & & \longrightarrow & & \mathcal{S}' & & \end{array} \text{ cartesian front and back squares} \right\}$$

Remark 3.1.3. 1. The category $\mathbf{LogSch}^{fs}_{st,\mathbb{Z}[1/d!]}$ is considered with the trivial log structure on Spec $\mathbb{Z}[1/d!]$. Since we are interested in covers with tame ramifications of degree d, we need to invert $d!$. Apart from that, we shall need to invert $d!$ while proving that the stack of log admissible covers is proper and nowhere else.

2. Except hypothesis (1), all the conditions in Definition 3.1.1 depend only on the underlying morphism of schemes. Hence, the fact that admissible covers pull back to admissible covers follows from base the change property of log-étale morphisms and the standard base change properties for properties (2) – (7).

3. For a log admissible cover $\zeta^{C \longrightarrow X}$ over S, we denote the source family of log curve over S (resp. target family of log curve over S) by ζ^C (resp. ζ^X).

4. The above definition of admissible cover generalises Harris and Mumfords's original definition in the following sense:

   - We are considering covers of arbitrary curves with fixed discrete invariants in our moduli space instead of fixing our target curve.
   - The ramification over the closed subscheme defined by the sections is not assumed to be simple.

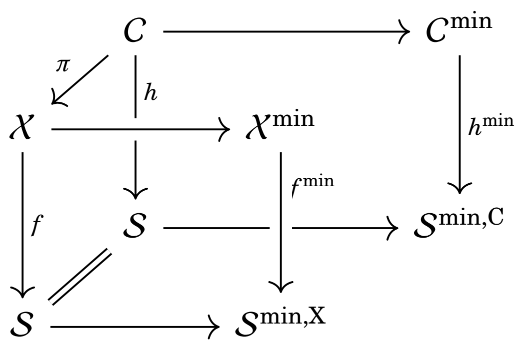
- The integers $l \leq d$ in the admissibility condition of Definition 3.1.1 are allowed to vary depending upon the nodal points.

3.1.1 Minimal log structures on admissible covers

In this section we construct the minimal objects in the fibered category of admissible covers following [21].

Let $\zeta^{C \longrightarrow X}$ be a log admissible cover over a base scheme S. For a schematic point $s \in S$, set $e := |C^s_{\mathbb{Z}}|$, $e' := |X^s_{\mathbb{Z}}|$, the number of nodes lying over $s \in S$ in C and X respectively as in Section 2.4. Note that by (5) of Definition 3.1.1, we have $e \geq e'$.

Let us denote by $\mathcal{S}^{\text{min, C}} = (S, \mathcal{M}_S^{\text{min,C}})$ to be the minimal log scheme determined by the nodes in the log curve $h : C \longrightarrow \mathcal{S}$ as constructed in Section 2.4. Similarly, let $\mathcal{S}^{\text{min,X}} = (S, \mathcal{M}_S^{\text{min,X}})$ be the minimal log scheme determined by the nodes in the log curve $f : \mathcal{X} \longrightarrow \mathcal{S}$. Let $\mathcal{M}_C^{\text{min}}$, $\mathcal{M}_X^{\text{min}}$ be the minimal log structures on the total spaces X and C respectively. Thus, we have a diagram of log curves with the front and back square cartesian



Moreover, recall from Theorem 2.3.1 that we have the following co-cartesian diagrams of fine saturated monoids:

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{q_P} & \overline{\mathcal{M}}^{\text{min,C}}_{S,\bar{s}} \cong \mathbb{N}^e \\ \Delta \downarrow & & \downarrow \\ \mathbb{N}^2 & \xrightarrow{(p_1^P, p_2^P)} & \overline{\mathcal{M}}^{\text{min}}_{C,\overline{P}} \cong \mathbb{N}^e \oplus_{\mathbb{N}} \mathbb{N}^2 \end{array} \qquad \begin{array}{ccc} \mathbb{N} & \xrightarrow{q'_Q} & \overline{\mathcal{M}}^{\text{min,X}}_{S,\bar{s}} \cong \mathbb{N}^{e'} \\ \Delta \downarrow & & \downarrow \\ \mathbb{N}^2 & \xrightarrow{(p_1^Q, p_2^Q)} & \overline{\mathcal{M}}^{\text{min}}_{X,\overline{Q}} \cong \mathbb{N}^{e'} \oplus_{\mathbb{N}} \mathbb{N}^2 \end{array}$$

where P and Q are nodes in the fibers $C_{\bar{s}}$ and $X_{\bar{s}}$, respectively, such that

$$\begin{cases} p_1^P(1) + p_2^P(1) = q_P(1) \\ p_1^Q(1) + p_2^Q(1) = q'_Q(1) \end{cases} \tag{3.1.1}$$

The structural morphism $\alpha_X^{\min} : \mathcal{M}_X^{\min} \longrightarrow \mathcal{O}_X$ preserves units, so $\mathcal{O}_X$ is generated as a sheaf of additive monoids by the image of $\alpha_X^{\min}$. Hence, using this and the relation in Equation (3.1.1), there exists a unique pair $\widetilde{x}, \widetilde{y} \in \mathcal{M}_{X,\overline{Q}}^{\min}$ with $\widetilde{x} + \widetilde{y} = \widetilde{b} \in \mathcal{M}_{S,\overline{s}}^{\min,\mathrm{X}}$ and such that

$$\begin{cases} \alpha_{X,\overline{Q}}^{\min} : \mathcal{M}_{X,\overline{Q}}^{\min} \longrightarrow \mathcal{O}_{X,\overline{Q}} \\ \qquad \widetilde{x} \longmapsto x \\ \qquad \widetilde{y} \longmapsto y \\ \alpha_{S,\overline{s}}^{\min,\mathrm{X}} : \mathcal{M}_{S,\overline{s}}^{\min,X} \longrightarrow \mathcal{O}_{S,\overline{s}} \\ \qquad \widetilde{b} \longmapsto a^l \end{cases} \tag{3.1.2}$$

To check the uniqueness of the pair, suppose there exists another pair $\widetilde{x}', \widetilde{y}' \in \mathcal{M}_{X,\overline{Q}}^{\min}$ satisfying the relations in (3.1.2). By the uniqueness of q'_Q, p_1^Q, p_2^Q, we have that $\widetilde{x}, \widetilde{x}'$ map to the same element in $\overline{\mathcal{M}}_{X,\overline{Q}}^{\min}$. Similarly $\widetilde{y}, \widetilde{y}'$ map to the same element in $\overline{\mathcal{M}}_{X,\overline{Q}}^{\min}$. Thus, $\widetilde{x} = \widetilde{x}' + c_1$ and $\widetilde{y} = \widetilde{y}' + c_2$ for some $c_1, c_2 \in \overline{\mathcal{O}}_{X,\overline{Q}}^*$. Since $\widetilde{x}+\widetilde{y}$, $\widetilde{x}'+\widetilde{y}' \in \mathcal{M}_{S,\overline{s}}^{\min,\mathrm{X}}$ and $\widetilde{x}+\widetilde{y} = \widetilde{x}'+\widetilde{y}' \in \overline{\mathcal{M}}_{S,\overline{s}}^{\min,\mathrm{X}}$, we have $c_1.c_2 \in \overline{\mathcal{O}}_{S,\overline{s}}^*$. Thus, by using Lemma 2.4.2, we have $c_1 = c_2 = 1$. Hence, $\widetilde{b}$ is also uniquely determined.

Similarly, there exist a unique pair $\widetilde{u}, \widetilde{v} \in \mathcal{M}_{C,\overline{P}}^{\min}$ with $\widetilde{u}+\widetilde{v} = \widetilde{a} \in \mathcal{M}_{S,\overline{s}}^{\min,\mathrm{X}}$ and such that

$$\begin{cases} \alpha_{C,\overline{P}}^{\min} : \mathcal{M}_{C,\overline{P}}^{\min} \longrightarrow \mathcal{O}_{C,\overline{P}} \\ \qquad \widetilde{u} \longmapsto u \\ \qquad \widetilde{v} \longmapsto v \\ \alpha_{S,\overline{s}}^{\min,\mathrm{C}} : \mathcal{M}_{S,\overline{s}}^{\min,C} \longrightarrow \mathcal{O}_{S,\overline{s}} \\ \qquad \widetilde{a} \longmapsto a \end{cases} \tag{3.1.3}$$

Note that $\widetilde{a}, \widetilde{u}, \widetilde{v}, \widetilde{x}, \widetilde{y}$ are non-units in the respective monoids. We have the canonical morphism induced by the log structures on the log curves

$$\mathbb{N}^e \oplus \mathbb{N}^{e'} \xrightarrow{(q_{P_1},\ldots,q_{P_e},q_{Q_1},\ldots,q_{Q_{e'}})} \mathcal{M}^{\min,C}_{S,\bar{s}} \oplus \mathcal{M}^{\min,X}_{S,\bar{s}} \longrightarrow \mathcal{O}_{S,\bar{s}}$$

where $(q_{P_1}, \ldots, q_{P_e})$ are the smoothing parameters of the nodes $(P_1, \ldots, P_e)$ in $C_{\bar{s}}$ and $(q_{Q_1}, \ldots, q_{Q_{e'}})$ are the smoothing parameters of the nodes $(Q_1, \ldots, Q_{e'})$ in $X_{\bar{s}}$.

We can lift $\widetilde{a}$ and $\widetilde{b}$ to a common étale neighbourhood of $\bar{s}$, which by an abuse of notation we denote by S. Furthermore, we can lift $\widetilde{x}, \widetilde{y}$ to an étale neighbourhood of Q lying over S, which by an abuse of notation we denote by C. Similarly, we can lift $\widetilde{u}, \widetilde{v}$ to an étale neighbourhood of P lying over S, which by an abuse of notation we denote by X. These étale neighbourhoods can be chosen such that $\widetilde{a}, \widetilde{b}, \widetilde{x}, \widetilde{y}, \widetilde{u}, \widetilde{v}$ satisfy the relations as in (3.1.2) and (3.1.3). In the étale neighbourhood of the point s, we define the minimal log admissible structure $\mathcal{M}^{\min,LA}_S$ on the base as the log structure associated with the morphism

$$\left(\mathcal{M}^{\min,C}_S \oplus_{\mathcal{O}^*_S} \mathcal{M}^{\min,X}_S\right)/\!\sim\; \longrightarrow \mathcal{O}_S \tag{3.1.4}$$

where '$\sim$' is defined by the minimal congruence relation stable under the monoid operation in $\mathcal{M}^{\min,C}_S \oplus_{\mathcal{O}^*_S} \mathcal{M}^{\min,X}_S$ that identifies all the local sections $(0, \widetilde{b}) \sim (\widetilde{a}, 0)$. In other words, the equivalence relation '$\sim$' is defined by the set

$$\{(\widetilde{b} + n_e, n_{e'}); (n_e, \widetilde{a} + n_{e'}) | n_e, n_{e'} \in \mathbb{N}\}$$

Thus, $(\mathbb{N}^e \oplus \mathbb{N}^{e'})/\!\sim\; \longrightarrow \mathcal{M}^{\min,LA}_S$ is a chart for the minimal log structure on the base S. These charts glue in the étale neighbourhoods in view of Lemma 2.4.2. Moreover, $(\mathbb{N}^e \oplus \mathbb{N}^{e'})/\!\sim$ is a fine saturated monoid as needed in the efinition of a log curve (see Definition 2.1.1).

An alternative description:

Alternatively, using the equivalent definition of admissibility stated in Remark 3.1.2, the congruence relation '$\sim$' identifies the extension of the smoothing parameters of the source and target curve in the étale neighbourhoods up to multiplicity, i.e.

$$(0, l \cdot \overline{q}_Q) \sim (\overline{q}_P, 0) \tag{3.1.5}$$

Thus, we have a local neat chart $\mathbb{N}^{e+e'-k} \longrightarrow \mathcal{M}_S^{\min,LA}$ where the 'k' components are obtained by identifying the smoothing parameters up to multiplicity as above.

Example 3.1.4. The intuitive idea in compactifying the Hurwitz space is to allow both the source and the target curve to be singular. Thus, a basic example of an admissible cover arises by considering smooth branched covers of a curve and then gluing together certain points away from the branched points. For instance, consider a degree two cover of $\underline{\pi} : \mathbb{P}^1_k \longrightarrow \mathbb{P}^1_k$ as shown in Figure 3.1.

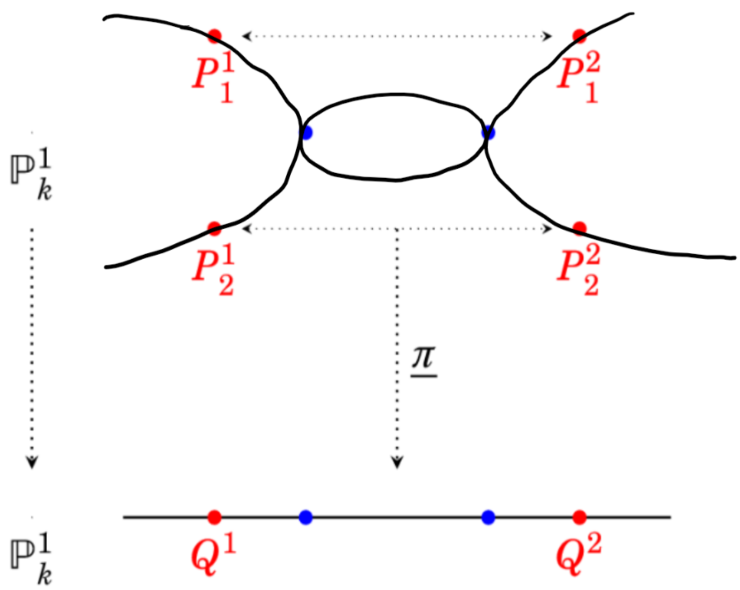


Figure 3.1: A degree two branched cover of $\mathbb{P}^1_k$

The marked points are labelled in blue over which it has tame ramification. In the base, we glue together the points Q^1 and Q^2 to obtain a node Q as in Figure 3.2. We denote the resulting stable curve by $X_{\overline{s}}$, which lies over a geometric point $\overline{s}$. Similarly, we glue the points P_1^2 and P_1^2 to obtain a node P_1 and glue the points P_2^1 and P_2^2 to obtain a node P_2 in the total space as in Figure 3.2. We denote the resulting stable curve by $C_{\overline{s}}$, which lies over a geometric point $\overline{s}$.

The nodal curve $C_{\overline{s}}$ has nodes P_1 and P_2 and the nodal curve $X_{\overline{s}}$ has a node Q such that $P_1 \longmapsto Q$ and $P_2 \longmapsto Q$ under $\underline{\pi}$. The marked points are labelled in blue in Figure 3.2 such that the cover $\underline{\pi} : C_{\overline{s}} \longrightarrow X_{\overline{s}}$ is branched over the marked points. The map $\underline{\pi} : C \longrightarrow X$ is of degree two with $l = 1$ for every node in the source and target curve as drawn in Figure 3.2 below. Thus, $\underline{\pi} : C \longrightarrow X$ satisfies conditions (2) – (6) in Definition 3.1.1.

Admissibility Condition 7 of the Definition 3.1.1 gives us étale locally a relation

$$A[x, y]/(xy - a^l) \longrightarrow A[u, v]/(uv - a)$$
$$x \longmapsto u^l$$
$$y \longmapsto v^l$$

over the nodes, where x, y, a are the local system of parameters as in Definition 3.1.1. In this case, $l = 1$.

The minimal log structure on s that comes from $C_{\overline{s}}$ is given by $\mathbb{N}^2$, determined by the smoothing parameters q_{P_1} and q_{P_2}. Similarly, the minimal log structure on s that comes from $X_{\overline{s}}$ is given by $\mathbb{N}$, determined by q_Q.

Thus, the minimal log admissible structure on s is given by $(\mathbb{N}^2 \oplus \mathbb{N})/\sim$ where $\sim$ identifies $q_{P_i} \sim q_{Q_j}$ if and only if $P_i \longmapsto Q_j$ under the map $C \longrightarrow X$. Thus, the minimal log structure on the curve is same as the target curve, i.e. $\mathbb{N} \cong \overline{\mathcal{M}}^{\min,LA}_{S,\overline{s}}$.

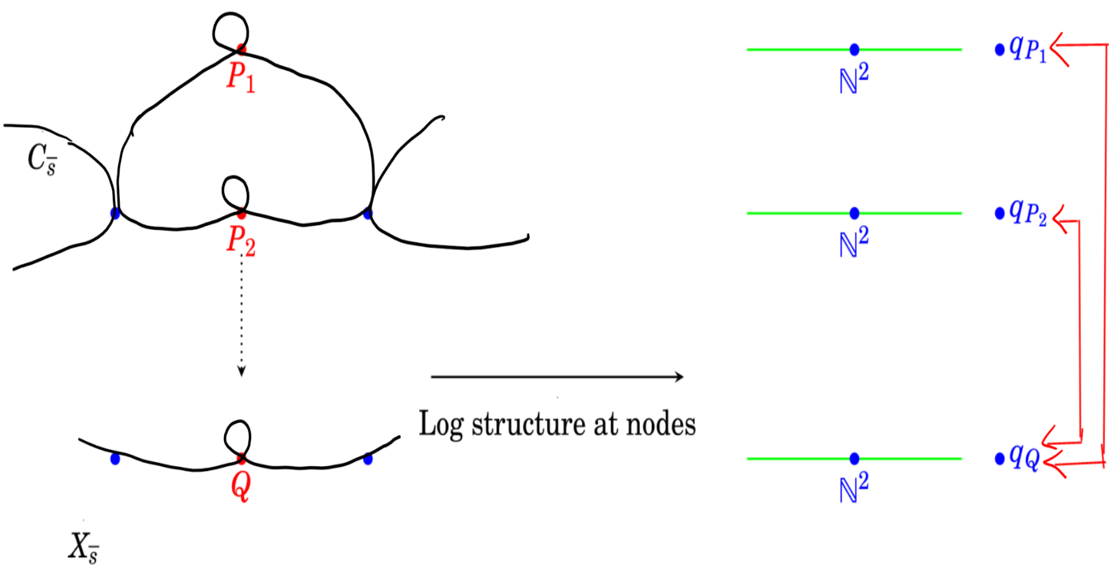


Figure 3.2: An admissible cover of degree two with a schematic representation of its minimal log structure on the base point. The blue dots on the left hand side diagram are the marked points

Minimal log structure on the total space

In general, having constructed the minimal log structure on the base, we now construct the minimal log structure on the total space of the admissible cover $\zeta^{C\longrightarrow X}$. We have the canonical morphisms of log schemes

$$c_X : \mathcal{S}^{\min,LA} \longrightarrow \mathcal{S}^{\min,\mathrm{X}}, \quad c_C : \mathcal{S}^{\min,LA} \longrightarrow \mathcal{S}^{\min,\mathrm{C}}$$

induced by

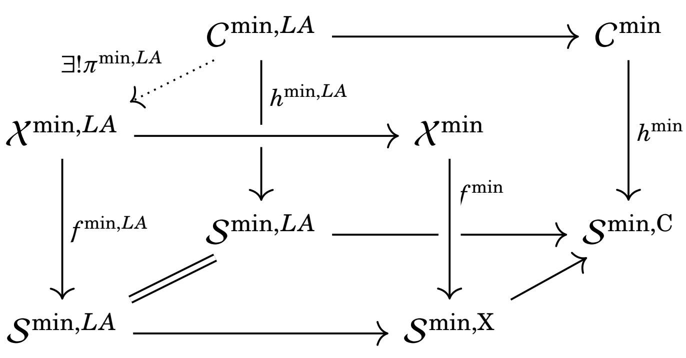
$$\mathcal{M}_S^{\min,C} \longrightarrow \left(\mathcal{M}_S^{\min,C} \oplus_{\mathcal{O}_S^*} \mathcal{M}_S^{\min,X}\right)/\!\sim\, \longrightarrow \mathcal{O}_S$$
$$\mathcal{M}_S^{\min,X} \longrightarrow \left(\mathcal{M}_S^{\min,C} \oplus_{\mathcal{O}_S^*} \mathcal{M}_S^{\min,X}\right)/\!\sim\, \longrightarrow \mathcal{O}_S$$

Define the log structure $\mathcal{M}_C^{\min,LA}$ (resp. $\mathcal{M}_X^{\min,LA}$) on the total space C (resp. X) étale locally (by the same abuse of notation as above) by the log structure associated with

$$(\underline{h}^{\min})^*\mathcal{M}_S^{\min,LA} \oplus_{(\underline{h}^{\min})^*\mathcal{M}_S^{\min,C}} \mathcal{M}_C^{\min} \longrightarrow \mathcal{O}_C \tag{3.1.6}$$

$$(\underline{f}^{\min})^*\mathcal{M}_S^{\min,LA} \oplus_{(\underline{f}^{\min})^*\mathcal{M}_S^{\min,X}} \mathcal{M}_X^{\min} \longrightarrow \mathcal{O}_X \tag{3.1.7}$$

Thus, we have a commutative diagram



where the front and back squares are cartesian in the category of fs log schemes (since $\mathcal{X}^{\min,LA} \longrightarrow \mathcal{S}^{\min,LA}$ and $\mathcal{C}^{\min,LA} \longrightarrow \mathcal{S}^{\min,LA}$ are integral saturated morphisms). There exists a unique morphism $\pi^{\min,LA} : \mathcal{C}^{\min,LA} \longrightarrow \mathcal{X}^{\min,LA}$ defined at the level of stalks in the following way:

Since the underlying schemes of the minimal log admissible cover remains the same as in the original family, we just need to define the morphism of log structures $\pi_{P,Q}^{\min,LA} : \mathcal{M}_{X,\overline{Q}}^{\min,LA} \longrightarrow \mathcal{M}_{C,\overline{P}}^{\min,LA}$ and verify $\pi_{P,Q}^{\min,LA}$ is log étale for every P and Q, where $P \in C_s$, $Q \in X_s$ and $\underline{\pi}^{\min,LA}(P) = Q$. Since we are working over integral monoids, it is enough to define morphisms $\overline{\pi}_{P,Q}^{\min,LA} : \overline{\mathcal{M}}_{X,\overline{Q}}^{\min,LA} \longrightarrow \overline{\mathcal{M}}_{C,\overline{P}}^{\min,LA}$. Moreover, recall that $\overline{\mathcal{M}}_{C/S,\overline{P}} \cong \overline{\mathcal{M}}_{X/S,\overline{Q}}$ by Corollary 2.3.3.

Case 1: If $\overline{\mathcal{M}}_{C/S,\overline{P}} \cong \overline{\mathcal{M}}_{X/S,\overline{Q}} \cong 0$, then $f^{\min,LA}$ and $h^{\min,LA}$ are strict morphisms in an étale neighbourhood of $\overline{P}$, resp. $\overline{Q}$. Thus, $\overline{\mathcal{M}}_{X,\overline{Q}}^{\min,LA} \longrightarrow \overline{\mathcal{M}}_{C,\overline{P}}^{\min,LA}$ is an isomorphism determined uniquely by the minimal log structure on the base.

Moreover, $C_P \longrightarrow X_Q$ is smooth of relative dimension zero by Theorem 2.3.1 and the definition of log admissible cover. Hence, $\pi_{P,Q}^{\min,LA}$ is log étale.

Case 2: If $\overline{\mathcal{M}}_{C/S,\overline{P}} \cong \overline{\mathcal{M}}_{X/S,\overline{Q}} \cong \mathbb{N}$, then by Theorem 2.3.1 the log structures $\mathcal{M}_{C/S,\overline{P}}$ (resp. $\mathcal{M}_{X/S,\overline{Q}}$) are étale locally given by divisorial log structures corresponding to the sections $\{s_i'\}_{i=1}^r$ (resp. $\{s_i\}_{i=1}^s$). By definition of a log admissible cover, the sections are compatible. Hence, the morphism of the divisorial log structures $\mathcal{M}_{X,\overline{Q}}^{\min} \longrightarrow \mathcal{M}_{C,\overline{P}}^{\min}$ is uniquely determined by the compatibility of the sections. Thus, we have a well defined morphism of fs monoids:

$$\overline{\pi}_{P,Q}^{\min,LA} : \overline{\mathcal{M}}_{X,\overline{Q}}^{\min,LA} \cong \overline{\mathcal{M}}_{X,\overline{Q}}^{\min} \oplus_{\mathbb{N}^{e'}} \overline{\mathcal{M}}_{S,\overline{s}}^{\min,LA} \longrightarrow \overline{\mathcal{M}}_{C,\overline{P}}^{\min,LA} \cong \overline{\mathcal{M}}_{C,\overline{P}}^{\min} \oplus_{\mathbb{N}^{e}} \overline{\mathcal{M}}_{S,\overline{s}}^{\min,LA}$$

where e and e' are the number of nodes in C_s and X_s respectively. It is an easy verification that $\overline{\pi}_{P,Q}^{\min,LA}$ preserves the relations that define the push-forward monoids. By definition of a log admissible cover, $\underline{\pi}$ exhibits a tame ramification over the closed subscheme defined by the sections. Hence, the above morphism satisfies the hypotheses of the chart criterion (Theorem 1.5.8).

Case 3: If $\overline{\mathcal{M}}_{C/S,\overline{P}} \cong \overline{\mathcal{M}}_{X/S,\overline{Q}} \cong \mathbb{Z}$, then P and Q are nodes in their respective fibers. In order to define $\overline{\pi}_{P,Q}^{\min,LA} : \overline{\mathcal{M}}_{X,\overline{Q}}^{\min} \oplus_{\mathbb{N}^{e'}} \overline{\mathcal{M}}_{S,\overline{s}}^{\min,LA} \longrightarrow \overline{\mathcal{M}}_{C,\overline{P}}^{\min} \oplus_{\mathbb{N}^{e}} \overline{\mathcal{M}}_{S,\overline{s}}^{\min,LA}$, it is enough to define the map

$$\overline{\mathcal{M}}_{X,\overline{Q}}^{\min} \longrightarrow \overline{\mathcal{M}}_{C,\overline{P}}^{\min}$$

Recall that in the co-cartesian diagrams

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{q_P} & \overline{\mathcal{M}}^{\min,C}_{S,\bar{s}} \cong \mathbb{N}^e \\ \Delta\downarrow & & \downarrow \\ \mathbb{N}^2 & \xrightarrow{(p_1^P,p_2^P)} & \overline{\mathcal{M}}^{\min}_{C,\overline{P}} \cong \mathbb{N}^e \oplus_{\mathbb{N}} \mathbb{N}^2 \end{array} \qquad \begin{array}{ccc} \mathbb{N} & \xrightarrow{q'_Q} & \overline{\mathcal{M}}^{\min,X}_{S,\bar{s}} \cong \mathbb{N}^{e'} \\ \Delta\downarrow & & \downarrow \\ \mathbb{N}^2 & \xrightarrow{(p_1^Q,p_2^Q)} & \overline{\mathcal{M}}^{\min}_{X,\overline{Q}} \cong \mathbb{N}^{e'} \oplus_{\mathbb{N}} \mathbb{N}^2 \end{array}$$

the parameters (p_1^Q, p_2^Q) are irreducible elements in the monoid $\overline{\mathcal{M}}^{\min}_{X,\overline{Q}}$. Hence, it is sufficient to define the map $\overline{\mathcal{M}}^{\min}_{X,\overline{Q}} \longrightarrow \overline{\mathcal{M}}^{\min}_{C,\overline{P}}$ on these elements. Thus, define

$$\begin{aligned} \overline{\mathcal{M}}^{\min}_{X,\overline{Q}} &\longrightarrow \overline{\mathcal{M}}^{\min}_{C,\overline{P}} \\ p_1^Q &\longmapsto l \cdot p_1^P \\ p_2^Q &\longmapsto l \cdot p_2^P \end{aligned}$$

The uniqueness of the above map can be argued as in the discussion after Equation 3.1.2 which basically uses Lemma 2.4.2. Moreover, by the chart criterion $\overline{\pi}^{\min,LA}_{P,Q}$ is log étale. Now, we can lift the maps $\overline{\pi}^{\min,LA}_{P,Q}$ to $\pi^{\min,LA}_{P,Q}$ since we are working over integral monoids and further check that these maps glue around the nodes by invoking Lemma 2.4.2.

In conclusion, the above construction gives a log admissible cover

$$\begin{array}{ccc} \mathcal{C}^{\min,LA} & \xrightarrow{\pi^{\min,LA}} & \mathcal{X}^{\min,LA} \\ & {\scriptstyle h^{\min,LA}} \searrow & \downarrow {\scriptstyle f^{\min,LA}} \\ & & \mathcal{S}^{\min,LA} \end{array}$$

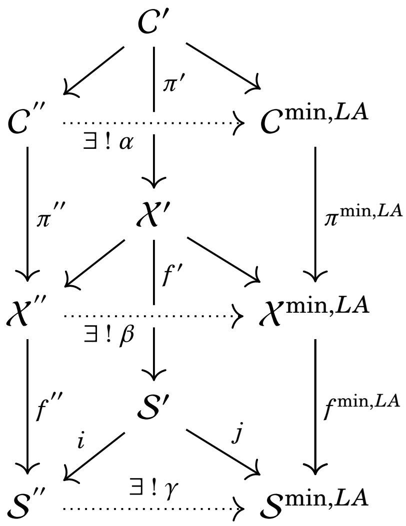
Example 3.1.5. Consider the example of a double admissible cover as in Example 3.1.4. Then the morphism of log structures at the nodes on the total space is given by the morphism of charts

$$\mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N} \oplus \mathbb{N}^e \longrightarrow \mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N} \oplus \mathbb{N}^e$$

$$(\widetilde{x}, \widetilde{y}, a) \longmapsto (l.\widetilde{u}, l.\widetilde{v}, a)$$

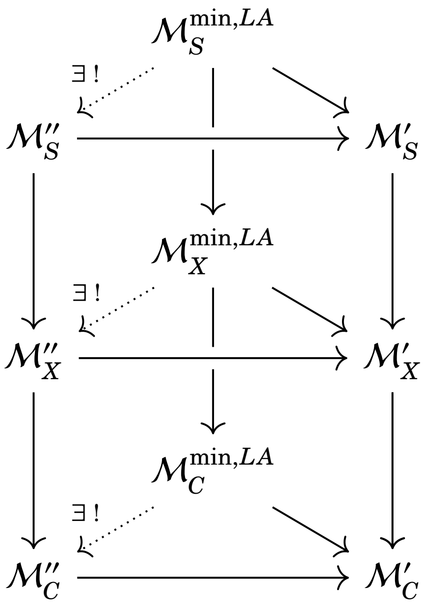
Theorem 3.1.6. *The log admissible cover constructed above is a minimal object* *in the fibered category* $\mathcal{LA}dm^{g,r}_{q,s,d} \longrightarrow \textbf{LogSch}^{fs}_{st,\mathbb{Z}[1/d!]}$ *in the sense of Definition 2.2.10.*

Proof. Given a commutative diagram of log admissible covers



$\underline{i} = \underline{j} = Id$

we need to uniquely complete the commutative diagram. As in the proof of Theorem 2.4.1, without loss of generality we may assume all the admissible covers have the same underlying schemes, and concentrate only on the log structures involved. Hence, it is enough to uniquely complete the dotted arrows of log structures in the following diagram



For every geometric point $\overline{s} \in S$, we constructed that

$$\overline{\mathcal{M}}_{S,\overline{s}}^{\min,LA} \cong \left(\mathbb{N}^e \oplus \mathbb{N}^{e'}\right)/\sim$$

where ' $\sim$' is as defined in Equation 3.1.5. Hence, the morphism

$$\overline{\mathcal{M}}_{S,\overline{s}}^{\min,LA} \longrightarrow \overline{\mathcal{M}}_{S,\overline{s}}$$

is uniquely determined by the images of the smoothing parameters

$$(q_{P_1}, \ldots, q_{P_e}, q_{Q_1}, \ldots, q_{Q_{e'}})$$

Since the smoothing parameters are unique, the uniqueness of $\overline{\mathcal{M}}_{S,\overline{s}}^{\min,LA} \longrightarrow \overline{\mathcal{M}}_{S,\overline{s}}$ follows. Since we are dealing with log curves, the uniqueness of the morphisms $\mathcal{M}_C^{\min,LA} \longrightarrow \mathcal{M}_C$ and $\mathcal{M}_X^{\min,LA} \longrightarrow \mathcal{M}_X$ in the diagram above follows from exactly the same argument as the uniqueness statements in Theorem 2.4.1. The existence statements follow an exact road map as in the proof of Theorem 3.1.8 below.

■

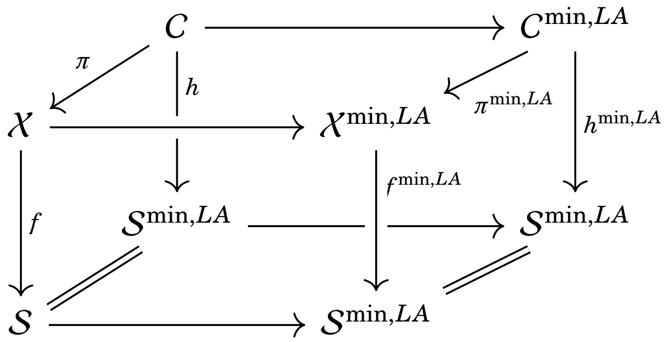
Remark 3.1.7. By construction, the minimal log structure $\mathcal{M}_S^{\min,LA}$ defines a DF log structure on the base S.

The next series of results show that the fibered category

$$\mathcal{LA}dm^{g,r}_{q,s,d} \longrightarrow \mathbf{LogSch}^{fs}_{st,\mathbb{Z}[1/d!]}$$

has enough minimal objects in the sense of Theorem 2.2.11.

Theorem 3.1.8. *For any log admissible cover $\zeta^{C \longrightarrow X}$, there exists a unique minimal log admissible cover $\zeta^{C^{\min,LA} \longrightarrow X^{\min,LA}}$ such that the diagram*



is a morphism of log admissible covers and $\underline{\mathcal{S}} = \underline{\mathcal{S}^{\min,LA}}$.

Proof. The morphism $\mathcal{S} \longrightarrow \mathcal{S}^{\min,LA}$ will be constructed étale locally on the base. Let $\overline{s}$ be a geometric point in S. Let $Q_1, \ldots, Q_{e'}$ be the nodes in X_s and let $P_1^1, \ldots, P_1^{j_1}, \ldots, P_{e'}^1, \ldots, P_{e'}^{j_{e'}}$ be the nodes in C_s such that $\Sigma_{n=1}^{e'} j_n = e$ and $\underline{\pi}(P_i^k) = Q_i \ \forall i, 1 \leq k \leq j_i$. As in the construction of the minimal log admissible covers, lift the smoothing parameters $q_{P_i^k}$ and q_{Q_i} to a common étale neighbourhood of $\overline{s}$ which by an abuse of notation we again call S. Then define $\mathcal{M}_S^{\min,LA}$ to be the log structure associated with the morphism

$$\left(\mathbb{N}^e \oplus \mathbb{N}^{e'}\right)/\!\sim\longrightarrow \mathcal{M}_S \longrightarrow \mathcal{O}_S$$

where '$\sim$' is the identification of the lifts of smoothing parameters as in the construction of minimal log admissible covers in Equation 3.1.5. Thus, we have a canonical morphism

$$\mathcal{M}_S^{\min,LA} \longrightarrow \mathcal{M}_S$$

Further, extend the parameters $(p_1^{Q_i}, p_2^{Q_i})$ to an étale neighbourhood of Q_i. Then define $\mathcal{M}_X^{\min,LA}$ to be the log structure associated with the morphism

$$\mathbb{N}^{e'} \oplus_{\mathbb{N}} \mathbb{N}^2 \oplus_{\mathbb{N}^{e'}} \left(\mathbb{N}^e \oplus \mathbb{N}^{e'}\right)/\sim \longrightarrow \mathcal{M}_X \longrightarrow \mathcal{O}_X$$

Similarly, define $\mathcal{M}_C^{\min,LA}$ to be the log structure associated with the morphism

$$\mathbb{N}^{e} \oplus_{\mathbb{N}} \mathbb{N}^2 \oplus_{\mathbb{N}^{e}} \left(\mathbb{N}^e \oplus \mathbb{N}^{e'}\right)/\sim \longrightarrow \mathcal{M}_C \longrightarrow \mathcal{O}_C$$

The morphism $\mathcal{M}_X^{\min,LA} \longrightarrow \mathcal{M}_C^{\min,LA}$ is defined as usual by defining

$$p_1^{Q_i} \longmapsto l \cdot p_1^{P_i^k}$$

$$p_2^{Q_i} \longmapsto l \cdot p_2^{P_i^k}$$

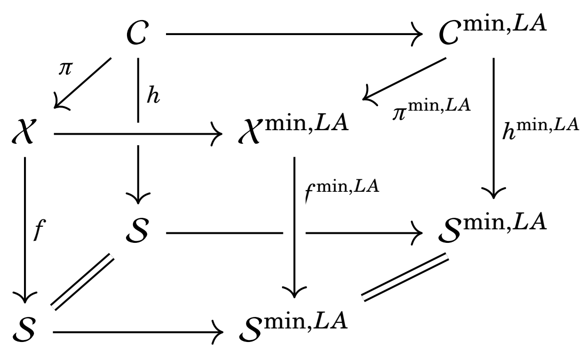
It is easy to see this morphism is log étale, as we have also done before. Other hypotheses of a log admissible cover depend upon the underlying schemes which remains the same in the construction of minimal log admissible covers. Hence, we are done.

For points in the closed subschemes defined by the sections, replace the log part given by $\mathbb{N}^e \oplus_{\mathbb{N}} \mathbb{N}^2$ with only $\mathbb{N}^e$. Similarly, replace the log part given by $\mathbb{N}^{e'} \oplus_{\mathbb{N}} \mathbb{N}^2$ with only $\mathbb{N}^{e'}$, to define the charts of log structures. Log étaleness is clear by tame ramification and other hypotheses are scheme theoretic. Hence, we are done.

Over the points in the smooth locus of the cover, the log structure on the total space is defined as the pull back of $\mathcal{M}_S^{\min,LA}$. Hence, we are done in this case. Next we need to verify that the charts constructed above indeed give us minimal log admissible structures, which one does stalk wise, very similarly to the proof of Theorem 2.4.4. The uniqueness of $\zeta^{C^{\min,LA} \longrightarrow X^{\min,LA}}$ follows since it is indeed a minimal object in the category of log admissible covers.

■

Theorem 3.1.9. *Consider a morphism of log admissible covers*



where the family on the right hand side is a minimal log admissible cover. Then the family on the left hand side is minimal if and only if $\mathcal{S} \longrightarrow \mathcal{S}^{\min,LA}$ *is a strict morphism of log schemes.*

Proof. This clearly follows from the definition of a minimal log admissible structure on the base.

■

The following corollary immediately follows from the uniqueness of the morphism $\pi^{\min,LA} : \mathcal{C}^{\min,LA} \longrightarrow \mathcal{X}^{\min,LA}$ constructed above.

Corollary 3.1.10. The canonical morphism

$$\mathcal{LA}dm^{g,r,\min}_{q,s,d} \longrightarrow \mathcal{LM}^{\min}_{q,s} \times_{(Sch)_{\mathbb{Z}[1/d!]}} \mathcal{LM}^{\min}_{g,r}$$

$$\zeta^{C \longrightarrow X} \longrightarrow (\zeta^{C}, \zeta^{X})$$

sending a minimal log admissible cover to its source and target family is faithful.

3.1.2 An alternate description of the admissibility condition

The moral of the construction of the minimal log structures on an admissible cover is that the tedious admissibility condition (7) in Definition 3.1.1 implies the existence of the relatively easy to handle log structures. Recall that over the nodes P and Q, the morphism of minimal log structures is given by

$$\overline{\pi}^{\min,LA}_{P,Q} : \overline{\mathcal{M}}^{\min}_{X,\overline{Q}} \oplus_{\mathbb{N}^{e'}} \overline{\mathcal{M}}^{\min,LA}_{S,\overline{s}} \longrightarrow \overline{\mathcal{M}}^{\min}_{C,\overline{P}} \oplus_{\mathbb{N}^{e}} \overline{\mathcal{M}}^{\min,LA}_{S,\overline{s}}$$

Over the log structures not coming from the base S, the above morphism is given by

$$\overline{\mathcal{M}}^{\min}_{X,\overline{Q}} \longrightarrow \overline{\mathcal{M}}^{\min}_{C,\overline{P}}$$
$$p_1^Q \longmapsto l \cdot p_1^P$$
$$p_2^Q \longmapsto l \cdot p_2^P$$

In the next set of definitions and results, we will see that morphisms of the above type in fact define the admissibility condition (7) in Definition 3.1.1. Thus, in the definition of a minimal log admissible cover, we can replace the admissibility condition (7) by a condition depending purely on log structures.

Definition 3.1.11. 1. A morphism of fs log schemes $f : (X, \mathcal{M}_X) \longrightarrow (Y, \mathcal{M}_Y)$ is said to be *irreducible of index* l at a geometric point $\overline{x} \in X$ if $\overline{f}^*_x$ is a monomorphism of monoids and for every irreducible element $m \in \overline{\mathcal{M}}_{Y,\underline{f}(\overline{x})}$, there exists an irreducible element $n \in \overline{\mathcal{M}}_{X,\overline{x}}$ such that $\overline{f}^*_x(n) = l \cdot m$, where $\overline{f}^*_x$ is the morphism

$$\overline{f}^*_x : \overline{\mathcal{M}}_{Y,\underline{f}(\overline{x})} \longrightarrow \overline{\mathcal{M}}_{X,\overline{x}}$$

For instance, if we have étale local charts such that $\overline{\mathcal{M}}_{Y,\underline{f}(\overline{x})} \cong \mathbb{N}^e$ and $\overline{\mathcal{M}}_{X,\overline{x}} \cong \mathbb{N}^{e'}$ and $e' \geq e$, then the morphism

$$\mathbb{N}^e \longrightarrow \mathbb{N}^{e'}$$
$$\epsilon_i \longmapsto l \cdot \epsilon_i$$

is an irreducible morphism, where ϵ_i's are the standard basis vectors. This is exactly the situation we are in the case of minimal log admissible covers.

Remark 3.1.12. Let $\pi : (C, \mathcal{M}_C) \longrightarrow (X, \mathcal{M}_X)$ be a morphism of log stable curves over a base $(S, \mathcal{M}_S)$. Assume that $\underline{\pi}^{-1}(X_{sing}) = C_{sing}$ and that π is *irreducible of index* $l^{P,Q}$ at all the nodal points of X over the part of the log structure not coming from the base S.

Furthermore, assume that for every node $P \in C$, there exists an (unordered) pair of elements $(\alpha, \beta) \in \Gamma(C, \overline{\mathcal{M}}_{C,P})$ such that $\alpha + \beta \in \Gamma(S, \overline{\mathcal{M}}_{S,s})$, where s is the image of P in S and, moreover, étale locally at P, there exist lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$ of the sections α and β respectively to $\mathcal{M}_C$ such that the images of $\widetilde{\alpha}$ and $\widetilde{\beta}$ under the morphism

$$\mathcal{M}_C \longrightarrow \mathcal{O}_C$$

generates the maximal ideal of $\overline{\mathcal{M}}_{C,P}$. We shall refer to such a pair (α, β) as a *log separating pair* for the minimal log stable curve $(C, \mathcal{M}_C) \longrightarrow (S, \mathcal{M}_S)$.

Moreover, assume also that the log stable curve $(X, \mathcal{M}_X) \longrightarrow (S, \mathcal{M}_S)$ admits a log separating pair.

Under the above hypotheses, it is easy to verify that we can indeed recover the admissibility condition (7) in Definition 3.1.1, i.e. étale locally around the nodes, we have the description

$$A[x,y]/(xy - a^l) \longrightarrow A[u,v]/(uv - a)$$
$$x \longmapsto u^{l^{P,Q}}$$
$$y \longmapsto v^{l^{P,Q}}$$

The irreducibilty hypothesis uniquely determines the minimal log structure constructed in Section 3.1.1 on the base S.

Thus, we have the following alternative definition of a minimal log admissible cover, purely relying on log structures:

Definition 3.1.13. Fix non-negative integers g, r, q, s, d such that $2g - 2 + r = d(2q - 2 + s)$. Let $C \longrightarrow S$ and $X \longrightarrow S$ be log curves of type (g, r) and (q, s) respectively. A *minimal log admissible cover of type* (g, r, q, s, d) is a commutative diagram of fine saturated log schemes

$$\zeta^{C\longrightarrow X} : \begin{array}{ccc} C & \xrightarrow{\pi} & X \\ & \searrow^{h} & \downarrow f \\ & & S \end{array}$$

with an underlying morphism of schemes

$$\underline{\zeta}^{C\to X} : \quad \begin{array}{ccc} C & \xrightarrow{\underline{\pi}} & X \\ & \underline{h} \searrow & \underline{f} \downarrow \uparrow \{s_i\}_{i=1}^{s} \\ \{s_i'\}_{i=1}^{r} \nwarrow & & S \end{array}$$

such that:

(A) Conditions (1) – (6) in the Definition 3.1.1 hold.

(B) For every node $P \in C$ and $Q \in X$ such that $\underline{\pi}(P) = Q$, the morphism $\pi : (C, \mathcal{M}_C) \longrightarrow (X, \mathcal{M}_X)$ is *irreducible of index* $l^{P,Q}$ for the part of the log structure not coming from $\mathcal{M}_S$.

(C) Étale locally the log curves $C \longrightarrow \mathcal{S}$ and $\mathcal{X} \longrightarrow \mathcal{S}$ admit log separating pairs as defined in Remark 3.1.12.

Following [21, Proposition §3.17], we now show that up to a finite étale covering of degree two, a log separating pair always exists étale locally on the base of a log stable curve.

Lemma 3.1.14. *Let $f : C \longrightarrow \mathcal{S}$ be a log stable curve. Let P be a node in C lying over $s \in S$. The, étale locally in a neighbourhood of $s \in S$, there exists a log separating pair $(\alpha, \beta) \in \Gamma(C, \overline{\mathcal{M}}_C)$.*

Proof. **Case 1**: P is étale locally a separating node in the curve C_s, i.e. blowing up the node at P gives rise to two connected components.

In an étale neighborhood of P, there exist elements $x, y \in m_{C,P} \subset \mathcal{O}_{C,P}$ and $t \in m_{S,\underline{f}(P)}$ such that $xy = t$, and such that x, y generate $m_{C,P}$. Then, in a neighborhood of P where x is defined and there are no nodes other than P, we let α to be the element of $\overline{\mathcal{M}}_C$ defined by $x \in \mathcal{M}_C$. Away from P, we define α as follows: Over the connected component where x is not identically zero, take α to be the trivial section of $\overline{\mathcal{M}}_C$. Over the connected component where x is identically zero, take α to be the section of $\overline{\mathcal{M}}_C$ defined by $t \in \mathcal{M}_C$. Similarly, define β by interchanging x and y in the argument above. By construction, (α, β) forms a separating pair at P.

Case 2: P is a non-separating node of the curve C_s. In this case, Mochizuki (see [21, §3.18]) constructs a finite strict étale covering of degree two $\phi : \widetilde{C} \longrightarrow C$ such that:

- $\underline{\phi}^{-1}(P)$ consists of two points P_1 and P_2.
- Over $C \setminus P$, $\widetilde{C}$ consists of two disjoint copies of $C \setminus P$.

The existence of a log separating pair for $\widetilde{C}$ follows from exactly the same argument as in Case 1.

■

3.2 The stack of log admissible covers

With the new interpretation of a minimal log admissible cover (Definition 3.1.13) invoking only log structures, we proceed towards our main goal of giving a stacky interpretation to the moduli space of log admissible cover. Since the minimal log structures we are dealing with are basically DF log structures (see Definition 1.6.1), a morphism of log structures boils down to morphisms of the line bundles that determine the DF log structure, as will see in Lemma 3.2.7. The moduli space of line bundles are classically well studied, hence, they are easier to deal with.

In view of the representability of the relative Picard functor $Pic_{X/S}$ (see Definition A.2.1), we will repeatedly use the following fact:

Lemma 3.2.1. *Let $f : X \longrightarrow S$ be a proper, flat map of noetherian schemes which is finitely presented and cohomologically flat (i.e. $\mathcal{O}_S \cong f_*\mathcal{O}_X$). Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be line bundles on X. Then, there exists an algebraic space of finite type $\mathfrak{Z} \longrightarrow S$ such that for any $T \longrightarrow S$, we have an isomorphism of line bundles $\pi_T^*\mathcal{L}_1 \cong \pi_T^*\mathcal{L}_2$ if and only if $T \longrightarrow S$ factors through $T \longrightarrow \mathfrak{Z} \longrightarrow S$, where $\pi_T : X \times_S T \longrightarrow T$.*

Theorem 3.2.2. *The category fibered in groupoids $\mathcal{LA}dm^{g,r,\min}_{q,s,d} \longrightarrow (Sch)_{\mathbb{Z}[1/d!]}$ is a stack over $(Sch)_{\mathbb{Z}[1/d!]}$ in the étale topology.*

Proof. The fact that the isomorphism functor $\mathcal{I}som_S(\zeta_1, \zeta_2) : (Sch/S)^{op} \longrightarrow (Sets)$ is a sheaf in the étale topology for every family of minimal log admissible covers ζ_1 and ζ_2 follows from Theorem 3.2.4 below.

To verify that the étale descent datum is effective for every family of log admissible cover, let $\{S_i \longrightarrow S\}_i$ be an étale cover of S and let $\zeta_i^{C\longrightarrow X}$ be elements of $\mathcal{LA}dm^{g,r,\min}_{q,s,d}(S_i)$ for each i. Let us assume that we have the cocycle condition $\phi_{ij} : \zeta^{C\longrightarrow X}_{i_{|S_i\times_S S_j}} \cong \zeta^{C\longrightarrow X}_{j_{|S_i\times_S S_j}}$ for every pair (i,j). Then there exists a unique $\zeta^C \in \mathcal{LM}_{g,r}(S)$ such that $\zeta^C_{|S_i} \cong \zeta^C_i$ holds for each i. Similarly, there exists a unique $\zeta^X \in \mathcal{LM}_{q,s}(S)$ such that we have $\zeta^X_{|S_i} \cong \zeta^X_i$ for each i. Now we need to argue that the local minimal log admissible covers extend uniquely to a log admssible cover $\zeta^{C\longrightarrow X}$. Since log structures are basically étale sheaves, local morphisms of log structures extend. For properties $(1)-(7)$ to extend in the Definition 3.1.1, we invoke the standard results on étale descent.

■

Remark 3.2.3. One can also argue directly that the isomorphism functor is a sheaf in the étale topology. The isomorphisms of log structures glue tautologically. Moreover, isomorphisms of the source and target stable curves glue by invoking the fact that the isomorphism functor for family of stable curves is a sheaf. Thus, we just need to argue that the glued up log curves indeed extend uniquely to a log admissible cover. This follows from the standard results on étale descent for properties $(1)-(7)$ in the Definition 3.1.1.

Theorem 3.2.4. *The diagonal morphism*

$$\Delta : \mathcal{LA}dm^{g,r,\min}_{q,s,d} \longrightarrow \mathcal{LA}dm^{g,r,\min}_{q,s,d} \times_{(Sch)_{\mathbb{Z}[1/d!]}} \mathcal{LA}dm^{g,r,\min}_{q,s,d}$$

is representable.

Equivalently, the isomorphism functor $\mathcal{I}som_S(\zeta^{C\longrightarrow X}, \zeta^{C'\longrightarrow X'}) : (Sch/S)^{op} \longrightarrow (Sets)$ is representable, hence it is a sheaf in the étale topology for every family of minimal log admissible covers $\zeta^{C\longrightarrow X}, \zeta^{C'\longrightarrow X'} \in \mathcal{LA}dm^{g,r,\min}_{q,s,d}(S)$.

Proof. We have the following canonical morphism of functors

$$\begin{array}{ccc}
\mathcal{I}som_S(\zeta^{C\longrightarrow X}, \zeta^{C'\longrightarrow X'}) & \xrightarrow{\ i\ } & \mathcal{I}som_S(\zeta^C, \zeta^{C'}) \times \mathcal{I}som_S(\zeta^X, \zeta^{X'}) \\
 & & \downarrow j \\
 & & \mathcal{I}som_S(\underline{\zeta}^C, \underline{\zeta}^{C'}) \times \mathcal{I}som_S(\underline{\zeta}^X, \underline{\zeta}^{X'}) \\
 & & \downarrow k \\
 & \searrow & \mathfrak{Hilb}_{C\times_S C'/S} \times \mathfrak{Hilb}_{X\times_S X'/S}
\end{array}$$

where $\mathfrak{Hilb}_{C\times_S C'/S}$ and $\mathfrak{Hilb}_{X\times_S X'/S}$ are the Hilbert functors as defined in Section A.2. The morphism k above is representable by schemes by Theorem A.2.2. The morphism j is representable by the openness of minimal log structures as proved in Lemma 2.2.13. Thus, it is sufficient to show that i is a representable morphism. Equivalently, it is enough to prove the following lemma as discussed after Lemma A.2.4.

■

Lemma 3.2.5. *Let $\zeta^{g,r} : C \longrightarrow S$ and $\zeta^{q,s} : X \longrightarrow S$ be minimal log stable curves of type (g,r) and (q,s) respectively and let $\pi : \zeta^{g,r} \longrightarrow \zeta^{q,s}$ be any morphism of families of log stable curves. Then there exists an algebraic space $\mathfrak{Z} \longrightarrow S$ such that for any morphism $T \longrightarrow S$, the pullback family $\zeta^{g,r}_T \longrightarrow \zeta^{q,s}_T$ is a minimal log admissible cover if and only if $T \longrightarrow S$ factors through $T \longrightarrow \mathfrak{Z} \longrightarrow S$.*

Proof. We separately verify that the conditions (1) – (7) in Definition 3.1.1 satisfy the universal property stated in the lemma and then take the intersection (in the étale topology) of all the algebraic spaces thus obtained.

Being a log étale morphism is an open condition on the target, in other words, the locus $\{s \in S | C_s \longrightarrow X_s \text{is log étale}\}$ is open in S. Hence, condition (1) satisfies the required universal property. By the same argument, condition (6) is also taken care of.

To verify condition (3), note that the locus $\{s \in S | C_s \longrightarrow X_s \text{is quasi-finite}\}$ is open in S, i.e. there exists a finite set of points $C_0 \subset C$ such that $C \setminus C_0 \longrightarrow X \setminus \underline{\pi}(C_0)$ is quasi-finite. Hence, $\mathfrak{Z} := S \setminus \underline{h}(C_0)$ is as required.

Condition (4) is taken care of by Corollary A.2.4.

The smooth locus of $C \longrightarrow S$ can be written as

$$\{s \in S \mid \Omega^1_{C/S,s}(D') \text{ is a locally free } \mathcal{O}_C\text{-module}\}$$

where $\Omega^1_{C/S}$ is the relative sheaf of dif and only iferentials of $C \longrightarrow S$ and D' is the closed subscheme defined by the sections s'_i. Similarly, the smooth locus of $X \longrightarrow S$ can be written as

$$\{s \in S \mid \Omega^1_{X/S,s}(D) \text{ is a locally free } \mathcal{O}_X\text{-module}\}$$

where $\Omega^1_{X/S}$ is the relative sheaf of dif and only iferentials of $X \longrightarrow S$ and D is the closed subscheme defined by the sections s_i. Moreover, we have a canonical morphism

$$\underline{\pi}^*\Omega^1_{X/S}(D) \longrightarrow \Omega^1_{C/S}(D')$$

Thus, condition (2), i.e. $\underline{\pi}^{-1}(X_{smooth}) = C_{smooth}$ holds if and only if

$$\{s \in S \mid \Omega^1_{C/S,s}(D') \text{ is a locally free } \mathcal{O}_C\text{-module}\}$$
$$\downarrow$$
$$\{s \in S \mid \Omega^1_{X/S,s}(D) \text{ is a locally free } \mathcal{O}_X\text{-module}\}$$

and $\underline{\pi}^*\Omega^1_{X/S}(D)_{|T} \longrightarrow \Omega^1_{C/S}(D')_{|T}$ is an isomorphism. The first condition is taken care of by Corollary A.2.4 and the last consdition is taken care of by the representability of the diagonal of the relative Picard functor (see Theorem A.2.9). In fact, this also proves the claim for condition (5).

Thus, it remains to prove that the log admissibility condition (7) of Definition 3.1.1 satisfies the scheme-like properties as in the lemma. This will be achieved in the next section while proving the algebraicity of the stack of minimal log admissible covers.

■

Remark 3.2.6. Instead of working with minimal log admissible covers, one can also work with the fibered category $\mathcal{LA}dm^{g,r}_{q,s,d} \longrightarrow \mathbf{LogSch}^{fs}_{st,\mathbb{Z}[1/d!]}$ and prove that the diagonal $\mathcal{LA}dm^{g,r}_{q,s,d} \longrightarrow \mathcal{LA}dm^{g,r}_{q,s,d} \times \mathcal{LA}dm^{g,r}_{q,s,d}$ is representable by a log algebraic space. In order to prove that the morphism j in the diagram above is representable, one has to resort to the more general result by J. Wise stated in Example A.2.7. The rest of the proof is analogous to the proof of Lemma 3.2.5 above.

3.2.1 Algebraicity: Representability of the canonical morphism

In order to prove that $\mathcal{LA}dm^{g,r,\min}_{q,s,d} \longrightarrow (Sch)_{\mathbb{Z}[1/d!]}$ is a log DM stack, we will prove that the canonical morphism

$$\mathcal{LA}dm^{g,r,\min}_{q,s,d} \longrightarrow \mathcal{LM}^{\min}_{q,s} \times_{(Sch)_{\mathbb{Z}[1/d!]}} \mathcal{LM}^{\min}_{g,r}$$

$$\zeta^{C\longrightarrow X} \longrightarrow (\zeta^C, \zeta^X)$$

sending a minimal log admissible cover to its source and target family is representable by an algebraic space of finite type, since $\mathcal{LM}^{\min}_{q,s} \times_{(Sch)_{\mathbb{Z}[1/d!]}} \mathcal{LM}^{\min}_{g,r}$ is a logarithmic DM stack of finite type. In other words, for any scheme S and a morphism $(\eta^{q,r}, \eta^{g,s}) : S \longrightarrow \mathcal{LM}^{\min}_{q,s} \times_{(Sch)_{\mathbb{Z}[1/d!]}} \mathcal{LM}^{\min}_{g,r}$ determined by minimal log curves $\eta^{q,s} \in \mathcal{LM}^{\min}_{q,s}(S)$ and $\eta^{g,r} \in \mathcal{LM}^{\min}_{g,r}(S)$, then the log stack $\mathfrak{W}$ in the fiber product below is representable by a log algebraic space.

$$\begin{array}{ccc} \mathfrak{W} & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle (\eta^{q,s},\eta^{g,r})} \\ \mathcal{LA}dm^{g,r,\min}_{q,s,d} & \longrightarrow & \mathcal{LM}^{\min}_{q,s} \times_{(Sch)_{\mathbb{Z}[1/d!]}} \mathcal{LM}^{\min}_{g,r} \end{array}$$

Note that by definition we have

$$\mathfrak{W}(T) = \{(\zeta^{C\longrightarrow X}, T \to S) \mid \zeta^{C\longrightarrow X} \in \mathcal{LA}dm^{g,r,\min}_{q,s,d}(T), \zeta^C = \eta^{g,r}_T,\ \zeta^X = \eta^{q,s}_T\}$$

Equivalently, this can be rewritten as

$$\mathfrak{W}(T) = \{(T \to S, \eta^{q,s} \xrightarrow{\pi} \eta^{g,r}) \mid \pi_T \text{ is a minimal log admissible cover}\}$$

In other words, to show that $\mathfrak{W}$ is representable by an algebraic space, it is enough to show the following two statements:

- The canonical morphism

$$\mathcal{LA}dm^{g,r,\min}_{q,s,d} \longrightarrow \mathcal{LM}^{\min}_{q,s} \times_{(Sch)_{\mathbb{Z}[1/d!]}} \mathcal{LM}^{\min}_{g,r}$$

$$\zeta^{C\longrightarrow X} \longrightarrow (\zeta^C, \zeta^X)$$

  sending a minimal log admissible cover to its source and target family is faithful.

- For any S-scheme T, the presheaf

  $T \mapsto \{(\zeta^{C\longrightarrow X}, \phi_C, \phi_X) \mid \zeta^{C\longrightarrow X} \in \mathcal{LA}dm^{g,r,\min}_{q,s,d}(T), \eta^{g,r}_T \xrightarrow{\phi_C} \zeta^C, \eta^{q,s}_T \xrightarrow{\phi_X} \zeta^X\}/\cong$

  is representable by an algebraic space.

Thus, we are reduced to proving the scheme-like properties for log admissible covers, exactly as in Lemma 3.2.5. In fact, we have checked the scheme-like properties for conditions (1) – (6) in the definition of a log admissible cover in Lemma 3.2.5. Thus, we only need to verify that the admissibility condition (7) in Definition 3.1.1 or equivalently, condition (B) in Definition 3.1.13 is scheme-like.

Recall from Remark 3.1.7 that the minimal log structure on the base of a log admissible cover defines a DF structure (see Definition 1.6.1). Hence, the locus where a morphism of minimal log curves extends to a minimal log admissible cover is determined by certain conditions on morphisms between the line bundles that determine the DF log structure. In view of the representability of the relative Picard functor (See Theorem A.2.9) and of constructible sheaves of sets by a quasi-finite algebraic space (see Theorem A.1.4), we expect scheme-like properties for the locus of minimal log admissible covers.

Lemma 3.2.7. *There exists an algebraic space $\mathfrak{Z}$ over S such that for any morphism of schemes $T \longrightarrow S$ and any morphism $\eta^{q,s} \xrightarrow{\pi} \eta^{g,r}$ of minimal log curves, the morphism $\eta^{q,s}_{|T} \xrightarrow{\pi_T} \eta^{g,r}_{|T}$ satisfies the admissibility condition (B) in Definition 3.1.13 if and only if $T \longrightarrow S$ factors through $\mathfrak{Z}$.*
In particular, the morphism $\eta^{q,s}_{|\mathfrak{Z}} \xrightarrow{\pi_{\mathfrak{Z}}} \eta^{g,r}_{|\mathfrak{Z}}$ satisfies the admissibility condition.

Proof. Let $\eta^{q,s} : (C, \mathcal{M}_C^{\min}) \longrightarrow (S, \mathcal{M}_S^{\min,C})$ and $\eta^{g,r} : (X, \mathcal{M}_X^{\min}) \longrightarrow (S, \mathcal{M}_S^{\min,X})$ be the given minimal log curves with a morphism

$$\begin{array}{ccc} (C, \mathcal{M}_C^{\min}) & \xrightarrow{\pi} & (X, \mathcal{M}_X^{\min}) \\ \downarrow & & \downarrow \\ (S, \mathcal{M}_S^{\min,C}) & \xrightarrow{\pi_S} & (S, \mathcal{M}_S^{\min,X}) \end{array}$$

which we want to extend to a log admissible cover over an algebraic space $\mathfrak{Z} \longrightarrow S$. Let $\bar{s} \in S$ be a geometric point with $Q_1, \dots, Q_{e'}$ nodes of $X_{\bar{s}}$ and $P_1, \dots, P_e$ nodes of $C_{\bar{s}}$. By the structure theorem of minimal log curves, the log structures define a DF structure étale locally around $\bar{s}$. Moreover, by Theorem 1.4.10 the sheaves $\overline{\mathcal{M}}_S^{\min,C}$ and $\overline{\mathcal{M}}_S^{\min,X}$ are constructible sheaves of sets, hence representable by quasi-finite étale algebraic spaces. In particular, the morphism

$$\overline{\pi}_S : \overline{\mathcal{M}}_S^{\min,X} \longrightarrow \overline{\mathcal{M}}_S^{\min,C}$$

of minimal log structures on the base S is representable by a quasi-finite algebraic space. In other words, for any morphism $T \longrightarrow \overline{\mathcal{M}}_S^{\min,C}$, the fiber product $\mathfrak{U}$ defined by the cartesian product

$$\begin{array}{ccc} \mathfrak{U} & \longrightarrow & T \\ \downarrow & & \downarrow \\ \mathbb{N}^{e'} \cong \overline{\mathcal{M}}_{S|Z_1}^{\min,X} & \xrightarrow{\overline{\pi}_S} & \mathbb{N}^{e} \cong \overline{\mathcal{M}}_S^{\min,C} \end{array}$$

is representable by an algebraic space. Hence, by Theorem A.1.4, there exists a quasi-finite étale algebraic space $Z_1 \longrightarrow S$ such that $T \longrightarrow S$ factors through Z_1. Now it is sufficient to work over Z_1 by restricting to the morphism

$$\overline{\pi}_{S|Z_1} : \overline{\mathcal{M}}_{S|Z_1}^{\min,X} \longrightarrow \overline{\mathcal{M}}_{S|Z_1}^{\min,C}$$

From Section 2.4, we have the isomorphism $\mathbb{N}^e \xrightarrow{\cong} \overline{\mathcal{M}}_S^{\min,C}$ (étale locally) lifting to a local chart of $\mathcal{M}_S^{\min,C}$. This determines a DF log structure on S given by the sequence of pairs $(\{\mathcal{L}_i, \gamma_i\}_{i=1}^{e})$ (see the proof of Theorem 1.6.2). Similarly, the isomorphism $\mathbb{N}^{e'} \xrightarrow{\cong} \overline{\mathcal{M}}_S^{\min,X}$ (étale locally) lifts to a local chart of $\mathcal{M}_S^{\min,X}$ determining a DF log structure on S given by the sequence of pairs $(\{\mathcal{L}'_i, \gamma'_i\}_{i=1}^{e'})$. The morphism $\overline{\pi}_{S|Z_1}$ is determined by the values of the standard basis $\{\epsilon_i\}_i^{e'} \in \mathbb{N}^{e'}$. In the commutative diagram,

$$\begin{array}{ccc} \mathcal{M}_{S|Z_1}^{\min,X} & \longrightarrow & \mathcal{M}_{S|Z_1}^{\min,C} \\ \downarrow{\scriptstyle Pr_X} & & \downarrow{\scriptstyle Pr_C} \\ \mathbb{N}^{e'} \cong \overline{\mathcal{M}}_{S|Z_1}^{\min,X} & \xrightarrow{\overline{\pi}_{S|Z_1}} & \mathbb{N}^{e} \cong \overline{\mathcal{M}}_{S|Z_1}^{\min,C} \end{array}$$

let $f_i := \overline{\pi}_{S|Z_1}(\epsilon_i)$ for all $1 \leq i \leq e'$. Denote the line bundles defined by the O_S^*-torsors $Pr_C^{-1}(f_i)$ by $\widetilde{\mathcal{L}}_i$ for all $1 \leq i \leq e'$. Also, let $\widetilde{\gamma}_i$ be the corresponding sections defined by $Pr_C^{-1}(f_i) \longrightarrow O_S$, for all $1 \leq i \leq e'$. Then, the morphism

$$\mathcal{M}_{S|Z_1}^{\min,X} \longrightarrow \mathcal{M}_{S|Z_1}^{\min,C}$$

is uniquely determined by the isomorphisms of the line bundles $\mathcal{L}_i \cong \widetilde{\mathcal{L}}_i$ and the equality of the sections $\gamma_i = \widetilde{\gamma}_i$, for all $1 \leq i \leq e'$. Let $Z_2 \longrightarrow Z_1$ be the algebraic space obtained from Lemma 3.2.1 defining the isomorphisms $\mathcal{L}_i = \widetilde{\mathcal{L}}_i$ for every

$1 \leq i \leq e'$.

We are interested in the locus where π extends to a minimal log admissible cover. $\pi : C \longrightarrow \mathcal{X}$ is a minimal log admissible cover if and only if the morphism $\pi' : \mathcal{M}_X^{\min} \longrightarrow \underline{\pi}^* \mathcal{M}_C^{\min}$ is determined uniquely by the relation $(0, \widetilde{b}) \sim (\widetilde{a}, 0)$ in $\mathcal{M}_S^{\min,C} \oplus_{O_S^*} \mathcal{M}_S^{\min,X}$, where $\widetilde{a}$ and $\widetilde{b}$ are as defined in Equation 3.1.3. This is equivalent to the fact that $\pi' : \mathcal{M}_X^{\min} \longrightarrow \underline{\pi}^* \mathcal{M}_C^{\min}$ is irreducible at every node. Consider the morphism Δ_{Z_1} étale locally defined as

$$\Delta_{Z_2} : \mathcal{M}_{S|Z_2}^{\min,X} \longrightarrow \mathcal{M}_{S|Z_2}^{\min,C}$$
$$\widetilde{b} \longmapsto \widetilde{a}$$

Since the smoothing parameters are irreducible elements in the monoid, it is enough to specify the value of Δ_{Z_2} on these smoothing parameters.

As in the previous paragraph, we have a commutative diagram,

$$\begin{array}{ccc} \mathcal{M}_{S|Z_1}^{\min,X} & \xrightarrow{\Delta_{Z_2}} & \mathcal{M}_{S|Z_1}^{\min,C} \\ \downarrow Pr_X & & \downarrow Pr_C \\ \mathbb{N}^{e'} \cong \overline{\mathcal{M}}_{S|Z_1}^{\min,X} & \xrightarrow{\overline{\Delta}_{Z_2}} & \mathbb{N}^{e} \cong \overline{\mathcal{M}}_{S|Z_1}^{\min,C} \end{array}$$

Let $f_i' := \overline{\Delta}_{S|Z_2}(\epsilon_i)$ for all $1 \leq i \leq e'$, where the ϵ_i's are as usual the standard basis. Denote the line bundles defined by the O_S^*-torsors $Pr_C^{-1}(f_i')$ by $\widetilde{\mathcal{L}}_i'$ for all $1 \leq i \leq e'$. Also, let $\widetilde{\gamma}_i'$ be the corresponding sections defined by $Pr_C^{-1}(f_i') \longrightarrow O_S$, for all $1 \leq i \leq e'$. Then the morphism

$$\Delta_{Z_2} : \mathcal{M}_{S|Z_2}^{\min,X} \longrightarrow \mathcal{M}_{S|Z_2}^{\min,C}$$

is uniquely determined by the isomorphism of the line bundles $\mathcal{L}_i \cong \widetilde{\mathcal{L}}_i'$ and the equality of the sections $\gamma_i = \widetilde{\gamma}_i'$, for all $1 \leq i \leq e'$. Let $Z_3 \longrightarrow Z_1$ be the algebraic space obtained from Lemma 3.2.1 defining the isomorphisms $\mathcal{L}_i = \widetilde{\mathcal{L}}_i'$ for every $1 \leq i \leq e'$. Set $Z_4 := Z_1 \times_S Z_2 \times_S Z_3$. By the above construction, the minimal log structure on Z_4 is obtained as

$$\mathcal{M}_{Z_4}^{\min,LA} := \left(\mathcal{M}_S^{\min,C} \oplus \mathcal{M}_S^{\min,X}\right)_{|Z_3} / \sim$$

where $\sim$ identifies the smoothing parameters of the source and target minimal curves as in the construction in Section 3.1.6. Moreover, $T \longrightarrow S$ factors through Z_4. Hence, we have the canonical morphisms

$$(Z_3, \mathcal{M}_{Z_3}^{\min,LA}) \longrightarrow (Z_3, \mathcal{M}_S^{\min,C}) \longrightarrow \mathcal{LM}_{q,s}^{\min}$$
$$(Z_3, \mathcal{M}_{Z_3}^{\min,LA}) \longrightarrow (Z_3, \mathcal{M}_S^{\min,X}) \longrightarrow \mathcal{LM}_{g,r}^{\min}$$

which determine minimal log curves $C' \longrightarrow (Z_3, \mathcal{M}_{Z_3}^{\min,LA})$ and $\mathcal{X}' \longrightarrow (Z_3, \mathcal{M}_{Z_3}^{\min,LA})$ respectively.

Hence, it is now enough to study the locus where the morphism of minimal log curves $C' \longrightarrow \mathcal{X}'$ extends to satisfy the minimal log admissibility condition and intersect the locus with Z_4. By definition 3.1.13, this boils down to verifying the scheme-like property for a log separating pair. Log separating pairs tautologically define a DF structure. Hence, the locus where the morphism of minimal log curves $C' \longrightarrow \mathcal{X}'$ extends to satisfy the minimal log admissibility condition can be computed in the exact same way as in the first part of the proof.

■

Since $\mathcal{LM}_{q,s}^{\min} \times_{(Sch)_{\mathbb{Z}[1/d!]}} \mathcal{LM}_{g,r}^{\min}$ is a log DM stack, the above lemma implies that the moduli space of minimal log admissible covers

$$\mathcal{LA}dm_{q,s,d}^{g,r,\min} \longrightarrow (Sch)_{\mathbb{Z}[1/d!]}$$

is a log DM stack. Hence, the stack

$$\mathcal{LA}dm_{q,s,d}^{g,r} \longrightarrow \mathbf{LogSch}_{st,\mathbb{Z}[1/d!]}^{fs}$$

is a log DM stack.

Remark 3.2.8 (Log étale cover). Note that by construction, the morphism $Z_4 \longrightarrow S$ is log étale by using the chart criterion (Theorem 1.5.8). In particular, the canonical morphism

$$\mathcal{LA}dm_{q,s,d}^{g,r,\min} \longrightarrow \mathcal{LM}_{q,s}^{\min}$$

is a log étale morphism.

Remark 3.2.9 (Representability by schemes). Since the algebraic space $\mathfrak{Z}$ constructed above is quasi-finite, the canonical morphism

$$\mathcal{LA}dm_{q,s,d}^{g,r,\min} \longrightarrow \mathcal{LM}_{q,s}^{\min} \times_{(Sch)_{\mathbb{Z}[1/d!]}} \mathcal{LM}_{g,r}^{\min}$$

is quasi-finite. Alternatively, one can argue that the number of possible branched covers of a curve is finite, hence, the morphism

$$\mathcal{LA}dm_{q,s,d}^{g,r,\min} \longrightarrow \mathcal{LM}_{q,s}^{\min}$$

is quasi-finite, which in turn implies that

$$\mathcal{LA}dm_{q,s,d}^{g,r,\min} \longrightarrow \mathcal{LM}_{q,s}^{\min} \times_{(Sch)_{\mathbb{Z}[1/d!]}} \mathcal{LM}_{g,r}^{\min}$$

is quasi-finite since $\mathcal{LM}_{q,s}^{\min} \times_{(Sch)_{\mathbb{Z}[1/d!]}} \mathcal{LM}_{g,r}^{\min} \longrightarrow \mathcal{LM}_{q,s}^{\min} \times_{(Sch)_{\mathbb{Z}[1/d!]}} \mathcal{LM}_{g,r}^{\min}$ is a separated morphism of log stacks. In Section 3.2.2 we will show that $\mathcal{LA}dm_{q,s,d}^{g,r,\min} \longrightarrow (Sch)_{\mathbb{Z}[1/d!]}$ is a proper stack, in particular the morphism

$$\mathcal{LA}dm_{q,s,d}^{g,r,\min} \longrightarrow \mathcal{LM}_{q,s}^{\min} \times_{(Sch)_{\mathbb{Z}[1/d!]}} \mathcal{LM}_{g,r}^{\min}$$

is separated. Hence, $\mathfrak{Z} \longrightarrow S$ constructed in Lemma 3.2.7 is separated.

Thus, by invoking the bootstrapping result in [30, Tag 03XX], we conclude that the canonical morphism

$$\mathcal{LA}dm_{q,s,d}^{g,r,\min} \longrightarrow \mathcal{LM}_{q,s}^{\min} \times_{(Sch)_{\mathbb{Z}[1/d!]}} \mathcal{LM}_{g,r}^{\min}$$

is representable by schemes. Moreover, the fact that the canonical morphism $\mathcal{LA}dm_{q,s,d}^{g,r,\min} \longrightarrow \mathcal{LM}_{q,s}^{\min}$ is quasi-finite implies that the dimension of the moduli stack $\mathcal{LA}dm_{q,s,d}^{g,r,\min}$ has dimension $3q-3+s$.

Remark 3.2.10 (Finite stabilizer group). Since $\mathcal{LA}dm_{q,s,d}^{g,r,\min}$ is a log DM stack, the stabilizer group (see Definition A.1.17) of a minimal log admissible cover $\zeta^{C \longrightarrow X}$ over $\operatorname{Spec} k$ in $\mathcal{LA}dm_{q,s,d}^{g,r,\min}(k)$ is a finite group scheme. In fact, this can be directly checked by using the fact the order of the stabiliser group is determined by the number of morphisms

$$\mathbb{N}^{e'} \oplus_{\mathbb{N}} \mathbb{N}^2 \oplus_{\mathbb{N}^{e'}} \left(\mathbb{N}^e \oplus \mathbb{N}^{e'}\right)/\sim \longrightarrow \mathbb{N}^{e'} \oplus_{\mathbb{N}} \mathbb{N}^2 \oplus_{\mathbb{N}^{e'}} \left(\mathbb{N}^e \oplus \mathbb{N}^{e'}\right)/\sim$$

such that the relations

$$p_1^Q \longmapsto l \cdot p_1^P$$
$$p_2^Q \longmapsto l \cdot p_2^P$$

hold. Since these parameters are fixed, there are only finitely many choices, hence the automorphism group is finite.

3.2.2 Properness of moduli space of admissible covers

In this section we show that the moduli stack of log admissible covers is proper. Since $\mathcal{LA}dm_{q,s,d}^{g,r} \longrightarrow \mathbf{LogSch}_{st,\mathbb{Z}[1/d!]}^{fs}$ is a noetherian log algebraic stack of finite type, we verify that it satisfies the weak valuative criterion for properness (see Theorem A.3.3). Since every log admissible cover can be uniquely pulled back from a minimal log admissible cover up to isomorphism, it suffices to verify Theorem A.3.3 for the stack of minimal log admissible covers

$$\mathcal{LA}dm_{q,s,d}^{g,r,\min} \longrightarrow (Sch)_{\mathbb{Z}[1/d!]}$$

Now we state a technical result which will be used in the the proof of the properness of the moduli space of log admissible cover. We shall refer to [21, §3.12] for the proof of Lemma 3.2.11.

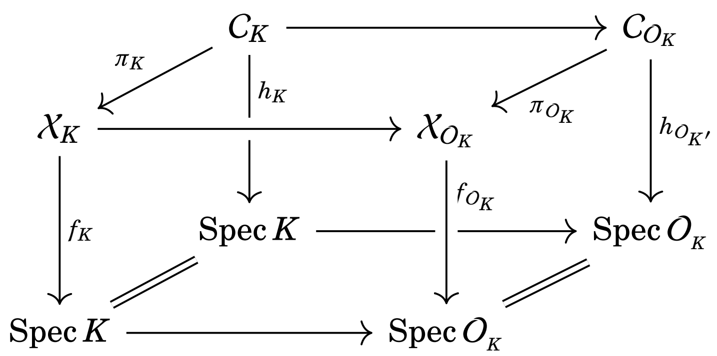
Lemma 3.2.11. *Let A be a strict henselian discrete valuation ring such that $n \in A^*$. Let $s \in A$ be a uniformizer. For each natural number $i \geq 1$, let R_i be the strict henselization of $A[X,Y]/(XY - s^i)$ at the maximal ideal $m = (x, y, s)$ (where x, y are the images of X, Y in R_i). Let $W_i = \operatorname{Spec} R_i \setminus m$. Then the étale fundamental group $\pi^1(W_n) \cong \mathbb{Z}/n\mathbb{Z}$, and the universal cover $\widetilde{W}_n$ of W_n is W_1, induced by the mapping*

$$\widetilde{W}_n \longrightarrow W_n$$
$$x \longmapsto x^n$$
$$y \longmapsto y^n$$

Theorem 3.2.12. *Let O_K be a discrete valuation ring with fraction field K such that $d!$ is invertible in O_K. Let $\zeta^{C_K \longrightarrow X_K}$ be a minimal log admissible cover over*

Spec K. *Then after possibly replacing O_K with a tamely ramified extension, the minimal log admissible cover $\zeta^{C_K \longrightarrow X_K}$ extends to a minimal log admissible cover $\zeta^{C_{O_K} \longrightarrow X_{O_K}}$ over* Spec O_K *uniquely up to unique isomorphism.*

In other words, there exists a unique minimal log admissible cover up to isomorphism $\zeta^{C_{O_K} \longrightarrow X_{O_K}}$ with a commutative diagram of minimal log admissible covers:



Proof. We will first deal with extending the underlying morphism of schemes to an admissible cover satisfying properties (2)–(6) in Definition 3.1.1 and then take care of the log structures since minimal log structures are uniquely determined over a point.

After possibly a separable field extension (which by an abuse of notation we gain refer to as K), the stable curve $X_K \longrightarrow \operatorname{Spec} K$ extends uniquely up to isomrophism to a stable curve $X_{O_K} \longrightarrow \operatorname{Spec} O_K$ (see [30, Tag 0E8C]). Now, set C_{O_K} to be the relative normalization of X_{O_K} in C_K. Precisely, if $a^* : O_{X_{O_K}} \longrightarrow a_* O_{C_K}$ is the morphism induced by $a : C_K \longrightarrow X_{O_K}$, then set

$$C_{O_K} := \underline{\operatorname{Spec}}_{X_{O_K}} \widetilde{O}_{X_{O_K}}$$

where $\widetilde{O}_{X_{O_K}}$ is the integral closure of $O_{X_{O_K}}$ in $a_* O_{C_K}$ (see [30, Tag 0BAK] for a detail definition of relative normalization of a scheme) and $\underline{\operatorname{Spec}}$ denotes the relative spectrum (see [30, Tag 01LL]). By the construction of the relative normalization, $C_{O_K} \longrightarrow \operatorname{Spec} O_K$ is a family of stable curves and $C_{O_K} \longrightarrow X_{O_K}$ satisfies properties (2) – (5) in the Definition 3.1.1.

After possibly replacing O_K by a tamely ramified extension, we can assume that $C_{O_K} \longrightarrow X_{O_K}$ has tame ramification over the closed subscheme defined by the sections. This follows from Abhyankar's lemma [30, Tag 0EXT].

Hence, it now remains to check that $C_{O_K} \longrightarrow X_{O_K}$ satisfies that admissibility condition (7) in the Definition 3.1.1. This follows from Lemma 3.2.11 stated above. Moreover, since C_K is a dense subset of C_{O_K} and $C_{O_K} \longrightarrow C_{O_K}$ is a separated morphism of schemes, the uniqueness of the extension follows.

$C_{O_K} \longrightarrow X_{O_K}$ satisfies that admissibility condition (7), hence, it has a minimal log structure as constructed in Section 3.1.1 which restricts to the minimal log structure on $C_K \longrightarrow X_K$. Moreover the uniqueness of the log structure follows from Theorem 3.1.9.

■

3.3 The main theorem

Thus, we are now ready to state the full fledged modular interpretation of the space of log admissible covers.

Theorem 3.3.1. *Fix non-negative integers g, r, q, s, d such that $2g-2+r = d(2q-2+s)$. Then the moduli space of log admissible covers*

$$\mathcal{LA}dm^{g,r}_{q,s,d} \longrightarrow \textbf{\textit{LogSch}}^{fs}_{st,\mathbb{Z}[1/d!]}$$

is a logarithmic DM stack, proper of finite type with a separated diagonal. The open substack $\mathcal{LA}dm^{g,r,\min}_{q,s,d} \longrightarrow (Sch)_{\mathbb{Z}[1/d!]}$ of minimal log admissible covers admits a finite log étale morphism

$$\mathcal{LA}dm^{g,r,\min}_{q,s,d} \longrightarrow \mathcal{LM}^{\min}_{q,s} \cong \overline{\mathcal{M}}_{q,s}$$

Moreover, $\mathcal{LA}dm^{g,r,\min}_{q,s,d} \longrightarrow (Sch)_{\mathbb{Z}[1/d!]}$ admits a projective coarse moduli space $LAdm^{g,r,\min}_{q,s,d}$, which is a finite étale scheme over the coarse moduli scheme $\overline{M}_{q,s}$ associated to $\overline{\mathcal{M}}_{q,s}$.

Proof. All but the last assertion the assertions in the theorem follow from the previous sections. The last assertion follows from the Keel-Mori theorem (Theorem A.4.2) and Zariski's main theorem for morphisms of algebraic spaces (see [30, Tag 05W7]) applied to the morphism $LAdm^{g,r,\min}_{q,s,d} \longrightarrow \overline{M}_{q,s}$ in the following commutative diagram.

$$\begin{array}{ccc} \mathcal{LA}dm_{q,s,d}^{g,r,\min} & \longrightarrow & LAdm_{q,s,d}^{g,r,\min} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{q,s} & \longrightarrow & \overline{M}_{q,s} \end{array}$$

Thus, $LAdm_{q,s,d}^{g,r,\min}$ is a projective algebraic space. Furthermore, we know that $\overline{M}_{q,s}$ is a projective scheme (see [19]). Then, using the fact that a finite-type, separated and quasi-finite morphism of algebraic spaces is representable by schemes (see [26, Proposition 3.1.] for details), we conclude that the coarse moduli space $LAdm_{q,s,d}^{g,r,\min}$ is indeed a projective scheme.

■

3.4 Comparison with the classical Hurwitz stack

Historically, admissible covers were defined in order to compactify the Hurwitz moduli space $\mathcal{H}^{d,b}$, which parametrises isomorphism classes of simple branched coverings of $\mathbb{P}^1$ with b branched points of a fixed degree d. The intuitive idea behind compactifying the moduli space of covers of curves is to allow both the source and target curves to become singular. This leads to the notion of admissibility.

In view of Theorem 3.3.1, restricting to the closed substack $\mathcal{LH}^{d,b} \longrightarrow (Sch)_{\text{ét}}$ of $\mathcal{LA}dm_{q,s,d}^{g,r,\min} \longrightarrow (Sch)_{\text{ét}}$ that consists of admissible covers of a fixed genus zero curve with simple ramification, we obtain a complete modular interpretation of the compactification of the classical Hurwitz stack, as introduced in [16, Section §4]. In other words, $\mathcal{LH}^{d,b}$ is a modular compactification of the open substack $\mathcal{H}^{d,b}$.

Moreover, by the main Theorem 3.3.1, there exists a unique extension $\mathcal{LH}^{d,b} \longrightarrow \overline{\mathcal{M}}_{g,b}$ of the finite étale morphism $\mathcal{H}^{d,b} \longrightarrow \mathcal{M}_{g,b}$ such that the following map of moduli stacks commutes:

$$\begin{array}{ccc} \mathcal{H}^{d,b} & \longrightarrow & \mathcal{M}_{g,b} \\ \downarrow & & \downarrow \\ \mathcal{LH}^{d,b} & \longrightarrow & \overline{\mathcal{M}}_{g,b} \cong \mathcal{LM}_{g,b}^{\min} \end{array}$$

In fact, Theorem 3.3.1 implies that the coarse moduli space associated to the stack $\mathcal{LH}^{d,b}$ recovers the coarse moduli space originally mentioned in Harris and Mumford's paper [16, Theorem 4].

APPENDIX A

Geometry of moduli stacks

In this chapter we mention the basic definitions and results concerning the geometry of moduli stacks following M. Olsson's book Algebraic spaces and stacks [23].

The main hindrance to the existence of a fine moduli space is the *presence of non-trivial automorphisms of families*. The non-trivial automorphisms prevent the moduli functor to enjoy sheaf like properties. This gives rise to the idea of formulating our moduli problem *without passing to isomorphism classes.* This requires to consider sheaf properties for our functor in the category of groupoids (a groupoid is a small category in which every morphism is an isomorphism) instead and this gives rise to the notion of a *moduli stack*. Morally, moduli stacks are *sheaves in groupoids*. In other words the right language to work with stacks is the two-categorical language of fibered categories (a category fibered in groupoids is something that has 'pull - backs' like vector bundles) and pseudo functors. We shall refer to [31, Chapter 2. and Chapter 3.] for a detailed survey on Grothendieck topologies and fibered categories. Formally speaking,

Definition A.0.1 (Stack fibered in groupoids)**.** A category fibered in groupoids $\mathfrak{X} \longrightarrow (Sch/S)$ is a stack if

1. The functors $\mathcal{I}som_S(x, y) : (Sch/S)^{op} \longrightarrow (Sets)$ which associates to any morphism $f : T \longrightarrow S$ the set of isomorphisms in $\mathfrak{X}(T)$ between f^*x and f^*y are sheaves in the étale topology for every $x, y \in \mathfrak{X}(S)$.
2. Every étale descent datum for objects in $\mathfrak{X}$ is effective.

We shall refer to [31, Chapter 4.] for a detailed definition of descent data for objects.

A.1 Algebraic spaces and stacks

Definition A.1.1 (Morphisms representable by schemes)**.**

1. A morphism $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ of stacks over $(Sch)_{\text{ét}}$ is representable by schemes if for every morphism $U \longrightarrow \mathfrak{Y}$ from a scheme U, the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} U$ is a scheme.
2. If **P** is a property of morphisms of schemes (e.g. smooth, étale), a morphism $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ of stacks representable by schemes has property **P** if for every morphism $U \longrightarrow \mathfrak{Y}$ from a scheme U, the morphism of schemes $\mathfrak{X} \times_{\mathfrak{Y}} U \longrightarrow U$ of schemes has property **P**.

We would like our moduli functor to enjoy sheaf like properties, i.e. a family of objects over a scheme should be uniquely determined by its restriction to a collection of open subschemes. Now we state *Grothendieck's representability criterion* which essentially says that *schemes are obtained by locally gluing up open subschemes*.

Theorem A.1.2. *Let $F : (Sch)^{op} \longrightarrow (Set)$ be a functor such that :*

1. *F is a Zariski sheaf (i.e a sheaf on (Sch) in the Zariski topology.*
2. *F has an open covering by a collection of representable open subfunctors.*

Then F is representable by a scheme.

In other words, considering sheaves in the Zariski topology that have an open cover by schemes, we recover the well known notion of schemes. An immediate generalisation of the above is the definition of *algebraic spaces*.

Definition A.1.3 (Algebraic space)**.** An algebraic space is a sheaf $\mathfrak{X}$ on the étale site $(Sch/S)_{\text{ét}}$ such that

1. $\mathfrak{X}$ has an open covering by a collection of representable open subfunctors in the étale topology.
2. There exists a scheme U and a surjective étale morphism $U \longrightarrow \mathfrak{X}$ representable by schemes, called an *étale atlas*.

Indeed schemes are algebraic spaces by the Yoneda embedding. Moreover, condition (3) in the definition of algebraic space guarantees that the diagonal of an algebraic space is representable by schemes.

A.1.1 Constructible sheaves and algebraic spaces

Recall that a sheaf $\mathfrak{X}$ on a scheme X in the étale topology is said to be *constructible* if every quasi-compact $U \subset X$ is a finite union of locally closed subschemes, $U = \bigcup_i Z_i$ such that $\mathfrak{X}_{Z_i}$ is finite and $\mathfrak{X}_{Z_i'}$ is a constant sheaf for some finite étale map $Z_i' \longrightarrow Z_i$. Equivalently, every irreducible closed subscheme $Z \subset X$ contains a non-empty open subscheme U such that $\mathfrak{X}_U$ is a locally constant sheaf.

Following [20], we have the following representability theorem for constructible sheaves.

Theorem A.1.4. *Let $\mathfrak{X}$ be a constructible sheaf on a scheme X in the étale topology. Then $\mathfrak{X}$ is representable by a finite algebraic space.*

A.1.2 Deligne-Mumford stacks

In order to do geometry over stacks we would like our moduli stacks to be as close to as possible to the world of schemes. Thus, one studies those special stacks $\mathcal{X} \longrightarrow (Sch/U)_{\text{ét}}$ which admit an étale (resp. smooth) atlas, i.e. a surjective representable étale (resp. smooth) morphism $X \longrightarrow \mathfrak{X}$. Such stacks are called as Deligne-Mumford (resp. algebraic) stacks. This leads to the following set of definitions.

Definition A.1.5 (Representable morphisms)**.**

1. A morphism $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ of stacks over $(Sch)_{\text{ét}}$ is representable if for every choice of a morphism $U \longrightarrow \mathfrak{Y}$ from a scheme U, the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} U$ is an algebraic space.

2. If **P** is a property of morphisms of schemes (e.g. smooth, étale) which is étale local on the source, a representable morphism $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ of stacks has property **P** if for every choice of a morphism $U \longrightarrow \mathfrak{Y}$ from a scheme U and of an étale atlas $V \longrightarrow \mathfrak{X} \times_{\mathfrak{Y}} U$, the composition $V \longrightarrow \mathfrak{X} \times_{\mathfrak{Y}} U \longrightarrow U$ is a morphism of schemes satisfying property **P**.

Definition A.1.6 (Deligne-Mumford stack)**.** A stack $\mathcal{X}$ is *Deligne-Mumford (DM)* if

- the diagonal $\Delta_{\mathcal{X}/U} : \mathcal{X} \longrightarrow \mathcal{X} \times_U \mathcal{X}$ is representable, and
- there exists a scheme S and an étale surjective morphism $S \longrightarrow \mathcal{X}$ (called an étale atlas).

Definition A.1.7 (Algebraic stack)**.** A stack $\mathcal{X}$ is an *algebraic stack* if

- the diagonal $\Delta_{\mathcal{X}/U} : \mathcal{X} \longrightarrow \mathcal{X} \times_U \mathcal{X}$ is representable and,
- there exists a scheme S and a smooth surjective morphism $S \longrightarrow \mathcal{X}$ (called a smooth atlas).

Example A.1.8. The moduli space of n-pointed stable stable curves $\overline{\mathcal{M}}_{g,n}$ is a DM stack over $\mathbb{Z}$ for $2g-2+n > 0$. The moduli space of smooth curves $\mathcal{M}_g$ is an open substack of $\overline{\mathcal{M}}_{g,0}$ and hence, is a DM stack over $\mathbb{Z}$ for all $g \geq 2$. See [10].

Remark A.1.9.

1. The atlas $\underline{S} \longrightarrow \mathcal{X}$ is a representable morphism.
2. Some sources add the extra hypothesis in the definition that the diagonal $\Delta_{\mathcal{X}/U} : \mathcal{X} \longrightarrow \mathcal{X} \times_U \mathcal{X}$ is quasi-compact and quasi-separated.
3. The surjectivity of the atlas $S \longrightarrow \mathcal{X}$ can be rephrased as requiring that there exists a family of objects ζ over S such that for any object η in $\mathfrak{X}$, $\eta \cong \zeta_x$ for some $x \in \mathfrak{X}(k)$, where k is an algebraically closed field.
4. {Algebraic schemes} $\subset$ {Algebraic spaces} $\subset$ {Algebraic stacks}.
5. The category of algebraic spaces and algebraic (resp. DM) stacks are closed under fiber products.

The following definitions lead to the geometric notions of stacks and morphisms of stacks.

Definition A.1.10 (Properties of algebraic spaces and stacks)**.** Let $\mathbf{P}$ be a smooth (resp. étale) local property of schemes, then an algebraic (resp. DM) stack $\mathfrak{X}$ has property $\mathbf{P}$ if U does, where $U \longrightarrow \mathfrak{X}$ is a smooth (resp. étale) atlas (eg. normal, locally noetherian, locally of finite type etc.).

Remark A.1.11. If U satisfies $\mathbf{P}$ for some atlas $U \longrightarrow \mathfrak{X}$, then V satisfies $\mathbf{P}$ for any other atlas $V \longrightarrow \mathfrak{X}$.

Definition A.1.12 (Properties of morphisms of algebraic spaces and stacks)**.** Let **P** be a property of schemes smooth (resp. étale) local on the source and target and stable under base change and composition. Then a morphism $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ of algebraic (resp. DM) stacks has **P** if there exists smooth (resp. étale) atlases $V \longrightarrow \mathfrak{Y}$ and $U \longrightarrow \mathfrak{X} \times_{\mathfrak{Y}} V$ in the cartesian diagram

$$\begin{array}{ccccc} U & & & & \\ & \searrow & & \searrow & \\ & & \mathfrak{X} \times_{\mathfrak{Y}} V & \longrightarrow & V \\ & & \downarrow & & \downarrow \\ & & \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

the composition $U \longrightarrow V$ has property **P**.

Remark A.1.13. If $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ satisfies **P** as above for some atlas, then one can show that it satisfies **P** for every atlas.

In particular, if $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is representable by schemes, then the morphism has property **P** if for every morphism $U \longrightarrow \mathfrak{Y}$ from a scheme U, the morphism of schemes $\mathfrak{X} \times_{\mathfrak{Y}} U \longrightarrow U$ of schemes has property **P**. This is in accordance with the definition A.1.1. For example, **P** could be étale, unramified, smooth, proper, separated, finite, affine, etc.

Remark A.1.14. If $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is a representable morphism of algebraic stacks, then the relative diagonal $\Delta_{\mathfrak{X}/\mathfrak{Y}} : \mathfrak{X} \longrightarrow \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is representable by schemes. This fact follows from theorem A.1.21.

More generally, for any morphism of algebraic stacks $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$, the diagonal $\Delta_{\mathfrak{X}/\mathfrak{Y}} : \mathfrak{X} \longrightarrow \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is representable.

The following definitions give rise to the notion of topology on a stack and topological properties of stacks and morphisms of stacks.

Definition A.1.15 (Topological space of an algebraic stack)**.** If $\mathfrak{X}$ is an algebraic stack, the topological space of $\mathfrak{X}$ is the set $|\mathfrak{X}| := \{x \in \mathfrak{X}(k) \mid k \text{ is a field}\}/\sim$, where $x_1 \sim x_2$, $x_1 \in \mathfrak{X}(k_1)$, $x_2 \in \mathfrak{X}(k_2)$ if there exist field extensions $k_1 \longrightarrow k_3$ and $k_2 \longrightarrow k_3$ such that $x_{1|k_3}$ and $x_{2|k_3}$ are isomorphic in $\mathfrak{X}(k_3)$.

Definition A.1.16 (Topological properties of stacks)**.**

1. A substack $\mathfrak{Y} \subset \mathfrak{X}$ is an open substack (resp. closed substack) if the inclusion is representable

by an open immersion (resp. by a closed immersion). Moreover, an open or closed substack is also an algebraic stack.

Now we can define the Zariski topology on $|\mathfrak{X}|$ by defining the open subsets to be the collection $|\mathfrak{U}|$ where $\mathfrak{U}$ varies over all the open substacks of $\mathfrak{X}$.

2. An algebraic stack $\mathfrak{X}$ is quasi-compact, connected, or irreducible if $|\mathfrak{X}|$ is.

A.1.3 Characterization of DM stacks

The geometry of a DM stack $\mathfrak{X}$ is encoded in its diagonal morphism $\mathfrak{X} \longrightarrow \mathfrak{X} \times_U \mathfrak{X}$. Indeed, for any U-scheme S and any morphism $(X_1, X_2) : S \longrightarrow \mathfrak{X} \times_U \mathfrak{X}$, where $X_1, X_2 \in \mathfrak{X}(S)$ correspond to the two projections, we have the following 2-cartesian diagram :

$$\begin{array}{ccc} \mathcal{I}som_S(X_1, X_2) & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle (X_1, X_2)} \\ \mathfrak{X} & \xrightarrow{\Delta_{\mathfrak{X}/U}} & \mathfrak{X} \times_U \mathfrak{X} \end{array}$$

where $\mathcal{I}som_S(X_1, X_2)$ is an algebraic space.

Definition A.1.17. If $S = \operatorname{Spec} K$, where K is a field and $X_1 = X_2 = X$, then the stabilizer of the point $X \in \mathfrak{X}(S)$ is defined as the group algebraic space $G_X := Aut_S(X)$.

For a DM stack, one can show that the diagonal is an unramified morphism and hence the algebraic space $\mathcal{I}som_S(X_1,X_2)$ is a finite reduced group algebraic space. Moreover, the unramifiedness of the diagonal characterises Deligne-Mumford stacks. Precisely,

Theorem A.1.18. *Let $\mathfrak{X}$ be a noetherian algebraic stack. Then the following are equivalent:*

1. *$\mathfrak{X}$ is Deligne–Mumford.*
2. *The diagonal $\mathfrak{X} \longrightarrow \mathfrak{X} \times_U \mathfrak{X}$ is unramified.*
3. *Every point of $\mathfrak{X}$ has a finite and reduced stabilizer group.*

In particular, the diagonal of a DM stack is quasi-finite and separated and hence quasi-affine by Zariski's main theorem.

Remark A.1.19. The diagonal morphism $\mathfrak{X} \longrightarrow \mathfrak{X} \times_U \mathfrak{X}$ of stacks need not be an immersion, unlike schemes or algebraic spaces. In fact, algebraic spaces are characterised by the fact that the diagonal is a monomorphism. Precisely,

Theorem A.1.20. *Let $\mathfrak{X}$ be a noetherian algebraic stack such that its diagonal is representable by schemes. Then the following are equivalent:*

1. *$\mathfrak{X}$ is an algebraic space.*
2. *The diagonal $\mathfrak{X} \longrightarrow \mathfrak{X} \times_U \mathfrak{X}$ is a monomorphism.*
3. *Every point of $\mathfrak{X}$ has a trivial stabilizer group.*

In general, the diagonal of a stack need not be representable by a scheme. Under certain separation axioms, we have the following fact:

Theorem A.1.21. *Let S be a scheme. Let $f : \mathfrak{X} \longrightarrow T$ be a morphism of algebraic spaces such that T is a scheme and f is locally quasi-finite and separated. Then $\mathfrak{X}$ is representable by schemes. See [30, Tag 03XX]*

A.2 The Hilbert scheme and its applications

The study of Hilbert schemes is central to understanding the diagonal of moduli stacks, hence we discuss that in the next section.

The Hilbert functor seeks to classify flat families of finitely presented closed subschemes of proper families. To be precise, let $f : X \longrightarrow S$ be a projective morphism of noetherian schemes of finite type. Implicitly, we have a fixed embedding $X \longrightarrow \mathbb{P}^n_S$, hence a fixed very ample line bundle $\mathcal{L}_{X/S}$ on X corresponding to which we can consider a Hilbert polynomial.

Consider the Hilbert functor

$$\mathfrak{Hilb}_{X/S} : (Sch/S)^{op} \longrightarrow (Sets)$$

$$T/S \longrightarrow \{\text{closed subschemes } Z \subset X_T := X \times_S T \mid Z \longrightarrow T \text{ is proper and flat}\}$$

For any polynomial P, we also define a subfunctor $\mathfrak{Hilb}^P_{X/S} \subset \mathfrak{Hilb}_{X/S}$

$$\mathfrak{Hilb}^P_{X/S} : (Sch/S)^{op} \longrightarrow (Sets)$$

$$T/S \longrightarrow \{Z \in \mathfrak{Hilb}_{X/S}(T) |\ P_{Z_t}(n) = P(n)\ \forall t \in T\}$$

where P_{Z_t} is the Hilbert polynomial corresponding to the closed fiber Z_t. Since the Hilbert polynomials are locally constant for proper flat families, we have a decomposition

$$\mathfrak{Hilb}_{X/S} = \bigsqcup_P \mathfrak{Hilb}^P_{X/S}$$

One of Grothendieck's significant results was to prove the representability of Hilbert functors by a projective scheme. Precisely,

Theorem A.2.1 (Grothendieck)**.** *The Hilbert functor $\mathfrak{Hilb}^P_{X/S} : (Sch/S)^{op} \longrightarrow (Sets)$ is representable by a finite type projective S scheme $hilb^P_{X/S}$ equipped with an S-very ample line bundle. Hence, $\mathfrak{Hilb}_{X/S}$ is representable by a projective scheme $hilb_{X/S} = \bigsqcup_P hilb^P_{X/S}$*

Proof. See [11, chapter 5].

■

In this section, we are interested in studying the application of Hilbert schemes to understand the properties of the diagonal of a moduli space $\mathfrak{X} \longrightarrow (Sch/S)$. In particular, we are interested in studying the automorphism group of families parametrized by the moduli problem.

Let X, Y be projective noetherian schemes of finite type over S. Consider the functor $\mathcal{H}om_S(X,Y)$ parametrizing morphisms of schemes, i.e.

$$\mathcal{H}om_S(X,Y) : (Sch/S)^{op} \longrightarrow (Sets)$$

$$T/S \longrightarrow \{\text{morphisms } X_T \longrightarrow Y_T\}$$

Theorem A.2.2. *Suppose X and Y are projective schemes over S with $X \longrightarrow S$ flat. Then the functor $\mathcal{H}om_S(X,Y)$ is a representable open subfunctor of the Hilbert functor $\mathfrak{Hilb}_{X\times_S Y/S}$. In particular, $\mathcal{H}om_S(X,Y)$ is representable by a quasi-projective scheme $Hom_S(X,Y)$ of $hilb_{X\times_S Y/S}$.*

As a corollary, we obtain that the functor

$$\mathcal{I}som_S(X,Y) : (Sch/S)^{op} \longrightarrow (Sets)$$

classifying isomorphisms $X_T \cong Y_T$ *is representable by a quasi-projective scheme* $Isom_S(X,Y)$ *of* $hilb_{X\times_S Y/S}$.

Proof. For every S-scheme T, define the morphism

$$\mathcal{H}om_S(X,Y)(T) \longrightarrow \mathfrak{Hilb}_{X\times_S Y/S}(T)$$
$$f : X_T \longrightarrow Y_T \longmapsto \Gamma_f$$

where $\Gamma_f = \mathrm{Img}(Id,f)$ is the graph of the morphism f obtained as the image of the following maps

$$X_T \xrightarrow{(Id,f)} X_T \times_T Y_T \xrightarrow{\sim} X \times_S Y \times_S T = (X \times_S Y)_T$$

Since we are working over separated schemes, the graph $\Gamma_f \cong X_T$ is a closed subscheme of $X \times_S T$ and it is projective and flat over T since $X \longrightarrow S$ is so. By the universal property of fiber product of schemes, a morphism is uniquely determined by its graph. Hence, $\mathcal{H}om_S(X,Y)$ is considered as a subfunctor of $\mathfrak{Hilb}_{X\times_S Y/S}$ by the identification

$$\mathcal{H}om_S(X,Y)(T) = \{\text{flat closed subscheme } Z \subset X \times_S Y \mid Z \cong (X \times_S Y)_T\}$$

In order to prove

$$\mathcal{H}om_S(X,Y)(T) \longrightarrow \mathfrak{Hilb}_{X\times_S Y/S}(T)$$

is a a representable open subfunctor, it suffices to show that for all S-schemes T and maps $\mathfrak{Z} : T \longrightarrow \mathfrak{Hilb}_{X\times_S Y/S}$ determined by the closed subscheme $\mathfrak{Z} \subset (X \times_S Y)_T$, the fiber product

$$\begin{array}{ccc} \mathcal{H}om_S(X,Y) \times_{\mathfrak{Hilb}_{X\times_S Y/S}} T & \longrightarrow & T \\ \downarrow & & \downarrow \\ \mathcal{H}om_S(X,Y) & \longrightarrow & \mathfrak{Hilb}_{X\times_S Y/S} \end{array}$$

is represented by an open subscheme of T. In other words, we need to show that for any T-scheme T', the set

$$\left(\mathcal{H}om_S(X,Y) \times_{\mathfrak{Hilb}_{X\times_S Y/S}} T\right)(T') = \{(T' \xrightarrow{\alpha} T, X_{T'} \xrightarrow{f} Y_{T'}) \mid \mathfrak{Z}_{T'} = \Gamma_f\}$$

is an open subscheme of T'. Hence, by the description of the graph Γ_f above as a unique flat closed subscheme $Z \subset (X \times_S Y)_T$ with $Z \cong X_T$, it is enough to show that the conditions $\mathfrak{Z}_{T'} \cong X'_T$ and $\mathfrak{Z}_{T'} \longrightarrow T'$ is flat are open conditions in the sense of the following important lemmas.

■

Lemma A.2.3. *Let $f : U \longrightarrow S$ and $g : V \longrightarrow S$ be flat projective morphisms of noetherian schemes. Let $\pi : U \longrightarrow V$ be a projective morphism such that $\pi \circ g = f$. Then S has uniquely determined open subschemes $S_2 \subset S_1 \subset S$ with the following universal properties:*

1. *For any S-scheme T, the morphism $\pi_T : U_T \longrightarrow V_T$ is flat if and only if the structure morphism $T \longrightarrow S$ factors uniquely as $T \longrightarrow S_1 \longrightarrow S$.*
2. *For any S-scheme T, the morphism $\pi_T : U_T \longrightarrow V_T$ is an isomorphism if and only if the structure morphism $T \longrightarrow S$ factors uniquely as $T \longrightarrow S_2 \longrightarrow S$.*

Sketch of proof: By the openness of the flat locus, the subset

$$U' := \{v \in V \mid \pi \text{ is flat at } v\} \subset U$$

is an open subset. Since f is proper, $f(U - U') \subset S$ is a closed subscheme. Then the open subscheme $S_1 := S - f(U - U') \subset S$ satisfies the required universal property by the fiber local property of flat morphisms (see [30, Tag 039C]).

Using (1), we can assume $S = S_1$ (hence, π is flat) and prove (2). Let $\mathcal{L} := \mathcal{L}_{U/V}$ be a very ample invertible sheaf on X corresponding to $U \longrightarrow \mathbb{P}^m_V \longrightarrow V$. Since, $U \longrightarrow V$ is projective, there exists $n \in \mathbb{N}$ such that $R^i\pi_*\mathcal{L}(n) = 0,$ forall $i > 0$ and $\mathcal{L}(n)$ is generated by global sections. By flatness of π and Grauerts's theorem (see [30, Tag 0AXD]), $\pi_*\mathcal{L}(n)$ is an invertible sheaf. Let $V' \subset V$ such that $\pi_*\mathcal{L}(n)_{|V'} \cong \mathcal{O}_{V'}$. Then $S_2 := S - g(V - V')$ satisfies the required universality.

■

An immediate corollary of the above lemma asserts that 'compatibility of sections of morphisms of schemes' satisfies a universal property. Precisely,

Corollary A.2.4. Let $f : X \longrightarrow S$ be a flat, projective morphism of noetherian schemes. Let C, D be closed subschemes of X flat over S. Then S has uniquely determined subscheme $S_1 \subset S$ such that for any morphism $T \longrightarrow S$, we have $C_T \subseteq D_T$ if and only if $T \longrightarrow S$ factors through S_1.

The above lemma can be generalised to any moduli problem parametrizing a family of geometric objects. In order to study the representability of the diagonal of a moduli space, we need to verify that all the properties of the family of geometric objects parametrized by the moduli problem satisfies a 'universal property' as above. Precisely,

Let $f : U \longrightarrow S$ and $g : V \longrightarrow S$ be a family of schemes with property ***P****. Let $\pi : U \longrightarrow V$ be a projective morphism such that $\pi \circ g = f$. Then S has a uniquely determined subscheme $S_1 \subset S$ with the following universal property: For any S-scheme T, the morphism $\pi_T : U_T \longrightarrow V_T$ has property* ***P*** *if and only if the structure morphism $T \longrightarrow S$ factors uniquely as $T \longrightarrow S_1 \longrightarrow S$.*

Example A.2.5 (Moduli space of stable curves). In this example, we show that if $\eta_1, \eta_2 \in \overline{\mathcal{M}}_{g,n}$ are families of n-pointed stable curves of genus g, then the isomorphism functor $\mathcal{I}som_S(\eta_1, \eta_2)$ is representable by a scheme.

For any scheme X/S, define a functor

$$\mathfrak{sechilb}_{X/S} : (Sch/S)^{op} \longrightarrow (Sets)$$

$$T/S \longmapsto \{(Z, s) \mid Z \in \mathfrak{hilb}_{X/S}(T),\ s : T \longrightarrow Z \text{ a section}\}$$

For every S-scheme T, we have a canonical map

$$\mathfrak{sechilb}_{X/S}(T) \longrightarrow \mathfrak{hilb}_{X/S}(T)$$

$$(Z, s) \longmapsto Z$$

Let $u : \mathfrak{U} \longrightarrow \text{hilb}_{X/S}$ be the universal object of the Hilbert functor. Given any $(Z, s) \in \mathfrak{sechilb}_{X/S}(T)$, by the definition of a fine moduli space, there exists a unique map $f : T \longrightarrow \text{hilb}_{X/S}$ such that $Z \cong T \times_{\text{hilb}_{X/S}} \mathfrak{U}$. Thus, by composition, we have a uniquely determined morphism $T \longrightarrow Z \cong T \times_{\text{hilb}_{X/S}} \mathfrak{U} \longrightarrow \mathfrak{U}$. Conversely, given a morphism $g : T \longrightarrow \mathfrak{U}$, there is a unique morphism $T \xrightarrow{g} \mathfrak{U} \xrightarrow{u} \text{hilb}_{X/S}$. Then the pair $Z := T \times_{\text{hilb}_{X/S}} \mathfrak{U}$ and $s := (Id, g) : T \longrightarrow Z$ determine an element in $\mathfrak{sechilb}_{X/S}(T)$.

$$\begin{array}{ccc} Z \cong T \times_{\mathrm{hilb}_{X/S}} \mathfrak{U} & \longrightarrow & \mathfrak{U} \\ {\scriptstyle s}\uparrow\downarrow & & \downarrow{\scriptstyle u} \\ T & \longrightarrow & \mathrm{hilb}_{X/S} \end{array}$$

In other words, the above discussion shows that the moduli functor $\mathfrak{sechilb}_{X/S}$ is representable by the universal object $\mathfrak{U}$ of the Hilbert functor. Iteratively, the functor

$$\mathfrak{sechilb}^n_{X/S} : (Sch/S)^{op} \longrightarrow (Sets)$$

$$T/S \longmapsto \{(Z, s_1, \ldots, s_n) \mid Z \in \mathfrak{hilb}_{X/S}(T),\ s_1, \ldots, s_n : T \longrightarrow Z \text{ ordered sections}\}$$

is represented by the scheme $\mathfrak{U}^{(n)} := \underbrace{\mathfrak{U} \times_{\mathrm{hilb}_{X/S}} \cdots \times_{\mathrm{hilb}_{X/S}} \mathfrak{U}}_{n \text{ times}}$ consisting of ordered copies of $\mathfrak{U}$. Let $(\mathfrak{V}^{(n)}, \sigma_1, \ldots, \sigma_n)$ be the universal object of $\mathfrak{sechilb}^n_{X/S}$. Moreover, imposing the additional condition that the sections are pairwise mutually disjoint, the functor $\mathfrak{sechilb}^n_{X/S}$ is represented by the complement of the pairwise diagonal of $\mathfrak{U} \times_{\mathrm{hilb}_{X/S}} \cdots \times_{\mathrm{hilb}_{X/S}} \mathfrak{U}$. In particular, if $X \longrightarrow S$ is an n-pointed stable curve of genus g, then geometrically connected fibers and ampleness of the relative dualising sheaf (stability) are open conditions, hence they satisfy the universal property mentioned above. The condition that each section s_i lies in the smooth locus of the fiber is taken care of by considering $\mathfrak{U}^{(n)} - \bigsqcup_{i=1}^n \sigma_i^{-1}(\mathfrak{V}^{(n)} - \mathfrak{V}^{(n),\mathrm{smooth}})$. The compatibility of sections in morphisms of family of stable curves is taken care of by Corollary A.2.4.

Thus, keeping track of the subschemes involved above, we get that $\mathcal{I}som_S(\eta_1, \eta_2)$ is representable by a noetherian group scheme of finite type over S. In particular, working over the field of complex numbers, $\mathcal{I}som_S(\eta_1, \eta_2)$ is representable by a complex Lie group.[1]

The following non-trivial examples generalise the moduli functor $\mathcal{I}som_S(\eta_1, \eta_2)$ to morphisms of stacks (resp. algebraic spaces) and logarithmic stacks (resp. algebraic spaces).

[1]For complex manifolds X, the result is true more generally for compact complex manifolds.

Example A.2.6 (HOM Stack). Let $\mathfrak{X}$ be an algebraic space of finite type, flat and proper over $\mathcal{S}$ and $\mathfrak{Y}$ be an algebraic stack of finite type with quasi-compact and quasi-separated diagonal and affine stabilizers. Then the functor

$$\mathcal{H}om_S(\mathfrak{X},\mathfrak{Y}) : (Sch/S)^{op} \longrightarrow (Sets)$$
$$\mathcal{T}/S \longrightarrow \{\text{morphisms } \mathfrak{X}_T \longrightarrow \mathfrak{Y}_T\}$$

is representable by an algebraic space over S.

This is the work of M. Aoki in [4].

Example A.2.7 (Logarithmic HOM Stack). Let $\mathfrak{X}$ be an algebraic space of finite type, flat and proper with fs log structure over a logarithmic scheme $\mathcal{S}$ and $\mathfrak{Y}$ be a log algebraic stack (in the sense of Definition 2.2.1) of finite type with quasi-compact and quasi-separated diagonal and affine stabilizers. Then the functor

$$\mathcal{H}om_{\mathcal{S}}(\mathfrak{X},\mathfrak{Y}) : (\mathbf{LogSch}^{fs}_{st,t}/\mathcal{S})^{op} \longrightarrow (Sets)$$
$$\mathcal{T}/\mathcal{S} \longrightarrow \{\text{morphisms } \mathfrak{X}_{\mathcal{T}} \longrightarrow \mathfrak{Y}_{\mathcal{T}}\}$$

is representable by a logarithmic algebraic space over $\mathcal{S}$.

This is the work of J. Wise in [32].

A.2.1 The relative Picard functor

For any scheme X, let $Pic(X) := H^1(X, \mathcal{O}_X^*)$ be the group of isomorphism class of line bundles on X. Consider the absolute Picard functor classifying line bundles:

$$Pic_{X/S} : (Sch/S)^{op} \longrightarrow (Sets)$$
$$T \longmapsto Pic(X \times_S T)$$

Remark A.2.8. The absolute Picard functor $Pic_{X/S}$ need not be representable. In fact, if there is a universal family $\mathcal{L} \longrightarrow \mathcal{M}$ for $Pic_{X/S}$, then for every scheme Y and a non-trivial class of $\mathcal{L}' \in Pic_{X/S}(Y)$, there exists a unique map $f : Y \longrightarrow \mathcal{M}$ such that $f^*\mathcal{L} \cong \mathcal{L}'$. In other words, this basically tells us that if $Y = \bigcup U_i$, where U_i's are the trivialising neighbourhoods of the line bundle $\mathcal{L}'$, then the line bundle $\mathcal{L}'_{|X\times_S Y}$ is trivial, a contradiction.

Thus, to obtain a full-fledged modular interpretation, one defines the relative Picard functor:

$$Pic^{rel}_{X/S} : (Sch/S)^{op} \longrightarrow (Sets)$$
$$T \longmapsto Pic(X \times_S T)/\pi_T^* Pic(T)$$

where $\pi_T : X \times_S T \longrightarrow T$ is the canonical projection.

Equivalently, the relative Picard functor can be defined as

$$T \longmapsto H^0(T, R^1 \pi_{T*} \mathbb{G}_m)$$

where $\pi_T : X \times_S T \longrightarrow T$ and $\mathbb{G}_m$ is the sheaf of units on $X \times_S T$. It was proven by M. Artin in [7, Theorem 7.3] that under the additional hypothesis of cohomological flattness, the relative Picard functor is representable by an algebraic space locally of finite type by using Artin's axioms [7, Theorem 5.3].

Theorem A.2.9. *Let $f : X \longrightarrow S$ be a proper, flat map of noetherian schemes which is finitely presented and cohomologically flat (i.e. $O_S \cong f_* O_X$). Then the relative Picard functor $Pic^{rel}_{X/S}$ is represented by an algebraic space locally of finite type over S.*

A.3 Properness for morphisms of algebraic stacks

Definition A.3.1.

1. A morphism $\mathfrak{X} \longrightarrow \mathfrak{Y}$ of algebraic stacks (not necessarily representable) is universally closed if for every morphism $\mathfrak{Y}' \longrightarrow \mathfrak{Y}$ of algebraic stacks, the morphism $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}' \longrightarrow \mathfrak{Y}'$ induces a closed map $|\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}'| \longrightarrow |\mathfrak{Y}'|$.

2. A morphism $\mathfrak{X} \longrightarrow \mathfrak{Y}$ of algebraic stacks (not necessarily representable) is separated if the diagonal morphism $\Delta_{\mathfrak{X}/\mathfrak{Y}} : \mathfrak{X} \longrightarrow \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is proper (see Remark A.1.14 which says that the diagonal is representable).

3. A morphism $\mathfrak{X} \longrightarrow \mathfrak{Y}$ of algebraic stacks (not necessarily representable) is proper if it is universally closed, separated and of finite type.

Remark A.3.2. The subtle point is that for a morphism of algebraic stacks $\mathfrak{X} \longrightarrow \mathfrak{Y}$, the diagonal $\Delta_{\mathfrak{X}/\mathfrak{Y}} : \mathfrak{X} \longrightarrow \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ need not be a monomorphism unlike in the case of morphism of schemes or algebraic spaces. The above definitions are consisitent with the usual definitions in the category of schemes. In fact, since proper monomorphisms of schemes are closed immersions, the diagonal is proper if and only if it is a closed immersion.

Theorem A.3.3 (Weak valuative criterion for properness)**.** *Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a finite type morphism of noetherian algebraic stacks. Consider a 2-commutative diagram*

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \mathfrak{Y} \end{array}$$

where R is a discrete valuation ring with fraction field K. Then $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is proper if and only if there exists extensions (which we can take of finite transcendence degree) $R \longrightarrow R'$ and $K \longrightarrow K'$ extending the following diagram uniquely up to 2-isomorphism.

$$\begin{array}{ccccc} \operatorname{Spec} K' & \longrightarrow & \operatorname{Spec} K & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} R' & \longrightarrow & \operatorname{Spec} R & \longrightarrow & \mathfrak{Y} \end{array}$$

A.4 Coarse moduli space: Keel-Mori theorem

In order to remove the stacky interpretation of the moduli problem, one considers the associated coarse moduli space, at the risk of losing the universal object. The coarse moduli space is in some sense, the closest approximation to an algebraic stack by an algebraic space. The Keel-Mori theorem guarantees the existence of the coarse moduli space under certain hypotheses.

Definition A.4.1 (Coarse moduli space)**.** A morphism $\pi : \mathfrak{X} \longrightarrow X$ from an algebraic stack to an algebraic space is a coarse moduli space if:

1. Points are in bijection with the objects being parametrized, i.e. for any algebraically closed field k, the induced map of k-valued points $\mathfrak{X}(k)/\sim \longrightarrow X(k)$,

from the set of isomorphism classes of objects of $\mathfrak{X}$ over k is bijective.

2. π is universal for maps to algebraic spaces, i.e. for any other map $f : \mathfrak{X} \longrightarrow Y$ to an algebraic space Y, f uniquely factors as

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\pi} & X \\ & {\scriptstyle f}\searrow & \downarrow \\ & & Y \end{array}$$

Theorem A.4.2 (Keel-Mori)**.** *Let $\mathfrak{X}$ be a Deligne-Mumford stack which is separated and of finite type over a noetherian algebraic space S. Then there exists a coarse moduli space $\pi : \mathfrak{X} \longrightarrow X$ with $O_X = \pi_* O_{\mathfrak{X}}$ such that*

1. *X is separated and of finite type over S.*
2. *π is a proper universal homeomorphism.*
3. *Coarse moduli spaces are stable under flat base changes.*

We shall refer to [23, Theorem 11.1.2] for the proof of the Keel-Mori theorem.

Bibliography

[1] Abramovich D., Chen Q., Gillam W.D., Huang Y., Olsson M., Satriano M. and Sun S. *Logarithmic geometry and moduli*. Handbook of moduli, Vol. I, 1–61, Adv. Lect. Math. (ALM), 24, International Press (2013), see also arXiv:1006.5870v1 [math.AG] (2010).

[2] Abramovich D., Corti A. and Vistoli A. *Twisted bundles and admissible covers*. Communications in Algebra, 31:8, 3547-3618, https://doi.org/10.1081/AGB-120022434.

[3] Alper J. *Artin algebraization and quotient stacks*. arXiv:1510.07804v1 [math.AG].

[4] Aoki M. *Hom stacks*. manuscripta math. 119, 37–56 (2006). https://doi.org/10.1007/s00229-005-0602-1.

[5] Arbarello E., Cornalba M. and Griffiths P. *Geometry of Algebraic curves - II*. Springer-Verlag Berlin Heidelberg, DOI 10.1007/978-3-540-69392-5.

[6] Artin M. *Algebraic approximation of structures over complete local rings*. Inst. Hautes Études Sci. Publ. Math. (1969), no. 36, 23–58, http://www.numdam.org/article/PMIHES-1969–36–23-0.pdf.

[7] Artin M. *Algebraization of formal moduli: I*. Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 21–71. MR 0260746.

[8] Artin M. *Algebraization of formal moduli: II – existence of modifications*. Annals of Mathematics 91 (1970), 88–135, JSTOR, www.jstor.org/stable/1970602.

[9] Artin M. *Versal deformations and algebraic stacks*. Invent Math 27, 165–189 (1974), https://doi.org/10.1007/BF01390174.

[10] Deligne P. and Mumford D. *The irreducibility of the space of curves of given genus*. Publ. Math. I.H.E.S. No.36, 75–100 (1969).

[11] Fantechi B., Göttshe L. Illusie L., Kleiman S.L., Nitsure N. and Vistoli A. *Fundamental algebraic geometry. Grothendieck's FGA explained*. Mathematical Surveys and Monographs 123, Amer. Math. Soc. 2005. x+339 pp, https://bookstore.ams.org/surv-123-s

[12] Fulton W. *Hurwitz schemes and moduli of algebraic curves*. Annals of Mathematics, 90(3), second series, 542-575, https://doi.org/10.2307/1970748.

[13] Gillam W.D. *Logarithmic stacks and minimality*. Internat. J. Math. Vol. 23, No. 07, 1250069 (2012), https://doi.org/10.1142/S0129167X12500693.

[14] Gross M. and Siebert B. *Logarithmic Gromov-Witten invariants*. J. Amer. Math. Soc., 26(2):451–510, 2013. https://doi.org/10.1090/S0894-0347-2012-00757-7.

[15] Hall J. and Rydh D. *Coherent Tannaka duality and algebraicity of Hom-stacks*. Algebra Number Theory 13 (7) 1633 - 1675, 2019. https://doi.org/10.2140/ant.2019.13.1633.

[16] Harris J. and Mumford D. *On the Kodaira dimension of the moduli space of curves*. Invent. Math. 67 (1982), no. 1, 23–88, https://www.jstor.org/stable/25099166.

[17] Kato F. *Log smooth deformation and moduli of log smooth curves*. Internat. J. Math. 11 (2000), no. 2, 215–232, https://doi.org/10.1142/S0129167X0000012X.

[18] Kato K. *Logarithmic structures of Fontaine-Illusie*. Algebraic Analysis, Geometry, and Number Theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 191–224.

[19] Knudsen F. F. *The Projectivity of the Moduli Space of Stable Curves, II: The Stacks* $\overline{\mathcal{M}}_{g,n}$. Math. Scand. 52 (1983), pp. 161-199, https://www.jstor.org/stable/24491475.

[20] Milne J. *Étale Cohomology*. Vol. 33. Princeton University Press; Princeton, NJ: 1980. (Princeton Mathematical Series).

[21] Mochizuki S. *The geometry of the compactification of the Hurwitz scheme*. Publ. Res. Inst. Math. Sci. 31 (1995), no. 3, 355–441; MR 96j:14017.

[22] Ogus A. *Lectures on logarithmic algebraic geometry*. (Cambridge Studies in Advanced Mathematics, pp. I-Iv). Cambridge: Cambridge University Press, https://doi.org/10.1017/9781316941614

[23] Olsson M. *Algebraic spaces and stacks*. American Mathematical Society Colloquium Publications, https://doi.org/10.1090/coll/062.

[24] Olsson M. *Log algebraic stacks and moduli of log schemes*. U.C. Berkeley thesis, May 2001.

[25] Olsson M. *Logarithmic geometry and algebraic stacks*. Annales Scientifiques de l'École Normale Supérieure Volume 36, Issue 5, September–October 2003, Pages 747-791, https://doi.org/10.1016/j.ansens.2002.11.001.

[26] Olsson M. and Starr J. *Quot Functors for Deligne-Mumford Stacks*. Communications in Algebra, 31:8, 4069-4096, 2003, DOI: 10.1081/AGB-120022454

[27] Romagny M. and Wewers S. *Hurwitz spaces*. Séminaires Congrès 13, 2006, p. 313–341, available at https://perso.univ-rennes1.fr/matthieu.romagny/articles/hurwitzspaces.pdf.

[28] Sernesi E. *Deformations of Algebraic Schemes*. Springer-Verlag Berlin Heidelberg Copyright Holder, DOI 10.1007/978-3-540-30615-3.

[29] Talpo M. and Vistoli A. *Deformation theory from the point of view of fibered categories*. Handbook of moduli. Vol. III, Adv. Lect. Math. (ALM), vol. 26, Int. Press, Somerville, MA, 2013, pp. 281–397.

[30] The Stacks Project Authors. *The Stacks project*. An open source textbook and reference work on algebraic geometry, avalibale at https://stacks.math.columbia.edu/.

[31] Vistoli A. *Notes on Grothendieck topologies, fibered categories and descent theory*. arXiv:math/0412512v4 [math.AG].

[32] Wise J. *Moduli of morphisms of logarithmic schemes*. Algebra Number Theory 10 (4) 695 - 735, 2016. https://doi.org/10.2140/ant.2016.10.695.