Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA “TULLIO LEVI-CIVITA”

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Double coset zeta function of some p-groups of maximal class

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Introduction

The study of zeta functions has long been a central theme in number theory and group theory, providing deep insights into the structure and properties of various algebraic objects. Among these, the subgroup zeta function emerges as a powerful tool for analysing the asymptotic behaviour and distribution of subgroups within algebraic structures. The double coset zeta function extends the concept of subgroup zeta functions by considering double cosets instead of single cosets, thus it offers a framework for studying symmetries and invariant properties. The aim of this thesis is to explore the double coset zeta function in depth, beginning with a detailed examination of the subgroup zeta function.

We will start by introducing the Riemann zeta function, which is one of the most famous and known functions in number theory and complex analysis. It contains deep properties of the integers, in particular, through its Euler Product identity. The Riemann zeta function finds one of its applications in group theory, in particular in the study of the subgroup zeta function. The subgroup zeta function counts subgroups of various indices, revealing rich structural information about the group. We will study in details two specific examples: the abelian group $\mathbb{Z}^d$, where d is a natural number, and the discrete Heisenberg group $\mathbb{H}_3$, which is not abelian. In these two examples the subgroup zeta function can be written as a product of Riemann zeta functions.

In order to define the double coset zeta function, we will first introduce the concept of double cosets, denoted as HgK, where H and K are subgroups of a group G and g is an element of G. The concept of double cosets was introduced in group theory to generalise the idea of single cosets and to study the structure and relationships between subgroups of a group. Double cosets arise naturally when analysing the symmetries and invariants that emerge from the actions of two subgroups simultaneously. Historically, their study began with Frobenius, who saw their potential in various applications, including counting problems and understanding group structures in greater detail. We will discuss how double cosets are related to single cosets, with a particular emphasis on the conditions under which they coincide - in particular, if and only if the subgroup is normal. After having analysed in detail some properties of double cosets, we will define the double coset zeta function and we will focus on calculating it for specific groups, including the dihedral group $D_{2^{n+1}}$ of order 2^{n+1}, the semidihedral group, and the quaternion group of the same order. We will derive a recursive formula for the dihedral group of order a power of 2. Indeed, we will establish a connection between the double coset zeta functions of groups of successive orders for the family $D_{2^{n+1}}$. Then we will study the double coset zeta function of the semidihedral and quaternion groups and we will se that it depends on the double coset zeta function of $D_{2^{n+1}}$.

Recalling that the pro-2-dihedral group is defined to be the inverse limit of the the family $D_{2^{n+1}}$, with $n \in \mathbb{N}$, we derive the double coset zeta function for it. Then, we extend the analysis to the dihedral group of order $2p^n$ for an odd prime p and its profinite completion, which is the pro-p-dihedral group.

Chapter 1

Dirichlet series and Riemann zeta function

To fully understand the results that will be presented in this thesis, it is essential to first introduce some fundamental concepts.

In this chapter, we will focus on Dirichlet series, which are among the most important tools in analytic number theory. In particular, we will study the Riemann Zeta function and examine some of its key properties. This groundwork will help to provide the necessary context for the discussions that follows.

1.1 Dirichlet series

Whenever a counting problem yields a sequence of non-negative integers $\{a_n\}_{n\in\mathbb{N}}$, we can gain insight by incorporating this sequence into a generating function. One of the most well-known generating functions is the Dirichlet series:

Definition 1.1. Given a sequence of non-negative integers $\{a_n\}_{n\in\mathbb{N}}$, we define the **Dirichlet series** as

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where s is a complex variable.

We do not look at it only as a formal sum, we also study the largest non-empty subset of $\mathbb{C}$ where it converges.

Remark 1.2. Let s be a complex variable. It can be expressed as $s = \sigma + it$, where σ and t are real numbers. Then $n^s = e^{s\log(n)} = e^{(\sigma+it)\log(n)} = n^\sigma e^{(it)\log(n)}$. This shows that $|n^s| = n^\sigma$, since $|e^{i\theta}| = 1$ when θ is real.

Taking a real number α, we define a *half-plane* as the set of points of the form $s = \sigma + it$ with $\sigma > \alpha$. We will show that for each Dirichlet series, there exists a half-plane $Re(s) > \sigma_c$ in which the series converges, and another half-plane $Re(s) > \sigma_a$ in which it converges absolutely. In the half-plane of convergence, the series represents an analytic function of the complex variable s.

1.2 The half-plane of absolute convergence of a Dirichlet series

First, we observe that if $Re(s) = \sigma \geq \alpha$, then $|n^s| = n^\sigma \geq n^\alpha$. This implies that

$$\left|\frac{a_n}{n^s}\right| \leq \frac{|a_n|}{n^\alpha}.$$

Therefore, if a Dirichlet series converges absolutely for $s = a + ib$, then by comparison, it also converges absolutely for $\mathrm{Re}(s) > a$. This observation leads to the following theorem.

Theorem 1.3. Suppose the series $\sum_{n=1}^{\infty} \left|\frac{a_n}{n^s}\right|$ neither converges for all s nor diverges for all s. Then there exists a real number σ_a, called the *abscissa of absolute convergence*, such that the series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges absolutely when $Re(s) > \sigma_a$, but does not converge absolutely when $Re(s) < \sigma_a$.

Proof. Let D be the set of all σ such that $\sum_{n=1}^{\infty} \left|\frac{a_n}{n^s}\right|$ diverges. D is non-empty because, by assumption, the series does not converge for all s. Moreover, D is bounded above because the series does not diverge for all s. Hence, D has a least upper bound, denoted by σ_a.

If $\sigma < \sigma_a$, then $\sigma \in D$ (otherwise, it would be an upper bound for D smaller than the least upper bound).

If $\sigma > \sigma_a$, then $\sigma \notin D$ (since σ_a is an upper bound for D). □

If the series $\sum_{n=1}^{\infty} \left|\frac{a_n}{n^s}\right|$ converges for all values of s, we define $\sigma_a = -\infty$. Conversely, if the series converges for no values of s, we define $\sigma_a = +\infty$.

An important example of Dirichlet series is the Riemann zeta function.

1.3 Riemann zeta function

In this chapter, we will delve into the Riemann zeta function, one of the most significant functions in number theory and complex analysis, due to its deep connections with the distribution of prime numbers and its appearance in various areas of mathematics, including mathematical physics.

The Reimann zeta function is a specific example of the Dirichlet series, defined by the infinite sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $Re(s) > 1$.

Remark 1.4. The Riemann zeta function is absolutely and uniformly convergent in the domain $\text{Re}(s) \geq 1 + \delta$ for any $\delta > 0$, which implies that it represents an analytic function in the half-plane $\text{Re}(s) > 1$. In particular, for $\text{Re}(s) = \sigma \geq 1 + \delta$, the series $\sum_{n=1}^{\infty} \left|\frac{1}{n^s}\right| = \sum_{n=1}^{\infty} \frac{1}{n^\sigma}$ is bounded by $\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}$, which guarantees the convergence.

The Riemann zeta function contains deep properties of the integers, particularly through its **Euler product identity**:

$$\zeta(s) = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Theorem 1.5. (*Euler product*) Let $s \in \mathbb{C}$. When $Re(s) > 1$, the Riemann zeta function $\zeta(s)$ satisfies the Euler product identity, so:

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - p^{-s}},$$

where p runs through all the prime numbers.

Proof. To prove the Euler product identity, recall that an infinite product $\prod_{n=1}^{\infty} a_n$ of complex numbers a_n converges if the sequence of partial products $P_N = \prod_{n=1}^{N} a_n$ has a nonzero limit. This condition is equivalent to requiring that the series $\sum_{n=1}^{\infty} \log(a_n)$ converges, where "log" denotes the principal branch of the logarithm. The product converges absolutely if the series $\sum_{n=1}^{\infty} |\log(a_n)|$ converges. In this case, any reordering of the product also converges to the same value.

Let

$$E(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}},$$

and let us analyse its logarithm

$$\log(E(s)) = \log\left(\prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}\right).$$

Recall that

$$\log(1-x) = -\sum_{n\geq 1} \frac{x^n}{n},$$

which yields to

$$\log(E(s)) = \sum_p \log\frac{1}{1-p^{-s}} = \sum_p \sum_{n\geq 1} \frac{(p^{-s})^n}{n} = \sum_p \sum_{n\geq 1} \frac{1}{np^{ns}}.$$

This sum converges absolutely for $Re(s) \geq 1+\delta$. Indeed, given that $|p^{ns}| = p^{n\sigma} \geq p^{(1+\delta)n}$, the sum of the corresponding geometric series leads to the following chain of inequalities:

$$\sum_p \sum_{n\geq 1} \frac{1}{np^{(1+\delta)n}} \leq \sum_p \sum_{n\geq 1} \left(\frac{1}{p^{1+\delta}}\right)^n = \sum_p \frac{1}{p^{1+\delta}-1} \leq 2\sum_p \frac{1}{p^{1+\delta}} \leq \sum_{n\geq 1} \frac{1}{n^{1+\delta}},$$

which converges.

The absolute convergence of the product then follows:

$$E(s) = \prod_p \frac{1}{1-p^{-s}} = \exp\left(\log\left(\sum_p \sum_{n\geq 1} \frac{1}{np^{ns}}\right)\right).$$

Expanding the factors gives:

$$\frac{1}{1-p^{-s}} = \sum_{n\geq 0} \frac{1}{p^{ns}},$$

which implies that for a fixed $N \in \mathbb{N}$ and for all prime numbers $p_1, \ldots, p_r \leq N$, we have the equality

$$\prod_{p\leq N} \frac{1}{1-p^{-s}} = \sum_{\nu_1,\ldots,\nu_r=0}^{\infty} \frac{1}{(p_1^{\nu_1}\ldots p_r^{\nu_r})^s} = \sum_n{}' \frac{1}{n^s},$$

where $\sum_n'$ denotes the sum over all natural numbers that are divisible only by the primes $p \leq N$. This sum includes the terms corresponding to all $n \leq N$, but also those n that are products of the primes $p_1, \ldots, p_r$ and are larger than N. Therefore, we can write:

$$\prod_{p\leq N} \frac{1}{1-p^{-s}} = \sum_{n\leq N} \frac{1}{n^s} + \sum_{n>N}{}' \frac{1}{n^s}.$$

Thus, we obtain:

$$\left|\prod_{p\leq N}\frac{1}{1-p^{-s}}-\zeta(s)\right|\leq\left|\sum_{\substack{n>N\\ p_i\nmid n}}\frac{1}{n^s}\right|\leq\sum_{n\geq N}\frac{1}{n^{1+\delta}},$$

where the right-hand side approaches 0 as $N\to\infty$, since it represents the remainder of a convergent series. This completes the proof of Euler's identity. □

Chapter 2

Subgroup growth and subgroup zeta function

Over the past few decades, there have been substantial progresses in the theory of zeta functions associated with groups and their connections to algebraic structures. This emerging field has opened up new paths for exploring the deep interrelations between algebra, number theory, and analysis. A key area of focus involves enumerative problems related to nilpotent groups. Grunewald, Segal, and Smith [3] initiated the study of zeta functions associated with the enumeration of subgroups of finite index within finitely generated torsion-free nilpotent groups.

In this chapter, we will delve into the study of subgroup growth, a fundamental aspect of group theory that explores how the number of subgroups with a given index grows, as the index grows. This area has significant connections with number theory and combinatorics and aims to understand the asymptotic behaviour of the finite index subgroups.

2.1 Subgroup growth

Consider the following function associated with a group G:

$$n \longmapsto a_n(G),$$

where $a_n(G)$ represents the number of subgroups of G with index n. This function is known as the *subgroup growth function* of G. It is well-defined when $a_n(G)$ is finite for all $n \in \mathbb{N}$. In particular, this condition holds when G is finitely generated. The subgroup growth function not only counts subgroups but also measures the algebraic complexity of the group. Studying these growth rates helps reveal significant characteristics of the group, such as whether it is virtually nilpotent.

Proposition 2.1. Let G be a finitely generated group. Then it has only finitely many subgroups of a given index n.

Proof. Let G be a finitely generated group, say $G = \langle g_1, \dots, g_r \rangle$. Fix $n \in \mathbb{N}$ and consider a subgroup H of G such that $|G : H| = n$. Since H has index n, G acts on the left cosets of H via left multiplication. This induces a homomorphism $\eta : G \to S_n$, determined uniquely by H.

Each homomorphism from G to S_n is defined by the images of the generators g_i for $i \in \{1, \dots, r\}$, with at most $n!$ possibilities for each g_i. Therefore, the total number of such homomorphisms is at most $(n!)^r$, implying that the number of subgroups of index n is bounded above by $(n!)^r$. □

It is also possible to define $s_n(G)$ as the total number of subgroups of G of index at most n, so

$$s_n(G) = \sum_{i=1}^{n} a_i(G).$$

We observe that if G is a finite group, then $s(G) = \sum_{i=1}^{+\infty} a_i(G)$ is the total number of subgroups of G.

Example 2.2. As a simple example we can consider the group of integers $\mathbb{Z}$. We have that $a_n(\mathbb{Z}) = 1$ and $s_n(\mathbb{Z}) = n$, for all $n \in \mathbb{N}$, while $s(\mathbb{Z})$ is not defined, since $\mathbb{Z}$ is an infinite group.

Let $R(G)$ denote the intersection of all subgroups of G with finite index and consider the quotient group $G/R(G)$. We observe that the number of subgroups of any given index n in G is preserved in the quotient. In particular, this implies that

$$a_n(G) = a_n(G/R(G)).$$

Consequently, we may assume without loss of generality that G is *residually finite*, meaning $R(G) = 1$.

Definition 2.3. A group G is said to have subgroup growth of *type* f if there exist positive constants a and b such that

$$\begin{aligned} &s_n(G) \leq f(n)^a \text{ for all } n, \\ &s_n(G) \geq f(n)^b \text{ for infinitely many } n; \end{aligned}$$

and G has growth of *strict type* f if the second inequality holds for *all* large n.

This definition raises three key questions:

1. What are the possible growth types?
2. Given a group G, what is its growth type?
3. Given a specific growth type, which groups exhibit it?

Let us examine some preliminary results.

Consider a group G and a subgroup H of G with index $n < \infty$. Then, there are n cosets of H, which, without loss of generality, can be represented as the set $\{1H, g_1H, \ldots, g_{n-1}H\}$, on which G can act transitively by multiplication. This action induces a homomorphism ϕ from G to the symmetric group S_n. The stabilizer of the element $1H$ under this action is the subgroup H itself, which is the preimage of all permutations that fix the first element of the set and permute the others freely. This means that $\mathrm{Stab}_G(1H) = \phi^{-1}(S_{n-1})$, so for each subgroup H of index n, there are $(n-1)!$ distinct transitive actions of G on $\{1, \ldots, n\}$. If we denote by $t_n(G)$ the number of transitive actions, then we have

$$a_n(G) = \frac{t_n(G)}{(n-1)!}.$$

If G is finitely generated, let d be the number of generators. Then the number of transitive homomorphisms from G to S_n is determined by the image of each generator. Consequently, we expect that $a_n(G) \leq \ n(n!)^{d-1}$.

Let us now consider the entire set of homomorphisms from G to S_n and denote the number of such homomorphisms by $h_n(G)$. We then obtain the following result:

Lemma 2.4. Let G be a group. Then

$$h_n(G) = \sum_{k=1}^{n} \binom{n-1}{k-1} t_k(G) h_{n-k}(G).$$

Proof. Let $h_{n,k}(G)$ be the number of homomorphisms from G to S_n in which the orbit of 1 under the action of G has order k. To select this orbit, we must choose $k-1$ elements from $n-1$ elements. There are $t_k(G)$ ways to act transitively on this orbit, and we can permute the remaining $n-k$ elements, giving us $h_{n-k}(G)$ ways to do this. Therefore, we have

$$h_n(G) = \sum_{k=1}^{n} h_{n,k}(G) = \sum_{k=1}^{n} \binom{n-1}{k-1} t_k(G) h_{n-k}(G).$$

□

Replacing $t_k(G)$ with $a_k(G)(k-1)!$, we obtain, as a corollary, a recursive formula for $a_n(G)$.

Corollary 2.5. Given a group G,

$$a_n(G) = \frac{h_n(G)}{(n-1)!} - \sum_{k=1}^{n-1} \frac{h_{n-k}(G)a_k(G)}{(n-k)!}.$$

We now turn our attention to free groups, which will be the focus of the subsequent discussion.
As we have seen from the result above, whenever we have a group G generated by d elements, we know $a_n(G) \leq n(n!)^{d-1}$. We can say more for free subgroups generated by d elements.

Theorem 2.6. [7] Let G be a free group with $d \geq 2$ generators, then

$$a_n \sim n(n!)^{d-1},$$

whereby $\sim$ we denote the asymptotic equivalence (i.e. $f \sim g$ if $\frac{f}{g} \to 1$ as $n \to \infty$).

The following corollary follows directly:

Corollary 2.7. Every finitely generated free group has subgroup growth of type n^n.

Proof. Observing that $n^{n/2} \leq n! \leq n^n$, and $a_n(G) \sim n(n!)^{d-1}$ for any free d-generated group G, the result follows. □

Since any d-generated group G is an image of the free group F_d, its growth is at most as fast as that of F_d. So, the fastest possible growth type is the **super-exponential type**, which is n^n.
Turning our attention to exponential subgroup growth, we introduce two key invariants for a group G:

$$\sigma(G) = \limsup \frac{\log s_n(G)}{n},$$

$$\sigma^-(G) = \liminf \frac{\log s_n(G)}{n},$$

where we are using the notation $\log x = \log_2 x$.

If $\sigma(G)$ is finite, then $s_n(G)$ grows at most exponentially.

If $\sigma(G) > 0$, then we have **exponential subgroup growth**, otherwise, i.e. if $\sigma(G) = 0$, we have **subexponential subgroup growth**.

Regarding the role of $\sigma^-(G)$, if we can prove for a group G that $\sigma(G) = \sigma^-(G)$, and the quantity is finite, then the group is said to exhibit a specific type of strict subgroup growth.

It is important to note that for finite groups, the sequence s_n remains constant. In the case of $G = \mathbb{Z}$, we find that $s_n = n$. This naturally raises the question:

is this the slowest possible growth type for infinite groups? The answer is indeed "yes", though proving this result is surprisingly difficult and relies on one of the most significant results in the field to date. Before delving into that, let us first introduce the concept of polynomial subgroup growth.

Definition 2.8. A group G has *polynomial subgroup growth* (**PSG**) if there is a constant $c > 0$ such that $s_n(G) \leq n^c$ for all $n \in \mathbb{N}$.

Here is one of the most famous theorem in geometric group theory:

Theorem 2.9. (Gromov's theorem) [7] A finitely generated group has polynomial growth if and only if is virtually nilpotent, i.e. it contains a nilpotent subgroup of finite index.

In this context, the term *growth* pertains to the growth of words obtained by a finite set of generators. But can this concept be extended to the growth of subgroups? While the notion of subgroup growth is meaningful even for groups that are not finitely generated, focusing on finitely generated groups allows us to cite a significant theorem. This theorem, due to A. Lubotzky, A. Mann, and D. Segal, provides a characterization of groups with "slow" subgroup growth [5], which we are not going to prove.

Theorem 2.10. (The PSG theorem) [7] Let G be a finitely generated residually finite group. G has polynomial subgroup growth if and only if it is *virtually solvable with finite rank*, i.e. it contains a solvable subgroup with finite index which is finitely generated.

An important point to notice is that the theorem does not provide a complete classification of all groups with polynomial subgroup growth.

2.2 The subgroup zeta function

When studying the subgroup structure of a finitely generated group G, one of the central problems is understanding how subgroups of a given finite index are distributed. Since we are interested in counting the number of subgroups of finite index in a finitely generated group G, a possible approach is to encode this counting sequence using a generating function. A natural choice is the ordinary generating function defined by

$$A_G(x) = \sum_{n \geq 1} a_n(G) x^n,$$

where $a_n(G)$ represents the number of subgroups of index n in G.

To encode the growth of subgroups in a finitely generated group there is another approach which involves the use of Dirichlet series. Instead of relying on the ordinary generating function, one can consider the subgroup zeta function.

Definition 2.11. The *subgroup zeta function* of G is defined to be

$$\zeta_G^{\leq}(s) = \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{H \leq_f G} [G:H]^{-s},$$

where s is a complex variable.

This function provides a different perspective by encoding the same counting sequence $a_n(G)$ in a manner that is particularly well-suited for analytic techniques. The use of Dirichlet series allows for the application of tools from analytic number theory, enabling the study of convergence properties, asymptotic behavior, and connections with other zeta functions.

This series can be viewed as a formal power series, but if $a_n(G) = O(n^c)$ for some constant c, indicating that G has polynomial subgroup growth, then the Dirichlet series converges in a right half-plane of the complex plane. In particular, this means that the series converges for complex numbers s where the real part of s is sufficiently large. More precisely:

Definition 2.12. We define the *abscissa of convergence* $\alpha(G)$ to be

$$\alpha(G) \coloneqq \inf \left\{ \alpha \in \mathbb{R} \mid \zeta_G^{\leq}(s) \text{ converges on the set } \{ s \in \mathbb{C} \mid Re(s) > \alpha \} \right\}.$$

This definition means that $\alpha(G)$ is the smallest real number such that the subgroup zeta function $\zeta_G^{\leq}(s)$ converges for all complex numbers s with a real part greater than $\alpha(G)$. In other words, it is the "boundary" where the zeta function starts to converge when moving from left to right on the real axis of the complex plane. Moreover, observing that [4]

$$\alpha(G) = \inf\{\alpha \mid s_n(G) = O(n^\alpha)\} = \limsup \frac{\log(s_n(G))}{\log(n)},$$

we can say that $\zeta_G^{\leq}(s)$ converges for $Re(s) > \alpha(G)$ and defines a holomorphic function on this half plane, while $\zeta_G^{\leq}(s)$ is divergent at $s = \alpha(G)$. In other words, for a PSG group G, the abscissa of convergence $\alpha(G)$ determines exactly the degree of polinomial subgroup growth of G.

Let G be a finitely generated group. If we assume G to be nilpotent, then $\zeta_G^{\leq}(s)$ has an *Euler product*, namely:

Proposition 2.13. Let G be a nilpotent group, then the subgroup zeta function of G is given by

$$\zeta_G^{\leq}(s) = \prod_{\text{p prime}} \zeta_{G,p}(s),$$

where $\zeta_{G,p}(s) = \sum_{i\geq 0} \frac{a_{p^i}(G)}{p^{is}}$.

The factors $\zeta_{G,p}(s)$ are called the *local zeta functions* of G, or just *local factors.*

Proof. The proof follows from the fact that a nilpotent group is isomorphic to the direct product of its Sylow subgroups. Choosing a subgroup H of G, we see that it must be isomorphic to the direct product of the intersection of H with each Sylow subgroup of G. It implies that $a_n(G)$ is a multiplicative function, which means that, using the factorization in prime, if $n = \prod_{i=1}^{k} p_i^{r_i}$,

$$a_n(G) = \prod_{i=1}^{k} a_{p_i^{r_i}}(G).$$

Therefore, letting p be a prime, we can find the local factors, which are defined by:

$$\zeta_{G,p}(s) = \sum_{t\geq 0} \frac{a_{p^t}(G)}{p^{ts}}.$$

We find the subgroup zeta function just multiplying all this factors together.

□

Example 2.14. Let $G = \mathbb{Z}$. For each $n \in \mathbb{N}$, there is only one subgroup of G of index n, which is $n\mathbb{Z}$. Therefore, in this case $a_n = 1$ for every n, and the subgroup zeta functions turns out to be

$$\zeta_G^{\leq}(s) = \sum_{n\geq 1} \frac{1}{n^s},$$

which is precisely the Riemann zeta function $\zeta(s)$.

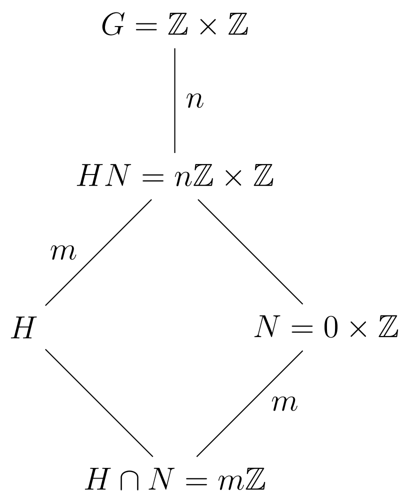
Let us consider $G = \mathbb{Z}^d$ the direct product of infinite d cyclic infinite groups, with d natural number, then we have the following:

Theorem 2.15. Let $d \in \mathbb{N}$ and $G = \mathbb{Z}^d$. Then

$$\zeta_G^{\leq}(s) = \zeta(s)\zeta(s-1)\ldots\zeta(s-d+1).$$

Proof. The idea is to use induction on d.

Let us first consider the case where $G = \mathbb{Z} \times \mathbb{Z}$ and take a subgroup H of G of finite index. The subgroup H is generated by two elements, meaning that there exist $x, y \in \mathbb{Z} \times \mathbb{Z}$ such that $H = \langle x, y \rangle$, where x and y are of the form $x = (x_1, x_2)$ and $y = (y_1, y_2)$.
Let us take $N = \{0\} \times \mathbb{Z}$, a normal subgroup of $\mathbb{Z} \times \mathbb{Z}$. Without loss of generality we can assume that either x_1 or y_1 is different from 0. Indeed, if both x_1 and y_1 were equal to zero, then both x and y would belong to N, implying that also $\langle x, y \rangle = H \subset N$. This leads to a contradiction. Therefore it is possible to assume without loss of generality that $x_1 \neq 0$. We have the following diagram:



We know that $HN/N \leq G/N \simeq \mathbb{Z}$, so HN/N is isomorphic to $n\mathbb{Z}$, for some $n \in \mathbb{N}$, and, by the Second Isomorphism Theorem for groups, it is also isomorphic to $H/H \cap N$. It means that it holds

$$\frac{HN}{N} \simeq \frac{H}{H \cap N} \simeq n\mathbb{Z}.$$

The preimages in H of the correspondent generator must be of the form $x = (n, a)$, with $a \in \mathbb{Z}$. Moreover, $H \cap N$ is a subgroup of N, which is isomorphic to $\mathbb{Z}$, so $H \cap N \simeq m\mathbb{Z}$. It means that $H \cap N$ is generated by the element $y = (0, m)$.

To summarize up to this point we have that whenever H is a subgroup of G of finite index, it is generated by two elements of the form $x = (n, a)$, $y = (0, m)$, with $a \in \mathbb{Z}$, and its index can be factorized as $n \cdot m$. In order to understand how many subgroup of G of index $n \cdot m$ there are, the goal becomes to understand how many possibilities we have for a.

If we consider the matrix defined as

$$M = \begin{pmatrix} n & a \\ 0 & m \end{pmatrix},$$

where every raw corresponds to a generator of H, we find all set of generators for H by multiplying M by a generic invertible 2×2 matrix with coefficient in $\mathbb{Z}$. In particular for any $k \in \mathbb{Z}$

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z}),$$

and by multiplying the two matrices we get:

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} n & a \\ 0 & m \end{pmatrix} = \begin{pmatrix} n & a + km \\ 0 & m \end{pmatrix}.$$

Since k is arbitrarily running in $\mathbb{Z}$, a can be seen as a possible reminder when dividing by k. Therefore, we find a bound for a that is

$$0 \leq \mathbf{a} \leq m - 1,$$

so there are m possible choices for a. Moreover, take $0 \leq a, a' \leq m - 1$, with $a \neq a'$. Setting $x = (n, a)$ and $x' = (n, a')$, we now show that the subgroups $\langle x, y \rangle$ and $\langle x', y \rangle$ are different. Assume by contradiction that

$$H = \langle x', y \rangle = \langle x, y \rangle.$$

If we take the element $x' - x = (0, a' - a) \in H$, then $a' - a = m \cdot t$ for some $t \in \mathbb{Z}$, but $|a' - a| < m$, so $a' = a$, which is a contradiction.

Summarizing, if H is a subgroup of $\mathbb{Z} \times \mathbb{Z}$ of finite index, there exist $n, m \in \mathbb{Z}$ such that the index of H can be factorized as $n \cdot m$, where n and m are such that the two generators of H are $x = (n, a)$ and $y = (0, m)$. Moreover, a can be chosen between $\{0, \ldots, m - 1\}$.

We can deduce that the subgroup zeta function is of the form

$$\zeta_G^{\leq}(s) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} m(nm)^{-s} = \sum_{n=1}^{\infty} n^{-s} \sum_{m=1}^{\infty} m^{1-s} = \zeta(s)\zeta(s-1).$$

To better understand the general case, we will first examine the case $d = 3$. Let $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^3$ and take H a subgroup of G of finite index. By a similar argument as above we observe that there exist $x_1, x_2, x_3 \in \mathbb{Z}^3$ such that $H = \langle x_1, x_2, x_3 \rangle$, with

$$x_1 = (n_1, a_1, a_2), x_2 = (0, n_2, b_1), x_3 = (0, 0, n_3).$$

Moreover the index of H can be factorized as $|\mathbb{Z}^3 : H| = n_1 \cdot n_2 \cdot n_3$. To find all the possible generators of H, we can fix a set of generators, as the one above, and consider a matrix M as the matrix where every raw corresponds to a generator:

$$M = \begin{pmatrix} n_1 & a_1 & a_2 \\ 0 & n_2 & b_1 \\ 0 & 0 & n_3 \end{pmatrix}.$$

We find all the sets of generators by multiplying M by all the matrices of $GL_3(\mathbb{Z})$. If we look at the upper left 2×2 matrix we know that there are n_2 possible choices for a_1, by what we have already seen. Looking at

$$\begin{pmatrix} 1 & 1 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} n_1 & a_1 & a_2 \\ 0 & n_2 & b_1 \\ 0 & 0 & n_3 \end{pmatrix} = \begin{pmatrix} n_1 & a_1 + n_2 & a_2 + b_1 + \alpha n_3 \\ 0 & n_2 & b_1 + \beta n_3 \\ 0 & 0 & n_3 \end{pmatrix},$$

we get a bound for both a_2 and b_1. The arbitrary of β gives n_3 possible choices for b_1 and the arbitrary of α leads to n_3 possible choices for a_2.

Iterating this procedure, we can analyse the general case. The matrix associated to the generators is

$$\begin{pmatrix} n_1 & a_1 & a_2 & \dots & a_{d-1} \\ 0 & n_2 & b_1 & \dots & b_{d-2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & n_d \end{pmatrix}.$$

Using the inductive hypothesis we obtain n_i possible choices for each one of the elements which lie above the element n_i in the i-th column.

It implies that the subgroup zeta function will be

$$\zeta_G^{\leq}(s) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} n_2 n_3^2 \dots n_d^{d-1} (n_1 \dots n_d)^{-s} = \sum_{n_1=1}^{\infty} n_1^{-s} \sum_{n_2=1}^{\infty} n_2^{1-s} \cdots \sum_{n_d=1}^{\infty} n_d^{(d-1)-s},$$

that is

$$\zeta_G^{\leq}(s) = \zeta(s)\zeta(s-1)\dots\zeta(s-(d-1)).$$

□

Things begins to get more unpredictable when we work with non abelian groups. In the following we will describe the subgroup zeta function of the **discrete Heisenberg group**, that is the free class-2 nilpotent group on 3 generators:

$$\mathbb{H}_3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}.$$

Set $G = \mathbb{H}_3$. The Heisenberg group is generated by 3 elements:

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which satisfy three relations:

$$[a, b] = c^{-1}, \text{ and } [a, c] = [b, c] = 1.$$

To show that $\mathbb{H}_3 = \langle a, b, c \rangle$, observe that for every $n \in \mathbb{Z}$

$$a^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}, b^n = \begin{pmatrix} 1 & n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, c^n = \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Taking an arbitrary element $g \in G$, it holds

$$g = \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} = a^x b^y c^z$$

for some $x, y, z \in \mathbb{Z}$. Moreover, observing that

$$a^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, b^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, c^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and computing the three commutators $[a, b]$, $[a, c]$ and $[b, c]$ we also get the relation described above. It follows that the presentation of $\mathbb{H}_3$ is

$$\langle a, b, c \mid [a, b] = c^{-1}, \text{ and } [a, c] = [b, c] = 1 \rangle.$$

We deduce that:

1. the centre $Z(G)$ is generated by the element c : $Z(G) = \langle c \rangle$;
2. the centre of the group coincides with the derived subgroup of G: $Z(G) = [G, G]$;
3. the centre of $\mathbb{H}_3$ is isomorphic to $\mathbb{Z}$.

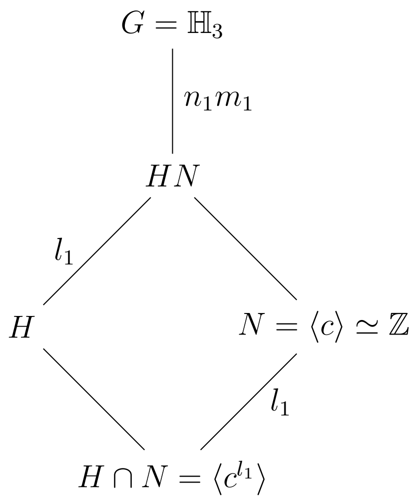
Our goal is to find the subgroup zeta function of the Heisenberg group.

Theorem 2.16. Let $G = \mathbb{H}_3$ be the discrete Heisenberg group. Then

$$\zeta_G^{\leq}(s) = \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(3s-3)}.$$

Proof. First of all the aim is to find the subgroup of finite index of G.

Let $N = Z(G)$ and let H be a subgroup of G with finite index. We can consider the following diagram:



Observe that $G/N \simeq \mathbb{Z} \times \mathbb{Z}$, by the isomorphism sending

$$a \to (1, 0);$$

$$b \to (0, 1).$$

If we look at the subgroup HN/N of G/N, it is isomorphic to a subgroup of $\mathbb{Z} \times \mathbb{Z}$ of finite index. It means that there exist $n_1, n_2, m_1 \in \mathbb{Z}$ such that the image of HN/N is generated by $x = (n_1, n_2)$ and $y = (0, m_1)$, and its index can be factorized as $n_1 \cdot m_1$. Therefore it holds that $HN/N \simeq \langle \bar{a}^{n_1} \bar{b}^{n_2}, \bar{b}^{m_1} \rangle$, which yields $HN = \langle a^{n_1} b^{n_2}, b^{m_1} \rangle N$. Since $H \cap N$ is isomorphic to a subgroup of $\mathbb{Z}$, say $l_1\mathbb{Z}$, we have that $H \cap N = \langle c^{l_1} \rangle$. Then H must be of the form

$$H = \langle a^{n_1} b^{n_2} c^{n_3}, b^{m_1} c^{m_2}, c^{l_1} \rangle,$$

with $0 \leq n_2 \leq m_1 - 1$ and $0 \leq n_3, m_2 \leq l_1 - 1$. Moreover, from the relation $[a, b] = c^{-1}$, recalling that $c \in Z(G)$, it holds that

$$\begin{aligned} [a^{n_1} b^{n_2} c^{n_3}, b^{m_1} c^{m_2}] &= [a^{n_1} b^{n_2}, b^{m_1}] \\ &= [a^{n_1}, b^{m_1}]^{b^{n_2}} \\ &= [a^{n_1}, b^{m_2}] \\ &= c^{-n_1 m_1}. \end{aligned}$$

Therefore, whenever H is a subgroup of G of finite index, there exist $n_1, n_2, n_3, m_1, m_2, l_1 \in \mathbb{Z}$, such that $H = \langle a^{n_1} b^{n_2} c^{n_3}, b^{m_1} c^{m_2}, c^{l_1} \rangle$ and the index of H can be factorized as

$n_1 \cdot m_1 \cdot l_1$. In particular we can add some more specific conditions, that are: $n_2 \in \{0, \ldots, m_1 - 1\}$, $n_3, m_2 \in \{0, \ldots l_1 - 1\}$ and $l_1 \mid n_1 m_1$. We can now describe the subgroup zeta function:

$$\zeta_G^{\leq}(s) = \sum_{n_1=1}^{\infty} \sum_{m_1=1}^{\infty} \sum_{l_1 | n_1 m_1} m_1 l_1^2 (n_1 m_1 l_1)^{-s}.$$

Since the group is nilpotent, we can use the Euler decomposition to simplify the subgroup zeta function. We look at the local factors. Let p be a prime, let $\alpha, \beta, \gamma \geq 0$ be natural numbers and $n_1 = p^\alpha, m_1 = p^\beta, l_1 = p^\gamma$. Then

$$\zeta_{G,p}^{\leq}(s) = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \sum_{\gamma=0}^{\alpha+\beta} p^{\beta+2\gamma} (p^{\alpha+\beta+\gamma})^{-s} = \sum_{\alpha=0}^{\infty} p^{\alpha(-s)} \sum_{\beta=0}^{\infty} p^{\beta(1-s)} \sum_{\gamma=0}^{\alpha+\beta} p^{\gamma(2-s)}.$$

The following identity:

$$\sum_{k=0}^{N} x^k = \frac{x^{N+1} - 1}{x - 1},$$

yields

$$\zeta_{G,p}^{\leq}(s) = \sum_{\alpha=0}^{\infty} p^{\alpha(-s)} \sum_{\beta=0}^{\infty} p^{\beta(1-s)} \frac{p^{(2-s)(\alpha+\beta+1)} - 1}{p^{(2-s)} - 1}.$$

Recall two results we are going to use:

1. $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$;
2. $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, when $-1 < x < 1$.

Let us analyse the local factors:

$$
\begin{aligned}
\zeta^{\leq}_{G,p}(s) &= \sum_{\alpha=0}^{\infty} p^{-\alpha s} \sum_{\beta=0}^{\infty} p^{\beta(1-s)} \frac{p^{(2-s)(\alpha+\beta+1)} - 1}{p^{2-s} - 1} \\
&= \frac{p^{2-s}}{p^{2-s}-1} \sum_{\alpha=0}^{\infty} p^{-\alpha(2-2s)} \sum_{\beta=0}^{\infty} p^{-\beta(3-2s)} - \frac{1}{p^{2-s}-1} \sum_{\alpha=0}^{\infty} p^{-\alpha s} \sum_{\beta=0}^{\infty} p^{-\beta(1-s)} \\
&= \frac{p^{2-s}}{(p^{2-s}-1)(1-p^{2-2s})(1-p^{3-2s})} - \frac{1}{(p^{2-s}-1)(1-p^{-s})(1-p^{1-s})} \\
&= \frac{(p^{2-s}(1-p^{-s})(1-p^{1-s}) - (1-p^{2-2s})(1-p^{3-2s})}{(p^{2-s}-1)(1-p^{2-2s})(1-p^{3-2s})(1-p^{-s})(1-p^{1-s})} \\
&= \frac{(p^{2-s}-1)(1-p^{3-3s})}{(p^{2-s}-1)(1-p^{2-2s})(1-p^{3-2s})(1-p^{-s})(1-p^{1-s})} \\
&= \frac{(1-p^{3-3s})}{(1-p^{2-2s})(1-p^{3-2s})(1-p^{-s})(1-p^{1-s})}.
\end{aligned}
$$

Therefore, setting $\zeta_p(s)$ the local factors of the Riemann zeta function, we obtain

$$\zeta^{\leq}_{G,p}(s) = \frac{\zeta_p(s)\zeta_p(s-1)\zeta_p(2s-2)\zeta_p(2s-3)}{\zeta_p(3s-3)}.$$

Knowing that

$$\zeta^{\leq}_{G}(s) = \prod_{\text{p prime}} \zeta^{\leq}_{G,p}(s),$$

and using the Euler product Theorem 1.5, we get

$$\zeta^{\leq}_{\mathbb{H}_3}(s) = \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(3s-3)},$$

the subgroup zeta function of the Heisenberg group. □

Chapter 3

Double cosets zeta function

A natural question that arises is what happens when we replace the number of cosets with the number of double cosets.

In this chapter, we will define the double coset zeta function and explore the information that it can provide. We will begin by introducing the concept of double coset and by investigating some of its key properties. Following this path, we will establish a general form for the double coset zeta function. With these foundations in place, we will then proceed to examine specific examples.

3.1 Double cosets

Let G be a group and let H and K be two subgroups of G.

Definition 3.1. An (H, K)-*double coset* of G is a subset of G of the form

$$HxK = \{hxk \mid h \in H, k \in K\},$$

where $x \in G$.

Equivalently, HxK is the equivalence class of x under the equivalence relation defined by:

$$x \sim y \text{ if and only if there exist } h \in H \text{ and } k \in K \text{ such that } hxk = y.$$

The set of all (H, K)-double cosets is denoted by $H\backslash G/K$. When $H = K$, we refer to these as H-double cosets.

A double coset can also be viewed as the orbit of an action of the group $H \times K$ on G. More precisely, let us consider the action of $H \times K$ on G given by the map

$$\begin{aligned}(H \times K) \times G &\to G\\ ((h, k), x) &\mapsto hxk^{-1}.\end{aligned}$$

For every $x \in G$, the orbit of x is given by:

$$\begin{aligned} O_x &= \{hxk^{-1} \mid h \in H, k \in K\} \\ &= \{hxk \mid h \in H, k \in K\} \text{ since } k \mapsto k^{-1} \text{ is an automorphism of } K \\ &= HxK. \end{aligned}$$

It follows that, under this action, every element of $H\backslash G/K$ corresponds exactly to an orbit. As a consequence the following holds:

Proposition 3.2. For all $x, y \in G$, the double cosets HxK and HyK are either equal or disjoint. Moreover, the group G can be expressed as the disjoint union of its double cosets:

$$G = \bigsqcup_{HxK \in H\backslash G/K} HxK.$$

Proof. This result follows directly from the fact that double cosets are orbits of the action of $H \times K$ on G. Since the set of the orbits of a group action is a partition of the set on which the group acts, it immediately implies that the double cosets HxK and HyK are either identical or disjoint for every $x, y \in G$. Therefore, the group G can be viewed as the disjoint union of these double cosets. □

Remark 3.3. There is a bijection between the sets $H\backslash G/K$ and $K\backslash G/H$. Such a bijection is given by

$$\begin{aligned} H\backslash G/K &\to K\backslash G/H \\ HxK &\mapsto Kx^{-1}H. \end{aligned}$$

Remark 3.4. If $H = \{1_G\}$, then the double coset space $H\backslash G/K$ simplifies to the coset space G/K, since for all $x \in G$, the double coset HxK reduces to xK. Similarly, if $K = \{1_G\}$, then $H\backslash G/K$ simplifies to the left coset space $H\backslash G$.

Lemma 3.5. A double coset HxK can be viewed as a union of certain left cosets from $H\backslash G$ and a union of certain right cosets from G/K. In particular:

$$HxK = \bigcup_{k \in K} Hxk = \bigsqcup_{Hxk \in H\backslash HxK} Hxk,$$

and

$$HxK = \bigcup_{h \in H} hxK = \bigsqcup_{hxK \in HxK/K} hxK.$$

Proof. Since H is a subgroup of G, we can express G as the disjoint union of the right cosets of H:

$$G = \bigsqcup_{Hg \in H\backslash G} Hg.$$

Therefore, for any $x \in G$, we have:

$$HxK = HxK \cap G \subseteq \bigcup_{Hg \in HxK} Hg \subseteq HxK.$$

By a similar argument we obtain the second result. □

The Lemma above proves how double cosets are related to both left and right cosets within the group G.

Lemma 3.6. Let the group H act on the quotient G/K by

$$\begin{aligned} H \times G/K &\to G/K \\ (h, xK) &\mapsto hxK. \end{aligned}$$

Denote by $H\backslash(G/K)$ the set of orbits under this action. Then, there exists a bijection between the sets $H\backslash G/K$ and $H\backslash(G/K)$.
Similarly, there exists a bijection between the sets $H\backslash G/K$ and $(H\backslash\ G)\ /\ K$.

Proof. Consider the map

$$\begin{aligned} \phi : H\backslash G/K &\to H\backslash(G/K) \\ HxK &\mapsto H(xK). \end{aligned}$$

Observe that ϕ is a well defined. Indeed, for two distinct $x, y \in G$ such that $HxK = HyK$, there exist $h \in H$ and $k \in K$ such that $y = hxk$. Thus, $\phi(HyK) = H(yK) = H(hxkK) = H(xK) = \phi(HxK)$.

To prove that ϕ is a bijection, we will first show that it is surjective and then that it is injective.

Surjectivity: Let $x \in G$, and consider the coset $xK \in G/K$. The orbit of xK in $H\backslash(G/K)$ is $H(xK)$. The double coset of x in $H\backslash G/K$ is HxK, and by definition, $\phi(HxK) = H(xK)$, so ϕ is surjective.

Injectivity: Suppose HxK and HyK are two double cosets in $H\backslash G/K$ such that $H(xK) = H(yK)$. Hence, there exists an element $h \in H$ such that $xK = hyK$, so $x = hyk$ for some $k \in K$. Therefore, $x \in HyK$, which implies that $HxK = HyK$. Thus, ϕ is injective.

Since ϕ is both injective and surjective, it is a bijection.

By a similar argument we get the bijection between $H\backslash G/K$ and $(H\backslash\ G)/K$. □

Proposition 3.7. Assume that H is a normal subgroup of G. Then the double coset space $H\backslash G/K$ is equal to the coset space G/HK. Similarly, if K is a normal subgroup of G, then $H\backslash G/K$ is equal to $HK\backslash G$.

In particular, if H is normal in G, then the double coset space $H\backslash G/H$ simplifies to G/H, which is the same as the left coset space $H \setminus G$.

Proof. First observe that if H is normal in G, then HK is a subgroup of G, allowing us to consider the coset space G/HK. Let $x \in G$. The double coset of x in $H\backslash G/K$ is HxK, while its coset in G/HK is xHK. As H is normal in G, we have that $Hx = xH$, hence $HxK = xHK$.

We conclude that $H\backslash\ G\ /\ K\ =\ G/HK$. □

The above proposition leads to the following

Corollary 3.8. Suppose that H or K is normal, then

1. For all $x \in G$, $|HxK| = |HK|$. In other words, every double cosets have the same cardinality.
2. If G is a finite group, then $|H\backslash G/K|$ divides $|G|$ and $|H\backslash G/K| = \frac{|G|}{|HK|}$.

From now on we will work only with subgroups of finite index of a group G.

Proposition 3.9. Let G be a group and H be a subgroup of G. Then, for every $g \in G$ it holds that $HgH = gH$ if and only if $g \in N_G(H)$.

Proof. If g lies in $N_G(H)$, then $HgH = gg^{-1}HgH = gH$.

For the converse, assume that $HgH = gH$. Thus, $Hg \subseteq gH$ and so $g^{-1}Hg \subseteq H$. Since $[G : H^g] = [G : H]$, then $H^g = H$, so $g \in N_G(H)$. □

Proposition 3.10. Given a subgroup H of a group G, we have that $|H\backslash G/H| \leq [G : H]$. The equality holds if and only if H is normal in G.

Proof. It follows from the Lemma 3.5 that $|H\backslash G/H| \leq [G : H]$.

We have already seen that when H is a normal subgroup of G, $HgH = gH$ for every $g \in G$.

For the converse, we assume that $|H\backslash G/H| = [G : H]$. Since every double coset is union of right cosets of H and the number of cosets is equal to the number of double cosets, it must hold that $HgH = gH$. From the Proposition 3.9, it follows that $g \in N_G(H)$ for every $g \in G$, i.e. H is a normal subgroup of G. □

We are going to give an example.

Example 3.11. Consider the symmetric group $G = S_n$ acting on the set $\{1, \ldots, n\}$ and the subgroup $H = \mathrm{Stab}_G(n)$. The subgroup H is isomorphic to S_{n-1} and the set of double cosets $H\backslash G/H$ consists of two distinct elements:

1. The trivial double coset H.

2. The double coset $H\gamma H$, where γ is a permutation that does not fix n.

Now consider the coset space G/H. It has order n because we can choose as representatives of the right cosets the elements $\{\gamma_1 H, \dots, \gamma_n H\}$, where γ_i is the permutation in G such that $\gamma_i(n) = i$.

We can now clearly observe that $|H\backslash G/H| = 2 < |G/H| = n$.

Proposition 3.12. Let $x \in G$. The number of right cosets of $H\backslash G$ contained in the double coset HxK is given by the index $[K : K \cap x^{-1}Hx]$. Similarly, the number of left cosets of G/K contained in the double coset HxK is given by the index $[H : H \cap\ xKx^{-1}]$.

Proof. Consider the action of H on the double coset HxK defined by:

$$\begin{aligned} H \times HxK &\to HxK \\ (h, h'xk) &\mapsto hh'xk. \end{aligned}$$

Denote the set of the orbits of HxK under this action as $H\backslash HxK$. Let us consider the transitive right action of K on $H\backslash HxK$ given by:

$$\begin{aligned} K \times H\backslash HxK &\to H\backslash HxK \\ (k, Hxk') &\mapsto Hxkk'. \end{aligned}$$

We will apply the Orbit-Stabilizer theorem to this second action. The element Hx belongs to the set $H\backslash HxK$. Consider its stabilizer under the given action:

$$\mathrm{Stab}_K(Hx) = \{k \in K \mid Hxk = Hx\}.$$

Let $k \in K$, we have $Hxk = Hx$ if and only if there exists $h \in H$ such that $xk = hx$. It holds if and only if there exists $h \in H$ such that $k = x^{-1}hx$. Thus if and only if $k \in x^{-1}Hx$. Then, we conclude that:

$$\mathrm{Stab}_K(Hx) = K \cap x^{-1}Hx.$$

Since the action is transitive, the orbit of Hx is the entire set $H\backslash HxK$. By the Orbit-Stabilizer theorem, we obtain:

$$|H\backslash HxK| = [K : \mathrm{Stab}_K(Hx)] = [K : K \cap x^{-1}Hx].$$

Therefore, the number of cosets in $H\backslash G$ that are contained in the double coset HxK is $[K : K \cap x^{-1}Hx]$.

The analogous result for the left cosets can be proved in a similar way. □

Corollary 3.13. Let G be a finite group and $x \in G$. The following relations hold:

- $|HxK| = [H : H \cap xKx^{-1}]\,|K| = |H|\,[K : K \cap x^{-1}Hx]$;
- $[G : H] = \sum_{HxK \in H\backslash G/K} [K : K \cap x^{-1}Hx]$ and $[G : K] = \sum_{HxK \in H\backslash G/K} [H : H \cap xKx^{-1}]$.

Moreover we have:

- $|HxK| = \frac{|H||K|}{|H \cap xKx^{-1}|} = \frac{|H||K|}{|K \cap x^{-1}Hx|}$;
- $[G : H] = \sum_{HxK \in H\backslash G/K} \frac{|K|}{|K \cap x^{-1}Hx|}$ and $[G : K] = \sum_{HxK \in H\backslash G/K} \frac{|H|}{|H \cap xKx^{-1}|}$.

Proof. In Lemma 3.5, we established that

$$HxK = \bigsqcup_{Hxk \in H\backslash HxK} Hxk.$$

For each $k \in K$, the size of the set Hxk is $|H|$. The previous proposition then tells us that there are $|K : K \cap x^{-1}Hx|$ cosets of $H\backslash G$ contained in the double coset HxK. Thus

$$\begin{aligned} |HxK| &= \sum_{Hxk \in H\backslash HxK} |Hxk| \\ &= [K : K \cap x^{-1}Hx] \cdot |H|. \end{aligned}$$

By similar argument, we get

$$\begin{aligned} |HxK| &= \sum_{hxK \in HxK/K} |hxK| \\ &= [H : H \cap xKx^{-1}] \cdot |K|. \end{aligned}$$

For the second assertion, recall that $[G : H]$ represents the number of cosets in $H\backslash G$, and since $G = \bigcup_{HxK \in H\backslash G/K} HxK$, each coset in $H\backslash G$ is uniquely contained in a double coset HxK. Therefore, to determine the number of cosets in $H\backslash G$, it suffices to sum the number of cosets of $H\backslash G$ contained in each double coset HxK. It follows that

$$[G : H] = \sum_{HxK \in H\backslash G/K} [K : K \cap x^{-1}Hx].$$

Similarly, we obtain:

$$[G : K] = \sum_{HxK \in H\backslash G/K} [H : H \cap xKx^{-1}].$$

In the case where G is finite, the remaining relations follow directly from Lagrange's theorem. □

Lemma 3.14. Let $x \in G$ and consider the stabilizer of x under the action of $H \times K$ defined as $\Gamma_x = \{(h,k) \in H \times K \mid hxk^{-1} = x\}$. Then, we can express Γ_x as follows:

$$\Gamma_x = \{(h, x^{-1}hx) \mid h \in H\} \cap (H \times K) = \{(xkx^{-1}, k) \mid k \in K\} \cap \ (H \ \times \ K).$$

Proof. The following equalities hold:

$$\begin{aligned}
\Gamma_x &= \{(h,k) \in H \times K \mid hxk^{-1} = x\} \\
&= \{(h,k) \in H \times K \mid h = xkx^{-1}\} \\
&= \{(xkx^{-1}, k) \mid k \in K, xkx^{-1} \in H\} \\
&= \{(xkx^{-1}, k), k \in K\} \cap (H \times K) \\
&= \{(h,k) \in H \times K \mid xhx^{-1} = k\} \\
&= \{(h, x^{-1}hx), h \in H\} \cap (H \times K).
\end{aligned}$$

□

Proposition 3.15. The following hold:

1. For all $x \in G$, the size of the double coset HxK is given by $|HxK| = [H \times K : \Gamma_x]$. In particular, if G is a finite group, then for any $x \in G$, it holds that $|HxK| = \frac{|H||K|}{|\Gamma_x|}$.

2. If G is a finite group, for all $(h,k) \in H \times K$, define $G^{(h,k)} = \{x \in G \mid hxk = x\}$. Then, it holds that $|H\backslash G/K| = \frac{1}{|H||K|} \sum_{(h,k)\in H\times K} |G^{(h,k)}|$.

Proof. The first assertion follows directly from the orbit-stabilizer theorem and the second one from the Cauchy-Frobenius Lemma. □

Remark 3.16. When G is a finite group, we can see from the point 2. of the Proposition 3.15 that the number of double cosets in $H\backslash G/K$ is the average number of elements in G that are fixed by the elements of $H \times K$. However, unlike Lagrange's Theorem for single cosets, the analogue for double cosets does not hold in general. In particular, the cardinality of a double coset does not necessarily divide the cardinality of G, and double cosets may vary in size, meaning they do not all have the same cardinality.

Let us consider an example.

Example 3.17. Consider the group $G = S_3$, the symmetric group on three elements, with subgroups $H = \langle (12) \rangle$ and $K = \langle (13) \rangle$. Let $e = 1_G$ denote the identity element. The double coset HeK is given by:

$$HeK = HK = \{e, (12), (13), (132)\}.$$

Thus, the size of the double coset $|HeK|$ is 4, which does not divide 6, the order of S_3. This remarks that the cardinality of a double coset does not necessarily divide the order of the whole group, in contrast to what happens with single cosets under Lagrange's Theorem.
Let us now examine the double coset $H(23)K$:

$$H(23)K = \{(23), (213)\}.$$

Observe that $H(23)K$ has order 2, while HeK has order 4. This shows that not all double cosets necessarily have the same cardinality.

Thanks to the Proposition 3.7, we remark that situation changes when either H or K is a normal subgroup of G.

3.2 Double coset zeta function

Let G be a group. Let H be a subgroup of G of finite index and $x \in G$. We define the weight of the H-double coset HxH to be

$$wt(HxH) \coloneqq \frac{r}{[G:H]},$$

where r is the number of disjoint right H-cosets whose union is the whole HxH.

Definition 3.18. The *the double coset zeta function* is defined as follows:

$$\zeta_G^{dc}(s) \coloneqq \sum_{H \leq_f G} \sum_{HxH \in H/G\backslash H} wt(HxH)^s.$$

Remark 3.19. Evaluating the subgroup zeta function of G at $s-1$ yields

$$\zeta_G^{\leq}(s-1) = \sum_{H <_f G} [G:H] \cdot [G:H]^{-s}.$$

When H is a normal subgroup of G, the number of double cosets equals the index $[G:H]$, and the corresponding weight is $[G:H]^{-1}$. This implies that, for normal subgroups of G, the contribution to the double coset zeta function and the contribution to the subgroup zeta function evaluated at $s-1$ are the same.

The aim is to study some relevant (families of) examples of groups of both finite and infinite order. Some of them are p-groups of maximal class. The first group we are going to study is the Dihedral group of order a power of 2:

$$D_{2^{n+1}} = \langle x, a \mid a^{2^n} = 1, x^2 = 1, a^x = y^{-1} \rangle, n \geq 2.$$

We will study its double coset zeta function and we will be able to give a recursive formula which establish a connection between the double coset zeta functions of groups of successive orders for the family $D_{2^{n+1}}$.

From the dihedral group of order 2^{n+1}, we will move to the pro-2-dihedral group:

$$C_2 \ltimes \mathbb{Z}_2,$$

which is defined to be the inverse limit of the family $D_{2^{n+1}}$, where $n \in \mathbb{N}$.

The result that we find for the pro-2-dihedral group can be generalised for the pro-p-dihedral group, with a general prime p. We will study the dihedral group of order $2p^n$ with $p \neq 2$,

$$D_{2p^n} = \langle x, a \mid a^{p^n} = 1, x^2 = 1, a^x = a^{-1} \rangle, n \geq 1,$$

and with the same kind of argument we will move to the group

$$C_2 \ltimes \mathbb{Z}_p.$$

Afterwards we are going to study the Semi-Dihedral group

$$SD_{2^{n+1}} = \langle x, a \mid a^{2^n} = 1, x^2 = 1, a^x = a^{-1+2^{n-1}} \rangle, n \geq 3,$$

and the Quaternion group

$$Q_{2^{n+1}} = \langle x, a \mid a^{2^n} = 1, x^2 = a^{2^{n-1}}, a^x = a^{-1} \rangle, n \geq 2.$$

For the last two groups, the point will be to use the relation between both $SD_{2^{n+1}}$ and $Q_{2^{n+1}}$ with D_{2^n}.

In order to study the double coset zeta function of a given group G, it is essential to first understand the subgroup lattice of the group in question. Later, one can proceed to examine the double cosets, their weight and, finally, analyse the double coset zeta function.

In the following few pages we will recall some results on nilpotent groups. In particular, we will recall what it means for a group to have maximal class and we will show that we are working with such groups when considering the dihedral, semidihedral and quaternion groups of order 2^{n+1}.

Let G be a group. The *lower central series* of G is defined inductively by means of

$$\gamma_1(G) = G;$$
$$\gamma_{i+1}(G) = [\gamma_i(G), G].$$

Recall 3.20. For any group G, $[\gamma_i, \gamma_j] \leq \gamma_{i+j}(G)$.

Definition 3.21. A group G is said to be *nilpotent* if $\gamma_{c+1}(G) = 1$ for some natural number c.

The smallest such c is called *the nilpotency class* of G. The groups of nilpotency class one are precisely the abelian groups. The property of being nilpotent may also be characterized in term of another series of G: the *upper central series*, which is defined recursively by means

$$Z_0(G) = 1$$
$$Z_{i+1}(G)/Z_i(G) = Z(G/Z_i(G)).$$

Observe that, for any subgroup H of G, $[H, G] \leq Z_i(G)$ if and only if $H \leq Z_{i+1}(G)$.

Lemma 3.22. Let G be a nilpotent group of class c. Then $\gamma_{c+1-i}(G) \leq Z_i(G)$ for all $i \in \{0, \dots, c\}$.

Proof. We argue by induction on i.

If $i = 0$, then $\gamma_{c+1}(G) = 1 = Z_0(G)$, so the result holds.

On the other hand,applying the inductive hypothesis,

$$[\gamma_{c+1-i}(G), G] = \gamma_{c+1-(i-1)}(G) \leq Z_{i-1}(G),$$

consequently $\gamma_{c+1-i}(G) \leq Z_i(G)$. □

Theorem 3.23. A group G is of nilpotency class c if and only if $Z_c(G) = G$, and $Z_{c-1}(G) \neq G$.

Proof. First of all, observe that $Z_c(G) = G$ implies that

$$\gamma_2(G) = [Z_c(G), G] \leq Z_{c-1}(G),$$

$$\gamma_3(G) \leq [Z_{c-1}(G), G] \leq Z_{c-2}(G),$$

and eventually

$$\gamma_{c+1}(G) \leq Z_0(G) = 1.$$

Thus G is nilpotent of class at most c.

On the other hand, according to the previous lemma, if G is nilpotent of class c, then $\gamma_1(G)$ is contained in $Z_c(G)$ and therefore $Z_c(G) = G$. Now the result follows. □

Thus, the class of nilpotency of a group G is the length of both the upper and the lower central series.

Recall 3.24. Any finite p-group has non trivial centre.

Let $G' = [G, G]$.

Theorem 3.25. Let G be a finite p-group of order $p^m \geq p^2$. Then:

1. The nilpotency class of G is at most $m - 1$;
2. If G has nilpotency class c, then $[G : Z_{c-1}(G)] \geq p^2$;
3. $[G : G'] \geq p^2$.

Proof. Let c be the nilpotency class of G. We are proving (2). Suppose by contradiction that $|G : Z_{c-1}(G)| = p$. If $c = 1$ this means that $|G| = p$, contrary to our assumption. Hence $c \geq 2$ and

$$\frac{G/Z_{c-2}(G)}{Z(G/Z_{c-2}(G))} = \frac{G/Z_{c-2}(G)}{Z_{c-1}(G)/Z_{c-2}(G)} \simeq \frac{G}{Z_{c-1}(G)},$$

is a cyclic group. Consequently $G/Z_{c-2}(G)$ is abelian and $Z_{c-1}(G) = G$, which is a contradiction. This proves (2).

Now (3) is a consequence of Lemma 3.22, which assures that $G' \leq Z_{c-1}(G)$. Finally, since the series

$$G = Z_c(G) > Z_{c-1}(G) > \cdots > Z(G) > Z_0(G) = 1$$

has c steps, it follows from (2) that $p^m = |G| \geq p^{c+1}$. Hence $c \leq m-1$ and (1) holds. □

Definition 3.26. Let p be a prime number and G be a p- group of order p^n. G is said to be a *group of maximal class* if its nilpotency class is $n-1$

In particular, a 2-group of order 2^{n+1} is of maximal class if its nilpotency class is n.

Remark 3.27. Let $G = D_{2^{n+1}}$ and $N = \langle a^{2^k} \rangle$, with $k \in \{2 \dots n\}$. Then N is normal subgroup of G, and $G/N \simeq D_{2^{k+1}}$.

Proof. To prove that N is a normal subgroup of G, it suffices to show that $\langle a^{2^k} \rangle$ it is closed under conjugation by x. The $(a^{2^k})^x = a^{-2^k}$ is the inverse of the generator of the subgroup, therefore it belongs to N. It implies that N is a normal subgroup of G.

To prove that $G/N \simeq D_{2^{k+1}}$, it is enough to look at the presentation, which is given by

$$\langle \bar{a}, \bar{x} \mid \bar{a}^{2^k} = 1, \bar{x}^2 = 1, \bar{a}^{\bar{x}} = \bar{a}^{-1} \rangle,$$

where $\bar{a} = aN$ and $\bar{x} = xN$. □

Remark 3.28. Let $n \geq 1$. The dihedral group $D_{2^{n+1}}$ is a 2-group of maximal class.

Proof. The order of $D_{2^{n+1}}$ is 2^{n+1}. Let us show that $D_{2^{n+1}}$ has maximal class by induction on n.

If $n = 1$, the group we are looking at is $D_4 = C_2 \times C_2$, which is abelian and so nilpotent of nilpotency class 1.

Let now assume that $G = D_{2^n}$ is a nilpotent group of maximal class. First, we describe the centre of G. Since G is generated by a and x, it is enough to find all the elements which commute with both a and x. Let $g \in G$. There exist $k \in \{0, \dots, 2^n - 1\}$ and $m \in \{0, 1\}$ such that $g = a^k x^m$. We want k and m such that $xg = gx$ and $ag = ga$. Using the relation $a^x = a^{-1}$, the first identity becomes $a^k = a^{-k}$, which is true if and only if $k = 0$ or $k = 2^{n-1}$. The second identity becomes $ax^m = x^m a$, which is true if and only if $m = 0$. It follows that $Z(G) = \langle a^{a^{2^{n-1}}} \rangle$, which has order 2. By the remark 3.27 the quotient $D_{2^{n+1}}/Z(D_{2^{n+1}})$ is isomorphic to D_{2^n}, which, by induction, is nilpotent of maximal class. It follows that $D_{2^{n+1}}$ is a nilpotent group of class n. □

Remark 3.29. Let $n \geq 3$. The semidihedral group $SD_{2^{n+1}}$ is a 2-group of maximal class.

Proof. The order of $G = SD_{2^{n+1}}$ is 2^{n+1}. Let us show that G is a group of maximal class.

The centre $Z(G)$ is the subgroup $\langle a^{2^{n-1}} \rangle$, which has order 2. Indeed, let $g \in G$, then there exist $i \in \{0, \dots, 2^n - 1\}$, $j \in \{0, 1\}$ such that $g = a^i x^j$. It is enough to find i and j such that g commutes with the two generators a and x. It means that the following identities must hold: $a^i x^j x = x a^i x^j$ and $a^i x^j a = a a^i x^j$.

The first identity is equivalent to $a^i x = x a^i$, which holds if and only if $i = 0$ or $i = 2^{n-1}$, because of the relation $a^x = a^{-1+2^{n-1}}$.

The second identity is equivalent to $x^j a = a x^j$, which leads to $j = 0$.

The quotient $G/Z(G)$ is isomorphic to D_{2^n}, since they have the same presentation. Applying the Remark 3.28 D_{2^n} is a 2-group of maximal class, which implies that also $SD_{2^{n+1}}$ has maximal class. □

Remark 3.30. Let $n \geq 2$. Then the Quaternion group $Q_{2^{n+1}}$ is a 2-group of maximal class.

Proof. The order of $G = Q_{2^{n+1}}$ is 2^{n+1}. Let us show that G is a group of maximal class.

The centre $Z(G)$ is the subgroup $\langle a^{2^{n-1}} \rangle$ of order 2. Indeed, let $g \in G$, then there exist $i \in \{0, \dots, 2^n - 1\}$, $j \in \{0, 1\}$ such that $g = a^i x^j$. Let us find i and j such that g commutes with the two generators a ad x. The following identities must hold: $a^i x^j x = x a^i x^j$ and $a^i x^j a = a a^i x^j$.

Using the relation $a^x = a^{-1}$, the first identity, which can be rewritten as $a^i x = x a^i$, holds if and only if $i = 0$ or $i = 2^{n-1}$.

The second identity is equivalent to $x^j a = a x^j$, which leads to $j = 0$.

The quotient $G/Z(G)$ is isomorphic to D_{2^n}, since they have the same presentation. The Remark 3.28 allows us to conclude that $Q_{2^{n+1}}$ has maximal class. □

3.3 Dihedral group of order a power of two

The first group we are going to analyse is the Dihedral group with order a power of two, which has the following presentation:

$$D_{2^{n+1}} = \langle a, x \mid a^{2^n} = 1 = x^2, a^x = a^{-1} \rangle,$$

with $n \geq 2$. This group consists of two generators: a, which has order 2^n, and x, which has order 2. The relation $a^x = a^{-1}$ indicates that conjugation by x inverts the element a.

There are two distinct types of subgroups in D_{2n+1}, characterized as follows:

1. **Subgroups of the form** $\langle a^{2^k} \rangle$, where $k \in \{0, \ldots, n-1\}$. These subgroups are normal in $D_{2^{n+1}}$ and their index is 2^{k+1}, as the order of a^{2^k} is 2^{n-k}.

2. **Subgroups of the form** $\langle a^d, a^r x \rangle$, where d is a divisor of 2^n, and $r \in \{0, \ldots, d-1\}$. Generally, these subgroups are not normal in $D_{2^{n+1}}$ and their index is d.

Let $G = D_{2^{n+1}}$ and $H = \langle a^{2^k} \rangle$ be a subgroup of G of the first form. Since H is normal in G, it holds that $HgH = gH$ for any $g \in G$. In this case the weight of the H-double coset of g is given by $wt(HgH) = \frac{1}{2^{k+1}}$. Moreover, for a fixed $k \in \{0, \ldots, n-1\}$, there are 2^{k+1} double cosets with the aforementioned weight. Therefore, the contribution of these subgroups to the double coset zeta function is

$$\sum_{k=0}^{n-1} (2^{k+1})^{1-s}.$$

Let $H = \langle a^d, a^r x \rangle$ be a subgroup of G of the second type, where $d \mid 2^n$ and $r \in \{0, \ldots, d-1\}$. Since d must divide 2^{n+1}, it follows that d must be of the form $d = 2^k$, with $k \in \{1, \ldots, n\}$.

For $k = 1$, $H = \langle a^2, a^r x \rangle$, where $r \in \{0, 1\}$. In this situation H is a normal subgroup of G, since it has index 2. Therefore, for any $g \in G$, the weight of the H-double coset is given by $wt(HgH) = \frac{1}{2}$. Additionally, there are two subgroups of this form (one corresponding to $r = 0$ and the other to $r = 1$), each of which generates two double cosets. As a result, the contribution to the double coset zeta function is

$$2 \cdot \; 2^{1-s}.$$

Let us now analyse the particular case where $k = n$ and $r = 0$, so the subgroup is $H = \langle x \rangle$.

By the Proposition 3.9, $HgH = gH$ if and only if $g \in N_G(H)$. The subgroup H is cyclic of order two, so $N_G(H) = C_G(H)$. Thus, we can just study the centralizer.

Claim 3.31. Let $G = D_{2^{n+1}}$ the dihedral group of order 2^{n+1}, and let x be a generator of G of order two, then $C_G(x) = \{1, x, a^{2^{n-1}}, xa^{2^{n-1}}\}$

Proof. Let g be an element of $D_{2^{n+1}}$. There exist $i \in \{1, \dots, 2^n\}$ and $\epsilon \in \{0, 1\}$ such that $g = a^i x^\epsilon$. The element g belongs to the centralizer $C_{D_{2^{n+1}}}(x)$ if and only if $a^i x^\epsilon x = x a^i x^\epsilon$. This equality leads to the condition $a^i = a^{-i}$, which holds if and only if $i = 0$ or $i = 2^{n-1}$. Consequently, we have $C_{D_{2^{n+1}}}(x) = \{1, x, a^{2^{n-1}}, xa^{2^{n-1}}\}$. □

There are two double cosets of H of weight $wt(HgH) = \frac{1}{2^n}$. They are $H = HxH$ and $Ha^{2^{n-1}}H = Ha^{2^{n-1}}xH$.

If $g \notin C_{D_{2^{n+1}}}(x)$, the corresponding double coset HgH has order 4, so it must be the union of two distinct right cosets of H.

Let m be the number of H-double cosets of weight $wt(HgH) = \frac{2}{2^n}$. Since G is the disjoint union of these H-double cosets, m must satisfy the equation:

$$2^{n+1} = 2 + 2 + 4m,$$

which implies that $m = 2^{n-1} - 1$. Thus, the contribute to the double coset zeta function given by the double coset of $H = \langle x \rangle$ is

$$2 \cdot \left(\frac{1}{2^n}\right)^s + \left(2^{n-1} - 1\right)\left(\frac{1}{2^{n-1}}\right)^s.$$

Let us now consider the general case where $k \in \{2, \dots, n\}$ and $r \in \{0, \dots, 2^k - 1\}$. Then H is the subgroup generated by a^d and $a^r x$, where $d = 2^k$. Let $N = \langle a^d \rangle$: it is a cyclic, normal subgroup of G and H can be expressed as $H = N\langle a^r x \rangle$. If we look at the element $a^r x$, it is an element of order 2 and $a^{a^r x} = xax = a^{-1}$. It means that $\langle a, a^r x \rangle$ give rise to the same presentation as $\langle a, x \rangle = D_{2^{n+1}}$. Therefore, for any $r \in \{0, \dots, 2^k - 1\}$ there exists an isomorphism sending

$$x \longmapsto a^r x$$
$$a \longmapsto a.$$

Let $\bar{H}$ be the image of H in the quotient $\bar{G} = G/N$. By the Remark 3.27 $\bar{G}$ is isomorphic to $D_{2^{k+1}}$. We can proceed just studying the $\bar{H}$-double coset of $\bar{G}$. For the isomorphism above it is equivalent to study the $\langle \bar{x} \rangle$-double coset of $D_{2^{k+1}}$, that is the same as the case we already analysed. Therefore, every subgroup $H = \langle a^d, a^r x \rangle$ gives to the double coset zeta function the same contribute as the subgroup $\langle \bar{x} \rangle$ in $D_{2^{k+1}}$. Since there are 2^k possible choices for the integer r, the contribution for the double coset zeta function is:

$$\sum_{k=2}^{n} 2^k \left(2 \cdot \left(\frac{1}{2^k}\right)^s + \left(2^{k-1} - 1\right)\left(\frac{1}{2^{k-1}}\right)^s\right).$$

To summarize, we can list the contributions of each type of subgroup to the double coset zeta function in a table.

Table 3.1: Table of the subgroups and contributions for the $\zeta^{dc}_{D_{2^{n+1}}}(s)$.

Subgroup H	**Choices**	**Contribution to** $\zeta^{dc}_{D_{2^{n+1}}}(s)$
$\langle a^{2^k}\rangle$	$k \in \{0,\dots,n-1\}$	$\sum_{k=0}^{n-1}(2^{k+1})^{1-s}$
$\langle a^{2^k}, a^r x\rangle$	$k \in \{2,\dots,n\}, r \in \{0,\dots,2^{k-1}\}$	$\sum_{k=2}^{n} 2^k \left(2\cdot\left(\frac{1}{2^k}\right)^s + (2^{k-1}-1)\left(\frac{1}{2^{k-1}}\right)^s\right)$
$\langle a^2, a^r x\rangle$	$r \in \{0,1\}$	$2\cdot 2^{1-s}$
$\langle a, x\rangle$	/	1

By summing all the contributes together we obtain the double coset zeta function of $D_{2^{n+1}}$:

$$\begin{aligned}\zeta^{dc}_G(s) &= 1 + 2^{2-s} + \sum_{k=0}^{n-1} 2^{1-s}\cdot(2^{1-s})^k + \sum_{k=2}^{n} 2\cdot(2^{1-s})^k + \sum_{k=2}^{n} 2\cdot(2^{2-s})^{k-1} - \sum_{k=2}^{n} 2\cdot(2^{1-s})^{k-1}\\ &= 1 + 2^{2-s} + 2^{1-s}\cdot\frac{1-(2^{1-s})^n}{1-2^{1-s}} + 2\cdot\frac{(2^{1-s})^2-(2^{1-s})^{n+1}}{1-2^{1-s}}\\ &+ 2\cdot\frac{2^{2-s}-(2^{2-s})^n}{1-2^{2-s}} - 2\cdot\frac{2^{1-s}-(2^{1-s})^n}{1-2^{1-s}}.\end{aligned}$$

If we set $p = 2$ and $q = 2^{-s}$, the double coset zeta function of $D_{2^{n+1}}$ becomes $\zeta^{dc}_{D_{2^{n+1}}}(s) = f_n(p,q)$, where

$$f_n(p,q) = 1+p^2q+pq\frac{1-(pq)^n}{1-pq}+p\frac{p^2q^2-(pq)^{n+1}}{1-pq}+p\frac{p^2q-(p^2q)^n}{1-p^2q}-p\frac{pq-(pq)^n}{1-pq} \in \mathbb{Q}(p,q)$$

is a rational function in p and q.

3.4 Recursive formula for the dihedral group of order 2^{n+1}

The next goal is to derive a recursive formula for the double coset zeta function of the dihedral group $D_{2^{n+1}}$. Let $N = \langle a^{2^{n-1}} \rangle$. By considering the quotient $D_{2^{n+1}}/N$, which is isomorphic to the dihedral group of order D_{2^n}, it is possible to establish a recursive formula for the double coset zeta function of the dihedral group of order a power of two.

Thanks to the correspondence between the subgroups of $D_{2^{n+1}}$ which contain N and the subgroups of D_{2^n}, their contribution to the double coset zeta function is the same. Therefore, we only need to determine the contribution given by the subgroups of the form $\langle a^r x \rangle$, where $r \in \{1, \dots, 2^n\}$, that are the proper subgroups of $D_{2^{n+1}}$ which do not contain N. Then we will add the contribution of the identity subgroup.

There are 2^n such subgroups of the form $\langle a^r x \rangle$. We already know the contribution of these subgroups to the double coset zeta function of $D_{2^{n+1}}$, which is:

$$2^n \left(2 \cdot 2^{-ns} + (2^{n-1} - 1)2^{-s(n-1)}\right).$$

If we substitute $p = 2$ and $q = 2^{-s}$, the expression becomes:

$$p^n(pq^n + (p^{n-1} - 1)q^{n-1}),$$

which simplifies to

$$p^{n+1}q^n + p^{n-1}q^{n-1} - q^{n-1}.$$

Moreover, the contribution of the identity subgroup is:

$$2^{(n+1)(1-s)} = p^{n+1}q^{n+1}.$$

By summing everything together, we obtain:

$$\zeta^{dc}_{D_{2^{n+1}}}(s) = \zeta^{dc}_{D_{2^n}}(s) + p^{n+1}q^n + p^{n-1}q^{n-1} - q^{n-1} + p^{n+1}q^{n+1},$$

a recursive formula for the computation of the double coset zeta function of the dihedral group of order a power of two.

3.5 Pro-2-dihedral group

In this section we will examine the pro-2-dihedral group, which is the inverse limit of the family $D_{2^{n+1}}$, with $n \in \mathbb{N}$. We can think to take

$$\zeta^{dc}_{D_{2^{n+1}}}(s) = f_n(p, q),$$

and let n go to infinity. In this way we obtain the double coset zeta function of the pro-2-dihedral group

$$C_2 \ltimes \mathbb{Z}_2.$$

Under the condition of convergence of $f_n(p, q)$, which is $Re(s) > 2$, when n goes to infinity, the double coset zeta function of the pro-2-dihedral group is:

$$\begin{aligned}\zeta^{dc}_{C_2 \ltimes \mathbb{Z}_2}(s) &= 1 + p^2 q + pq\frac{1}{1-pq} + p\frac{p^2q^2}{1-pq} + p\frac{p^2 q}{1-p^2 q} - p\frac{pq}{1-pq} \\ &= \frac{1-p^2q-pq+p^3q^2+p^2q-p^4q^2-p^3q^2+p^5q^3+pq-p^3q^2+p^3q^2-p^5q^3+p^3q-p^4q^2-p^2q+p^4q^2}{(1-pq)(1-p^2q)} \\ &= \frac{1-p^4q^2+p^3q-p^2q}{(1-pq)(1-p^2q)}.\end{aligned}$$

As we have seen in the Remark 3.19, we can compare the double coset zeta function $G = C_2 \ltimes \mathbb{Z}_2$ with its subgroup zeta function evaluated at $s-1$.

We start analysing the function $\zeta^{\leq}_{G}(s-1)$ of the group $G = D_{2^{n+1}}$. Recall that the subgroups of the form $\langle a^{2^k} \rangle$, with $k \in \{0, \dots, n-1\}$, and $\langle a^2, a^r x \rangle$, with $r \in \{0, 1\}$ are normal in G, and we have already analysed their contribute to the double coset zeta function. The only subgroups that we need to examine are the ones of the form $\langle a^{2^k}, a^r x \rangle$, where $k \in \{2, \dots, n\}$ and $r \in \{0, \dots, 2^k - 1\}$.

Let $k \in \{2, \dots, n\}$ and $H = \langle a^{2^k}, a^r x \rangle$. Then $[G : H] = 2^k$. Since the integer $r \in \{0, \dots, 2^k - 1\}$, there are 2^k subgroups of this form. Therefore, the contribution to the subgroup zeta function evaluated at $s-1$ that we obtain is

$$\sum_{k=2}^{n} 2^k \cdot \left(\frac{1}{2^k}\right)^{s-1} = \sum_{k=2}^{n} (2^{2-s})^k.$$

On the next table we can read the contribution that every type of subgroup gives to the subgroup zeta function of $D_{2^{n+1}}$.

Table 3.2: Table of the subgroups and contributions for the $\zeta^{\leq}_{D_{2^{n+1}}}(s)$.

Subgroup H	**Choices**	**Contribution to** $\zeta^{\leq}_{D_{2^{n+1}}}(s-1)$
$\langle a^{2^k} \rangle$	$k \in \{0, \dots, n-1\}$	$\sum_{k=0}^{n-1} (2^{k+1})^{1-s}$
$\langle a^{2^k}, a^r x \rangle$	$k \in \{0, \dots, n-1\}, r \in \{0, \dots, 2^{k-1}\}$	$\sum_{k=0}^{n-1} \left(2^{2-s}\right)^k$
$\langle a, x \rangle$	/	1

In this case we have

$$\begin{aligned}\zeta^{\leq}_{D_{2^{n+1}}}(s-1) &= 1 + \sum_{k=0}^{n-1} (2^{k+1})^{1-s} + \sum_{k=0}^{n-1} (2^{2-s})^k \\ &= 1 + 2^{1-s} \cdot \frac{1-(2^{1-s})^n}{1-2^{1-s}} + \frac{1-(2^{2-s})^n}{1-2^{2-s}}.\end{aligned}$$

By setting $p = 2$ and $q = 2^{-s}$, we obtain $\zeta^{\leq}_{D_{2^{n+1}}}(s-1) = f_n(p,q)$, where

$$f_n(p,q) = 1 + pq\frac{1-(pq)^n}{1-pq} + \frac{1-(p^2q)^n}{1-p^2q}$$

and letting n go to infinity we get the subgroup zeta function of the pro-2-dihedral group

$$\begin{aligned}\zeta^{\leq}_{C_2 \ltimes \mathbb{Z}_2}(s-1) &= \frac{(1-pq)(1-p^2q) + pq(1-p^2q) + 1 - pq}{(1-pq)(1-p^2q)} \\ &= \frac{2-pq-p^2q}{(1-pq)(1-p^2q)}.\end{aligned}$$

We observe that the denominator is equal to the denominator of the double coset zeta function, while the numerator differs. In particular, the degree of the numerator in the subgroup zeta function is lower.

3.6 Dihedral group of order $2p^n$ and pro-p-dihedral group

In this section, we aim to generalise the previous results to the pro-p-dihedral group, where p is an odd prime ($p \neq 2$). We begin by considering the dihedral group of order $2p^n$, which is defined as follows:

$$D_{2p^n} = \langle a, x \mid a^{p^n} = 1, x^2 = 1, a^x = a^{-1} \rangle.$$

This group consists of two generators: a, which has order p^n, and x, which has order 2. The relation $a^x = a^{-1}$ indicates that conjugation by x inverts the element a.

There are two distinct types of subgroups within D_{2p^n}, characterized as follows:

1. **Subgroups of the form** $\langle a^{p^k} \rangle$, where $k \in \{0, \ldots, n-1\}$.These are normal subgroups of D_{2p^n}. As the order of a^{p^k} is p^{n-k}, $[D_{2p^n} : \langle a^{p^k} \rangle] = 2p^k$.

2. **Subgroups of the form** $\langle a^d, a^r x \rangle$, where d is a divisor of p^n, and $r \in \{0, \ldots, d-1\}$. Generally, these subgroups are not normal in D_{2p^n} and their index is d.

About the subgroups of the first type, $\langle a^{p^k} \rangle$, since they are normal, the double cosets are cosets. There are $[D_{2p^n} : \langle a^{p^k} \rangle] = 2p^k$ cosets, each of weight $[D_{2p^n} : \langle a^{p^k} \rangle]^{-1} = 2p^{-k}$. Therefore, the contribution of these subgroups to the double coset zeta function is

$$\sum_{k=0}^{n-1} 2p^k \cdot (2p^k)^{-s}.$$

Let us now examine the second type of subgroups. As a first step, we analyse the subgroup $H = \langle x \rangle$, so the particular case in which $k = 0$ and $r = 0$.

Let $G = D_{2p^n}$. By the Remark 3.9 it holds that $HgH = gH$ if and only if $g \in N_G(H)$. Since H is cyclic of order two, its normalizer coincides with its centralizer, so we can analyse $C_G(H)$.

Remark 3.32. Let $G = D_{2p^n}$ and let x and a be two generators of G respectively of order 2 and p^n, then $C_G(x) = \{1, x\}$.

Proof. Let g be an arbitrary element of D_{2p^n}. We can express g as $g = x^\epsilon a^i$, where $\epsilon \in \{0, 1\}$ and $i \in \{0, \ldots, p^n - 1\}$. We can then rewrite the previous condition as $x x^\epsilon a^i = x^\epsilon a^i x$. This condition is equivalent to requiring that $x a^i = a^i x$, which means that $a^i = (a^i)^x$. Given the relation $a^x = a^{-1}$ and considering that a has odd order, it follows that i must be 0. □

Therefore, when $g = 1$ or $g = x$, we have $HgH = gH = H$, which means that there is a double coset in D_{2p^n} with weight $wt(HgH) = \frac{1}{p^n}$.
For $g \in D_{2p^n} \setminus \{1, x\}$, there are $\frac{|D_{2p^n}|-2}{4} = \frac{p^n-1}{2}$ double cosets. Since every double coset has order 4, each of them is the union of two disjoint right cosets. Consequently, their weight is $wt(HgH) = \frac{2}{p^n}$.

Thus, the contribute that H gives to the double coset zeta function is

$$\left(\frac{1}{p^n}\right)^s + \frac{p^n-1}{2}\left(\frac{2}{p^n}\right)^s.$$

Consider now the subgroup $H = \langle a^{p^k}, a^r x\rangle$, where $k \in \{0, \dots, n-1\}$ and $r \in \{0, \dots, p^k - 1\}$.

Let $N = \langle a^{p^k}\rangle$, which is a cyclic normal subgroup of D_{2p^n}. We can write H as $H = N\langle a^r x\rangle$. Let $\bar{G} = G/N$, it is isomorphic to D_{2p^k}, since they have the same presentation. Moreover, the element $a^r x$ has order 2 and $a^{a^r x} = a^{-1}$. It means that $\langle a, a^r x\rangle$ give rise to the same presentation as $\langle a, x\rangle = D_{2^{n+1}}$. Therefore, for any $r \in \{0, \dots, p^k - 1\}$ there exists an isomorphism sending

$$\begin{aligned} x &\longmapsto a^r x \\ a &\longmapsto a. \end{aligned}$$

To understand the structure of the H-double cosets, we can study the $\bar{H}$-double cosets in $\bar{G}$. Thanks to what we said above, we know that it is the same as studying the $\langle\bar{x}\rangle$-double cosets, so we are in the case we already examinated. Together with the fact that there are p^k subgroups of this form, due to the possible choices for the integer r, the contribute to the double coset zeta function arising from the subgroup of the form $H = \langle a^{p^k}, a^r x\rangle$, where $k \in \{0, \dots, n-1\}$ and $r \in \{0, \dots, p^k - 1\}$ is

$$\sum_{k=0}^{n-1} p^k \left(\left(\frac{1}{p^k}\right)^s + \frac{p^k-1}{2}\left(\frac{2}{p^k}\right)^s\right).$$

We can now list the contributions into a table and sum them all together.

Table 3.3: Table of the subgroups and contributions for the $\zeta^{dc}_{D_{2p^n}}(s)$.

Subgroup H	**Choices**	**Contribution to** $\zeta^{dc}_{D_{2p^n}}(s)$
$\langle a^{p^k}\rangle$	$k \in \{0,\dots,n-1\}$	$\sum_{k=0}^{n-1}(2p^k)^{1-s}$
$\langle a^{p^k}, a^r x\rangle$	$k \in \{0,\dots,n-1\}$ and $r \in \{0,\dots,2^k-1\}$	$\sum_{k=0}^{n-1} p^k \left(\left(\frac{1}{p^k}\right)^s + \frac{p^k-1}{2}\left(\frac{2}{p^k}\right)^s\right)$
$\langle a, x\rangle$	/	1

$$\zeta^{dc}_{D_{2p^n}}(s) = 1 + \sum_{k=0}^{n-1}(2p^k)^{1-s} + \sum_{k=0}^{n-1} p^k \left(\left(\frac{1}{p^k}\right)^s + \frac{p^k-1}{2}\left(\frac{2}{p^k}\right)^s\right)$$

$$1 + \sum_{k=0}^{n-1} 2^{1-s}p^{k(1-s)} + \sum_{k=0}^{n-1}\left(p^{k(1-s)} + \frac{p^k-1}{2}\cdot\frac{2^s}{p^{ks}}\right)$$

$$= 1 + 2^{1-s}\sum_{k=0}^{n-1} p^{k(1-s)} + \sum_{k=0}^{n-1} p^{k(1-s)} + \sum_{k=0}^{n-1} 2^{s-1}\left(p^{k(1-s)} - p^{k(-s)}\right)$$

$$= 1 + 2^{1-s}\cdot\frac{1-(p^{1-s})^n}{1-p^{1-s}} + \frac{1-(p^{1-s})^n}{1-p^{1-s}} + 2^{s-1}\frac{1-(p^{1-s})^n}{1-p^{1-s}} - 2^{s-1}\frac{1-(p^{-s})^n}{1-p^{-s}}.$$

For $Re(s) > 1$, when n goes to infinity we obtain:

$$\zeta^{dc}_{C_2 \ltimes \mathbb{Z}_p} = 1 + \frac{2^{1-s}}{1-p^{1-s}} + \frac{1}{1-p^{1-s}} + \frac{2^{s-1}}{1-p^{1-s}} - \frac{2^{s-1}}{1-p^{-s}},$$

the double coset zeta function of the pro-p-dihedral group.

3.7 Semidihedral group

In this section we will examine the Semidihedral group, defined as

$$SD_{2^{n+1}} = \langle a, x \mid a^{2^n} = 1, x^2 = 1, a^x = a^{2^{n-1}-1} \rangle,$$

where $n \geq 3$.
A subgroup H of $SD_{2^{n+1}}$ can take one of the following forms:

1. **Subgroup of the form** $\langle a^{2^k} \rangle$, where $k \in \{0, \dots, n-1\}$. They are normal subgroups of $SD_{2^{n+1}}$ of order 2^{n-k}, therefore the index is 2^{k+1}.

2. **Subgroup of the form** $\langle a^d, a^r x \rangle$, where d is a divisor of 2^n, and $r \in \{0, \dots, d-1\}$. The cosets of $H = \langle a^d, a^r x \rangle$ can be represented by $\{1H, aH, \dots, a^{d-1}H\}$, so $[SD_{2^{n+1}} : H] = d$.

Let us consider $\langle a^{2^{n-1}} \rangle$, the cyclic normal subgroup of $SD_{2^{n+1}}$ generated by $a^{2^{n-1}}$.

Recall 3.33. Let $n \geq 3$. Then $SD_{2^{n+1}} / \langle a^{2^{n-1}} \rangle \simeq D_{2^n}$.

There are two types of subgroups of $G = SD_{2^{n+1}}$: those that contain $N = \langle a^{2^{n-1}} \rangle$ and those that do not. By looking at the image in the quotient G/N, the subgroups of $SD_{2^{n+1}}$ that contain N correspond exactly to the subgroups of the dihedral group D_{2^n}. Thus, their contribution for the double coset zeta function remains the same.

The double coset zeta function can be decomposed into two parts: the first part corresponds to the subgroups of the first type, so it is the same as the double coset zeta function of the dihedral group. The second part corresponds to the subgroups of the second type. Therefore, we only need to study the subgroups of $SD_{2^{n+1}}$ that do not contain the element $a^{2^{n-1}}$. These subgroups are the identity subgroup and the ones of the form $\langle a^k x \rangle$, where $k \in \{1, \dots, 2^n\}$.

Looking at the element

$$(a^k x)^2 = a^k \cdot a^{k \cdot 2^{n-1} - k} = a^{k \cdot 2^{n-1}},$$

we observe that

- when k is odd, $(a^k x)^2 = a^{2^{n-1}}$, so N is contained in $\langle a^k x \rangle$;
- when k is even, $(a^k x)^2 = 1$.

Therefore, we only need to study the case in which k is even. In such a case, $\langle a^k x \rangle \simeq \langle x \rangle$, which is the cyclic group of order two. Moreover, by the relation $a^{a^k x} = a^{2^{n-1}-1}$, we get that $\langle a, a^r x \rangle$ has the same presentation of $\langle a, x \rangle = SD_{2^{n+1}}$. This implies that studying the $\langle a^r x \rangle$-double coset of $SD_{2^{n+1}}$ is the same as studying the $\langle x \rangle$-double coset of $SD_{2^{n+1}}$.

Let $g \in G$, and $H = \langle x \rangle$. The H-double coset of g is the set $HgH = \{g, x, gx, xgx\}$, so $HgH = gH$ if and only if $g \in C_G(x)$.

Claim 3.34. Let $G = SD_{2^{n+1}}$ and let x be a generator of G of order 2. Then $C_G(x) = \{1, x, a^{2^{n-1}}, a^{2^{n-1}}x\}$.

Proof. Let $g \in SD_{2^{n+1}}$, there exist $k \in \{0, \ldots, 2^n - 1\}$ and $m \in \{0, 1\}$ such that $g = a^k x^m$. We want to find k and m such that $g \in C_G(H)$.

- If $m = 0$, then $a^k x = xa^k$ if and only if $xa^k x = a^k$, but, since $xa^k x = a^{k(-1+2^{n-1})}$ and $a^{2^n} = 1$, we get $k = 0$ or $k = 2^{n-1}$.
- If m=1, then $a^k xx = xa^k x$ if and only if $a^k = a^{k(2^{n-1}-1)}$, which implies $k = 0$ or $k = 2^{n-1}$.

Thus, the claim is proved. □

The claim above implies that, for this type of subgroup, there are 2 double cosets that are just cosets and their weight is 2^{-n}. The other double cosets must be the disjoint union of two different right cosets and their weight is $\frac{2}{2^n}$. Using the fact that G can be seen as the disjoint union of these H-double cosets, the number of double cosets of weight 2^{1-n} is $2^{n-1} - 1$. The contribute that they give to the double coset zeta function is

$$2^{n-1}\left(2\left(\frac{1}{2^n}\right)^s + (2^{n-1} - 1)\left(\frac{1}{2^{n-1}}\right)^s\right).$$

Moreover, the identity subgroup is normal with index 2^{n+1}, so it contributes to the zeta function with $2^{(n+1)(1-s)}$.

By summing everything together, we obtain the total contribution of the second part, which is:

$$\begin{aligned}
&2^{n-1}\left(2\left(\frac{1}{2^n}\right)^s + (2^{n-1} - 1)\left(\frac{1}{2^{n-1}}\right)^s\right) + 2^{(n+1)(1-s)} \\
&= 2^n \cdot 2^{-ns} + 2^{n-1} \cdot 2^{-s(n-1)} - 2^{n-1} \cdot 2^{-s(n-1)} + 2^{(n+1)(1-s)} \\
&= 2^{n(1-s)} + 2^{n-1} \cdot 2^{2-s} - 2^{n-1} \cdot 2^{1-s} + 2^{(n+1)(1-s)}.
\end{aligned}$$

Setting now $p = 2$ and $q = 2^{-s}$ we get

$$\begin{aligned}
&p^n q^n + p^{n+1} q - p^n q + p^{n+1} q^{n+1} \\
&= p^n q(q^{n-1} + p - 1 + pq^n).
\end{aligned}$$

So the double coset zeta function of the Semidihedral Group is

$$\zeta^{dc}_{SD_{2^{n+1}}}(s) = \zeta^{dc}_{D_{2^n}}(s) + p^n q(q^{n-1} + p + pq^n - 1),$$

where $p = 2$ and $q = 2^{-s}$.

3.8 Quaternion group

In this section we will analyse the Quaternion group, which is defined to be

$$Q_{2^{n+1}} = \langle a, x \mid a^{2^n} = 1, x^2 = a^{2^{n-1}}, a^x = a^{-1} \rangle,$$

with $n \geq 2$.

Let H be a subgroup of $Q_{2^{n+1}}$. Then H can be one of the following:

1. **Subgroup of the form** $\langle a^{2^k} \rangle$, where $k \in \{0, \dots, n-1\}$. They are normal subgroup of $Q_{2^{n+1}}$ of order 2^{n-k}. Therefore $[Q_{2^{n+1}} : H] = 2^{k+1}$.

2. **Subgroup of the form** $\langle a^d, a^r x \rangle$, where d is a divisor of 2^n, and $r \in \{0, \dots, d-1\}$. Their cosets can be represented by $\{1H, aH, \dots, a^{d-1}H\}$, so $[Q_{2^{n+1}} : H] = d$.

Let us consider $\langle a^{2^{n-1}} \rangle$, the cyclic normal subgroup of $Q_{2^{n+1}}$ generated by the element $a^{2^{n-1}}$.

Recall 3.35. Let $n \geq 2$, then $Q_{2^{n+1}} / \langle a^{2^{n-1}} \rangle \simeq D_{2^n}$.

The subgroups of $Q_{2^{n+1}}$ that contain $\langle a^{2^{n-1}} \rangle$ corresponds to the subgroups of the dihedral group of order 2^n. Therefore, the respective contribute to the double coset zeta function is also the same. We can focus on studying the subgroups that do not contain the element $a^{2^{n-1}}$.

Claim 3.36. There are no nontrivial subgroups of $Q_{2^{n+1}}$ which does not contain $a^{2^{n+1}}$.

Proof. It suffices to show that $a^{2^{n-1}}$ belongs to all the subgroups of $Q_{2^{n+1}}$ of the form $\langle a^r x \rangle$. Given the relation $x^2 = a^{2^{n-1}}$, we can use the relation $ax = xa^{-1}$ to deduce that

$$(a^k x)^2 = a^k x a^k x = a^k x^2 a^{-k} = a^{2^{n-1}}.$$

□

This implies that the only subgroup of $Q_{2^{n+1}}$ that does not contain the element under consideration is the identity subgroup 1_G, whose contribution is $2^{(n+1)(1-s)}$, since it is a normal subgroup of index 2^{n+1}.

Therefore,

$$\zeta^{dc}_{Q_{2^{n+1}}}(s) = \zeta^{dc}_{D_{2^n}}(s) + 2^{(n+1)(1-s)}$$

is the double coset zeta function of the Quaternion group.

Bibliography

[1] B. Klopsch, N. Nikolov and C. Voll, *Lectures on Profinite Topics in Group Theory*, edito da Dan Segal, 16 Agosto 2010.

[2] G. A. Fernández-Alcober, *An introduction to finite p-groups: regular p-groups and groups of maximal class*, Matemática Contemporânea, Vol. 20 (2001).

[3] F.J. Grunewald, D. Segal, and G.C. Smith, *Subgroups of finite index in nilpotent groups*, Invent. Math. **93** (1988).

[4] A. Lubotzky and D. Segal, *Subgroup Growth*, Ferran Sunyer i Balaguer Award winning monograph, Birkhäuser, 2003.

[5] A. Lubotzky, A. Mann and D. Segal, *Finitely generated groups of polynomial subgroup growth*, Israel J. Math. 82 (1993).

[6] J. Neukirch, *Algebraic Number Theory*, tradotto dal tedesco da Norbert Schappacher, Springer, 1999.

[7] A. Rajaraman, *Subgroup growth: An introduction*, Course paper, SFU (2011).