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Schwarzian Theories and Sachdev-Ye-Kitaev model

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Introduction

The Schwarzian derivative is at the heart of a number of recent publications, concerning various fields of theoretical physics, from measure theory to uniformization theory and quantum gravity. It is defined as

$$\{f, z\} := \frac{f'''(z)}{f'(z)} - \frac{3}{2}\Big(\frac{f''(z)}{f'(z)}\Big)^2, \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a three times differentiable function. The same formula defines the Schwarzian derivative of a holomorphic function $f : \mathbb{C} \to \mathbb{C}$.
Typically, the Schwarzian derivative emerges due to its peculiar symmetry properties, namely it is invariant under the action of the group $\mathrm{PSL}(2, \mathbb{C})$ acting by linear fractional transformations

$$f(z) \to f_\gamma(z) := \frac{af(z) + b}{cf(z) + d}, \qquad \gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}). \tag{2}$$

This means that $\{f_\gamma, z\} = \{f, z\}$. In this thesis we study two interesting realizations of this symmetry-induced emergence of Schwarzian derivatives.

In sections 1.1 to 1.3 we relate the Schwarzian derivative to the Sachdev-Ye-Kitaev model. It describes the thermodynamic limit $N \to \infty$ of a system composed by N Majorana fermions $\{\chi_i\}_{i=1}^N$, in zero spatial dimension, interacting through the Hamiltonian density

$$H(t) = \frac{1}{4!} \sum_{ijkh}^{N} J_{ijkh} \chi_i(t)\chi_j(t)\chi_k(t)\chi_l(t). \tag{3}$$

The couplings J_{ijkh} are time independent and randomly chosen according to a specific Gaussian probability density function, which makes the computations of the disorder averaged physical quantities particularly simple in the thermodynamic limit.
In the infrared limit, which corresponds to high values of the effective coupling constant of the theory J, the statistically relevant configurations in the $N \to \infty$ limit are small fluctuations around the minima of an effective action $I = NS_{\mathrm{IR}}$, which displays a high degree of symmetry. Indeed, the minima of S_{IR} constitute an infinite-dimensional manifold, invariant under an action of the group of reparameterizations f of the real line

$$f(t) \in \mathrm{Rep}(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \ s.t. \ f \text{ differentiable and invertible}\}. \tag{4}$$

This symmetry is both explicitly broken by the non IR contributions to the theory, and spontaneously broken to $\mathrm{PSL}(2, \mathbb{R})$ by the choice of a particular minimum of the action.
In the IR limit, this proliferation of energetically inexpensive modes causes the functional integrals of the theory to diverge. If we give up the low energy approximation, the symmetry of the effective action gets broken by the high energy corrections to S_{IR}, so that the minimum of the exact action NS is unique, solving the troubles with divergences. On the other hand, computations become significantly more difficult, and finding an explicit solution was (so far!) impossible.

An intermediate step between the completely general situation described by S and the extreme approximation S_{IR} consists in studying how the high energy corrections split the minima of S_{IR}. We are then studying a restriction of the theory to the soft modes of the theory, *i.e.* the field configurations that minimize the approximated action S_{IR}. There are very strong indications that this restricted theory (written in terms of the $f(t) \in \mathrm{Rep}(\mathbb{R})$ parameterizing the space of minima) is governed by a Schwarzian action

$$S_{\mathrm{restricted}}[f] \propto \int_{\mathbb{R}} \{f, t\} dt, \tag{5}$$

one of the strongest being that one can argue that the action should be invariant under the action (2) of $\mathrm{PSL}(2, \mathbb{R})$. We will call this sort of theory "Schwarzian theory".
In recent years the SYK model has been intensely studied [1, 12, 13, 17, 22] due to its link to Jackiw-Teitelboim dilaton gravity theory, which is an instance of nearly-Anti de Sitter geometry in two dimensions arising in a number of gravity models in the near horizon region. The first hints to a possible involvement of the SYK model in the context of AdS/CFT correspondence came from the study of the out of time order 4-point correlation function. It turns out that at low energy its growth saturates the *chaos bound*, *i.e.* the relevant 4-point functions grow exponentially at a rate which is the maximum allowed for the theory to possibly admit an holographic bulk description in terms of Einstein gravity (see for example [16]). Another interesting fact is that the symmetry breaking pattern (conformal $\rightarrow$ $\mathrm{PSL}(2, \mathbb{R})$) characterizing the SYK model is also realized in any nearly-Anti de Sitter gravity theory, when passing from pure AdS to nAdS via the coupling to the dilaton, regardless of the precise matter content of the theory. Following those hints, it was possible to prove [13, 17] that the soft mode of the SYK model is indeed the holographic dual for a dilaton gravity theory on a AdS background. There is yet to understand whether a bulk interpretation for the full SYK model exists, and if so, what kind of string theory might describe its matter content. These unanswered questions motivate to develop a general technique to study the model in a functional integral approach.

Giving a rigorous meaning to functional integration over the domain $\mathrm{Rep}(\mathbb{R})$ in a theory governed by such an action requires the introduction of a new measure. The Schwarzian derivative has been known since a long time [21] to be involved in the quasi-invariance properties of the Wiener measure. In more recent years, Belokurov and Shavgulidze [3, 5] have devised a method that allows to use the Wiener measure to induce a measure μ on the space $\mathrm{Rep}(\mathbb{R})$. The striking feature of this construction is that the measure μ happens to be particularly adapted to computing integrals in a field theory with Schwarzian-like actions. This can be seen from the (formal) representation of μ in terms of Feynman path integrals, which reads

$$\mu_\sigma(d\phi) = \int \exp\Big[\frac{1}{\sigma^2} \int_0^1 \{\phi, t\} dt\Big] d\phi. \tag{6}$$

Thanks to this feature, it is possible to compute many functional integrals of physical interest explicitly, reducing them to ordinary Lebesgue multiple integrals. In section (1.4) we go over the construction of Belokurov and Shavgulidze, and show how the explicit calculations are carried over.

Another context where the $\mathrm{PSL}(2, \mathbb{C})$ symmetry brings the Schwarzian derivative into play is that of the Quantum Hamilton-Jacobi equation. Traditionally this term refers to the Quantum Theory of Motion developed by David Bohm and Louis de Broglie (see [10] for a comprehensive review) with the aim to preserve concepts such as *determinism* and *trajectory* in a theory that could give account for the empirical results traditionally explained in terms of Quantum Mechanics. In this framework the Quantum Hamilton-Jacobi equation is identified with the system of coupled differential equations

$$\begin{cases} \frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + V = 0 \\ \frac{\partial R^2}{\partial t} + \nabla\Big(\frac{R^2 \nabla S}{m}\Big) = 0 \end{cases} \tag{7}$$

which is equivalent to the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \Big(-\frac{\hbar^2}{2m}\nabla^2 + V\Big)\psi \tag{8}$$

once you decompose the wave function as

$$\psi(x) = R(x)e^{\frac{i}{\hbar}S_0(x)}\,; \quad (R, S_0) \in \mathbb{R}^2. \tag{9}$$

Although the first line in eq. (7) surely resembles the classical Hamilton-Jacobi equation for the principal function S, with the addition of a *quantum potential*

$$Q := \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}, \tag{10}$$

it is still a bit problematic to identify the two concepts. For example it is clear that the system (7), being comprised of two equations, is qualitatively very different from the classical Hamilton-Jacobi equation. Moreover, in general, taking the classical $\hbar \to 0$ limit of a solution S of (7) does not give a solution of the classical Hamilton-Jacobi equation.
In sections 2 and 2.4 we will see how a more recent proposal by M. Matone and A. E. Faraggi [7, 8] for the quantum generalization of the Hamilton-Jacobi equation can be justified in terms of an extremely minimal and elegant set of assumptions. Moreover, we discuss how such an equation is not affected by the above objections, *i.e.* gives results completely consistent with the one of the classical Hamilton-Jacobi theory in the $\hbar \to 0$ limit. The proposed quantum generalization, for a theory describing one particle in one dimension subjected to the Hamiltonian

$$H = \frac{p^2}{2m} + V(q) \tag{11}$$

is given by

$$V(q) - E = -\frac{\hbar^2}{2m}\{e^{\frac{2i}{\beta}S_0}; q\}. \tag{12}$$

Equation (12) is obtained imposing that when making a coordinate transformation $q_a \to q_b(q_a)$ under which the potential $W(q) := V(q) - E$ transforms as a projective connection

$$W^a(q^a) = \Big(\frac{\partial q^b}{\partial q^a}\Big)^2 W^b(q^b) + \{q^b(q^a), q^a\} \tag{13}$$

the solution of the to be found Quantum Stationary Hamilton-Jacobi equation transforms as a scalar:

$$S^b(q^b) = S^a(q^a). \tag{14}$$

Again, as explained in section (2.3.2) the symmetry properties of the Schwarzian derivative play a central role in deriving eq. (12). In section (2.4) we will study the solutions of eq. (12) and show how they allow a first principle derivation of some key features of Quantum Mechanics, such as tunneling and the quantization of the energy spectrum for bounded systems. To that end, we stress that we will not be relying on the probabilistic interpretation of the wave function, *i.e.* we find a energy quantized spectrum without imposing $\psi \in \mathrm{L}_2(\mathbb{R})$, which is indeed a *consequence* of our approach.

In the last section of this thesis, we analyze how a very strict analogy exists between the framework of the Quantum Stationary Hamilton-Jacobi equation and the theory of uniformization of Riemann surfaces. In particular we will illustrate how the same techniques used to study the solutions of eq. (12) can be applied to find explicitly the uniformization map $J_H^{-1} : \Sigma \to \mathcal{H}$ from a Riemann surface with genus $g \geq 2$ to its universal covering space $\mathcal{H} := \{z \in \mathbb{C}\, s.t.\, \mathrm{Im}\, z > 0\}$. The analogy is inspired by the fact that the uniformization map turns out to satisfy the Schwarzian equation

$$\{J_H^{-1}, z\} = T(z), \tag{15}$$

where $T(z)$ is the classical Liouville stress-energy tensor associated to the (unique) hyperbolic metric on Σ. Establishing such a stringent correspondence between aspects of geometry and fundamental results of Quantum Mechanics, such as energy quantization, might hint to some yet to be discovered common language to describe Quantum Mechanics and Gravity.

Chapter 1

The Sachdev-Ye-Kitaev model and functional integration

1.1 The Sachdev-Ye-Kitaev model

The model we are considering describes the statistics of a system of N Majorana fermions in zero spatial dimension, $D = 1 + 0$. In the most widespread version, the interaction is all to all and couples together a number q of particles at each vertex. The Hamiltonian density is given by

$$H(t) = \frac{i^{\frac{q}{2}}}{q!} \sum_{a_1 < \ldots < a_q}^{N} J_{a_1 \ldots a_q} \chi_{a_1} \cdots \chi_{a_q}(t), \tag{1.1.1}$$

where the couplings $J_{a_1 \ldots a_q}$ are independent random variables extracted according to a Gaussian distribution. We are considering a quenched disorder situation, *i.e.* the couplings are random and time independent. With the aim of simplifying the large N behavior of the system, we define their probability density function (pdf) as follows:

$$P(J_{a_1 \ldots a_q}) = \sqrt{\frac{N^{(q-1)}}{2\pi (q-1)! J^2}} \exp\Big[- \frac{N^{(q-1)}}{2(q-1)! J^2} J^2_{a_1 \ldots a_q} \Big]. \tag{1.1.2}$$

It has mean $\mu = 0$ and variance

$$\mathrm{Var}[J] = \frac{(q-1)! J^2}{N^{(q-1)}}. \tag{1.1.3}$$

An apparently equivalent version is given by

$$H(t) = \frac{i^{\frac{q}{2}}}{q!} \sum_{a_1 \ldots a_q}^{N} J_{a_1 \ldots a_q} \chi_{a_1} \cdots \chi_{a_q}(t), \tag{1.1.4}$$

where we have removed the ordering prescription in the summation. Consider reordering the anti-commuting fields in the Hamiltionian (1.1.4):

$$\begin{aligned} H(t) =& \frac{i^{\frac{q}{2}}}{q!} \sum_{a_1 \ldots a_q}^{N} J_{a_1 \ldots a_q} \chi_{a_1} \cdots \chi_{a_q}(t) \\ =& \frac{i^{\frac{q}{2}}}{q!} \sum_{a_1 < \ldots < a_q}^{N} \chi_{a_1} \cdots \chi_{a_q}(t) \sum_{\sigma} J_{\sigma(a_1 \ldots a_q)} \,\mathrm{sgn}\, \sigma \end{aligned}$$

$$=\frac{i^{\frac{q}{2}}}{q!}\sum_{a_1<...<a_q}^{N}\chi_{a_1}\cdots\chi_{a_q}(t)\alpha_{a_1<...<a_q} \tag{1.1.5}$$

where σ indicates a permutation of the indexes $(a_1, ..., a_q)$ and we defined

$$\alpha_{a_1<...<a_q} := \sum_{\sigma} J_{\sigma(a_1...a_q)}\,\text{sgn}\,\sigma. \tag{1.1.6}$$

In the second summand in eq. (1.1.5) we excluded the case in which two ore more indexes are equal, since it does not contribute due to the anticommutativity of the fermion fields. We then find that the theory defined by (1.1.1) is not exactly equivalent to that defined by (1.1.4), since it is effectively driven by the couplings α which are not distributed exactly as the Js are. Since the Js are independent Gaussian variables with equal variance, we have that the αs are Gaussian variables with null mean and variance

$$\text{Var}[\alpha] = q!\frac{(q-1)!J^2}{N^{(q-1)}}. \tag{1.1.7}$$

The factor $q!$ comes from counting the number of permutations of q different objects. We will see that the interesting features of this model are due to the particular dependence of the variance of the couplings on the number of fermions N. Since in both cases (1.1.1) and (1.1.4) this dependence is the same, for fixed q the two models are essentially equivalent. They differ however in the large $q \to \infty$ behavior, which is of some physical importance as well (see [16]).

A third variant of the model is obtained from the Hamiltonian (1.1.4) but not considering the couplings as independent variables. Instead they are taken to be the completely antisymmetric random tensor distributed as:

$$\begin{cases} P(J_{a_1...a_q}) = \sqrt{\frac{N^{(q-1)}}{2\pi(q-1)!J^2}}\exp\left[-\frac{N^{(q-1)}}{2(q-1)!J^2}J^2_{a_1...a_q}\right] & \text{for } a_1 < ... < a_q \\ J_{a_1...a_q} = 0 & \text{if any two indexes are equal} \\ J_{a_1...a_q} = J_{\sigma(a_1...a_q)}\,\text{sgn}(\sigma) & \text{if } a_1, ..., a_q \text{ are pairwise different and not ordered} \end{cases} \tag{1.1.8}$$

where σ is the permutation that sorts the indexes increasingly. In order to better compare this variant with the one we will be using, we set $q = 4$. Then condition given by eq. (1.1.8) can be rewritten as

$$\begin{cases} P(J_{ijkh}) = \sqrt{\frac{N^{(3)}}{2\pi(3)!J^2}}\exp\left[-\frac{N^3}{2(3)!J^2}J^2_{a_1...a_q}\right] & \text{for } i < ... < h \\ P(J_{ijkh}) = \delta(J_{ijkh} - 0) & \text{if any two indexes are equal} \\ P(J_{ijkh}) = \delta(J_{ijkh} - J_{\sigma(ijkh)}\,\text{sgn}(\sigma)) & \text{if } ijkh \text{ are pairwise different and not ordered} \end{cases}$$

Note that the δ distribution can be considered as a the limit case of a Gaussian distribution (with variance tending to zero), allowing us to extend to it all the well known results for Gaussian pdfs. For fixed q, this version is equivalent to the previous ones, with some minor differences. For instance, it allows non zero mean value for the effective couplings α, since the antisymmetry of J_{ijkh} balances the anticommutativity of the fermion fields in the definition of α. On the contra side, the covariance matrix of the couplings becomes more complicated, which makes averaging over disorder more cumbersome. The non zero elements are

$$\langle J_{ijkh}J_{\sigma(ijkh)}\rangle = \frac{3!J^2}{N^3}\,\text{sgn}(\sigma) \qquad i < j < k < h. \tag{1.1.9}$$

where σ is any permutation.

In what follows we will mostly consider the original version of the model proposed by Kitaev [12], in which $q = 4$, the Hamiltonian is and the couplings are independent random variables distributed according to eq. (1.1.2). This means that the covariance matrix is given by

$$\langle J_{abcd} J_{ijkh} \rangle = \frac{3! J^2}{N^3} \delta_{ai} \delta_{bj} \delta_{ck} \delta_{dh}. \tag{1.1.10}$$

The corresponding Lagrangian density is found by Legendre transformation of H

$$L(t) = \frac{1}{2} \chi_i(t) \partial_t \chi_i(t) - \frac{1}{4!} \sum_{ijkh}^{N} J_{ijkh} \chi_i \chi_j \chi_k \chi_l(t) \tag{1.1.11}$$

and the associated (euclidean) generating functional is

$$Z[J] = \int D\chi \exp \Big[\int \big(-\frac{1}{2} \sum_{\beta=1}^{N} \chi_\beta(t) \partial_t \chi_\beta(t) + \sum_{ijkh}^{N} \frac{1}{4!} J_{ijkh} \chi_i \chi_j \chi_k \chi_l(t) + \sum_{\alpha=1}^{N} \chi_\alpha(t) J_\alpha(t) \big) dt \Big] =: e^{W[J]}$$

where we have used a collective notation $D\chi := \prod_{i=1}^{N} D\chi_i$. In the functional approach the χ_i fields as well as the external sources J_i are of course real Grassman-valued fields. From now on we will use an Einstein-like notation, where repeated indexes are summed over.

1.2 The Green functions of the theory

The SYK model is characterized by the fact that in the large N limit it is possible to write closed relations that allow to compute non perturbatively (in the strong coupling limit) the disorder-averaged interacting 2,4 and 6 point Green functions at leading order in $\frac{1}{N}$.

1.2.1 The free propagator of the theory

The free 2-point function is defined as $G^0_{ij}(t_i, t_j) := \langle T[\chi_i(t_i) \chi_j(t_j)] \rangle$, where $\langle ... \rangle$ denotes the vacuum expectation value and T the time ordering operator. It can be written in terms of the inverse of the operator in the quadratic part of the Lagrangian. In fact, the generating functional of the free theory is

$$Z^0[J] = \int D\chi \exp \Big[\int \big(\frac{1}{2} \chi_i(t) A_{ij} \chi_j(t) + \chi_\alpha(t) J_\alpha(t) \big) dt \Big],$$

where we have set $A_{ij} := -\partial_t \delta_{ij}$. Using the integration rules for Gaussian integrals over real Grassman variables, and choosing the normalization so that $Z_0[0] = 1$, we get

$$Z^0[J] = \exp \Big[\frac{1}{2} \int d\tau_1 d\tau_2 J_i(\tau_1) (A_{ij})^{-1}(\tau_1, \tau_2) J_j(\tau_2) \Big].$$

Differentiating twice the free generating functional we get the free propagator

$$\begin{aligned} G^0_{ij}(t_1, t_2) &= \frac{\delta}{\delta J_i(t_1)} \frac{\delta}{\delta J_j(t_2)} Z^0[J] \Big|_{J=0} \\ &= \frac{\delta}{\delta J_i(t_1)} \Big\{ \Big(\frac{1}{2} \int d\tau_2 (A_{j\beta})^{-1}(t_2, \tau_2) J_\beta(\tau') - \frac{1}{2} \int d\tau_1 J_\alpha(\tau) (A_{\alpha j})^{-1}(\tau_1, t_2) \Big) \\ &\quad \times \exp \Big[\frac{1}{2} \int d\tau_1 d\tau_2 J_\alpha(\tau_1) (A_{\alpha\beta})^{-1}(\tau_1, \tau_2) J_\beta(\tau_2) \Big] \Big\} \Big|_{J=0} \\ &= \frac{1}{2} \Big((A_{ji})^{-1}(t_2, t_1) - (A_{ij})^{-1}(t_1, t_2) \Big). \end{aligned} \tag{1.2.1}$$

The minus sign in the second line follows by differentiating with respect to Grassman fields. To find the inverse of A_{ij}, we use the Fourier transform $\mathcal{F}[\cdot]$

$$\mathcal{F}[\phi](\omega) := \phi(\omega) := \int_{\mathbb{R}} \phi(t) e^{-i\omega t} dt.$$

It gives

$$-\partial_t \delta_{ij} (A_{ij})^{-1}(t) = \delta(t) \Longleftrightarrow A_{ij}^{-1}(\omega) = \frac{1}{\omega}\delta_{ij},$$

where the indexes are not summed. To get back to t-space we use Jordan's lemma,

$$A_{ij}^{-1}(t_1, t_2) = \int_{-\infty}^{\infty} \frac{i}{2\pi} \frac{e^{i\omega(t_1 - t_2)}}{\omega + i\epsilon} \delta^{ij} d\omega = -\theta(t_1 - t_2)\delta^{ij}$$

with $\theta(\cdot)$ the Heaviside step function. Plugging this result into eq. (1.2.1) we obtain

$$G_{ij}^0(t_1, t_2) = \frac{1}{2} \operatorname{sgn}(t_1 - t_2)\delta_{ij} =: G^0(t_1 - t_2)\delta_{ij}.$$

1.2.2 Diagrammatic approach to the 2-point Green function

In the following we use the above results to compute loop corrections to the propagator. The model exhibits its interesting features in the disorder averaged correlation functions, *i.e.* we average the correlation functions with respect to the probability distribution of the coupling constants

$$\Big\langle G^{a_1 \dots a_n}(t_1 \dots t_n) \Big\rangle := \int dJ P(J) \int D\chi \prod_{i=a_1}^{a_n} \chi_{a_i}(t_i) \exp\Big[-\int \big(\frac{1}{2}\chi_i \partial_t \chi_i - \sum_{ijkl}^{N} \frac{1}{4!} J_{ijkh} \chi_i \chi_j \chi_k \chi_l(t) \big) \Big]. \tag{1.2.2}$$

Again, we are using a collective notation $dJ := \prod_{i,j,k,l=1}^{N} dJ_{ijkl}$ and $P(J) := \prod_{ijkh=1}^{N} P(J_{ijkh})$. Of course, the correlation functions can be understood in terms of Feynman diagrams, where the diagrams with n vertexes are obtained from the n^{th} order term in the power series expansion of the exponential in the correlation function. Also the process of disorder averaging admits a diagrammatic representation. In particular, when we average over disorder a given term in the loop expansion of $\Big\langle G^{a_1 \dots a_n}(t_1 \dots t_n) \Big\rangle$ we have to compute an integral of the following type

$$\Big\langle G^{a_1 \dots a_n}(t_1 \dots t_n) \Big\rangle \propto \int dJ P(J) J_{abcd} \dots J_{ijkh} \tag{1.2.3}$$

where each J comes from an interaction vertex. Since $P(J)$ is a Gaussian, this computation amounts (Isserlis theorem) to summing over all the possible partition in pairs of the coupling constants in the integrand, with each pair (contraction), contributing by a factor

$$\underbrace{J_{abcd} J_{ijkh}} \equiv \langle J_{abcd} J_{ijkh} \rangle = \frac{3!}{N^3 J^2} \delta_{ia} \delta_{jb} \delta_{kc} \delta_{hd}. \tag{1.2.4}$$

This is analogous to applying Wick's theorem to express the expectation value of a product of fields in terms of Feynman diagrams. If we represent pictorially each contraction of the constants by dotted lines, we can break formula (1.2.2) down to a sum of diagrams where we indicate the contractions chosen for both the χ fields and the coupligns. For example, let us compute the 2-loop contribution to the 2-point correlation function. It is given by

$$\Big\langle G^{ij}_{2-loop}(t_i,t_j)\Big\rangle=\frac{1}{2(4!)^2}\int dJP(J)\int D\chi\exp\Big[-\int\frac{1}{2}\chi_h\partial_t\chi_h\Big]\int d\tau_1 d\tau_2\chi_i(t_i)\chi_j(t_j)$$
$$\times\sum_{pqrs}^{N}J_{pqrs}\chi_p\chi_q\chi_r\chi_s(\tau_1)\sum_{abcd}^{N}J_{abcd}\chi_a\chi_b\chi_c\chi_d(\tau_2). \tag{1.2.5}$$

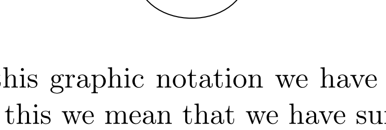
Applying Isserlis theorem (in this case there is only one pair of couplings, *i.e.* only one possible contraction as shown in figure (1.2.8)) and using expression (1.2.4) for the covariance of the coupling distribution we get

$$\Big\langle G^{ij}_{2-loop}(t_i,t_j)\Big\rangle=\frac{1}{2(4!)^2}\int D\chi\exp\Big[-\int\frac{1}{2}\chi_h\partial_t\chi_h\Big]\int d\tau_1 d\tau_2\chi_i(t_i)\chi_j(t_j)$$
$$\times\sum_{abcd}^{N}\sum_{pqrs}^{N}\frac{3!}{N^3J^2}\delta_{ap}\delta_{bq}\delta_{cr}\delta_{ds}\chi_p\chi_q\chi_r\chi_s(\tau_1)\chi_a\chi_b\chi_c\chi_d(\tau_2)$$
$$=\frac{3!}{2(4!)^2}\frac{1}{N^3J^2}\int D\chi\exp\Big[-\int\frac{1}{2}\chi_h\partial_t\chi_h\Big]\int d\tau_1 d\tau_2\chi_i(t_i)\chi_j(t_j)$$
$$\times\sum_{a\neq b\neq c\neq d}^{N}\chi_a\chi_b\chi_c\chi_d(\tau_1)\chi_a\chi_b\chi_c\chi_d(\tau_2). \tag{1.2.6}$$

Note that in the last line we used the anti-commuting nature of the χ's to drop the terms with one or more coincident indexes. Now, according to Wick's theorem, we have to sum over the contractions of the fields. The only contractions that give a non zero result are those in which the external fields are paired with two vertex fields with the same index. This produces a symmetry factor $S = 8$. The contractions in which the external fields are paired together correspond to disconnected diagrams, which are not physically relevant and can be disposed of by normalizing appropriately the generating functional. The final result is

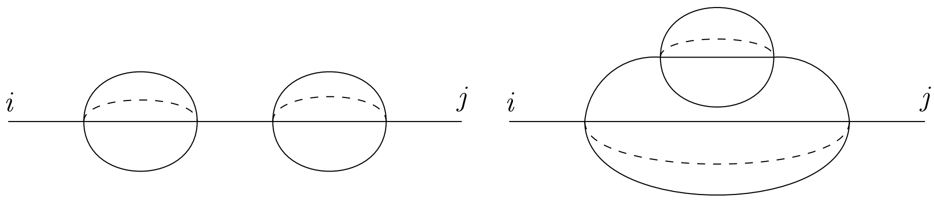
$$\begin{aligned}\Big\langle G^{ij}_{2-loop}(t_i,t_j)\Big\rangle &= \frac{8}{2(4!)^2}\int d\tau_1 d\tau_2 G^0(t_i-\tau_1)G^0(\tau_2-t_j)\big(G^0(\tau_1-\tau_2)\big)^3\frac{3!}{N^3J^2}\delta_{aa}\delta_{bb}\delta_{cc}\delta_{ij}\\ &= \frac{8}{2(4!)^2}\frac{3!J^2}{N^3}N^3\delta_{ij}\int d\tau_1 d\tau_2 G^0(t_i-\tau_1)G^0(\tau_2-t_j)\big(G^0(\tau_1-\tau_2)\big)^3\\ &= \frac{J^2}{24}\delta_{ij}\int d\tau_1 d\tau_2 G^0(t_i-\tau_1)G^0(\tau_2-t_j)\big(G^0(\tau_1-\tau_2)\big)^3.\end{aligned} \tag{1.2.7}$$

Diagrammatically the hole calculation is represented by

$$\Big\langle G^{ij}_{2-loop}(t_i,t_j)\Big\rangle\sim$$ 

(1.2.8)

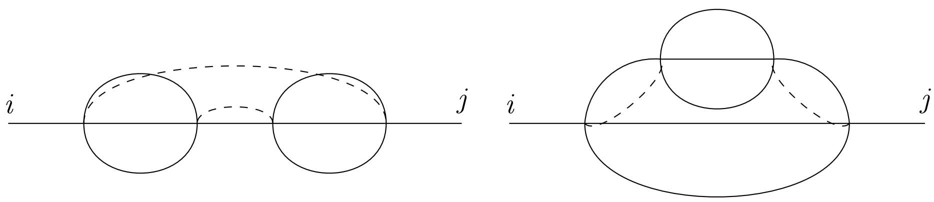
We will call this a "melon diagram". In this graphic notation we have not specified to which fermion the internal lines correspond. By this we mean that we have summed over all possible propagators.



(a)

(b)

Figure 1.1: Two 4-loop contributions to the leading order in $1/N$ of the 2-point green function $\langle G^{ij}_{\text{4-loop}}(t_i, t_j)\rangle$.



(a)

(b)

Figure 1.2: Two subleading $O(N^{-2})$ contributions to the 4-loop 2-point function $\langle G^{ij}_{\text{4-loop}}(t_i, t_j)\rangle$

It is also useful to introduce a compact notation for the convolution of two functions

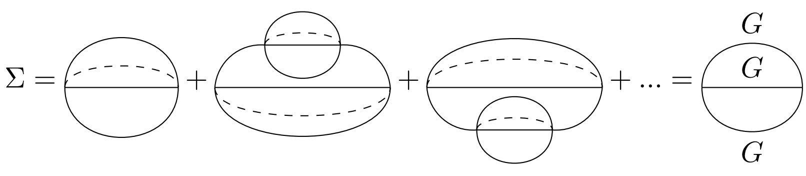
$$AB(t,t') := A(t,x) * B(x,t') = \int dx A(t,x)B(x,t').$$

Eq. (1.2.7) than reads

$$\left\langle G^{ij}_{2-loop}(t_i, t_j)\right\rangle = \frac{J^2}{24}\delta_{ij} G^0(t_i - \tau_1) * \left(G^0(\tau_1 - \tau_2)\right)^3 * G^0(\tau_2 - t_j).$$

The result has no N dependence. Similar calculations show that only a simple class of disorder-averaged diagrams contribute at leading order in $1/N$ to the 2-point function. The leading order contribution to the N-loop function comes from those diagrams obtained by dressing the propagators in the leading order $(N-2)$-loop function with melon insertions where the two new couplings are also paired together by a dotted line, so that a similar cancellation as in the last line of (1.2.7) takes place. For example, in the 4-loop calculation, the diagrams in figure (1.1) are of order $O(N^0)$, while those in figure 1.2a and 1.2b are of order $O(1/N^2)$. The final lesson is that the leading order contribution to the 2-point function is given by the diagrams in figure 1.3.

Denote by Σ the One Particle Irreducible (1PI) truncated part of the interacting propagator. By definition it is given by the sum of the graphs in figure (1.3) which cannot be split into subgraphs by removing one internal line, convoluted on the left and on the right side with the inverse propagator. At leading order in $1/N$ we have



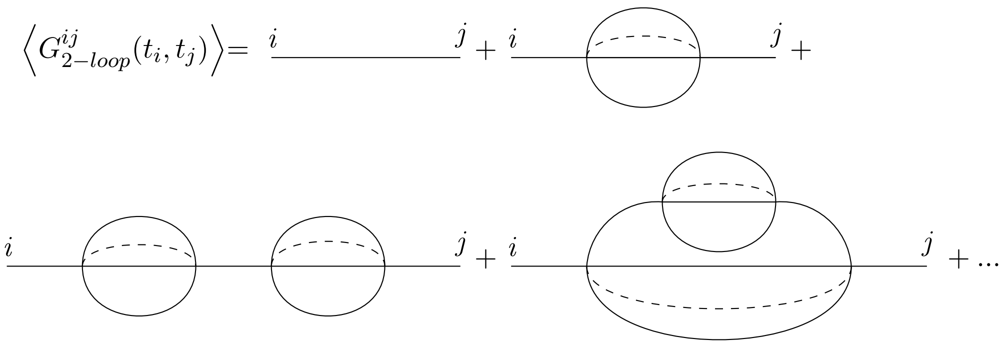


Figure 1.3: The leading order contribution to the 2-point correlation function

where the G's indicate propagators in the interacting theory. This is algebraically stated as

$$\Sigma(t_1,t_2) = J^2 G(t_1,t_2)^3. \tag{1.2.9}$$

Note that this result is non perturbative. By the definition of Σ, it follows that

$$\begin{aligned} G(t_1,t_2) &= G_0(t_1,t_2) + G_0 * \Sigma * G_0(t_1,t_2) + ... \\ &= G_0[1+\Sigma * G_0 + ...] \\ &= G_0[1-\Sigma * G_0]^{-1} \\ &= [G_0^{-1} - \Sigma]^{-1} \\ &= [\delta(t_1-t_2)\partial_{t_2} - \Sigma(t_1,t_2)]^{-1} \end{aligned} \tag{1.2.10}$$

or equivalently, in frequency space,

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega). \tag{1.2.11}$$

To sum up, the exact propagator and 1PI 2-point function solve the equations

$$\Sigma(\tau,\tau') = J^2 G(\tau,\tau')^3 \qquad \text{and} \qquad \frac{1}{G(\omega)} = -i\omega - \Sigma(\omega). \tag{1.2.12}$$

They are too hard to solve exactly, but in the infrared limit, when the frequency is much smaller than the characteristic energy scale of the theory J, or equivalently $\tau - \tau' >> J^{-1}$, we can neglect the $-i\omega$ in the second equation, reducing it to

$$\Sigma^c(\tau,\tau') = J^2 G^c(\tau,\tau')^3 \quad \text{and} \quad \int G^c(\tau,t)\Sigma^c(t,\tau')dt = G^c * \Sigma^c(\tau,\tau') = -\delta(\tau-\tau'). \tag{1.2.13}$$

The apex is meant to stress that Σ^c and G^c solve the approximated equation (1.2.13). The key feature of this equations is their conformal reparametrization symmetry. Namely, given a solution (G,Σ) also

$$\begin{aligned} G^c_f(\tau,\tau') :=& [f'(\tau)f'(\tau')]^{\frac{1}{4}} G^c(f(\tau),f(\tau')) \\ \Sigma^c_f(\tau,\tau') :=& [f'(\tau)f'(\tau')]^{\frac{3}{4}} \Sigma^c(f(\tau),f(\tau')) \end{aligned} \tag{1.2.14}$$

is a solution $\forall\, f(\tau) \in \mathrm{Rep}(\mathbb{R}) := \{f(\tau) : \mathbb{R} \to \mathbb{R} \text{ s.t. } f \text{ is monotonic and differentiable}\}$. This is the transformation law of a 2-point function in a conformal field theory where the fields have conformal dimension $\Delta = \frac{1}{4}$.
It is immediate to see that (1.2.14) is a solution of the first equation in (1.2.13). To prove that it also solves the second equation, we set $s := f(t)$, so that

$$\begin{aligned}
\int dt \quad G^c_f(\tau,t)\Sigma^c_f(t,\tau') &= \int dt \quad [f'(\tau)f'(t)]^{\frac{1}{4}} G(f(\tau),f(t))[f'(t)f'(\tau')]^{\frac{3}{4}}\Sigma^c(f(t),f(\tau')) \\
&= \int ds \quad G^c\big(f(\tau),s\big)\Sigma^c\big(s,f(\tau')\big)f'(\tau')\Big[\frac{f'(\tau)}{f'(\tau')}\Big]^{\frac{1}{4}} f'\big(t(s)\big)(f^{-1})'(s) \\
&= -\delta\big(f(\tau)-f(\tau')\big)f'(\tau')\Big[\frac{f'(\tau)}{f'(\tau')}\Big]^{\frac{1}{4}} = -\delta(\tau-\tau')f'(\tau')^{-1}f'(\tau')\Big[\frac{f'(\tau)}{f'(\tau')}\Big]^{\frac{1}{4}} \\
&= -\delta(\tau-\tau').
\end{aligned}$$

An explicit solution of the approximate equations (1.2.13) is given by

$$G^c(\tau-\tau') = -\frac{k}{J^{\frac{1}{2}}}\frac{\mathrm{sgn}(\tau-\tau')}{|\tau-\tau'|^{\frac{1}{2}}} \quad ; \quad \Sigma^c(\tau-\tau') = -k^3 J^{\frac{1}{2}}\frac{\mathrm{sgn}(\tau-\tau')}{|\tau-\tau'|^{\frac{3}{2}}} \tag{1.2.15}$$

where $k = (4\pi)^{-\frac{1}{4}}$. As a consistency check note that in frequency space we get

$$G^c(\omega) = -i\sqrt{2\pi}b\,\mathrm{sgn}\,\omega|\omega J|^{-\frac{1}{2}} \qquad \text{and} \qquad \Sigma^c(\omega) = -i\sqrt{8\pi}b^3\,\mathrm{sgn}\,\omega|\omega J|^{\frac{1}{2}}$$

which confirms that it was appropriate to neglect the $i\omega$ term with respect to Σ^c in the IR regime in eq. (1.2.12). We remark that eq. (1.2.15) approximates the behavior of the complete 2-point function in the IR limit $\tau-\tau' >> J^{-1}$. On the other hand, also the UV behavior of the 2-point function is known, since in this case the theory becomes approximately free, so we get $G(\tau) \sim G^0(\tau)$ when $\tau-\tau' << J^{-1}$.

The conformal reparametrization symmetry of eq. (1.2.13) implies that for any given solution, there is an infinite family of related minima of the action functional, given by

$$\begin{aligned}
G^c_f(\tau,\tau') &= -\frac{k}{J^{\frac{1}{2}}}\frac{\mathrm{sgn}(f(\tau)-f(\tau'))f'(\tau)^{\frac{1}{4}}f'(\tau')^{\frac{1}{4}}}{|f(\tau)-f(\tau')|^{\frac{1}{2}}} \\
\Sigma^c_f(\tau,\tau') &= -k^3 J^{\frac{1}{2}}\frac{\mathrm{sgn}(f(\tau)-f(\tau'))f'(\tau)^{\frac{1}{4}}f'(\tau')^{\frac{1}{4}}}{|f(\tau)-f(\tau')|^{\frac{3}{2}}}
\end{aligned} \tag{1.2.16}$$

$\forall\, f(\tau) \in \mathrm{Rep}(\mathbb{R})$.
Note however that our explicit solution singles out a subgroup of the reparametrization group: (1.2.15) is in fact invariant under the $\mathrm{SL}(2,\mathbb{R})$ subgroup acting as $\tau \to \frac{a\tau+b}{c\tau+d}$, as can be verified by

explicit calculation

$$
\begin{aligned}
G^c_{\frac{at+b}{ct+d}}(\tau,\tau') &= -\frac{k}{J^{\frac{1}{2}}}\frac{\mathrm{sgn}\Big(\frac{a\tau+b}{c\tau+d}-\frac{a\tau'+b}{c\tau'+d}\Big)\Big(\frac{1}{c\tau+d}\Big)^{\frac{1}{2}}\Big(\frac{1}{c\tau'+d}\Big)^{\frac{1}{2}}}{|\frac{a\tau+b}{c\tau+d}-\frac{a\tau'+b}{c\tau'+d}|^{\frac{1}{2}}} \\
&= -\frac{k}{J^{\frac{1}{2}}}\frac{\mathrm{sgn}(\tau-\tau')\mathrm{sgn}\Big((c\tau+d)(c\tau'+d)\Big)\Big(\frac{1}{c\tau+d}\frac{1}{c\tau'+d}\Big)^{\frac{1}{2}}}{|\tau-\tau'|^{\frac{1}{2}}} \\
&\quad \times |(c\tau+d)(c\tau'+d)|^{\frac{1}{2}} \\
&= -\frac{k}{J^{\frac{1}{2}}}\frac{\mathrm{sgn}(\tau-\tau')\mathrm{sgn}\Big((c\tau+d)(c\tau'+d)\Big)\Big(\frac{1}{c\tau+d}\frac{1}{c\tau'+d}\Big)^{\frac{1}{2}}}{|\tau-\tau'|^{\frac{1}{2}}} \\
&\quad \times \Big((c\tau+d)(c\tau'+d)\Big)^{\frac{1}{2}}\mathrm{sgn}\Big((c\tau+d)(c\tau'+d)\Big) \\
&= -\frac{k}{J^{\frac{1}{2}}}\frac{\mathrm{sgn}(\tau-\tau')}{|\tau-\tau'|^{\frac{1}{2}}},
\end{aligned}
$$

and similarly for $\Sigma^c(\tau,\tau')$. This also means that the reparametrizations $\tau\to f(\tau)$ and $\tau\to\frac{af(\tau)+b}{cf(\tau)+d}$ are equivalent, namely

$$
G^c_{\frac{af(t)+b}{cf(t)+d}}(\tau,\tau') = G^c_{f(t)}(\tau,\tau').
$$

To summarize, the 2-point function in the Sachdev-Ye-Kitaev model exhibits a conformal reparametrization symmetry, which is not exact, but emergent in the IR limit. This symmetry is spontaneously broken by our choice of the conformal solution (1.2.15) down to $\mathrm{SL}(2,\mathbb{R})$. As we will see in section 1.2.3, the large N behavior of the theory can be studied in the functional approach by stationary phase methods. In that context we find that the symmetry properties just described will bring into play an infinite dimensional manifold of Goldstone modes described by the quotient $\mathrm{Rep}(\mathbb{R})/\mathrm{SL}(2,\mathbb{R})$.

1.2.3 Functional approach to the 2-point green function

We now set out to determine an explicit expression for the generating functional of the theory. To simplify the notation we introduce the following definitions

$$
\int\prod_{\alpha=1}^{N} D\chi^\alpha := \int D\chi, \qquad \int\prod_{ijkl}^{N} dJ_{ijkl} := \int dJ \qquad \text{and} \qquad \chi_i\chi_j\chi_k\chi_l(\tau) := \chi_{ijkl}(\tau)
$$

Moreover, repeated indexes are to be summed over. The disorder averaged generating functional reads

$$
\langle Z\rangle_J[\phi] = \int dJ\int D\chi P(J_{ijkl})\exp\Big[\int -\frac{1}{2}\chi_h\partial_\tau\chi_h(\tau) + \frac{1}{4!}J_{ijkl}\chi_{ijkl}(\tau) + \phi_i(\tau)\chi_i(\tau)d\tau\Big]
$$

where $P(J_{ijkl})$ is the probability density function for the couplings defined in section (1.1) and the ϕ_i are the Grassman-valued external currents.
To compute the disorder average we perform the Gaussian integration over the couplings J_{ijkl}.

Completing the square in the exponential we get

$$\begin{aligned}
\langle Z\rangle_J[\phi] &= \int D\chi \int \prod_{ijkl}^{N} dJ_{ijkl}\sqrt{\frac{N^3}{12\pi J^2}}\exp\Big[-\frac{N^3 J_{ijkl}^2}{12J^2}+\frac{1}{4!}\int J_{ijkl}\chi_{ijkl}(\tau)-\frac{1}{2}\chi_a\partial_\tau\chi_a(\tau)+\phi_a(\tau)\chi_a(\tau)d\tau\Big]\\
&= \int D\chi \int \prod_{ijkl}^{N} dJ_{ijkl}\sqrt{\frac{N^3}{12\pi J^2}}\exp\Big[-\Big(J_{ijkl}\sqrt{\frac{N^3}{12J^2}}-\frac{1}{2\cdot 4!}\sqrt{\frac{12J^2}{N^3}}\int\chi_{ijkl}(\tau)d\tau\Big)^2\Big]\\
&\qquad\times\exp\Big[\frac{3J^2}{(4!)^2N^3}\int\chi_{ijkl}(\tau)\chi_{ijkl}(\tau')d\tau d\tau'+\int-\frac{1}{2}\chi_a\partial_\tau\chi_a(\tau)+\phi_a(\tau)\chi_a(\tau)d\tau\Big]\\
&= \int D\chi\prod_{ijkl}\exp\Big[\frac{3J^2}{(4!)^2N^3}\int\chi_{ijkl}(\tau)\chi_{ijkl}(\tau')d\tau d\tau'+\int-\frac{1}{2}\chi_a\partial_\tau\chi_a(\tau)+\phi_a(\tau)\chi_a(\tau)d\tau\Big]\\
&= \int D\chi\exp\Big[\frac{J^2}{N^3 8}\int\big(\chi_i(\tau)\chi_i(\tau')\big)^4 d\tau d\tau'+\int-\frac{1}{2}\chi_a\partial_\tau\chi_a(\tau)+\phi_a(\tau)\chi_a(\tau)d\tau\Big]
\end{aligned}$$

This functional integral can be rewritten in a more manageable form by plugging in the integral representation of the $\delta[\cdot]$ functional

$$\begin{aligned}
1 &= \int DG\delta\big[G(\tau,\tau')-\sum_{i=1}^{N}\frac{\chi_i(\tau)\chi_i(\tau')}{N}\big]\\
&= \int DGD\Sigma\exp\Big[-\frac{1}{2}\int d\tau d\tau'\Sigma(\tau,\tau')\Big(G(\tau,\tau')-\sum_{i=1}^{N}\frac{\chi_i(\tau)\chi_i(\tau')}{N}\Big)\Big].
\end{aligned}$$

We get

$$\begin{aligned}
\langle Z\rangle_J[\phi] &= \int D\chi DGD\Sigma\exp\Big[-\frac{1}{2}\int d\tau d\tau'\Sigma(\tau,\tau')\Big(G(\tau,\tau')-\sum_{i=1}^{N}\frac{\chi_i(\tau)\chi_i(\tau')}{N}\Big)\Big]\\
&\qquad\times\exp\Big[\frac{J^2N}{8}\int G(\tau,\tau')^4 d\tau d\tau'+\int-\frac{1}{2}\chi_a\partial_\tau\chi_a(\tau)+\phi_a(\tau)\chi_a(\tau)d\tau\Big].
\end{aligned}$$

Now we can perform the integral over the Majorana fields, which appear at most quadratically:

$$\begin{aligned}
\langle Z\rangle_J[\phi] &= \int DGD\Sigma\exp\Big[\frac{N}{2}\Big(\log\det\big(\partial_\tau-\Sigma(\tau,\tau')\big)+\int\frac{J^2}{4}G(\tau,\tau')^4-\Sigma(\tau,\tau')G(\tau,\tau')d\tau d\tau'\Big)\Big]\\
&\qquad\times\exp\Big[\frac{1}{2}\int\phi_a(t)\big(\partial_\tau-\Sigma(\tau,\tau')\big)^{-1}\phi_a(t)dt\Big]\\
&= \int DGD\Sigma\exp\Big[-NS[G,\Sigma]\Big]\exp\Big[\frac{1}{2}\int\phi_a(t)\big(\partial_\tau-\Sigma(\tau,\tau')\big)^{-1}\phi_a(t)dt\Big].
\end{aligned}\tag{1.2.17}$$

where we have defined

$$S[G,\Sigma] := -\frac{1}{2}\Big(\log\det\big(\partial_\tau-\Sigma(\tau,\tau')\big)+\int\frac{J^2}{4}G(\tau,\tau')^4-\Sigma(\tau,\tau')G(\tau,\tau')d\tau d\tau'\Big).\tag{1.2.18}$$

The factor N in the last line of (1.2.17) allows us to approximate the large N limit of the functional integral by the steepest descent method. In order to compute the leading order contribution to the 2-point correlation function, which we have proven to be $O(N^0)$, the leading $O(N^0)$ order approximation of eq. (1.2.17) will suffice. It is usually called saddle point approximation. Apart from some irrelevant multiplicative constant, it is obtained by plugging into the integrand in (1.2.17) the configuration (Σ_0, G_0) that minimizes the functional (1.2.18):

$$\begin{aligned}\langle Z\rangle_J[\phi] &= \int DGD\Sigma \exp\Big[-NS[G,\Sigma]\Big] \exp\Big[\frac{1}{2}\int \phi_a(t)\big(\partial_\tau - \Sigma(\tau,\tau')\big)^{-1}\phi_a(t)dt\Big] \\ &= \exp\Big[\frac{1}{2}\int \phi_a(t)\big(\partial_\tau - \Sigma_0(\tau,\tau')\big)^{-1}\phi_a(t)dt\Big] + O(N^{-1}). \end{aligned} \tag{1.2.19}$$

To find (Σ_0, G_0) we set to zero the functional derivatives of (1.2.18) with respect to G and Σ:

$$\begin{aligned} 0 &= \frac{\delta}{\delta\Sigma(t,t')}\Big[\log\det\big(\partial_\tau - \Sigma(\tau,\tau')\big) + \int \frac{J^2}{4}G(\tau,\tau')^4 - \Sigma(\tau,\tau')G(\tau,\tau')d\tau d\tau'\Big] \\ &= \frac{\delta}{\delta\Sigma(t,t')}\Big[\log\int D\chi \exp\Big(\int \chi(\tau)\big(\Sigma(\tau,\tau') - \partial_{\tau'}\big)\chi(\tau')\Big) \\ &\qquad + \int \frac{J^2}{4}G(\tau,\tau')^4 - \Sigma(\tau,\tau')G(\tau,\tau')d\tau d\tau'\Big] \\ &= \frac{\int D\chi\chi(t)\chi(t')\exp\Big(\int\chi(\tau)\big(\Sigma(\tau,\tau') - \partial_{\tau'}\big)\chi(\tau')\Big)}{\int D\chi\exp\Big(\int\chi(\tau)\big(\Sigma(\tau,\tau') - \partial_{\tau'}\big)\chi(\tau')\Big)} - G(t,t') \\ &= (-\Sigma(\tau,\tau') + \partial_{\tau'})^{-1} - G(t,t') \end{aligned}$$

and

$$\begin{aligned} 0 &= \frac{\delta}{\delta G(t,t')}\Big[\log\det(\partial_\tau - \Sigma(\tau,\tau')) + \int \frac{J^2}{4}G(\tau,\tau')^4 - \Sigma(\tau,\tau')G(\tau,\tau')d\tau d\tau'\Big] \\ &= J^2G(t,t')^3 - \Sigma(t,t'). \end{aligned}$$

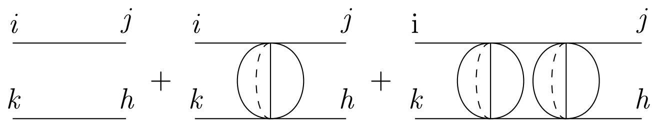
We then find the same equations as section (1.2.2):

$$\Sigma_0(\tau,\tau') = J^2G_0(\tau,\tau')^3 \qquad \text{and} \qquad G_0(\tau,\tau') = (\partial_{\tau'} - \Sigma_0(\tau,\tau'))^{-1}. \tag{1.2.20}$$

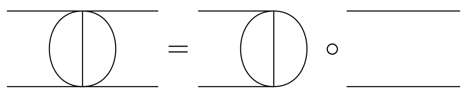
Indeed, differentiating twice with respect to the external currents eq. (1.2.19) we see that the $O(N^0)$ approximation of the 2-point interacting function G is given by G_0. Then equations (1.2.20) are simply telling us that the bilocal fields (Σ_0, G_0) that extremize the functional (1.2.18) are exactly the leading order 2-point function and self energy of section (1.2.2).

1.2.4 Diagrammatic approach to the 4-point Green function

Analogously to what happens for the 2-point function, only a restricted class of diagrams contributes to the leading order in N of the 4-point function. Namely, it can be shown (see for example [16]) that the leading order contribution is given by

$\langle G_{ijkh}(t_1, ..., t_4)\rangle =$  $+ ... - (i \leftrightarrow j).$ (1.2.21)

In this picture the dotted lines represent as usual the disorder average, while the continuous lines represent $\langle G^{ij}_{2-loop}(t_i, t_j)\rangle$, the leading $O(N^0)$ contribution to the interacting 2-point function. It is given by (1.3). The appearance of $\langle G^{ij}_{2-loop}(t_i, t_j)\rangle$ in (1.2.21) tells us that in order to single out the leading order contribution coming from a particular diagram we have to decompose the



(1.2.25)

Figure 1.4: The action of the Kernel on the diagram $\mathcal{F}_n$

diagram in 2-point 1PI subdiagrams, then choose the pairings of the coupling constants within each subdiagam so that we maximize its order, and then contract the remaining couplings in the most efficient way. Following this procedure one finds that, once we neglect disconnected diagrams, the leading order contribution is given by "ladder" diagrams as shown in (1.3). They are $O(N^{-1})$. Since $\langle G^{ij}_{2-loop}(t_i,t_j)\rangle \propto \delta_{ij}$, we see by (1.2.21) that the $O(N^{-1})$ contribution is non zero only if the indeces of the four fields in $\langle G_{ijkh}(t_1,...,t_4)\rangle$ are equal in pairs.
Following [16], we will focus on the 4-point function averaged on the i,j labels:

$$\begin{aligned}\big\langle \chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4)\big\rangle_{i,j} &:= \frac{1}{N^2}\sum_{i,j}\langle \chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4)\rangle \\ &= G(\tau_1-\tau_2)G(\tau_3-\tau_3)+\mathcal{F}(\tau_1,...,\tau_4)+O(\frac{1}{N^2}).\end{aligned}$$

The first term is a disconnected contribution while $\mathcal{F}$ is the $O(\frac{1}{N})$ contribution and corresponds to the i,j-avaraged sum of the ladder diagrams.
Denoting with $\mathcal{F}_n$ the "ladder with n rungs", it is clear that each $\mathcal{F}_n$ can be obtained acting on the right on $\mathcal{F}_{n-1}$ with the kernel defined by

$$K(\tau_1,...,\tau_4) := -3J^2 G(\tau_1-\tau_3)G(\tau_2-\tau_4)G(\tau_3-\tau_4)^3. \tag{1.2.22}$$

The right and left actions of the kernel on a generic function $F:\mathbb{R}^n\to\mathbb{R}$ are defined as:

$$K|\circ : (K(\tau_1,...,\tau_4),F(t_1,...,t_n)) \to K|F(\tau_1,\tau_2,t_3,...,t_n)) := \int d\tau_3 d\tau_4 K(\tau_1,...,\tau_4)F(\tau_3,\tau_4,t_3,...,t_n)$$
$$\circ|K : (K(\tau_1,...,\tau_4),F(t_1,...,t_n)) \to F|K(\tau_1,\tau_2,t_3,...,t_n)) := \int d\tau_3 d\tau_4 F(\tau_1,...,\tau_4)K(\tau_3,\tau_4,t_3,...,t_n).$$

A graphic representation of the right action is given in figure (1.4). In this notation, we have

$$\mathcal{F}_{n+1}(\tau_1,...,\tau_4) = \int dt dt' K(\tau_1,\tau_2,t,t')\mathcal{F}_n(t,t',\tau_3,\tau_4) = K|\mathcal{F}_n(\tau_1,...,\tau_4), \tag{1.2.23}$$

and we can write

$$\mathcal{F} = \sum_n \mathcal{F}_n = \sum_n K^n \mathcal{F}_0 = \frac{1}{1-K}\mathcal{F}_0. \tag{1.2.24}$$

One could be tempted to compute the IR behavior (*i.e.* the conformal limit) of the 4-point function using instead of the exact (leading order in N) propagator G, it's (conformal) IR approximation G^c given by (1.2.15). This would introduce a "conformal Kernel"

$$K^c(\tau_1,...,\tau_4) := -3J^2 G^c(\tau_1-\tau_3)G^c(\tau_2-\tau_4)G^c(\tau_3-\tau_4)^3.$$

This path is in general not safe, because the conformal kernel has 1 as eigenvalue, making the rightmost equivalence in (1.2.4) false. It is indeed possible to study the complete spectrum of the conformal kernel using techniques based on his symmetry properties under the group $SL(2,\mathbb{R})$, see for example [16]. The full discussion is not needed here, we just want to show that 1 is in the spectrum. To this end, consider a reparametrization in the form $\tau\mapsto\tau+\epsilon(\tau)$ and the related transformation of a conformally symmetric solution (Σ^c, G^c) given by (1.2.14). We have shown that this leaves equations (1.2.13) invariant. In particular, at the first order in ϵ we get

$$\delta G_c * \Sigma_c + G_c * \delta\Sigma_c = 0 \tag{1.2.26}$$

where

$$\delta G_c(\tau,\tau') = \big(\frac{1}{4}\epsilon'(\tau) + \frac{1}{4}\epsilon'(\tau') + \epsilon(\tau)\partial_\tau + \epsilon(\tau')\partial_{\tau'}\big)G_c(\tau,\tau'),$$
$$\delta\Sigma_c(\tau,\tau') = \big(\frac{3}{4}\epsilon'(\tau) + \frac{3}{4}\epsilon'(\tau') + \epsilon(\tau)\partial_\tau + \epsilon(\tau')\partial_{\tau'}\big)\Sigma_c(\tau,\tau').$$

Since $\Sigma_c^{-1} = G_c$, eq. (1.2.26) implies

$$0 = \big(\delta G_c * \Sigma_c^{+} G_c * \delta\Sigma_c\big) * \Sigma_c^{-1} = \delta G_c + G_c * \delta\Sigma_c * G_c.$$

On the other hand, by (1.2.13)

$$\Sigma^c(\tau,\tau') = J^2 G^c(\tau,\tau')^3 \Longrightarrow \delta\Sigma^c(\tau,\tau') = 3J^2 G^c(\tau,\tau')^2 \delta G_c(\tau,\tau'),$$

so that it holds

$$0 = \delta G_c + G_c * (3J^2 G_c^2 \delta G_c) * G_c = (1 - K_c)|\delta G_c.$$

This shows that in the spectrum of K_c, seen as a linear operator acting on the right as in eq. (1.2.23), contains 1. In fact any function $h : \mathbb{R}^4 \to \mathbb{R}$ in the form

$$h(\tau,\tau',t,t') = \delta G_c(\tau,\tau') \times f(t,t')$$

is an eigenfunction with corresponding eigenvalue 1, $\forall\,\epsilon(\tau)$. The fact that we used conformal invariance to prove that the conformal approximation cannot give safe results has a rather deep meaning, as will be evident after we study the functional approach to the 4-point Green function.

1.2.5 Functional approach to the 4-point Green function

We have seen in section 1.2.4 that the high degree of symmetry of the theory in the infrared approximation leads to divergences in the correlation functions. Let's see how this reflects in the functional approach. Since we have proven that the leading order contribution to the connected 4-point green function is of order $O(N^{-1})$, we cannot limit ourselves to the saddle point approximation of the generating functional, but we need to include further corrections around the saddle. In the light of the concluding comments to section (1.2.3) we will indicate by (Σ, G) the extremal points of the functional (1.2.18) and by $(\tilde\Sigma, \tilde G)$ a generic field configuration. Then we set

$$\tilde G = G + \frac{g}{|G|} \qquad \text{and} \qquad \tilde\Sigma = \Sigma + |\Sigma|\sigma$$

and concentrate on the deviations (g,σ) from the extremal. Note that this definition leaves the integration measure invariant: $D\tilde\Sigma D\tilde G = D\sigma Dg$. Then the effective action (1.2.18) reads

$$\begin{aligned} S[\Sigma,G] =& \log\det(\partial_\tau - \tilde\Sigma(\tau,\tau')) + \int \frac{J^2}{4}\tilde G(\tau,\tau')^4 - \tilde\Sigma(\tau,\tau')\tilde G(\tau,\tau')d\tau d\tau' \\ &= \log\det(\partial_\tau - \Sigma(\tau,\tau') - |G|\sigma) \\ &+ \int \frac{J^2}{4}\big(G(\tau,\tau') + \frac{g}{|G|}\big)^4 - \big(\Sigma(\tau,\tau') + |G|\sigma\big)\big(G(\tau,\tau') + \frac{g}{|G|}\big)d\tau d\tau' \end{aligned}$$

We now expand S to second order in (σ, g). The second term is easily truncated, giving

$$\int \frac{J^2}{4}\tilde{G}(\tau,\tau')^4 - \tilde{\Sigma}(\tau,\tau')\tilde{G}(\tau,\tau')d\tau d\tau' \approx \int \sigma(\tau_1,\tau_2)g(\tau_1,\tau_2) - \frac{3J^2}{2}g(\tau_1,\tau_2)^2 d\tau_1 d\tau_2.$$

For the first term we proceed as follows:

$$\begin{aligned}
&\log\det(\partial_\tau - \Sigma(\tau,\tau') - |G|\sigma) = \mathrm{Tr}\log(\partial_\tau - \Sigma(\tau,\tau') - |G|\sigma) \\
&= \mathrm{Tr}\log[(\partial_\tau - \Sigma) * (1 - (\partial_\tau - \Sigma)^{-1} * |G|\sigma)] = \mathrm{Tr}\log[(\partial_\tau - \Sigma) * (1 - G * |G|\sigma)] \\
&\simeq \mathrm{Tr}\Big[-\frac{1}{2} G * |G|\sigma * G * |G|\sigma \Big]
\end{aligned}$$

where in the second line we used the property $(\partial_\tau - \Sigma)^{-1} = G$ and expanded the logarithm, neglecting all σ-independent terms. Thus we find

$$\begin{aligned}
-NS[G,\Sigma] =& \frac{N}{2}\Big(\log\det(\partial_\tau - \tilde{\Sigma}(\tau,\tau')) + \int \frac{J^2}{4}\tilde{G}(\tau,\tau')^4 - \tilde{\Sigma}(\tau,\tau')\tilde{G}(\tau,\tau')d\tau d\tau' \Big) \\
\simeq & -\frac{N}{12J^2}\int \sigma(\tau_1,\tau_2)\tilde{K}(\tau_1,...,\tau_4)\sigma(\tau_3,\tau_3)d\tau_1...d\tau_4 \\
&+\frac{N}{2}\int \sigma(\tau_1,\tau_2)g(\tau_1,\tau_2) - \frac{3J^2}{2}\big(g(\tau_1,\tau_2)\big)^2 d\tau_1 d\tau_2
\end{aligned}$$

where $\tilde{K}$ is the symmetric kernel defined by

$$\tilde{K}(\tau_1,...,\tau_4) := |G(\tau_1,\tau_2)|^1 K(\tau_1,...,\tau_4)|G(\tau_1,\tau_2)|^{-1}. \tag{1.2.27}$$

Its spectral properties are strictly related to those of K. In particular, if $h(\tau_1,...,\tau_4)$ is an eigenfunction of K, then $\tilde{h}(\tau_1,...,\tau_4) := |G(\tau_1,\tau_2)|h(\tau_1,...,\tau_4)$ is an eigenfunction of $\tilde{K}$ with the same eigenvalue.
Since the i,j-averaged 4-point function can be written as

$$\frac{1}{N^2}\sum_{i,j}\langle\chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4)\rangle = \int d\tilde{\Sigma}d\tilde{G}e^{-NS[\tilde{G},\tilde{\Sigma}]}\tilde{G}(\tau_1,\tau_2)\tilde{G}(\tau_3,\tau_4),$$

we might as well integrate out the σ field in our approximation, obtaining

$$\frac{1}{N^2}\sum_{i,j}\langle\chi_i(\tau_1)\chi_i(\tau_2)\chi_j(\tau_3)\chi_j(\tau_4)\rangle = \int dg e^{-NS[g]}g(\tau_1,\tau_2)g(\tau_3,\tau_4) \tag{1.2.28}$$

where

$$S[g] = \frac{3J^2}{4}g * (\tilde{K}^{-1} - 1) * g. \tag{1.2.29}$$

This makes it evident that in the conformal approximation $\tilde{K} \to \tilde{K}_c$, which corresponds to expanding around a solution of the approximated saddle-point equations (1.2.13), the divergence of the 4-point function is a consequence of the reparametrization symmetry in the conformal approximation: the value of the action is zero on the entire manifold spanned by the first order approximation of conformally reparametrized propagators δG_c:

$$S\big[|G_c|\delta G_c\big] = \frac{3J^2}{4}|G_c|\delta G_c * (\tilde{K}_c^{-1} - 1) * |G_c|\delta G_c = 0.$$

This traces back to the fact that if we neglect the ∂_τ term in eq. (1.2.18) the action $S(G,\Sigma)$ becomes invariant under the transformation (1.2.14).

1.3 The Schwarzian action for reparametrizations

In the previous section we have seen that, in order to calculate the IR $J >> 1$ approximation of the 4-point correlation function, the brute force substitution $K \to K_c$ is not appropriate. We have stressed how this is related to the role of the infinitesimal conformal reparametrizations of the conformal propagator δG_c, seen as eigenfunctions of the conformal propagator K_c with eigenvalue 1.
In order to avoid nonsensical results, one needs to study in greater detail the spectral properties of K. The detailed calculation has been worked out in [16]. It turns out that δG_c retains an important role also in this context. In particular, one can show that δG_c is also an eigenfunction of K, with eigenvalue $1 + O(J^{-1})$. When we compute a path integration with action (1.2.29), as in the case of eq. (1.2.28), we get an enhanced contribution from the elements in the domain in the form δG_c for some $\epsilon(t)$. This is because even though they are not exact minima of the action functional, and therefore should be irrelevant in the $N \to \infty$ limit, the value of the action functional on these configurations is killed by the 'almost 1" eigenvalue, and can be made parametrically small in the limit $J >> 1$. The precise result obtained by Maldacena is that the value of the action functional on the reparametrizations of the conformal propagator scales as $1/J$. This means that the IR limit $J >> 1$ and the $N >> 1$ in expressions like (1.2.28) compete, making these configurations relevant. In particular, the brute substitution $K \to K_c$ is equivalent to taking the $J >> 1$ limit first, while the appropriate approach in this context is to take first the $N >> 1$ limit.
The restriction of the action functional to the set $\{G_f\}$ defined in eq. (1.2.16) will be called 'action for fluctuations" and indicated by

$$S[f] := \frac{3J^2}{4} G_f * (\tilde{K}^{-1} - 1) * G_f. \tag{1.3.1}$$

There is still an open debate on what should the action for fluctuations exactly look like. Many authors have proposed it should be the integral of a Schwarzian derivative, which raises naturally by symmetry considerations:

$$S[f] = -\frac{1}{g^2} \int \Big[\frac{f'''}{f'} - \frac{3}{2} \Big(\frac{f''}{f'} \Big)^2 \Big] dt = -\frac{1}{g^2} \int \{ f(t), t \} dt. \tag{1.3.2}$$

In fact, as already pointed out in section 1.2.2, the prototype (1.2.15) solutions of the low energy theory are exactly $SL(2, \mathbb{R})$ invariant, meaning that

$$G_{\frac{af(t)+b}{cf(t)+d}}(\tau, \tau') = G_{f(t)}(\tau, \tau') \, \forall f. \tag{1.3.3}$$

As a consequence, the action for fluctuations should be invariant under $f(\tau) \to \frac{af(\tau)+b}{cf(\tau)+d}$, and it is reasonable to expect it to be the simplest action (lowest order in derivatives) exhibiting this invariance. It is readily checked that (1.3.2) does the job. On the more quantitative side, Maldacena managed to calculate the leading order value in a $\epsilon(t)$ expansion of the action for fluctuations $S[f(t) = t + \epsilon(t)]$, finding exactly the leading order approximation of eq. (1.3.2). Although the infinitesimal result improves our confidence in the action (1.3.2), a rigorous method to extend this result to finite (*i.e.* not infinitesimal) deviations ΔG_c from the conformal propagator is still missing. In order to ensure consistency with the leading order result in [16], we must have $1/g^2 \propto 1/J$.
A rigorous study of this kind of theory is possible within the path integral approach due to some recent developments in infinite dimensional measure theory, that will be illustrated in section (1.4).

1.3.1 Finite temperature case

So far we have discussed the SYK model as a QFT defined on the real line $\mathbb{R}$. As we will see, the most natural setting for a functional integral approach to its soft modes governed by the action (1.3.2) is that of a 'finite temperature" field theory. This means using a circle S^1 of length $\beta = 1/KT$, where T is the temperature, instead of $\mathbb{R}$. We then need to find solutions to eq. (1.2.13) defined on the circle, or equivalently, periodic solutions

$$G^c_\beta(t,t') \text{ s.t. } G^c_\beta(t+\beta, t'+\beta) = G^c_\beta(t,t'). \tag{1.3.4}$$

Luckily, the symmetry property (1.2.14) of the conformal propagator allows to transition immediately to this slightly different language. This is done by noting that such periodic solutions can be constructed choosing in (1.2.14) any function f which is periodic and invertible when restricted to a period. A convenient way to parametrize the space of periodic functions is given by

$$f(t) := -\cot\frac{\pi\phi(t)}{\beta}. \tag{1.3.5}$$

where ϕ is a periodic function with period β:

$$\phi \in \text{Diff}^1(S^1) = \{f : [0,\beta] \to [0,\beta] \quad s.t. \quad f \text{ is differentiable and invertible,} \\ f(0)=0,\ f(\beta)=\beta,\ f'(0)=f'(\beta)\}$$

The function (1.3.5) introduces a compactification of the codomain $\mathbb{R} \to S^1$. In terms of the ϕ variable, the action (1.3.2) reads

$$S[\phi] := -\frac{1}{g^2}\int_0^\beta \{-\cot\frac{\pi\phi(\tau)}{\beta};\tau\}d\tau = -\frac{1}{g^2}\int_0^\beta \Big(\{\phi,\tau\} + \frac{\pi}{\beta}\phi'^2(\tau)\Big)d\tau. \tag{1.3.6}$$

Note that the condition on ϕ is not rigorously sufficient to guarantee the existence of $\{\phi,\tau\}$, that require that ϕ', ϕ'' and ϕ''' exist. In what follows the Schwarzian derivative will be intended in the generalized sense typical of path integration, namely as a short-hand notation for its finite difference approximation.

The Schwarzian action came into our analysis due to its symmetry properties. It is then worth stressing that they naturally translate into symmetry properties of (1.3.6), which is invariant under the action on ϕ of the group $\text{SL}(2,\mathbb{R})$ defined implicitly by

$$\phi(t) \xrightarrow{\gamma} \tilde{\phi(t)} \quad t.c \quad \tilde{f}(t) = \gamma \circ f(t) = \frac{af(t)+b}{cf(t)+d}, \gamma \in SL(2,\mathbb{R}) \tag{1.3.7}$$

due to the properties of the Schwarzian derivative:

$$\{f\circ g(t), t\} = (g'(t))^2\{f,g\} + \{g,t\} \quad \text{and} \quad \{\frac{at+b}{ct+d}, t\} = 0. \tag{1.3.8}$$

We will refer to the action (1.3.6) as *Schwarzian action*. It turns out that this kind of theory is exactly solvable. By this we mean that it is possible to compute in closed form both the free energy and the green functions.

1.4 Functional integration in the Sachdev-Ye-Kitaev model

In this section we will discuss a method to explicitly carry out functional integration in the Schwarzian theory, *i.e.* integrals like

$$I_F = \int_{\text{Diff}^1(S^1)} F[\phi]e^{S[\phi]}, \tag{1.4.1}$$

where

$$S[\phi] = \exp\Big[\frac{1}{g^2}\int_0^\beta \{\phi,\tau\} + \frac{\pi}{\beta}\phi'^2(\tau)d\tau\Big]d\phi, \tag{1.4.2}$$

and $F[\cdot]$ is some functional. It is based on a specific measure defined on $\text{Diff}^1(S^1)$ which will be introduced in the next subsection. Then we proceed applying this technique to the calculation of the partition function and the correlation functions of the theory. Before we set to work, we apply some cosmetic adjustments to the action. Setting

$$t = \frac{\tau}{\beta}, \quad \varphi(t) = \frac{\phi(\tau)}{\beta}, \quad \sigma = \sqrt{2\pi}g$$

one gets

$$S[\varphi] = \exp\Big[\frac{1}{\sigma^2}\int_0^1 \{\varphi,\tau\} + 2\pi^2\varphi'^2(t)dt\Big]d\varphi. \tag{1.4.3}$$

1.4.1 Mathematical preliminaries

The basis of the following work is given by the notorious Wiener measure, which we now briefly recall. An extensive account can be found in [14].

The Wiener measure

Consider the space of continuous functions defined on $[0,1]$ preserving 0, that is

$$C_0([0,1]) := \{f : [0,1] \to \mathbb{R} \quad s.t. \quad f(0) = 0,\ f \text{ continuous}\}. \tag{1.4.4}$$

Given a finite set $F := \{t_1, .., t_n\}$ with $0 < t_1 < ... < t_n = 1$, and a set U belonging to the Borel σ-algebra of $\mathbb{R}^n$, $\mathcal{B}(\mathbb{R}^n)$, we define a set $\mathcal{C}_F^U \in C_0([0,1])$ *cylindrical with gates* (F,U) if it is of the form

$$\mathcal{C}_F^U = \{x \in C_0([0,1]) \text{ s.t. } (f(t_1), ..., f(t_n)) \in U\}. \tag{1.4.5}$$

We can define the measure of a cylindrical set by

$$\mu_F^U(\mathcal{C}_F^U) := \Big[(2\pi)^n t_1(t_2-t_1)\cdots(t_n-t_{n-1})\Big]^{-\frac{1}{2}}\int_U \exp\Big[-\frac{x_1^2}{2t_1} + ... - \frac{(x_n-x_{n-1})^2}{2(t_n-t_{n-1})}\Big]dx. \tag{1.4.6}$$

This measure is denumerably additive, and can be therefore be extended uniquely to the σ-algebra generated by the sets $\{\mathcal{C}_\circ^\circ\}$ as stated in the following

Theorem 1.4.1. *(Daniell-Kolmogorov extension). There exists a unique probability measure μ_w on the σ-algebra $(C_0([0,1]),\mathcal{C})$ generated by the sets $\{\mathcal{C}_\circ^\circ\}$, called the Wiener measure, such that for every finite F, $\mu_w(A) = \mu_F^U(A)$ if $A = \mathcal{C}_F^U$.*

The space $(C_0([0,1])$ is a normed space (indeed, it is a Banach space) endowed with the supremum norm

$$||X \in C_0([0,1])|| := \sup_{0\le t\le 1} |X(t)|.$$

We can use this norm to make it a topological space with the topology T built by countable union and finite intersections of "open balls" $B_r^c \subset C_0([0,1])$

$$B_r^c := \{x \in C_0([0,1]) \text{ s.t. } ||x-c|| < r\};\ c \in C_0([0,1]),\ r \in \mathbb{R}.$$

The σ-algebra $\big((C_0([0,1]),T\big)$ generated by T is called *Borel σ-algebra* and denoted by $\mathcal{B}\big((C_0([0,1])\big)$. A relevant result is that $(C_0([0,1]),\mathcal{C})$ indeed coincides with $\mathcal{B}\big((C_0([0,1])\big)$.

Integration with respect to the Wiener measure

We call a function $F : C_0([0,1]) \to \mathbb{R}$ measurable if

$$F^{-1}(A) \in (C_0([0,1]), \mathcal{C}) \ \forall\ A \in \mathcal{B}(\mathbb{R}).$$

To build a theory of integration with respect to the measure μ_w, we start by defining the integral of the *cylindrical functions*, i.e. functions of the form

$$F[x \in C_0([0,1])] = f\big(x(t_1), ..., v(t_n)\big) \tag{1.4.7}$$

for some measurable $f : \mathbb{R}^n \to \mathbb{R}$ and some $0 < t_1 < ... < t_n < 1$. Note that they are measurable functions. Their integral is defined as

$$\int\limits_{C_0([0,1])} F[x]\mu_w(dx) := \Big[(2\pi)^n t_1(t_2-t_1)\cdots(t_n-t_{n-1})\Big]^{-\frac{1}{2}} \int_{\mathbb{R}^n} f(x_1, ..., x_n) \exp\Big[-\frac{x_1^2}{2t_1}+...-\frac{(x_n - x_{n-1})^2}{2(t_n - t_{n-1})}\Big] d^n x. \tag{1.4.8}$$

The integral of a general measurable function $F[\cdot]$ is defined by approximating it in terms of cylindrical functions. This is done by splitting arbitrarily the interval $[0,1]$ into n parts $0 < t_1 < ... < t_n = 1$ and then considering for each function $f \in C_0([0,1])$ its poligonal approximations $\{f_n\}$ specified by the conditions

$$f_n(0) = 0,\ f_n(t_1) = f(t_1),\ \dots,\ f_n(t_n) = f(t_n). \tag{1.4.9}$$

To any measurable, continuous and limited function $F[\cdot]$ we can then associate its cylindrical approximations $\{F_n[\cdot]\}$ defined by

$$F_n[f] := F[f_n]. \tag{1.4.10}$$

Since each $F_n[\cdot]$ is cylindrical, its integral is defined by (1.4.8). Moreover, it can be proven that the limit

$$\lim_{n\to\infty} \int_{C_0([0,1])} F_n[f]\mu_w(df) \tag{1.4.11}$$

exists and is independent on how we choose $t_1, ..., t_n$, so it is safe to define

$$\int_{C_0([0,1])} F[f]\mu_w(df) := \lim_{n\to\infty} \int_{C_0([0,1])} F_n[f]\mu_w(df). \tag{1.4.12}$$

One striking feature of the Wiener measure is that in the $n \to \infty$ limit, the exponent in eq. (1.4.8) looks like the Riemann sum approximation for the integral of the derivative of $x(t)$ squared:

$$-\frac{x_1^2}{2t_1} + ... - \frac{(x_n - x_{n-1})^2}{2(t_n - t_{n-1})} = -\frac{1}{2}\Big(\frac{x_1^2}{(t_1)^2}(t_1) + ... + \frac{(x_n - x_{n-1})^2}{(t_n - t_{n-1})^2}(t_n - t_{n-1})\Big) \sim -\frac{1}{2}\int_0^1 \big(x'(t)\big)^2 dt, \tag{1.4.13}$$

so that often integration with respect to the Wiener measure is indicated, in the original spirit of Feynman path integral, by

$$\int\limits_{C_0([0,1])} F[x]\mu_w(dx) =: \int\limits_{C_0([0,1])} F[x] \exp\Big(-\frac{1}{2}\int_0^1 \big(x'(t)\big)^2 dt\Big) dx.$$

This formal rewriting is of course merely a concise way of representing the limiting procedure described in eq. (1.4.8). In fact, the Wiener measure is concentrated on the paths which are nowhere differentiable, *i.e.* the sets of functions which are differentiable even at a single point has 0 Wiener measure, so the rightmost equality in expression (1.4.13) makes sense almost μ_σ-nowhere. Moreover, one is in general not allowed to commute the integration and the limit in eq. (1.4.12).

Wiener measure with arbitrary variance

A mild generalization of what stated above is given by introducing a parameter $\sigma^2 \in \mathbb{R}^+$ in the expression (1.4.6), in the form of

$$\mu^U_{F,\sigma}(\mathcal{C}^U_F) := \Big[(2\pi\sigma^2)^n t_1(t_2-t_1)\cdots(t_n-t_{n-1})\Big]^{-\frac{1}{2}} \int_U \exp\Big[-\frac{x_1^2}{2\sigma^2 t_1} + ... - \frac{(x_n-x_{n-1})^2}{2\sigma^2(t_n-t_{n-1})}\Big]dx.$$

This defines the Wiener measure with variance σ, which we will be denoting by w_σ. This terminology is related to the fact that the Wiener measure is indeed a Gaussian measure, but this property is not important for what follows.

The measure on $\mathcal{B}\big(\mathbf{Diff}^1_+([0,1])\big)$ and $\mathcal{B}\big(\mathbf{Diff}^1(S^1)\big)$

The presence of the term

$$\frac{2\pi^2}{\sigma^2}\int_0^1 \phi'(t)^2 dt$$

in the action of (1.4.3) may suggest that our path integrals should be considered as a functional integration with respect to the Wiener measure

$$w_k(d\phi) = \exp\Big[-\frac{1}{k}\int_0^1 \phi'(t)^2 dt\Big]d\phi \tag{1.4.14}$$

analytically continued to $k = \frac{i\sigma}{\sqrt{2\pi}}$. For example the partition function would look like

$$\begin{aligned} Z = \int e^{-S[\phi]}d\phi &= \int\limits_{\mathrm{Diff}^1(S^1)} \exp\Big[\frac{1}{\sigma^2}\int_0^1 \{\phi,\tau\} + 2\pi^2\phi'^2(t)dt\Big]d\phi \\ &= \int\limits_{\mathrm{Diff}^1(S^1)} \exp\Big[\frac{1}{k^2}\int_0^{2\pi}\{\phi,\tau\}d\tau\Big]w_k(d\phi)\Big|_{k=\frac{i\sigma}{\sqrt{2\pi}}}. \end{aligned} \tag{1.4.15}$$

This line of reasoning cannot be used to make sense of the theory. The reason is that the Wiener measure we are considering is defined as a measure on the space $C([0,1])$ of continuous functions on the intervall $[0,1]$. Our integration domain turns out to have null Wiener measure, making every integral vanish. We need to build a measure adapted to our integration domain.

To do so, following [3], we start from the Weiner measure w_σ on the space

$$C_0([0,1]) := \{f:[0,1]\to\mathbb{R} \quad s.t. \quad f(0)=0,\ f \text{ continuous}\}. \tag{1.4.16}$$

Now consider the map

$$A:\mathrm{Diff}^1_+([0,1]) \to C_0([0,1])\,; \quad A[f](t) := \log\Big(\frac{f'(t)}{f'(0)}\Big), \tag{1.4.17}$$

where we have defined

$$\begin{aligned}\mathrm{Diff}^1_+([0,1]) := \{f:[0,1]\to[0,1] \quad s.t.\ & f \text{ is monotonically increasing, } f\in C^1([0,1]),\ f^{-1}\in C^1([0,1]), \\ & \text{and } f(0)=0,\ f(1)=1\}.\end{aligned}$$

Map A is invertible, with inverse

$$A^{-1}[\xi](t) = \frac{\int_0^t \exp[\xi(\tau)]d\tau}{\int_0^1 \exp[\xi(\tau)]d\tau}, \tag{1.4.18}$$

and it can be used to induce a measure on the space $\mathrm{Diff}^1_+([0,1])$. The result of Belokurov and Shavgulidze is the following:

Theorem 1.4.2. *The set function $\mu_\sigma(\cdot)$ defined on the Borel σ-algebra $\mathcal{B}\big(\mathit{Diff}^1_+([0,1])\big)$ by*

$$\mu_\sigma(X) := w_\sigma(A(X)),\ X \in \mathcal{B}\big(\mathit{Diff}^1_+([0,1])\big) \tag{1.4.19}$$

is a positive measure.

It turns out that this measure has interesting properties of quasi-invariance under the action of the group

$$\mathrm{Diff}^3_+([0,1]) := \{f : [0,1] \to [0,1] \quad s.t.\ \ f \in \mathrm{Diff}^1([0,1]) \text{ and } f \text{ is three times differentiable}\}.$$

The precise statement is the following:

$$\int\limits_{\mathrm{Diff}^1_+([0,1])} F(\phi)\mu_\sigma(d\phi) = \int\limits_{\mathrm{Diff}^1_+([0,1])} F(f\circ\phi)\times$$
$$\frac{1}{\sqrt{f'(0)f'(1)}}\exp\Big[\frac{1}{2\sigma^2}\Big(\frac{f''(0)}{f'(0)}\phi'(0)-\frac{f''(1)}{f'(1)}\phi'(1)\Big)+\frac{1}{\sigma^2}\int_0^1\{f,t\}(\phi'(t))^2dt\Big]\mu_\sigma(d\phi)$$
$$\forall\, f \in \mathrm{Diff}^3_+([0,1]). \tag{1.4.20}$$

The proof of this equality is sort of technical, and can be found in [5]. It is based upon discretization of space and a suitable limiting procedure. In the final analysis, the appearance of the Schwarzian derivative in equation (1.4.20) can be understood as a consequence of its involvement in similar quasi-invariance properties of the Wiener measure under the action of $\mathrm{Diff}^3_+([0,1])$, which were first noted in [21]. On the other hand, as remarked by M. Kac (see [11]), integration with respect to the Wiener measure is strictly related to Quantum Mechanics in the path integral approach, at least in its better mathematically defined version of *Euclidean* path integral. As a consequence, the Schwarzian derivative should be considered as rather deeply nested in the fiber of general Quantum Mechanics, although in what follows we will concentrate the exposition on the particular case of Schwarzian-like actions.

An analogous procedure allows to define a measure μ_σ on the space $\mathcal{B}\big(\mathrm{Diff}^1(S^1)\big)$, where

$$\mathrm{Diff}^1(S^1) := \{f : [0,1] \to [0,1] \quad s.t.\, f \text{ is invertible, } f \in C^1([0,1]),\ f^{-1} \in C^1([0,1]),$$
$$\text{and } f(0) = 0,\ f(1) = 1,\ f'(0) = f'(1)\}.$$

This time we have an additional condition with respect to the $\mathrm{Diff}^1_+([0,1])$ case, namely

$$f'(0) = f'(1).$$

Via the map (1.4.17) this translate into the condition

$$\xi(0) = \xi(1) = 0,$$

so that ξ belongs to the space of Brownian bridges, *i.e.* Brownian motions with two fixed points rather than one. This space will be denoted by

$$C_{x,y}([0,1]) := \{f : [0,1] \to \mathbb{R} \quad s.t. \quad f(0) = x,\ f(1) = y,\ f \text{ continuous}\}. \tag{1.4.21}$$

It is then natural to utilize the Wiener measure w^{b}_σ adapted to $C_{0,0}([0,1])$ in order to induce a measure on $\mathcal{B}\big(\mathrm{Diff}^1(S^1)\big)$, rather than the one adapted to $C_0([0,1])$. The Brownian bridge measure w^{b}_σ is related to the standard Wiener measure by the equation

$$w_\sigma(d\xi) = w^{\mathrm{b}}_\sigma(d\xi)\frac{1}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{x^2}{2\sigma^2}\Big)dx. \tag{1.4.22}$$

This means that for any Lebesgue measurable set $U \subset \mathbb{R}$, the measure $w_\sigma(X)$ of the set $X_U \subset C_0([0,1])$ defined by the gate condition

$$X_U := \{\xi(t) \in C_0([0,1]) \quad s.t. \quad \xi(1) \in U\} \tag{1.4.23}$$

is given by

$$w_\sigma(X_U) = \int_U dx \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{x^2}{2\sigma^2}\Big) \int_{C_{0,x}([0,1])} w_\sigma^{\mathrm{b}}(d\xi). \tag{1.4.24}$$

As one should expect, the two μ_σ measures on $\mathcal{B}\big(\mathrm{Diff}^1(S^1)\big)$ and $\mathcal{B}\big(\mathrm{Diff}^1_+([0,1])\big)$ are strictly related. In fact it holds:

$$\begin{aligned}\frac{1}{\sqrt{2\pi}\sigma} \int_{\mathrm{Diff}^1(S^1)} F[\phi]\mu_\sigma(d\phi) &= \frac{1}{\sqrt{2\pi}\sigma} \int_{C_{0,0}([0,1])} F[\phi(\xi)] w_\sigma^{\mathrm{b}}(d\xi) = \int_{C_0([0,1])} \delta(\xi(1)) F[\phi(\xi)] w_\sigma(d\xi) \\ &= \int_{\mathrm{Diff}^1_+([0,1])} \delta\Big(\log \frac{\phi'(1)}{\phi'(0)}\Big) F[\phi]\mu_\sigma(d\phi) = \int_{\mathrm{Diff}^1_+([0,1])} \delta\Big(\frac{\phi'(1)}{\phi'(0)} - 1\Big) F[\phi]\mu_\sigma(d\phi).\end{aligned} \tag{1.4.25}$$

The second equality rests upon eq. (1.4.22), while in the fourth is the functional generalization of

$$\int_0^{+\infty} \delta(\log x) f(x) dx = f(1) = \int_{-\infty}^{+\infty} \delta(x-1) f(x) dx.$$

The measures defined above are relevant to our case since it can be proven (see [3]) that the following representation in terms of the Feynman path integral holds for the measure μ_σ on $\mathrm{Diff}^1(S^1)$:

$$\mu_\sigma(d\phi) = \int \exp\Big[\frac{1}{\sigma^2} \int_0^1 \{\phi, t\} dt\Big] d\phi. \tag{1.4.26}$$

This is analogous to the formal representation for the Wiener measure

$$w_\sigma(d\phi) = \int \exp\Big[-\frac{1}{\sigma^2} \int_0^1 \phi'(t)^2 dt\Big] d\phi \tag{1.4.27}$$

we have discussed in section (1.4.1), and both the integral and derivative are to be considered in a Riemann approximation prospective.
It sounds then reasonable to interpret path integrals in Schwarzian theory rigorously as integrals over $\mathrm{Diff}^1(S^1)$ endowed with the measure μ_σ.

1.4.2 The partition function in the Schwarzian theory

Let us put this machinery at work by calculating the partition function for the Schwarzian theory:

$$Z = \int_{\mathrm{Diff}^1(S^1)} \exp\Big[\frac{1}{\sigma^2} \int_0^1 \Big(\{\phi, \tau\} + 2\pi^2 \phi'^2(\tau) d\tau\Big]\Big) d\phi\Big|_{\sigma=\sqrt{2\pi g}} = \int_{\mathrm{Diff}^1(S^1)} \exp\Big[\frac{1}{\sigma^2} \int_0^1 2\pi^2 \phi'^2 d\tau\Big] \mu_\sigma(d\phi)\Big|_{\sigma=\sqrt{2\pi g}}.$$

As it is, this integral is divergent. We will see that this divergence can be traced back to the $SL(2,\mathbb{R})$ symmetry of the theory, and therefore can be reabsorbed regularizing the integral and dividing it by the volume of the $SL(2,\mathbb{R})$ group. To regularize the integral we introduce a parameter α

$$Z \longrightarrow Z_\alpha := \int\limits_{\text{Diff}^1(S^1)} \exp\Big[\frac{1}{\sigma^2}\int_0^1 2\alpha^2\phi'^2 d\tau\Big]\mu_\sigma(d\phi). \tag{1.4.28}$$

The limit $\alpha \to \pi$ restores the original theory. To compute the integral, it is convenient to use eq. (1.4.25) to switch to an integration over $\text{Diff}^1_+([0,1])$. The regularized partition function then reads

$$Z_\alpha := \sqrt{2\pi}\sigma \int\limits_{\text{Diff}^1_+([0,1])} \delta\Big(\frac{\phi'(1)}{\phi'(0)} - 1\Big) \exp\Big[\frac{1}{\sigma^2}\int_0^1 2\alpha^2\phi'^2 d\tau\Big]\mu_\sigma(d\phi). \tag{1.4.29}$$

In order to explicitly carry out the integration, we proceed along the following steps:

- We use the quasi-invariance of the measure μ_σ on $\mathcal{B}\big(\text{Diff}^1_+([0,1])\big)$ to simplify the integral. Specifically, we exploit the freedom in the choice of $F[\cdot]$ and $f(\cdot)$ in the Radon-Nikodym derivative formula

$$\int\limits_{\text{Diff}^1_+([0,1])} F(\phi)\mu_\sigma(d\phi) = \int\limits_{\text{Diff}^1_+([0,1])} F(f\circ\phi)\times$$
$$\frac{1}{\sqrt{f'(0)f'(1)}}\exp\Big[\frac{1}{2\sigma^2}\Big(\frac{f''(0)}{f'(0)}\phi'(0) - \frac{f''(1)}{f'(1)}\phi'(1)\Big) + \frac{1}{\sigma^2}\int_0^1 \{f,t\}(\phi'(t))^2 dt\Big]\mu_\sigma(d\phi)$$

  to match the right hand side with the integral in (1.4.29), and then compute it as the left hand side.

- We normalize the result dividing it by the α-regularized Haar volume of $\text{SL}(2,\mathbb{R})$, which is given by

$$V_\alpha = \exp\Big(-\frac{2(\pi^2-\alpha^2)}{\sigma^2}\Big)\frac{2\pi\sigma^2}{\pi^2-\alpha^2} \tag{1.4.30}$$

- We compute the integral mapping it into an integral over $C_0([0,1])$ with respect to the Wiener measure w_σ via the map

$$\phi(t) =: \frac{\int_0^t \exp[\xi(\tau)]d\tau}{\int_0^1 \exp[\xi(\tau)]d\tau} \tag{1.4.31}$$

  It turns out that the integral we get can be explicitly evaluated.

The details of the computation can be found in appendix A. The final result is

$$Z = \frac{\pi}{\sigma^2}\exp\Big(\frac{2\pi^2}{\sigma^2}\Big) = \frac{1}{2g^2}\exp\Big(\frac{\pi}{g^2}\Big). \tag{1.4.32}$$

Note that this result is different from the one obtained in the original work of Shavgulidze and Belokurov [2], which is missing a factor $\sqrt{2\pi}\sigma$. Their result reads

$$Z = \frac{1}{4\pi g^3}\exp\Big(\frac{\pi}{g^2}\Big). \tag{1.4.33}$$

This is also the result obtained by Witten and Stanford in [23] following a very different path integration approach. To understand the discrepancy between (1.4.32) and (1.4.33), notice that in our calculation the coefficient in (1.4.32) is dependent on the normalization one chooses to get rid of the divergence as one sends $\alpha \to \pi$. This is different from what happens with the exponential factor, which comes directly from the path integral computation, and which is common between (1.4.32) and (1.4.33). The most intuitive (although not very elegant) way to make our line of reasoning match Witten and Stanford's result is choosing a different $\sigma' \neq \sigma$ in the expression for the regularized volume of $\mathrm{SL}(2,\mathbb{R})$

$$V_\alpha := \int_{\mathrm{SL}(2,\mathbb{R})} \exp\Big(-\frac{2(\pi^2-\alpha^2)}{\sigma'^2}\int_0^1 \big(\phi'(t)\big)^2 dt\Big)\mu_H(d\phi) \tag{1.4.34}$$

In particular, choosing $\sigma'^2 \propto \sigma^3$ adjusts the g dependence in eq. (1.4.32).

1.4.3 Correlation functions in the Schwarzian theory

This line of reasoning can be extended to the calculation of the correlation functions of the theory. In particular, in the context of the SYK model, we are interested in the α-regularized average of the conformal propagator given by the first line of eq. (1.2.16), in the theory defined by the Schwarzian action. This corresponds to studying the infrared sector of the theory. Upon the finite temperature substitution (1.3.5) it turns into

$$G_\phi(t,0) = \frac{\big[\phi'(t)\phi'(0)\big]^{\frac{1}{4}}}{|\sin\pi[\phi(t)-\phi(0)]|^{\frac{1}{2}}}, \tag{1.4.35}$$

so that

$$\langle G(0,t)\rangle_\alpha := \int_{\mathrm{Diff}^1(S^1)} \frac{\big[\phi'(t)\phi'(0)\big]^{\frac{1}{4}}}{|\sin\pi[\phi(t)-\phi(0)]|^{\frac{1}{2}}} \exp\Big(\frac{2\alpha^2}{\sigma^2}\int_0^1 (\phi'(\tau))^2 d\tau\Big)\mu_\sigma(d\phi) \tag{1.4.36}$$

The α-regularization has been introduced for the same reason as in section (1.4.2). Also for this computation it is convenient to use eq. (1.4.25) and rewrite $\langle G_\alpha(0,t)\rangle$ as an integral over $\mathrm{Diff}^1_+([0,1])$:

$$\langle G(0,t)\rangle_\alpha = \sqrt{2\pi}\sigma \int_{\mathrm{Diff}^1_+([0,1])} \delta\Big(\frac{\phi'(1)}{\phi'(0)}-1\Big)\frac{\big[\phi'(t)\phi'(0)\big]^{\frac{1}{4}}}{|\sin\pi[\phi(t)-\phi(0)]|^{\frac{1}{2}}} \exp\Big(\frac{2\alpha^2}{\sigma^2}\int_0^1 (\phi'(\tau))^2 d\tau\Big)\mu_\sigma(d\phi). \tag{1.4.37}$$

To calculate this we follow same strategy as for the partition function, that is

- We use the quasi-invariance property (1.4.20) with an appropriate $F[\cdot]$ and $f(\cdot)$ to simplify the integral. This time we address the more general problem of choosing $F_\Psi[\cdot]$ so that we can integrate any given functional $\Psi[\cdot]$, not only the conformal propagator.
- As in the case of the partition function, we normalize the divergence in the integral dividing by the α-regularized Haar volume of $\mathrm{SL}(2,\mathbb{R})$.
- We compute the integral in the $C_0([0,1])$ domain. This time the calculation is more complicated, due to the fact that we are averaging the propagator instead of the constant functional $\Psi[\phi]\equiv 1$ as in the partition function case. The key point is that all the interesting quantities in a QFT (take for example the green functions) are quantum averages of very special functionals, namely functionals that depend only on the value of the field and its derivative at a finite set of points. This allows to rewrite their functional integrals in terms of a *master functional integral* that can be explicitly computed. Functional integration then boils down to multiple Lebesgue integrals, that can be computed numerically.

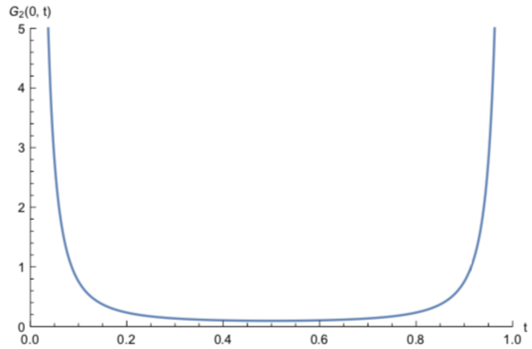


Figure 1.5: The conformal propagator averaged in the Schwarzian theory

The details of the calculation can be found in appendix B. The result is given by

$$\begin{aligned}\langle G(0,t)\rangle =&\frac{2\sqrt{2}\pi}{\sigma}[t(1-t)]^2\int_0^1[z(z-1)]^{-3}dz\int_0^\infty dx_0\int_0^\infty x_t dx_t\int_0^\infty dx_1\delta\Big(\frac{x_1}{x_0}-1\Big)\frac{(x_t x_0)^{\frac{1}{4}}}{z^{\frac{1}{2}}}\exp\Big(\frac{8}{\sigma^2}x_0\Big)\\&\times\epsilon_{\sqrt{t}\sigma}(\frac{t}{z}x_0,\frac{t}{z}x_t)\epsilon_{\sqrt{1-t}\sigma}(\frac{1-t}{1-z}x_t,\frac{1-t}{1-z}x_1)\\=&\frac{2\sqrt{2}\pi}{\sigma}[t(1-t)]^2\int_0^1 z^{-\frac{7}{2}}(z-1)^{-3}dz\int_0^\infty\int_0^\infty dx_0dx_t x_t^{\frac{5}{4}}x_1^{\frac{5}{4}}\exp\Big(\frac{8}{\sigma^2}x_0\Big)\epsilon_{\sqrt{t}\sigma}(\frac{t}{z}x_0,\frac{t}{z}x_t)\epsilon_{\sqrt{1-t}\sigma}(\frac{1-t}{1-z}x_t,\frac{1-t}{1-z}x_0),\end{aligned}$$

where the master integrals are

$$\begin{aligned}\epsilon_\sigma(u,v):=&\int_{Diff^1([0,1])}\delta(\psi'(0)-u)\delta(\psi'(1)-v)\mu_\sigma(d\psi)=\Big(\frac{2}{\pi\sigma^2}\Big)^{\frac{3}{2}}\frac{1}{\sqrt{uv}}\exp\Big\{\frac{2}{\sigma^2}(\pi^2-v-u)\Big\}\times\\&\times\int_0^{+\infty}\exp\Big\{-\frac{2}{\sigma^2}(2\sqrt{uv}\cosh\tau+\tau^2)\Big\}\sin\Big(\frac{4\pi\tau}{\sigma^2}\Big)\sinh\tau d\tau.\end{aligned}\tag{1.4.38}$$

As explained in the concluding remarks in section (1.4.2), one might be interested in using a slightly different renormalization, namely substituting $\sigma\to\sigma^{\frac{3}{2}}$ in the computation of the regularized Haar volume of $\mathrm{SL}(2,\mathbb{R})$. This reflects in a different prefactor, namely

$$\begin{aligned}\langle G(0,t)\rangle =&\frac{2\sqrt{\pi}}{\sigma^2}[t(1-t)]^2\int_0^1[z(z-1)]^{-3}dz\int_0^\infty dx_0\int_0^\infty x_t dx_t\int_0^\infty dx_1\delta\Big(\frac{x_1}{x_0}-1\Big)\frac{(x_t x_0)^{\frac{1}{4}}}{z^{\frac{1}{2}}}\exp\Big(\frac{8}{\sigma^2}x_0\Big)\\&\times\epsilon_{\sqrt{t}\sigma}(\frac{t}{z}x_0,\frac{t}{z}x_t)\epsilon_{\sqrt{1-t}\sigma}(\frac{1-t}{1-z}x_t,\frac{1-t}{1-z}x_1).\end{aligned}$$

As one can easily verify by plugging in the explicit expressions (1.4.38), $\langle G(0,t)\rangle$ is divergent both in $t_1=0$ and $t_1=1$. The integral can be computed numerically, and its form is shown in figure 1.5. We stress that the techniques exemplified in this section can be applied to compute any correlation function in the Schwarzian theory. See for example [4].

Chapter 2

The Quantum Hamilton-Jacobi equation

Interestingly, the Schwarzian derivative appears in a variety of contexts in modern theoretical physics. As exemplified by the derivation of the action for fluctuations in section (1.3), it is often the case that Möbius symmetry is one of the profound reason for its appearance. Another context in which the Schwarzian derivative has recently been studied is that of the foundations of Quantum Mechanics. As we are about to illustrate, also in this case Möbius symmetry retains a central role. Before going into the details of the construction elaborated by M. Matone and A. E. Faraggi (see [7, 8]), we begin by recalling the essential points in Hamilton-Jacobi theory, which will be useful to understand what follows.

2.1 Classical Hamilton-Jacobi Equation

In the context of Hamiltonian dynamics a *canonical transformation* is defined as a bijective differentiable map

$$f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}\,, \quad (q,p) \to (Q,P) = f(p,q) \tag{2.1.1}$$

such that $\forall H(q,p)$ the Hamilton equations of motion

$$\dot{q}(t) = \frac{\partial H}{\partial p} \quad , \quad \dot{p}(t) = -\frac{\partial H}{\partial q} \tag{2.1.2}$$

are mapped into the Hamiltonian equations defined by some $\tilde{H}(Q,P)$. What we mean is that $\big(q(t), p(t)\big)$ solves eq. (2.1.2) *iff* $f(p(t), q(t))$ solves the Hamilton equations for $\tilde{H}(P,Q)$. Note that we are using a compact notation where $p, q, ...$ etc. all represent vectors in $\mathbb{R}^n$.

Among the possible techniques for generating such transformations, there is the generating function method. It is known that given a function $F(q,P,t)$ of $2n+1$ variables such that

$$\det\Big(\frac{\partial^2 F}{\partial q_i \partial P_j}\Big) \neq 0 \tag{2.1.3}$$

the equations

$$p = \frac{\partial F}{\partial q}(q,P) \quad , \quad Q = -\frac{\partial F}{\partial P}(q,P) \tag{2.1.4}$$

define implicitly a time dependent canonical transformation $(p,q) \to (P,Q)$. The Hamiltonian $\tilde{H}(P,Q)$ is given by

$$\tilde{H}(P,Q,t) = H\Big(p(P,Q), q(P,Q)\Big) + \frac{\partial F}{\partial t}\Big(q(P,Q), P, t\Big). \tag{2.1.5}$$

It is now natural to ask which generating function maps the system H into the system with the most easily solvable dynamics, namely $\tilde{H} \equiv 0$. The answer to this question is given by the following theorem:

Theorem 2.1.1. *Let $S(\alpha, q, t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a family of solutions of the Hamilton-Jacobi equation*

$$H\Big(\frac{\partial S}{\partial q}(\alpha, q, t), q, t\Big) + \frac{\partial S}{\partial t}(\alpha, q, t) = 0, \tag{2.1.6}$$

where $\alpha \in \mathbb{R}^n$ is considered as a parameter, such that

$$det\Big(\frac{\partial^2 S}{\partial q_i \partial \alpha_j}\Big) \neq 0. \tag{2.1.7}$$

Then S, if we identify the parameters α with the new momenta P, generates via (2.1.4) a (time dependent) canonical transformation mapping H to $\tilde{H}(P, Q) \equiv 0$. Equation (2.1.6) is called the Hamilton-Jacobi equation *(HJE) associated to $H(p, q, t)$, and the family of solutions $S(\alpha, q, t)$ is called a* complete integral *for the Hamilton-Jacobi equation.*

For a time independent Hamiltonian the formalism simplifies, resulting in the following

Corollary 2.1.1. *If H(p,q) is time independent, then $S(q, Q, t) := S_0(q, Q) - Et$, where S_0 is a solution of*

$$H\Big(\frac{\partial S_0}{\partial q}(P, q), q\Big) - E = 0, \tag{2.1.8}$$

generates via (2.1.4) a (time dependent) canonical transformation mapping H to $\tilde{H}(P, Q) \equiv 0$. Equation (2.1.8) is called Stationary Hamilton-Jacobi equation *(SHJE) associated to $H(p, q)$.*

We will deal only with time independent hamiltonians in the form $H(p, q) = p^2/2m + V(q)$, so that (2.1.8) takes the form

$$\frac{1}{2m}\Big(\frac{\partial S_0}{\partial q}\Big)^2 + W = 0\,, \quad W := V(q) - E. \tag{2.1.9}$$

The fact that we denoted a complete integral of the Hamilton-Jacobi equation by the symbol S, which is typically associated to the action functional of the theory, is no coincidence. It is in fact possible to show that the action, seen as a function of the initial and final points of the classical trajectory is indeed a complete integral of the Hamilton-Jacobi equation. Namely, it holds:

Theorem 2.1.2. *Given two points q_0, q in the configuration space of a Lagrangian system with Lagrangian $\mathcal{L}(q, \dot{q}, t)$, the function $S(q_0, q, t)$ defined by*

$$S(q_0, q, t) := \int_0^t \mathcal{L}((t), \dot{q}(t), t)dt, \tag{2.1.10}$$

where $q(t)$ is the unique solution of the Lagrange equation with conditions $q(0) = q_0$, $q(t) = q$, is a complete integral of the Hamilton-Jacobi equation associated to the Hamiltonian of the system.

The canonical transformation $\varphi : (p, q) \to (p_0, q_0)$ induced by $S(q_0, q, t)$ via

$$p = \frac{\partial S}{\partial q} \quad , \quad p_0 = -\frac{\partial S}{\partial q_0} \tag{2.1.11}$$

is a very peculiar one, in the sense that $p(q_0, q, t)$ and $p_0(q_0, q, t)$ are the final and initial momenta of the trajectory in (2.1.10). This means that φ trivializes the dynamics by "rewinding" the Hamiltonian flow.

We have seen that being able to integrate the Hamiltonian flow we can reconstruct a complete integral of the HJE. This is by no means an easy task, and often one is not able neither to solve the dynamics nor to find by some other means a complete integral to the HJE (for example, by separation of variables). We might though be able to find a solution to the HJE without an explicit dependence on the parameters P, that is a single function $S(p, t)$ such that

$$H\Big(\frac{\partial S}{\partial q}(q,t), q, t\Big) + \frac{\partial S}{\partial t}(q,t) = 0. \tag{2.1.12}$$

When this less ambitious request is met, it is still possible to use S to help finding *some* solutions to the Hamilton equations by the ansatz

$$\big(q(t), p(t)\big) = \big(q(t), \frac{\partial S}{\partial q}(q(t), t)\big). \tag{2.1.13}$$

In fact, differentiating (2.1.12) with respect to q you get

$$\frac{\partial^2 S}{\partial q \partial t}(q,t) + \frac{\partial H}{\partial p}\big(q, \frac{\partial S}{\partial q}(q,t), t\big)\frac{\partial^2 S}{\partial q^2}(q,t) = -\frac{\partial H}{\partial q}\big(q, \frac{\partial S}{\partial q}(q,t), t\big). \tag{2.1.14}$$

On the other hand, the ansatz (2.1.13) implies

$$\frac{\partial p}{\partial t}(t) = \frac{\partial^2 S}{\partial q \partial t}\Big(q(t), t\Big) + \frac{\partial H}{\partial p}\Big(q(t), \frac{\partial S}{\partial q}\big(q(t), t\big), t\Big)\frac{\partial^2 S}{\partial q^2}\big(q(t), t\big), \tag{2.1.15}$$

so that, by (2.1.14),

$$\frac{\partial p}{\partial t}(t) = -\frac{\partial H}{\partial q}\big(q(t), \frac{\partial S}{\partial q}(q(t), t), t\big) = -\frac{\partial H}{\partial q}\big(q(t), p(t), t\big). \tag{2.1.16}$$

Equation (2.1.13) is then consistent with the Hamilton-Jacobi equations, and can be used to write $p(t)$ as a function of $q(t)$ to solve them. It is evident that this strategy does not give the most general solution to the dynamic. In fact, by (2.1.13) it is clear that we only find trajectories such that

$$p(0) = \big(q(t), \frac{\partial S}{\partial q}(q(0), 0)\big), \tag{2.1.17}$$

which is a constraint on the contour conditions for the Cauchy problem given by the Hamilton equations. Different solutions S corresponds to different constraints to the initial conditions.

After this classical introduction, let us move to Quantum Mechanics.

2.2 Bohm formulation of the Quantum Theory of Motion

What is most commonly referred to as *Quantum Hamilton-Jacobi Equation* (QHJE) for a particle subjected to a potential $V(x)$ is the system of coupled differential equations that arise when solving the Schrödinger equation

$$i\hbar\frac{\partial \psi}{\partial t} = \Big(-\frac{\hbar^2}{2m}\nabla^2 + V\Big)\psi \tag{2.2.1}$$

in the form

$$\psi(x) = R(x,t)e^{\frac{i}{\hbar}\hat{S}_0(x,t)}\,, \quad (R, \hat{S}_0) \in \mathbb{R}^2. \tag{2.2.2}$$

Substituting eq. (2.2.2) into (2.2.1) you get

$$i\hbar\Big[\frac{\partial R}{\partial t} + \frac{iR}{\hbar}\frac{\partial S}{\partial t}\Big] = -\frac{\hbar^2}{2m}\Big\{\nabla^2 R - \frac{R}{\hbar^2}(\nabla S)^2 + i\Big[\frac{2}{\hbar}\nabla R\cdot\nabla S + \frac{R}{\hbar}\nabla^2 S\Big]\Big\} + VR. \tag{2.2.3}$$

Separating (2.2.3) into real and imaginary part we obtain the following coupled equations:

$$\begin{cases} \frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + V = 0 \\ \frac{\partial R^2}{\partial t} + \nabla\Big(\frac{R^2\nabla S}{m}\Big) = 0 \end{cases} \tag{2.2.4}$$

In the one dimensional stationary case (call the coordinate q), *i.e.* if we look for solutions such that $\partial_t\psi(q,t) \propto \psi(q,t)$, eq. (2.2.1) implies that $\psi(x,t)$ is for all t a solution of the eigenvalue equation, or stationary Schrödinger equation

$$\Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)\Big)\psi(q,t) = E\psi(q,t) \qquad \forall\, t, \tag{2.2.5}$$

and its time dependence is given by

$$\psi(q,t) = \psi(q,0)e^{-i\frac{Et}{\hbar}} =: \psi_0(q)e^{-i\frac{Et}{\hbar}}. \tag{2.2.6}$$

If we now write $\psi_0(q)$ in the form

$$\psi_0(q) = R_0(q)e^{i\frac{\hat{S}_0}{\hbar}}\,, \qquad (R_0, \hat{S}_0) \in \mathbb{R}^2 \tag{2.2.7}$$

and plug eq. (2.2.6) into eq. (2.2.5), we get the stationary analogue of (2.2.4)

$$\begin{cases} \frac{1}{2m}\Big(\frac{\partial \hat{S}_0}{\partial q}\Big) + V - E - \frac{\hbar^2}{2mR_0}\frac{\partial^2 R_0}{\partial^2 q} = 0 \\ \frac{\partial}{\partial q}\big(R_0^2\frac{\partial \hat{S}_0}{\partial q}\big) = 0 \end{cases} \tag{2.2.8}$$

The relation with the variables introduced in (2.2.2) is given by

$$R(q,t) = R_0(q), \quad S(q,t) = S_0(q) - tE. \tag{2.2.9}$$

The term

$$\hat{Q} := -\frac{\hbar^2}{2mR}\frac{\partial^2 R}{\partial^2 q} \tag{2.2.10}$$

is called *Quantum Potential*, meaning that the coefficient $\hbar^2/2m$ goes to 0 when the $\hbar \to 0$ limit is considered, reducing the first equation in (2.2.4) to the classical Hamilton-Jacobi equation. This interesting fact led Bohm and others (see for example [10]) to identify the phase S defined by eq. (2.2.2) as the analogue of the classical action, building a bridge between the classical and the quantum world. The ultimate goal of the Quantum Theory of Motion they developed is furnishing an interpretation of the phenomena predicted by Quantum Mechanics which does no give up the classical concept of trajectory, and explains the probabilistic features of Quantum Mechanics in terms of uncertainty on the initial state of the system studied. The theory is articulated in the following axioms:

1. A system is composed by a wave $\psi(q,t)$ propagating in space-time and a particle, which moves along a trajectory influenced by the wave.

2. The wave ψ is a solution of the Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = \Big(-\frac{\hbar^2}{2m}\nabla^2 + V\Big)\psi. \tag{2.2.11}$$

3. In analogy with the classical result (2.2.2), the trajectory of the particle is a solution of

$$\frac{\partial q}{\partial t} = \frac{1}{m}\nabla S\big(q(t),t\big), \tag{2.2.12}$$

where S is the phase of the wave, as defined in (2.2.2).

Note that equation (2.2.2) defines the phase only modulo additive constant terms, that is $S' := S + nh$ is equivalent to S, so $S(q,t)$ is a multivalued function. Moreover, S is not defined whenever $R = 0$. The two aspects are related: moving along a continuous non intersecting loop γ in q-space we might have

$$\int_\gamma \nabla S \cdot d\gamma = nh. \tag{2.2.13}$$

This implies that S must be discontinuous somewhere within γ, and those discontinuity points can only lie on $R = 0$, because of the smoothness of ψ. We will see later that the dynamic implied by the given axioms does not conflict with the previous observations, *i.e.* trajectory never cross $R = 0$ surfaces.

The solution of the Schrödinger equation is determined by the chosen contour condition $\psi(q,0)$. This is equivalent to fixing $R(q,0)$ and $S(q,0)$. By eq. (2.2.12) this is equivalent to fixing the initial conditions for the evolution of a family of trajectories, each beginning at a point q at $t = 0$ with momentum $p = \nabla S(q,0)$. This is analogous to what happens when an arbitrary solution for the classical HJE is chosen as in (2.1.17), and corresponds to concentrating our attention on the evolution of a section of the cotangent bundle of the system, defined by

$$(q,p) \text{ s.t. } p = \nabla S(q,0)$$

rather then on the hole cotangent bundle. Once $S(q,0)$ has been chosen and $S(q,t)$ has been determined, the axioms given so far define a consistent deterministic theory of motion, in which the trajectories can be computed once the initial condition $q(0)$ is fixed. As anticipated, in order to give account of the probabilistic nature of quantum mechanics in this framework, one needs to assume a lack of knowledge on the initial position of the material particle that, together with the wave ψ, constitutes the system. Then, in a completely classical sense, one assumes that the initial condition $q(0) := q_0$ is distributed accordingly to a pdf $P_0(q_0)$. In order for the predictions of the theory to be consistent with Quantum Mechanics, this $P_0(q_0)$ must be strictly related to the wave function:

$$P_0(q_0) = R^2(q_0,0). \tag{2.2.14}$$

Here we have assumed that ψ is normalized according to $\int \psi(q,t)\psi^*(q,t)dq = 1\,\forall t$, so that $\int R^2(q,0)dq = 1$, as it should if R is to be a pdf. This assumption entails a striking consequence: the second equation in (2.2.4)

$$\frac{\partial R^2}{\partial t} + \nabla\Big(\frac{R^2\nabla S}{m}\Big) = 0 \tag{2.2.15}$$

is nothing other than the continuity equation in classical statistical mechanics for the probability distribution in a system with velocity field $v(q) = (1/m)\nabla S(q)$. Since this is exactly our case, we find that $R^2(q,t)$ represents the probability density of finding the particle in q at *any* time t (not only at $t=0$, which was our original assumption), given the initial distribution $R^2(q,0)$. Let us stress that, although in agreement with the quantum mechanical statistical interpretation of $R^2 = \psi\psi^*$, this result has a hole different meaning, since it is based on the concepts of trajectory and equation of motion, which are absent in Quantum Mechanics.

In Bohmian theory of motion one starts from Quantum Mechanics, in the form of the Schrödinger equation, and then stumbles upon the analogous to the Hamilton-Jacobi equation (2.2.4). Although very intriguing, this approach does not underline the fact that trying to establish a rigorous correspondence between (2.2.4), the trajectories it entails and classical Hamilton-Jacobi theory is actually impossible. For one thing, note that (2.2.4) is a system of coupled differential equations, therefore qualitatively different from the single classical HJE. Moreover, Bohmian trajectories do not in general reduce to classical trajectories in the $\hbar \to 0$ limit. In particular, if the Schroedinger equation admits real solutions, by (2.2.2) $S(q)$ is a constant, and therefore implies a trivial dynamic $\dot{q} = 0$ for any value of $\hbar$. Notably this is the case for the harmonic oscillator, which of course is classically described by a periodic and not necessarily trivial dynamic. In the next section we will try and solve these issues by a radically different approach. Namely, we will forget all about Quantum Mechanic and, starting from a purely classical (almost geometrical) point of view, we will build a generalization of the HJE. Only later we will realize how this generalization is actually suited to be the basis of a Quantum Mechanic.

2.3 Quantum Mechanics from a modified Hamilton-Jacobi Equation

The idea behind Hamilton-Jacobi equation is indeed pretty natural: being interested in solving the dynamical problem given by H, we want to associate to the system H a system $\tilde{H}$ which poses an easier (namely the easiest) dynamical problem, in such a way that from its solution we can answer the original question (via inverse mapping). This brought up the HJ equation for the generating function. What if we iterate the process? We are now interested in finding the solution to the HJ problem posed by H. It would be nice to be able to associate a general H to some other $\tilde{H}$ posing a less prohibitive HJ problem and to be able to retrieve from this the original solution. Being optimistic, we try the association $H^a \to H^b$ defined by a generic invertible function $q^a \to q^b(q^a)$ via the condition

$$S^b(q^b) := S^a(q^a(q^b)). \tag{2.3.1}$$

More formally, given a system defined by the potential $W^a(q^a)$, we associate it to the system defined by $W^b(q^b)$ such that iff $S_0^a(q^a)$ solves the CSHJE

$$\frac{1}{2m}\Big(\frac{\partial S_0^a}{\partial q^a}\Big)^2 + W^a(q^a) = 0, \tag{2.3.2}$$

then $S_0^b(q^b) := S_0^a(q^a(q^b))$ solves

$$\frac{1}{2m}\Big(\frac{\partial S_0^b}{\partial q^b}\Big)^2 + W^b(q^b) = 0. \tag{2.3.3}$$

Our dream would be to be able to find for any system a transformation associating it to the most trivial HJ problem, corresponding to the free Hamiltonian with vanishing energy: $W(q) \equiv 0$. But we are not so lucky: plugging (2.3.1) into (2.3.2) and using (2.3.2) we find

$$W^b(q^b) = \Big(\frac{\partial q^b}{\partial q^a}\Big)^2 W^a(q^a), \tag{2.3.4}$$

or equivalently

$$W^b(dq^b)^2 = W^a(dq^a)^2. \tag{2.3.5}$$

It is then clear that $W^0(q) \equiv 0$ is a fixed point in our transformation $W^a \to W^b$, and it is therefore not possible to map any system into the simplest one using our rule.
Nevertheless, we are not ready to give up on this idea yet, so we inquire on what kind of modifications to the CSHJE are needed in order for it to be treated accordingly to the strategy explained above. That is, we want write a modified HJE (MHJE) such that the following *Equivalence Principle* holds:

For any pair W^a, W^b there exists an invertible function $q^a(q^b)$ such that $S^b(q^b) = S^a\big(q^a(q^b)\big)$, where S^a and S^b solve the MHJE associated to W^a and W^b respectively.

We will call those functions *v-transformations*. Somewhat surprisingly, we will find that the modification needed is indeed naturally related do quantum-mechanical aspects, so we will refer to the MHJE as *Quantum Stationary Hamilton-Jacobi Equation* (QSHJE).

2.3.1 Introducing the Quantum Stationary Hamilton-Jacobi Equation

Without loss of generality, we can write the to be found QSHJE as

$$\frac{1}{2m}\Big(\frac{\partial S_0^a}{\partial q}\Big)^2 + W^a(q) + Q^a(q) = 0. \tag{2.3.6}$$

$Q^a(q)$ is the corrective term we wish to identify, the apex stresses that it will most likely depend on the system we are considering, namely on the particular W^a.
The same passages that led to eq. (2.3.4) now give

$$W^a(q^a) + Q^a(q^a) = \Big(\frac{\partial q^b}{\partial q^a}\Big)^2\Big(W^b(q^b) + Q^b(q^b)\Big) \tag{2.3.7}$$

that is, $W + Q$ transforms as a quadratic differential under *v-transformations*. This equation has the same meaning of eq.(2.3.4): it defines, this time in a possibly pretty implicit way due to the presence of the Q terms, the transformation law $W^b \to W^a$ of the system under the action of an arbitrary map $q^a(q^b)$. Is this transformation surjective? We want to find Q to make it so, *i.e.* we want Q such that given any two systems W^a, W^b eq. (2.3.7) admits a solution $q^b(q^a)$.

Since we have already ruled out a possible 2-differential transformation property of W, there must be an inhomogeneous term

$$W^a(q^a) =: \Big(\frac{\partial q^b}{\partial q^a}\Big)^2 W^b(q^b) + \big(q^b(q^a); q^a\big). \tag{2.3.8}$$

By eq. (2.3.8) we mean that the parenthesis $(\cdot,\cdot)$ on the rhs is a function of q^a defined as

$$\big(q^b(q^a); q^a\big) := W^a(q^a) - \Big(\frac{\partial q^b}{\partial q^a}\Big)^2 W^b\big(q^b(q^a)\big), \tag{2.3.9}$$

where $q^b(q^a)$ is the particular function that solves $S_0^b(q^b) := S_0^a(q^a(q^b))$, namely $q^b = S_0^{b^{-1}} \circ S_0^a$. The notation ASSUMES that $(\cdot,\cdot)$ depends only on the function $q^b(q^a)$ and not specifically on the couple (W^a, W^b). Which is to say, for any couple of systems related by the same $q^b(q^a)$

$$W^a(q^a) - \Big(\frac{\partial q^b}{\partial q^a}\Big)^2 W^b(q^b)$$

is the same function of q^a (or q^b). This assumption, that we will show to be consistent with the EP, is the second pillar of this formulation.

Eq. (2.3.7) then gives the transformation rule of Q

$$Q^a(q^a) = \Big(\frac{\partial q^b}{\partial q^a}\Big)^2 Q^b(q^b) - \big(q^b(q^a); q^a\big). \tag{2.3.10}$$

If we specialize eq. (2.3.9) to the case $W^b \equiv 0$ we get, with a slight change of notation

$$W(q) = \big(q^0(q); q\big), \tag{2.3.11}$$

that is, the set of theories that can be transformed into the free one is given by the image of $\big(\cdot, q\big)$. Since we are considering invertible functions $q^b(q^a)$, we see that asking to be able to move between any two potentials $W^a \leftrightarrow W^b$ is equivalent to asking to be able to move to $W^0 \equiv 0$ from any W. So $\big(\cdot, \cdot\big)$ must be such that eq. (2.3.11) admits a solution $\forall\, W$.

2.3.2 The cocycle condition and the Schwarzian derivative

Lets consider the couple of equations

$$W^a(q^a) = \Big(\frac{\partial q^b}{\partial q^a}\Big)^2 W^b(q^b) + \big(q^b(q^a); q^a\big) \quad \wedge \quad W^b(q^b) = \Big(\frac{\partial q^a}{\partial q^b}\Big)^2 W^a(q^a) + \big(q^a(q^b); q^b\big) \tag{2.3.12}$$

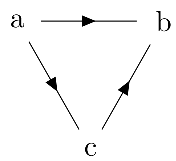
They correspond to the obvious fact that if $q^a(q^b)$ sends $W^a \to W^b$, then the inverse function $q^b(q^a)$ sends $W^b \to W^a$. Together they imply

$$(q^b; q^a) = -\Big(\frac{\partial q^b}{\partial q^a}\Big)^2 (q^a; q^b). \tag{2.3.13}$$

In particular

$$(q; q) = 0. \tag{2.3.14}$$

Now consider transforming W^a into W^b by an intermediate step W^c, according to the following commuting diagram:



Algebraically, it amounts to

$$\begin{aligned} W^b(q^b) &= \Big(\frac{\partial q^a}{\partial q^b}\Big)^2 W^a(q^a) + \big(q^a(q^b); q^b\big), \\ W^b(q^b) &= \Big(\frac{\partial q^c}{\partial q^b}\Big)^2 W^c(q^c) + \big(q^c(q^b); q^b\big) \\ &= \Big(\frac{\partial q^c}{\partial q^b}\Big)^2 \Big[\Big(\frac{\partial q^a}{\partial q^c}\Big)^2 W^a(q^a) + \big(q^a(q^c); q^c\big)\Big] + \big(q^c(q^b); q^b\big), \end{aligned} \tag{2.3.15}$$

that is

$$\big(q^a(q^b(q^c)); q^c\big) = \Big(\frac{\partial q^b}{\partial q^c}\Big)^2 \big(q^a; q^b(q^c)\big) - \big(q^b(q^c); q^c\big).$$

This notation is a bit cumbersome, so we will simplify it where there is no risk of confusion to

$$(q^a; q^c) = \Big(\frac{\partial q^b}{\partial q^c}\Big)^2 (q^a; q^b) - (q^b; q^c) \tag{2.3.16}$$

or, using (2.3.13),

$$(q^a; q^c) = \Big(\frac{\partial q^b}{\partial q^c}\Big)^2 \Big[(q^a; q^b) - (q^c; q^b)\Big]. \tag{2.3.17}$$

From (2.3.13) and (2.3.17) alone it is possible to individuate the explicit expression for $(\circ; q)$ in an essentially unique way. To this end, we start by proving the following lemma:

Lemma 2.3.1. *Condition (2.3.13) imply the Möbius invariance of* $(q^a; q^b)$

$$\big(\gamma(q^a); q^b\big) = (q^a; q^b), \tag{2.3.18}$$

$$\textit{where } \gamma(q) = \frac{Aq + B}{Cq + D}, \quad AD - BC \neq 0, \quad A, B, C, D \in \mathbb{C}.$$

Proof. We will proceed in three steps:

1. First of all, recall that we are looking for a QSHJE that depends only on the derivative of the action S_0. Since the scalar condition eq. (2.3.1) gives for the trivializing coordinate $q^0(q) = S_0^{0^{-1}} \circ S_0(q)$, eq. (2.3.11) implies that $(f(x); x)$ MUST depend only on the derivatives of f. It follows that

   $$(q + B; q) = (q; q).$$

   On the other hand, by (2.3.13), $(q; q + B) = -\Big[\frac{\partial}{\partial q}(q - B)\Big]^2 (q + B; q)$, so we find

   $$(q + B; q) = (q; q) = (q, q + B). \tag{2.3.19}$$

2. Using again (2.3.13) and (2.3.14) we have $(Aq; q) = 0 = (q; Aq)$. From this and the cocycle condition (2.3.17) we get

   $$(q^a; Aq^b) = A^{-2}\big((q^a; q^b) - (Aq^b; q^b)\big) = A^{-2}(q^a; q^b). \tag{2.3.20}$$

   Together with eq. (2.3.13) it implies

   $$\begin{cases} (Aq^a; q^b) = -A^2(\partial_{q^b} q^a)^2 (q^b; Aq^a) \\ (q^b; A^{-1}q^c) = A^2(q^b; c^c) \end{cases} \Rightarrow (Aq^a; q^b) = -(\partial_{q^b} q^a)^2 (q^b; q^a) = (q^a; q^b). \tag{2.3.21}$$

3. Consider now $f(q) := q^{-2}(q; q^{-1})$, and notice that by (2.3.13) and (2.3.21) it holds $f(Aq) = -f(q^-1)$. Setting $A = 1$ implies

   $$(q; q^{-1}) = 0 = (q^{-1}; q). \tag{2.3.22}$$

   Finally, using (2.3.17) and (2.3.22) we get

   $$\big((q^a)^{-1}; q^b\big) = (q^a; q^b). \tag{2.3.23}$$

Since translations, dilatations and inversion generate the Möbius group, points 1 , 2 and 3 prove the statement. □

This property points us towards a candidate for $(q^a; q^b)$: the Schwarzian derivative satisfies (2.3.18):

$$\{\gamma(q^a); q^b\} = \{q^a; q^b\} \quad ; \quad \gamma(q) = \frac{Aq+B}{Cq+D}, \quad AD-BC \neq 0, \quad A,B,C,D \in \mathbb{C}.$$

We can even go further, since M. Matone [8] managed to prove that indeed conditions (2.3.13) and (2.3.18), which are direct consequences of the EP, essentially define the Schwarzian derivative. The proof of this theorem requires the following lemma:

Lemma 2.3.2. *The cocycle condition (2.3.17) admits a non trivial solution $(q^a; q^b)$ depending only on the derivatives of q^a only if, in the case of an infinitesimal transformation*

$$q^a = q^b + \epsilon^{ab}(q^b) := q^b + \epsilon\, f^{ab}(q^b),$$

it holds

$$(q^a; q^b) = c_1 \epsilon^{ab'''}(q) + \mathcal{O}(\epsilon^2), \ \textit{with } c_1 \neq 0.$$

Proof. Consider the case in which $q^a(q) = q + \epsilon f(q)$, with ϵ an infinitesimal parameter. Since $(q^a; q)$ depends only on the first and higher order derivatives of $q^a(\cdot)$, we have

$$(q + \epsilon f(q); q) = \epsilon^h c_1 f^{(k)}(q) + \mathcal{O}(\epsilon^{h+1}). \tag{2.3.24}$$

for some c_1, possibly equal to zero. Here $f^{(k)}(q) := \partial_q^k f(q)$ denotes the k-th derivative. Consider for a start the case $h = 1$. Other possibilities will be discussed at the end of the proof. Eq. (2.3.24) also assumes that $(q^a; q^b)$ is not identically null. This must be the case, for we seek compatibility with the EP.
Let us fix the value of k: note that due to Lemma (2.3.1) it holds

$$(Aq + A\epsilon f(q); Aq) = (q + \epsilon f(q); Aq) = A^{-2}(q + \epsilon f(q); q). \tag{2.3.25}$$

On the other hand, setting $F(Aq) = Af(q) \Leftrightarrow F(q) = Af(A^{-1}q)$, by eq. (2.3.24) we have

$$\begin{aligned}(Aq + A\epsilon f(q); Aq) &= (Aq + F(Aq); Aq) = \epsilon c_1 \partial_{Aq} F(Aq) + \mathcal{O}(\epsilon^2) \\ &= \epsilon c_1 A^{1-k} f^{(k)}(q) + \mathcal{O}(\epsilon^2).\end{aligned} \tag{2.3.26}$$

Consistency then requires $K = 3$ and $(q + \epsilon f(q); q) = \epsilon c_1 f^{(3)}(q) + \mathcal{O}(\epsilon^2)$.

The scaling property (2.3.26) generalizes to higher order in ϵ. In particular the contribution of order ϵ^n will be a sum of terms like

$$c_{i_1 \dots i_n} \epsilon \partial_{Aq}^{(i_1)} F(Aq) \cdots \epsilon \partial_{Aq}^{(i_n)} F(Aq) = c_{i_1 \dots i_n} \epsilon^n A^{n - \sum i_k} \partial^{(i_1)} f(q) \cdots \partial^{(i_n)} f(q) \tag{2.3.27}$$

so that by (2.3.25)

$$\sum_{k=1}^{n} i_k = 2 + n. \tag{2.3.28}$$

On the other hand, since $\left(q^a(q^b); q^b\right)$ depends only on the derivatives of q^a, we have $i^k > 1$, $k \in [1, n]$. It follows that either

$$i_3 = 3, \quad i_{k \neq 3} = 1 \tag{2.3.29}$$

or

$$i_k = i_j = 2, \quad i_{h \neq (k,j)} = 1 \tag{2.3.30}$$

so that

$$(q+\epsilon f(q);q)=\sum_{n=1}^{\infty}\epsilon^n\big(c_3 f^{(3)}f^{(1)^{n-1}}+d_n f^{(2)^2}f^{(1)^{n-2}}\big)\,,\quad d_1=0. \tag{2.3.31}$$

We now impose the cocycle condition, which implies that either $c_1\neq 0$ or $(q+\epsilon f(q);q)=0$. Define

$$q^b=v^{ba}(q^a)\,,\quad q^c=v^{cb}(q^b)=v^{cb}\circ v^{ba}(q^a)\quad q^c=v^{ca}(q^a). \tag{2.3.32}$$

Note that $v^{ab}={v^{ba}}^{-1}$ and $v^{ca}=v^{cb}\circ v^{ba}$. Now consider the infinitesimal form of these relations:

$$q^b=q^a+\epsilon^{ba}(q^a)\,,\quad q^c=q^b+\epsilon^{cb}(q^b)=q^b+\epsilon^{cb}(q^a+\epsilon^{ba}(q^a))\,,\quad q^c=q^a+\epsilon^{ca}(q^a). \tag{2.3.33}$$

Confronting the first eq. in (2.3.33) with $q^b=q^a-\epsilon^{ba}(q^a)$ we get

$$\epsilon^{ba}+\epsilon^{ab}\circ(\mathbb{1}+\epsilon^{ba})=0, \tag{2.3.34}$$

so that at first order

$$\epsilon^{ab}=-\,\epsilon^{ba}\,. \tag{2.3.35}$$

More generally, eq. (2.3.33) implies

$$\epsilon^{ca}(q^a)=\epsilon^{cb}(q^b)+\epsilon^{ba}(q^a)=\epsilon^{cb}(q^b)-\epsilon^{ab}(q^b), \tag{2.3.36}$$

so that

$$\epsilon^{ca}=\epsilon^{cb}\circ(\mathbb{1}+\epsilon^{ba})+\epsilon^{ba}=(\mathbb{1}+\epsilon^{cb})\circ(\mathbb{1}+\epsilon^{ba})-\mathbb{1}. \tag{2.3.37}$$

At first order this reads

$$\epsilon^{ca}=\epsilon^{cb}+\epsilon^{ba} \tag{2.3.38}$$

Consider now the case in which $\epsilon^{xy}(q^y)=\epsilon f^{xy}(q^y)$ with ϵ infinitesimal and impose the cocycle condition (2.3.17). Since

$$(q^a;q^b)=c_1{\epsilon^{ab}}'''(q^b)+\mathcal{O}^{ab}(\epsilon^2) \tag{2.3.39}$$

it reads

$$c_1{\epsilon^{ac}}'''(q^c)+\mathcal{O}^{ac}(\epsilon^2)=(1+{\epsilon^{bc}}'(q^c))^2\big(c_1{\epsilon^{ab}}'''(q^b)-c_1{\epsilon^{cb}}'''(q^b)+\mathcal{O}^{ab}(\epsilon^2)-\mathcal{O}^{cb}(\epsilon^2))\big), \tag{2.3.40}$$

where $\mathcal{O}^{xy}(\epsilon^2)$ denotes the second order contribution coming from f^{xy}. This implies $c_1\neq 0$. For, if $c_1=0$, eq. (2.3.40) reads

$$\mathcal{O}^{ac}(\epsilon^2)=\mathcal{O}^{ab}(\epsilon^2)-\mathcal{O}^{cb}(\epsilon^2)). \tag{2.3.41}$$

But, form (2.3.31) we get

$$\mathcal{O}^{ab}(\epsilon^2)=c_2{\epsilon^{ab}}'''(q^b){\epsilon^{ab}}'(q^b)+d_2{\epsilon^{ab}}''^2(q^b)+\mathcal{O}^{ab}(\epsilon^3), \tag{2.3.42}$$

which makes (2.3.41) inconsistent with (2.3.38).

A similar analysis allows to exclude the case in which the lowest order contribution to $(\cdot;q)$ is of order $\mathcal{O}(\epsilon^h)$, $h\geq 2$, because in this case condition (2.3.17) cannot be consistent with the linearity of (2.3.38). So our starting assumption $h=1$ is justified.

□

We are now ready to show that

Theorem 2.3.1. *Conditions (2.3.13) and (2.3.18) univocally define the Schwarzian derivative.*

$$(q^a;q^b)=-\frac{\beta^2}{4m}\{q^a;q^b\}. \tag{2.3.43}$$

The peculiar form of the multiplicative constant in eq. (2.3.43) will be made clearer in section (2.3.4).

Proof. Note that the cocycle condition (2.3.17), and consequently lemma (2.3.2), are satisfied by

$$[q^a; q^b] := (q^a; q^b) - c_1\{q^a; q^b\}. \tag{2.3.44}$$

Hence, with the same notation as in (2.3.24), it holds

$$[q^a; q^b] = \tilde{c}_1\, \epsilon\, f^{(3)}(q) + \mathcal{O}(\epsilon^2). \tag{2.3.45}$$

However, since $[q^a; q^b] = \epsilon\, f^{(3)}(q) + \mathcal{O}(\epsilon^2))$, we have $\tilde{c}_1 = 0$ and by the lemma

$$[q^a; q^b] \equiv 0.$$

□

This result allows us to give an explicit expression for the transformation properties of any system

$$W^a(q^a) = \Big(\frac{\partial q^b}{\partial q^a}\Big)^2 W^b(q^b) - \frac{\beta^2}{4m}\{q^b; q^a\}. \tag{2.3.46}$$

As we already explained, this is equivalent to considering only the case $W^b \equiv 0$, which reads

$$W(q) = -\frac{\beta^2}{4m}\{q^0; q\}. \tag{2.3.47}$$

2.3.3 Writing the Quantum Stationary Hamilton-Jacobi Equation

The central result (2.3.43) is a consequence of the cocycle condition (2.3.17) alone, which in turn comes from our assumption on the form of $(\circ\,; q)$. Now we need to ask: is there a suitable modification Q to the QSHJE such that its solutions are consistent with the EP and with eq. (2.3.46)? We will find that the answer is Yes! and that the solution is unique.

To this end, consider the known property of the Schwarzian derivative

$$\{f \circ g; x\} = \{f; g(x)\}\Big(\frac{\partial g}{\partial x}\Big)^2 + \{g; x\} \tag{2.3.48}$$

which implies

$$\{e^{i\alpha h}; x\} = \Big(\frac{\partial h}{\partial x}\Big)^2 \frac{\alpha^2}{2} + \{h; x\}, \qquad \alpha \equiv \text{constant}. \tag{2.3.49}$$

This allows us to rewrite a squared derivative as

$$\Big(\frac{\partial S_0}{\partial q}\Big)^2 = \frac{\beta^2}{2}\big(\{e^{\frac{2iS_0}{\beta}}; q\} - \{S_0; q\}\big), \tag{2.3.50}$$

and plugging (2.3.50) into the QSHJE (2.3.6) we get

$$\frac{\beta^2}{2m}\Big(\{e^{\frac{2i}{\beta}S_0}; q\} - \{S_0; q\}\Big) + W + Q = 0. \tag{2.3.51}$$

Given those cosmetic adjustment we can write our consistency condition as follows.

For each couple of systems (W^a, W^b) there exists an invertible map $q^a(q^b)$ such that

$$\frac{\beta^2}{2m}\Big(\{e^{\frac{2i}{\beta}S_0^a};q^a\}-\{S_0^a;q^a\}\Big)+W^a+Q(S^a,q^a)=0,\quad \frac{\beta^2}{2m}\Big(\{e^{\frac{2i}{\beta}S_0^b};q^b\}-\{S_0^b;q^b\}\Big)+W^b+Q(S^b,q^b)=0$$
$$S^b(q^b)=S^a\big(q^a(q^b)\big),\quad W^a(q^a)=\Big(\frac{\partial q^b}{\partial q^a}\Big)^2W^b(q^b)-\frac{\beta^2}{4m}\{q^b;q^a\}. \tag{2.3.52}$$

The first two equations define the relation between S and W, the third is nothing but the EP and the last one comes from the cocycle condition. Now proceed as follows:

$$\begin{aligned}
(2.3.52)\Rightarrow 0 &= \frac{\beta^2}{2m}\Big(\{e^{\frac{2i}{\beta}S_0^b};q^b\}-\{S_0^b;q^b\}\Big)+W^b+Q(S^b,q^b)\\
&=\frac{\beta^2}{2m}\Big(\{e^{\frac{2i}{\beta}S_0^a\left(q^a(q^b)\right)};q^b\}-\{S_0^b;q^b\}\Big)+W^b+Q(S^b,q^b)\\
&=\frac{\beta^2}{2m}\Big[\{e^{\frac{2i}{\beta}S_0^a};q^a(q^b)\}\Big(\frac{\partial q^a}{\partial q^b}\Big)^2+\{q^a;q^b\}-\{S_0^b;q^b\}\Big]+W^b+Q(S^b,q^b)\\
&=\Big[\frac{\beta^2}{2m}\{S_0^a;q^a\}-W^a-Q(S^a,q^a)\Big]\Big(\frac{\partial q^a}{\partial q^b}\Big)^2+\Big(\frac{\partial q^a}{\partial q^b}\Big)^2\Big[\{q^a;q^b\}-\{S_0^b;q^b\}\Big]\\
&\quad+W^b+Q(S^b,q^b)\\
&=\Big[\frac{\beta^2}{2m}\{S_0^a;q^a\}-Q(S^a,q^a)-\Big(\frac{\partial q^b}{\partial q^a}\Big)^2W^b(q^b)+\frac{\beta^2}{4m}\{q^b;q^a\}\Big]\Big(\frac{\partial q^a}{\partial q^b}\Big)^2\\
&\quad+\Big(\frac{\partial q^a}{\partial q^b}\Big)^2\Big[\{q^a;q^b\}-\{S_0^b;q^b\}\Big]+W^b+Q(S^b,q^b).
\end{aligned} \tag{2.3.53}$$

In the second line we use the scalar property of S, in the third line the properties of the Schwarzian derivative, in the fourth we used the first equation in (2.3.52) and finally the fifth equivalence comes from the last equation in (2.3.52). Via simple algebra we get from (2.3.53)

$$Q(S_0^b,q^b)-Q(S_0^a,q^a)\Big(\frac{\partial q^a}{\partial q^b}\Big)^2=\frac{\beta^2}{2m}\Big[\{S_0^b;q^b\}-\{S_0^a,q^a\}\Big(\frac{\partial q^a}{\partial q^b}\Big)^2\Big]. \tag{2.3.54}$$

The structure of this equation is that of

$$f(x)-f\circ g(x)\equiv h(x)-h\circ g(x), \tag{2.3.55}$$

from which follows $f=h$, that is

$$Q(S_0,q)=\frac{\beta^2}{2m}\{S_0;q\}. \tag{2.3.56}$$

With this result, our finally completely specified QSHJE reads

$$\frac{1}{2m}\Big(\frac{\partial S_0}{\partial q}\Big)^2+W(q)+\frac{\beta^2}{2m}\{S_0;q\}=0 \tag{2.3.57}$$

or, from (2.3.51)

$$W(q)=-\frac{\beta^2}{2m}\{e^{\frac{2i}{\beta}S_0};q\}. \tag{2.3.58}$$

2.3.4 Why Quantum Hamiltom-Jacobi equation, and how to solve it

From eq. (2.3.57) we immediately see that the parameter β regulates the deviation of our theory from the CSHJE. Its dimensions are those of an action, and you would be tempted to identify it with the Plank constant $\beta = \hbar$. We will now show that such an identification rests upon much more profound bases than this suggestion.

Consider the following identity:

$$h'^{1/2}\frac{\partial}{\partial x}\frac{1}{h'}\frac{\partial}{\partial x}h'^{1/2} = \frac{\partial^2}{\partial x^2} + \frac{1}{2}\{h, x\} \tag{2.3.59}$$

where h' denotes the first derivative and h is any function with nowhere vanishing derivative. This shows that a possible basis for the two-dimensional kernel of the second order differential operator on the rhs is given by

$$\frac{\partial}{\partial x}h'^{1/2}f_1(x) = \text{const} \quad \text{and} \quad \frac{1}{h'}\frac{\partial}{\partial x}h'^{1/2}f_2(x) = \text{const} \tag{2.3.60}$$

that is

$$f_1 = (h')^{-1/2} \quad \text{and} \quad f_2 = (h')^{-1/2}h. \tag{2.3.61}$$

Their ratio is $f_2/f_1 = h$. If instead we take the ratio f of two generic independent elements of the kernel, we have

$$f = \frac{Af_1 + Bf_2}{Cf_1 + Df_2} = \frac{Af_1/f_2 + B}{Cf_1/f_2 + D}, \qquad AD - BC \neq 0. \tag{2.3.62}$$

The condition on the coefficients, which comes from the linear independence of numerator and denumerator, implies that f is a Möbius transform of h. Then, due to the symmetry properties of the Schwarzian derivative it holds

$$f = \frac{Af_1 + Bf_2}{Cf_1 + Df_2} \Leftrightarrow \{f; x\} = \{h; x\}. \tag{2.3.63}$$

Consider now the two differential equations

$$\begin{cases} \{h, x\} = V(x) \\ \left(\frac{\partial^2}{\partial x^2} + \frac{1}{2}\{h, x\}\right)\psi_{1,2} = 0 \end{cases} \Leftrightarrow \begin{cases} \{h, x\} = V(x) \\ \left(\frac{\partial^2}{\partial x^2} + \frac{1}{2}V(x)\right)\psi_{1,2} = 0 \end{cases} \tag{2.3.64}$$

where ψ_1 and ψ_2 are any two independent solutions. From what we said above this is equivalent to

$$\begin{cases} h = \frac{A\frac{\psi_1}{\psi_2} + B}{C\frac{\psi_1}{\psi_2} + C} \\ \left(\frac{\partial^2}{\partial x^2} + \frac{1}{2}V(x)\right)\psi_{1,2} = 0 \end{cases} \tag{2.3.65}$$

where $AD - BC \neq 0$. This allows us to map the nonlinear third-order differential problem posed by $\{h; x\} = V(x)$ to the linear (simpler) one of finding the kernel of the linear differential operator in the last line of (2.3.65).

Applying this technique to the QSHJE, we find that

$$W(q) = -\frac{\beta^2}{2m}\{e^{\frac{2i}{\beta}S_0}; q\} \tag{2.3.66}$$

if and only if

$$e^{\frac{2i}{\beta}S_0} = \gamma\Big(\frac{\psi^D}{\psi}\Big), \tag{2.3.67}$$

where γ is an arbitrary Möbius transformation and ψ^D, ψ solve the Schrödinger-like equation

$$\Big(- \frac{\beta^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\Big)\psi = E\psi. \tag{2.3.68}$$

With this in mind, we are finally convinced to identify $\beta = \hbar$, and we will refer to the limit $\beta \to 0$ as the *classical limit*. This justifies the name *Quantum Hamilton-Jacobi Equation*.

It is also possible to proceed the other way around, solving the QSHJE and then recovering the solutions to the associated Schrödinger equation. In fact we know that any solution S_0 is in the form (2.3.67), from which follows that

$$\psi_1 := \frac{e^{\frac{2i}{\beta}S_0}}{\sqrt{S_0'}} \qquad \text{and} \qquad \psi_2 := \frac{e^{-\frac{2i}{\beta}S_0}}{\sqrt{S_0'}} \tag{2.3.69}$$

solve (2.3.68) and form a basis for the kernel:

$$\psi = \frac{1}{\sqrt{S_0'}}\Big(Ae^{\frac{2i}{\beta}S_0} + Be^{-\frac{2i}{\beta}S_0}\Big). \tag{2.3.70}$$

2.3.5 Solutions of the QSHJE and trivializing map

Now that we found a HJ theory where the EP can be consistently implemented, we turn to studying its solutions, giving a more precise meaning to our initial goal, that is finding the trivializing coordinate $q^0(q^a)$ that maps $W^a \to W^0 \equiv 0$. As in standard HJ theory, where solving the dynamical problem automatically leads to the solution of the CHJE, we will see that solving the QHJE is equivalent to finding such a *trivializing map*.

In the previous section we found that the solution to the QSHJE is not unique. Namely, eq. (2.3.67) reads

$$e^{\frac{2i}{\hbar}S_0} = \gamma\Big(\frac{\psi^D}{\psi}\Big),$$

with γ an arbitrary Möbius transformation. This fact is reflected in the arbitrariness we faced in the solving procedure: choosing instead of ψ and ψ^D two different independent solutions

$$\tilde{\psi}^D = A\psi^D + B\psi\,, \qquad \tilde{\psi} = C\psi^D + D\psi\,, \quad AD - BC \neq 0 \tag{2.3.71}$$

gives

$$e^{\frac{2i}{\hbar}\tilde{S}_0} = \gamma\Big(\frac{\tilde{\psi}^D}{\tilde{\psi}}\Big) = \gamma\Big(\frac{A\psi^D + B\psi}{C\psi^D + D\psi}\Big). \tag{2.3.72}$$

The condition $AD - BC \neq 0$ that guaranties the independence of ψ and ψ^D identifies the rightmost term in eq. (2.3.72) as a composition of Möbius maps, which is of course a Möbius map. In other words, throughout the next sections we need not to consider the full arbitrariness in both the choice of the γ transform in eq. (2.3.67) and in the choice of the basis $\{\psi^D, \psi\}$, which would be redundant. In what follows we will then concentrate, without loss of generality, on the $\{\psi^D, \psi\}$ such that $\omega := \psi^D/\psi \in \mathbb{R}$.

The fact that the solution to the QSHJE is not unique compels us to revise our definition of trivializing map. In fact, the definition

$$q^b(q^a) \text{ s.t. } S^b(q^b) = S^a\big(q^a(q^b)\big), \tag{2.3.73}$$

coming from the EP, together with the multiplicity of solutions, suggests to relate the trivializing map to the particular solutions we chose to the QSHJE for the systems W^a and W^b rather than to W^a and W^b themselves.
This can also be seen explicitly as follows: imposing $S^b(q^b) = S^a\big(q^a(q^b)\big)$ and using the properties of the Schwarzian derivative we have

$$\{e^{\frac{2i}{\hbar}S_0^b}; q^b\} = \{e^{\frac{2i}{\hbar}S_0^a(q^a(q^b))}; q^b\} = \Big(\frac{\partial q^a}{\partial q^b}\Big)^2 \Big[\{e^{\frac{2i}{\hbar}S_0^a}; q^a\} - \{q^b; q^a\}\Big]. \tag{2.3.74}$$

By (2.3.6) this is equivalent to

$$W^b(q^b) = \Big(\frac{\partial q^a}{\partial q^b}\Big)^2 \Big[\{W^a(q^a) + \frac{\hbar^2}{4m}\{q^b; q^a\}\Big]. \tag{2.3.75}$$

We see that

$$W^b(q^0) \equiv 0 \qquad \Leftrightarrow \{q^0; q^a\} = \{e^{\frac{2i}{\hbar}S_0^a}; q^a\}, \tag{2.3.76}$$

that is

$$q^0 = \frac{Ae^{\frac{2i}{\hbar}S_0^a} + B}{Ce^{\frac{2i}{\hbar}S_0^a} + D}. \tag{2.3.77}$$

Hence, requiring $W^b \equiv 0$ is a weaker condition than $S^b(q^b) = S^a\big(q^a(q^b)\big)$: the solution is not unique, and the plurality of solutions corresponds to the plurality of solutions $S^0(q^0)$ for the system $W^0 \equiv 0$. In particular, given a solution S_0^a for the QSHJE for the potential W^a, the trivializing coordinates $q^0(q^a)$ mapping S^a, via $S^0(q^0) := S^a\big(q^a(q^0)\big)$, into some solution of the QSHJE for $W^0 \equiv 0$ are all the Möbius transformation of S_0^a.

The plurality of solutions to the QSHJE can be understood noticing that it is a third order differential equation, so there are three integration constants that specify a S_0. Since S_0 should be a real function, it must hold

$$1 = \Big|e^{\frac{2i}{\hbar}S_0(q)}\Big| = \Big|\frac{A\psi^D + B\psi}{C\psi^D + D\psi}\Big| = \Big|\frac{A}{C}\Big|\Big|\frac{\omega(q) + \frac{B}{A}}{\omega(q) + \frac{D}{C}}\Big|, \quad q \in \mathbb{R}. \tag{2.3.78}$$

Differentiating with respect to q we get a condition on the coefficients A, B, C and D, namely

$$\frac{B}{A} = \frac{\bar{D}}{\bar{C}} \tag{2.3.79}$$

which in turn implies, setting $B/A =: il$

$$\Big|\frac{\omega(q) + \frac{B}{A}}{\omega(q) + \frac{D}{C}}\Big| = \Big|\frac{\omega(q) + i\bar{l}}{\omega(q) - il}\Big| = 1 \Rightarrow \Big|\frac{A}{C}\Big| = 1. \tag{2.3.80}$$

To summarize, without loss of generality our $S_0 \in \mathbb{R}$ is given by

$$e^{\frac{2i}{\hbar}S_0\{\delta\}} = e^{i\alpha}\frac{\omega + i\bar{l}}{\omega - il} \tag{2.3.81}$$

where $\{\delta\} := \{\alpha, l\}$ with α a real integration constant and $l = l_1 + il_2$. We will refer to the family of solutions parametrized in this way as *delta moduli*. Note that in terms of δ parameters, the necessary condition for $\{S_0; q\}$ to be defined is $l_1 \neq 0$.

All this reasoning is of course compatible with a possible different choice of the functions ψ^D, ψ that build up $w(q)$, provided the new $\tilde{w}(q) \in \mathbb{R}$. Changing w with this constraint corresponds to a *real* Möbius transform of w, *i.e.* a PSL$(2, \mathbb{R})$ transform

$$w \to \tilde{w} := \frac{aw + b}{cw + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}). \tag{2.3.82}$$

It is immediate to find that in terms of δ moduli this reflects in

$$e^{\frac{2i}{\hbar}S_0\{\delta\}} \to e^{\frac{2i}{\hbar}S_0\{\delta'\}} = e^{i\alpha}\frac{a + i\bar{l}c}{a - ilc}\frac{\omega + i(\bar{l}d - ib)(a + i\bar{l}c)}{\omega - i(ld + ib)(a - ilc)}, \tag{2.3.83}$$

that is

$$e^{i\alpha'} = e^{i\alpha}\frac{a + i\bar{l}c}{a - ilc}, \quad l' = \frac{ld + ib}{a - ilc}. \tag{2.3.84}$$

It is then possible to write explicitly the trivializing map $q^0(q)$ in terms of the δ moduli chosen for the systems W^0 and W. Choosing

$$\psi^{D^0}(q^0) := q^0 \text{ and } \psi^0 := 1 \tag{2.3.85}$$

from eq. (2.3.73) we get

$$e^{i\alpha_0}\frac{q^0 + i\bar{l}_0}{q^0 - il_0} = e^{i\alpha}\frac{\omega + i\bar{l}}{\omega - il}, \tag{2.3.86}$$

that is

$$q^0 = \frac{(l_0 e^{i\beta} + \bar{l}_0 e^{-i\beta})\omega + il_0\bar{l}e^{i\beta} - i\bar{l}_0 l e^{-i\beta}}{2\omega\sin\beta + le^{-i\beta} + \bar{l}e^{i\beta}}, \tag{2.3.87}$$

where $\beta = (\alpha - \alpha_0)/2$. Note that since we restricted our attention to $S_0 \in \mathbb{R}$, the trivializing map is real, as it should be. These relations will be put into use in the next section, where we focus on the Quantum Mechanics stemming from the QSHJE.

2.4 The physics behind this geometrical approach

In the next sections we explore some interesting features of the QSHJE. To begin with, we analyze its relation to the older quantum stationary Hamilton-Jacobi equation (OQSHJE) referred to section 2.2. This will show how our version is much better suited to generalize the HJE, for its classical limit is indeed the HJE, whereas this is not always true for the OQSHJE. Then we press on, studying what kind of quantum picture one gets if he takes seriously the analogies between the QSHJE, classical Quantum Mechanics and classical HJ theory. We will start by identifying (*à la* Bohm) the conjugate momentum with $p = \nabla S$, which will introduce a kind of Quantum tunneling. Then we will study the spectral properties of the *energy* of a system, understood as the E eigenvalue of the associated Schrödinger equation. Surprisingly, we will find that the EP alone is enough to impose a discrete energy spectrum. This is very different from Quantum Mechanics where the discrete spectrum stems from imposing a probabilistic interpretation of the wave function, in the form of $\psi \in L^2(\mathbb{R})$. Then we will inquire about the possibility of introducing trajectories in this framework, based on a more general assumption than $\dot{q} = p/m$, which characterizes Bohmian mechanics.

2.4.1 The classical limit

Let us recall the origin of the OQSHJE, *i.e.* that imposing the definition

$$\psi(q) = R(q)e^{\frac{i}{\hbar}\hat{S}_0(q)}, \quad (R, \hat{S}_0) \in \mathbb{R}^2 \tag{2.4.1}$$

turns the stationary Schrödinger equation for the wave function

$$\Big(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)\Big)\psi(q) = E\psi(q) \tag{2.4.2}$$

into

$$\begin{cases} \frac{1}{2m}\Big(\frac{\partial \hat{S}_0}{\partial q}\Big)^2 + V - E - \frac{\hbar^2}{2mR}\frac{\partial^2 R}{\partial^2 q} = 0 \\ \frac{\partial}{\partial q}\Big(R^2\frac{\partial \hat{S}_0}{\partial q}\Big) = 0 \end{cases} \tag{2.4.3}$$

We will use the notation $\hat{Q} := -\frac{\hbar^2}{2mR}\frac{\partial^2 R}{\partial^2 q}$ and refer to the second equation in (2.4.3) as *continuity equation.*

There are many issues in identifying this as the quantum analog of the HJE. First of all, the first line of (2.4.3) makes sense only together with the second line, which in turn doesn't depend on $\hbar$. So there is no way to turn the second line into a trivial condition by taking the classical limit, and even in this limit you have to deal with two equations, which is very different from Classical Mechanics.
The second issue concerns the $\hbar \to 0$ limit. To begin with, we have to agree on what we mean by "limit" for a system of differential equation depending on a parameter a (SDE_a). If we interpret the SDE as an implicit way of defining a set of functions, namely its solutions, a natural definition of $\lim_{a\to 0} SDE_a =: SDE_0$ would be a system of differential equations whose solutions are the $a \to 0$ limits of the solutions of SDE_a. Finding such a SDE_0 is a more involved procedure than just taking the $a \to 0$ limit of the equations defining SDE_a. The reason is that each equation is sensible not only on the explicitly written a parameters but also tho the ones coming implicitly from the solution of the other equations. Eq. (2.4.3) is an example of this: it is true that if we neglect the $\hat{Q}$ term in the first line it turns into the CHJE. On the other hand, any system allowing real eigenfunctions for the Schrödinger eq. (2.4.2) will have, from the definition (2.4.1), a constant value for $\hat{S}_0$:

$$\hat{S}_0(q) \equiv k\hbar\pi,\, k \in \mathbb{N} \quad \Longrightarrow \lim_{\hbar\to 0}\hat{S}_0(q) = 0. \tag{2.4.4}$$

The classical limit of the solutions is then trivial, whereas the CHJE might admit far from trivial solutions. A notable example of this is the quantum harmonic oscillator. In this case the CHJE reads

$$\frac{\partial S_0^{cl}}{\partial q} = \pm m\omega(a^2 - q^2)^{\frac{1}{2}}, \tag{2.4.5}$$

where $a = (2E/m\omega^2)^{\frac{1}{2}}$ is the amplitude of motion. Instead, in the quantum case we have

$$\hat{S}_0 = 0 \tag{2.4.6}$$

and

$$\hat{Q} = \frac{\hbar^2}{2mR}\frac{\partial^2 R}{\partial^2 q} = E_n - V, \tag{2.4.7}$$

where $E_n = (n+1/2)\hbar\omega$ spans the spectrum of H. The two descriptions have nothing in common!

Our formulation is not plagued by those problems. First of all, note that in our formulation of the QSHJE, any value of E is allowed. This is because we are not starting from a probabilistic interpretation of the wave function, i.e. we are not requiring $\psi \in L^2(\mathbb{R})$. The eigenvalue problem for the Schrödinger operator has then no quantized spectrum. We will come back to this very important point later, for now what is relevant is that in taking the classical limit we can keep E fixed, since it does not depend on $\hbar$. The QSHJE (2.3.6) than implies

$$Q = \frac{\hbar^2}{2m}\{S_0; q\} = \frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V - E, \tag{2.4.8}$$

which reduces to the CHJE when we take $\hbar \to 0$.

Lets now inquire on the relation between S_0 and $\hat{S}_0, R$. As just mentioned, the natural setting for the older version of th QSHJE (2.3.6) is that of $\psi \in L^2(\mathbb{R})$, so what follows is referred to energies belonging to the quantized spectrum. There are two possibilities:

- The Schrödinger equation admits a solution ψ such that $\bar{\psi} \not\propto \psi$. In this case, we can choose $\psi^D := \bar{\psi}$, so that
$$\psi = Re^{\frac{i}{\hbar}\hat{S}_0}, \qquad \psi^D = \bar{\psi} = Re^{-\frac{i}{\hbar}\hat{S}_0} \tag{2.4.9}$$
and by (2.3.67)
$$S_0 = \hat{S}_0 + \pi k\hbar,\ k \in \mathbb{Z}. \tag{2.4.10}$$
The two formulations are in fact pretty closely related, since the continuity equation in (2.4.3) gives
$$R \propto 1/\sqrt{\hat{S}_0'} = 1/\sqrt{S_0'} \tag{2.4.11}$$
and by (2.3.67) we get
$$Q = \frac{\hbar^2}{4m}\{S_0; q\} = -\frac{\hbar^2}{2mR}\frac{\partial^2 R}{\partial^2 q} = \hat{Q}, \tag{2.4.12}$$
so that the quantum potentials coincide on the solutions.
- The Schrödinger equation has only solutions such that $\bar{\psi} \propto \psi$. In this case the difference between the two formulations becomes relevant: as anticipated $\hat{S}_0$ is constant, the continuity equation degenerates and up to a normalization constant we have
$$\psi = R. \tag{2.4.13}$$
The first equation in (2.4.3) now reads
$$\hat{Q} = \frac{\hbar^2}{2mR}\frac{\partial^2 R}{\partial^2 q} = E - V. \tag{2.4.14}$$
On the other hand $\psi = R$ and
$$\psi^D := R\int_{q^0}^{q} R^{-2}dx \tag{2.4.15}$$
form a base for the kernel of the Schrödinger operator, and identifying them with those in (2.3.67) we get the relation between S_0 and R:
$$S_0 = \frac{\hbar}{2i}\log\int_{q^0}^{q} R^{-2}dx. \tag{2.4.16}$$

By (2.3.6) and (2.4.14) it holds

$$\left(\frac{\partial S_0}{\partial q}\right)^2 + \frac{\hbar^2}{2}\{S_0; q\} = -\frac{\hbar^2}{2mR}\frac{\partial^2 R}{\partial^2 q}, \tag{2.4.17}$$

that is

$$\left(\frac{\partial S_0}{\partial q}\right)^2 + Q = \hat{Q}. \tag{2.4.18}$$

We see that in this case the two formulations are intrinsically different.

At the heart of the difference between the two formulations of the QSHJE is the definition (2.4.1):

Old QSHJE	New QSHJE
$\psi = Re^{\frac{i}{\hbar}\hat{S}_0}$	$\psi = \frac{1}{\sqrt{S_0'}}\left(Ae^{\frac{2i}{\beta}S_0} + Be^{-\frac{2i}{\beta}S_0}\right)$

Considering eq. (2.4.11) we see that the two formulations would be equivalent if in the definition (2.4.1) we had used

$$\psi = R\Big(Ae^{\frac{2i}{\beta}S_0} + Be^{-\frac{2i}{\beta}S_0}\Big). \tag{2.4.19}$$

This way the reality of ψ would translate into a condition on the coefficients A, B rather than on $\hat{S}_0$, allowing non trivial solutions. This is related to the fact that, not being subjected to the condition $\psi \in L^2(\mathbb{R})$, in the new QSHJE we are dealing with two independent solutions of the Schrödinger equation rather than one.

2.4.2 The canonical variables of the system

Now we proceed, in analogy with Bohmian (and classical) mechanics, identifying the canonical momentum of the system with

$$p(q) := \frac{\partial S}{\partial q}(q). \tag{2.4.20}$$

As in classical mechanics, the choice of a particular solution to the QSHJE, that is, the choice of a particular module δ, selects a subclass of the possible $(q, p(q))$. From eq. (2.3.81) we have

$$p = \frac{\hbar W(l+\bar{l})}{2|\psi^D - il\psi|^2} \tag{2.4.21}$$

where

$$W = \psi'\psi^D - \psi\psi^{D\prime} = const \in \mathbb{R}\setminus\{0\} \tag{2.4.22}$$

is the Wronskian. It is a constant since ψ^D and ψ are linearly independent. Both eq. (2.3.81) and eq. (2.4.21) can be written in a very concise way defining

$$\phi = \sqrt{2}\frac{e^{-i\frac{\alpha}{2}}(\psi^D - il\psi)}{\hbar^{\frac{1}{2}}|W(l+\bar{l})|^{\frac{1}{2}}}, \tag{2.4.23}$$

so that

$$e^{\frac{2i}{\hbar}S_0\{\delta\}} = \frac{\bar{\phi}}{\phi}. \tag{2.4.24}$$

Since

$$\frac{\hbar}{2i}(\phi\bar{\phi}' - \phi'\bar{\phi}) = \mathrm{sgn}[W(l+\bar{l})] := \epsilon, \tag{2.4.25}$$

we have

$$p = \frac{\hbar}{2i}\partial_q \log\frac{\bar{\phi}}{\phi} = \epsilon|\phi|^{-2}. \tag{2.4.26}$$

From eq. (2.4.26) we see that ϵ sets the direction of motion. We also see that α does not affect the momentum $p(q)$, so that all Möbius transformations like (2.3.82) that leave l unchanged do not affect the conjugate momentum distribution. Since the action of those transformations on the δ-moduli is given by eq. (2.3.84), we have that

$$l' = l \Longleftrightarrow l = i\frac{d-a\pm\sqrt{(d-a)^2+4bc}}{2c}. \tag{2.4.27}$$

Another interesting feature is that from (2.4.24) and (2.4.25) we get

$$\phi = \frac{\epsilon^{\frac{1}{2}}}{\sqrt{S_0'\{\delta\}}}e^{-\frac{i}{\hbar}S_0\{\delta\}}, \tag{2.4.28}$$

so that by (2.4.26)

$$p = \Big(\frac{1}{\sqrt{S_0'\{\delta\}}}\Big)^{-2}. \tag{2.4.29}$$

We have seen in section (2.4.1) that, in the case where the QSHJE and the OQSHJE are equivalent, the square modulus of the wave function is given (cfr. eq. (2.4.11)) by

$$|\psi|^2 = \Big(\frac{1}{\sqrt{S_0'\{\delta\}}}\Big)^{2}. \tag{2.4.30}$$

Then the classical and our formulation of Quantum Mechanics agree in that the regions in which it is less likely to detect the particle are those where the momentum is higher, which are traversed more quickly.

In this context, it is useful to stress how the EP

$$S^a(q^a) = S^b\big(q^b(q^a)\big) \tag{2.4.31}$$

induces a stringent correspondence between the canonical variables in Classical and Quantum mechanics. In fact, we have

$$p^b = \frac{\partial}{\partial q^b}S^b(q^b) = \frac{\partial}{\partial q^b}S^a\big(q^a(q^b)\big) = \frac{\partial q^a}{\partial q^b}\frac{\partial}{\partial q^a}S^a(q^a) = \Big(\frac{\partial q^b}{\partial q^a}\Big)^{-1}p^a. \tag{2.4.32}$$

This means that the EP is such that a v-transformation of q^a induce a transformation on p^a which is exactly its point extension (p.e.) to the phase space:

Classical Mechanics	Quantum Mechanics
$(q^a,p^a) \overset{\text{p.e. of } q^b(q^a)}{\longrightarrow} \Big(q^b(q^a), \Big(\frac{\partial q^b}{\partial q^a}\Big)^{-1}p^a\Big)$	$(q^a,p^a) \overset{v\text{-transformation}}{\longrightarrow} \Big(q^b(q^a), \Big(\frac{\partial q^b}{\partial q^a}\Big)^{-1}p^a\Big)$

2.4.3 Tunneling

Identifying the conjugate momentum as $p = \partial_q S_0$, originates a kind of quantum tunneling effect. While in classical Hamiltonian mechanics the SHJE

$$p = \pm\sqrt{2m(E - V)} \tag{2.4.33}$$

implies that the regions where $E - V(q) < 0$ are forbidden to classical trajectories, the QSHJE furnishes a completely different picture. This is due to the presence of the quantum potential: the analogue of eq. (2.4.33) is in fact

$$p = \pm\sqrt{2m(E - V - Q)} \tag{2.4.34}$$

which implies $p(q) \in \mathbb{R}\,\forall\, q$. This is particularly transparent from eq. (2.4.26), that is

$$p = \epsilon|\phi|^{-2}.$$

A related fact is that the quantum potential Q is never trivial, as can be seen by its definition (2.3.56) together with the fact that S_0 is never trivial, even in the case in which we have only real solutions to the associated Schrödinger equation.
A comment is in order: to consider the region "not forbidden" we should also check that the velocity $\dot{q}$ is real. Trying to define a velocity in this quantum mechanical context is not easy, as we will see in section (2.4.5). We anticipate that, at least at a formal level, this is the case, since eq. (2.4.59) implies

$$\dot{q} = \frac{1}{\partial_E p} \tag{2.4.35}$$

which is real.

2.4.4 Energy quantization

A suggestive feature of the QSHJE is that, without any further assumption other than the EP, we retrieve energy quantization. In this respect, note that in the Copenhagen interpretation of Quantum Mechanics, discrete energy spectra arise when trying to solve the eigenvalue equation

$$\mathcal{H}\psi = E\psi \tag{2.4.36}$$

imposing $\psi \in L^2(\mathbb{R})$. This last assumption in turn comes from the probabilistic interpretation of the wave-function, which is an addition to an otherwise already consistent theory, specified by the Schrödinger equation.
To see how energy quantization is inherent in our formulation, start by considering the conditions we need to impose on ψ^D and ψ in order for the QSHJE

$$\{\omega; q\} = -\frac{4m}{\hbar^2}W(q) \tag{2.4.37}$$

to be well posed. For a start, it is clear that we must have

$$\omega \in C^2(\mathbb{R}),\ \partial_q\omega \neq 0,\ \partial_q^2\omega \text{ differentiable in } \mathbb{R} \tag{2.4.38}$$

in order for the Schwarzian derivative to be well defined. This will not be all though. The reason is that although conditions (2.4.38) guarantee the existence of the Schwarzian derivative, they are not strong enough to guarantee all the symmetry properties of $\{\omega, q\}$ that we used in the derivation of the QSHJE, which are at the heart of the physics. To see this, note that the property of the Schwarzian derivative

$$\{f \circ i; x\} = \{f; i(q)\}\Big(\frac{\partial i}{\partial q}\Big)^2 + \{i; q\} \tag{2.4.39}$$

with $i(q) := 1/q$ implies

$$\{\omega; i(q)\} = \Big(\frac{\partial i}{\partial q}\Big)^{-2}\{\omega \circ i; q\} = q^4\{\omega \circ i; q\}. \tag{2.4.40}$$

Composition with i than gives

$$\{\omega \circ i; q^{-1}\} = q^4\{\omega; q\}. \tag{2.4.41}$$

This means that on the base of the sole Equivalence Principle we could have obtained indifferently eq. (2.4.37) or

$$\{\omega \circ i; q^{-1}\} = -\frac{4m}{\hbar^2}q^4 W(q). \tag{2.4.42}$$

In particular this means that $\{\omega \circ i; q^{-1}\}$ must be well behaved at $q = 0$, and since the inversion i maps $0^{\pm} \to \pm\infty$, we are forced to require the continuity of ω not only on $\mathbb{R}$ but on the extended line $\hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$:

$$\omega \in C^2(\hat{\mathbb{R}}),\ \partial_q\omega \neq 0,\ \partial_q^2\omega \text{ differentiable in } \hat{\mathbb{R}} \tag{2.4.43}$$

even though the physical system is defined on $\mathbb{R}$. This is related to the fact that the most natural environment to study the Möbius group is the Riemann sphere, or equivalently the complex plane with the point at infinity added. Similarly, the symmetry property

$$\{\omega, q\} = -\Big(\frac{\partial \omega}{\partial q}\Big)^2\{q; \omega\} \tag{2.4.44}$$

is implemented only if ω is univalent, and this property must again hold in $\hat{\mathbb{R}}$:

$$\omega(+\infty) = \begin{cases} \omega(-\infty) \text{ for } \omega(\infty) \neq \pm\infty \\ -\omega(-\infty) \text{ for } \omega(\infty) = \pm\infty \end{cases} \tag{2.4.45}$$

Since as we have seen ω is a Möbius transformation of the trivializing map, and since you can connect any two systems passing through the system $W^0 \equiv 0$, these properties reflect on the $q^a(q^b)$ transformations, which must be singlevalued and therefore locally invertible in $\hat{\mathbb{R}}$.

We are now going to show that condition (2.4.45), which comes directly form the EP, implies the following result on energy spectra:

Theorem 2.4.1. *If $V(q) - E < 0\ \forall\, q \in \mathbb{R}$, then there are no solutions ψ_1, ψ_2 to the Schrödinger equation such that their ratio $\omega = \psi_1/\psi_2$ satisfies the conditions (2.4.43) and (2.4.45), that is, ω is never a self-homeomorphism of $\mathbb{R}$. On the other hand, if V is a confining potential, that is $V(\pm\infty) > E$, and*

$$V(q) - E \geq \begin{cases} P_-^2 \textit{ for } q < q_- \\ P_+^2 \textit{ for } q > q_+ \end{cases} \quad \textit{for some } p_- < p_+ \in \mathbb{R} \textit{ and } P_+, P_- > 0 \tag{2.4.46}$$

then ω is a local self-homeomorphism of $\hat{\mathbb{R}}$ if and only if the corresponding Schrödinger equation has an $L^2(\mathbb{R})$ solution.

Proof. First we prove that the existence of an $L^2(\mathbb{R})$ solution is a necessary condition to meet the gluing requests (2.4.45). To this end, note that applying Wronskian arguments to the solutions of the stationary Schrödinger equation, it can be shown that (see [19])

Lemma 2.4.1. *If V satisfies conditions (2.4.46), then as $q \to +\infty$*

- *There is a solution to the SE that vanishes at least as $e^{-P_+ q}$.*
- *Any other independent solution must diverge at least as $e^{P_+ q}$.*

Similarly, as $q \to -\infty$

- *There is a solution to the SE that vanishes at least as $e^{-P_- q}$.*
- *Any other independent solution must diverge at least as $e^{P_- q}$.*

Note that lemma (2.4.1) refers to real solutions of the SE, but this is no obstacle, since the solutions are real modulo a constant phase. In fact, if $\psi \in L^2(\mathbb{R})$ is a solution, it is unique up to a constant. But also $\bar{\psi}$ is a solution, so it must be $\bar{\psi} = c\psi$ for some $c \in \mathbb{C}$, which implies

$$\psi = e^{i\alpha(q)}|\psi|(q) \text{ with } e^{2i\alpha(q)} = c. \tag{2.4.47}$$

Lemma (2.4.1) imply that if a $\psi_1 \in L^2(\mathbb{R})$ solution exists for the SE, it must decay at least exponentially at $\pm\infty$, and any other independent solution ψ_2 must diverge at least exponentially at $\pm\infty$. To see this, consider the $q \to +\infty$ behavior of ψ_1 first. From lemma (2.4.1) we know that an exponentially decaying $\psi_{+,0}$ solution exists. But ψ_1 cannot be linearly independent from $\psi_{+,0}$ because it does not diverge as $q \to \infty$. The $q \to -\infty$ behavior is treated with an analogous argument, proving that ψ_1 must decay at least exponentially at $\pm\infty$. Now, since ψ_1 decays exponentially at $\pm\infty$, from lemma (2.4.1) we see that any other independent solution must diverge exponentially at $\pm\infty$.
Forming linear combinations of ψ_1 and ψ_2 we see that if the SE admits a $L^2(\mathbb{R})$ solution, any solution might

- Diverge both at $-\infty$ and $+\infty$ at least as $e^{-P_- q}$ and $e^{P_+ q}$ respectively.
- Vanish both at $-\infty$ and $+\infty$ at least as $e^{P_- q}$ and $e^{-P_+ q}$ respectively.

On the other hand, if the SE does not admit any $L^2(\mathbb{R})$ solution, analogous reasoning imply that any two independent solutions ψ_1, ψ_2 must have the following asymptotic behavior

- ψ_1 diverges at $-\infty$ at least as $e^{-P_- q}$ and vanishes at $+\infty$ at least as $e^{-P_+ q}$
- ψ_2 vanishes at $-\infty$ at least as $e^{P_- q}$ and diverges at $+\infty$ at least as $e^{P_+ q}$

so that, forming their linear combinations, we see that if the SE does not admit a $L^2(\mathbb{R})$ solution, any solution might:

- Diverges both at $-\infty$ and $+\infty$ at least as $e^{-P_- q}$ and $e^{P_+ q}$ respectively.
- Diverges at $-\infty$ at least as $e^{-P_- q}$ and vanishes at $+\infty$ at least as $e^{-P_+ q}$.
- Vanishes at $-\infty$ at least as $e^{P_- q}$ and diverges at $+\infty$ at least as $e^{P_+ q}$.

Consider now

$$\omega_{12}(q) := \frac{\psi_2}{\psi_1}(q), \tag{2.4.48}$$

so that

$$\lim_{q\to-\infty}\omega_{12}(q) = 0 \quad \text{and} \quad \lim_{q\to+\infty}\omega_{12}(q) = \infty. \tag{2.4.49}$$

Since considering two arbitrary independent solutions ψ^D, ψ instead of ψ_1, ψ_2 amounts to a Möbius transformation $\omega_{12}(q) \to \omega = \gamma \circ \omega_{12}(q)$ we see that a general solution $\omega(q)$ cannot satisfy the gluing conditions (2.4.45) the SE does not admit a $L^2(\mathbb{R})$ solution.

Now we show that under the hypothesis of the theorem, if the Schrödinger equation admits an $L^2(\mathbb{R})$ solution, then ω satisfies (2.4.45). Let $\psi \in L^2(\mathbb{R})$ be such a solution, and consider

$$\psi^D(q) := \psi \int_{q_0}^{q} \psi^{-2}(x)dx, \tag{2.4.50}$$

which is a second independent solution of the Schrödinger equaiton. Note that

$$\psi \in L^2(\mathbb{R}) \Rightarrow \lim_{q\to\pm\infty}\psi(q) = 0 \tag{2.4.51}$$

so that $\psi^{-2} \notin L^2(\mathbb{R})$ and consequently

$$\lim_{q\to\pm\infty}\int_{q_0}^{q}\psi^{-2}(x)dx = \pm\infty. \tag{2.4.52}$$

Moreover, we have already proved that being independent form ψ, ψ^D must be divergent at $\pm\infty$. The ratio of two general solutions will then be

$$\omega = \frac{A\psi^D + B\psi}{C\psi^D + D\psi}, \quad AD - BC \neq 0. \tag{2.4.53}$$

Then:

- If $AC \neq 0$, since ψ^D is divergent at infinity, we have

$$\lim_{q\to\pm\infty}\omega(q) = \frac{A}{C}, \tag{2.4.54}$$

  so that $\omega(\infty)$ is finite and $\omega(\infty) = \omega(-\infty)$.

- In the case $C = 0$ we get

$$\lim_{q\to\pm\infty}\omega(q) = \lim_{q\to\pm\infty}\frac{A\psi^D}{D\psi} = \pm\infty \cdot \operatorname{sgn}\frac{A}{D}. \tag{2.4.55}$$

  In this case then $\omega(\infty)$ is not finite, and $\omega(\infty) = -\omega(-\infty)$.

- In the case $A = 0$ we get

$$\lim_{q\to\pm\infty}\omega(q) = \lim_{q\to\pm\infty}\frac{B\psi}{C\psi^D} = 0, \tag{2.4.56}$$

  so that $\omega(\infty)$ is finite and $\omega(\infty) = \omega(-\infty)$.

So the gluing conditions (2.4.45) are always met, and this concludes the proof. □

Although it is is surely fascinating to be able to imply energy quantization from a first principle, independently from a probabilistic interpretation of the wave function, we will see in the next section how this feature essentially forbids one to consider the concept of trajectory in this approach to Quantum Mechanics.

2.4.5 Criticism to the concept of Quantum trajectories

We conclude this section with some remarks on the possibility of introducing classical trajectories in our picture. One of the most striking features of Bohmian Mechanics is that eq. (2.2.4) can be seen as the continuity equation for a system with density R^2 evolving along the velocity field $\nabla S/m$. This observation is at the origin of the identification $m\dot{q} = \nabla S = p$, from which one deduces the trajectories in the Quantum Theory of Motion. Yet this assumption is somewhat problematic, because mechanical and conjugate momenta do not generally coincide in Classical Mechanics. In particular, its quite clear that it doesn't make sense to import it in our theory: the Quantum Hamilton-Jacobi equation is associated to an highly non trivial Hamiltonian, as a result of the Schwarzian derivative term. It is nevertheless possible (at least on a formal level) to consider the concept of trajectory in this framework. Lets start by looking back at classical Hamiltonian mechanics. For our reasoning we only need do concentrate on the stationary case $\partial_t H = 0$ in one dimension $q \in \mathbb{R}$. The HJ equation then reads

$$H\Big(q, \frac{\partial S_0}{\partial q}\Big) = E. \tag{2.4.57}$$

Differentiating with respect to E we find

$$1 = \frac{\partial}{\partial E} H\Big(q, \frac{\partial S_0}{\partial q}\Big) = \frac{\partial H}{\partial p}\Big(q, \frac{\partial S_0}{\partial q}\Big)\frac{\partial^2 S_0}{\partial q \partial E}. \tag{2.4.58}$$

Imposing the Hamilton equations of motion and $p(t) = \nabla S_0\big(q(t)\big)$ we get

$$\dot{q}\frac{\partial^2 S_0}{\partial q \partial E} = 1, \tag{2.4.59}$$

which is the equation of motion for the family of trajectories specified by $p = \nabla S_0$. This equation can be integrated along the trajectory, giving

$$t(q) - t_0 = \int_{q_0}^{q} \frac{\partial^2 S_0}{\partial x \partial E} dx = \frac{\partial S_0}{\partial E}, \tag{2.4.60}$$

where we have fixed the constant of integration so that $t(q_0) = 0$.
It has been proposed (see [9]) to use this technique to introduce time parametrization in the quantum theory of motion. At least formally, using the QSHJE (2.3.57) we find

$$t - t_0 = \Big(\frac{m}{2}\Big)^{1/2} \int_{q_0}^{q} dx \frac{1 - \partial_E Q}{(E - V - Q)^{1/2}}. \tag{2.4.61}$$

This allows to compute the acceleration as

$$\ddot{q} = \frac{2(E - \mathcal{V})\partial_E \partial_q \mathcal{V}}{m(1 - \partial_E \mathcal{V})^3} - \frac{\partial_q \mathcal{V}}{m(1 - \partial_E \mathcal{V})^2} \tag{2.4.62}$$

where $\mathcal{V} := V + Q$ denotes the effective potential.

Although more natural and less arbitrary then simply assuming $p = m\dot{q}$ as it is done in Bohmian mechanics, this approach presents some rather deep critical issues. In fact, as we have seen in section (2.4.4), quantized spectra are a characteristic feature of the QSHJE, that actually emerges directly from the EP, without any need for a probabilistic interpretation of the wave function. This makes the concept of derivative with respect to E badly defined, underminig the foundations of the trajectory analysis carried out above. Some tentative approach to define an appropriate generalization of the concept of derivative with respect to E which can be used in this context has been made, see for example [6]. Another suggestive idea is that of sticking to the concept discreet spectrum, and to substitute the right hand side of eq. (2.4.60) with a finite difference approximation. This would introduce a discretization in time, which renders a consistent classical limit in the case in which the spacing between adjacent energy level goes to zero in the $\hbar \to 0$ limit.

Chapter 3

A bridge to Uniformization theory

It is interesting to see how the techniques used in the previous chapter in the context of the QSHJE can be applied to the study of Riemann surfaces and to the Uniformization problem. Those two apparently very different worlds share a surprising lot of common features. The reason is that, as we have stressed many times, the EP and the projective connection-like transformation law of the potential W that build the foundations of the QSHJE give a *geometrical* connotation to the theory. Before we go deeper into the analogy, some definitions and results are in order. We will give for granted the basic notions of Differential Geometry, concentrating on what makes the Riemann surfaces so special.

3.1 Riemann surfaces

Definition 3.1.1. *(Riemann surface) A Riemann surface* $(S, \{\phi_\alpha\})$ *is a Hausdorff topological space* S *together with a countable set of maps*

$$\{\phi_\alpha : U_\alpha \to \tilde{U}_\alpha \text{ s.t. } \phi_\alpha \text{ is an homomorphism}\},$$

where $\{U_\alpha\}$ *covers* S *(*i.e. $S = \cup_\alpha U_\alpha$*),* $\tilde{U}_\alpha$ *are open subsets in* $\mathbb{C}$ *and the transition maps* $f_{\alpha\beta} := \phi_\beta \circ \phi_\alpha^{-1}$ *are holomorphic in their domain of definition.*

The set $\{\phi_\alpha\}$ is called an *atlas for* S. The maps ϕ_α are referred to as *local charts*, their domain U_α are called *patches* and the image $\phi_\alpha(x) = z \in \tilde{U}_\alpha$ of a point $x \in U_\alpha$ is called *coordinate of* x. To make the notation easier we will often avoid mentioning the atlas when referring to a surface, *i.e.* we will just write S instead of $(S, \{\phi_\alpha\})$.
Maps between Riemann surfaces are classified with respect to the regularity of their representation in coordinates:

Definition 3.1.2. *A map* f *between two Riemann surfaces* $(X, \{\phi_\alpha\})$ *and* $(Y, \{\psi_\beta\})$ *is called holomorphic iff for each* α, β *the composite* $\phi_\alpha \circ f \circ \psi_\beta^{-1}$ *is holomorphic in its domain of definition.*

Two Riemann surfaces are called *equivalent* iff there exists an holomorphic bijection with a holomorphic inverse between them. It is then natural to take the quotient of the set of Riemann surfaces by this equivalence relation, so that deforming an atlas by composition with holomorphic functions

$$\phi_\alpha \to \tilde{\phi}_\alpha = f_\alpha \circ \phi_\alpha,\ f_\alpha : \mathbb{C} \to \mathbb{C},\ f_\alpha \text{ holomorphic}$$

or simply adding some extra charts compatible with the preexisting ones, no longer generates a distinct differential structure on S.

3.2 Topological classification of surfaces

Definition 3.2.1. *(Homeomorphism) Two topological spaces X, Y are called homeomorphic or topologically equivalent iff there is a bijection between the two that is compatible with the respective topologies,* i.e. *a continuous invertible function $F : X \to Y$ with a continuous inverse.*

The "topological classification problem" amounts to identifying, given a set of topological spaces $T = \{X_i\}$, the number of topologically inequivalent families contained in T. The case that matters to us is that of

$$T = \{\text{Riemann surfaces}\},$$

although the results are more general than this, and refer to the wider class of smooth 2-dimensional surfaces. It turns out that their topological classification is extremely simple, and relies only on the following topological invariants.

Definition 3.2.2. *(Orientation) We call a Riemann surface "orientable" iff there exists an atlas where every transition function has positive Jacobian.*

Definition 3.2.3. *(Compactness) We call a Riemann surface $(S, \{\phi_\alpha\})$ "compact" iff the underling topological space S is compact,* i.e. *iff each of its open covers has a finite subcover.*

Definition 3.2.4. *(Genus) The genus of a connected Riemann surface is defined as the maximum number of closed non intersecting loops along which you can cut the surface without making the resulting manifold disconnected.*

Given this terminology, we can state the

Theorem 3.2.1. *(Topological classification for 2-dimensional surfaces) Any compact Riemann surface $(X, \{\phi_\alpha\})$ is homeomorphically equivalent to one of the following:*

- *The sphere S^2 (if it is orientable and it has genus $g_X = 0$)*
- *A connected sum of n tori (if it is orientable and it has genus $g_X = n$)*
- *A connected sum of n real projective plains (if it is not orientable and it has genus $g_X = n$)*

Another topological invariant that will play a role in the forthcoming analysis is the Euler characteristic $\chi(S)$ of a Riemann surface. Consider a triangulation of the surface S, *i.e.* a covering of S by images by continuous functions of the fundamental triangle

$$\triangle := \{(x, y) \in \mathbb{R}^2 \text{ s.t. } 0 \leq x, y;\ x + y \leq 1\}$$

which satisfies the following intuitive costraints:

- Every point $p \in S$ which is not on an edge or a vertex belongs to one and only one triangle on S, and this triangle is a neighbourhood of p
- Every point $p \in S$ on an edge which is not a vertex belongs exactly to two triangles $\triangle_p^1$ and $\triangle_p^2$, and $\triangle_p^1 \cup \triangle_p^2$ is a neighborhood of p
- Every point $p \in S$ in a vertex belongs to a finite number of triangles, whose union is a neighborhood of p and which can be numbered in such a way that each one shares exactly one vertex with the following.

By vertex (resp. edge) of a triangle on S we mean the image of an vertex (resp. edge) of the fundamental triangle $\triangle$. Without giving too many details, it is a fact that a surface admits a finite triangulation iff it is compact, and in that case any triangulation is finite. Moreover, the Euler characteristic of S, defined as

$$\chi(S) = \#\text{triangles} + \#\text{vertexes} - \#\text{edges}$$

is independent on the particular triangulation chosen, and for a compact orientable surface of genus g and RRRRR is given by

$$\chi(S) = 2 - 2g - r.$$

3.3 Coverings, fundamental groups and Uniformization theorem

Definition 3.3.1. *(Covering map) Given two topological spaces M and M', a map $F : M \to M'$ is called a* covering map *if, around each point $y \in M'$, there is an open neighborhood V such that $F^{-1}(V)$ is a disjoint union of open sets $U_\alpha \in M$ and $F|_{U_\alpha}$ is a homeomorphism from U_α to V.*

This structure is often informally referred to as a "pile of disks over V".

Definition 3.3.2. *(Universal covering) A covering $F : M \to M'$ is called "universal covering" iff M is simply connected.*

The reason for calling a covering with simply connected domain "universal" comes from the following important result:

Theorem 3.3.1. *(On the factorization of the universal covering map) If the mapping $f : M \to M'$ is a universal cover of the space M' and the mapping $g : N \to M'$ is any cover of the space M' where the covering space N is connected, then there exists a covering map $\tilde{f} : M \to N$ such that $f = \tilde{f} \circ g$. Moreover the covering map $\tilde{f}$ is unique in the following sense: choose any points $m \in M$, $m' \in M'$, $n \in N'$ such that $f(m) = m'$ and $g(n) = m'$. Then imposing $\tilde{f}(m) = n$ makes $\tilde{f}$ unique.*

From this it is clear that the universal covering of a surface, if it exists, is unique up to homeomorphy. Now, given a Riemann surface, does a universal covering exist? The answer is given by the following theorem:

Theorem 3.3.2. *(On the existence of the universal covering) Every compact Riemann surface $(M, \{\phi_\alpha\})$ admits an universal covering. More specifically*

- *If M has genus $g_M = 0$, it is its own universal covering*
- *If M has genus $g_M = 1$, its universal covering is the complex number plane $\mathbb{C}$*
- *If M has genus $g_M \geq 2$, its universal covering is the upper half plane*

$$\mathcal{H} := \{z \in \mathbb{C} \ s.t. \ \operatorname{Im} z > 0\}$$

Note that strictly speaking in this theorem we have left the field of bare topology, introducing a complex structure on M. Indeed one finds that the covering maps are analytic (*i.e.* holomorphic in the sense of Riemann surfaces), and that the universal covering is unique in the sense of Riemann surfaces (*i.e.* up to holomorphic bijection).

Another widespread concept in the study of Riemann surfaces is that of *fundamental group*, which we now introduce. Consider the space of loops $\mathcal{L}_{x_0}$ centered in point $x_0 \in M$, where M is a surface, *i.e.* the set of continuous functions

$$\{\gamma : [0,1] \to M \, s.t. \, \gamma(0) = \gamma(1) = x_0\}$$

Define on $\mathcal{L}_{x_0}$ the homotopycal equivalence relation $\overset{h}{\sim}$, which identifies paths that can be continuously shrunk to the single point x_0. The number of equivalence classes in $[\mathcal{L}_{x_0}] := \mathcal{L}_{x_0}/\overset{h}{\sim}$ "counts" the number of holes in M. It is immediate to prove the following

Definition/Theorem 3.3.1. *(Fundamental group of a surface) Given a surface M the set of equivalence classes $\pi_1(M, x_0) := \{[\mathcal{L}_{x_0}]\}$ of homotopycally equivalent loops centered in $x_0 \in M$ has a group structure, with the operation*

$$* : [\mathcal{L}_{x_0}] \times [\mathcal{L}_{x_0}] \to [\mathcal{L}_{x_0}] \;\; s.t. \;\; [\gamma_1] * [\gamma_2] := [\gamma_{1,2}],$$

where we have defined

$$\gamma_{1,2}(t) := \begin{cases} \gamma_1(2t) & 0 \le t \le 1/2 \\ \gamma_2(2t-1) & 1/2 \le t \le 1 \end{cases}$$

Moreover, the group $\pi_1(M, t_0)$ obtained starting from any different point $y_0 \in M$ is isomorphic to $\pi_1(M, x_0)$, so we can safely refer this structure generically to the surface and call it the "fundamental group of M", denoted by $\pi_1(M)$.

Loosely speaking, the concept of fundamental group is useful to identify how "flexible" the structure of the covering spaces of M is. More precisely, given a (not necessarily universal) covering $F : N \to M$, we can ask if there exists topological isomorphisms of the covering space $\phi : N \to N$ that cannot be seen at the level of M, *i.e.* such that

$$F \circ \phi = F.$$

These maps are called covering transformations, or deck transformation. Morally they "shuffle" the pile of disks over the neighborhood of each point $x \in M$. The covering transformations form a group, denoted by $\mathrm{Aut}(F)$, which acts as a permutation of the points in the fiber $F^{-1}(x)$ for any fixed $x \in M$. Obviously $\mathrm{Aut}(F)$ is a subgroup of the holomorphic automorphisms of the covering space, indicated by $\mathrm{Aut}(N)$. In the following analysis we will be dealing with surfaces with genus $g \ge 2$. Therefore the classification given by theorem (3.3.2) draws our attention to the holomorphic automorphisms group of the upper half plane $\mathcal{H}$. This group is given by the subgroup of the Möbius group

$$\mathrm{Aut}(\mathcal{H}) = \{\phi : \mathcal{H} \to \mathcal{H} \text{ s.d. } \phi(z) = \frac{az+b}{cz+d} \qquad a, b, c, d \in \mathbb{R}, \text{ and } ad - bc = 1\}.$$

$\mathrm{Aut}(\mathcal{H})$ is therefore isomorphic to the quotient $\mathrm{PSL}(2, \mathbb{R}) := \mathrm{SL}(2, \mathbb{R})/\{\pm \mathbb{1}\}$, where $\mathrm{SL}(2, \mathbb{R})$ is the group of 2×2 matrices with real entries and unitary determinant.

If $F : N \to M$ is the fundamental covering of M, the relation between $\mathrm{Aut}(F)$ and $\pi_1(M)$ is extremely simple: they are isomorphic. Very sketchy, this can be understood in terms of the so called "lifting" of paths in M: if you take any point $x \in M$, its fiber $F^{-1}(x)$ consist of a number n of points $\{F_i^{-1}(x)\}$ (actually n is independent on x). Since the $\{F_i^{-1}(x)\}$ are "distant from each other", *i.e.* they belong to different non intersecting disks of the pile, it is possible to associate to any closed loop $\gamma(t)$ in M based in $x \in M$, n continuous curves in N, call them $\{F_i^{-1}(\gamma(t))\}$, each one starting from a point in $\{F_i^{-1}(x)\}$ and specified requiring that

$$\lim_{\Delta t \to 0} F_i^{-1}\big(\gamma(t+\Delta t)\big) = F_i^{-1}\big(\gamma(t)\big)\ \forall t \in [0.1] \text{ and } \forall\, i.$$

The lifted paths also end in the fiber $\{F^{-1}(x)\}$, since $\gamma(0) = \gamma(1) = x$, but not necessarily in the same point they started from. The interesting result is that permutation in the fiber of x generated by a loop γ is depends only on the homotopy class of γ, thus establishing a correspondence with $\pi_1(M)$.

All these notions are laid down to make the following fundamental theorem on the classification of Riemann surfaces more believable.

Theorem 3.3.3. *(Uniformization theorem) Every Riemann surface is (holomorphically equivalent to) the quotient of its universal cover by a free action of its fundamental group. Equivalently, any Riemann surface is one of the following:*

- *the Riemann Sphere* S^2
- $\mathbb{C}$ *or* $\mathbb{C}/\mathbb{Z} = \mathbb{C} \setminus \{0\}$ *or* $\mathbb{C}/\Lambda$ *for some lattice* Λ
- *a quotient* $\mathcal{H}/\Gamma$*, where* $\Gamma \subset PSL(2,\mathbb{R})$ *is a discrete subgroup acting freely on* $\mathcal{H}$

The "uniformization problem" for a given Riemann surface $(M, \{\phi_\alpha\})$ of genus $g \geq 2$ consists in finding the group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ and a holomorphic bijection $F : M \leftrightarrow \mathcal{H}/\Gamma$ that makes M and $\mathcal{H}/\Gamma$ equivalent. Alternatively, we can look for the universal covering map $J_H : \mathcal{H} \to M$ and for its automorphism group $\Gamma = \mathrm{Aut}(J_H)$.
To find such an isomorphism different techniques have been developed. On of them was studied by Poincaré [20] and is based on the peculiar properties of the conformal equivalence classes of metrics on Riemann surfaces. Finding the explicit form of J_H will bring the Schwarzian derivative into play, as we are about to illustrate.

3.4 Conformal classes of Riemann surfaces

In the following we use the standard notations of differential geometry:

- $T(M)$ indicates the tangent vector bundle over the Riemann surface M and $T_x(M)$ indicates the tangent space over $x \in M$
- $T^*(M)$ indicates the cotangent vector bundle over the Riemann surface M $T_x^*(M)$ indicates the cotangent space over $x \in M$

Definition 3.4.1. *(Tensor field) A tensor field over a Riemann surface is a smooth function that associates to every point* $p \in M$ *a tensor* $T(p) \in T_p(M) \otimes \cdots \otimes T_p(M) \otimes T_p^*(M) \otimes \cdots \otimes T_p^*(M)$.

If $x, y \in \mathbb{C}$ indicate local coordinates in some patch of M, the derivatives $\partial_1 := \partial_x$, $\partial_2 := \partial_y$ form a local basis for $T(M)$, and their dual vectors $d^1 := dx$, $d^2 := dy$ form a local basis for $T^*(M)$. With respect to this basis any tensor field can be represented as a sum

$$T(x,y) = f^{a_1 \dots a_n}_{b_1 \dots b_m}(x,y)\partial_{a_1} \otimes \dots \otimes \partial_{a_n} \otimes d^{b_1} \otimes \dots \otimes d^{b_m} \qquad a_i, b_i \in \{0,1\}$$

where $f^{a_1 \dots a_n}_{b_1 \dots b_m}(x,y)$ are smooth functions defined in a neighborhood of x.

Definition 3.4.2. *(Metric) A Riemann metric $g(M)$ over a Riemann surface M is a section of the bundle $T^*(M) \otimes T^*(M)$ which is definite positive and symmetric, i.e. in local coordinates is represented as $g(x,y) = f_{ij} d^i \otimes d^j$ with $f_{ij} = f_{ji}$.*

From now on we will not indicate the direct product between differentials. If we define

$$z := x + iy \quad \Rightarrow \quad dz := dx + idy;\; d\bar{z} := dx - idy$$

we obtain a new local base for $T^*(M) \otimes T^*(M)$. With this notation, the following result holds

Theorem 3.4.1. *Given a Riemann metric g on a Riemann surface M, it is possible to choose local coordinates such that in a neighborhood $U_p \subset M$ of any point $p \in M$ the metric in local coordinates reads*

$$g(z,\bar{z}) = \rho(z,\bar{z}) dz d\bar{z} =: \rho(z,\bar{z}) |dz|^2.$$

The positiveness of the metric allows to write locally the metric as

$$g(z,\bar{z}) = e^{\phi(z,\bar{z})} |dz|^2. \tag{3.4.1}$$

Such coordinates are called “isothermal”, and the only non vanishing components are

$$g_{z\bar{z}} = g_{\bar{z}z} = \frac{1}{2}\rho. \tag{3.4.2}$$

3.5 Hyperbolic metric on $g \geq 2$ Riemann surfaces

On the universal cover of $g \geq 2$ Riemann surfaces, which is the upper half complex plane $\mathcal{H}$, we can define the Poincaré metric

$$g^{\mathcal{P}}(z,\bar{z}) = \frac{|dz|^2}{(\operatorname{Im} z)^2}. \tag{3.5.1}$$

Its curvature is

$$R_g = -g^{z\bar{z}} \partial_z \partial_{\bar{z}} \log g_{z\bar{z}} = -(\operatorname{Im} z)^2 \partial_z \partial_{\bar{z}} \frac{1}{(\operatorname{Im} z)^2} = -1. \tag{3.5.2}$$

Here we indicated by $g^{\circ\circ}$ the inverse of $g_{\circ\circ}$. The upper half plain equpped with the Poincaré metric is called *hyperbolic plain* $(\mathcal{H}, g^P)$. A generic Riemann surface M is called an *hyperbolic surface* if it is isometric to the hyperbolic plane.

The striking feature of the hyperbolic plane is that its self isometry group coincides exactly with its automorphism group $\operatorname{Aut}(\mathcal{H}) = \operatorname{PSL}(2,\mathbb{R})$, as can be readily checked. This means that the quotient operation in theorem (3.3.3), when applied to $g \geq 2$ surfaces, is compatible with the metric, and the quotient space inherits a $R = -1$ curvature metric. It follows that any surface M that has $\mathcal{H}$ as universal covering has a $R = -1$ curvature metric, which is given by the push-forward of the Poincaré metric onto M via the covering map $J_H : \mathcal{H} \to M$

$$ds^2(z,\bar{z}) = \Big(\frac{|J_H^{-1\prime}(z)|}{\operatorname{Im} J_H^{-1}(z)}\Big)^2 |dz|^2 =: e^{\phi(z)} |dz|^2. \tag{3.5.3}$$

Here and in what follows we indicate the derivative ∂_z by an apex

$$\partial_z f =: f' \tag{3.5.4}$$

If we write any metric in terms of the exponential factor introduced in (3.4.1), the curvature reads

$$R(z,\bar{z}) = -g^{z\bar{z}} \partial_z \partial_{\bar{z}} \log g_{z\bar{z}} = -2 e^{-\phi(z,\bar{z})} \partial_z \partial_{\bar{z}} \phi(z,\bar{z}), \tag{3.5.5}$$

The condition $R = -1$ the reads

$$\partial_z \partial_{\bar{z}} \phi(z, \bar{z}) = \frac{1}{2} e^{\phi(z,\bar{z})}, \tag{3.5.6}$$

which is the Liouville equation. It follows from the discussion above that this equation admits a solution. Moreover the solution is unique. So the study of compact Riemann surfaces with genus $g \geq 2$ is essentially equivalent to the study of compact hyperbolic surfaces.

3.6 How to find the uniformization map

From eq. (3.5.3) it is already clear that knowing the explicit form of the hyperbolic metric on a surface M is equivalent to finding inverse covering map $J_H^{-1} : M \to \mathcal{H}$. Concretely, the uniformizing map is found by means of the Schwarzian equation

$$\{J_H^{-1}; z\} = \partial_z^2 \phi - \frac{1}{2}(\partial_z \phi)^2. \tag{3.6.1}$$

The right end side is the classical Liouville stress-energy tensor

$$T^F(z) = \partial_z^2 \phi - \frac{1}{2}(\partial_z \phi)^2. \tag{3.6.2}$$

Proof. To prove eq. (3.6.1) we proceed in two steps. First, notice that by (3.5.3), $\phi_{\bar{z}}$ is a Möbius transform of the uniformization map J_H^{-1}. To see this, consider

$$\begin{aligned}
\partial_{\bar{z}} \phi &= \partial_{\bar{z}} \log \frac{J_H^{-1\prime} \overline{J_H^{-1\prime}}}{(\mathrm{Im}\, J_H^{-1})^2} = \partial_{\bar{z}} \log J_H^{-1\prime} + \partial_{\bar{z}} \log \overline{J_H^{-1\prime}} - 2\partial_{\bar{z}} \log(J_H^{-1} - \overline{J_H^{-1}}) \\
&= \frac{\partial_{\bar{z}} J_H^{-1\prime}}{J_H^{-1\prime}} + \frac{\partial_{\bar{z}} \overline{J_H^{-1\prime}}}{\overline{J_H^{-1\prime}}} - 2\frac{\partial_{\bar{z}}(J_H^{-1} - \overline{J_H^{-1}})}{(J_H^{-1} - \overline{J_H^{-1}})} = \frac{\overline{J_H^{-1\prime\prime}}}{\overline{J_H^{-1\prime}}} - 2\frac{\overline{J_H^{-1\prime}}}{(J_H^{-1} - \overline{J_H^{-1}})} \\
&= \frac{\overline{J_H^{-1\prime\prime}}(J_H^{-1} - \overline{J_H^{-1}}) - 2\overline{J_H^{-1\prime}}^2}{\overline{J_H^{-1\prime}}(J_H^{-1} - \overline{J_H^{-1}})} = \frac{\overline{J_H^{-1\prime\prime}} J_H^{-1} - (2\overline{J_H^{-1\prime}}^2 + \overline{J_H^{-1\prime\prime} J_H^{-1}})}{\overline{J_H^{-1\prime}} J_H^{-1} - \overline{J_H^{-1\prime} J_H^{-1}}}
\end{aligned} \tag{3.6.3}$$

The third equality follows from the holomorphicity of J_H^{-1}. The last expression in eq. (3.6.3) can be interpreted as a generalized Möbius transform of J_H^{-1}, where the coefficients are not constant:

$$\partial_{\bar{z}} \phi = \frac{a(\bar{z}) J_H^{-1} + b(\bar{z})}{c(\bar{z}) J_H^{-1} + d(\bar{z})}, \text{ where}$$

$$\begin{aligned}
a(\bar{z}) &= \overline{J_H^{-1\prime\prime}} & b(\bar{z}) &= -2\big(\overline{J_H^{-1\prime}}\big)^2 - \overline{J_H^{-1\prime\prime} J_H^{-1}} \\
c(\bar{z}) &= \overline{J_H^{-1\prime}} & d(\bar{z}) &= -\overline{J_H^{-1\prime} J_H^{-1}}
\end{aligned} \tag{3.6.4}$$

Still, the invariance property of the Schwarzian derivative

$$\left\{\frac{af(z) + b}{cf(z) + d}; z\right\} = \{f, z\} \tag{3.6.5}$$

relies only on the weaker condition $\partial_z a = \partial_z b = \partial_z c = \partial_z d = 0$, *i.e.* the coefficients need not be constant, but rather antiholomorphic functions, which is met by (3.6.4). This interesting observation, first noted in [18], allows us to write

$$\{J_H^{-1}, z\} = \{\partial_{\bar{z}} \phi, z\}. \tag{3.6.6}$$

It is now a matter of simple calculations to prove (3.6.1): using Liouville equation (3.5.6) we get

$$\{\partial_{\bar z}\phi, z\} = \frac{\partial_z^3\partial_{\bar z}\phi}{\partial_z\partial_{\bar z}\phi} - \frac{3}{2}\Big(\frac{\partial_z^2\partial_{\bar z}\phi}{\partial_z\partial_{\bar z}\phi}\Big)^2 = \frac{\partial_z^2 e^\phi}{e^\phi} - \frac{3}{2}\Big(\frac{\partial_z e^\phi}{e^\phi}\Big)^2 = \partial_z^2\phi - \frac{1}{2}(\partial_z\phi)^2$$
$$\Longrightarrow \{J_H^{-1}; z\} = \partial_z^2\phi - \frac{1}{2}(\partial_z\phi)^2. \tag{3.6.7}$$

□

This is of course to be compared with the QSHJE

$$\{e^{\frac{2i}{\beta}S_0}; q\} = -\frac{2m}{\hbar^2}W(q). \tag{3.6.8}$$

The analogy between the two equations hints to a strict correspondence between the QSHJE and the Uniformization problem. In the case of the QSHJE we did not have a natural notion of transformation of the potential W nor of the solution S_0 under a change of variable. So, we focused on the change of variable $q^a \to q^b(q^a)$ such that the transformation law $W^a(q^a) \to W^b(q^b)$ was that of a projective connection,

$$W^a(q^a) = \Big(\frac{\partial q^b}{\partial q^a}\Big)^2 W^b(q^b) - \frac{\hbar^2}{4m}\{q^b; q^a\} \tag{3.6.9}$$

and imposed that under this transformation S_0 behaved as a scalar

$$S^b(q^b) = S^a\big(q^a(q^b)\big). \tag{3.6.10}$$

This uniquely determined equation (3.6.8).
In the case of the uniformization equation (3.6.1), of course, there is a natural and definite transformation property of the involved objects under a change of variable. The unifromizing map J_H^{-1}, being an holomorphic diffeomorphism between complex manifolds, transforms as a scalar

$$J_H^{-1}(z^b) = J_H^{-1}\big(z^a(z^b)\big). \tag{3.6.11}$$

The Liouville stress-energy tensor transforms as a projective connection, as can be seen proceeding backwards, starting from eq. (3.6.1) and using the properties of the Schwarzian derivative:

$$T^F(q^a) = \Big(\frac{\partial q^b}{\partial q^a}\Big)^2 T^F(q^b) - \{q^b; q^a\}. \tag{3.6.12}$$

In order to solve eq. (3.6.1) we can apply the same techniques developed in section (2.3.4). Namely, we have

$$J_H^{-1}(z) = \frac{a\psi_1(z) + b\psi_2(z)}{c\psi_1(z) + d\psi_2(z)}, \qquad ad - bc \neq 0, \tag{3.6.13}$$

where ψ_1 and ψ_2 are solutions of the Schrödinger-like equation

$$\Big(\frac{\partial^2}{\partial z^2} + \frac{1}{2}T^F(z)\Big)\psi_{1,2}(z) = 0. \tag{3.6.14}$$

When faced with the problem given by eq. (3.6.1), one could wonder if there is some coordinate system in which $T^F \equiv 0$, where the calculations are easier. Indeed such a trivializing coordinate system exists, and is given by

$$w = \frac{AJ_H^{-1}(z) + B}{CJ_H^{-1}(z) + D}. \tag{3.6.15}$$

To see this explicitly, consider the identity

$$h'^{1/2}\frac{\partial}{\partial z}\frac{1}{h'}\frac{\partial}{\partial z}h'^{1/2} = \frac{\partial^2}{\partial z^2} + \frac{1}{2}\{h, z\} \tag{3.6.16}$$

applied to eq. (3.6.14). It reads

$$(J_H^{-1})'^{1/2}\frac{\partial}{\partial z}\frac{1}{J_H^{-1\prime}}\frac{\partial}{\partial z}(J_H^{-1})'^{1/2}\psi_{1,2} = 0. \tag{3.6.17}$$

If we now switch to a trivializing coordinate w, eq. (3.6.14) turns into the trivial equation

$$w'^{\frac{3}{2}}\partial_w^2\phi = 0. \tag{3.6.18}$$

At the level of the uniformization map equation, switching to the trivializing coordinates corresponds to

$$\{J_H^{-1}; z\} = \partial_z^2\phi - \frac{1}{2}(\partial_z\phi)^2 \longrightarrow \{w; w\} = 0 \tag{3.6.19}$$

This behavior is rooted in the geometrical nature of the uniformization problem. Namely, any biholomorphic function between complex manifolds can be seen equivalently as a change of coordinates on one of the two. In this case, the uniformizing map $J_H^{-1} : \Sigma \to \mathcal{H}$ also furnishes the change in local coordinates that allows to rewrite the associated Schrödinger differential equation in the most economic way. This sheds light on one of the key points in the discussion in section 2.3.5, namely that solving the QSHJE also furnished the building block to write the *trivializing coordinate* $q^0(q)$ mapping $W \to W^0 = 0$, as specified in eq. (2.3.77).
Now, the uniformizing map is a solution of eq. (3.6.1), not just *any* solution. In particular, by definition its image must be contained in the upper half plain $\mathcal{H}$. This is a condition on the coefficients in eq. (3.6.13). Once such a solution is found, any $\mathrm{PSL}(2,\mathbb{R})$ transformation will give an equally valid uniformizing map. This fact is related to $\mathrm{PSL}(2,\mathbb{R})$ being the automorphism group of $\mathcal{H}$ and to the fact that the universal covering map is unique *modulo automorphism of the covering space*. A similar symmetry breaking $\mathrm{PSL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{R})$ characterized also the analysis of the QSHJE. In that context, it is determined by the request that S_0 and consequently the v-transforms must be real functions, rather than complex functions (cfr. section 2.3.5).

To conclude, we have shown a very surprising fact: starting from what we are now allowed to call "a geometrist point of view" it was possible to build a generalization of the Hamilton-Jacobi equation which in turn appears to be deeply related to fundamental aspects in Quantum Mechanics. This chain of reasoning might allude to some common origin of Quantum Mechanics and Gravitation.

Conclusions

In this thesis we analyzed two examples in which the $\mathrm{PSL}(2,\mathbb{C})$ symmetry leads to the emergence of the Schwarzian derivative.

The first instance is given by the Sachdev-Ye-Kitaev model, where the IR approximation of the theory exhibits a conformal symmetry, which is broken down to a $\mathrm{PSL}(2,\mathbb{R})$ symmetry when one chooses an extremal point for the action to carry out a saddle point approximation. Introducing leading order corrections to the IR limit of the theory brings into play the Schwarzian derivative, which acts as an effective action describing the behavior of fluctuations around the conformally symmetric extremal point of the IR theory. We then analyzed how a rigorous method can be developed to treat a Quantum Field Theory based on the Schwarzian action in the path integral approach. The method is based on some recent progress made in infinite dimensional measure theory by Belokurov and Shavgulidze [3, 5]. Their construction furnishes a measure theory on the space of the reparametrizations of the circle $\mathrm{Diff}(S^1)$, which is the space parameterizing the IR excitations of the SYK model described above. Along with this measure theory comes a theory of functional integration that turns out to be particularly fitted for the computation of physically relevant quantities in the Schwarzian theory. We carried on explicitly the computation of the Free Energy of the theory and the 2-point correlation function in the Schwarzian theory, obtaining numerical results. As explained at the end of section 1.4, our results which are mildly different from the original ones from Belokurov and Shavgulidze. In order to restore the agreement between the results obtained via the path integral approach and those obtained by other means in the literature [23], the most reasonable proposal seems to modify the regularization prescription adopted in the original papers by Belokurov and Shavgulidze.

The second instance is given by the Quantum Hamilton-Jacobi equation. The formulation we describe was proposed by M. Matone and A. E. Faraggi [7, 8], many years after Bohm introduced the concept for the first time. The interesting point is that, while it can still be compared with the Bohm formulation, the modern formulation has a completely different origin. Namely, it rests upon the Equivalence Principle, *i.e.* the request that any Hamiltonian system H can be mapped into the simplest one $H^0 = p^2/2m$ via a specific action of a change of coordinate $q \to q^0(q)$, in a way that is compatible with the solutions S, S^0 of the QSHJE associated to H and H^0. Namely we impose a scalar transformation property of the principal function

$$S^0(q^0) = S(q(q^0)).$$

From this assumption the form of the QSHJE is uniquely determined. For an Hamiltonian $H = p^2/2m + V(q)$ it reads

$$V(q) - E = -\frac{\beta^2}{2m}\{e^{\frac{2i}{\hbar}S_0}; q\}. \tag{3.6.20}$$

Being formulated in these abstract terms, the Equivalence Principle shows a sort of *geometrical* nature. It was in fact possible to unveil a pretty stringent analogy between the study of the QSHJE and the problem of Uniformization for Riemann surfaces. Again, one of the red lines connecting all the pieces together is symmetry: $\mathrm{PSL}(2,\mathbb{C})$ symmetry is a characterizing feature of both the QSHJE and the equation that embodies the *hyperbolic metric approach* to the uniformization of Riemann surfaces with genus $g \geq 2$. Moreover, in both cases this symmetry is broken down to $\mathrm{PSL}(2,\mathbb{R})$ by the details of the problem. In the QSHJE this happens requiring that the principal function S should be a real function, in the Uniformization problem this happen because the automorphism group of the covering space of surfaces with genus $g \geq 2$ is $\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{PSL}(2,\mathbb{C})$.

The physical interest in the QSHJE comes from the fact that it allows a suggestive first principle derivation of some key features of Quantum Mechanics, above all energy quantization. As illustrated in detail in section 2.4.4, in the context of QSHJE quantized spectra arise independently from a probabilistic interpretation of the wave function ψ, *i.e.* without requiring $\psi \in \mathrm{L}^2(\mathbb{R})$. Instead, they are a necessary consequence of the $\mathrm{PSL}(2,\mathbb{R})$ symmetry characterizing the Schwarzian derivative and the QSHJE problem. It is indeed remarkable that such a fundamental aspect of Quantum Mechanics can be derived from an essentially geometrical point of view. This might suggest some hidden connection between Quantum Mechanics and Gravity via the Schwarzian derivative which, in the formulation we presented, appears as a kind of quantum potential (see eq. (2.3.56)). On the other hand, the Schwarzian derivative emerges in Quantum Mechanics even on a path integral level. In fact, as mentioned in section 1.4.1, the Euclidean version of Feynman's approach is deeply related to integration with respect to the Wiener measure, which in turn involves the Schwarzian derivative. The theoretical reasons underlying this fascinating intertwining are still to be found.

Appendix A

Computation of the partition function in Schwarzian theory

We want to compute

$$Z \longrightarrow Z_\alpha := \int\limits_{\mathrm{Diff}^1(S^1)} \exp\Big[\frac{1}{\sigma^2}\int_0^1 2\alpha^2\phi'^2 d\tau\Big]\mu_\sigma(d\phi). \tag{A.1}$$

To this end, we use eq. (1.4.20) for the Radon-Nikodym derivative of the measure on $\mathcal{B}\big(\mathrm{Diff}^1_+([0,1])\big)$. We want to choose f so that we get on the right hand side something similar to the integrand in (1.4.29). We must then solve the equation

$$\{f,t\} = 2\alpha^2 \Rightarrow f(t) = (c_2\tan(\alpha(c_1+t)) + c_3 \tag{A.2}$$

For convenience we choose

$$f(t) = \frac{1}{2}\Big[\frac{1}{\tan\frac{\alpha}{2}}\tan\Big(\alpha(t-\frac{1}{2})\Big)+1\Big], \tag{A.3}$$

which implies

$$\{f,t\} = 2\alpha, \quad f'(0) = f'(1) = \frac{\alpha}{\sin\alpha}, \quad \frac{f''(1)}{f'(1)} = -\frac{f''(0)}{f'(0)} = 2\alpha\tan\frac{\alpha}{2}.$$

Plugging this into (1.4.20) we find

$$\int\limits_{\mathrm{Diff}^1_+([0,1])} F[\phi]\mu_\sigma(d\phi) = \frac{\sin\alpha}{\alpha}\int\limits_{\mathrm{Diff}^1_+([0,1])} F[f\circ\phi]\exp\Big(-\frac{2\alpha}{\sigma^2}\tan\frac{\alpha}{2}\phi'(0) - \frac{2\alpha}{\sigma^2}\tan\frac{\alpha}{2}\phi'(1)\Big)$$
$$\times\exp\Big(\frac{1}{\sigma^2}\int 2\alpha^2\phi'(t)dt\Big)\mu_\sigma(d\phi). \tag{A.4}$$

Now we accommodate $F[\cdot]$ to make the right hand side equal to the partition function (1.4.29):

$$F[f\circ\phi]\exp\Big(-\frac{2\alpha}{\sigma^2}\tan\frac{\alpha}{2}\phi'(0) - \frac{2\alpha}{\sigma^2}\tan\frac{\alpha}{2}\phi'(1)\Big) = \delta\Big(\frac{\phi'(t)}{\phi'(0)} - 1\Big).$$

We write it as

$$F[f\circ\phi] = F_1[f\circ\phi]F_2[f\circ\phi] \tag{A.5}$$

and look for solutions of the form

$$\begin{cases} F_1[f\circ\phi] = \delta\Big(\frac{\phi'(1)}{\phi'(0)} - 1\Big) \\ F_2[f\circ\phi]\exp\Big(-\frac{2\alpha}{\sigma^2}\tan\frac{\alpha}{2}\phi'(0) - \frac{2\alpha}{\sigma^2}\tan\frac{\alpha}{2}\phi'(1)\Big) = 1 \end{cases} \tag{A.6}$$

Since in eq. (A.5) F_2 is multiplied by the delta functional F_1, we can simplify the system (A.6) to

$$\begin{cases} F_1[f\circ\phi] = \delta\Big(\frac{\phi'(1)}{\phi'(0)} - 1\Big) \\ F_2[f\circ\phi]\exp\Big(-\frac{4\alpha}{\sigma^2}\tan\frac{\alpha}{2}\phi'(0)\Big) = 1 \end{cases} \tag{A.7}$$

To find $F_i[\cdot]$, note that

$$(f\circ\phi)'(t) = f'\circ\phi(t))\phi'(t) \Rightarrow \begin{cases} (f\circ\phi)'(0) = f'(0)\phi'(0) = \frac{\alpha}{\sin\alpha}\phi'(0) \\ (f\circ\phi)'(1) = f'(1)\phi'(1) = \frac{\alpha}{\sin\alpha}\phi'(1) \end{cases}, \tag{A.8}$$

so that the solution is given by

$$\begin{cases} F_1[f\circ\phi] = \delta\Big(\frac{(f\circ\phi)'(1)}{(f\circ\phi)'(0)} - 1\Big) \\ F_2[f\circ\phi] = \exp\Big(-\frac{4\alpha}{\sigma^2}\tan\frac{\alpha}{2}(f\circ\phi)'(0)\Big) \end{cases} \tag{A.9}$$

Putting all together, eq. (A.4) reads

$$\begin{aligned} Z_\alpha &= \sqrt{2\pi}\sigma \int_{\mathrm{Diff}^1_+([0,1])} \delta\Big(\frac{\phi'(1)}{\phi'(0)} - 1\Big)\exp\Big(\frac{1}{\sigma^2}\int 2\alpha^2\phi'(t)dt\Big)\mu_\sigma(d\phi) \\ &= \sqrt{2\pi}\sigma\frac{\alpha}{\sin\alpha} \int_{\mathrm{Diff}^1_+([0,1])} \delta\Big(\frac{\phi'(1)}{\phi'(0)} - 1\Big)\exp\Big(\frac{8}{\sigma^2}\sin^2\frac{\alpha}{2}\phi'(0)\Big)\mu_\sigma(d\phi) \end{aligned} \tag{A.10}$$

As anticipated, in the $\alpha \to \pi$ limit the partition function is divergent. To get rid of this divergence we will renormalize it with respect to the opportunely regularized volume of the $\mathrm{SL}(2,\mathbb{R})$ group equipped with the Haar measure. This is morally equivalent to integrating not over the full $\mathrm{Diff}^1_+([0,1])$ group but only over the quotient $\mathrm{Diff}^1_+([0,1])/\mathrm{SL}(2,\mathbb{R})$, which is perfectly sensible in our SYK model since, as already discussed, Möbius conformal reparametrizations actually act as the identity on the fluctuations around the conformal saddle, and thus lead to over-counting of the configurations in the theory. For the regularized volume we have

$$V_\alpha := \int_{SL(2,\mathbb{R})} \exp\Big(-\frac{2(\pi^2-\alpha^2)}{\sigma^2}\int_0^1 \big(\phi'(t)\big)^2 dt\Big)\mu_H(d\phi). \tag{A.11}$$

Note that the regularization of the volume is the same as the one we used for Z_α.
The details of the calculation can be found in [15]. The final result is

$$V_\alpha = \exp\Big(-\frac{2(\pi^2-\alpha^2)}{\sigma^2}\Big)\frac{2\pi\sigma^2}{\pi^2-\alpha^2}, \tag{A.12}$$

giving for the partition function

$$\begin{aligned} Z := \lim_{\alpha\to\pi}\frac{Z_\alpha}{V_\alpha} &= \lim_{\alpha\to\pi}\frac{\pi^2-\alpha^2}{2\pi\sigma^2}\sqrt{2\pi}\sigma\frac{\alpha}{\sin\alpha}\int_{\mathrm{Diff}^1_+([0,1])} \delta\Big(\frac{\phi'(1)}{\phi'(0)} - 1\Big)\exp\Big(\frac{8}{\sigma^2}\sin^2\frac{\alpha}{2}\phi'(0)\Big)\mu_\sigma(d\phi) \\ &= \sqrt{2\pi}\sigma\frac{\pi}{\sigma^2}\int_{\mathrm{Diff}^1_+([0,1])} \delta\Big(\frac{\phi'(1)}{\phi'(0)} - 1\Big)\exp\Big(\frac{8}{\sigma^2}\phi'(0)\Big)\mu_\sigma(d\phi). \end{aligned}$$

Now we use the definition of the measure μ_σ to compute Z as an integral over the Wiener measure: under the substitution

$$\phi(t) =: \frac{\int_0^t \exp[\xi(\tau)]d\tau}{\int_0^1 \exp[\xi(\tau)]d\tau} \tag{A.13}$$

the measure μ_σ goes into the Wiener measure w_σ over $\mathcal{B}\big(C_0([0,1])\big)$, and we get

$$\begin{aligned} Z &= \sqrt{2\pi}\sigma\frac{\pi}{\sigma^2}\int\limits_{C_0([0,1])} \delta\Big(\frac{\exp[\xi(1)]}{\int_0^1 \exp[\xi(\tau)]d\tau}\frac{\int_0^1 \exp[\xi(\tau)]d\tau}{1} - 1\Big)\exp\Big(\frac{8}{\sigma^2}\frac{1}{\int \exp[\xi(\tau)]d\tau}\Big)w_\sigma(d\xi) \\ &= \sqrt{2\pi}\sigma\frac{\pi}{\sigma^2}\int\limits_{C_0([0,1])} \delta\Big(\exp[\xi(1)] - 1\Big)\exp\Big(\frac{8}{\sigma^2}\frac{1}{\int \exp[\xi(\tau)]d\tau}\Big)w_\sigma(d\xi) \\ &= \sqrt{2\pi}\sigma\frac{\pi}{\sigma^2}\int\limits_{C_0([0,1])} \delta\Big(\xi(1) - 0\Big)\exp\Big(\frac{8}{\sigma^2}\frac{1}{\int \exp[\xi(\tau)]d\tau}\Big)w_\sigma(d\xi). \end{aligned} \tag{A.14}$$

This kind of integral can be computed due to the following

Lemma A.0.1. *It holds:*

$$\int\limits_{C_0([0,1])} \delta\Big(\xi(1)-0\Big)\exp\Big(\frac{-2\beta^2}{\sigma^2(\beta+1)}\frac{1}{\int_0^1 \exp[\xi(t)dt}]\Big)w_\sigma(d\phi) = \frac{1}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{4(\log(\beta+1))^2}{2\sigma^2}\Big). \tag{A.15}$$

Proof. To prove the statement we need to borrow another classical result, concerning the quasi invariance of the Wiener measure with respect to the transformation

$$\phi \to K_f[\phi] := A \circ f \circ A^{-1} \circ \phi,$$

where A is the operator defined in (1.4.17) and $f \in \mathrm{Diff}^1([0,1])$. Defining

$$w_\sigma^f(X) := w_\sigma(K_f X) \quad \forall \quad X \subset C([0,1]),$$

we have that w_σ^f is absolutely continuous with respect to w_σ, with Radon-Nicodym derivative given by

$$\begin{aligned} \frac{dw_\sigma^f}{dw_\sigma}(\xi) =& \frac{1}{\sqrt{f'(0)f'(1)}}\exp\Big(\frac{1}{\sigma^2}\Big[\frac{f''(0)}{f'(0)} - \frac{f''(1)}{f'(1)}e^{\phi(1)}\Big]\frac{1}{\int_0^1 e^{\phi(\tau)}d\tau}\Big) \\ &\times \exp\Big(\frac{1}{\sigma^2}\int\Big\{f, \frac{\int_0^t e^{\phi(\tau)}d\tau}{\int_0^t e^{\phi(\tau)}d\tau}\Big\}\frac{e^{2\phi(t)}}{(\int_0^t e^{\phi(t)}dt)^2}dt\Big). \end{aligned} \tag{A.16}$$

Now consider the transformation induced by $\tilde{f}(t) := \frac{(\beta+1)t}{\beta t+1}$. It is a Möbius transform, implying $\{\tilde{f}, t\} \equiv 0$. Formula (A.16) then reads

$$\frac{dw_\sigma^{\tilde{f}}}{dw_\sigma}(\xi) = \exp\Big(-\frac{\beta}{\sigma^2}\Big[1 - \frac{\xi(1)}{\beta+1}\Big]\frac{1}{\int_0^1 e^{\xi(t)}dt}\Big). \tag{A.17}$$

Then we have

$$\begin{aligned} \int\limits_{C_0([0,1])} \delta[\xi(1)-0]w_\sigma^{\tilde{f}}(d\xi) &= \int\limits_{C_0([0,1])} \delta[\xi(1)-0]\exp\Big(-\frac{\beta}{\sigma^2}\Big[1-\frac{\xi(1)}{\beta+1}\Big]\frac{1}{\int_0^1 e^{\xi(t)}dt}\Big)w_\sigma(d\xi) \\ &= \int\limits_{C_0([0,1])} \delta[\xi(1)-0]\exp\Big(-\frac{\beta^2}{\sigma^2(\beta+1)}\frac{1}{\int_0^1 e^{\xi(t)}dt}\Big)w_\sigma(d\xi). \end{aligned} \tag{A.18}$$

On the other hand, the very definition of $w_\sigma^{\tilde{f}}(d\xi)$ leads to

$$\begin{aligned}\int\limits_{C_0([0,1])} \delta[\xi(1)-0]w_\sigma^{\tilde{f}}(d\xi) &= \int\limits_{C_0([0,1])} \delta\circ K_{\tilde{f}}^{-1}[\xi(1)-0]w_\sigma(d\xi)\\ &= \int\limits_{C_0([0,1])} \delta[\eta(1)+2\log(\beta-1)-0]w_\sigma(d\eta)\\ &= \frac{1}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{4(\log(\beta+1))^2}{2\sigma^2}\Big). \end{aligned} \tag{A.19}$$

Putting together eqs. (A.18) and (A.19) we find

$$\int\limits_{C_0([0,1])} \delta[\xi(1)-0]\exp\Big(-\frac{2\beta^2}{\sigma^2(\beta+1)}\frac{1}{\int_0^1 e^{\xi(t)}dt}\Big)w_\sigma(d\xi) = \frac{1}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{4(\log(\beta+1))^2}{2\sigma^2}\Big),$$

which proves the statement. □

The general formula (A.15) reduces to our functional integral in (A.14) if we select $\beta=-2$, giving

$$Z = \frac{\pi}{\sigma^2}\exp\Big(\frac{2\pi^2}{\sigma^2}\Big) = \frac{1}{2g^2}\exp\Big(\frac{\pi}{g^2}\Big). \tag{A.20}$$

Appendix B

Computation of the averaged conformal propagator in Schwarzian theory

We want to compute

$$\langle G(0,t)\rangle_\alpha = \sqrt{2\pi}\sigma \int_{\mathrm{Diff}^1_+([0,1])} \delta\Big(\frac{\phi'(1)}{\phi'(0)} - 1\Big) \frac{\big[\phi'(t)\phi'(0)\big]^{\frac{1}{4}}}{|\sin\pi[\phi(t)-\phi(0)]|^{\frac{1}{2}}} \exp\Big(\frac{2\alpha^2}{\sigma^2}\int_0^1 (\phi'(\tau))^2 d\tau\Big)\mu_\sigma(d\phi). \tag{B.1}$$

To do so, we follow same strategy as for the partition function, that is we use the quasi-invariance property (1.4.20) with the same function f as in (A.3):

$$\int F[\phi]\mu_\sigma(d\phi) = \frac{\sin\alpha}{\alpha}\int F[f(\phi)]\exp\Big(-\frac{4\alpha}{\sigma^2}\tan\frac{\alpha}{2}\phi'(0)\Big)\times\exp\Big(\frac{2\alpha^2}{\sigma^2}\int_0^1 \phi'(t)dt\Big)\mu_\sigma(d\phi). \tag{B.2}$$

Now we address the general problem of choosing the functional $F_\Psi[\cdot]$ in such a way that, for an arbitrary functional $\Psi[\cdot]$, it holds

$$\begin{aligned}&\int \Psi[\phi]\times\exp\Big(\frac{2\alpha^2}{\sigma^2}\int_0^1\phi'(t)dt\Big)\mu_\sigma(d\phi)\\ &= \int F_\Psi[f(\phi)]\exp\Big(-\frac{4\alpha}{\sigma^2}\tan\frac{\alpha}{2}\phi'(0)\Big)\times\exp\Big(\frac{2\alpha^2}{\sigma^2}\int_0^1\phi'(t)dt\Big)\mu_\sigma(d\phi),\end{aligned} \tag{B.3}$$

namely

$$\Psi[\phi] = F_\Psi[f(\phi)]\exp\Big(-\frac{4\alpha}{\sigma^2}\tan\frac{\alpha}{2}\phi'(0)\Big). \tag{B.4}$$

We get

$$\begin{aligned}&\Psi[f^{-1}\circ\phi] = F_\Psi[\phi]\exp\Big(-\frac{4\alpha}{\sigma^2}\tan\frac{\alpha}{2}\big(f^{-1}\circ\phi\big)'(0)\Big)\\ &\Leftrightarrow F_\Psi[\phi] = \Psi[f^{-1}\circ\phi]\exp\Big(\frac{4\alpha}{\sigma^2}\tan\frac{\alpha}{2}\big(f^{-1}\circ\phi\big)'(0)\Big).\end{aligned} \tag{B.5}$$

In the case at hand we have

$$\big(f^{-1}\circ\phi\big)(t) = \frac{1}{\alpha}\arctan[\tan\frac{\alpha}{2}(2\phi(t)-1)] + \frac{1}{2}] \tag{B.6}$$

and

$$\Psi[\phi] = \delta\Big(\frac{\phi'(1)}{\phi'(0)} - 1\Big)\frac{\big[\phi'(t)\phi'(0)\big]^{\frac{1}{4}}}{|\sin\pi[\phi(t)-\phi(0)]|^{\frac{1}{2}}},$$

so that, after some computations, eq. (B.2) gives

$$\langle G(0,t)\rangle_\alpha = \sqrt{2\pi}\sigma\frac{\alpha}{\sin\alpha}\int_{\mathrm{Diff}^1([0,1])}\delta\Big(\frac{\phi'(1)}{\phi'(0)}-1\Big)\frac{\big(\phi'(t)\phi'(0)\big)^{\frac{1}{4}}}{|\pi\phi(t)|^{\frac{1}{2}}}\exp\Big(\frac{8}{\sigma^2}\phi'(0)\Big)\mu_\sigma(d\phi) \tag{B.7}$$

Normalizing by the Haar volume of $\mathrm{SL}(2,\mathbb{R})$ we find

$$\langle G(0,t)\rangle = \frac{2\sqrt{2}\pi}{\sigma}\int_{\mathrm{Diff}^1([0,1])}\delta\Big(\frac{\phi'(1)}{\phi'(0)}-1\Big)\frac{\big(\phi'(t)\phi'(0)\big)^{\frac{1}{4}}}{|\pi\phi(t)|^{\frac{1}{2}}}\exp\Big(\frac{8}{\sigma^2}\phi'(0)\Big)\mu_\sigma(d\phi). \tag{B.8}$$

The integrand is a functional depending only on the variables $\phi(\cdot)$ and $\phi'(\cdot)$ evaluated in a finite number of points, namely it is a function of $\phi(t,)\,\phi'(0)$, $\phi'(t)$, and $\phi'(1)$. This feature allows us to reduce the functional integral with respect to the measure μ_σ to a multiple Lebesgue integral. To this end, consider again the substitution

$$\phi(t) =: \frac{\int_0^t \exp[\xi(\tau)]d\tau}{\int_0^1 \exp[\xi(\tau)]d\tau}$$

turning the measure μ_σ into the Wiener measure. The precise form of the integrand in the ξ variable is unimportant, the real point is to perform the change of variable

$$\begin{cases}\xi(\tau) =: \eta_1(\frac{\tau}{t}), & \tau\in[0,t]\\ \xi(\tau) =: \eta_1(1)+\eta_2(\frac{\tau-t}{1-t}), & \tau\in[t,1]\end{cases} \tag{B.9}$$

Note the η_1 and η_2 are both defined on $[0,1]$.
Under this change of variables the Wiener measure factorizes in the product of two Wiener measures. To prove this, note that

$$\begin{aligned} w_\sigma(d\xi) &= \exp\Big[\frac{1}{\sigma^2}\int_0^1(\xi'(\tau))^2d\tau\Big]d\xi = \exp\Big[\frac{1}{\sigma^2}\Big(\int_0^t(\xi'(\tau))^2d\tau+\int_t^1(\xi'(\tau))^2d\tau\Big)\Big]d\xi\\ &= \exp\Big[\frac{1}{\sigma^2}\Big(\frac{1}{t}\int_0^t(\eta_1'(\tau))^2d\tau+\frac{1}{t-1}\int_t^1(\eta_2'(\tau))^2d\tau\Big)\Big]d\eta_1d\eta_2\\ &= w_{\sigma\sqrt{t}}(d\eta_1)w_{\sigma\sqrt{t-1}}(d\eta_2). \end{aligned} \tag{B.10}$$

If we now get back to the μ_σ space via

$$\psi_i := \frac{\int_0^t \exp[\eta_i(\tau)]d\tau}{\int_0^1 \exp[\eta_i(\tau)]d\tau} \tag{B.11}$$

we obtain the following factorization property:

$$\int_{\mathrm{Diff}^1([0,1])}F[\phi]\mu_\sigma(d\phi) = \int_{\mathrm{Diff}^1([0,1])}\int_{\mathrm{Diff}^1([0,1])}F\big[\phi[\psi_1,\psi_2]\big]\mu_{\sigma\sqrt{t}}(d\psi_1)\mu_{\sigma\sqrt{t-1}}(d\psi_2) \tag{B.12}$$

where

$$\begin{cases}\phi[\psi_1,\psi_2](\tau) = \frac{t\psi_2'(0)\psi_1(\frac{\tau}{t})}{t\psi_2'(0)+(1-t)\psi_1'(1)} & 0\le\tau\le t,\\ \phi[\psi_1,\psi_2](\tau) = \frac{t\psi_2'(0)+(1-t)\psi_1'(1)\psi_2(\frac{\tau-t}{1-t})}{t\psi_2'(0)+(1-t)\psi_1'(1)} & t\le\tau\le 1.\end{cases} \tag{B.13}$$

In order to compute (B.8), we set

$$F[\phi] = g\big(\phi(t), \phi'(0), \phi'(t), \phi'(1)\big) = \delta\Big(\frac{\phi'(1)}{\phi'(0)} - 1\Big)\frac{\big(\phi'(t)\phi'(0)\big)^{\frac{1}{4}}}{|\pi\phi(t)|^{\frac{1}{2}}}\exp\Big(\frac{8}{\sigma^2}\phi'(0)\Big) \tag{B.14}$$

and the ϕ variables are expressed in terms of ψ by means of (B.13):

$$\phi(t) = \frac{t\psi_2'(0)}{t\psi_2'(0) + (1-t)\psi_1'(1)} \qquad \phi'(t) = \frac{\psi_2'(0)\psi_1'(1)}{t\psi_2'(0) + (1-t)\psi_1'(1)}$$
$$\phi'(0) = \frac{\psi_2'(0)\psi_1'(0)}{t\psi_2'(0) + (1-t)\psi_1'(1)} \qquad \phi'(1) = \frac{\psi_2'(1)\psi_1'(1)}{t\psi_2'(0) + (1-t)\psi_1'(1)}$$

Now lets insert twice 1 in the integrand in the form of

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dudv\delta(\psi'(0) - u)\delta(\psi'(1) - v)). \tag{B.15}$$

Using the $\delta(\cdot)$ properties we can trade the ψ, ψ' in the integrand g for u and v and move g out of the functional integration, reducing the integral to the Lebesgue integral

$$\langle G(0,t)\rangle = \frac{2\sqrt{2}\pi}{\sigma}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} du_1dv_1du_2dv_2 g\big(\phi(t), \phi'(0), \phi'(t), \phi'(1)\big)\epsilon_{\sqrt{t}\sigma}(u_1, v_1)\epsilon_{\sqrt{1-t}\sigma}(u_2, v_2). \tag{B.16}$$

In this formula we have defined the master integrals

$$\epsilon_\sigma(u, v) := \int_{Diff^1([0,1])} \delta(\psi'(0) - u)\delta(\psi'(1) - v)\mu_\sigma(d\psi) \tag{B.17}$$

and the arguments of g are meant as functions of u_1, v_1, u_2, v_2. Since this inversion is cumbersome, we prefer to work with variables defined by

$$z = \phi(t)(u_1, v_1, u_2, v_2), \qquad x_0 = \phi'(0)(u_1, v_1, u_2, v_2),$$
$$x_t = \phi'(t)(u_1, v_1, u_2, v_2), \qquad x_1 = \phi'(1)(u_1, v_1, u_2, v_2),$$
$$\Longrightarrow u_1 = \frac{t}{z_1}x_0, \quad v_1 = \frac{t}{z_1}x_t, \quad u_2 = \frac{1-t}{1-z_1}x_t, \quad v_2 = \frac{1-t}{1-z_1}x_1, \tag{B.18}$$

obtaining

$$\langle G(0,t)\rangle = \frac{2\sqrt{2}\pi}{\sigma}[t(1-t)]^2\int_0^1 [z(z-1)]^{-3}dz\int_0^\infty dx_0\int_0^\infty x_t dx_t\int_0^\infty dx_1$$
$$\times g(z, x_0, x_t, x_1)\epsilon_{\sqrt{t}\sigma}(\frac{t}{z}x_0, \frac{t}{z}x_t)\epsilon_{\sqrt{1-t}\sigma}(\frac{1-t}{1-z}x_t, \frac{1-t}{1-z}x_1). \tag{B.19}$$

Note that we adapted the domain of integration according to the conditions imposed on z, x_0, x_t, x_1 by the constraints on the field ϕ. The master integrals can be explicitly computed, (see [4]) and read

$$\epsilon_\sigma(u, v) = \Big(\frac{2}{\pi\sigma^2}\Big)^{\frac{3}{2}}\frac{1}{\sqrt{uv}}\exp\Big\{\frac{2}{\sigma^2}(\pi^2 - v - u)\Big\}$$
$$\times\int_0^{+\infty}\exp\Big\{-\frac{2}{\sigma^2}(2\sqrt{uv}\cosh\tau + \tau^2)\Big\}\sin\Big(\frac{4\pi\tau}{\sigma^2}\Big)\sinh\tau d\tau. \tag{B.20}$$

From this and the definition of g given in eq. (B.14), it follows that the averaged propagator is given in terms of the Lebesgue integral

$$\begin{aligned}\langle G(0,t)\rangle =&\frac{2\sqrt{2}\pi}{\sigma}[t(1-t)]^2\int_0^1[z(z-1)]^{-3}dz\int_0^\infty dx_0\int_0^\infty x_t dx_t\int_0^\infty dx_1\delta\Big(\frac{x_1}{x_0}-1\Big)\frac{\big(x_t x_0\big)^{\frac{1}{4}}}{z^{\frac{1}{2}}}\exp\Big(\frac{8}{\sigma^2}x_0\Big)\\ &\times\epsilon_{\sqrt{t}\sigma}(\frac{t}{z}x_0,\frac{t}{z}x_t)\epsilon_{\sqrt{1-t}\sigma}(\frac{1-t}{1-z}x_t,\frac{1-t}{1-z}x_1)\\ =&\frac{2\sqrt{2}\pi}{\sigma}[t(1-t)]^2\int_0^1 z^{-\frac{7}{2}}(z-1)^{-3}dz\int_0^\infty\int_0^\infty dx_0dx_tx_t^{\frac{5}{4}}x_1^{\frac{5}{4}}\exp\Big(\frac{8}{\sigma^2}x_0\Big)\epsilon_{\sqrt{t}\sigma}(\frac{t}{z}x_0,\frac{t}{z}x_t)\epsilon_{\sqrt{1-t}\sigma}(\frac{1-t}{1-z}x_t,\frac{1-t}{1-z}x_0).\end{aligned}$$

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