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MAGNUS SERIES AND DEFLATION FOR FEYNMAN INTEGRALS

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Abstract

This thesis aims at presenting the mathematical methods of *Magnus Series Expansion* and *Eigenvalue Deflation*, to improve the analytic evaluation of multi-loop Feynman integrals through the *Differential Equations* approach.

In the first part of the work, the formalism of multi-loop integrals in dimensional regularization and the relations among them are introduced, discussing how *integration-by-parts identities*, Lorentz invariance identities and Euler's scaling equation can be used to derive first order differential equations for Feynman integrals. The analytic properties of the solutions are then investigated by introducing the concepts of iterated integrals and uniform transcendentality.

The central part of the thesis is dedicated to the mathematical systematization of the Magnus Series Expansion and of the Eigenvalue Deflation, employed to addreess the determination of the solution of a system of differential equations by means of algebraic techniques. An original derivation of the Eigenvalue Deflation method, based upon the operations of deflation and balance transformation, is here presented.

The final part consists of the detailed application of both Magnus Series and Eigenvalue Deflation methods to the one-loop box diagram, to the two-loop ladder diagram and to the non-trivial three-loop ladder diagram, which enter the evaluation of $2 \to 2$ scattering process among massless partons up to the next-to-next-to-next leading order in Quantum Electrodynamics and Quantum Chromodynamics. The analytic expressions of the corresponding integrals, as well as the ones corresponding to their subdiagrams, previously known in the literature, are hereby re-derived one order higher and in a simpler way.

The presented approaches can be applied in a wider context, ranging from high-precision collider phenomenology to the study of formal aspects of scattering amplitudes in gauge theories.

Sommario

Questo lavoro di tesi di laurea magistrale ha lo scopo di presentare i metodi di *Espansione in Serie di Magnus* e di *Deflazione di Autovalori*, sviluppati al fine di migliorare il calcolo analitico di integrali di Feynman a molti loop utilizzando il metodo delle *Equazioni Differenziali per Integrali di Feynman*.

Nella prima parte del lavoro vengono introdotti il formalismo degl'integrali a molti loop in regolarizzazione dimensionale e le relazioni notevoli fra questi, mostrando come le *identità di integrazione per parti*, le identità d'invarianza di Lorentz e l'equazione di scala di Euler possono essere utilizzate per costruire equazioni differenziali del primo ordine fa gl'integrali. Le proprietà analitiche delle soluzioni sono quindi trattate, introducendo i concetti di integrali ripetuti e di uniforme trascendenza.

La parte centrale della tesi è dedicata alla sistematizzazione matematica delle tecniche di Espansione in Serie di Magnus e di Deflazione di Autovalori, utilizzate per ricondurre la determinazione delle soluzioni di un sistema di equazioni differenziali a tecniche algebriche. Viene infine presentata una nuova derivazione del metodo di Deflazione di Autovalori, basata sul concetto di *Deflazione* e di trasformazione di bilancio.

La parte conclusiva si occupa dell'applicazione dettagliata di entrambi i metodi di Serie di Magnus e Deflazione di Autovalori ai diagrammi non banali *1-loop box*, *2-loop ladder* e *3-loop ladder*, che contribuiscono ai processi di diffusione $2 \to 2$ fra partoni non massivi in correzioni fino al terzo ordine in processi di Elettrodinamica Quantistica e Cromodinamica Quantistica. Le espressioni analitiche dei diagrammi sopra menzionati, insieme a quelle dei loro sottodiagrammi, già note in letteratura, sono qui ricalcolate a più alto ordine e facendo uso di tecniche algebriche.

Le modalità di calcolo affrontate nel presente lavoro possono essere applicate in ampi contesti, dalla fenomenologia ad alta precisione degl'acceleratori allo studio di proprietà formali di ampiezze di processi nelle teorie di *gauge*.

Contents

Introduction

This thesis describes the method of *Differential Equations for Feynman Integrals* [1–12], giving particular relevance to the recently developed techniques of *Magnus Series Expansion* [13, 14] and *Eigenvalue Deflation* [15, 16], suitable for determining solutions by means of purely algebraic approaches.

The necessity to investigate and understand the components of atomic and subatomic world has lead to the formulation of *quantum field theory*, the framework unifying quantum mechanics and special relativity, able to precisely describe all but one fundamental interactions and constituents of matter known till now. A cardinal aspect of quantum field theory, derived to describe and compute the different possible ways in which particles evolve during a scattering or decaying process, is represented by *Feynman diagrams*. Given a set of external legs (the only observable properties of the Feynman diagram), two classes of interactions are possible: interactions represented by *tree-level diagrams*, in which the cut of any internal line will split into two parts the process under study, and interactions involving *loop diagrams*, namely structures containing closed internal paths and not subdivisible into separate graphs cutting an arbitrary line. While tree level diagrams represent the main contribution to most of the scattering processes and can be computed obtaining a finite result, loop diagrams are *corrections* proportional to $\hbar^l$ (with l the number of independent loops in the diagram), containing also integrals that diverge in $D = 4$ dimensions. Therefore, to regulate their bad behaviour, both in the Ultraviolet and Infrared regions, regularization prescriptions are mandatory. Through this work, the 't Hooft–Veltman continuous-dimensional regularization is adopted, and accordingly, results will be given as Laurent series in ϵ, being $\epsilon = (4 - D)/2$ the regularization parameter.

Thanks to both the increasing precision of collider experiments and to the developments of supersymmetric theories (like supergravity) in the last three decades, the necessity to compute corrections beyond the tree level has became pressing, both to provide highly accurate theoretical predictions of scattering amplitudes and to allow the study of formal properties of quantum field theory, as degrees of divergence of form factors corrections. Since the direct evaluation of multiloop processes in terms of ϵ series of analytic functions is prohibitive, the scattering amplitudes are at first rewritten as a sum of terms composed by two parts: a *tensorial structure*, carrying informations about Lorentz indexes, spin, and quantum numbers, and a *form factor*, containing integrals on the loop momenta. These scalar integrals are closely connected to the Feynman diagrams from which they come: the numerator may contain all scalar products with at least one loop variable, whereas the denominator is formed by all the propagators present in the diagram itself. To compute these *Feynman integrals* they are reduced in terms of a

basis of *Master Integrals*, which are evaluated. Several techniques to compute master integrals have been developed, as difference [17, 18] and differential [1, 4–7, 9, 19] equations, Mellin–Barnes integration [20–24], asymptotic expansions [20, 25–27], sector decomposition [20, 28–33], complex integration and contour deformation [34], and many more.

The method of *Differential Equations for Feynman Integrals* was originally proposed by Kotikov [1] in 1991, whose basic idea was to consider the Feynman integrals as functions of one of the propagator masses, writing a differential equation for the integral in that variable. The value of the original expression could be found by solving the differential problem. The advantages of the approach were soon realized [2, 3, 35, 36] and generalized by Remiddi [4] to the differentiation with respect to any variable kinematic invariant of the process. A demonstration of the power of this method has been given by Gehrmann and Remiddi with the evaluation of non-trivial two-loop corrections to $\gamma^* \to 3$ jets [5–7], which results are still considered as state-of-the-art.

Differential equations have been widely used in different contexts of quantum field theory and particle phenomenology, allowing to obtain results previously unreachable in jet physics, QED corrections to lepton form factors, Bhabha scattering, QCD corrections to lepton form factors and top-physics, forward-backward asymmetry of heavy quarks, Higgs physics, electroweak theory, Sudakov form factors, semileptonic decay, static parameters and gauge boson properties. The effort to achieve analytic solutions of differential equations for Feynman integrals has stimulated new developments on the mathematical side [8, 37–40], especially concerning transcendental functions: on this side, a generalization of Nielsen's polylogarithms, the so-called *Harmonic Polylogarithms* [8], has been introduced to cast the results in an analytical form.

Recently, Henn [10] suggested a new way to evaluate systems of differential equations: instead of reducing the system of first order differential equations to a single differential equation of high order, the system itself have to be solved through iterated integrals. Moreover, starting with an ϵ-factorized system, the solutions can be written in the even simpler form of iterated integrals over a rational kernel.

At this point, the problem of how to obtain an ϵ-factorized form for the system of differential equations arises. Although a general algorithm to determine whether an ϵ-factorized form for a given basis exists or not, and in a positive case able to give such form, has not yet found or negated, algebraic techniques [13, 16] to manipulate differential equations system in order to find a suitable form have become available in the last couple of years.

The leading path under the approach described up to here is to solve the problem of Feynman integrals simplifying (under a conceptual point of view) the tools used to evaluate the solution: the first step is to pass from *integration* to *differentiation*, a much more easier procedure (analytic functions always admit analytic derivatives, whilst only in rare cases it is possible to express the integral of an analytic function in terms of other simple analytic functions). From there, the necessity to find an ϵ-factorized form moves the attention from *differential* properties to *algebraic* ones, allowing to work with matrices and linear algebra instead of directly trying to integrate functions.

A disadvantage of this passage from integral to differential, and to algebraic techniques is that not all the systems of differential equations satisfy the constraints for the application of the methods mentioned above [13, 16]. Therefore, the systematization of these methods in a mathematical framework is essential for a better comprehension of their limits and for a possible

generalization to a wider class of systems of the algorithms known so far.

The present thesis aims at presenting the method of *Differential Equations for Feynman Diagrams*, as well as the most recent techniques to find an ϵ-factorized form for the system of differential equations associated to a given basis of master integrals. The ϵ-factorization allows then to write the solution in terms of a Dyson series of *Harmonic Polylogarithms* [8]. The two algebraic techniques examined here are *Magnus Series Expansion* [13, 14] and *Eigenvalue Deflation* [16].

The work is divided into three parts. The first part is focused on the presentation of the differential equation method, from the construction of the equations to the mathematical form of their solutions. The second part illustrates the algebraic methods of Magnus series expansion and eigenvalue deflation to find a canonical form for the system of differential equations under consideration. The last part presents the application of the two methods previously introduced to the solution of the 1-loop box, 2-loop ladder and 3-loop ladder massless graphs.

In the first part, Chapter 1 presents the notions of *topology* and *Feynman integral*, showing the process of tensor decomposition to pass from Feynman diagrams to Feynman integrals [9, 20]. After the derivation of a wide range of properties and relations of Feynman integrals [9, 20, 41, 42], integration-by-parts identities [41, 43] (the fundamental tool for the construction of differential and algebraic relations among integrals) are illustrated, showing how they also include most of the relations among different Feynman integrals.

Chapter 2 is divided into two main sections, after a brief introduction to the Laporta algorithm, corresponding to two different methods to evaluate Feynman integrals: the first one shows integral evaluation via Feynman parameters [20, 44] for sunset topologies, in order to find a generic expression for sunset and bubble integrals [41], while the second one presents a detailed discussion of the method of *differential equations for Feynman integrals* [4, 9, 20], as well as of the fixing of *boundary conditions* [4, 9–12], and of the existence and uniqueness of the solutions [4, 9, 20], concluding with the notion of *canonical* system [10].

In Chapter 3, following the procedure indicated in [10, 11, 13, 14], the solution of a canonical system of differential equations [10] is written in terms of a *Dyson series* of iterated integrals. Introducing the notion of *fuchsian form*, the solution of the differential equations can be written in terms of *harmonic polylogarithms*, functions built up using iterated integrations on a rational kernel [8]. The concept of *transcendental weight* (related to the number of iterated integrations in the definition of a function) is then presented [8, 12, 14] and applied to series expansions in order to characterize *uniform transcendental* functions. In conclusion, a selection of exact and guiding criteria for the *a priori* determination of uniform transcendental Feynman integrals is presented [10–14].

The second part opens with the presentation of the *Magnus series expansion* [13, 14] method in Chapter 4. After the proof of Magnus theorem [13, 45], the concept underlying the definition of the interaction picture in quantum perturbation theory is applied to a ϵ-linear system of differential equations in a single variable, finding a change of basis for the master integrals which casts the system in an ϵ-factorized form. Generalization to multi-scale problems is also explained.

In Chapter 5, after recalling the properties of Jordan block form, eigenvalue reduction [16] and deflation [15] are explained, as well as their generalization to the differential equation framework, for one-parameter problems. Here a new derivation of the algorithms presented in [16] is obtained, starting from the deflation methods mentioned above. These algorithms allow to eliminate non-

fuchsian poles in a wide range of cases, thanks to the property of eigenvalue deflation to reduce the rank of a matrix, and to reduce all the eigenvalues to an ϵ-homogeneous form, thanks to the eigenvalue reduction process. The application of deflation and reduction is possible thanks to the introduction of *balance transformations* [16]. From there, a similarity transformation allows to land on a canonical system.

The last part of the work presents the results of the application of Magnus series expansion and eigenvalue deflation to the 1-loop box, 2-loop ladder and 3-loop ladder graphs [10, 11, 13, 14]: in Chapter 6 the graphs are evaluated up to order ϵ^7, an order greater by one than the available results in literature [10, 11]. For each one of these graphs, starting from the topologies determined using the Laporta algorithm via `Reduze2` code [17, 46, 47], a basis of master integrals and its corresponding differential system in $z = t/s$ are determined. With the application of Magnus series, a transformation into a canonical fuchsian form is retrieved. Fixing the boundary conditions with analysis of the pseudothresholds and of the complex behaviour of the basis, a solution in terms of harmonic polylogarithms is found for all the masters, in terms of a small set of integrals evaluated at fixed point [11, 14]. The evaluated integrals are proven to be uniformly transcendent.

In Chapter 7, 1-loop box, 2-loop ladder and a subsystem of the 3-loop ladder differential system are rewritten in a canonical fuchsian form using the eigenvalue deflation method, starting from the Laporta basis of master integrals: at first, the systems are transformed into fuchsian ones, then a form with ϵ-homogeneous eigenvalues is reached. Finally, a similarity transformation allows to reach a fuchsian ϵ-factorized form. The comparison between the resulting bases of master integrals and the ones obtained via Magnus series reduction is realized through integration-by-parts identities.

Appendix A, presents a discussion of the `Mathematica` code used to determine the basis of harmonic polylogarithms thanks to their property of shuffle algebra, followed by the functions used to generate their series expansions around $x_0 = 0; 1; -1$ up to order $o_0\left((x - x_0)^{15}\right)$.

The integrals computed in this work were already considered in [10, 11, 16]. The derivation presented in the following constitute an alternative and independent evaluation by means of the new methods of Magnus Series Expansion and Eigenvalue Deflation. Moreover, results are given up to the seventh order in ϵ, one order higher than the previously known expansions. As said before, the passage from a differential problem to a purely algebraic one allowed by using Magnus series expansion and eigenvalue deflation represents a huge simplification in terms of typology of needed operations, and this simplification is even more important with respect to the original integral problem.

The illustrated techniques and results can be used in the evaluation of QED and QCD corrections to $2 \to 2$ scattering processes, such as $pp \to jj$ (dijet production), known at NNLO [48], or to $2 \to 1$ processes, like $gg \to H$ (Higgs production via gluon fusion) in the limit $m_{\text{top}} \to +\infty$, known at NNNLO within the threshold expansion approximation [49], and more generally for the 3-loop vertex form factors (in the massless approximation).

Besides the application to high-accuracy phenomenology, the illustrated results and methods can be applied in the context of scattering amplitudes within supersymmetric theories ad supergravity [50], for the study of formal properties of quantum field theory, such as the study of divergences and renormalizability conditions, as well as to investigate properties of scattering amplitudes which are not manifest in the Lagrangian.

Part I

Feynman integrals

The analytical integration of a function is rarely an easy task: among the (Lebesgue) integrable functions a very small number of expressions admits a primitive written in terms of simple functions, and, even in that case, the way to find out such primitive may be tortuous and computationally expensive.

On the other hand, derivation is an easier operation: assumed a function is at least $\mathcal{C}^1$ it is always possibile to find a precise form for its derivative. Moreover, knowing the exact form of the derivative of a function, it is possible to establish differential relations between this one and other known functions, finding an explicit expression for it by solving the so-generated differential equation.

A handy one-dimensional example from [9] to explain the idea above is the following: finding analytic form for

$$I_n(\alpha) = \int_0^{+\infty} \mathrm{e}^{-\alpha x^2} x^n \,\mathrm{d}x, \quad n \in \mathbb{N}. \tag{1}$$

The computation of the integral with $n = 0$ or $n = 1$ is not complicated: for $n = 0$, (1) is the gaussian integral $I_0(\alpha) = \int_0^{+\infty} \mathrm{e}^{-\alpha x^2} \,\mathrm{d}x = \frac{1}{2}\sqrt{\pi/\alpha}$, while for $n = 1$ it has the form $I_1(\alpha) = \int_0^{+\infty} \mathrm{e}^{-\alpha x^2} x \,\mathrm{d}x = \frac{1}{2\alpha}$. For $n \geq 2$ the expression becomes more complicated, and techniques like integration by parts must be applied before integration.

A different way to compute the integral for $n \geq 2$ is looking for a differential relation linking the already known functions $I_0(\alpha)$ and $I_1(\alpha)$ to the unknown $I_n(\alpha)$ with greater n. In particular, the following relation holds:

$$I_n(\alpha) = -\frac{\mathrm{d}}{\mathrm{d}\alpha} I_{n-2}(\alpha). \tag{2}$$

Iterating this relation all the integrals with $n \geq 2$ can be expressed in terms of $I_0(\alpha)$ or $I_1(\alpha)$:

- n even

$$I_n(\alpha) = -\frac{\mathrm{d}}{\mathrm{d}\alpha} I_{n-2}(\alpha) = \left(-\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^2 I_{n-4}(\alpha) = \ldots = \left(-\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^{\frac{n}{2}} I_0(\alpha); \tag{3}$$

- n odd

$$I_n(\alpha) = -\frac{\mathrm{d}}{\mathrm{d}\alpha} I_{n-2}(\alpha) = \left(-\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^2 I_{n-4}(\alpha) = \ldots = \left(-\frac{\mathrm{d}}{\mathrm{d}\alpha}\right)^{\frac{n-1}{2}} I_1(\alpha). \tag{4}$$

In this way the value of all the integrals is determined starting from the direct computation of just two of them, the so-called *master integrals* for the class of functions I_n.

This example sums up the leading path on which the method of *solving Feynman integrals via differential equations* is based: given a scalar integral associated to a Feynman diagram, differential relations linking the integral under examination to "simpler" ones can be determined differentiating with respect to the not-integrated variables. Solving these differential equation with suitable boundary conditions (known values of the master integrals at certain points, or at least regularity properties) will then lead to an analytic expression for the desired functions (usually in the form of a series in some regularization parameter).

Chapter 1

From topologies to integral relations

In the first section, the notions of *topology* and *Feynman integral* are presented, following [9, 20]. The second part of the chapter describes a wide range of properties and relations of the Feynman integrals [9, 20, 41, 42], introducing integration-by-parts identities [41, 43] and their relations to Lorentz-invariance identities. They are the fundamental tool for the construction of differential and algebraic relations among integrals. Integration-by-parts identities also include most of the relations that can be constructed to relate different Feynman integrals.

1.1 Diagrams and integrals

1.1.1 Tensor decomposition

The generic structure of an l-loop one-particle irreducible Feynman diagram with g generic external legs $\mathcal{F}^{(l)}_{\lambda;\mathbf{s}}(p_1;\ldots;p_g)$ (λ is the vector of bosonic indices, $\mathbf{s}$ is the vector of fermionic indices) is composed by a *tensorial part*, carrying information related to the nature of the particles and interactions involved, and by an *integral function*, resulting from the summation of all the possible values of the momenta characterizing the loop propagators. As a result of the bare application of the Feynman rules to compute the Feynman amplitude of a chosen process, this two elements are often mixed together.

In order to focus on the integration techniques without worrying about tensor calculus or peculiar problems of the interactions involved, the primary interest is to achieve a factorization of the tensorial part from the integral functions. This process goes under the name of *tensorial decomposition*, and allows to reach a result of the form:

$$\mathcal{F}^{(l)}_{\lambda;\mathbf{s}}(p_1;\ldots;p_g) = E^{\mu}_{\lambda}(p_1;\ldots;p_g)\sum_{h=1}^{H} T_{h;\mathbf{s};\mu}(p_1;\ldots;p_g) f^{(l)}_{h}(p_1;\ldots;p_g), \tag{1.1}$$

where:

- $E^{\mu}_{\lambda;\mathbf{s}}(p_1;\ldots;p_g)$ includes the polarization vectors of the bosonic external legs, and it is independent of the nature of the internal lines or vertexes of the diagram (μ is the vector of the Lorentz indices);

- $T_{h;\mathbf{s};\mu}(p_1;\ldots;p_g)$ is a projector containing all the tensorial and coupling information of the k-th term of the sum plus the nature of the fermionic part of the external legs;

- $f^{(l)}_h(p_1;\ldots;p_g)$ is a scalar function of the external momenta in which the loop calculations are enclosed, called *form factor*.

Thanks to tensorial reduction it is then possible to project out all the indices (in particular, it is possible to move out all the Lorentz ones) from the loop integration variables on the external momenta.

The fact that the decomposition (1.1) always exists is a consequence of the Lorentz covariance and gauge invariance of scattering amplitudes. The same argument in a reversed way can be used to determine the exact form of all the possible projectors available for the process under study, through relations as Ward–Takahashi identity or BRST (notice that this property of invariance allows to determine projectors independently from the internal propagators of the Feynman diagram investigated or loop number).

1.1.2 Retrieving Feynman integrals

Once the tensorial part has been factorized out and the Wick rotation has been performed on all the loop momenta, the form factor of a diagram with l loops, g external legs and d internal lines can be written as:

$$f^{(l)}_h(p_1;\ldots;p_g) = C_h \int \ldots \int \frac{\prod_{i=1}^{N_h} \mathcal{S}_i^{n_i}}{\prod_{j=1}^{M_h} \mathcal{D}_j} \frac{\mathrm{d}^D k_l}{(2\pi)^{D-2}} \cdots \frac{\mathrm{d}^D k_1}{(2\pi)^{D-2}}, \tag{1.2}$$

where

- C_h is a numerical coefficient;

- S_i is any scalar product formed by either a contraction between a loop momentum and an external one, or by a contraction between two loop momenta. $n_i \in (\mathbb{N} \cup \{0\})$ and N_h is given by[1]

$$N_h = l(g-1) + \frac{l(l+1)}{2} = l\left(g + \frac{l}{2} - \frac{1}{2}\right); \tag{1.3}$$

  Notice that N_h is independent from the nature of the propagators or vertices.

- $\mathcal{D}_j = (q_j^2 + \mathcal{M}_j^2)$ is the factor (from now on, with a little abuse of terminology, *denominator*) corresponding to the j-th inner propagator with momentum q_j^{μ} and mass $\mathcal{M}_j$.

[1] $l(g-1)$ comes from all the possible independent contractions between external legs and loop momenta (there are only $g-1$ independent external momenta for momentum conservation). $l(l+1)/2$ is the number of possible contractions between two loop momenta, taking into account also self-contractions like $k_p \cdot k_p$.

The notation is not yet minimal: among the denominators there could be repeated expressions, and the scalar products in the numerator are not independent from the denominators.

To eliminate the redundancies among the M_h denominators all the identical expressions are gathered together, giving out the structure

$$\prod_{j=1}^{M_h} \mathcal{D}_j \quad \rightarrow \quad \prod_{j=1}^{R_h} \mathcal{D}_j^{m_j} = \prod_{\mathbb{D}} \mathcal{D}_j^{m_j} \tag{1.4}$$

($\mathbb{D}$ is the set of the R_h independent denominators; each denominator is evaluated to a corresponding power $m_j \in \mathbb{N}$ resulting from the rearranging of the original expression).

Looking at the numerator, it can be seen that R_h of the scalar products can be expressed in terms of the independent denominators, leaving only $Q_h = N_h - M_h$ *irreducible scalar products*, composing a set indicated as $\mathbb{S}$. It is still possible to express the elements of such set in terms of propagators, at the cost of adding denominators not present in $\mathbb{D}$).

The form factor shows now the structure:

$$f_h^{(l)}(p_1; \ldots; p_g) = C_h \sum_{t=1}^{T} \int \cdots \int \frac{\prod_{\mathbb{S}} \mathcal{S}_i^{n_{i;t}}}{\prod_{\mathbb{D}} \mathcal{D}_j^{m_{j;t}}} \frac{\mathrm{d}^D k_l}{(2\pi)^{D-2}} \cdots \frac{\mathrm{d}^D k_1}{(2\pi)^{D-2}}. \tag{1.5}$$

The following step is to focus only on each integral at a time, completely forgetting the quantum field theory structure of the process, that is now completely embedded in the tensorial part. Doing so, first of all it is possible to associate to an integral of the form

$$\int \cdots \int \frac{\prod_{\mathbb{S}} \mathcal{S}_i^{n_{i;t}}}{\prod_{\mathbb{D}} \mathcal{D}_j^{m_{j;t}}} \frac{\mathrm{d}^D k_l}{(2\pi)^{D-2}} \cdots \frac{\mathrm{d}^D k_1}{(2\pi)^{D-2}} \tag{1.6}$$

a structure called *topology*, describing how the momentum flows through the minimal set of propagators.

Definition 1 (Topology)**.** Given a set of loop momenta $\mathcal{K} = \{k_1^\mu; \ldots; k_l^\mu\}$, a topology is a couple of sets $(\mathbb{D}; \mathbb{E})$, where $\mathbb{D}$ is the set of different denominators of the form $\mathcal{D}_i = K_i^2 + \mathcal{M}_i^2$ ($\mathcal{M}_i$ is the mass of the i-th propagator, and K_i is a function of at least one of the loop momenta), and $\mathbb{E}$ is the set of the independent external legs $(p_e; \mathcal{M}_e)$ (where p_e is the momentum and $\mathcal{M}_e$ is the mass). This couple $(\mathbb{D}; \mathbb{E})$ satisfies the following drawing rules to generate a graph:

- at each element in $\mathbb{D}$ is associated an oriented line with two extremal points (called *internal line*). The associated momentum flows along this line according to its orientation;
- at each element in $\mathbb{E}$ is associated an oriented line with one extremal point (called *external line*). The associated momentum flows along this line according to its orientation;
- two or more lines can be joint at their extremal points creating vertices. At each vertex momentum conservation must be satisfied, namely the total momentum entering the vertex must be equal to the total momentum exiting the vertex:

$$\sum_{i \text{ entering}} K_i = \sum_{j \text{ exiting}} K_j \quad \text{for each vertex;} \tag{1.7}$$

- the graph is connected.[2]

Given the essential constituents of the diagram (the denominators), and using a principle analogous to the one underlying Kirchhoff's first law, a topology gives a minimal graph representation that can be drawn following the enlisted rules. Notice that this representation is completely separated from the original Feynman diagram, possibly resulting in different organization of the lines or in vertices with five or more lines attached.

Definition 2 (Subtopology)**.** A topology $\mathcal{A}$ is said to be a *subtopology* of $\mathcal{B}$ if

$$\mathbb{D}_{\mathcal{A}} \subset \mathbb{D}_{\mathcal{B}} \quad \text{and} \quad \mathbb{E}_{\mathcal{A}} = \mathbb{E}_{\mathcal{B}} \tag{1.8}$$

At a level of graph representation, a subtopology of a given topology is pictured by a graph that can be obtained from the one of the topology by shrinking one or more internal lines (i.e. erasing the internal lines one by one and merging together the vertices to which the extremal points of each line were jointed). Given a topology, the set of all its subtopologies is called a *subtopology tree*.

The concept of topology is fundamental in multi-loop calculations: given a form factor, all the integrals in the sum of (1.5) share the same topology (at most some of them will belong to the subtopology tree); moreover, all the integrals generated starting from a diagram $\mathcal{F}$ are part of the same subtopology tree.[3]

It is now possible to bring back scalar products and powers of the denominators, retrieving the expression (1.6), that goes under the name of *Feynman integral*:

Definition 3 (Feynman integral (FI))**.** A *Feynman integral* is an integral expression written in terms of elements of $\mathbb{D}$ at the denominator (with powers $m_j \in \mathbb{N}$) and elements of $\mathbb{S}$ at the numerator:

$$\int \cdots \int \frac{\prod_{\mathbb{S}} \mathcal{S}_i^{n_i}}{\prod_{\mathbb{D}} \mathcal{D}_j^{m_j}} \frac{\mathrm{d}^D k_l}{(2\pi)^{D-2}} \cdots \frac{\mathrm{d}^D k_1}{(2\pi)^{D-2}} \tag{1.9}$$

A pictorial representation of the Feynman integral is obtained from the topology graph, putting a number of dots on the lines equal to $m_j - 1$, and reporting the factors in the numerator as coefficients of the graph in the form $\mathcal{S}^i$.[4]

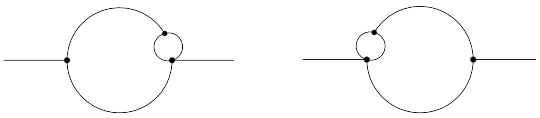
So, while topologies are the common ground for Feynman integrals belonging to the same Feynman diagram and for the relations between them (like "skeletons", showing the essential underlying momentum structure), the Feynman integrals themselves (from now on, FI) are the elementary blocks for multiloop Feynman diagrams evaluation, carrying all and only the information about the complete structure of the loops.

Obviously, starting from a FI a unique topology can always be determined, whilst the inverse path is not unique. Being more precise, from FI to topologies the results are unique module propagator inversions in a set of chained propagators, as in:

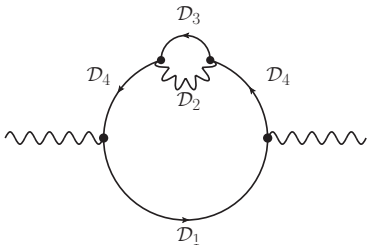
[2] Starting from any vertex it is possible to reach a different one only moving along the internal lines.

[3] The operations of tensor decomposition and reduction to $\mathbb{S}$ for the numerators can only delete some of the denominators, not adding new ones.

[4] The usage to put dots on a line to indicate the power to which the corresponding denominator is risen mimics the fact that having a power m_j on a denominator is like chaining m_j of such lines in sequence.



Example 1 (From Feynman diagrams to Feynman integrals). From [9]. Consider the following QED Feynman diagram with incoming momentum p^μ from a photonic eternal leg:



related to the integral (using Euclidean metric):

$$\int \frac{\gamma^\mu[-\mathrm{i}(\not{p}-\not{k}_1)+m]\gamma^\nu[-\mathrm{i}\not{k}_1+m]\gamma^\rho[-\mathrm{i}(\not{k}_1-\not{k}_2)+m]\gamma_\rho[-\mathrm{i}\not{k}_1+m]}{[k_1^2+m^2][(k_1-p)^2+m^2][(k_1-k_2)^2+m^2]k_2^2[(k_1-p)^2+m^2]}\,\frac{\mathrm{d}^D k_2}{(2\pi)^{D-2}}\,\frac{\mathrm{d}^D k_1}{(2\pi)^{D-2}} \tag{1.10}$$

The sets $\mathbb{D}$ and $\mathbb{S}$ are composed as follows:

$\mathbb{D}$	$\mathbb{S}$	$\mathbb{E}$
$\mathcal{D}_1 = k_1^2 + m^2$	$\mathcal{S} = k_2 \cdot p$	$\mathcal{E} = (p^\mu; p^2)$
$\mathcal{D}_2 = k_2^2$		
$\mathcal{D}_3 = (k_1 - k_2)^2 + m^2$		
$\mathcal{D}_4 = (k_1 - p)^2 + m^2$		

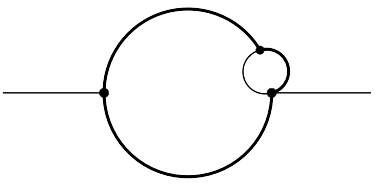
Table 1.1: Sets related to (1.10).

One of the denominators ($\mathcal{D}_1$) is present twice, while among the five scalar products only one is irreducible, and it is chosen to be $\mathcal{S}$ (usually the choice of the elements of $\mathbb{S}$ is not unique). The other scalar products can be rewritten as:

$k_1^2 = \mathcal{D}_1 - m^2$
$k_2^2 = \mathcal{D}_2$
$k_1 \cdot k_2 = \frac{1}{2}(\mathcal{D}_1 + \mathcal{D}_2 - \mathcal{D}_3)$
$k_1 \cdot p = \frac{1}{2}(\mathcal{D}_1 - \mathcal{D}_4 + p^2)$

Table 1.2: Substitutions for reducible scalar products.

The resulting topology has a graph of the following form:



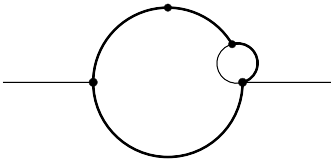
Notice that a peculiar difference between denominators is the value (more generally, the presence) of the mass factor m^2: to distinguish between massive and massless lines the former will be drawn thicker, the latter thinner.

After the tensor decomposition and the substitution of the reducible scalar products the resulting sum of FI will show different topologies, all belonging to the same subtopoloy tree, namely the subtopology tree of the above topology.

The simpler FI belonging to the topology just pictured is:

$$\int \frac{1}{\mathcal{D}_1^2 \mathcal{D}_2 \mathcal{D}_3 \mathcal{D}_4} \frac{\mathrm{d}^D k_2}{(2\pi)^{D-2}} \frac{\mathrm{d}^D k_1}{(2\pi)^{D-2}} \tag{1.11}$$

where the denominators have the same powers owned in the Feynman diagram. The pictorial representation of such a FI is then:



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1.2 Topology relations

As seen before, given a Feynman diagram, or even just one of its form factors, all the FIs in the expression will share the same generator topology (i.e. will belong to the set composed by a given topology and by the elements in its subtopology tree), so in order to completely evaluate the process under study it will be necessary to solve each one of the FIs generated. Denoting with $I_{(t;q;r;s)}$ the set of FIs with t denominators in $\mathbb{D}$, q irreducible scalar products in $\mathbb{S}$, $r = \sum_{j=1}^{\operatorname{card}\mathbb{D}}(m_j - 1)$ and $s = \sum_{i=1}^{\operatorname{card}\mathbb{S}} n_i$, the number of FIs generated is

$$N(I_{(t;q;r;s)}) = \binom{r+t-1}{t-1}\binom{s+q-1}{q-1}. \tag{1.12}$$

Often, especially increasing loop number l, r and s are different from 0, resulting in a quite large number of FIs to be evaluated. The question whether the FIs belonging to a common topology plus related subtopology tree are all independent from each other or not arises then naturally. Fortunately, there is a important number of identities and properties regarding FIs that allows to reduce the number of effectively necessary evaluated integrals (as a reference for this section, consider [9, 41]).

1.2.1 Dimensional regularization

Proposition 1. *Given a FI of the form*

$$I = \int \dots \int f(k_1; \dots; k_l) \frac{\mathrm{d}^D k_l}{(2\pi)^{D-2}} \cdots \frac{\mathrm{d}^D k_1}{(2\pi)^{D-2}} \tag{1.13}$$

where all the loop momenta are massless (i.e. $k_i^2 = \mathcal{M}_1^2 = 0 \, \forall i = 1; \dots; l$) and $f(k_1; \dots; k_l)$ is a continuous scalar function solely of the loop momenta with mass dimension different from $-D$, it must be

$$I = 0 \tag{1.14}$$

for dimensional regularization.

Proof. First of all:

$$J = \int (k^2)^\alpha \, \mathrm{d}^D k = 0 \quad \forall \alpha \in \mathbb{R} \text{ in } D \to 4. \tag{1.15}$$

This has been proven in [42] for $\alpha \in \mathbb{R}$. Another, more naive, way to proof this equivalence is to use dimensional analysis: J has a mass dimension of $[m]^{D+2\alpha}$, so the result must contain a well-dimensioned mass term. Since J does not depend on any external dimensioned variable, and no dimensioned scalar variable is related to it (the only variables present are the loop momenta), J must be null. This strictly applies when $D + 2\alpha \neq 0$, while for $\alpha = -D/2 \to -2$ the factor $\frac{1}{(k^2)^2}$ can be seen as $\frac{1}{k^2}\frac{1}{k^2}$, depicting a vacuum bubble formed by two equal propagators. When $\alpha \in (-\mathbb{N})$ and $\alpha \neq -2$, the corresponding graph is a vacuum bubble with $-\alpha$ propagators all equal to k^2 and it is 0 in dimensional regularization, so the case with $\alpha = -2$ should not be different, since it differs from the previous ones just for a minor (and not null) number of propagators.

A similar argument holds also to prove that $I = 0$: since no dependance from dimensioned external variables is present in the FI, until it has not null mass dimension the only possibile result for the FI is 0. For mass dimension of f equal to $-D$ the structure of f must be investigated, looking for odd terms in the momentum variables or dispositions of the denominators allowing argument as the one above to prove $J = 0$ in $\alpha = -D/2$. □

Thanks to this property all diagrams involving massless tadpoles or completely massless vacuum structure automatically vanish.

Another interesting consequence of the previous proposition is:

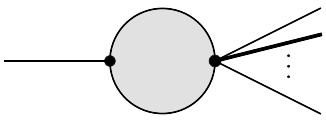
Lemma 1. *A completely massless FI having:*

- *mass dimension* $[m]^\gamma$ *and* $\gamma \neq 0$*;*
- *only two vertices on which external lines are attached;*
- *on one of these two vertices only one external line attached with related momentum* p^μ *satisfying* $p^2 = 0$*;*

is null.

Proof. As for the main proposition, the only scalar dimensional parameter related to the FI is the squared external momentum $p^2 = 0$. Since this variable is 0, so it has to be the FI. □

As a consequence, topologies with massless internal lines having only two point on which external lines can attach, just one massless external line attached to one of these are related to null FIs. Such FIs are of the generic form:



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1.2.2 Redefinition of loop momenta

Since loop momenta are the variables of integration, it is always possible perform a change of coordinates (usually through linear functions) to redefine the flows of the momenta in the internal lines without changing the result of the integration.[5]

The first, and most important, consequence of that invariance is the formulation of an equivalence relation on the set of topologies: all the topologies equal up to a change of coordinates are enclosed in the same class.

Definition 4 (Independent topologies)**.** Two topologies are said to be *independent* if, starting from one of them, is impossible to obtain the other one via change of coordinates on the loop momenta.

Most of the relations found redefining loop momenta may appear trivial, thanks to the intuitive graph representation that can be associated to FIs. Nevertheless, operating on loop momenta is still possible to relate different graphs and to find properties of the integrals themselves or of their sub-integrals.

Starting from a FI having a subgraph without external lines that is connected to the rest of the graph in just two vertices it is always possible to chose k_1^μ as the momentum entering in the sub-graph and $\mathcal{K} = \{k_2; \dots; k_A\}$ as the set of loop momenta only flowing inside the sub-graph. Denoting with n the sum of the powers of the numerator scalar products in which elements of $\mathcal{K}$ are present, indicating with J the FI obtained from the original I by detaching the subgraph and attaching it back swapping the vertices, it is found that:

$$J = (-1)^n I. \tag{1.16}$$

In fact, swapping the vertices results in a graph equal to the original one, except for the inversed subgraph and the related incoming momentum, that changes verse of flow of k_1 and of all the elements of $\mathcal{K}$: $k_1 \to -k_1; \dots; k_A \to -k_A$.

This technique is also useful in getting rid of some loop variables present in scalar products.

Last but not least, by imposing the identity between the original FI and the expression resulting from the redefinition of loop variables, a set of identities relating different FIs (not all belonging to the same topology, but also to similar ones) is obtained, as explained in the next session.

1.2.3 Factorization

If for a FI exists a choice of the loop variables such that the integrand is a factorized expression in the loop variables

$$f(k_1; \dots; k_l) = f_1(\{k_i\}_I) f_2(\{k_j\}_J) \tag{1.17}$$

($\{k_i\}_I$ and $\{k_j\}_J$ are two partition of the set of loop variables $\{k_1; \dots; k_l\}$) then the FI can be considered as a product of two separated FIs.

A remarkable class of factorized FI is the one which graph representation shows a sub-graph connected to the rest of the graph on only one vertex: factorizing the FI consists in separating

[5] An immediate consequence is the fact that odd functions of the loop momenta give FIs equal to 0.

the two sub-graphs on the vertex of connection, then adding to it in both new graphs an external line with incoming momentum equal to

$$\sum_{i\in A_n \text{ entering}} q_i^\mu - \sum_{j\in A_n \text{ exiting}} q_j^\mu \tag{1.18}$$

where A is the set of removed internal lines attached to the vertex in which the separation has been performed to obtain the new graphs ($n = 1; 2$, since incoming and outgoing momenta differs for the detatched graph, indicated by 2, and for the parent one, indicated by 1).

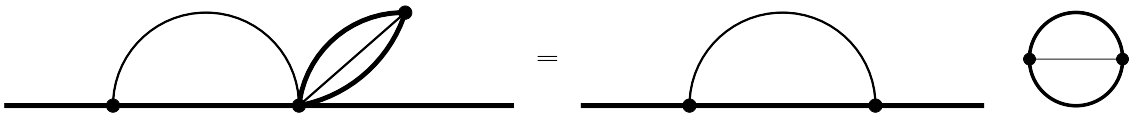
Example 2 (Factorization). From [41].The FI (p^μ is the momentum flowing from the left to the right with $p^2 = m^2$)

$$\int\int\int \frac{1}{k_1^2[(p-k_1)^2+m_1^2](k_2^2+m_2^2)(k_3^2+m_3^2)(k_2+k_3)^2}\,\frac{\mathrm{d}^D k_3}{(2\pi)^{D-2}}\,\frac{\mathrm{d}^D k_2}{(2\pi)^{D-2}}\,\frac{\mathrm{d}^D k_1}{(2\pi)^{D-2}} \tag{1.19}$$

can be factorized as

$$\int \frac{1}{k_1^2[(p-k_1)^2+m_1^2]}\,\frac{\mathrm{d}^D k_1}{(2\pi)^{D-2}} \int\int \frac{1}{(k_2^2+m_2^2)(k_3^2+m_3^2)(k_2+k_3)^2}\,\frac{\mathrm{d}^D k_3}{(2\pi)^{D-2}}\,\frac{\mathrm{d}^D k_2}{(2\pi)^{D-2}}. \tag{1.20}$$

Using FI graphs:



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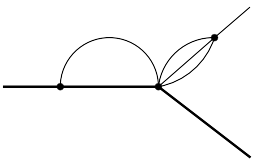
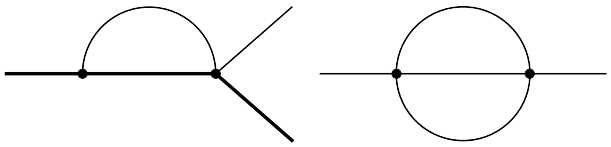
Considering now



this graph can be factorized in



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Thanks to the fact that the second graph of the factorized form is null, satisfying lemma 1, the original FI is 0.

1.2.4 Partial fractioning

A useful technique to separate a topology into a sum of its subtopologies is the *partial fractioning*. At the cost to add more scalar products in the numerator it is possible to rewrite an integral as:

$$\int \frac{1}{\mathcal{D}_1\mathcal{D}_2}\,\mathrm{d}^D\mathcal{K} = \int \frac{f_1(\mathcal{K})}{\mathcal{D}_1}\,\mathrm{d}^D\mathcal{K} + \int \frac{f_1(\mathcal{K})}{\mathcal{D}_2}\,\mathrm{d}^D\mathcal{K}. \tag{1.21}$$

Note that with this technique it is not possible to completely remove one or more loop momenta from one of the obtained integrals, otherwise they will be null, for dimensional regularization.

1.2.5 Integration-by-parts identities

The most wide and useful set of relations used to relate integrals comes from the property of invariance of the FIs under redefinition of loop momenta; these identities are called *integration-by-parts identities*.

As explained in [41], a generic FI is invariant under the transformation

$$k_i'^{\mu} = A_{ij}k_j^{\mu} + B_{ij}p_j^{\mu} \quad i = 1;\ldots;l \tag{1.22}$$

acting on the loop momenta. A is a $l \times l$ invertible matrix, while B is a $l \times g$ rectangular matrix.

The invariance under this kind of substitutions corresponds to an invariance under the action of the Lee group

$$GL(l) \ltimes \mathbb{R}^l. \tag{1.23}$$

Consider now the associated infinitesimal transformation (also the integration measure can change):

$$k_i'^{\mu} = k_i^{\mu} + \alpha_{ij}q_j^{\mu} \quad i \leq l, \tag{1.24}$$

$$\alpha = \begin{pmatrix} \tilde{\alpha}_{l\times l} & \hat{\alpha}_{l\times g} \\ \mathbb{O}_{g\times l} & \mathbb{O}_{e\times g} \end{pmatrix}, \tag{1.25}$$

$$f(\mathcal{K}';\mathcal{P}) = f(\mathcal{K};\mathcal{P}) + \frac{\partial f(\mathcal{Q})}{\partial q_i^{\nu}}\alpha_{ij}q_j^{\nu} = \quad i \leq l, \tag{1.26}$$

$$\mathrm{d}^Dk_1'\ldots\mathrm{d}^Dk_l' = \prod_{i=1}^{l}(1 + D\alpha_{ii})\,\mathrm{d}^Dk_1\ldots\mathrm{d}^Dk_l \tag{1.27}$$

where $\mathcal{K}$ is the set of loop momenta, $\mathcal{P}$ is the set of external momenta, $\mathcal{Q} = \mathcal{K} \cup \mathcal{P}$ ordered as $k_1^{\mu};\ldots;k_l^{\mu};p_1^{\mu};\ldots;p_g^{\mu}$, $q_i^{\mu} \in \mathcal{Q}$ and α is the $(l+g)\times(l+g)$ matrix of infinitesimal increases, with $\tilde{\alpha}_{l\times l}$ and $\hat{\alpha}_{l\times g}$ generally non-zero rectangular sub-matrices.[6]

[6] α has all elements with $i > l$ equal to 0 because the transformation under which the integral is invariant involves only the loop momenta k_i^{μ}.

To explain (1.27) consider the variation of the volume measure under a change of coordinates:

$$\mathrm{d}^D\mathcal{K}' = \left|\det\frac{\partial\mathcal{K}'}{\partial\mathcal{K}}\right|\mathrm{d}^D\mathcal{K} = |\det(\mathbb{I}+\hat{\alpha})|\;\mathrm{d}^D\mathcal{K} = |1 + D\,\mathrm{tr}\,\hat{\alpha}|\;\mathrm{d}^D\mathcal{K} = (1 + D\,\mathrm{tr}\,\hat{\alpha})\,\mathrm{d}^D\mathcal{K} \tag{1.28}$$

where the chain of equivalences holds at first order in α_{ij}. $\hat{\alpha}$ is the sub-matrix obtained from α removing the

The transformation acting on the integral produces then (always at the first order)

$$\delta I = \int \alpha_{ij} \left(D\delta_{ij} + q_j^\nu \frac{\partial}{\partial q_i^\nu} \right) f(\mathcal{Q}) \, \mathrm{d}^D k_1 \dots \mathrm{d}^D k_l = 0 \quad i \leq l, \tag{1.30}$$

allowing to isolate the generator operator

$$O_{ij} = D\delta_{ij} + q_j^\nu \frac{\partial}{\partial q_i^\nu} = \frac{\partial}{\partial q_i^\nu} \cdot q_j^\nu \quad i \leq l, \tag{1.31}$$

with commutation rules

$$[O_{ij}; O_{qr}] = \delta_{ir} O_{qj} - \delta_{qj} O_{ir}, \tag{1.32}$$

giving the structure constant of the Lie algebra.

To conclude the discussion, considering that $1 \leq i \leq l$ identifies the loop variable set, the action of the operator on a FI can be rewritten as:

$$\boxed{\int \frac{\partial}{\partial k_i^\mu} \left(q_j^\mu f(\mathcal{Q}) \right) \mathrm{d}^D \mathcal{K} = 0} \tag{1.33}$$

This is the classical form of the integration-by-parts (IBP) identities.

Some remarks on the IBP identities are enlisted below.

1. Given a FI, $l(l + g - 1)$ IBP-ids can be generated.

2. All the FI comparing in a set of IBP-ids belong to the same topology plus related subtopology tree: applying the IBP operator the result may differ for some missing denominators (resulting in a subtopology) or for some scalar products (which presence do not alter the topology), whereas new denominators cannot appear; in particular, before involving subtopologies due to simplifications, IBP-ids involve integrals $I_{(t;q;r;s)}$ (derivative acting on the IBP momentum), $I_{(t;q;r;s-1)}$ (differentiating deletes an irreducible scalar product, substituting it with a mass term), $I_{(t-1;q;r;s-1)}$ (irreducible scalar product changed to a reducible one, without completely erasing a denominator), $I_{(t+1;q;r;s)}$ (derivative acting on one of the denominators) and $I_{(t+1;q;r;s+1)}$ (differentiating one of the denominators generates an irreducible scalar product in the numerator).

3. IBP-ids *cannot* relate integrals with different loop number: if an IBP completely eliminates the presence of a loop momentum the corresponding integral would be factorizable as $\int \mathrm{d}k_1 \int f(k_2; \dots; k_l) \, \mathrm{d}k_2 \dots \mathrm{d}k_l$, but then it will be null due to dimensional regularization (see proposition 1).

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elements with $i > l$ and $j > l$ and the dimension factor D arises in the trace due to the fact that each loop momentum has D components. Finally, the module can be dropped considering α_{ij} as infinitesimal quantities.

Due to the presence of the trace, the infinitesimal variation operates only if $i = j$ in α_{ij}, so:

$$\mathrm{d}^D \mathcal{K}' = (1 + D \operatorname{tr} \hat{\alpha}) \, \mathrm{d}^D \mathcal{K} = \prod_{i=1}^{l} (1 + D\alpha_{ii}) \, \mathrm{d}^D k_i \tag{1.29}$$

still at first order in α_{ij}.

4. Given a topology, the IBP-ids that can be written are infinite, due to the infinite number of possible powers of denominators and irreducible scalar products.[7]

5. An alternative way to demonstrate the truth of the IBP relations is via the divergence theorem: starting from the simple case

$$\int \frac{\partial}{\partial k^\mu}\left(\frac{k^\mu}{(k^2+m^2)^\alpha}\right)\mathrm{d}^D k \tag{1.34}$$

(where the integral is over $\mathbb{R}^D$) it is possible to apply the divergence theorem and pass to spherical coordinates:

$$\int_{\partial\mathbb{R}^D}\frac{\|k\|^{D-1}}{(k^2+m^2)^\alpha}k\cdot\hat{n}\,\mathrm{d}\Omega(D)=\lim_{r\to+\infty}\frac{r^D}{(r^2+m^2)^\alpha}\Omega(D) \tag{1.35}$$

(where $\Omega(D)=\frac{S_D(\rho)}{\rho^D}$ is the solid-angle in D dimensions). This expression vanish for sufficiently small dimensions $D<2\alpha$.[8]

Example 3 (1-loop massive vacuum diagram). Consider the 1-loop massive vacuum diagram FI, also known as *tadpole* ($n\in\mathbb{N}$):

$$I_n=\int\frac{1}{(k^2+m^2)^n}\frac{\mathrm{d}^D k}{(2\pi)^{D-2}}. \tag{1.36}$$

The only IBP for it is

$$\int\frac{\partial}{\partial k^\mu}\frac{k^\mu}{(k^2+m^2)^n}\frac{\mathrm{d}^D k}{(2\pi)^{D-2}}=0 \tag{1.37}$$

$$\Downarrow$$

$$(D-2n)\int\frac{1}{(k^2+m^2)^n}\frac{\mathrm{d}^D k}{(2\pi)^{D-2}}=-2nm^2\int\frac{1}{(k^2+m^2)^{n+1}}\frac{\mathrm{d}^D k}{(2\pi)^{D-2}} \tag{1.38}$$

Iterating such identity (and remembering that $\Gamma(n+1)=n!$) it is possible to relate all the I_n to I_1:

$$I_n=\frac{\Gamma\left(n-\frac{D}{2}\right)}{(m^2)^{n-1}\Gamma(n)\Gamma\left(1-\frac{D}{2}\right)}I_1; \tag{1.39}$$

to indicate the fact that, knowing only the value of I_1, it is possible to retrieve $I_n\,\forall n\in\mathbb{N}$, I_1 is called the *master integral* of the topology.

1.2.6 Lorentz-invariance identities

Another important class of identities is derived considering the nature of the FIs with respect to the Lorentz group: being Lorentz scalars, FIs are invariant under the action of the Lorentz group $O(1;3)$ acting on the momenta carried by the external lines. As a result:

$$p_e'^\mu=p_e^\mu+\omega^\mu{}_\nu p_e^\nu, \tag{1.40}$$

$$I(\mathcal{P}_e')=I(\mathcal{P})+\frac{\partial I(\mathcal{P})}{\partial p_e^\mu}\omega^\mu{}_\nu p_e^\nu \tag{1.41}$$

[7] This apparently irrelevant comment is quite important: not always starting from a given FI the most useful IBP-ids can be determined, while starting from FI with higher values of powers can bring to useful simplifications, as for the case of the Lorentz invariance identities (see section 1.2.6).

[8] In operative cases, thanks to the high number of denominators, this condition is always satisfied, so the IBP-ids are proven to be correct.

where $\omega^{\mu}{}_{\nu}$ is the generator of the infinitesimal Lorentz transformation (a totally antisymmetric tensor) and $\mathcal{P}_e^{\mu} = (p_1^{\mu}; \ldots; p_e'^{\mu}; \ldots; p_g^{\mu})$.

Using the invariance of FIs and the fact that $\omega_{\mu\nu} = \omega_{[\mu\nu]} = \frac{1}{2}(\omega_{\mu\nu} - \omega_{\nu\mu}) \neq 0$ a relation is found:

$$\left(p_{e\nu}\frac{\partial}{\partial p_e^{\mu}} - p_{e\mu}\frac{\partial}{\partial p_e^{\nu}}\right) I(\mathcal{P}) = 0, \quad e = 1; \ldots; g; \tag{1.42}$$

summing over all the external momenta:

$$\sum_{e=1}^{g}\left(p_{e\nu}\frac{\partial}{\partial p_e^{\mu}} - p_{e\mu}\frac{\partial}{\partial p_e^{\nu}}\right) I(\mathcal{P}) = 0. \tag{1.43}$$

Finally, contracting with all the antisymmetric couples of a maximal set of external *independent* momenta:

$$2\left(p_i^{[\mu}p_j^{\nu]}\right)\sum_{e=1}^{g}\left(p_{e[\nu}\frac{\partial}{\partial p_e^{\mu]}}\right) I(\mathcal{P}) = 0 \quad \forall i; j = 1; \ldots; g \text{ independent.} \tag{1.44}$$

These expressions go under the name of *Lorentz-invariance identities* (LI).

This result can also be seen as the application of the generators of the Lorentz rotation (the angular momentum $L_{\mu\nu} = p_{e[\nu}\frac{\partial}{\partial p_e^{\mu]}}$) to a Lorentz scalar.

Concerning LI-ids, a remarkable property can be demonstrated ([43]):

Proposition 2. *The LI-ids of a certain topology can be expressed as a linear combination of IBP-ids of the same topology or of its subtopology tree.*

Proof. First of all, notice that th differentiation with respect to the external momenta can commute with the integral on the loop momenta, so it can be applied directly to the integrand. Adding and subtracting the expression $2\left(p_a^{[\mu}p_b^{\nu]}\right)\sum_{i=1}^{l}\left(k_{i[\nu}\frac{\partial}{\partial k_i^{\mu]}}\right)$ to the LI operator:

$$2\left(p_a^{[\mu}p_b^{\nu]}\right)\left(\sum_{j=1}^{l+g} q_{j[\nu}\frac{\partial}{\partial q_j^{\mu]}} - \sum_{i=1}^{l} k_{i[\nu}\frac{\partial}{\partial k_i^{\mu]}}\right) f(\mathcal{Q}) = 0. \tag{1.45}$$

Since $f(\mathcal{Q})$ is a Lorentz scalar function of q_j^{μ}, the first operator gives 0 acting on f (it is essentially the generator of the rotations in the space $(k_1; \ldots; k_l; p_1; \ldots; p_g)$), the expression becomes:

$$2\int\left(p_a^{[\mu}p_b^{\nu]}\right)\sum_{i=1}^{l} k_{i[\nu}\frac{\partial}{\partial k_i^{\mu]}} f(\mathcal{Q})\,\frac{\mathrm{d}^D\mathcal{K}}{(2\pi)^{l(D-2)}} = \sum_{i=1}^{l}\int\frac{\partial}{\partial k_i^{\mu}}\left[(p_a^{\mu}p_b\cdot k_i - p_b^{\mu}p_a\cdot k_i)\, f(\mathcal{Q})\right]\frac{\mathrm{d}^D\mathcal{K}}{(2\pi)^{l(D-2)}}. \tag{1.46}$$

Thus LI-id can be expressed in term of IBP-ids of the same topology (the added derivative of k_i with respect to k_i^{μ} vanishes due to antisymmetry), by adding a scalar product on the numerator (that can at most reduce the FI topology to a subtopology of the original one). □

1.2.7 Are IBP identities enough?

After having shown that LI-ids are just a particular set of IBP-ids, a question may arise: does the IBP mechanism generate all the meaningful relations for a given topology?

At least it is not true for the discrete ones, like redefinition of loop momenta, partial fractioning or factorization of graphs, due to the fact that IBP-ids originates from a continuous and connected group, whilst the swapping loop momenta and the rewriting of fractions are encoded by discrete rules.

Even restricting the domain of the question to Lie groups the answer is not clear: on one hand it seems that a group of relations among FIs, called *Larin identities*, is not related to the IBP-ids, and on the other hand the IBP-ids are infinite in number for a given topology, so new structure may arise with higher values of the parameters.

Another important aspect of the investigated relations is that they are not all independent: as seen with LI-ids and IBP-ids, a huge part of the relations is redundant.

In any case, despite the fact that the question on the origin of relations among FIs is very significant as a theoretical aspect, another important question is: in the case that all "continuous" relations are generated by IBP mechanism, is this useful to compute all such relations with only this one mechanism?

At present days most of the recursive work, like the determination of the relations among FIs, is assigned to computers. If the same relations that could be obtained by immediate evaluation (like the LI-ids, related to manifest invariance of the expressions) were obtained by iterated computation of more complex structures (like IBP, involving differentiation and algebraic rearranging of the terms) the only result, especially for bigger topologies, will be a really huge demand of resources in terms of computational time and computer hardware.

To avoid unnecessary waste of resources, all current programs for the evaluation of relations among FIs (like `Reduze` [46, 47], or `FIRE` [51]) use different approaches to the generation of relations, like IBP, Lorentz invariance or change-of-coordinates invariance.

Chapter 2

Feynman integrals evaluation

Two ways of evaluating Feynman integrals are presented. At first, integral evaluation via Feynman parameters [20, 44] for sunset topologies is used to find a generic expression for sunset and bubble integrals [41]. The subsequent part presents a detailed discussion of the method of *differential equations for Feynman integrals* [4, 9, 20], as well as of the fixing of *boundary conditions* [4, 9–12], and of the existence and uniqueness of the solutions [4, 9, 20]. Finally, the notion of *canonical* system [9, 10] is introduced starting from ϵ-factorized systems.

2.1 Master integrals and Laporta algorithm

Using the relations investigated in the previous chapter it is possible to express the FIs of a specific topology in terms of either other FIs of the same topology or of FIs belonging to the subtopology tree, or even of FIs of different topologies. Usually, the number of such fundamental FIs is greatly inferior to the total number of initial integrals, and the knowledge of this relatively restricted set of functions allows to obtain, with just algebraic operations, the values of all the others. These "fundamental" FIs are called *master integrals* (MI) of the system.

The amount of MIs for a given topology may vary drastically from several, to one, to none: in this last case, the topology is said to be *reducible*, and can be completely expressed in terms of MIs belonging to different topologies (usually located in the subtopology tree of the orginal FI).

Unfortunately, it has not yet discovered a criterion able to determine if, starting from a FI or a set of FIs, a given set of MIs is truly a *basis* (i.e. is a maximal set of independent functions with respect to all the possible algebraic relations among FIs). Nevertheless, starting from even thousands of FIs, the number of MIs is usually about tens, so in any case an enormous simplification of the calculations needed has been performed. It is even possible that some MIs that appear to be independent are in fact related via identities (usually, IBP-ids) only derived at higher orders; it would not be surprising, since relations like IBP-ids are infinite in number and not fully understood.

Fortunately, at least it has been proven that the number of MIs for a problem starting from a Feynman diagram is always finite (see [52]).

The set of MIs is not fixed or unique, and the choice of a suitable basis is not an easy problem: it is necessary to avoid uselessly complicated FIs, and to find expression as regular as possible for $D \to 4$ to obtain a smart form for the FI under investigation. This is not an easy task to complete without some sort of automated classification of the "complexity" of a FI, due to the overwhelming number of function to examine. Moreover, once the basis has been chosen, the relations among FIs must be solved in order to express all the other integrals in terms of the MIs. A parallel problem is the fact that not all the relations determined before are independent, so the problem to select the easiest set of independent ones has to be considered.

The Laporta algorithm introduces a solution for the problem (for a detailed discussion, see [17]): the core of the procedure is the determination of a "weight" function for FIs, an increasing function of the exponents n_i and m_j, such as FIs with higher powers have higher weight; once this is done the most weighty FIs are expressed in terms of the less weighty ones, trying to minimize the total weight of the independent FIs. In this way, a set of MIs is determined.

Once a set of MIs is fixed, their evaluation can be performed in two different ways.

- **Integral evaluation**: for simple integrals a direct calculation can be performed, usually thanks to Feynman parametrization. Feynman parametrization is often useful also on more complex integrals to investigate some properties of the FIs without direct evaluation, like properties of their series expansion.

- **Differential evaluation**: solving differential relations among MIs (determined differentiating with respect to external parameters and simplifying the results using algebraic relations, like IBP-ids) and imposing boundary conditions a series expansion in $\epsilon = \frac{4-D}{2}$ of the integrals is obtained.

2.2 Integral evaluation

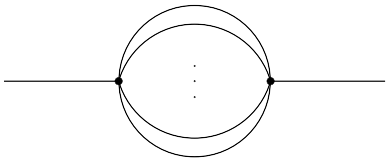
Given an expression of the form $\frac{1}{A_1^{\alpha_1} \dots A_n^{\alpha_n}}$, with $\alpha_i \in \mathbb{N}$, it is possible to rewrite it as:

$$\frac{1}{A_1^{\alpha_1} \dots A_n^{\alpha_n}} = \frac{\Gamma\left(\sum_{i=1}^n \alpha_i\right)}{\prod_{j=1}^n \Gamma(\alpha_i)} \int_0^1 \dots \int_0^1 \frac{\delta\left(\sum_{i=1}^n x_i - 1\right) \prod_{k=1}^n x_k^{\alpha_k - 1}}{[\sum_{l=1}^n x_l A_l]^{\sum_{m=1}^n \alpha_m}} \, \mathrm{d}x_n \dots \, \mathrm{d}x_1. \tag{2.1}$$

This technique is called *Feynman parametrization* (x_i are the *Feynman parameters*, for more details, see [20]). For $n = 2$, the expression simplifies as

$$\frac{1}{A_1^{\alpha_1} A_2^{\alpha_2}} = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 \frac{x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1}}{[xA_1 + (1-x)A_2]^{\alpha_1 + \alpha_2}} \, \mathrm{d}x. \tag{2.2}$$

Feynman parameters are useful to evaluate selected elements belonging to the set of *sunset graphs*. This set contains graphs with topology



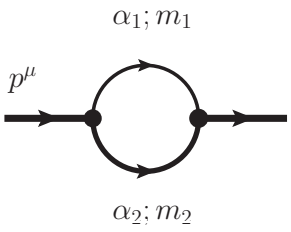
with either massive or massless lines.

2.2.1 Bubble integrals

A bubble integral is a loop integral of the form:

$$I = \int \frac{\mathrm{d}^D k}{[(k-p)^2 - m_1^2]^{\alpha_1}[k^2 - m_2^2]^{\alpha_2}}. \tag{2.3}$$

corresponding to the graph



.

Reintroducing the imaginary prescription for the pole and rearranging the terms in the denominator ($p \cdot p = s$, the arrows indicate the flowing of the momentum):

$$I = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 \int \frac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{[(k-px)^2 - (-x(1-x)s + x(m_1^2-m_2^2)+m_2^2) + \mathrm{i}\eta]^{\alpha_1+\alpha_2}} \,\mathrm{d}^D k \,\mathrm{d}x. \tag{2.4}$$

Operating the change of variable $k'^\mu = k^\mu + p^\mu x$ ($\mathrm{d}^D k' = \mathrm{d}^D k$), and performing Wick's rotation:

$$I = \mathrm{i}\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 \int \frac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{[-(k'^2+\Omega) - \mathrm{i}\eta]^{\alpha_1+\alpha_2}} \,\mathrm{d}^D k' \,\mathrm{d}x, \tag{2.5}$$

then

$$I = \mathrm{i}(-1)^{\alpha_1+\alpha_2}\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 \int \frac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{[k'^2+\Omega+\mathrm{i}\eta]^{\alpha_1+\alpha_2}} \,\mathrm{d}^D k' \,\mathrm{d}x, \tag{2.6}$$

with

$$\Omega = \Omega(m_1^2; m_2^2; s; x) = (-x(1-x)s + x(m_1^2-m_2^2) + m_2^2). \tag{2.7}$$

The integral in the loop momentum can be solved as:

$$\int \frac{\mathrm{d}^D k'}{(k'^2+\Omega)^{\alpha_1+\alpha_2}} = (4\pi)^{\epsilon-2}\frac{\Omega^{2-\alpha_1-\alpha_2-\epsilon}\Gamma(\alpha_1+\alpha_2-2+\epsilon)}{\Gamma(\alpha_1+\alpha_2)}, \tag{2.8}$$

so I has the structure:

$$I = \mathrm{i}(-1)^{\alpha_1+\alpha_2}(4\pi)^{\epsilon-2}\frac{\Gamma(\alpha_1+\alpha_2-2+\epsilon)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^1 \Omega^{2-\alpha_1-\alpha_2-\epsilon}x^{\alpha_1-1}(1-x)^{\alpha_2-1}\,\mathrm{d}x. \tag{2.9}$$

According to the masses and the momenta associated with the lines of the graph, several expression can be determined.

- Massless bubble with massless external legs: $m_1^2 = m_2^2 = 0 = s$: the integral is equal to 0 for dimensional regularization (proposition 1).
- Massless bubble: $m_1 = m_2 = 0$ implies $\Omega = -x(1-x)s$, so the integral becomes

$$(-s)^{2-\alpha_1-\alpha_2-\epsilon}\int_0^1 x^{2-\alpha_1-1-\epsilon}(1-x)^{2-\alpha_2-1-\epsilon}\,\mathrm{d}x = \\ = (-s)^{2-\alpha_1-\alpha_2-\epsilon}\frac{\Gamma(2-\alpha_1-\epsilon)\Gamma(2-\alpha_2-\epsilon)}{\Gamma(4-\alpha_1-\alpha_2-2\epsilon)}. \tag{2.10}$$

- Massless external legs: $s = 0$ and $m_1 = m_2 = m \neq 0$ implies $\Omega = m^2$, and the integral simplifies as

$$(m^2)^{2-\alpha_1-\alpha_2-\epsilon}\int_0^1 x^{\alpha_1-1}(1-x)^{\alpha_2-1}\,\mathrm{d}x = (m^2)^{2-\alpha_1-\alpha_2-\epsilon}\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}; \tag{2.11}$$

Considering $m_1 \neq 0$ or $m_2 \neq 0$ generally causes the integral to be not solvable in terms of simple expressions of known functions.

- Equal-mass massive bubble: $m_1 = m_2 = m \neq 0$, so $\Omega = -x(1-x)s + m^2$.
- Half-massive bubble: $m = m_1 \neq m_2 = 0$, so $\Omega = -x(1-x)s + xm^2$. Assuming also $s = 0$, so $\Omega = xm^2$, an integrable expression is found:

$$(m^2)^{2-\alpha_1-\alpha_2-\epsilon}\int_0^1 x^{2-\alpha_2-1-\epsilon}(1-x)^{\alpha_2-1}\,\mathrm{d}x = \\ = (m^2)^{2-\alpha_1-\alpha_2-\epsilon}\frac{\Gamma(2-\alpha_2-\epsilon)\Gamma(\alpha_2)}{\Gamma(2-\epsilon)}; \tag{2.12}$$

  moreover, when $m_1^2 = m^2 = s$ ($\mathbf{p} = 0$), when $\Omega = x^2m^2$:

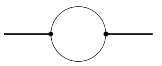
$$(m^2)^{2-\alpha_1-\alpha_2-\epsilon}\int_0^1 x^{4-\alpha_1-2\alpha_2-1-2\epsilon}(1-x)^{\alpha_2-1}\,\mathrm{d}x = \\ = (m^2)^{2-\alpha_1-\alpha_2-\epsilon}\frac{\Gamma(4-\alpha_1-2\alpha_2-2\epsilon)\Gamma(\alpha_2)}{\Gamma(4-\alpha_1-\alpha_2-2\epsilon)}. \tag{2.13}$$

All the integrals above share the same constant factor $\mathrm{i}(4\pi)^{\epsilon-2}$, that is then redundant. Moreover, it is useful to obtain dimensionless terms depending on ϵ. To reach such results it is possible to perform a smart redefinition of the integration measure, like:

$$\boxed{\mathrm{d}_0^D k = \frac{(\mu^2)^\epsilon \, \mathrm{d}^D k}{\mathrm{i}\pi^{2-\epsilon}},} \tag{2.14}$$

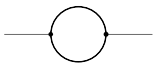
where μ is an auxiliary parameter with mass dimension $[m]^1$. Rewriting also the expression $(-1)^{\alpha_1+\alpha_2}(-s)^{2-\alpha_1-\alpha_2-\epsilon}$ as $(-s)^{-\epsilon}s^{2-\alpha_1-\alpha_2}$ the integrals assume the form:

- $m_1 = m_2 = 0$:



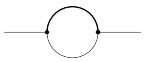
$$= \left(-\frac{s}{\mu^2}\right)^{-\epsilon} s^{2-\alpha_1-\alpha_2} \frac{\Gamma(\alpha_1+\alpha_2-2+\epsilon)\Gamma(2-\alpha_1-\epsilon)\Gamma(2-\alpha_2-\epsilon)}{\Gamma(4-\alpha_1-\alpha_2-2\epsilon)\Gamma(\alpha_1)\Gamma(\alpha_2)} \tag{2.15}$$

- $s = 0$ and $m_1 = m_2 = m \neq 0$:



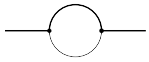
$$= \left(\frac{m^2}{\mu^2}\right)^{-\epsilon} \left(-m^2\right)^{2-\alpha_1-\alpha_2} \frac{\Gamma(\alpha_1+\alpha_2-2+\epsilon)}{\Gamma(\alpha_1+\alpha_2)} \tag{2.16}$$

- $m_1 \neq 0$ and $m_2 = 0 = s$:



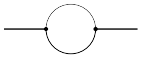
$$= \left(\frac{m^2}{\mu^2}\right)^{-\epsilon} \left(-m^2\right)^{2-\alpha_1-\alpha_2} \frac{\Gamma(\alpha_1+\alpha_2-2+\epsilon)\Gamma(-\alpha_2-2-\epsilon)}{\Gamma(\alpha_1)\Gamma(-2+\epsilon)} \tag{2.17}$$

- $m_1^2 = s \neq 0$ and $m_2 = 0$:



$$= \left(\frac{m^2}{\mu^2}\right)^{-\epsilon} \left(-m^2\right)^{2-\alpha_1-\alpha_2} \frac{\Gamma(\alpha_1+\alpha_2-2+\epsilon)}{\Gamma(\alpha_1)\Gamma(4-\alpha_1-\alpha_2-2\epsilon)} \tag{2.18}$$

- $m_1 \neq 0$ or $m_2 \neq 0$:



$$= (-1)^{\alpha_1+\alpha_2} \frac{\Gamma(\alpha_1+\alpha_2-2+\epsilon)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^1 \Omega^{2-\alpha_1-\alpha_2-\epsilon} x^{\alpha_1-1}(1-x)^{\alpha_2-1} \, \mathrm{d}x \tag{2.19}$$

Example 4 (Tadpole). A tadpole is a FI of the form:

$$I(m^2) = \int \frac{\mathrm{d}_0^D k}{(k^2+m^2)^\alpha}. \tag{2.20}$$

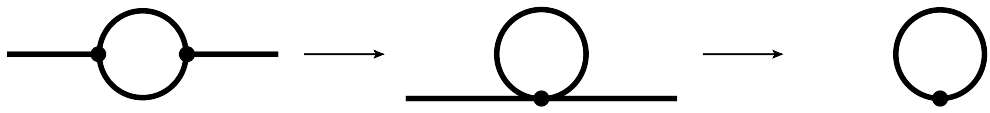
It is possible to follow the algorithm used above to determine the explicit expression for the bubbles, where now $\Omega = m^2$, $k'^\mu = k^\mu - p^\mu$ and the Feynman parametrization in no more necessary. The result is:

$$I(m^2) = \left(m^2\right)^{2-\alpha-\epsilon} \frac{\Gamma(\alpha-2+\epsilon)}{\Gamma(\alpha)}. \tag{2.21}$$

Another way to derive the analytic expression for the massive vacuum diagram is by starting from the integral

$$\int \frac{\mathrm{d}_0^D k}{[k^2-m^2]^{\alpha_1}[(k-p)^2-m^2]^{\alpha_2}} \tag{2.22}$$

and putting $\alpha_1 = \alpha$ and $\alpha_2 = 0$ the loop does not depend on external momentum anymore, resulting equivalent to a vacuum tadpole:



It is possible to proceed as above without Feynman parameters. A third way consists of considering integral (2.22) with $p^2 = s = 0$, using equation (2.16), putting $\alpha_1 + \alpha_2 = \alpha$: thanks to the fact that the external momentum in Feynman parametrization is present only as a squared parameter, the formula above is valid also in the case of a momentum with all components separately vanishing, reproducing the vacuum bubble. The expression is then:

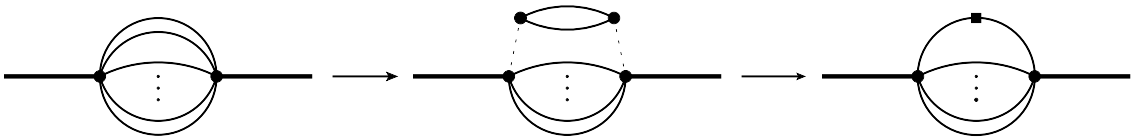
$$\left(\frac{m^2}{\mu^2}\right)^{-\epsilon} \left(-m^2\right)^{2-\alpha} \frac{\Gamma(\alpha - 2 + \epsilon)}{\Gamma(\alpha)}, \tag{2.23}$$

as before.

2.2.2 Sunset integrals

A smart way to evaluate an l-loop (or n-propagator, with $n = l + 1$) massless sunset integral is to detach a couple of internal lines at a time and substitute them with the expression of the corresponding massless bubble seen above.

To perform this substitution, first of all a change of loop variables is needed to associate to the couple of internal lines momenta of the form $(k; k + k_1)$ (k denotes a combination of loop momenta not containing k_1). It is then possible to substitute this bubble with expression (2.15), where $s = k^2$:



Example 5 (2-loop massless sunset). Consider a 2-loop sunset integral, with internal lines carrying momenta k_1^μ, $(k_2 - k_1)^\mu$ and $(p - k_2)^\mu$ (where p^μ is the external momentum), with powers α_1, α_2 and α_3 respectively. The detachable couple, with external momentum k_2^μ, is $(k_1^\mu; (k_2 - k_1)^\mu)$, reducible using the expression (2.15):

$$\begin{aligned}
&\int\int \frac{\mathrm{d}^D k_1 \mathrm{d}^D k_2}{[k_1^2]^{\alpha_1}[(k_2 - k_1)^2]^{\alpha_2}[(p - k_2)^2]^{\alpha_3}} = \\
&\quad = \left(-\mu^2\right)^{\epsilon} \frac{\Gamma(\alpha_1 + \alpha_2 - 2 + \epsilon)\Gamma(2 - \alpha_1 - \epsilon)\Gamma(2 - \alpha_2 - \epsilon)}{\Gamma(4 - \alpha_1 - \alpha_2 - 2\epsilon)\Gamma(\alpha_1)\Gamma(\alpha_2)} \int \frac{\mathrm{d}^D k_2}{[k_2^2]^{\alpha_1 + \alpha_2 - 2 + \epsilon}[(p - k_2)^2]^{\alpha_3}} = \\
&\quad = \left(-\frac{s}{\mu^2}\right)^{-2\epsilon} s^{4 - \alpha_1 - \alpha_2 - \alpha_3} \frac{\Gamma(2 - \alpha_1 - \epsilon)\Gamma(2 - \alpha_2 - \epsilon)\Gamma(2 - \alpha_3 - \epsilon)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)} \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3 - 4 + 2\epsilon)}{\Gamma(4 - \alpha_1 - \alpha_2 - \alpha_3 - 2\epsilon)}.
\end{aligned} \tag{2.24}$$

It is possible to generalize the result obtained in the previous example to a massless sunset with an arbitrary number of loops (to confront the final result, see [41]):

Proposition 3 (Massless sunset FI)**.** *Given a massless sunset FI with incoming momentum p^μ and n internal lines with powers $\{\alpha_n\}_{1;\dots;n}$, its analytic expression has the form:*

$$\left(-\frac{p^2}{\mu^2}\right)^{-(n-1)\epsilon} \left(p^2\right)^{2(n-1)-\sum_{j=1}^n \alpha_j} \frac{\Gamma\left(\sum_{l=1}^n \alpha_l + (n-1)(\epsilon-2)\right)}{\Gamma\left(-\sum_{m=1}^n \alpha_m + n(2-\epsilon)\right)} \prod_{k=1}^n \frac{\Gamma(2-\alpha_k-\epsilon)}{\Gamma(\alpha_k)}. \tag{2.25}$$

Proof. The demonstration is carried out by induction.

First of all, the formula is verified for the basic case $n = 2$ (1-loop massless bubble): here it reproduces the analytic result (2.15):

$$\left(-\frac{p^2}{\mu^2}\right)^{-\epsilon} \left(p^2\right)^{2-\alpha_1-\alpha_2} \frac{\Gamma(\alpha_1+\alpha_2-2+\epsilon)\Gamma(2-\alpha_1-\epsilon)\Gamma(2-\alpha_2-\epsilon)}{\Gamma(4-\alpha_1-\alpha_2-2\epsilon)\Gamma(\alpha_1)\Gamma(\alpha_2)} \tag{2.26}$$

For a generic n-propagator massless sunset ($(n-1)$-loop massless sunset), assuming that the first $n-1$ internal lines have been reduced to a single one using the method explained in example 5, the equation for the final step is:

$$\begin{aligned}
&\left(-\frac{p^2}{\mu^2}\right)^{-(n-2)\epsilon} \left(p^2\right)^{2(n-2)-\sum_{j=1}^{n-1}\alpha_j} \\
&\qquad \frac{\Gamma\left(\sum_{l=1}^{n-1}\alpha_l + (n-2)(\epsilon-2)\right)}{\Gamma\left(-\sum_{m=1}^{n-1}\alpha_m + (n-1)(2-\epsilon)\right)} \prod_{k=1}^{n-1} \frac{\Gamma(2-\alpha_k-\epsilon)}{\Gamma(\alpha_k)} \left(-\frac{p^2}{\mu^2}\right)^{-\epsilon} \left(p^2\right)^{2-\alpha_n} \\
&\frac{\Gamma\left(\sum_{h=1}^{n-1}\alpha_h + \alpha_n - 2 + (n-2)(\epsilon-2) + \epsilon\right) \Gamma\left(-\sum_{q=1}^{n-1}\alpha_q + 2 + (n-2)(2-\epsilon) - \epsilon\right) \Gamma(2-\alpha_n-\epsilon)}{\Gamma\left(4-\sum_{y=1}^{n}\alpha_y + (n-2)(2-\epsilon) - 2\epsilon\right) \Gamma\left(\sum_{z=1}^{n-1}\alpha_z + (n-2)(\epsilon-2)\right) \Gamma(\alpha_n)} = \\
&\qquad = \left(-\frac{p^2}{\mu^2}\right)^{-(n-1)\epsilon} \left(p^2\right)^{2(n-1)-\sum_{j=1}^n \alpha_j} \frac{\Gamma\left(\sum_{l=1}^n \alpha_l + (n-1)(\epsilon-2)\right)}{\Gamma\left(-\sum_{m=1}^n \alpha_m + n(2-\epsilon)\right)} \prod_{k=1}^n \frac{\Gamma(2-\alpha_k-\epsilon)}{\Gamma(\alpha_k)},
\end{aligned} \tag{2.27}$$

and this proves the thesis. □

2.3 Differential evaluation

2.3.1 Constructing differential equations

As explained in [4, 9], a given FI is a scalar integral depending on a set of not integrated parameters $\mathcal{S}$, as momenta of the external lines, masses of all the lines, Mandelstam invariants. It can therefore be differentiated with respect to any of the elements of $\mathcal{S}$, obtaining relations among the given FI and other FIs, these last ones all belonging to the subtopology tree of the first.[1]

Since scalar equations can be solved and manipulated easily with respect to integration, the focus is set to retrieve scalar differential equations – for that, the variable of differentiation will be only scalar quantities.

$\mathcal{S}$ can be divided into two subsets, based on the nature of the variables: a subset of internal variables and a subset of external ones.

[1] As already said, differentiating can only vary the powers of the elemnts of $\mathbb{D}$ and $\mathbb{S}$, without inserting new elements.

Internal variables formed by all the non-zero masses of the internal lines, the construction of the differential equations is straightforward:

$$\frac{\partial I(D;\mathcal{S})}{\partial\left(m_i^2\right)} = C_{ij} J_j(D;\mathcal{S}) \tag{2.28}$$

where J_j are FIs that differ from I just for some increased powers in the denominators, therefore belonging to the same topology of I. This is due to the fact that differentiating with respect to m_i^2 with denominators of the form $(\mathcal{K} + m_j^2)$ does not generate, in the numerator, terms depending on momenta.

External variables formed by quantities containing external momenta or masses. Due to the fact that inside the FIs also scalar products of the form $p_e \cdot k_i$ are present, the derivative must be expanded:

$$\frac{\partial I(D;\mathcal{S})}{\partial s_\alpha} = \frac{\partial p_i^\mu}{\partial s_\alpha}\frac{\partial I(D;\mathcal{S})}{\partial p_i^\mu} = \left[\frac{\partial s_\alpha}{\partial p_i^\mu}\right]^{-1}\frac{\partial I(D;\mathcal{S})}{\partial p_i^\mu} = \\ = A_{\alpha j}(D;\mathcal{S}) J_j(D;\mathcal{S}) + B_{\alpha k}(D;\mathcal{S}) K_k(D;\mathcal{S}), \tag{2.29}$$

where the term $[\partial s_\alpha/\partial p_i^\mu]^{-1}$ is obtained starting from the expressions $\partial I/\partial p_i^\mu = [\partial s_\alpha/\partial p_i^\mu]\partial I/\partial s_\alpha$ and inverting these relations. The left-hand side of one of these expressions is a scalar quantity, so it is possible to express the right-hand side in terms of denominators and irreducible scalar products, possibly multiplied by mass coefficients. The integrals J_j, as above, will belong to the same topology, but with different powers, while K_k are integrals only of the subtopology tree, specifically of the trench with the same number of loop of the original, otherwise the FIs will be null for dimensional regularization.

The same procedure can be applied to a vector of MIs, all depending on a common set of kinematic variables, obtaining a system of differential relations that can be studied at once.

Most of the integrals J_j and K_k are not MIs for a given topology, therefore, applying again the identities of reduction to MIs, it is possible to express the right-hand side of the relations in terms of MIs only.

To conclude, through differentiation on the kinematic invariants it is possible to write *differential equations* (DE) for a set $\mathbf{I}$ of MIs, with structure:

$$\frac{\partial \mathbf{I}(D;\mathcal{S})}{\partial s_\alpha} = M_{\alpha i}(D;\mathcal{S})\mathbf{I}_i(D;\mathcal{S}) + C_{\alpha j}(D;\mathcal{S})\mathbf{J}_j(D;\mathcal{S}) \tag{2.30}$$

with

- $\mathbf{I}$ the vector of MIs under study;
- $\mathbf{J}$ a vector of MIs belonging to subtopologies of $\mathbf{I}$;
- M and C matrices of rational functions of the kinematic parameters (they can be singular at particular configurations $\tilde{\mathcal{S}}$).

The system for $\mathbf{I}$ is not homogeneous, so a bottom-up approach to the MIs becomes natural: starting with the evaluation of the MIs related to the simplest subtopologies one proceed with the more complex ones, having determined all the K_k for a given "topology complexity" thanks to the "simpler" one(s).

Another way to solve the problem is to start from a basis of MIs wide enough to include all the FIs comparing in the DEs, each one with its own DE, then solving the whole system at once (this is usually the way to proceed, when a MI with a wide number of sub-FIs must be evaluated); this approach is often used when there are more than one (independent) MI for a given topology, since MIs of the same topology have usually entangled DEs. Matematically speaking, due to the fact that a MIs can be related via DEs only to FIs belonging to his subtopology or topology, it is always possible to put the matrix of the coefficients in a block-triangular form, where the blocks correspond to interactions between MIs with the same topology:

$$\frac{\partial \mathbf{I}(D;\mathcal{S})}{\partial s_\alpha} = M_{\alpha i}(D;\mathcal{S})\mathbf{I}_i(D;\mathcal{S}). \tag{2.31}$$

Mixing the two methods is also possible, mainly when topologies with more than one MI are involved: the MIs related to the topology under study are treated in a differential way, while all the other MIs are considered known.

Euler's scaling equation

Another important source of differential relations is the *Euler's scaling equation for homogeneous functions* (see [4, 9]).

Proposition 4. *An l-loop FI $I(D;p_1;\ldots;p_{g-1};m_1;\ldots;m_l)$ depending on dimension D, independent external momenta p^μ and internal masses m_i, is an homogeneous function of degree $lD - 2\sum_{j=1}^{l}\alpha_j$, where α_j are the powers of the denominators $\mathcal{D}_j$. So the integral obeys to the* Euler's scaling equation*:*

$$\left(lD - 2\sum_{j=1}^{l}\alpha_j\right) I = \left(\sum_{j=1}^{l} m_j \frac{\partial}{\partial m_j} + \sum_{e=1}^{g-1} p_e^\mu \frac{\partial}{\partial p_e^\mu}\right) I \tag{2.32}$$

Proof. A change of loop momenta $k'^\mu = \frac{k^\mu}{\lambda}$ ($\mathrm{d}^D k' = \lambda^{lD}\,\mathrm{d}^D k$) determines the expression:

$$I(D;p_1;\ldots;p_{g-1};m_1;\ldots;m_l)\lambda^{lD-2\sum_{j=1}^{l}\alpha_j} = I(D;\lambda p_1;\ldots;\lambda p_{g-1};\lambda m_1;\ldots;\lambda m_l) \tag{2.33}$$

Differentiating both sides of the relation with respect to λ one obtains:

$$\begin{aligned}
&\left(lD - 2\sum_{j=1}^{l}\alpha_j\right) I(D;p_1;\ldots;p_{g-1};m_1;\ldots;m_l)\lambda^{lD-2\sum_{k=1}^{l}\alpha_k-1} = \\
&\quad = \left(\sum_{j=1}^{l}\frac{\partial(\lambda m_j)}{\partial\lambda}\frac{\partial}{\partial(\lambda m_j)} + \sum_{e=1}^{g-1}\frac{\partial(\lambda p_e^\mu)}{\partial\lambda}\frac{\partial}{\partial(\lambda p_e^\mu)}\right) I(D;\lambda p_1;\ldots;\lambda p_{g-1};\lambda m_1;\ldots;\lambda m_l) = \\
&\quad\quad = \left(\sum_{j=1}^{l} m_j\frac{\partial}{\partial(\lambda m_j)} + \sum_{e=1}^{g-1} p_e^\mu\frac{\partial}{\partial(\lambda p_e^\mu)}\right) I(D;\lambda p_1;\ldots;\lambda p_{g-1};\lambda m_1;\ldots;\lambda m_l).
\end{aligned} \tag{2.34}$$

Now taking the limit $\lambda \to 1$:

$$\left(lD - 2\sum_{j=1}^{l}\alpha_j\right) I = \left(\sum_{j=1}^{l} m_j \frac{\partial}{\partial m_j} + \sum_{e=1}^{g-1} p_e^\mu \frac{\partial}{\partial p_e^\mu}\right) I. \tag{2.35}$$

So a FI is a homogeneous function of degree $lD - 2\sum_{j=1}^{l}\alpha_j$. □

Also the scaling equation is related to the IBP-ids:

Proposition 5. *The Euler's scaling equation for FIs belonging to a certain topology can be written in terms of IBP-ids of the same FIs.*

Proof. It is possible to apply the scaling operator (2.32) directly on the integrand function, since all the derivatives commute with the integration on loop momenta:

$$\left(lD - 2\sum_{j=1}^{l}\alpha_j - \sum_{k=1}^{l} m_k \frac{\partial}{\partial m_k} - \sum_{e=1}^{g-1} p_e^\mu \frac{\partial}{\partial p_e^\mu}\right) f(D; p_1; \dots; p_{g-1}; m_1; \dots; m_l; k_1; \dots; k_l) = 0. \tag{2.36}$$

Now summing and subtracting $\sum_{i=1}^{l} k_i^\mu \frac{\partial f}{\partial k_i^\mu}$:

$$\left(lD - 2\sum_{j=1}^{l}\alpha_j - \sum_{k=1}^{l} m_k \frac{\partial}{\partial m_k} - \sum_{h=1}^{l+g-1} q_h^\mu \frac{\partial}{\partial q_h^\mu} + \sum_{i=1}^{l} k_i^\mu \frac{\partial}{\partial k_i^\mu}\right) f = 0. \tag{2.37}$$

The integrand f is an homogeneous function of degree $-2\sum_{j=1}^{l}\alpha_j$:

$$f(D; \lambda\mathcal{P}; \lambda m; \lambda\mathcal{K}) = \lambda^{-2\sum_{j=1}^{l}\alpha_j} f(D; \mathcal{P}; m; \mathcal{K}). \tag{2.38}$$

So:

$$\left(\sum_{k=1}^{l} m_k \frac{\partial}{\partial m_k} + \sum_{h=1}^{l+g-1} q_h^\mu \frac{\partial}{\partial q_h^\mu}\right) f = -2\sum_{j=1}^{l}\alpha_j f \tag{2.39}$$

thanks to the Euler's theorem for homogeneous functions. Substituting the expression into (2.37):

$$\left(lD + \sum_{i=1}^{l} k_i^\mu \frac{\partial}{\partial k_i^\mu}\right) f = 0 \quad \to \quad \left(\sum_{i=1}^{l} \frac{\partial}{\partial k_i^\mu} k_i^\mu\right) f = 0. \tag{2.40}$$

This is an IBP-id performed on the integrand of the initial FI. □

2.3.2 Boundary conditions

To complete the Cauchy problem *boundary conditions* (BC) are needed, in order to select the desired solution.

Traditionally BCs are obtained knowing the exact value of the solution at a certain point inside the domain and imposing it to fix the value of the constant parameters. This possibility, when available, is the easiest one, and fixes uniquely the solution on an interval of the domain.

Problems may arise when the domain is not connected, due to the presence of singular points in the DEs: in this case, it is important to distinguish between singularities present both in the DEs and in the MIs, called *thresholds*, and singularities present only in the DEs, called *pseudothresholds*. This last type of singularities are fictional, and generates subdivisions in the domain of the solutions that must be resewed imposing regularity on the points of divergence. To find such points, analysis on the integral form of the FIs must be performed, verifying that

the diagrams are still well-defined (or, at least, collapse on well-defined ones), even not knowing the exact values. It is also possible to impose constraints to the values of the MIs in certain regions of space, like forcing the MIs to be real in specific parts of the domain.

The two methods are complementary: obviously fixing the solution in the classical way is more simple than imposing continuity or reality, but on the other hand the complexity of evaluation of MIs by integration or related methods suggests to reduce as more as possible the set of exact values needed as BCs.

A mid-way approach consists of using the pseudothresholds of the DEs to find useful relations between MIs at certain points: given a DE

$$\frac{\partial I_0(x)}{\partial x} = P_{x_0}(x) \sum_{i \in \mathcal{I}} I_i(x) + \sum_{j \in \mathcal{J}} I_j(x), \tag{2.41}$$

where $P_{x_0}(x)$ indicates a function with a pseudothreshold in x_0 of degree n, multiplying both sides of the DE for $(x - x_0)$ and taking the limit $x \to x_0$ a relation is found:[2]

$$0 = \lim_{x \to x_0} \left[(x - x_0)^n P_{x_0}(x)\right] \sum_{i \in \mathcal{I}} I_i(x_0). \tag{2.42}$$

This relation determines a BC on the pseudothreshold point, imposing regularity and a precise value of the FIs at the same time.

Regarding the true thresholds, such points cannot be included in the domain. It is still possible though to expand the function bypassing the singularity thanks to analytic continuation on the complex plane, to connect different pieces of the domain.

Quadruple cut

Since a Feynman diagram has non-zero imaginary part when at least one of its loop particles goes on-shell, and when $k^2 - m^2 = 0$ the corresponding denominator generates a singularity, a good way to determine the true thresholds of a problem is to perform the maximum possible amount of cuts on a give diagram, and to find the singularity in the outcome expression, which will correspond to the singularities of the diagram.

Two properties of cuts on FIs will be useful:

1. the thresholds of a given FI are a subset of the thresholds of its topology;
2. thresholds of FIs belonging to subtopologies of a given FI are a subset of the thresholds of the latter one;

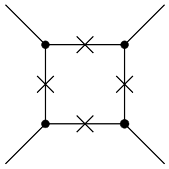
The first property descends from the fact that dotted propagator do not generate new points of singularity if compared to the undotted counterparts (maybe the singularity will present a higher pole, but not new ones), and irreducible scalar products in the denominator can at most eliminate some zeroes of the denominator, resulting harmless (at least for finite values of the kinematic parameters).

[2] $\lim_{x \to x_0} \left[(x - x_0)^n \frac{\partial I_0(x)}{\partial x}\right] = 0$, because both $I(x)$ and its derivative are regular in x_0, $\lim_{x \to x_0} \left[(x - x_0)^n I_j(x)\right] = 0$ because all the integrals are regular in x_0.

The second point comes from the fact that, removing denominator to pass into a subtopology FI, singular points present in the original topology can be eliminated, whilst new singular point cannot arise.

Thanks to the properties enlisted above, once that the singular points of the main FI are determined, all the MIs used to determine it will present a subset of the found singularities.

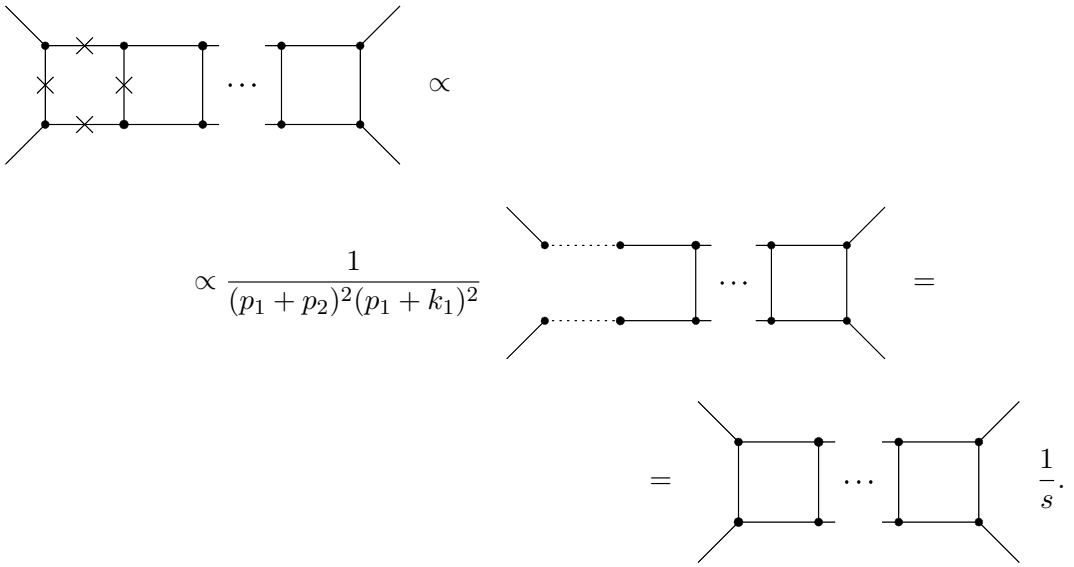
For the problems investigated in this work then, it is sufficient to determine the singularities of the n-loop ladders. For the 1-loop box, the quadruple cut is evaluated in appendix B, resulting in[3]



$$\propto \frac{1}{st} = \frac{1}{(p_1+p_2)^2(p_1+p_3)^2}. \tag{2.43}$$

It is then possible to see that, for the 1-loop box, the thresholds lies in $s = 0$ and in $t = 0$. Its subtopologies, namely the s-bubble and the t-bubble, have respectively singularities in s and t, as expected (confront the results with [10–12]).

Regarding the generic n-loop ladder, consider a quadruple cut of one of its extremal sub-boxes:

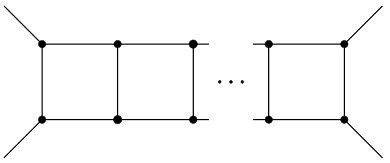


$$\propto \frac{1}{(p_1+p_2)^2(p_1+k_1)^2} \cdots = \cdots \frac{1}{s}. \tag{2.44}$$

The neat effect of a quadruple cut is then the reduction of a n-loop ladder to a $(n-1)$-loop ladder multiplied by $1/s$. Iterating this procedure for all the new extremal boxes, the final result

[3]External momenta $p_1^\mu; p_2^\mu; p_3^\mu; p_4^\mu$ are all incoming, the left and right upper legs carry p_1^μ and p_3^μ, respectively, the left and right lower ones carry p_2^μ and p_4^μ, respectively.

is:



$$\Rightarrow \frac{1}{s^n t}. \tag{2.45}$$

It is then possible to say that the thresholds are located in $s = 0$ and in $t = 0$, and all the subtopologies have as singular points a subset (proper or not) of the set of thresholds of the parent n-loop ladder (see [12]).

2.3.3 ϵ dependence

With all the ingredients above, an exact Cauchy problem in D is obtained, so in principle it would be possible to express the solutions in terms of exact functions of the dimension D.

Usually, while working with wide systems of DEs it is easier (if not mandatory, due to the large number of equations involved, and to the functions arising in the solution, like hypergeometric functions) to work with series expansions of the MIs in the dimensional regularization parameter $\epsilon = \frac{4-D}{2}$. Moreover, as it will be explained in section 2.3.4, a stronger formulation of existence and uniqueness of the solutions holds in expanded DEs. Since loop integral are divergent, $\mathbf{I}$ will show a Laurent expansion in ϵ. In dimensional regularization divergencies are caused only by finite poles, so the Laurent expansion has no more terms below $N_0 < 0$:

$$\mathbf{I}(D;\mathcal{S}) = \sum_{n=N_0}^{N} \epsilon^n \mathbf{I}^{(n)}(\mathcal{S}) + o_0\left(\epsilon^N\right). \tag{2.46}$$

Starting with a single DE (2.30), and expanding the terms in it:

$$\sum_{n=N_0}^{+\infty} \epsilon^n \frac{\partial I^{(n)}(\mathcal{S})}{\partial s_\alpha} = \sum_{i=0}^{+\infty} \epsilon^i M_\alpha^{(i)}(\mathcal{S}) \sum_{n=N_0}^{+\infty} \epsilon^n I^{(n)}(\mathcal{S}) + \sum_{j=J_0}^{+\infty} \epsilon^j C_{\alpha c}^{(j)}(\mathcal{S}) \sum_{m=M_0}^{+\infty} \epsilon^m \mathbf{J}_c^{(m)}(\mathcal{S}) \tag{2.47}$$

a system of chained DEs in the Laurent expansions of I, $\mathbf{J}$, M and C is obtained:

$$\frac{\partial I^{(N_0)}(\mathcal{S})}{\partial s_\alpha} = M_\alpha^{(0)}(\mathcal{S}) I^{(N_0)}(\mathcal{S}) + \sum_{j+m=N_0} C_{\alpha c}^{(j)}(\mathcal{S}) \mathbf{J}_c^{(m)}(\mathcal{S}) \tag{2.48}$$

$$\frac{\partial I^{(N_0+1)}(\mathcal{S})}{\partial s_\alpha} = M_\alpha^{(0)}(\mathcal{S}) I^{(N_0+1)}(\mathcal{S}) + M_\alpha^{(1)} I^{(N_0)}(\mathcal{S}) + \sum_{j+m=N_0+1} C_{\alpha c}^{(j)}(\mathcal{S}) \mathbf{J}_c^{(m)}(\mathcal{S}) \tag{2.49}$$

$$\dots$$

Note that the homogeneous DE at every order is always the same: $\frac{\partial I^{(i)}(\mathcal{S})}{\partial s_\alpha} = M_\alpha^{(0)}(\mathcal{S}) I^{(i)}(\mathcal{S})$, so it is necessary to solve it just once to cover all orders.

In particular, the coefficient M cannot exhibit poles in ϵ: if it was the case, each term of the MI woud be linked to the next one, and the homogeneous part of the DE would read

$I^{(n)}(\mathcal{S}) = C(\mathcal{S})I^{(n+h)}(\mathcal{S})$, causing the series to have no inferior limit. In fact, starting from an arbitrary n (if the MI is not a constant, at least one order must be non zero), the order $n-h$ will evolve according to the expression of that order; iterating the same procedure to the orders $n-ih$ the non-zero terms of the Laurent series will have no inferior limit, against the hypothesis of finite-pole integrals. Considering $\mathbf{I}$ to be a vector of MIs the above property holds only if the maximal ϵ pole of all the MIs has the same order, otherwise ϵ poles can be present in the matrix to match different starting orders N_{0i}.

To avoid ϵ poles in M the basis of MIs can be modified by multiplying each single MI by ϵ^h in order to eliminate the poles:

$$E\frac{\partial \mathbf{I}(\epsilon;\mathcal{S})}{\partial s_\alpha} = E[M_\alpha(\epsilon;\mathcal{S})\mathbf{I}(\epsilon;\mathcal{S}) + C_{\alpha c}(\epsilon;\mathcal{S})\mathbf{J}_c(\epsilon;\mathcal{S})]; \tag{2.50}$$

$$\frac{\partial \tilde{\mathbf{I}}(\epsilon;\mathcal{S})}{\partial s_\alpha} = EM_\alpha(\epsilon;\mathcal{S})E^{-1}\tilde{\mathbf{I}}(\epsilon;\mathcal{S}) + EC_{\alpha c}(\epsilon;\mathcal{S})E^{-1}\tilde{\mathbf{J}}_c(\epsilon;\mathcal{S}); \tag{2.51}$$

$$\frac{\partial \tilde{\mathbf{I}}(\epsilon;\mathcal{S})}{\partial s_\alpha} = \tilde{M}_\alpha(\epsilon;\mathcal{S})\tilde{\mathbf{I}}(\epsilon;\mathcal{S}) + \tilde{C}_{\alpha c}(\epsilon;\mathcal{S})\tilde{\mathbf{J}}_c(\epsilon;\mathcal{S}); \tag{2.52}$$

where E is a diagonal matrix with elements of the form ϵ^n, such that $\tilde{\mathbf{I}}$ has no poles in ϵ, as $\tilde{M}$.

2.3.4 Existence and uniqueness of the solutions

First of all, the basis of MIs used to write the DEs is obtained starting from tensor decomposition performed on actual Feynman diagrams involved in scattering amplitudes, so the MIs have to exist, usually diverging for $\epsilon \to 0$. Considering now the ϵ-divergent-free version of this basis, the MIs have to exist with finite values. So, at least, there exists a choice of functions to be determined via DEs that is well defined.

The problem now arises from the DE system perspective: given a set of DEs and a set of BCs, can they determine a unique solution?

Due to the fact that (a basis of) MIs can depend on more that one single parameter, and that for each one a (system of) DE(s) is determined, the problem separates according to the number of needed parameters (single-scale or multi-scale problems) and according to the number of parameters with respect to differentiation is performed (complete systems of DE or partial systems of DEs).

One-scale problems

Considering a one-scale problem (consequently, a complete system of DE), the equations are linear DEs of the first order. The system takes the form

$$\frac{\mathrm{d}\mathbf{y}(x)}{\mathrm{d}x} = M(x)\mathbf{y}(x) + \mathbf{b}(x), \tag{2.53}$$

defined on the open set $D = \mathbb{R} \setminus \{x : M(x) \text{ is singular}\}$.

For each open interval $A \in D$ of the x variable, the theorem of global existence and uniqueness of the solutions holds ($M(x)y(x) + b(x)$ is a Lipschitz continuous function in y for each compact in A).

Regarding the behaviour of the solutions on the frontier of D, an analysis can be performed distinguishing between threshold and pseudothresholds: consider $I_1 \subseteq D$ and $I_2 \subseteq D$, separated by x_0, frontier point of both the intervals, and a solution $\mathbf{y}_1(x)$ in I_1, already fixed using the BC of the problem. If x_0 is a pseudothreshold for the problem, it can be imposed that $\lim_{I_1 \ni x \to x_0} y_1(x) = \lim_{I_2 \ni x \to x_0} y_2(x)$, providing a BC for I_2. Iterating the procedure it is possible to joint all the intervals separated by pseudothresholds.[4]

On the other hand, if an interval is delimited on a side by a threshold, it is still possible to extend the solution thanks to analytic continuation, bypassing the pole with a prescription that expands $\mathbf{y}$ in the complex plane, while before $\mathbf{y} \in \mathbb{R}^n$. This property is quite useful when dealing with expressions like $\log x$, where $x = 0$ is a true threshold for the function, and BCs are fixed in $x < 0$.

Multi-scale problems, complete systems

Passing to multi-scale problems, the system of DE transforms into a system of partial DEs (PDE) on $\mathbf{y}$ (a "system of systems of DEs"), of the form:

$$\begin{cases} \frac{\partial \mathbf{y}(\mathbf{x})}{x_1} = M_1(\mathbf{x})\mathbf{y}(\mathbf{x}) + \mathbf{b}_1(\mathbf{x}) \\ \frac{\partial \mathbf{y}(\mathbf{x})}{x_2} = M_2(\mathbf{x})\mathbf{y}(\mathbf{x}) + \mathbf{b}_2(\mathbf{x}) \\ \dots \\ \frac{\partial \mathbf{y}(\mathbf{x})}{x_n} = M_n(\mathbf{x})\mathbf{y}(\mathbf{x}) + \mathbf{b}_n(\mathbf{x}) \end{cases} . \tag{2.54}$$

In complete generality, for a PDE problem only the Cauchy—Kovalevskaya theorem for analytic DEs holds, assuring local existence and uniqueness of an analytic solution. Working with analytic functions as the MIs, this theorem will assure local existence and uniqueness, but it is possible to say more on the solutions.

Considering the peculiar structure of the system under examination, in which the first-order derivatives with respect to each variable are separated from the other ones (i.e. each PDE for $\mathbf{y}$ contains only derivatives in *one* variable), it is possible to proceed solving one PDE at a time, then substituting its solution in the remaining ones, proceeding for each one as in the ordinary DEs case: solving e.g. $\partial \mathbf{y}/\partial x_1$ a solution of the form $\Phi_1(\mathbf{x}) = \Phi_1(F_1(\mathbf{x}); \mathbf{C}_1(x_2; \dots; x_n))$ is obtained (F_1 is the function obtained integration with respect to x_1, while $\mathbf{C}$ is the constant function with respect to x_1); substituting now $\Phi_1(\mathbf{x})$ to $\mathbf{y}(\mathbf{x})$ and solving the PDE a function $\Phi_2(F_2(\mathbf{x}); \mathbf{C}_2(x_3; \dots; x_n))$ is found, where x_2 has moved from $\mathbf{C}$ to Φ; iterating this procedure one arrives to a form $\Phi_n(\mathbf{x}) = \Phi_n(F_n(\mathbf{x}); \mathbf{C}_n)$, that now can be fixed using BCs for $\mathbf{y}$.

It is then possible to work with each single PDE at a time, as if each one was an ordinary DE with respect to the variable of differentiation, considering all the other variables as constants. Therefore, it is possible to apply theorems of existence and uniqueness of the solutions variable by variable, as in the ordinary case.

To verify existence and uniqueness of the solution the simple *a priori* check on the coefficient of the system is not sufficient: Φ_i can introduce expression in $x_{i+1}; \dots; x_n$ no more satisfying the

[4] Notice that a pseudothreshold must always be the separation point between two open intervals, otherwise the extension of the solution to the point will generate a DE with a solution in a semi-closed set, which is impossible.

hypothesis of the theorem of global existence and uniqueness of the solutions. Therefore, after each substitution of the solution the hypothesis must be controlled.

It is possible to obtain a more constrained result using ϵ expanded DEs for a basis of "ϵ-pole free" MIs: the ϵ dependence enters no more directly in the DEs, and it is possible to solve the problem order by order, starting from the lowest one. Thanks to the fact that the solutions, according to the nature of the coefficients, are of the form $\Phi_i(\mathbf{x}) = \mathbf{F}_i(\mathbf{x})\mathbf{C}_i(x_{i+1}; \dots; x_n)$ or $\Phi_i(\mathbf{x}) = \mathbf{F}_i(\mathbf{x}) + \mathbf{C}_i(x_{i+1}; \dots; x_n)$ (the first one if the same ϵ-order of the integral is present both differentiated and as a simple term, the second if it only appears under differentiation) the condition of Lipschitz continuity remains satisfied after the substitutions, if matched at the beginning for every different PDE system (except al least for some isolated points, due to possible inversion problem of $\mathbf{F}$, that can be treated as pseudothresholds).

Finally, considering an ϵ factorized structure, order by order of the form

$$\begin{cases} \frac{\partial \mathbf{y}^{(i)}(\mathbf{x})}{x_1} = M_1^{(1)}(\mathbf{x})\mathbf{y}^{(i-1)}(\mathbf{x}) \\ \frac{\partial \mathbf{y}^{(i)}(\mathbf{x})}{x_2} = M_2^{(1)}(\mathbf{x})\mathbf{y}^{(i-1)}(\mathbf{x}) \\ \dots \\ \frac{\partial \mathbf{y}^{(i)}(\mathbf{x})}{x_n} = M_n^{(1)}(\mathbf{x})\mathbf{y}^{(i-1)}(\mathbf{x}) \end{cases} \tag{2.55}$$

(ϵ-homogeneous structure), ir follows that all solutions have the form $\Phi_i(\mathbf{x}) = \mathbf{F}_i(\mathbf{x})\mathbf{C}_i(x_{i+1}; \dots; x_n)$, so it is necessary to check the validity of the theorems of existence and uniqueness only once on the original system, thanks to the fact that $\mathbf{F}_i(\mathbf{x})$ is a constant with respect to the x_{i+1} differentiation (at least some isolated points may be excluded during the operation of substitution due to some inversions, but from direct inspection of the integral form of the MIs regularity can be imposed at these points, jointing back the new sets into the original one).

Multi-scale problems, partial systems

Even if $\mathbf{y}$ depends on a set $\mathbf{x}$ of parameters, but only PDEs with respect to a subset $(x_1; \dots; x_h)$ are available, it is possible to obtain a unique solution, assumed that for all the available PDEs global existence and uniqueness theorems hold, as explained above.

First of all, existence is guaranteed by the theorem of existence and uniqueness of the solutions applied to the system of PDEs (in the sense explained above), plus the BC, in the form of a function or constraint depending on $x_{h+1}; \dots; x_n$.

Concerning uniqueness: at first sight, knowing only a subset of all the PDEs for $\mathbf{y}$, it will be possible to find a unique solution only with respect to the variables $x_1; \dots; x_h$, leaving as free parameters the remaining ones, obtaining in principle $\mathbb{R}^{n-h}$ different solutions. Consider although how the BC is determined: the BC $\Psi(x_{h+1}; \dots; x_n)$ is a certain vector of MIs or a constraint. If a vector of MIs, it is determined either via integral evaluation (and related tools), or via DE method; in the first case, it depends uniquely from $x_{h+1}; \dots; x_n$, in the second one it has the form $\Psi(\Omega(x_{h+1}; \dots; x_k); \mathbf{C}_k(x_{k+1}; \dots; x_n))$, and also Ψ is a function determined either via integral methods or differential methods, so the same conclusions apply also to it, and so on until the constant term does not depend on parts of $\mathbf{x}$ and is completely fixed; at

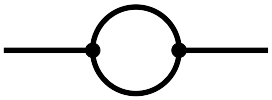
this point, going back to function Φ, the solution is completely determined without ambiguities, each function inheriting existence and uniqueness from the inner ones. In the case the BC is a constraint, ambiguities can arise if it does not bind the behaviour of the MIs with respect to all the variables; in that case, a more suitable constraint must be used.

2.3.5 Solving DEs

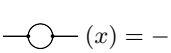
A natural technique to solve a non-homogeneous DE (i.e. with an inhomogeneous part), either exactly in D or in ϵ expansion, is the *Euler's method of variation of the constants*: by solving the related homogeneous DE a solution depending on a constant is obtained (the homogeneous DE is a first order DE); promoting the constant to a function, reinserting the present solution into the original DE and solving for this new function the complete solution is retrieved. Note that, thanks to the peculiar structure of the PDEs to which MIs obey (i.e. PDEs with one derivative in each equation), even these equations can be solved using this method.

This approach can be used both with solution in exact D dimension, and with ϵ expanded solutions.

Example 6 (Massive bubble DE). A massive bubble FI $F(x)$ with topology



satisfies the DE[5]

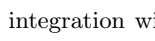
$$\frac{\mathrm{d}}{\mathrm{d}x}$$  $$= -\frac{\epsilon x+2}{x^2+4x}$$ $$+ \frac{2(1-\epsilon)}{m^2 x(x+4)}$$, (2.56)

where is the simple tadpole (as presented in example 4, with $\alpha = 1$), and x is the dimensionless s Mandelstam variable ($x = s/\mu^2$). Solving directly the equation will lead to expressions containing hypergeometric functions of the form ${}_2F_1(\dots)$, so a better way to proceed consists of expanding both the known function (the tadpole, of which the inhomogeneous term consists) and the unknown function (the massive bubble) in series of ϵ, then solving the DE order by order, starting from the lowest one.

The kernel of the iterated integration will be the function $C\sqrt{\frac{4+x}{x}}$. As it will be seen later, it is better to have a kernel of the form $C \log y$, in order to have iterated integrations of logarithmic functions. It is possible to perform a change of variables, passing to the *Landau parametrization*:

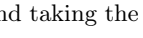
$$-\frac{s}{m^2} = x = \frac{(y-1)^2}{y}. \tag{2.57}$$

The DE then reads:

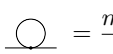
$$\frac{\mathrm{d}}{\mathrm{d}y}$$  $$= -\frac{(y-1)^2\epsilon+2y}{y(y^2-1)}$$ $$- \frac{2(\epsilon-1)}{m^2(y^2-1)}$$; (2.58)

where $:=$ $(x(y))$.

[5]The present equations have been automatically determined using the computer code `Reduze` [46, 47].

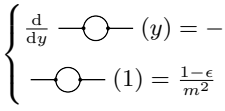
A useful limit for the BC is $s = x = 0$, corresponding to $y = 1$: in this limit the massive bubble reproduces the result (2.16) (two propagators with zero incoming squared momentum, $\alpha_1 = \alpha_2 = 1$. Otherwise, multiplying both sides of (2.58) by $y - 1$ and taking the limit $y \to 1$,  is found; the right-hand side is nothing else, thanks to IBP-ids, than the dotted tadpole  .

The ϵ expansion of the simple tadpole ($\alpha = 1$) will be used as BC, and has the form:



$$= \frac{m^2}{\epsilon} - m^2 \left(\log \frac{m^2}{\mu^2} + \gamma - 1 \right) + o_0(\epsilon^0) \tag{2.59}$$

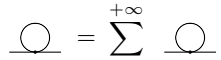
In conclusion, the Cauchy problem has the form:



$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}y} \quad (y) = -\frac{(y-1)^2\epsilon+2y}{y(y^2-1)} \quad (y) - \frac{2(\epsilon-1)}{m^2(y^2-1)} \\ \quad (1) = \frac{1-\epsilon}{m^2} \end{cases} . \tag{2.60}$$

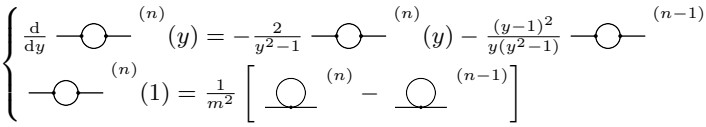
It is now possible to expand  and  (y) in ϵ and solve the DE order by order in ϵ, starting from the highest pole and rising the power of ϵ. The expansions can be written as:

$$= \sum_{i=p<0}^{+\infty} \quad ^{(i)}(y) \tag{2.61}$$



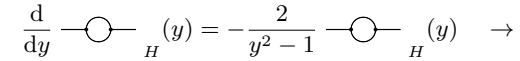
$$= \sum_{i=-1}^{+\infty} \quad ^{(i)} \tag{2.62}$$

The general term of the DE expanded in ϵ has the form:



$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}y} \quad ^{(n)}(y) = -\frac{2}{y^2-1} \quad ^{(n)}(y) - \frac{(y-1)^2}{y(y^2-1)} \quad ^{(n-1)}(y) + \frac{2}{m^2(y^2-1)} \quad ^{(n)} - \frac{2}{m^2(y^2-1)} \quad ^{(n-1)} \\ \quad ^{(n)}(1) = \frac{1}{m^2} \left[\quad ^{(n)} - \quad ^{(n-1)} \right] \end{cases} . \tag{2.63}$$

To evaluate the equation the method of variation of the constants will be used: in particular, the homogeneous equation has the same form at every order, so it has to be computed just once

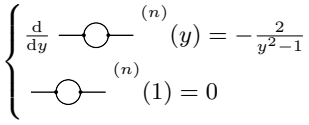


$$\frac{\mathrm{d}}{\mathrm{d}y} \quad _H(y) = -\frac{2}{y^2-1} \quad _H(y) \quad \to \quad \quad _H(y) = -C\frac{y+1}{y-1}. \tag{2.64}$$

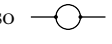
Notice that to determine the non-homogeneous part it is necessary to take into account also the degree of ϵ to which each term of the integrals is related.

Fixed the homogeneous part, it is possible to proceed order by order in ϵ.

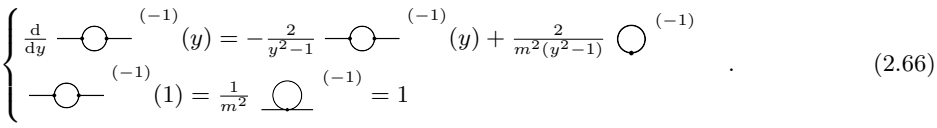
- ϵ^n and $n < -1$ Assuming that the massive bubble has finite order poles, and considering $-n$ as the highest order, still lower than the simple pole of the tadpole, the Cauchy problem assumes the form:



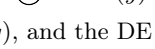
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}y} \quad ^{(n)}(y) = -\frac{2}{y^2-1} \quad ^{(n)}(y) \\ \quad ^{(n)}(1) = 0 \end{cases} , \tag{2.65}$$

  so the DE is equal to the associated homogeneous one. Due to the fact that $_H(y)$ has a pole in $y = 1$, $C^{(n)} = 0 \, \forall n < -1$, so  $^{(n)}(y) = 0 \, \forall n < -1$.

- ϵ^{-1} At the lowest order of  , the Cauchy problem has the form

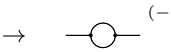


(2.66)

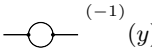
Using the variation of constants method:  (promotion of the constant of the homogeneous solution to a function of y), and the DE becomes

$$\frac{2C(y)-(y^2-1)C'(y)}{(y-1)^2} = -\frac{2-[(y+1)C(y)]}{(y^2-1)(y-1)} \quad \rightarrow \quad -\frac{(y+1)}{y-1}C'(y) = 0 \tag{2.67}$$

which complete solution is

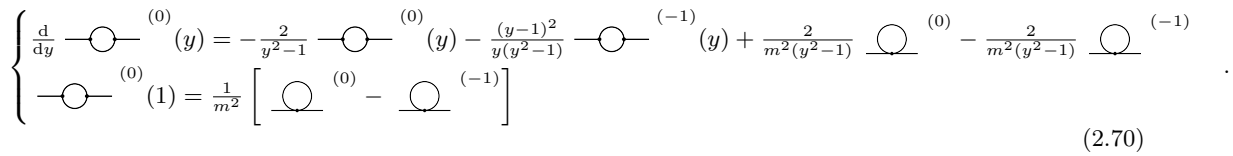
$$C^{(-1)}(y) = C \quad \rightarrow \quad \text{  } = \frac{C(y+1)+2}{1-y}. \tag{2.68}$$

This solution must be equal to 1 in the $y \to 1$ limit, so the divergence must be eliminated. Multiplying the expression for $y-1$ and taking the limit $y \to 1$ the coefficient of the divergent term will be put in evidence; this term must be zero when evaluated in $y = 1$. So $C = -1$, giving the solution:

$$\text{  } = 1. \tag{2.69}$$

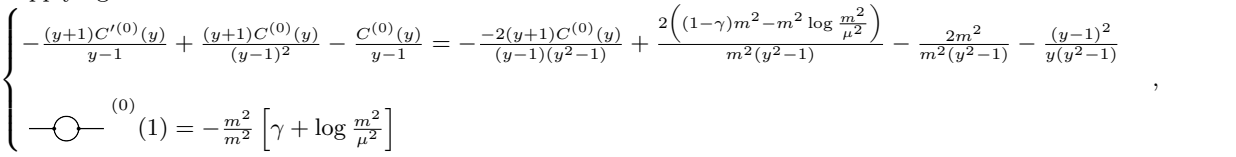
This solution has the same behaviour that the simple pole of the tadpole, and does not depend on kinematic parameters or masses, as expected from the BC.

- ϵ^0 Now also the previous orders enters in the DE:



(2.70)

Applying the variation of the constants method:



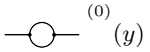
which gives:

$$C'^{(0)}(y) = \frac{2y\log\frac{m^2}{\mu^2}+y(y+2\gamma-2)+1}{y(y+1)^2}. \tag{2.71}$$

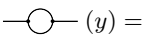
So:

$$C^{(0)}(y) = C - \frac{2\left(\log\frac{m^2}{\mu^2}+\gamma-2\right)}{y+1} + \log(y); \tag{2.72}$$

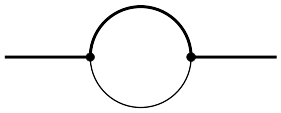
and asking for regularity in $y = 1$ the function $F^{(0)}$ is determined:

$$\text{  } = -\frac{(y+1)}{y-1}\log(y) - \log\frac{m^2}{\mu^2} - \gamma + 2. \tag{2.73}$$

The final result up to order ϵ^0 is then:

$$\text{  } = \frac{1}{\epsilon} - \frac{(y+1)}{y-1}\log(y) - \log\frac{m^2}{\mu^2} - \gamma + 2 + o_0(\epsilon^0). \tag{2.74}$$

Example 7 (Half-massive bubble DE). Consider a bubble with one massive and one massless propagator:



First of all, an undotted version will be considered, obeying the DE:

$$\frac{\mathrm{d}}{\mathrm{d}x}\ \text{—◯—}\ (x) = \frac{(-x\epsilon+\epsilon-1)}{x(x+1)}\ \text{—◯—}\ (x) + \frac{1}{x(x+1)}\ \text{◯—}\ , \tag{2.75}$$

where $x = -\frac{s}{m^2}$, and ◯— is the dotted vacuum tadpole, whit expression:

$$\text{◯—} = \frac{1}{\epsilon} - \log\frac{m^2}{\mu^2} - \gamma + o_0(\epsilon^0). \tag{2.76}$$

Looking at the DE, it seems that there could be two possible suitable points to fix the BC: $x = -1$ and $x = 0$, but the first one will give $s = p^2 = m^2$, a point in which the integral under consideration is singular, so only $x = 0$ remains.[6]

Another point of interest will be $x = 1$: in this point the half-massive bubble is well defined, and it will be useful for fixing the value of coefficients of orders less than -1.

With this set up, the Cauchy problem reads:

$$\text{—◯—}\ (y) = \sum_{i=p<0}^{+\infty} \text{—◯—}^{(i)}(y) \tag{2.77}$$

$$\text{◯—} = \sum_{i=-1}^{+\infty} \text{◯—}^{(i)} \tag{2.78}$$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\ \text{—◯—}\ (x) = \frac{-x\epsilon+\epsilon-1}{x(x+1)}\ \text{—◯—}\ (x) + \frac{1}{x(x+1)}\ \text{◯—} \\ \text{—◯—}\ (0) = \frac{1}{1-\epsilon}\ \text{◯—} \end{cases}, \tag{2.79}$$

or, order by order:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\ \text{—◯—}^{(n)}(x) = -\frac{1}{x(x+1)}\ \text{—◯—}^{(n)}(x) - \frac{(x-1)}{x(x+1)}\ \text{—◯—}^{(n-1)}(x) + \frac{1}{x(x+1)}\ \text{◯—}^{(n)} \\ \text{—◯—}^{(n)}(0) = \text{—◯—}^{(n-1)}(0) + \text{◯—}^{(n)} \end{cases}. \tag{2.80}$$

The solution of the associated homogeneous DE is:

$$\frac{\mathrm{d}}{\mathrm{d}x}\ \text{—◯—}_H(x) = -\frac{1}{x(x+1)}\ \text{—◯—}_H(x) \rightarrow \text{—◯—}_H(x) = C\frac{x+1}{x}. \tag{2.81}$$

- ϵ^n and $n < -1$ In the hypothesis of finite pole, starting from the lowest order N of —◯— (x), the Cauchy problem reduces to:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\ \text{—◯—}^{(N)}(x) = -\frac{1}{x(x+1)}\ \text{—◯—}^{(N)}(x) \\ \text{—◯—}^{(N)}(0) = 0 \end{cases} \tag{2.82}$$

[6] Singularities in a single loop integral can be found cutting the propagators and considering the tree diagram coming out as a decay diagram and evaluating the invariant mass of the process. The pole lies on that value of the squared incoming momentum.

this problem has a solution equal to the one of the homogeneous DE. Fulfilling the BC $C = 0$, $\left(\text{bubble}\right)^{(N)}(x) = 0$. The same procedure can be carried on iteratively till $n = -1$, so under ϵ^{-1} there are no coefficients different from 0:

$$\left(\text{bubble}\right)^{(n)} = 0 \text{ if } n < -1. \tag{2.83}$$

- ϵ^{-1} The Cauchy problem has the form:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\left(\text{bubble}\right)^{(-1)}(x) = -\frac{1}{x(x+1)}\left(\text{bubble}\right)^{(-1)}(x) + \frac{1}{x(x+1)} \\ \left(\text{bubble}\right)^{(-1)}(0) = 1 \end{cases} \rightarrow \tag{2.84}$$

$$\rightarrow C^{(-1)}(x) = \frac{x}{1+x} \rightarrow \left(\text{bubble}\right)^{(-1)}(x) = 1.$$

- ϵ^0 The Cauchy problem has the form:

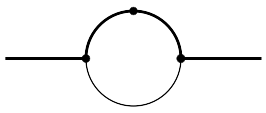
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\left(\text{bubble}\right)^{(0)}(x) = -\frac{1}{x(x+1)}\left(\text{bubble}\right)^{(0)}(x) - \frac{x-1}{x(x+1)} - \frac{1}{x(x+1)}\left(\gamma + \log\frac{m^2}{\mu^2}\right) \\ \left(\text{bubble}\right)^{(0)}(0) = 1 - \gamma - \log\frac{m^2}{\mu^2} \end{cases} \rightarrow \tag{2.85}$$

$$\rightarrow C^{(0)} = -\log(x-1) - \frac{x\left(\log\frac{m^2}{\mu^2} + \gamma - 2\right)}{x+1} \rightarrow \left(\text{bubble}\right)^{(0)}(x) = 2 - \log\frac{m^2}{\mu^2} - \gamma - \frac{(x+1)\log(x+1)}{x}$$

In conclusion:

$$\left(\text{bubble}\right)(x) = \frac{1}{\epsilon} + 2 - \log\frac{m^2}{\mu^2} - \gamma - \frac{(x+1)\log(x+1)}{x} + o_0(\epsilon^0). \tag{2.86}$$

Example 8 (Half-massive bubble with massive dotted propagator). Consider now a half-massive bubble with the massive propagator dotted:



.

The corresponding DE is:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\left(\text{bubble}\right)(x) = -\frac{(x-1)\epsilon+x+1}{x(x+1)}\left(\text{bubble}\right)(x) - \frac{\epsilon}{m^2x(x+1)}\left(\text{dotted}\right) \\ \left(\text{bubble}\right)(0) = \frac{\epsilon}{m^2(\epsilon-1)}\left(\text{dotted}\right) \end{cases} ; \tag{2.87}$$

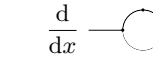
with its order-by-order Cauchy problem:

$$\left(\text{bubble}\right)(y) = \sum_{i=p<0}^{+\infty}\left(\text{bubble}\right)^{(i)}(y) \tag{2.88}$$

$$\left(\text{dotted}\right) = \sum_{i=-1}^{+\infty}\left(\text{dotted}\right)^{(i)} \tag{2.89}$$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\left(\text{bubble}\right)^{(n)}(x) = -\frac{1}{x}\left(\text{bubble}\right)^{(n)}(x) - \frac{(x-1)}{x(x+1)}\left(\text{bubble}\right)^{(n-1)}(x) - \frac{1}{m^2x(x+1)}\left(\text{dotted}\right)^{(n-1)} \\ \left(\text{bubble}\right)^{(n)}(0) = \left(\text{bubble}\right)^{(n-1)}(0) - \frac{1}{m^2}\left(\text{dotted}\right)^{(n-1)} \end{cases} ; \tag{2.90}$$

and the associated homogeneous DE:

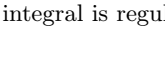
$$\frac{\mathrm{d}}{\mathrm{d}x}\,\text{  }_{H}(x) = -\frac{1}{x}\,\text{}_{H}(x) \;\rightarrow\; \text{}_{H}(x) = \frac{C}{x}. \tag{2.91}$$

So it is possible to proceed order by order using the variation of constants method (to fix the constants is sufficient to impose regularity in $x = 0$, regular point by direct inspection of the integral).

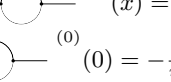
- ϵ^n and $n < 0$ In the hypothesis of finite pole, starting from the lowest order, the Cauchy problem reduces to:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\,\text{}^{(n)}(x) = -\frac{1}{x}\,\text{}^{(n)}(x) \\ \text{}^{(n)}(0) = 0 \end{cases}, \tag{2.92}$$

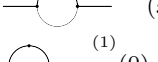
  this problem has a solution equal to the one from the homogeneous DE, and thanks to the fact that in $x = 0$ the integral is regular, the constant must vanish to avoid singularities. Therefore

$$\text{  }^{(n)} = 0 \text{ if } n < 0. \tag{2.93}$$

- ϵ^0 The Cauchy problem has the form:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\,\text{  }^{(0)}(x) = -\frac{1}{x}\,\text{}^{(0)}(x) - \frac{1}{m^2x(x+1)} \\ \text{}^{(0)}(0) = -\frac{1}{m^2} \end{cases} \rightarrow\; \text{}^{(0)}(x) = -\frac{\log(x+1)}{m^2x} \tag{2.94}$$

- ϵ^1 The Cauchy problem has the form:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\,\text{  }^{(1)}(x) = -\frac{1}{x}\,\text{}^{(1)}(x) + \frac{(x-1)}{x(x+1)}\frac{\log(x+1)}{m^2x} + \frac{1}{m^2x(x+1)}\left(\gamma + \log\frac{m^2}{\mu^2}\right) \\ \text{}^{(1)}(0) = -\frac{1}{m^2} + \frac{\gamma+\log\frac{m^2}{\mu^2}}{m^2} \end{cases} \tag{2.95}$$

$$\downarrow$$

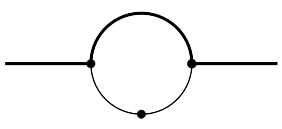
$$\text{}^{(1)}(x) = \frac{\log(x+1)\left(\log\frac{m^2}{\mu^2} - \log(x+1) + \gamma\right) - \mathrm{Li}_2(-x)}{m^2x} \tag{2.96}$$

  where $\mathrm{Li}_2 = \int_0^x \frac{1}{t}\log(1+t)\,\mathrm{d}t$ is the dilogarithm function.

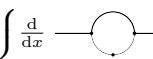
Gathering the terms:

$$\text{}(x) = -\frac{\log(x+1)}{m^2x} + \epsilon\frac{\log(x+1)\left(\log\frac{m^2}{\mu^2} - \log(x+1) + \gamma\right) - \mathrm{Li}_2(-x)}{m^2x} + o_0(\epsilon) \tag{2.97}$$

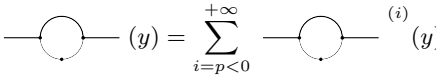
Example 9 (Half-massive bubble with massless dotted propagator). Considering the same topology, but with the massless propagator dotted instead of the massive one:



Proceeding as above:

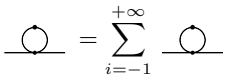
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}\,\text{  }(x) = -\frac{(x-2)x(\epsilon+1)+\epsilon-1}{x(x^2-1)}\,\text{}(x) - \frac{1}{m^2x(x^2-1)}\,\text{} \\ \text{}(0) = -\frac{1}{m^2(\epsilon-1)}\,\text{} \end{cases}; \tag{2.98}$$

order by order:



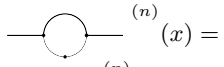
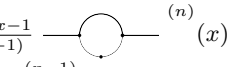
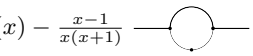
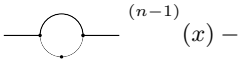
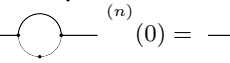
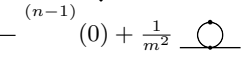
$$= \sum_{i=p<0}^{+\infty}$$

$$(y) \tag{2.99}$$

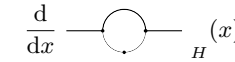
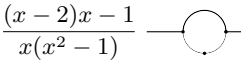
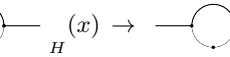


$$= \sum_{i=-1}^{+\infty}$$

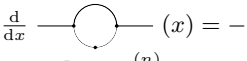
$$\tag{2.100}$$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}$$  $$(x) = -\frac{(x-2)x-1}{x(x^2-1)}$$  $$(x) - \frac{x-1}{x(x+1)}$$  $$(x) - \frac{1}{m^2x(x^2-1)}$$  $$\\ $$  $$(0) =$$  $$(0) + \frac{1}{m^2}$$ $$\end{cases} . \tag{2.101}$$

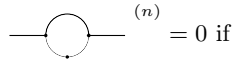
The asociated homogeneous DE:

$$\frac{\mathrm{d}}{\mathrm{d}x}$$  $$(x) = -\frac{(x-2)x-1}{x(x^2-1)}$$  $$(x) \rightarrow$$  $$(x) = C\frac{1-x}{x^2+x}. \tag{2.102}$$

- ϵ^n and $n < -1$ In the hypothesis of finite pole, starting from the lowest order, the Cauchy problem reduces to:

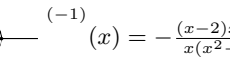
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}$$ $$(x) = -\frac{(x-2)x-1}{x(x^2-1)}$$  $$(x) \\ $$ $$(0) = 0 \end{cases} , \tag{2.103}$$

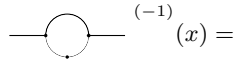
this problem has a solution equal to the one of the homogeneous DE, and thanks to the fact that in $x = 0$ the integral is regular, the constant must vanish to avoid singularities. Therefore



$$= 0 \text{ if } n < -1. \tag{2.104}$$

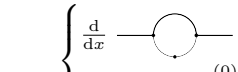
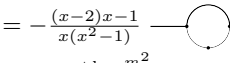
- ϵ^{-1} The Cauchy problem has the form:

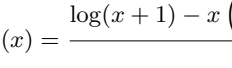
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}$$  $$(x) = -\frac{(x-2)x-1}{x(x^2-1)}$$  $$(x) - \frac{1}{m^2x(x^2-1)} \\ $$  $$(0) = \frac{1}{m^2} \end{cases} \rightarrow \tag{2.105}$$

$$\rightarrow$$ 

$$= \frac{1}{m^2(x+1)}$$

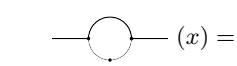
- ϵ^0 The Cauchy problem has the form:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}$$  $$(x) = -\frac{(x-2)x-1}{x(x^2-1)}$$  $$(x) - \frac{x-1}{x(x+1)}\frac{1}{m^2(x+1)} + \frac{\gamma-\log\frac{m^2}{\mu^2}}{m^2x(x^2-1)} \\ $$  $$(0) = \frac{1}{m^2} - \frac{\gamma+\log\frac{m^2}{\mu^2}}{m^2} \end{cases} \rightarrow \tag{2.106}$$

$$\rightarrow$$ 

$$= \frac{\log(x+1) - x\left(\log\frac{m^2}{\mu^2}\right) - \gamma x + \log(x+1)}{m^2x(x+1)}$$

Putting all together:



$$= \frac{1}{\epsilon}\frac{1}{m^2(x+1)} + \frac{\log(x+1) - x\left(\log\frac{m^2}{\mu^2}\right) - \gamma x + \log(x+1)}{m^2x(x+1)} + o_0(\epsilon^0). \tag{2.107}$$

It is interesting to have a look to the divergencies in the two cases of dotted bubbles: while the integral with massive dotted propagator does not shows poles in ϵ, the other one has a pole of order 1. The justification for

this different behaviour does not lie in the UV area (that does not cause problems, having a maximal power in the denominator of order k^j, with $j > 4$), but in the IR behaviour. The integral in the massive dotted propagator case has an asymptotic behaviour in 0 (chosen euclidean variables and performed Wick's rotation) of the form:

$$\int \frac{\mathrm{d}^D k}{k^2((k-p)^2+m^2)^2} = \int \frac{\kappa^{D-1}\,\mathrm{d}\kappa\,\mathrm{d}\Omega_D}{\kappa^2(m^4+p^4+2p^2m^2)+o_0(\kappa^2)} \sim_0^* \int \frac{\kappa^{D-3}\,\mathrm{d}\kappa\,\mathrm{d}\Omega_D}{m^2}, \tag{2.108}$$

converging in in $\kappa = 0$ iff $D > 2$, so there is no pole in the $\epsilon \to 0$ limit.

Regarding the integral with a massless dotted propagator, the behaviour is different:

$$\int \frac{\mathrm{d}^D k}{k^4((k-p)^2+m^2)} = \int \frac{\kappa^{D-1}\,\mathrm{d}\kappa\,\mathrm{d}\Omega_D}{\kappa^4(m^2+p^2)+o_0(\kappa^4)} \sim_0^* \int \frac{\kappa^{D-5}\,\mathrm{d}\kappa\,\mathrm{d}\Omega_D}{m^2} \tag{2.109}$$

this time the integral converges in $\kappa = 0$ iff $D > 4$, so it develops a pole in $\epsilon = 0$.

Notice, as a conclusion, that a similar behaviour does not occur in the case of the massive bubble: at every order, there is no divergence in $\kappa = 0$, thanks to the mass term $m^{2(\alpha_1+\alpha_2)}$ from $[k^2+m_1^2]^{\alpha_1}[(p-k)^2+m_2^2]^{\alpha_2}$ (the divergence encoded by ϵ is in the UV region), while in the massless bubble is finite in $\kappa = 0$ without dotted propagators, and develops divergences rising the denominators.

The previous examples illustrate application of the method of variation of the constants in quite general cases. In order to make the resolution of the DE more compact three further aspects can be considered:

Definition 5 (Canonical system)**.** A canonical system of DEs is an ϵ-expanded system of DEs with the following properties:

ϵ **finiteness** MIs are chosen to be free from ϵ poles (the Laurent expansion starts form $N_0 = 0$) and to obtain a matrix of coefficients M_α not singular in ϵ, so the whole system will be regular in $D \to 4$ ($\epsilon \to 0$);

completeness the system of DEs is a homogeneous system of DEs, namely:

$$\frac{\partial \mathbf{I}(\epsilon;\mathcal{S})}{\partial s_\alpha} = M_\alpha(\epsilon;\mathcal{S})\mathbf{I}(\epsilon;\mathcal{S}). \tag{2.110}$$

This expression does not contain non-homogeneous terms;

ϵ-homogeneity M_α is homogeneous in ϵ.

These requests allow to cast the system at ϵ^i order in the form:

$$\frac{\partial \mathbf{I}^{(i)}(\mathcal{S})}{\partial s_\alpha} = M_\alpha^{(1)}(\mathcal{S})\mathbf{I}^{(i-1)}(\mathcal{S}), \tag{2.111}$$

with associated homogeneous DE

$$\frac{\partial \mathbf{I}^{(i)}(\mathcal{S})}{\partial s_\alpha} = 0 \quad \to \quad \mathbf{I}^{(i)}(\mathcal{S}) = \mathbf{C}^{(i)}, \tag{2.112}$$

and equation for the constants of the form:

$$\begin{aligned}
\frac{\partial \mathbf{C}^{(0)}(\mathcal{S})}{\partial s_\alpha} = 0 \rightarrow & \mathbf{I}^{(0)}(\mathcal{S}) = \mathbf{C}^{(0)}, \\
\frac{\partial \mathbf{C}^{(1)}(\mathcal{S})}{\partial s_\alpha} = M_\alpha^{(1)}(\mathcal{S})\mathbf{I}^{(0)}(\mathcal{S}) \rightarrow & \mathbf{I}^{(1)}(\mathcal{S}) = \mathbf{C}^{(1)} + \int_{s_\alpha} M_\alpha^{(1)}(\mathbf{X}_1)\mathbf{C}^{(0)}\,\mathrm{d}\tau_1, \\
\frac{\partial \mathbf{C}^{(2)}(\mathcal{S})}{\partial s_\alpha} = M_\alpha^{(1)}(\mathcal{S})\mathbf{I}^{(1)}(\mathcal{S}) \rightarrow & \mathbf{I}^{(2)}(\mathcal{S}) = \mathbf{C}^{(2)} + \int_{s_\alpha} M_\alpha^{(1)}(\mathbf{X}_1)\mathbf{C}^{(1)}\,\mathrm{d}\tau_1 + \\
& + \int_{s_\alpha} M_\alpha^{(1)}(\mathbf{X}_1) \int_{\tau_1} M_\alpha^{(1)}(\mathbf{X}_2)\mathbf{C}^{(0)}\,\mathrm{d}\tau_2\mathrm{d}\tau_1, \\
& \dots
\end{aligned} \tag{2.113}$$

where $\mathbf{X}_i = (x_1; \dots; x_{i-1}; \tau_i; x_{i+1}; \dots; x_n)$, and it is possible to proceed as the system has a single independent variable for the reasons explained in 2.3.4.

So with the constraints on the matrix M_α the solution order by order reduces to a series of terms with increasing nested integrations, generating a regular iterated structure. This will be the starting point to understand how to properly express the solutions.

Chapter 3

Iterated integrals and uniform transcendentality

Following the procedure indicated in [10, 11, 13, 14], the solution of a canonical system of differential equations [10] is written in terms of a *Dyson series* of iterated integrals. Introducing the notion of *fuchsian form*, it is possible to write the solution of the differential equations in terms of *harmonic polylogarithms*, functions built up using iterated integrations on a rational kernel [8]. The concept of *transcendental weight* is presented [8, 12, 14] and applied to series expansions in order to characterize *uniform transcendental* functions. In conclusion, a selection of exact and guiding criteria for the *a priori* determination of uniform transcendental Feynman integrals is presented [10–14].

From now on the analysis will focus on canonical systems only.

3.1 Dyson Series

The final result of (2.113) can be seen as a consequence of the fact that (starting with one-scale problems) the differential problem

$$\begin{cases} \frac{\mathrm{d}\mathbf{y}(x;\epsilon)}{\mathrm{d}x} = \epsilon m(x)\mathbf{y}(x;\epsilon) \\ \mathbf{y}(x_0;\epsilon) = \mathbf{y}_0(\epsilon) \end{cases} \tag{3.1}$$

is equivalent to the Volterra integral problem

$$\mathbf{y}(x;\epsilon) = \mathbf{y}_0(\epsilon) + \int_{x_0}^{x} m(\tau)\mathbf{y}(\tau;\epsilon)\,\mathrm{d}\tau. \tag{3.2}$$

Considering now that (the matrix of coefficients is $M(x;\epsilon) = \epsilon m(x)$)

- generally speaking, matrices does not commute with their integrals, namely: $\int_x m(\tau)\,\mathrm{d}\tau m(x) \neq m(x)\int_x m(\tau)\,\mathrm{d}\tau$,

- the operator $T = \mathbf{y}_0(\epsilon) + \int_{x_0}^{x} m(\tau)\ \mathrm{d}\tau$ is a contraction in the space of the solutions of the differential problem wrote above,

it is possible to write the solution as a series in T:

$$\begin{aligned}\mathbf{y}(x) =&\mathbf{y}_0(\epsilon) + \epsilon \int_{x_0}^{x} m(\tau_1)\,\mathrm{d}\tau_1 \mathbf{y}_0(\epsilon) + \epsilon^2 \int_{x_0}^{x}\int_{x_0}^{\tau_1} m(\tau_1)m(\tau_2)\,\mathrm{d}\tau_2\mathrm{d}\tau_1 \mathbf{y}_0(\epsilon) + o_0(\epsilon^2) + \ldots = \quad (3.3)\\ =&\left(1 + \sum_{n=1}^{N} \epsilon^n \int_{x_0}^{x}\int_{x_0}^{\tau_1} \ldots \int_{x_0}^{\tau_{n-1}} m(\tau_1)m(\tau_2)\ldots m(\tau_n)\,\mathrm{d}\tau_n \ldots \mathrm{d}\tau_2\mathrm{d}\tau_1\right)\mathbf{y}_0(\epsilon) + o_0\left(\epsilon^N\right) = \quad (3.4)\\ =&\mathcal{D}\mathrm{e}^{\int_{x_0}^{x} m(\tau)\,\mathrm{d}\tau}\mathbf{y}_0. \quad (3.5)\end{aligned}$$

The result obtained is equal to the Dyson series obtained considering $M(x;\epsilon)$ as a perturbation term of the hamiltonian $H = H_0 + M$; for this reason, the equation (3.3) is called the *Dyson series* of the ϵ-factorized DE (3.1).

3.1.1 ϵ factorization

The dependence from ϵ in (3.3) is not yet fully factorized, as

$$\mathbf{y}(x;\epsilon) = \sum_{n=0}^{N} \epsilon^n \mathbf{y}^{(n)}(x) + o_0\left(\epsilon^N\right), \quad (3.6)$$

$$\mathbf{y}_0(\epsilon) = \sum_{n=0}^{N} \epsilon^n \mathbf{y}_0^{(n)} + o_0\left(\epsilon^N\right). \quad (3.7)$$

for the sake of simplicity the functions are considered to be finite in ϵ, but the results that will be obtained will be valid also for ϵ-divergent functions, since it is possible to pass from the divergent form of the function to the finite one multiplying by a suitable power of ϵ.

First of all, it is important to properly understand what expression (3.3) means: for this aim, the process of antiderivation has to be separated from the process of evaluation of the primitive at the end points of the interval of integration

$$\int_{x_0}^{x} m(\tau)\,\mathrm{d}\tau = \left[\int m(\tau)\,\mathrm{d}\tau\right]_{x_0}^{x}. \quad (3.8)$$

So the equation (3.3) can be written as:

$$\mathbf{y}(x) = \left[1 + \sum_{n=1}^{N} \mathcal{M}^{(n)}(x)\right]\mathbf{y}_0; \quad (3.9)$$

$$\mathcal{M}^{(n)}(x) = \begin{cases}\left[\int m(\tau_n)\mathcal{M}^{(n-1)}(\tau_n)\,\mathrm{d}\tau_n\right]_{x_0}^{x} & \text{if } n > 1\\ \left[\int m(\tau_1)\,\mathrm{d}\tau_1\right]_{x_0}^{x} & \text{if } n = 1\end{cases} \quad (3.10)$$

($\int$ indicates indefinite integration, or antiderivation) .It is possible to see that, at every order n, the matrix $\mathcal{M}^{(n-1)}$ is *evaluated* at the end points, while $\int m(\tau_n)\mathcal{M}^{(n-1)}(\tau_n)\,\mathrm{d}\tau_n$ is just an antiderivation, evaluated in a second moment ($\left[\int m(\tau_n)\mathcal{M}^{(n-1)}(\tau_n)\,\mathrm{d}\tau_n\right]_{x_0}^{x}$). Furthermore, $\mathcal{M}^{(n)}(x)$ can be decomposed considering the following notation:

$$\mathbf{M}^{(i)}(x) = \left[\int \dots \int m(\tau_1)\dots m(\tau_i)\,\mathrm{d}\tau_i \dots \mathrm{d}\tau_1\right]_x ; \tag{3.11}$$

expanding in ϵ the Dyson series, and collecting the terms with the same power, the following relations are obtained:

$$\mathbf{y}^{(0)}(x) = \mathbf{y}_0^{(0)}; \tag{3.12}$$

$$\mathbf{y}^{(1)}(x) = \mathbf{y}_0^{(1)} + \mathbf{M}^{(1)}(x)\mathbf{y}_0^{(0)} - \mathbf{M}^{(1)}(x_0)\mathbf{y}_0^{(0)}; \tag{3.13}$$

$$\begin{aligned}\mathbf{y}^{(2)}(x) = {} & \mathbf{y}_0^{(2)} + \mathbf{M}^{(1)}(x)\mathbf{y}_0^{(1)} - \mathbf{M}^{(1)}(x_0)\mathbf{y}_0^{(1)} + \mathbf{M}^{(2)}(x)\mathbf{y}_0^{(0)} - \mathbf{M}^{(2)}(x_0)\mathbf{y}_0^{(0)} + \\ & - \mathbf{M}^{(1)}(x)\mathbf{M}^{(1)}(x_0)\mathbf{y}_0^{(0)} + \left[\mathbf{M}^{(1)}(x_0)\right]^2 \mathbf{y}_0^{(0)}; \end{aligned} \tag{3.14}$$

$$\dots$$

finding a Dyson series of the form:

$$\mathbf{y}(x) = \sum_{i=0}^{N} \epsilon^i \sum_{j=0}^{i} \mathcal{M}^{(j)}(x)\mathbf{y}_0^{(i-j)} + o_0\left(\epsilon^N\right), \tag{3.15}$$

where $\mathcal{M}^{(0)}(x) = 1$.

It is now possible to redefine the constant terms $\mathbf{y}_0^{(i)}$ as:

$$\tilde{\mathbf{y}}_0^{(0)} = \mathbf{y}_0^{(0)}; \tag{3.16}$$

$$\tilde{\mathbf{y}}_0^{(1)} = \mathbf{y}_0^{(1)} - \mathbf{M}^{(1)}(x_0)\mathbf{y}_0^{(0)} = \mathbf{y}_0^{(1)} - \mathbf{M}^{(1)}(x_0)\tilde{\mathbf{y}}_0^{(0)}; \tag{3.17}$$

$$\begin{aligned}\tilde{\mathbf{y}}_0^{(2)} = {} & \mathbf{y}_0^{(2)} - \mathbf{M}^{(1)}(x_0)\mathbf{y}_0^{(1)} - \mathbf{M}^{(2)}(x_0)\mathbf{y}_0^{(0)} + \left[\mathbf{M}^{(1)}(x_0)\right]^2 \mathbf{y}_0^{(0)} = \\ & = \mathbf{y}_0^{(2)} - \mathbf{M}^{(1)}(x_0)\tilde{\mathbf{y}}_0^{(1)} - \mathbf{M}^{(2)}(x_0)\tilde{\mathbf{y}}_0^{(0)}; \end{aligned} \tag{3.18}$$

$$\dots$$

$$\tilde{\mathbf{y}}_0^{(n)} = \mathbf{y}_0^{(n)} - \sum_{i=1}^{n} \mathbf{M}^{(i)}(x_0)\tilde{\mathbf{y}}_0^{(n-i)}. \tag{3.19}$$

This nested redefinition is well written, as i-th term is obtained by integrating the previous one, and the term $\tilde{\mathbf{y}}_0^{(i-1)}$ multiplies always a term of the form $[\mathbf{M}_1(x) - \mathbf{M}_1(x_0)]$, as $\tilde{\mathbf{y}}_0^{(i-1)}$ is the only

constant term in each iteration. Rewriting the expressions in terms of $\mathbf{y}_0^{(i)}$:

$$\mathbf{y}^{(0)}(x) = \tilde{\mathbf{y}}_0^{(0)}; \tag{3.20}$$

$$\mathbf{y}^{(1)}(x) = \tilde{\mathbf{y}}_0^{(1)} + \mathbf{M}^{(1)}(x)\tilde{\mathbf{y}}_0^{(0)}; \tag{3.21}$$

$$\mathbf{y}^{(2)}(x) = \tilde{\mathbf{y}}_0^{(2)} + \mathbf{M}^{(1)}(x)\tilde{\mathbf{y}}_0^{(1)} + \mathbf{M}^{(2)}(x)\tilde{\mathbf{y}}_0^{(0)}; \tag{3.22}$$

$$\dots$$

$$\mathbf{y}^{(n)}(x) = \tilde{\mathbf{y}}_0^{(n)} + \sum_{i=1}^{n} \mathbf{M}^{(i)}(x_0)\tilde{\mathbf{y}}_0^{(n-i)} \tag{3.23}$$

In this way the solution of the DE only depends on the operation of antiderivation, and the values of the integrals in x_0 are absorbed in $\tilde{\mathbf{y}}_0^{(n)}$ together with the constant $\mathbf{y}_0$, and can be fixed imposing the BCs.

Concluding, the expression of the solution in terms of Dyson series takes the form:

$$\mathbf{y}(x) = \sum_{i=0}^{N} \epsilon^i \sum_{j=0}^{i} \mathbf{M}^{(j)}(x)\tilde{\mathbf{y}}_0^{(i-j)} + o_0\left(\epsilon^N\right) = \mathcal{D}\mathrm{e}^{\int m(\tau)\,\mathrm{d}\tau}, \tag{3.24}$$

where $\mathbf{M}^{(0)}(x) = 1$.

What has been said above is still valid when working on a multi-scale problem, with some precautions. The most important one is the fact that the operation of antiderivation carried on to solve one by one the PDEs is *separated* from the operation of integration.

Regarding the operation of antiderivation, the properties explained above are still valid, one variable at a time, as explained in the previous chapter for existence and uniqueness of the solutions.

To evaluate the primitives, remember that all calculations are usually carried out in simple connected spaces, so the path of integration is irrelevant, until the extremal points are fixed. It is then possible to divide the line of integration in jointed straight lines parallel to one of the axis at a time:

$$\begin{aligned}\gamma(\mathbf{x_0} \to \mathbf{x}) = & \\ &= [(x_{1;0} + \alpha[x_1 - x_{1;0}]; x_{2;0}; \dots; x_{n;0})]_{\alpha\in[0;1]} \cdot \\ &\cdot [(x_1; x_{2;0} + \alpha[x_2 - x_{2;0}]; x_{3;0}; \dots; x_{n;0})]_{\alpha\in[0;1]} \cdot \\ &\dots \\ &\cdot [(x_1; \dots; x_{n-1;0}; x_{n;0} + \alpha[x_n - x_{n;0}])]_{\alpha\in[0;1]} \cdot\end{aligned} \tag{3.25}$$

This allows to rewrite, according to the necessities, the result in various forms. One form that is useful when working with two variables is:

$$\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{f}(\mathbf{x})\,\mathrm{d}^g\mathbf{x} = \sum_{i=1}^{g} \int_{(x_1;\dots;x_{i-1};x_{i;0};x_{i+1;0};\dots;x_{g;0})}^{(x_1;\dots;x_{i-1};x_i;x_{i+1;0};\dots;x_{g;0})} \mathbf{f}(x_1;\dots;x_{i-1};\tau;x_{i+1;0};\dots;x_{n;0})\,\mathrm{d}\tau, \tag{3.26}$$

obtained summing and subtracting terms like

$$\left[\int \mathbf{f}(x_1;\ldots;x_{i-1};\tau;x_{i+1;0};\ldots;x_{n;0})\,\mathrm{d}\tau\right]_{(x_1;\ldots;x_i;x_{i+1;0};\ldots;x_{g;0})} \tag{3.27}$$

with $i = 1;\ldots;g-1$.

3.1.2 Fuchsian form

The last step concerns a property of the coefficients of the DEs that will be fundamental for the introduction of the harmonic polylogarithms, the class of functions in which the solutions are usually expressed:

Definition 6 (Fuchsian system)**.** A system of DEs

$$\begin{cases} \frac{\partial \mathbf{y}(\mathbf{x})}{\partial x_1} = M_1(\mathbf{x})\mathbf{y}(\mathbf{x}) \\ \ldots \\ \frac{\partial \mathbf{y}(\mathbf{x})}{\partial x_n} = M_n(\mathbf{x})\mathbf{y}(\mathbf{x}) \end{cases} \tag{3.28}$$

is said to be fuchsian if all its matrices of coefficients $M_i(\mathbf{x})$ are composed only by simple poles in all the $x_i \in \mathbf{x}$, and only rational coefficients are present.

Rarely a system of DEs is already fuchsian. In order to pass to a fuchsian system a redefinition of the MIs is necessary, usually inserting dots or irreducible scalar products. Good choices for basis of MIs are the so-called uniform transcendental bases, that will be introduced in section 3.3.

3.2 Harmonic polylogarithms

From the beginning of the study on scattering amplitudes, functions like Euler's dilogarithms $\mathrm{Li}_2(x)$ appeared in the result of radiation emission and scattering (one of the first paper in which such functions appear is [53]). For multi-loop calculation with loop number $l > 2$ a generalization of such dilogarithms is necessary, so Nielsen polylogarithms were used. A further generalization of such structure has been made, introducing *harmonic polylogarithms* (HPL), and their version for multi-scale problems, *G-polylogarithms* (G-HPL). This class of functions represents a useful mathematical support for the study of multi-loop problems, especially massless. For a detailed introduction see e.g. [8].

Here HPLs of one variable will be considered. Most of the properties and definitions are still valid in the multi-scale case, variable by variable.

3.2.1 Basic definitions

HPLs originates from nested sum, as for Nielsen polylogarithms, of the form:

$$H_{m_1;\ldots;m_k} = \mathrm{Li}_{m_1;\ldots;m_k}(x;1_{k-1};\ldots;1_1) = \sum_{i_1=1}^{+\infty}\sum_{i_2=1}^{i_1-1}\cdots\sum_{i_k=1}^{i_{k-1}-1}\frac{x^{i_1}}{\prod_{j=1}^{k} i_j^{m_j}} \tag{3.29}$$

(for more details, see [54]). Such a series expression, that for $k = 1$ and $m_1 = 1$ generates logarithms like $-\log(1-x)$, is centered in $x = 0$ and has a radius of convergence $r = 1$. These functions admit also an integral representation, which interval of validity, subsequently to the finite radius of convergence, will be $]-1;1[$.

Speaking about integral representation, HPLs are defined following a nested integration structure, totally analogous to the one used in the Dyson series. First of all, a finite set of integrands, the *integration kernel* is chosen. By integration on a fixed interval $[x_0; x]$ (here, $x_0 = 0$) a first generation of functions (HPLs of weight 1) is generated. To compute HPLs of weight increased by 1, the previous HPLs are multiplied by one of the functions of the integration kernel, then integrated on the interval above. Iterating this process, starting from HPL of weight one, HPLs on a desired weight can be generated.

Definition 7 (HPL kernel of integration)**.** [8]

$$f(0;x) = \frac{1}{x}, \tag{3.30}$$

$$f(1;x) = \frac{1}{1-x}, \tag{3.31}$$

$$f(-1;x) = \frac{1}{1+x} \tag{3.32}$$

form the kernel of integration for the HPLs.

Definition 8 (Transcendental weight)**.** The transcendental weight w (also, *weight*, for simplicity) of an HPL is the number of nested integrations carried out using the HPL kernel of integration, in the form:

$$\int_{x_0}^{x} f(a_w;\tau_w) \int_{x_0}^{\tau_w} f(a_{w-1};\tau_{w-1}\dots \int_{x_0}^{\tau_2} f(a_1;\tau_1)\,\mathrm{d}\tau_1 \dots \mathrm{d}\tau_{w-1}\mathrm{d}\tau_w. \tag{3.33}$$

Operatively, the weight of a HPL is equal to the length of the list of indices $(m_w;\dots;m_1)$ in $H(m_w;\dots;m_1;x)$.

In general, a function $F_w : \mathbb{C}^n \to \mathbb{C}$ is a function of transcendental weight w if it satisfies the relation:

$$\mathrm{d}F_w(\mathbf{x}) = \sum_{i=1}^{N} F_{w-1}^{(i)}(\mathbf{x})\,\mathrm{d}\log R_i(\mathbf{x}) \tag{3.34}$$

where $F_{w-1}^{(i)}(\mathbf{x})$ are transcendental functions of weight $w-1$ (function of weight 0 are natural constants) and $R_i(\mathbf{x})$ are rational functions.

Observation 1. Thanks to linearity property of integration, a sum of terms all with same transcendental weight w has transcendental weight w.

Observation 2. Given a function $f(\mathbf{x})$ of weight w, the function $1/f(\mathbf{x})$ is considered to have weight $-w$ for consistency.

The relation connecting HPLs of higher weight with lower HPLs is a differential expression of the form:

$$\frac{\mathrm{d}H(m_w;\dots;m_1;x)}{\mathrm{d}x} = f(m_w;x)H(m_{w-1};\dots;m_1;x). \tag{3.35}$$

At weight $w = 1$ there are three different HPLs:

$$H(0;x) = \log x, \tag{3.36}$$

$$H(1;x) = \int_0^x \frac{\mathrm{d}\tau}{1-\tau} = -\log(1-x), \tag{3.37}$$

$$H(-1;x) = \int_0^x \frac{\mathrm{d}\tau}{1+\tau} = \log(1+x). \tag{3.38}$$

Few clarifications about the form of the HPLs are mandatory.

First of all, thanks to the fact that the integral representation is valid only inside the region of convergence (i.e. in $]-1;1[$) and to the explicit form of the nested sum series, the result of the two integrals enlisted above can be written without the symbol of absolute value. For coherence with respect to the series expressions above, $H(0;x)$ is defined as $\log x$ without any absolute value.[1]

Due to the definition of $H(0;x)$, in the subinterval $]-1;0[$ an imaginary factor $\mathrm{i}\pi$ appears. This divide the interval of definition of the HPLs into two parts: $]-1;0[$ and $]0;1[$, the latest is called *Euclidean region*, in which all the HPLs are real.

Finally, to preserve the notion of weight, also to $H(0;x)$ will be associated weight 1 (it comes from a "formal integration" of $f(0;x)$).

Inverting equation (3.35) and taking into account the properties enlisted above, the generic HPL is determined thanks to the iterative formula:

$$H(m_w;\ldots;m_1;x) = \begin{cases} \frac{1}{w!}\log^w x & \text{if } m_k = 0\,\forall k \\ \int_0^x f(m_w;\tau)H(m_{w-1};\ldots;m_1;\tau)\,\mathrm{d}\tau & \text{if } \exists\, m_k \neq 0 \end{cases}. \tag{3.39}$$

It follows immediately from simple integration that also

$$H(1_w;\ldots;1_1;x) = \frac{1}{w!}\left[-\log(1-x)\right]^w; \tag{3.40}$$

$$H(-1_w;\ldots;-1_1;x) = \frac{1}{w!}\log^w(1+x). \tag{3.41}$$

Observation 3. Since HPLs are encoded by an ordered list of three numbers $(-1;0;1)$ and a variable x, the number of HPLs grows with the weight w as 3^w.

Example 10 ($w = 2$ HPLs). For $w = 2$ there are $3^2 = 9$ HPLs. These can be rewritten by direct integration

[1] Notice that the expression $H(0;x) = \int_0^x \frac{1}{\tau}\,\mathrm{d}\tau = \log x$ is not well-defined: first of all the integral in $\tau = 0$ is divergent, not returning the logarithmic expression, moreover it is not related to any of the polylogarithmic series, being $\mathrm{Li}_1(1-x) = -\log x$, without the absolute value.

in terms of Euler's dilogarithms, as:

$$H(0;0;x) = \frac{1}{2}\log^2 x, \tag{3.42}$$

$$H(0;1;x) = \mathrm{Li}_2\, x, \tag{3.43}$$

$$H(0;-1;x) = -\mathrm{Li}_2(-x), \tag{3.44}$$

$$H(1;0;x) = -\log x \log(1-x) - \mathrm{Li}_2\, x, \tag{3.45}$$

$$H(1;1;x) = \frac{1}{2}\log^2(1-x), \tag{3.46}$$

$$H(1;-1;x) = \mathrm{Li}_2\,\frac{1-x}{2} - \log 2 \log(1-x) - \mathrm{Li}_2\,\frac{1}{2}, \tag{3.47}$$

$$H(-1;0;x) = \log x \log(1+x) + \mathrm{Li}_2(-x), \tag{3.48}$$

$$H(-1;1;x) = \mathrm{Li}_2\,\frac{1+x}{2} - \log 2 \log(1+x) - \mathrm{Li}_2\,\frac{1}{2}, \tag{3.49}$$

$$H(-1;-1;x) = \frac{1}{2}\log^2(1+x). \tag{3.50}$$

3.2.2 Properties of HPLs

For a close examination of the properties of HPLs see [8]. Here only the most relevant ones will be presented.

In any case, almost all of the left properties (mainly involving change of variables not used in this work), can be derived using the fundamental theorem of calculus, in the form:

$$H(\mathbf{m};x) = H(\mathbf{m};a) + \int_a^x \frac{\mathrm{d}H(\mathbf{m};\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau. \tag{3.51}$$

- If $\mathbf{m} \neq (0;\ldots;0)$ the HPLs vanish at $x = 0$, thanks to the fact that at least one actual integration is involved: $\int_0^0 \phi(\tau)\,\mathrm{d}\tau = 0$, regardless of ϕ.
- If $m_w \neq 1$ then $H(\mathbf{m};1)$ is finite. It is also finite if $\mathbf{m} = (1;0;\ldots;0)$, otherwise it has a logaritmic divergence $\log^p(1-x)$ in $x = 1$, with maximum power of the logarithm p equal to the number of leftmost consecutive terms equal to 1.
- The following proposition on the change of sign of the argument holds:

  Proposition 6 (Change of sign relation)**.** *If* $m_1 \neq 0$

  $$H(\mathbf{m};-x) = (-1)^{\sum_{i=1}^w |m_i|} H(-\mathbf{m};x) \tag{3.52}$$

  Proof. The proof is carried out by induction. First of all, $H(1;-x) = -\log(1+x) = -H(-1;x)$. Now $H(\mathbf{m};-x) = \int_0^{-x} f(m_w;\tau)H(m_{w-1};\ldots;m_1;\tau)\,\mathrm{d}\tau$, performing the change of variables $\tau = -\eta$: $\int_0^{-x} f(m_w;\tau)H(m_{w-1};\ldots;m_1;\tau)\,\mathrm{d}\tau = -\int_0^x f(m_w;-\eta)\,H(m_{w-1};\ldots;m_1;-\eta)\,\mathrm{d}\eta = -\int_0^x f(m_w;-\eta)(-1)^{\sum_{i=1}^{w-1}|m_j|}$ $H(-m_{w-1};\ldots;-m_1;\eta)\,\mathrm{d}\eta$. Now:

  $$f(m_w;-\eta) = \begin{cases} f(m_w;\eta) & \text{if } m_w \neq 0 \\ -f(m_w;\eta) & \text{if } m_w = 0 \end{cases} \tag{3.53}$$

  So:

  $$-\int_0^x f(m_w;-\eta)(-1)^{\sum_{i=1}^{w-1}|m_j|} H(-m_{w-1};\ldots;-m_1;\eta)\,\mathrm{d}\eta = (-1)^{\sum_{i=1}^w |m_i|} H(-\mathbf{m};x). \tag{3.54}$$

  □

- Using integration by parts, it is possible to write:

$$H(m_1;\ldots;m_w;x) = \sum_{p=1}^{w}(-1)^{p+1}H(m_p;m_{p-1};\ldots;m_1;x)H(m_{p+1};\ldots;m_w;x), \qquad (3.55)$$

where $H(x) \equiv 1$ and $\int_0^x \frac{1}{\tau}\,\mathrm{d}\tau \equiv \log x$. The proof is a direct calculation:

$$\begin{aligned} H(m_1;\ldots;m_w;x) &= \int_0^x f(m_1;\tau)H(m_2;\ldots;m_w)\,\mathrm{d}\tau = \\ &= H(m_1;x)H(m_2;\ldots;m_w;x) - \int_0^x f(m_2;\tau)H(m_1;\tau)H(m_3;\ldots;m_w;\tau)\,\mathrm{d}\tau = \\ &= H(m_1;x)H(m_2;\ldots;m_w;x) - H(m_2;m_1;x)H(m_3;\ldots;m_w;x) + \ldots + \\ &+ (-1)^{p+1}H(\_p;\ldots;m_1;x)H(m_{p+1};\ldots;m_1;x) + \ldots + (-1)^{w}H(m_w;\ldots;m_1;x). \end{aligned} \qquad (3.56)$$

- When $x < 0$ imaginary terms may appear, due to the presence of the term $H(0;x) = \log x$. While $H(\pm 1;x)$ are real in all the interval of convergence $]-1;1[$, $H(0;x)$ develops an imaginary part, following the prescription for analytic continuation:

$$H(0;x) = \log(x+\mathrm{i}0) \quad \Rightarrow \quad H(0;x) = H(0;-x) + \mathrm{i}\pi = \log(-x) + \mathrm{i}\pi. \qquad (3.57)$$

Thanks to the fact that an integral with real functions on real intervals cannot develop imaginary parts, and all except $H(0;\ldots;0;x)$ HPLs are evaluated via integration, only terms obtained iterating integration on HPLs with $\mathbf{m} = 0$ can generate imaginary terms: imaginary parts then come only from HPLs with one or more rightmost consecutive indices equal to 0, thanks to the prescription (3.57) for the analytic continuation.

- Thanks to equation (3.57), to maintain the relation among terms of same weight, since a sum of terms has weight w if all the terms have weight w, π must be considered having weight 1: $W(\pi) = 1$. Moreover, from relation (3.55) it follows that products between functions of weight a and b are functions with weight $a+b$ (it will be proven in proposition 7).

3.2.3 Shuffle algebra and MHPLs

The most fruitful property of HPLs is their nature to be a *shuffle algebra*.

Definition 9 (Shuffle product)**.** Given two ordered set $A = (a_1;\ldots;a_n)$ and $B = (b_1;\ldots;b_m)$, the *shuffle product* $A \sqcup\!\sqcup B$ is the sum of all the ordered sets composed by all the elements of A and B, ordered without changing the order among elements originating from the same set:

$$A \sqcup\!\sqcup B = \sum_{\substack{a_i \text{ ordered as in } A \\ b_j \text{ ordered as in } B}} (\{a_1\ldots;a_i;\ldots;a_n;b_1;\ldots;b_j;\ldots;b_m\}). \qquad (3.58)$$

A good analogy can be made with a card deck: splitting it into two parts and shuffling them together, the resulting set maintains order among cards from the same half-deck.

Observation 4 (Shuffle algebra of nested integrals [8, 54]). Nested integrals

$$I(\tau_1;\ldots;\tau_n;x)=\int_{x_0}^{x}\ldots\int_{x_0}^{\tau_{n-1}} f_1(\tau_1)\ldots f_n(\tau_n)\,\mathrm{d}\tau_n\ldots\mathrm{d}\tau_1 \tag{3.59}$$

(where $f_i(\tau_i)$ are functions possibly divergent in the interval $[x_0;x]$, depending only on τ_i among integration variables) form a shuffle algebra.

First of all, consider the simple case (immediate by graphical comparision):

$$I(\tau_1;x)I(\tau_2;x)=\int_0^x \mathrm{d}\tau_1\int_0^x \mathrm{d}\tau_2=\int_0^x\int_0^{\tau_1}\mathrm{d}\tau_2\mathrm{d}\tau_1+\int_0^x\int_0^{\tau_2}\mathrm{d}\tau_1\mathrm{d}\tau_2= \\ =H(\tau_1;\tau_2;x)+H(\tau_2;\tau_1;x) \tag{3.60}$$

depicted in figure 3.1.

This simple case of integrals showing a shuffle algebra property can be generalized in several ways.

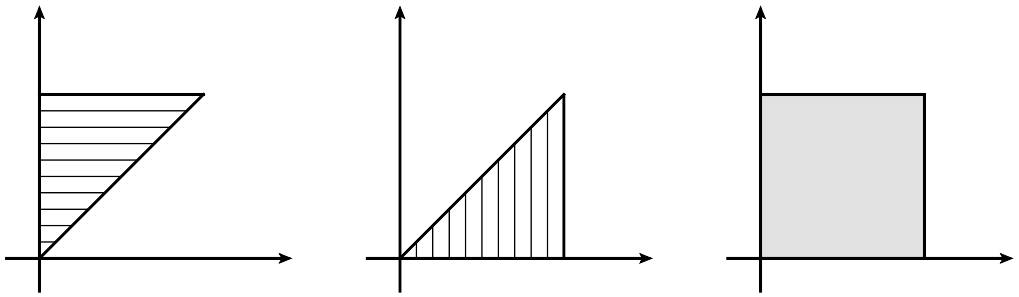


Figure 3.1: From left to right: $\int_0^x\int_0^{\tau_1}\mathrm{d}\tau_2\mathrm{d}\tau_1$, $\int_0^x\int_0^{\tau_2}\mathrm{d}\tau_1\mathrm{d}\tau_2$, $\int_0^x\mathrm{d}\tau_1\int_0^x\mathrm{d}\tau_2$.

First of all, consider two integrals $I(\mathbf{a};x)$ and $I(\mathbf{b};x)$, working on the region $\{0\leq a_1\leq x\}\times\ldots\times\{0\leq a_n\leq a_{n-1}\}=A\subset\mathbb{R}^n$ and $\{0\leq b_1\leq x\}\times\ldots\times\{0\leq b_m\leq\ b_{m-1}\}=B\subset\mathbb{R}^m$ respectively: their shuffle product will determine nested integrals on a region in the volume $(A\times B)\subset\mathbb{R}^{n+m}$. Consider now the effect on the integration regions of the "swapping" of the nested integrals: in general, the new evaluated region does not overlap the previous one, still remaining a subset of the region $\{0\leq\tau_1\leq x\}\times\ldots\times\{0\leq\tau_i\leq\tau_{i-1}\}$. In particular, if two integrals with integration intervals belonging to different initial sets are swapped, the new integral will still cover a region inside $A\times B$ (that is because all parameters are not greater than the previous ones in order of integration, so the set cannot exit from the borders of $A\times B$). Otherwise, if the swapped integrals belong to the same initial set, the resulting nested expression will no more be contained in $A\times B$, because the order of the integrals of each set has been changed, resulting in a different initial set. So for the equivalence only integrals with the same order on the separated lists of parameters must be considered: these are exactly the expressions generated by a shuffle product.

Now the problem is to understand if summing together the shuffle products all and only the region $A\times B$ is evaluated. Considering the swapping between two indices of different sets, *without*

changing the order of integration for the variables coming from the same set, the new region does not overlap the previous one. That can be seen, intuitively, from figure 3.1: swapping x_i with x_{i+1} the relation $x_{i+1} \leq x_i$ becomes $x_i \leq x_{i+1}$, determining a new, not overlapping, region. This argument is also valid with x_i and x_j, $i > j$; moreover, for the same reason different nested integrals derived from the same shuffle product cannot have overlapping domains of integration. So all the possible nested integrals generated from the shuffle product are necessary in the sum.

The fact that the whole region $A \times B$ is covered by the sum of all the integrals from the shuffle product can be proven by taking a point $\mathbf{y} = (a_1 \dots; a_n; b_1 \dots; b_m)$ in $A \times B$, and taking an integral of the shuffle product. First of all, the coordinates of one set, take A, generates a plane $(a_1; \dots; a_n; \tau_1; \dots; \tau_m)$ in $A \times B$ that passes in all the sub-domains of integration (all the intervals start from 0 and arrive at x, even if bounded by integral relations), so when an integration region containing $(b_1; \dots; b_m)$ is found, it automatically contains $(a_1; \dots; a_n; b_1; \dots; b_m)$, so it is possible to solve the problem working only with b coordinates. Operatively, to find such a set, one starts from the most internal integral: if the corresponding b coordinate of the point does belong to the interval of integration, it is possible to pass to the upper one, otherwise one swaps this one with the nearest external one belonging to a different initial set. Proceed in this way until all the b coordinates belong to the intervals of integration of the B part, and the domain of the integral determined will be the region to which the point belong. This procedure can be carried out for each point, so $A \times B$ is fully covered by the disjoint union of integration regions:

$$A \times B = \bigsqcup_{I \in A \sqcup\!\sqcup B} D_I. \tag{3.61}$$

In this way it is shown that volume integrals admit a structure of shuffle algebra. This argument is also valid assigning to each point of an interval a weight, i.e. changing the integrand from 1 to $\phi(\tau_i)$. Admitting functions in the integrals only depending on the variable of integration (regardless for their finiteness in the integration domains) does not change the core of the argument used above, so in general nested integrals form a shuffle algebra.

Since HPLs are a particular case of nested integrals, they also admit a structure of shuffle algebra (note that this structure is closed, i.e. the shuffle product of two HPLs is a sum of HPLs).

Example 11 (Dyson series). Consider the expression (3.3) for the Dyson series. A generic element of the series has the form:

$$\int_{x_0}^{x} A(\tau_1) \int_{x_0}^{\tau_1} A(\tau_2) \dots \int_{x_0}^{\tau_{n-1}} A(\tau_n)\, \mathrm{d}\tau_n \dots \mathrm{d}\tau_2 \mathrm{d}\tau_1 \mathbf{y}_0, \tag{3.62}$$

which involves nested integrals. It is then possible to use the properties of shuffle algebra of nested integration; the order of the matrices must be considered.

Consider first of all the case of matrices commuting with their own integral, and split the expression above in two blocks of integration: $\int_{x_0}^{x} A(\tau_1)\, \mathrm{d}\tau_1$ and $\int_{x_0}^{x} A(\tau_2) \dots \int_{x_0}^{\tau_{n-1}} A(\tau_n)\, \mathrm{d}\tau_n \dots \mathrm{d}\tau_2$. It is possible to rewrite the

initial integral using the shuffle algerba properties of the nested integration:

$$\begin{aligned}\int_{x_0}^{x} A(\tau_1)\int_{x_0}^{\tau_1} A(\tau_2)\dots\int_{x_0}^{\tau_{n-1}} A(\tau_n)\,\mathrm{d}\tau_n\dots\mathrm{d}\tau_2\mathrm{d}\tau_1+\\ +\int_{x_0}^{x} A(\tau_2)\int_{x_0}^{\tau_1} A(\tau_1)\dots\int_{x_0}^{\tau_{n-1}} A(\tau_n)\,\mathrm{d}\tau_n\dots\mathrm{d}\tau_1\mathrm{d}\tau_2+\\ +\dots+\\ +\int_{x_0}^{x} A(\tau_2)\dots\int_{x_0}^{\tau_{n-1}} A(\tau_n)\int_{x_0}^{\tau_{n-1}} A(\tau_1)\,\mathrm{d}\tau_1\mathrm{d}\tau_n\dots\mathrm{d}\tau_2=\\ =\int_{x_0}^{x} A(\tau_1)\,\mathrm{d}\tau_1\int_{x_0}^{x} A(\tau_2)\dots\int_{x_0}^{\tau_{n-1}} A(\tau_n)\,\mathrm{d}\tau_n\dots\mathrm{d}\tau_2. \quad (3.63)\end{aligned}$$

Moreover, the equation can be written, performing a change of variables of the form $(\tau_2\to\eta_1;\dots;\tau_n\to\eta_{n-1};\tau_1\to\eta_n)$, $(\eta_1\to\tau_1;\dots;\eta_n\to\tau_n)$, as:

$$\begin{aligned}n\int_{x_0}^{x} A(\tau_1)\int_{x_0}^{\tau_1} A(\tau_2)\dots\int_{x_0}^{\tau_{n-1}} A(\tau_n)\,\mathrm{d}\tau_n\dots\mathrm{d}\tau_2\mathrm{d}\tau_1=\\ =\int_{x_0}^{x} A(\tau_1)\,\mathrm{d}\tau_1\int_{x_0}^{x} A(\tau_2)\dots\int_{x_0}^{\tau_{n-1}} A(\tau_n)\,\mathrm{d}\tau_n\dots\mathrm{d}\tau_2. \quad (3.64)\end{aligned}$$

Iterating the procedure on the block $\int_{x_0}^{x} A(\tau_2)\dots\int_{x_0}^{\tau_{n-1}} A(\tau_n)\,\mathrm{d}\tau_n\dots\mathrm{d}\tau_2$ one arrives at the final form:

$$\int_{x_0}^{x} A(\tau_1)\int_{x_0}^{\tau_1} A(\tau_2)\dots\int_{x_0}^{\tau_{n-1}} A(\tau_n)\,\mathrm{d}\tau_n\dots\mathrm{d}\tau_2\mathrm{d}\tau_1=\frac{1}{n!}\left[\int_{x_0}^{x} A(\tau)\,\mathrm{d}\tau\right]. \quad (3.65)$$

It is then possible to write the Dyson series as a true exponential, of the form:

$$\mathbf{y}(x)=\mathrm{e}^{\int_{x_0}^{x} A(\tau)\,\mathrm{d}\tau}\mathbf{y}_0. \quad (3.66)$$

For generic squared matrices the order must be considered: in order to follow the original order in nested integrations, in the product the matrices will be ordered according to their arguments, from the greater to the smaller, i.e. in a time-ordered way:

$$\mathbf{y}(x)=\sum_{k=0}^{+\infty}\frac{1}{k!}\mathcal{T}\left[\int_{x_0}^{x} A(\tau_1)\,\mathrm{d}\tau_1\dots\int_{x_0}^{x} A(\tau_k)\,\mathrm{d}\tau_k\right]\mathbf{y}_0=\mathcal{D}\mathrm{e}^{\int_{x_0}^{x} A(\tau)\,\mathrm{d}\tau}\mathbf{y}_0. \quad (3.67)$$

Observation 5 (HPLs shuffle algebra).

$$H(\mathbf{m};x)H(\mathbf{n};x)=\sum_{\mathbf{p}=\mathbf{a}\sqcup\sqcup\mathbf{b}} H(\mathbf{p};x) \quad (3.68)$$

Another way to prove the existence of the shuffle algebra for HPLs is shown in [8], using integration by parts and induction.

The shuffle algebra structure inherited by the HPLs allows to write relations between sums of HPLs of weight w and products of HPLs of minor individual weight p and q ($p+q=w$) of the *same argument*. This relations already embed the integration by parts relations, as shuffle relations are originated by integration by parts (another way to see this fact is by directly extract the integration by parts identities from a convenient set of shuffle relations); these relations contain all the possible relations among HPLs with the same argument x.

Construction of the basis of independent HPLs

Shuffle relations, for a given weight and a given set of a points (called alphabet), form a set with[2]

$$\sum_{p=1}^{\left[\frac{w+1}{2}\right]} a^p a^{w-p} = a^w \left[\frac{w+1}{2}\right] \tag{3.69}$$

relations. Considered that the total number of HPLs at weight w are a^w, the set found has quite a great amount of redundant relations. Moreover, not all the HPLs of a given weight can be expressed in terms of HPLs of lower weight: such HPLs are called *master HPLs* (MHPL). Obviously their choice is not fixed, but some prescriptions should be considered in the construction of such set:

- HPLs with rightmost consecutive 0s generates imaginary parts in $x < 0$, due to the presence of $\log^a x$ terms in the integration;
- HPLs with leftmost consecutive 1s diverge in $x = 1$ as $\log^p(1-x)$;
- HPLs with leftmost consecutive -1 diverge in $x = -1$ as $\log^p(1+x)$.

So the preferred path to follow is to separate i-generators and divergencies as powers of, respectively, $\log x$, $-\log(1-x)$ and $\log(1+x)$. The set of MHPL at each weight will be composed preferably by HPLs with ± 1 in rightmost position and 0 in leftmost position.

The number of MHPLs has been found by direct evaluation, and it is shown in table 3.1.

Weight w	Number of HPLs	Number of MHPLs
1	3	3
2	9	3
3	27	8
4	81	18
5	243	48
6	729	116
7	2187	312
8	6561	810

Table 3.1: MHPLs.

The values of the right column of table 3.1 can also be determined thanks to the Witt formula:

$$N(w;a) = \frac{1}{w}\sum_{d|w} \mu(d) a^{\frac{n}{d}} \tag{3.70}$$

where d are the divisors of w and μ is the Moebius function. See [14] for more details.

In appendix A codes to find MHPLs and relations, together with some example at low weights are enlisted.

[2]For HPLs described in this chapter, the alphabet consists of $\{0; 1; -1\}$.

3.3 Uniform transcendentality

HPLs are a good way to express the terms generated by the integrations on the matrix of coefficients for fuchsian canonical systems. Using the concept of weight it can be seen how ϵ^n term contains only functions of weight n, when HPLs are used; therefore, assigning to ϵ transcendental weight $W(\epsilon) = -1$, all the terms of the series have weight 0. This choice is also supported by the fact that dimensional regularization is equivalent to cutoff regularizations, in which role of ϵ is played by logarithmical structures like $\log \Lambda$ (see [12]). Although, not always only HPLs are present in the series expansions of the functions used for graph evaluation, so also series with non null weight can exist. It is then useful to give the following

Definition 10 (ϵ uniform transcendental function)**.** A function $F(\mathbf{x}; \epsilon)$ is ϵ uniform transcendental (ϵ-UT, here simply UT) if all the term of its series expansion in ϵ have the same transcendental weight (called *total weight*), provided $W(\epsilon) = -1$. A UT function is called *pure UT* function if it has total weight 0 and all the terms of its expansion contain only functions with definite transcendental weight.

Considering fuchsian canonical systems fixes only half of the problem, since the final results depends also on the BC $\mathbf{y}_0$. It is then important to find also UT BCs: constraints usually do not modify the transcendental structure of the functions, while limit functions may introduce non-UT terms. Usually functions involved in BCs are other FIs or combinations of them; it follows that BCs are sums, products and inversions of functions like HPLs, $x^{P(\epsilon)}$, $\zeta(n)$ and $\Gamma(P(\epsilon))$.

Proposition 7. *The product of two functions with weights w_1 and w_2 is a function of weight $w_1 + w_2$.*

Proof. First of all it is possible to use the property of shuffle algebra of nested integrations: rewriting the functions as nested integrations starting from weight 0 integrands, and shuffling, the expression can be rewritten as:

$$\int_{\mathcal{A}_{w_1}} a(\tau)\, \mathrm{d}R_a(\tau) \int_{\mathcal{B}_{w_2}} b(\eta)\, \mathrm{d}R_b(\eta) = \sum_{a \sqcup\!\sqcup b} \int_{(\mathcal{A}_{w_1}; \mathcal{B}_{w_2})} b(\eta) a(\tau)\, \mathrm{d}R_b(\eta) \mathrm{d}R_a(\tau). \tag{3.71}$$

Each term of the sum now is a nested integration with differentials of the form $\mathrm{d}\log R$, starting from a function of weight 0: they are then all terms of weight $w_1 + w_2$, so their sum has weight $w_1 + w_2$. □

Proposition 8. *Sums and products of UT functions are still UT. If $f(\mathbf{x}; \epsilon)$ is UT, also $1/f(\mathbf{x}; \epsilon)$ is UT.*

Proof. For sums:

$$f(\mathbf{x}; \epsilon) + g(\mathbf{x}; \epsilon) = \sum_{i=0}^{+\infty} \epsilon^i f^{(i)}(\mathbf{x}) + \sum_{j=0}^{+\infty} \epsilon^j g^{(j)}(\mathbf{x}) = \sum_{k=0}^{+\infty} \epsilon^k \left[f^{(k)}(\mathbf{x}) + g^{(k)}(\mathbf{x}) \right]. \tag{3.72}$$

For products:

$$f(\mathbf{x}; \epsilon) g(\mathbf{x}; \epsilon) = \left[\sum_{i=0}^{+\infty} \epsilon^i f^{(i)}(\mathbf{x}) \right] \left[\sum_{j=0}^{+\infty} \epsilon^j g^{(j)}(\mathbf{x}) \right] = \sum_{i=0}^{\infty} \epsilon^i \sum_{j=0}^{i} f^{(j)}(\mathbf{x}) g^{(i-j)}(\mathbf{x}), \tag{3.73}$$

And $W(f^{(j)}(\mathbf{x})) = j + a$, $W(g^{(i-j)}(\mathbf{x})) = i - j + b$, so $W(f^{(j)}(\mathbf{x}) g^{(i-j)}(\mathbf{x})) = i + a + b$, $W(f(\mathbf{x}; \epsilon) g(\mathbf{x}; \epsilon)) = a + b$ as seen above.

For inversions:

$$\frac{1}{f(\mathbf{x};\epsilon)} = \sum_{i=0}^{+\infty} \epsilon^i \left[\frac{\mathrm{d}^i}{\mathrm{d}\epsilon^i} \frac{1}{f(\mathrm{x};\epsilon)}\right]_{\epsilon=0} = \sum_{i=0}^{+\infty} \epsilon^i \sum_{(k_1;\ldots;k_I)\in\mathcal{K}} \frac{f^{(k_1)}(\mathbf{x})\ldots f^{(k_I)}(\mathbf{x})}{f^{I+1}(\mathbf{x};0)} \tag{3.74}$$

(FIs have a finite non-zero parts in $\epsilon \to 0$, so ϵ poles cannot appear by simple inversion) where $\mathcal{K}$ is the set of lists $(k_1;\ldots;k_I)$ of any length I such that $\sum_{j=1}^{I} k_j = i$. It can then be seen that each term of the ϵ series has weight $I \cdot a + \sum_{j=1}^{I} k_j - a \cdot (I+1) = i - a$. □

Observation 6. Terms of the form $x^{P(\epsilon)}$, where $P(\epsilon)$ is a polynomial in ϵ, are UT expressions:

$$x^{a_0+a_1\epsilon+\ldots+a_n\epsilon^n} = x^{a_0}\mathrm{e}^{a_1\epsilon\log x}\ldots\mathrm{e}^{a_n\epsilon^n\log x}, \tag{3.75}$$

and each exponential is a UT function, expanded in $\epsilon^i \log^i x$, and a product of UT functions is still UT. Moreover, only the term x^{a_0} has no definite weight, so $x^{P(\epsilon)}$ will be a pure UT function only if the polynomial has no constant term.

Observation 7. Concerning ζ functions, the following relation holds:

$$\zeta(2n) = \frac{2^{2n-1}\pi^{2n}\,|B_{2n}|}{(2n)!}, \quad n \in \mathbb{N}, \tag{3.76}$$

and B_{2n} is the $(2n)$-th Bernoulli number. So all even arguments of the ζ can be expressed in terms of numbers with weight 0 (Bernoulli numbers are not related to HPLs or weighted functions) and power of π, a constant with weight 1; is therefore immediate to associate to $\zeta(2n)$ weight $2n$. It is finally reasonable to expand the association also to $\zeta(2n+1)$, resulting in $W(\zeta(n)) = n$, if $n \in \mathbb{N}$.

The major role in analytic expressions of FIs is played by Euler's gamma functions, defined as:

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t. \tag{3.77}$$

It can be seen using integration by parts that $z\Gamma(z) = \Gamma(z+1)$, and, for $n \in \mathbb{N}$, $\Gamma(n) = (n-1)!$.

To determine if the gamma function is ϵ-UT varying the argument, each term of the series expansion in ϵ must be inspected, checking if it has definite weight and if all the terms have the same weight.

Before expanding the gamma function, a particular constant must be investigated. The Euler–Mascheroni constant, indicated as γ, is a key valor in the evaluation of Euler's gamma function expansions, and admits the following expressions:

$$\gamma = \sum_{n=0}^{+\infty} (-1)^n \frac{\log n}{n\log 2}; \tag{3.78}$$

$$\gamma = -\int_0^{+\infty} \mathrm{e}^{-x}\log x\,\mathrm{d}x = \int_{y=0}^{y=1} \frac{y}{y-1}\mathrm{e}^{\frac{y}{y-1}}\log\frac{y}{1-y}\,\mathrm{d}\log\frac{y}{1-y}, \tag{3.79}$$

where the change of variable has been performed to reduce the interval of integration into a compact set.[3] From the first expression it is *reasonable* (but not yet proven) *to assign* the weight

[3] The use of compact sets for integration of weighted functions has been inherited from the definition of *period* (see [55]).

$W(\gamma) = 1$. The second expression is the typical form used to define a function with weight n; assumed γ with weight 1, equation (3.79) tells that $\frac{y}{y-1}\mathrm{e}^{\frac{y}{y-1}} \log \frac{y}{1-y}$ must have weight 0 for coherence, so $W\left(\frac{y}{y-1}\mathrm{e}^{\frac{y}{y-1}}\right) = -1$.

Rewriting now the Euler's gamma definition as:

$$\Gamma(1+\epsilon) = \int_0^{+\infty} t^{\epsilon}\mathrm{e}^{-t}\,\mathrm{d}t = \sum_{n=0}^{+\infty} \frac{\epsilon^n}{n!} \int_{t=0}^{t=+\infty} \log^n t\,\mathrm{e}^{-t} t\,\mathrm{d}\log t = \\ = \sum_{n=0}^{+\infty} \frac{\epsilon^n}{n!} \int_{y=0}^{y=1} \log^n \frac{y}{1-y}\,\frac{y}{y-1}\mathrm{e}^{\frac{y}{y-1}}\,\mathrm{d}\log\frac{y}{1-y} \quad (3.80)$$

the integrand is a product of the function $\frac{y}{y-1}\mathrm{e}^{\frac{y}{y-1}}$ having weight $w_1 = -1$ with the function $\log^n \frac{y}{1-y}$ with weight $w_2 = n$, so the integral is a function with weight $w_1 + w_2 + 1 = n$. It is then proven that $\Gamma(1+\epsilon)$ is a pure UT function with weight 0. It follows moreover that also $\Gamma(1+b\epsilon)$ is a pure UT function with $w = 0$.

Consider now $\Gamma(b\epsilon)$: it is related to the previous case by $\Gamma(1+b\epsilon) = b\epsilon\Gamma(b\epsilon)$. Considering that the only effect of a multiplication by ϵ on a UT function is to shift all the coefficients on the next power of ϵ, $\Gamma(b\epsilon)$ is still a UT function, but with weight 1. This reasoning fails with $\Gamma(a+b\epsilon)$ terms, when $a \neq 0; 1$: the multiplying factor $(a-1)+b\epsilon$ mixes the terms of the series expansion with different weights, so they cannot be proven to be UT with this method, nevertheless there could be some UT functions of that form.

The result cannot be generalized to ϵ^c: in that case the series expansion will feature terms of the form ϵ^{cn}, resulting in a different growth of the power of ϵ and of the weight of each coefficient. This is not a suitable behaviour, considering also the fact that in a canonical Dyson series the weight grows as fast as the power of ϵ.

In conclusion, assuming $W(\gamma) = 1$, $\Gamma(b\epsilon)$ and $\Gamma(1+b\epsilon)$ with $c \in \mathbb{N}$ are UT function, with total weight 1 and 0, respectively.

3.3.1 UT graphs

Once established all the properties seen previously, it is possible to inspect the MIs, determining if they are UT, and in some cases finding the correct UT MI for a given topology.

However, most of the time *a priori* criteria to determine the UT nature of FIs are not available, if not for simple topologies, and the only possible way is to guess the UT form for a given topology; the definitive test for UT properties will be reaching canonical fuchsian form and UT BCs. To restrict the field of possible choices, it is useful to reverse some theorems or conjectures about UT functions: if a UT function shows a certain property, usually graphs with that property can be truly UT functions. Due to often long evaluation time to determine a canonical fuchsian form (if it exists for the chosen basis) and to fix UT BCs (that may not exist at all), this strategy turns out to be a valid ally to address the research.

For all the rest of the chapter, use of $\mathrm{d}_0^D k$ integration measure (2.14) is understood.

3.3.2 Exact criteria

Sunset graph

Consider again the sunset integral of section 2.2.2:

$$S(\alpha_1;\dots;\alpha_n) = \left(-\frac{p^2}{\mu^2}\right)^{-(n-1)\epsilon} \left(p^2\right)^{2(n-1)-\sum_{j=1}^n \alpha_j} \frac{\Gamma\left(\sum_{l=1}^n \alpha_l + (n-1)(\epsilon-2)\right)}{\Gamma\left(-\sum_{m=1}^n \alpha_m + n(2-\epsilon)\right)} \prod_{k=1}^n \frac{\Gamma(2-\alpha_k-\epsilon)}{\Gamma(\alpha_k)}. \quad (3.81)$$

Thanks to proposition 8, it is possible to inspect the UT of each factor of $S(\alpha_1;\dots;\alpha_n)$ at a time:

$\left(-\frac{p^2}{\mu^2}\right)^{-(n-1)\epsilon}$ UT with total weight 0;

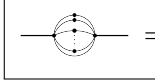
$\frac{\Gamma\left(\sum_{l=1}^n \alpha_l+(n-1)(\epsilon-2)\right)}{\Gamma\left(-\sum_{m=1}^n \alpha_m+n(2-\epsilon)\right)}$ is UT if the constant terms in each gamma function must be 0 or 1, generating the system of conditions

$$\begin{cases} \sum_{l=1}^n \alpha_l - 2(n-1) = 1 \text{ or } 0 \\ -\sum_{m=1}^n \alpha_m + 2n = 1 \text{ or } 0 \end{cases} \rightarrow \begin{cases} \sum_{l=1}^n \alpha_l = 2n-1 \text{ or } 2n-2 \\ \sum_{m=1}^n \alpha_m = 2n-1 \text{ or } 2n \end{cases} \rightarrow \quad (3.82)$$
$$\rightarrow \sum_{l=1}^n \alpha_l = 2n-1$$

$\prod_{k=1}^n \frac{\Gamma(2-\alpha_k-\epsilon)}{\Gamma(\alpha_k)}$ implies that $2-\alpha_k = 1$ or $0\,\forall k = 1;\dots;n$, resulting in $\alpha_k = 1$ or $2\,\forall k = 1;\dots;n$.

Putting together all the conditions the only possible solutions are permutations of the list $(2;\dots;2;1)$. Thus the UT form for the n-propagator sunset topology is a FI with all propagators dotted except for one.

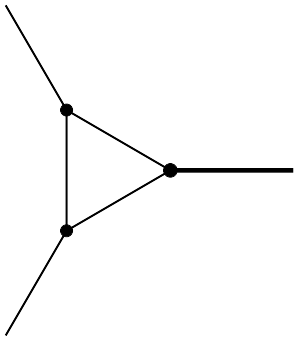
The generic formula for the UT sunset with $n \geq 2$ propagators, indicated as $\mathcal{S}_n$, is:



$$= \left(-\frac{p^2}{\mu^2}\right)^{(1-n)\epsilon} \left(p^2\right)^{-1} \frac{\Gamma\left(1+(n-1)\epsilon\right)}{\Gamma(1-n\epsilon)} \Gamma(1-\epsilon)\Gamma^{n-1}(-\epsilon). \quad (3.83)$$

Observation 8. $\mathcal{S}_n$ is a UT function, but it is not a pure UT function: the term $\left(p^2\right)^{-1}$ has no definite weight, and will remain as an overall factor in the series expansion. A pure UT version of the sunset graph is instead:

$$\hat{\mathcal{S}}_n = p^2\mathcal{S}_n = \left(-\frac{p^2}{\mu^2}\right)^{(1-n)\epsilon} \frac{\Gamma\left(1+(n-1)\epsilon\right)}{\Gamma(1-n\epsilon)} \Gamma(1-\epsilon)\Gamma^{n-1}(-\epsilon). \quad (3.84)$$



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Triangle graph

Consider the *massless triangle graph*, with massive external line momentum $p^2 \neq 0$, and massless external line momenta $p_1^2 = p_2^2 = 0$:

The UT FI for the massless triangle topology has all undotted propagators. To prove this, consider:

$$\int \frac{\mathrm{d}_0^D k}{k^2(k-p_1)^2(k+p_2)^2}, \tag{3.85}$$

and perform Feynman parametrization:

$$\begin{aligned}\int \frac{\mathrm{d}_0^D k_1}{k^2(k-p_1)^2(k+p_2)^2} &= \\ &= \int\int_0^1 \int_0^{1-y} \frac{2\,\mathrm{d}x\mathrm{d}y}{[xk^2 + y(k+p_2)^2 + (1-x-y)(k-p_1)^2 + \mathrm{i}\eta]^3}\,\mathrm{d}_0^D k = \\ &= -\mathrm{i}\left(\mu^2\right)^\epsilon \int\int_0^1 \int_0^{1-y} \frac{2\,\mathrm{d}x\mathrm{d}y}{[(k+(x-1)p_1 - yp_2)^2 - y^2p^2 - 2y(1-x)p_1\cdot p + \mathrm{i}\eta]^3}\,\mathrm{d}^D k. \end{aligned} \tag{3.86}$$

At this point $y^2p^2 - 2y(1-x)p_1\cdot p + \mathrm{i}\eta = \Omega(x;y)$, Wick-rotate the integral, and change the variable of integration into $k'^\mu = k^\mu + (x-1)p_1^\mu - yp_2^\mu$. Using equation 2.8:

$$\begin{aligned}\left(\mu^2\right)^\epsilon \Gamma(1+\epsilon) \int_0^1 \int_0^{1-y} \Omega^{-1-\epsilon}\,\mathrm{d}x\mathrm{d}y &= \\ &= \left(\mu^2\right)^\epsilon \Gamma(1+\epsilon) \int_0^1 \int_0^{1-y} (y^2p^2 - 2y(1-x)p_1\cdot p)^{-1-\epsilon}\,\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.87}$$

The coefficient of the integral is UT, and also the integrand is UT, but the integration could remove this condition in the second factor. Integrating first in x one obtains

$$\int_0^{1-y} \Omega^{-1-\epsilon}\,\mathrm{d}x = -\frac{(y(p^2y - 2p_1\cdot p))^{-\epsilon} + (y^2(p^2 - 2p_1\cdot p))^{-\epsilon}}{2\epsilon y p_1\cdot p}. \tag{3.88}$$

Rewriting the integral as a series in ϵ:

$$-\frac{(y(p^2y-2p_1\cdot p))^{-\epsilon}+(y^2(p^2-2p_1\cdot p))^{-\epsilon}}{2\epsilon y p_1\cdot p}= \\ =-\frac{1}{2\epsilon y p_1\cdot p}\sum_{n=0}^{+\infty}\frac{(-\epsilon)^n}{n!}\log^n\left[y(p^2y-2p_1\cdot p)(y^2(p^2-2p_1\cdot p))\right]. \quad (3.89)$$

Now, rearranging the term $1/y$ with dy:

$$-\frac{1}{2\epsilon p_1\cdot p}\sum_{n=0}^{+\infty}\frac{(-\epsilon)^n}{n!}\int_0^1\log^n[y(p^2y-2p_1\cdot p)(y^2(p^2-2p_1\cdot p))]\,\mathrm{d}\log y. \quad (3.90)$$

Each term of the expansion has definite weight, and the structure has total weight 1.

Propagator substitution

Generalizing the idea expressed in section 2.2.2, the following proposition can be formulated:

Proposition 9. *Given a UT FI and substituting one of its undotted propagators with a 2-points UT FI with mass dimension $[m]^{2(1+b\epsilon)}$, a new UT FI is obtained.*

Proof. Consider a UT FI. Substituting a simple propagator with a UT two-point FI has the neat effect to multiply the original structure by a bunch of terms independent from the kinematics that is UT, and to substitute the expression of the propagator k_1^2 with $\left(k_1^2\right)^{1+b\epsilon}$. Using Feynman parametrization on the specific loop:

$$\frac{\Gamma\left(\epsilon-2+\sum_{i=1}^{N_1}\alpha_i\right)}{\prod_{j=1}^{N_1}\Gamma(\alpha_j)}\int_0^1\dots\int_0^{1-\sum_{i=1}^{N_1-2}x_i}\Omega_1^{-\epsilon+2-\sum_{i=1}^{N_1}\alpha_i}\,\mathrm{d}\mathbf{x}. \quad (3.91)$$

The modifications occur in:

- $\Omega_1^{-\epsilon+2-\sum_{i=1}^{N_1}\alpha_i}$ becoming $\Omega_1^{-\epsilon+2-\sum_{i=1}^{N_1}\alpha_i+b\epsilon}$, with the same constant term form;
- $\Gamma\left(\epsilon-2+\sum_{i=1}^{N_1}\alpha_i\right)$ changing into $\Gamma\left(\epsilon-2+\sum_{i=1}^{N_1}\alpha_i+b\epsilon\right)$, the constant term has not changed;
- $\Gamma(\alpha_{k_1})=\Gamma(1)$ becoming $\Gamma(1+b\epsilon)$, UT.

The loop integral differs then only in the ϵ coefficient, not modifying the constant structure of the whole integral. So the overall function is still UT. □

Two two-point functions are said to be *chained* if they are jointed in the point each one has one of the cumulative external legs connected to.

Lemma 2. *Given a UT FI figuring a propagator with h dots, it is possible to substitute it with h chained two-point UT FIs, where each one of the h FIs has mass dimension $[m]^{2(1+b\epsilon)}$. Indicated with $\tilde{\alpha}_i=1+\tilde{b}_i\epsilon$ the power of the squared momentum of the new sub-graphs, multiplying the new FI by $\prod_{k=1}^{h-1}\left(k+\epsilon\sum_{i=1}^{h}\tilde{b}_i\right)$ a UT function is retrieved.*

Proof. Proceeding as in proposition 9, and using Feynman parametrization on the loop with the substituted propagator, associated with the power $\alpha_i = h$:

$$\frac{\Gamma\left(\epsilon - 2 + \sum_{i=1}^{N_1} \alpha_i\right)}{\prod_{j=1}^{N_1} \Gamma(\alpha_j)} \int_0^1 \dots \int_0^{1-\sum_{i=1}^{N_1-2} x_i} \Omega_1^{-\epsilon+2-\sum_{i=1}^{N_1} \alpha_i} \, \mathrm{d}\mathbf{x}. \tag{3.92}$$

with the substitution this time $\alpha_i = h \Rightarrow h + \epsilon \sum_{i=1}^{h} \tilde{b}_i$. Everything follows as in proposition 9, with the exception that this time $\Gamma(h)$ at the denominator goes into $\Gamma\left(h + \epsilon \sum_{i=1}^{h} \tilde{b}_i\right)$, which is not UT for $h \neq 1$. Although, remembering $\Gamma(z) = (z-1)\Gamma(z-1)$:

$$\Gamma\left(h + \epsilon \sum_{i=1}^{h} \tilde{b}_i\right) = \left(h - 1 + \epsilon \sum_{i=1}^{h} \tilde{b}_i\right)\left(h - 2 + \epsilon \sum_{i=1}^{h} \tilde{b}_i\right) \dots \left(1 + \epsilon \sum_{i=1}^{h} \tilde{b}_i\right) \Gamma\left(1 + \epsilon \sum_{i=1}^{h} \tilde{b}_i\right). \tag{3.93}$$

So, multiplying the structure by $\left(h - 1 + \epsilon \sum_{i=1}^{h} \tilde{b}_i\right)\left(h - 2 + \epsilon \sum_{i=1}^{h} \tilde{b}_i\right) \dots \left(1 + \epsilon \sum_{i=1}^{h} \tilde{b}_i\right)$, a UT function is found. □

These last proven results concerning substitutions will be almost only used in this work to replace propagators with sunsets in order to recognize UT versions of MIs.

3.3.3 Guiding criteria

Differential analysis and BCs

BCs, either as values of the MIs (exact or constrained) at some kinematic points, or as pseudothresholds, play a fundamental role in the determination of the UT structure. Even if the system is fuchsian and canonical, and the basis has no singularities in ϵ, no UT structure can be reached if the BCs are not UT. The complete check of the UT nature of a MI basis is therefore made up by two conditions:

- DE system in canonical fuchsian form;
- UT BCs.

Note that the presence of just one of the two properties does not assure the presence of the other one: it is possible, for example, to find a canonical fuchsian form that does not admit BCs all UT. Only when working with UT functions fuchsian canonical form assures the UT behaviour and vice versa.

To find a suitable MI basis, it is possible to relax a bit the condition above, and consider just the necessity to have a canonical fuchsian system: in this way the set of possible FIs is greatly restricted, although some non-UT functions may still be present.

Mass dimension

A guiding relation can be found considering the mass dimension of UT graphs with a given number of external legs, and looking for FIs having a mass dimension equal to the one of proven UT graphs with the same number of external legs, following the scheme (in $\epsilon \to 0$, at least in simple cases):

- 2-point UT FIs: $[m]^{-2}$ (from UT version of the sunset);

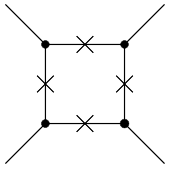
- 3-point UT FIs: $[m]^{-2}$ (from UT version of the 1-loop triangle);
- 4-point UT FIs: $[m]^{-4}$ or less (from UT version of the 1-loop box).

To match the generic n-point FI with those mass dimensions, dots or scalar products can be added to the functions under examination, giving a way to choose the MIs. Usually one of the MIs for a given topology follows this scheme, but this is no more than a rule of thumb.

Quadruple cut

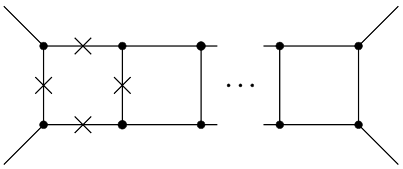
As stated in [12], if a FI is UT, also its cuts will be UT. It is possible to reverse this relation to determine a criterion to find UT diagrams: if the cut of a diagram determines a UT graph, the original FI is a good candidate for the UT basis.

Consider, as example, the 1-loop box FI: performing a quadruple cut (as explained in appendix B), the result is a UT function (since it does not depend on ϵ):



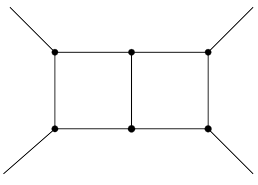
$$\propto \frac{1}{st}. \tag{3.94}$$

A similar procedure can be applied also to the n-loop ladder: iterating the quadruple cutting on one of the extremal sub-boxes since all the propagators have been cut, the resulting function is a weight 0 expression, since it does not present logarithms or π factors:



$$\propto \frac{1}{s^n t}. \tag{3.95}$$

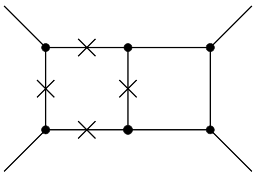
To find other suitable candidates to be UT it is possible to insert auxiliary propagators in the numerator. To find the correct ones, again quadruple cuts are useful. Consider the following FI of the 2-loop ladder topology:[4]



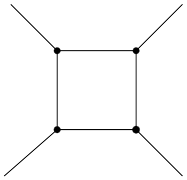
$$(p_1 + k_2)^2. \tag{3.96}$$

[4]External momenta $p_1^\mu; p_2^\mu; p_3^\mu; p_4^\mu$ are all incoming, the left and right upper legs carry p_1^μ and p_3^μ, respectively, the left and right lower ones carry p_2^μ and p_4^μ, respectively. See the graph at the beginning of section 6.2.

Cutting the left sub-box:

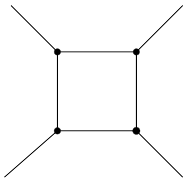


$$(p_1+k_2)^2 \propto$$



$$\frac{(p_1+k_2)^2}{s}, \tag{3.97}$$

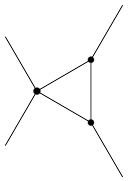
but, since



$$= \int \frac{\mathrm{d}^D k}{(k+p_1)^2 k^2 (k-p_3)^2 (k+p_1+p_2)^2} \tag{3.98}$$

the cut is nothing else than

$$\frac{1}{s}\int \frac{(p_1+k_2)^2}{(k_2+p_1)^2 k_2^2 (k_2-p_3)^2 (k_2+p_1+p_2)^2}\,\mathrm{d}^D k_2 = \frac{1}{s}$$



, (3.99)

which is a UT FI, as proven in section 3.3.2.

Remember in any case that the properties shown here are *guiding* criteria to find UT functions, not rigorous ones. They not always lead to truly UT functions, and often the paths proposed to reach UT FIs are not compatible.

Part II

Canonical systems

In previous chapters DEs have been introduced, showing a different and fruitful way to determine FIs, otherwise hardy evaluable via direct integration. Among the systems of DEs, particularly suitable structures are fuchsian canonical systems, admitting a Dyson series expansion in ϵ that can be written in terms of HPLs. All the terms of the series share the same total weight (considered $w(\epsilon) = -1$). Moreover, MIs can be chosen to have a vector of UT solutions, expressed in terms of HPLs, π and $\zeta(n)$, without poles in ϵ. Finally, evaluation through Dyson series of such a system is straightforward, involving only repeated integrations.

The hope is, starting from a given set of topologies, to find a suitable UT basis of MIs and to construct over it a fuchsian canonical system. Once these conditions are satisfied, the system can be quite easily solved.

In this part of the work two recent algorithms to find fuchsian canonical structure will be described.

The first method, introduced in [13], starting from [45], allows a system with a UT basis without ϵ divergences to be put into a canonical form, starting from linear structures in ϵ, i.e. the DE system has a coefficient matrix with each term depending on ϵ through the expression $a+b\epsilon$, $a, b \in \mathbb{R}$. The idea undergoing the method is the same of the Dirac picture in perturbation theory: through a transformation obtained from the unperturbed hamiltonian it is possible to express the time dependence of the states only using ϵH_{1I}, where H_{1I} is the perturbation in interaction picture. To determine the transformation, an equivalent formulation of the Dyson series, the *Magnus series*, is used. This procedure works also with systems of PDEs, simply iterating the procedure each partial derivation at a time.

The second method does not require fuchsian form or UT basis *a priori*, but can work only with single-scale problems. The procedure, introduced for the very first time in [16], is here derived starting from a technique of manipulation of eigenvalues called *eigenvalue deflation* [15], re-deriving a large part of the algorithms. This technique allows to modify and even nullify the eigenvalues of a given matrix, and to diminish its rank. Adapting this method, originally born to numerically compute eigenvalues, to DEs, it is possible, given a suitable non-fuchsian ϵ-dependent system, to reduce it to a fuchsian form. Starting then with a fuchsian system, it is possible to operate on its eigenvalues, passing to an ϵ-homogeneous structure for them (also here, starting from a suitable form). At the end, a similarity transformation allows to put the system into a canonical fuchsian form.

The two methods usually produce different basis of functions for the fuchsian canonical system, that can be linked by IBP-ids.

In analogy with previous studies [4, 9–11, 13, 14, 16], in the following discussion only DE systems with 0 total weight will be considered. This will allow highlighting remarkable properties of the solutions.

Chapter 4

Magnus series expansion

The method of Magnus series expansion, developed in [13] and used in [14], is presented. After the proof of Magnus theorem [13, 45], the concept underlying the definition of the interaction picture in quantum perturbation theory is applied to a ϵ-linear system of differential equations in a single variable, finding a change of basis for the master integrals which casts the system in a canonical (ϵ-factorized) form. Generalization to multi-scale problems is also explained.

The leading concept of the Magnus series can be introduced considering perturbation theory in elementary quantum mechanics, as shown in [13].

Consider a system governed by a Hamiltonian operator $H(t) = H_0(t) + \epsilon H_1(t)$, where ϵ is small, so the system can be analyzed as a result of a small perturbation on a known structure $H_0(t)$.

The time evolution equation for the states of this system, in Schroedinger picture, is:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle_S = \left[H_0(t) + \epsilon H_{1S}(t)\right]\left|\psi(t)\right\rangle_S \tag{4.1}$$

It is possible to write the time evolution of the system using the *interaction picture*, in which, starting from Schrödinger picture:

- states are written as $\left|\psi(t)\right\rangle_S = B(t)\left|\psi(t)\right\rangle_I$ and evolve according only to $\epsilon H_{1I}(t)$: $\mathrm{i}\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle_I = \epsilon H_{1I}(t)\left|\psi(t)\right\rangle_I$;
- observables are written as $A_S(t) = B(t)A_I(t)B^\dagger(t)$, and evolve following only $H_0(t)$: $\mathrm{i}\hbar\frac{\partial}{\partial t}A_I(t) = [A_I(t); H_0(t)]$;

$B(t)$ is a transformation matrix, for hypothesis commuting with $H_0(t) = H_{0I}(t)$. To determine

$B(t)$, time-evolution equations of the states can be used:

$$\begin{aligned}
\mathrm{i}\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle_I &= \mathrm{i}\hbar\frac{\partial}{\partial t}\left[B^\dagger(t)\left|\psi(t)\right\rangle_S\right] = \mathrm{i}\hbar\frac{\partial}{\partial t}\left[B^\dagger(t)U_S(t)\left|\psi(t_0)\right\rangle_S\right] = \\
&= \left[\mathrm{i}\hbar\frac{\partial B^\dagger(t)}{\partial t}U_S(t) + \mathrm{i}\hbar B^\dagger(t)\frac{\partial U_S(t)}{\partial t}\right]\left|\psi(t_0)\right\rangle_S = \\
&= \left[\mathrm{i}\hbar\frac{\partial B^\dagger(t)}{\partial t}B(t)U_I(t)B^\dagger(t) + B^\dagger(t)\left(H_0(t)+\epsilon H_{1S}(t)\right)B(t)U_I(t)B^\dagger(t)\right]\left|\psi(t_0)\right\rangle_S = \\
&= \left[-\mathrm{i}\hbar B^\dagger(t)\frac{\partial B(t)}{\partial t} + H_0(t) + \epsilon H_{1I}(t)\right]U_I(t)B^\dagger(t)\left|\psi(t_0)\right\rangle_S = \\
&= \left[-\mathrm{i}\hbar B^\dagger(t)\frac{\partial B(t)}{\partial t} + H_0(t) + \epsilon H_{1I}(t)\right]\left|\psi(t)\right\rangle_I \equiv \epsilon H_{1I}(t)\left|\psi(t)\right\rangle_I ; \qquad (4.2)
\end{aligned}$$

so:

$$\mathrm{i}\hbar\frac{\partial B(t)}{\partial t} = B(t)H_0(t). \qquad (4.3)$$

The solution of such DE is:

$$\boxed{B(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} H_0(\tau)\,\mathrm{d}\tau}.} \qquad (4.4)$$

As said above, in the interaction picture states evolve following only the perturbation in the Hamiltonian, in other words their DE is written in a canonical form for ϵ.

It is possible to use this idea to transform a given system of DEs linear in ϵ (i.e. of the form $\partial_i \mathbf{y}(\mathbf{x}) = M(\mathbf{x})\mathbf{y}(\mathbf{x})$ with $M(\mathbf{x}) = A_0(\mathbf{x}) + \epsilon A_1(\mathbf{x})$) into a system with a canonical form, through a matrix evaluated similarly to $B(t)$.

4.1 Magnus series

Following [13, 56], consider the derivative of $\Omega^k(x)$ with respect to x

$$\mathrm{d}_x\Omega^k = \mathrm{d}_x(\Omega\ldots\Omega) = (\mathrm{d}_x\Omega)\,\Omega^{k-1} + \Omega\,(\mathrm{d}_x\Omega)\,\Omega^{k-2} + \ldots + \Omega^{k-1}\,(\mathrm{d}_x\Omega)\,, \qquad (4.5)$$

and the action of the expression $\left(H\frac{\mathrm{d}}{\mathrm{d}\Omega}\right)$ on Ω (where H is an operator not depending on Ω)

$$\begin{aligned}
\left(H\frac{\mathrm{d}}{\mathrm{d}\Omega}\right)\Omega^k &= H\frac{\mathrm{d}\Omega}{\mathrm{d}\Omega}\Omega^{k-1} + \Omega H\frac{\mathrm{d}\Omega}{\mathrm{d}\Omega}\Omega^{k-2} + \ldots + \Omega^{k-1}H\frac{\mathrm{d}\Omega}{\mathrm{d}\Omega} = \\
&= H\Omega^{k-1} + \Omega H\Omega^{k-2} + \ldots + \Omega^{k-1}H. \qquad (4.6)
\end{aligned}$$

Defining the action of the operator $\left(\frac{\mathrm{d}}{\mathrm{d}\Omega} \quad H\right)$ on Ω^k as

$$\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}\Omega} \quad H\right)\Omega^k := \left(\frac{\mathrm{d}}{\mathrm{d}\Omega}\Omega^k\right)H &= \left(\frac{\mathrm{d}\Omega}{\mathrm{d}\Omega}H\right)\Omega^{k-1} + \Omega\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\Omega}H\right)\Omega^{k-2} + \ldots + \Omega^{k-1}\left(\frac{\mathrm{d}\Omega}{\mathrm{d}\Omega}H\right) = \\
&= H\Omega^{k-1} + \Omega H\Omega^{k-2} + \ldots + \Omega^{k-1}H, \qquad (4.7)
\end{aligned}$$

it is possible to write:

$$\mathrm{d}_x\Omega^k = \left(\frac{\mathrm{d}\Omega}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}\Omega}\right)\Omega^k = \left(\frac{\mathrm{d}}{\mathrm{d}\Omega}\Omega^k\right)\frac{\mathrm{d}\Omega}{\mathrm{d}x}. \qquad (4.8)$$

Note in particular that equations (4.6) and (4.7) reduce to $kH\Omega^{k-1}$ when $[H;\Omega]=0$. Defining

$$\mathrm{ad}_\Omega H := [\Omega; H] \tag{4.9}$$

it is possible to write $\left(\mathrm{d}\Omega^k/\mathrm{d}\Omega\right) H$ as:

$$\left(\frac{\mathrm{d}}{\mathrm{d}\Omega}\Omega^k\right) H = \sum_{i=0}^{k-1} \binom{k}{i+1} \mathrm{ad}_\Omega^i H\Omega^{k-i-1}, \tag{4.10}$$

where $\mathrm{ad}_\Omega^i H = \mathrm{ad}_\Omega \mathrm{ad}_\Omega \dots \mathrm{ad}_\Omega H$ i times, with $\mathrm{ad}_\Omega^0 H = H$. The proof for the formula is carried out by induction, starting from $(\mathrm{d}\Omega/\mathrm{d}\Omega) H = H$ and $\Omega \,\mathrm{ad}_\Omega^i H - \mathrm{ad}_\Omega^i H\Omega = \mathrm{ad}_\Omega^{i+1} H$.

Given the definition of *exponential of a matrix*:

$$\mathrm{e}^\Omega := \sum_{k=0}^{+\infty} \frac{\Omega^k}{k!}, \tag{4.11}$$

the following lemmas hold.

Lemma 3 (Derivative of the exponential)**.** *The derivative of the exponential of a matrix is*

$$\left(\frac{\mathrm{d}}{\mathrm{d}\Omega}\mathrm{e}^\Omega\right) H = \mathrm{d}\exp\mathrm{ad}_\Omega H\mathrm{e}^\Omega, \tag{4.12}$$

where H is a generic operator not depending on Ω and

$$\mathrm{d}\exp\mathrm{ad}_\Omega H := \sum_{k=0}^{+\infty} \frac{\mathrm{ad}_\Omega^k H}{(k+1)!} \tag{4.13}$$

Proof.

$$\left(\frac{\mathrm{d}}{\mathrm{d}\Omega}\mathrm{e}^\Omega\right) H = \sum_{k=0}^{+\infty} \frac{1}{k!} \left(\frac{\mathrm{d}}{\mathrm{d}\Omega}\Omega^k\right) H = \sum_{k=1}^{+\infty} \frac{1}{k!} \sum_{i=0}^{k-1} \binom{k}{i+1} \mathrm{ad}_\Omega^i H\Omega^{k-i-1} =$$
$$= \sum_{k=1}^{+\infty} \sum_{i=0}^{k-1} \frac{\mathrm{ad}_\Omega^i H}{(i+1)!} \frac{\Omega^{k-i-1}}{(k-i-1)!} = \sum_{k=1}^{+\infty} \frac{\Omega^{k-1}}{(k-1)!} + \frac{\mathrm{ad}_\Omega H}{2} \frac{\Omega^{k-2}}{(k-2)!} + \dots + \frac{\mathrm{ad}_\Omega^{k-1}}{k!}, \tag{4.14}$$

where terms of the form $\frac{\mathrm{ad}_\Omega^n H}{n!}$ start to be present when $k = n+1$;

$$\sum_{i=0}^{+\infty} \frac{\mathrm{ad}_\Omega^i H}{(i+1)!} \mathrm{e}^\Omega = \mathrm{d}\exp\mathrm{ad}_\Omega H\mathrm{e}^\Omega. \tag{4.15}$$

□

Lemma 4 (Inverse of the exponential)**.** *If the eigenvalues of* ad_Ω *are different from* $2\mathrm{i}k\pi$*,* $k \in (\mathbb{Z} \setminus \{0\})$*,* $\mathrm{d}\exp\mathrm{ad}_\Omega$ *is invertible, with:*

$$\mathrm{d}\exp\mathrm{ad}_\Omega^{\leftarrow} H = \sum_{k=0}^{+\infty} \frac{\beta_k}{k!} \mathrm{ad}_\Omega^k H. \tag{4.16}$$

β_k are the Bernoulli numbers, satisfying:

$$\frac{t}{\mathrm{e}^t - 1} = \sum_{k=0}^{+\infty} \frac{\beta_k}{k!} t^k. \tag{4.17}$$

Proof. Proceeding in a formal way:

$$\mathrm{d}\exp\mathrm{ad}_\Omega H = \sum_{k=0}^{+\infty} \frac{\mathrm{ad}_\Omega^k H}{(k+1)!} = \frac{\sum_{k=1}^{+\infty} \frac{1}{k!} \mathrm{ad}_\Omega^k}{\mathrm{ad}_\Omega} H = \frac{\sum_{k=0}^{+\infty} \frac{1}{k!} \mathrm{ad}_\Omega^k - 1}{\mathrm{ad}_\Omega} H = \frac{\exp\mathrm{ad}_\Omega - 1}{\mathrm{ad}_\Omega} H. \tag{4.18}$$

$$\mathrm{d}\exp\mathrm{ad}_\Omega^{\leftarrow} H = (\mathrm{d}\exp\mathrm{ad}_\Omega)^{-1} H = \frac{\mathrm{ad}_\Omega}{\exp\mathrm{ad}_\Omega - 1} H = \sum_{k=0}^{+\infty} \frac{\beta_k}{k!} \mathrm{ad}_\Omega^k H \tag{4.19}$$

If the eigenvalues are multiple of $2\mathrm{i}\pi$ the exponential of ad_Ω is not invertible, as $\exp\mathrm{ad}_\Omega = 1$. □

It is now possible to demonstrate the Magnus theorem.

Proposition 10 (Magnus theorem)**.** *Given the linear system of DEs*

$$\begin{cases} \frac{\mathrm{d}Y(x)}{\mathrm{d}x} = A(x)Y(x) \\ Y(x_0) = Y_0 \end{cases}, \tag{4.20}$$

its solution can be written as

$$Y(x) = \mathrm{e}^{\Omega[A](x;x_0)} Y_0 = \mathrm{e}^{\Omega} Y_0 \tag{4.21}$$

where $\Omega[A](x;x_0)$ can be computed solving the DE

$$\begin{cases} \mathrm{d}_x\Omega[A](x;x_0) = \mathrm{d}\exp\mathrm{ad}_{\Omega[A](x;x_0)}^{\leftarrow} A(x) \\ \Omega[A](x_0;x_0) = 0 \end{cases}. \tag{4.22}$$

Proof. See [13]. Consider the derivative of $Y(x) = \mathrm{e}^{\Omega} Y_0$ ($\mathbf{y}_0$ can be extracted from the operation of derivation since it is constant):

$$\frac{\mathrm{d}Y(x)}{\mathrm{d}x} = \left(\frac{\mathrm{d}}{\mathrm{d}\Omega}\mathrm{e}^{\Omega}\right)\frac{\mathrm{d}\Omega}{\mathrm{d}x} Y_0 = \mathrm{d}\exp\mathrm{ad}_\Omega(\mathrm{d}_x\Omega)\mathrm{e}^{\Omega} Y_0 = \mathrm{d}\exp\mathrm{ad}_\Omega(\mathrm{d}_x\Omega) Y(x); \tag{4.23}$$

so it must be

$$\mathrm{d}\exp\mathrm{ad}_\Omega(\mathrm{d}_x\Omega) = A(x); \tag{4.24}$$

applying $\mathrm{d}\exp\mathrm{ad}_\Omega^{\leftarrow}$ to both members of the relation, and consdering that $Y(x_0) = \mathrm{e}^{\Omega[A](x_0;x_0)} Y_0 = \mathbf{y}_0$, so $\Omega[A](x_0;x_0) = 0$:

$$\begin{cases} \mathrm{d}_x\Omega[A](x;x_0) = \mathrm{d}\exp\mathrm{ad}_{\Omega[A](x;x_0)}^{\leftarrow} A(x) \\ \Omega[A](x_0;x_0) = 0 \end{cases}. \tag{4.25}$$

□

The solution can be rewritten as a *Magnus series*:

$$Y(x) = \mathrm{e}^{\Omega[A](x;x_0)} Y_0, \tag{4.26}$$

$$\Omega[A](x;x_0) = \sum_{n=1}^{+\infty} \Omega_n[A](x;x_0), \tag{4.27}$$

$$\Omega_n[A](x;x_0) = \sum_{j=1}^{n-1} \frac{\beta_j}{j!} \int_{x_0}^{x} S_n^{(j)}(\tau)\,\mathrm{d}\tau. \tag{4.28}$$

The integrands $S_n^{(j)}$ can be computed recursively:

$$\begin{cases} S_n^{(1)}(\tau) = [\Omega_{n-1}[A](\tau;x_0);A(\tau)]\,, \\ S_n^{(j)}(\tau) = \sum_{k=j-1}^{n-j}\left[\Omega_m[A](\tau;x_0);S_{n-m}^{(j-1)}(\tau)\right] \quad \text{if } 2\le j\le n-2 \\ S_n^{(n-1)}(\tau) = \mathrm{ad}^{n-1}_{\Omega_1[A](\tau;x_0)}A(\tau) \end{cases}. \tag{4.29}$$

The first terms of the Magnus series are:

$$\Omega_1[A](x;x_0) = \int_{x_0}^{x} A(\tau_1)\,\mathrm{d}\tau_1, \tag{4.30}$$

$$\Omega_2[A](x;x_0) = \frac{1}{2}\int_{x_0}^{x}\int_{x_0}^{\tau_1} [A(\tau_1);A(\tau_2)]\,\mathrm{d}\tau_2\mathrm{d}\tau_1, \tag{4.31}$$

$$\Omega_3[A](x;x_0) = \frac{1}{6}\int_{x_0}^{x}\int_{x_0}^{\tau_1}\int_{x_0}^{\tau_2} [A(\tau_1);[A(\tau_2);A(\tau_3)]]+[A(\tau_3);[A(\tau_2);A(\tau_1)]]\,\mathrm{d}\tau_3\mathrm{d}\tau_2\mathrm{d}\tau_1; \tag{4.32}$$

$$\dots$$

in particular, if $A(x)$ commutes with its own intergal $\int_{x_0}^{x}A(\tau)\,\mathrm{d}\tau$, the series is truncated at its first term, resulting in $\mathrm{e}^{\int_{x_0}^{x}A(\tau)\,\mathrm{d}\tau}$.

Magnus series is equivalent to Dyson series to express solutions of Cauchy problems of the form (4.20), and it is possible to pass from one representation to the other considering

$$\sum_{n=0}^{\infty}\int_{x_0}^{x}\int_{x_0}^{\tau_1}\dots\int_{x_0}^{\tau_{n-1}} A(\tau_1)A(\tau_2)\dots A(\tau_n)\,\mathrm{d}\tau_n\dots\mathrm{d}\tau_2\mathrm{d}\tau_1 Y_0 = \mathrm{e}^{\sum_{j=0}^{+\infty}\Omega_j[A](x;x_0)}Y_0, \tag{4.33}$$

from which:

$$Y_k = \Omega_k + \sum_{i=2}^{k}\frac{1}{i!}Q_n^{(i)}, \tag{4.34}$$

$$Y_k = \int_{x_0}^{x}\int_{x_0}^{\tau_1}\dots\int_{x_0}^{\tau_{n-1}} A(\tau_1)A(\tau_2)\dots A(\tau_n)\,\mathrm{d}\tau_n\dots\mathrm{d}\tau_2\mathrm{d}\tau_1 Y_0, \tag{4.35}$$

$$Q_k^{(i)} = \sum_{m=1}^{k-i+1} Q_m^{(1)}Q_{k-m}^{(i-1)}, \quad Q_k^{(1)} = \Omega_n[A](x;x_0), \quad Q_k^{(k)} = \Omega_1^n[A](x;x_0). \tag{4.36}$$

Both Magnus and Dyson series have positive aspects useful to represent solutions of linear DEs. Dyson approach allows to write a series in terms of powers of the matrix of coefficients (considering a canonical system, in terms on powers of ϵ) and it has a more immediate interpretation, in terms of Volterra integration problem (therefore it is easier to evaluate using computers). Magnus series, on the other hand, preserves unitarity properties and allows to obtain finite series, which can be exactly computed, if $\Omega_i[A](x;x_0)=0$.

As a paradigmatic example, consider a vectorial DE with $A(x)$ diagonal: its Magnus series is truncated at $\Omega_1[A](x;x_0)$, resulting in $\mathbf{y}(x) = \mathrm{e}^{\Omega[A](x;x_0)}\mathbf{y}_0 = \mathrm{e}^{\Omega_1[A](x;x_0)}\mathbf{y}_0 = \mathrm{e}^{\int_{x_0}^{x}A(\tau)\,\mathrm{d}\tau}\mathbf{y}_0$; this

expression is perfectly consistent with the one obtained using the Dyson series (see example 11):

$$\mathbf{y}(x) = \sum_{n=0}^{+\infty} \frac{1}{n!} \left[\int_{x_0}^{x} A(\tau)\, \mathrm{d}\tau \right]^n \mathbf{y}_0 = \mathrm{e}^{\int_{x_0}^{x} A(\tau)\, \mathrm{d}\tau} \mathbf{y}_0. \tag{4.37}$$

4.2 Reduction to canonical form

Consider a system of first order linear DEs

$$\frac{\mathrm{d}\mathbf{y}(x;\epsilon)}{\mathrm{d}x} = M(x;\epsilon)\mathbf{y}(x;\epsilon), \tag{4.38}$$

with linear ϵ dependence in the matrix of coefficients:

$$M(x;\epsilon) = A_0(x) + \epsilon A_1(x). \tag{4.39}$$

Proceeding in the same way as in the time-evolution case, one imposes that

$$\frac{\mathrm{d}\mathbf{g}(x;\epsilon)}{\mathrm{d}x} = \epsilon \hat{A}_1(x)\mathbf{g}(x;\epsilon), \tag{4.40}$$

with

$$\mathbf{y}(x;\epsilon) = B(x)\mathbf{g}(x;\epsilon), \quad A_1(x) = B(x)\hat{A}_1(x)B^\dagger(x), \tag{4.41}$$

obtaining:

$$B^\dagger(x)\frac{\mathrm{d}B(x)}{\mathrm{d}x} = B^\dagger(x)A_0(x)B(x). \tag{4.42}$$

Inverting the previous relations:

$$\mathbf{g}(x;\epsilon) = B^\dagger(x)\mathbf{y}(x;\epsilon), \tag{4.43}$$

$$\epsilon\hat{A}_1(x) = \hat{A}(x;\epsilon) = \epsilon B^\dagger(x)A_1(x)B(x) = B^\dagger(x)A(x;\epsilon)B(x) - B^\dagger(x)\frac{\mathrm{d}B(x)}{\mathrm{d}x}. \tag{4.44}$$

The matrix $B(x)$ obeys the DE

$$\frac{\mathrm{d}B(x)}{\mathrm{d}x} = A_0(x)B(x). \tag{4.45}$$

Note that this time $A_0(x)$ and $B(x)$ do not commute, since $A_0(x)$ in general does not commute with its integral matrix. Instead of a simple exponential the Magnus series is then used:

$$B(x) = \mathrm{e}^{\Omega[A_0](x;x_0)}. \tag{4.46}$$

Once the changes are performed, the system assumes the canonical form

$$\frac{\mathrm{d}\mathbf{g}(x;\epsilon)}{\mathrm{d}x} = \epsilon \hat{A}_1(x)\mathbf{y}(x;\epsilon). \tag{4.47}$$

Using this procedure, a generic ϵ-linear DE system can be cast in a canonical form. Nothing else is known about the nature of the poles, if working with generic MIs. For example, starting

from a non-fuchsian ϵ-linear expression and changing basis to the canonical form, the new DE could remain non-fuchsian, or even become fuchsian (it happens for n-loop ladders, as shown in chapter 6).

The situation becomes more interesting when working with UT functions: in this case, canonical form implies fuchsian form, so it is only necessary to reach canonical form to obtain a system solvable in terms of HPLs.

Proposition 11. *Given a system of DEs for UT MIs not singular in ϵ, if the system is canonical it is also fuchsian.*

Proof. If the system is canonical, the ϵ^0 term does not evolve (otherwise a term ϵ^{-1} will be present in the basis, but the MIs are not singular in ϵ). Then the basis is composed by pure UT functions: the total weight is 0, because 0 is the weight of the first term and the functions are UT, and the first term is composed by constants of weight 0. This last property is due to the BCs: if the BCs would not be constant for the 0 order, figuring a term with no definite weight, in order 1 the primitive of that term will be present, again with no definite weight, resulting in a structure not factorable and with no fixed total weight, against the hypothesis of UT function. So a UT basis with no ϵ divergences, described by a canonical system of DEs forces that system to be also fuchsian.

The relation is not a bijection: think for example to the massless UT bubble, that is UT, and has a fuchsian DE which is not canonical: $\partial \mathcal{S}_2/\partial s = -(1+\epsilon)/s$. This is due to the presence of the term $1/s$ in the analytic expression of the function. □

As for the Dyson series, the exponentiated integrals in Magnus series are considered as a simple antiderivation operation, namely only the evaluation in x is considered. This is possible thanks to redefinitions in the initial values $\mathbf{y}_0$, as Magnus and Dyson series are related as shown in section 4.1.

4.3 Multi-scale problems

The path illustrated for single-scale problem from ϵ-linear form to canonical form is still valid, variable by variable, for multi-scale problems. In case of more than one variable, the construction of the transformation is performed each variable at a time, using as matrix $A(\mathbf{x};\epsilon) = A_0(\mathbf{x}) + \epsilon A_1(\mathbf{x})$ the corresponding matrix of coefficients.

Starting from a system of PDE systems (for simplicity, PDEs), each one linear in ϵ:

1. select the matrix of coefficients of the first PDE: $A_1(\mathbf{x};\epsilon) = A_{1\,0}(\mathbf{x}) + \epsilon A_{1\,1}(\mathbf{x})$ from $\partial_{x_1}\mathbf{y}(\mathbf{x};\epsilon) = A_1(\mathbf{x};\epsilon)\mathbf{y}(\mathbf{x};\epsilon)$;
2. construct the matrix of change of basis $B_i(\mathbf{x}) =$ starting from $A_{1\,0}(\mathbf{x})$ with respect to the variable x_1;
3. apply $B_1(\mathbf{x})$ to the base of MIs ($\mathbf{g}(\mathbf{x};\epsilon) = B^\dagger(\mathbf{x})\mathbf{y}(\mathbf{x};\epsilon)$), and to to all the $A_i(\mathbf{x};\epsilon)$;
4. iterate these points until all the PDEs have been processed.

Note that this algorithm produces a canonical system of PDEs. This is guaranteed because with the i-th iteration the PDE in x_i is written in a canonical form, and with the $(i+1)$-th iteration the previous PDE cannot gain terms either independent from ϵ (the matrix $\hat{A}_{i1}$ is multiplied by ϵ) or depending on ϵ^n, $n > 1$ (the transformation does not depend on ϵ).

With more variables the definition of UT functions does not change, and if the basis is composed by UT MIs without divergences in ϵ, proposition 11 still holds, guaranteeing, since the algorithm generates canonical systems, also the fuchsianity of the results.

Chapter 5

Eigenvalues deflation

After recalling the properties of Jordan block form, eigenvalue reduction [16] and deflation [15] are explained, as well as their generalization to the differential equation framework, for one-parameter problems. Following [16], with the introduction of *balance transformations* it is possible to adapt the deflation in order to eliminate non-fuchsian poles in a wide range of cases, and to adapt eigenvalue reduction to cast a well-chosen fuchsian system into one with all eigenvalues proportional to ϵ. From there, a similarity transformation allows to land on a canonical system.

5.1 Eigenvalues and eigenvectors

Given a real square matrix A, $n \times n$, the expression:

$$Au_i = a_i u_i \tag{5.1}$$

(where u_i is a vector and a_i a scalar value) determines u_i, called the *right eigenvector* of A, with associated *eigenvalue* a_i. On the other hand, considering

$$v_j^\dagger A = a_j v_j^\dagger \tag{5.2}$$

the *left eigenvector* v_j with associated eigenvalue a_j is defined. Since $\left(v_j^\dagger A\right)^\dagger = A^\dagger v_j$ and $\det\left[(A - a\mathbb{I})^\dagger\right] = \left(\det\left[(A - a\mathbb{I})^T\right]\right)^* = [\det(A - a\mathbb{I})]^*$, if a_i admits n_i right eigenvectors, a_i^* admits n_i left eigenvectors; the set of right eigenvectors associated to a specific eigenvalue may differ from the corresponding set of left eigenvectors.

A fundamental property of left and right eigenvectors is that a left and a right eigenvector associated to different eigenvalues are orthogonal. Consider in fact the defining equation for a left eigenvector u_i of the eigenvalue a_i: $Au_i = a_i u_i$. Taking the adjoint of this expression and

multiplying it from the right by v_j (left eigenvector) one obtains:

$$u_i^\dagger A^\dagger v_j = a_i^* u_i^\dagger v_j; \tag{5.3}$$
$$a_j u_i^\dagger v_j = a_i^* u_i^\dagger v_j; \tag{5.4}$$
$$(a_j - a_i^*) u_i^\dagger v_j = 0. \tag{5.5}$$

So either $a_j = a_i^*$, meaning that the eigenvectors are related to the same eigenvalue, or $u_i^\dagger v_j = 0$, giving u_i orthogonal to v_j.

A fundamental ingredient for the method of eigenvalues deflation is the Jordan form. A matrix in Jordan form is a matrix like:

$$A = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_n \end{pmatrix}, \qquad B_i = \begin{pmatrix} a_i & 1 & & \\ & \ddots & \ddots & \\ & & a_i & 1 \\ & & & a_i \end{pmatrix}. \tag{5.6}$$

Where nothing is indicated in the matrices, 0 is understood. Each B_i is a square matrix of dimension K_i, and $\sum_{i=1}^{I} K_i = N$ is the dimension of the matrix A.

Thanks to Jordan theorem, it is always possible in $\mathbb{C}$ to find a matrix in Jordan form similar to a given matrix. For simplicity (and also because all the systems of interest in this work are of this form) only matrices similar to real Jordan matrices will be considered here.

Speaking about eigenvectors, it is possible to expand this concept for Jordan matrices defining the so-called *generalized eigenvectors*.

As seen in (5.6), each block B_i is associated to an eigenvalue a_i (different blocks can have the same eigenvalue), a single right proper eigenvector $u_i^{(0)} = (1; 0; \ldots; 0)^T$ and a single left proper eigenvector $v_i^{(0)} = (0; \ldots; 0; 1)^T$. Consider now the vector $u_i^{(k)} = (0_1; \ldots; 0_k; 1_{k+1}; 0_{k+2}; \ldots; 0_{K_i})^T$, $k > 0$, the vector with 0 in all entries exept the $(k+1)$-th, equal to 1:

$$B_i u_i^{(k)} = \begin{pmatrix} a_i & 1 & & \\ & \ddots & \ddots & \\ & & a_i & 1 \\ & & & a_i \end{pmatrix} \begin{pmatrix} \vdots \\ 0 \\ 1_{k+1} \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ 0 \\ 1_k \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} \vdots \\ 0 \\ a_i \cdot 1_{k+1} \\ 0 \\ \vdots \end{pmatrix} = u_i^{(k-1)} + a_i u_i^{(k)}; \tag{5.7}$$

its behaviour is similar to the one of an eigenvector, if not for the term $u_i^{(k-1)}$. For each block in a Jordan matrix there are one regular eigenvector, and $K-1$ vectors as the one above, which are the generalized eigenvectors. The same happens for the left eigenvectors. To come back to the complete matrix A it will be only necessary to add $K_1 + \ldots + K_{i-1}$ zeroes before the first element of the generalized eigenvectors for B_i and $K_{i+1} + \ldots + K_I$ zeroes after the last element.

It is therefore possible to sum up the rules for generalized eigenvectors as follows:

$$u_i^{(k)} = \begin{pmatrix} 0_1 \\ \vdots \\ 0_k \\ 1_{k+1} \\ 0_{k+2} \\ \vdots \\ 0_{K_i} \end{pmatrix}; \tag{5.8}$$

$$w_j^{(k)\dagger} = \begin{pmatrix} 0_{K_j} & \dots & 0_{k+2} & 1_{k+1} & 0_k & \dots & 0_1 \end{pmatrix}; \tag{5.9}$$

$$Au_i^{(0)} = a_i u_i^{(0)}; \quad Au_i^{(k)} = a_i u_i^{(k)} + u_i^{(k-1)} \quad \text{with } 0 < k < K_i; \tag{5.10}$$

$$v_j^{(0)\dagger} A = a_j v_j^{(0)\dagger}; \quad v_j^{(k)\dagger} A = a_j v_j^{(k)\dagger} + v_j^{(k-1)\dagger} \quad \text{with } 0 < k < K_j; \tag{5.11}$$

$$v_j^{(k)\dagger} u_i^{(h)} = \delta_{i;j}\delta_{k+h;K_i-1}. \tag{5.12}$$

Notice that for a Jordan matrix a basis can be defined using the whole set of eigenvectors, proper and generalized together, and for each matrix similar to a real Jordan matrix generalized eigenvectors are still present and well-defined, guaranteeing the presence of N linearly independent vectors to use as a basis.

5.2 Deflation

Proposition 12 (Eigenvalue reduction)**.** *[16]. Given a matrix A with eigenvectors $u_1; \dots; u_n$ and associated eigenvalues $a_1; \dots; a_n$ (some of the a_i may be equal), the transformation*

$$\tilde{A} = A - \lambda u_i x_i^\dagger \tag{5.13}$$

(where $x_i^\dagger u_i = 1$) operates on the eigenvalues lowering a_i by λ and leaving all the other eigenvalues unchanged.

Proof. First of all, pass in the basis of generalized right eigenvectors of A (this is always possible, and a similarity transformation does not change the eigenvalues). In this basis (for sake of simplicity, take $u = (1; 0; \dots; 0)^T$), $u_i x_i^\dagger = (1; 0; \dots; 0)^T, x_i^\dagger = (x_i; 0 \dots; 0)^\dagger$, so $x_i^\dagger = (1; x_{i2}^*; \dots)$ since $x_i^\dagger u_i = x_{i1}^* = 1$ for hypothesis, and A has a Jordan form (5.6). So:

$$A - \lambda u_i x_i^\dagger = \begin{pmatrix} a_1 - \lambda & c_1 - \lambda x_{i2}^* & \dots & \dots & -\lambda x_{iN} \\ 0 & a_i & c_i & & \\ \vdots & & & \ddots & \\ 0 & & & & a_I \end{pmatrix} \tag{5.14}$$

and the characteristic equation for this matrix assumes the form $(a_1 - \lambda - a)(a_1 - a)^{K_1 - 1}(a_2 - a)^{K_2} \dots (a_I - a)^{K_I}$, giving eigenvalues $a_1 - \lambda; a_1$ (if $K_1 > 1); \dots; a_I$. □

Notice that is also possible to perform an eigenvalue reduction using the left eigenvectors of A, namely

$$\tilde{A} = A - \lambda x_i v_i^\dagger \tag{5.15}$$

still diminishes a_i by λ, leaving all the other eigenvalues unchanged. The proof is specular to the one above: assuming for simplicity that $v_i^\dagger = 0; \dots; 0; 1$, and passing to the base of left generalized eigenvectors,

$$A - \lambda x_i v_i^\dagger = \begin{pmatrix} a_1 & c_1 & & & -\lambda x_{i1} \\ 0 & a_i & c_i & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & a_i & c_{N-1} - \lambda x_{i(N-1)} \\ 0 & \dots & \dots & 0 & a_I - \lambda \end{pmatrix}, \tag{5.16}$$

and, as above, a_I is decreased by λ, while all the other eigenvalues are left unchanged.

With a little modification is also possible to immediately set a_i to 0.

Proposition 13 (Deflation)**.** *[15]. The transformation*

$$\hat{A} = \left(\mathbb{I} - u_i x_i^\dagger\right) A \tag{5.17}$$

(with the same notation used in proposition 12) transforms a_i into 0, not modifying the other eigenvalues or the eigenvector u_i. Moreover, $\operatorname{rank} \hat{A} = \operatorname{rank} A - 1$.

Proof. From [15]. Applying u_i from the right on the right hand side of the equation:

$$\left(\mathbb{I} - u_i x_i^\dagger\right) A u_i = a_i \left(u_i - u_i x_i^\dagger u_i\right) = a_i \left(u_i - u_i\right) = 0; \tag{5.18}$$

so the couple has passed from $(u_i; a_i)$ to $(u_i; 0)$. For the other eigenvalues consider the adjoint of the left multiplication by v_j, $j \neq i$:

$$\left[v_j^\dagger \left(\mathbb{I} - u_i x_i^\dagger\right) A\right]^\dagger = \left[\left(\mathbb{I} - u_i x_i^\dagger\right) A\right]^\dagger v_j = A^\dagger \left(\mathbb{I} - x_i u_i^\dagger\right) v_j \tag{5.19}$$

where $v_j^\dagger u_i = \delta_{ij}$ as in (5.12) (a similarity transformation do not change the result), so $v_j^\dagger u_i = 0$

$$A^\dagger v_j = \left(v_j^\dagger A\right)^\dagger = a_j \tag{5.20}$$

because a_j is real. So eigenvalues different from a_i are not touched by the transformation.

Consider now, in the basis of right generalized eigenvectors (so with A in Jordan form), the product (for easier notation, $i = 1$, but the demonstration is valid $\forall i$):

$$\begin{aligned} x_1^\dagger A = \left[v_1^{(K_1-1)\dagger} + x_{1;2}^* v_1^{(K_1-2)\dagger} + \dots + x_{1;N}^* v_n^{(0)\dagger}\right] A = \\ = \left(a_1 v_1^{(K_1-1)\dagger} + v_1^{(K_1-2)\dagger}\right) + x_{1;2} \left(a_1 v_1^{(K_1-2)\dagger} + v_1^{(K_1-3)\dagger}\right) + \dots \end{aligned} \tag{5.21}$$

($x_1 = (1; x_{1;2}; \dots; x_{1;N})^T$ in right generalized eigenvectors basis, since $x_1^\dagger u_1 = 1$); the first parenthesis generates a term of the form $\begin{pmatrix} a_1 & 1 & 0 & 0 & \dots \\ & & \mathbb{O} & & \end{pmatrix}$ that eliminates the Jordan structure of the first row, while the rest of the parenthesis generates non-zero terms in entries where $a_i \neq 0$; notice that rows and columns containing null Jordan cells are formed by 0: the product (5.21) gives 0 for eigenvectors related to null eigenvalues in 1×1 blocks (i.e. rows made up only by 0), and the corresponding Jordan cells are always 1×1.

The maximum minor of A will contain at least all the upper diagonal terms of B_1; eliminating one of them by acting with a deflation, the determinant of the corresponding minor in $\hat{A}$ is 0, since an entire column is now

null. Eliminating raw and column of the nullified terms from the minor of order h, a new minor is retrieved, with dimension $h-1$ (this matrix has non-zero determinant, since the originating one was a minor). The extra terms introduced with the other x^*_{1j} cannot avoid the downgrade of the rank, since they are all located on a row, off from the null columns (that remain null). □

Regarding the eigenvectors u_j, v_j and v_i, their form can be modified by the operation of *deflation*. For example, the relation between the new u'_j and the old u_j is:

$$u_j = (a_j - a_i)\, u'_j + a_i \left(x_i^\dagger u'_j\right) u_1. \tag{5.22}$$

Different types of deflation can be constructed, according to the choice of the vector x_i (from v_i to normalized columns of A) and to the form of the matrix A. See [15] for more details.

Lemma 5. *The transformation*

$$\hat{A} = \left(\mathbb{I} - \sum_{i\in I} u_i x_i^\dagger\right) A \tag{5.23}$$

with $x_i^\dagger u_j = \delta_{ij}$ and the same notation as in proposition 13, transforms a_i in 0 $\forall i \in I$, and $\operatorname{rank}\hat{A} = \operatorname{rank} A - \operatorname{card} I$.

Proof.

$$\begin{aligned}\prod_{i\in I}\left(\mathbb{I} - u_i x_i^\dagger\right) &= \left(\mathbb{I} - u_{i_1} x_{i_1}^\dagger\right)\left(\mathbb{I} - u_{i_2} x_{i_2}^\dagger\right) \prod_{i\in(I\setminus\{i_1;i_2\})}\left(\mathbb{I} - u_i x_i^\dagger\right) = \\ &= \left(\mathbb{I} - u_{i_1} x_{i_1}^\dagger - u_{i_2} x_{i_2}^\dagger\right) \prod_{i\in(I\setminus\{i_1;i_2\})}\left(\mathbb{I} - u_i x_i^\dagger\right) = \ldots = \mathbb{I} + \sum_{i\in I} u_i x_i^\dagger,\end{aligned} \tag{5.24}$$

since $u_{i_1} x_{i_1}^\dagger u_{i_2} x_{i_2}^\dagger = u_{i_1}\delta_{i_1 i_2} x_{i_2}^\dagger = 0$. So the transformation $(\mathbb{I}-\mathbb{P})A$ with $\mathbb{P} = \sum_{i\in I} u_i x_i^\dagger$ is equivalent to the chain of deflations $(\mathbb{I}-\mathbb{P}_{i_1})\ldots(\mathbb{I}-\mathbb{P}_{i_{\operatorname{card} I}})A$, $\mathbb{P}_i = u_1 x_i^\dagger$, each one reducing to 0 the corresponding eigenvalue and reducing by 1 the rank of the outgoing matrix with respect to the rank of the incoming. □

Using deflation is therefore possible to eliminate one by one all the eigenvalues of a matrix, while using eigenvalue reduction it is possible to diminish the eigenvalues of an arbitrary constant λ.

Since all matrices considered in the present work are similar to real Jordan ones, proposition 13 applies to a generic matrix, up to similarity transformations (which cannot modify the rank, or the eigenvalues).

It is therefore possible to establish an algorithm to reduce the rank of a given matrix to zero via rank deflation:

1. given a matrix M similar to a real Jordan matrix A, through a similarity transformation transform M into A;[1] define $A_0 := A$;

[1] This procedure can be carried out immediately thanks to calculus programs; for example, in `Mathematica` the command to jordanize a matrix is `JordanDecomposition`.

2. construct the projector
$$\mathbb{P} = \sum_{i=1}^{I} u_i^{(0)} x_i^{\dagger}, \tag{5.25}$$
with $u_i^{(0)}$ right proper eigenvector and $x_i : x_i^{\dagger} v_i^{(0)} = 0$, both related to the eigenvalue a_i of the block B_i of A_0;

3. perform
$$A_1 = (\mathbb{I} - \mathbb{P})A_0; \tag{5.26}$$

4. define $A_0 := A_1$;

5. repeat from point 2 to point 5 until A_1 is a null matrix (the result will be reached in $\max\{K_i, 1 \leq i \leq I\}$ iterations).

5.3 Deflation for DEs

As explained in [16], given a DE system, the main purpose is to cast it in a canonical fuchsian form. The matrix of coefficients of such DE system has a form:
$$A(z;\epsilon) = \sum_{p=1}^{P} \sum_{q=1}^{Q_p} \frac{A_{(p;q)}(\epsilon)}{(z - z_p)^q}, \tag{5.27}$$
namely it is a sum of residues in the poles $\{x_1; \ldots; x_P\}$, each one with different degrees $\{1; \ldots; Q_p\}$, $p \in P$, each one greater or equal to 0.

The first aim is to reach a fuchsian form, i.e. to eliminate the matrices of residues of the poles with $q > 1$; since only the null matrix has rank 0, lowering the rank of the undesired matrices down to 0 will eliminate the corresponding poles.

Once the system is fuchsian, a canonical form must be found. In order to do so, consider the fact that if a matrix $M(\tau)$ is proportional to τ, the expression $M(\tau)/\tau$ will stay the same for all the values of τ:
$$\frac{M(\tau_1)}{\tau_1} = \frac{M(\tau_2)}{\tau_2}; \tag{5.28}$$
then, starting from a fuchsian matrix $B(x;\epsilon)$, imposing that its similarity transformation multiplied by $1/\epsilon$ is independent from ϵ, one obtains
$$\frac{T^{-1}(\epsilon) B_p(\epsilon) T(\epsilon)}{\epsilon} = \frac{T^{-1}(\mu) B_p(\mu) T(\mu)}{\mu}, \tag{5.29}$$
from which a class of transformations $\tilde{T}(\epsilon;\mu;\mathbf{t})$ (where $\mathbf{t}$ is a vector of parameters independent from both ϵ and μ) can be determined. Fixing all the elements of $\tilde{T}$ in terms only of ϵ, $T(\epsilon)$ is found.

This procedure is by the way only possible if all the eigenvalues of the fuchsian matrix $B(x;\epsilon)$ are proportional to ϵ, since a similarity transformation preserves the eigenvalues and a completely

ϵ-factorized matrix has eigenvalues of the form $b\epsilon$ ($B(x;\epsilon) = \epsilon\tilde{B}(x)$ and $\tilde{B}u_i = b_i u_i \Rightarrow \epsilon\tilde{B}(x)u_i = \epsilon b_i u_i$). It is then necessary to find a way to force all the eigenvalues to be proportional to ϵ: the procedure of eigenvalue reduction solves only partially the problem, namely allows to reach a form in which all the single matrices of residues in the, so it must have eigenvalues proportional to ϵ. In conclusion, imposing that all the matrices of the residues have ϵ-proportional eigenvalues will assure the existence of $T(\epsilon)$.

Although, all the techniques enlisted in section 5.2 cannot be used as they have been presented when dealing with DEs. The main reason is that they have not the correct transformation form, i.e. $P^{-1}A(z)P - P^{-1}\mathrm{d}_z P$. Moreover, with the bare elimination of entire raws or columns from A, some MIs will be cut off from the DE, resulting in a harder (if not impossible) choice of suitable BCs to solve the problem.

The main effort then is to rewrite the previous transformation as proper transformation for DE systems, and retrieving the results enlisted in [16]. To obtain these new expressions, some constraints have to be performed on the form of the matrices; the final result is nevertheless useful for the evaluation in the present work.

5.3.1 Balances

Definition 11 (Balance)**.** A balance between the points z_1 and z_2 is a transformation of the form:

$$\mathcal{B}(\mathbb{P}; z_1; z_2|z) := \begin{cases} \mathbb{I} - \mathbb{P} + \frac{z-z_2}{z-z_1}\mathbb{P} & \text{if } z_1 \neq \infty \neq z_2 \\ \mathbb{I} - \mathbb{P} - (z - z_2)\mathbb{P} & \text{if } z_1 = \infty \neq z_2 \\ \mathbb{I} - \mathbb{P} - \frac{1}{z-z_1}\mathbb{P} & \text{if } z_1 \neq \infty = z_2 \end{cases} \tag{5.30}$$

where $\mathbb{P}$ is a projector.

The inverse of a balance is also a balance, and precisely:

$$\mathcal{B}(\mathbb{P}; z_1; z_2|z)\mathcal{B}(\mathbb{P}; z_2; z_1|z) = \mathbb{I} \quad \Rightarrow \quad \mathcal{B}^{-1}(\mathbb{P}; z_1; z_2|z) = \mathcal{B}(\mathbb{P}; z_2; z_1|z) \tag{5.31}$$

(it can be proven by direct calculation, remembering that $\mathbb{P}\mathbb{P} = \mathbb{P}$).

Balances act as transformations in the usual way:

$$A'(z) = \mathcal{B}^{-1}A(z)\mathcal{B} - \mathcal{B}^{-1}\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}z}, \qquad \mathcal{B} := \mathcal{B}(\mathbb{P}; z_1; z_2|z). \tag{5.32}$$

These transformations are called *balance transformations*, with balance $\mathcal{B}$.

5.3.2 Reduction to fuchsian form

Proposition 14 (Rank reduction for maximum pole)**.** *[16]. Given a matrix* A_0

$$A_0 := A(z;\epsilon) = \sum_{p=1}^{P}\sum_{q=1}^{Q_p}\frac{A_{(p;q)}(\epsilon)}{(z - z_p)^q}, \tag{5.33}$$

consider a non-fuchsian pole z_1, with matrix of coefficients of the maximum pole in $z = z_1$ $M := A_{0(1;Q_1)}$ and $Q := A_{0(p;Q_1-1)}$. M_0 must be nilpotent. *Using a similarity transformation, pass from A_0 to $\hat{A}_0$, such as M_0 is sent to $\hat{M}_0$, Jordan matrix with non-zero blocks in the upper left corner, and Q is sent to $\hat{Q}$. Construct the projector*

$$\mathbb{P} = \sum_{i \in I} u_i^{(0)} x_i^\dagger, \tag{5.34}$$

with $u_i^{(0)}$ right proper eigenvector associated to the block B_i of $\hat{M}_0$ with eigenvalue a_i and auxiliary vector x_i, such as:

- $q_{k;j} = 0 \quad \forall j \in I$ and $\forall k \in (\mathcal{N} \setminus I)$, *where* $\mathcal{N} = \{1; 2; \ldots; N\}$;
- $\sum_{j \in (\mathcal{N} \setminus \{i\})} \left(x_i^\dagger\right)_j q_{j;k} = 0 \quad \forall i; k \in I$;
- $x_i^\dagger u_j = \delta_{ij}$;
- *the lines of $\mathcal{B}^{-1}\hat{Q}\mathcal{B}$ corresponding to the null blocks of $\hat{M}_0$ are null;*
- *if z_2 is a singular point of $\hat{A}_0$, x_i is a left proper eigenvector $\tilde{v}_i^\dagger$ of $\hat{A}_{0(2;Q_2)}(\epsilon)$.*

Perform on the whole *$\hat{A}_0$ the balance transformation*

$$\hat{A}_1 := \mathcal{B}^{-1}\hat{A}_0\mathcal{B} - \mathcal{B}^{-1}\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}z}, \quad \mathcal{B} = \mathcal{B}(\mathbb{P}; z_1; z_2|z), \tag{5.35}$$

obtaining $\hat{M}_1 := \mathcal{B}^{-1}\hat{M}_0\mathcal{B} - \mathcal{B}^{-1}\mathrm{d}_z\mathcal{B} = \hat{A}_{1(p;q)}$. Then $\operatorname{rank} M_1 = \operatorname{rank} M_0 - \operatorname{card} I$, *and neither new poles are generated, nor degrees of already present poles increased.*

Proof. First of all, it is necessary to prove that the balance transformation (5.35) generates the same effect of the transformation (5.26) on M_0 and does not generate non-fuchsian terms where are not yet present or should be eliminated by the transformation itself.

For convenience both z_1 and z_2 will be finite, since the proof will apply with the same steps to the case in which one of them is infinite. Consider the derivative term:

$$\mathcal{B}^{-1}\mathrm{d}_z\mathcal{B} = \left(\mathbb{I} - \mathbb{P} + \mathbb{P}\frac{z - z_1}{z - z_2}\right)\mathbb{P}\frac{z_2 - z_1}{(z - z_1)^2} = \mathbb{P}\left(\frac{1}{z - z_2} - \frac{1}{z - z_1}\right); \tag{5.36}$$

this term has the only effect to add fuchsian expressions, so it is not problematic.

Consider now the other term of the transformation:

$$\mathcal{B}^{-1}A_0\mathcal{B} = \mathcal{B}^{-1}\left[\frac{\hat{M}_0}{(z - z_1)^{Q_1}} + \frac{\hat{Q}}{(z - z_1)^{Q_1 - 1}} + \ldots + \frac{\hat{A}_{0(1;1)}(\epsilon)}{z - z_1} + \frac{\hat{A}_{0(2;Q_2)}(\epsilon)}{(z - z_2)^{Q_2}} + \ldots + \sum_{p=3}^{P}\sum_{q=1}^{Q_p}\frac{\hat{A}_{0(p;q)}(\epsilon)}{(z - z_p)^q}\right]\mathcal{B} \tag{5.37}$$

and examine it term by term.

- $\sum_{p=3}^{P}\sum_{q=1}^{Q_p}\hat{A}_{0(p;q)}(\epsilon)/(z - z_p)^q$:

  $$\frac{\mathcal{B}^{-1}\hat{A}_{0(p;q)}(\epsilon)\mathcal{B}}{(z - z_p)^q} = \left[\hat{A}_{0(p;q)} + \mathbb{P}\hat{A}_{0(p;q)}(\mathbb{I} - \mathbb{P})\frac{z_2 - z_1}{z - z_2} + (\mathbb{I} - \mathbb{P})\hat{A}_{0(p;q)}\mathbb{P}\frac{z_1 - z_2}{z - z_1}\right]\frac{1}{(z - z_p)^q} \tag{5.38}$$

  this term neither does create new poles nor increase the degree of existing ones, since

  $$\frac{1}{(z - a)(z - b)^n} = \frac{1}{(x - a)(a - b)^n} - \sum_{j=1}^{n}\frac{1}{(a - b)^{n+1-j}(x - b)^j}, \quad n \in \mathbb{N} \tag{5.39}$$

  (the proof is carried out by induction).

- $\hat{A}_{0(2;Q_2)}(\epsilon)/(z-z_2)^{Q_2}$:

$$\frac{\mathcal{B}^{-1}\hat{A}_{0(2;Q_2)}(\epsilon)\mathcal{B}}{(z-z_2)^{Q_2}} = \left[\hat{A}_{0(2;Q_2)} + \mathbb{P}\hat{A}_{0(2;Q_2)}(\mathbb{I}-\mathbb{P})\frac{z_2-z_1}{z-z_2} + (\mathbb{I}-\mathbb{P})\hat{A}_{0(2;Q_2)}\mathbb{P}\frac{z_1-z_2}{z-z_1}\right]\frac{1}{(z-z_2)^{Q_2}}; \quad (5.40)$$

the term $(z_1-z_2)/[(z-z_1)(z-z_2)^{Q_2}]$ does not generate higher poles than the ones already present, thanks to (5.39); problems arise with the term $(z_1-z_2)/(z-z_2)^{Q_2+1}$, that introduces a higher order pole in the second point of balance (this will result in an inconsistent method, which simply moves the non-fuchsian poles, instead of eliminating them); to avoid this result two possibilities ar available:

  - if z_2 is a pole, $\mathbb{P}\hat{A}_{0(2;Q_2)}(\mathbb{I}-\mathbb{P})$ must be equal to 0. Choosing $x_i^\dagger = \tilde{v}_i^\dagger$, proper eigenvector of $\hat{A}_{0(2;Q_2)}$, the expression becomes:

$$\mathbb{P}\hat{A}_{0(2;Q_2)}(\mathbb{I}-\mathbb{P}) = \sum_{i=1}^{I} u_i v_i^\dagger \hat{A}_{0(2;Q_2)}\left(\mathbb{I}-\sum_{j=1}^{I}u_j v_j^\dagger\right) = \sum_{i=1}^{I} u_i v_i^\dagger a_i\left(\mathbb{I}-\sum_{j=1}^{I}u_j v_j^\dagger\right) =$$
$$= \sum_{i=1}^{I} u_i v_i^\dagger a_i - \sum_{i=1}^{I}\sum_{j=1}^{I} u_i\delta_{ij}v_j^\dagger = \sum_{i=1}^{I} u_i v_i^\dagger a_i - \sum_{i=1}^{I} u_i v_i^\dagger a_i = 0, \quad (5.41)$$

    and no higher poles than $(z-z_2)^{-Q_2}$ are generated;

  - if z_2 is a regular point (so $Q_2 = 0$ is not in the list $i \in [1;I]$), the expression generates a fuchsian point $(z-z_2)^{-1}$, and no further conditions are necessary.

- $\hat{M}_0/(z-z_1)^{Q_1}$

$$\frac{\mathcal{B}^{-1}\hat{M}_0\mathcal{B}}{(z-z_1)^{Q_1}} = \left[\hat{M}_0 + \mathbb{P}\hat{M}_0(\mathbb{I}-\mathbb{P})\frac{z_2-z_1}{z-z_2} + (\mathbb{I}-\mathbb{P})\hat{M}_0\mathbb{P}\frac{z_1-z_2}{z-z_1}\right]\frac{1}{(z-z_1)^{Q_1}}; \quad (5.42)$$

since $\hat{M}_0$ is nilpotent, and $\mathbb{P} = \sum_{i=1}^{I} u_i^{(0)} x_i^\dagger$, $\hat{M}_0\mathbb{P} = 0$. The equation then can be rewritten as:

$$\frac{\mathcal{B}^{-1}\hat{M}_0\mathcal{B}}{(z-z_1)^{Q_1}} = \left[\hat{M}_0 + \mathbb{P}\hat{M}_0\frac{z_2-z_1}{z-z_2}\right]\frac{1}{(z-z_1)^{Q_1}} =$$
$$= (\mathbb{I}-\mathbb{P})\hat{M}_0\frac{1}{(z-z_1)^{Q_1}} + \mathbb{P}\hat{M}_0\frac{1}{(z-z_2)(z-z_1)^{Q_1-1}}; \quad (5.43)$$

the first term of the last line is a deflation, and since the eigenvalues of $\hat{M}_0$ are null ($\hat{M}_0$ is nilpotent by hypothesis), its effect is to lower the rank of the matrix by I, so to substitute the first row of each Jordan block B_i, $i \in I$ with a row of zeroes; the second term do not introduce forbidden poles, thanks to (5.39), and has the effect to "traslate" the pole from z_1 to z_2. The nett effect of this part is to have a matrix which rows and columns containing 1×1 null Jordan blocks are completely formed by 0 as coefficient of the pole $(z-z_1)^{-Q_1}$.

- $\hat{Q}/(z-z_1)^{Q_1-1}$:

$$\frac{\mathcal{B}^{-1}\hat{Q}\mathcal{B}}{(z-z_1)^{Q_1-1}} = \left[\hat{Q} + \mathbb{P}\hat{Q}(\mathbb{I}-\mathbb{P})\frac{z_2-z_1}{z-z_2} + (\mathbb{I}-\mathbb{P})\hat{Q}\mathbb{P}\frac{z_1-z_2}{z-z_1}\right]\frac{1}{(z-z_1)^{Q_1-1}}; \quad (5.44)$$

$\mathbb{P}\hat{Q}(\mathbb{I}-\mathbb{P})$ does not generate poles with undesired degrees, also this time thanks to (5.39), while $(\mathbb{I}-\mathbb{P})\hat{Q}\mathbb{P}$ increases the degree of the pole to Q_1. The key point here is to find a way not to alter the new rank of $\hat{M}_1$, so raws and columns corresponding to null singular Jordan blocks must be null. Consider the second

term of the sum, and assume to have a projector of the form $\mathbb{P} = u_1 x_1^\dagger$:

$$(\mathbb{I}-\mathbb{P})\hat{Q}\mathbb{P} = \begin{pmatrix} 0 & -x_{1;2}^* & \dots & -x_{1;N}^* \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix} \hat{Q} \begin{pmatrix} 1 & x_{1;2}^* & \dots & x_{1;N}^* \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -x_1^\dagger \hat{Q}_{\_1} & -x_{1;2}^* x_1^\dagger \hat{Q}_{\_1} & \dots & -x_{1;N}^* x_1^\dagger \hat{Q}_{\_1} \\ q_{2;1} & q_{2;2} & \dots & q_{2;N} \\ \vdots & \vdots & \ddots & \vdots \\ q_{N;1} & q_{N;2} & \dots & q_{N;N} \end{pmatrix} \begin{pmatrix} 1 & x_{1;2}^* & \dots & x_{1;N}^* \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -x_1^\dagger \hat{Q}_{\_1} & -x_{1;2}^* x_1^\dagger \hat{Q}_{\_1} & \dots & -x_{1;N}^* x_1^\dagger \hat{Q}_{\_1} \\ q_{2;1} & x_{1;2}^* q_{2;2} & \dots & x_{1;N}^* q_{2;N} \\ \vdots & \vdots & & \vdots \\ q_{N;1} & x_{1;2}^* q_{N;2} & \dots & x_{1;N}^* q_{N;N} \end{pmatrix}; \quad (5.45)$$

the condition now is that the first column of $\hat{Q}$ is a null vector: this set to 0 the first row and column, so summing this result to (5.43) the resulting matrix has the same rank as the initial two. In the case of $\mathbb{P}$ made up by sums of eigenvectors, the condition of null column must be applied simultaneously for all the lines containing a null Jordan block, since the matrices will have the form (supposing that all the null Jordan blocks in $\hat{M}_0$ have been set in the upper left corner of the matrix, ordered with i; this is always possible, since it is only a swapping of the blocks; $\hat{Q}$ is ordered in the same way):

$$\hat{M}_0 = \begin{pmatrix} 0_{i_1} & \dots & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & 0_{i_{\text{card } I}} & 0 \\ 0 & \dots & 0 & \mathbb{B} \end{pmatrix}, \quad (5.46)$$

$$\hat{Q} = \begin{pmatrix} -x_{i_1}^\dagger \hat{Q}_{\_ i_1} & \dots & -x_{i_{\text{card } I}}^\dagger \hat{Q}_{\_ \text{ card } I} & -\sum_I x_{i;\text{card } I+1}^* x_{i_1}^\dagger \hat{Q}_{\_ i} & \dots & -\sum_I x_{i;N}^* x_{i_1}^\dagger \hat{Q}_{\_ i} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ -x_{i_{\text{card } I}}^\dagger \hat{Q}_{\_ i_1} & \dots & -x_{i_{\text{card } I}}^\dagger \hat{Q}_{\_ i_{\text{card } I}} & -\sum_I x_{i;\text{card } I+1}^* x_{\text{card } I}^\dagger \hat{Q}_{\_ i} & & -\sum_I x_{i;N}^* x_{i_{\text{card } I}}^\dagger \hat{Q}_{\_ i} \\ q_{\text{card } I+1;i_1} & \dots & q_{\text{card } I+1;\text{card } I} & \sum_I x_{i;\text{card } I+1}^* q_{\text{card } I;i} & \dots & \sum_I x_{i;N}^* q_{\text{card } I;i} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ q_{N;i_1} & \dots & q_{N;\text{card } I} & \sum_I x_{i;\text{card } I+1}^* q_{N;i} & \dots & \sum_I x_{i;N}^* q_{N;i} \end{pmatrix} \quad (5.47)$$

(the form of the matrices can be verified by direct calculation). So the column of $\hat{Q}$ related to the eigenvectors of $\hat{M}_0$ have to satisfy

$$q_{i;j} = 0 \quad \forall j \in I \text{ and } i \in (\mathcal{N} \setminus I), \quad (5.48)$$

$$\sum_{j \in (\mathcal{N} \setminus \{i\})} (x_i^\dagger)_j q_{j;k} = 0 \quad \forall i; k \in I. \quad (5.49)$$

Moreover all the lines in $\hat{Q}$ corresponding to null lines in $\hat{M}_0$ have to be null, in order to not increase the rank of the matrix resulting from their sum. In this way the sum of the two matrices will preserve the null columns and rows, not increasing the rank of the new matrix of the pole.

□

Example 12 (Rank reduction, one eigenvector). To better understand the expression (5.47), consider the case in of a $\mathbb{P} = u_1 x_1^\dagger$, with $u_1 = (1; 0; \dots; 0)^T$. In this case, after the application

of the balance, the matrices forming the new term in the pole will ave the form (remember that $\hat{M}_0$ is a matrix in Jordan form):

$$\hat{M}_0 = \begin{pmatrix} 0 & & \dots & \dots & 0 \\ & m_{2;2} & m_{2;3} & & \\ \vdots & & \ddots & \ddots & \\ & & & & m_{N-1;N} \\ 0 & & & & m_{N;N} \end{pmatrix} \tag{5.50}$$

$$\begin{pmatrix} 0 & -x_2^* & \dots & \dots & -x_N^* \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & \\ 0 & & & & 1 \end{pmatrix} Q \begin{pmatrix} 1 & x_2^* & \dots & x_N^* \\ 0 & \dots & \dots & 0 \\ \vdots & \ddots & & \\ 0 & & & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -\sum_{i=2}^N x_i^* q_{i;1} & \dots & -\sum_{i=2}^N x_i^* q_{i;N} \\ q_{2;1} & \dots & q_{2;N} \\ \vdots & \ddots & \vdots \\ q_{N;1} & \dots & q_{N;N} \end{pmatrix} \begin{pmatrix} 1 & x_2^* & \dots & x_N^* \\ 0 & \dots & \dots & 0 \\ \vdots & \ddots & & \\ 0 & & & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} -\sum_{i=2}^N x_i^* q_{i;1} & -x_2^* \sum_{i=2}^N x_i^* q_{i;1} & \dots & -x_N^* \sum_{i=2}^N x_i^* q_{i;1} \\ q_{2;1} & x_2^* q_{2;1} & \dots & x_N^* q_{2;1} \\ \vdots & \vdots & \ddots & \vdots \\ q_{N;1} & x_2 q_{N;1} & \dots & x_N^* q_{N;1} \end{pmatrix} \tag{5.51}$$

To have a decrease in the rank of the sum of these two matrices, the row and the column corresponding to the 1×1 block of $a_1 \to 0$ must be null. It follows then that the quantities $q_{j;1}, j > 1$ must be all equal to 0, so also $\sum_{i=2}^N x_i^* q_{i;1} = 0$; this will also assure that the first row will be null. Moreover, further conditions on the rows and columns not related to the previous one arise from the necessity that the lines corresponding to null lines in $\hat{M}_0$ do not gain non null elements (this would increase the rank of the final matrix with respect to $\hat{M}_0$).

Example 13 (Rank reduction, two eigenvectors). Consider $\mathbb{P} = u_1 x_1^\dagger + u_N y_N^\dagger$. Now:

$$\hat{M}_0 = \begin{pmatrix} 0 & & & \dots & & 0 \\ & m_{2;2} & m_{2;3} & & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & & m_{N-2;N-1} & \vdots \\ & & & & m_{N-1;N-1} & \\ 0 & & & \dots & & 0 \end{pmatrix} \tag{5.52}$$

$$\begin{pmatrix} 0 & -x_2^* & & \dots & & -x_N^* \\ 0 & 1 & 0 & \dots & & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & & & & 1 & 0 \\ -y_1^* & & \dots & & -y_{N-1}^* & 0 \end{pmatrix} Q \begin{pmatrix} 1 & x_2^* & & \dots & & x_N^* \\ 0 & 0 & 0 & \dots & & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & & & & 0 & 0 \\ y_1^* & & \dots & & y_{N-1}^* & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} -\sum_{i=2}^N x_i^* q_{i;1} & \dots & -\sum_{i=2}^N x_i^* q_{i;N} \\ q_{2;1} & \dots & q_{2;N} \\ \vdots & \ddots & \vdots \\ q_{N-1;1} & \dots & q_{N-1;N} \\ -\sum_{i=1}^{N-1} y_i^* q_{i;1} & \dots & -\sum_{i=1}^{N-1} y_i^* q_{i;N} \end{pmatrix} \begin{pmatrix} 1 & x_2^* & & \dots & & x_N^* \\ 0 & 0 & 0 & \dots & & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & & & & 0 & 0 \\ y_1^* & & \dots & & y_{N-1}^* & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} -\sum_{i=2}^N x_i^* v_{i1} & \dots & -\sum_{i=2}^N x_i^* v_{iN} \\ v_{2;1} & \dots & v_{2;N} \\ \vdots & \ddots & \vdots \\ v_{N-1;1} & \dots & v_{N-1;N} \\ -\sum_{i=1}^{N-1} y_i^* v_{i1} & \dots & -\sum_{i=1}^{N-1} y_i v_{iN} \end{pmatrix} \tag{5.53}$$

where $v_{ij} := x_j^* q_{i;1} + y_j^* q_{i;N}$, remembering that $x_1^* = 1$ and $y_N^* = 1$. Both the first row and column have to be null, in order to not increase the rank of the final matrix, so:

$$q_{i;1} = 0 \quad \forall i \neq 1; N, \tag{5.54}$$

$$q_{i;N} = 0 \quad \forall i \neq 1; N, \tag{5.55}$$

$$\sum_{i=2}^{N} x_i^* q_{i;1} = 0, \tag{5.56}$$

$$\sum_{i=2}^{N} x_i^* q_{i;N} = 0, \tag{5.57}$$

$$\sum_{i=1}^{N-1} y_i^* q_{i;1} = 0, \tag{5.58}$$

$$\sum_{i=1}^{N-1} y_i^* q_{i;N} = 0, \tag{5.59}$$

due to the generality of x_j^* and y_j^*. Furthermore, the lines corresponding to null ones in $\hat{M}_0$ have to be null.

The proposition above can be iterated multiple times, and if the hypothesis remain satisfied all over the iterations, the rank of the non-fuchsian poles can be reduced to 0, eliminating the non-fuchsian pole from the complete matrix of coefficients. It is then possible to summarize the needed operations in the following algorithm.

Algorithm 1 (Reduction to fuchsian form). Given a matrix $A(z;\epsilon)$

$$A(z;\epsilon) = \sum_{p=1}^{P} \sum_{q=1}^{Q_p} \frac{A_{(p;q)}(\epsilon)}{(z - z_p)^q}, \tag{5.60}$$

if the following algorithm arrives at conclusion, a system in fuchsian form is reached.

1. Consider a non-fuchsian pole z_1, with matrix of coefficients of the maximum pole in $z = z_1$ $M := A_{(1;Q_1)}(\epsilon)$. M must be *nilpotent*. Then:
2. $A_0 := A(z;\epsilon), \quad M_0 := M = A_{0(1;Q_1)} \quad Q := A_{0(p;Q_1-1)}$;
3. choose a point $z_2 \neq z_1$;
4. using a similarity transformation, pass from A_0 to $\hat{A}_0$, such as M_0 is sent to $\hat{M}_0$, Jordan matrix with non-zero blocks at the beginning of the diagonal, and Q is sent to $\hat{Q}$;
5. construct the projector
$$\mathbb{P} = \sum_{i \in I} u_i^{(0)} x_i^\dagger, \tag{5.61}$$
with $u_i^{(0)}$ right proper eigenvector associated to the block B_i of $\hat{M}_0$ with eigenvalue a_i, and auxiliary vector x_i such as:
   - $q_{k;j} = 0 \quad \forall j \in I$ and $\forall k \in (\mathcal{N} \setminus I)$, where $\mathcal{N} = \{1; 2; \ldots; N\}$,
   - $\sum_{j \in (\mathcal{N} \setminus \{i\})} \left(x_i^\dagger\right)_j q_{j;k} = 0 \quad \forall i; k \in I$,
   - $x_i^\dagger u_j = \delta_{ij}$,
   - the lines of $\mathcal{B}^{-1}\hat{Q}\mathcal{B}$ corresponding to the null blocks of $\hat{M}_0$ are null;
   - if z_2 is a singular point of $\hat{A}_0$, x_i is a left proper eigenvector $\tilde{v}_i^\dagger$ of $\hat{A}_{0(2;Q_2)}(\epsilon)$;
6. perform on the *whole* $\hat{A}_0$ the balance transformation
$$\hat{A}_1 := \mathcal{B}^{-1}\hat{A}_0\mathcal{B} - \mathcal{B}^{-1}\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}z}, \quad \mathcal{B} = \mathcal{B}(\mathbb{P}; z_1; z_2|z), \tag{5.62}$$
obtaining $\hat{M}_1 := \mathcal{B}^{-1}\hat{M}_0\mathcal{B} - \mathcal{B}^{-1}\mathrm{d}_z\mathcal{B} = \hat{A}_{1(p;q)}$;
7. $\hat{A}_0 := \hat{A}_1$, $\hat{M}_0 := \hat{M}_1$;
8. repeat from point 3 to here until the procedure cannot be iterated anymore;
9. repeat from point 1 to here until the procedure cannot be iterated anymore;
10. conclusion.

Some observation on the hypothesis and on the outcome of the algorithm are enlisted.

- It is not always possible to reduce a matrix to a fuchsian form, depending on the form of the lower degrees of a given pole (in particular, on the columns related to eigenvectors of the pole to be erased) and on the nilpotency of the matrix itself.
- Conditions satisfied in the first iteration of the algorithm may not be satisfied anymore in further iterations, due to the mixing of the different matrices of the poles.
- Using a regular point as z_2, the construction of $\mathbb{P}$ is easier, at the cost of introducing a new fuchsian pole.

5.3.3 Reduction to ϵ-homogeneous eigenvalues

Once a fuchsian form has been reached for the matrix of coefficients, the eigenvalues must be normalized to be proportional to ϵ in order to have a suitable matrix for the similarity transformation to a canonical form.

Since the transformation corresponding to a deflation for DEs is a balance, the aim is to find a way to use balances to adjust the eigenvalues in a way similar to the one of proposition 12.

First of all, the case of a matrix with only two singular points, both fuchsian, is considered. Despite the specificity of the case under consideration, in all the evaluations performed here singularities consist of no more than two singular points, both fuchsian.

Proposition 15 (Eigenvalue balance, two fuchsian points)**.** *[16].Consider $A(z;\epsilon)$, matrix of coefficients of a DE system with only two singular points, both fuchsian*

$$A(z;\epsilon) = \frac{A_1(\epsilon)}{z - z_1} + \frac{A_2(\epsilon)}{z - z_2}. \tag{5.63}$$

Choose two points $(\tau_1;\tau_2)$ in the set $\{z_1;z_2;\infty\}$; call E the matrix of residues associated to τ_1, and F the one associated to τ_2 (remember that the matrix of residues at $z=\infty$ is $A_\infty(\epsilon) = -A_1(\epsilon) - A_2(\epsilon)$). Named u_i the right eigenvectors of E with associated eigenvalues e_i and v_j the left eigenvectors of F with associated eigenvalues f_j, performing the balance transformation (5.32) with $\mathcal{B} := \mathcal{B}(\mathbb{P};\tau_1;\tau_2|z)$ and

$$\mathbb{P} = u_i^{(0)} v_j^{(0)\dagger} \quad \text{and} \quad v_j^{(0)\dagger} u_i^{(0)} = 1 \tag{5.64}$$

the eigenvalue e_i is increased by one and the eigenvalue f_j is decreased by one, namely

$$e_i' = e_i + 1, \qquad f_j' = f_j - 1, \tag{5.65}$$

while all the other eigenvalues in all the matrices remain unchanged.

Proof. $\tau_1 \neq \infty \neq \tau_2$ **case.**

Consider $\tau_1 \neq \infty \neq \tau_2$ for simplicity; the case in which one of the points is infinite can be faced following the same path presented in the following lines.

Perform the balance transformation:

$$\mathcal{B}^{-1}\left(\frac{E}{z-\tau_1} + \frac{F}{z-\tau_2}\right)\mathcal{B} - \mathcal{B}^{-1}\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}z} = \mathcal{B}^{-1}\left(\frac{E}{z-\tau_1} + \frac{F}{z-\tau_2} - \frac{\mathrm{d}\mathcal{B}}{\mathrm{d}z}\mathcal{B}^{-1}\right)\mathcal{B}. \tag{5.66}$$

At first, consider the term $-(\mathrm{d}\mathcal{B}/\mathrm{d}z)\mathcal{B}^{-1}$:

$$-\frac{\mathrm{d}\mathcal{B}}{\mathrm{d}z}\mathcal{B}^{-1} = \left(\frac{1}{z-\tau_1} - \frac{1}{z-\tau_2}\right)\mathbb{P}. \tag{5.67}$$

The expression (5.66) can be written as

$$\mathcal{B}^{-1}\left(\frac{\mathcal{E}}{z-\tau_1} + \frac{\mathcal{F}}{z-\tau_2}\right)\mathcal{B} = \frac{1}{z-\tau_1}\mathcal{B}^{-1}\mathcal{E}\mathcal{B} + \frac{1}{z-\tau_2}\mathcal{B}^{-1}\mathcal{F}\mathcal{B}, \tag{5.68}$$

where $\mathcal{E} = E + \mathbb{P}$ and $\mathcal{F} = F - \mathbb{P}$. Concerning the eigenvalues of $\mathcal{E}$ and $\mathcal{F}$, the expressions

$$\mathcal{E} = E + u_i v_j^\dagger, \tag{5.69}$$

$$\mathcal{F}^\dagger = F^\dagger - v_j u_i^\dagger \tag{5.70}$$

are exactly eigenvalue reductions from proposition 12 with $\lambda_E = -1$ and $\lambda_F = 1$, so $\mathcal{E}$ has the same eigenvalues of E, except for e_i associated to u_i that is increased by 1, and $\mathcal{F}^\dagger$ has the same eingenvalues of F, except f_j associated to v_j, that is lowered by 1 (remember that the eigenvalues are supposed to be real); since the eigenvalues of a matrix and of its transpose are the same, the logic holds also for $\mathcal{F}$ and F.

Consider now the nature of $\mathcal{B}$: $\mathcal{B}$ is an invertible matrix, so the transformation $\mathcal{B}^{-1}A\mathcal{B}$ is a similarity transformation, conserving the eigenvalues of the matrix A. It is not yet sufficient to say that the eigenvalues of the matrix of residues are the same: the balance transformation introduces in $\mathcal{E}$ terms with poles $z - \tau_2$, and in $\mathcal{F}$ poles in $z - \tau_2$. To find the relation between the eigenvalues, consider that if A and B are matrices with the same eigenvalues, also cA and cB have the same eigenvalues; then:

$$(z-\tau_1)\left(\frac{\mathcal{E}}{z-\tau_1}+\frac{\mathcal{F}}{z-\tau_2}\right) \quad \text{and} \quad (z-\tau_1)\left(\frac{\mathcal{E}'}{z-\tau_1}+\frac{\mathcal{F}'}{z-\tau_2}\right) \tag{5.71}$$

have the same eigenvalues. Taking the limit $z \to \tau_1$ follows that

$$\mathcal{E} \quad \text{and} \quad \mathcal{E}' \tag{5.72}$$

have the same eigenvalues. The same procedure can be carried out for $\mathcal{F}$ and $\mathcal{F}'$.

The neat result of an eigenvalue balance with projector $\mathbb{P} = u_i v_j^\dagger$ is then to increase the eigenvalue e_i of the matrix E and to decrease the eigenvalue f_j of the matrix F, leaving all the other eigenvalues unchanged.

$\tau_1 = \infty$ or $\tau_2 = \infty$ case.

Regarding the residues at infinity with a balance between two finite points:

$$A'_\infty = -\mathcal{E}' - \mathcal{F}' = -B^{-1}(\mathcal{E}+\mathcal{F})\mathcal{B} = -\mathcal{B}^{-1}(E+\mathbb{P}+F-\mathbb{P})\mathcal{B} = -\mathcal{B}^{-1}(E+F)\mathcal{B} \tag{5.73}$$

since a similarity transformation does not change the eigenvalues, the residue at infinity preserves its original eigenvalues.

For a balance with infinity, like $\mathcal{B}(\mathbb{P};\tau_1;\infty|z) = \mathbb{I} - \mathbb{P} - \frac{1}{z-\tau_1}\mathbb{P}$, $-(\mathrm{d}\mathcal{B}/\mathrm{d}z)\mathcal{B}^{-1} = \frac{1}{z-\tau_1}\mathbb{P}$, $\mathbb{P} = u_{1;i}v^\dagger_{\infty;j}$, so at finite only E has $e_i \to e_i + 1$, while at infinity:

$$A'_\infty = -\mathcal{E}' - \mathcal{F}' = -B^{-1}(\mathcal{E}+\mathcal{F})\mathcal{B} = -\mathcal{B}^{-1}(E+\mathbb{P}+F)\mathcal{B}; \tag{5.74}$$

the eigenvalues of $-\mathcal{B}^{-1}(E+\mathbb{P}+F)\mathcal{B}$ are the same of $-E-F-\mathbb{P})\mathcal{B}$. $\mathbb{P}$ is constructed with a left eigenvector of $A_\infty = -E-F$, so it acts like an eigenvalue reduction on the eigenvalue a_j of A_∞, lowering it by one, as expected. The same happens, with $u_{\infty;i}v^\dagger_{1;j}$ instead of $-u_{1;i}v^\dagger_{\infty;j}$, with $\mathcal{B}(\mathbb{P};\infty;\tau_2|z)$, rising a_i at infinity and lowering f_j at the corresponding finite pole. □

The results can be compared with the ones presented in [16].

The proposition can be generalized to a matrix of coefficients with an arbitrary number of poles, all fuchsian.

Lemma 6 (Eigenvalue reduction, fuchsian points)**.** *Given $A(z;\epsilon$ of the form*

$$A(z;\epsilon) = \sum_{m=1}^{M} \frac{A_m(\epsilon)}{z - z_m}, \tag{5.75}$$

choose two points $(\tau_1;\tau_2)$ in the set $\{\{z_m\}_M;\infty\}$; call E the matrix of residues associated to τ_1, and F the one associated to τ_2 (remember that the matrix of residues at $z = \infty$ is $A_\infty(\epsilon) = -\sum_{m=1}^{M} A_m(\epsilon)$). Named u_i the right eigenvectors of E with associated eigenvalues e_i and v_j the left eigenvectors of F with associated eigenvalues f_j, peforming the balance transformation (5.32) with $\mathcal{B} := \mathcal{B}(\mathbb{P};\tau_1;\tau_2|z)$ and

$$\mathbb{P} = u_i^{(0)} v_j^{(0)\dagger} \quad \text{and} \quad v_j^{(0)\dagger} u_i^{(0)} = 1 \tag{5.76}$$

the eigenvalue e_i is increased by one and the eigenvalue f_j is decreased by one, namely

$$e_i' = e_i + 1, \qquad f_j' = f_j - 1, \tag{5.77}$$

while all the other eigenvalues in all the matrices remain unchanged.

Proof. Proceeding as in proposition 15, E and F change their eigenvalues as $e_i \to e_i + 1$ and $f_j \to f_j - 1$, and A evolves as

$$A'(z;\epsilon) = \mathcal{B}^{-1}\left(\frac{E+\mathbb{P}}{z-\tau_1} + \frac{F-\mathbb{P}}{z-\tau_2} + \sum_{\bar{m}} \frac{A_{\bar{m}}}{z-z_{\bar{m}}}\right)\mathcal{B} = \tag{5.78}$$

$$= \frac{\mathcal{E}'}{z-\tau_1} + \frac{\mathcal{F}'}{z-\tau_2} + \sum_{\bar{m}} \frac{\mathcal{A}'_{\bar{m}}}{z-z_{\bar{m}}} \tag{5.79}$$

so all the $A_{\bar{m}}$ preserve their eigenvalues, thanks to $\mathcal{B}$ being a similarity transformation. From this follows also that the residue at infinity does not change if the balance is performed between two finite points.Finally, if the balance is performed between a finite point and the infinite, it is possible to proceed as in proposition 15. □

Notice that, while eigenvalues have a simple behaviour under the balance, the same is not true for the eigenvectors: both the eigenvectors of modified eigenvalues and unmodified ones are altered.

Using lemma 6 it is possible, moving unit by unit quantities between eigenvalues of different residues, to force an eigenvalue to assume the form $a + b(\epsilon)$, where $a \in \left[-\frac{1}{2}; \frac{1}{2}\right]$.

Consider now the case in which both the residues of the singular points and the residue at infinity have eigenvalues of the form $a_{\tau;i} = n_{\tau;i} + \epsilon b_{\tau;i}$ ($\tau \in \{z_1; \ldots; z_M; \infty\} = \mathcal{M}$, $n_{\tau,i} \in \mathbb{Z}$), such as

$$\sum_{\tau\in\mathcal{M}} \sum_{i=1}^{N_\tau} c_{\tau;i} n_{\tau;i} = 0, \tag{5.80}$$

with $c_{\tau;i}$ the dimension of the Jordan block corresponding to the eigenvalue $a_{\tau;i}$. In this case there is the possibility to balance all the eigenvectors in order to eliminate their non-homogeneous part in ϵ, using the claim above.

The reason for which also the dimension of each Jordan block is considered lies in the fact that each time a balance among two eigenvalues is performed, if both or just one of them belong to a non-unit Jordan block, the corresponding blocks are split into a 1×1 block with modified eigenvalue and into a block $(n-1) \times (n-1)$ preserving the original eigenvalue: this is due intuitively to the fact that the projector $\mathbb{P}$ is built with the proper eigenvector of the block, so its action focuses only on the first term of the diagonal of the block. So, to completely balance one eigenvalue, the balance have to be performed a number of times equal to the dimension of the corresponding Jordan block.

Operatively, the following algorithm is established.

Algorithm 2 (Reduction to ϵ-homogeneous eigenvalues). Starting with a matrix $A(z;\epsilon)$ with singular points, only fuchsian, with matrices of residues satisfying ($c_{\tau;i}$ is the dimension of the Jordan block corresponding to the eigenvalue $a_{\tau;i} = n_{\tau;i} + \epsilon b_{\tau;i}, n \in \mathbb{Z}$)

$$\sum_{\tau\in\{\{z_m\}_M;\infty\}} \sum_{i=1}^{N_\tau} c_{\tau;i} n_{\tau;i} = 0 : \tag{5.81}$$

1. evaluate eigenvectors and corresponding eigenvalues of all the matrices of residues ($\{A_m\}_M$ and A_∞);
2. choose a ordered couple of eigenvalues to balance $(a_1; a_2)$, where the first will be increased by 1 and the second decreased by 1, such as the right eigenvector u of a_1 and the left eigenvector v of a_2 satisfy $v^\dagger u = 1$;
3. perform the balance transformation (5.32) with $\mathbb{P} = uv^\dagger$;
4. repeat from point 1 to here until all the eigenvalues are proportional to ϵ.

It can happen sometimes that no couple of eigenvalues with non-homogeneous parts can be balanced, due to the fact that there are no couples with $v^\dagger u \neq 0$. In these cases, a third eigenvalue of the matrix of residues that would not be used in the desired balance, but has a suitable associated left or right eigenvector (according to the balance needed), can be used as a "bypass" eigenvalue to transfer the unit. Remember however that the transformed eigenvectors of the bypass eigenvalue can be modified by the balance, so a new final eigenvector might have to be chosen to complete the passage.

Regarding cases with non-fuchsian poles, problems arise. The balances can operate only on matrix associated to fuchsian points, since the part responsible for the eigenvalues reduction is generated by $\mathrm{d}_x\mathcal{B}\mathcal{B}^{-1}$, and the term is fuchsian, so not able to modify the matrices of deeper singularities. The condition of fuchsianity of the poles is then a mandatory requirement.

5.3.4 Factoring out ϵ

If the algorithm of eigenvalue reduction allows to obtain a matrix $A(z;\epsilon)$ in which all the eigenvalues are proportional to ϵ, it is possible, via a similarity transformation, to find a basis in which the whole matrix of coefficients will be ϵ-factorized. Since the matrix of coefficients, after the successful application of algorithm 1 and 2, has the form

$$A(z;\epsilon) = \sum_{m=1}^{M} \frac{\tilde{A}_m(\epsilon)}{z - z_m}, \tag{5.82}$$

where $\tilde{A}_m(\epsilon)$ has all eigenvalues proportional to ϵ, if each $\tilde{A}_m(\epsilon)$ has an ϵ-factorized form, the whole matrix $A(z;\epsilon)$ will present an ϵ-factorized structure.

To obtain this result, consider a similarity transformation of matrix $T(\epsilon)$: this matrix have to be independent from z, to not alter the fuchsian structure and the residue form, and is applied to the whole $A(z;\epsilon)$. So

$$T^{-1}(\epsilon)A(z;\epsilon)T(\epsilon) = \sum_{m=1}^{M} \frac{T^{-1}(\epsilon)\tilde{A}_m(\epsilon)T(\epsilon)}{z - z_m}, \tag{5.83}$$

and to obtain an ϵ-factorized $\mathcal{A}(z;\epsilon)$, T have to satisfy to the following system:

$$\begin{cases} T^{-1}(\epsilon)A_1(\epsilon)T(\epsilon) & = \epsilon\mathcal{A}_1 \\ \vdots & \vdots \\ T^{-1}(\epsilon)A_M(\epsilon)T(\epsilon) & = \epsilon\mathcal{A}_M \end{cases}. \tag{5.84}$$

Dividing all the equation by ϵ the system becomes independent from ϵ, since the matrices $\mathcal{A}_i$ do not depend on ϵ:

$$\begin{cases} \frac{1}{\epsilon}T^{-1}(\epsilon)A_1(\epsilon)T(\epsilon) & = \mathcal{A}_1 \\ \vdots & \vdots \\ \frac{1}{\epsilon}T^{-1}(\epsilon)A_M(\epsilon)T(\epsilon) & = \mathcal{A}_M \end{cases} . \tag{5.85}$$

The ϵ-independent structure just found then does not vary substituting ϵ with another value, like μ, so a new set of equations can be written:

$$\begin{cases} \frac{1}{\epsilon}T^{-1}(\epsilon)A_1(\epsilon)T(\epsilon) & = \frac{1}{\mu}T^{-1}(\mu)A_1(\mu)T(\mu) \\ \vdots & \vdots \\ \frac{1}{\epsilon}T^{-1}(\epsilon)A_M(\epsilon)T(\epsilon) & = \frac{1}{\mu}T^{-1}(\mu)A_M(\mu)T(\mu) \end{cases} . \tag{5.86}$$

Multiplying from the left by $T(\epsilon)$ and from the right by $T^{-1}(\mu)$ both the sides of each equation, and naming $\tilde{T}(\epsilon;\mu) := T(\epsilon)T^{-1}(\mu)$, the following system must be solved:

$$\begin{cases} \frac{1}{\epsilon}A_1(\epsilon)\tilde{T}(\epsilon;\mu) & = \frac{1}{\mu}\tilde{T}(\epsilon;\mu)A_1(\mu) \\ \vdots & \vdots \\ \frac{1}{\epsilon}A_M(\epsilon)\tilde{T}(\epsilon;\mu) & = \frac{1}{\mu}\tilde{T}(\epsilon;\mu)A_M(\mu) \end{cases} . \tag{5.87}$$

Solving this system, in general a class of solutions $\tilde{T}(\epsilon;\mu;\mathbf{t})$ is found. Since the only needed result is to find one similarity transformation that factorizes out ϵ, it is possible to choose arbitrary values for μ (real constants b or expressions like $b\epsilon$) and the vector of parameters $\mathbf{t}$ (real constants $\mathbf{b}$), with the only constraint that the matrix must remain invertible.

In this way the similarity transformation $T(\epsilon)$ is determined, and the matrix $\mathcal{A}(z;\epsilon) = T^{-1}(\epsilon)A(z;\epsilon)T(\epsilon)$ is canonical, as sum of canonical matrices.

5.4 Overview

The eigenvalue deflation method consists of three main steps:

- reduction to fuchsian form (section 5.3.2),
- eigenvalue balancing (section 5.3.3),
- transformation to a canonical form (section 5.3).

Each one of these parts has quite important constraints for its succesfull application, nevertheless for some of the cases investigated in this work (1-loop box, 2-loop ladder and a part of the 3-loop ladder) it has been possible, starting from the Laporta basis, to reach a canonical form for the DEs.

It is not yet clear if the conditions imposed for the algorithm of reduction to fuchsian form can be relaxed, maybe through a keen work to tune matrix Q of the coefficients of the immediate lower degree of the pole of the reduction.

In conclusion, it must be stressed how the adjustment of the eigenvalue deflation to DEs, although supported by [16], is not yet fully understood, also for the limited set of its application so far.

Part III

Multiloop ladder graphs

The calculations of the 1-loop box, 2-loop ladder and 3-loop ladder massless graphs are presented, discussing how a canonical system of differential equations can be built, both by Magnus exponential matrix and by eigenvalue deflation.

First of all, using `Reduze2` (see [46, 47] for the documentation) the list of the topologies comparing in the complete system of DEs for each graph listed above is determined. Starting from these sets, an extensive study of the properties of each topology has been performed in order to identify suitable candidates for the roles of UT functions.

Using again `Reduze2`, the systems of DEs in $(s;t)$ variables for the basis chosen above is determined.

In order to use Magnus series on easier DE systems, the DEs, originally dependent on $(s;t)$, are transformed into systems depending on $(s;z=t/s)$, and the s-dependance is eliminated through a redefinition of the MIs. ϵ singularities in MIs are then eliminated through a second redefinition of the functions of the basis; these results have been compared to the ones in [10, 11]. Magnus series expansion is performed, arriving to a canonical, and also fuchsian, form for the DE system in z. At this point, BCs are fixed considering that $z=-1$ (equivalent to $u=0$) is a pseudothreshold for the MIs and that for $z>0$ the solutions must be real; after that these requirements are satisfied, the series expansion in ϵ of just 1, 2 and 4 MIs is necessary, respectively for 1-loop, 2-loop and 3-loop problems. The solutions of the three systems are obtained up to order ϵ^7, one order higher than the results presented in the literature [10, 11].

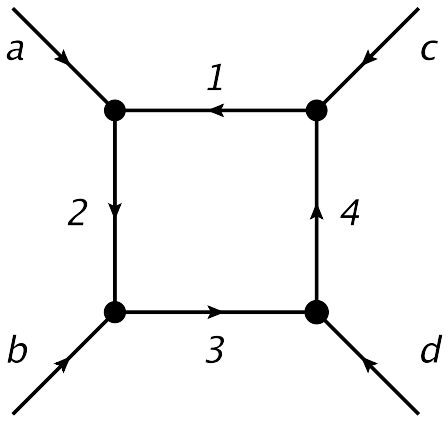
Regarding the method of eigenvalue deflation, it is applied to the most elementary version of MI basis, obtained eliminating dots if unnecessary for topology copies independence. After the elimination of non-fuchsian poles and the normalization of eigenvalues, canonical fuchsian basis for 1-loop and 2-loop problems are found, and compared with the ones obtained with Magnus series expansion through IBP-ids. For the 3-loop ladder problem, only the sub-system composed by 3-loop ladders is investigated and its canonical fuchsian form determined.

Chapter 6

Evaluation using Magnus series

Using Magnus series expansion method [13, 14] (and chapter 4), the 1-loop box, 2-loop ladder and 3-loop ladder Feynman integrals are evaluated [10, 11, 13, 14], up to order ϵ^7, corresponding to a transcendental weight of the HPLs of 7. For each one of these graphs, starting from the topologies determined using the Laporta algorithm [17, 46, 47], a basis of master integrals and its corresponding differential system in $z = t/s$ are determined. With the application of Magnus series, a transformation into a canonical fuchsian form is retrieved. Fixing the boundary conditions with analysis of the pseudothresholds and of the complex behaviour of the basis, a solution in terms of harmonic polylogarithms is found for all the masters, in terms of a small set of integrals evaluated at fixed point [11, 14]. The evaluated integrals are proven to be uniformly transcendent.

6.1 1-looop box



- External momenta ($p_i^2 = 0$)

  a p_1^μ
  b p_2^μ
  c p_3^μ
  d p_4^μ

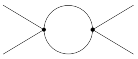
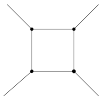
- Propagators

  1 $(k - p_1)^2$
  2 k^2
  3 $(k + p_2)^2$
  4 $(k + p_2 + p_4)^2$

THESIS_1L_MI shows the basis of MIs, and THESIS_1L_DE presents the DEs in $s$and t, both from `Reduze`. THESIS_1L_SOL contains the determination of the canonical form and of the BCs, while THESIS_1L_BC contains the construction of the series expansion of the solutions.

6.1.1 Laporta basis

`Reduze` determines for this topology the following subtopology tree (the notation for the subtopology is the same one used in `Reduze`, see [46, 47] for more details):

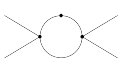
- `INT["nm1", 2, 5]` =  : t-bubble;
- `INT["nm1", 2, 10]` =  : s-bubble;
- `INT["nm1", 4, 15]` =  : 1-loop box.

6.1.2 UT basis

t and s-bubbles

The UT structure for these topologies is obtained dotting one of the propagators, as explained in section 3.3.2.


$$= \left(-\frac{t}{\mu^2}\right)^{-\epsilon} t^{-1} \frac{\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \Gamma(1-\epsilon)\Gamma(-\epsilon) \tag{6.1}$$

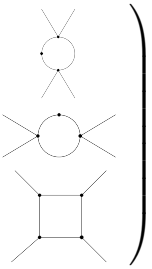

$$= \left(-\frac{s}{\mu^2}\right)^{-\epsilon} s^{-1} \frac{\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} \Gamma(1-\epsilon)\Gamma(-\epsilon) \tag{6.2}$$

1-loop box

The 1-loop box will be left unchanged.

The basis of MIs is then:

$$\mathcal{I}(s;t) = \left(\quad \right). \tag{6.3}$$



6.1.3 DEs reduction

$(s;t)$ **DEs**

The equation are determined using `Reduze`.

$$\frac{\partial}{\partial s}\mathcal{I}(s;t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\epsilon+1}{s} & 0 \\ \frac{2}{s(s+t)} & -\frac{2}{s(s+t)} & -\frac{s+t+t\epsilon}{s(s+t)} \end{pmatrix} \mathcal{I}(s;t) \tag{6.4}$$

$$\frac{\partial}{\partial t}\mathcal{I}(s;t) = \begin{pmatrix} -\frac{\epsilon+1}{t} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{2}{t(s+t)} & \frac{2}{t(s+t)} & -\frac{\epsilon s+s+t}{t(s+t)} \end{pmatrix} \mathcal{I}(s;t) \tag{6.5}$$

$(s;z)$ **DEs**

A different choice of variables for the problem consists of the couple $(s;z)$, where $z = t/s$. This new choice will cast the matrix of coefficients for the PDEs in s in a diagonal form, only depending on s. This change allows to find a redefinition of the MIs which eliminates the s dependence. In this way, the solution can be written int terms of HPLs in the z variable only.

First of all, consider the following change of coordinates:

$$s = s \qquad t = t(s;z) = sz. \tag{6.6}$$

In the new coordinates the couple of PDEs has the form:

$$\frac{\partial}{\partial s}\mathcal{I}(s;z) = \frac{\partial}{\partial s}\left[\mathcal{I}(s;t)\right]_{t=sz} = \left[\frac{\partial \mathcal{I}(s;t)}{\partial s} + \frac{\partial \mathcal{I}(s;t)}{\partial t} z\right]_{t=sz} = \begin{pmatrix} -\frac{\epsilon+1}{s} & 0 & 0 \\ 0 & -\frac{\epsilon+1}{s} & 0 \\ 0 & 0 & -\frac{\epsilon+2}{s} \end{pmatrix} \mathcal{I}(s;z); \tag{6.7}$$

$$\frac{\partial}{\partial z}\mathcal{I}(s;z) = \left[\frac{\partial \mathcal{I}(s;t)}{\partial t} s\right]_{t=sz} = \begin{pmatrix} -\frac{\epsilon+1}{z} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{2}{sz^2+sz} & \frac{2}{sz^2+sz} & -\frac{z+\epsilon+1}{z^2+z} \end{pmatrix} \mathcal{I}(s;z) \tag{6.8}$$

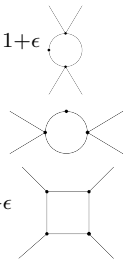
where $\mathcal{I}(s;t(s;z))$ is indicated as $\mathcal{I}(s;z)$, with a little abuse of notation.

It is now possible to solve the PDE system in s, obtaining solutions of the form:

$$\mathcal{I}(s;z) = \begin{pmatrix} (-s)^{-1-\epsilon} C_1(z) \\ (-s)^{-1-\epsilon} C_2(z) \\ (-s)^{-2-\epsilon} C_3(z) \end{pmatrix}. \tag{6.9}$$

Redefining now the MIs as

$$I(z) = S(s)\mathcal{I}(s;z) = \begin{pmatrix} (-s)^{1+\epsilon} \\ (-s)^{1+\epsilon} \\ (-s)^{2+\epsilon} \end{pmatrix} \tag{6.10}$$



the PDEs assume the form:

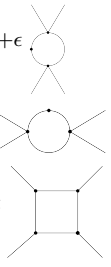
$$\frac{\partial I(z)}{\partial s} = S(s)\frac{\partial \mathcal{I}(s;z)}{\partial s}S^{-1}(s) + \frac{\mathrm{d}S(s)}{\mathrm{d}s} = \mathbb{O}, \tag{6.11}$$

$$\frac{\partial I(z)}{\partial s} = S(s)\frac{\partial \mathcal{I}(s;z)}{\partial z}S^{-1}(s) = \begin{pmatrix} -\frac{\epsilon+1}{z} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2}{z(z+1)} & -\frac{2}{z(z+1)} & -\frac{z+\epsilon+1}{z(z+1)} \end{pmatrix}. \tag{6.12}$$

Elimination of ϵ poles

The next step consists of the elimination of the poles in ϵ related to the MIs. It is known from section 3.3.2 that bubbles have simple poles in ϵ, so a prefactor ϵ is applied to them. For the box integral, a good way to find the prefactor is to look at the PDE in z (from now on, *the* DE of the problem): since there is an ϵ factor only in the box coefficient for the box DE, it is reasonable to think that the box has a pole of order 2 in ϵ. Moreover, multiplying the box by ϵ^2, the DE has ϵ-independent terms only on the diagonal, greatly simplifying the construction of the transformation to the canonical form. The basis will then assume the following form:

$$\begin{pmatrix} \epsilon(-s)^{1+\epsilon} \\ \epsilon(-s)^{1+\epsilon} \\ \epsilon^2(-s)^{2+\epsilon} \end{pmatrix}, \tag{6.13}$$



with matrix of coefficients:

$$\tilde{A}(z;\epsilon) = \begin{pmatrix} -\frac{\epsilon+1}{z} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2\epsilon}{z(z+1)} & -\frac{2\epsilon}{z(z+1)} & -\frac{z+\epsilon+1}{z(z+1)} \end{pmatrix}. \tag{6.14}$$

Canonical fuchsian form

It is now possible to use Magnus series expansion to pass in canonical form. The matrix of coefficients can be separated as:

$$\tilde{A}(z;\epsilon) = A_0(z) + \epsilon A_1(z), \tag{6.15}$$

with

$$A_0(z) = \begin{pmatrix} -\frac{1}{z} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{z} \end{pmatrix}, \qquad A_1(z) = \begin{pmatrix} -\frac{1}{z} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2}{z(z+1)} & -\frac{2}{z(z+1)} & -\frac{1}{z(z+1)} \end{pmatrix} \tag{6.16}$$

Since $A_0(z)$ has a diagonal structure, the Magnus series is truncated at first term, resulting in a transformation matrix:

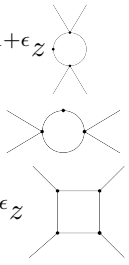
$$B(z) = \mathrm{e}^{\int A_0(z)\,\mathrm{d}z} = \begin{pmatrix} \frac{1}{z} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{z} \end{pmatrix}, \tag{6.17}$$

which gives the canonical matrix of coefficients:

$$\begin{aligned} A(z;\epsilon) = B^\dagger(z)\tilde{A}(z;\epsilon)B(z) - B^\dagger\frac{\mathrm{d}B(z)}{\mathrm{d}z} &= \begin{pmatrix} -\frac{\epsilon}{z} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2\epsilon}{z} - \frac{2\epsilon}{z+1} & -\frac{2\epsilon}{z+1} & \frac{\epsilon}{z+1} - \frac{\epsilon}{z} \end{pmatrix} = \\ &= \epsilon\left[\frac{1}{z}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{pmatrix} + \frac{1}{1+z}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & -2 & 1 \end{pmatrix}\right]. \end{aligned} \tag{6.18}$$

Note that this system is also fuchsian, so the choice of MI for the 1-loop box topology is not negated. Moreover, the s-bubble does not evolve, as it should be once the dependance from s has been eliminated. The definitive basis of functions for the 1-loop box problem is then:

$$\mathbf{I}(z) = \begin{pmatrix} \epsilon(-s)^{1+\epsilon}z \\ \epsilon(-s)^{1+\epsilon} \\ \epsilon^2(-s)^{2+\epsilon}z \end{pmatrix}. \tag{6.19}$$



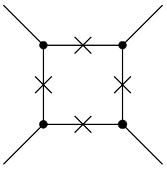
6.1.4 BCs

Analysis of the divergences

To fix part of the BCs for the problem, thresholds for a given FI must be found, in order to identify the pseudothresholds, and to eliminate these last ones from the solutions.

To determine the singular points of the FI, a quadruple cut is performed, as explained in section 2.3.2.

For the case under consideration, only maximal cut of the 1-loop box have to be considered. Cutting all the propagators, the result has the form



$$\propto \frac{1}{st}, \tag{6.20}$$

so the divergences occur in $s = 0$ or in $t = 0$. In particular, in $u = 0 \Rightarrow s = -t$ all the MIs are regular.

In order to identify the solution of the problem, two conditions are considered:

- finiteness of the solutions in $z = -1$;
- reality of the solutions for $z \geq 0$.

Since $u = 0$ is not a singular point of the system, it is possible to multiply both members of the DE system and take the limit $z \to -1$, obtaining:

$$0 = \lim_{z\to-1}\left[(z+1)\frac{\mathrm{d}\mathbf{I}(z;\epsilon)}{\mathrm{d}z}\right] = \lim_{z\to-1}\left[(z+1)A(z;\epsilon)\mathbf{I}(z;\epsilon)\right]; \tag{6.21}$$

solving the resulting purely algebraic equations, it is possible to find relations among the MIs in $z = 1$.

At this point, expressing the solution in $z = -1$ as $\mathbf{I}(-1;\epsilon) = \mathcal{A}(-1;\epsilon)\mathbf{I}_0(\epsilon)$, where $\mathcal{A}(z;\epsilon)$ is the Dyson series of $A(z;\epsilon)$ in $z = -1$,it is possible to extract relations among $\mathbf{I}_0^{(i)}$ order by order in ϵ. Starting from the expressions of $\mathbf{I}(z;\epsilon)$ order by order

$$\mathbf{I}^{(0)}(z) = \mathbf{I}_0^{(0)}, \tag{6.22}$$

$$\mathbf{I}^{(1)}(z) = \mathbf{I}_0^{(1)} + \mathcal{A}^{(1)}(z)\mathbf{I}_0^{(0)}, \tag{6.23}$$

$$\mathbf{I}^{(2)}(z) = \mathbf{I}_0^{(2)} + \mathcal{A}^{(1)}(z)\mathbf{I}_0^{(1)} + \mathcal{A}^{(2)}(z)\mathbf{I}_0^{(0)}, \tag{6.24}$$

$$\dots$$

and substituting them into the limit, one obtains:

$$\lim_{z\to-1}\left[(z+1)A(z)\right]\mathbf{I}_0 = 0, \tag{6.25}$$

$$\lim_{z\to-1}\left[(z+1)A(z)\right]\mathbf{I}_0^{(1)} + \lim_{z\to-1}\left[(z+1)A(z)\mathcal{A}^{(1)}(z)\right]\mathbf{I}_0^{(0)} = 0, \tag{6.26}$$

$$\lim_{z\to-1}\left[(z+1)A(z)\right]\mathbf{I}_0^{(2)} + \lim_{z\to-1}\left[(z+1)A(z)\mathcal{A}^{(1)}(z)\right]\mathbf{I}_0^{(1)} + \lim_{z\to-1}\left[(z+1)A(z)\mathcal{A}^{(2)}(z)\right]\mathbf{I}_0^{(0)} = 0, \tag{6.27}$$

$$\dots$$

These relations must be satisfied at all orders in ϵ. Due to the presence of possible diverging terms generated by HPLs with kernel $F(1;z)$, series expansions in z around $z_0 = -1$ of the HPLs are evaluated and substituted in the expressions, imposing that both the finite and the divergent terms are 0 in the expressions. Note that, according to the expansion enlisted in section A.2, divergent terms in $z+1$ have only form $\log^n(z+1)$, thanks to the fact that all the exponents of the $(z-1)^n$ terms are positive and $(z+1)\log^n(z+1)$ is 0 in $z \to -1$ limit. With this request, at order ϵ^n relations fixing coefficients of order n are determined.

Notice that this approach produces the same results of imposing the solutions $\mathbf{I}(z;\epsilon)$ to be finite in $z = -1$: multiplying by $z+1$ and taking the limit $z \to -1$ only coefficients of divergent parts remain in the relations, and are imposed to be equal to zero. In fact, the first method works with coefficients of the terms $1/(z+1)$ set to zero, while the second on the coefficients of the terms $\log(1+z)$ imposed to be zero, but since the integral of the matrix of singularities in $z = -1$ is the matrix of $\log(z+1)$, the two paths lead to the same results, but at different orders: while the first method determines at order ϵ^n conditions for $\mathbf{I}_0^{(n)}$, the second needs order ϵ^{n+1} for the same results; since series expansions and HPLs are available only up to order ϵ^8, gaining an order is quite important.

In expressions related to order n it has been found that, investigating for null coefficients in lower orders, no additional relations are generated, so to find all the possible relations for a given order n, ϵ^n terms are sufficient. One may expect that terms of the form $\log(z+1)$ will appear at order ϵ^n, $n \geq 1$, but working on systems in which previous order relations have been substituted eliminates such occurrence. Consider, as example, the order ϵ^1:

$$\mathbf{I}^{(1)}(z) = \int A(\tau_1)\mathbf{I}_0^{(0)}\,\mathrm{d}\tau_1 + \mathbf{I}_0^{(1)}; \tag{6.28}$$

substituting the relation for order 0, all the terms containing $\log(1+z)$ in $\int A(\tau_1)\mathbf{I}_0^{(0)}\,\mathrm{d}\tau_1$ are suppressed, avoiding the formation of terms $\log^2(1+z)$ from integration. Moreover, after the substitution of the internal relations for $\mathbf{I}_0^{(0)}$, the expression is in general different from the one for ϵ^0, resulting in possibly different constraints for $\mathbf{I}_0^{(1)}$, with respect to $\mathbf{I}_0^{(0)}$ (usually, from ϵ^1, imaginary terms may be present, due to $\mathcal{A}^{(1)}(-1)$).

Not all the equations are linearly independent: in particular, for the 1-loop box problem just one condition can be extracted at each order from the elimination of the pseudothreshold.

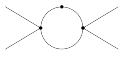
The second condition that can be applied is the fact that all the MIs must be real for $z \geq 0$. This condition emerges from the fact that the interval $z \in [0;1]$ corresponds to an Euclidean region for the MIs, therefore they have to be real (as explained in [10, 11]). This condition can be applied to the relations obtained from the finiteness in $z = -1$: initial conditions are evaluated in $z = 0$, and imposing that all the coefficients of the series expansion of each MI have to be real for $z > 0$, also $\mathbf{I}_0(\epsilon) = \mathbf{I}(0;\epsilon)$ has to be real. It follows that the relations among the terms of $\mathbf{I}_0(\epsilon)$ must be set to zero, providing a new and independent set of conditions. The imaginary parts come only from the terms $\log^n z$, evaluated in $z = -1$, and due to the fact that the imaginary unit i is always multiplied by π, factors (explicitly or in the form of $\zeta(n)$), the relations for $\mathbf{I}_0^{(n)}$ will be determined at one order higher in ϵ than the previous ones; so at order ϵ^{n+1}.

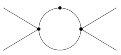
Also in this case, investigating terms of order less than $n-1$ (or $n-2$, respectively) does not provide more information; the proof of this follows the same steps to the one illustrated before for finiteness in $z=-1$.

In this case, all the previous relations from the appropriate order have linearly independent imaginary parts, so as much new relations as the previous ones are retrieved: in the 1-loop box case, one relation.

Notice that all the constraint found solving the relations obtained here have definite weight, equal to the one of the $\mathbf{I}_0^{(n)}$ term with maximum weight present in them.

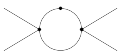
Thanks to the previous analysis, the actual number of known MIs necessary for the evaluation is reduced to 1. Solving the relations, the UT s-bubble turns out to be the *master MI* (MMI) for this problem:

$\left(\epsilon(-s)^{1+\epsilon}\right.$  $\left.\right).$ (6.29)

To simplify the problem, a suitable integration measure is chosen: this measure normalizes the value of $\epsilon(-s)^{1+\epsilon}$  to 1, so

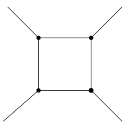
$$\mathrm{d}_B^D k = \frac{(\mu^2)^\epsilon}{\mathrm{i}\pi^{2-\epsilon}} \frac{\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)} \mathrm{d}^D k. \tag{6.30}$$

The MMI assumes the form:

$\epsilon^2(-s)^{2+2\epsilon}$  $= 1.$ (6.31)

It is only at this point of the analysis that it is truly possible to say that the chosen MIs are pure UT functions: their coefficients are determined through relations with weight equal to the one of the term determined, and the parent function is a pure UT FI. Here the proof of pure UT nature of the function concludes, and the results obtained here are pure UT results, even if not all expression are known and have been investigated separately (like polygamma functions $\psi^{(n)}(a)$).

6.1.5 1-loop box graph

$\epsilon^2(-s)^{2+\epsilon} z$  $= \sum_{i=0}^{7} \epsilon^i f_i(z) + o_0\left(\epsilon^7\right);$ (6.32)

- $f_0 = 4$;
- $f_1(z) = -2H(0;z)$;
- $f_2(z) = -\pi^2$;
- $f_3(z) = (1/3)H^3(0;z) - H(-1;z)[\pi^2 + H^2(0;z)] + H(0;z)[\pi^2 + 2H(0;-1;z)] - 2[H(0;0;-1;z) + \zeta(3)]$;

- $f_4(z) = -\pi^4/15+2H(0;0;-1;-1;z)+2H(0;0;0;-1;z)+1/6(-3H(-1;z)^2(\pi^2+H(0;z)^2)-H(0;z)(H(0;z)^3+3H(0;z)(\pi^2+2H(0;-1;z))+12(H(0;-1;-1;z)-\zeta(3)))+2H(-1;z)(2H(0;z)^3+3H(0;z)(\pi^2+2H(0;-1;z))-6(H(0;0;-1;z)+\zeta(3))));$

- $f_5(z) = +1/60(3H(0;z)^5 - 10H(-1;z)^3(\pi^2 + H(0;z)^2) + 10H(0;z)^3(\pi^2 + 2H(0;-1;z)) + 4H(0;z)(\pi^4+30H(0;-1;-1;-1;z))-H(-1;z)(4(\pi^4-30H(0;0;-1;-1;z)-30H(0;0;0;-1;z))+15H(0;z)(H(0;z)^3 + 2H(0;z)(\pi^2 + 2H(0;-1;z)) + 8H(0;-1;-1;z) - 8\zeta(3))) + 60H(0;z)^2(H(0;-1;-1;z)-\zeta(3))+10H(-1;z)^2(2H(0;z)^3+3H(0;z)(\pi^2+2H(0;-1;z))-6(H(0;0;-1;z)+\zeta(3)))-120(H(0;0;-1;-1;-1;z)+H(0;0;0;-1;-1;z)+H(0;0;0;0;-1;z)+\zeta(5)));$

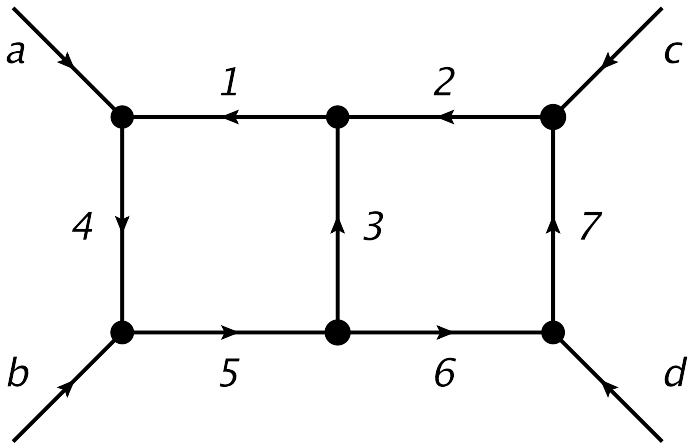
- $f_6(z) = -((2\pi^6)/315) + 2H(0;0;-1;-1;-1;-1;z) + 2H(0;0;0;-1;-1;-1;z) + 2H(0;0;0;0;-1;-1;z) + 2H(0;0;0;0;0;-1;z) + 1/360 (-15H(-1;z)^4(\pi^2+H(0;z)^2)-3H(-1;z)^2(4(\pi^4-30H(0;0;-1;-1;z)-30H(0;0;0;-1;z))+15H(0;z)(H(0;z)^3 + 2H(0;z)(\pi^2 + 2H(0;-1;z)) + 8H(0;-1;-1;z) - 8\zeta(3))) + 20H(-1;z)^3(2H(0;z)^3+3H(0;z)(\pi^2+2H(0;-1;z))-6(H(0;0;-1;z)+\zeta(3)))-H(0;z)(4H(0;z)^5+15H(0;z)^3(\pi^2+2H(0;-1;z))+12H(0;z)(\pi^4+30H(0;-1;-1;-1;z))+120H(0;z)^2(H(0;-1;-1;z)-\zeta(3))+720(H(0;-1;-1;-1;-1;z)-\zeta(5)))+12H(-1;z)(2H(0;z)^5+5H(0;z)^3(\pi^2+2H(0;-1;z))+2H(0;z)(\pi^4+30H(0;-1;-1;-1;z))+30H(0;z)^2(H(0;-1;-1;z)-\zeta(3))-60(H(0;0;-1;-1;-1;z)+H(0;0;0;-1;-1;z) + H(0;0;0;0;-1;z) + \zeta(5))));$

- $f_7(z) = 1/2520(5H(0;z)^7 - 21H(-1;z)^5(\pi^2 + H(0;z)^2) + 21H(0;z)^5(\pi^2 + 2H(0;-1;z)) + 28H(0;z)^3(\pi^4 + 30H(0;-1;-1;-1;z)) + 16H(0;z)(\pi^6 + 315H(0;-1;-1;-1;-1;-1;z)) - 7H(-1;z)^3(4(\pi^4-30H(0;0;-1;-1;z)-30H(0;0;0;-1;z))+15H(0;z)(H(0;z)^3+2H(0;z)(\pi^2+2H(0;-1;z)) + 8H(0;-1;-1;z) - 8\zeta(3))) + 210H(0;z)^4(H(0;-1;-1;z) - \zeta(3)) + 35H(-1;z)^4(2H(0;z)^3+3H(0;z)(\pi^2+2H(0;-1;z))-6(H(0;0;-1;z)+\zeta(3)))-H(-1;z)(16(\pi^6-315H(0;0;-1;-1;-1;-1;z) - 315H(0;0;0;-1;-1;-1;z) - 315H(0;0;0;0;-1;-1;z) - 315H(0;0;0;0;0;-1;z))+7H(0;z)(H(0;z)(12(\pi^4+30H(0;-1;-1;-1;z))+5H(0;z)(H(0;z)^3+3H(0;z)(\pi^2 + 2H(0;-1;z)) + 24(H(0;-1;-1;z) - \zeta(3)))) + 720(H(0;-1;-1;-1;-1;z) - \zeta(5))))+2520H(0;z)^2(H(0;-1;-1;-1;-1;z)-\zeta(5))+42H(-1;z)^2(2H(0;z)^5+5H(0;z)^3(\pi^2+2H(0;-1;z))+2H(0;z)(\pi^4+30H(0;-1;-1;-1;z))+30H(0;z)^2(H(0;-1;-1;z)-\zeta(3))-60(H(0;0;-1;-1;-1;z) + H(0;0;0;-1;-1;z) + H(0;0;0;0;-1;z) + \zeta(5))) - 5040(H(0;0;-1;-1;-1;-1;-1;z)+H(0;0;0;-1;-1;-1;-1;z)+H(0;0;0;0;-1;-1;-1;z)+H(0;0;0;0;0;-1;-1;z) + H(0;0;0;0;0;0;-1;z) + \zeta(7)));$

As expected, the 1-loop box has a ϵ^2 pole.

The results for the other graphs of the basis are shown in THESIS_1L_BC.

6.2 2-loop ladder



- External momenta ($p_i^2 = 0$)

  a p_1^μ
  b p_2^μ
  c p_3^μ
  d p_4^μ

- Propagators

  1 k_1^2
  2 k_2^2
  3 $(k_1 - k_2)^2$
  4 $(k_1 + p_1)^2$
  5 $(k_1 + p_1 + p_2)^2$
  6 $(k_2 + p_1 + p_2)^2$
  7 $(k_2 - p_3)^2$

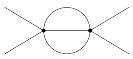
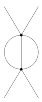
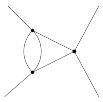
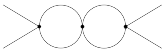
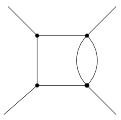
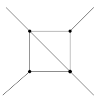
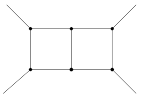
- Auxiliary propagators

  8 $(k_1 - p_3)^2$
  9 $(k_2 + p_1)^2$

THESIS_2L_MI shows the basis of MIs, and THESIS_2L_DE presents the DEs in $s$and t, both from `Reduze`. THESIS_2L_SOL contains the determination of the canonical form and of the BCs, while THESIS_2L_BC contains the construction of the series expansion of the solutions.

6.2.1 Laporta basis

`Reduze` determines for this topology the following subtopology tree (the notation for the subtopology is the same one used in `Reduze`, see [46, 47] for more details):

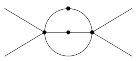
- `INT["nm2", 3, 22]` =  : s-sunset;
- `INT["nm2", 3, 76]` =  : t-sunset;
- `INT["nm2", 3, 76]` =  : s-triangle;
- `INT["nm2", 4, 51]` =  : double s-bubble;
- `INT["nm2", 5, 93]` =  : 2-loop box;
- `INT["nm2", 5, 94]` =  : N-box;
- `INT["nm2", 7, 127]` =  , considered *twice*: 2-loop ladder.

There are then seven different topologies, for a total of eight MIs.

6.2.2 UT basis

t and s-sunsets

The UT structure for these topologies is obtained dotting two of the propagators, as explained in section 3.3.2.


$$= \left(-\frac{s}{\mu^2}\right)^{-2\epsilon} s^{-1} \frac{\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)} \Gamma(1-\epsilon)\Gamma^2(-\epsilon) \tag{6.33}$$


$$= \left(-\frac{t}{\mu^2}\right)^{-2\epsilon} t^{-1} \frac{\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)} \Gamma(1-\epsilon)\Gamma^2(-\epsilon) \tag{6.34}$$

s-triangle

The simple triangle is proven to be UT. Using proposition 9 it is possible to substitute one of its propagators with a UT bubble. The UT MI related to this topology is then obtained dotting one of the bubble propagators.

double s-bubble

As seen in the previous chapter, a bubble FI is UT when one of its propagators is dotted. This topology is a product of two bubbles, so dotting one propagator for each of them a UT function is obtained.

2-loop box

Using proposition 9 it is sufficient to dot one of the bubble propagators.

N-box

The N-box is UT. The proof has been produced by Amedeo Primo.

Consider the explicit form of the FI:

$$\int\int \frac{\mathrm{d}k_2\mathrm{d}k_1}{(k_2)^2(k_2-p_3)^2(k_1-k_2)^2(k_2+p_{12})^2(k_1+p_1)^2}, \tag{6.35}$$

where $p_{12}=p_1+p_2$. Introducing a Feynman parameter to sum up $(k_2)^2$ and $(k_2-p_3)^2$, the expression $[(k_2-p_3)^2\alpha+(k_2)^2(1-\alpha)]^2(k_1-k_2)^2$ represents a UT bubble with incoming momentum $(k_1-\alpha p_3)^\mu$. It is then possible to integrate over k_2^μ, obtaining:

$$\frac{\Gamma(1+\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\int_0^1\int \frac{\mathrm{d}k_1\mathrm{d}\alpha}{[(k_1-\alpha p_3)^2]^{1+\epsilon}(k_1+p_{12})^2(k_1+p_1)^2}. \tag{6.36}$$

It is now possible to shift the loop variable as $k_1'^\mu=k_1^\mu+p_{12}^\mu$, and introduce two additional Feynman parameters x and y. Rearranging the terms it is now possible to reduce the FI to a tadpole:

$$\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\int_0^1\int_0^1\int_0^{1-x} \frac{\mathrm{d}y\mathrm{d}x\mathrm{d}\alpha}{[s(1-\alpha)(1-x)+[t+u(1-\alpha)]y]^{1+2\epsilon}} \tag{6.37}$$

where $s=(p_1+p_2)^2$, $t=(p_1+p_3)^2$ and $u=(p_1+p_4)^2$ with $s+t+u=0$.

Performing the change of variables

$$\begin{aligned} x &= z_1,\\ y &= z_2(1-z_1),\\ \alpha &= z_3, \end{aligned} \tag{6.38}$$

all the integration intervals are set to $[0;1]$. In this way:

$$\frac{\Gamma(1+2\epsilon)\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\int_0^1 \frac{\mathrm{d}z_1}{z_1^\epsilon(1-z_1)^{1+2\epsilon}}\int_0^1\int_0^1 \frac{\mathrm{d}z_3\mathrm{d}z_2}{[sz_3+(t+uz_3)z_2]^{1+2\epsilon}}. \tag{6.39}$$

The first integral is an Euler's beta function, namely $\int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t=B(x;y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. Integrating over z_3:

$$-\frac{\Gamma^2(1-\epsilon)\Gamma(-\epsilon)\Gamma(-2\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1-3\epsilon)}\int_0^1 \frac{s^{-2\epsilon}(1-z_2)^{-2\epsilon}-t^{-2\epsilon}z_2^{-2\epsilon}}{s-(s+t)z_2}\,\mathrm{d}z_2. \tag{6.40}$$

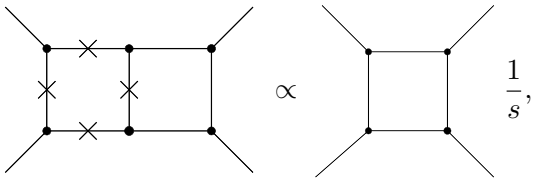
Defining $X = s - (s+t)z_2$, the integral is rewritten as:

$$\frac{\Gamma^2(1-\epsilon)\Gamma(-\epsilon)\Gamma(-2\epsilon)}{\Gamma(1-2\epsilon)\Gamma(1-3\epsilon)}\frac{s^{-2\epsilon}}{s+t}\int_{X=s}^{X=t}\left[(1-z_2)^{-2\epsilon} - \left(\frac{t}{s}\right)^{-2\epsilon} z_2^{-2\epsilon}\right] \mathrm{d}\log X \tag{6.41}$$

since the prefactor is UT (even if not pure UT), and the integrand has definite weight $w = n+1$ at each term of its series expansion in ϵ^n (the integral has the structure shown in definition 8), the function is UT. Note also that the factor $s+t$ present in the denominator of the prefactor will be the coefficient arising from Magnus series expansion.

2-loop ladder

Performing a quadruple cut on one of the loop of the 2-loop ladder topology, a function proportional to a 1-loop box is retrieved. Proceeding as shown in section 3.3.3,

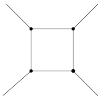


$$\propto \frac{1}{s}, \tag{6.42}$$

since, cutting the first box, the t channel is nothing else than $(p_1+k_2)^2$.

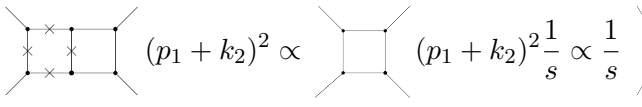
Since a maximal cut on one of the loops of the 2-loop ladder topology results in a box multiplied by s, it is reasonable to use, as one of the MIs for this topology, the simple 2-loop ladder FI.

For the other copy, power counting is useful: since the simple 2-loop ladder has mass dimension $[m]^{-6}$, dots would not be a good choice. Instead, a numerator will increase the mass dimension to $[m]^{-4}$, as it usually is for 4-point functions. To choose the appropriate irreducible scala product (in a massless case scalar product or numerator are truly synonyms), it is possible to perform a quadruple cut on one of the loops, and look for a numerator that simplifies the FI, reducing it to a known UT function. So, cutting for example the right loop, as for the analysis of the divergences, one obtains:



$$\frac{1}{s} \propto \frac{1}{s}\int \frac{\mathrm{d}k_2}{(k_2)^2(k_2+p_1)^2(k_2+p_1+p_2)^2(k_2-p_3)^2}. \tag{6.43}$$

If a denominator is removed, a UT triangle FI is retrieved. Since numerators can only show irreducible scalar products, only $(k_2+p_1)^2$ can be used, resulting in:



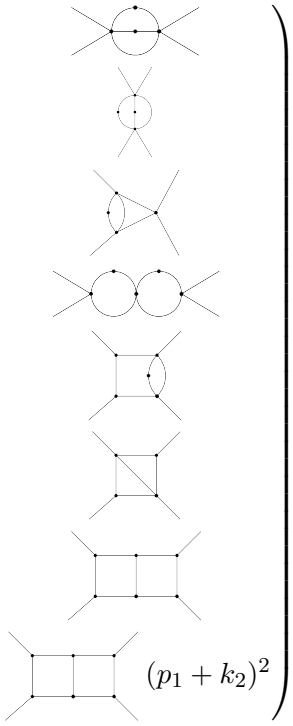
$$(p_1+k_2)^2 \propto (p_1+k_2)^2\frac{1}{s} \propto \frac{1}{s}, \tag{6.44}$$

where the proportionality indicates that the members are equal up to a numerical constant.

Also the auxiliary propagator $(k_1-p_3)^2$ can be used for the 2-loop ladder (it would have been present if maximal cut had been performed on the loop on the right).

The basis of MIs is then:

$$\mathcal{I}(s;t) = \left(\right. \quad \left. (p_1+k_2)^2 \right) . \tag{6.45}$$



6.2.3 DEs reduction

$(s;t)$ **DEs**

The equation are determined using `Reduze`.

$$\frac{\partial}{\partial s}\mathcal{I}(s;t) = A_s(s;t)\mathcal{I}(s;t), \tag{6.46}$$

$$\frac{\partial}{\partial t}\mathcal{I}(s;t) = A_t(s;t)\mathcal{I}(s;t); \tag{6.47}$$

$$A_s = \begin{pmatrix} -\frac{2\epsilon+1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2\epsilon+1}{s} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2(\epsilon+1)}{s} & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2s(s+t)} & \frac{3\epsilon}{s(s+t)} & 0 & -\frac{s+t+t\epsilon}{s(s+t)} & 0 & 0 & 0 \\ -\frac{1}{2(s+t)\epsilon} & \frac{t}{2s(s+t)\epsilon} & 0 & 0 & 0 & -\frac{s+2t\epsilon}{s(s+t)} & 0 & 0 \\ \frac{3(s-t)}{s^2t(s+t)\epsilon} & \frac{3}{s^2(s+t)\epsilon} & -\frac{6}{s^2(s+t)} & \frac{2}{s(s+t)\epsilon} & \frac{4}{s(s+t)} & -\frac{12\epsilon}{s^2t} & -\frac{2(s+t+t\epsilon)}{s(s+t)} & \frac{2\epsilon}{s(s+t)} \\ \frac{3}{s(s+t)\epsilon} & \frac{9t}{2s^2(s+t)\epsilon} & -\frac{3}{s(s+t)} & \frac{s+2t}{s(s+t)\epsilon} & \frac{4t}{s(s+t)} & -\frac{18\epsilon}{s^2} & \frac{t\epsilon}{s+t} & -\frac{3\epsilon s+2s+2t+2t\epsilon}{s(s+t)} \end{pmatrix}, \tag{6.48}$$

$$A_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2\epsilon+1}{t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2t(s+t)} & -\frac{3\epsilon}{t(s+t)} & 0 & -\frac{2\epsilon s+s+t+t\epsilon}{t(s+t)} & 0 & 0 & 0 \\ \frac{s}{2t(s+t)\epsilon} & -\frac{1}{2(s+t)\epsilon} & 0 & 0 & 0 & -\frac{t+2s\epsilon}{t(s+t)} & 0 & 0 \\ -\frac{3(s-t)}{st^2(s+t)\epsilon} & -\frac{3}{st(s+t)\epsilon} & \frac{6}{st(s+t)} & -\frac{2}{t(s+t)\epsilon} & -\frac{4}{t(s+t)} & \frac{12\epsilon}{st^2} & -\frac{2\epsilon s+s+t}{t(s+t)} & -\frac{2\epsilon}{t(s+t)} \\ -\frac{3}{t(s+t)\epsilon} & -\frac{9}{2s(s+t)\epsilon} & \frac{3}{t(s+t)} & -\frac{s+2t}{t(s+t)\epsilon} & -\frac{4}{s+t} & \frac{18\epsilon}{st} & -\frac{s\epsilon}{s+t} & \frac{s\epsilon}{t(s+t)} \end{pmatrix}. \quad (6.49)$$

$(s;z)$ DEs

As explained in the previous chapter:

$$\frac{\partial}{\partial s}\mathcal{I}(s;z) = A_s(s;z)\mathcal{I}(s;z) \quad (6.50)$$

$$\frac{\partial}{\partial z}\mathcal{I}(s;z) = A_z(s;z)\mathcal{I}(s;z) \quad (6.51)$$

$$A_s = \begin{pmatrix} -\frac{2\epsilon+1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2\epsilon+1}{s} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2\epsilon+1}{s} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2(\epsilon+1)}{s} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{2(\epsilon+1)}{s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{2\epsilon+1}{s} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2\epsilon+3}{s} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2(\epsilon+1)}{s} \end{pmatrix}, \quad (6.52)$$

$$A_z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2\epsilon+1}{z} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2sz(z+1)} & -\frac{3\epsilon}{sz(z+1)} & 0 & -\frac{\epsilon z+z+2\epsilon+1}{z(z+1)} & 0 & 0 & 0 \\ \frac{1}{2z(z+1)\epsilon} & -\frac{1}{2(z+1)\epsilon} & 0 & 0 & 0 & -\frac{z+2\epsilon}{z(z+1)} & 0 & 0 \\ \frac{3(z-1)}{s^2z^2(z+1)\epsilon} & -\frac{3}{s^2z(z+1)\epsilon} & \frac{6}{s^2z(z+1)} & -\frac{2}{sz(z+1)\epsilon} & -\frac{4}{sz(z+1)} & \frac{12\epsilon}{s^2z^2} & -\frac{z+2\epsilon+1}{z(z+1)} & -\frac{2\epsilon}{sz(z+1)} \\ -\frac{3}{sz(z+1)\epsilon} & -\frac{9}{2s(z+1)\epsilon} & \frac{3}{sz(z+1)} & -\frac{2z+1}{z(z+1)\epsilon} & -\frac{4}{z+1} & \frac{18\epsilon}{sz} & -\frac{s\epsilon}{z+1} & \frac{\epsilon}{z(z+1)} \end{pmatrix}, \quad (6.53)$$

where $\mathcal{I}(s;t(s;z))$ is indicated with $\mathcal{I}(s;z)$, with a little abuse of notation.

Redefining now the MIs to eliminate s dependence as

$$I(z) = \left(\begin{array}{l} (-s)^{1+2\epsilon} \\ (-s)^{1+2\epsilon} \\ (-s)^{1+2\epsilon} \\ (-s)^{2+2\epsilon} \\ (-s)^{2+2\epsilon} \\ (-s)^{1+2\epsilon} \\ (-s)^{3+2\epsilon} \\ (-s)^{2+2\epsilon} \qquad (p_1+k_2)^2 \end{array} \right). \tag{6.54}$$

the PDEs assume the form:

$$A_s = \mathbb{O}, \tag{6.55}$$

$$A_z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2\epsilon+1}{z} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2z(z+1)} & \frac{3\epsilon}{z(z+1)} & 0 & -\frac{\epsilon z+z+2\epsilon+1}{z(z+1)} & 0 & 0 & 0 \\ \frac{1}{2z(z+1)\epsilon} & -\frac{1}{2(z+1)\epsilon} & 0 & 0 & 0 & -\frac{z+2\epsilon}{z(z+1)} & 0 & 0 \\ \frac{3(z-1)}{z^2(z+1)\epsilon} & -\frac{3}{z(z+1)\epsilon} & \frac{6}{z(z+1)} & \frac{2}{z(z+1)\epsilon} & \frac{4}{z(z+1)} & \frac{12\epsilon}{z^2} & -\frac{z+2\epsilon+1}{z(z+1)} & \frac{2\epsilon}{z(z+1)} \\ \frac{3}{z(z+1)\epsilon} & \frac{9}{2(z+1)\epsilon} & -\frac{3}{z(z+1)} & -\frac{2z+1}{z(z+1)\epsilon} & -\frac{4}{z+1} & -\frac{18\epsilon}{z} & \frac{\epsilon}{z+1} & \frac{\epsilon}{z(z+1)} \end{pmatrix}. \tag{6.56}$$

Note how MIs of the same topology generates block structures, instead of triangular forms.

Elimination of ϵ poles

Knowing that sunsets and double bubbles have ϵ^{-2} poles, and fixing the other coefficients in order to restrict terms independent from ϵ in the diagonal of the matrix, basis and DE become:

$$\left(\begin{array}{l} \epsilon^2(-s)^{1+2\epsilon} \\ \epsilon^2(-s)^{1+2\epsilon} \\ \epsilon^3(-s)^{1+2\epsilon} \\ \epsilon^2(-s)^{2+2\epsilon} \\ \epsilon^3(-s)^{2+2\epsilon} \\ \epsilon^4(-s)^{1+2\epsilon} \\ \epsilon^4(-s)^{3+2\epsilon} \\ \epsilon^4(-s)^{2+2\epsilon} \qquad (p_1+k_2)^2 \end{array}\right), \tag{6.57}$$

$$\tilde{A}(z;\epsilon) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2\epsilon+1}{z} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3\epsilon}{2z(z+1)} & \frac{3\epsilon}{z(z+1)} & 0 & -\frac{\epsilon z+z+2\epsilon+1}{z(z+1)} & 0 & 0 & 0 \\ \frac{\epsilon}{2z(z+1)} & -\frac{\epsilon}{2(z+1)} & 0 & 0 & 0 & -\frac{z+2\epsilon}{z(z+1)} & 0 & 0 \\ \frac{3(z-1)\epsilon}{z^2(z+1)} & -\frac{3\epsilon}{z(z+1)} & \frac{6\epsilon}{z(z+1)} & \frac{2\epsilon}{z(z+1)} & \frac{4\epsilon}{z(z+1)} & \frac{12\epsilon}{z^2} & -\frac{z+2\epsilon+1}{z(z+1)} & \frac{2\epsilon}{z(z+1)} \\ \frac{3\epsilon}{z(z+1)} & \frac{9\epsilon}{2(z+1)} & -\frac{3\epsilon}{z(z+1)} & -\frac{(2z+1)\epsilon}{z(z+1)} & -\frac{4\epsilon}{z+1} & -\frac{18\epsilon}{z} & \frac{\epsilon}{z+1} & \frac{\epsilon}{z(z+1)} \end{pmatrix}. \tag{6.58}$$

Canonical fuchsian form

It is now possible to use Magnus series expansion to pass in canonical form. Rewriting $\tilde{A}(z;\epsilon)$ as $\tilde{A}(z;\epsilon) = A_0(z) + \epsilon A_1(z)$, the matrix of the terms not depending on ϵ is

$$A_0(z) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{z} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{z+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{z} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{6.59}$$

this matrix is diagonal, so the Magnus series is truncated at first term, resulting in a matrix for the transformation

$$B(z) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{z} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{z+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{z} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.60}$$

After the transformation, the canonical matrix of coefficients $A(z)$ is:

$$A(z;\epsilon) = \epsilon \left[\frac{1}{z} A_0 + \frac{1}{z+1} A_{-1} \right] \tag{6.61}$$

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 & -2 & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & -2 & 0 & 0 \\ -3 & -3 & 0 & 0 & 4 & 12 & -2 & 0 \\ 3 & \frac{9}{2} & -3 & -1 & -4 & -18 & 1 & 1 \end{pmatrix}; \tag{6.62}$$

$$A_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 3 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 6 & 3 & 6 & 2 & -4 & -12 & 2 & 2 \\ -3 & -\frac{9}{2} & 3 & -1 & 4 & 18 & -1 & -1 \end{pmatrix}. \tag{6.63}$$

Note that this system is also fuchsian, so the choice of MI for the 1-loop box topology is not negated. Moreover, the s-sunset and the double s-bubble do not evolve, as it should be once the

dependance from s has been eliminated. The definitive basis of functions for the 2-loop ladder problem is then:

$$\mathbf{I}(z) = \begin{pmatrix} \epsilon^2(-s)^{1+2\epsilon} \\ \epsilon^2(-s)^{1+2\epsilon} z \\ \epsilon^3(-s)^{1+2\epsilon} \\ \epsilon^2(-s)^{2+2\epsilon} \\ \epsilon^3(-s)^{2+2\epsilon} z \\ \epsilon^4(-s)^{1+2\epsilon}(1+z) \\ \epsilon^4(-s)^{3+2\epsilon} z \\ \epsilon^4(-s)^{2+2\epsilon} \qquad (p_1+k_2)^2 \end{pmatrix}. \tag{6.64}$$

6.2.4 BCs

Analysis of the divergences

As discussed in section 2.3.2, quadruple cuts are performed on the 2-loop ladder topology, loop by loop:

$$\propto \quad \frac{1}{s}, \tag{6.65}$$

$$\frac{1}{s} \propto \frac{1}{s^2 t}, \tag{6.66}$$

since, cutting the first box, the t channel is $(p_1 + k_2)^2$. The divergences occur in $s = 0$ or in $t = 0$. In particular, in $u = 0 \Rightarrow s = -t$ all the MIs are regular.

In order to identify the solution to the problem, as explained in the previous chapter, two conditions are considered:

- finiteness of the solutions in $z = -1$ (giving three conditions);
- reality of the solutions for $z \geq 0$ (giving other three conditions).

The number of MMIs is two. Solving the relations, s-sunset and double s-bubble can be choosen as MMIs:

$$\left(\begin{array}{c} \epsilon^2(-s)^{1+2\epsilon} \\ \epsilon^2(-s)^{2+2\epsilon} \end{array} \right) \tag{6.67}$$

As before, to simplify the problem a suitable integration measure is chosen: the measure normalizes the value of $\left[\epsilon(-s)^{1+\epsilon}\right]^2 = \epsilon^2(-s)^{2+2\epsilon}$ to 1, so

$$\mathrm{d}_B^D k_1 \mathrm{d}_B^D k_2 = \left[\frac{(\mu^2)^\epsilon}{\mathrm{i}\pi^{2-\epsilon}} \frac{\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}\right]^2 \mathrm{d}^D k_1 \mathrm{d}^D k_2. \tag{6.68}$$

The MMIs assume the form:

$$\epsilon^2(-s)^{1+2\epsilon} = -\frac{\Gamma(1+2\epsilon)\Gamma^2(1-2\epsilon)}{\Gamma(1-3\epsilon)\Gamma^2(1+\epsilon)\Gamma(1-\epsilon)}; \tag{6.69}$$

$$\epsilon^2(-s)^{2+2\epsilon} = 1. \tag{6.70}$$

The basis is proven to be pure UT.

6.2.5 2-loop ladder graph

$$\epsilon^4(-s)^{3+2\epsilon} z = \sum_{i=0}^{7} \epsilon^i f_i(z) + o_0\left(\epsilon^7\right); \tag{6.71}$$

- $f_0 = -4$;
- $f_1(z) = 5H(0;z)$;
- $f_2(z) = (11\pi^2)/6 - 2H(0;z)^2$;
- $f_3(z) = 2H(-1;z)(\pi^2+H(0;z)^2) - 2/3H(0;z)(7\pi^2+H(0;z)^2+6H(0;-1;z)) + 4H(0;0;-1;z) + 3\zeta(3)$;
- $f_4(z) = (187\pi^4)/180 - 4H(0;0;-1;-1;z) - 44H(0;0;0;-1;z) + 1/3(4H(0;z)^4+3H(-1;z)^2(\pi^2+H(0;z)^2) + H(0;z)^2(17\pi^2 - 6H(0;-1;z)) - 20\pi^2 H(0;-1;z) + 6H(0;z)(2H(0;-1;-1;z) + 12H(0;0;-1;z) - 3\zeta(3)) - 2H(-1;z)(4H(0;z)^3+H(0;z)(5\pi^2+6H(0;-1;z)) - 6(H(0;0;-1;z)+\zeta(3))))$;

- $f_5(z) = -(14/15)H(0;z)^5+1/3H(-1;z)^3(\pi^2+H(0;z)^2)-8/9H(0;z)^3(5\pi^2-6H(0;-1;z))-4/3\pi^2H(0;-1;-1;z)+H(0;z)(-((101\pi^4)/45)+34/3\pi^2H(0;-1;z)+10H(0;-1;z)^2-4H(0;-1;-1;-1;z)-32H(0;0;-1;-1;z))+4(H(0;0;-1;-1;-1;z)+5H(0;0;-1;0;-1;z)+29H(0;0;0;-1;-1;z)+28H(0;0;0;0;-1;z))-6H(0;z)^2(H(0;-1;-1;z)+4H(0;0;-1;z)-\zeta(3))-4H(0;-1;z)(5H(0;0;-1;z)+18\zeta(3))-1/3H(-1;z)^2(4H(0;z)^3+H(0;z)(5\pi^2+6H(0;-1;z))-6(H(0;0;-1;z)+\zeta(3)))+1/18H(-1;z)(23\pi^4-72H(0;0;-1;-1;z)-1008H(0;0;0;-1;z)+12(3H(0;z)^4+H(0;z)^2(7\pi^2-6H(0;-1;z))-13\pi^2H(0;-1;z)+6H(0;z)(H(0;-1;-1;z)+8H(0;0;-1;z)+7\zeta(3))))+27\zeta(5);$

- $f_6(z) = (1493\pi^6)/3780+4/9H(0;z)^6+1/12H(-1;z)^4(\pi^2+H(0;z)^2)+1/9H(0;z)^4(23\pi^2-42H(0;-1;z))+28H(0;-1;-1;z)H(0;0;-1;z)+1/90\pi^2(-229\pi^2H(0;-1;z)+150H(0;-1;z)^2+2640H(0;-1;-1;-1;z))+20/3(\pi^2+3H(0;-1;z))H(0;0;-1;-1;z)+H(0;z)^2((217\pi^4)/90-34/3\pi^2H(0;-1;z)-10H(0;-1;z)^2+34H(0;-1;-1;-1;z)+32H(0;0;-1;-1;z))+4(\pi^2+10H(0;-1;z))H(0;0;0;-1;z)-4H(0;0;-1;-1;-1;-1;z)-20H(0;0;-1;-1;0;-1;z)-68H(0;0;-1;0;-1;-1;z)-260H(0;0;0;-1;-1;-1;z)-40H(0;0;0;-1;0;-1;z)-304H(0;0;0;0;-1;-1;z)-272H(0;0;0;0;0;-1;z)+4/3H(0;z)^3(4H(0;-1;-1;z)+12H(0;0;-1;z)-3\zeta(3))+8(10H(0;-1;-1;z)+9H(0;0;-1;z))\zeta(3)+27\zeta(3)^2-1/9H(-1;z)^3(4H(0;z)^3+H(0;z)(5\pi^2+6H(0;-1;z))-6(H(0;0;-1;z)+\zeta(3)))+1/36H(-1;z)^2(23\pi^4-72H(0;0;-1;-1;z)-1008H(0;0;0;-1;z)+12(3H(0;z)^4+H(0;z)^2(7\pi^2-6H(0;-1;z))-13\pi^2H(0;-1;z)+6H(0;z)(H(0;-1;-1;z)+8H(0;0;-1;z)+7\zeta(3))))+H(0;z)(-2(\pi^2+14H(0;-1;z))H(0;-1;-1;z)+4H(0;-1;-1;-1;-1;z)+8H(0;-1;0;-1;-1;z)+48H(0;0;-1;-1;-1;z)+6(\pi^2+10H(0;-1;z))\zeta(3)-18\zeta(5))-2/45H(-1;z)(24H(0;z)^5+90H(0;z)^3(\pi^2-2H(0;-1;z))+10H(0;z)(4\pi^4-6H(0;-1;z)(7\pi^2+9H(0;-1;z))+9H(0;-1;-1;-1;z)+72H(0;0;-1;-1;z))+90H(0;z)^2(5H(0;-1;-1;z)+8H(0;0;-1;z)+4\zeta(3))+15(23\pi^2H(0;-1;-1;z)+2(5\pi^2+36H(0;-1;z))H(0;0;-1;z)+(-7\pi^2+150H(0;-1;z))\zeta(3)-6(H(0;0;-1;-1;-1;z)+12H(0;0;-1;0;-1;z)+50H(0;0;0;-1;-1;z)+36H(0;0;0;0;-1;z)+4\zeta(5))));$

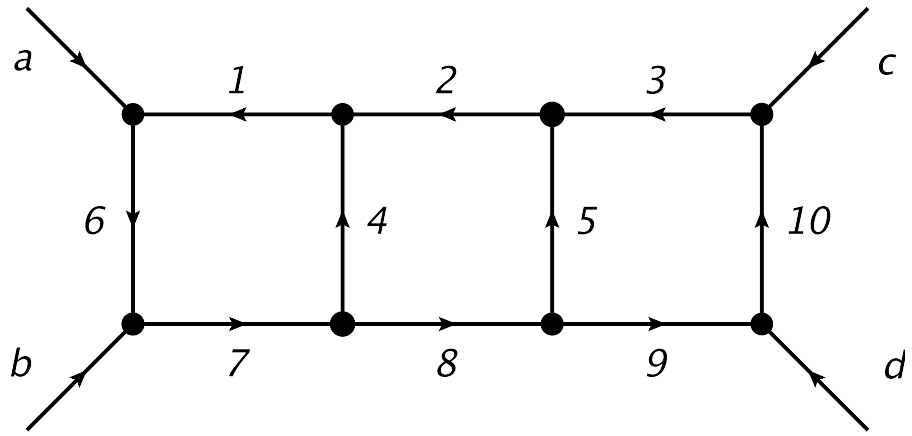
- $f_7(z) = -(52/315)H(0;z)^7+1/60H(-1;z)^5(\pi^2+H(0;z)^2)+H(0;z)^5(-((52\pi^2)/45)+8/3H(0;-1;z))+77/30\pi^4H(0;-1;-1;z)-38/3\pi^2H(0;-1;z)H(0;-1;-1;z)+6/5\pi^4H(0;0;-1;z)+12\pi^2H(0;-1;z)H(0;0;-1;z)+32H(0;-1;z)^2H(0;0;-1;z)+124H(0;0;-1;z)H(0;-1;-1;-1;z)-76H(0;-1;-1;z)H(0;0;-1;-1;z)+32H(0;0;-1;z)H(0;0;-1;-1;z)-4/135H(0;z)^3(58\pi^4-15H(0;-1;z)(17\pi^2+15H(0;-1;z))+1440H(0;-1;-1;-1;z)+720H(0;0;-1;-1;z))-152H(0;-1;-1;z)H(0;0;0;-1;z)+212/3\pi^2H(0;-1;-1;-1;-1;z)+148/3\pi^2H(0;-1;0;-1;-1;z)+148\pi^2H(0;0;-1;-1;-1;z)-20H(0;-1;z)H(0;0;-1;-1;-1;z)-20\pi^2H(0;0;-1;0;-1;z)-64H(0;-1;z)H(0;0;-1;0;-1;z)-52\pi^2H(0;0;0;-1;-1;z)-184H(0;-1;z)H(0;0;0;-1;-1;z)-16\pi^2H(0;0;0;0;-1;z)-80H(0;-1;z)H(0;0;0;0;-1;z)+4(H(0;0;-1;-1;-1;-1;-1;z)+5H(0;0;-1;-1;-1;0;-1;z)+29H(0;0;-1;-1;0;-1;-1;z)+41H(0;0;-1;0;-1;-1;-1;z)+16H(0;0;-1;0;-1;0;-1;z)+36H(0;0;-1;0;0;-1;-1;z)+137H(0;0;0;-1;-1;-1;-1;z)+46H(0;0;0;-1;-1;0;-1;z)+142H(0;0;0;-1;0;-1;-1;z)+388H(0;0;0;0;-1;-1;-1;z)+20H(0;0;0;0;-1;0;-1;z)+188H(0;0;0;0;0;-1;-1;z)+$

$$160H(0;0;0;0;0;0;-1;z)) + (-((13\pi^4)/5) + 38\pi^2 H(0;-1;z) + 18H(0;-1;z)^2 + 72H(0;-1;-1;-1;z) - 96H(0;0;-1;-1;z) + 96H(0;0;0;-1;z))\zeta(3) + H(0;z)^4(-(10/3)H(0;-1;-1;z) - 8H(0;0;-1;z) + 2\zeta(3)) + H(0;z)(-((568\pi^6)/945) - 16/3\pi^2 H(0;-1;z)^2 - 32/3H(0;-1;z)^3 + 38H(0;-1;-1;z)^2 + H(0;-1;z)((148\pi^4)/45 - 124H(0;-1;-1;-1;z) - 32H(0;0;-1;-1;z)) - 2/3\pi^2(91H(0;-1;-1;-1;z) + 8H(0;0;-1;-1;z)) - 4H(0;-1;-1;-1;-1;-1;z) + 144H(0;-1;0;-1;-1;-1;z) + 256H(0;0;-1;-1;-1;-1;z) - 112H(0;0;-1;0;-1;-1;z) - 336H(0;0;0;-1;-1;-1;z) + H(0;-1;-1;z)(64H(0;0;-1;z) - 52\zeta(3)) - 144H(0;0;-1;z)\zeta(3)) - 1/36H(-1;z)^4(4H(0;z)^3 + H(0;z)(5\pi^2 + 6H(0;-1;z)) - 6(H(0;0;-1;z) + \zeta(3))) + 1/108H(-1;z)^3(23\pi^4 - 72H(0;0;-1;-1;z) - 1008H(0;0;0;-1;z) + 12(3H(0;z)^4 + H(0;z)^2(7\pi^2 - 6H(0;-1;z)) - 13\pi^2 H(0;-1;z) + 6H(0;z)(H(0;-1;-1;z) + 8H(0;0;-1;z) + 7\zeta(3)))) + 2/3H(0;z)^2((7\pi^2 + 18H(0;-1;z)) H(0;-1;-1;z) + 9(11H(0;-1;-1;-1;-1;z) + 8H(0;-1;0;-1;-1;z) + 20H(0;0;-1;-1;-1;z) - 2(\pi^2 + 2H(0;-1;z))\zeta(3) - 3\zeta(5))) - 6(11\pi^2 + 70H(0;-1;z))\zeta(5) - 1/45H(-1;z)^2(24H(0;z)^5 + 90H(0;z)^3(\pi^2 - 2H(0;-1;z)) + 10H(0;z)(4\pi^4 - 6H(0;-1;z)(7\pi^2 + 9H(0;-1;z)) + 9H(0;-1;-1;-1;z) + 72H(0;0;-1;-1;z)) + 90H(0;z)^2(5H(0;-1;-1;z) + 8H(0;0;-1;z) + 4\zeta(3)) + 15(23\pi^2 H(0;-1;-1;z) + 2(5\pi^2 + 36H(0;-1;z))H(0;0;-1;z) + (-7\pi^2 + 150H(0;-1;z))\zeta(3) - 6(H(0;0;-1;-1;-1;z) + 12H(0;0;-1;0;-1;z) + 50H(0;0;0;-1;-1;z) + 36H(0;0;0;0;-1;z) + 4\zeta(5)))) + 1/1890H(-1;z)(5(169\pi^6 - 1512H(0;0;-1;-1;-1;-1;z) - 36288H(0;0;-1;-1;0;-1;z) - 54432H(0;0;-1;0;-1;-1;z) - 184464H(0;0;0;-1;-1;-1;z) - 54432H(0;0;0;-1;0;-1;z) - 272160H(0;0;0;0;-1;-1;z) - 133056H(0;0;0;0;0;-1;z)) + 42(20H(0;z)^6 + 10H(0;z)^4(11\pi^2 - 30H(0;-1;z)) + 180\pi^2 H(0;-1;z)^2 + 9H(0;z)^2(11\pi^4 - 100H(0;-1;z)(\pi^2 + H(0;-1;z)) - 20H(0;-1;-1;-1;z) - 40H(0;0;-1;-1;z)) + H(0;-1;z)(-187\pi^4 + 4320H(0;0;-1;-1;z) + 6480H(0;0;0;-1;z)) + 120H(0;z)^3(9H(0;-1;-1;z) + 8H(0;0;-1;z) + \zeta(3)) - 30(\pi^2(13H(0;-1;-1;-1;z) + 50H(0;0;-1;-1;z) + 12H(0;0;0;-1;z)) + H(0;-1;-1;z)(72H(0;0;-1;z) - 6\zeta(3)) + 3(2H(0;0;-1;z) - 3\zeta(3))(10H(0;0;-1;z) + 3\zeta(3))) + 60H(0;z)((22\pi^2 + 36H(0;-1;z))H(0;-1;-1;z) + 2(5\pi^2 - 12H(0;-1;z))H(0;0;-1;z) + (-7\pi^2 + 78H(0;-1;z))\zeta(3) + 3(H(0;-1;-1;-1;-1;z) - 36H(0;-1;0;-1;-1;z) - 64H(0;0;-1;-1;-1;z) + 28H(0;0;-1;0;-1;z) + 84H(0;0;0;-1;-1;z) + 55\zeta(5))))) + 159\zeta(7);$$

Here the 2-loop ladder has a ϵ^{-4} pole.

The results for the other graphs of the basis are shown in THESIS_2L_BC.

6.3 3-loop ladder



- External momenta ($p_i^2 = 0$)

  a p_1^μ
  b p_2^μ
  c p_3^μ
  d p_4^μ

- Propagators

  1 k_1^2
  2 k_2^2
  3 k_3^2
  4 $(k_1 - k_2)^2$
  5 $(k_2 - k_3)^2$
  6 $(k_1 + p_1)^2$
  7 $(k_1 + p_1 + p_2)^2$
  8 $(k_2 + p_1 + p_2)^2$
  9 $(k_3 + p_1 + p_2)^2$
  10 $(k_3 - p_3)^2$

- Auxiliary propagators

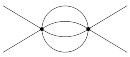
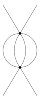
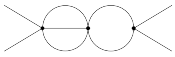
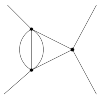
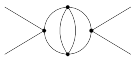
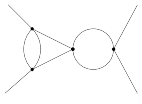
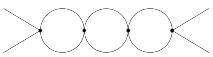
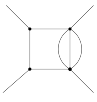
  11 $(k_1 - p_3)^2$
  12 $(k_2 - p_3)^2$

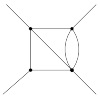
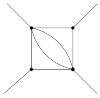
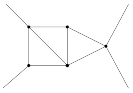
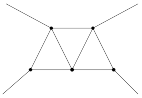
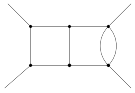
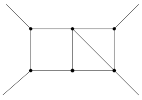
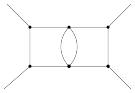
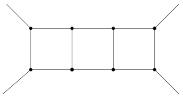
13 $(k_2 + p_1)^2$
14 $(k_3 + p_1)^2$
15 $(k_3 - k_1)^2$

THESIS_3L_MI shows the basis of MIs, and THESIS_3L_DE presents the DEs in $s$and t, both from `Reduze`. THESIS_3L_SOL contains the determination of the canonical form and of the BCs, while THESIS_3L_BC contains the construction of the series expansion of the solutions.

6.3.1 Laporta basis

`Reduze` determines for this topology the following subtopology tree (the notation for the subtopology is the same one used in `Reduze`, see [46, 47] for more details):

- `INT["nm3", 4, 92]` =  : s-sunset;
- `INT["nm3", 4, 568]` =  : t-sunset;
- `INT["nm3", 5, 157]` =  : nested bubbles;
- `INT["nm3", 5, 188]` =  : 3-loop triangle;
- `INT["nm3", 5, 213]` =  : sunset bubble;
- `INT["nm3", 5, 316]` =  : sunset triangle;
- `INT["nm3", 6, 349]` =  : 3-loop sefl-energy.
- `INT["nm3", 6, 430]` =  : triangle bubble;
- `INT["nm3", 6, 455]` =  : triple bubble;
- `INT["nm3", 6, 633]` =  : 3-loop box;

- `INT["nm3", 6, 634]` =  : bubble box with diagonal;
- `INT["nm3", 6, 636]` =  : bubble diagonal box;
- `INT["nm3", 6, 698]` =  , considered *twice*: double bubble box;
- `INT["nm3", 7, 382]` =  : 3-point FI;
- `INT["nm3", 7, 701]` =  , considered *twice*: trapezoid;
- `INT["nm3", 8, 763]` =  , considered *twice*: side-bubble ladder;
- `INT["nm3", 8, 765]` =  , considered *twice*: diagonal ladder;
- `INT["nm3", 8, 893]` =  , considered *twice*: central-bubble ladder;
- `INT["nm3", 10, 1023]` =  , considered *three times*: 3-loop ladder;

There are then nineteen different topologies, for a total of twenty-six MIs.

6.3.2 UT basis

Sunsets and 2-loop UT graphs with UT FIs as propagators

All inner sunsets are replaced with their corresponding UT version.

In particular, s and t-sunsets are now UT, together with triangles-like graphs and most of the box and ladder integrals (3-loop triangle, sunset triangle, triple bubble, 3-loop box, bubble box with diagonal, bubble diagonal box).

Nested bubbles

Thanks to lemma 2, starting from a UT bubble, it is possible to substitute the dotted internal line with two chained UT bubbles and multiply by $1 + 2\epsilon$ to find a UT function.

3-loop self-energy

First of all, the internal bubble is dotted in one propagator, to find a local UT structure. Considering now the matrix of coefficients, in order to eliminate off-diagonal presence of terms not depending on ϵ, the MI is multiplied by $-1+2\epsilon$. This choice also matches what is shown in [11, 14].

Double bubble box

First of all, the bubbles are dotted in one propagator to obtain their UT versions. The first MIs is so obtained, and it is known to be UT. For the second one, consider the topology generated once the bubbles are replaced with their analytic structures: a box topology is obtained. A numerator is then added, in order to collapse to a triangular structure: $(k_2+p_1)^2$ is used (equivalenty, $(k_2-p_3)^2$ is valid); in this way, a triangular structure is retrieved (on the original FI, only for $\epsilon \to 0$).

3-point FI

The graph does not show any symmetrical structure, so there are no preferred internal lines to dot. Moreover, the mass dimension of this MI is $[m]^{-2}$, correct for a three-point function not resulting from chaining FIs.

Trapezoid

The first MI for this topology is chosen to be the simple trapezoid FI, without dots or numerators. For the second MI, considering that the mass dimension is $[m]^{-2}$ instead of $[m]^{-4}$, a dotted version is a good candidate. Since the structure is symmetrical, if a dot have to placed, the upper central propagator is the best choice.

Side-bubble ladder

As for the double bubble box, first of all a UT structure is reached, and it is the first MI. For the second one, proceed as for the 2-loop ladder, using the numerator $(k_2+p_1)^2$.

Diagonal ladder

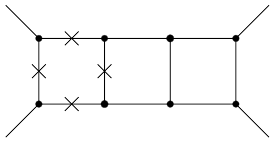
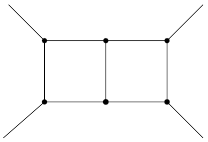
The mass dimension is $[m]^{-4}$, so the undotted version is a good candidate. Since adding numerators will increase the mass dimension, a dot can be a better choice, so the diagonal is dotted.

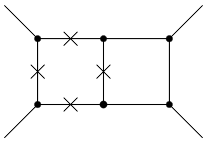
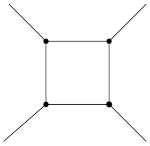
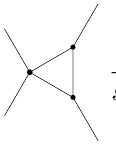
Central-bubble ladder

As explained above, the bubble is dotted in one propagator, obtaining the first MI. On this FI, the numerator $(k_3+p_1)^2$ is added, retrieving the second MI.

3-loop ladder

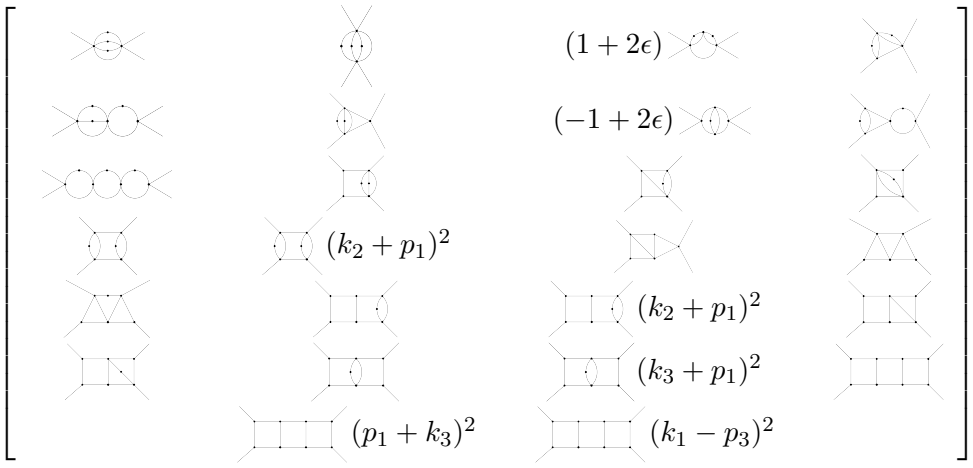
As for the 2-loop ladder, scalar product are introduced to find the two extra MIs: in particular, following the method of maximal cuts, $(p_1+k_3)^2$ and $(k_1-p_3)^2$ are introduced. See section 3.3.3 for a general discussion. In the present case, the quadruple cut is applied twice, each time on the sub-box at the same end, once starting from the left (the version with numerator $(p_1+k_3)^2$), once starting from the right (the version with numerator $(k_1-p_3)^2$). For example:


$(p_1+k_3)^2 \propto$

$\frac{(p_1+k_3)^2}{s};$ (6.72)


$\frac{(p_1+k_3)^2}{s} \propto$

$\frac{(p_1+k_3)^2}{s^2} =$

$\frac{1}{s^2}$ (6.73)

The basis of MIs is then:[1]

$$\mathcal{I}(s;t) = \tag{6.74}$$



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[1] From now on, the MIs of the 3-loop ladder problem will be shown in a table, but it must be intended as a vector, with ordered elements starting from the highest row to the lowest one, from left to right.

6.3.3 DEs reduction

$(s;t)$ DEs

The equation are determined using `Reduze`.

$$\frac{\partial}{\partial s}\mathcal{I}(s;t) = A_s(s;t)\mathcal{I}(s;t), \tag{6.75}$$

$$\frac{\partial}{\partial t}\mathcal{I}(s;t) = A_t(s;t)\mathcal{I}(s;t); \tag{6.76}$$

$$A_s(s;t) = \begin{pmatrix} S_1 & S_2 & S_3 & S_4 \end{pmatrix}, \tag{6.77}$$

$$A_t(s;t) = \begin{pmatrix} T_1 & T_2 & T_3 & T_4 \end{pmatrix}. \tag{6.78}$$

$$S_1 = \begin{pmatrix}
-\frac{3\epsilon+1}{s} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{3\epsilon+1}{s} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{3\epsilon+1}{s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{3\epsilon+2}{s} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{3\epsilon+1}{s} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{3\epsilon+1}{s} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{4}{3s(s+t)} & 0 & 0 & 0 & \frac{4\epsilon}{s(s+t)} & 0 \\
0 & \frac{t}{6s(s+t)\epsilon} & 0 & \frac{1}{s+t} & 0 & 0 & 0 \\
-\frac{1}{3(s+t)\epsilon} & \frac{t}{3s(s+t)\epsilon} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{s(s+t)} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{t}{4s(s+t)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{t}{12s(s+t)\epsilon^2} & -\frac{2(2\epsilon+1)}{(s+t)\epsilon^2} & -\frac{2}{(s+t)\epsilon} & 0 & 0 & 0 \\
0 & \frac{t}{3s^2(s+t)\epsilon} & -\frac{8(2\epsilon+1)}{s(s+t)\epsilon} & -\frac{8}{s(s+t)} & 0 & 0 & 0 \\
0 & \frac{2}{s^2(s+t)\epsilon} & 0 & -\frac{8(3s-2t)}{3s^2t(s+t)} & 0 & -\frac{4}{s^2(s+t)} & 0 \\
0 & \frac{2t}{s^2(s+t)\epsilon} & 0 & -\frac{40}{9s(s+t)} & 0 & -\frac{2}{3s(s+t)} & 0 \\
\frac{17}{54s(s+t)\epsilon^2} & \frac{23t}{27s^2(s+t)\epsilon^2} & -\frac{14(2\epsilon+1)}{3s(s+t)\epsilon^2} & -\frac{56}{9s(s+t)\epsilon} & -\frac{1}{6(s+t)\epsilon^2} & -\frac{1}{6s(s+t)\epsilon} & 0 \\
\frac{12s-t}{9s^2t(s+t)\epsilon} & \frac{4(3s+7t)}{9s^3(s+t)\epsilon} & -\frac{12(2\epsilon+1)}{s^2(s+t)\epsilon} & -\frac{40}{3s^2(s+t)} & \frac{2}{s(s+t)\epsilon} & -\frac{7}{s^2(s+t)} & 0 \\
\frac{4s-t}{3s^2t(s+t)\epsilon} & \frac{4}{3s^2(s+t)\epsilon} & 0 & 0 & 0 & -\frac{6}{s^2(s+t)} & -\frac{3(2\epsilon-1)}{s^2(s+t)} \\
\frac{19s+9t}{9s^2(s+t)\epsilon} & \frac{20t}{9s^2(s+t)\epsilon} & 0 & 0 & 0 & -\frac{2}{s^2} & -\frac{(s+3t)(2\epsilon-1)}{s^2(s+t)} \\
-\frac{17}{9s^3(s+t)\epsilon^2} & 0 & -\frac{28(2\epsilon+1)}{3s^3(s+t)\epsilon^2} & -\frac{40}{3s^3(s+t)\epsilon} & \frac{7}{s^2(s+t)\epsilon^2} & \frac{7}{s^3(s+t)\epsilon} & \frac{6(2\epsilon-1)}{s^3(s+t)\epsilon} \\
-\frac{41s+32t}{18s^3(s+t)\epsilon^2} & \frac{4(6s-7t)t}{9s^4(s+t)\epsilon^2} & -\frac{2(7s-4t)(2\epsilon+1)}{3s^3(s+t)\epsilon^2} & -\frac{4(17s-8t)}{3s^3(s+t)\epsilon} & \frac{7s+10t}{2s^2(s+t)\epsilon^2} & \frac{7s+12t}{2s^3(s+t)\epsilon} & \frac{3(s+2t)(2\epsilon-1)}{s^3(s+t)\epsilon} \\
\frac{7}{6s^2(s+t)\epsilon^2} & \frac{28t}{9s^3(s+t)\epsilon^2} & -\frac{22(2\epsilon+1)}{3s^2(s+t)\epsilon^2} & -\frac{20}{3s^2(s+t)\epsilon} & \frac{3(s+2t)}{2s^2(s+t)\epsilon^2} & -\frac{9}{2s^2(s+t)\epsilon} & 0
\end{pmatrix};$$

$$S_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3\epsilon+2}{s} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{3(\epsilon+1)}{s} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{s+t+t\epsilon}{s(s+t)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{s+2t\epsilon}{s(s+t)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{s+3t\epsilon}{s(s+t)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{s+t+2t\epsilon}{s(s+t)} & \frac{4\epsilon}{s(s+t)} \\
0 & 0 & 0 & 0 & 0 & \frac{t\epsilon}{2(s+t)} & -\frac{3\epsilon s+s+t+2t\epsilon}{s(s+t)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{2}{s} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2}{s(s+t)} & 0 & \frac{3}{2s(s+t)} & -\frac{24\epsilon}{s^2t} & 0 & \frac{2}{s(s+t)} & 0 \\
-\frac{2(2s+3t)}{3s(s+t)} & 0 & \frac{t}{s(s+t)} & -\frac{24\epsilon}{s^2} & 0 & \frac{2t}{s(s+t)} & -\frac{8}{3s(s+t)} \\
0 & 0 & \frac{t}{s(s+t)\epsilon} & -\frac{20}{3s^2} & -\frac{2}{s^2} & 0 & 0 \\
0 & 0 & \frac{3}{s^2} & -\frac{16\epsilon}{s^3} & -\frac{12(s+t)\epsilon}{s^3t} & 0 & 0 \\
0 & 0 & \frac{3}{s(s+t)} & 0 & -\frac{12\epsilon}{s^2t} & 0 & 0 \\
0 & 0 & \frac{3t}{s(s+t)} & 0 & -\frac{20\epsilon}{s^2} & 0 & 0 \\
\frac{6}{s^2(s+t)\epsilon} & -\frac{2}{s(s+t)\epsilon^2} & 0 & 0 & 0 & 0 & 0 \\
\frac{3s+2t}{s^2(s+t)\epsilon} & -\frac{s+2t}{s(s+t)\epsilon^2} & -\frac{3t^2}{s^3(s+t)\epsilon} & -\frac{16(3s-t)}{s^4} & \frac{12(s+t)}{s^4} & \frac{4t}{s^2(s+t)\epsilon} & 0 \\
\frac{3(s+2t)}{s^2(s+t)\epsilon} & -\frac{s+2t}{s(s+t)\epsilon^2} & \frac{3t}{s^2(s+t)\epsilon} & -\frac{16}{s^3} & -\frac{12}{s^3} & 0 & 0
\end{pmatrix};$$

$$S_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3\epsilon+1}{s} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{3(2s+t)\epsilon}{s(s+t)} & \frac{s}{s+t} & 0 & 0 & 0 & 0 \\
0 & \frac{12t\epsilon^2}{s^2(s+t)} & -\frac{2(s+t+2t\epsilon)}{s(s+t)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{2(s+t+t\epsilon)}{s(s+t)} & \frac{3\epsilon}{s(s+t)} & 0 & 0 \\
0 & 0 & 0 & \frac{2t\epsilon}{3(s+t)} & -\frac{4\epsilon s+2s+2t+3t\epsilon}{s(s+t)} & 0 & 0 \\
-\frac{2\epsilon}{3s(s+t)} & \frac{8t\epsilon}{3s^2(s+t)} & \frac{2}{s+t} & 0 & 0 & -\frac{4\epsilon s+2s+t+5t\epsilon}{s(s+t)} & \frac{t}{3(s+t)} \\
-\frac{4\epsilon^2}{s^2(s+t)} & \frac{16t\epsilon^2}{s^3(s+t)} & \frac{4\epsilon}{s(s+t)} & 0 & 0 & -\frac{12\epsilon^2}{s^2} & -\frac{2s+2t+t\epsilon}{s(s+t)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{4\epsilon}{s^3(s+t)} & 0 & 0 & \frac{4}{s(s+t)} & 0 & 0 & -\frac{4}{s(s+t)} \\
\frac{2(s+4t)\epsilon}{s^3(s+t)} & -\frac{16t^2\epsilon}{s^4(s+t)} & -\frac{4t}{s^2(s+t)} & \frac{6t}{s(s+t)} & \frac{6t}{s^2(s+t)} & \frac{12t\epsilon}{s^3} & -\frac{2t(3s+t)}{s^2(s+t)} \\
-\frac{6\epsilon}{s^2(s+t)} & \frac{16t\epsilon}{s^3(s+t)} & \frac{4}{s(s+t)} & \frac{2t}{s(s+t)} & 0 & -\frac{12\epsilon}{s^2} & -\frac{2t}{s(s+t)}
\end{pmatrix};$$

$$S_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\frac{2s+2t+3t\epsilon}{s(s+t)} & \frac{3\epsilon}{s(s+t)} & 0 & 0 & 0 \\
\frac{2t\epsilon}{s+t} & -\frac{2(2\epsilon s+s+t+t\epsilon)}{s(s+t)} & 0 & 0 & 0 \\
\frac{2}{s(s+t)} & 0 & -\frac{3(s+t+t\epsilon)}{s(s+t)} & \frac{2\epsilon}{s(s+t)} & \frac{2\epsilon}{s(s+t)} \\
\frac{2t}{s(s+t)} & 0 & \frac{2t\epsilon}{s+t} & -\frac{4\epsilon s+3s+3t+3t\epsilon}{s(s+t)} & \frac{2t\epsilon}{s(s+t)} \\
\frac{3t}{s(s+t)} & -\frac{3}{s(s+t)} & \frac{t\epsilon}{s+t} & -\frac{\epsilon}{s+t} & -\frac{3(\epsilon+1)}{s}
\end{pmatrix}.$$

$$T_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{3\epsilon+1}{t} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{4}{3t(s+t)} & 0 & 0 & 0 & -\frac{4\epsilon}{t(s+t)} & 0 \\
0 & -\frac{1}{6(s+t)\epsilon} & 0 & -\frac{s}{t(s+t)} & 0 & 0 & 0 \\
\frac{s}{3t(s+t)\epsilon} & -\frac{1}{3(s+t)\epsilon} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{t(s+t)} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{4(s+t)} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{12(s+t)\epsilon^2} & \frac{2s(2\epsilon+1)}{t(s+t)\epsilon^2} & \frac{2s}{t(s+t)\epsilon} & 0 & 0 & 0 \\
0 & -\frac{1}{3s(s+t)\epsilon} & \frac{8(2\epsilon+1)}{t(s+t)\epsilon} & \frac{8}{t(s+t)} & 0 & 0 & 0 \\
0 & -\frac{2}{st(s+t)\epsilon} & 0 & \frac{8(3s-2t)}{3st^2(s+t)} & 0 & \frac{4}{st(s+t)} & 0 \\
0 & -\frac{2}{s(s+t)\epsilon} & 0 & \frac{40}{9t(s+t)} & 0 & \frac{2}{3t(s+t)} & 0 \\
-\frac{17}{54t(s+t)\epsilon^2} & -\frac{23}{27s(s+t)\epsilon^2} & \frac{14(2\epsilon+1)}{3t(s+t)\epsilon^2} & \frac{56}{9t(s+t)\epsilon} & \frac{s}{6t(s+t)\epsilon^2} & \frac{1}{6t(s+t)\epsilon} & 0 \\
-\frac{12s-t}{9st^2(s+t)\epsilon} & -\frac{4(3s+7t)}{9s^2t(s+t)\epsilon} & \frac{12(2\epsilon+1)}{st(s+t)\epsilon} & \frac{40}{3st(s+t)} & -\frac{2}{t(s+t)\epsilon} & \frac{7}{st(s+t)} & 0 \\
-\frac{4s-t}{3st^2(s+t)\epsilon} & -\frac{4}{3st(s+t)\epsilon} & 0 & 0 & 0 & \frac{6}{st(s+t)} & \frac{3(2\epsilon-1)}{st(s+t)} \\
-\frac{19s+9t}{9st(s+t)\epsilon} & -\frac{20}{9s(s+t)\epsilon} & 0 & 0 & 0 & \frac{2}{st} & \frac{(s+3t)(2\epsilon-1)}{st(s+t)} \\
\frac{17}{9s^2t(s+t)\epsilon^2} & 0 & \frac{28(2\epsilon+1)}{3s^2t(s+t)\epsilon^2} & \frac{40}{3s^2t(s+t)\epsilon} & -\frac{7}{st(s+t)\epsilon^2} & -\frac{7}{s^2t(s+t)\epsilon} & -\frac{6(2\epsilon-1)}{s^2t(s+t)\epsilon} \\
\frac{41s+32t}{18s^2t(s+t)\epsilon^2} & -\frac{4(6s-7t)}{9s^3(s+t)\epsilon^2} & \frac{2(7s-4t)(2\epsilon+1)}{3s^2t(s+t)\epsilon^2} & \frac{4(17s-8t)}{3s^2t(s+t)\epsilon} & -\frac{7s+10t}{2st(s+t)\epsilon^2} & -\frac{7s+12t}{2s^2t(s+t)\epsilon} & -\frac{3(s+2t)(2\epsilon-1)}{s^2t(s+t)\epsilon} \\
-\frac{7}{6st(s+t)\epsilon^2} & -\frac{28}{9s^2(s+t)\epsilon^2} & \frac{22(2\epsilon+1)}{3st(s+t)\epsilon^2} & \frac{20}{3st(s+t)\epsilon} & -\frac{3(s+2t)}{2st(s+t)\epsilon^2} & \frac{9}{2st(s+t)\epsilon} & 0
\end{pmatrix};$$

$$T_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{3\epsilon s+s+t+2t\epsilon}{t(s+t)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\epsilon t+t+3s\epsilon}{t(s+t)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{t+3s\epsilon}{t(s+t)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{3\epsilon s+s+t+t\epsilon}{t(s+t)} & -\frac{4\epsilon}{t(s+t)} \\
0 & 0 & 0 & 0 & 0 & -\frac{s\epsilon}{2(s+t)} & -\frac{\epsilon}{s+t} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{t} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2}{t(s+t)} & 0 & -\frac{3}{2t(s+t)} & \frac{24\epsilon}{st^2} & 0 & -\frac{2}{t(s+t)} & 0 \\
\frac{2(2s+3t)}{3t(s+t)} & 0 & -\frac{1}{s+t} & \frac{24\epsilon}{st} & 0 & -\frac{2}{s+t} & \frac{8}{3t(s+t)} \\
0 & 0 & -\frac{1}{(s+t)\epsilon} & \frac{20}{3st} & \frac{2}{st} & 0 & 0 \\
0 & 0 & -\frac{3}{st} & \frac{16\epsilon}{s^2t} & \frac{12(s+t)\epsilon}{s^2t^2} & 0 & 0 \\
0 & 0 & -\frac{3}{t(s+t)} & 0 & \frac{12\epsilon}{st^2} & 0 & 0 \\
0 & 0 & -\frac{3}{s+t} & 0 & \frac{20\epsilon}{st} & 0 & 0 \\
-\frac{6}{st(s+t)\epsilon} & \frac{2}{t(s+t)\epsilon^2} & 0 & 0 & 0 & 0 & 0 \\
-\frac{3s+2t}{st(s+t)\epsilon} & \frac{s+2t}{t(s+t)\epsilon^2} & \frac{3t}{s^2(s+t)\epsilon} & \frac{16(3s-t)}{s^3t} & -\frac{12(s+t)}{s^3t} & -\frac{4}{s(s+t)\epsilon} & 0 \\
-\frac{3(s+2t)}{st(s+t)\epsilon} & \frac{s+2t}{t(s+t)\epsilon^2} & -\frac{3}{s(s+t)\epsilon} & \frac{16}{s^2t} & \frac{12}{s^2t} & 0 & 0
\end{pmatrix};$$

$$T_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3\epsilon s-s-t}{t(s+t)} & -\frac{s^2}{t(s+t)} & 0 & 0 & 0 & 0 \\
0 & -\frac{12\epsilon^2}{s(s+t)} & \frac{(t-3s)\epsilon}{t(s+t)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{3\epsilon s+s+t+t\epsilon}{t(s+t)} & -\frac{3\epsilon}{t(s+t)} & 0 & 0 \\
0 & 0 & 0 & -\frac{2s\epsilon}{3(s+t)} & \frac{s\epsilon}{t(s+t)} & 0 & 0 \\
\frac{2\epsilon}{3t(s+t)} & -\frac{8\epsilon}{3s(s+t)} & -\frac{2s}{t(s+t)} & 0 & 0 & \frac{2\epsilon t-t+s\epsilon}{t(s+t)} & -\frac{s}{3(s+t)} \\
\frac{4\epsilon^2}{st(s+t)} & -\frac{16\epsilon^2}{s^2(s+t)} & -\frac{4\epsilon}{t(s+t)} & 0 & 0 & \frac{12\epsilon^2}{st} & -\frac{3\epsilon s+s+t+2t\epsilon}{t(s+t)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{4\epsilon}{s^2t(s+t)} & 0 & 0 & -\frac{4}{t(s+t)} & 0 & 0 & \frac{4}{t(s+t)} \\
-\frac{2(s+4t)\epsilon}{s^2t(s+t)} & \frac{16t\epsilon}{s^3(s+t)} & \frac{4}{s(s+t)} & -\frac{6}{s+t} & -\frac{6}{s(s+t)} & -\frac{12\epsilon}{s^2} & \frac{2(3s+t)}{s(s+t)} \\
\frac{6\epsilon}{st(s+t)} & -\frac{16\epsilon}{s^2(s+t)} & -\frac{4}{t(s+t)} & -\frac{2}{s+t} & 0 & \frac{12\epsilon}{st} & \frac{2}{s+t}
\end{pmatrix};$$

$$T_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\frac{3\epsilon s+s+t}{t(s+t)} & -\frac{3\epsilon}{t(s+t)} & 0 & 0 & 0 \\
-\frac{2s\epsilon}{s+t} & \frac{(s-t)\epsilon}{t(s+t)} & 0 & 0 & 0 \\
-\frac{2}{t(s+t)} & 0 & -\frac{3\epsilon s+s+t}{t(s+t)} & -\frac{2\epsilon}{t(s+t)} & -\frac{2\epsilon}{t(s+t)} \\
-\frac{2}{s+t} & 0 & -\frac{2s\epsilon}{s+t} & \frac{s\epsilon}{t(s+t)} & -\frac{2\epsilon}{s+t} \\
-\frac{3}{s+t} & \frac{3}{t(s+t)} & -\frac{s\epsilon}{s+t} & \frac{s\epsilon}{t(s+t)} & 0
\end{pmatrix}.$$

$(s;z)$ DEs, elimination of ϵ poles and canonical fuchsian form

Changing variables the PDE in s can be immediately solved, and the basis of MIs can be redefined eliminating the dependence on s. Each term of the basis is then multiplied by an ϵ^n term to eliminate poles in ϵ and to confine ϵ-independent terms on only on the diagonal. The new basis has then the form:

$$\begin{bmatrix}
\epsilon^3(-s)^{1+3\epsilon} & \epsilon^3(-s)^{1+3\epsilon} & \epsilon^3(-s)^{1+3\epsilon}(1+2\epsilon) \\
\epsilon^4(-s)^{1+3\epsilon} & \epsilon^3(-s)^{2+3\epsilon} & \epsilon^4(-s)^{1+3\epsilon} \\
\epsilon^4(-s)^{1+3\epsilon}(1-2\epsilon) & \epsilon^4(-s)^{2+3\epsilon} & \epsilon^3(-s)^{3+3\epsilon} \\
\epsilon^4(-s)^{2+3\epsilon} & \epsilon^5(-s)^{1+3\epsilon} & \epsilon^5(-s)^{1+3\epsilon} \\
\epsilon^4(-s)^{2+3\epsilon} & \epsilon^4(-s)^{1+3\epsilon}\ (k_2+p_1)^2 & \epsilon^6(-s)^{1+3\epsilon} \\
\epsilon^6(-s)^{1+3\epsilon} & \epsilon^5(-s)^{2+3\epsilon} & \epsilon^5(-s)^{3+3\epsilon} \\
\epsilon^5(-s)^{1+3\epsilon}\ (k_2+p_1)^2 & \epsilon^6(-s)^{2+3\epsilon} & \epsilon^5(-s)^{3+3\epsilon} \\
\epsilon^5(-s)^{3+3\epsilon} & \epsilon^5(-s)^{2+3\epsilon}\ (k_3+p_1)^2 & \epsilon^6(-s)^{4+3\epsilon} \\
\epsilon^6(-s)^{3+3\epsilon}\ (p_1+k_3)^2 & \epsilon^6(-s)^{3+3\epsilon}\ (k_1-p_3)^2 &
\end{bmatrix}.$$

The matrix for the change of basis to the canonical form is constructed using only $A_0(z)$, of

the matrix $A(z;\epsilon) = A_0(z) + \epsilon A_1(z)$. This matrix is diagonal, with non-zero terms:

$$D_{2;2} = D_{10;10} = D_{13;13} = D_{16;16} = D_{18;18} = D_{21;21} = D_{22;22} = D_{24;24} = -\frac{1}{z},$$
$$D_{11;11} = D_{12;12} = D_{20;20} = -\frac{1}{1+z}; \tag{6.79}$$

resulting in a diagonal matrix for the transformation, with elements on the diagonal:
$\left(1; \frac{1}{z}; 1; 1; 1; 1; 1; 1; 1; 1; \frac{1}{z}; \frac{1}{z+1}; \frac{1}{z+1}; \frac{1}{z}; 1; 1; \frac{1}{z}; 1; \frac{1}{z}; 1; \frac{1}{z+1}; \frac{1}{z}; \frac{1}{z}; 1; \frac{1}{z}; 1; 1\right)$

After these transformations the matrix of coefficients of the system has the form:

$$A(z) = \epsilon \left[\frac{1}{z} A_0 + \frac{1}{z+1} A_{-1}\right] \tag{6.80}$$

A_0 and A_{-1} are presented in table 6.1 and table 6.2, respectively.

The basis of MIs is of the form:

$$\left[\begin{array}{ccc}
\epsilon^3(-s)^{1+3\epsilon} & \epsilon^3(-s)^{1+3\epsilon} z & \epsilon^3(-s)^{1+3\epsilon}(1+2\epsilon) \\
\epsilon^4(-s)^{1+3\epsilon} & \epsilon^3(-s)^{2+3\epsilon} & \epsilon^4(-s)^{1+3\epsilon} \\
\epsilon^4(-s)^{1+3\epsilon}(1-2\epsilon) & \epsilon^4(-s)^{2+3\epsilon} & \epsilon^3(-s)^{3+3\epsilon} \\
\epsilon^4(-s)^{2+3\epsilon} z & \epsilon^5(-s)^{1+3\epsilon}(1+z) & \epsilon^5(-s)^{1+3\epsilon}(1+z) \\
\epsilon^4(-s)^{2+3\epsilon} z & \epsilon^4(-s)^{1+3\epsilon} (k_2+p_1)^2 & \epsilon^6(-s)^{1+3\epsilon} \\
\epsilon^6(-s)^{1+3\epsilon} z & \epsilon^5(-s)^{2+3\epsilon} & \epsilon^5(-s)^{3+3\epsilon} z \\
\epsilon^5(-s)^{1+3\epsilon} (k_2+p_1)^2 & \epsilon^6(-s)^{2+3\epsilon}(1+z) & \epsilon^5(-s)^{3+3\epsilon} z \\
\epsilon^5(-s)^{3+3\epsilon} z & \epsilon^5(-s)^{2+3\epsilon} (k_3+p_1)^2 & \epsilon^6(-s)^{4+3\epsilon} z \\
\epsilon^6(-s)^{3+3\epsilon} (p_1+k_3)^2 & \epsilon^6(-s)^{3+3\epsilon} (k_1-p_3)^2 &
\end{array}\right].$$

6.3.4 BCs

Analysis of the divergences

As discussed in section 2.3.2, maximal cuts are performed on the 3-loop ladder topology, loop by loop, obtaining:

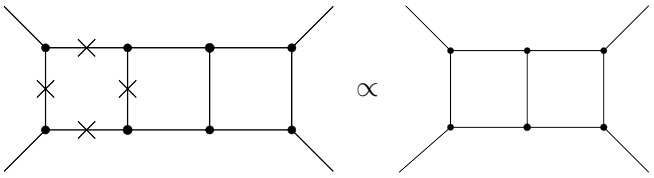
$$\frac{1}{s^3 t} \tag{6.81}$$

Table 6.1: Matrix A_0.

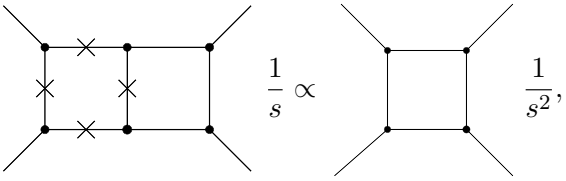
$$A_0 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&\frac{4}{3}&0&0&0&0&0&0&0&-3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-\frac{1}{6}&0&-1&0&0&0&0&0&0&-3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\frac{1}{3}&-\frac{1}{3}&0&0&0&0&0&0&0&0&0&-3&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&-3&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-\frac{1}{4}&0&0&0&0&0&0&0&0&0&0&\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&3&0&0&0&0&0&0&0&0&0&0\\
0&\frac{1}{3}&-8&-8&0&0&0&0&0&0&0&0&0&0&0&12&-3&0&0&0&0&0&0&0&0&0\\
0&-2&0&8&0&0&0&0&0&\frac{3}{2}&24&0&2&0&0&0&0&-3&0&0&0&0&0&0&0&0\\
0&2&0&-\frac{40}{9}&0&-\frac{2}{3}&0&\frac{4}{3}&0&-1&-24&0&-2&-\frac{8}{3}&0&0&0&\frac{2}{3}&1&0&0&0&0&0&0&0\\
\frac{17}{54}&\frac{23}{27}&-\frac{14}{3}&-\frac{56}{9}&\frac{1}{6}&-\frac{1}{6}&0&0&0&-1&-\frac{20}{3}&-2&0&0&-\frac{2}{3}&\frac{8}{3}&-2&0&0&1&\frac{1}{3}&0&0&0&0&0\\
-\frac{4}{3}&-\frac{4}{3}&0&0&0&0&0&0&0&3&0&12&0&0&0&0&0&0&0&0&-3&0&0&0&0&0\\
-\frac{4}{3}&-\frac{4}{3}&0&0&0&0&0&0&0&3&0&12&0&0&0&0&0&0&0&0&0&-3&0&0&0&0\\
\frac{19}{9}&\frac{20}{9}&0&0&0&-2&1&0&0&-3&0&-20&0&0&0&0&0&0&0&0&0&2&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&0&0&-4&2&0&-3&0&0\\
\frac{41}{18}&-\frac{8}{3}&\frac{14}{3}&\frac{68}{3}&\frac{7}{2}&-\frac{7}{2}&3&3&1&0&48&-12&4&0&-2&0&0&-6&0&0&6&-2&0&2&1&0\\
-\frac{7}{6}&-\frac{28}{9}&\frac{22}{3}&\frac{20}{3}&\frac{3}{2}&\frac{9}{2}&0&3&1&3&16&12&0&0&6&-16&4&-2&0&-12&2&-3&-3&1&1&0
\end{pmatrix};$$

$$A_{-1} = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-\frac{4}{3}&0&0&0&4&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0&0&0&2&4&0&0&0&0&0&0&0&0&0&0&0&0\\
0&\frac{1}{4}&0&0&0&0&0&0&0&0&0&0&-\frac{1}{2}&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-\frac{1}{12}&2&2&0&0&0&0&0&0&2&0&0&0&0&-3&1&0&0&0&0&0&0&0&0&0\\
0&-\frac{1}{3}&8&8&0&0&0&0&0&0&0&0&0&0&0&-12&4&0&0&0&0&0&0&0&0&0\\
0&2&0&-\frac{40}{3}&0&4&0&-2&0&-\frac{3}{2}&-24&0&-2&0&0&0&0&2&3&0&0&0&0&0&0&0\\
0&-2&0&\frac{40}{9}&0&\frac{2}{3}&0&\frac{2}{3}&0&1&24&0&2&\frac{8}{3}&0&0&0&-\frac{2}{3}&-1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
\frac{13}{9}&-\frac{16}{9}&12&\frac{40}{3}&2&7&0&0&0&0&16&0&0&0&4&-16&4&0&0&-12&1&0&0&0&0&0\\
\frac{5}{3}&\frac{4}{3}&0&0&0&6&-3&0&0&-3&0&-12&0&0&0&0&0&0&0&0&0&3&3&0&0&0\\
-\frac{10}{9}&-\frac{20}{9}&0&0&0&0&2&0&0&3&0&20&0&0&0&0&0&0&0&0&0&-2&-2&0&0&0\\
-\frac{17}{9}&0&-\frac{28}{3}&-\frac{40}{3}&-7&7&-6&-6&-2&0&0&0&0&0&4&0&0&-4&0&0&4&-2&0&3&2&2\\
-\frac{1}{2}&\frac{52}{9}&-\frac{22}{3}&-\frac{100}{3}&\frac{3}{2}&-\frac{5}{2}&3&-1&1&-3&-64&0&-4&0&-6&16&-4&6&6&12&-4&2&0&-2&-1&-2\\
\frac{7}{6}&\frac{28}{9}&-\frac{22}{3}&-\frac{20}{3}&\frac{3}{2}&-\frac{9}{2}&0&3&1&-3&-16&-12&0&0&-6&16&-4&2&0&12&-2&3&3&-1&-1&0
\end{pmatrix}.$$

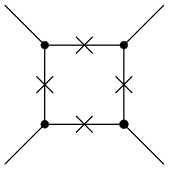
Table 6.2: Matrix A_{-1}.



$$\propto \frac{1}{s}, \tag{6.82}$$



$$\frac{1}{s} \propto \frac{1}{s^2}, \tag{6.83}$$



$$\frac{1}{s^2} \propto \frac{1}{s^3 t}, \tag{6.84}$$

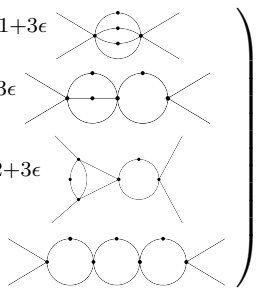
The divergences occur in $s = 0$ or in $t = 0$. In particular, in $u = 0 \Rightarrow s = -t$ all the MIs are regular.

In order to identify the solution to the problem, as explained for the 1-loop box, two conditions are considered:

- finiteness of the solutions in $z = -1$ (giving eleven conditions);
- reality of the solutions for $z \geq 0$ (giving other eleven conditions).

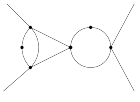
The number of MMIs is four. Solving the relations, the basis of MMIs can be chosen as:

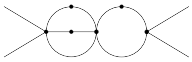
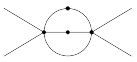
$$\left(\begin{array}{c} \epsilon^3(-s)^{1+3\epsilon} \\ \epsilon^3(-s)^{2+3\epsilon} \\ \epsilon^4(-s)^{2+3\epsilon} \\ \epsilon^3(-s)^{3+3\epsilon} \end{array} \right) . \tag{6.85}$$



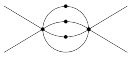
As before, to simplify the problem a suitable integration measure is chosen: the measure normalizes the value of $\left[\epsilon(-s)^{1+\epsilon}\right]^3 = \epsilon^3(-s)^{3+3\epsilon}$ to 1, so

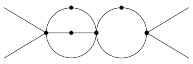
$$\mathrm{d}_B^D k_1 \mathrm{d}_B^D k_2 \mathrm{d}_B^D k_3 = \left[\frac{(\mu^2)^\epsilon}{\mathrm{i}\pi^{2-\epsilon}} \frac{\Gamma(1-2\epsilon)}{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}\right]^3 \mathrm{d}^D k_1 \mathrm{d}^D k_2 \mathrm{d}^D k_3. \tag{6.86}$$

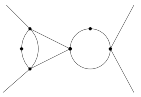
The third MMI is recovered from the 2-loop ladder problem. Notice that, thanks to the normalization factor, the expression of $\epsilon^4(-s)^{2+3\epsilon}$  is equal to the expression of

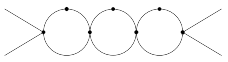
$\epsilon^3(-s)^{1+2\epsilon}$  , and the expression $\epsilon^3(-s)^{2+3\epsilon}$  to $\epsilon^2(-s)^{1+2\epsilon}$  : the bubble of the 3-loop graph is eliminated by the normalization, leaving only the other structure with 2-loop normalization. Moreover, since the BCs are fixed at $z = 0$, for the triangular MMI only the expansion in terms of initial values in $z = 0$ is necessary (all other MMI do not evolve, so their expansions coincide with the expansion of their initial values).

The MMIs assume the form:

$$\epsilon^3(-s)^{1+3\epsilon} \quad \text{  } = \frac{\Gamma(1+3\epsilon)\Gamma^3(1-2\epsilon)}{\Gamma(1-4\epsilon)\Gamma^3(1+\epsilon)\Gamma^2(1-\epsilon)} \tag{6.87}$$

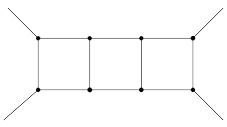
$$\epsilon^3(-s)^{2+3\epsilon} \quad \text{  } = -\frac{\Gamma(1+2\epsilon)\Gamma^2(1-2\epsilon)}{\Gamma(1-3\epsilon)\Gamma^2(1+\epsilon)\Gamma(1-\epsilon)}; \tag{6.88}$$

$$\begin{aligned}\epsilon^4(-s)^{2+3\epsilon} \quad \text{  } &= \frac{1}{4} + \frac{\pi^2}{12}\epsilon^2 + \frac{1}{4}\left[3\psi^{(2)}(1) + 2\zeta(3)\right]\epsilon^3 + \\ &+ \frac{1}{48}\left[72\zeta(5) + 4\pi^2\left(2\zeta(3) + 3\psi^{(2)}(1)\right) + 21\psi^{(4)}(1)\right]\epsilon^5 + \\ &+ \left[\frac{1}{8}\left(2\zeta(3) + 3\psi^{(2)}(1)\right)^2 - \frac{13\pi^6}{540}\right]\epsilon^6 + \\ &+ \left[\frac{9\zeta(7)}{2} + \frac{1}{48}\pi^2\left(24\zeta(5) + 7\psi^{(4)}(1)\right) + \frac{49\psi^{(6)}(1)}{480}\right]\epsilon^7 + o_0\left(\epsilon^7\right)\end{aligned} \tag{6.89}$$

$$\epsilon^3(-s)^{3+3\epsilon} \quad \text{  } = 1. \tag{6.90}$$

The basis is proven to be pure UT. Notice that, even not knowing the weight of $\psi^{(n)}(a)$ functions, the fact that they appear in the coefficient of a specific term ϵ^n of a known pure UT function assures that they have a well-defined and fixed weight, equal to n.

6.3.5 3-loop ladder graph

$$\epsilon^6(-s)^{4+3\epsilon}z \quad \text{  } = \sum_{i=0}^{7}\epsilon^i f_i(z) + o_0\left(\epsilon^7\right); \tag{6.91}$$

- $f_0 = 16/9$;
- $f_1(z) = -11/3H(0;z)$;
- $f_2(z) = -((19\pi^2)/18) + 3H(0;z)^2$;
- $f_3(z) = 1/18(-9(H(0;z)^3 + 3H(-1;z)(\pi^2 + H(0;z)^2) - 3H(0;z)(3\pi^2 + 2H(0;-1;z)) + 6H(0;0;-1;z)) - 38\zeta(3))$;

- $f_4(z) = -((43\pi^4)/30)+3H(0;0;-1;-1;z)+51H(0;0;0;-1;z)+1/4(-6H(0;z)^4-3H(-1;z)^2(\pi^2+H(0;z)^2)+32\pi^2H(0;-1;z)+5H(0;z)^2(-7\pi^2+2H(0;-1;z))-4/3H(0;z)(9H(0;-1;-1;z)+84H(0;0;-1;z)-5\zeta(3))+2H(-1;z)(6H(0;z)^3+H(0;z)(7\pi^2+6H(0;-1;z))-6(H(0;0;-1;z)+\zeta(3))));$

- $f_5(z) = 81/40H(0;z)^5-1/4H(-1;z)^3(\pi^2+H(0;z)^2)+1/4H(0;z)^3(43\pi^2-38H(0;-1;z))-8\pi^2H(0;-1;-1;z) - 37/3\pi^2H(0;0;-1;z) + 12H(0;-1;z)H(0;0;-1;z) + H(0;z)((46\pi^4)/9 - 49/3\pi^2H(0;-1;z) - 6H(0;-1;z)^2 + 3H(0;-1;-1;-1;z)+52H(0;0;-1;-1;z)+24H(0;0;0;-1;z))-3(H(0;0;-1;-1;-1;z)+4H(0;0;-1;0;-1;z) + 41H(0;0;0;-1;-1;z) + 81H(0;0;0;0;-1;z)) + ((65\pi^2)/9 + 96H(0;-1;z))\zeta(3) + 1/2H(0;z)^2 (-5H(0;-1;-1;z) + 76H(0;0;-1;z) + 9\zeta(3)) + 1/4H(-1;z)^2(6H(0;z)^3 + H(0;z)(7\pi^2 + 6H(0;-1;z))-6(H(0;0;-1;z)+\zeta(3)))-1/8H(-1;z)(27H(0;z)^4+H(0;z)^2(58\pi^2-68H(0;-1;z))+8(2\pi^4-14\pi^2H(0;-1;z)-3H(0;0;-1;-1;z)-87H(0;0;0;-1;z))+8H(0;z)(3H(0;-1;-1;z)+52H(0;0;-1;z) + 49\zeta(3))) + 13\zeta(5);$

- $f_6(z) = -((10705\pi^6)/6804)-63/40H(0;z)^6-1/16H(-1;z)^4(\pi^2+H(0;z)^2)+739/90\pi^4H(0;-1;z)-21\pi^2H(0;-1;z)^2+9/16H(0;z)^4(-17\pi^2+22H(0;-1;z))+4H(0;0;-1;z)(-27H(0;-1;-1;z)+4H(0;0;-1;z)) - 76\pi^2H(0;-1;-1;-1;z) + 1/3\pi^2H(0;0;-1;-1;z) + 36H(0;-1;z) H(0;0;-1;-1;z) + H(0;z)^2(-((533\pi^4)/60) + 57/2\pi^2H(0;-1;z) - 6H(0;-1;z)^2 - 163/2H(0;-1;-1;-1;z)-78H(0;0;-1;-1;z)-36H(0;0;0;-1;z))+35\pi^2H(0;0;0;-1;z)-216H(0;-1;z)H(0;0;0;-1;z) + 3H(0;0;-1;-1;-1;-1;z) - 36H(0;0;-1;-1;0;-1;z) + 36H(0;0;-1;0;-1;-1;z) + 195H(0;0;0;-1;-1;-1;z) + 216H(0;0;0;-1;0;-1;z) + 1011H(0;0;0;0;-1;-1;z) + 951H(0;0;0;0;0;-1;z) + 1/2H(0;z)^3(19H(0;-1;-1;z) - 68H(0;0;-1;z) - 23\zeta(3)) - 4(48H(0;-1;-1;z) + 119H(0;0;-1;z))\zeta(3) - (358\zeta(3)^2)/9 + 1/12H(-1;z)^3(6H(0;z)^3 + H(0;z)(7\pi^2 + 6H(0;-1;z)) - 6(H(0;0;-1;z) + \zeta(3))) - 1/16H(-1;z)^2(27H(0;z)^4 + H(0;z)^2(58\pi^2 - 68H(0;-1;z)) + 8(2\pi^4 - 14\pi^2H(0;-1;z) - 3H(0;0;-1;-1;z) - 87H(0;0;0;-1;z)) + 8H(0;z)(3H(0;-1;-1;z) + 52H(0;0;-1;z) + 49\zeta(3)))+1/60H(-1;z)(162H(0;z)^5+15H(0;z)^3(37\pi^2-86H(0;-1;z))+2H(0;z)(107\pi^4-10H(0;-1;z)(115\pi^2 + 108H(0;-1;z)) + 90H(0;-1;-1;-1;z) + 1560H(0;0;-1;-1;z)-2880H(0;0;0;-1;z))+30H(0;z)^2(55H(0;-1;-1;z)+188H(0;0;-1;z)+117\zeta(3)) + 20(66\pi^2H(0;-1;-1;z) + (47\pi^2 + 216H(0;-1;z))H(0;0;-1;z) + (-43\pi^2 + 468H(0;-1;z))\zeta(3) - 9(H(0;0;-1;-1;-1;z)+24H(0;0;-1;0;-1;z)+101H(0;0;0;-1;-1;z)+49H(0;0;0;0;-1;z)-19\zeta(5)))) + 1/3H(0;z)((97\pi^2 + 324H(0;-1;z)) H(0;-1;-1;z) + 4(\pi^2 + 90H(0;-1;z))H(0;0;-1;z) - 9H(0;-1;-1;-1;-1;z) - 12(36H(0;-1;0;-1;-1;z) + 85H(0;0;-1;-1;-1;z) + 38H(0;0;-1;0;-1;z) + 90H(0;0;0;-1;-1;z)) + 3(-35\pi^2 + 24H(0;-1;z))\zeta(3) - 471\zeta(5));$

- $f_7(z) = 513/560H(0;z)^7-1/80H(-1;z)^5(\pi^2+H(0;z)^2)+9/80H(0;z)^5(59\pi^2-94H(0;-1;z))-499/90\pi^4H(0;-1;-1;z) + 18\pi^2H(0;-1;z)H(0;-1;-1;z) - 179/15\pi^4H(0;0;-1;z) - 48\pi^2H(0;-1;z)H(0;0;-1;z) - 154H(0;-1;z)^2 H(0;0;-1;z) - 156H(0;0;-1;z)H(0;-1;-1;-1;z) + 156H(0;-1;-1;z) H(0;0;-1;-1;z) + 40H(0;0;-1;z)H(0;0;-1;-1;z) + 264H(0;-1;-1;z) H(0;0;0;-1;z)-168H(0;0;-1;z)H(0;0;0;-1;z)+H(0;z)^3((101\pi^4)/10-65/2\pi^2H(0;-1;z)+$

$$21H(0;-1;z)^2 + 317/2H(0;-1;-1;-1;z) + 78H(0;0;-1;-1;z) + 36H(0;0;0;-1;z)) -$$
$$56\pi^2H(0;-1;-1;-1;-1;z) - 132\pi^2H(0;-1;0;-1;-1;z) - 1105/3\pi^2H(0;0;-1;-1;-1;z) -$$
$$84H(0;-1;z)H(0;0;-1;-1;-1;z) + 220/3\pi^2H(0;0;-1;0;-1;z) +$$
$$308H(0;-1;z)H(0;0;-1;0;-1;z) + 101\pi^2H(0;0;0;-1;-1;z) + 1080H(0;-1;z)$$
$$H(0;0;0;-1;-1;z) - 87\pi^2H(0;0;0;0;-1;z) +$$
$$588H(0;-1;z)H(0;0;0;0;-1;z) - 3H(0;0;-1;-1;-1;-1;-1;z) +$$
$$84H(0;0;-1;-1;-1;0;-1;z) + 12H(0;0;-1;-1;0;-1;-1;z) -$$
$$60H(0;0;-1;0;-1;-1;-1;z) - 308H(0;0;-1;0;-1;0;-1;z) -$$
$$976H(0;0;-1;0;0;-1;-1;z) - 267H(0;0;0;-1;-1;-1;-1;z) - 1080H(0;0;0;-1;-1;0;-1;z) -$$
$$3504H(0;0;0;-1;0;-1;-1;z) +$$
$$224H(0;0;0;-1;0;0;-1;z) - 8307H(0;0;0;0;-1;-1;-1;z) + 308H(0;0;0;0;-1;0;-1;z) -$$
$$2351H(0;0;0;0;0;-1;-1;z) - 3591H(0;0;0;0;0;0;-1;z) -$$
$$3/8H(0;z)^4(33H(0;-1;-1;z) - 60H(0;0;-1;z) - 37\zeta(3)) + ((307\pi^4)/30 + 2(\pi^2 - 137H(0;-1;z))$$
$$H(0;-1;z) - 72H(0;-1;-1;-1;z) + 740H(0;0;-1;-1;z) + 1044H(0;0;0;-1;z))\zeta(3) +$$
$$1/48H(-1;z)^4(6H(0;z)^3 + H(0;z)(7\pi^2 + 6H(0;-1;z)) - 6(H(0;0;-1;z) + \zeta(3)))$$
$$+ H(0;z)((52357\pi^6)/11340 + 59\pi^2H(0;-1;z)^2 + 154/3H(0;-1;z)^3 - 78H(0;-1;-1;z)^2 +$$
$$24H(0;0;-1;z)^2 + 539/3\pi^2H(0;-1;-1;-1;z) + 8\pi^2H(0;0;-1;-1;z) + H(0;-1;z)(-((277\pi^4)/18) +$$
$$156H(0;-1;-1;-1;z) - 176H(0;0;-1;-1;z) + 120H(0;0;0;-1;z)) + 3H(0;-1;-1;-1;-1;-1;z) -$$
$$72H(0;-1;0;-1;-1;-1;z) - 92H(0;0;-1;-1;-1;-1;z) + 136H(0;0;-1;-1;0;-1;z) +$$
$$552H(0;0;-1;0;-1;-1;z) + 1560H(0;0;0;-1;-1;-1;z) - 176H(0;0;0;-1;0;-1;z) -$$
$$160H(0;0;0;0;-1;-1;z) + 464H(0;0;-1;z)\zeta(3) + (32\zeta(3)^2)/3 + 8H(0;-1;-1;z)(5H(0;0;-1;z) +$$
$$41\zeta(3))) - 1/48H(-1;z)^3(27H(0;z)^4 + H(0;z)^2(58\pi^2 - 68H(0;-1;z)) + 8(2\pi^4 - 14\pi^2H(0;-1;z) -$$
$$3H(0;0;-1;-1;z) - 87H(0;0;0;-1;z)) + 8H(0;z)(3H(0;-1;-1;z) + 52H(0;0;-1;z) +$$
$$49\zeta(3))) + 1/120H(-1;z)^2(162H(0;z)^5 + 15H(0;z)^3(37\pi^2 - 86H(0;-1;z)) + 2H(0;z)(107\pi^4 -$$
$$10H(0;-1;z)(115\pi^2 + 108H(0;-1;z)) + 90H(0;-1;-1;-1;z) + 1560H(0;0;-1;-1;z) -$$
$$2880H(0;0;0;-1;z)) + 30H(0;z)^2(55H(0;-1;-1;z) + 188H(0;0;-1;z) + 117\zeta(3)) +$$
$$20(66\pi^2H(0;-1;-1;z) + (47\pi^2 + 216H(0;-1;z))H(0;0;-1;z) + (-43\pi^2 + 468H(0;-1;z))\zeta(3) -$$
$$9(H(0;0;-1;-1;-1;z) + 24H(0;0;-1;0;-1;z) + 101H(0;0;0;-1;-1;z) + 49H(0;0;0;0;-1;z) -$$
$$19\zeta(5)))) + (191\pi^2 + 828H(0;-1;z))\zeta(5) + 1/2H(0;z)^2(-(105\pi^2 + 344H(0;-1;z))H(0;-1;-1;z) -$$
$$4(\pi^2 + 100H(0;-1;z))H(0;0;-1;z) - 101H(0;-1;-1;-1;-1;z) + 232H(0;-1;0;-1;-1;z) +$$
$$412H(0;0;-1;-1;-1;z) + 472H(0;0;-1;0;-1;z) + 1048H(0;0;0;-1;-1;z) + (145\pi^2 -$$
$$188H(0;-1;z))\zeta(3) + 825\zeta(5)) - 1/15120H(-1;z)(20(1535\pi^6 - 2268H(0;0;-1;-1;-1;-1;z) -$$
$$54432H(0;0;-1;-1;0;-1;z) - 108864H(0;0;-1;0;-1;-1;z) - 392364H(0;0;0;-1;-1;-1;z) -$$
$$317520H(0;0;0;-1;0;-1;z) - 1381212H(0;0;0;0;-1;-1;z) - 1248156H(0;0;0;0;0;-1;z)) +$$
$$21(1215H(0;z)^6 + 405H(0;z)^4(15\pi^2 - 46H(0;-1;z)) + 72H(0;z)^2(72\pi^4 - 5H(0;-1;z)(131\pi^2 +$$
$$74H(0;-1;z)) + 275H(0;-1;-1;-1;z) - 460H(0;0;-1;-1;z) - 2120H(0;0;0;-1;z)) +$$
$$360H(0;z)^3(101H(0;-1;-1;z) + 220H(0;0;-1;z) + 87\zeta(3)) + 8(2610\pi^2H(0;-1;z)^2 +$$
$$H(0;-1;z)(-1147\pi^4 + 6480H(0;0;-1;-1;z) + 37800H(0;0;0;-1;z)) + 30(-420H(0;0;-1;z)^2 +$$
$$\pi^2(66H(0;-1;-1;-1;z) - 373H(0;0;-1;-1;z) - 441H(0;0;0;-1;z)) + 252(H(0;-1;-1;z) +$$
$$5H(0;0;-1;z))\zeta(3) + 528\zeta(3)^2)) + 240H(0;z)(137\pi^2H(0;-1;-1;z) + 8(35\pi^2 -$$
$$51H(0;-1;z))H(0;0;-1;z) + 3(3H(0;-1;-1;-1;-1;z) - 72H(0;-1;0;-1;-1;z) -$$
$$92H(0;0;-1;-1;-1;z) + 416H(0;0;-1;0;-1;z) + 1152H(0;0;0;-1;-1;z) + 544H(0;0;0;0;-1;z) +$$
$$(29\pi^2 + 68H(0;-1;z))\zeta(3) + 737\zeta(5))))) - (509\zeta(7))/3;$$

Here the 3-loop ladder has a ϵ^{-6} pole, as expected.

The results for the other graphs of the basis are shown in THESIS_3L_BC.

Chapter 7

Evaluation using eigenvalue deflation

1-loop box, 2-loop ladder and a subsystem of the 3-loop ladder differential systems are rewritten in a canonical fuchsian form using tje eigenvalue deflation method [16] (and chapter 5), starting from the Laporta basis of master integrals (except for topologies with multiple masters). First, the systems are transformed into fuchsian ones [16](using algorithm 1 of chapter 5), then a form with ϵ-homogeneous eigenvalues is reached [16] (using algorithm 2 of chapter 5). Finally, a similarity transformation allows to reach a fuchsian ϵ-factorized form. The comparison between the resulting bases of master integrals and the ones obtained via Magnus series expansion is realized through integration-by-parts identities.

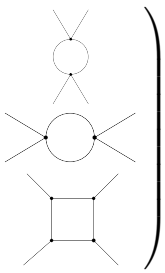
7.1 1-loop box

The integral to be evaluated is the one presented in section 6.1.

See THESIS_1L_ER for the explicit calculations.

Starting from the Laporta basis:

$$\mathcal{I}(s;t) = \left(\right. \quad \left. \right) , \tag{7.1}$$



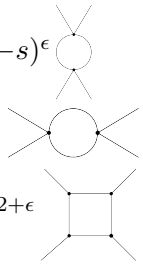
the equations in $(s;t)$ are of the form:

$$\frac{\partial}{\partial s}\mathcal{I}(s;t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{\epsilon}{s} & 0 \\ \frac{2(2\epsilon-1)}{st(s+t)} & -\frac{2(2\epsilon-1)}{s^2(s+t)} & -\frac{s+t+t\epsilon}{s(s+t)} \end{pmatrix} \mathcal{I}(s;t); \tag{7.2}$$

$$\frac{\partial}{\partial t}\mathcal{I}(s;t) = \begin{pmatrix} -\frac{\epsilon}{t} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{2(2\epsilon-1)}{t^2(s+t)} & \frac{2(2\epsilon-1)}{st(s+t)} & -\frac{\epsilon s+s+t}{t(s+t)} \end{pmatrix} \mathcal{I}(s;t). \tag{7.3}$$

Passing now to $(s;z)$ variables, and absorbing the s-dependance through a change of variables as in 6.1.3, the DEs assume the aspect:

$$I(z) = \begin{pmatrix} (-s)^{\epsilon} \\ (-s)^{\epsilon} \\ (-s)^{2+\epsilon} \end{pmatrix}, \tag{7.4}$$



$$\frac{\partial I(z)}{\partial s} = \mathbb{O}; \tag{7.5}$$

$$\frac{\partial I(z)}{\partial z} = \begin{pmatrix} -\frac{\epsilon}{z} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{2(2\epsilon-1)}{z} - \frac{2(2\epsilon-1)}{z^2} - \frac{2(2\epsilon-1)}{z+1} & \frac{2(2\epsilon-1)}{z} - \frac{2(2\epsilon-1)}{z+1} & \frac{-\epsilon-1}{z} + \frac{\epsilon}{z+1} \end{pmatrix}. \tag{7.6}$$

7.1.1 Reduction to fuchsian form

A pole in $z = 0$ with degree 2 have to be eliminated. Jordanizing the matrix of coefficients of $1/z^2$, the matrix of coefficients of the system assumes the form:

$$A(z;\epsilon) = \begin{pmatrix} \frac{-\epsilon-1}{z} + \frac{\epsilon}{z+1} & \frac{1}{z^2} + \frac{1}{z+1} - \frac{1}{z} & \frac{2(2\epsilon-1)}{z} - \frac{2(2\epsilon-1)}{z+1} \\ 0 & -\frac{\epsilon}{z} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.7}$$

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q(\epsilon) = \begin{pmatrix} -\epsilon-1 & -1 & 2(2\epsilon-1) \\ 0 & -\epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.8}$$

M is nilpotent and has eigenvalue $u_1 = (1;0;0)^T$, and the corresponding column of Q, $q_{i;1} = (-\epsilon-1;0;0)^T$ is null in all entries except in $Q_{1;1}$, as required by algorithm 1, while the transformed Q (from expression 5.51) assumes the form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.9}$$

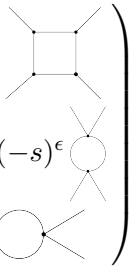
It is possible to choose as second point of balance $z = \infty$, since A can be written in terms of only poles at finite:

$$\mathcal{B} = \begin{pmatrix} -\frac{1}{z} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathbb{P} = u_1 x_1^\dagger = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}. \tag{7.10}$$

The final result, obtained after one iteration, is:

$$A = \begin{pmatrix} \frac{\epsilon}{z+1} - \frac{\epsilon}{z} & \frac{1}{z+1} - \frac{1}{z} & -\frac{2(2\epsilon-1)}{z+1} \\ 0 & -\frac{\epsilon}{z} & 0 \\ 0 & 0 & 0 \end{pmatrix}; \tag{7.11}$$

$$I(z) = \begin{pmatrix} -(-s)^{2+\epsilon} z \\ -2(2\epsilon-1)(-s)^{\epsilon} \\ (-s)^{\epsilon} \end{pmatrix}. \tag{7.12}$$



As can be seen, the matrix is now fuchsian in all its singular points.

7.1.2 Reduction to ϵ-homogeneous eigenvalues

There are three residues matrices to check for ϵ-homogeneous eigenvalues:

- residues in $z = 0$

$$A_0 = \begin{pmatrix} -\epsilon & -1 & 0 \\ 0 & -\epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.13}$$

  with eigenvalues $\{(-\epsilon)_{\mathbf{2}}; 0\}$ (the subscript indicates that the eigenvalue belongs to a Jordan block of the indicated dimension; if no subscript is present, the eigenvalue is linked to a 1×1 block);

- residues in $z = -1$

$$A_{-1} = \begin{pmatrix} \epsilon & 1 & -2(2\epsilon-1) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.14}$$

  with eigenvalues $\{\epsilon; 0; 0\}$;

- residues at infinity

$$A_\infty = -A_0 - A_{-1} = \begin{pmatrix} 0 & 0 & 4\epsilon-2 \\ 0 & \epsilon & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.15}$$

  with eigenvalues $\{\epsilon; 0_{\mathbf{2}}\}$.

The matrix has then already ϵ-homogeneous eigenvalues in all its residues.

7.1.3 Factoring out ϵ

As explained in section 5.3.4, to determine the transformation matrix $\tilde{T}$ to pass into a canonical form, the following system must be solved:

$$\begin{cases} \frac{1}{\epsilon} A_0(z;\epsilon)\tilde{T}(\epsilon;\mu) = \frac{1}{\mu}\tilde{T}(\epsilon;\mu)A_0(z;\mu) \\ \frac{1}{\epsilon} A_{-1}(z;\epsilon)\tilde{T}(\epsilon;\mu) = \frac{1}{\mu}\tilde{T}(\epsilon;\mu)A_{-1}(z;\mu) \end{cases} ; \tag{7.16}$$

the result is a matrix of the form:

$$\tilde{T}(\epsilon;\mu;t) = \begin{pmatrix} t & 0 & 0 \\ 0 & \frac{\epsilon t}{\mu} & 0 \\ 0 & 0 & \frac{\epsilon(2\mu-1)t}{(2\epsilon-1)\mu} \end{pmatrix}. \tag{7.17}$$

Choosing $\mu = t = 1$, the matrix becomes

$$\tilde{T}(\epsilon) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \frac{\epsilon}{2\epsilon-1} \end{pmatrix}. \tag{7.18}$$

This matrix is invertible, and its application to A results in a canonical system of DE:

$$A = \epsilon \begin{pmatrix} \frac{1}{z+1} - \frac{1}{z} & \frac{1}{z+1} - \frac{1}{z} & -\frac{2}{z+1} \\ 0 & -\frac{1}{z} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} -z(-s)^{2+\epsilon} \\ \frac{2(1-2\epsilon)}{\epsilon}(-s)^{\epsilon} \\ \frac{(2\epsilon-1)}{\epsilon}(-s)^{\epsilon} \end{pmatrix}. \tag{7.19}$$

Using IBP-Ids it is possible to rewrite the basis above in terms of UT diagrams, obtaining:

$$I = \begin{pmatrix} -(-s)^{2+\epsilon} z \\ \frac{2}{\epsilon}(-s)^{1+\epsilon} z \\ -\frac{1}{\epsilon}(-s)^{1+\epsilon} \end{pmatrix}. \tag{7.20}$$

Multiplying the MIs by ϵ^2, dividing the second one by 2 and changing sing to the first and the third MIs, the basis determined using Magnus series expansion is retrieved:

$$A = \epsilon \begin{pmatrix} \frac{1}{z+1} - \frac{1}{z} & \frac{2}{z} - \frac{2}{z+1} & -\frac{2}{z+1} \\ 0 & -\frac{1}{z} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} \epsilon^2(-s)^{2+\epsilon} z \\ \epsilon(-s)^{1+\epsilon} z \\ \epsilon(-s)^{1+\epsilon} \end{pmatrix}. \tag{7.21}$$

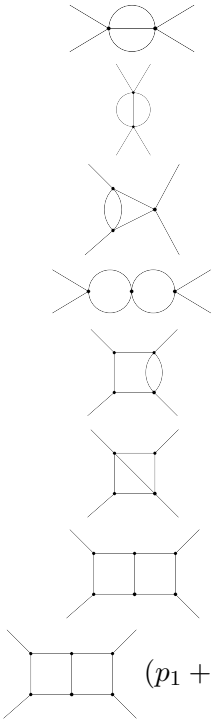
In this case the eigenvalue deflation method has determined a system identical to the one found using Magnus series expansion method, up to constants or global powers of ϵ. From there, the BC fixing and the evaluation of the solutions follows exactly the path presented in section 6.1.

7.2 2-loop ladder

For the explicit calculations, see THESIS_2L_ER.

Here the integral presented in section 6.2 is faced using eigenvalue deflation techniques. Starting from the basis:

$$\mathcal{I}(s;t) = \left(\begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \\ (p_1+k_2)^2 \end{array} \right) \tag{7.22}$$



the equation in $(s;t)$ have the following matrices of coefficients:

- $A_s(s;t;\epsilon)$

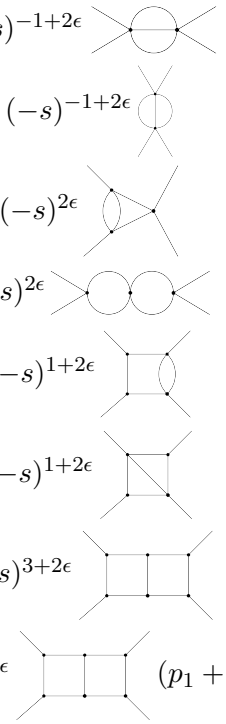
$$\left(\begin{array}{cccccccc} -\frac{2\epsilon-1}{s} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2\epsilon}{s} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2\epsilon}{s} & 0 & 0 & 0 & 0 \\ 0 & \frac{(3\epsilon-2)(3\epsilon-1)}{st(s+t)\epsilon} & \frac{3\epsilon-1}{s(s+t)} & 0 & -\frac{s+t+t\epsilon}{s(s+t)} & 0 & 0 & 0 \\ -\frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{s^2(s+t)\epsilon^2} & \frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{st(s+t)\epsilon^2} & 0 & 0 & 0 & -\frac{s+2t\epsilon}{s(s+t)} & 0 & 0 \\ \frac{6(s-t)(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{s^4t(s+t)\epsilon^2} & \frac{6(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{s^2t^2(s+t)\epsilon^2} & \frac{3(2s+t)(2\epsilon-1)(3\epsilon-1)}{s^3t(s+t)\epsilon} & \frac{2(2\epsilon-1)^2}{s^3(s+t)\epsilon} & \frac{12(2\epsilon-1)}{st(s+t)} & -\frac{12\epsilon}{s^2t} & -\frac{2(s+t+t\epsilon)}{s(s+t)} & \frac{2\epsilon}{s(s+t)} \\ \frac{6(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{s^3(s+t)\epsilon^2} & \frac{9(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{s^2t(s+t)\epsilon^2} & \frac{15(2\epsilon-1)(3\epsilon-1)}{2s^2(s+t)\epsilon} & \frac{(s+2t)(2\epsilon-1)^2}{s^3(s+t)\epsilon} & \frac{12(2\epsilon-1)}{s(s+t)} & -\frac{18\epsilon}{s^2} & \frac{t\epsilon}{s+t} & -\frac{3\epsilon s+2s+2t+2t\epsilon}{s(s+t)} \end{array} \right);$$

- $A_t(s;t;\epsilon)$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{2\epsilon-1}{t} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{(3\epsilon-2)(3\epsilon-1)}{t^2(s+t)\epsilon} & -\frac{3\epsilon-1}{t(s+t)} & 0 & -\frac{(2s+t)\epsilon}{t(s+t)} & 0 & 0 & 0 \\
\frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{st(s+t)\epsilon^2} & -\frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{t^2(s+t)\epsilon^2} & 0 & 0 & 0 & -\frac{t+2s\epsilon}{t(s+t)} & 0 & 0 \\
-\frac{6(s-t)(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{s^3t^2(s+t)\epsilon^2} & -\frac{6(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{st^3(s+t)\epsilon^2} & -\frac{3(2s+t)(2\epsilon-1)(3\epsilon-1)}{s^2t^2(s+t)\epsilon} & -\frac{2(2\epsilon-1)^2}{s^2t(s+t)\epsilon} & -\frac{12(2\epsilon-1)}{t^2(s+t)} & \frac{12\epsilon}{st^2} & -\frac{2\epsilon s+s+t}{t(s+t)} & -\frac{2\epsilon}{t(s+t)} \\
-\frac{6(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{s^2t(s+t)\epsilon^2} & -\frac{9(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{st^2(s+t)\epsilon^2} & -\frac{15(2\epsilon-1)(3\epsilon-1)}{2st(s+t)\epsilon} & -\frac{(s+2t)(2\epsilon-1)^2}{s^2t(s+t)\epsilon} & -\frac{12(2\epsilon-1)}{t(s+t)} & \frac{18\epsilon}{st} & -\frac{s\epsilon}{s+t} & \frac{s\epsilon}{t(s+t)}
\end{pmatrix}.$$

Passing to $(s;z)$ variables and absorbing the s-dependence, the system becomes:

$$I(z) = \begin{pmatrix}
(-s)^{-1+2\epsilon} & \\
(-s)^{-1+2\epsilon} & \\
(-s)^{2\epsilon} & \\
(-s)^{2\epsilon} & \\
(-s)^{1+2\epsilon} & \\
(-s)^{1+2\epsilon} & \\
(-s)^{3+2\epsilon} & \\
(-s)^{2+2\epsilon} & (p_1+k_2)^2
\end{pmatrix} = \phi, \tag{7.23}$$



$$A_s(s;z;\epsilon) = \mathbb{O}; \tag{7.24}$$

$$A_z(z;\epsilon) = \frac{A_1(\epsilon)}{z} + \frac{A_2(\epsilon)}{z^2} + \frac{A_3(\epsilon)}{z^3} + \frac{A_0(\epsilon)}{z+1} \tag{7.25}$$

- $A_1(\epsilon)$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1-2\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{9\epsilon^2-9\epsilon+2}{\epsilon} & 3\epsilon-1 & 0 & -2\epsilon & 0 & 0 & 0 \\
\frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & \frac{18\epsilon^3-27\epsilon^2+13\epsilon-2}{\epsilon^2} & 0 & 0 & 0 & -2\epsilon & 0 & 0 \\
\frac{12\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)}{\epsilon^2} & -\frac{6\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)}{\epsilon^2} & -\frac{3\left(6\epsilon^2-5\epsilon+1\right)}{\epsilon} & \frac{2(2\epsilon-1)^2}{\epsilon} & 12(2\epsilon-1) & 0 & -2\epsilon-1 & 2\epsilon \\
\frac{6(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & -\frac{9\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)}{\epsilon^2} & -\frac{15(2\epsilon-1)(3\epsilon-1)}{2\epsilon} & -\frac{(2\epsilon-1)^2}{\epsilon} & 12(2\epsilon-1) & -18\epsilon & 0 & \epsilon
\end{pmatrix}$$

- $A_2(\epsilon)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-9\epsilon^2+9\epsilon-2}{\epsilon} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-18\epsilon^3+27\epsilon^2-13\epsilon+2}{\epsilon^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{6\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)}{\epsilon^2} & \frac{6\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)}{\epsilon^2} & \frac{6\left(6\epsilon^2-5\epsilon+1\right)}{\epsilon} & 0 & -12(2\epsilon-1) & 12\epsilon & 0 & 0 \\ 0 & \frac{9\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)}{\epsilon^2} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.26}$$

- $A_3(\epsilon)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{6\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)}{\epsilon^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.27}$$

- $A_0(\epsilon)$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-9\epsilon^2+9\epsilon-2}{\epsilon} & 1-3\epsilon & 0 & \epsilon & 0 & 0 & 0 \\ -\frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & \frac{-18\epsilon^3+27\epsilon^2-13\epsilon+2}{\epsilon^2} & 0 & 0 & 0 & 2\epsilon-1 & 0 & 0 \\ -\frac{12\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)}{\epsilon^2} & \frac{6\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)}{\epsilon^2} & \frac{3\left(6\epsilon^2-5\epsilon+1\right)}{\epsilon} & -\frac{2(2\epsilon-1)^2}{\epsilon} & -12(2\epsilon-1) & 0 & 2\epsilon & -2\epsilon \\ -\frac{6(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & \frac{9\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)}{\epsilon^2} & \frac{15(2\epsilon-1)(3\epsilon-1)}{2\epsilon} & -\frac{(2\epsilon-1)^2}{\epsilon} & -12(2\epsilon-1) & 0 & \epsilon & -\epsilon \end{pmatrix}$$

7.2.1 Reduction to fuchsian form

Elimination of $1/z^3$

The pole in $z = 0$ presents terms up to degree 3. Starting from the pole of order 3, and jordanizing its matrix of coefficients, the system presents a matrix M for $1/z^3$ and a matrix Q for $1/z^2$ of the form:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.28}$$

$$Q(\epsilon) = \begin{pmatrix} 0 & -1 & 0 & 12\epsilon & -12(2\epsilon-1) & 0 & \frac{6\left(6\epsilon^2-5\epsilon+1\right)}{\epsilon} & -\frac{6\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)}{\epsilon^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon}{6(2\epsilon-1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.29}$$

M is nilpotent, and $q_{i;1}$, the column of $Q(\epsilon)$ corresponding to the eigenvector $u_1 = (1;0;0;0;0;0;0;0)^T$ of M, to be used in the projector, is null. Here a bit more is risked, and the projector is constructed as

$$\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.30}$$

using all the right proper eigenvectors of M. In this way, the matrix Q transforms into:

$$\begin{pmatrix} -\sum_{i\in I}(x_i^\dagger)_1 \sum_{j\in(\mathcal{N}\setminus I)}^N (x_1^\dagger)_j q_{j;i} & \dots & -\sum_{i\in I}(x_i^\dagger)_N \sum_{j\in(\mathcal{N}\setminus I)}^N (x_1^\dagger)_j q_{j;i} \\ \sum_{i\in I} q_{2;i}(x_i^\dagger)_1 & \dots & \sum_{i\in I} q_{2;i}(x_i^\dagger)_N \\ -\sum_{i\in I}(x_i^\dagger)_1 \sum_{j\in(\mathcal{N}\setminus I)}^N (x_3^\dagger)_j q_{j;i} & \dots & -\sum_{i\in I}(x_i^\dagger)_N \sum_{j\in(\mathcal{N}\setminus I)}^N (x_3^\dagger)_j q_{j;i} \\ \vdots & & \vdots \\ -\sum_{i\in I}(x_i^\dagger)_1 \sum_{j\in(\mathcal{N}\setminus I)}^N (x_N^\dagger)_j q_{j;i} & \dots & -\sum_{i\in I}(x_i^\dagger)_N \sum_{j\in(\mathcal{N}\setminus I)}^N (x_N^\dagger)_j q_{j;i} \end{pmatrix}, \tag{7.31}$$

equal to

$$\begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}, \tag{7.32}$$

so algorithm 1 can be applied, and infinite is chosen as second point for the balance transformation.

After the application of algorithm 1, the system has no more $1/z^3$ poles, and presents the form:

$$A(z;\epsilon) = \frac{A_1(\epsilon)}{z} + \frac{A_2(\epsilon)}{z^2} + \frac{A_0(\epsilon)}{z+1} \tag{7.33}$$

$$A_1 = \begin{pmatrix} -2\epsilon & 1 & 2\epsilon & 0 & 12(2\epsilon-1) & \frac{2(2\epsilon-1)^2}{\epsilon} & -\frac{3\left(6\epsilon^2-5\epsilon+1\right)}{\epsilon} & \frac{12\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)}{\epsilon^2} \\ 0 & 1-2\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & \epsilon+1 & -18\epsilon & 12(2\epsilon-1) & -\frac{(2\epsilon-1)^2}{\epsilon} & -\frac{15(2\epsilon-1)(3\epsilon-1)}{2\epsilon} & \frac{6(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} \\ 0 & -\frac{1}{6} & 0 & 1-2\epsilon & 0 & 0 & 0 & \frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} \\ 0 & -\frac{\epsilon}{6(2\epsilon-1)} & 0 & 0 & 1-2\epsilon & 0 & 3\epsilon-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.34}$$

$$A_2 = \begin{pmatrix} 0 & -1 & 0 & 12\epsilon & -12(2\epsilon-1) & 0 & \frac{6\left(6\epsilon^2-5\epsilon+1\right)}{\epsilon} & -\frac{6\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)}{\epsilon^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.35}$$

$$A_0 = \begin{pmatrix} 2\epsilon & -1 & -2\epsilon & 0 & -12(2\epsilon-1) & -\frac{2(2\epsilon-1)^2}{\epsilon} & \frac{3\left(6\epsilon^2-5\epsilon+1\right)}{\epsilon} & -\frac{12\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)}{\epsilon^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon & -\frac{3}{2} & -\epsilon & 0 & -12(2\epsilon-1) & -\frac{(2\epsilon-1)^2}{\epsilon} & \frac{15(2\epsilon-1)(3\epsilon-1)}{2\epsilon} & -\frac{6(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} \\ 0 & \frac{1}{6} & 0 & 2\epsilon-1 & 0 & 0 & 0 & -\frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} \\ 0 & \frac{\epsilon}{6(2\epsilon-1)} & 0 & 0 & \epsilon & 0 & 1-3\epsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{7.36}$$

Elimination of $1/z^2$

Proceeding as before, the matrix of coefficients of $1/z^2$ is jordanized, resulting in:

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.37}$$

$$Q(\epsilon) = \begin{pmatrix} -2\epsilon & -1 & \frac{6(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & \frac{3(2\epsilon-1)(3\epsilon-1)}{\epsilon} & \frac{2(2\epsilon-1)^2}{\epsilon} & 0 & 12\epsilon & 2\epsilon \\ 0 & 1-2\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{\epsilon}{6(2\epsilon-1)} & \frac{(3\epsilon-2)(3\epsilon-1)}{\epsilon} & 0 & 0 & 1 & -\frac{2\epsilon^2}{2\epsilon-1} & 0 \\ 0 & \frac{1}{6} & \frac{2(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & -\frac{(2\epsilon-1)(3\epsilon-1)}{\epsilon} & 0 & 2(2\epsilon-1) & 1-4\epsilon & 0 \\ 0 & -\frac{3}{2} & -\frac{3(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & \frac{3(2\epsilon-1)(3\epsilon-1)}{2\epsilon} & -\frac{(2\epsilon-1)^2}{\epsilon} & -6(2\epsilon-1) & 0 & \epsilon+1 \end{pmatrix}.$$

Also here, M is nilpotent, and choosing $u_1 = (1;0;0;0;0;0;0;0)^T$ as a vector to construct $\mathbb{P}$, the corresponding column in Q (the first one) is null except in the first entry. Is then possible to apply algorithm 1, and after one iteration the non-fuchsian pole is eliminated, again balancing with infinity.

The final result is:

$$A(z;\epsilon) = \frac{A_1(\epsilon)}{z} + \frac{A_0(\epsilon)}{z+1} \tag{7.38}$$

with

$$A_1 = \begin{pmatrix} 1-2\epsilon & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-2\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{\epsilon}{6(2\epsilon-1)} & \frac{(3\epsilon-2)(3\epsilon-1)}{\epsilon} & 0 & 0 & 1 & -\frac{2\epsilon^2}{2\epsilon-1} & 0 \\ 0 & \frac{1}{6} & \frac{2(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & -\frac{(2\epsilon-1)(3\epsilon-1)}{\epsilon} & 0 & 2(2\epsilon-1) & 1-4\epsilon & 0 \\ -\epsilon & -\frac{3}{2} & -\frac{3(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & \frac{3(2\epsilon-1)(3\epsilon-1)}{2\epsilon} & -\frac{(2\epsilon-1)^2}{\epsilon} & -6(2\epsilon-1) & 0 & \epsilon+1 \end{pmatrix},$$

$$A_0 = \begin{pmatrix} 2\epsilon & 1 & -\frac{6(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & -\frac{3(2\epsilon-1)(3\epsilon-1)}{\epsilon} & -\frac{2(2\epsilon-1)^2}{\epsilon} & 0 & -12\epsilon & -2\epsilon \\ 0 & 0 & -\frac{12(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon} & 12(2\epsilon-1)(3\epsilon-1) & 0 & -12\epsilon(2\epsilon-1) & 12\epsilon(2\epsilon-1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\epsilon}{6(2\epsilon-1)} & -\frac{(3\epsilon-2)(3\epsilon-1)}{\epsilon} & 0 & 0 & -\epsilon & \frac{2\epsilon^2}{2\epsilon-1} & 0 \\ 0 & -\frac{1}{6} & -\frac{2(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & \frac{(2\epsilon-1)(3\epsilon-1)}{\epsilon} & 0 & -2(2\epsilon-1) & 4\epsilon-1 & 0 \\ \epsilon & \frac{3}{2} & \frac{3(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & -\frac{3(2\epsilon-1)(3\epsilon-1)}{2\epsilon} & -\frac{(2\epsilon-1)^2}{\epsilon} & 6(2\epsilon-1) & -18\epsilon & -\epsilon \end{pmatrix},$$

The vector of MIs has now the following form:

$$\begin{pmatrix} (-s)^{3+2\epsilon} z^2 \phi_7 \\ \frac{6(-s)^{2\epsilon-1}\left(z\left(s\epsilon\left(2s\epsilon((2\epsilon-1)\phi_5-\epsilon\phi_6)+\left(6\epsilon^2-5\epsilon+1\right)\phi_3\right)+\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)\phi_1\right)+\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)\phi_2\right)}{\epsilon^2} \\ -(-s)^{-1+2\epsilon} z \phi_1 \\ -z(-s)^{2\epsilon}\phi_3 \\ -(-s)^{2\epsilon} z \phi_4 \\ -(-s)^{1+2\epsilon} z \phi_5 \\ -z(-s)^{1+2\epsilon}\phi_6 \\ -z(-s)^{2+2\epsilon}\phi_8 \end{pmatrix} \tag{7.39}$$

7.2.2 Reduction to ϵ-homogeneous eigenvalues

The residues are the following:

- residues in $z = 0$

$$\begin{pmatrix} 1-2\epsilon & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-2\epsilon & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{\epsilon}{6(2\epsilon-1)} & \frac{(3\epsilon-2)(3\epsilon-1)}{\epsilon} & 0 & 0 & 1 & -\frac{2\epsilon^2}{2\epsilon-1} & 0 \\ 0 & \frac{1}{6} & \frac{2(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & -\frac{(2\epsilon-1)(3\epsilon-1)}{\epsilon} & 0 & 2(2\epsilon-1) & 1-4\epsilon & 0 \\ -\epsilon & -\frac{3}{2} & -\frac{3(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & \frac{3(2\epsilon-1)(3\epsilon-1)}{2\epsilon} & -\frac{(2\epsilon-1)^2}{\epsilon} & -6(2\epsilon-1) & 0 & \epsilon+1 \end{pmatrix},$$

  with eigenvalues $\{1; 1; 1; 1+\epsilon; 1-2\epsilon; (1-2\epsilon)_{\mathbf{3}}\}$ (the subscript indicates that the corresponding Jordan block is $n \times n$, otherwise the block is 1×1);

- residues in $z = -1$

$$\begin{pmatrix} 2\epsilon & -1 & -2\epsilon & 0 & -12(2\epsilon-1) & -\frac{2(2\epsilon-1)^2}{\epsilon} & \frac{3\left(6\epsilon^2-5\epsilon+1\right)}{\epsilon} & -\frac{12\left(18\epsilon^3-27\epsilon^2+13\epsilon-2\right)}{\epsilon^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \epsilon & -\frac{3}{2} & -\epsilon & 0 & -12(2\epsilon-1) & -\frac{(2\epsilon-1)^2}{\epsilon} & \frac{15(2\epsilon-1)(3\epsilon-1)}{2\epsilon} & -\frac{6(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} \\ 0 & \frac{1}{6} & 0 & 2\epsilon-1 & 0 & 0 & 0 & -\frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} \\ 0 & \frac{\epsilon}{6(2\epsilon-1)} & 0 & 0 & \epsilon & 0 & 1-3\epsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

  with eigenvalues $\{0; 0; 0; 0; 0; \epsilon; \epsilon; 2\epsilon-1\}$;

- residues at infinity

$$\begin{pmatrix} -1 & 0 & \frac{6(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & 3\left(6\epsilon-5+\frac{1}{\epsilon}\right) & 8\epsilon-8+\frac{2}{\epsilon} & 0 & 12\epsilon & 2\epsilon \\ 0 & 2\epsilon-1 & \frac{12(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon} & -12\left(6\epsilon^2-5\epsilon+1\right) & 0 & 12\epsilon(2\epsilon-1) & -12\epsilon(2\epsilon-1) & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8\epsilon-8+\frac{2}{\epsilon} & 0 & 18\epsilon & -1 \end{pmatrix}$$

  with eigenvalues $\{0; -1; -1; (-1)_{\mathbf{3}}; \epsilon-1; 2\epsilon-1\}$.

All the eigenvalues are of the form $b\epsilon + n, n \in \mathbb{Z}$, and their sum (each one multiplied by the dimension of his Jordan block) returns a quantity proportional to ϵ, so it is possible to balance them to find residues with eigenvalues all proportional to ϵ.

To reduce the eigenvalues to an ϵ-homogeneous expression, the following couples are balanced (see THESIS_2L_ER for the ordered eigenvalues and the explicit calculations, "m→n" indicates that the m-th eigenvalue is decreased by 1 and the n-th eigenvalue is increased by 1):

1. $2_0 \to 3_\infty$ (from $(1; -1)$ to $(0; 0)$);
2. $3_0 \to 3_\infty$ (from $(1; -1)$ to $(0; 0)$);

3. $4_0 \to 4_\infty$ (from $(1+\epsilon;-1)$ to $(\epsilon;0)$);

4. $5_0 \to 8_{-1}$ (from $(1-2\epsilon;-1+2\epsilon)$ to $(2\epsilon;2\epsilon)$);

5. $5_0 \to 5_\infty$ (from $(1-2\epsilon;-1)$ to $(-2\epsilon;0)$);

6. $4_0 \to 8_\infty$ (from $(1-2\epsilon;-1+2\epsilon)$ to $(-2\epsilon;2\epsilon)$);

7. $8_0 \to 7_\infty$ (from $(1-2\epsilon;-1+\epsilon)$ to $(-2\epsilon;\epsilon)$);

8. $4_0 \to 6_\infty$ (from $(1;-1)$ to $(0;0)$).

After this chain, the residues have the form:

- residues in $z=0$

$$\begin{pmatrix} -2\epsilon & \frac{4}{3} & \frac{28(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & -\frac{14(2\epsilon-1)(3\epsilon-1)}{\epsilon} & 0 & 16(2\epsilon-1) & -4\epsilon & 0 \\ 0 & -4\epsilon & -\frac{36(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon} & 24(2\epsilon-1)(3\epsilon-1) & 0 & -24\epsilon(2\epsilon-1) & 24\epsilon^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\epsilon}{6(2\epsilon-1)} & \frac{(3\epsilon-2)(3\epsilon-1)}{\epsilon} & 0 & 0 & 0 & -\frac{2\epsilon^2}{2\epsilon-1} & 0 \\ 0 & 0 & -\frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & \frac{(2\epsilon-1)(3\epsilon-1)}{\epsilon} & 0 & 0 & -2\epsilon & 0 \\ -\epsilon & -\frac{3}{2} & -\frac{3(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & \frac{3(2\epsilon-1)(3\epsilon-1)}{2\epsilon} & -\frac{(2\epsilon-1)^2}{\epsilon} & -10(2\epsilon-1) & 22\epsilon & \epsilon \end{pmatrix},$$

  with eigenvalues $\{0;0;0;\epsilon;-2\epsilon;(-2\epsilon)_{\mathbf{3}}\}$;

- residues in $z=-1$

$$\begin{pmatrix} 2\epsilon & 1 & -\frac{6(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & -\frac{3(2\epsilon-1)(3\epsilon-1)}{\epsilon} & -\frac{2(2\epsilon-1)^2}{\epsilon} & 12(2\epsilon-1) & -24\epsilon & -2\epsilon \\ 0 & 0 & 0 & 0 & 0 & -24\epsilon(2\epsilon-1) & 24\epsilon^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\epsilon}{6(2\epsilon-1)} & -\frac{(3\epsilon-2)(3\epsilon-1)}{\epsilon} & 0 & 0 & -\epsilon & \frac{2\epsilon^2}{2\epsilon-1} & 0 \\ 0 & -\frac{1}{6} & -\frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & 0 & 0 & -3(2\epsilon-1) & 4\epsilon & 0 \\ \epsilon & \frac{3}{2} & \frac{3(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & -\frac{3(2\epsilon-1)(3\epsilon-1)}{2\epsilon} & -\frac{(2\epsilon-1)^2}{\epsilon} & 10(2\epsilon-1) & -22\epsilon & -\epsilon \end{pmatrix}$$

  with eigenvalues $\{0;0;0;0;0;\epsilon;\epsilon;2\epsilon\}$;

- residues at infinity

$$\begin{pmatrix} 0 & -\frac{7}{3} & -\frac{22(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & 17\left(6\epsilon-5+\frac{1}{\epsilon}\right) & 8\epsilon-8+\frac{2}{\epsilon} & 28-56\epsilon & 28\epsilon & 2\epsilon \\ 0 & 4\epsilon & \frac{36(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon} & -24(2\epsilon-1)(3\epsilon-1) & 0 & 48\epsilon(2\epsilon-1) & -48\epsilon^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon & 0 & 0 \\ 0 & \frac{1}{6} & 36\epsilon-\frac{4}{\epsilon^2}-54+\frac{26}{\epsilon} & -6\epsilon+5-\frac{1}{\epsilon} & 0 & 6\epsilon-3 & -2\epsilon & 0 \\ 0 & 0 & 0 & 0 & 8\epsilon-8+\frac{2}{\epsilon} & 0 & 0 & 0 \end{pmatrix}$$

  with eigenvalues $\{0;0;0_{\mathbf{4}};\epsilon;2\epsilon\}$.

The basis of MIs is now:

$$\begin{pmatrix} -(-s)^{1+2\epsilon}\left(s^2 z\phi_7 - 14(z+1)\phi_6\right) \\ \frac{6(-s)^{2\epsilon}\left(2s^2 z(z+1)\epsilon^3\phi_6+(\epsilon(6\epsilon-5)+1)(sz\epsilon\phi_3+(3\epsilon-2)(z\phi_1+\phi_2))\right)}{sz\epsilon^2} \\ (-s)^{-1+2\epsilon}\phi_1 \\ (-s)^{2\epsilon}\phi_3 \\ (-s)^{2\epsilon}\phi_4 \\ (-s)^{1+2\epsilon}\phi_5 \\ \frac{(-s)^{1+2\epsilon}((2\epsilon-1)\phi_5-(z+1)\epsilon\phi_6)}{\epsilon} \\ (-s)^{2+2\epsilon}\phi_8 \end{pmatrix}. \tag{7.40}$$

$2_0 \to 3_\infty$ **(from** $(1;-1)$ **to** $(0;0)$**)**

As example, the first balance is reported entirely (all the balances are present in detail in THESIS_2L_ER).

2_0 has associated left eigenvector (it must be decreased) v, while 3_∞ has associated right eigenvector (it must be increased) u:

$$u = \begin{pmatrix} 0 \\ 9\left(6\epsilon + \frac{1}{\epsilon} - 5\right) \\ \frac{\epsilon}{4-6\epsilon} \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \quad v^\dagger = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{7.41}$$

$$v^\dagger u = 1; \quad \mathbb{P} = uv^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9\left(6\epsilon - 5 + \frac{1}{\epsilon}\right) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\epsilon}{4-6\epsilon} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{7.42}$$

$$\mathcal{B}(\mathbb{P};\infty;0|z) = \mathbb{I} - \mathbb{P} - z\mathbb{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{9(z+1)\left(6\epsilon^2-5\epsilon+1\right)}{\epsilon} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{(z+1)\epsilon}{6\epsilon-4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.43}$$

The new residues are:

- residues in $z = 0$

$$\begin{pmatrix} 1-2\epsilon & -1 & 0 & \frac{9(2\epsilon-1)(3\epsilon-1)}{\epsilon} & 0 & 0 & 0 & 0 \\ 0 & 1-2\epsilon & 0 & \frac{9\left(12\epsilon^3-16\epsilon^2+7\epsilon-1\right)}{\epsilon} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{\epsilon}{2(3\epsilon-2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{\epsilon}{6(2\epsilon-1)} & \frac{(3\epsilon-2)(3\epsilon-1)}{\epsilon} & 1-3\epsilon & 0 & 1 & -\frac{2\epsilon^2}{2\epsilon-1} & 0 \\ 0 & \frac{1}{6} & \frac{2(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & -\frac{(2\epsilon-1)(3\epsilon-1)}{2\epsilon} & 0 & 2(2\epsilon-1) & 1-4\epsilon & 0 \\ -\epsilon & -\frac{3}{2} & -\frac{3(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & \frac{12(2\epsilon-1)(3\epsilon-1)}{\epsilon} & -\frac{(2\epsilon-1)^2}{\epsilon} & -6(2\epsilon-1) & 0 & \epsilon+1 \end{pmatrix},$$

  with eigenvalues $\{0; 1; 1; 1+\epsilon; 1-2\epsilon; (1-2\epsilon)_{\mathbf{3}}\}$;

- residues in $z = -1$

$$\begin{pmatrix} 2\epsilon & 1 & -\frac{6(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & -\frac{3(2\epsilon-1)(3\epsilon-1)}{\epsilon} & -\frac{2(2\epsilon-1)^2}{\epsilon} & 0 & -12\epsilon & -2\epsilon \\ 0 & 0 & -\frac{12(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon} & 12\left(6\epsilon^2-5\epsilon+1\right) & 0 & -12\epsilon(2\epsilon-1) & 12\epsilon(2\epsilon-1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\epsilon}{6(2\epsilon-1)} & -\frac{(3\epsilon-2)(3\epsilon-1)}{\epsilon} & 0 & 0 & -\epsilon & \frac{2\epsilon^2}{2\epsilon-1} & 0 \\ 0 & -\frac{1}{6} & -\frac{2(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & \frac{(2\epsilon-1)(3\epsilon-1)}{\epsilon} & 0 & -2(2\epsilon-1) & 4\epsilon-1 & 0 \\ \epsilon & \frac{3}{2} & \frac{3(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & -\frac{3(2\epsilon-1)(3\epsilon-1)}{2\epsilon} & -\frac{(2\epsilon-1)^2}{\epsilon} & 6(2\epsilon-1) & -18\epsilon & -\epsilon \end{pmatrix}$$

  with eigenvalues $\{0; 0; 0; 0; 0; \epsilon; \epsilon; -1+2\epsilon\}$;

- residues at infinity

$$\begin{pmatrix} -1 & 0 & \frac{6(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon^2} & -36\epsilon+30-\frac{6}{\epsilon} & 8\epsilon-8+\frac{2}{\epsilon} & 0 & 12\epsilon & 2\epsilon \\ 0 & 2\epsilon-1 & \frac{12(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{\epsilon} & -180\epsilon^2+204\epsilon-75+\frac{9}{\epsilon} & 0 & 12\epsilon(2\epsilon-1) & -12\epsilon(2\epsilon-1) & 0 \\ 0 & 0 & -1 & \frac{\epsilon}{4-6\epsilon} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3\epsilon-1 & 0 & \epsilon-1 & 0 & 0 \\ 0 & 0 & 0 & -3\epsilon+\frac{5}{2}-\frac{1}{2\epsilon} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{21}{2}\left(-6\epsilon+5-\frac{1}{\epsilon}\right) & 8\epsilon-8+\frac{2}{\epsilon} & 0 & 18\epsilon & -1 \end{pmatrix}$$

  with eigenvalues $\{0_{\mathbf{2}}; -1; -1_{\mathbf{3}}; -1+\epsilon; -1+2\epsilon\}$.

So the neat result consists in an eigenvalue from residues in $z = 0$ diminished from 1 to 0 (in this case moreover the eigenvector of this particular eigenvalue is preserved by the transformation), and in an eigenvalue from residues in infinity increased from -1 to 0 (in this case the eigenvector is not preserved).

7.2.3 Factoring out ϵ

As explained in section 5.3.4, to determine the transformation matrix $\tilde{T}$ to pass into a canonical form, the following system must be solved:

$$\begin{cases} \frac{1}{\epsilon}A_0(z;\epsilon)\tilde{T}(\epsilon;\mu) = \frac{1}{\mu}\tilde{T}(\epsilon;\mu)A_0(z;\mu) \\ \frac{1}{\epsilon}A_{-1}(z;\epsilon)\tilde{T}(\epsilon;\mu) = \frac{1}{\mu}\tilde{T}(\epsilon;\mu)A_{-1}(z;\mu) \end{cases} ; \tag{7.44}$$

once taken $\mu = t = 1$ in the result, the matrix of the transformation is:

$$\begin{pmatrix}
\frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{2\epsilon^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 9\epsilon - \frac{27}{2} + \frac{13}{2\epsilon} - \frac{1}{\epsilon^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 - \frac{2}{\epsilon} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{9}{4} + \frac{1}{4-8\epsilon} - \frac{1}{\epsilon} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{9}{2} + \frac{1}{\epsilon^2} - \frac{9}{2\epsilon} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{2\epsilon^3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(2\epsilon-1)(3\epsilon-2)(3\epsilon-1)}{2\epsilon^3}
\end{pmatrix}.$$

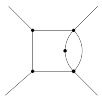
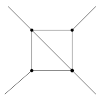
This matrix is invertible, and its application to A results in a canonical system of DE:

$$A = \epsilon \begin{pmatrix}
\frac{2}{z+1} - \frac{2}{z} & \frac{1}{z+1} + \frac{4}{3z} & \frac{56}{z} - \frac{12}{z+1} & -\frac{6}{z+1} - \frac{28}{z} & -\frac{2}{z+1} & \frac{12}{z+1} + \frac{16}{z} & -\frac{24}{z+1} - \frac{4}{z} & -\frac{2}{z+1} \\
0 & -\frac{4}{z} & -\frac{72}{z} & \frac{48}{z} & 0 & -\frac{24}{z+1} - \frac{24}{z} & \frac{24}{z+1} + \frac{24}{z} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{6z} - \frac{1}{6(z+1)} & \frac{2}{z} - \frac{2}{z+1} & 0 & 0 & -\frac{1}{z+1} & \frac{2}{z+1} - \frac{2}{z} & 0 \\
0 & -\frac{1}{6(z+1)} & -\frac{2}{z+1} - \frac{2}{z} & \frac{2}{z} & 0 & -\frac{3}{z+1} & \frac{4}{z+1} - \frac{2}{z} & 0 \\
\frac{1}{z+1} - \frac{1}{z} & \frac{3}{2(z+1)} - \frac{3}{2z} & \frac{6}{z+1} - \frac{6}{z} & \frac{3}{z} - \frac{3}{z+1} & -\frac{1}{z+1} - \frac{1}{z} & \frac{10}{z+1} - \frac{10}{z} & \frac{22}{z} - \frac{22}{z+1} & \frac{1}{z} - \frac{1}{z+1}
\end{pmatrix}.$$

Using IBP-Ids it is possible to rewrite the basis for this problem in terms of UT diagrams, obtaining:

$$I = \begin{pmatrix}
\epsilon^4(-s)^{3+2\epsilon} z \\
\epsilon^2(-s)^{1+2\epsilon} z \\
\epsilon^2(-s)^{1+2\epsilon} \\
\epsilon^3(-s)^{1+2\epsilon} \\
\epsilon^2(-s)^{2+2\epsilon} \\
\epsilon^4(-s)^{1+2\epsilon}(1+z) \\
\epsilon^2(-s)^{2\epsilon}\left[2\epsilon s^2 z \quad + 3\left(\epsilon(6\epsilon-5)+1\right)\right] \\
\epsilon^4(-s)^{2+2\epsilon} \quad (p_1+k_2)^2
\end{pmatrix}. \tag{7.45}$$

$$A = \epsilon \begin{pmatrix} \frac{2}{z+1}-\frac{2}{z} & \frac{3}{z}-\frac{3}{z+1} & \frac{3}{z}-\frac{6}{z+1} & \frac{6}{z+1}-\frac{12}{z} & -\frac{2}{z+1} & \frac{12}{z}-\frac{12}{z+1} & \frac{2}{z+1}-\frac{2}{z} & \frac{2}{z+1} \\ 0 & -\frac{2}{z} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2z} & -\frac{1}{2z} & 0 & 0 & \frac{2}{z+1}-\frac{2}{z} & 0 & 0 \\ 0 & \frac{3}{z}-\frac{3}{z+1} & 0 & \frac{12}{z+1}-\frac{12}{z} & 0 & 0 & \frac{1}{z+1}-\frac{2}{z} & 0 \\ \frac{1}{z}-\frac{1}{z+1} & \frac{9}{2(z+1)}-\frac{9}{2z} & \frac{3}{z+1}-\frac{3}{z} & \frac{15}{z}-\frac{15}{z+1} & \frac{1}{z+1}+\frac{1}{z} & \frac{18}{z+1}-\frac{18}{z} & \frac{2}{z}-\frac{2}{z+1} & \frac{1}{z}-\frac{1}{z+1} \end{pmatrix} \tag{7.46}$$

It is interesting to notice that the seventh function in the MI vector seems to be not UT: since $\epsilon^2(-s)^{2\epsilon},\ 2\epsilon s^2 z$  and  are UT (as proven in sections 6.2 and 3.3), but $3\,(\epsilon(6\epsilon-5)+1)$ is not, the result is not *a priori* UT. So, even if the matrix of coefficients is in a canonical fuchsian form, as said in chapter 4, until BCs are not fixed, there is no certainty that the solution will be expressed in terms of only UT functions. Even if the determination of the solution for the current problem is not evaluated here, it can be predicted that the system will not allow the same MMIs used for the basis of chapter 6, but will instead involve the seventh MI of the present basis.

7.3 Reduced 3-loop ladder

As last application, the 3×3 matrix of the sub-problem

$$\frac{\partial}{\partial z} \begin{pmatrix} (-s)^{4+3\epsilon} & \\ (-s)^{3+3\epsilon} & (p_1+k_3)^2 \\ (-s)^{3+3\epsilon} & (k_1-p_3)^2 \end{pmatrix} = \begin{pmatrix} \frac{3\epsilon}{z+1}+\frac{-3\epsilon-1}{z} & \frac{2\epsilon}{z}-\frac{2\epsilon}{z+1} & \frac{2\epsilon}{z}-\frac{2\epsilon}{z+1} \\ \frac{2\epsilon}{z+1} & \frac{\epsilon}{z}-\frac{\epsilon}{z+1} & -\frac{2\epsilon}{z+1} \\ \frac{\epsilon}{z+1} & \frac{\epsilon}{z}-\frac{\epsilon}{z+1} & 0 \end{pmatrix} \begin{pmatrix} (-s)^{4+3\epsilon} & \\ (-s)^{3+3\epsilon} & (p_1+k_3)^2 \\ (-s)^{3+3\epsilon} & (k_1-p_3)^2 \end{pmatrix}$$

is studied. Details and calculations can be found in THESIS_3L_ER.

7.3.1 Reduction to fuchsian form

The system is already fuchsian, so it is possible to pass to algorithm 2 directly, to find ϵ-homogeneous eigenvalues.

7.3.2 Reduction to ϵ-homogeneous eigenvalues

There are three residues matrices to check for ϵ-homogeneous eigenvalues:

- residues in $z = 0$

$$A_0 = \begin{pmatrix} -3\epsilon-1 & 2\epsilon & 2\epsilon \\ 0 & \epsilon & 0 \\ 0 & \epsilon & 0 \end{pmatrix} \tag{7.47}$$

  with eigenvalues $\{\epsilon; -1-3\epsilon; 0\}$;

- residues in $z = -1$

$$A_{-1} = \begin{pmatrix} 3\epsilon & -2\epsilon & -2\epsilon \\ 2\epsilon & -\epsilon & -2\epsilon \\ \epsilon & -\epsilon & 0 \end{pmatrix} \tag{7.48}$$

  with eigenvalues $\{\epsilon; \epsilon; 0\}$;

- residues at infinity

$$A_\infty = \begin{pmatrix} 1 & 0 & 0 \\ -2\epsilon & 0 & 2\epsilon \\ -\epsilon & 0 & 0 \end{pmatrix} \tag{7.49}$$

  with eigenvalues $\{1; 0_{\mathbf{2}}\}$.

The non-homogeneous parts are all integer, and their sum gives 0, so algorithm 2 can be applied to reduce the system:

1. $1_\infty \to 2_0$ (from $(1; -1-3\epsilon)$ to $(0; -3\epsilon)$).

The resulting system has the form:

$$A(z;\epsilon) = \begin{pmatrix} \frac{3\epsilon}{z+1} - \frac{3\epsilon}{z} & -\frac{2\epsilon}{z+1} & -\frac{2\epsilon}{z+1} \\ \frac{2\epsilon}{z+1} - \frac{2\epsilon}{z} & \frac{\epsilon}{z} - \frac{\epsilon}{z+1} & -\frac{2\epsilon}{z+1} \\ \frac{\epsilon}{z+1} - \frac{\epsilon}{z} & \frac{\epsilon}{z} - \frac{\epsilon}{z+1} & 0 \end{pmatrix}; \quad I(z) = \begin{pmatrix} -(-s)^{4+3\epsilon} z \\ (-s)^{3+3\epsilon} \\ (-s)^{3+3\epsilon} \end{pmatrix} \begin{matrix} \\ (p_1+k_3)^2 \\ (k_1-p_3)^2 \end{matrix}. \tag{7.50}$$

Multiplying by ϵ^6 the vector of MIs, and changing sign to the first one, the subsystem of DE for the three version of the 3-loop ladder topology from the 3-loop ladder problem of section 6.3 is retrieved:

$$A(z;\epsilon) = \begin{pmatrix} \frac{3\epsilon}{z+1} - \frac{3\epsilon}{z} & \frac{2\epsilon}{z+1} & \frac{2\epsilon}{z+1} \\ \frac{2\epsilon}{z} - \frac{2\epsilon}{z+1} & \frac{\epsilon}{z} - \frac{\epsilon}{z+1} & -\frac{2\epsilon}{z+1} \\ \frac{\epsilon}{z} - \frac{\epsilon}{z+1} & \frac{\epsilon}{z} - \frac{\epsilon}{z+1} & 0 \end{pmatrix}; \quad I(z) = \begin{pmatrix} \epsilon^6(-s)^{4+3\epsilon} z \\ \epsilon^6(-s)^{3+3\epsilon} \\ \epsilon^6(-s)^{3+3\epsilon} \end{pmatrix} \begin{matrix} \\ (p_1+k_3)^2 \\ (k_1-p_3)^2 \end{matrix}. \tag{7.51}$$

Chapter 8

Conclusions

Feynman integrals in dimensional regularization obey differential equations with respect to the masses appearing in their propagators, as well as with respect to the kinematic invariants built from the momenta associated to their external legs. The method of Differential Equations for Feynman integrals is one of the mostly common used computational technique in high-energy theoretical physics for evaluating high-order virtual corrections to scattering processes. Accordingly, the evaluation of the multi-loop integrals proceed by solving the system of differential equations they obey, rather than addressing their direct integration.

This thesis focused on the discussion and on the mathematical systematization of two of the most recent ideas to solve differential equations for Feynman integrals, namely Magnus Series Expansion and Eigenvalue Deflation. When applicable, they yield a dramatic simplification of the algorithms for determining the solutions, turning the solving strategy into a purely algebraic problem.

Algebra is ubiquitous. On the one side, it controls the transformation to bring the systems of differential equations into a canonical form where the dependence on the space-time dimension is factorized from the kinematics, by means of matrix similarity transformations, rank reductions and eigenvalue balances. On the other side, it controls the shape of the solution, which, for canonical systems, can be naturally expressed in terms of iterated integrals as Dyson series expansion, or equivalently as Magnus series expansion, in the space-time dimensions.

Elaborating on the methodology present in the literature, the mathematical aspects of the Magnus transform were discussed in details, while the algorithm of Eigenvalue Deflation for differential equations was derived anew starting from the concepts of Deflation and Balance Transformation.

The two techniques were applied to a certain class of integrals entering the virtual corrections for $2 \to 2$ scattering processes among massless particles in QCD, such as dijet production, $pp \to jj$ (known at NNLO), and to $2 \to 1$ processes, like Higgs production via gluon fusion in the heavy top limit (known at NNNLO within the threshold expansion approximation), and more generally for the 3-loop vertex form factors in the massless approximation (known at NNNLO). In particular, the thesis contains the detailed calculation of the 1-loop box graph, the 2-loop ladder graph and the non-trivial 3-loop ladder graph, and of all integrals related to their subdiagrams. The corresponding results were given as Laurent series around four space-time dimensions up to

order 7 in the expansion parameter, that is one order higher than the known results present in the literature. The coefficients of the Laurent series are pure transcendental functions, expressed in terms of Harmonic Polylogarithms. This achievement shows how the method can be iterated at will, returning all the needed orders to the proper evaluation of a graph.

Besides their application to high-accuracy collider phenomenology, the illustrated methods and results can be used for the study of formal properties of scattering amplitudes. More generally, they can be employed for investigating properties of quantum field theory which cannot be deduced simply from the structure Lagrangian. There are properties that emerge only through direct calculations, as it happened for the iterative behaviour of supersymmetric amplitudes, as well as for the existence of dualities between supersymmetry and supergravity amplitudes, recently exploited for investigation of the divergent behaviour of supergravity.

Concluding, the problems still open at the end of this work are numerous, even with respect to only the method of differential equations. Some selected topics in this vaste outlook are:

- The search for uniform transcendental functions, and the notion of uniform transcendentality itself are not fully understood: given a Feynman integral, there is no algorithm to find its uniform transcendental form. The same Euler's gamma γ is not yet rigorously proven to have transcendental weight 1. Moreover, it is not known if the homogeneous transcendentality plays a fundamental role in the evaluation of the solutions: to use Magnus Series Expansion, uniform transcendental functions are warmly required, while for Eigenvalue Deflation this property appear not to be essential.

- In the present work, Magnus Series have always shown a finite number of terms in their construction. Studying massive sunsets, Magnus Series with an infinite number of terms appear. It is then important to understand if these series admit a sum in terms of known finite function to perform a change of basis in order to reach canonical form.

- The eigenvalue deflation technique presents not negligible constraints for the class of system to which it can be applied. It is then important to understand if these constraints can be relaxed, allowing a wider class of systems to be reduced to canonical form using the algorithms derived in this work. Moreover, differently from Magnus Series Expansion, the Eigenvalue Deflation method can be applied only to single-scale problems. A generalization to just two-scale problems of this completely algebraic method will allow to cast a wide class of systems into a canonical form without the huge preliminary study on the transcendental properties of the master integrals.

Part IV

Appendices

Appendix A

HPLs evaluation

Here the algorithms to manage HPLs up to weight 8 are presented, implemented using `Mathematica` v.10.2.

First of all, the functions to order the HPLs of a given weight following the hints given in section 3.2.3, and the function to extract MHPLs and their relations with the other HPLs, are explained.

A.1 Construction of MHPLs and HPL tables

A.1.1 Organizer function

This function, given an alphabet `alph` (i.e. a list of symbols, in this case $\{0; 1; -1\}$), and a weight `w` generates all the possible `w`-disposition with repetitions of the element of `alph`, and organize them in an ordered list following the section 3.2.3. The alphabet is used as a set of signs forming a `w`-base system, as

`alph` (`w` = 3)		Base `w` = 3
0	$\Leftrightarrow$	0
1	$\Leftrightarrow$	1
-1	$\Leftrightarrow$	2

so, in `alph` $= \{0; 1; -1\}$, terms ending with 0s will be multiple of the length $L = 3$ of `alph`, terms starting with 1 and not ending with 0 will be of the form $3^{w-1} + 3^i + j$, $i = 1; \ldots; w-2$ and $j = 1; 2$, terms starting with -1 and not ending with 0 will be of the form $2 \cdot 3^{w-1} + 3^i + j$, $i = 1; \ldots; w-2$ and $j = 1; 2$. The list is then generated ordering numbers.

To allow an easier check with the existing results, HPLs with leftmost -1 have not always been reduced, so it is possible to find them in the MHPL basis. This does not affect the validity of the results in any way, because extra divergencies generated by these terms are compensated by suitable counterterms of the form $[H(-1; x)]^i$.

```
ORGANIZER[alph_List,w_]:=
Module[{tmp,avar,bvar,cut,svar,svar2,i,j,k,FLAG,sep,che,list1,aaa},
```

```
  For[i=1,i<w,i++,
    If [Length[avar]==0,
     H avar={(Length[alph]^i-1)/(Length[alph]-1)},
      avar=Join[{(Length[alph]^i-1)/(Length[alph]-1)},avar]
    ]
  ];
(*Terms of the form 0;...;0;1;...;1*)
  For[i=1,i<w-1,i++,
    For[j=1,j<w-i,j++ ,
      avar=Join[{(2Length[alph]^(i+j)-Length[alph]^i-1)/(Length[alph]-1)},avar]
    ]
  ];
(*Terms of the form 0;...;0;-1;...;-1;1;...;1*)
  For[i=1,i<w,i++,
    avar=Join[{2 (Length[alph]^i-1)/(Length[alph]-1)},avar]
  ];
(*Terms of the form 0;...;0;-1;...;-1*)
  For[i=2,i<Length[alph]^(w-1),i++,
    FLAG=0;
    For[j=1,j<=Length[avar],j++,
      If[avar[[j]]==i||Mod[i,Length[alph]]==0,
        FLAG=1;
        j=Length[avar]
      ]
    ];
    If[FLAG==0,
      avar=Join[{i},avar]
    ]
  ];
(*Terms of the form 0;...;0;a_1;...;a_n*)
  For[i=1,i<w,i++,
    avar=Join[{(2Length[alph]^w-Length[alph]^i-1)/(Length[alph]-1)},avar]
  ];
(*0 and 1 at the top of the list*)
  For[i=0,i<(Length[alph]^(w-1)),i++,(*rightmost 0*)
    aaa=i(Length[alph]);
    If[i==0,
      bvar={aaa},
      bvar=Join[bvar,{aaa}]
    ]
  ];
  For[i=0,i<Length[alph]^(w-2),i++,(*leftmost 1*)
    For[j=1,j<Length[alph],j++,
      If[j==1,
        If[Length[alph]^(w-1)+i*Length[alph]+1=!=(Length[alph]^w-1)/
                        (Length[alph]-1),
          bvar=Join[bvar,{Length[alph]^(w-1)+i*Length[alph]+j}]],
        bvar=Join[bvar,{Length[alph]^(w-1)+i*Length[alph]+j}]
      ]
    ]
  ];
  bvar=Join[bvar,{(Length[alph]^w-1)/(Length[alph]-1)}];
  cut=Length[bvar];
  avar=Join[bvar,avar];
```

```
  For[i=Length[alph]^w-1,i>=0,i--,
    FLAG=0;
    For[j=1,j<=Length[avar],j++,
      If[avar[[j]]==i,
        FLAG=1;
        j=Length[avar]+2
      ]
    ];
    If[FLAG==0,
      avar=Insert[avar,i,cut+1]
    ]
  ];
  svar=IntegerDigits[avar,Length[alph],w];
  For[i=1,i<=Length[svar],i++,(*Labelling H functions*)
    For[j=1,j<=w,j++,(*Labelling function indices*)
      For[k=0,k<Length[alph],k++,(*Labelling position in the alphabet*)
        If[svar[[i,j]]==k,svar[[i,j]]=alph[[k+1]];k=Length[alph]
        ]
      ]
    ]
  ];
  Clear[H];
  Clear[x];
  svar=H/@svar;
  svar=svar/.H[{list1__}]->H[{list1},x];
  Return[svar]
];
```

A.1.2 Building shuffle relations

This group of functions generates the relations obtained by all the possible shuffle products with a given total weight.

- `shuffle` generates the shuffle of two lists.
- `PERM` generates all the possible HPLs of given weight.
- `BUILDER` generates an ordered list of all the shuffle products at weight w; the first element of the output is the list of products, the second one is the list of corresponding sums obtained from the shuffle.

The function `shuffle` is from [57].

```
shuffle[s1_,s2_]:=
Module[{p,tp,accf,ord,shuffle},
  p=Permutations@Join[1&/@s1,0&/@s2];
  tp=BitXor[p,1];
  accf=Accumulate[#\[Transpose]]\[Transpose] #&;
  ord=accf[p]+(accf[tp]+Length[s1]) tp;
  shuffle=Outer[Part,{Join[s1,s2]},ord,1]//First;
  Return[shuffle]
];
```

```
PERM[alph_List,m_]:=
Module[{tmp,n},
  tmp=alph;
  Do[ tmp=Join[tmp,alph],{n,m-1}];
  Return[Permutations[tmp,{m}]]
];
```

```
BUILDER[list_List,w_]:=
Module[{list1,list2,list1H,list2H,listMOLT,listSUM,somma,a,b,c,el,soluz,lista4,
                        lista5,lista7},
  If[w==1,Return["ERROR"],
    For[a=1,a<(w+1)/2,a++,
      list1=PERM[list,a];
      list2=PERM[list,w-a];
      Clear[H];
      Clear[x];
      list1H=list1;
      list2H=list2;
      list1H=H/@list1;
      list1H=list1H/.H[{lista4__}]->H[{lista4},x];
      list2H=H/@list2;
      list2H=list2H/.H[{lista5__}]->H[{lista5},x];
      If[a==w/2,
        For[b=1,b<=Length[list1],b++,
          For[c=1,c<=b,c++,
            If[a==1&&b==1&&c==1,listMOLT={list1H[[b]]*list2H[[c]]},
            listMOLT=Join[listMOLT,{list1H[[b]]*list2H[[c]]}]
          ];
          If[a==1&&b==1&&c==1,listSUM={shuffle[Part[list1,b],Part[list2,c]]},
            listSUM=Join[listSUM,{shuffle[Part[list1,b],Part[list2,c]]}]]
          ];
        ],
        For[b=1,b<=Length[list1],b++,
          For[c=1,c<=Length[list2],c++,
            If[a==1&&b==1&&c==1,
              listMOLT={list1H[[b]]*list2H[[c]]},
              listMOLT=Join[listMOLT,{list1H[[b]]*list2H[[c]]}]
            ];
            If[a==1&&b==1&&c==1,listSUM={shuffle[Part[list1,b],Part[list2,c]]},
              listSUM=Join[listSUM,{shuffle[Part[list1,b],Part[list2,c]]}]
            ]
          ];
        ]
      ]
    ];
    For[a=1,a<=Length[listSUM],a++,
    listSUM[[a]]=H/@listSUM[[a]];
    listSUM[[a]]=listSUM[[a]]/.H[{lista7__}]->H[{lista7},x];
    somma=0;
      For[b=1,b<=Length[listSUM[[a]]],b++,
        somma=somma+listSUM[[a,b]];
      ];
      listSUM[[a]]=somma;
    ];
```

```
    Return[{listMOLT,listSUM}]
  ]
];
```

A.1.3 Solving equations

`BUILDER3` is a function using Gauss' algorithm to solve linear equations. The necessity to build such a function instead of using the command `Solve` already present in `Mathematica` is due essentially to two reasons:

- `Solve` needs several hours to only solve $w = 5$ relations (time needed grows more than linearly with the weight), while Gauss' algorithm solves all the relations up to $w = 8$ in few hours;
- Since the number of MHPLs is known, but the single elements are not, it is only possible to have a preference list for the reduction; `Solve` does not take into account the order of the equations in the system, so no control is possible without knowing the exact elements of the basis; Gauss' algorithm allows to initially solve first equation, related to desired reducible HPLs, assuring more control on the choice of the basis of MHPLs.

```
SOLVER3[listL_List,el_List]:=
Module[{tmp,i,j,k,sol,sys,sysold,ren,kk,sysres,elc,elvar},
  sys=Table[listL[[1]][[k]]-listL[[2]][[k]],{k,1,Length[listL[[1]]]}];
  sysres=Table[Coefficient[sys[[j]],el[[i]]],{j,1,Length[sys]},
                        {i,1,Length[el]}];
  For[i=1,i<=Length[sysres],i++,
    sysres[[i]]=Join[sysres[[i]],{listL[[1,i]]}]
  ];
  sysres=RowReduce[sysres];
  elc=Join[el,{1}];
  For[i=1,i<=Length[sys],i++,
    If[sysres[[i,Length[sysres[[1]]]]]==0,
      sysres=Drop[sysres,-(Length[sysres]-i+1)];
      i=Length[sys]+1
    ]
  ];
  Print["System reduced."];
  elvar=0;
  For[i=1,i<=Length[sysres],i++,
    For[j=1,j<=Length[el],j++,
      If[sysres[[i,j]]=!=0,
        If[Length[elvar]==0,
          elvar={el[[j]]},
          elvar=Join[elvar,{el[[j]]}]
        ];
        j=Length[el]+2
      ]
    ]
  ];
  sol=Flatten[Solve[sysres.elc==0,elvar]];
  Return[sol]
```

```
];
```

A.1.4 Retrieving MHPLs

```
REN[listold_List,list1_List]:=
Module[{tmp,i},
  tmp=0;
  For[i=1,i<=Length[list1],i++,
    If[listold[[i]]==list1[[i]],
      If[Length[tmp]==0,
        tmp={list1[[i]]},
        tmp=Join[tmp,{list1[[i]]}]
      ]
    ]
  ];
  Return[tmp]
];
```

To find MHPLs and relation at weight w MHPLs of the previous weight must be known, together with the shuffle relations. A possible code that can be written to use the previous functions is:

```
F1[p]:=
Module[{tmp},
  If[i==1,
    hpl[[i]]={H[{0},x],H[{1},x],H[{-1},x]};
    hpls[[i]]=hpl[[i]];
    rn[[i]]=hpl[[i]],
    hpl[[i]]=ORGANIZER[alfa,i];
    shuffle[[i]]=BUILDER[alfa,i];
    shuffle[[i]]=shuffle[[i]]/.subp[[i-1]]//Simplify;
    subp[[i]]=SOLVER3[shuffle[[i]],hpl[[i]]];
    hpls[[i]]=hpl[[i]]/.subp[[i]]//Simplify;
    rn[[i]]=REN[hpl[[i]],hpls[[i]]]
  ]
];
```

where `subp[[i-1]]` is the list of relations of the previous weight, already evaluated.

With the enlisted functions MHPLs and relations have been determined up to weight 8.[1]

Example 14 (HPLs at $w = 1; 2; 3$). MHPLs and substitution tables for $w = 1; 2; 3$ are enlisted.

$w = 1$ **MHPLs** There are 3 MHPLs:

$H(0;x)$
$H(1;x)$
$H(-1;x)$

Relations There are no relations, all the HPLs at $w = 1$ are MHPLs.

[1] See ancillary files THESIS_HPL and THESIS_MHPL for the functions, and THESIS_HPL_LIST for the results.

$w = 2$ **MHPLs** There are 3 MHPLs:

$H(-1;1;x)$
$H(0;-1;x)$
$H(0;1;x)$

Relations There are 6 relations:

$H(0;0;x)$	$\to$	$\frac{1}{2}H^2(0;x)$
$H(1;0;x)$	$\to$	$H(0;x)H(1;x) - H(0;1;x)$
$H(-1;0;x)$	$\to$	$H(-1;x)H(0;x) - H(0;-1;x)$
$H(1;-1;x)$	$\to$	$H(-1;x)H(1;x) - H(-1;1;x)$
$H(1;1;x)$	$\to$	$\frac{1}{2}H^2(1;x)$
$H(-1;-1;x)$	$\to$	$\frac{1}{2}H^2(-1;x)$

$w = 3$ **MHPLs** There are 8 MHPLs:

$H(-1;1;1;x)$
$H(-1;-1;1;x)$
$H(0;1;-1;x)$
$H(0;-1;-1;x)$
$H(0;0;-1;x)$
$H(0;-1;1;x)$
$H(0;1;1;x)$
$H(0;0;1;x)$

Relations There are 19 relations:

$H(0;0;0;x)$	$\to$	$\frac{1}{6}H^3(0;x)$
$H(0;1;0;x)$	$\to$	$H(0;x)H(0;1;x) - 2H(0;0;1;x)$
$H(0;-1;0;x)$	$\to$	$H(0;x)H(0;-1;x) - 2H(0;0;-1;x)$
$H(1;0;0;x)$	$\to$	$\frac{1}{2}H(1;x)H^2(0;x) - H(0;1;x)H(0;x) + H(0;0;1;x)$
$H(1;1;0;x)$	$\to$	$\frac{1}{2}H(0;x)H^2(1;x) - H(0;1;x)H(1;x) + H(0;1;1;x)$
$H(1;-1;0;x)$	$\to$	$H(-1;x)H(0;x)H(1;x) - H(0;-1;x)H(1;x) - H(0;x)H(-1;1;x) + H(0;-1;1;x)$
$H(-1;0;0;x)$	$\to$	$\frac{1}{2}H(-1;x)H^2(0;x) - H(0;-1;x)H(0;x) + H(0;0;-1;x)$
$H(-1;1;0;x)$	$\to$	$H(0;x)H(-1;1;x) - H(-1;x)H(0;1;x) + H(0;1;-1;x)$
$H(-1;-1;0;x)$	$\to$	$\frac{1}{2}H(0;x)H^2(-1;x) - H(0;-1;x)H(-1;x) + H(0;-1;-1;x)$
$H(1;0;1;x)$	$\to$	$H(1;x)H(0;1;x) - 2H(0;1;1;x)$
$H(1;0;-1;x)$	$\to$	$H(1;x)H(0;-1;x) - H(0;-1;1;x) - H(0;1;-1;x)$
$H(1;1;-1;x)$	$\to$	$\frac{1}{2}H(-1;x)H^2(1;x) - H(-1;1;x)H(1;x) + H(-1;1;1;x)$
$H(1;-1;1;x)$	$\to$	$H(1;x)H(-1;1;x) - 2H(-1;1;1;x)$
$H(1;-1;-1;x)$	$\to$	$\frac{1}{2}H(1;x)H^2(-1;x) - H(-1;1;x)H(-1;x) + H(-1;-1;1;x)$
$H(1;1;1;x)$	$\to$	$\frac{1}{6}H^3(1;x)$
$H(-1;0;1;x)$	$\to$	$H(-1;x)H(0;1;x) - H(0;-1;1;x) - H(0;1;-1;x)$
$H(-1;0;-1;x)$	$\to$	$H(-1;x)H(0;-1;x) - 2H(0;-1;-1;x)$
$H(-1;1;-1;x)$	$\to$	$H(-1;x)H(-1;1;x) - 2H(-1;-1;1;x)$
$H(-1;-1;-1;x)$	$\to$	$\frac{1}{6}H^3(-1;x)$

A.2 Series expansion

In order to fix BC for the l-loop ladder problems, the MHPLs have been expanded around the possible points of divergence, i.e. $x_0 = 0; 1; -1$, up to $o_0\left((x-x_0)^{15}\right)$.

To perform the expansion in x_0 for a MHPL $H(a_1;\ldots;a_w;x)$ with $w \geq 2$, the integral representation $\int_0^x f(a_1;\tau)H(a_2;\ldots;a_n;\tau)\,\mathrm{d}\tau$ will be used (note that $H(0;\ldots;0;x)$ is not part of the MHPLs, except for $H(0;x)$, so the integral representation is legit); for $w = 1$ MHPLs the expansion will be carried out on the well-known primitive.

Starting from the integral form, $f(a_1;\tau)$ will be substituted with the corresponding series expansion around x_0, as $H(a_2;\ldots;a_n;\tau)$, once it has been expressed in terms of MHPLs. After that, the expression will be integrated term by term. For $x_0 = 0$ $\int_0^{x_0}\mathrm{d}\tau$ will be used, whilst for $x_0 = 1$ the integration path is divided into $\int_0^1\mathrm{d}\tau$ and $\int_1^{x_0}\mathrm{d}\tau$; the first one is always a finite integral in the set of MHPLs, and its values are expressed in terms of ζ functions and constants. The case $x_0 = -1$ will be reabsorbed in $x_0 = 1$ using proposition 6. Note finally that the divergent parts, using this approach, are encoded only in terms like $H(\mathbf{0};x)$ for 0 and $H(\mathbf{1};x)$ for 1 ($\lim_{x\to x_0}(x-x_0)\log^p(x-x_0) = 0$).

First of all, series expansion in $x_0 = 0$ will be evaluated, followed by expansions in $x_0 = 1$ and $x_0 = -1$ at the end.

```
CloseKernels;
LaunchKernels[8]
```

```
Clear[listint0];
Clear[listint1];
w=8;
ord=15;
```

To overcome long-time evaluations with direct use of the command `Integrate`, the integration has been performed on a set of selected terms of the form $x^p\log^q x$, that, with a suitable change of variable x, are the only ones that can be present in the series expansion both around 0 and around 1 ($\log^q x$ is not expanded in the point 0, so it will multiply the expansion of the regular part). This set will provide a list of substitution used instead of direct integration. Notice, as expected, that the value of the primitive of each term of the series is always 0 when evaluated in 0 in expansions around 0 and when evaluated in 1 in expansion around 1, except for $H(0;x)$ and $H(1;x)$, respectively.

```
listint0=0;
For[k=0,k<=ord,k++,
  For[j=1,j<=w,j++,
    tm=Integrate[\[Xi]^k Log[\[Xi]]^j,\[Xi]];
    If[k==0&&j==1,
      listint0={x^k Log[x]^j->tm},
      listint0=Join[listint0,{x^k Log[x]^j->tm}]
    ]
  ]
];
For[k=1,k<=ord,k++,
  tm=Integrate[\[Xi]^k,\[Xi]];
  listint0=Join[listint0,{x^k->tm}];
];
```

```
SetDirectory[NotebookDirectory[]];
subp=ToExpression[Import["Substitutions8.txt","List"]];
rn=ToExpression[Import["MHPL8.txt","List"]];
```

The functions SMONT0 and SMONT1 order the expansions and perform integration, respectively in 0 and 1.

```
SMONT0[w_,HE_List(*,fE_List*),ord_,kk_,LL_]:=
Module[{tmp,cont,ext,ww,j,val0,val\[Xi],valx},
  tmp=LL;
  cont=LL/.H[lista__,x]->lista;
  If[Length[cont]==0,
    ext=tmp,
    If[Length[cont]==1,
      ext=tmp;
      ext=ext/.Normal[HE[[kk,1]]];
      ext=ext/.Normal[fE[[kk]]],
      ext=tmp/.H[lista__,x]->f[{First[lista]},x]H[Drop[lista,{1}],x];
      For[j=w,j>=1,j--,
        ext=ext/.subp[[j]]
      ];
      For[ww=1,ww<=w,ww++,
        ext=ext/.Normal[HE[[kk,ww]]]
      ];
      ext=ext/.Normal[fE[[kk]]];
      ext=ext+O[x]^(ord+1);
      ext=Normal[ext];
      Clear[x];
      Clear[xx];
      ext=ext//Expand;
      ext=ext/.q_ x^n_/;n>=ord:>0;
      ext=ext/.x^n_/;n>=ord:>0;
      ext=ext*xx//Expand;
      ext=ext/.x^n_ xx->x^n;
      ext=ext/.x xx->x;
      ext=ext/.x^n_ Log[x]^m_ xx->x^n Log[x]^m;
      ext=ext/.Log[x]^m_ xx->Log[x]^m;
      ext=ext/.Log[x] xx->Log[x];
      ext=ext/.x Log[x] xx->x Log[x];
      ext=ext/.x Log[x]^m_ xx->Log[x]^m;
      ext=ext/.x^n_ Log[x] xx->x^n Log[x];
      ext=ext/.listint0;
      ext=ext/.xx->\[Xi];
      val0=ext/.Log[\[Xi]]->1;
      val0=val0/.\[Xi]->0;
      val\[Xi]=ext/.\[Xi]->x;
      ext=-val0+val\[Xi]+O[x]^(ord+1);
      ext=ext//Expand;
      ext=ext+O[x]^(ord+1);
    ]
  ];
  Return[ext]
];
```

```
SMONT1[w_,HE_List,fE_List,ord_,kk_,LL_]:=
```

```
Module[{tmp,cont,ext,agg,ww,j,val1,val\[Xi],val0},
  tmp=LL;
  agg=tmp/.H[lista__,x]->H[lista,1];
  cont=LL/.H[lista__,x]->lista;
  If[Length[cont]==0,
    ext=tmp,
    If[Length[cont]==1,
      ext=tmp;
      ext=ext/.Normal[HE[[kk,1]]];
      ext=ext/.Normal[fE[[kk]]],
      ext=tmp/.H[lista__,x]->f[{First[lista]},x]H[Drop[lista,{1}],x];
      For[j=w,j>=1,j--,
        ext=ext/.subp[[j]]
      ];
      For[ww=1,ww<=w,ww++,
        ext=ext/.Normal[HE[[kk,ww]]]
      ];
      ext=ext/.Normal[fE[[kk]]];
      ext=ext+O[x,1]^(ord+1);
      ext=Normal[ext];
      Clear[x];
      ext=ext/.x->1-\[Xi];
      ext=ext//Expand;
      ext=ext/.q_ \[Xi]^n_/;n>=ord:>0;
      ext=ext/.\[Xi]^n_/;n>=ord:>0;
      ext=ext*xx//Expand;
      ext=ext/.\[Xi]^n_ xx->\[Xi]^n;
      ext=ext/.\[Xi] xx->\[Xi];
      ext=ext/.\[Xi]^n_ Log[\[Xi]]^m_ xx->\[Xi]^n Log[\[Xi]]^m;
      ext=ext/.Log[\[Xi]]^m_ xx->Log[\[Xi]]^m;
      ext=ext/.Log[\[Xi]] xx->Log[\[Xi]];
      ext=ext/.\[Xi] Log[\[Xi]] xx->\[Xi] Log[\[Xi]];
      ext=ext/.\[Xi] Log[\[Xi]]^m_ xx->Log[\[Xi]]^m;
      ext=ext/.\[Xi]^n_ Log[\[Xi]] xx->\[Xi]^n Log[\[Xi]];
      Clear[xx];
      ext=ext/.\[Xi]->x;
      ext=ext/.listint0;
      ext=ext/.xx->\[Xi];
      val1=ext/.Log[\[Xi]]->1;
      val1=val1/.\[Xi]->0;
      val1=-val1;
      val\[Xi]=ext/.\[Xi]->1-x;
      val\[Xi]=-val\[Xi];
      ext=agg-val1+val\[Xi];
      ext=ext/.x->1-\[Xi];
      ext=ext//Expand;
      ext=ext/.\[Xi]->1-x;
      ext=ext+O[x,1]^(ord+1)
    ]
  ];
  Return[ext]
];
```

This function put the MHPLs in integral representation and apply the functions of series

expansion.

```
EXPANDERPAR2[w_,ord_,HE_,p_,HH_]:=
Module[{tmp,tmp2,tmp3(*,HE*),i,j,k,h,kk,App,lista,ku,listb,listc,App1,App2,ww,
                          LApp,LApp2,add,pro,ii,jj,HE2,ind,HE22},
  Clear[lista];
  Clear[H];
  Clear[f];
  HE2=HE;
  Clear[listb];
  Clear[listc];
  j=w;
  k=p+2;
  Clear[H];
  Clear[f];
  kk=k;
  If[k==1,
    tmp=HH/.H[lista__,x]->lista;
    App=(-1)^Sum[tmp[[i]]^2,{i,1,Length[tmp]}]HH;
    App=App/.H[lista__,x]->H[-lista,x];
    kk=k+2,
    App=1*HH;
    App=App/.H[lista__,x]->H[lista,x]
  ];
  For[ind=j,ind>=1,ind--,
    App=App/.subp[[ind]]
  ];
  LApp=MonomialList[App];
  LApp=Map[FactorList,LApp,{1}];
  If[kk==2,
    LApp=Map[SMONT0[j,HE2,fE,ord,kk,#]&,LApp,{3}]
  ];
  If[kk==3,
    LApp=Map[SMONT1[j,HE2,fE,ord,kk,#]&,LApp,{3}]
  ];
  tmp3=0;
  For[ii=1,ii<=Length[LApp],ii++,
    tmp2=1;
    For[jj=1,jj<=Length[LApp[[ii]]],jj++,
      tmp2=tmp2*LApp[[ii,jj,1]]^LApp[[ii,jj,2]]
    ];
    tmp3=tmp3+tmp2;
  ];
  If[k==3,
    tmp3=tmp3+O[x,1]^(ord+1),
    If[k==2,
      tmp3=tmp3+O[x]^(ord+1),
      If[k==1,
        tmp3=tmp3//Normal;
        tmp3=tmp3/.x->(-x);
        tmp3=tmp3/.x->y-1;
        tmp3=tmp3//Simplify;
        tmp3=tmp3//Expand;
        tmp3=tmp3+O[y]^(ord+1);
```

```
        tmp3=tmp3//Normal;
        tmp3=tmp3/.y->x+1;
        tmp3=tmp3+O[x,-1]^(ord+1),
        Print["ERROR"]
      ]
    ]
  ];
  tmp=HH/.H[lista__,x]->lista;
  HE22=H[tmp,x]->tmp3;
  Return[HE22]
];
```

Series expansions of the f and of the MHPLs at $w = 1$. The last part loads expansions in previous points and weights (expansions in different points are used only for $x_0 = -1$, where also series in $x_0 = 1$ are used).

```
HE={{H[{-1},x]->Log[1+x],
H[{0},x]->(Normal[Series[Log[x]+I \[Pi],{x,1,ord}]]/.x->(-x))+O[x,-1]^(ord+1),
H[{1},x]->(Normal[Series[-Log[1+x],{x,1,ord}]]/.x->(-x))+O[x,-1]^(ord+1)},
{H[{-1},x]->Series[Log[1+x],{x,0,ord}],
H[{0},x]->Log[x],
H[{1},x]->Series[-Log[1-x],{x,0,ord}]},
{H[{-1},x]->Series[Log[1+x],{x,1,ord}],
H[{0},x]->Series[Log[x],{x,1,ord}],
H[{1},x]->-Log[1-x]}};
Clear[iii];
For[i=1,i<=3,i++,
  For[j=2,j<=w,j++,
    If[j==2,
      HE[[i]]=Join[{HE[[i]]},{iii->iii}],
      HE[[i]]=Join[HE[[i]],{iii->iii}]
    ]
  ]
];
SetDirectory[NotebookDirectory[]];
(*Expansions of previous points and weights loaded.*)
HE[[3]]=ToExpression[Import["HPL_in_1.txt","List"]];
HE[[1]]=ToExpression[Import["HPL_in_-1_PAR7.txt","List"]];
```

```
fE={{f[{-1},x]->1/(1+x),
f[{0},x]->Series[1/x,{x,-1,ord}],
f[{1},x]->Series[1/(1-x),{x,-1,ord}]},
{f[{-1},x]->Series[1/(1+x),{x,0,ord}],
f[{0},x]->1/x,
f[{1},x]->Series[1/(1-x),{x,0,ord}]},
{f[{-1},x]->Series[1/(1+x),{x,1,ord}],
f[{0},x]->Series[1/x,{x,1,ord}],
f[{1},x]->1/(1-x)}};
```

This block contains evaluation information, like the weight, indicated as `index`.

```
(*Block with evaluation information.*)
index=8;
HEP=ParallelTable[EXPANDERPAR2[index,ord,HE,-1,rn[[index,h]]],{h,1,100}];
```

```
HEP=HEP//Flatten;
SetDirectory[NotebookDirectory[]];
Export["HPL_-1_8_001-100.txt", HEP];
```

```
Quit[];
```

Function `EXPANDERMOD2` is analogous to `EXPANDERPAR2`, without upgrades for parallel calculations. All the functions and the series expansions of all the MHPLs around 1, 0 and -1, up to order 15 and weight 8 can be found in THESIS_SERIES_TABLES.nb, THESIS_SERIES_DEVELOPER.m and THESIS_SERIES.nb. Random samplings of the results have been compared with previously known expansions, for points 0 and 1, up to order 10 and weight 7.

Example 15 (Series expansions for $H(0;1;-1;x)$).

- Around $x \to 0$

$$
\begin{aligned}
&H(0;1;-1;x) = \\
&\quad = \frac{1}{4}x^2 + \frac{1}{18}x^3 + \frac{5}{96}x^4 + \frac{7}{300}x^5 + \frac{47}{2160}x^6 + \frac{37}{2940}x^7 + \frac{319}{26\,880}x^8 + \frac{533}{68\,040}x^9 + \frac{1879}{252\,000}x^{10} + \\
&\quad + \frac{1627}{304\,920}x^{11} + \frac{20\,417}{3\,991\,680}x^{12} + \frac{18\,107}{4\,684\,680}x^{13} + \frac{263\,111}{70\,630\,560}x^{14} + \frac{237\,371}{81\,081\,000}x^{15} + o_0\left(x^{15}\right). \qquad \text{(A.1)}
\end{aligned}
$$

- Around $x \to 1$

$$
\begin{aligned}
&H(0;1;-1;x) = \\
&\quad = (1-x)\log 2\log(1-x) + \frac{1}{2}(1-x)^2\log 2\log(1-x) + \frac{1}{3}(1-x)^3\log 2\log(1-x) + \\
&\quad + \frac{1}{4}(1-x)^4\log 2\log(1-x) + \frac{1}{5}(1-x)^5\log 2\log(1-x) + \frac{1}{6}(1-x)^6\log 2\log(1-x) + \\
&\quad + \frac{1}{7}(1-x)^7\log 2\log(1-x) + \frac{1}{8}(1-x)^8\log 2\log(1-x) + \frac{1}{9}(1-x)^9\log 2\log(1-x) + \\
&\quad + \frac{1}{10}(1-x)^10\log 2\log(1-x) + \frac{1}{11}(1-x)^{11}\log 2\log(1-x) + \frac{1}{12}(1-x)^{12}\log 2\log(1-x) + \\
&\quad + \frac{1}{13}(1-x)^{13}\log 2\log(1-x) + \frac{1}{14}(1-x)^{14}\log 2\log(1-x) + \frac{1}{15}(1-x)^{15}\log 2\log(1-x) + \\
&\quad + H(0;1;-1;1) + [-H(-1;1;1) + \log 2](x-1) + \left[-\frac{1}{4} + \frac{1}{2}H(-1;1;1) - \frac{\log 2}{4}\right](x-1)^2 + \\
&\quad + \left[\frac{3}{16} - \frac{1}{3}H(-1;1;1) + \frac{\log 2}{9}\right](x-1)^3 + \left[-\frac{83}{576} + \frac{1}{4}H(-1;1;1) - \frac{\log 2}{16}\right](x-1)^4 + \\
&\quad + \left[\frac{1337}{11\,520} - \frac{1}{5}H(-1;1;1) + \frac{\log 2}{25}\right](x-1)^5 + \left[-\frac{33\,497}{345\,600} + \frac{1}{6}H(-1;1;1) - \frac{\log 2}{36}\right](x-1)^6 + \\
&\quad + \left[\frac{5587}{67\,200} - \frac{1}{7}H(-1;1;1) + \frac{\log 2}{49}\right](x-1)^7 + \left[-\frac{136\,919}{1\,881\,600} + \frac{1}{8}H(-1;1;1) - \frac{\log 2}{64}\right](x-1)^8 + \\
&\quad + \left[\frac{35\,054\,939}{541\,900\,800} - \frac{1}{9}H(-1;1;1) + \frac{\log 2}{81}\right](x-1)^9 + \left[-\frac{946\,522\,553}{16\,257\,024\,000} + \frac{1}{10}H(-1;1;1) - \frac{\log 2}{100}\right](x-1)^{10} + \\
&\quad + \left[\frac{946\,538\,429}{17\,882\,726\,400} - \frac{1}{11}H(-1;1;1) + \frac{\log 2}{121}\right](x-1)^{11} + \\
&\quad + \left[-\frac{114\,531\,943\,709}{2\,360\,519\,884\,800} + \frac{1}{12}H(-1;1;1) - \frac{\log 2}{144}\right](x-1)^{12} + \\
&\quad + \left[\frac{458\,129\,108\,861}{10\,228\,919\,500\,800} - \frac{1}{13}H(-1;1;1) + \frac{\log 2}{169}\right](x-1)^{13} +
\end{aligned}
$$

$$+\left[-\frac{77\,423\,915\,447\,309}{1\,861\,663\,349\,145\,600}+\frac{1}{14}H(-1;1;1)-\frac{\log 2}{196}\right](x-1)^{14}+$$
$$+\left[\frac{38\,711\,978\,428\,267}{997\,319\,651\,328\,000}-\frac{1}{15}H(-1;1;1)+\frac{\log 2}{225}\right](x-1)^{15}+o_0\left((x-1)^{15}\right). \quad (A.2)$$

- Around $x \to -1$

$$H(0;1;-1;x) =$$
$$= -H(-1;1;1)(1+x)+\frac{1}{8}(1+x)^2(3-4H(-1;1;1)-2\log(1+x))+$$
$$+\frac{1}{144}(1+x)^3[37-48H(-1;1;1)-30\log(1+x)]+$$
$$+(1+x)^4\left[\frac{107}{576}-\frac{1}{4}H(-1;1;1)-\frac{1}{6}\log(1+x)\right]+$$
$$+(1+x)^5\left[\frac{8257}{57\,600}-\frac{1}{5}H(-1;1;1)-\frac{131}{960}\log(1+x)\right]+$$
$$+(1+x)^6\left[\frac{13\,369}{115\,200}-\frac{1}{6}H(-1;1;1)-\frac{661}{5760}\log(1+x)\right]+$$
$$+(1+x)^7\left[\frac{953}{9800}-\frac{1}{7}H(-1;1;1)-\frac{1327}{13\,440}\log(1+x)\right]+$$

$$+(1+x)^8\left[\frac{314\,543}{3\,763\,200}-\frac{1}{8}H(-1;1;1)-\frac{1163}{13\,440}\log(1+x)\right]+$$
$$+(1+x)^9\left[\frac{357\,205\,771}{4\,877\,107\,200}-\frac{1}{9}H(-1;1;1)-\frac{148\,969}{1\,935\,360}\log(1+x)\right]+$$
$$+(1+x)^{10}\left[\frac{1\,059\,178\,397}{16\,257\,024\,000}-\frac{1}{10}H(-1;1;1)-\frac{447\,047}{6\,451\,200}\log(1+x)\right]+$$
$$+(1+x)^{11}\left[\frac{11\,538\,639\,919}{196\,709\,990\,400}\frac{-1}{11}H(-1;1;1)-\frac{44\,711}{709\,632}\log(1+x)\right]+$$
$$+(1+x)^{12}\left[\frac{125\,893\,736\,459}{2\,360\,519\,884\,800}-\frac{1}{12}H(-1;1;1)-\frac{983\,705}{17\,031\,168}\log(1+x)\right]+$$
$$+(1+x)^{13}\left[\frac{6\,501\,060\,475\,493}{132\,975\,953\,510\,400}-\frac{1}{13}H(-1;1;1)-\frac{7\,869\,871}{147\,603\,456}\log(1+x)\right]+$$
$$+(1+x)^{14}\left[\frac{84\,007\,545\,221\,459}{1\,861\,663\,349\,145\,600}-\frac{1}{14}H(-1;1;1)-\frac{102\,309\,709}{2\,066\,448\,384}\log(1+x)\right]+$$
$$+(1+x)^{15}\left[\frac{20\,892\,179\,156\,921}{498\,659\,825\,664\,000}-\frac{1}{15}H(-1;1;1)-\frac{40\,924\,141}{885\,620\,736}\log(1+x)\right]+$$
$$+H(0;-1;1;1)+o_0\left((x+1)^{15}\right). \quad (A.3)$$

A.3 HPLs at fixed points

When evaluated at $x = 0; 1; -1$, if not divergent, HPLs of weight w return constant values with same weight.

To evaluate these particular values, properties like change of variables are used (see [8] for a complete list of properties). The values needed here are only the ones related to $x = 1$, but they are not evaluated explicitly in this work. A list of these values up to weight 7 can be found at the bottom of the file THESIS_HPL_LIST.

Example 16 (HPLs in $x = 1$). For $w = 1$, $w = 2$ and $w = 3$ in $x = 1$ one has:

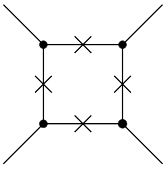
$H(0;1) = 0$
$H(1;1) = \lim_{x\to 1}\left[-\log(1-x)\right]$
$H(-1;1) = \log 2$
$H(-1;1;1) = -\frac{\pi^2}{12} - \frac{\log^2 2}{2}$
$H(0;-1;1) = -\frac{\pi^2}{12}$
$H(0;1;1) = \frac{\pi^2}{6}$
$H(-1;1;1;1) = -\frac{\pi^2 \log 2}{12}$
$H(-1;-1;1;1) = -\frac{1}{6}\log^3 2 + \frac{\zeta(3)}{8}$
$H(0;1;-1;1) = -\zeta(3) + \frac{\pi^2 \log 2}{4}$
$H(0;-1;-1;1) = \frac{\zeta(3)}{8}$
$H(0;0;-1;1) = \frac{3\zeta(3)}{4}$
$H(0;-1;1;1) = -\frac{\pi^2 \log 2}{4} + \frac{13\zeta(3)}{8}$
$H(0;1;1;1) = \zeta(3)$
$H(0;0;1;1) = \zeta(3)$

Notice how the weight of the explicit values of the HPLs at a fixed point is the same of the HPLs at a generic point x.

Appendix B

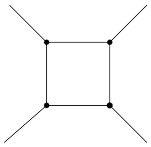
Quadruple cut of the 1-loop box

The quadruple cut operated on a 1-loop box is a fundamental ingredient in several parts of the present work. Here a derivation of the relation



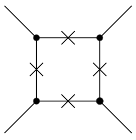
$$\propto \frac{1}{st}. \tag{B.1}$$

is presented, using the spinor elicity formalism. As explained in [44], a cut corresponds to put the momentum of a propagator on-shell; mathematically speaking, this operation corresponds to substitute a propagator with a Dirac delta function. Considering a box graph with flowing momenta as depicted in section 6.1, the quadruple cut transforms the integral



$$= \int \frac{\mathrm{d}^D k}{k^2 (k+p_1)^2 (k+p_1+p_2)^2 (k-p_3)^2} \tag{B.2}$$

into



$$= \int \delta\left(k^2\right) \delta\left((k+p_1)^2\right) \delta\left((k+p_1+p_2)^2\right) \delta\left((k-p_3)^2\right) \mathrm{d}^D k. \tag{B.3}$$

Using the elicity spinor formalism (see [58]), it is possible to write the external momenta (all massless and incoming) as:

- upper-left momentum $p_1^\mu = \frac{1}{2}\langle 1\gamma^\mu 1] = \frac{1}{2}[1\gamma^\mu 1\rangle$;
- lower-left momentum $p_2^\mu = \frac{1}{2}\langle 2\gamma^\mu 2] = \frac{1}{2}[2\gamma^\mu 2\rangle$;
- upper-right momentum $p_3^\mu = \frac{1}{2}\langle 3\gamma^\mu 3] = \frac{1}{2}[3\gamma^\mu 3\rangle$;

- lower-right momentum $p_4^\mu = \frac{1}{2}\langle 4\gamma^\mu 4] = \frac{1}{2}[4\gamma^\mu 4\rangle$;

and loop momentum k^μ as:

$$k^\mu = \alpha_1 \frac{[1\gamma^\mu 1\rangle}{2} + \alpha_2 \frac{[2\gamma^\mu 2\rangle}{2} + \alpha_3 \frac{[1\gamma^\mu 2\rangle}{2} + \alpha_4 \frac{[2\gamma^\mu 1\rangle}{2}. \tag{B.4}$$

As expected:

$$p_i \cdot p_i = \frac{1}{4}[i\gamma^\mu i\rangle[i\gamma_\mu i\rangle = 0; \tag{B.5}$$

and the only non-zero scalar products are:

$$2p_1 \cdot p_2 = 2p_3 \cdot p_4 = s = \quad \langle 12\rangle[21], \tag{B.6}$$
$$2p_1 \cdot p_3 = 2p_2 \cdot p_4 = t = \quad \langle 13\rangle[31], \tag{B.7}$$
$$2p_1 \cdot p_4 = 2p_2 \cdot p_3 = u = \quad -s - t = \langle 14\rangle[41]. \tag{B.8}$$

A change of variables from k^μ to $(\alpha_1; \ldots; \alpha_4)$ is performed.

First of all, it is necessary to express the arguments of the Dirac deltas in terms of spinors. After some algebraic calculations and applications of momentum conservation:

$$k^2 = (\alpha_1\alpha_2 - \alpha_3\alpha_4)s; \tag{B.9}$$
$$(k + p_1)^2 = (\alpha_1\alpha_2 + \alpha_2 - \alpha_3\alpha_4)s; \tag{B.10}$$
$$(k + p_1 + p_2)^2 = (\alpha_1\alpha_2 + \alpha_1 + \alpha_2 + 1 - \alpha_3\alpha_4)s; \tag{B.11}$$
$$(k - p_3)^2 = (\alpha_1\alpha_2 + \alpha_2 - \alpha_3\alpha_4)s + (\alpha_1 + \alpha_2)t + \alpha_3[13]\langle 32\rangle + \alpha_4[23]\langle 31\rangle. \tag{B.12}$$

The deltas impose to solve the following system:

$$\begin{cases} (\alpha_1\alpha_2 - \alpha_3\alpha_4)s = 0 \\ (\alpha_1\alpha_2 + \alpha_2 - \alpha_3\alpha_4)s = 0 \\ (\alpha_1\alpha_2 + \alpha_1 + \alpha_2 + 1 - \alpha_3\alpha_4)s = 0 \\ (\alpha_1\alpha_2 + \alpha_2 - \alpha_3\alpha_4)s + (\alpha_1 + \alpha_2)t + \alpha_3[13]\langle 32\rangle + \alpha_4[23]\langle 31\rangle = 0 \end{cases}; \tag{B.13}$$

which solutions are:

$$\mathcal{S}_1 = \begin{cases} \alpha_1 = -1 \\ \alpha_2 = 0 \\ \alpha_3 = t/([13]\langle 32\rangle) \\ \alpha_4 = 0 \end{cases} \quad \text{or} \quad \mathcal{S}_2 = \begin{cases} \alpha_1 = -1 \\ \alpha_2 = 0 \\ \alpha_3 = 0 \\ \alpha_4 = t/([23]\langle 31\rangle) \end{cases}. \tag{B.14}$$

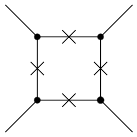
Now consider the variation associated to the integration measure:

$$\mathrm{d}^D k = \sqrt{\left|\det\left(\frac{\partial k^\mu}{\partial \alpha_i}\frac{\partial k_\mu}{\partial \alpha_j}\right)\right|}\,\mathrm{d}\alpha_1\mathrm{d}\alpha_2\mathrm{d}\alpha_3\mathrm{d}\alpha_4, \tag{B.15}$$

$$\left(\frac{\partial k^\mu}{\partial \alpha_i}\frac{\partial k_\mu}{\partial \alpha_j}\right) = \begin{pmatrix} 0 & \frac{s}{2} & 0 & 0 \\ \frac{s}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{s}{2} \\ 0 & 0 & -\frac{s}{2} & 0 \end{pmatrix}, \tag{B.16}$$

$$\sqrt{\left|\det\left(\frac{\partial k^\mu}{\partial \alpha_i}\frac{\partial k_\mu}{\partial \alpha_j}\right)\right|} = \frac{s^2}{4}. \tag{B.17}$$

It is then possible to rewrite the expression (B.3) as:



$$= \left(\frac{1}{|\det J(\alpha)|_{\mathcal{S}_1}} + \frac{1}{|\det J(\alpha)|_{\mathcal{S}_2}}\right)\frac{s^2}{4}, \tag{B.18}$$

where $J(\alpha) = \frac{\partial D_i}{\partial \alpha_j}$ has the following form:

$$J(\alpha) = \begin{pmatrix} \alpha_2 s & \alpha_1 s & -\alpha_4 s & -\alpha_3 s \\ \alpha_2 s & (\alpha_1+1)s & -\alpha_4 s & -\alpha_3 s \\ (\alpha_2+1)s & (\alpha_1+1)s & -\alpha_4 s & -\alpha_3 s \\ \alpha_2 s + t & (\alpha_1+1)s + t & -\alpha_4 s + [13]\langle 32\rangle & -\alpha_3 s + [23]\langle 31\rangle \end{pmatrix}. \tag{B.19}$$

When evaluated on the solutions, $J(\alpha)$ becomes:

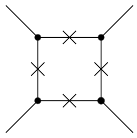
$$[J(\alpha)]_{\mathcal{S}_1} = \begin{pmatrix} 0 & -s & 0 & -\frac{st}{[13]\langle 32\rangle} \\ 0 & 0 & 0 & -\frac{st}{[13]\langle 32\rangle} \\ s & 0 & 0 & -\frac{st}{[13]\langle 32\rangle} \\ t & t & [13]\langle 32\rangle & -\frac{st}{[13]\langle 32\rangle} + [23]\langle 31\rangle \end{pmatrix}, \tag{B.20}$$

$$[J(\alpha)]_{\mathcal{S}_2} = \begin{pmatrix} 0 & -s & -\frac{st}{[23]\langle 31\rangle} & 0 \\ 0 & 0 & -\frac{st}{[23]\langle 31\rangle} & 0 \\ s & 0 & -\frac{st}{[23]\langle 31\rangle} & 0 \\ t & t & -\frac{st}{[23]\langle 31\rangle} + [13]\langle 32\rangle & [23]\langle 31\rangle \end{pmatrix}, \tag{B.21}$$

with

$$[\det J(\alpha)]_{\mathcal{S}_1} = -s^3 t, \qquad [\det J(\alpha)]_{\mathcal{S}_2} = s^3 t. \tag{B.22}$$

Substituting the expressions above in (B.18):



$$= \left(\frac{1}{s^3 t} + \frac{1}{s^3 t}\right)\frac{s^2}{4} = \frac{1}{2}\frac{1}{st}. \tag{B.23}$$

Bibliography

[1] A. V. Kotikov. "Differential equations method. New technique for massive Feynman diagram calculation". In: *Phys. Lett. B* 254 (1991).

[2] A. V. Kotikov. "Differential equations method: The Calculation of N (N $>$ 2) points diagrams". In: *ITF-91-26E* (1991).

[3] A. V. Kotikov. "Differential equations method: the calculation of vertex-type Feynman diagrams". In: *Phys. Lett. B* 259.314 (1991).

[4] Ettore Remiddi. "Differential Equations for Feynman Graph Amplitudes." In: *Nuovo Cimento* A110 (1997). URL: http://arxiv.org/abs/hep-th/9711188v2.

[5] Thomas Gehrmann and Ettore Remiddi. "Differential Equations for Two-Loop Four-Point Functions". In: *Nucl. Phys.* B580 (2000), pp. 485–518. URL: http://arxiv.org/abs/hep-ph/9912329v2.

[6] Thomas Gehrmann and Ettore Remiddi. "Two-Loop Master Integrals for $\gamma^* \to 3$ Jets: The planar topologies". In: *Nucl. Phys.* B601 (2001), pp. 248–286. URL: http://arxiv.org/abs/hep-ph/0008287v2.

[7] Thomas Gehrmann and Ettore Remiddi. "Two-Loop Master Integrals for $\gamma^* \to 3$ Jets: The non-planar topologies". In: *Nucl. Phys.* B601 (2001), pp. 287–317. URL: http://arxiv.org/abs/hep-ph/0101124v2.

[8] Ettore Remiddi and Jozef A. V. Vermaseren. "Harmonic Polylogarithms". In: *Int.J.Mod.Phys.* (2000). URL: http://arxiv.org/abs/hep-ph/9905237v1.

[9] Mario Argeri and Pierpaolo Mastrolia. "Feynman diagrams and differential equations". In: *Int. J. Mod. Phys.* (2007). URL: http://arxiv.org/abs/0707.4037v1.

[10] Johannes M. Henn. "Multiloop integrals in dimensional regularization made simple". In: *Phys.Rev.Lett.* 110 (2013). URL: http://arxiv.org/abs/1304.1806.

[11] Johannes M. Henn, Alexander V. Smirnov, and Vladimir A. Smirnov. "Analytic results for planar three-loop four-point integrals from a Knizhnik–Zamolodikov equation". In: *JHEP* 1307.128 (2013). URL: http://arxiv.org/abs/1306.2799v2.

[12] Johannes M. Henn. *Lectures on differential equations for Feynman integrals.* 2014. URL: http://arxiv.org/abs/1412.2296v3.

[13] Mario Argeri, Stefano Di Vita, Pierpaolo Mastrolia, Edoardo Mirabella, Johannes Schlenk, Ulrich Schubert, and Lorenzo Tancredi. "Magnus and Dyson Series for Master Integrals". In: *JHEP* 1403.082 (214). URL: http://arxiv.org/abs/1401.2979v2.

[14] Stefano Di Vita, Pierpaolo Mastrolia, Ulrich Schubert, and Valery Yundin. "Three-loop master integrals for ladder-box diagrams with one massive leg". In: *JHEP* 148 (2014). URL: http://arxiv.org/abs/1408.3107v1.

[15] Tien-Hao Chang. *The inverse power method and deflation.* URL: http://zoro.ee.ncku.edu.tw/na2007/res/NA09.pdf.

[16] Roman N. Lee. "Reducing differential equations for multiloop master integrals". In: *JHEP* 1504.108 (2015). URL: http://arxiv.org/abs/1411.0911v3.

[17] Stefano Laporta. "High precision calculation of multiloop Feynman integrals by difference equations". In: *Int.J.Mod.Phys.* A15 (2000). URL: http://arxiv.org/abs/hep-ph/0102033v1.

[18] Roman N. Lee, Alexander V. Smirnov, and Vladimir A. Smirnov. "Dimensional recurrence relations: an easy way to evaluate higher orders of expansion in ϵ". In: *Nucl. Phys. Proc. Suppl.* 205-206 (2010). URL: http://arxiv.org/abs/1005.0362v2.

[19] Roberto Bonciani, Pierpaolo Mastrolia, and Ettore Remiddi. "Vertex diagrams for the QED form factors at the 2-loop level". In: *Nucl. Phys.* B661 (2003), pp. 289–343. URL: http://arxiv.org/abs/hep-ph/0301170v4.

[20] Vladimir A. Smirnov. *Analytic Tools for Feynman Integrals.* Springer Tracts in Modern Physics. Springer-Verlag, 2012.

[21] Vladimir A. Smirnov. "Analytical Results for Dimensionally Regularized Massless On-shell Double Box". In: *Phys. Lett.* B460 (1999). URL: http://arxiv.org/abs/hep-ph/9905323v2.

[22] Jan B. Tausk. "Non-planar massless two-loop Feynman diagrams with four on-shell legs". In: *Phys. Lett.* B469 (1999). URL: http://arxiv.org/abs/hep-ph/9909506v1.

[23] Michal Czakon. "Automatized analytic continuation of Mellin-Barnes integrals". In: *2006* 175 (Comput. Phys. Commun.). URL: http://arxiv.org/abs/hep-ph/0511200v2.

[24] Alexander V. Smirnov and Vladimir A. Smirnov. "On the Resolution of Singularities of Multiple Mellin-Barnes Integrals". In: *Eur. Phys. J.* C62 (2009). URL: http://arxiv.org/abs/0901.0386v1.

[25] Martin Beneke and Vladimir A. Smirnov. "Asymptotic expansion of Feynman integrals near threshold". In: *Nucl. Phys.* B522 (1998). URL: http://arxiv.org/abs/hep-ph/9711391v1.

[26] Vladimir A. Smirnov. 1990.

[27] Vladimir A. Smirnov. Berlin: Springer, 2006.

[28] Thomas Binoth and Heinrich Gundrun. "An automatized algorithm to compute infrared divergent multi-loop integrals". In: *Nucl. Phys.* B585 (2000). URL: http://arxiv.org/abs/hep-ph/0004013v2.

[29] Thomas Binoth and Heinrich Gundrun. "Numerical evaluation of multi-loop integrals by sector decomposition". In: *Nucl. Phys.* B680 (2004). URL: http://arxiv.org/abs/hep-ph/0305234v1.

[30] Stefan Weinzierl and Christian Bogner. "Resolution of singularities for multi-loop integrals". In: *Comput. Phys. Commun.* (2008). URL: http://arxiv.org/abs/0709.4092v2.

[31] Heinrich Gundrun. "Sector Decomposition". In: *Int. J. Mod. Phys.* (2008). URL: http://arxiv.org/abs/0803.4177v2.

[32] Toshiaki Yokoyama. *Recursive calculation of connection formulas for systems of differential equations of Okubo normal form.* 2009. URL: http://arxiv.org/abs/0912.1058.

[33] Jonathon Carter and Heinrich Gundrun. "SecDec: A general program for sector decomposition". In: *Comput. Phys. Commun.* (2011). URL: http://arxiv.org/abs/1011.5493v2.

[34] Anastasiou Charalampos, Stefan Beerli, and Daleo Alejandro. "Evaluating multi-loop Feynman diagrams with infrared and threshold singularities numerically". In: *JHEP* (2007). URL: http://arxiv.org/abs/hep-ph/0703282v2.

[35] E. E. Boos and A. I. Davydychev. In: *Theor. Math. Phys.* (1991).

[36] F. A. Lunev. "Differential Equations for Definition and Evaluation of Feynman Integrals". In: *Phys. Rev.* D50 (1994). URL: http://arxiv.org/abs/hep-th/9407174v1.

[37] David J. Broadhurst. "Solving differential equations for 3-loop diagrams: relation to hyperbolic geometry and knot theory". In: *OUT-4102-74* (). URL: http://arxiv.org/abs/hep-th/9806174v2.

[38] Thomas Gehrmann and Ettore Remiddi. "Numerical Evaluation of Harmonic Polylogarithms". In: *Comput. Phys. Commun.* (2001). URL: http://arxiv.org/abs/hep-ph/0107173v2.

[39] Thomas Gehrmann and Ettore Remiddi. "Numerical Evaluation of Two-Dimensional Harmonic Polylogarithms". In: *Comput. Phys. Commun.* (2002). URL: http://arxiv.org/abs/hep-ph/0111255v1.

[40] Thomas Gehrmann and Ettore Remiddi. "Analytic Continuation of Massless Two-Loop Four-Point Functions". In: *Nucl. Phys.* B640 (2002). URL: http://arxiv.org/abs/hep-ph/0207020.

[41] Andrey G. Grozin. "Integration by parts - An introduction". In: *International Journal of Modern Physics A* (2011). URL: http://arxiv.org/abs/1104.3993v4.

[42] John C. Collins. *Renormalization.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1984.

[43] Roman N. Lee. "Group structure of the integration-by-parts identities and its application to the reduction of multiloop integrals". In: *JHEP* 031 (2008). URL: http://arxiv.org/abs/0804.3008v2.

[44] Michael E. Peskin and Daniel V. Shroeder. *An Introduction to Quantum Field Theory.* Addison-Wesley Advanced Book Program, 1995.

[45] Wilhelm Magnus. "On the Exponential Solution of Differential Equations for a Linear Operator". In: *Communications on Pure and Applied Mathematics* 7.4 (1954), pp. 649–673.

[46] A. von Manteuffel and C. Studerus. *Reduze 2 – Distributed Feynman Integral Reduction.* URL: http://arxiv.org/abs/1201.4330v1.

[47] A. von Manteuffel and C. Studerus. *Reduze 2 Tutorial.* URL: https://reduze.hepforge.org/download/reduze2-tutorial-20120619.pdf.

[48] James Currie, Aude Gehrmann-De Ridder, E. W. N. Glover, and Joao Pires. "NNLO QCD corrections to jet production at hadron colliders from gluon scattering". In: *JHEP* (2014). URL: http://arxiv.org/abs/1310.3993.

[49] Charalampos Anastasiou, Claude Duhr, Falko Dulat, Franz Herzog, and Bernhard Mistlberger. "Higgs boson gluon-fusion production in N3LO QCD". In: *Phys.Rev.Lett.* (2015). URL: http://arxiv.org/abs/1503.06056.

[50] Zvi Bern, John Joseph M. Carrasco, Lance Jenkins Dixon, Henrick Johansson, David Ariel Kosower, and Radu Silviu Roiban. "Three-loop Superfiniteness of N=8 Supergravity". In: *Phys.Rev.Lett.* 98 (2007). URL: http://arxiv.org/abs/hep-th/0702112v2.

[51] Alexander V. Smirnov. "Algorithm FIRE – Feynman Integral REduction". In: *JHEP* 10 (2008). URL: http://arxiv.org/abs/0807.3243v3.

[52] Vladimir A. Smirnov and A. V. Petukhov. "The number of master integrals is finite". In: *Lett. Math. Phys.* 97 (2011). URL: http://arxiv.org/abs/1004.4199v3.

[53] G. Racah. In: *Nuovo Cimento* 11 (1934), p. 461.

[54] Stefan Weinzierl. *Feynman integrals and multiple polylogarithms.* 2007. URL: http://arxiv.org/abs/0705.0900v2.

[55] Christian Bogner and Stefan Weinzierl. "Periods and Feynman integrals". In: *J. Math. Phys.* 50 (2009). URL: http://arxiv.org/abs/0711.4863v2.

[56] Ernst Hairer. *Solving Differential Equations on Manifolds.* Lecture notes. 2011. URL: http://www.unige.ch/~hairer/poly-sde-mani.pdf.

[57] *Shuffle product of two lists.* URL: http://mathematica.stackexchange.com/questions/41614/shuffle-product-of-two-lists.

[58] *Calculating scattering amplitudes efficiently.* SLAC. 1996. URL: http://arxiv.org/pdf/hep-ph/9601359v2.pdf.

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