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R. K. LUNEBURG’S APPROACH TO GEOMETRICAL OPTICS

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Abstract

We will elaborate on a little section of Professor R. K. Luneburg's mathematical theory for geometric optics in this work. We shall pay particular attention to his approach to Maxwell's equations solving in non-homogeneous media. We are particularly interested in the solutions of Maxwell's equations in the case where the optical properties of the medium are deduced from the - presumed to be discontinuous - functions ε and μ (dielectric constant and magnetic permeability). With this formulation, a physical-mathematical model relevant to optical systems (lens systems) can be built.

The differential form of Maxwell's equations is introduced at the beginning of the talk, after which certain electromagnetic field considerations are made. Together with a set of second order partial differential equations, whose unknowns are the vector components of the electric and magnetic fields, Chapter 1 also establishes the *energy balance law* for electromagnetic waves. Only in two situations can this system be solved simply: either the wave propagating medium is homogeneous, in which case the wave equations for electromagnetic fields can be derived, or the medium is not homogeneous but its optical properties vary as a function of a single variable (ε to μ depend on a single spatial component). Other than those circumstances, the system obtained is far more complex, making it more difficult to figure out its solutions.

We cannot work with the differential form of Maxwell's equations in order to follow Luneburg's method, which is to examine the discontinuous jumps of the field on its wavefronts, because these equations are valid only for fields of class C^1, i.e. for smooth functions. Furthermore, the differential form does not allow us to define conditions on the boundary of discontinuity surfaces. In order to address discontinuous electromagnetic fields, the topic of Maxwell's equations is continued in Chapter 2 with the derivation of their integral form. The general criteria that field discontinuities must satisfy on any space-time hypersurface (designated with $\mathbb{R}^4$) where the field is discontinuous are then derived from these equations. Specifically, these equations need to hold on the electromagnetic field's wavefronts and so we can derive the *eikonal equation* from them. This is a first-order partial differential equation which allows us to determine the orthogonal trajectories of the wavefronts - also referred to as *light rays* - by employing the *characteristic method*.

Light rays have the property of being invariant by reparametrization, which indicates that the curve's geometric features are independent of the parameterization chosen. As a result, Chapter 3 will demonstrate that defining an arc-length parameterization that enables us to provide a variational formulation of the problem is always possible. In fact, it will be demonstrated that the functional related to the variational problem is the integral of the *index of refraction* (optical path length), and that the wavefronts' orthogonal trajectories make up the set of its stationary points. Known as *Fermat's Principle*, this finding states that a light ray will always choose the path that minimizes its length. However, as demonstrated once more in Chapter 3, the length of an optical path is proportional to the time required to travel that path. This leads us to the conclusion that the path a light ray follows is always the one with the shortest travel time.

Chapter 4 gives the general conditions for the transport of discontinuities along light rays. Explicit formulas are then derived from these conditions to find the discontinuity values at any point in the trajectory; the only "constraint" is that the discontinuity value must be known at least at some point in the light ray. If this fact is known, on the other hand, we may use these formulas to accurately forecast the value that an electromagnetic field will take on at any given point in space-time, allowing us to determine the field's evolution.

The figures used in this paper were adapted appropriately to fit the discussion that follows from Luneburg's lecture collection [1].

Contents

Chapter 1

Introduction to electromagnetic fields

A quick introduction to electromagnetic fields and their energy will be given in this chapter. We will also discuss the Maxwell's equations, which the electric field $\vec{E}$ and the magnetic field $\vec{H}$ must satisfy in accordance with Maxwell's electromagnetic theory. We will in the end derive a system of second order partial differential equations for $\vec{E}$ and $\vec{H}$, which may be substituted in specific situations by a more straightforward system of wave equations or modified wave equations.

1.1 Electromagnetic equations

Let's start the discussion introducing some relevant objects necessary for the treatment of electromagnetic fields. Let $\vec{E}$ and $\vec{H}$ be two vector fields defined as follows:

$$\begin{aligned} \vec{E}\colon \mathbb{R}^3 \times \mathbb{R} &\to \mathbb{R}^3 \\ (x, y, z, t) &\mapsto \vec{E}\,(x, y, z, t)\colon = (E_1, E_2, E_3) \end{aligned}$$

$$\begin{aligned} \vec{H}\colon \mathbb{R}^3 \times \mathbb{R} &\to \mathbb{R}^3 \\ (x, y, z, t) &\mapsto \vec{H}\,(x, y, z, t)\colon = (H_1, H_2, H_3) \end{aligned}$$

They are the *electric field* $\vec{E}$ and the *magnetic field* $\vec{H}$, and together they represent the electromagnetic field.

Then, let ε and μ be two scalar fields defined as:

$$\begin{aligned} \varepsilon\colon \mathbb{R}^3 &\to \mathbb{R} \\ (x, y, z) &\mapsto \varepsilon(x, y, z) \end{aligned}$$

$$\begin{aligned} \mu\colon \mathbb{R}^3 &\to \mathbb{R} \\ (x, y, z) &\mapsto \mu(x, y, z) \end{aligned}$$

They are the *dielectric constant* ε and the *magnetic permeability* μ. These scalar fields describe the physical properties of the medium in which the electromagnetic field propagates.[1]

Maxwell's equations Maxwell's electromagnetic theory states that, having supposed $\vec{j} = 0$ and $\rho = 0$, an electromagnetic field must satisfy the following set of equations:

$$\vec{\nabla} \cdot (\varepsilon \vec{E}) = 0 \tag{1.1}$$

$$\vec{\nabla} \cdot (\mu \vec{H}) = 0 \tag{1.2}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}\left(\frac{\mu}{c}\vec{H}\right) = 0 \tag{1.3}$$

$$\vec{\nabla} \times \vec{H} - \frac{\partial}{\partial t}\left(\frac{\varepsilon}{c}\vec{E}\right) = 0 \tag{1.4}$$

where c is the speed of light in vacuum, $\rho : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is the electric charge density and $\vec{j} : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is the electric current.

Since the first two equations (1.1) - (1.2) are constantly zero, it is clear that the time derivative is equal to zero too. Hence we can write:

$$\frac{\partial}{\partial t}\left(\vec{\nabla} \cdot (\varepsilon \vec{E})\right) = 0 \tag{1.5}$$

$$\frac{\partial}{\partial t}\left(\vec{\nabla} \cdot (\mu \vec{H})\right) = 0 \tag{1.6}$$

Instead, regarding the two equations (1.3) - (1.4), as long as we are considering media that don't change with the time, we can rewrite them as:

$$\vec{\nabla} \times \vec{E} + \frac{\mu}{c}\frac{\partial \vec{H}}{\partial t} = 0 \tag{1.7}$$

$$\vec{\nabla} \times \vec{H} - \frac{\varepsilon}{c}\frac{\partial \vec{E}}{\partial t} = 0 \tag{1.8}$$

which represent a system of six linear differential equations of first-order.

Let us observe that the equations (1.5) - (1.6) are constantly zero and this states that the elctromagnetic field does not contain a source of electricity or magnetism. Nevertheless they are not independent of (1.7) - (1.8).

Index of recfraction Let us conclude this section with the introduction of the *index of refraction* of the medium, defined as:

$$n = \sqrt{\mu\varepsilon}$$

This index is very important in Optics and - as we will see - it is crucial to determine the velocity of the electromagnetic field.

[1]The medium is assumed not to change with the time, hence it does not depend on the variable t.

1.2 Energy of an electromagnetic field

If we make the scalar product of $\vec{H}$ with (1.7) and $\vec{E}$ with (1.8), and then we subtract one from the other, we get the relation:

$$\vec{E} \cdot (\vec{\nabla} \times \vec{H}) - \vec{H} \cdot (\vec{\nabla} \times \vec{E}) - \frac{1}{c} \left(\varepsilon \vec{E} \cdot \vec{E}_t - \mu \vec{H} \cdot \vec{H}_t \right) = 0 \tag{1.9}$$

where we denote with $\vec{E}_t = \partial \vec{E} / \partial t$ and with $\vec{H}_t = \partial \vec{H} / \partial t$. Hence, considering the following vector identity:

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \tag{1.10}$$

we obtain from (1.9) the equation:

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \frac{1}{2c} \frac{\partial}{\partial t} (\varepsilon E^2 + \mu H^2) = 0 \tag{1.11}$$

where we denote with $E = \|\vec{E}\|$ and with $H = \|\vec{H}\|$. Let us observe that the previous equation is equivalent to:

$$\frac{c}{4\pi} \vec{\nabla} \cdot (\vec{E} \times \vec{H}) + \frac{\partial}{\partial t} \frac{1}{8\pi} (\varepsilon E^2 + \mu H^2) = 0 \tag{1.12}$$

Therefore, if we define the *distribution of the electromagnetic energy* as:

$$W(x, y, z, t) = \frac{1}{8\pi} (\varepsilon E^2 + \mu H^2) \tag{1.13}$$

and we also define the *Poynting's radiation vector* as:

$$\vec{S}(x, y, z, t) = \frac{c}{4\pi} (\vec{E} \times \vec{H}) \tag{1.14}$$

then, the relation between W and $\vec{S}$ is given by the equation:

$$\frac{\partial W}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0 \tag{1.15}$$

which represents the *balance law* for the energy of the electromagnetic field.

If we integrate this equation over a generic domain $\Omega \subset \mathbb{R}^3$ enclosed by a closed boundary $\partial\Omega$, from the Gauss' divergence theorem we have that:

$$\int_\Omega \left(\frac{\partial W}{\partial t} + \vec{\nabla} \cdot \vec{S} \right) dx\, dy\, dz = \frac{\partial}{\partial t} \int_\Omega W \; dx\, dy\, dz \; + \int_{\partial\Omega} \vec{S} \cdot \vec{n} \; d\sigma = 0 \tag{1.16}$$

where $\vec{n}$ is the unit outward pointing unit normal to the surface $\partial\Omega$. The first one of the two last integrals represents the change per unit time of the total energy contained in the domain Ω, while the second one gives the amount of energy which has left the domain through its surface. This means we can interpret $\vec{S}$ as the vector field of *energy flux* of the electromagnetic field and in particular - if we consider an infinitesimal element $d\sigma$ of $\partial\Omega$ - the energy flux through this one is given by:

$$dI = \vec{S} \cdot \vec{n}\, d\sigma$$

In optics dI is called *illumination* of the surface element $d\sigma$ and represents the energy flux per unit area.

1.3 Maxwell's equations in a varying medium

Now we are going to determinate a system of second order partial differential equations for $\vec{E}$ and $\vec{H}$ from Maxwell's equations (1.7) - (1.8). Let's begin making the time derivative of these two equations, which become respectively:

$$\vec{\nabla} \times \vec{H}_t - \frac{\varepsilon}{c}\, \vec{E}_{tt} = 0 \tag{1.17}$$

$$\vec{\nabla} \times \vec{E}_t + \frac{\mu}{c}\, \vec{H}_{tt} = 0 \tag{1.18}$$

and then, again from (1.7) - (1.8), we extrapolate $\vec{H}_t = \frac{c}{\mu}\, \vec{\nabla} \times \vec{E}$ and $\vec{E}_t = \frac{c}{\varepsilon}\, \vec{\nabla} \times \vec{H}$ to replace them in (1.17) - (1.18). In this way we get the equations:

$$\mu\, \vec{\nabla} \times \left(\frac{1}{\mu}\, \vec{\nabla} \times \vec{E} \right) + \frac{\varepsilon\mu}{c^2}\, \vec{E}_{tt} = 0 \tag{1.19}$$

$$\varepsilon\, \vec{\nabla} \times \left(\frac{1}{\varepsilon}\, \vec{\nabla} \times \vec{H} \right) + \frac{\varepsilon\mu}{c^2}\, \vec{H}_{tt} = 0 \tag{1.20}$$

and using the vector identity:

$$\vec{\nabla} \times (\vec{\nabla} f \times \vec{A}) = -f\, \Delta\vec{A} + f\, \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) + (\vec{\nabla} f) \times (\vec{\nabla} \times \vec{A}) \tag{1.21}$$

- where f is an arbitrary scalar function, $\vec{A}$ is an arbitrary scalar field and $\Delta\vec{A} := \vec{\nabla}\cdot\vec{\nabla}\, \vec{A} = \frac{\partial^2 \vec{A}}{\partial x^2} + \frac{\partial^2 \vec{A}}{\partial y^2} + \frac{\partial^2 \vec{A}}{\partial z^2}$ is the Laplace operator - they become:

$$\frac{\varepsilon\mu}{c^2}\, \vec{E}_{tt} - \Delta\vec{E} = (\vec{\nabla} \times \vec{E}) \times \left(\mu\ \vec{\nabla}\frac{1}{\mu} \right) - \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) \tag{1.22}$$

$$\frac{\varepsilon\mu}{c^2}\, \vec{H}_{tt} - \Delta\vec{H} = (\vec{\nabla} \times \vec{H}) \times \left(\varepsilon\ \vec{\nabla}\frac{1}{\varepsilon} \right) - \vec{\nabla}(\vec{\nabla} \cdot \vec{H}) \tag{1.23}$$

Now let us define the vectors:

$$\vec{p} = \frac{1}{\varepsilon}\, \vec{\nabla}\varepsilon = \vec{\nabla}\big(\log \varepsilon \big) \tag{1.24}$$

$$\vec{q} = \frac{1}{\mu}\, \vec{\nabla}\mu = \vec{\nabla}\big(\log \mu \big) \tag{1.25}$$

From Maxwell's equations (1.1) - (1.2) it follows that:

$$\vec{\nabla} \cdot (\, \varepsilon \vec{E}\,) = \varepsilon \, \vec{\nabla} \cdot \vec{E} + \vec{E} \cdot \vec{\nabla} \varepsilon = 0 \tag{1.26}$$

$$\vec{\nabla} \cdot (\, \mu \vec{H}\,) = \mu \, \vec{\nabla} \cdot \vec{H} + \vec{H} \cdot \vec{\nabla} \mu = 0 \tag{1.27}$$

Hence we can write the divergences of $\vec{E}$ and $\vec{H}$ as:

$$\vec{\nabla} \cdot \vec{E} = -\vec{E} \cdot \left(\frac{1}{\varepsilon} \vec{\nabla} \varepsilon \right) = -\vec{E} \cdot \vec{p} \tag{1.28}$$

$$\vec{\nabla} \cdot \vec{H} = -\vec{H} \cdot \left(\frac{1}{\mu} \vec{\nabla} \mu \right) = -\vec{H} \cdot \vec{q} \tag{1.29}$$

and so - introducing the index of refraction n and replacing $\vec{\nabla} \cdot \vec{E}$ and $\vec{\nabla} \cdot \vec{H}$ in (1.22) - (1.23) - these last ones become:

$$\frac{n^2}{c^2} \, \vec{E}_{tt} - \Delta \vec{E} = \vec{\nabla}(\vec{E} \cdot \vec{p}) + (\vec{\nabla} \times \vec{E}) \times \vec{q} \tag{1.30}$$

$$\frac{n^2}{c^2} \, \vec{H}_{tt} - \Delta \vec{H} = \vec{\nabla}(\vec{H} \cdot \vec{q}) + (\vec{\nabla} \times \vec{H}) \times \vec{p} \tag{1.31}$$

We observe that $\vec{p}$, $\vec{q}$ and n are determined by the properties of the medium. Moreover they are not independent of each over, but are related by the equation:

$$\frac{1}{2} \, (\vec{p} + \vec{q}) = \vec{\nabla}\big(\log n \big) \tag{1.32}$$

Homogeneous media In the case of a homogeneous medium, for definetion of $\vec{p}$ and $\vec{q}$ they both are equal to zero; while $n = \sqrt{\varepsilon\mu}$ is a constant function. consequently the equations (1.30) - (1.31) become:

$$\frac{n^2}{c^2} \, \vec{E}_{tt} - \Delta \vec{E} = 0 \tag{1.33}$$

$$\frac{n^2}{c^2} \, \vec{H}_{tt} - \Delta \vec{H} = 0 \tag{1.34}$$

and therefore each component of $\vec{E}$ and $\vec{H}$ satisfies the ordinary wave equation. This means that the velocity module of an electromagnetic wave is:

$$v = \frac{c}{n} \tag{1.35}$$

and therefore - as we had already anticipated - the propagation velocity of an electromagnetic field depends directly on the index of refraction; namely on ε and μ.

Non-homogeneous media Indeed, in case of a non-homogeneous medium, the set of equations (1.30) - (1.31) is more complicated and - since n is no longer a constant, but a function which varies as x, y, z vary - the six equations of the system no longer yield one equation in each component, because the first-order operators on the right sides involve all the components of the vectors in each equation.

However we shall see that for the propagation of a light signal only the second order terms are significant in (1.30) - (1.31) and nevertheless the wave velocity of an electromagnetic wave in a non-homogeneous medium is still equal to the ratio c/n.

1.3.1 Stratified media

Now we study the case of a stratified medium in which ε and μ are two scalar functions depending only on one variable[2]; for instance z.

Let us suppose $\mu = 1$ and $\sqrt{\varepsilon} = n(z)$. Then it is clear that $\vec{q} = 0$ and $\vec{p} = 2\,\vec{\nabla}\big(\log n\big) = \big(0, 0, \frac{2n'(z)}{n(z)}\big)$. Therefore - if we denote with $p = \|\,\vec{p}\,\| = 2\,(n'/n)$ - it follows that $\vec{p}\cdot\vec{E} = pE_3$ and consequently:

$$\vec{\nabla}\,(\vec{p}\cdot\vec{E}) = \left(p\,\frac{\partial E_3}{\partial x},\, p\,\frac{\partial E_3}{\partial y},\, \frac{\partial\big(pE_3\big)}{\partial z}\right) \tag{1.36}$$

$$(\vec{\nabla}\times\vec{H})\times\vec{p} = p\left(\frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x},\, \frac{\partial H_2}{\partial z} - \frac{\partial H_3}{\partial y},\, 0\right) \tag{1.37}$$

Hence, if we replace these ones in (1.30) - (1.31), we obtain the equations:

$$\frac{n^2}{c^2}\,\frac{\partial^2\vec{E}}{\partial t^2} - \Delta\vec{E} = \left(p\,\frac{\partial E_3}{\partial x},\, p\,\frac{\partial E_3}{\partial y},\, \frac{\partial\big(p\,E_3\big)}{\partial z}\right) \tag{1.38}$$

$$\frac{n^2}{c^2}\,\frac{\partial^2\vec{H}}{\partial t^2} - \Delta\vec{H} = p\left(\frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x},\, \frac{\partial H_2}{\partial z} - \frac{\partial H_3}{\partial y},\, 0\right) \tag{1.39}$$

which - in terms of n - are nothing but:

$$\frac{n^2}{c^2}\,\frac{\partial^2\vec{E}}{\partial t^2} - \Delta\vec{E} = \left(2\,\frac{n'}{n}\frac{\partial E_3}{\partial x},\, 2\,\frac{n'}{n}\frac{\partial E_3}{\partial y},\, \frac{\partial}{\partial z}\left(2\,\frac{n'}{n}E_3\right)\right) \tag{1.40}$$

$$\frac{n^2}{c^2}\,\frac{\partial^2\vec{H}}{\partial t^2} - \Delta\vec{H} = 2\,\frac{n'}{n}\left(\frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x},\, \frac{\partial H_2}{\partial z} - \frac{\partial H_3}{\partial y},\, 0\right) \tag{1.41}$$

[2]This case is of particular interest for the study of waves propagation through thin and multilayer films, which can be - for example - glasses or lenses.

We have thus obtained two second order partial differential equations for E_3 and H_3 in which none of the other components of $\vec{E}$ and $\vec{H}$ appears and therefore we are able to determine them thanks to the equations:

$$\frac{n^2}{c^2}\,\frac{\partial^2 E_3}{\partial t^2} - \Delta E_3 = 2\,\frac{\partial}{\partial z}\left(\frac{n'}{n}\,E_3\right) \tag{1.42}$$

$$\frac{n^2}{c^2}\,\frac{\partial^2 H_3}{\partial t^2} - \Delta H_3 = 0 \tag{1.43}$$

After that E_3 and H_3 have been determined from (1.42) - (1.43), we can replace them in the remaining equations of (1.40) - (1.41), which become *modified wave equations* for E_1, E_2 and H_1, H_2; modified in the sense that the right side is not zero, but a known function.

Through this method we are able to find explicit solutions for many problems concerned to stratified media, such as the case of films producing low reflection.

Chapter 2

Discontinuities of electromagnetic fields

In this chapter we are going to introduce the concept of *wavefront* of an electromagnetic field, defined as discontinuity surface of $\vec{E}$ and $\vec{H}$. Moreover we will make some considerations about the general conditions which $\vec{E}$ and $\vec{H}$ have to satisfy on these surfaces and how the wavefronts propagate in space. With this purpose we need first to determine the integral form of Maxwell's equations, in such a way that we can establish the boundary conditions for the electromagnetic field on a discontinuity surface.

2.1 Integral form of Maxwell's equations

The scalar functions $\varepsilon = \varepsilon(x, y, z)$ and $\mu = \mu(x, y, z)$ are not necessarily continuous. In any case - for the following discussion - we will assume that they are *sectionally smooth*; namely that every finite domain of $\mathbb{R}^3$ can be divided into a finite number of subsets in which ε and μ are continuous functions and their derivatives are continuous too.

Maxwell's equations (1.1) - (1.4) represent the conditions that an electromagnetic field satisfies in every part of the space where ε, μ and $\vec{E}$, $\vec{H}$ are smooth fields.[1] However, they are not sufficient to establish the boundary conditions for $\vec{E}$ and $\vec{H}$ on a discontinuous surface. So it could be advantageous to replace the differential form of Maxwell's equations (1.1) - (1.4) with certain integral relations; because if ε, μ and $\vec{E}$, $\vec{H}$ are smooth fields, this integral form is equivalent to the differential one, but in case of discontinuous fields the integral equations apply equally well and - unlike the differential form - establish definite boundary conditions for the electromagnetic fields.

Therefore our purpose is to determine this integral form of the Maxwell's equations, but to do this we need first to introduce a theorem that will be useful for the determination of them: the *Divergence Theorem n-dimensional.*

[1] We suppose $\vec{j}$ and ρ equal to zero.

Theorem 2.1.1. - Divergence theorem for bounded open subsets of $\mathbb{R}^n$
Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with C^1 boundary and let $\xi : \mathbb{R}^n \to \mathbb{R}$ be a scalar field. If $\xi \in C^1(O)$, with $\mathrm{O} \subset \overline{\Omega}$ open neighbourhood, then for each $i \in \{1, \dots, n\}$:

$$\int_\Omega \frac{\partial \xi}{\partial x_i}\, dx = \int_{\partial\Omega} \xi n_i \, d\sigma^{n-1}$$

where $\vec{n}\colon \partial\Omega \to \mathbb{R}^n$ is the outward pointing unit normal vector to $\partial\Omega$, $dx = dx_1 \dots dx_n$ and $d\sigma^{n-1}$ is the infinitesimal measure of surface. In other terms we have the vector identity:

$$\int_\Omega \vec{\nabla}\xi \, d\vec{x} = \int_{\partial\Omega} \xi\vec{n}\, d\sigma^{n-1} \ \in \mathbb{R}^n$$

The proof is omitted[2].

Now let us consider a generic domain $\Omega \subset \mathbb{R}^3 \times \mathbb{R} \cong \mathbb{R}^4$ - referred to the coordinates (x, y, z, t) - such that $\partial\Omega$ is a closed and smooth three-dimensional hypersurface and the outward pointing unit normal $\vec{n}\colon = (n_x, n_y, n_z, n_t)$ of $\partial\Omega$ varies continuously, that is $\vec{n} \in C^1(\partial\Omega)$. We also consider the natural inclusion of $\vec{E}$ and $\vec{H}$ in $\mathbb{R}^4$:

$$\begin{aligned} i\colon \mathbb{R}^3 &\hookrightarrow \mathbb{R}^4 \\ \vec{E} &\mapsto \hat{\vec{E}} := (E_1, E_2, E_3, 0) \\ \vec{H} &\mapsto \hat{\vec{H}} := (H_1, H_2, H_3, 0) \end{aligned}$$

By applying the four-dimensional divergence theorem to the equation (1.1) we obtain the relation:

$$\int_\Omega \vec{\nabla} \cdot (\varepsilon\vec{E})\, dx\, dy\, dz\, dt = \int_{\partial\Omega} \varepsilon\hat{\vec{E}} \cdot \vec{n}\, d\sigma^3 = 0 \tag{2.1}$$

However, since $\hat{E}_4 = 0 \ \ \forall (x, y, z, t) \in \mathbb{R}^4$, if we define the vector field $\vec{N} := (n_x, n_y, n_z) \in \mathbb{R}^3$, then it is true that:

$$\hat{\vec{E}} \cdot \vec{n} = \vec{E} \cdot \vec{N} \tag{2.2}$$

and so:

$$\int_\Omega \vec{\nabla} \cdot (\varepsilon\vec{E})\, dx\, dy\, dz\, dt = \int_{\partial\Omega} \varepsilon\vec{E} \cdot \vec{N}\, d\sigma^3 = 0 \tag{2.3}$$

In the same way, from the equation (1.2) we obtain:

$$\int_\Omega \vec{\nabla} \cdot (\mu\vec{H})\, dx\, dy\, dz\, dt = \int_{\partial\Omega} \mu\hat{\vec{H}} \cdot \vec{n}\, d\sigma^3 = \int_{\partial\Omega} \mu\vec{H} \cdot \vec{N}\, d\sigma^3 = 0 \tag{2.4}$$

Therefore - by the arbitrariness of Ω - we can say that the integrals (2.3) - (2.4) are *constantly zero for any closed hypersurface* in the space-time $\mathbb{R}^4$ and we have thus derived the integral form of Maxwell's equations (1.1) - (1.2).

[2]For the proof see [2, pp. 259-261, 270-272]

For the other two Maxwell's equations we proceed as follows: let $\mathcal{B} = \{\vec{i}, \vec{j}, \vec{k}\}$ be an orthonormal basis of $\mathbb{R}^3$; through a straightforward computation it can be demonstrated that:

$$\vec{\nabla} \times \vec{E} = \vec{i} \times \left(\frac{\partial \vec{E}}{\partial x}\right) + \vec{j} \times \left(\frac{\partial \vec{E}}{\partial y}\right) + \vec{k} \times \left(\frac{\partial \vec{E}}{\partial z}\right) \tag{2.5}$$

and consequently:

$$\begin{aligned}
\int_\Omega \vec{\nabla} \times \vec{E}\, dx\, dy\, dz\, dt &= \int_\Omega \left[\vec{i} \times \left(\frac{\partial \vec{E}}{\partial x}\right) + \vec{j} \times \left(\frac{\partial \vec{E}}{\partial y}\right) + \vec{k} \times \left(\frac{\partial \vec{E}}{\partial z}\right)\right] dx\, dy\, dz\, dt &(2.6)\\
&= \int_\Omega \vec{i} \times \left(\frac{\partial \vec{E}}{\partial x}\right) dx\, dy\, dz\, dt + \int_\Omega \vec{j} \times \left(\frac{\partial \vec{E}}{\partial y}\right) dx\, dy\, dz\, dt &(2.7)\\
&\quad + \int_\Omega \vec{k} \times \left(\frac{\partial \vec{E}}{\partial z}\right) dx\, dy\, dz\, dt &(2.8)
\end{aligned}$$

Now, if we consider only the first integral of (2.7), it is true that:

$$\begin{aligned}
\int_\Omega \vec{i} \times \left(\frac{\partial \vec{E}}{\partial x}\right) dx\, dy\, dz\, dt &= \int_\Omega \left(0, -\frac{\partial E_3}{\partial x}, \frac{\partial E_2}{\partial x}\right) dx\, dy\, dz\, dt &(2.9)\\
&= \int_\Omega \frac{\partial}{\partial x} \big(0, -E_3, E_2\big)\, dx\, dy\, dz\, dt &(2.10)\\
&= \int_\Omega \frac{\partial}{\partial x} (\vec{i} \times \vec{E})\, dx\, dy\, dz\, dt &(2.11)\\
&= \int_{\partial\Omega} (\vec{i} \times \vec{E})\, n_x\, d\sigma^3 &(2.12)
\end{aligned}$$

where the last equality is a consequence of the four-dimensional divergence theorem. Proceeding in same way for the other two integrals we also get the relations:

$$\int_\Omega \vec{j} \times \left(\frac{\partial \vec{E}}{\partial y}\right) dx\, dy\, dz\, dt = \int_{\partial\Omega} (\vec{j} \times \vec{E})\, n_y\, d\sigma^3 \tag{2.13}$$

$$\int_\Omega \vec{k} \times \left(\frac{\partial \vec{E}}{\partial z}\right) dx\, dy\, dz\, dt = \int_{\partial\Omega} (\vec{k} \times \vec{E})\, n_z\, d\sigma^3 \tag{2.14}$$

and so, the equation (2.6) becomes:

$$\begin{aligned}
\int_\Omega \vec{\nabla} \times \vec{E}\, dx\, dy\, dz\, dt &= \int_{\partial\Omega} [(\vec{i} \times \vec{E})\, n_x + (\vec{j} \times \vec{E})\, n_y + (\vec{k} \times \vec{E})\, n_z]\, d\sigma^3 &(2.15)\\
&= \int_{\partial\Omega} [(n_x\vec{i} \times \vec{E}) + (n_y\vec{j} \times \vec{E}) + (n_z\vec{k} \times \vec{E})]\, d\sigma^3 &(2.16)\\
&= \int_{\partial\Omega} \big(n_x\vec{i} + n_y\vec{j} + n_z\vec{k}\,\big) \times \vec{E}\, d\sigma^3 &(2.17)\\
&= \int_{\partial\Omega} \vec{N} \times \vec{E}\, d\sigma^3 &(2.18)
\end{aligned}$$

Therefore, by integrating Maxwell's equation (1.3) on a generic domain Ω and by applying the divergence theorem to the term $\partial\vec{H}/\partial t$, we can conclude that:

$$\int_{\Omega}\left(\vec{\nabla}\times\vec{E}+\frac{\mu}{c}\frac{\partial\vec{H}}{\partial t}\right)dx\,dy\,dz\,dt=\int_{\partial\Omega}\left(\vec{N}\times\vec{E}+\frac{\mu n_t}{c}\,\vec{H}\right)d\sigma^3=0 \tag{2.19}$$

With a similar reasoning for the equation (1.4) we even obtain:

$$\int_{\Omega}\left(\vec{\nabla}\times\vec{H}-\frac{\varepsilon}{c}\frac{\partial\vec{E}}{\partial t}\right)dx\,dy\,dz\,dt=\int_{\partial\Omega}\left(\vec{N}\times\vec{H}-\frac{\varepsilon n_t}{c}\,\vec{E}\right)d\sigma^3=0 \tag{2.20}$$

As before, by the arbitrariness of Ω we can state that the integrals (2.19) - (2.20) are *constantly zero for any closed hypersurface* in the space-time $\mathbb{R}^4$.

We have thus derived the integral form of Maxwell's equations:

$$\int_{\partial\Omega}\varepsilon\vec{E}\cdot\vec{N}\,d\sigma^3=0 \tag{2.21}$$

$$\int_{\partial\Omega}\mu\vec{H}\cdot\vec{N}\,d\sigma^3=0 \tag{2.22}$$

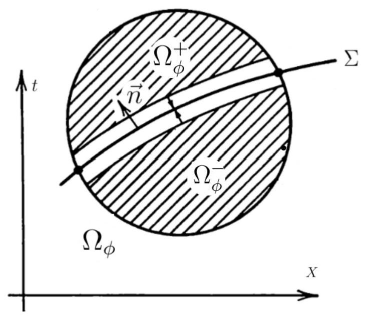
$$\int_{\partial\Omega}\left(\vec{N}\times\vec{E}+\frac{\mu n_t}{c}\,\vec{H}\right)d\sigma^3=0 \tag{2.23}$$

$$\int_{\partial\Omega}\left(\vec{N}\times\vec{H}-\frac{\varepsilon n_t}{c}\,\vec{E}\right)d\sigma^3=0 \tag{2.24}$$

which are satisfied for any closed and smooth hypersurface in $\mathbb{R}^4$ and which involve only the vector fields $\vec{E}$, $\vec{H}$ and the scalar fields ε, μ; and not their derivatives. Moreover, it is important to point out that the integral and the differential forms of Maxwell's equations are equivalent if the fields are continuous; but if they are not it is necessary to define boundary conditions.

2.2 General conditions for discontinuities

Now we apply the integral Maxwell's equations (2.21) - (2.24) to the case of a surface where $\vec{E}$ and $\vec{H}$ or ε and μ are discontinuous.



Let $\phi\colon D\subseteq\mathbb{R}^4\to\mathbb{R}$ be a scalar field, $\Sigma=\{(x,y,z,t)\in\mathbb{R}^4\,|\,\phi(x,y,z,t)=0\}\subset\mathbb{R}^4$ be a fixed discontinuity hypersurface and let $\Omega_\phi\subset\mathbb{R}^4$ be a generic compact support set such that $\Omega_\phi\cap\Sigma\neq\emptyset$ and Σ divides Ω_ϕ into two parts: Ω_ϕ^+ and Ω_ϕ^-. Then we denote with $\Sigma_\phi=\Omega_\phi\cap\Sigma$ and we suppose that the outward pointing normal $\vec{n}$ to Σ_ϕ points towards Ω_ϕ^+.

Since Σ is a discontinuity surface for $\vec{E}$ and $\vec{H}$ or ε and μ, obviously Σ_ϕ is too; therefore we define as $\vec{E}_+, \vec{H}_+, \varepsilon_+$ and μ_+ the fields values in a neighbourhood of Σ_ϕ belonging to Ω^+_ϕ, and we denote with $\vec{E}_-, \vec{H}_-, \varepsilon_-$ and μ_- the fields values in a neighbourhood of Σ_ϕ belonging to Ω^-_ϕ. Moreover, we define the outward pointing unit normal to Σ_ϕ as:

$$\vec{n}\Big|_{\Sigma_\phi} := \frac{(\phi_x\ ,\ \phi_y\ ,\ \phi_z\ ,\ \phi_t)}{\sqrt{{\phi_x}^2 + {\phi_y}^2 + {\phi_z}^2 + {\phi_t}^2}} = \frac{(\vec{\nabla}_{\vec{X}}\,\phi\ ,\ \phi_t)}{\|\vec{\nabla}\phi\|} \tag{2.25}$$

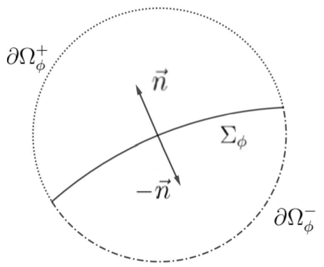
where $\vec{\nabla}_{\vec{X}}\,\phi := (\phi_x\ ,\ \phi_y\ ,\ \phi_z)$ and $\phi_i = \frac{\partial \phi}{\partial i}$ for $i \in \{x, y, z, t\}$.

Discontinuity conditions for the equations (2.21) - (2.22) Since $\partial\Omega_\phi :\ = \partial\Omega^+_\phi + \partial\Omega^-_\phi$ is a closed hypersurface[3] of $\mathbb{R}^4$, from the integral Maxwell's equation (2.21) it follows that:

$$0 = \int_{\partial\Omega_\phi} \varepsilon\vec{E}\cdot\vec{N}\ d\sigma^3 \tag{2.26}$$

$$= \int_{\partial\Omega^+_\phi} \varepsilon\vec{E}\cdot\vec{N}\ d\sigma^3 + \int_{\partial\Omega^-_\phi} \varepsilon\vec{E}\cdot\vec{N}\ d\sigma^3 \tag{2.27}$$

However, also $\Gamma^- :\ = \partial\Omega^-_\phi + \Sigma_\phi$ is a closed hypersurface, then we can write:



$$0 = \int_{\Gamma^-} \varepsilon\vec{E}\cdot\vec{N}\ d\sigma^3 \tag{2.28}$$

$$= \int_{\partial\Omega^-_\phi} \varepsilon\vec{E}\cdot\vec{N}\ d\sigma^3 + \int_{\Sigma_\phi} \varepsilon\vec{E}\cdot\vec{N}\ d\sigma^3 \tag{2.29}$$

$$= \int_{\partial\Omega^-_\phi} \varepsilon\vec{E}\cdot\vec{N}\ d\sigma^3 + \int_{\Sigma_\phi} \varepsilon_-\vec{E}_-\cdot\frac{\vec{\nabla}_{\vec{X}}\,\phi}{\|\vec{\nabla}\phi\|}\ d\sigma^3 \tag{2.30}$$

and in the same way for $\Gamma^+ :\ = \partial\Omega^+_\phi + \Sigma_\phi$:

$$0 = \int_{\Gamma^+} \varepsilon\vec{E}\cdot\vec{N}\ d\sigma^3 \tag{2.31}$$

$$= \int_{\partial\Omega^+_\phi} \varepsilon\vec{E}\cdot\vec{N}\ d\sigma^3 + \int_{\Sigma_\phi} \varepsilon\vec{E}\cdot\vec{N}\ d\sigma^3 \tag{2.32}$$

$$= \int_{\partial\Omega^+_\phi} \varepsilon\vec{E}\cdot\vec{N}\ d\sigma^3 + \int_{\Sigma_\phi} \varepsilon_+\vec{E}_+\cdot\left(- \frac{\vec{\nabla}_{\vec{X}}\,\phi}{\|\vec{\nabla}\phi\|}\right) d\sigma^3 \tag{2.33}$$

$$= \int_{\partial\Omega^+_\phi} \varepsilon\vec{E}\cdot\vec{N}\ d\sigma^3 - \int_{\Sigma_\phi} \varepsilon_+\vec{E}_+\cdot\frac{\vec{\nabla}_{\vec{X}}\,\phi}{\|\vec{\nabla}\phi\|}\ d\sigma^3 \tag{2.34}$$

[3]We assumed that Ω is a compact support set of $\mathbb{R}^4$, then it is closed and bounded.

Therefore we can subtract (2.30) and (2.34) from (2.27), to obtain thus the relation:

$$\int_{\Sigma_\phi} \left(\varepsilon_+ \vec{E}_+ \;-\; \varepsilon_- \vec{E}_-\right) \cdot \frac{\vec{\nabla}_{\vec{X}}\, \phi}{\|\vec{\nabla}\phi\|} \;\; d\sigma^3 = 0 \tag{2.35}$$

and since (2.35) must be true for any Σ_ϕ, by the arbitrariness of Ω_ϕ it follows that:

$$[\, \varepsilon \vec{E}\,] \cdot \vec{\nabla}_{\vec{X}}\, \phi = 0 \tag{2.36}$$

where $[\, \varepsilon \vec{E}\,] := \varepsilon_+ \vec{E}_+ - \varepsilon_- \vec{E}_-$ denote the size of the discontinuity on Σ_ϕ. In a similar way for the equation (2.22), we also derive:

$$[\, \mu \vec{H}\,] \cdot \vec{\nabla}_{\vec{X}}\, \phi = 0 \tag{2.37}$$

Discontinuity conditions for the equations (2.23) - (2.24) Now we consider the Maxwell's integral equation (2.23) and - as we have done previously - we compute the value of the integral along the closed hypersurfaces $\partial\Omega_\phi$, Γ^- and Γ^+. Therefore we get the following relations:

- On $\partial\Omega_\phi$: $= \partial\Omega_\phi^+ \,+\, \partial\Omega_\phi^-$:

$$\begin{aligned} 0 &= \int_{\partial\Omega_\phi} \big(\, \vec{N} \times \vec{E} + \frac{\mu n_t}{c}\, \vec{H}\, \big)\; d\sigma^3 && (2.38)\\ &= \int_{\partial\Omega_\phi^+} \big(\, \vec{N} \times \vec{E} + \frac{\mu n_t}{c}\, \vec{H}\, \big)\; d\sigma^3 + \int_{\partial\Omega_\phi^-} \big(\, \vec{N} \times \vec{E} + \frac{\mu n_t}{c}\, \vec{H}\, \big)\; d\sigma^3 && (2.39) \end{aligned}$$

- On Γ^-: $= \partial\Omega_\phi^- \,+\, \Sigma_\phi$:

$$\begin{aligned} 0 &= \int_{\Gamma^-} \big(\, \vec{N} \times \vec{E} + \frac{\mu n_t}{c}\, \vec{H}\, \big)\; d\sigma^3 && (2.40)\\ &= \int_{\partial\Omega_\phi^-} \big(\, \vec{N} \times \vec{E} + \frac{\mu n_t}{c}\, \vec{H}\, \big)\; d\sigma^3 + \int_{\Sigma_\phi} \big(\, \vec{N} \times \vec{E} + \frac{\mu n_t}{c}\, \vec{H}\, \big)\; d\sigma^3 && (2.41)\\ &= \int_{\partial\Omega_\phi^-} \big(\, \vec{N} \times \vec{E} + \frac{\mu n_t}{c}\, \vec{H}\, \big)\; d\sigma^3 + \int_{\Sigma_\phi} \frac{1}{\|\vec{\nabla}\phi\|} \left(\vec{\nabla}_{\vec{X}}\, \phi \times \vec{E}_- \;+\; \frac{\mu_-\, \phi_t}{c}\, \vec{H}_- \right) d\sigma^3 && (2.42) \end{aligned}$$

- On $\Gamma^+ : = \partial\Omega_\phi^+ + \Sigma_\phi$:

$$0 = \int_{\gamma^+} \big(\vec{N} \times \vec{E} + \frac{\mu n_t}{c} \vec{H} \big) \, d\sigma^3 \tag{2.43}$$

$$= \int_{\partial\Omega_\phi^+} \big(\vec{N} \times \vec{E} + \frac{\mu n_t}{c} \vec{H} \big) \, d\sigma^3 + \int_{\Sigma_\phi} \big(\vec{N} \times \vec{E} + \frac{\mu n_t}{c} \vec{H} \big) \, d\sigma^3 \tag{2.44}$$

$$= \int_{\partial\Omega_\phi^+} \big(\vec{N} \times \vec{E} + \frac{\mu n_t}{c} \vec{H} \big) \, d\sigma^3 + \int_{\Sigma_\phi} \frac{1}{\|\vec{\nabla}\phi\|} \Big((-\vec{\nabla}_{\vec{X}} \, \phi) \times \vec{E}_+ + \frac{\mu_+(-\phi_t)}{c} \vec{H}_+ \Big) \, d\sigma^3 \tag{2.45}$$

$$= \int_{\partial\Omega_\phi^+} \big(\vec{N} \times \vec{E} + \frac{\mu n_t}{c} \vec{H} \big) \, d\sigma^3 - \int_{\Sigma_\phi} \frac{1}{\|\vec{\nabla}\phi\|} \Big(\vec{\nabla}_{\vec{X}} \, \phi \times \vec{E}_+ + \frac{\mu_+ \, \phi_t}{c} \vec{H}_+ \Big) \, d\sigma^3 \tag{2.46}$$

At this point we subtract (2.42) and (2.46) from (2.39), obtaining thus the equation:

$$\int_{\Sigma_\phi} \frac{1}{\|\vec{\nabla}\phi\|} \Big[\vec{\nabla}_{\vec{X}} \, \phi \times \Big(\vec{E}_+ - \vec{E}_- \Big) \; + \; \frac{\phi_t}{c} \Big(\mu_+ \vec{H}_+ - \mu_- \vec{H}_- \Big) \Big] \, d\sigma^3 = 0 \tag{2.47}$$

and since (2.47) must be true for any Σ_ϕ, by the arbitrariness of Ω_ϕ it follows that:

$$\vec{\nabla}_{\vec{X}} \, \phi \, \times [\, \vec{E} \,] \, + \, \frac{\phi_t}{c} [\, \mu \vec{H} \,] = 0 \tag{2.48}$$

where we denote with $[\vec{E}] = \vec{E}_+ - \vec{E}_-$ and $\big[\mu\vec{H}\big] = \mu_+\vec{H}_+ - \mu_-\vec{H}_-$ the sizes of the discontinuities on Σ_ϕ. With the same proceeding we have done for (2.23), we derive from (2.24) the condition:

$$\vec{\nabla}_{\vec{X}} \, \phi \, \times [\, \vec{H} \,] \, - \, \frac{\phi_t}{c} [\, \varepsilon \vec{E} \,] = 0 \tag{2.49}$$

In conclusion we have demonstrated that:

An electromagnetic field which is discontinuous on a hypersurface $\Sigma = \{(x, y, z, t) \in \mathbb{R}^4 \mid \phi(x, y, z, t) = 0\}$ *must satisfy the conditions:*

$$\vec{\nabla}_{\vec{X}} \, \phi \cdot [\, \varepsilon \vec{E} \,] = 0 \tag{2.50}$$

$$\vec{\nabla}_{\vec{X}} \, \phi \cdot [\, \mu \vec{H} \,] = 0 \tag{2.51}$$

$$\vec{\nabla}_{\vec{X}} \, \phi \, \times [\, \vec{E} \,] \, + \, \frac{\phi_t}{c} [\, \mu \vec{H} \,] = 0 \tag{2.52}$$

$$\vec{\nabla}_{\vec{X}} \, \phi \, \times [\, \vec{H} \,] \, - \, \frac{\phi_t}{c} [\, \varepsilon \vec{E} \,] = 0 \tag{2.53}$$

We observe that (2.50) - (2.53) represent a system of linear differential equations which are the counterpart of the differential form of the Maxwell's equations (1.1) - (1.4).

2.2.1 Discontinuities of ε and μ

Now let us consider the special case where $\vec{E}$ and $\vec{H}$ are discontinuous as a consequence of discontinuities of ε and μ.[4]

Let $\psi\colon D \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a scalar field and let $\Psi = \{(x, y, z) \in \mathbb{R}^3 \mid \psi(x, y, z) = 0\}$ be a refracting surface so that ε and μ are discontinuous on it and the equation $\psi(x, y, z) = 0$ represents a cylindrical hypersurface in the space-time $\mathbb{R}^4$. Since $\phi_t \equiv 0$, then $\vec{\nabla}_{\vec{X}}\, \phi \equiv \vec{\nabla}_{\vec{X}}\, \psi$ and the conditions (2.50) - (2.53) become:

$$\vec{\nabla}_{\vec{X}}\, \psi \cdot [\, \varepsilon\vec{E}\,] = 0 \tag{2.54}$$
$$\vec{\nabla}_{\vec{X}}\, \psi \cdot [\, \mu\vec{H}\,] = 0 \tag{2.55}$$
$$\vec{\nabla}_{\vec{X}}\, \psi\ \times [\, \vec{E}\,] = 0 \tag{2.56}$$
$$\vec{\nabla}_{\vec{X}}\, \psi\ \times [\, \vec{H}\,] = 0 \tag{2.57}$$

and we observe that:

- the vectors $\frac{\vec{\nabla}_{\vec{X}}\, \psi \times \vec{H}}{\|\vec{\nabla}_{\vec{X}}\, \psi\|}$ and $\frac{\vec{\nabla}_{\vec{X}}\, \psi \times \vec{E}}{\|\vec{\nabla}_{\vec{X}}\, \psi\|}$ are linearly related to the tangential components of $\vec{H}$ and $\vec{E}$ to Ψ;
- the quantities $\frac{\vec{\nabla}_{\vec{X}}\, \psi \cdot (\varepsilon\vec{E})}{\|\vec{\nabla}_{\vec{X}}\, \psi\|}$ and $\frac{\vec{\nabla}_{\vec{X}}\, \psi \cdot (\mu\vec{H})}{\|\vec{\nabla}_{\vec{X}}\, \psi\|}$ are the normal components of $\varepsilon\vec{E}$ and $\mu\vec{H}$ to Ψ.

Hence, by the arbitrariness of Ψ, we can conclude that *the tangential components of $\vec{E}$ and $\vec{H}$ and the normal components of $\varepsilon\vec{E}$ and $\mu\vec{H}$ are continuous on a discontinuous surface of ε and μ.*

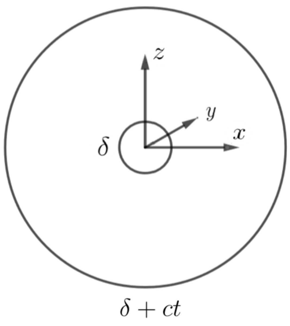
2.3 Propagation of discontinuities

2.3.1 Wavefronts

Unlike what we have seen until now, the discontinuities of an electromagnetic field can also appear without being caused by a discontinuous distribution of substance. For example, let us suppose $\varepsilon = \mu = 1$ and we assume that at $t = 0$ the vector fields $\vec{E}(x, y, z, 0)$ and $\vec{H}(x, y, z, 0)$ are different from zero only in a small sphere of radius $\delta > 0$ around the origin. Then - since the sphere expands with time - at a given $t > 0$ the vectors $\vec{E}$ and $\vec{H}$ are different from zero in a larger sphere of radius $\delta + ct$. In other words, if we denote with $\delta + ct$ the surface which separates the parts of the space which are still at rest from those that are penetrated by the original impulse, this one travels over the space as time passes. A surface of this type is called *wavefront* and in the above example the wavefronts are given by the equation:

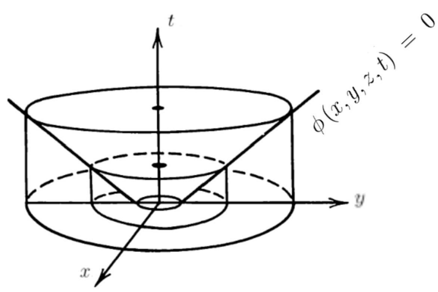
$$\phi\,(x, y, z, t) = \sqrt{x^2 + y^2 + z^2} - \delta - ct = 0 \tag{2.58}$$

[4] An example of this case is any system of glass lenses.



If $\vec{E}$ and $\vec{H}$ are different from zero at $t = 0$ on the boundary of the original sphere of radius δ, then this sphere will be a surface on which the electromagnetic field is discontinuous. Moreover we expect that at time $t > 0$ the values of $\vec{E}$ and $\vec{H}$ are different from zero even on the boundary of the wavefront (2.58), namely the electromagnetic field is discontinuous on the new wavefront too. This means that we can define a wavefront more generally as *any surface of the three dimensional space* $\mathbb{R}^3$ - referred to the coordinates (x, y, z) - *on which, at a given time* t*, the electromagnetic field is discontinuous.*

Wavefronts as hypersurfaces of space-time $\mathbb{R}^4$ Another way to define a wavefront - instead of using a set of surfaces of $\mathbb{R}^3$ depending on the parameter t - is to interpret $\Sigma_t := \{(x, y, z) \in \mathbb{R}^3 : \phi(x, y, z, t) = 0\}$ as a hypersurface in the space-time $\mathbb{R}^4$. Hence, in our example, instead of representing the wavefronts (2.58) as spheres on $\mathbb{R}^3$ space, we can consider a hypercone on $\mathbb{R}^4$ of equation:



$$\sqrt{x^2 + y^2 + z^2} - ct = \delta \tag{2.59}$$

in which the vector fields $\vec{E}$ and $\vec{H}$ are discontinuous and the cross sections of the hypercone $\phi(x, y, z, t) = 0$ with the hyperplanes $t = const.$ represent the above set of wavefronts.

2.3.2 Characteristic Equation

We may expect that a hypersurface - similar to the example above - must have certain characteristics to determine the discontinuity surfaces of an electromagnetic field. Thanks to the general conditions (2.50) - (2.53) for discontinuities, we can derive these conditions.

Let $\Sigma_t \subset \mathbb{R}^3$ be a hypersurface of equation $\phi(x, y, z, t) = 0$ on which $\vec{E}$ and $\vec{H}$ are discontinuous and suppose that ε and μ are continuous in a neighbourhood of Σ_t. Then we define on Σ_t the vectors:

$$\begin{aligned} [\vec{E}] &= \vec{E}_+ - \vec{E}_- \\ [\vec{H}] &= \vec{H}_+ - \vec{H}_- \end{aligned} \tag{2.60}$$

which measure the jump of $\vec{E}$ and $\vec{H}$ on Σ_t. Then, from (2.50) - (2.53), it follows that:

$$\vec{\nabla}_{\vec{X}}\, \phi \cdot [\vec{E}] = 0 \tag{2.61}$$

$$\vec{\nabla}_{\vec{X}}\, \phi \cdot [\vec{H}] = 0 \tag{2.62}$$

$$\vec{\nabla}_{\vec{X}}\, \phi \times [\vec{E}] + \frac{\mu}{c}\, \phi_t\, [\vec{H}] = 0 \tag{2.63}$$

$$\vec{\nabla}_{\vec{X}}\, \phi \times [\vec{H}] - \frac{\varepsilon}{c}\, \phi_t\, [\vec{E}] = 0 \tag{2.64}$$

We observe that the relations (2.63) - (2.64) represent a system of six linear homogeneous equations for $[\vec{E}]$ and $[\vec{H}]$, which have non-trivial solutions only if the determinant of the associated matrix is zero. Therefore, if we take the vector product of $\vec{\nabla}_{\vec{X}}\,\phi$ with one of the previous equations - for instance (2.63) - and then we replace $\vec{\nabla}_{\vec{X}}\,\phi\,\times[\vec{H}] = \frac{\varepsilon}{c}\,\phi_t[\vec{E}]$ (from 2.64), we obtain:

$$0 = \vec{\nabla}_{\vec{X}}\,\phi\,\times\big(\vec{\nabla}_{\vec{X}}\,\phi\,\times[\vec{E}]\big) + \frac{\mu\phi_t}{c}\big(\vec{\nabla}_{\vec{X}}\,\phi\times[\vec{H}]\big) \tag{2.65}$$

$$= \vec{\nabla}_{\vec{X}}\,\phi\,\times\big(\vec{\nabla}_{\vec{X}}\,\phi\,\times[\vec{E}]\big) + \frac{\mu\phi_t}{c}\left(\frac{\varepsilon\phi_t}{c}\,[\vec{E}]\right) \tag{2.66}$$

$$= \vec{\nabla}_{\vec{X}}\,\phi\,\times\big(\vec{\nabla}_{\vec{X}}\,\phi\,\times[\vec{E}]\big) + \frac{\varepsilon\mu}{c^2}\,\phi_t^2\,[\vec{E}] \tag{2.67}$$

Therefore, by applying the vector identity $(\vec{A}\times\vec{B})\times\vec{C} = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{B}\cdot\vec{C})\vec{A}$ to (2.67) it follows that:

$$\big(\vec{\nabla}_{\vec{X}}\,\phi\cdot[\vec{E}]\big)\,\vec{\nabla}_{\vec{X}}\,\phi - \|\vec{\nabla}_{\vec{X}}\,\phi\|^2\,[\vec{E}] + \frac{\varepsilon\mu}{c^2}\,\phi_t^2\,[\vec{E}] = 0 \tag{2.68}$$

and since $\vec{\nabla}_{\vec{X}}\,\phi\cdot[\vec{E}] = 0$, we can conclude:

$$\left(\|\vec{\nabla}_{\vec{X}}\,\phi\|^2 - \frac{\varepsilon\mu}{c^2}\,\phi_t^2\right)[\vec{E}] = 0 \tag{2.69}$$

With a similar proceeding for (2.64) we also get:

$$\left(\|\vec{\nabla}_{\vec{X}}\,\phi\|^2 - \frac{\varepsilon\mu}{c^2}\,\phi_t^2\right)[\vec{H}] = 0 \tag{2.70}$$

Therefore we can state that, if $[\vec{E}]$ and $[\vec{H}]$ are different from zero - namely $\vec{E}$ and $\vec{H}$ are discontinuous on Σ - then the scalar field $\phi = \phi\,(x,y,z,t)$ must satisfy the equation:

$$\|\vec{\nabla}_{\vec{X}}\,\phi\|^2 = \frac{\varepsilon\mu}{c^2}\,\phi_t^2 \tag{2.71}$$

that is:

$$\phi_x^2 + \phi_y^2 + \phi_z^2 = \frac{\varepsilon\mu}{c^2}\,\phi_t^2 \tag{2.72}$$

The relation (2.71) is called *characteristic equation* of Maxwell's differential equations; while every hypersurface $\Sigma_t = \{(x,y,z,t)\in\mathbb{R}^4 \mid \phi(x,y,z,t) = 0\}$ which satisfies the characteristic equation for any $(x,y,z,t)\in\Sigma_t$ is called *characteristic surface.*

2.3.3 Eikonal Equation and mutual position of $\vec{E}$ and $\vec{H}$

The Eikonal Equation: The characteristic equation (2.71) is not a true differential equation for the scalar field ϕ, because it does not have to be satisfied $\forall(x,y,z,t)\in$ *Dom*(ϕ), but only for those points for which $\phi\,(x,y,z,t) = 0$. However - without loss of generality - we can assume that the characteristic surface is given by the equation:

$$\phi\,(x,y,z,t) :\, = \psi\,(x,y,z) - ct = 0 \tag{2.73}$$

where ψ is a scalar field independent of t.[5]
In this way the characteristic equation becomes:

$$\psi_x^2 + \psi_y^2 + \psi_z^2 = \frac{\varepsilon\mu}{c^2}\, c^2 = \varepsilon\mu \tag{2.74}$$

and introducing the index of refraction $n = \sqrt{\varepsilon\mu}$ we obtain what is the *Eikonal Equation*:

$$\|\vec{\nabla}\psi\|^2 = n^2 \tag{2.75}$$

which - unlike the characteristic equation - is a partial differential equation for ϕ and so it must be satisfied $\forall (x, y, z, t) \in \mathbb{R}^4$.

Mutual position of $\vec{E}$ and $\vec{H}$: Let us suppose $\phi = \psi - ct$. Then the equations (2.61) - (2.64) become:

$$\vec{\nabla}\psi \cdot [\vec{E}] = 0 \tag{2.76}$$

$$\vec{\nabla}\psi \cdot [\vec{H}] = 0 \tag{2.77}$$

$$\vec{\nabla}\psi \times [\vec{E}] - \mu[\vec{H}] = 0 \tag{2.78}$$

$$\vec{\nabla}\psi \times [\vec{H}] + \varepsilon[\vec{E}] = 0 \tag{2.79}$$

Thanks to the equations (2.76) - (2.77), we can state that $[\vec{E}]$ and $[\vec{H}]$ are orthogonal to $\vec{\nabla}\psi$; while from (2.78) - (2.79) we can respectively derive:

$$[\vec{H}] = \frac{1}{\mu}\, \vec{\nabla}\psi \times [\vec{E}] \quad \perp \quad [\vec{E}] \tag{2.80}$$

$$[\vec{E}] = -\frac{1}{\varepsilon}\, \vec{\nabla}\psi \times [\vec{H}] \quad \perp \quad [\vec{H}] \tag{2.81}$$

Hence it follows that:

$$[\vec{E}] \cdot [\vec{H}] = 0 \tag{2.82}$$

We can thus conclude that the vector fields $[\vec{E}]$ and $[\vec{H}]$ are tangential to the wavefronts Σ_t and perpendicular to each other.

On the other hand - at the beginning of this discussion - we have defined the wavefront as the boundary of a three dimensional space within which the electromagnetic field is different from zero and outside it is zero. This means that $[\vec{E}]$ and $[\vec{H}]$ are equal to the vectors $\vec{E}$ and $\vec{H}$ on the wavefront $\Sigma_t \colon = \{(x, y, z) \in \mathbb{R}^3 \mid \psi\,(x, y, z) = ct\}$, because on one side of Σ_t the vector fields are zero.

[5] We pay attention to the fact that, if $\vec{E}$ and $\vec{H}$ are discontinuous on $\phi = \psi - ct = 0$, then the wavefronts of the electromagnetic field are defined by the level sets $\Sigma_t \colon = \{(x, y, z) \in \mathbb{R}^3 \mid \psi\,(x, y, z) = ct\}$ $\forall t \in \mathbb{R}$.

So we have demonstrated that *on a wavefront Σ_t the vector fields $\vec{E}$ and $\vec{H}$ are tangential to the surface Σ_t and perpendicular to each other.*

Moreover - using the vector identity $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\,\vec{B} - (\vec{A} \cdot \vec{B})\,\vec{C}$ and denoting with $\nabla\psi = \left\|\vec{\nabla}\psi\right\|$ - on a wavefront Σ_t it is true that:

$$\vec{\nabla}\psi \times \vec{E} = \vec{\nabla}\psi \times \left(-\frac{1}{\varepsilon}\,\vec{\nabla}\psi \times \vec{H} \right) \tag{2.83}$$

$$= -\frac{1}{\varepsilon}\left(\big(\vec{\nabla}\psi \cdot \vec{H}\big)\vec{\nabla}\psi - \nabla\psi^2 \vec{H} \right) = \frac{\nabla\psi^2}{\varepsilon}\,\vec{H} \tag{2.84}$$

Similarly:

$$\vec{\nabla}\psi \times \vec{H} = \vec{\nabla}\psi \times \left(\frac{1}{\mu}\,\vec{\nabla}\psi \times \vec{E} \right) \tag{2.85}$$

$$= \frac{1}{\mu}\left(\big(\vec{\nabla}\psi \cdot \vec{E}\big)\vec{\nabla}\psi - \nabla\psi^2 \vec{E} \right) = -\frac{\nabla\psi^2}{\mu}\,\vec{E} \tag{2.86}$$

Therefore - by the right hand rule - we can state that *on a wavefront Σ_t the vector fields $\vec{E}$, $\vec{H}$ and $\vec{\nabla}\psi$ are oriented as in the following figure:*

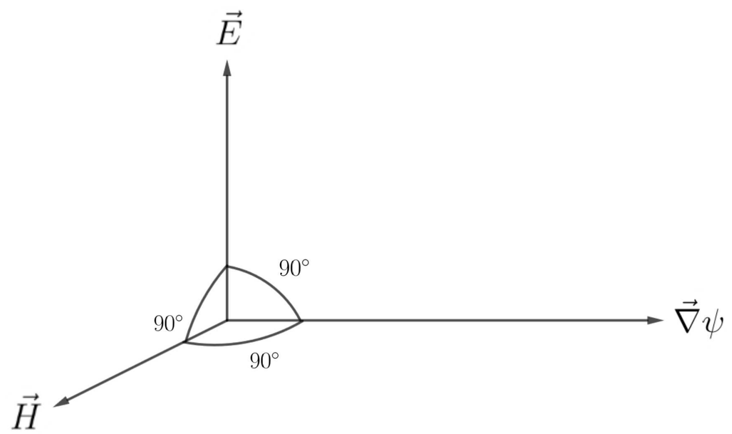


Figure 2.1: Mutual position of the vector fields $\vec{E}$, $\vec{H}$ and $\vec{\nabla}\psi$ on a wavefront Σ_t.

Chapter 3

The Fermat's Principle

In this chapter we are going to derive an important principle of geometrical optics which concerns the optical path of the wavefronts' orthogonal trajectories, namely the integral curves of the eikonal equation. This result is known as the *Fermat's Principle* and - with the aim of proving it - we will first make some considerations about the orthogonal trajectories of the electromagnetic wavefronts, which are even called *light rays*.

3.1 Characteristics of eikonal equation

In general, the problem of integrating a scalar PDE[1] of first-order can be reduced to the problem of integration of a system of ordinary differential equations, that is:

$$(CS)\ \begin{cases} \dot{\vec{X}}(s) = \dfrac{d}{ds}\,\vec{X}(s) = \vec{\nabla}_{\vec{P}}\,F\big(\vec{e}(s)\big) \\ \dot{Z}(s) = \dfrac{d}{ds}\,Z(s) = \vec{\nabla}_{\vec{P}}\,F\big(\vec{e}(s)\big)\cdot\vec{P}(s) \\ \dot{\vec{P}}(s) = \dfrac{d}{ds}\,\vec{P}(s) = -\vec{\nabla}_{\vec{X}}\,F\big(\vec{e}(s)\big) - \dfrac{\partial}{\partial Z}F\big(\vec{e}(s)\big)\,\vec{P}(s) \end{cases} \tag{3.1}$$

where $\vec{X}(s)$, $\vec{P}(s)\colon \mathbb{R}\to\mathbb{R}^n$ and $Z(s)\colon\mathbb{R}\to\mathbb{R}$ are smooth functions ($s\in\mathbb{R}$ is an evolution parameter), $\vec{e}(s) = \big(\vec{X}(s), Z(s), \vec{P}(s)\big)\in\mathbb{R}^{2n+1}$ and:

$$\begin{aligned} F\colon D\subset\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n &\longrightarrow \mathbb{R} \\ \big(\vec{X}, Z, \vec{P}\,\big) &\longmapsto F = F\big(\vec{X}, Z, \vec{P}\,\big) \end{aligned}$$

is a smooth function such that $F\big(\vec{X}, Z, \vec{P}\,\big) = 0$ is a PDE.
The system (3.1) is known as *characteristic system* of the PDE $F\big(\vec{X}, Z, \vec{P}\,\big) = 0$, while its solutions $\vec{e}(s)$ are called *integral characteristic curves*. The components $\big(\vec{X}(s), Z(s)\big)\in\mathbb{R}^{n+1}$ of a solution $\vec{e}(s)$ are even known as *characteristic* of the PDE, while $\vec{X}(s)$ is the *projected characteristic*.

[1]Partial Differential Equation

As we have seen in the previous chapter, the eikonal equation:

$$\psi_x^2 + \psi_y^2 + \psi_z^2 = n^2 \tag{3.2}$$

is a PDE of first-order whose solutions allow us to determine the equation which describes the discontinuity surfaces of an electromagnetic field, namely the wavefronts $\Sigma_t = \{(x, y, z) \in \mathbb{R}^3 \mid \psi(x, y, z) = 0\}$. However we can reduce the eikonal equation into its characteristic system (3.1) to find thus its characteristics $\big(\vec{X}(s), \psi(\vec{X}(s))\big)$ and so even the projected characteristics $\vec{X}(s) = \big(x(s), y(s), z(s)\big)$ of the points belonging to the wavefronts. Indeed, the following theorem holds:

Theorem 3.1.1. *Let $u = u(\vec{X})$ be a smooth scalar field that is a solution of a PDE defined by $F = 0$, that is:*

$$F\big(\vec{X}, u(\vec{X}), \vec{\nabla} u(\vec{X})\big) = 0 \quad \forall \vec{X} \in Dom(u) \tag{3.3}$$

and then we suppose that $\vec{X}(s)$ solves $(CS)_1$, namely:

$$\dot{\vec{X}}(s) = \vec{\nabla}_{\vec{P}} F\big(\vec{X}(s), u(\vec{X}(s)), \vec{\nabla} u(\vec{X}(s))\big) \tag{3.4}$$

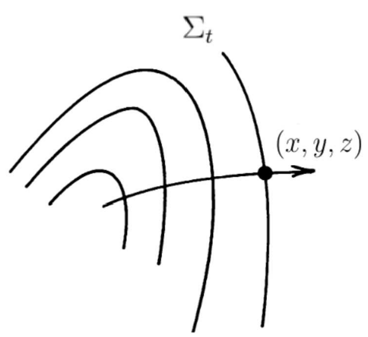
Then,

$$\vec{e}(s) = \big(\vec{X}(s), Z(s), \vec{P}(s)\big) = \big(\vec{X}(s), u(\vec{X}(s)), \vec{\nabla} u(\vec{X}(s))\big) \tag{3.5}$$

is an integral characteristic curve of the characteristic system (CS).

We will not go into details, but with this method - called *Method of Characteristics* - we would find that the projected characteristics of the eikonal equation are nothing but the *orthogonal trajectories* of the wavefronts Σ_t, which are even known as *light rays.*

3.2 Second order equations for orthogonal trajectories



Let us consider a set of wavefronts $\Sigma_t = \{(x, y, z) \in \mathbb{R}^3 : \psi(x, y, z) = ct\}$. An orthogonal trajectory of this surfaces at any point $(x, y, z) \in \Sigma_t$ is normal to the wavefront itself through this point. This means that - if we denote with $\alpha \in \mathbb{R}$ the evolution parameter of the curve $(x, y, z) = \big(x(\alpha), y(\alpha), z(\alpha)\big)$ - the complete manifold of orthogonal trajectories through the given set of wavefronts Σ_t must coincide with the solutions of the differential equations:

$$\frac{dx}{d\alpha} = \lambda \psi_x \qquad \frac{dy}{d\alpha} = \lambda \psi_y \qquad \frac{dz}{d\alpha} = \lambda \psi_z \tag{3.6}$$

where $\lambda = \lambda(x, y, z, \alpha)$ is an arbitrary factor which describes the parametric representation of the trajectory without influencing its geometrical form.

If we differentiate $\frac{1}{\lambda}\frac{dx}{d\alpha} = \psi_x$ with respect to α we obtain:

$$\frac{d}{d\alpha}\left(\frac{1}{\lambda}\frac{dx}{d\alpha}\right) = \psi_{xx}\,\frac{dx}{d\alpha} + \psi_{yx}\,\frac{dy}{d\alpha} + \psi_{zx}\,\frac{dz}{d\alpha} \tag{3.7}$$

$$= \lambda\left(\psi_{xx}\,\psi_x + \psi_{yx}\,\psi_y + \psi_{zx}\,\psi_z\right) \tag{3.8}$$

$$= \frac{\lambda}{2}\frac{\partial}{\partial x}\left(\psi_x^2 + \psi_y^2 + \psi_z^2\right) \tag{3.9}$$

and from (3.2) it follows that:

$$\frac{1}{\lambda}\frac{d}{d\alpha}\left(\frac{1}{\lambda}\frac{dx}{d\alpha}\right) = \frac{1}{2}\frac{\partial n^2}{\partial x} \tag{3.10}$$

If we make the same proceeding for $\frac{1}{\lambda}\frac{dy}{d\alpha} = \psi_y$ and $\frac{1}{\lambda}\frac{dz}{d\alpha} = \psi_z$ we also find the equations:

$$\frac{1}{\lambda}\frac{d}{d\alpha}\left(\frac{1}{\lambda}\frac{dy}{d\alpha}\right) = \frac{1}{2}\frac{\partial n^2}{\partial y} \tag{3.11}$$

$$\frac{1}{\lambda}\frac{d}{d\alpha}\left(\frac{1}{\lambda}\frac{dz}{d\alpha}\right) = \frac{1}{2}\frac{\partial n^2}{\partial z} \tag{3.12}$$

We have thus obtained a set of second order differential equations (3.10)-(3.12) which gives us a two-parameter manifold of solutions to determine the orthogonal trajectories of the wavefronts.

Invariance of solutions by reparametrization It is important to remark that the choice of the function λ does not influence the geometric form of the integral curves of (3.10) - (3.12), because any particular solution $\vec{X}(\alpha) = \big(x(\alpha), y(\alpha), z(\alpha)\big)$ can be reparameterized so that $\lambda \equiv 1$. Indeed, if we consider for instance the first component of $\vec{X}(\alpha)$, we have from (3.6) that:

$$\frac{d}{d\alpha}\,x(\alpha) = \lambda\big(\vec{X}(\alpha), \alpha\big)\;\psi_x\big(\vec{X}(\alpha)\big) \tag{3.13}$$

Hence, if we set $\alpha = f(\beta)$ it follows that:

$$\frac{d}{d\beta}\left[x\big(f(\beta)\big)\right] = \left(\frac{d}{d\alpha}\,x(\alpha)\right)\left(\frac{d}{d\beta}\,f(\beta)\right) \tag{3.14}$$

$$= \lambda\big(\vec{X}\big(f(\beta)\big), f(\beta)\big)\;\psi_x\big(\vec{X}\big(f(\beta)\big)\big)\left(\frac{d}{d\beta}\,f(\beta)\right) \tag{3.15}$$

and if $f(\beta)$ is a solution for the equation:

$$\frac{d}{d\beta} f(\beta) = \Big(\lambda\big(\vec{X}\big(f(\beta)\big), f(\beta)\big)\Big)^{-1} \tag{3.16}$$

we obtain that:

$$\frac{d}{d\beta}\left[x\big(f(\beta)\big)\right] = \psi_x\big(\vec{X}\big(f(\beta)\big)\big) \tag{3.17}$$

In the same way for the components $y(\alpha)$ and $z(\alpha)$, we also derive:

$$\frac{d}{d\beta}\left[y\big(f(\beta)\big)\right] = \psi_y\big(\vec{X}\big(f(\beta)\big)\big) \tag{3.18}$$

$$\frac{d}{d\beta}\left[z\big(f(\beta)\big)\right] = \psi_z\big(\vec{X}\big(f(\beta)\big)\big) \tag{3.19}$$

Hence, we have demonstrated that any integral curve of (3.10) - (3.12) can be transformed into a solution $\big(x(\beta), y(\beta), z(\beta)\big)$ for which $\lambda \equiv 1$ and so the set of equations become:

$$\frac{d^2x}{d\beta^2} = \frac{1}{2}\frac{\partial n^2}{\partial x} \qquad \frac{d^2y}{d\beta^2} = \frac{1}{2}\frac{\partial n^2}{\partial y} \qquad \frac{d^2z}{d\beta^2} = \frac{1}{2}\frac{\partial n^2}{\partial z}$$

Let us observe that - since we are doing a reparametrization of the curve - this change of parameter does not affect its geometric shape, and so $\big(x(\alpha), y(\alpha), z(\alpha)\big)$ and $\big(x(\beta), y(\beta), z(\beta)\big)$ are equivalent orthogonal trajectories of Σ_t.

3.3 Mechanic interpretation of light rays

We fix $\lambda \equiv 1$ and let $\tau \in \mathbb{R}$ be the evolution parameter of a generic orthogonal trajectory $\vec{\kappa}(\tau) = \big(x(\tau), y(\tau), z(\tau)\big)$. Then the first-order differential equations (3.6) become:

$$\frac{dx}{d\tau} = \psi_x \qquad \frac{dy}{d\tau} = \psi_y \qquad \frac{dz}{d\tau} = \psi_z \tag{3.20}$$

while the second order ones (3.10) - (3.12) become:

$$\frac{d^2x}{d\tau^2} = \frac{\partial}{\partial x}\left(\frac{1}{2}n^2\right) = F_x \tag{3.21}$$

$$\frac{d^2y}{d\tau^2} = \frac{\partial}{\partial y}\left(\frac{1}{2}n^2\right) = F_y \tag{3.22}$$

$$\frac{d^2z}{d\tau^2} = \frac{\partial}{\partial z}\left(\frac{1}{2}n^2\right) = F_z \tag{3.23}$$

that is $\ddot{\vec{\kappa}}(\tau) = \vec{F}\big(\vec{\kappa}(\tau)\big)$. If we define the scalar field U:

$$\begin{aligned} U\colon \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ (x,y,z) &\longmapsto U(x,y,z) := -\frac{n^2}{2} = -\frac{\varepsilon\mu}{2} \end{aligned}$$

it is clear that $\vec{F} = -\vec{\nabla}U$; hence we can interpret U as the *potential field* of the light rays. Moreover - in according to (3.20) - we can rewrite the eikonal equation as:

$$\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2 = n^2 \tag{3.24}$$

and by denoting with $\vec{v} = \vec{v}(\tau) = \left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right)$ the *velocity* of a generic point $(x, y, z) \in \Sigma_t$, from (3.24) it follows that any orthogonal trajectory $\vec{\kappa}(\tau)$ must satisfy the relation:

$$E = \frac{1}{2}\,\|\vec{v}\|^2 + U = \frac{1}{2}\,n^2\big(\vec{\kappa}(\tau)\big) + U\big(\vec{\kappa}(\tau)\big) = 0 \tag{3.25}$$

Therefore we can consider the light rays as the *paths of particles moving along the potential field U with a total energy equal to zero.*

3.4 Fermat's principle of geometrical optics

Now let us fix $\lambda = 1/n$ and let $s \in \mathbb{R}$ be the evolution parameter of a generic orthogonal trajectory $\vec{\chi}(s) = \big(x(s), y(s), z(s)\big) \in \mathbb{R}^3$. From (3.6) it follows that:

$$\frac{d\vec{\chi}}{ds} = \lambda\vec{\nabla}\psi = \frac{1}{n}\vec{\nabla}\psi \tag{3.26}$$

that is:

$$n\,\frac{dx}{ds} = \psi_x \qquad\qquad n\,\frac{dy}{ds} = \psi_y \qquad\qquad n\,\frac{dz}{ds} = \psi_z \tag{3.27}$$

If we take the derivative of (3.26) with respect to s, we have that:

$$\frac{d}{ds}\left(n\,\frac{d\vec{\chi}}{ds}\right) = \frac{d}{ds}\vec{\nabla}\psi = \left(\frac{d\psi_x}{ds}, \frac{d\psi_y}{ds}, \frac{d\psi_z}{ds}\right) \tag{3.28}$$

and considering only the first component of the vector, it follows that:

$$\frac{d\psi_x}{ds} = \psi_{xx}\,\dot{x} + \psi_{xy}\,\dot{y} + \psi_{xz}\,\dot{z} \tag{3.29}$$

$$= \psi_{xx}\,\frac{1}{n}\psi_x + \psi_{xy}\,\frac{1}{n}\psi_y + \psi_{xz}\,\frac{1}{n}\psi_z \tag{3.30}$$

$$= \frac{1}{n}\left[\frac{1}{2}\,\frac{\partial}{\partial x}\big(\psi_x^2 + \psi_y^2 + \psi_z^2\big)\right] \tag{3.31}$$

$$= \frac{1}{2n}\frac{\partial n^2}{\partial x} = \frac{\partial n}{\partial x} \tag{3.32}$$

Making the same proceeding for the other components of (3.28), we also obtain the relations:

$$\frac{d\psi_y}{ds} = \frac{\partial n}{\partial y} \qquad\qquad \frac{d\psi_z}{ds} = \frac{\partial n}{\partial z} \tag{3.33}$$

and from (3.27) we get the equations:

$$\frac{d}{ds}\left(n\,\frac{dx}{ds}\right)=\frac{\partial n}{\partial x}\qquad\qquad \frac{d}{ds}\left(n\,\frac{dy}{ds}\right)=\frac{\partial n}{\partial y}\qquad\qquad \frac{d}{ds}\left(n\,\frac{dz}{ds}\right)=\frac{\partial n}{\partial z}\tag{3.34}$$

We claim that the relations (3.34) represent the *Euler-Lagrange equations* associated to the variational problem which we are going to introduce; but before proving this, let us demonstrate that any regular curve admit an arc length parameterization.

Let $\vec{\chi}=\vec{\chi}(\sigma)\in\mathbb{R}^3$ be a regular three-dimensional curve referred to the parameter $\sigma\in[0,1]$; we can define the curve length as:

$$s(\sigma)=\int_0^{\sigma}\left\|\frac{d\vec{\chi}}{d\sigma'}\right\|\,d\sigma'\tag{3.35}$$

Since $\vec{\chi}(\sigma)=\big(x(\sigma),y(\sigma),z(\sigma)\big)$ is regular, it follows that:

$$\left\|\frac{d\vec{\chi}}{d\sigma}\right\|=\sqrt{\left(\frac{dx}{d\sigma}\right)^2+\left(\frac{dy}{d\sigma}\right)^2+\left(\frac{dz}{d\sigma}\right)^2}\neq 0\quad\forall\sigma\in[0,1]\tag{3.36}$$

and so:

$$\frac{d}{d\sigma}\,s(\sigma)=\left\|\frac{d\vec{\chi}}{d\sigma}\right\|>0\quad\forall\sigma\in[0,1]\tag{3.37}$$

Consequently $s(\sigma)$ is invertible and its inverse function is $\sigma=\sigma(s)$. Hence, if we denote with $\tilde{\chi}(s)=\vec{\chi}\big(\sigma(s)\big)\in\mathbb{R}^3$ the curve referred to the parameter s, we can observe that:

$$\frac{d\tilde{\chi}}{ds}=\frac{d\vec{\chi}}{d\sigma}\frac{d\sigma}{ds}=\frac{d\vec{\chi}}{d\sigma}\left(\frac{ds}{d\sigma}\right)^{-1}=\frac{d\vec{\chi}}{d\sigma}\frac{1}{\left\|\frac{d\vec{\chi}}{d\sigma}\right\|}\tag{3.38}$$

Therefore $\left\|\frac{d\tilde{\chi}}{ds}\right\|=1$ and s is an arc parameter for the curve $\vec{\chi}$.

Proposition 3.4.1. *Let us suppose that s is the arc parameter for the light ray $\vec{\chi}(s)=\big(x(s),y(s),z(s)\big)\in\mathbb{R}^3$ and let us suppose that $\vec{\chi}(s)$ is a C^1 curve with end-points $P_0,P_1\in\mathbb{R}^3$. Then, the relations (3.34) represent the Euler-Lagrange equations associated to the variational problem:*

$$\mathcal{F}\left[\vec{\chi}(\cdot)\right]=\int_{P_0}^{P_1} n\big(\vec{\chi}(s)\big)\,ds\tag{3.39}$$

Proof. Let us suppose that exist $s_0,s_1\in\mathbb{R}^3$ such there $\vec{\chi}(s_0)=P_0$ and $\vec{\chi}(s_1)=P_1$. Moreover, we consider an evolution parameter $\sigma\in\mathbb{R}$ such that $s=s(\sigma)$ and $\vec{\chi}=\vec{\chi}\big(s(\sigma)\big)$. Then exist $\sigma_0,\sigma_1\in\mathbb{R}$ such that $s(\sigma_0)=s_0$ and $s(\sigma_1)=s_1$ and if we suppose that $\frac{ds}{d\sigma}>0$, from the theorem for change of variable on integrals it follows that:

$$\mathcal{F}\left[\vec{\chi}(\cdot)\right]=\int_{P_0}^{P_1} n\big(\vec{\chi}(s)\big)\,ds=\int_{\vec{\chi}(s(\sigma_0))}^{\vec{\chi}(s(\sigma_1))} n\big(\vec{\chi}(s)\big)\,ds=\int_{\sigma_0}^{\sigma_1} n\big(\vec{\chi}\big(s(\sigma)\big)\big)\,\frac{ds}{d\sigma}\,d\sigma\tag{3.40}$$

Since s is an arc parameter, it is true that:

$$\frac{ds}{d\sigma} = \left\| \frac{d\vec{\chi}}{d\sigma} \right\| = \sqrt{\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2} \tag{3.41}$$

and so:

$$\mathcal{F}\left[\,\vec{\chi}(\cdot)\right] = \int_{\sigma_0}^{\sigma_1} n\big(\vec{\chi}\big(s(\sigma)\big)\big) \sqrt{\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2}\ d\sigma \tag{3.42}$$

Now, if we denote with $\vec{q}\,(\sigma) = \vec{\chi}\big(s(\sigma)\big)$, with $\vec{q}\,'(\sigma) = \frac{d\vec{\chi}}{d\sigma}$ and with $\mathcal{L}$ the integrating function of (3.42), we can rewrite the last relation as:

$$\mathcal{F}\left[\vec{q}(\cdot)\right] = \int_{\sigma_0}^{\sigma_1} \mathcal{L}\big(\,\vec{q}\,(\sigma), \vec{q}\,'(\sigma)\big)\ d\sigma \tag{3.43}$$

Let us show that, $\forall k \in \{x, y, z\}$, the following relation holds:

$$\frac{d}{d\sigma}\left(\frac{\partial \mathcal{L}}{\partial {q_k}'}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0 \tag{3.44}$$

We suppose $k = x$, then it follows that

$$\frac{d}{d\sigma}\left(\frac{\partial \mathcal{L}}{\partial {q_x}'}\right) = \frac{d}{d\sigma}\left[n\big(\vec{q}(\sigma)\big)\left(2\sqrt{\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2}\,\right)^{-1}\left(2\,\frac{dx}{d\sigma}\right)\right] \tag{3.45}$$

and

$$\frac{\partial \mathcal{L}}{\partial q_x} = \frac{\partial n}{\partial x}\sqrt{\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2} \tag{3.46}$$

Now - for the arbitrariness of the parameter σ and since we have demonstrated that any regular curve can be reparametrized to arc length - we can suppose that σ is an arc parameter, which means $s(\sigma) = s(s) = s$ and then:

$$\left\| \frac{d\vec{\chi}}{d\sigma} \right\| = \sqrt{\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{dy}{d\sigma}\right)^2 + \left(\frac{dz}{d\sigma}\right)^2} = 1 \tag{3.47}$$

Hence it follows that:

$$\frac{d}{d\sigma}\left(\frac{\partial \mathcal{L}}{\partial {q_x}'}\right) - \frac{\partial \mathcal{L}}{\partial q_x} = \frac{d}{d\sigma}\left(n\big(\vec{\chi}(s)\big)\,\frac{dx}{ds}\right) - \frac{\partial}{\partial x}\,n\big(\vec{\chi}(s)\big) = 0 \tag{3.48}$$

which coincides with the first equation of (3.34). In the same way, (3.44) holds even for $k = y$ and $k = z$. □

We have thus demonstrated that the equation (3.34) are nothing but the Euler-Lagrange equation associated to variational problem (3.39) and so - for the *Hamilton*

Variational Principle - we can say that any orthogonal trajectory $\vec{\chi}$, which is a solution of the set of equations (3.34), is a three-dimensional curve which stationarizes the functional:

$$\mathcal{F}\left[\,\vec{\chi}(\cdot)\right] = \int_{P_0}^{P_1} n\big(\vec{\chi}(s)\big)\; ds \tag{3.49}$$

where $\mathcal{F}$ represents the optical length of the light ray and the quantity:

$$ds = \left\| \frac{d\vec{\chi}}{d\sigma} \right\| d\sigma \tag{3.50}$$

represents the arc-length.

This result is known as *Fermat's principle of geometrical optics* and it means that the light ray between two points P_0 and P_1 is the curve for which the optical path assume an extreme value, and more precisely a minimum value.
If we interpret ds as the infinitesimal space variation of $\vec{\chi}(s) = \big(x, y, z\big)(s)$ over the unit of time and denote with $v = v(x, y, z) = \frac{ds}{dt}$ the velocity module of the point $\big(x, y, z\big)(s)$ in the medium - since the index of refraction n can be thought of as the ratio c/v - then the optical path of a light ray can be also defined as:

$$\mathcal{F}\left[\,\vec{\chi}(\cdot)\right] = \int n\big(\vec{\chi}(s)\big)\; ds = c\int \frac{ds}{v\big(\vec{\chi}(s)\big)} = c\int dt \tag{3.51}$$

Hence we have demonstrated that the optical path of a light ray is proportional to the time necessary to travel from P_0 and P_1 and - for the Fermat's principle - this means that *a light ray is a curve on which the optical path requires the minimal travel time.*

Chapter 4

Transport equations for discontinuities

In this chapter we shall derive certain differential relations which allow us to determine the discontinuities of an electromagnetic field along a given light ray in the case that the discontinuity is known at one point of the ray. For this discussion we will assume that the scalar fields $\varepsilon = \varepsilon(x, y, z)$ and $\mu = \mu(x, y, z)$ are continuous functions.

4.1 Differentiation along a light ray

Let us consider a set of wavefronts $\Sigma_t = \{(x, y, z) \in \mathbb{R}^3 : \psi(x, y, z) = ct\}$ and let $\vec{\kappa}(\tau) = \big(x(\tau), y(\tau), z(\tau)\big)$ be a given light ray of Σ_t (we suppose $\lambda = 1$). Moreover let $F : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable scalar field and we denote with:

$$F(\tau) = F\big(x(\tau), y(\tau), z(\tau)\big) \tag{4.1}$$

the scalar field F computed along the light ray $\vec{\kappa}$. Then we can differentiate $F(\tau)$ with respect to τ, obtaining thus the equation:

$$\frac{dF}{d\tau} = \frac{\partial F}{\partial x}\frac{dx}{d\tau} + \frac{\partial F}{\partial y}\frac{dy}{d\tau} + \frac{\partial F}{\partial z}\frac{dz}{d\tau} \tag{4.2}$$

which - for the relations (3.20) - becomes:

$$\frac{dF}{d\tau} = F_x\psi_x + F_y\psi_y + F_z\psi_z \tag{4.3}$$

where we denote with $F_\nu = \partial F/\partial\nu$, for $\nu \in \{x, y, z\}$. Let us observe that:

$$\frac{dF}{d\tau} = \frac{\partial F}{\partial \tau} \tag{4.4}$$

and so we can state that the differential operator is equivalent to:

$$\frac{\partial}{\partial \tau} = \psi_x\frac{\partial}{\partial x} + \psi_y\frac{\partial}{\partial y} + \psi_z\frac{\partial}{\partial z} \tag{4.5}$$

This last one can be interpreted as the differential operator which differentiate a function - with respect to τ - along a light ray $\vec{\kappa}(\tau) = \big(x(\tau), y(\tau), z(\tau)\big)$, as long as $\psi = \psi(x, y, z)$ is a solution of the eikonal equation:

$$\psi_x^2 + \psi_y^2 + \psi_z^2 = n^2 \tag{4.6}$$

With this hypothesis - if we suppose $F(x, y, z) = \psi(x, y, z)$ - it follows that:

$$\frac{\partial \psi}{\partial \tau} = \psi_x^2 + \psi_y^2 + \psi_z^2 = n^2 \tag{4.7}$$

and since the differential operator (4.5) can be also applied component-wise to a vector field, we can even compute:

$$\frac{\partial}{\partial \tau}\vec{\nabla}\psi = \left(\frac{\partial}{\partial \tau}\,\psi_x, \frac{\partial}{\partial \tau}\,\psi_y, \frac{\partial}{\partial \tau}\,\psi_z\right) \tag{4.8}$$

Indeed, if we consider only the first component of the vector, we have that:

$$\frac{\partial}{\partial \tau}\,\psi_x = \psi_x\,\psi_{xx} + \psi_y\,\psi_{xy} + \psi_z\,\psi_{xz} \tag{4.9}$$

$$= \frac{1}{2}\,\frac{\partial}{\partial x}\left(\psi_x^2 + \psi_y^2 + \psi_z^2\right) \tag{4.10}$$

$$= \frac{1}{2}\,\frac{\partial}{\partial x}\,n^2 \tag{4.11}$$

and making the same proceeding for the other components we even obtain:

$$\frac{\partial}{\partial \tau}\,\psi_y = \frac{1}{2}\frac{\partial}{\partial y}\,n^2 \tag{4.12}$$

$$\frac{\partial}{\partial \tau}\,\psi_z = \frac{1}{2}\frac{\partial}{\partial z}\,n^2 \tag{4.13}$$

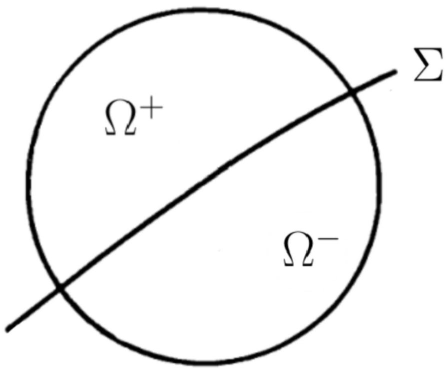
Therefore, we can conclude that:

$$\frac{\partial}{\partial \tau}\vec{\nabla}\psi = \frac{1}{2}\vec{\nabla}\,n^2 \tag{4.14}$$

4.2 Variable t as implicit function

Let us consider a hypersurface $\Sigma \subset \mathbb{R}^4$ defined by the equation:

$$\phi(x, y, z, t) = \psi(x, y, z) - ct = 0 \tag{4.15}$$



and let Ω^+, $\Omega^- \subset \mathbb{R}^4$ be two domains separated by Σ. Then let us consider an electromagnetic field and suppose that Σ is a discontinuous surface for it, while $\vec{E}$ and $\vec{H}$ are continuous functions with continuous derivatives in the individual domains Ω^+ and Ω^-; that is $\vec{E}$, $\vec{H} \in C^1(\Omega^+)$ and $\vec{E}$, $\vec{H} \in C^1(\Omega^-)$. Moreover we denote with:

- $\vec{E}_+$ and $\vec{H}_+$ the boundary values of the electromagnetic field in a neighbourhood of Σ belonging to Ω^+;
- $\vec{E}_-$ and $\vec{H}_-$ the boundary values of the electromagnetic field in a neighbourhood of Σ belonging to Ω^-;
- $[\vec{E}] = \vec{E}_+ - \vec{E}_-$ and $[\vec{H}] = \vec{H}_+ - \vec{H}_-$ the discontinuities of the electromagnetic field on Σ.

For any point $(x, y, z, t) \in \Sigma$ it is true that:

$$\frac{\partial}{\partial t}\,\phi(x, y, z, t) = -c \neq 0 \tag{4.16}$$

Hence - for the implicit function theorem - we can redefine every point of Σ denoting t with a function on the variables $(x, y, z) \in \mathbb{R}^3$ and Σ can be defined as:

$$\Sigma := \left\{(x, y, z, t) \in \mathbb{R}^4 \,|\, \phi(x, y, z, t) = 0\right\} = \left\{\left(x, y, z, \frac{1}{c}\,\psi(x, y, z)\right) \in \mathbb{R}^4 \,\Big|\, (x, y, z) \in \mathbb{R}^3\right\}$$

Moreover, on Σ also $\vec{E}$ and $\vec{H}$ can be defined as functions depending only on (x, y, z). So we can denote with:

$$\left.\begin{aligned}\vec{E}^*_+(x, y, z) &= \vec{E}_+\left(x, y, z, \frac{1}{c}\,\psi(x, y, z)\right)\\ \vec{H}^*_+(x, y, z) &= \vec{H}_+\left(x, y, z, \frac{1}{c}\,\psi(x, y, z)\right)\end{aligned}\right. \quad \text{on } \Sigma$$

and with:

$$\left.\begin{aligned}\vec{E}^*_-(x, y, z) &= \vec{E}_-\left(x, y, z, \frac{1}{c}\,\psi(x, y, z)\right)\\ \vec{H}^*_-(x, y, z) &= \vec{H}_-\left(x, y, z, \frac{1}{c}\,\psi(x, y, z)\right)\end{aligned}\right. \quad \text{on } \Sigma$$

Now, let us consider $\vec{E}^*_+$. Its partial derivatives are:

$$\frac{\partial \vec{E}^*_+}{\partial x} = \frac{\partial \vec{E}_+}{\partial x} + \frac{1}{c}\,\frac{\partial \vec{E}_+}{\partial t}\,\psi_x \tag{4.17}$$

$$\frac{\partial \vec{E}^*_+}{\partial y} = \frac{\partial \vec{E}_+}{\partial y} + \frac{1}{c}\,\frac{\partial \vec{E}_+}{\partial t}\,\psi_y \tag{4.18}$$

$$\frac{\partial \vec{E}^*_+}{\partial z} = \frac{\partial \vec{E}_+}{\partial z} + \frac{1}{c}\,\frac{\partial \vec{E}_+}{\partial t}\,\psi_z \tag{4.19}$$

and by a straightforward computation it can be demonstrated that:

$$\vec{\nabla} \times \vec{E}^*_+ = \vec{\nabla} \times \vec{E}_+ + \frac{1}{c}\,\vec{\nabla}\psi \times \left(\frac{\partial \vec{E}_+}{\partial t}\right) \tag{4.20}$$

In the same way for $\vec{H}^*_+$ we also derive:

$$\vec{\nabla} \times \vec{H}^*_+ = \vec{\nabla} \times \vec{H}_+ + \frac{1}{c}\,\vec{\nabla}\psi \times \left(\frac{\partial \vec{H}_+}{\partial t}\right) \tag{4.21}$$

However, from Maxwell's equations (1.3) - (1.4) we know that:

$$\vec{\nabla} \times \vec{E} = -\frac{\mu}{c}\frac{\partial \vec{H}}{\partial t} \qquad\qquad \vec{\nabla} \times \vec{H} = \frac{\varepsilon}{c}\frac{\partial \vec{E}}{\partial t} \tag{4.22}$$

and so - if we replace the ones in (4.20) and (4.21) - we obtain the relations:

$$c\vec{\nabla} \times \vec{E}^*_+ = \vec{\nabla}\psi \times \left(\frac{\partial \vec{E}_+}{\partial t}\right) - \mu\,\frac{\partial \vec{H}_+}{\partial t} \tag{4.23}$$

$$c\vec{\nabla} \times \vec{H}^*_+ = \vec{\nabla}\psi \times \left(\frac{\partial \vec{H}_+}{\partial t}\right) + \varepsilon\,\frac{\partial \vec{E}_+}{\partial t} \tag{4.24}$$

which represent a non-homogeneous system of six linear equations for the six components of the vectors $\partial\vec{E}/\partial t$ and $\partial\vec{H}/\partial t$.
The matrix of this system is the same one that is associated to the set of equations:

$$\vec{\nabla}\psi \times [\vec{E}] - \mu[\vec{H}] = 0 \tag{4.25}$$

$$\vec{\nabla}\psi \times [\vec{H}] + \varepsilon[\vec{E}] = 0 \tag{4.26}$$

and as we have seen in Chapter 2, Subsection 2.3.2, we already know that the determinat of this matrix is zero on Σ. Therefore the equations (4.23) - (4.24) admit non-trivial solutions, but only if the right sides of the equations (4.23) - (4.24) satisfy certain conditions.

4.3 Transport of discontinuities along a light ray

Let us determine these conditions. We form the vector product of $\vec{\nabla}\psi$ with (4.24), which become:

$$c\vec{\nabla}\psi \times \left(\vec{\nabla} \times \vec{H}^*_+\right) = \vec{\nabla}\psi \times \left(\vec{\nabla}\psi \times \left(\frac{\partial \vec{H}_+}{\partial t}\right)\right) + \varepsilon\vec{\nabla}\psi \times \left(\frac{\partial \vec{E}_+}{\partial t}\right) \tag{4.27}$$

From the vector identity $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$ it follows that:

$$c\vec{\nabla}\psi \times \left(\vec{\nabla} \times \vec{H}^*_+\right) = \left(\frac{\partial \vec{H}_+}{\partial t}\cdot\vec{\nabla}\psi\right)\vec{\nabla}\psi - \left\|\vec{\nabla}\psi\right\|^2\frac{\partial \vec{H}_+}{\partial t} + \varepsilon\vec{\nabla}\psi \times \left(\frac{\partial \vec{E}_+}{\partial t}\right) \tag{4.28}$$

$$= \left(\frac{\partial \vec{H}_+}{\partial t}\cdot\vec{\nabla}\psi\right)\vec{\nabla}\psi + \varepsilon\left(\vec{\nabla}\psi \times \left(\frac{\partial \vec{E}_+}{\partial t}\right) - \mu\,\frac{\partial \vec{H}_+}{\partial t}\right) \tag{4.29}$$

where - in the last equality - we used the relation $\|\vec{\nabla}\psi\|^2 = n^2 = \varepsilon\mu$. Then, by using the equation (4.23) we can conclude that

$$\vec{\nabla}\psi \times \left(\vec{\nabla} \times \vec{H}^*_+\right) - \varepsilon\vec{\nabla} \times \vec{E}^*_+ = \frac{1}{c}\left(\frac{\partial \vec{H}_+}{\partial t} \cdot \vec{\nabla}\psi\right)\vec{\nabla}\psi \tag{4.30}$$

This last equation means that the vector $\vec{\nabla}\psi \times \left(\vec{\nabla} \times \vec{H}^*_+\right) - \varepsilon\vec{\nabla} \times \vec{E}^*_+$ has the same direction of $\vec{\nabla}\psi$; namely it is normal to the wavefront Σ.

The same considerations can be done for $\vec{E}^*_-$ and $\vec{H}^*_-$ to conclude that also

$$\vec{\nabla}\psi \times \left(\vec{\nabla} \times \vec{H}^*_-\right) - \varepsilon\vec{\nabla} \times \vec{E}^*_- \tag{4.31}$$

is normal to Σ; and so - if we consider the discontinuities of $\vec{E}$ and $\vec{H}$ on the wavefront - even the vector

$$\vec{\nabla}\psi \times \left(\vec{\nabla} \times [\vec{H}^*]\right) - \varepsilon\vec{\nabla} \times [\vec{E}^*] \tag{4.32}$$

is normal to Σ.

This statement can be formulated in the equation:

$$\varepsilon\vec{\nabla} \times [\vec{E}^*] - \vec{\nabla}\psi \times \left(\vec{\nabla} \times [\vec{H}^*]\right) = \mathcal{R}\,\vec{\nabla}\psi \tag{4.33}$$

where $\mathcal{R} = \mathcal{R}(x, y, z)$ is a scalar field which can be determined explicitly by forming the scalar product of $\vec{\nabla}\psi$ with the equation (4.33). It follows that:

$$\mathcal{R} = \frac{1}{\mu}\left(\vec{\nabla}\psi \cdot \left(\vec{\nabla} \times [\vec{E}^*]\right)\right) \tag{4.34}$$

Moreover - if we introduce from (4.26) the relation $[\vec{E}^*] = -\frac{1}{\varepsilon}\,\vec{\nabla}\psi \times [\vec{H}^*]$ - then (4.33) become:

$$\vec{\nabla} \times \left(\frac{1}{\varepsilon}\,\vec{\nabla}\psi \times [\vec{H}^*]\right) + \frac{1}{\varepsilon}\,\vec{\nabla}\psi \times \left(\vec{\nabla} \times [\vec{H}^*]\right) = -\frac{\mathcal{R}}{\varepsilon}\,\vec{\nabla}\psi \tag{4.35}$$

which is a differential equation of first-order in the discontinuity $[\vec{H}^*]$.
Now, let us consider the vector identity:

$$\vec{\nabla} \times f\vec{A} = f\vec{\nabla} \times \vec{A} + \vec{\nabla}f \times \vec{A} \tag{4.36}$$

where f and $\vec{A}$ are respectly a scalar field and a vector field. Then:

$$\vec{\nabla} \times \left(\frac{1}{\varepsilon}\vec{\nabla}\psi\right) = \frac{1}{\varepsilon}\vec{\nabla} \times \vec{\nabla}\psi + \vec{\nabla}\frac{1}{\varepsilon} \times \vec{\nabla}\psi \tag{4.37}$$

$$= \vec{\nabla}\frac{1}{\varepsilon} \times \vec{\nabla}\psi \tag{4.38}$$

and since $[\vec{H}^*]$ and $\vec{\nabla}\psi$ are perpendicular vectors, we obtain the relation:

$$[\vec{H}^*] \times \left(\vec{\nabla} \times \left(\frac{1}{\varepsilon}\vec{\nabla}\psi\right)\right) = [\vec{H}^*] \times \left(\vec{\nabla}\frac{1}{\varepsilon} \times \vec{\nabla}\psi\right) \tag{4.39}$$

$$= \left([\vec{H}^*] \cdot \vec{\nabla}\psi\right)\vec{\nabla}\frac{1}{\varepsilon} - \left([\vec{H}^*] \cdot \vec{\nabla}\frac{1}{\varepsilon}\right)\vec{\nabla}\psi \tag{4.40}$$

$$= -\left([\vec{H}^*] \cdot \vec{\nabla}\frac{1}{\varepsilon}\right)\vec{\nabla}\psi \tag{4.41}$$

which states that the vector $[\vec{H}^*] \times \left(\vec{\nabla} \times \left(\frac{1}{\varepsilon}\vec{\nabla}\psi\right)\right)$ has the same direction of $\vec{\nabla}\psi$. Hence we can rewrite (4.35) as:

$$\vec{\nabla} \times \left(\frac{1}{\varepsilon}\vec{\nabla}\psi \times [\vec{H}^*]\right) + \frac{1}{\varepsilon}\vec{\nabla}\psi \times \left(\vec{\nabla} \times [\vec{H}^*]\right) + [\vec{H}^*] \times \left(\vec{\nabla} \times \left(\frac{1}{\varepsilon}\vec{\nabla}\psi\right)\right) = \mathcal{R}'\vec{\nabla}\psi \tag{4.42}$$

where $\mathcal{R}' = \mathcal{R}'(x, y, z)$ is a certain scalar function. The left side of the equation can be finally transformed with the aid of the vector identity:

$$\begin{gathered}\vec{\nabla} \times \left(\vec{A} \times \vec{B}\right) + \vec{A} \times \left(\vec{\nabla} \times \vec{B}\right) + \vec{B} \times \left(\vec{\nabla} \times \vec{A}\right) = \\ = -\,2\,\frac{\partial \vec{B}}{\partial \alpha} - (\vec{\nabla} \cdot \vec{A})\,\vec{B} + (\vec{\nabla} \cdot \vec{B})\,\vec{A} + \vec{\nabla}(\vec{A} \cdot \vec{B})\end{gathered} \tag{4.43}$$

Indeed, if we denote with:

$$\vec{A} = \frac{1}{\varepsilon}\,\vec{\nabla}\psi \qquad\qquad \vec{B} = [\vec{H}^*]$$

and with:

$$\frac{\partial}{\partial \alpha} = \frac{1}{\varepsilon}\left(\psi_x\,\frac{\partial}{\partial x} + \psi_y\,\frac{\partial}{\partial y} + \psi_z\,\frac{\partial}{\partial z}\right) = \frac{1}{\varepsilon}\frac{\partial}{\partial \tau}$$

since $\vec{\nabla}\psi \cdot [\vec{H}^*] = 0$, then the equation (4.42) becomes:

$$-\frac{2}{\varepsilon}\,\frac{\partial[\vec{H}^*]}{\partial \tau} - \vec{\nabla} \cdot \left(\frac{1}{\varepsilon}\vec{\nabla}\psi\right)[\vec{H}^*] = -\mathcal{R}^*\vec{\nabla}\psi \tag{4.44}$$

where $\mathcal{R}^* = \mathcal{R}^*(x, y, z)$ is a new scalar factor which can be computed by taking the scalar product of (4.44) with $\vec{\nabla}\psi$. In this way we have that:

$$n^2\mathcal{R}^* = \frac{2}{\varepsilon}\frac{\partial[\vec{H}^*]}{\partial \tau} \cdot \vec{\nabla}\psi = -\frac{2}{\varepsilon}\,[\vec{H}^*] \cdot \frac{\partial}{\partial \tau}\vec{\nabla}\psi = -\frac{1}{\varepsilon}[\vec{H}^*] \cdot \vec{\nabla}n^2$$

where in the last equality we used the relation (4.14). So, since $\vec{\nabla}n^2 = 2n\,\vec{\nabla}n$, we can conclude that:

$$\mathcal{R}^* = -\frac{2}{\varepsilon n}\,[\vec{H}^*] \cdot \vec{\nabla}n \tag{4.45}$$

and the equation (4.35) becomes:

$$\frac{\partial[\vec{H}^*]}{\partial \tau} + \frac{\varepsilon}{2}\,\vec{\nabla} \cdot \left(\frac{1}{\varepsilon}\vec{\nabla}\psi\right)[\vec{H}^*] + \frac{[\vec{H}^*] \cdot \vec{\nabla}n}{n}\,\vec{\nabla}\psi = 0 \tag{4.46}$$

At this point we introduce the notation:

$$\Delta_\varepsilon \psi := \varepsilon \, \vec{\nabla} \cdot \left(\frac{1}{\varepsilon} \vec{\nabla} \psi \right) = \varepsilon \left[\left(\frac{\psi_x}{\varepsilon} \right)_x + \left(\frac{\psi_y}{\varepsilon} \right)_y + \left(\frac{\psi_z}{\varepsilon} \right)_z \right] \tag{4.47}$$

and finally we get the equation:

$$\frac{\partial [\vec{H}^*]}{\partial \tau} + \frac{1}{2} \, \Delta_\varepsilon \psi \, [\vec{H}^*] + \frac{[\vec{H}^*] \cdot \vec{\nabla} n}{n} \, \vec{\nabla} \psi = 0 \tag{4.48}$$

A similar relation can be found for the discontinuity $[\vec{E}^*]$ by replacing ε and $[\vec{H}^*]$ by μ and $-[\vec{E}^*]$ in (4.48).

Therefore we have found that - given a discontinuity surface perpendicular to a light ray - the discontinuities $[\vec{E}^*]$ and $[\vec{H}^*]$ have to satisfy the differential equations:

$$\frac{\partial [\vec{E}^*]}{\partial \tau} + \frac{1}{2} \, \Delta_\mu \psi \, [\vec{E}^*] + \frac{\vec{\nabla} n \cdot [\vec{E}^*]}{n} \, \vec{\nabla} \psi = 0 \tag{4.49}$$

$$\frac{\partial [\vec{H}^*]}{\partial \tau} + \frac{1}{2} \, \Delta_\varepsilon \psi \, [\vec{H}^*] + \frac{\vec{\nabla} n \cdot [\vec{H}^*]}{n} \, \vec{\nabla} \psi = 0 \tag{4.50}$$

4.4 Exponential representation of discontinuities

On a given light ray, the equations (4.49) - (4.50) represent a system of ordinary differential equations. Indeed we have demonstrated that the differential operator $\partial/\partial\tau$ differentiates a function in the direction of an orthogonal trajectory.
Now let us introduce the functions:

$$\zeta_\mu(\tau, \psi) := \frac{1}{2} \int_0^\tau \Delta_\mu \psi \, d\tau' \qquad\qquad \zeta_\varepsilon(\tau, \psi) := \frac{1}{2} \int_0^\tau \Delta_\varepsilon \psi \, d\tau' \tag{4.51}$$

and we define the vectors:

$$\vec{P} = \vec{P}(\tau) = [\vec{E}^*] \, e^{\zeta_\mu(\tau,\psi)} \qquad\qquad \vec{Q} = \vec{Q}(\tau) = [\vec{H}^*] \, e^{\zeta_\varepsilon(\tau,\psi)} \tag{4.52}$$

Let us consider just the vector $\vec{P}$ and take the derivative with rescpect to τ; we get the relation:

$$\frac{d\vec{P}}{d\tau} = \frac{d}{d\tau} \left([\vec{E}^*] \, e^{\zeta_\mu(\tau,\psi)} \right) = \frac{d[\vec{E}^*]}{d\tau} \, e^{\zeta_\mu(\tau,\psi)} + [\vec{E}^*] \, \frac{d}{d\tau} \, e^{\zeta_\mu(\tau,\psi)} \tag{4.53}$$

From the chain rule and from that $\big(\dot{x}(\tau), \dot{y}(\tau), \dot{z}(\tau)\big) = \vec{\nabla}\psi$, the first addend becomes:

$$\frac{d[\vec{E}^*]}{d\tau} \, e^{\zeta_\mu(\tau,\psi)} = \left(\frac{\partial [\vec{E}^*]}{\partial x} \frac{dx}{d\tau} + \frac{\partial [\vec{E}^*]}{\partial y} \frac{dy}{d\tau} + \frac{\partial [\vec{E}^*]}{\partial z} \frac{dz}{d\tau} \right) e^{\zeta_\mu(\tau,\psi)} \tag{4.54}$$

$$= \left(\frac{\partial}{\partial x} \, \psi_x + \frac{\partial}{\partial y} \, \psi_y + \frac{\partial}{\partial z} \, \psi_z \right) [\vec{E}^*] \, e^{\zeta_\mu(\tau,\psi)} \tag{4.55}$$

$$= \frac{\partial [\vec{E}^*]}{\partial \tau} \, e^{\zeta_\mu(\tau,\psi)} \tag{4.56}$$

while the second one - for the derivation rule of integral functions - becomes:

$$[\vec{E}^*]\,\frac{d}{d\tau}\,e^{\zeta_\mu(\tau,\psi)} = [\vec{E}^*]\,e^{\zeta_\mu(\tau,\psi)}\left(\frac{1}{2}\,\Delta_\mu\psi\big|_{\tau'=\tau}\right) \tag{4.57}$$

Hence we have demonstrated that:

$$\frac{d\vec{P}}{d\tau} = \frac{\partial[\vec{E}^*]}{\partial\tau}\,e^{\zeta_\mu(\tau,\psi)} + \frac{1}{2}\,\Delta_\mu\psi\,[\vec{E}^*]\,e^{\zeta_\mu(\tau,\psi)} \tag{4.58}$$

However, if we multiply the equation (4.49) for $e^{\zeta_\mu(\tau,\psi)}$, it follows that:

$$\frac{\partial[\vec{E}^*]}{\partial\tau}\,e^{\zeta_\mu(\tau,\psi)} + \frac{1}{2}\,\Delta_\mu\psi\,[\vec{E}^*]\,e^{\zeta_\mu(\tau,\psi)} + \frac{\vec{\nabla}n\cdot[\vec{E}^*]}{n}\vec{\nabla}\psi\,e^{\zeta_\mu(\tau,\psi)} = 0 \tag{4.59}$$

and so, introducing $\vec{P}$ and replacing (4.58), we get the equation:

$$\frac{d\vec{P}}{d\tau} + \frac{\vec{\nabla}n\cdot\vec{P}}{n}\,\vec{\nabla}\psi = 0 \tag{4.60}$$

With a similar proceeding for $\vec{Q}$, we obtain from (4.50) the equation:

$$\frac{d\vec{Q}}{d\tau} + \frac{\vec{\nabla}n\cdot\vec{Q}}{n}\,\vec{\nabla}\psi = 0 \tag{4.61}$$

Now, let us observe that $\vec{P}$ and $\vec{Q}$ have the same directions respectively as $[\vec{E}^*]$ and $[\vec{H}^*]$, but different lengths. This means that - since $[\vec{E}^*]$ and $[\vec{H}^*]$ are perpendicular to $\vec{\nabla}\psi$ - also $\vec{P}$ and $\vec{Q}$ are perpendicular to $\vec{\nabla}\psi$; and from $[\vec{E}^*]\cdot[\vec{H}^*] = 0$ it follows that $\vec{P}\cdot\vec{Q} = 0$ too. Therefore, from (4.60) - (4.61) it follows that:

$$\frac{d\vec{P}}{d\tau} = -\frac{\vec{\nabla}n\cdot\vec{P}}{n}\,\vec{\nabla}\psi \quad \perp \quad \vec{P} \quad \Longrightarrow \quad \vec{P}\cdot\frac{d\vec{P}}{d\tau} = 0 \tag{4.62}$$

$$\frac{d\vec{Q}}{d\tau} = -\frac{\vec{\nabla}n\cdot\vec{Q}}{n}\,\vec{\nabla}\psi \quad \perp \quad \vec{Q} \quad \Longrightarrow \quad \vec{Q}\cdot\frac{d\vec{Q}}{d\tau} = 0 \tag{4.63}$$

This means that the lengths of the vectors $\vec{P}$ and $\vec{Q}$ do not change on a given light ray and so - without loss of generality - we can assume $\|\vec{P}\| = \|\vec{Q}\| = 1$ and interpret them as unit vectors which determine the directions of the vectors $[\vec{E}^*]$ and $[\vec{H}^*]$.

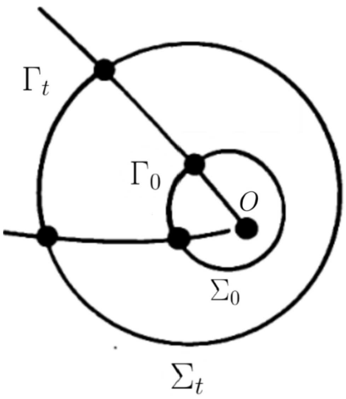
Therefore - if $\vec{P}$ and $\vec{Q}$ have been found as solutions of the equations (4.60) - (4.61) - then we can compute $[\vec{E}^*]$ and $[\vec{H}^*]$ as:

$$[\vec{E}^*] = \big\|\,[\vec{E}^*_0]\,\big\|\,\vec{P}e^{-\zeta_\mu(\tau,\psi)} \tag{4.64}$$

$$[\vec{H}^*] = \big\|\,[\vec{H}^*_0]\,\big\|\,\vec{Q}e^{-\zeta_\varepsilon(\tau,\psi)} \tag{4.65}$$

where $\vec{E}^*_0 = \vec{E}^*\big|_{\tau=0}$ and $\vec{H}^*_0 = \vec{H}^*\big|_{\tau=0}$.

Let us observe that from (4.64) - (4.65) it is clear that $[\vec{E}^*]$ and $[\vec{H}^*]$ are zero on the whole light ray if they are zero on one particular point $\tau = 0$ of the trajectory. This means that the light rays determine the region of the space where directed light signals can be seen.



Indeed - if we suppose that from the origin O a light signal is released at the time $t_0 = 0$ and we assume that the discontinuities $[\vec{E}^*]$ and $[\vec{H}^*]$ which represent the signal are different from zero only on a section Γ_0 of the wavefront $\Sigma_0 = \{(x, y, z, \frac{1}{c}\,\psi(x, y, z)) \in \mathbb{R}^4 \,|\, \psi(x, y, z) = 0\}$ - from (4.64) - (4.65) it follows that only on the corresponding section $\Gamma_t \subset \Sigma_t$ the discontinuities $[\vec{E}^*]$, $[\vec{H}^*]$ will be observed at a time $t > 0$. This section is determined by all the light rays which pass through Γ_0.

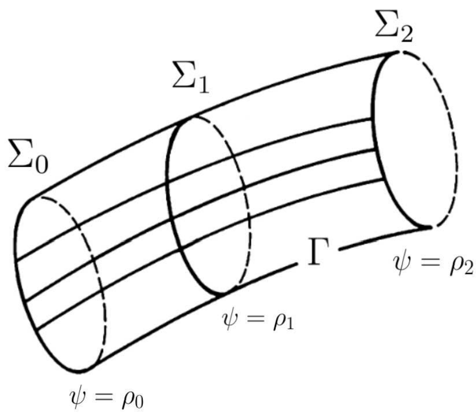
Meaning of the exponential factor At first we observe that:

$$\Delta_\varepsilon\psi = \Delta\psi - \frac{1}{\varepsilon}\left(\varepsilon_x\psi_x + \varepsilon_y\psi_y + \varepsilon_z\psi_z\right) = \Delta\psi - \frac{\partial}{\partial\tau}\left(\log\varepsilon\right) \tag{4.66}$$

$$\Delta_\mu\psi = \Delta\psi - \frac{1}{\mu}\left(\mu_x\psi_x + \mu_y\psi_y + \mu_z\psi_z\right) = \Delta\psi - \frac{\partial}{\partial\tau}\left(\log\mu\right) \tag{4.67}$$

Then let us consider a "tube" of light rays which we identify with a domain $\Omega \subset \mathbb{R}^3$ enclosed by a surface $\partial\Omega := \Sigma_1 \cup \Sigma_2 \cup \Gamma$ where:

- $\Sigma_1 = \{(x, y, z) \in \mathbb{R}^3 \,|\, \psi(x, y, z) = \rho_1\}$ with $\rho_1 \in \mathbb{R}$;
- $\Sigma_2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, \psi(x, y, z) = \rho_2\}$ with $\rho_2 \in \mathbb{R}$;
- Γ is the cylindrical wall formed by the light rays which pass through both $\partial\Sigma_1$ and $\partial\Sigma_2$.



Hence for the Gauss' theorem we have:

$$\int_\Omega \Delta\psi\; dx\,dy\,dz = \int_\Omega \vec{\nabla}\cdot\vec{\nabla}\psi\; dx\,dy\,dz = \int_{\partial\Omega} \vec{\nabla}\psi\cdot\vec{N}\; d\sigma^2$$

where $\vec{N} = \vec{N}(x, y, z)$ is the outward pointing unit normal to $\partial\Omega$. If we denote with $\vec{N}_i = \vec{N}\big|_{\Sigma_i}$ for $i = 1, 2$ and with $\vec{N}_3 = \vec{N}\big|_\Gamma$, it follows that:

$$\int_\Omega \Delta\psi\; dx\,dy\,dz = \int_{\Sigma_1} \vec{\nabla}\psi\cdot\vec{N}_1\; d\sigma^2 + \int_{\Sigma_2} \vec{\nabla}\psi\cdot\vec{N}_2\; d\sigma^2 + \int_\Gamma \vec{\nabla}\psi\cdot\vec{N}_3\; d\sigma^2 \tag{4.68}$$

$$= -\int_{\Sigma_1} \|\vec{\nabla}\psi\|\|\vec{N}_1\|\; d\sigma^2 + \int_{\Sigma_2} \|\vec{\nabla}\psi\|\|\vec{N}_2\|\; d\sigma^2 \tag{4.69}$$

because $\vec{N}_1$ and $\vec{N}_2$ have the same direction of $\vec{\nabla}\psi$ but only $\vec{N}_2$ has the same orientation of $\vec{\nabla}\psi$; while $\vec{N}_3$ is perpendicular to $\vec{\nabla}\psi$, hence $\vec{\nabla}\psi\cdot\vec{N}_3 = 0$.

Now let us consider a linear transformation $\mathcal{K} = \mathcal{K}(\tau)$ which measures the expansion of an infinitesimal surface element of an arbitrarily chosen wavefront Σ_0 as τ varies; and we suppose that:

$$d\sigma^2 \Big|_{\Sigma_1} = \mathcal{K}_1 \, d\sigma^2 \Big|_{\Sigma_0} \qquad\qquad d\sigma^2 \Big|_{\Sigma_2} = \mathcal{K}_2 \, d\sigma^2 \Big|_{\Sigma_0}$$

with $\mathcal{K}_1, \mathcal{K}_2 \in \mathbb{R}$. If we denote $\nu_i = \|\vec{\nabla}\psi\| \|\vec{N}_i\|$ for $i = 1, 2$, then it follows that:

$$\int_\Omega \Delta\psi \; dx\, dy\, dz = \int_{\Sigma_2} \nu_2 \; d\sigma^2 - \int_{\Sigma_1} \nu_1 \; d\sigma^2 \tag{4.70}$$

$$= \int_{\Sigma_0} \nu_2 \mathcal{K}_2 - \nu_1 \mathcal{K}_1 \; d\sigma^2 \tag{4.71}$$

$$= \int_{\Sigma_0} \int_{\tau_1}^{\tau_2} \frac{d(\nu\mathcal{K})}{d\tau} \; d\tau \, d\sigma^2 \tag{4.72}$$

Hence we can express the volume element as follows:

$$dx\, dy\, dz = \nu\mathcal{K} \; d\tau \, d\sigma^2 \tag{4.73}$$

to obtain thus the relation:

$$\int_\Omega \Delta\psi \; dx\, dy\, dz = \int_\Omega \frac{1}{\nu\mathcal{K}} \frac{\partial(\nu\mathcal{K})}{\partial\tau} \; dx\, dy\, dz \tag{4.74}$$

Hence - for the arbitrary of Ω - we find that:

$$\Delta\psi = \frac{1}{\nu\mathcal{K}} \frac{\partial(\nu\mathcal{K})}{\partial\tau} = \frac{\partial}{\partial\tau} \log\left(\nu\mathcal{K}\right) \tag{4.75}$$

and so we can conclude that:

$$\Delta_\varepsilon\psi = \frac{\partial}{\partial\tau} \log\left(\frac{\nu\mathcal{K}}{\varepsilon}\right) \tag{4.76}$$

$$\Delta_\mu\psi = \frac{\partial}{\partial\tau} \log\left(\frac{\nu\mathcal{K}}{\mu}\right) \tag{4.77}$$

Therefore we can define ζ_ε and ζ_μ as follow:

$$\zeta_\varepsilon(\tau, \psi) = \frac{1}{2} \int_0^\tau \Delta_\varepsilon\psi \; d\tau' = \frac{1}{2} \log\left(\frac{\nu\mathcal{K}/\varepsilon}{\nu_0\mathcal{K}_0/\varepsilon_0}\right) \tag{4.78}$$

$$\zeta_\mu(\tau, \psi) = \frac{1}{2} \int_0^\tau \Delta_\mu\psi \; d\tau' = \frac{1}{2} \log\left(\frac{\nu\mathcal{K}/\mu}{\nu_0\mathcal{K}_0/\mu_0}\right) \tag{4.79}$$

and consequently we find that:

$$e^{-\zeta_\varepsilon(\tau,\psi)} = \sqrt{\frac{\nu_0\mathcal{K}_0\varepsilon}{\nu\mathcal{K}\varepsilon_0}} \tag{4.80}$$

$$e^{-\zeta_\mu(\tau,\psi)} = \sqrt{\frac{\nu_0\mathcal{K}_0\mu}{\nu\mathcal{K}\mu_0}} \tag{4.81}$$

and by replacing these last relations in (4.64) - (4.65), we obtain the equations:

$$\frac{\mathcal{K}}{\nu}\varepsilon\big\|[\vec{E}^*]\big\|^2 = \frac{\mathcal{K}_0}{\nu_0}\varepsilon_0\big\|[\vec{E}_0^*]\big\|^2 \tag{4.82}$$

$$\frac{\mathcal{K}}{\nu}\mu\big\|[\vec{H}^*]\big\|^2 = \frac{\mathcal{K}_0}{\nu_0}\mu_0\big\|[\vec{H}_0^*]\big\|^2 \tag{4.83}$$

which state that the quantities $\frac{\mathcal{K}}{\nu}\varepsilon\|[\vec{E}^*]\|^2$ and $\frac{\mathcal{K}}{\nu}\mu\|[\vec{H}^*]\|^2$ are constant along a light ray.

Hence we have demonstrated that we are able to determine the modules of the discontinuities $[\vec{E}^*]$ and $[\vec{H}^*]$ along a given light ray of set of wavefronts Σ_t without the necessity to integrate $\Delta_\varepsilon\psi$ and $\Delta_\mu\psi$, but simply by calculating the ratio $\mathcal{K}/\mathcal{K}_0$ of the corresponding surface elements of the wavefronts.

Bibliography

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