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Finite symmetry groups and orbifolds of two-dimensional toroidal conformal field theories

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Introduction

The existence of more than the four observed space-time dimensions is an old idea, that has nevertheless crawled through the decades and reached the present times: proving to be a fertile intuition and bringing us a deep insight into the very structure of Nature, it is undoubtedly a recurrent feature in Theoretical Physics. Just to cite a relevant example, it is well known that String Theories require a precise number of space-time dimensions, for a reason of consistency. To make contact with reality, these extra dimensions are hence required to be compactified on some manifold. Among the many possible compactifications, the simplest ones are the compactifications on tori.

Despite their "simplicity" , toroidal compactifications of String and Conformal Field Theory are of great interest: they are both toy models and building blocks for more interesting theories. For example, toroidal compactification is the conceptual framework where to build heterotic string theory. Moreover, compactifying different String and Superstring Theories we may realize that some of them are identified under dualities. Besides being useful tools for various applications, these dualities are the hint that the different consistent Superstrings Theories are actually limits of a unique theory.

Other kind of dualities play a large role in our Thesis: different toroidal 2-dimensional Conformal Field Theories are indeed linked by an intricate web of dualities. In special points of the moduli space these dualities map the model into itself (mapping non-trivially fields into other fields): in this case we call them self-dualities, and they are symmetries of the model. The symmetries born from self-duality are of remarkable interest: if we consider their action on the states of the Conformal Field Theory, we discover that the associated symmetry groups act projectively on them. This should not sound too surprising, as a projective realization of a symmetry is not a rare feature in Physics. However, this topic is sometimes overlooked in literature: on one hand, there exist very abstract and generic descriptions of this theme, of little use for practical applications. On the other hand, the problems correlated with this subject are often treated with ad hoc metods, valid only for the cases considered. What is absent is a general treatment of this topic that could also give practical recipes to deal with the largest possible class of concrete examples: the spirit behind this Thesis is to try to fill this gap.

The symmetry action of a self-duality group on the states of the theory will be called a *lift* of that symmetry: there are in general more possible lift choices, and they generically imply an increment in the group order associated to the symmetry. This issue was also recently discussed in an article [12] by J. Harvey and G. Moore. To give an answer to some of the questions raised by this article is one of the principal aims of this Thesis. In particular, finding general results on the conditions that ensure the existence of an order-preserving lift for self-duality groups will be our main task: a task that will be accomplished in Chapter 4.

Outside the mere study of the symmetries of a given toroidal theory, this topic is relevant for orbifold construction. Orbifold construction is a more complicated example

of compactification, obtained by the quotient of a Conformal Field Theory by one of its symmetry groups. This should be a symmetry of the Conformal Field Theory: if this symmetry comes from a self-duality, then the knowlege of its possible lifts (and, in particular, their order) becomes an important issue, opening also the possibility of shedding a light on the conditions that make the orbifold construction consistent.
The structure of this Thesis is as follows.

In Chapter 1, we introduce Conformal Field Theories. Starting from the conformal group and the notion of conformal invariance, we describe the properties of these theories and the role of vertyex operators, following steps of increasing abstraction. The main references for this Chapter are [4], [11], [8] and [6].

In Chapter 2, we present String and Superstring Theories. Some of the topics we describe here are handled in a slightly heuristic style, as the aim of this Chapter is to give motivation and physical insight into the main subjects of the thesis, that are described in the last two Chapters. This Chapter is based upon [17], [18], [2] and [20].

Indeed, with Chapter 3 we start delving into the core of the Thesis. We will discuss the question of compactification, describing in particular many features of toroidal and orbifold compactification of String and Conformal Field Theories. Among these topics, we discuss enhanced symmetries and dualities. The main sources for this Chapter are [17] and [2].

Chapter 4 is the original contribution of this Thesis. Finally, we face here the problem of symmetry lift, finding general conditions that allow to choose lifts that preserve the naïve order of the self-duality. We focus here on the case of cyclic self-duality groups. In the last part of this Chapter, we discuss some physically relevant examples, notably we draw some interesting conclusion on the lifts of the cyclic symmetries of heterotic string theories on $\mathbb{T}^4$ (that are dual to sigma models on K3), whose self-dualities were originally classified in [9].

Chapter 1

Introduction to two dimensional Conformal Field Theory

In this Chapter we introduce the notion of conformal invariance, as the starting point of our review of Conformal Field Theories. After a brief review of the conformal group in the generic d-dimensional case, we focus on the two-dimensional case, naturally defined in a complex manifold. As we will show, the two dimensional case is "special" : the local symmetry algebra is infinite-dimensional, and this remarkable fact makes the operator formalism particularly suitable for the study of these theories. In Section 1.2 we discuss the implications of the conformal invariance, mainly in the framework of radial quantization, and we introduce Virasoro algebra as the symmetry algebra of the quantum theory. In Section 1.3, we focus on the structure of the quantum state space of a Conformal Field Theory and we introduce the notion of vertex operator. We present here a more abstract description of Conformal Field Theories, that has the merit to highlight the role and the properties of the vertex operators.

There are many possible approaches to Conformal Field Theories. We believe the exposition we have chosen could enlighten the interplay between the algebraic structure of the theory with the underlying physical concepts, as well as build a solid ground for the comprehension of the main topics of the Thesis. For the sake of brevity, some of the topics that are not directly related with the subject of this Thesis (although important) will not be covered. Among them, it is imperative to mention the hermitian structure of a two dimensional Conformal Field Theory.

The final Section of the Chapter is devoted to the study of concrete examples, that would be relevant for the following.

1.1 The conformal group

From a general point of view, Conformal Field Theories are Quantum Field Theories (either defined in Euclidean space or Minkowski space) characterized by the invariance under *local conformal transformations*, i.e. local transformations that preserves angles. More precisely, given a metric $g_{\mu\nu}$, a local conformal transformation is a mapping $x \to x'$ that leaves the metric invariant up to a scale factor:

$$g'_{\mu\nu}(x') = \Lambda^2(x) g_{\mu\nu}(x) \tag{1.1}$$

We do not require such a transformation to be everywere defined and invertible. If the transformation is a bijection between the space-time and itself, then it is a global conformal transformation. The global conformal transformations form a group. In the following, we will consider theories defined in Euclidean space, for a matter of convenience. Moreover, we will fix the metric to be the flat one, $\eta_{\mu\nu}$, since this is always locally possible thanks to the conformal symmetry.

Let us determine the global conformal group in d-dimensions. Under an arbitrary infinitesimal change of coordinates $x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$, the metric transforms as:

$$\eta'_{\mu\nu} = \eta_{\mu\nu} - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) \,. \tag{1.2}$$

Requiring that the considered transformation is conformal is equivalent to requiring that $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$ is proportional to $\eta_{\mu\nu}$:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = K \eta_{\mu\nu} \tag{1.3}$$

The proportionality factor K can be expressed by contracting μ and ν in Equation (1.3):

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \partial_\rho \epsilon^\rho \eta_{\mu\nu}. \tag{1.4}$$

Note that, comparing Equation (1.4) with Equation (1.1), we deduce $\Lambda^2(x) = 1 - \frac{2}{d}\partial_\rho \epsilon^\rho$.

If we apply ∂_σ to Equation (1.4) and write down a suitable linear combination of copies of Equation (1.4) with permutated indices we obtain

$$\partial_\mu \partial_\nu \epsilon_\sigma = \frac{1}{d} \left(\eta_{\nu\sigma} \partial_\mu + \eta_{\sigma\mu} \partial_\nu - \eta_{\mu\nu} \partial_\sigma \right) \partial_\rho \epsilon^\rho. \tag{1.5}$$

Applying ∂^σ to Equation (1.5):

$$\left[\eta_{\mu\nu} \Box + (d-2) \partial_\mu \partial_\nu \right] \partial_\rho \epsilon^\rho = 0. \tag{1.6}$$

If $d = 1$, this equation is actually trivial, so every 1-dimensional theory is locally conformal invariant under any smooth transformation. The case $d = 2$ will be studied in detail in Section 1.1.1. Let us focus on the case $d > 2$.

Contracting μ and ν in Equation (1.6), we get:

$$(d-1) \Box \partial_\rho \epsilon^\rho = 0, \tag{1.7}$$

and plugging back Equation (1.7) into Equation (1.6) we obtain

$$\left[(d-2) \partial_\mu \partial_\nu \right] \partial_\rho \epsilon^\rho = 0, \tag{1.8}$$

so Equation (1.8) tells us that $\partial_\rho \epsilon^\rho$ is at most linear in x. Combining this fact with Equation (1.5), we conclude that ϵ^μ is at most quadratic in x:

$$\epsilon^\alpha(x) = a^\alpha + B^{\alpha\beta} x_\beta + C^{\alpha\beta\gamma} x_\beta x_\gamma, \quad C^{\alpha\beta\gamma} = C^{\alpha\gamma\beta}.$$

Plugging this expression into Equation (1.4), we get the following constraint for ϵ:

$$B^{\mu\nu} + B^{\nu\mu} + 2 \left(C^{\mu\nu\rho} + C^{\nu\mu\rho} \right) x_\rho = \frac{2}{d} \left(B^\sigma{}_\sigma + 2 C^\sigma{}_\sigma{}^\rho x_\rho \right) \eta^{\mu\nu}. \tag{1.9}$$

1. We have no constraints on a^μ. $\epsilon^\mu(x) = a^\mu$ is an infinitesimal translation.

2. For $x = 0$, Equation (1.4) becomes

$$B^{\mu\nu} + B^{\nu\mu} = \frac{2}{d} B^{\sigma}{}_{\sigma} \eta^{\mu\nu},$$

that is equivalent to say that $B^{\mu\nu}$ splits into the following way:

$$B^{\mu\nu} = \omega^{\mu\nu} + \lambda\eta^{\mu\nu}, \quad \text{with } \omega^{\mu\nu} + \omega^{\nu\mu} = 0. \tag{1.10}$$

$\epsilon^\mu(x) = \omega^{\mu\nu} x_\nu$ is an infinitesimal rotation, while $\epsilon^\mu(x) = \lambda x^\mu$ correspond to a dilatation.

3. Finally, when ϵ^μ is quadratic in x we have the infinitesimal form of a *special conformal transformation* (SCT): $\epsilon^\mu(x) = 2(b^\nu x_\nu)x^\mu - b^\mu x^2$.

The finite transformations correspond to:

$$\begin{aligned} \text{translations:} \quad & x'^\mu = x^\mu + a^\mu, && (1.11) \\ \text{rotations:} \quad & x'^\mu = M^\mu{}_\nu x^\nu, \quad M \in O(d), && (1.12) \\ \text{dilatations:} \quad & x'^\mu = \lambda x^\mu, && (1.13) \\ \text{SCT:} \quad & x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b^\nu x_\nu + b^2 x^2}. && (1.14) \end{aligned}$$

Remark 1.1.1. The special conformal transformation (1.14) can be written as

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - b^\mu,$$

that makes clear that the SCT is nothing but a translation, preceded and followed by an "inversion" $x^\mu \mapsto \frac{x^\mu}{x^2}$.

Remark 1.1.2. The conformal symmetry algebra for $d > 2$ is generated by

$$\begin{aligned} \text{translations:} \quad & P_\mu = -i\partial_\mu, && (1.15) \\ \text{rotations:} \quad & L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu), && (1.16) \\ \text{dilatations:} \quad & D = -ix^\mu \partial_\mu, && (1.17) \\ \text{SCT:} \quad & K_\mu = -i(2x_\mu x^\nu \partial_\mu - x^2 \partial_\mu). && (1.18) \end{aligned}$$

Altough this algebra seems quite messy, it can be shown that it is isomorphic to the Lie algebra $\mathfrak{so}(1 + d, 1)$.

1.1.1 Conformal group in two dimensions

We will now focus on the case of two dimensional Conformal Field Theory. Consider Equation (1.4) with $d = 2$: it is equivalent to the set of equations

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2, \tag{1.19}$$
$$\partial_1 \epsilon_2 = -\partial_2 \epsilon_1. \tag{1.20}$$

Let us define the complex variables

$$z = x_1 + ix_2, \tag{1.21}$$
$$\bar{z} = x_1 - ix_2, \tag{1.22}$$

and, accordingly to this definition, $\bar{\epsilon} = \epsilon_1 + i\epsilon_2$ and $\epsilon = \epsilon_1 - i\epsilon_2$. If we call $\partial := \partial_z = \frac{\partial}{\partial z}$ and $\bar{\partial} := \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}}$, the derivatives in the new variables are:

$$\begin{aligned} \partial_1 &= \partial + \bar{\partial}, \\ \partial_2 &= i(\partial - \bar{\partial}). \end{aligned}$$

We can then rewrite Equations (1.19) and (1.20) as

$$\bar{\partial}\epsilon = 0, \tag{1.23}$$

$$\partial\bar{\epsilon} = 0, \tag{1.24}$$

that is exactly equivalent to say that ϵ and $\bar{\epsilon}$ are respectively holomorphic and antiholomorphic in the complex variables $(z, \bar{z})$.

Remark 1.1.3. As we have implicitly done, the variables z and $\bar{z}$ should be regarded as independent. The correct approach is to extend the original coordinates x^1 and x^2 to the complex plane, and then Equations (1.21) and (1.22) are simply a change of coordinates. The "physical" space is recovered on the two dimensional-submainfold $z^* = \bar{z}$, called *real surface*. The advantages of this "natural" framework will be evident in the following.

Remark 1.1.4. In the complex coordinates $z, \bar{z}$ the flat metric tensor becomes

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

and the completely antisymmetric tensor becomes

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{pmatrix}, \quad \epsilon^{\mu\nu} = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix},$$

where the indices take the values z and $\bar{z}$, in that order.

Two dimensional Conformal Field Theories are hence naturally described by a theory on a complex surface, parametrized by a pair of suitable complex coordinates $z, \bar{z}$. Here and in the following, we will consider only compact and closed complex surfaces. These surfaces are topologically classified by their *genus* g, i.e. the number of their handles. In the case of $g = 0$, the *Riemann sphere*, every point but one can be parametrized by a complex coordinate z in the complex plane. We refer to the Riemann sphere by writing $\mathbb{C} \cup \infty$. Our Euclidean space would hence be the union of two (holomorphic and antiholomorphic, we will say) Riemann spheres, described by z and $\bar{z}$ (respectively).

In this description, conformal transformations are identified with holomorphic and antiholomorphic mappings, respectively $z \to w(z)$ and $\bar{z} \to \bar{w}(\bar{z})$. Considering the holomorphic map, we write the infinitesimal form of such a transformation as

$$z \to z + \epsilon(z). \tag{1.25}$$

Since we are interested in local transformations, we look for holomorphic maps $\epsilon(z)$ that are defined in a neighborhood of $z = 0$. Hence, we can write them as Laurent series:

$$\epsilon(z) = \sum_{n=-\infty}^{+\infty} c_n z^{n+1}. \tag{1.26}$$

We will now compute the generators of such a transformation. The action of this infinitesimal transformation, and of its antiholomorphic counterpart, on a field Φ that is not affected by the transformation, *i.e.* $\Phi'(z', \bar{z}') = \Phi(z, \bar{z})$, is given by:

$$\begin{aligned}\delta\Phi(z', \bar{z}') :=& \Phi'(z', \bar{z}') - \Phi(z', \bar{z}') = \Phi'(z', \bar{z}') - \Phi(z, \bar{z}) + \Phi(z, \bar{z}) - \Phi(z', \bar{z}') \\ =& - (\epsilon(z')\partial' + \bar{\epsilon}(\bar{z}')\bar{\partial}')\Phi(z', \bar{z}'),\end{aligned}$$

that we can write as

$$\delta\Phi = \sum_{n=-\infty}^{+\infty} (c_n \ell_n + \bar{c}_n \bar{\ell}_n)\Phi, \tag{1.27}$$

with

$$\ell_n = -z^{n+1}\partial, \quad \bar{\ell}_n = -\bar{z}^{n+1}\bar{\partial}. \tag{1.28}$$

ℓ_n and $\bar{\ell}_n$ are generetors of two isomorphic infinite-dimensional algebras (sometimes called Witt algebra):

$$\begin{aligned}[\ell_n, \ell_m] &= (n-m)\ell_{n+m} \\ \left[\bar{\ell}_n, \bar{\ell}_m\right] &= (n-m)\bar{\ell}_{n+m} \\ \left[\ell_n, \bar{\ell}_m\right] &= 0.\end{aligned} \tag{1.29}$$

Remark 1.1.5. The real surface $z^* = \bar{z}$ is invariant under the subalgebra generated by $\ell_n + \bar{\ell}_n$ and $i(\ell_n - \bar{\ell}_n)$.

As we have hinted, the algebra we described is associated to the local symmetries of the theory. To be globally defined, we require a holomorphic transformation $z \mapsto w(z)$ to be a bijection between the Riemann sphere $\mathbb{C} \cup \infty$ and itself. In general, the obstructions to the global definition of a transformation can be tracked down to the fact that the generators of the local algebra are not globally well defined. Holomorphic conformal transformation are generated by vector fields

$$\sum_n a_n \ell_n = \sum_n a_n z^{n+1} \partial_z.$$

If we require non-singularity at $z = 0$, we have to impose $a_n = 0$ for $n < -1$. To study the behavior of the field in a neighbour of ∞, we employ the change of variable $w = \frac{1}{z}$: the vector fields become

$$\sum_n a_n \left(\frac{1}{w}\right)^{n+1} \frac{dw}{dz} \partial_w = -\sum_n a_n \left(\frac{1}{w}\right)^{n-1} \partial_w.$$

The field is well defined for $w \to 0$ if and only if $a_n = 0$ for every $n > 1$. Hence, the holomorphic tranformations that are globally defined on the Riemann sphere are the ones generated by ℓ_n, $n = -1, 0, +1$. The analogous statement holds for the antiholomorphic transformations.

Remark 1.1.6. The subalgebra generated by ℓ_n, $n = -1, 0, +1$ is isomorphic to the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \simeq \mathfrak{so}(3, 1)$, according to the generic conclusion of Remark 1.1.2.

The global symmetry algebra is hence generated by $\{\ell_{-1}, \ell_0, \ell_{+1}\} \cup \{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_{+1}\}$. We can give an explicit description of the global conformal transformations. Without restricting to the real surface, it can be shown that in general a finite conformal transformation can be written as

$$z \mapsto f(z) = \frac{az+b}{cz+d}, \quad \bar{z} \mapsto \bar{f}(\bar{z}) = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}, \tag{1.30}$$

with $a, b, c, d, \bar{a}, \bar{b}, \bar{c}, \bar{d} \in \mathbb{C}$, $ad - bc = 1 = \bar{a}\bar{d} - \bar{b}\bar{c}$.

The transformations $z \mapsto f(z)$ are the only holomorphic, bijective maps between the Riemann sphere and itself. We can associate to the complex parameters a, b, c, d the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$$

The maps composition $f_1 \circ f_2$ correpond to the matrix product $A_2 A_1$. Indeed, the maps $z \mapsto f(z)$ form the group $SL(2, \mathbb{C})/\mathbb{Z} \simeq SO(3, 1)$, also known as the *conformal projective group*. The quotient by $\mathbb{Z}$ means that the transformation is unaffected by reversing the sign of a, b, c, d.

Remark 1.1.7. Restricting on the real surface, and according to the classification of the previous Section 1.1, we identify $\ell_{-1} + \bar{\ell}_{-1}$, $i(\ell_{-1} - \bar{\ell}_{-1})$ as the generator of translations respectively along x^1 and x^2, $\ell_0 + \bar{\ell}_0$ as the generator of dilatations, $i(\ell_0 - \bar{\ell}_0)$ as the generator of rotations, $\ell_1 + \bar{\ell}_1$, $i(\ell_1 - \bar{\ell}_1)$ as the generator of special conformal transformations.

We anticipate that the symmetry algebra of the quantum theory will be a central extension of the Witt algebra, the famous Virasoro algebra. The only difference will actually be the presence of an addictional central charge term in the commutator relations (1.29). Anyway, the subalgebra related to the global conformal symmetry will remain the same. In the quantum theory, the Hilbert space of states will host a representation of Virasoro algebra, and in particular a representation of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, in terms of operators $\{L_{-1}, L_0, L_{+1}\} \cup \{\bar{L}_{-1}, \bar{L}_0, \bar{L}_{+1}\}$, representations of the generators $\{\ell_{-1}, \ell_0, \ell_{+1}\} \cup \{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_{+1}\}$. The action of such operators on the physical states are useful to characterize their properties: suppose we work in a base of eigenstates of L_0 and $\bar{L}_0$. Consider such an eigenstate: we call its eigenvalues h and $\bar{h}$ of (respectively) L_0 and $\bar{L}_0$ the *conformal dimensions* of such state. Since $L_0 + \bar{L}_0$ represents dilatations and $i(L_0 - \bar{L}_0)$ represent rotations, we will define the *scaling dimension* of the state as $\delta = h + \bar{h}$ and the *spin* of the state as $s = h - \bar{h}$. We will return on these concept in the following Sections.

1.2 Conformal invariance in two dimensions

Before giving a more general and formal description of a Conformal Field Theory in two dimensions, it will be useful to discuss about the notion and the implications of the conformal invariance in quantum field theory. This step is not strictly mandatory for the purposes of our work, but it is somewhat useful to become familiar with the many features of Conformal Field Theory in two dimensions. Moreover, this first discussion of conformal invariance will be helpful to relate the formal description of Section 1.3 to a physical insight of the theory. In the following exposition, we will assume the knowledge of some general results in Field Theory. Anyway, where needed, we will briefly recall these results, although we will not delve deep into the technical details of their motivations.

1.2.1 Fields and correlation functions

Let us start introducing the notions of *quasi-primary field*, *primary field* and *secondary* (or *descendent*) *field*.

1. A quasi-primary field is a field ϕ that under a global conformal transformation $(z, \bar{z}) \mapsto (w, \bar{w})$ transforms according to:

$$\phi'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}), \tag{1.31}$$

where h and $\bar{h}$ are the conformal dimensions of the field.

2. Broadly speaking, primary field is a field ϕ that transforms according to Equation (1.31) under *any* local conformal transformation. In more precise terms, this is to say that the variation of a primary field under a local conformal transformation close to the identity $(z, \bar{z}) \mapsto (z + \epsilon, \bar{z} + \bar{\epsilon})$ is:

$$\delta_{\epsilon,\bar{\epsilon}}\phi = \phi' - \phi = -(h\phi\partial_z\epsilon + \epsilon\partial_z\phi) - (\bar{h}\phi\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}}\phi). \tag{1.32}$$

Observe that such expression holds also for quasi-primary fields, under global conformal transformations.

3. A field that is not primary is usually called a secondary field.

Remark 1.2.1. All primary fields are quasi-primary, but the inverse statement is not true (a remarkable example will be the stress-energy tensor). Also, there exist secondary fields that are not quasi primary.

It follows that a correlation function of (quasi-)primary fields $\phi_1, \dots, \phi_n$ with conformal dimensions $(h_1, \bar{h}_1), \dots, (h_n, \bar{h}_n)$ transforms according to:

$$\langle \phi_1(w_1, \bar{w}_1) \dots \phi_n(w_n, \bar{w}_n) \rangle = \prod_{j=1}^{n} \left(\frac{dw}{dz}\right)^{-h_j}_{w=w_j} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}_j}_{\bar{w}=\bar{w}_j} \langle \phi_1(z_1, \bar{z}_1) \dots \phi_n(z_n, \bar{z}_n) \rangle, \tag{1.33}$$

and this relation is enough to fix two- and three-point correlation functions, but not the four-point correlation functions. The reason lies in the possibility of performing a global conformal transformation: we have the freedom fo send three point z_1, z_2, z_3 on the Riemann sphere to, say, $0, 1, \infty$, but an additional fourth pont z_4 would remain unfixed.

1.2.2 Stress-energy tensor and conformal transformations

As a classical Field Theory invariant under Poincaré group, Noether theorem grants a two dimensional Conformal Field Theory a stress-energy tensor that satisfies a continuity equation when the fields configuration obeys the classical equations of motion, under the assumption that such equations of motion come from a variational principle. That stress-energy tensor can be chosen to be symmetric and, because of the additional dilatation invariance, it can be also chosen traceless (under certain conditions, satisfied by a large class of physical theories). In the x^μ coordinates description, we would have

$$\partial_\mu T^{\mu\nu} = 0, \quad T^\mu{}_\mu = 0 \tag{1.34}$$

Writing everything in the complex coordinates z and $\bar{z}$, we have

$$T_{zz} = \frac{1}{4}\left(T_{11} - T_{22} - 2iT_{12}\right), \quad T_{\bar{z}\bar{z}} = \frac{1}{4}\left(T_{11} - T_{22} + 2iT_{12}\right), \tag{1.35}$$

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}(T_{11} + T_{22}) = \frac{1}{4}T^\mu{}_\mu = 0, \tag{1.36}$$

$$\partial_z T_{\bar{z}\bar{z}} = 0, \quad \partial_{\bar{z}} T_{zz} = 0. \tag{1.37}$$

This is exactly to say that the only non-zero component of the stress-energy tensor

$$-2\pi T_{zz}(z,\bar{z}) = T(z), \quad -2\pi T_{\bar{z}\bar{z}}(z,\bar{z}) = \bar{T}(\bar{z}) \tag{1.38}$$

are respectively holomorphic and antiholomorphic. As we will discover later on, this is only one of the many examples of the separation of the holomorphic and antiholomorphic degrees of freedom inside the framework of two dimensional Conformal Field Theory.

If we consider an infinitesimal confomal transformation, described by $x^\mu \mapsto x^\mu + \epsilon$, the conserved Noether current is given by $j^\mu = T^{\mu\nu}\epsilon_\nu$. Under very general assumption, associated to a conserved current there is a conserved charge, given by the integration along the space directions of the time component of the current. From the point of view of the Quantum Theory, the conserved charge Q generates the infinitesimal transformations of any field ϕ:

$$\delta_\epsilon \phi = \epsilon\,[Q,\phi] = [Q_\epsilon,\phi]$$

We are studying two dimensional Conformal Field Theories in an Euclidean framework, hence the identification of the "time" and "space" directions is quite arbitrary. In the next Subsection we will consider a particularly useful option, namely *radial quantization*, and we will write down the conserved conformal charge according to this choice.

1.2.3 Operator formalism: radial quantization

The infinite dimension of the symmetry algebra encourage the use of the operator formalism, rather than the path integral approach. In the operator formalism, we distinguish a "time" direction from the "space" direction(s). However, in the Euclidean this choice is somewhat arbitrary, and this is at the heart of the idea of radial quantization.

In order to make this choice more natural, we start with a theory defined on a Minkowski space-time. Let this space-time be a cylinder, and define the time direction as the axial direction of the cylinder. The space is then compactified, and is described by a coordinate x that ranges between 0 and L, with the points (x,t) and $(x+L,t)$ identified. The time ranges from $-\infty$ (the remote past) to $+\infty$ (the remote future). We then continue our theory in the Euclidean space, and we describe the space-time points (t,x) by a single complex coordinate $t-ix$. As in Subsection 1.1.1, we shall think of a complexified theory, where the coordinates $t-ix$ and $t+ix$ are independent. We then map our Euclidean space to the Riemann sphere, by means of the maps

$$z = e^{\frac{2\pi(t-ix)}{L}}, \quad \bar{z} = e^{\frac{2\pi(t+ix)}{L}} \tag{1.39}$$

Observe that points at the remote past $t=-\infty$ are mapped to $z=0$, while points at the remote future $t=+\infty$ are send to the point at infinity ∞ of the Riemann sphere. We will assume the existence of a vacuum state $|0\rangle$, and we will interpret fields as operator that creates asymptotic states:

$$|\phi_{in}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})\,|0\rangle\,.$$

As we know, to write correlation functions as vacuum expectation values for products of fields, we have to introduce the concept of time ordering. In the framework of radial quantization, *time ordering* is naturally translated into the notion of radial ordering: if we consider two holomorphic bosonic fields $a(z), b(w)$, their radially ordered product is defined to be

$$\mathcal{R}\left(a(z)b(w)\right) = \begin{cases} a(z)b(w) & \text{if } |z|>|w|, \\ b(w)a(z) & \text{if } |z|<|w|. \end{cases} \tag{1.40}$$

If $a(z)$ and $b(w)$ are fermionic, a minus sign should be added when commuting the two fields. Consider now the operators defined as

$$A = \oint dz\, a(z), \quad B = \oint dz\, b(z),$$

where the integral is performed along a closed curve taken at fixed time (fixed radius). The following important relations can be shown:

$$[A, b(w)] = \oint_w dz\, \mathcal{R}\left(a(z)b(w)\right), \tag{1.41}$$

$$[A, B] = \oint_0 dw \oint_w dz\, \mathcal{R}\left(a(z)b(w)\right), \tag{1.42}$$

where $[\cdot,\cdot]$ denotes the commutator, and where the integral over z is performed around w, and the integral over w is taken around 0. We will consider the radial order implicit for the following operator products.

If we have an expression for the radially ordered product of the fields, or at least if we know the analytical structure of this product, we will compute easily such commutator, with the help of the residue theorem. The notion of *Operator Product Expansion*, or briefly *OPE*, replies to this necessity. If in a correlation function there appears a product between two fields $A(z)$ and $B(w)$, this product can be replaced by a suitable expansion:

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{(AB)_n(w)}{(z-w)^n}, \tag{1.43}$$

where the fields $(AB)_n$ are non-singular for z close to w. We do not require Equation (1.43) to have any operatorial meaning: it is meaningful only inside a correlation function.

In Section 1.4 we will compute explicitly many OPEs, and in particular we will find the following general expression for the product $T(z)T(w)$:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{1.44}$$

where the symbol $\sim$ means "equal up to regular terms as $w \to z$". The constant c, called *central charge* for a reason that will be clear later, depends on the specific model considered. We will not give a general motivation for this OPE, as it would go beyond the scopes of this introduction. Obviously, there exist an analogous expression for the antiholomorphic component of the stress-energy tensor:

$$\bar{T}(\bar{z})\bar{T}(\bar{w}) \sim \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{\bar{z}-\bar{w}}. \tag{1.45}$$

Armed with the OPE (1.44) and (1.45), and with the help of Equation (1.42), we are going to discuss the generators of the quantum conformal symmetry. Accordingly to the final considerations of Subsection 1.2.2, we introduce the conformal charge associated to an infinitesimal conformal transformation $z \to z + \epsilon(z)$, $\bar{z} \mapsto \bar{z} + \bar{\epsilon}(\bar{z})$, as

$$Q_{\epsilon,\bar{\epsilon}} = \frac{1}{2\pi i}\oint dz\, \epsilon(z)T(z) + \frac{1}{2\pi i}\oint d\bar{z}\, \bar{\epsilon}(\bar{z})\bar{T}(\bar{z}). \tag{1.46}$$

If we expand $T(z)$ and $\bar{T}(\bar{z})$ in mode operators as

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-2-n} L_n, \quad L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1} T(z), \tag{1.47}$$

$$\bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-2-n} \bar{L}_n, \quad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^{n+1} \bar{T}(\bar{z}), \tag{1.48}$$

and recalling that

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} z^{n+1}\epsilon_n, \quad \bar{\epsilon}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{n+1}\bar{\epsilon}_n,$$

we can write the conformal charge as

$$Q_{\epsilon,\bar{\epsilon}} = \sum_{n\in\mathbb{Z}} \left(\epsilon_n L_n + \bar{\epsilon}_n \bar{L}_n\right) \tag{1.49}$$

where the generators of the conformal symmetry $\{L_n\}$ and $\{\bar{L}_n\}$ obey the following algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \tag{1.50}$$

$$\left[\bar{L}_n, \bar{L}_m\right] = (n-m)\bar{L}_{n+m} + \frac{\bar{c}}{12}n(n^2-1)\delta_{n+m,0} \tag{1.51}$$

$$\left[L_n, \bar{L}_m\right] = 0 \tag{1.52}$$

This is the celebrated Virasoro algebra, and it is a central extension (to be precise, the only central extension) of Witt algebra. For example, let us show in detail the derivation of Equation (1.50), with the help of the residue theorem and some complex calculus:

$$\begin{aligned}
[L_n, L_m] =& \frac{1}{2\pi i}\oint_0 dw\, \frac{1}{2\pi i}\oint_w dz\, z^{n+1}w^{m+1}T(z)T(w) \\
=& \frac{1}{2\pi i}\oint_0 dw\, w^{m+1}\frac{1}{2\pi i}\oint_w dz\, z^{n+1}\left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular term}\right] \\
=& \frac{1}{2\pi i}\oint_0 dw\, w^{m+1}\left[\frac{c}{12}(n+1)n(n-1)w^{n-2} + 2(n+1)w^n T(w) + w^{n+1}\partial T(w)\right] \\
=& \frac{1}{2\pi i}\oint_0 dw\, w^{m+1}\left[\frac{c}{12}n(n^2-1)w^{n-2} + 2(n+1)w^n T(w) - (n+m+2)w^n T(w)\right] \\
=& \frac{1}{2\pi i}\oint_0 dw\, w^{m+n+1}\left[\frac{c}{12}n(n^2-1)w^{-2} + (n-m)T(w)\right] \\
=& \frac{c}{12}n(n^2-1)\delta_{n+m,0} + (n-m)\frac{1}{2\pi i}\oint_0 dw\, w^{m+n+1}T(w) \\
=& \frac{c}{12}n(n^2-1)\delta_{n+m,0} + (n-m)L_{n+m}.
\end{aligned}$$

As Witt algebra, Virasoro algebra contain $\mathfrak{su}(2)$ as a subgroup. The operator $L_0 + \bar{L}_0$ generates the translations $(z, \bar{z}) \mapsto \lambda(z, \bar{z})$, that are nothing but the time translations in the radial quantization formalism. Hence, the Hamiltonian of the theory should be proportional to $L_0 + \bar{L}_0$.

1.3 General structure of a Conformal Field Theory in two dimensions

In this Section, we make another step in the direction of abstraction, as we describe in a general and slightly scketchy way what are our requests for a two dimensional Conformal Field Theory. We will emphasize the role of vertex operators, as they will play an important role in the description of the main topics of this thesis. Many concepts introduced in Section 1.2 will be rephrased into a more formal language.

In general, to define a Quantum Field Theory we have to identify the space state and the correlation functions. The space of states of the theory is a Hilbert space $\mathcal{H}$, and the correlation functions are defined for states that belongs to a dense subspace $\mathcal{F}$ of $\mathcal{H}$. Usually, $\mathcal{F}$ is taken as the Fock space of finite occupation number of some set of harmonic oscillators. The states ψ of $\mathcal{F}$ are biunivocally associated to the fields $V(\psi, z, \bar{z})$, that are called vertex operators. The correlation functions take the form

$$\langle V(\psi_1, z_1, \bar{z}_1) \ldots V(\psi_n, z_n, \bar{z}_n)\rangle. \tag{1.53}$$

An important property of the correlation function is *locality*: we require that the correlation function do not depend on the order of the fields $V(\psi_i, z_i, \bar{z}_i)$ (in the bosonic case).

As we know, the symmetry algebra of a two dimensional conformal field theory is infinite-dimensional, and it is given by the direct sum of two Virasoro algebras. The states space of the theory is the space of a representation of the symmetry algebra. Each independent Virasoro algebra is related to a particular subspace of $\mathcal{F}$, that we will call respectively $\mathcal{F}_0$ and $\bar{\mathcal{F}}_0$. A state ψ that belongs to $\mathcal{F}_0$ is characterize by the property that, for any collection of states $\psi_i \in \mathcal{F}$, the correlation function

$$\langle V(\psi, z, \bar{z}) V(\psi_1, z_1, \bar{z}_1) \ldots V(\psi_n, z_n, \bar{z}_n)\rangle. \tag{1.54}$$

does not depend on $\bar{z}$. Conversely, ψ belongs to $\bar{\mathcal{F}}_0$ if the correlation function (1.54) does not depend on z. In these two cases, such correlation functions are respectively holomorphic and antiholomorphic functions on Riemann sphere, and they define respectively the meromorphic and the antimeromorphic theories. In the next Subsection 1.3.1, we will focus on the meromorphic theory, and we will give an accurate description of it. Obviously, the methods proposed to study the meromorphic theory can be used as well to study the antimeromorphic theory.

Let us review now the notion of OPE: as we know, the idea is to expand the product of two fields in terms of a sum of single fields, and the OPE is meaningful when we consider the product of fields to appear inside a correlation function. If $\psi_1, \psi_2 \in \mathcal{F}$, then we have the general expression

$$\begin{aligned} &V(\psi_1, z_1, \bar{z}_1) V(\psi_2, z_2, \bar{z}_2) = \\ &\qquad = \sum_i (z_1 - z_2)^{\Delta_i} (\bar{z}_1 - \bar{z}_2)^{\bar{\Delta}_i} \sum_{r,s \geq 0} V(\phi^i_{r,s}, z_2, \bar{z}_2)(z_1 - z_2)^r (\bar{z}_1 - \bar{z}_2)^s, \end{aligned} \tag{1.55}$$

where $\phi^i_{r,s} \in \mathcal{F}$. From Equation (1.55), we deduce the associativity of the OPE. If ψ_1, ψ_2 belong to the meromorphic theory $\mathcal{F}_0$, then the OPE (1.55) does not depend on $\bar{z}_1$ and $\bar{z}_2$, and $\phi^i_{r,s}$ belongs to $\mathcal{F}_0$. We have that the OPE define a certain associative algebraic structure on the meromorphic theory, namely a *vertex operator algebra* (not an algebra, because of the kind of dependence of the OPE on complex parameters z_1 and z_2). From the associativity, we have that $\mathcal{F}$ form a representation of the vertex operator algebra, and the same hold true for the antimeromorphic theory. Hence, we can decompose $\mathcal{F}$ or the whole space $\mathcal{H}$ as the direct sum of irreducible representationsof the two vertex operator algebras:

$$\mathcal{H} = \bigoplus_{(j,\bar{j})} \mathcal{H}_{(j,\bar{j})} \tag{1.56}$$

where $\mathcal{H}_{(j,\bar{j})}$ is an irreducible representation of the two vertex operator algebras.

Remark 1.3.1. In some case of interest, only a finite number of representations appears in the decomposition (1.56): we then speak of *finite theories*.

For many conformal field theories, the spaces $\mathcal{H}_{(j,\bar{j})}$ are the tensor product of an irreducible representation $\mathcal{H}_j$ of the meromorphic vertex operator algebra and an irreducible representation $\mathcal{H}_{\bar{j}}$ of the antimeromorphic vertex operator algebra. In these cases, we write

$$\mathcal{H} = \bigoplus_{j,\bar{j}} M_{j\bar{j}} \left(\mathcal{H}_j \otimes \mathcal{H}_{\bar{j}}\right), \tag{1.57}$$

where $M_{j\bar{j}} \in \mathbb{N}$ specifies the multiplicity in which the tensor product $\mathcal{H}_j \otimes \mathcal{H}_{\bar{j}}$ appears in $\mathcal{H}$ The two theories, meromorphic and antimeromorphic, contain all the information on the symmetries of the theory. In the next Subsection, we will study their structure, in an appropriate mathematical framework.

1.3.1 Meromorphic Conformal Field Theory

Closely following [6], we describe now the meromorphic theory.

Definition 1.3.1. A *meromorphic conformal field theory* $(\mathcal{H}, \mathcal{F}, \mathbf{V}, |0\rangle, \psi_L)$ is composed by a Hilbert state space $\mathcal{H}$, a dense subspace $\mathcal{F}$ of $\mathcal{H}$, a set $\mathbf{V}$ of linear operators $V(\psi, z)$ in one-to-one correspondence with the states ψ of $\mathcal{F}$. There exist two special states belonging to $\mathcal{F}$, the *vacuum* $|0\rangle$ and the *conformal state* ψ_L. The theory must satisfy the Properties we will specify in the following.

The operators $V(\psi, z)$ of $\mathbf{V}$ are called *vertex operators*.

Property 1.3.1. *Let us define the moments L_n of the vertex operator associated to the conformal state as:*

$$V(\psi_L, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}. \tag{1.58}$$

We require that L_n form a representation of Virasoro algebra:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0},$$

with $L_n^\dagger = L_{-n}$ and $L_n |0\rangle = 0$ for every $n \geq -1$.

Remark 1.3.2. Comparing Equation (1.58) with Equation (1.47), we observe that $V(\psi_L, z)$ coincides with the stress-energy tensor $T(z)$.

Two general requirements for the vertex operators are given by the following two Properties:

Property 1.3.2. *If ψ belongs to $\mathcal{F}$, then*

$$V(\psi, z)|0\rangle = e^{zL_{-1}}\psi. \tag{1.59}$$

Property 1.3.3 (Locality of vertex operators)**.** *For every $\psi, \phi \in \mathcal{F}$, with at least one of them bosonic, we require the matrix elements associated to the product $V(\psi, z)V(\phi, w)$ are well defined for $|z| > |w|$, and the function obtained by analytic continuation of such matrix element is regular except for possible poles at $z, w = 0, \infty$ and $z = w$. We require that*

$$V(\psi, z)V(\phi, w) = V(\phi, w)V(\psi, z), \tag{1.60}$$

where both sides of Equation (1.60) *are to be considered analytical extended function, as specified above.*
If ψ, ϕ are both fermionic, then we require a relative minus sign between the two sides of Equation (1.60)*.*

If we want to write correlation function between fields as vacuum expectation value, we have to apply radial ordering to the fields. This notion of locality is hence rather natural: for example, in the case of bosonic field, the result of the correlation function should not depend on the order in which the fields appear in its expression.
If an operator $U(z)$ commutes with all the vertex operators in the sense of Equation (1.60), we say that $U(z)$ is local with respect to the systems of vertex operators $\mathbf{V}$. Property 1.3.2 implies two important results, namely the *uniqueness theorem* and the *duality theorem*

Proposition 1.3.1 (Uniqueness theorem)**.** *If U(z) satisfies*

$$U(z)\left|0\right\rangle = V(z,\psi)\left|0\right\rangle$$

for some $\psi \in \mathcal{F}$, and $U(z)$ is local with respect to $\mathbf{V}$, then $U(z) = V(\psi, z)$.

Proof. Consider $\phi \in \mathcal{F}$. We have:

$$\begin{aligned} U(z)e^{wL_{-1}}\phi =& U(z)V(\phi,w)\left|0\right\rangle = V(\phi,w)U(z)\left|0\right\rangle = V(\phi,w)V(\psi,z)\left|0\right\rangle \\ =& V(\psi,z)V(\phi,w)\left|0\right\rangle = V(\psi,z)e^{wL_{-1}}\phi, \end{aligned}$$

and then taking the limit $w \to 0$ we deduce $U(z) = V(\psi, z)$ on $\mathcal{F}$, and from the density of $\mathbf{F}$ we conclude $U(z) = V(\psi, z)$. □

Proposition 1.3.2. *Property 1.3.2 holds if and only if*

$$[L_{-1}, V(\psi,z)] = \frac{d}{dz}V(\psi,z), \tag{1.61}$$

$$\lim_{z\to 0} V(\psi,z)\left|0\right\rangle = \psi. \tag{1.62}$$

Proof. If Equation (1.61) holds, then when we apply it to $|0\rangle$ we get

$$\frac{d}{dz}V(\psi,z)\left|0\right\rangle = L_{-1}V(\psi,z)\left|0\right\rangle,$$

since $L_{-1}|0\rangle = 0$. From Equation (1.62), we conclude $V(\psi, z)|0\rangle = e^{zL_{-1}}\psi$.
Conversely, let Property 1.3.2 hold. Equation (1.62) is obviously true. Moreover,

$$\frac{d}{dz}V(\psi,z)\left|0\right\rangle = \frac{d}{dz}e^{zL_{-1}}\psi = L_{-1}e^{zL_{-1}}\psi = L_{-1}V(\psi,z)\left|0\right\rangle = [L_{-1}, V(\psi,z)]\left|0\right\rangle.$$

The derivative $\frac{d}{dz}V(\psi, z)$ is local with respect to $\mathbf{V}$, and then we conclude for the uniqeness theorem that also Equation (1.61) holds. □

Remark 1.3.3. Note that for the uniqueness theorem we can also write

$$\frac{d}{dz}V(\psi,z) = V(L_{-1}\psi, z)$$

Remark 1.3.4 (Translation property)*.* Observe that Equation (1.61) can be written as

$$e^{wL_{-1}}V(\psi,z)e^{-wL_{-1}} = V(\psi, z+w). \tag{1.63}$$

Proposition 1.3.3 (Duality theorem)**.** *If $\psi, \phi \in \mathcal{F}$, then*

$$V(\psi, z)V(\phi, w) = V(V(\psi, z-w)\phi, w). \tag{1.64}$$

Proof. The product $V(\psi, z)V(\phi, w)$ is local with respect to $\mathbf{V}$, since each factor is local with respect to $\mathbf{V}$. We have:

$$\begin{aligned} V(\psi, z)V(\phi, w)\left|0\right\rangle =& V(\psi, z)e^{wL_{-1}}\phi = e^{wL_{-1}}e^{-wL_{-1}}V(\psi, z)e^{wL_{-1}}\phi \\ =& e^{wL_{-1}}V(\psi, z-w)\phi = V(V(\psi, z-w)\phi, w)\left|0\right\rangle, \end{aligned}$$

and then the proof follow from the uniqueness theorem. □

Remark 1.3.5. In this framework, the duality theorem is the true motivation behind the OPE for the vertex operators, as we will see.

Remark 1.3.6. The uniqueness theorem brings other interesting results: for example, by looking at the action on the vacuum, it is immediate to observe that $V(\psi, z)$ is linear in ψ and $V(\left|0\right\rangle, z) = 1$

The operator L_0 is self-adjoint, hence $\mathcal{F}$ splits into a direct sum of L_0-eigenspace:

$$\mathcal{F} = \bigoplus \mathcal{F}_h,$$

where $\psi \in \mathcal{F}_h$ if $L_0\psi = h\psi$. We now recall the important notion of quasi-primary state, as *highest weight* $\mathfrak{su}(2)$ *states.*

Definition 1.3.2. A quasi-primary state or highest weight $\mathfrak{su}(2)$ state is a state $\psi \in \mathcal{F}$ of L_0 such that $L_1\psi = 0$. The associated vertex operator $V(\psi, z)$ is called a quasi-primary field.

Remark 1.3.7. Remember that $\mathfrak{su}(2)$ is isomorphic to the subalgebra generated by $\{L_0, L_{\pm 1}\}$. We speak of highest weight because the state is killed by all positive modes (in this case, only one: L_1).

The following request deals with the operator L_0 and its spectrum:

Property 1.3.4. *The spectrum of L_0 is bounded below and, in the bosonic case, integer.*

Property 1.3.4 is an intuitive assuption, since in the complete theory the Hamiltonian is given by $L_0 + \bar{L}_0$. We want it to be bounded below, and since the meromorphic and antimeromorphic sectors are independent, we have to require L_0 and $\bar{L}_0$ to be separately bounded below.

Property 1.3.5. *If $\psi \in \mathcal{F}_h$ then*

$$w^{L_0}V(\psi, z)w^{-L_0} = V(w^h\psi, wz), \tag{1.65}$$

Remark 1.3.8. Equation (1.65) is equivalent to

$$[L_0, V(\psi, z)] = \left(z\frac{d}{dz} + h\right)V(\psi, z). \tag{1.66}$$

Remark 1.3.9. For linearity,

$$w^{L_0}V(\psi, z)w^{-L_0} = V(w^{L_0}\psi, wz), \tag{1.67}$$

for every $\psi \in \mathcal{F}$

Quite remarkably,

Proposition 1.3.4. *The eigenvalues of L_0 are non-negative.*

The eigenvalue of L_0 associated with an eigenstate ψ is referred as the *conformal weight* of ψ.

Proposition 1.3.5. *If a state ψ has conformal weight 0 then $L_{\pm 1}\psi = 0$. In other words: a state has conformal weight 0 if and only if is $\mathfrak{su}(2)$ invariant.*

Proof. Consider a state ψ of conformal weight h. If we apply L_1 to that state, we get a state of conformal weight $h-1$

$$L_0 L_1 \psi = L_1 L_0 \psi - [L_1, L_0]\,\psi = h L_1 \psi - L_1 \psi = (h-1) L_1 \psi,$$

hence if a state ψ has conformal weight 0 it must be such that $L_1\psi = 0$ (i.e. ψ is quasi-primary), to avoid the presence of a state of negative conformal weight. On the other hand, if we apply $[L_1, L_{-1}] = 2L_0$ to a quasi-primary state ψ, and we compute the scalar product $(\cdot,\cdot)$ of ψ with this new state, we get

$$\|L_{-1}\psi\| = 2h\|\psi\|,$$

hence if the conformal weight of ψ is 0 then $L_1\psi = 0$ too. □

We require then

Property 1.3.6. *The vacuum is the only $\mathfrak{su}(2)$ invariant state.*

The space $\mathcal{F}$ hence splits into a direct sum of $\mathfrak{su}(2)$ representations, each one generated by the action of L_{-1} on a quasi-primary state.

Definition 1.3.3. A primary state or highest weight Virasoro state is a state $\psi \in \mathcal{F}$ of L_0 such that

$$L_n\psi = 0 \quad \text{for every } n > 0. \tag{1.68}$$

The associated vertex operator $V(\psi, z)$ is called a primary field.

Obviously, a primary state is also a quasi-primary state. Even in this case, $\mathcal{F}$ splits into a direct sum of Virasoro representations, each one generated by the action of L_n, $n < 0$, on a primary state. It is useful to observe that it is sufficient to check Equation (1.68) only for $n = 1, 2$, as it will be inductively satisfied. Indeed, if $m > 1$, $[L_1, L_m]$ is proportional to L_{m+1}, hence if Equation (1.68) holds for $n = 1, \ldots, m$, it hold also for L_{m+1}, since

$$L_{m+1}\psi \propto [L_1, L_m]\,\psi = 0.$$

Let us focus now our attention back to the vertex operator associated to a field ψ of conformal weight h_ψ: we define its *moments* $V_n(\psi)$ as

$$V(\psi, z) = \sum_{n\in\mathbb{Z}} V_n(\psi) z^{-n-h_\psi}. \tag{1.69}$$

The Property 1.3.2 translates into

$$V_{-h_\psi}(\psi)\,|0\rangle = \psi, \quad V_n\,|0\rangle = 0 \quad \text{for } n > -h_\psi. \tag{1.70}$$

In a similar way, Equation (1.66) becomes

$$[L_0, V_n(\psi)] = -nV_n(\psi). \tag{1.71}$$

Consider the fields ψ, ϕ, with conformal weights h_ψ, h_ϕ. We can deduce the OPE associated to $V(\psi, z)V(\phi, w)$ from the duality theorem:

$$\begin{aligned} V(\psi, z)V(\phi, w) =&V(V(\psi, z-w)\phi, w) \\ &=V(\psi, z) = \sum_{n\in\mathbb{Z}} z^{-n-h_\phi-h_\psi} V(\phi_n, w), \end{aligned}$$

where $\phi_n = V_{h_\phi - n}(\psi)\phi$. Observe that the terms of the sum associated with $n < 0$ are acually null: in fact,

$$\begin{aligned} L_0\phi_n =&L_0 V_{h_\phi-n}(\psi)\phi = \left\{\left[L_0, V_{h_\phi-n}(\psi)\right] + V_{h_\phi-n}(\psi)L_0\right\}\phi \\ =&\{-(h_\phi - n) + h_\phi\}V_{h_\phi-n}\phi = n\phi_n, \end{aligned}$$

hence we should require $\phi_n = 0$ if $n < 0$, otherwise we would have a negative conformal weight for the state $\phi_n |0\rangle$.

If we compute the OPE $T(z)T(w)$, where $T(z) = V(\psi_L, z)$ and $\psi_L = L_{-2}|0\rangle$, we have only four singular terms, associated to $V_{2-n}(\psi_L)\psi_L$, $n = 0, 1, 2, 3$. It is possible to work out them easily, remembering that $V_m(\psi_L) = L_m$, and if we do so the OPE assume the familiar form

$$T(z)T(w) = \frac{(2\|\psi_L\|)/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + O(1). \tag{1.72}$$

If we remember that L_n are a representation of a Virasoro algebra with central charge c, we deduce $2\|\psi_L\| = c$, and hence we recover the usual OPE for the product of the stress-energy tensor with itself.

1.4 Examples

In this Section we present a brief showcase of examples, with a double aim: to explain how the general theory we have exposed applies to concrete simple cases, and to present the building blocks of more complicated and interesting applications. We will start with a discussion of the free bosonic theory, in which we will recognize many of the feature we have discussed from a general point of view in the past Sections. While the bosonic case will be studied in a quite complete manner, the other examples (free Majorana fermions and $b-c$ ghost system) will be discussed only from the point of view of the OPEs between the fields of the theory.

1.4.1 Free boson

As a Quantum Field Theory, a Conformal Field Theory describe massless fields, since the presence of a massive term would break the dilatation invariance. The easiest possible exampe is the free, massless boson. Consider such a theory in $\mathbb{R}^2$, defined by the action:

$$S = \frac{g}{2}\int dx^2\, \partial_\mu\phi(x)\partial^\mu\phi(x).$$

The two-point correlation function for this theory is, up to a constant,

$$\langle\phi(x)\phi(y)\rangle = -\frac{1}{4\pi g}\log{(x-y)^2},$$

or, in complex coordinates,

$$\langle \phi(z,\bar{z})\phi(w,\bar{w})\rangle = -\frac{1}{4\pi g}\left(\log\left(z-w\right)+\log\left(\bar{z}-\bar{w}\right)\right). \tag{1.73}$$

If we derive the fields for z, w or $\bar{z}, \bar{w}$, we discover that the holomorphic and antiholomorphic degree of freedom splits:

$$\partial_z\phi(z,\bar{z})\partial_w\phi(w,\bar{w}) \sim -\frac{1}{4\pi g(z-w)^2}, \tag{1.74}$$

$$\partial_{\bar{z}}\phi(z,\bar{z})_{\bar{w}}\phi(w,\bar{w}) \sim -\frac{1}{4\pi g(\bar{z}-\bar{w})^2}. \tag{1.75}$$

The symmetry of these OPEs under the exchange of z, w reflects the bosonic nature of the fields. As usual, we can write the stress-energy tensor with the help of Noether theorem, and the form of its holomorphic component in complex coordinates is

$$T(z) = g\colon \partial\phi\partial\phi\colon . \tag{1.76}$$

In the quantum theory, its expression is normal ordered, as we impose the vanishing of its vacuum expectation value. Thanks to Wick theorem and OPE (1.74), writing down the following two relevant OPEs is just a simple exercise:

$$T(z)\partial\phi(w) \sim -\frac{1}{2\pi g(z-w)^2}, \tag{1.77}$$

$$T(z)T(w) \sim \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{1.78}$$

We will now consider bosons on a cylinder of circumference L, and apply the radial quantization procedure to them (we will always work in the Heisemberg picture). As expectable, we will quantize the system by writing it as a sum of decoupled harmonic oscillators. Consider periodic bounday conditions: $\phi(x+L,t) = \phi(x,t)$. We can hence Fourier expand the bosonic field in the space variable:

$$\phi(x,t) = \sum_{n\in\mathbb{Z}} e^{\frac{2\pi i n}{L}}\phi_n(t), \tag{1.79}$$

$$\phi_n(t) = \frac{1}{L}\int dx\, e^{-\frac{2\pi i n}{L}}\phi(x,t). \tag{1.80}$$

Notice that, since $\phi(x,t)$ is real, $\phi_{-n}(t)$ is the complex conjugate of $\phi_n(t)$. The lagrangian of the theory,

$$L = \frac{g}{2}\int dx\, [(\partial_t\phi(x,t))^2 - (\partial_x\phi(x,t))^2],$$

now reads

$$L = \frac{gL}{2}\sum_{n\in\mathbb{Z}}\left[\dot{\phi}_n\dot{\phi}_{-n} - \left(\frac{2\pi n}{L}^2\phi_n\phi_{-n}\right)\right]. \tag{1.81}$$

The momentum canonically conjugated to ϕ_n is

$$\pi_n = gL\dot{\phi}_{-n},$$

and performing a Legendre transformation we get the Hamiltonian

$$H = \frac{1}{2gL}\sum_{n\in\mathbb{Z}}\left[\pi_n\pi_{-n} + (2\pi n g)^2\phi_n\phi_{-n}\right]. \tag{1.82}$$

We have succeded in describing the systems as a sum of decoupled harmonic oscillators, with frequencies $\omega_n = \frac{2\pi|n|}{L}$. Note that the presence of a null frequency is related to the fact that the mass term is null. Observe also that the Hamiltonian of the system does not depend on ϕ_0, and this classically implies that π_0 is a constant of motion (this will hold also in the quantum theory). In the quantum theory, the caninical variables ϕ_n, π_n (at fixed time $t = 0$) are promoted to operators, with $\phi_{-n}^\dagger = \phi_n$, $\pi_{-n}^\dagger = \pi_n$, and we require the commutation relation $[\phi_n, \pi_m] = i\delta_{nm}$. We search for suitable creation and annihilation operators: for $n \in \mathbb{Z}$, we define

$$a_n = \frac{1}{\sqrt{4\pi g}}(\pi_{-n} - 2\pi i g n \phi_n), \quad \tilde{a}_n = \frac{1}{4\pi g}(\pi_n - 2\pi i g n \phi_{-n}). \tag{1.83}$$

Observe that $\tilde{a}_0 = a_0 = \frac{\pi}{\sqrt{4\pi g}}$. These operators satisfy:

$$[a_n, a_m] = n\delta_{n+m,0}, \quad [a_n, \tilde{a}_m] = 0, \quad [\tilde{a}_n, \tilde{a}_m] = n\delta_{n+m,0}.$$

The Hamiltonian of the system can be rewrited as

$$H = \frac{2\pi}{L}\left[\sum_{n\in\mathbb{Z}}\left(a_{-n}a_n + \frac{a_0^2}{2}\right) + \sum_{n\in\mathbb{Z}}\left(\tilde{a}_{-n}\tilde{a}_n + \frac{\tilde{a}_0^2}{2}\right)\right]. \tag{1.84}$$

We will return to the form of this Hamiltonian when we will discuss the representation of Virasoro algebra in this theory.

From the commutation relation

$$[H, a_{-n}] = \frac{2\pi}{L} n a_n$$

we have that when we apply a_{-n}, $n > 0$, to an eigenstate of H of eigenvalue E, we get another eigenstate of H, with eigenvalue $E + \frac{2n\pi}{L}$. We can rewrite the mode expansion (1.79) at time $t = 0$ in terms of a_n and $\tilde{a}_n$, and obtain the field operator at arbitrary time t by the usual time evolution rule. What we get is the following:

$$\phi(x,t) = \phi_0 + \frac{t}{gL}\pi_0 + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}(a_n e^{\frac{2\pi i n(x-t)}{L}} - \tilde{a}_{-n} e^{\frac{2\pi i n(x+t)}{L}}). \tag{1.85}$$

Replacing t with $-i\tau$, we now go to the Euclidean space, where we adopt the complex coordinates (1.39):

$$\phi(z,\bar{z}) = \phi_0 - \frac{i}{4\pi g}\pi_0 \log(z\bar{z}) + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}(a_n z^{-n} + \tilde{a}_n \bar{z}^{-n}). \tag{1.86}$$

Deriving:

$$i\partial\phi(z) = \frac{1}{\sqrt{4\pi g}}\sum_n a_n z^{-1-n}, \tag{1.87}$$

and analogously for the antiholomorphic derivative. The reason behind the decoupling of the holomorphic and antiholomorphic degree of freedom lies in the periodic boundary conditions of the theory. With a quite pictorical, yet arbitrary, terminology, we interpret a_n as the creation/annihilations operators of right-moving excitations, and $\tilde{a}_n$ as the creation/annihilations operators of left-moving excitations. Now, we can write the stress-energy tensor of the theory in terms of creation and annihilation operators, and by means of Equation (1.47) we can recognize Virasoro algebra. Straightforwardly, the holomorphic component of the stress-energy tensor is:

$$T(z) = -2\pi g \colon \partial\phi(z)\partial\phi(z) \colon = \frac{1}{2}\sum_{n,m\in\mathbb{Z}} z^{-2-n-m} \colon a_n a_m \colon , \tag{1.88}$$

and hence Virasoro algebra is represented by the operators

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} a_{n-m}a_m, \quad L_0 = \sum_{n>0} a_{-n}a_n + \frac{1}{2}a_0^2. \tag{1.89}$$

As we have anticipated, the Hamiltonian is proportional to $L_0 + \bar{L}_0$:

$$H = \frac{2\pi}{L}(L_0 + \bar{L}_0). \tag{1.90}$$

Let us have a glimpse of the state space of the theory. As we have said, π_0 (and, equivalently, $a_0 = \bar{a}_0$) is still a constant of motion in the quanum theory, since it commutes with the Hamiltonian. Hence, its continuous eigenvalue α, that we interpret as the momentum of the center of mass of the system, is a good quantum number. Also, creation and annihilation operators commute with π_0, hence after the application of a creation or annihilation operator on an eigenstate of π_0, we will find an eigenstate associated to the same eigenvalue. This means that we can consider a Fock space build from a continuous, one (real) parameter, family of ground states $|\alpha\rangle$ that are eigenstates of a_0 associated to the eigenvalue α. We have:

$$a_0 |\alpha\rangle = \alpha |\alpha\rangle , \quad a_n |\alpha\rangle = \bar{a}_n |\alpha\rangle = 0 \quad \text{for } n > 0, \tag{1.91}$$

meaning that $a_n, \bar{a}_n$, $n > 0$, are destruction operators. We can also state that α is the highest weight state with respect to the creation/annihilation operator algebra. The Fock space is built by applying creation operators to the ground states: the generic state is

$$a_{-1}^{n_1}a_{-2}^{n_2}\ldots\bar{a}_{-1}^{\bar{n}_1}\bar{a}_{-2}^{\bar{n}_2}\ldots|\alpha\rangle , \tag{1.92}$$

with $n_i, \bar{n}_j$ non-negative integers. The conformal dimensions of this generic Fock state, i.e. the eigenvalues $h, \bar{h}$ of $L_0, \bar{L}_0$, are

$$h = \frac{\alpha^2}{2} + \sum_k kn_k, \quad \bar{h} = \frac{\alpha^2}{2} + \sum_k k\bar{n}_k \tag{1.93}$$

Finally, let us consider an explicit description of Vertex operators in this theory. Consider the family, labeled by $\alpha \in \mathbb{R}$, of field defined by

$$V_\alpha(z,\bar{z}) = \colon e^{i\alpha\phi(z,\bar{z})} \colon$$

From an identity that comes from the application of Hadamard formula,

$$\colon e^{a\phi_1} \colon \colon e^{b\phi_2} \colon = e^{ab\langle\phi_1\phi_2\rangle} \colon e^{a\phi_1 + b\phi_2} \colon$$

and OPE (1.73) it can be deduced that the OPE for the Vertex operator product is

$$V_\alpha(z,\bar{z})V_\beta(w,\bar{w}) \sim [(z-w)(\bar{z}-\bar{w})]^{\frac{\alpha\beta}{4\pi g}} V_{\alpha\beta}(w,\bar{w})\delta_{\alpha+\beta,0} + \ldots \tag{1.94}$$

The presence of the factor $\delta_{\alpha+\beta,0}$ comes from general requirements to the two-point correlation functions. Moreover, it can be shown that the ground states $|\alpha\rangle$ can be obtained from the vacuum of the theory $|0\rangle$ by applying the Vertex operator V_α:

$$V_\alpha(0,0)|0\rangle = |\alpha\rangle . \tag{1.95}$$

1.4.2 Free Majorana fermion

The action of two dimensional, Majorana fermionic theory in Euclidean space is

$$S = \frac{1}{2}g \int d^2x \, \Psi^\dagger \gamma_0 \gamma^\mu \partial_\mu \Psi, \tag{1.96}$$

where the matrices γ^μ, $\mu = 0, 1$, form a representation of Clifford algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$$

A possible choice is

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The classical equation of motion for Ψ is $\gamma^0 \gamma^\mu \partial_\mu \Psi = 0$. If we remember the definition of ∂_z and $\partial_{\bar{z}}$ and write the doublet Ψ as $(\psi, \bar{\psi})$, we discover that this equation is nothing more than

$$\partial_{\bar{z}} \psi = 0, \quad \partial_z \bar{\psi} = 0,$$

i.e. ψ and $\bar{\psi}$ are respectively holomorphic and antiholomorphic.

For fermions on a cylinder, we can choose two possible boundary conditions: Ramond (R) condition:

$$\psi(x + 2\pi L, t) = \psi(x, t), \tag{1.97}$$

or Neveu-Schwarz (NS) condition:

$$\psi(x + 2\pi L, t) = -\psi(x, t). \tag{1.98}$$

The boundary condition chosen affects the mode expansion: in general, we have

$$\psi(z) = \sum_{r \in \mathbb{Z} + \nu} \psi_r z^{r + \frac{1}{2}}, \tag{1.99}$$

where $\nu = 0$ for Ramond boundary condition and $\nu = \frac{1}{2}$ for NS boundary condition.

Remark 1.4.1. The conformal dimension of ψ is $\frac{1}{2}$, and not 0 as in the bosonic case. This implies that the field is affected by the map from the cylinder to the complex plane.

Using gaussian integration, we compute the two point correlation function of the theory, and we discover that the OPE between the fermionic fields is

$$\psi(z)\psi(w) \sim \frac{1}{2\pi g(z - w)} \tag{1.100}$$

As we can suspect from the different mode expansions, the two-point correlation functions are different for different periodicity conditions. However, in the limit $z \to w$ they both take the form above. Analogously to the bosonic case, the OPE reflects the fermionic nature of the fields involved.

From Noether theorem, we can deduce the stress-energy tensor, and if we put it in complex coordinates we find that its holomorphic component is

$$T(z) = -\pi g \colon \psi(z) \partial \psi(x), \tag{1.101}$$

and thanks to Wick theorem we obtain the following OPEs:

$$T(z)\psi(w) \sim \frac{\frac{1}{2}\psi(w)}{(z-w)^2} + \frac{\partial\psi(w)}{z-w}, \tag{1.102}$$

$$T(z)T(w) \sim \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{1.103}$$

We remark that Equation (1.102) tell us that ψ is a primary field of conformal dimension $\frac{1}{2}$. Observe that the semi-integer value for the conformal dimension would not have been allowed in a bosonic theory.

1.4.3 Ghost system

The $b-c$ ghost systems,

$$S = \frac{g}{2}\int d^2x\, b_{\mu\nu}\partial^\mu c^\nu, \tag{1.104}$$

notably appears in the covariant quantization of String theory, as a result of some change of variables in a fermionic integral. $b_{\mu\nu}$ and c^μ are anticommuting fields, and $b_{\mu\nu}$ is a traceless symmetric tensor. Their equations of motion are

$$\partial^\mu b_{\mu\nu} = 0, \quad \partial^\mu c^\nu + \partial^\nu c^\mu - \eta^{\mu\nu}\partial_\lambda c^\lambda = 0.$$

If we choose the complex coordinates $z, \bar{z}$, we have that the traceless antisymmetric tensor has only two components different from zero, $b = b^{zz}$ and $\tilde{b} = b^{\bar{z}\bar{z}}$. If we define $c = c^z, \tilde{c} = c^{\bar{z}}$, the equations of motion become

$$\bar{\partial} b = 0, \quad \partial\tilde{b} = 0, \tag{1.105}$$

$$\bar{\partial} c = 0, \quad \partial\tilde{c} = 0. \tag{1.106}$$

If we compute the two point correlation function, we discover the OPE

$$b(z)c(w) \sim \frac{1}{\pi g(z-w)}. \tag{1.107}$$

We can introduce a symmetric traceless stress-energy tensor, whose holomorphic component in complex coordinates reads

$$T(z) = \pi g\colon (2\partial c b + c\partial b)\colon. \tag{1.108}$$

Again, using Wick theorem one can deduce the following OPEs:

$$T(z)b(w) \sim \frac{2b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w}, \tag{1.109}$$

$$T(z)c(w) \sim \frac{-c(w)}{(z-w)^2} + \frac{\partial c(w)}{z-w}, \tag{1.110}$$

$$T(z)T(w) \sim \frac{-26/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}. \tag{1.111}$$

Chapter 2

Elements of String and Superstring Theory

String Theory is a consistent theoretical framework that provides a quantum description of Physics, including gravity. Heuristically speaking, the foundamental objects of the theory are *strings*, one-dimensional objects. Strings enjoy more degrees of freedom than point particles, as they can vibrate. According to the theory, the great variety of particles that we observe in Nature comes from the massless string modes, that after compactification may acquire mass through several mechanisms (Higgs mechanism, non-perturbative effects...). At an energy scale higher than the one experimentally investigated today, the tower of states that emerges from the spectrum of string oscillations should appear. There are many others remarkable points that make String Theory interesting. For example, the number of space-time dimensions come directly from the theory, as a self-consistency requirement. Also, String Theory is safe from many of the divergences that we come across in Quantum Field Theory, as the interaction between more-than-zero-dimensional objects is less localized that point-particle interaction. Moreover, the study of String Theory has brought to light many concepts and tools, that have found applications in other areas of Physics and even Mathematics. Last but not least, String Theory, as we will discuss soon, is strictly correlated with two dimensional Conformal Field Theory: motivations and applications of the topics of this Thesis can be found inside the playground of this fertile theory.

A comprehensive and detailed exposition of String and Superstring Theory is well beyond the scopes and the possibilities of this Chapter. What we present here is a general introduction to the theory, as well as an excursion through selected topics, that will be relevant for the aims of this Thesis. We will start with a review of Bosonic String Theory, its quantization and its spectrum. We will discuss the theoretical problems that force us to move to Superstrings, in order to obtain a coherent description of Physics, and we will briefly discuss the various consistent Superstring Theories (type I, type IIA, type IIB, and the two Heterotic theories).

As completeness is not among the scopes of this Chapter, we will not delve deep inside the various topics presented here. There exist many excellent textbooks and notes that are devoted to String and Superstring theories, for exampe [17], [18] and [2]. For a comprehensive presentation of these subject, we recommend them as references.

2.1 Bosonic String Theory

The Bosonic theory is the simplest example of String Theory. Despite being, as we will discuss, an unsatisfactory theory under some points of view, it has a great didactical value as a toy model. Moreover, many of the typical stringy features that are also common to more realistic theories yet emerge in the Bosonic theory. Starting from a suitable action, we will discuss the Bosonic string, both classical and quantum, underlining the connection to two dimensional Conformal Field Theory.

2.1.1 Classical Bosonic String

The starting point of our journey through String Theory is the introduction of a suitable action for more-than-zero-dimensional objects. String are object that moves inside a D-dimensional flat Minkowski space-time, parametrizated by coordinates X^μ. Following the main String textbooks, we will adopt the mostly positive metric convention, $\eta_{\mu\nu} = (-,+,+,\dots,+)$. Point-like particles can be described with one single parameter (the proper time), as they draw curves, *wordlines*, while they move through space-time. For one-dimensional objects, we need two parameters, that are conventionally called τ and σ, as they describe surfaces in their motion, called *worldsheets*. With a slight abuse of notation, we will also refer to the domain of (τ, σ) with the term worldsheet. Conventionally, an index written as a latin letter is a worldsheet index, and we will often refer to worldsheet parameters as σ^a, $a = 0, 1$, with $\sigma^0 = \tau$ and $\sigma^1 = \sigma$. The objects that represent foundamental strings are then maps $X^\mu(\tau, \sigma)$, with $\mu = 0, \dots, D-1$, and an appropriate action should be a functional of $X^\mu(\tau, \sigma)$. Reminescent of the parametrization invariance of point-particle theories, we require that the action of the theory depends only on the embedding in space-time of the world-sheet. A correct guess is that such action is proportional to the area swept by the string, and straightforwardly walking this way we will get the Nambu-Goto action. We will however introduce an equivalent action, namely Brink-Di Vecchia-Howe-Deser-Zumino or Polyakov action. Despite being less physically intuitive, such action is much more neat and useful for the quantization of the theory, at the price of introducing a worldsheet metric $\gamma_{ab}(\tau, \sigma)$ as an auxiliary field. We will consider the signature of γ_{ab} as $(-,+)$, and if we call γ the determinant of γ_{ab}, the expression of Polyakov action is

$$S_P[X, \gamma] = -\frac{1}{4\pi\alpha'} \int_M d\tau d\sigma \left(\sqrt{-\gamma}\right) \gamma^{ab} \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu. \tag{2.1}$$

where α' is a constant and M denotes the worldsheet. This action enjoys many symmetries:

- Poincaré (space-time) invariance:

$$X'^\mu(\tau, \sigma) = \Lambda^\mu_\nu X^\nu(\tau, \sigma) + a^\mu, \tag{2.2}$$
$$\gamma'_{ab}(\tau, \sigma) = \gamma_{ab}(\tau, \sigma), \tag{2.3}$$

  with $\Lambda \in O(1, D-1)$ and a constant.

- Diffeomorphism invariance:

$$\sigma'^a = \sigma'^a(\tau, \sigma), \tag{2.4}$$
$$X'^\mu(\tau', \sigma') = X^\mu(\tau, \sigma), \tag{2.5}$$
$$\partial_a \sigma'^c \partial_d \sigma'^c \gamma'_{cd}(\tau', \sigma') = \gamma_{ab}(\tau, \sigma), \tag{2.6}$$

where $\sigma'^a(\tau,\sigma)$ is an arbitrary diffeomorphism.

- Weyl invariance:

$$X'^\mu(\tau,\sigma) = X^\mu(\tau,\sigma), \tag{2.7}$$
$$\gamma'_{ab}(\tau,\sigma) = e^{2\omega(\tau,\sigma)}\gamma_{ab}(\tau,\sigma), \tag{2.8}$$

with $\omega(\tau,\sigma)$ an arbitrary function.

Polyakov action defines a two-dimensional (Conformal) Quantum Field Theory on the worldsheet. The variation of the action with respect to the worldsheet metric give us the following stress-energy tensor:

$$T^{ab} = -\frac{1}{\alpha'}(\partial^a X^\mu \partial^b X_\mu - \frac{1}{2}\gamma^{ab}\partial_c X^\mu \partial^c X_\mu). \tag{2.9}$$

As a consequence of Weyl invariance, we have that the stress-energy tensor is traceless. $T^a_a = 0$. The variation of S with respect to γ and X bring (respectively) the classical equations of motion

$$T_{ab} = 0, \tag{2.10}$$
$$\partial_a\left[\left(\sqrt{-\gamma}\right)\gamma^{ab}\partial_b X^\mu\right] = \left(\sqrt{-\gamma}\right)\partial^\sigma\partial_\sigma X^\mu. \tag{2.11}$$

To obtain such equation, we have required that the surface term that comes out in the variation of S does vanish. If we consider τ as a "time" variable and σ as a "space" variable, it is reasonable to conside the coordinate region

$$-\infty < \tau < +\infty, \quad 0 < \sigma < L$$

To ensure the vanishing of the boundary terms, we may require Neumann boundary conditions (open strings):

$$\partial_a X^\mu(\tau,0) = \partial_a X^\mu(\tau,L) = 0, \tag{2.12}$$

or periodic boundary conditions (closed strings):

$$X^\mu(\tau,0) = X^\mu(\tau,L), \tag{2.13}$$
$$\partial_a X^\mu(\tau,0) = \partial_a X^\mu(\tau,L), \tag{2.14}$$
$$\gamma^{ab}(\tau,0) = \gamma^{ab}(\tau,L). \tag{2.15}$$

These are the only two boundary conditions consistent with Poincaré invariance and equations of motion. It is relevant to mention also Dirichlet boundary condition for open strings, i.e. fixed open string endpoints. This boundary condition has very important applications inside the theory, such as D-branes.

For the purposes of a path integral formulation, we will consider the Euclidean version of the theory, connected to the Minkowski one by analitic continuation. Precisely, we will have a positive defined metric $g(\sigma^1,\sigma^2)$ on the worldsheet, and the Euclidean action is

$$S_P{}^E = \frac{1}{4\pi\alpha'}\int_M d^2\sigma\ (\sqrt{g})\, g^{ab}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu.$$

In the future, we will omit the label E, as the Euclidean formulation will be understood.

We may ask whether Polyakov action is the most general action that enjoys its symmetries. The answer is no: we can include two extra terms permitted by the symmetries:

$$S'_P = S_P + \lambda\chi, \tag{2.16}$$

with

$$\chi = \frac{1}{4\pi}\int_M d^2\sigma\ (\sqrt{g})\,R + \frac{1}{2\pi}\int_{\partial M} ds\,k, \tag{2.17}$$

where R is the Ricci scalar associated to the metric g, and

$$k = -t^a n_b \nabla_a t^b \tag{2.18}$$

where t^a is a unit vector tangent to the boundary and n^a is a unit, outward pointing vector orthogonal to t^a. The variation of the integrand of the first term of χ under a local Weyl is a total derivative. The surface term is included in the case of a worldsheet with boundaries, in order to write an invariant expression.

2.1.2 Quantization

Everything until now was strictly classical. Now, we try to quantize the theory, following the BRST quantization method. This is not the only possible choice, but it enjoys several conceptual advantages. Remarkably, this approach is an explicitly covariant quantization procedure, and works more or less the same as in general gauge theories. First, we will rephrase Polyakov action, following the Faddev-Popov formalism. Indeed, if we want to define a path-integral from Polyakov action, we may be tempted to write

$$Z = \int [dX\,dg] e^{-S}.$$

Just as in gauge theory, this object is tremendously ill defined, since we are integrating over equivalent configurations, connected by transformations belonging to the group diff $\times$ Weyl. This group plays the role of a local gauge symmetry group. Faddev-Popov methods help us to get rid of this overcounting, by factorizing the path integral in an integration over gauge-inequivalent configurations and in the (infinite) volume of the group diff $\times$ Weyl. First of all, we have to choose a gauge-fixing condition. It can be shown that there is enough gauge freedom to eliminate the integration over the metric, fixing it to a certain *fiducial metric* $\hat{g}$. A possible choice is the unit metric:

$$\hat{g}_{ab} = \delta_{ab},$$

or, if we want to employ the only diffeomorphism invariance to reduce the gauge freedom, the covariant metric:

$$\hat{g}_{ab} = e^{2\omega(\sigma)}\delta_{ab}.$$

Remark 2.1.1. There is anyway a residual symmetry freedom. In the case of the unit metric, such gauge freedom coincides exactly with holomorphic reparametrization - that is, conformal invariance. The conformal symmetry emerges as the subgroup of the group diff $\times$ Weyl that preserves the unit metric.

After the choice of the fiducial metric, we can follow the Faddev-Popov method to separate the functional integration on the gauge-inequivalent configurations and the volume of the gauge group: as said above, the calculation is nearly identical to the one performed in the case of Yang-Mills theories. Also in this case, the price to pay is the introduction of *reparametrization ghosts*, namely the Grassmann fields c^a and b_{ab}, with the latter being traceless. We have that c^a corresponds to infinitesimal reparametrizations and b_{ab} to variations perpendicular to the gauge slice. The Polyakov path integral is turned into:

$$Z[\hat{g}] = \int [dX\, db\, dc] e^{-S'_P - S_g}, \tag{2.19}$$

where

$$S_g[b, c] = \frac{1}{2\pi} \int d^2\sigma \left(\sqrt{\hat{g}}\right) b_{ab} \nabla^a c^b. \tag{2.20}$$

It is possible to give a path integral formulation of the gauge fixing, by the introduction of the gauge-fixing action S_{gf},

$$Z = \int [dX\, db\, dc\, dB\, d\gamma]\, e^{-S'_P - S_g - S_{gf}}, \tag{2.21}$$

where

$$S_{gf}[B, \gamma] = \frac{i}{4\pi} \int d^2\sigma\, (\sqrt{g})\, B^{ab}(\delta_{ab} - g_{ab}) \tag{2.22}$$

Observe that the ghost action is the same one considered in Section 1.4.3.

Looking back to the procedure we have outlined and to Equation (2.19), we may be tempted to inquire if the path integral is truly invariant under different choices for the fiducial metric. Indeed, the answer is *almost never*: it holds only for a certain space-time background. This consistency condition is actually the one that fixes the number of space-time dimensions D of our theory. Explicitly, such independece from the fiducial metric chosen is translated into the condition

$$\langle \mathcal{O} \rangle_{g'} = \langle \mathcal{O} \rangle_g \tag{2.23}$$

with

$$\langle \mathcal{O} \rangle_g = \int [dX\, db\, dc] e^{-S[X,b,c,g]} \mathcal{O}, \tag{2.24}$$

where $\mathcal{O}$ is some fields expression. We are considering a non-trivial metric on the worldsheet, in general: in such a theory, it is known that the stress-energy tensor can be defined as an operator by the infinitesimal variation of the path integral (2.24) with respect to the metric:

$$\delta \langle \mathcal{O} \rangle_g = -\frac{1}{4\pi} \int d^2\sigma \left(\sqrt{g(\sigma)}\right) \delta g_{ab}(\sigma) \langle T^{ab}(\sigma) \mathcal{O} \rangle_g. \tag{2.25}$$

If we specialize to an infinitesimal Weyl transformation, we end up with

$$\delta_{\text{Weyl}} \langle \mathcal{O} \rangle_g = -\frac{1}{2\pi} \int d^2\sigma \left(\sqrt{g(\sigma)}\right) \delta\omega(\sigma) \langle T^a_a(\sigma) \mathcal{O} \rangle_g.$$

If the stress-energy tensor is traceless, the path integral is invariant under Weyl transformations. In a flat worldsheet theory, this is verified. Moreover, the trace of the stress-energy tensor should be Poincaré- and diff-invariant, because these symmetries have to be preserved as well. To satisfy these requirements, the trace of stress-energy tensor have to be proportional to the Ricci scalar R that comes from the worldsheet metric:

$$T^a_a = AR,$$

and we should require that the proportionality constant A vanishes. After some calculation, one discovers that A is proportional to the total central charge of the flat-metric worldsheet theory.

We can see now that the question has been turned into a Conformal Field Theory problem - that is, the total central charge of the theory has to vanish. Recalling the examples of Section 1.4, we have that the sum of the conformal charges of the bosonic field X^μ and of the ghost system is $D-26$, and hence Weyl-invariance of the theory entails $D=26$.

In general, the value of the trace of the stress-energy tensor is called Weyl or conformal anomaly. The term *anomaly* is used to refer to something that, if different from zero, precludes the preservation of a certain symmetry at the quantum level.

2.1.3 BRST symmetry and Hilbert Space

Proceding with the analogy with the quantization of gauge theories, we are ready to discuss the state space of the theory. Ghost fields apparently violate the spin-statistic Theorem, and thus we expect no ghosts inside the physical state space. Moreover, the Hilbert space product defined in the state space should be positive defined. The problem of finding a condition that can tell the difference between physical and unphysical states was addressed also in Yang-Mills theories (even in the abelian case, namely QED). Conceptually, there is little difference, as we can emply the same theoretical machinery to deal with this issue, introducing the BRST symmetry and its conserved charge. BRST symmetry acts on the fields of the theory as follows:

$$\delta X^\mu = i\epsilon(c\partial + \tilde{c}\bar{\partial})X^\mu \tag{2.26}$$

$$\delta b = i\epsilon(T^X + T^g) \tag{2.27}$$

$$\delta c = i\epsilon c\partial c \tag{2.28}$$

$$\delta\tilde{b} = i\epsilon(\bar{T}^X + \bar{T}^g) \tag{2.29}$$

$$\delta\tilde{c} = i\epsilon\tilde{c}\bar{\partial}\tilde{c}, \tag{2.30}$$

where T^X and T^g are the stress-energy tensor associated to (respectively) the bosonic field and the ghost system. It is a symmetry of the theory, and its parameter ϵ should be Grassmannian, in order to preserve the commuting or anticommuing nature of the expressions that appear at both sides of the equal sign. There is a conserved ghost number, whose value is -1 for b and ϵ, $+1$ for c and 0 for the other fields. The holomorphic component of Noether current associated to this symmetry is

$$j = cT^X + \frac{1}{2} : cT^g : + \frac{3}{2}\partial^2 c, \tag{2.31}$$

and the conserved charge is

$$Q = \frac{1}{2\pi i}\oint (dz\, j + d\bar{z}\,\tilde{j}). \tag{2.32}$$

It can be shown that Q is nilpotent, i.e. $Q^2=0$, if $D=26$.

As in Yang-Mills theory, from imposing the gauge-invariance of the scattering amplitudes one obtains the condition

$$Q\left|\psi\right\rangle = 0 \tag{2.33}$$

on physical states $\left|\psi\right\rangle$. Let us refer the space of the states that obey condition (2.33) with $\tilde{\mathcal{H}}$. Observe that states whose form is $Q\left|\phi\right\rangle$ trivially satisfy this relation, but their norm is null. They form a subspace of $\tilde{\mathcal{H}}$, that we will call $\tilde{\mathcal{H}}_0$.

The states $\left|\psi\right\rangle$ and $\left|\psi\right\rangle + Q\left|\phi\right\rangle$ are equivalent. Consider a physical state $\left|\psi'\right\rangle$, then

$$\left\langle\psi'\right|(\left|\psi\right\rangle + Q\left|\phi\right\rangle) = \left\langle\psi'\right\rangle\psi + \left\langle\psi'\right|Q\left|\phi\right\rangle = \left\langle\psi'\right\rangle\psi.$$

This hints that the right definition of the physical Hilbert space of the theory is:

$$\mathcal{H} = \overline{\tilde{\mathcal{H}}/\tilde{\mathcal{H}}_0}, \tag{2.34}$$

that is what with a more mathematical terminology is called the cohomology of Q.

Remark 2.1.2. Since $\tilde{\mathcal{H}}/\tilde{\mathcal{H}}_0$ is a pre-Hilbert space, we have to consider its completion with respect to the norm induced by its inner product, $\overline{\tilde{\mathcal{H}}/\tilde{\mathcal{H}}_0}$, to end up with an Hilbert space.

2.1.4 Spectrum of the Bosonic String

Once the theory is quantized and we have identified the Hilbert space of the theory, we can study its spectrum. This is a crucial step, since it allow us to make contact with the observed Nature. Moreover, this study will enlight the flaws of the Bosonic theory, and the necessity to go further. We will start with the description of open string spectrum.

Neumann boundary condition translates into the following constraint for the modes of the open string expansion

$$\alpha_n^\mu = \tilde{\alpha}_n^\mu. \tag{2.35}$$

The mode expansion of the open string hence become

$$X^\mu(z,\bar{z}) = x^\mu - i\alpha' p^\mu \log\left(|z|^2\right) + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{\alpha_n^\mu}{n}\left(\frac{1}{z^n} + \frac{1}{\bar{z}^n}\right). \tag{2.36}$$

There are also boundary conditions for the ghost fields, although the BRST formalism is not the simplest framework to discuss them. Anyway, expressed in terms of modes, these are

$$c_n = \tilde{c}_n, \quad b_n = \tilde{b}_n. \tag{2.37}$$

We have seen that physical states must obey the condition $Q\,|\psi\rangle = 0$. However, this is not the only constraint that the states that appear in the physical spectrum of the theory have to obey. Indeed, we have to require that the physical states are annihilated by the zero mode of the b ghost:

$$b_0\,|\psi\rangle = 0. \tag{2.38}$$

This is a kinematic condition, and the theory of string scattering amplitudes gives a proper explanation for this constraint (and also for the analogous one we present for the closed string). Since the justification of this requirement goes beyond the scopes of this Chapter, we will accept it without proof. From this additional condition follows the so called *mass-shell condition*:

$$L_0\,|\psi\rangle = \{Q, b_0\}\,|\psi\rangle = 0, \tag{2.39}$$

where

$$L_0 = \alpha'(p^\mu p_\mu + M^2), \tag{2.40}$$

with

$$\alpha' M^2 = N - 1 = \sum_{n=1}^{\infty}\left(N_{bn} + N_{cn} + \sum_{\mu=0}^{25} N_{\mu n}\right) - 1, \tag{2.41}$$

where $N_{bn} = b_{-n}c_n$, $N_{bn} = c_{-n}b_n$ and $N_{\mu n} = \frac{1}{n}\alpha_{-n}^\mu \alpha_n^\mu$ (without summation over μ). We can write a generic state by the application of creation mode operators on an open string vacuum (with the redefinition $\alpha_n^0 = \alpha_{-n}^0$):

$$|N,k\rangle = \sum_{A\in\mathcal{A}} C_A \left[\prod_{\mu=0}^{25}\prod_{n=1}^{\infty} \frac{(\alpha_{-n}^\mu)^{N_{\mu n}}}{\sqrt{n^{N_{\mu n}} N_{\mu n}!}}\right]\left[\prod_{n=0}^{\infty}(c_{-n})^{N_{cn}}\right]\left[\prod_{n=1}^{\infty}(b_{-n})^{N_{bn}}\right]|0,k\rangle\,. \tag{2.42}$$

Where $A \in \mathcal{A}$ labels the possible choice of quantum numbers $\{N_{\mu n}, N_{cn}, N_{bn}\}$ and C_A are normalization coefficients. The quantum number N is called the level of the state. The vacuum $|0, k\rangle$ should be regarded not as the space-time vacuum of Quantum Field Theory, but as a 0-level open string state with momentum k. Observe that the Grassmannian nature of the ghosts only allow the values $\{0, 1\}$ for the quantum numbers N_{cn} and N_{bn}. Finally, the presented state does not automatically respect the physical conditions above, that should be checked "by hands".

Let us discuss now the states that appear in the open string spectrum, level by level

$N = 0$. We have to check two possible states:

$$|0, k\rangle \,, \quad c_0 \,|0, k\rangle$$

with $k^2 = \frac{1}{\alpha'}$. The second state fails to satisfy the b_0-condition, hence we drop it. The first one obeys both these conditions, then the only acceptable 0-level states have this form. For every k there is a different cohomology class. The condition $k^2 = \frac{1}{\alpha'}$ implies that these states are associated to scalar fields with squared mass $M^2 = -\frac{1}{\alpha'}$. This is obviusly a problematic point, and we will discuss it in the next Section.

$N = 1$. States at this level are massless. We have to select from the states of the form

$$|1, k\rangle = (e \cdot \alpha_{-1} + C_b b_{-1} + C_c c_{-1}) \,|0, k\rangle \,,$$

where e^μ is a 26-vector and C_b, C_c are complex constants. These are $26 + 2$ independent states. It can be shown that the physical states are only

$$|1, k\rangle = e \cdot \alpha_{-1} \,|0, k\rangle \,, \quad \text{with } k^2 = e \cdot k = 0,$$

and two states $e \cdot \alpha_{-1} \,|0, k\rangle$, $e' \cdot \alpha_{-1} \,|0, k\rangle$ are in the same cohomology class if $e'_\mu - e_\mu$ is proportional to k_μ. The vector e_μ is called *polarization vector*. These states are massless, their momenta are orthogonal to their polarization vectors and they can be written on a basis with $(26-2)$-rotational symmetry: for these reasons, they are identified with *photons*.

This analysis can be applied to the higher level states. There is an important general feature, that is valid at every level: the states that remain after projecting away the states with time- and longitudinal-$X^\mu(z)$ modes, as well as the states with ghost modes, are the physical ones. This result goes under the name of *no-ghost theorem*, and turns out to be a powerful tool in the study of the spectrum, as we can write the physical states without any computation.

Now, let us have a look at the closed string spectrum. The analysis is actually very similar to the previous case, with some important differences. First, we have to take into accout also the antiholomorphic degrees of freedom. The physical states obey

$$Q \,|\psi\rangle = 0, \quad b_0 \,|\psi\rangle = \tilde{b}_0 \,|\psi\rangle = 0, \tag{2.43}$$

that imply the mass-shell condition

$$L_0 \,|\psi\rangle = \bar{L}_0 \,|\psi\rangle = 0 \tag{2.44}$$

with

$$L_0 = \frac{\alpha'}{4}p^2 + N - 1, \tag{2.45}$$

$$\bar{L}_0 = \frac{\alpha'}{4}p^2 + \tilde{N} - 1. \tag{2.46}$$

Note the different coefficient before p^2, different from the one of the open string case: this is simply because the relation between the momentum p and the zero mode of the string is in the closed string expansion is different. From the physical conditions follows the *level-matching condition*:

$$(L_0 - \bar{L}_0)\, |N, \tilde{N}, k\rangle = (N - \tilde{N})\, |N, \tilde{N}, k\rangle = 0, \tag{2.47}$$

that means that the quantum numbers N and $\tilde{N}$ are equal for physical states. We will refer to $N = \tilde{N}$ as the *level* of the state $|N, \tilde{N}, k\rangle$. Let us briefly review the states that appear in the physical spectrum of the closed string. Again, we have the *no-ghost theorem* valid at any level: what remains after projecting out states with time- and longitudinal-$X^\mu(z)$ and $\tilde{X}^\mu(z)$ modes, as well as the states with ghost and anti-ghost modes, is the physical spectrum.

$N = 0$. Analogously to the open string case, we have only the scalar tachyon $|0, 0, k\rangle$ with $k^2 = \frac{4}{\alpha'}$.

$N = 1$. For $k^\mu = (k^0, k^0, 0, \ldots, 0)$, a basis for the $N = 1$ states is

$$\alpha^i_{-1}\alpha^j_{-1}\, |0, 0, k\rangle\,, \quad \text{with } k^2 = 0 \text{ and } i, j = 2, \ldots, 25. \tag{2.48}$$

These states transform as a 2-tensor under $SO(24)$, hence it is a reducible representation: we can decompose it into a symmetric traceless tensor, an antisymmetric tensor and a scalar. These are irreducible representations, and do not mix when we consider different inertial observers. Physically this means that the most generic level-1 closed string state is

$$\begin{aligned} &\sum_{i,j=2}^{25} C_{ij}\alpha^i_{-1}\alpha^j_{-1}\, |0, 0, k\rangle \\ &= \sum_{i,j=2}^{25} \left[\left(C_{(ij)} - \frac{1}{24}\,\mathrm{Tr}\,(C_{ij})\right) + C_{[ij]} + \frac{1}{24}\delta_{ij}\,\mathrm{Tr}\,(C_{ij})\right] \alpha^i_{-1}\alpha^j_{-1}\, |0, 0, k\rangle\,. \end{aligned} \tag{2.49}$$

The symmetric state is called the *graviton*, the antisymmetric state is called the *axion* and the scalar is called the *dilaton*.

2.1.5 What's wrong with Bosonic String?

As we have explained, tachyons do appear in both open and closed Bosonic String Theory spectrum. Tachyons are commonly considered as harbingers of inconsistance for a theory, as the presence of "faster than light" particles lead to severe violation of causality. However, the open and closed string vacua are of different nature.

To explain this point, let us consider a lagrangian describing a scalar field ϕ:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi).$$

As we know, not all stationary points are good to perturbatively expand the potential V: if we tentatively expand around a maximum, we can found excitations ("particles") whose squared mass is negative (namely, tachyons). This does not tell us that the theory is necessary inconsistent: in this case, it means only that we are expanding around a "bad" point, as tachyons are signal of instability around the stationary point considered. This is the picture of the open string vacuum: it does not represent a true minimum of the theory. The closed string vacuum is different, as nowadays it seems that there is no stable minimum in such theory. The tachyonic vacuum here is a true inconsistency.

Moreover, we have seen that the theory does not produce any fermionic particle, and this strikes against the presence of such particles in Nature. These two issues suggest us that the Bosonic theory on its own is not a good description of the Foundamental Physics, and force us to find something more realistic, namely Superstring Theory.

2.2 Superstring Theory

The issues we have highlighted show that, despite its many merits, the Bosonic String theory fails in describing reality. To exit this stalemate, a possible way is including worldsheet fermions and supersymmetry in the theory, and thus come to Superstring Theory. As we will see, this is enough to get rid of the tachyons and to produce space-time spinors.

It is known that there are only five consistent Superstring theories. However, these theories are linked by a number of non-perturbative dualities: this hints that they are actually different "perturbative" limits of a unique theory, the *M Theory*. The topics presented below will be exposed in an even more heuristic way, and are reported only to give a proper physical contextualization to the subjects discussed in the next Chapters of the present Thesis.

2.2.1 Introduction to Superstrings

We want to write a theory that, along all the symmetries of the Bosonic theory, includes local worldsheet supersymmetry, a symmetry that exchanges bosons and fermions. Our fields will be $X^\mu(\tau, \sigma)$, as in the Bosonic theory, and the two-dimensional worldsheet Majorana fermions $\psi^\mu_\alpha(\tau, \sigma)$. The worldsheet fermions have no geometrical interpretation, in the sense that they are internal degrees of freedom, like the spin, associated to every point of the worldsheet. Also the metric $g_{\mu,\nu}$ (say, the *graviton*) acquires a supersymmetric (fermionic) counterpart, χ^a_α (the *gravitino*). Such theory enjoys a large number of worldsheet local symmetries (supersymmetry, Weyl and super-Weyl transformations, two-dimensional Lorentz transformations and reparametrization), along the global, space-time Poincaré symmetry. After fixing the gauge freedom from Weyl invariance and reparametrization invariance, we are left with a residual symmetry that is bigger than the one (conformal symmetry) we have found for the bosonic string case, as it includes also superconformal symmetry.

Again, it is possible to properly gauge-fix the theory via a path integral approach, paying attention to several remarkable differences with respect to the previous case. As above, in the path integral gauge-fixing approach we have to introduce ghost fields, but not only the ones we have discussed before: since the gauge group is bigger, also their supersymmetric counterparts appear, the fields β and γ. Again, the cancellation of the conformal anomaly fixes the number of space-time dimensions of the theory: according to this computation, there should be 10 space-time dimensions.

Among the new features introduced by adding wordsheet fermions to the theory, we have that closed string fermions are not bound to satisfy the same periodicity conditions of bosons: as we have discussed in Section 1.4.2, they have to obey Ramond or Neveu-Schwarz conditions, respectively:

$$R:\ \psi^\mu(\tau,\sigma+2\pi)=\psi^\mu(\tau,\sigma),\quad NS:\ \psi^\mu(\tau,\sigma+2\pi)=-\psi^\mu(\tau,\sigma),$$

We have the same periodicity conditions for the antiholomorphic (right-moving) fermionc fields, hence we have $2\times 2=4$ sectors, characterized by the conditions:

$$(R,R),\ (R,NS),\ (NS,R),\ (NS,NS).$$

If we look at the spectrum generated by a single set of NS modes, we discover the existance of a unique vacuum, $|0\rangle_{\mathrm{NS}}$. Differently, in the case of Ramond modes, we find 16 vacua, and the zero modes form a representation of Dirac gamma matrix algebra. We can introduce a fermion number operator $e^{\pi i F}$, that is a symmetry of the theory and is such that the fermionic vacua are eigenvector of it, with eigenvalue ± 1. We can decompose further NS and R sectors with respect to the fermionic number, and our notation will be NS$\pm$ and R$\pm$.

BRST invariance is generalized to Superstring Theory, and it is an actual mean to quantize the theory. Although there are more complications and subtleties in this procedure, we will not discuss them. The Hilbert space of states of the theory is again the cohomology of the BRST charge Q, and the physical states have to respect

$$b_0\,|\psi\rangle=\tilde{b}_0\,|\psi\rangle=0.$$

In addition to this, for the Ramond sector we have the condition

$$\beta_0\,|\psi\rangle=0 \tag{2.50}$$

or its antiholomorphic version, that should hold for the states that appear in the physical spectrum of the theory (in the Neveu-Schwarz sector, there is no zero mode for β and $\tilde{\beta}$). As before, $L_0=\{Q,b_0\}$, hence we have the mass-shell condition

$$L_0\,|\psi\rangle=\bar{L}_0\,|\psi\rangle=0. \tag{2.51}$$

L_0 is the zero mode of the holomorphic generator of the conformal transformations. Hence, it is not surprising that there is an analogous relation in the supersymmetric theory, that is $G_0=[Q,\beta_0]$, where G_0 is the zero mode of the expansion of the additional (fermionic) holomorphic generator of superconformal transformations. As remarked before, this zero mode exist only in the Ramond sector, where we have then another physicity condition,

$$G_0\,|\psi\rangle=0 \tag{2.52}$$

or its antiholomorphic counterpart.

As we have said before, Superstring Theory allows us to go beyond the flaws of the bosonic theory. Having introduced wordsheet fermions allows the presence of space-time fermions in the spectrum. Moreover, we want to build consistent theories, and in particular get rid of the nasty presence of unphysical tachyons, and this is possible thanks to an operation called *GSO projection*. Let us briefly describe it in the closed theory. In the closed theory, we have to consider both holomorphic an aniholomorphic (left- and right-moving) parts of the theory. Spacetime bosons comes from having the worldsheet fermions in the same sector (NS or R) for both left- and right-movers, while the remaining two possibilities lead to spacetime fermions. Not all sectors are allowed for a consistent theory: it turns out that there exist only two consistent closed superstring theories, namely type IIA and IIB, with sectors

$$\begin{aligned}\text{IIA}: \ & (\text{NS}+,\text{NS}+),\ (\text{R}+,\text{NS}+),\ (\text{NS}+,\text{R}-),\ (\text{R}+,\text{R}-);\\ \text{IIB}: \ & (\text{NS}+,\text{NS}+),\ (\text{R}+,\text{NS}+),\ (\text{NS}+,\text{R}+),\ (\text{R}+,\text{R}+).\end{aligned}$$

The projection of the full spectrum on the eigenspaces of the fermion number operator is called GSO projection. For the sake of completeness, we mention that also for a theory that contains open superstrings we have constraints that reduce the number of possible theories: indeed, there is only one consistent theory, called type I. This theory contains also closed, unoriented superstrings, and this is intuitively reasonable, as two open strings can interact and form a closed string state.

2.2.2 Heterotic String

The theoretical building block to properly discuss heterotic string is toroidal compactification. Such topic is developed in detail in the following Chapter, hence the reader is invited to read through Section 3.3.1 before proceding with this Section.

We have considere left- and right-moving coordinates as independent chiral bosons, so we can drop one of them. We build the heterotic string as a theory where the left-moving sector is the one of the bosonic, 26-dimensional String Theory, combined with the right-moving sector of the 10-dimensional superstring. The left-moving sector consists of 10 uncompactified bosons fields $X_L^\mu(\tau+\sigma)$, $\mu = 0,\dots,9$ and 16 internal bosons, $X_L^I(\tau+\sigma)$, $I = 1,\dots,16$, compactified on a 16-dimensional torus. The right-moving sector is formed by 10 uncompactified bosons $X_R^\mu(\tau-\sigma)$, $\mu = 0,\dots,9$, along with their supesymmetric fermionic partners $\psi_R^\mu(\tau-\sigma)$. In the end, we have to add the left- and right-moving ghosts b, c and the right-moving superconformal ghosts β, γ. The theory is effectively 10-dimensional, as the chiral bosons play the role of internal degrees of freedom, needed to cancel the conformal anomaly.

Since we do not want to lose the geometrical interpretation of the string, we require that the uncompactified bosonic coordinates, left- and right- moving, have common center of mass and momentum, with continuous spectrum. The momentum spectrum of the chiral compact bosons is instead discrete, and it is composed by the vectors of a 16-dimensional lattice Γ_{16} . As we can expect, modular invariance imposes precise constraints on the lattice Γ_{16}. Namely, Γ_{16} should be an even, self-dual, Euclidean lattice in 16 dimensions, and we know that there exist only two lattices that obey these requirements:

1. The product lattice $\Gamma_{E_8}\otimes\Gamma_{E_8}$, where Γ_{E_8} is the root lattice of E_8.
2. The lattice $\Gamma_{D_{16}}$, that is the weight lattice of $\text{Spin}(32)/\mathbb{Z}_2$. Such lattice contains the root lattice of $SO(32)$.

Both lattices contains exactly 480 vectors of squared norm equal to two, that are, respectively, the roots of $E_8\times E_8$ and $SO(32)$. For a reason that will be explained in Section 3.3.1, the gauge group of these theories are, respectively, $E_8\times E_8$ and $SO(32)$. These two groups have both dimension 496. More in detail, if we look at the left-moving sector of the theory, we have the massless, vector states $\tilde{\alpha}^{\mu}_{-1}\,|0\rangle$ and $\tilde{\alpha}^{m}_{-1}\,|0\rangle$, where $|0\rangle$ is the usual tachyonic vacuum of the bosonic theory. The latter vector states are the Cartan (commuting) subalgebra of the gauge group of the theory (they are the left-moving part of $U(16)$ gauge boson). Then, if we look at the massless states build from the solitonic vacua characterized by non-trivial Kaluza Klein (internal) momentum, we find the states $|p_L \cdot p_L = 2\rangle$, $N_L = 0$, that generate the non-Abelian gauge bosons of the gauge group.

Chapter 3

Compactifications of String Theory

In general, fields are maps between a manifold (space-time in the context of Quantum Field Theory, worldsheet for String Theory) and another one, called target space (in the case of String Theory, space-time). Until now, we have implicitly considered theories whose fields live in a non-compact manifold, eventually $\mathbb{R}^d$. In this Chapter we will study the case of compact target space, or at least target space with some dimensions compactified.

As String Theory predicts more dimensions than the 1+3 we are used to, compactifications are needed to make contact with reality. There is a great variety of possible compactifications: from the easiest ones, as toroidal compactification, to very complicated cases, as theories defined on general Calabi-Yau manifolds. Usually, the easiest compactifications are rather unrealistic, anyway they can be seen as both toy models and building blocks for more realistic theories.

Having the target space compactified opens our possibilities to a remarkable variety of physical properties. For example, in the case of String Theory, a string can "wind" along a compact dimension, and this is a feature that would not be possible without some dimension compactified. The identical feature can be studied in two dimensional Conformal Field theory with toroidal target space.

Aside from these physical motivations, toroidal conformal field theory theories are interesting for many other reasons. They are completely solvable, i.e. one can in principle compute all the correlation functions of the theory. This is because toroidal theories are essentially free theories, as their equations of motion are linear in the fields. Despite this, they host interesting mathematical structures: for instance, different toroidal models are related by an intricate web of dualities, i.e. equivalence between theories. In this "landcape" of different models, there are theories that are mapped in a non-trivial way into themself by some duality. In this case, the duality is manifestly a symmetry of these theories.

After dealing with toroidal compactifications, both from the Stringy and the Conformal Field Theory point of view, we will discuss the more complicated and realistic case of orbifold theories. Orbifold theories are obtained by "quotienting" a Conformal Field Theory by a symmetry group of the theory. Such compactifications are interesting also inside the playground of String Theory, for many different reasons: for example, they allow to break symmetries.

3.1 An introduction: Kaluza-Klein compactification

The idea of adding extra, compact dimensions to the observed four spacetime dimensions historically dates back to 1914 [15], and was initially suggested as a mean to unify gravitation and electromagnetism.

Consider a $(d+1)$ dimensional spacetime, described by coordinates x^M, $M = 0, \dots, d$, with x^d periodic:

$$x^d \sim x^d + 2\pi R,$$

and the other coordinates x^μ, $\mu = 0, \dots, d-1$, noncompact. Indices that refer to the noncompact coordinates will be represented with greek letters, while generic indices will be written as capital latin letters. The metric G_{MN} splits then into three components: $G_{\mu\nu}$, $G_{\mu 4}$ and G_{44}. These three are, from the point of view of the d dimensional spacetime, respectively a tensor, a vector and a scalar. We parametrize the metric G_{MN} as

$$G_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + g_{dd}(dx^d + A_\mu dx^\mu)^2. \tag{3.1}$$

Additionally, we impose that $g_{\mu\nu}$, g_{dd} and A_μ only depend on the noncomplact coordinates. Under these hypothesis, the metric (3.1) is the most general one that is invariant under translations of the compact coordinate x^d. This form still allows reparametrizations

$$x'^\mu = x'^\mu(x^\nu)$$

and

$$x'^d = x^d + \lambda(x^\nu).$$

Under the latter, A_μ transforms as

$$A'_\mu = A_\mu - \partial_\mu \lambda, \tag{3.2}$$

that is the form of a gauge transformation. d dimensional gauge transformations are then encoded into the $d+1$ symmetry group.

Let us consider $G_{dd} = 1$ for simplicity. Let $\phi(x^M)$ be a scalar massless field, and make explicit the x^d dependence by a mode expansion:

$$\phi(x^M) = \sum_{n=-\infty}^{+\infty} \phi_n(x^\mu) e^{ix^d \frac{n}{R}}.$$

In this expression, it appears the quantized momentum associated to the compact dimension, $p_d = \frac{n}{R}$. The equation of motion

$$\partial_M \partial^M \phi = 0$$

becomes

$$\partial_\mu \partial^\mu \phi_n = \frac{n^2}{R^2} \phi_n. \tag{3.3}$$

The momentum along the compactified dimension is quantized:

$$p_d^2 = -p^\mu p_\mu = \frac{n^2}{R^2}$$

This is not surprising: even in non-relativistic point particle quantum mechanics, compact dimensions are associated with discrete momentum spectrum. In the case of circle compactification, this happens because we require the states of the theory to be invariant under the action of the translation operator along the compact dimension, $e^{2\pi i R p_d}$. We will call $-p^\mu p_\mu$ the d dimensional mass squared, since an observer that does not see the compactified dimension identifies the mass squared operator with this expression.

The modes ϕ_n can be interpreted as an infinite tower of d dimensional fields, each one associated with a d dimensional mass $\frac{n}{R}$. One surprising feature is that, starting from a massless theory in $d+1$ dimensions, what emerges is a *tower* of massive states in d dimensions. The d dimensional mass increases as the radius R decreases: the x^d dependence of the theory is hence not observable for small energies (compared to $\frac{1}{R}$), and the Physics is effectively described as d dimensional. At energies above $\frac{1}{R}$, the tower of Kaluza-Klein states appear, revealing the presence of compactified dimensions. It is easy to guess that the same phenomenon is verified also in the case of several compactified dimensions, as we will discuss in the following Sections. Moreover, if we consider a theory that describe strings, the discretization of the momentum of the center of mass will not be the only emerging feature: the nature of strings will allow properties that are impossible for point particles.

3.2 Free boson compactified on a circle

Let us present the topic of this Chapter with the most simple example of Conformal Field Theory with compact target space: a bosonic field compactified on a circle of radius R. To conform our notation to the one usually employed in String Theory, we will denote the bosonic field and the conjugate momentum respectively as X and p. On the circle, X is identified with $X + 2\pi R$: hence, we can adapt the periodic boundary condition studied in the bosonic example of Section 1.4 in the following way

$$X(x+L,t) = X(x,t) + 2\pi m R, \tag{3.4}$$

with m integer. It means that as the space coordinate x circles once around the space-time cylinder, the field "winds" m times around the target space. For this reason, we will call m the *winding number*. Another effect of the compactification, already present in the point particle case discussed in the previous section, is the discretization of the spectrum of the momentum of the center of mass. In this context, it is custom to rescale and write $\frac{n}{R}$ instead of p_0, and the spectrum of the operator n is integer. With a slight abuse of notation, we will refer to n as the momentum. The mode expansion of the bosonic field thus becomes

$$X(x,t) = X_0 + \frac{t}{gRL} n + \frac{2\pi R x}{L} m + \frac{i}{\sqrt{4\pi g}} \sum_{n\neq 0} \frac{1}{n} (a_n e^{\frac{2\pi i n(x-t)}{L}} - \tilde{a}_{-n} e^{\frac{2\pi i n(x+t)}{L}}). \tag{3.5}$$

that is, in complex coordinates,

$$X(z,\bar{z}) = X_0 - i\left(\frac{1}{4\pi g R} n + \frac{R}{2} m\right)\log(z) + \frac{i}{\sqrt{4\pi g}} \sum_{n\neq 0} \frac{1}{n} a_n z^{-n} + \tag{3.6}$$

$$- i\left(\frac{1}{4\pi g R} n - \frac{R}{2} m\right)\log(\bar{z}) + \frac{i}{\sqrt{4\pi g}} \sum_{n\neq 0} \frac{1}{n} \tilde{a}_n \bar{z}^{-n}. \tag{3.7}$$

We can introduce Vertex operators for this theory. As appear from the mode expansion, we can split the holomorphic and the antiholomorphic degrees of freedom:

$$X(z,\bar{z}) = X_L(z) + X_R(\bar{z}), \tag{3.8}$$

with

$$X_L(z) = x_L - i\frac{1}{4\pi g}p_L \log(z) + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}a_n z^{-n}, \tag{3.9}$$

$$X_R(\bar z) = x_R - i\frac{1}{4\pi g}p_R \log(\bar z) + \frac{i}{\sqrt{4\pi g}}\sum_{n\neq 0}\frac{1}{n}\tilde a_n \bar z^{-n}, \tag{3.10}$$

where (thanks to Noether theorem) the left- and right-moving momenta were identified as

$$p_L = \frac{n}{R} + 2\pi g R m, \tag{3.11}$$

$$p_R = \frac{n}{R} - 2\pi g R m, \tag{3.12}$$

and x_L, x_R are independent operators such that

$$[x_L, p_L] = [x_R, p_R] = 1.$$

We observe that in the noncompact theory we would have $p_L = p_R$. The Virasoro generators $L_0, \bar L_0$ are given by

$$L_0 = \frac{\alpha'}{4}p_L^2 + \sum_{n=1}^{\infty} a_{-n}a_n,$$

$$\bar L_0 = \frac{\alpha'}{4}p_R^2 + \sum_{n=1}^{\infty} \tilde a_{-n}\tilde a_n.$$

Let us recall that the state that we label as $|0; k_L, k_R\rangle$ is characterized as an eigenstate of p_L and p_R, with eigenvectors k_L and k_R, and it is annihilated by every positive mode oscillator. Heuristically, we can guess that the state $|0; k_L, k_R\rangle$ is created from the vacuum by the action of the Vertex operator

$$V_{k_L,k_R}(z,\bar z) =: e^{ik_L X_L(z) + ik_R X_R(\bar z)} : ,$$

and the OPE for these fields is

$$V_{k_L,k_R}(z,\bar z)V_{k'_L,k'_R}(w,\bar w) \sim (z-w)^{\frac{1}{4\pi g}k_L k'_L}(\bar z - \bar w)^{\frac{1}{4\pi g}k_R k'_R}V_{k_L+k'_L,k_R+k'_R}(w,\bar w).$$

However, the previous OPE is non-local (in the sense of Equation (1.60)): if we compute the same product with the fields commutated, we will get the additional phase factor

$$e^{i\pi(nm'+mn')}.$$

To get rid of this issue, we have to include in the actual expression of the Vertex operator an extra factor. A possible choice is:

$$V_{k_L,k_R}(z,\bar z) = e^{i\pi\frac{1}{8\pi g}(k_L-k_R)(p_L+p_R)} : e^{ik_L X_L(z)+ik_R X_R(\bar z)} : , \tag{3.13}$$

and when we compute the OPE with the fields commutated these new extra factors produce a phase term

$$e^{i\pi(nm'-mn')}, \tag{3.14}$$

that makes the OPE local, compensating the phase term that menaces locality. The phase term (3.14) is what we will call a *cocycle*. These phase factors can be regarded as technicalities, and indeed in many situations they are fully negligible. However, there are some situations where we need to take them into accout. For example, they allow us to compute the relative sign between certain scattering amplitudes. The presence of these phase factors inside the OPE of two Vertex operators is actually of crucial importance for the aims of this Thesis, but for a completely different reason: they lie at the heart of the concept of *non-trivial symmetry lift* in toroidal Conformal Field Theory, as we will discuss in detail and generality in Chapter 4.

3.2.1 Closed strings and T-Duality

Another relevant observation is that our theory depends only on one parameter, R (the only modulus of our theory). Naïvely, we may think that for every R we obtain a different model, but this is not true (as we can recognize also from the partition function of the theory). We will explain this remarkable feature in the context of String theory. To make contact with the usual notation, define

$$\frac{\alpha'}{2} = \frac{1}{4\pi g}.$$

Let us consider 26-dimensional bosonic string theory, with one dimension (say, X^{25}) periodic, and the others noncompact as usual. The mass-shell condition can be written as

$$M_L^2 = \frac{1}{2}\left(\frac{n}{R} + \frac{mR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N-1), \tag{3.15}$$

$$M_R^2 = \frac{1}{2}\left(\frac{n}{R} - \frac{mR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(\tilde{N}-1), \tag{3.16}$$

$$M^2 = \frac{n^2}{R^2} + \frac{m^2R^2}{\alpha'^2} + \frac{2}{\alpha'}(N+\tilde{N}-2), \tag{3.17}$$

$$M_L^2 = M_R^2, \quad \text{or, equivalently,} \quad 0 = nm + N - \tilde{N}, \tag{3.18}$$

where M is the mass as seen from the point of view of a 25-dimensional observer, that does not see the compactified dimension: $M^2 = -p_\mu p^\mu$. Let us have a look at the first relation, without entering in the details of the spectrum. The four terms can be interpreted respectively as the discrete compact momentum contribution, the potential energy term associated to the winding, the usual oscillators contribution and the zero-point energy term. We expect that, at very big radius R, we recover the noncompact theory. Indeed, if R goes to ∞, compact momenta approach a continuous spectrum (just like the momentum of a particle inside a very large box), while winding states become infinitely massive (it requires an infinite energy to wind along an infinite circumference). However, if we consider the opposite limit $R \to 0$ we discover something unexpected: states of compact momentum become infinitely massive, but winding states form approximatively a continuum (heuristically, it does not take much energy to wrap around a small circle). Even in the limit $R \to 0$, the theory resembles the non-compact one, albeit with the roles of n and m exchanged. This is not a mere coincidence, since there is an actual duality between the theory at small and big radius, given by:

$$R \to R' = \frac{\alpha'}{R}, \quad n \leftrightarrow m. \tag{3.19}$$

This is what is called a *T-duality*. Reversing winding and momentum is exactly the same as mapping

$$p_L^{25} \to p_L^{25}, \quad p_R^{25} \to -p_R^{25}. \tag{3.20}$$

The qualitative discussion of the small and big radius limits hints us that the spectrum is left unchanged by this transformation. Moreover, consider the field

$$X'^{25}(z,\bar{z}) = X_L^{25}(z) - X_R^{25}(\bar{z}).$$

If we replace X with X', we will obtain a theory with the same OPEs and stress energy tensor, the only difference being the reverse sign in front of p_R, just as the theory with radius R' instead of R. This tells us that the two theories connected by the T-duality are exactly the same, one described in terms of X an the other written in terms of X'.

Summarizing, by means of the circle compactification our theory acquired a new feature: if we exchange the operators n and m and map R into $\frac{1}{2\pi g R}$, we obtain the same model. This symmetry is the simplest example of T-duality, a characteristic feature of string theory compactified on tori: such structure is absent in the point particle case, as a particle on a torus lacks of the winding number. The points R and $\frac{\alpha'}{R}$ on the moduli space of the theory are identified, hence the moduli space of this simple theory can be taken as $]0,\sqrt{\alpha'}]$, or equivalently $[\sqrt{\alpha'},+\infty[$. The value $R=\sqrt{\alpha'}$ is special, since the model is mapped in a non-trivial way (since we exchange winding number and momentum) into itself. In general, points in the moduli space that are mapped into themselves by a duality are called self-dual points, and such symmetries are called *self-duality* of a theory. T-dualities and self-dualities are among the main subjects of this Thesis, and we would address them in a more general context in the next Section 3.3. The presence of light states even in the limit $R \to 0$ is completely dissimilar from what happens for point particles: the properties we have explored shed a light on how string-like object perceive small distance geometry differently from point particles.

3.2.2 Enhanced symmetries at self-dual radius

We can go further in the analysis of the symmetries that emerge at $R = \sqrt{\alpha'}$. To accomplish this, we have to discuss in some detail the spectrum of the theory. First, we review the purely 25-dimensional states, with $n = m = 0$.

1. The tachyon $|0\rangle$ at $M^2 = -\frac{4}{\alpha'}$ is still present.

2. For $M^2 = 0$, we have the 25-dimensional graviton, antisymmetric tensor and dilaton:

$$|G^{\mu\nu}\rangle = \alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|0\rangle, \quad \mu,\nu = 0,\dots,24 \tag{3.21}$$

3. Two massless vector states, emerging from compactification:

$$|V_1^{\mu}\rangle = \alpha^{\mu}\tilde{\alpha}^{25}|0\rangle, \quad |V_2^{\mu}\rangle = \alpha^{25}\tilde{\alpha}^{\mu}|0\rangle. \tag{3.22}$$

   These vector are actually gauge bosons, and the gauge symmetry associated with them is $U(1)_L \times U(1)_R$.

4. A massless scalar, obtain with the application of two internal oscillators

$$|\phi\rangle = \alpha_{-1}^{25}\tilde{\alpha}_{-1}^{25}|0\rangle. \tag{3.23}$$

   The vacuum expectation value of this field is an internal degree of freedom of the 26-dimensional metric, namely the radius R. This is consonant with the usual definition of moduli as vacum expectation values.

These were the massless states without compactified momentum and winding number. In corrispondence of nontrivial n and m there are soliton vacua $|n,m\rangle$, and acting with the oscillators on them we will get other interesting states. Focus on the four sectors characterized by $n^2 = m^2 = 1$: these are $n = m = \pm 1$, $n = -m = \pm 1$. The squared mass is

$$M^2 = \frac{1}{R^2} + \frac{R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2), \tag{3.24}$$

and it is in general different from zero for any allowed values of N and $\tilde{N}$. However, at the self-dual radius $R = \sqrt{\alpha'}$ something special happens: the squared mass become

$$M^2 = \frac{2}{\alpha'}(N + \tilde{N} - 1), \tag{3.25}$$

and we see that several states, that were massive for generic R, become massless. Precisely, they are

1. $n = m = \pm 1$, $N = 1$, $\tilde{N} = 0$: two vectors and two scalars,

$$|V_a^\mu\rangle = \alpha^\mu |\pm 1, \pm 1\rangle\,, \quad |\phi_a\rangle = \alpha^{25} |\pm 1, \pm 1\rangle\,, \quad a = 1, 2. \tag{3.26}$$

2. $n = m = \pm 1$, $N = 0$, $\tilde{N} = 1$: two vectors and two scalars,

$$|V'^\mu_a\rangle = \alpha^\mu |\pm 1, \mp 1\rangle\,, \quad |\phi'_a\rangle = \alpha^{25} |\pm 1, \mp 1\rangle\,, \quad a = 1, 2. \tag{3.27}$$

Together with the massless vectors (3.22), these new four massless vectors form the adjoint representation of $SU(2)_L \times SU(2)_R$. Such symmetry group is broken to $U(1)_L \times U(1)_R$ outside the self-dual point, and the newly introduced vector states become massive: this is just like a stringy Higgs effect. At the self-dual radius, we have what we will call an *enhanced symmetry*: this is purely a String theory phenomenon. Indeed, while the massless vectors (3.22) are also common to point-particle theories compactified on a circle, the states with non-trivial winding number are an exclusive stringy feature.

Remark 3.2.1. Observe that the massless states (3.22) correspond to the $U(1)_L \times U(1)_R$ Cartan subalgebra of $SU(2)_L \times SU(2)_R$

Remark 3.2.2. There appear other massless states at self-dual radius, namely the four massless scalars $|0, \pm 2\rangle\,, |\pm 2, 0\rangle$. Along with the massless scalar (3.23), they form the $(3, 3)$ representation of $SU(2)_L \times SU(2)_R$.

3.3 Toroidal compactification of several dimensions: Narain compactification

After the simple example discussed in the previous Section, we are ready to consider a two dimensional Conformal Field Theory with several dimensions compactified, and describe what T-dualities are in this more general case. Generic compactification can be described in a general and elegant way, called Narain compactification. The periodicity condition is straightforwardly generalized to

$$x^m \sim x^m + 2\pi L^m, \tag{3.28}$$

where

$$L^m = \sum_{i=1}^{k} a_i e_i^m, \tag{3.29}$$

with e_i being independent vectors, and a_i integers. This is equivalent to say that L^m is a vector of the lattice Λ defined by the linear combination with integer coefficients of the vectors e_i, and X^m are compactified on the k-dimensional torus associated to the lattice $2\pi\Lambda$.

Geometrically speaking, a k-dimensional torus is identified with the quotient of $\mathbb{R}^d$ by a lattice of maximal rank. Up to a diffeomorphism, it is equivalent to the product of k circles. Consider the canonically conjugate variables, namely the center of mass position x^m and momentum p_m: the latter, as usual, generates the translation of the former, and the translation operator is $e^{ix^m p_m}$. To be single-valued under the identification (3.28), we need $L^m p_m$ to be integer, or, in other words, we need the momentum to lie in the dual lattice of Λ, Λ^*. Remember that the dual lattice is defined as

$$\Lambda^* = \{\lambda \in \mathbb{R}^k | \lambda \bullet \mu \in \mathbb{Z} \, \text{for every } \mu \text{ in } \Lambda\}.$$

Λ will be referred to as the *winding vector lattice*. The basis $\{e_i\}$ induces a metric g on Λ:

$$g_{ij} = e_i \cdot e_j,$$

where $\cdot$ denotes the usual scalar product in $\mathbb{R}^k$. Analogously, if we denote with $\{e^{*j}\}$ its conjugate basis (i.e., $e_i \cdot e^{*j} = \delta_i^j)$), on Λ^* we define the metric

$$(g^*)_{ij} = g^{ij} = e^{*i} \cdot e^{*j} = (g^{-1})_{ij}.$$

In a mathematical terminology, g_{ij} is called Gramian matrix. The volumes of the unit cell of lattices Λ and Λ^* are respectively $\sqrt{\det g}$ and $\sqrt{\det g^*}$.

Return now to the closed string problem: the condition $X^m(\tau, \sigma)$ has to obey is

$$X^m(\tau, \sigma + 2\pi) = X^m(\tau, \sigma) + 2\pi L^m, \quad \text{with } L^m \in \Lambda. \tag{3.30}$$

Here, L^m is what encodes the winding of the string. Let us write down the mode expansion for $X^m(\tau, \sigma)$, separating the holomorphic and antiholomorphic dependencies:

$$X(z, \bar{z}) = X_L(z) + X_R(\bar{z}), \tag{3.31}$$

with

$$X_L^m(z) = x_L^m - i\frac{\alpha'}{2} p_L^m \log(z) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} a_n^m z^{-n}, \tag{3.32}$$

$$X_R^m(\bar{z}) = x_R^m - i\frac{\alpha'}{2} p_R^m \log(\bar{z}) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{a}_n^m \bar{z}^{-n}, \tag{3.33}$$

where right- and left-moving momenta are defined as

$$p_L^m = p^m + \frac{1}{\alpha'} L^m, \tag{3.34}$$

$$p_R^m = p^m - \frac{1}{\alpha'} L^m. \tag{3.35}$$

Remembering that the mass seen by an observer that does not perceive the compactified dimensions is $M^2 = -\sum_{\mu=0}^{25-k} p^\mu p_\mu$, the mass-shell condition is

$$M_L^2 = \frac{1}{2}\sum_{m=1}^{k}\left(p^m + \frac{1}{\alpha'}L^m\right)^2 + \frac{2}{\alpha'}(N-1), \tag{3.36}$$

$$M_L^2 = \frac{1}{2}\sum_{m=1}^{k}\left(p^m - \frac{1}{\alpha'}L^m\right)^2 + \frac{2}{\alpha'}(N-1), \tag{3.37}$$

$$M^2 = \sum_{m=1}^{k}\left[(p^m)^2 + \frac{1}{\alpha'^2}(L^m)^2\right] + \frac{2}{\alpha'}(N+\tilde{N}-2), \tag{3.38}$$

$$M_L^2 = M_R^2, \quad \text{or, equivalently,} \quad 0 = \sum_{m=1}^{k}(p^m L^m) + N - \tilde{N}. \tag{3.39}$$

From right- and left-moving momenta (or, more precisely, from their eigenvalues k_L and k_R), we can build a new lattice. The set of the eigenvectors (k_L, k_R) of (p_L, p_R) is the lattice

$$\Gamma = \left\{(\mu+\lambda, \mu-\lambda) \in \mathbb{R}^{2k} \colon \mu \in \Lambda^*,\ \lambda \in \frac{2\pi}{\alpha'}\Lambda\right\}. \tag{3.40}$$

The moments (k_L, k_R) are vectors of the maximal rank lattice Γ, embedded into the momentum space $\mathbb{R}^{2k}$. We will call Γ the Narain lattice. It is useful to introduce dimensionless momenta, $l_{L,R} := \sqrt{(\frac{\alpha'}{2})}k_{L,R}$.

Consider the winding state vertex operator

$$V_{k_L,k_R}(z,\bar{z}) =: e^{ik_L\cdot X_L(z)}e^{ik_R\cdot X_R(\bar{z})} : .$$

For a reason of locality, we require the single-valuedness of the OPE

$$: e^{ik_L\cdot X_L(z)+ik_R\cdot X_R(\bar{z})} : \ : e^{ik'_L X_L(0)+ik'_R X_R(0)} : \ \sim z^{l_L\cdot l'_L}\bar{z}^{l_R\cdot l'_R} : e^{i(k_L+k'_L)\cdot X_L(0)+i(k_R+k'_R)\cdot X_R(0)} : .$$

To avoid extra phase factors to appear when one vertex operator circles the other, we have to impose

$$(l_L, l_R) \bullet (l'_L, l'_R) := l_L \cdot l'_L - l_R \cdot l'_R \in \mathbb{Z}. \tag{3.41}$$

We have introduced a bilinear form, the $\bullet$ product, that has (k,k) signature in the momentum space $\mathbb{R}^{2k}$. A lattice that obeys condition (3.41) is said to be integer, and condition (3.41) can be rephrased into

$$\Gamma \subset \Gamma^*,$$

where Γ^* is the dual lattice of Γ^*. This is not enough to ensure consistency, since the constraints imposed by modular invariance are stronger. Indeed, it can be proved that for modular invariance the lattice Γ should be self-dual, i.e. $\Gamma = \Gamma^*$, and even, i.e. $\lambda \bullet \lambda \in 2\mathbb{Z}$ for ever $\lambda \in \Gamma$

Remark 3.3.1. An even, self-dual lattice necessarily has the difference between the right and left dimensions that is an integer multiple of 8.

Every even, self-dual lattice can be obtained from another even, self-dual lattice with a $O(k,k,\mathbb{R})$ rotation: if we fix an arbitrary even, self-dual lattice Γ_0, we have

$$\Gamma = R\Gamma_0, \quad \text{with } R \in O(k,k,\mathbb{R}).$$

An important observation is that $O(k,k,\mathbb{R})$ is not a symmetry of the theory, since, in general, such rotations mix the left momenta and the right momenta, while the products $l_L \cdot l'_L$ and $l_R \cdot l'_R$ appear separately in the OPEs (and in the mass-shell condition, if we look at the string spectrum). Hence, many $O(k,k,\mathbb{R})$ rotations lead to different theories. What is truly a simmetry of the theory is the subgroup of $O(k,k,\mathbb{R})$ that describes independent rotations of the positive- and negative-definite subspaces of $\mathbb{R}^{2k}$, namely $O(k,\mathbb{R}) \times O(k,\mathbb{R})$. Hence, the space of inequivalent theories is describe by the quotient

$$O(k,\mathbb{R}) \times O(k,\mathbb{R}) \setminus O(k,k,\mathbb{R}),$$

up to some discrete identifications. Indeed, if we apply to the reference lattice Γ_0 a transformation that has the only effect of permuting its points, we will end up with an equivalent theory. Such transformation should be a bijection of the lattice with itself, and should be regarded as an element of a subgroup of $O(k,k,\mathbb{R})$, hence it will be an automorphism of the lattice Γ_0. We will denote the automorphism group of Γ_0 with $O(\Gamma_0)$, and we will call it the T-duality group of the theory. The group $O(\Gamma_0)$ is often called $O(k,k,\mathbb{Z})$.

Now we can draw our conclusion: a theory defined by

$$R\Gamma_0, \quad R \in O(k,k,\mathbb{R})$$

is equivalent to the theory defined by

$$R'RR''\Gamma_0, \quad R' \in O(k,\mathbb{R}) \times O(k,\mathbb{R}),\ R'' \in O(\Gamma_0)$$

The identification $R \sim R'RR''$ let us conclude that the true space of inequivalent theories is

$$O(k,\mathbb{R}) \times O(k,\mathbb{R}) \setminus O(k,k,\mathbb{R})/O(\Gamma_0). \tag{3.42}$$

We will call it the *moduli space* of the theory.

To make contact with Physics, let us review briefly the spectrum of the theory. The sector without compactified momentum and winding number contains again a $(26-k)$ dimensional tachyon, a massless graviton, antisymmetric field and dilaton. Because of the compactification, there are $2k$ massless vectors,

$$|V_1^{\mu m}\rangle = \alpha^\mu \tilde{\alpha}^m |0\rangle, \quad |V_2^{\mu m}\rangle = \alpha^m \tilde{\alpha}^\mu |0\rangle. \tag{3.43}$$

that are the massless gauge bosons of $U(1)_L^k \times U(1)_R^k$. Finally, we have k^2 massless scalar fields

$$|\phi^{mn}\rangle = \alpha_{-1}^m \tilde{\alpha}_{-1}^n |0\rangle. \tag{3.44}$$

Just as the circle compactification case, these massless scalars are associated with the moduli of the toroidal compactification. We have that $\frac{1}{2}k(k+1)$ of them form the internal graviton components, and their vacuum expectation values are constant parameters (moduli) G_{mn} that encode the shape of the k-dimensional torus. The other $\frac{1}{2}k(k-1)$ are the antisymmetric field components, that may acquire non-zero vacum expectation vale B_{mn}.

The enhanced symmetry at self-dual points of the moduli spaces appears again as a feature of this compactification. However, we will discuss this issue in the next Section, where we present a generalization of this model.

3.3.1 A generalization

Here we present a (nearly) straightforward generalization of the Narain construction. What we will explain here has a remarkable theoretical importance, since it is the key to the heterotic string construction. Let us recall back the mode expansion for the model compactified on a k-dimensional torus:

$$X_L^m(\tau+\sigma) = x_L^m + (\tau+\sigma)p_L^m + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}a_n^m e^{-in(\tau+\sigma)}, \tag{3.45}$$

$$X_R^m(\tau-\sigma) = x_R^m + (\tau-\sigma)p_R^m + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{a}_n^m e^{-in(\tau-\sigma)}, \tag{3.46}$$

Since we have interpreted X^m as a coordinate on a k-dimensional manifold, we had to (tacitly) imposte that left- and right-moving modes have common center of mass and common momentum. However, this constraint is not compulsory if we regard X_L and X_R as chiral, independent fields of a generic two-dimensional worldsheet boson theory. The closed string condition $X_{L,R}^m(\tau,\sigma+2\pi) = X_{L,R}^m(\tau,\sigma)$ entails that each chiral boson should be compactified on a torus, but in this generalized framework we are free to choose different tori for the two chiral bosons. Accepting this, we abandon to the geometrical picture of the compactification of the string. The proper way to regard this theory is as a String theory in $(26-k)$ space-time dimensions, and chiral bosons X_L, X_R are internal degree of freedom that are needed to cancel the conformal anomaly.

Let us fix some notation: we have considered the following identifications in the target space of the chiral fields

$$x_L^m \sim x_L^m + 2\pi\lambda_L^m, \tag{3.47}$$

$$x_R^m \sim x_R^m + 2\pi\lambda_R^m, \tag{3.48}$$

where λ_L and λ_R are respectively vectors of the lattices Λ_L and Λ_R. From this, we conclude that

$$p_L \in \Lambda_L, \quad p_R \in \Lambda_R, \tag{3.49}$$

or, with a more pictorial expression, p_L and p_R are winding vectors. Without delving deep into some technical issues, we can safely write down the following commutation relations

$$\left[x_{L,R}^m, p_{L,R}^n\right] = i\delta^{mn}, \tag{3.50}$$

$$\left[x_{L,R}^m, p_{R,L}^n\right] = 0. \tag{3.51}$$

The momenta $p_{L,R}$ generate the translations of $x_{L,R}$. To ensure that the operator $e^{ix_{L,R}^m p_{L,R}^m}$ is single valued under the identification (3.47), we have to require

$$p_L \in \Lambda_L^*, \quad p_R \in \Lambda_R^*. \tag{3.52}$$

We have then discovered that

$$p_L \in \Lambda_L \cap \Lambda_L^* =: \Gamma_L, \quad p_R \in \Lambda_R \cap \Lambda_R^* =: \Gamma_R. \tag{3.53}$$

Again, the vectors (p_L, p_R) form a lattice $\Gamma = \Gamma_L \oplus \Gamma_R$, and if we equip it with the product

$$(p_L, p_R) \bullet (p'_L, p'_R) = p_L \cdot p'_L - p_R \cdot p'_R$$

we have that modular invariance forces Γ to be even and self-dual.

Let us finally discuss the spectrum of this theory and the symmetries it reveals. With obvious definitions of N_L, N_R, the mass-shell condition is

$$\alpha' m_{L,R}^2 = p_{L,R} \cdot p_{L,R} + 2(N_{L,R} - 1), \tag{3.54}$$

$$p_L \cdot p_L - p_R \cdot p_R + 2(N_L - N_R) = 0. \tag{3.55}$$

Again, we have the massless $U(1)_L^k \times U(1)_R^k$ gauge boson vectors (3.43). Other massless $(26-k)$-dimensioal vectors appear if there exist lattice vectors $(p_L, p_R) \in \Gamma$ such that

$$p_L \cdot p_L = 2, \quad p_R = 0, \tag{3.56}$$

or

$$p_L = 0, \quad p_R \cdot p_R = 2. \tag{3.57}$$

In this case there appear the massless vectors

$$|V_L^\mu\rangle = \alpha_{-1}^\mu \, |p_L \cdot p_L = 2, \, p_R = 0\rangle \, , \tag{3.58}$$

$$|V_R^\mu\rangle = \tilde{\alpha}_{-1}^\mu \, |p_L = 0, \, p_R \cdot p_R = 2\rangle \, . \tag{3.59}$$

For each momentum $p_{L,R}$ such that $p_{L,R} \cdot p_{L,R} = 2$ there is a massless vector $|V_L^\mu\rangle$, and such vectors correspond to the non-commuting generators of some non-Abelian Lie group $G_{L,R}$, with G_L and G_R different, in general. Together with the gauge boson vectors (3.43), these massless vectors form a representation of $G_L \times G_R$, and the gauge boson vectors (3.43) correspond to the $U(1)_L^k \times U(1)_R^k$ Cartan subalgebra of $G_L \times G_R$.

Remark 3.3.2. From what we have said, $p_{L,R} \cdot p_{L,R} = 2$ vectors must be roots of $G_{L,R}$, and $G_{L,R}$ must be simply laced. The only possible groups are hence D_n, A_n, E_n, with $n = 6, 7, 8$ (whose rank is n), or products of them.

3.3.2 Heterotic string compactified on $\mathbb{T}^k$

We will now briefly review a topic that has interesting connections with some of the results presented in the following Chapter, the toroidal compactification of the heterotic string. As we have said in Section 2.2.2, the 16 chiral bosons of the only two possible heterotic strings are already compactified, and they can be considered as internal degrees of freedom, needed to get rid of the conformal anomaly. Now, we will consider the yet totally uncompactified coordinates in the ten dimensional spacetime compactified on a k-dimensional torus. We will not delve deep into the technical details of this construction, since the final result shares several analogies with the kind of Narain compactification we have explored in this Chapter. As we can expect, along the ones related to the chiral bosons, the momenta associated to the coordinates compactified on $\mathbb{T}^k$ develop a discrete spectrum (p_L, p_R). Precisely, the vector (p_L, p_R) belongs now to an even, self-dual lattice, with a bilinear form of signature $(k, k+16)$, defined by

$$(p_L, p_R) \bullet (p_L', p_R') = p_L \cdot p_L' - p_R \cdot p_R'.$$

Every latticewith this properties can be obtained from another even, self-dual, signature $(k, k+16)$ lattice Γ_0 with a $O(k, k+16, \mathbb{R})$ rotation. The moduli space of the theory is very similar to the one we have described above:

$$O(k, \mathbb{R}) \times O(k, \mathbb{R} + 16) \setminus O(k, k+16, \mathbb{R}) / O(\Gamma_0). \tag{3.60}$$

Interestingly, the two possible heterotic theories, once compactified, are different points on the same moduli space: this transformation can be understood as the result of a continuous rotation $O(k+16)$ that acts only on the right momenta subspace of Γ_0. Also in this even more complicated model, the enhanced symmetry phenomenon is present: again, the states of momentum $p_L \cdot p_L = 2$ correspond to non-Abelian gauge bosons. The existance of those states permits of enlarge the symmetry group, beyond the original $E_8 \times E_8$ or $SO(32)$. The enhancement only occurs when $p_L \cdot p_L = 2$, $p_R = 0$, and not vice versa.

3.4 Orbifold compactification

Let us consider the usual inversion map on $\mathbb{R}^k$,

$$x \mapsto -x.$$

The group generated by this transformation is isomorphic to $\mathbb{Z}_2$. Consider again a torus $\mathbb{T}^d = \mathbb{R}^k/\Lambda$, and consider the action of the transformation map on the points of the torus. Even in this case, the inversion map is obviously still invertible. If $e_1, \ldots, e_k$ are independent generators of Λ, we can parametrize a point x^μ on the torus by

$$x = a_1 e_1 + \ldots a_k e_k,$$

with $a_1, \ldots, a_k \in [0, 1[$. It is easy to see that there are exactly 2^d fixed points under the inversion map, and such points are the ones obtained for $a_i \in \{0, \frac{1}{2}\}$. If we identify the points on the torus that are obtained each from the other by the inversion map, we obtain the orbifold $\mathbb{T}^d/\mathbb{Z}_2$. This object is not a manifold, since after the identification we have (conic) singularities in correspondence of the 2^d fixed points.

What we have presented is the geometrical description of a toroidal orbifold. In general, if we have a theory (two dimensional Conformal Field Theory or String Theory) compactified on a torus $\mathbb{T}^d$, we can obtain an orbifold theory whose target space is the quotient $\mathbb{T}^d/\mathbb{Z}_2$.

For such a theory, we want the states of the orbifold theory to be single valued, and hence $\mathbb{Z}_2$-invariant. To accomplish this aim, we may try to perform a projection on the G-invariant states of the original theory. Unfortunately, acting in this way, the modular invariance of the partition function of the theory is generally lost. However, there is a general theoretical recipe (that we call generically *orbifold construction*) that allows us to build a consistent orbifold theory, and to avoid this inconvenience.

3.4.1 Twisted sectors and orbifold construction

Orbifold compactification brings new physical features. In the case of String Theory, it is easy to figure out the existence of a new type of closed strings: if we have a string on the torus whose endpoints are respectively x and $-x$, such string becomes obviously closed in the associated orbifold theory. These new kinds of states represent a new sector of the theory, called *twisted sector*. Consider, as the simplest example, the orbifold theory on $\mathbb{S}^1/\mathbb{Z}_2$. The twisted sector is given by

$$X(\tau, \sigma + 2\pi) = -X(\tau, \sigma), \tag{3.61}$$

where σ^1 denotes, as usual, the "space" dimension, along the $\mathbb{S}^1$ circle. In the case of the $\mathbb{T}^d/\mathbb{Z}_2$ thery, there are 2^d twisted secors, one for each orbifold sigularity. Apparently, the presence of such states is a threat for the locality of the theory, as we will explain below. However, the key to overcome the theoretical difficulties we have highlighted in the previous Section lies in the introduction of twisted sectors.

What we will discuss here is not restricted to the $\mathbb{Z}_2$ symmetry case explained above, as it holds for general Conformal Field Theories and for general finite symmetries. Orbifold construction allows to build new Conformal Field Theories from others Conformal Field Theories. Let G be a (finite) symmetry group of a two dimensional Conformal Field Theory. G acts on the fields of the theory, and we require it to preserve the OPEs, the stress-energy tensor and the vacuum. Heuristically, we want to build a new theory by restricting to the fields that are invariant under G. This seems to be reasonable, since Virasoro algebra should be preserved by the invariance of the stress-energy tensor, and the OPE of two G-invariant fields is still G-invariant. However, as said in the previous Section, a pedestrian application of such a plan in general produces theories whose partition function is not modular invariant anymore, and hence inconsistent. To overcome this problem, we have to add fields: these are the twisted sectors discussed above. Twisted strings are defined as (closed string) worldsheet fields that are periodic up to some transformation $g \in G$:

$$\phi(\tau, \sigma + 2\pi) = g\phi(\tau, \sigma). \tag{3.62}$$

Note that twisted states are in general not invariant under the action of G. There are in genereal many twisted sectors, potentially one for every element of G. After introducing "by force" the twisted fields, the theory loses its locality, in the sense that branch cuts begin to appear in the correlation fuctions. The qualitative explanation is the following: when an arbitrary field of the theory circles around a g-twisted field, it is transformed by the G element g. We need not to worry about this fact, since if we restrict now to G invariant fields/states, a transformation for a G element is simply irrelevant.

The general procedure can hence be summarized in two subsequential steps:

- add the twisted sectors;
- project on the G invariant states.

After the projection, we end up with a local theory whose partition fuction is, as we will now explain, modular invariant. Consider the original partition function of the theory:

$$Z = \mathrm{Tr}\left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\right),$$

where $q = e^{2\pi i \tau}$. Tentatively, let us insert in the trace a projector onto the g-invariant states,

$$\mathrm{Tr}\left[P_G\left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\right)\right], \quad P_G = \frac{1}{|G|}\sum_{g \in G} g.$$

If we consider this partition function from a path integral perspective, we realize that the presence of g in the trace makes the field aperiodic in "time" by a g transformation. This means that we are summing over fields that are periodic in "space", albeit g-twisted in "time". Since modular transformations can exchange the role of "space" and "time" in the path integration, we see that $SL(2, \mathbb{C})$ invariance is broken to a subgroup.

For simplicity, we will consider G an abelian group (for non-abelian symmetries this construction is more delicated). We can avoid this problem by summing over all possible periodicity conditions, in space and time: this is exactly the same as summing over the g-twisted sectors. The correct choice for the partition function is then

$$Z = \sum_g \mathrm{Tr}_{g\text{-twisted sector}}\left[P_G\left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\right)\right], \tag{3.63}$$

or equivalently

$$Z = \frac{1}{|G|} \sum_{g,h \in G} Z_{g,h} \quad \text{where} \quad Z_{g,h} = \mathrm{Tr}_{g\text{-twisted sector}} \left(h q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right). \tag{3.64}$$

Observe that the previous (inconsistent) partition function was simply the sum in Equation (3.64) restricted to the $Z_{1,h}$ terms (we were considering only the 1-twisted sector, i.e. the theory without twisted sectors).

3.5 Other compactifications and dualities

Despite of the theoretical relevance of toroidal compactifications, these are rather unrealistic for many applications, and we have to turn to more complicated examples. In Superstring Theory, an important possibility open by the compactification operation is to break supersymmetries. Observe that, in generale, compactifying on a manifold with a Ricci-flat metric ensures the Weyl invariance of the worldsheet theory, and moreover it is coherent with the equation of motion of (super)gravity if all fields except the metric are set to zero. Supersymmetry-breaking features of a certain m-dimensional manifold $\mathcal{M}$ are related to the holonomy group associated to the mainfold. Under parallel transport along a closed curve on $\mathcal{M}$, a vector v is mapped into a rotated vector Uv, where $U \in O(m)$ or, in the case of oriented manifold, $U \in SO(m)$. These rotations form a subgroup of $O(m)$ or $SO(m)$, called holonomy group. Quite intuitively, for a simply connected group to have a trivial holonomy group is equivalent to have no curvature. In the context of compactifications, the so called *Calabi-Yau manifolds* are of great theoretical importance. They are complex manifolds, so their real dimension is even. The Calabi-Yau manifold CY_n is defined as $2n$-dimensional compact Riemannian manifold with $SU(n) \in SO(2n)$ holonomy group. These are Ricci-flat manifolds, and for $n = 1$ there is only one Calabi-Yau manifold, that is the torus $\mathbb{T}^2$. Also for $n = 2$ there is a unique Calabi-Yau manifold, the K3 manifold: we will mention this object in the next Chapter.

Compactifications are important also because they shed a light on the relations between “different” string theories: there are indeed a great variety of dualities between compactified theories, as we have alluded in the past Chapter. Just to mention a couple of relevant examples, the T-duality we have discussed above acts on type II superstrings on a circle, modifying the GSO projection and exchanging type IIB theory compactified with radius R with type IIA theory compactified with radius $\frac{\alpha'}{R}$. Another case is the string-string, non perturbative duality between type IIA superstring on K3 and heterotic string on $\mathbb{T}^4$, that will be briefly discussed in the next Chapter, as it plays an important role in 4.5. The existance of these dualities are hints that the various consistent theories are the descriptions of the same theory, under different limits. We will not go further in this discussion about general compactifications and dualities, as it would go beyond the scopes of this Thesis.

Chapter 4

Lift of the group action on Narain moduli space to the CFT state space

The topic of this Chapter is the description of group actions on two-dimensional conformal field theories defined by sigma models with toroidal target spaces. In particular, our aim is to investigate some of the problems that arise when a self-duality, i.e. non-trivial equivalences of a model with itself, is lifted to a symmetry of the conformal field theory state space. Toroidal models are among the simplest and best studied conformal field theories and they are one of the main building blocks in many string and superstring compactifications. They are simple enough to be completely solvable, i.e. one can in principle compute all the correlation functions. At the same time, they have a rather rich and complicated structure. For example, apparently different toroidal models are related by an intricate web of dualities. A well-known example of these dualities is the T-duality that set an equivalence between conformal field theories with circle compactification of small and big radius. As we have discussed in the previous Chapter, such duality plays an important role in string theory.

The symmetry action of a self-duality group on the states of the theory will be called a *lift* of that symmetry: there is in general some arbitrariness in the choice of the lift, and it generically implies an increment in the group order associated to the symmetry. As remarked in the Introduction, there exist very abstract descriptions of this subject in literature, but they are of little use for the study of concrete physical models. At the opposite extreme, problems correlated with this topic are often treated with ad hoc methods, valid only for the cases considered. The aim of this Chapter is to fill this gap, proving general results useful to deal with the largest possible class of concrete examples. In particular, our main task will be finding conditions that ensure the existence of an order-preserving lift for self-duality groups.

After a brief review of Conformal Field Theory defined by sigma models with toroidal target space, we will introduce the discussion on the possible arising of the "increased order" phenomenon, and we will study in particular the case of the lift of an order two self-duality to the state space of the theory (Theorem 1). We will find general results that allow to state whether the order of an arbitrary cyclic self-duality is preserved by a consistent choice of lift. We will discover that if all order 2 cyclic symmetries of a model admit lifts that are still of order 2, then *all* cyclic symmetries admit lifts that preserve their order (Theorem 3). We will apply these results to cases of physical relevance, and in particular to the study of the symmetries of heterotic strings on $\mathbb{T}^4$, which are related by a non-perturbative duality to non-linear sigma models on K3 surfaces. (Section 4.5 and 4.6).

4.1 Torus model construction

In this section, we present the bosonic sigma model constrution on a d-dimensional torus T^d. Although this construction can be interpreted in a geometrical way, here we will review it from an abstract point of view, as presented, e.g., in [19]. The field content of this model is given by:

- d real left-moving $U(1)$-currents $j^a(z) = i\partial\phi^a(z)$, along with their right-moving analogues $\bar{j}^a(\bar{z}) = i\bar{\partial}\bar{\phi}^a(\bar{z})$. The holomorphic fields have the following mode expansion:

$$j^a(z) = \sum_{n\in\mathbb{Z}} \alpha^a_n z^{-n-1}, \quad \text{with} \quad \left[\alpha^a_n, \alpha^b_m\right] = n\delta^{ab}\delta_{n,-m}, \tag{4.1}$$

  that correspond to the OPE

$$j^a(z)j^b(w) \sim \frac{\delta^{ab}}{(z-w)^2}. \tag{4.2}$$

  We have analogous relations for the anti-holomorphic fields $\bar{j}^a$ and for their modes $\bar{\alpha}^a_n$;

- the fields $V_\lambda(z,\bar{z})$. These are the vertex operator associated with eigenstates $|\lambda\rangle$ of α^a_0 and $\bar{\alpha}^b_0$, with eigenvalues λ^a_L and λ^b_R, $a, b = 1 \dots d$. The vectors

$$\lambda = (\lambda_L, \lambda_R) := (\lambda^1_L, \dots \lambda^d_L, \lambda^1_R, \dots \lambda^d_R)$$

  form an *even*, *unimodular* lattice, $\Gamma^{(d,d)}$, known as *winding-momentum* or *Narain* lattice, with signature (d,d) and quadratic form

$$\lambda \bullet \mu = \sum_{a=1}^{d} (\lambda^a_L\mu^a_L - \lambda^a_R\mu^a_R). \tag{4.3}$$

  We can give the following definition for the vertex operators:

$$V_{(\lambda_L,\lambda_R)}(z,\bar{z}) =: \exp\left[i\sum_{a=1}^{d}\lambda^a_L\phi^a(z) + i\sum_{a=1}^{d}\lambda^a_R\bar{\phi}^a(\bar{z})\right] : \sigma_{(\lambda_L,\lambda_R)}, \tag{4.4}$$

  with the operators σ_λ satisfying $\sigma_\lambda\sigma_\mu = \epsilon(\lambda,\mu)\sigma_{\lambda+\mu}$ for every $\lambda, \mu \in \Gamma^{d,d}$, where $\epsilon(\lambda,\mu) \in \{\pm 1\}$ is a function from $\Gamma^{d,d} \times \Gamma^{d,d}$ to $\{\pm 1\}$ that obeys:

$$\begin{gathered}\epsilon(\lambda,\mu) = (-1)^{\lambda\bullet\mu}\epsilon(\mu,\lambda), \\ \epsilon(\lambda,\mu)\epsilon(\lambda+\mu,\nu) = \epsilon(\lambda,\mu+\nu)\epsilon(\mu,\nu), \\ \epsilon(\lambda,0) = 1 = \epsilon(0,\lambda).. \end{gathered} \tag{4.5}$$

  The function $\epsilon(\lambda,\mu)$ is what we call a *2-cocycle*, and the second condition of (4.5) is called *cocycle condition*. The conditions (4.5) identify $\epsilon(\lambda,\mu)$ up to a 2-coboundary $\frac{v(\lambda)v(\mu)}{v(\lambda+\mu)}$, with an arbitrary $v(\lambda) \in \{\pm 1\}$, $v(0) = 1$. We have the following OPEs:

$$j^a(z)V_\lambda(w,\bar{w}) \sim \frac{\lambda^a_L}{z-w}V_\lambda(w,\bar{w}), \tag{4.6}$$

$$\bar{j}^a(\bar{z})V_\lambda(w,\bar{w}) \sim \frac{\lambda^a_R}{\bar{z}-\bar{w}}V_\lambda(w,\bar{w}), \tag{4.7}$$

$$V_\lambda(z,\bar{z})V_\mu(w,\bar{w}) \sim \epsilon(\lambda,\mu)(z-w)^{\lambda_L\cdot\mu_L}(\bar{z}-\bar{w})^{\lambda_R\cdot\mu_R}V_{\lambda+\mu}(w,\bar{w}) + \dots. \tag{4.8}$$

The OPE (4.8) justifies why we have to introduce the operators $\sigma_{(\lambda_L,\lambda_R)}$ and the 2-cocycles: indeed, the OPE (4.8) is valid when $|z| > |w|$, i.e. when the operators on the left hand side are radially ordered. When $|w| > |z|$ the operators on the left hand side appear in the opposite order, hence the right hand side gain a factor $(-1)^{\lambda \bullet \mu}$. The presence of a 2-cocycle that obeys the first of the conditions (4.5) compensate this factor, allowing the right hand side to remain the same after the analytic continuation.

The real vector space $\Gamma^{d,d} \otimes \mathbb{R} \simeq \mathbb{R}^{d,d}$ splits into a positive-defined subspace and a negative-defined subspace, respectively spanned by vectors of the form $(v,0)$ and $(0,w)$. The relative positions of these two subspaces with respect to the lattice $\Gamma^{d,d}$ uniquely identify the model. We have hence the following moduli space, called Narain moduli space:

$$\mathcal{N} = O(d,\mathbb{R}) \times O(d,\mathbb{R}) \setminus O(d,d,\mathbb{R})/O(\Gamma^{d,d}), \tag{4.9}$$

where $O(\Gamma^{d,d})$ is the automorphism group of $\Gamma^{d,d}$.

We can build more general Conformal Field Theories, by considering d_L holomorphic currents and d_R anti-holomorphic ones, and having the lattice $\Gamma^{d,d}$ replaced by an even, unimodular lattice Γ^{d_L,d_R}. The sigma model construction can still be adapted to this case, and the Narain moduli space becomes simply

$$\mathcal{N} = O(d_L,\mathbb{R}) \times O(d_R,\mathbb{R}) \setminus O(d_L,d_R,\mathbb{R})/O(\Gamma^{d_L,d_R}). \tag{4.10}$$

We will not discuss here in any detail this construction. In the following, we will assume Γ to be the more general Γ^{d_L,d_R}. We remark that the case $d_R - d_L = 16$ is relevant in heterotic string theory.

4.2 Group action on CFT state space

We deal now with the problem of the lift of a symmetry on the lattice to a symmetry on the Conformal Field Theory (CFT) state space. Let us consider an automorphism $g \in O(\Gamma)$, acting on the even, unimodular lattice $\Gamma = \Gamma^{d_L,d_R}$. We want to describe the action of g on a vertex operator V_λ. Naively, we might expect that the lift of g, $\hat{g}$, acts like:

$$\hat{g}(V_\lambda) = V_{g(\lambda)}.$$

However, after a short calculation we realize that this definition may not respect the OPE (4.8), because of the presence of the cocyles $\epsilon(\lambda,\mu)$. A more careful definition is:

$$\hat{g}(V_\lambda) = \xi_g(\lambda) V_{g(\lambda)}, \tag{4.11}$$

where $\xi_g(\lambda)$ is some phase factor. Imposing consistency with the OPE (4.8):

$$\hat{g}(V_\lambda)\hat{g}(V_\mu) = \hat{g}(V_\lambda V_\mu), \tag{4.12}$$

we find the condition:

$$\epsilon(g(\lambda),g(\mu)) = \epsilon(\lambda,\mu)\frac{\xi_g(\lambda+\mu)}{\xi_g(\lambda)\xi_g(\mu)}. \tag{4.13}$$

The presence of the factor $\xi_g(\lambda)$ in the definition of the lift may give birth to some remarkable effects. Consider for example a g of order 2, i.e. a g such that $g^2 = 1$.

$$\hat{g}^2(V_\lambda) = \xi_g(\lambda)\xi_g(g(\lambda))V_{g^2(\lambda)} = \xi_g(\lambda)\xi_g(g(\lambda))V_\lambda, \tag{4.14}$$

hence $\hat{g}^2 = 1$ if and only if

$$\xi_g(\lambda)\xi_g(g(\lambda)) = 1 \quad \text{for every } \lambda \in \Gamma. \tag{4.15}$$

This condition is equivalent to

$$\xi_g(\lambda) = (-1)^{\frac{1}{2}\lambda \bullet \lambda} \quad \text{for every} \quad \lambda \in (1+g)\Gamma, \tag{4.16}$$

as we discover after a brief calculation: using (4.13) and (4.5) we obtain

$$\begin{aligned}\xi_g(\lambda)\xi_g(g(\lambda)) &= \frac{\epsilon(\lambda, g(\lambda))}{\epsilon(g(\lambda), g^2(\lambda))}\xi_g(\lambda + g(\lambda)) = (-1)^{\lambda \bullet g(\lambda)}\xi_g(\lambda + g(\lambda)) \\ &= (-1)^{\frac{1}{2}(\lambda + g(\lambda))\bullet(\lambda + g(\lambda))}\xi_g(\lambda + g(\lambda)), \quad \text{for every } \lambda \in \Gamma.\end{aligned}$$

These observations lead to a rather natural question: is there a choice of $\xi_g(\lambda)$ such that $\hat{g}$ can be defined to be of order 2? Or, equivalently, does there exist a $\xi_g(\lambda)$ that satisfies both (4.13) and (4.16)? Since we can always choose $\xi_g(\lambda) \in \{\pm 1\}$, in the "worst" case $\hat{g}$ can be defined to be of order 4.

The message of this example is that symmetries of given order defined on Γ do not lift as one might intuitively expect on the CFT state space, in general. In fact, they may lift to symmetries of higher order on the CFT state space. What we want to discover are general conditions that ensure the existence of a lift that preserves the original order of the symmetry. Observe that if such a lift exists, it will only be a special choice among the other permitted lifts, since all lifts that obey condition (4.13) are consistent with the OPE (4.8). In the first part of the following Section, we focus on the lift of an order two automorphism of an even unimodular lattice, and later we will work out explicitly some relevant examples. As we will discover with Theorem 3, among the cyclic symmetry groups, the description of the order two cases is of special importance.

4.3 General results

In order to discuss the lift of the group action on Narain moduli space to the CFT state space, in the present section we build a general theoretical framework. The results obtained will be then applied to concrete examples in the following sections. Despite all of them are derived from scratch, some of these general results are probably known by experts. For instance, Proposition 4.3.1 was derived also by Dolan and Goddard, and Proposition 4.3.5 should be part of the standard lore of the subject. However, the most important results we present are, to the best of our knowledge, new.

Let Γ be an even unimodular lattice. We can describe an explicit choice of the 2-cocycles $\epsilon(\lambda, \mu)$:

Proposition 4.3.1. *Let $\{e_1, \ldots, e_D\}$ be a basis of Γ. Let us associate to every element of this basis e_i an operator γ_i such that $\gamma_i^2 = 1$, $\gamma_i\gamma_j = (-1)^{e_i \bullet e_j}\gamma_j\gamma_i$. Let $\lambda = \sum_i a_i e_i$ be a vector of the lattice Γ, and define $\gamma_\lambda = \gamma_1^{a_1} \ldots \gamma_D^{a_D}$. Define:*

$$\gamma_\lambda \gamma_\mu = \epsilon(\lambda, \mu)\gamma_{\lambda+\mu}.$$

Then $\epsilon(\lambda, \mu)$ obeys the properties (4.5).

Proof. Let λ, μ, ν be vectors of the lattice Γ.
It is easy to observe that $\epsilon(\lambda, \mu)$ belongs to $\{\pm 1\}$. Note that:

$$\epsilon(\mu, \lambda)\gamma_\mu\gamma_\lambda = \gamma_{\lambda+\mu} = \epsilon(\lambda, \mu)\gamma_\lambda\gamma_\mu.$$

From

$$\gamma_\mu\gamma_\lambda = \gamma_\lambda\gamma_\mu(-1)^{\sum_{i,j} a_i b_j e_i \bullet e_j} = \gamma_\lambda\gamma_\mu(-1)^{\lambda\bullet\mu},$$

it follows

$$\epsilon(\lambda, \mu) = (-1)^{\lambda\bullet\mu}\epsilon(\mu, \lambda).$$

To prove the equation

$$\epsilon(\lambda, \mu)\epsilon(\lambda + \mu, \nu) = \epsilon(\lambda, \mu + \nu)\epsilon(\mu, \nu)$$

we have simply to observe that

$$\begin{aligned}
\gamma_{\lambda+\mu+\nu} &= \gamma_{\lambda+\mu}\gamma_\nu\epsilon(\lambda + \mu, \nu) = \gamma_\lambda\gamma_\mu\gamma_\nu\epsilon(\lambda, \mu)\epsilon(\lambda + \mu, \nu),\\
\gamma_{\lambda+\mu+\nu} &= \gamma_\lambda\gamma_{\mu+\nu}\epsilon(\lambda, \mu + \nu) = \gamma_\lambda\gamma_\mu\gamma_\nu\epsilon(\lambda, \mu + \nu)\epsilon(\mu, \nu).
\end{aligned}$$

Finally, from $\gamma_0 = 1$ we have trivially $\epsilon(\lambda, 0) = 1 = \epsilon(0, \lambda)$. □

This particular choice of 2-cocycles enjoys several additional properties. In the following Propositions, we will consider always the 2-cocycles defined by the procedure described in Proposition 4.3.1.

Proposition 4.3.2. *Consider the 2-cocycles associated to the basis $\{e_1, \dots, e_D\}$. Let $\lambda = \sum_i a_i e_i$ and $\mu = \sum_i b_i e_i$ be two vectors of the lattice Γ. Then $\epsilon(\lambda, \mu)$ depends only on the parity of the coefficients a_i and b_i. In particular, if n is even then $\epsilon(n\lambda, \mu) = \epsilon(0, \mu) = 1$, and if n is odd then $\epsilon(n\lambda, \mu) = \epsilon(\lambda, \mu)$.*

Proof. If we consider the procedure described in Proposition 4.3.1, we observe that ${\gamma_i}^{a_i} = \gamma_i^{\pi(a_i)}$, where

$$\pi(a) = \begin{cases} 0 & \text{if } a \text{ is even,} \\ 1 & \text{if } a \text{ is odd.} \end{cases}$$

It is trivial then to conclude that $\epsilon(\lambda, \mu) = \epsilon(\tilde{\lambda}, \tilde{\mu})$, where $\tilde{\lambda} = \sum_i \pi(a_i)e_i$ and $\tilde{\mu} = \sum_i \pi(b_i)e_i$. □

It follows that knowing all 2^{2D} 2-cocycles associated to the possible parities of the coefficients is enough for computing all $\epsilon(\lambda, \mu)$.

Proposition 4.3.3. *Let $\Gamma = \Gamma_A \oplus \Gamma_B$, $\lambda = \lambda_A + \lambda_B$, $\mu = \mu_A + \mu_B$, where $\lambda_A, \mu_A \in \Gamma_A$, $\lambda_B, \mu_B \in \Gamma_B$. Consider the basis $\{e_1, \dots e_N\}$ of Γ_A, the basis $\{\tilde{e}_{N+1}, \dots \tilde{e}_D\}$ of Γ_B, the basis $\{e_1, \dots e_N, \tilde{e}_{N+1}, \dots \tilde{e}_D\}$ of Γ and the 2-cocycles associated to these bases. Then:*

$$\epsilon(\lambda_A + \lambda_B, \mu_A + \mu_B) = \epsilon(\lambda_A, \mu_A)\epsilon(\lambda_B, \mu_B).$$

Proof. Following the procedure described in Proposition 4.3.1, let us associate to the basis elment e_i the operator γ_i and to the basis element $\tilde{e}_j$ the operator $\tilde{\gamma}_j$. From $e_i \bullet \tilde{e}_j = 0$, we deduce that γ_i and $\tilde{\gamma}_j$ commute.
Every element of Γ can be written uniquely as the sum of an element of Γ_A and an

element of Γ_B. Let us consider $\lambda_A = \sum_{i=1}^N a_i e_i$, $\lambda_B = \sum_{i=N+1}^D \tilde{a}_i \tilde{e}_i$, $\mu_A = \sum_{i=1}^N b_i e_i$, $\mu_B = \sum_{i=N+1}^D \tilde{b}_i \tilde{e}_i$. We can compute $\epsilon(\lambda_A + \lambda_B, \mu_A + \mu_B)$:

$$
\begin{aligned}
&\gamma_1^{a_1} \dots \gamma_N^{a_N} \tilde{\gamma}_{N+1}^{\tilde{a}_{N+1}} \dots \tilde{\gamma}_D^{\tilde{a}_D} \gamma_1^{b_1} \dots \gamma_N^{b_N} \tilde{\gamma}_{N+1}^{\tilde{b}_{N+1}} \dots \tilde{\gamma}_D^{\tilde{b}_D} \\
=&\gamma_1^{a_1} \dots \gamma_N^{a_N} \gamma_1^{b_1} \dots \gamma_N^{b_N} \tilde{\gamma}_{N+1}^{\tilde{a}_{N+1}} \dots \tilde{\gamma}_D^{\tilde{a}_D} \tilde{\gamma}_{N+1}^{\tilde{b}_{N+1}} \dots \tilde{\gamma}_D^{\tilde{b}_D} \\
=&\epsilon(\lambda_A, \mu_A)\epsilon(\lambda_B, \mu_B)\gamma_1^{a_1+b_1} \dots \gamma_N^{a_N+b_N} \tilde{\gamma}_{N+1}^{\tilde{a}_{N+1}+\tilde{b}_{N+1}} \dots \tilde{\gamma}_D^{\tilde{a}_D+\tilde{b}_D},
\end{aligned}
$$

from which follows

$$
\epsilon(\lambda_A + \lambda_B, \mu_A + \mu_B) = \epsilon(\lambda_A, \mu_A)\epsilon(\lambda_B, \mu_B).
$$

□

Proposition 4.3.4. *Let n be an integer. Consider the lattice $\Gamma(n)$: we can associate univocally a basis of Γ, $\{e_1, \dots, e_D\}$, to a basis of $\Gamma(n)$, $\{\tilde{e}_1, \dots, \tilde{e}_D\}$, with the invertible linear map $\phi\colon e_i \mapsto \sqrt{n} e_i := \tilde{e}_1$.*

- *If n is even, then the 2-cocycles of $\Gamma(n)$ associated with this basis are all equal to 1.*

- *If n is odd, then the 2-cocycles of $\Gamma(n)$ associated with this basis are related to the 2-cocycles of Γ associated with the correspondent basis by: $\epsilon(\phi(\lambda), \phi(\mu)) = \epsilon(\lambda, \mu)$ for every $\lambda, \mu \in \Gamma$.*

Proof. Observe that $\tilde{e}_i \bullet \tilde{e}_j = n e_i \bullet e_j$. Consider the procedure described in Proposition 4.3.1. If n is even, then all $\tilde{\gamma}_i$ associated with $\{\tilde{e}_1, \dots, \tilde{e}_D\}$ commute, and the 2-cocycles are all simply 1. If n is odd, then all $\tilde{\gamma}_i$ associated with $\{\tilde{e}_1, \dots, \tilde{e}_D\}$ have the same commutation relation of the operators γ_i associated with the corresponding basis $\{e_1, \dots, e_D\}$, since $(-1)^{\tilde{e}_i \bullet \tilde{e}_j} = (-1)^{e_i \bullet e_j}$. Then we conclude that $\epsilon(\phi(\lambda), \phi(\mu)) = \epsilon(\lambda, \mu)$ for every $\lambda, \mu \in \Gamma$. □

Consider now an automorphism of the lattice, $g \in O(\Gamma)$. Define:

$$
\text{the invariant sublattice} \quad \Gamma^g = \{v \in \Gamma | g(v) = v\}\,; \tag{4.17}
$$

$$
\text{the coinvariant sublattice} \quad \Gamma_g = \{w \in \Gamma | w \bullet v = 0 \,\text{for any}\, v \in \Gamma^g\}\,. \tag{4.18}
$$

The properties of these two sublattices will be relevant for our scopes, as we will soon realize. The problem that we address now is the choice of a lift of this symmetry on the CFT state space that keeps the order of the symmetry as low as possible.We present here a showcase of propositions, that will provide us useful tools for studying the lift phenomena. The following results are independent of the choice of the 2-cocycles. Our most important results are Theorem 1 and Theorem 3, that will be powerful enough to deal with our most complicated examples, and Theorem 2, that gives a good characterization of the existence of an order-preserving lift.We will consider first the case of the lift of a symmetry of order 2.

Theorem 1. *Let Γ be an even, unimodular lattice, and let g be an automorphism of Γ of order 2. If $\lambda_1 \bullet \lambda_2$ is even for every $\lambda_1, \lambda_2 \in \Gamma^g$, then there exists a lift of g, $\hat{g}$, that is still of order two.*

Proof. Consider a basis of Γ^g, and complete it to a basis of Γ. Since Γ^g is a primitive sublattice of Γ, every vector v of Γ can be written in an unique way as the sum of a vector $\lambda \in \Gamma^g$ and a vector μ, belonging to the sublattice spanned by the other basis elements. Consider an arbitrary function $\xi_g^0 \colon \Gamma \to \{\pm 1\}$ that satisfies (4.13) (it always exist). Inspired by equation (4.13), for a generic $v = \lambda + \mu \in \Gamma$ define:

$$\xi_g(\lambda + \mu) = (-1)^{\frac{1}{2}\lambda \bullet \lambda} \xi_g^0(\mu) \frac{\epsilon(\lambda, g(\mu))}{\epsilon(\lambda, \mu)}. \tag{4.19}$$

Observe that:

$$\begin{aligned} \xi_g(0 + \mu) &= \xi_g^0(\mu), \\ \xi_g(\lambda + 0) &= (-1)^{\frac{1}{2}\lambda \bullet \lambda}, \end{aligned}$$

hence we have that if the new ξ_g satisfies (4.13), then we can conclude that with this choice we would have an order 2 lift $\hat{g}$. Let us prove that our definition satisfies (4.13). Consider two vectors of Γ, $\lambda_1 + \mu_1$ and $\lambda_2 + \mu_2$, decomposed as above. We must show that

$$\frac{\xi_g(\lambda_1 + \mu_1 + \lambda_2 + \mu_2)}{\xi_g(\lambda_1 + \mu_1)\xi_g(\lambda_2 + \mu_2)} = \frac{\epsilon(\lambda_1 + g(\mu_1), \lambda_2 + g(\mu_2))}{\epsilon(\lambda_1 + \mu_1, \lambda_2 + \mu_2)}. \tag{4.20}$$

Let us focus on the first member of (4.20). Using the definition (4.19)

$$\begin{aligned} &\frac{\xi_g(\lambda_1 + \mu_1 + \lambda_2 + \mu_2)}{\xi_g(\lambda_1 + \mu_1)\xi_g(\lambda_2 + \mu_2)} = \\ &= \frac{\epsilon(\lambda_1 + \lambda_2, g(\mu_1) + g(\mu_2))}{\epsilon(\lambda_1 + \lambda_2, \mu_1 + \mu_2)} \frac{\epsilon(\lambda_1, \mu_1)}{\epsilon(\lambda_1, g(\mu_1))} \frac{\epsilon(\lambda_2, \mu_2)}{\epsilon(\lambda_2, g(\mu_2))} \frac{\xi_g^0(\mu_1 + \mu_2)}{\xi_g^0(\mu_1)\xi_g^0(\mu_2)} \frac{(-1)^{\frac{1}{2}(\lambda_1 + \lambda_2)^2}}{(-1)^{\frac{1}{2}\lambda_1^2}(-1)^{\frac{1}{2}\lambda_2^2}}. \end{aligned}$$

Note that

$$\frac{(-1)^{\frac{1}{2}(\lambda_1 + \lambda_2)^2}}{(-1)^{\frac{1}{2}\lambda_1^2}(-1)^{\frac{1}{2}\lambda_2^2}} = (-1)^{\lambda_1 \bullet \lambda_2} = 1,$$

because $\lambda_1 \bullet \lambda_2$ is even. Remember that ξ_g^0 satisfies (4.13):

$$\frac{\xi_g(\lambda_1 + \mu_1 + \lambda_2 + \mu_2)}{\xi_g(\lambda_1 + \mu_1)\xi_g(\lambda_2 + \mu_2)} = \frac{\epsilon(\lambda_1 + \lambda_2, g(\mu_1) + g(\mu_2))}{\epsilon(\lambda_1 + \lambda_2, \mu_1 + \mu_2)} \frac{\epsilon(\lambda_1, \mu_1)}{\epsilon(\lambda_1, g(\mu_1))} \frac{\epsilon(\lambda_2, \mu_2)}{\epsilon(\lambda_2, g(\mu_2))} \frac{\epsilon(g(\mu_1), g(\mu_2))}{\epsilon(\mu_1, \mu_2)}.$$

Applying the second property of (4.5), the first member of (4.20) becomes

$$\begin{aligned} &\frac{\epsilon(\lambda_1 + g(\mu_1), \lambda_2 + g(\mu_2))}{\epsilon(\lambda_1 + \mu_1, \lambda_2 + \mu_2)} \frac{\epsilon(\lambda_1, \lambda_2)}{\epsilon(\lambda_1, \lambda_2)} \frac{\epsilon(\lambda_2, g(\mu_1) + g(\mu_2))}{\epsilon(\lambda_2, \mu_1 + \mu_2)} \frac{\epsilon(\lambda_1, g(\mu_1))}{\epsilon(\lambda_1, \mu_1)} \frac{\epsilon(\mu_1, \lambda_2 + \mu_2)}{\epsilon(g(\mu_1), \lambda_2 + g(\mu_2))} \\ &\frac{\epsilon(\lambda_1, \mu_1)}{\epsilon(\lambda_1, g(\mu_1))} \frac{\epsilon(\lambda_2, \mu_2)}{\epsilon(\lambda_2, g(\mu_2))} \frac{\epsilon(g(\mu_1), g(\mu_2))}{\epsilon(\mu_1, \mu_2)} \\ =&\frac{\epsilon(\lambda_1 + g(\mu_1), \lambda_2 + g(\mu_2))}{\epsilon(\lambda_1 + \mu_1, \lambda_2 + \mu_2)} \frac{\epsilon(\lambda_2, g(\mu_1) + g(\mu_2))}{\epsilon(\lambda_2, \mu_1 + \mu_2)} \frac{\epsilon(\lambda_1, g(\mu_1))}{\epsilon(\lambda_1, \mu_1)} \frac{\epsilon(\lambda_2, \mu_1 + \mu_2)}{\epsilon(\lambda_2, g(\mu_1) + g(\mu_2))} \\ &\frac{\epsilon(\mu_1, \mu_2)}{\epsilon(g(\mu_1), g(\mu_2))} \frac{\epsilon(\lambda_2, g(\mu_2))}{\epsilon(\lambda_2, \mu_2)} \frac{\epsilon(\lambda_1, \mu_1)}{\epsilon(\lambda_1, g(\mu_1))} \frac{\epsilon(\lambda_2, \mu_2)}{\epsilon(\lambda_2, g(\mu_2))} \frac{\epsilon(g(\mu_1), g(\mu_2))}{\epsilon(\mu_1, \mu_2)} \\ =&\frac{\epsilon(\lambda_1 + g(\mu_1), \lambda_2 + g(\mu_2))}{\epsilon(\lambda_1 + \mu_1, \lambda_2 + \mu_2)}, \end{aligned}$$

i.e. it is equal to the second member of (4.20). Hence, our choice (4.19) respects equation (4.13) and the corresponding lift $\hat{g}$ is of order 2. $\square$

The basic idea of the proof was to impose the condition $\xi_g(\lambda) = (-1)^{\frac{1}{2}\lambda\bullet\lambda}$ on the whole Γ^g. In the end, we have discovered that this procedure works fine. Note that this is only a sufficient condition: the inverse of Theorem 1 is false. A possible counterexample is the case discussed in Subsection 4.6.1.
Observe that for every $\lambda_1, \lambda_2 \in (1+g)\Gamma$ we have that $\lambda_1 \bullet \lambda_2$ is always even. Indeed, there exist $\mu_1, \mu_2 \in \Gamma$ such that $\lambda_1 = \mu_1 + g(\mu_1)$ and $\lambda_2 = \mu_2 + g(\mu_2)$. We have:

$$(\mu_1 + g(\mu_1)) \bullet (\mu_2 + g(\mu_2)) = 2\left(\mu_1 \bullet \mu_2 + \mu_1 \bullet g(\mu_2)\right).$$

Now, we inquire into more general results. First of all, we characterize the possible choice of ξ_g that satisfy Equation 4.13, or equivalently the set of the possible lifts of an automorphism $g \in O(\Gamma)$. We will assume that there always exist a function $\xi_g : \Gamma \to \{\pm 1\}$ that satisfies Equation 4.13. For the sake of generality, we allow ξ_g to be $\mathbb{Z}_N$- or even generically $U(1)$-valued. In the following, H can be either $\mathbb{Z}_N$ or $U(1)$.

Proposition 4.3.5. *Let g be an automorphism of Γ, and let $\xi_g : \Gamma \to H$ satisfies Equation 4.13. A function $\xi'_g : \Gamma \to H$ satisfies Equation 4.13 if and only if there exist a group homomorphism ρ that maps Γ into H such that $\xi'_g(\lambda) = \rho(\lambda)\xi_g(\lambda)$ for every $\lambda \in \Gamma$.*

Proof. Suppose that $\xi'_g : \Gamma \to H$ satisfies Equation 4.13. Define:

$$\rho(\lambda) := \frac{\xi'_g(\lambda)}{\xi_g(\lambda)} \in H.$$

Then, ρ is a group homomorphism between Γ and H:

$$\rho(\lambda+\mu) = \frac{\xi'_g(\lambda+\mu)}{\xi_g(\lambda+\mu)} = \frac{\epsilon(g(\lambda), g(\mu))}{\epsilon(\lambda, \mu)} \frac{\epsilon(\lambda, \mu)}{\epsilon(g(\lambda), g(\mu))} \frac{\xi'_g(\lambda)\xi'_g(\mu)}{\xi_g(\lambda)\xi_g(\mu)} = \rho(\lambda)\rho(\mu).$$

Suppose now that there exists a group homomorphism ρ between Γ and H. Define:

$$\xi'_g(\lambda) := \rho(\lambda)\xi_g(\lambda) \in H.$$

Then, ξ'_g satisfies Equation 4.13:

$$\frac{\xi'_g(\lambda+\mu)}{\xi'_g(\lambda)\xi'_g(\mu)} = \frac{\xi_g(\lambda+\mu)}{\xi_g(\lambda)\xi_g(\mu)} \frac{\rho(\lambda+\mu)}{\rho(\lambda)\rho(\mu)} = \frac{\epsilon(g(\lambda), g(\mu))}{\epsilon(\lambda, \mu)}.$$

□

Proposition 4.3.6. *Let g and h be automorphisms of Γ. If $\hat{g}$ and $\hat{h}$ are choices for the lift or g and h, then the composition $\hat{g} \circ \hat{h}$ is a choice for the lift of $g \circ h$.*

Proof. Consider $\hat{g}\,(V_\lambda) = \xi_g(\lambda)V_\lambda$ and $\hat{h}\,(V_\lambda) = \xi_h(\lambda)V_\lambda$. Let us verify that $\hat{g} \circ \hat{h}$ satisfies (4.13). $\hat{g} \circ \hat{h}\,(V_\lambda) = \xi_g(h(\lambda))\xi_h(\lambda)V_{g\circ h(\lambda)}$.

$$\begin{aligned}\frac{\xi_g(h(\lambda+\mu))\xi_h(\lambda+\mu)}{\xi_g(h(\lambda))\xi_h(\lambda)\xi_g(h(\mu))\xi_h(\mu)} &= \frac{\xi_g(h(\lambda+\mu))}{\xi_g(h(\lambda))\xi_g(h(\mu))} \frac{\xi_h(\lambda+\mu)}{\xi_h(\lambda)\xi_h(\mu)} \\ =\frac{\epsilon(g\circ h(\lambda), g\circ h(\mu))}{\epsilon(h(\lambda), h(\mu))} \frac{\epsilon(h(\lambda), h(\mu))}{\epsilon(\lambda, \mu)} &= \frac{\epsilon(g\circ h(\lambda), g\circ h(\mu))}{\epsilon(\lambda, \mu)}.\end{aligned}$$

□

The groups we consider are all generated by a finite set of elements. Proposition 4.3.6 motivates us to inquire whether, once the lift of the generators is given, the lift of every element of the group remains defined by

$$\widehat{g \circ h} = \hat{g} \circ \hat{h}, \tag{4.21}$$

or, equivalently,

$$\xi_{g \circ h} = \xi_g(h(\lambda))\xi_h(\lambda). \tag{4.22}$$

This is obviously true for the cyclic groups case, where we have only one generator, say g. Indeed, once we have defined the lift of g, $\hat{g}$, the lift of an arbitrary element g^m is set to be $\hat{g}^m$. For now, our focus remains on cyclic groups of arbitrary order n.

Theorem 2. *Let g be an automorphism of order n.*

1. *If there exist a lift of g, $\tilde{g} \leftrightarrow \xi'_g$, of order n, then every lift $\hat{g} \leftrightarrow \xi_g$ of g is such that*

$$\xi_{g^n} = \rho\left(\lambda + g(\lambda) + \cdots + g^{n-1}(\lambda)\right), \tag{4.23}$$

   where ρ is a group homomorphism between Γ and H.

2. *If there exist a lift of g, $\hat{g} \leftrightarrow \xi_g$, such that*

$$\xi_{g^n} = \rho\left(\lambda + g(\lambda) + \cdots + g^{n-1}(\lambda)\right),$$

   where ρ is a group homomorphism between Γ and H, then there exist a lift of g, $\tilde{g} \leftrightarrow \xi'_g$, that is still of order n.

Proof. This result is a direct consequence of Proposition 4.3.5:

1. For Proposition 4.3.5, every possible lift is such that $\xi_g(\lambda) = \rho(\lambda)\xi'_g(\lambda)$, for some group homomorphism ρ between Γ and H.

$$\begin{aligned}
\xi_{g^n}(\lambda) =& \xi_g(\lambda)\xi_g(g(\lambda))\dots\xi_g(g^{n-1}(\lambda)) \\
=& \rho\left(\lambda + g(\lambda) + \cdots + g^{n-1}(\lambda)\right)\xi'_g(\lambda)\xi'_g(g(\lambda))\dots\xi'_g(g^{n-1}(\lambda)) \\
=& \rho\left(\lambda + g(\lambda) + \cdots + g^{n-1}(\lambda)\right)\xi'_{g^n}(\lambda) = \rho\left(\lambda + g(\lambda) + \cdots + g^{n-1}(\lambda)\right).
\end{aligned}$$

2. For Proposition 4.3.5, a possible lift of g is $\xi'_g(\lambda) := \frac{\xi_g(\lambda)}{\rho(\lambda)}$, since $\lambda \mapsto \frac{1}{\rho(\lambda)}$ is a homomorphism between Γ and H. This lift is of order n:

$$\begin{aligned}
\xi'_{g^n}(\lambda) =& \xi'_g(\lambda)\xi'_g(g(\lambda))\dots\xi'_g(g^{n-1}(\lambda)) \\
=& \left[\rho\left(\lambda + g(\lambda) + \cdots + g^{n-1}(\lambda)\right)\right]^{-1}\xi_g(\lambda)\xi_g(g(\lambda))\dots\xi_g(g^{n-1}(\lambda)) \\
=& \left[\rho\left(\lambda + g(\lambda) + \cdots + g^{n-1}(\lambda)\right)\right]^{-1}\xi_{g^n}(\lambda) = 1.
\end{aligned}$$

□

Consider a lift of $g^n = 1$. In general, it will satisfy (4.13):

$$\frac{\xi_{g^n}(\lambda + \mu)}{\xi_{g^n}(\lambda)\xi_{g^n}(\mu)} = \frac{\epsilon(g^n(\lambda), g^n(\mu))}{\epsilon(\lambda, \mu)} = 1,$$

hence

$$\xi_{g^n}(\lambda+\mu) = \xi_{g^n}(\lambda)\xi_{g^n}(\mu), \tag{4.24}$$

that is equal to say that ξ_{g^n} is a group homomorphism between the lattice Γ and H. Another obvious property is that if g is an automorphism of order n and ξ_{g^n} is defined by $\xi_{g^n}(\lambda) = \xi_g(\lambda)\xi_g(g(\lambda))\ldots\xi_g(g^{n-1}(\lambda))$, then we have $\xi_{g^n}(g(\lambda)) = \xi_{g^n}(\lambda)$.
These two easy properties lead us to a remarkable result: in the case of an odd order automorphism g, we can always choose a lift given by a $\mathbb{Z}_2$-valued ξ that preserve the original order of the symmetry.

Proposition 4.3.7. *Let g be an automorphism of odd order n. Then, there exist a lift of g defined by $\mathbb{Z}_2$-valued ξ' that is still of order n.*

Proof. Consider an arbitrary lift, defined by a function $\xi_g : \Gamma \to \{\pm 1\}$ that satisfies Equation 4.13. Define:

$$\xi'_g(\lambda) := \xi_{g^n}(\lambda)\xi_g(\lambda) \quad \text{for every } \lambda \in \Gamma, \tag{4.25}$$

where $\xi_{g^n}(\lambda)$ is defined by $\xi_{g^n}(\lambda) = \xi_g(\lambda)\xi_g(g(\lambda))\ldots\xi_g(g^{n-1}(\lambda))$. Since ξ_{g^n} is a homomorphism between Γ and $\mathbb{Z}_2$, for Proposition 4.3.5 we have that ξ'_g satisfies Equation 4.13 and defines a lift of g. This lift is still of order n:

$$\begin{aligned}
\xi'_{g^n}(\lambda) =& \xi'_g(\lambda)\xi'_g(g(\lambda))\ldots\xi'_g(g^{n-1}(\lambda)) \\
=& \underbrace{\xi_g(\lambda)\xi_g(g(\lambda))\ldots\xi_g(g^{n-1}(\lambda))}_{=\xi_{g^n}(\lambda)}\xi_{g^n}(\lambda)\xi_{g^n}(g(\lambda))\ldots\xi_{g^n}(g^{n-1}(\lambda)) \\
=& \xi_{g^n}(\lambda)\xi_{g^n}(\lambda)\xi_{g^n}(g(\lambda))\ldots\xi_{g^n}(g^{n-1}(\lambda)) \\
=& \xi_{g^n}(\lambda)\underbrace{\xi_{g^n}(\lambda)\xi_{g^n}(\lambda)\ldots\xi_{g^n}(\lambda)}_{n \text{ times}} \\
=& [\xi_{g^n}(\lambda)]^{(n+1)} = 1, \quad \text{for every } \lambda \in \Gamma.
\end{aligned}$$

□

Let us focus now on the cyclics groups of even order. We will start with groups of order 2^n, and we will then show that if all cyclic groups whose order is a power of 2 admit a lift that preserves their order, then every cyclic group admits a lift that preserves its order. Let us introduce a technical lemma:

Lemma 4.3.1. *Let g be a generic automorphism, and let ξ_g define a lift of g. Let n be a positive integer, and consider the lift of g^{2^n} defined by*

$$\xi_{g^{2^n}}(\lambda) = \xi_g(\lambda)\ldots\xi_g(g^{2^n-1}(\lambda)). \tag{4.26}$$

Then, we have

$$\xi_{g^{2^n}}(\lambda) = \frac{\epsilon(\lambda, g(\lambda)+\cdots+g^{2^n-1}(\lambda)}{\epsilon(g(\lambda)+\cdots+g^{2^n-1}(\lambda), g^{2^n}(\lambda))}\xi_g(\lambda+\cdots+g^{2^n-1}(\lambda)). \tag{4.27}$$

Proof. First of all, let us introduce a simple (yet quite ugly) notation, in order to avoid to write nearly illegible equations. Inside the arguments of ξ_g, $\xi_{g^{2^n}}$ and ϵ, we will simply replace $g^m(\lambda)$ with m. With this notation, Equation (4.27) can be written as

$$\xi_{g^{2^n}}(0) = \frac{\epsilon(0, 1+\cdots+(2^n-1))}{\epsilon(1+\cdots+(2^n-1), 2^n)}\xi_g(0+\cdots+(2^n-1)). \tag{4.28}$$

We will prove this Equation by induction. For $n = 1$, the statement trivially holds, since for the lift consistency condition (4.13) we have

$$\xi_{g^2}(0) = \xi_g(0)\xi_g(1) = \frac{\epsilon(0,1)}{\epsilon(1,2)}\xi_g(0+1).$$

To fulfill the inductive step, we will prove that if Equation (4.27) holds for $n-1$, then it holds for n, with $n > 1$. The strategy is to gather the ξ_g factors that appear in Equation (4.26) in two blocks:

$$\xi_{g^{2^n}}(\lambda) = \left[\xi_g(0)\dots\xi_g(2^{n-1}-1)\right]\left[\xi_g(2^{n-1})\dots\xi_g(2^n-1)\right].$$

Then, if we assume that Equation (4.27) holds for $n-1$, we can apply it to each block:

$$\begin{aligned}\xi_{g^{2^n}}(0) =& \frac{\epsilon(0, 1+\dots+(2^{n-1}-1))}{\epsilon(1+\dots+(2^{n-1}-1), 2^{n-1})}\xi_g(0+\dots+(2^{n-1}-1))\\ &\cdot \frac{\epsilon(2^{n-1}, (2^{n-1}+1)+\dots+(2^n-1))}{\epsilon((2^{n-1}+1)+\dots+(2^n-1), 2^n)}\xi_g(2^{n-1}+\dots+(2^n-1)).\end{aligned}$$

We can use the lift consistency condition (4.13) to compute the product

$$\begin{aligned}&\xi_g(0+\dots+(2^{n-1}-1))\xi_g(2^{n-1}+\dots+(2^n-1))\\ =&\frac{\epsilon(0+\dots+(2^{n-1}-1), 2^{n-1}+\dots+(2^n-1))}{\epsilon(1+\dots+(2^{n-1}), (2^{n-1}+1)+\dots+2^n)}\xi_g(0+\dots+(2^n-1)).\end{aligned}$$

The last two equations allow us to write

$$\begin{aligned}\xi_{g^{2^n}}(0) =& \frac{\epsilon(0, 1+\dots+(2^{n-1}-1))}{\epsilon(1+\dots+(2^{n-1}-1), 2^{n-1})}\\ &\cdot \frac{\epsilon(0+\dots+(2^{n-1}-1), 2^{n-1}+\dots+(2^n-1))}{\epsilon(1+\dots+(2^{n-1}), (2^{n-1}+1)+\dots+2^n)}\\ &\cdot \frac{\epsilon(2^{n-1}, (2^{n-1}+1)+\dots+(2^n-1))}{\epsilon((2^{n-1}+1)+\dots+(2^n-1), 2^n)}\xi_g(0+\dots+(2^n-1)).\end{aligned}$$

Now, we are going to use the cocycle property

$$\epsilon(\lambda,\mu)\epsilon(\lambda+\mu,\nu) = \epsilon(\lambda,\mu+\nu)\epsilon(\mu,\nu). \tag{4.29}$$

We get

$$\begin{aligned}\xi_{g^{2^n}}(0) =& \frac{\epsilon(0, 1+\dots+(2^n-1))}{\epsilon(1+\dots+(2^n-1), 2^n)}\\ &\cdot \frac{\epsilon(1+\dots+(2^{n-1}-1), 2^{n-1}+\dots+(2^n-1))}{\epsilon(1+\dots+2^{n-1}, (2^{n-1}+1)\dots(2^n-1))}\\ &\cdot \frac{\epsilon(2^{n-1}, (2^{n-1}+1)+\dots+(2^n-1))}{\epsilon(1+\dots+(2^{n-1}-1), 2^{n-1})}\xi_g(0+\dots+(2^n-1)).\end{aligned}$$

Using again the propery (4.29) on the denominator of this expression, we finally end up

with

$$\begin{aligned}\xi_{g^{2^n}}(0) =& \frac{\epsilon(0, 1+\cdots+(2^n-1))}{\epsilon(1+\cdots+(2^n-1), 2^n)} \\ &\cdot \frac{\epsilon(1+\cdots+(2^{n-1}-1), 2^{n-1}+\cdots+(2^n-1))}{\epsilon(1+\cdots+(2^{n-1}-1), 2^{n-1}\ldots(2^n-1))} \\ &\cdot \frac{\epsilon(2^{n-1}, (2^{n-1}+1)+\cdots+(2^n-1))}{\epsilon(2^{n-1}, (2^{n-1}+1)\cdots+(2^n-1))}\xi_g(0+\cdots+(2^n-1)) \\ =& \frac{\epsilon(0, 1+\cdots+(2^n-1))}{\epsilon(1+\cdots+(2^n-1), 2^n)}\xi_g(0+\cdots+(2^n-1)).\end{aligned}$$

□

Now we can easily show that

Proposition 4.3.8. *Let g be an automorphism of order 2^n. A lift of g defined by ξ_g is still of order 2^n if and only if*

$$\xi_g(\lambda) = (-1)^{\frac{1}{2^n}\lambda\bullet\lambda} \quad \textit{for every } \lambda \in (1+\cdots+g^{2^n-1})\Gamma. \tag{4.30}$$

Proof. This is a direct consequence of Lemma 4.3.1. We have

$$\xi_g^{2^n}(\lambda) = \xi_g(\lambda)\ldots\xi_g(g^{2^n-1}(\lambda)) = 1$$

for every λ in Γ if and only if

$$1 = \frac{\epsilon(\lambda, g(\lambda)+\cdots+g^{2^n-1}(\lambda)}{\epsilon(g(\lambda)+\cdots+g^{2^n-1}(\lambda), g^{2^n}(\lambda))}\xi_g(\lambda+\cdots+g^{2^n-1}(\lambda)).$$

But $g^{2^n}(\lambda) = \lambda$, and applying the property $\epsilon(\lambda,\mu) = (-1)^{\lambda\bullet\mu}\epsilon(\mu,\lambda)$ the previous equation becomes

$$\begin{aligned}1 =& \frac{\epsilon(\lambda, g(\lambda)+\cdots+g^{2^n-1}(\lambda))}{\epsilon(g(\lambda)+\cdots+g^{2^n-1}(\lambda), \lambda)}\xi_g(\lambda+\cdots+g^{2^n-1}(\lambda)) \\ =& (-1)^{\lambda\bullet(g(\lambda)+\cdots+g^{2^n-1}(\lambda))}\xi_g(\lambda+\cdots+g^{2^n-1}(\lambda))\end{aligned} \tag{4.31}$$

Focus on the right side of the last equation. Since Γ is even,

$$(-1)^{\lambda\bullet(g(\lambda)+\cdots+g^{2^n-1}(\lambda))} = (-1)^{\lambda\bullet(\lambda+g(\lambda)+\cdots+g^{2^n-1}(\lambda))},$$

and since g is an automorphism

$$\lambda\bullet(\lambda+g(\lambda)+\cdots+g^{2^n-1}(\lambda)) = \frac{1}{2^n}(\lambda+g(\lambda)+\cdots+g^{2^n-1}(\lambda))\bullet(\lambda+g(\lambda)+\cdots+g^{2^n-1}(\lambda)),$$

hence Equation (4.31) is equivalent to

$$1 = (-1)^{\frac{1}{2^n}(\lambda+g(\lambda)+\cdots+g^{2^n-1}(\lambda))\bullet(\lambda+g(\lambda)+\cdots+g^{2^n-1}(\lambda))}\xi_g(\lambda+\cdots+g^{2^n-1}(\lambda)),$$

and this conclude our proof. □

Hence, the property of a lift to preserve the order 2^n lies into its behaviour on a precise sublattice of the fixed lattice Γ^g. This clearly includes the condition (4.16) as a particular ($n = 1$) case.

Remark 4.3.1. It is worth to observe that if $\lambda \in (1+\cdots+g^{2^n-1})\Gamma$, then $\lambda \bullet \lambda$ is an integer multiple of 2^n, since there exist μ such that $\lambda = (1+\cdots+g^{2^n-1})\mu$, and hence

$$(1+\cdots+g^{2^n-1})\mu \bullet (1+\cdots+g^{2^n-1})\mu = 2^n \mu \bullet (1+\cdots+g^{2^n-1})\mu.$$

Finally, let us discuss the generic even case.

Proposition 4.3.9. *Let g be an automorphism of order $2^n m$, where m is odd. If for every cyclic groups of order 2^n there exist a lift that is still of order 2^n, then there exist a lift of g that is of order $2^n m$.*

Proof. Let us define $p := 2^n$. p and m are coprime integers, that is equivalent to say that there exist integers a, b such that $ap + bm = 1$. Hence, we have

$$(g^p)^a (g^m)^b = g,$$

and this tells us that we can write g as the composition of two commuting automorphisms, $g_1 = (g^p)^a$ and $g_2 = (g^m)^b$, such that $g_1^m = 1$ and $g_2^p = 1$. If there exist lifts of g_1 and g_2, defined by ξ_{g_1} and ξ_{g_2}, that preserve the respective orders, then we can define a lift of g that is still of order $2^n m$. Indeed, for Proposition 4.3.6 we have that $\xi_g(\lambda) := \xi_{g_1}(g_2(\lambda))\xi_{g_2}(\lambda)$ defines a lift of g. Let us show that this lift preserves the order of g:

$$\begin{aligned}
\xi_g(\lambda)\dots\xi_g(g^{pm-1}(\lambda)) =& \xi_{g_1}(g_2(\lambda))\dots\xi_{g_1}((g_1g_2)^{pm-1}g_2(\lambda))\xi_{g_2}(\lambda)\dots\xi_{g_2}((g_1g_2)^{pm-1}(\lambda)) \\
=& \xi_{g_1}(g_2(\lambda))\xi_{g_1}(g_1g_2(\lambda))\dots\xi_{g_1}(g_1^{m-1}g_2(\lambda)) \\
& \cdot \xi_{g_1}(g_2g_2(\lambda))\xi_{g_1}(g_1g_2g_2(\lambda))\dots\xi_{g_1}(g_1^{m-1}g_2g_2(\lambda)) \\
& \dots \\
& \cdot \xi_{g_1}(g_2^{p-1}g_2(\lambda))\xi_{g_1}(g_1g_2^{p-1}g_2(\lambda))\dots\xi_{g_1}(g_1^{m-1}g_2^{p-1}g_2(\lambda)) \\
& \cdot \xi_{g_2}(\lambda)\xi_{g_2}(g_2(\lambda))\dots\xi_{g_2}(g_2^{p-1}(\lambda)) \\
& \cdot \xi_{g_2}(g_1(\lambda))\xi_{g_2}(g_1g_2(\lambda))\dots\xi_{g_2}(g_1g_2^{p-1}(\lambda)) \\
& \dots \\
& \cdot \xi_{g_2}(g_1^{m-1}(\lambda))\xi_{g_2}(g_1^{m-1}g_2(\lambda))\dots\xi_{g_1}(g_1^{m-1}g_2^{p-1}(\lambda)),
\end{aligned} \tag{4.32}$$

and every line of the last passage is equal to 1, since $\xi_{g_1}(\mu)\dots\xi_{g_1}(g_1^{m-1}(\mu)) = 1$ and $\xi_{g_2}(\mu)\dots\xi_{g_2}(g_2^{p-1}(\mu)) = 1$ hold for every μ.

As a final remark, it is not completely obvious that we can organize the ξ factors as in the last passage, and the very reason lies in the fact that p and m are coprime integers. Indeed, if we consider two elements A and B of some abelian group, and A and B have respectively orders m and p, we can organize the group elements $(AB)^k$, $k = 1, \dots, mp-1$, in the following matrix:

$$\begin{pmatrix}
A^0B^0 & A^1B^1 & \dots & A^{m-1}B^{m-1} \\
A^0B^m & A^1B^{m+1} & \dots & A^{m-1}B^{2m-1} \\
\dots & \dots & \dots & \dots \\
A^0B^{(p-1)m} & A^1B^{(p-1)m+1} & \dots & A^{m-1}B^{pm-1}
\end{pmatrix}.$$

Reading the columns, we observe that modding the powers that appears on B for p we obtain every elements of $\{0, 1, \dots, p-1\}$, without repetitions (actually, to prove this statement, it is sufficient to show that there are no repetitions, since we are considering p

integers to be modded by p). To prove by contradiction this statement, consider the first column (this is enough, since the powers that appear on the q-th column are simply shifted by $q-1$). The powers that appear are $0, m \mod p, \ldots, (p-1)m \mod p$. Suppose that $r_1 = k_1 m \mod p$ and $r_2 = k_2 m \mod p$ are equal for some integers $k_1, k_2 \in 0, \ldots, p-1$, and without loss of generality assume $k_1 \geq k_2$. Then there will be integers q_1, q_2 such that

$$\begin{aligned} k_1 m &= q_1 p + r_1, \\ k_2 m &= q_2 p + r_2, \\ r_1 &= r_2, \end{aligned}$$

and hence

$$(k_1 - k_2)m = (q_1 - q_2)p.$$

$(k_1 - k_2) < p$, and if $k_1 \neq k_2$ we would have that the least common multiple of m and p is smaller or equal than $(k_1 - k_2)m$. But this is impossible, since m and p are coprime integers, and their least common multiple is mp, that is strictly greater than $(k_1 - k_2)m$.

We can then shuffle every column in the following way:

$$\begin{pmatrix} A^0B^0 & A^1B^0 & \ldots & A^{m-1}B^0 \\ A^0B^1 & A^1B^1 & \ldots & A^{m-1}B^1 \\ \ldots & \ldots & \ldots & \ldots \\ A^0B^{p-1} & A^1B^{p-1} & \ldots & A^{m-1}B^{p-1} \end{pmatrix},$$

and this shows that it is possible to obtain the desired order in the last passage of Equation (4.32). □

Now we have nearly the whole picture about cyclic groups. Let us highlight what we have learned so far about the general question:

Proposition 4.3.10. *Let Γ be an even, unimodular lattice. Every cyclic group of automorphisms of Γ admits a lift that preserves its order if and only if every cyclic group of automorphisms of Γ whose order is a power of 2 admits a lift that preserves its order.*

Before going on with the exposition of our true general result, let us find some conditions for a group of order 2^n that assure the existence of a lift that is still of order 2^n. We can adapt one of our earlier results, Theorem 1, by simply taking into account that the condition (4.16) is straightforwardly generalized by Proposition 4.3.8. If we consider the case of a group of order 2^n, and follow the original proof of Theorem 1, it is immediate to conclude

Proposition 4.3.11. *Let Γ be an even, unimodular lattice, and let g be an automorphism of Γ of order 2^n. If $\lambda_1 \bullet \lambda_2$ is a multiple of 2^n for every $\lambda_1, \lambda_2 \in \Gamma^g$, then there exists a lift of g, $\hat{g}$, that is still of order 2^n.*

But this is not the end of the story: take back the condition (4.30). For every λ in Γ, we have

$$\begin{aligned} &(\lambda + \cdots + g^{2^n-1}(\lambda)) \bullet (\lambda + \cdots + g^{2^n-1}(\lambda)) \\ =&2^n(\lambda \bullet \lambda + \lambda \bullet g(\lambda) + \cdots + \lambda \bullet g^{2^{n-1}-1}(\lambda) \\ &\quad + \lambda \bullet g^{2^{n-1}}(\lambda) + \lambda \bullet g^{2^{n-1}+1}(\lambda) + \cdots + \lambda \bullet g^{2^n-1}(\lambda)) \\ =&2^n(\lambda \bullet \lambda + \lambda \bullet g(\lambda) + \cdots + \lambda \bullet g^{2^{n-1}-1}(\lambda) \\ &\quad + \lambda \bullet g^{2^{n-1}}(\lambda) + g^{2^{n-1}-1}(\lambda) \bullet \lambda + \cdots + g(\lambda) \bullet \lambda) \\ =&2^n(\lambda \bullet \lambda + 2(\lambda \bullet g(\lambda) + \cdots + \lambda \bullet g^{2^{n-1}-1}(\lambda)) + \lambda \bullet g^{2^{n-1}}(\lambda)), \end{aligned}$$

and remembering that $\lambda \bullet \lambda$ is even, we have

$$(-1)^{\frac{1}{2^n}(\lambda+\cdots+g^{2^n-1}(\lambda))\bullet(\lambda+\cdots+g^{2^n-1}(\lambda))} = (-1)^{\lambda\bullet g^{2^{n-1}}(\lambda)} = (-1)^{\frac{1}{2}(\lambda+g^{2^{n-1}}(\lambda))\bullet(\lambda+g^{2^{n-1}}(\lambda))},$$

and hence we can replace condition (4.30) with an equivalent one, that is

$$\xi_g(\lambda) = (-1)^{\frac{1}{2}\lambda\bullet\lambda} \quad \text{for every } \lambda \in (1+g^{2^{n-1}})\Gamma. \tag{4.33}$$

Clearly, $g^{2^{n-1}}$ is an order two symmetry, and this condition strongly reminds us of condition (4.16). Observe that

Lemma 4.3.2. *Let g be an automorphism of Γ. Then*

1. $(1+\cdots+g^{2^n-1})\Gamma \subset (1+g^{2^{n-1}})\Gamma$*;*

2. $\Gamma^g \subset \Gamma^{g^{2^{n-1}}}$.

Proof. Easily:

1. consider $\lambda \in \Gamma$.

$$(1+\cdots+g^{2^n-1})\lambda = (1+g^{2^{n-1}})(1+\cdots+g^{2^{n-1}-1})\lambda \in (1+g^{2^{n-1}})\Gamma;$$

2. consider $\lambda \in \Gamma^g$. $g(\lambda)=\lambda$ implies $g^{2^{n-1}}(\lambda)=\lambda$.

□

We are now tempted to suppose that if all order two symmetries admit lifts that preserve their order, then the same thing should be possible for the order 2^n symmetries. Indeed,

Proposition 4.3.12. *Let g be an automorphism of order 2^n. If $g^{2^{n-1}}$ admits a lift that is still of order 2, defined by $\xi_g^{2^{n-1}}$, then there exist a lift of g that is still of order 2^n.*

Proof. We will explicitly build the choice of lift ξ_g that preserve the order 2^n. The way we will follow is extremely close to the strategy adopted in the proof of Theorem 1.

Consider a basis of Γ^g, and complete it to a basis of Γ. Since Γ^g is a primitive sublattice of Γ, every vector v of Γ can be written in an unique way as the sum of a vector $\lambda \in \Gamma^g$ and a vector μ, belonging to the sublattice spanned by the other basis elements. Consider an arbitrary function $\xi_g^0\colon \Gamma \to \{\pm 1\}$ that satisfies (4.13) (it always exists). For a generic $v = \lambda + \mu \in \Gamma$ define:

$$\xi_g(\lambda+\mu) = \xi_{g^{2^{n-1}}}(\lambda)\xi_g^0(\mu)\frac{\epsilon(\lambda, g(\mu))}{\epsilon(\lambda,\mu)}. \tag{4.34}$$

Observe that:

$$\xi_g(0+\mu) = \xi_g^0(\mu),$$
$$\xi_g(\lambda+0) = \xi_{g^{2^{n-1}}}(\lambda).$$

We have that $\xi_{g^{2^{n-1}}}$ defines a lift that is of order two, hence $\xi_{g^{2^{n-1}}}(\lambda) = (-1)^{\frac{1}{2}\lambda\bullet\lambda}$ for $\lambda \in (1+g^{2^{n-1}})\Gamma$, but this is exactly (4.33)! Hence, we have that if ξ_g satisfies (4.13), then we can conclude that with this choice we would have an order 2^n lift of g. Let us prove that our definition satisfies (4.13). Consider two vectors of Γ, $\lambda_1+\mu_1$ and $\lambda_2+\mu_2$, decomposed as above. We must show that

$$\frac{\xi_g(\lambda_1+\mu_1+\lambda_2+\mu_2)}{\xi_g(\lambda_1+\mu_1)\xi_g(\lambda_2+\mu_2)}=\frac{\epsilon(\lambda_1+g(\mu_1),\lambda_2+g(\mu_2))}{\epsilon(\lambda_1+\mu_1,\lambda_2+\mu_2)}. \tag{4.35}$$

Let us focus on the first member of (4.35). Using the definition (4.34)

$$\begin{aligned}&\frac{\xi_g(\lambda_1+\mu_1+\lambda_2+\mu_2)}{\xi_g(\lambda_1+\mu_1)\xi_g(\lambda_2+\mu_2)}=\\ =&\frac{\epsilon(\lambda_1+\lambda_2,g(\mu_1)+g(\mu_2))}{\epsilon(\lambda_1+\lambda_2,\mu_1+\mu_2)}\frac{\epsilon(\lambda_1,\mu_1)}{\epsilon(\lambda_1,g(\mu_1))}\frac{\epsilon(\lambda_2,\mu_2)}{\epsilon(\lambda_2,g(\mu_2))}\frac{\xi^0_g(\mu_1+\mu_2)}{\xi^0_g(\mu_1)\xi^0_g(\mu_2)}\frac{\xi_{g^{2^{n-1}}}(\lambda_1+\lambda_2)}{\xi_{g^{2^{n-1}}}(\lambda_1)\xi_{g^{2^{n-1}}}(\lambda_2)}.\end{aligned}$$

Note that

$$\frac{\xi_{g^{2^{n-1}}}(\lambda_1+\lambda_2)}{\xi_{g^{2^{n-1}}}(\lambda_1)\xi_{g^{2^{n-1}}}(\lambda_2)}=1,$$

because this is exactly (4.13) for $\xi_{g^{2^{n-1}}}$ on $\Gamma^{g^{2^{n-1}}}$, and λ_1,λ_2 are vectors of Γ^g, that for Lemma 4.3.2 is a subset of $\Gamma^{g^{2^{n-1}}}$.

Remember that ξ^0_g satisfies (4.13):

$$\frac{\xi_g(\lambda_1+\mu_1+\lambda_2+\mu_2)}{\xi_g(\lambda_1+\mu_1)\xi_g(\lambda_2+\mu_2)}=\frac{\epsilon(\lambda_1+\lambda_2,g(\mu_1)+g(\mu_2))}{\epsilon(\lambda_1+\lambda_2,\mu_1+\mu_2)}\frac{\epsilon(\lambda_1,\mu_1)}{\epsilon(\lambda_1,g(\mu_1))}\frac{\epsilon(\lambda_2,\mu_2)}{\epsilon(\lambda_2,g(\mu_2))}\frac{\epsilon(g(\mu_1),g(\mu_2))}{\epsilon(\mu_1,\mu_2)}.$$

The rest of the calculations are perfectly identical to the ones of the proof of Theorem 1, and it results that our claim (4.35) is true. ☐

We can now summarize almost everything we learned in the following

Theorem 3. *Let Γ be an even, unimodular lattice. Every cyclic group of automorphisms of Γ admits a lift that preserves its order if and only if every cyclic group of automorphisms of Γ whose order is 2 admits a lift that preserves its order.*

4.4 Example: $\Gamma=\Gamma^{2,2}$

In this section we discuss three simple examples of symmetry lift on $\Gamma^{2,2}$, the lattice associated to the two-dimensional torus. Our aim is to find a lift of the lowest possible order. These examples are easy enough to be solved in an elementary and explicit way.

More generally, let us consider $\Gamma^{d,d}$, and let us choose a basis $\{e_1,\dots e_d\}$ such that:

$$e_i\bullet e_j=\begin{pmatrix}0&\mathbb{I}_d\\ \mathbb{I}_d&0\end{pmatrix} \tag{4.36}$$

Following the procedure described in Proposition 4.3.1 we can easily compute the 2-cocycles associated to this basis: consider $\lambda,\mu\in\Gamma^{d,d}$, $\lambda=\sum_{i=1}^{2d}a_ie_i$, $\mu=\sum_{i=1}^{2d}b_ie_i$, $\gamma_\lambda=\gamma_1^{a_1}\dots\gamma_{2d}^{a_{2d}}$, $\gamma_\mu=\gamma_1^{b_1}\dots\gamma_{2d}^{b_{2d}}$. Observe that only the pairs (γ_i,γ_{d+i}), $i=1,\dots,d$, anticommute, while any other pair of operators commute.

$$\begin{aligned}\gamma_1^{a_1}\dots\gamma_{2d}^{a_{2d}}\gamma_1^{b_1}\dots\gamma_{2d}^{b_{2d}}=&(-1)^{b_1a_{d+1}}\gamma_1^{a_1+b_1}\dots\gamma_{2d}^{a_{2d}}\gamma_1^{b_2}\dots\gamma_{2d}^{b_{2d}}\\ =&\dots=(-1)^{\sum_{i=1}^d b_ia_{d+i}}\gamma_1^{a_1+b_1}\dots\gamma_{2d}^{a_{2d}+b_{2d}}.\end{aligned}$$

Then the following explicit expression for the 2-cocycles holds:

$$\epsilon(\lambda,\mu) = (-1)^{\sum_{i=1}^{d} b_i a_{d+i}}. \tag{4.37}$$

It is easy to check that the first equation of (4.5) is satisfied:

$$\begin{aligned}\epsilon(\lambda,\mu)(-1)^{\lambda\bullet\mu} =&(-1)^{\sum_{i=1}^{d} b_i a_{d+i}+\sum_{i=1}^{d}(b_i a_{d+i}+a_i b_{d+i})} = (-1)^{2\sum_{i=1}^{d} b_i a_{d+i}+\sum_{i=1}^{d} a_i b_{d+i}}\\ =&(-1)^{\sum_{i=1}^{d} a_i b_{d+i}} = \epsilon(\mu,\lambda).\end{aligned}$$

Let us come back to the special case $d = 2$. Our explicit expression for cocycles (4.37) becomes

$$\epsilon(\lambda,\mu) = (-1)^{b_1 a_3 + b_2 a_4}.$$

Let us consider the following automorphisms, defined through their action on the basis elements:

1. $e_1 \to e_3$, $e_2 \to e_2$, $e_3 \to e_1$, $e_4 \to e_4$;
2. $e_1 \to -e_1$, $e_2 \to e_2$, $e_3 \to -e_3$,$e_4 \to e_4$;
3. $e_i \to -e_i$, $i = 1,\dots,4$.

The last two cases, 2 and 3, are actually trivial: since we have $\epsilon(g(\lambda), g(\mu)) = \epsilon(\lambda,\mu)$, equation (4.13) becomes

$$\xi_g(\lambda+\mu) = \xi_g(\lambda)\xi_g(\mu),$$

hence we can choose $\xi_g(\lambda)$ to be simply 1 for every $\lambda \in \Gamma^{2,2}$. This choice trivially leads to a lift $\hat{g}$ of order 2.

The first case is less obvious: equation (4.13) becomes

$$\frac{\xi_g(\lambda+\mu)}{\xi_g(\lambda)\xi_g(\mu)} = \frac{\epsilon(g(\lambda), g(\mu))}{\epsilon(g(\lambda), g(\mu))} = \frac{(-1)^{b_3 a_1 + b_2 a_4}}{(-1)^{b_1 a_3 + b_2 a_4}} = (-1)^{b_3 a_1 + b_1 a_3}.$$

As we know, to have an order two lift is equivalent to require $\xi_g(\lambda)\xi_g(g(\lambda)) = 1$ for every $\lambda \in \Gamma^{2,2}$. After some attemps, one can guess that a good choice could be

$$\xi_g(\lambda) = (-1)^{a_1 a_3}. \tag{4.38}$$

Let us prove it. With this choice,

$$\frac{\xi_g(\lambda+\mu)}{\xi_g(\lambda)\xi_g(\mu)} = (-1)^{(a_1+b_1)(a_3+b_3)-a_1a_3-b_1b_3} = (-1)^{b_3 a_1 + b_1 a_3},$$

so it respect equation (4.13). It is also a lift of order 2:

$$\xi_g(\lambda)\xi_g(g(\lambda)) = (-1)^{a_1 a_3}(-1)^{a_3 a_1} = 1.$$

Observe that case 1 has an interesting physical interpretation: e_1 and e_3 can be interpreted as winding and momentum with respect to one of the circles of $\mathbb{T}^2$, and e_2 and e_4 as the ones of the other circle. This symmetry is hence the T-duality on the first circle, and we have proved that this symmetry admits an order 2 lift. Observe that in [12] this lift is not recognized as a T-duality, since it acts with a minus sign on $V_{e_1+e_3}$. However, following most of the literature, we do not impose such restriction.

4.5 Symmetries of Heterotic String Theory on $\mathbb{T}^4$

The previous example was only a warm-up: this time we will deal with a much more complicated case. Moreover, we will discuss it in detail, since it is directly related with a model of physical relevance. We will consider the lattice $\Gamma^{4,20}$: let us explain first the physical motivation that makes this lattice and its symmetries interesting.

In the article [9] Gaberdiel, Hohenegger and Volpato classified the symmetries of sigma models of type IIA superstring on K3. As we have remarked in 3.5, there is a non-perturbative, string-string duality between type IIA superstring on K3 and the heterotic string on the four dimensional torus $\mathbb{T}^4$. In the previous Chapter we have observed that the moduli space of heterotic string theory compactified on $\mathbb{T}^4$ is exactly

$$O(4,\mathbb{R}) \times O(20,\mathbb{R}) \setminus O(4,20,\mathbb{R})/O(\Gamma^{4,20}),$$

that, thanks to the duality, is also the moduli space of non-linear sigma models on K3 [1] [14]. The symmetry group of the two dual theories should be the same. In [9], Gaberdiel, Hohenegger and Volpato focused on that symmetries of K3 sigma models that preserve the superconformal algebra (and spectral flow operators). What they discovered is that such symmetry group of sigma models of type IIA superstring on K3 is a subgroup of $O(\Gamma^{4,20})$. This can sound surprising: from the string-string duality and the discussion presented at the begining of this Chapter, we expect that what acts on the states of the Conformal Field Theory is an extension of $O(\Gamma^{4,20})$, or, equivalently, that the elements $O(\Gamma^{4,20})$ have a projective action on the vertex operators of the theory. This puzzle can be easily understood: since we are considered sigma model, the only objects we are taking in account in the Conformal Field Theory on the type IIA side of the duality are fundamental strings. Now, there are physical reasons why the group action on the fundamental strings of the theory should be trivial: heuristically speaking, the extension of the symmetry group is related to certain gauge groups, and the fundamental strings are chargeless under these gauge groups. There exist charged objects inside the complete theory, and they are extended and generically massive (D-branes, precisely). To discuss the true groups that act on the states of the theory, in principle one has to work out the group actions on these charged objects, and this is in general a difficult task. Nevertheless, thanks to the string-string duality, the charged, extended objects of the theory are mapped to fundamental strings of the heterotic theory: our plan hence is to study the lift of the symmetry groups by looking at the group action on the vertex operators of the heterotic theory. This is a smarter way, and the general result we have proven will allow us to draw precise conclusions.

Remark 4.5.1. The "charge" we have mentioned in the type IIA theory is the RR charge, and it corresponds, under the string-string duality, to the winding number of the heterotic string.

To fulfill our scope, we first present the main points of the theoretical construction that lead to the classification of the symmetries of the model we are interested in, following the proof given by Gaberdiel, Hohenegger and Volpato in [9]. Then, we will employ the machinery we have developed in this Chapter to give an aswer to our problem: we will show that, for all cyclic symmetry groups of our theory, there exist a choice of lift that preserves their order. This will be done in Section 4.6.

4.5.1 Symmetry groups classification

The construction we present here is quite heuristic, since we want only to explain the main steps and ideas of the proof given by Gaberdiel, Hohenegger and Volpato in [9], without discussing the many technical details. First, let us introduce some reasonable physical restrictions on the self-dualities we are interested in. These symmetry groups appear as subgroups of $O(\Gamma^{4,20}) \cap (O(4,\mathbb{R}) \times O(20,\mathbb{R}))$. The first restriction is that such symmetries have to fix *pointwise* (not only setwise) the 4-dimensional subspace of given signature, that from now will be called Π. These symmetries act only (as a rotation) on the 20-dimensional subspace of given signature. This is physically equivalent to ask that these symmetries commute with space-time supersymmetries. Supersymmetries are related to gravitinos, that appear in the heterotic picture as massless fields in the Ramond sector, and for our requirement the symmetry action is trivial on gravitinos. Let us explain in greater detail: suppose that this restriction does not hold. In this case, the O(4) rotation would affect the α_0^i modes, but these are connected to the fermionic modes by supersymmetry. To preserve the OPEs, we have to act also on the Ramond sector. The zero modes ϕ_0^i obey Clifford algebra $\{\phi_0^i, \phi_0^j\} = 2\delta^{ij}$, i.e. they are Dirac matrices that act on the Ramond sector. Conside a Romond state $|u\rangle$, and let $|v\rangle$ be $\psi_0^i\,|u\rangle$ for some index i. If we act with a symmetry g, we end up with

$$|gv\rangle = g(\psi_0^i)\,|gu\rangle\,.$$

Hence, to fix the gravitinos, we have to keep fixed the fermionic modes, and hence the 4-dimensional subspace of given signature Π.

Remark 4.5.2. From the point of view of the Conformal Field Theory associated to type IIA superstring on K3, this first requirement is equivalent to ask our symmetries to respect the superconformal algebra and the spectral flow operators.

We enforce a second restriction: we want to avoid points of enhanced symmetry. This is because in the Conformal Field Theory associated to type IIA superstring on K3 we are excluding D-branes, and this is possible because, in the perturbative string limit, they are generically very massive objects. However, in those points these object become massless, and our Conformal Field Theory would become singular. Mathematically, this requirement is translated into the statement that the subspace Π is not ortogonal to any vector of $\Gamma^{4,20}$ with squared norm -2. This is rephrased in the condition that if $(0, p_R) \in \Gamma^{4,20}$ then $p_R \cdot p_R \neq 2$.

These are our conditions. Now, let us find the self-dualities that obey them. The difficulties start from the very beginning: the group $O(\Gamma^{4,20})$ is discrete, but infinite. The key of the proof is to bring ourselves back to the case of a finite group. Π will be surely fixed by a certain subgroup G_Π of $O(\Gamma^{4,20})$ (generically, the trivial group $\{1\}$; for "special" Π, a bigger one). We have then the sublattice fixed by this group, Γ^{G_Π}, of signature $(4, d)$, and the sublattice

$$\Gamma_{G_\Pi} := \left(\Gamma^{G_\Pi}\right)^\perp \cap \Gamma^{4,20},$$

of signature $(0, 20 - d)$, as follows from $\Pi \subset \Gamma^{G_\Pi} \otimes \mathbb{R}$. In particular, $\Pi \perp \Gamma_{G_\Pi}$. Applying the second restriction, we require that Γ_{G_Π} does not contain vector of squared lenght -2 (Γ_{G_Π} is negative defined). Then, the next, crucial, technical step is to prove that a lattice with the properties of Γ_{G_Π} can be primitively embedded in the negative defined Leech lattice $\Lambda(-1)$. The Leech lattice Λ is defined as the unique even, self-dual, positive defined, 24-dimensional lattice without vectors of square lenght 2.

Now: Γ_{G_Π} is the lattice where G_Π effectively acts, since Γ^{G_Π} is fixed. Moreover, the action of G_Π can be extended to a subgroup of the automorphism group of the Leech lattice. Such group is finite, and the automorphisms of the Leech lattice have been all classified. What we have said above can be rephrased in the following way: Γ_{G_Π} is isomorphic to a primitive sublattice Λ_{G_Π} of Λ, and the action of G_Π can be extended to the whole Λ, in a way that left the sublattice

$$\Lambda^{G_\Pi} := (\Lambda_{G_\Pi})^\perp \cap \Lambda$$

fixed. At this point, G_Π is isomorphic to a subgroup of $O(\Lambda)$, a well known group called Conway group Co_0. The conclusion is the following:

Theorem 4 (Gaberdiel, Hohenegger, Volpato)**.** *Every self-duality group G is isomorphic to a subgroup of $O(\Lambda)$ that fixes a sublattice Λ^G of dimension $(4+d)$ (at least, 4).*

A complete classification of the subgroups of $O(\Lambda)$ satisfying these properties, for every d, was given by Höhn and Mason [13]. The classification of the cyclic groups can be found in [16]: these are 42, and there exist only 3 cyclic groups of order 2.

Summarizing, we have started from the sublattice Π, and we have foung G_Π as a subgroup of $O(\Gamma)$. We may ask if the inverse statement does hold: if we start from a subgroup G of $O(\Gamma)$, is there a model that realizes that symmetry (i.e., an appropriate subspace Π with the right properties)? The answer is: yes, and there can even be more than one model that realizes the subgroup as a self-duality that respects our restrictions. The proof is somewhat similar to the one described above, in the sense that its steps are followed in the inverse order. With this remark we conclude this short digression on the classification of the symmetries of Heterotic String Theory on $\mathbb{T}^4$. In the next section, we will study how these symmetries (in particular, the cyclic ones) are realized on the states of the Conformal Field Theory. More precisely, we will give an answer to the question whether there exists a choice of lifts that preserve the order of the cyclic groups of self-dualities.

4.6 Example: $\Gamma = \Gamma^{4,20}$

Lead by the motivations explained in the previous Section, let us consider

$$g \in O(\Gamma^{4,20}) \cap (O(4,\mathbb{R}) \times O(20,\mathbb{R}))\,.$$

We are interested in the automorphisms g such that

- Γ^g has $(4,n)$ signature;
- Γ_g does not have any vector v of lenght $v^2 = -2$.

These properties are invariant under conjugation: $g \to hgh^{-1}$, $h \in O(\Gamma^{4,20})$. Hence we have to discuss only one example for every conjugacy class. If we focus on symmetries of order 2, we have only three classes. If they admit lifts that are still of order 2, then for Theorem 3 we will conclude that every cyclic symmetry of $\Gamma^{4,20}$ admits a lift that preserves its order.

4.6.1 Order 2: first case

For this case we have an explicit description of the action of g.

$$\Gamma^{4,20} = \Gamma^{4,4} \oplus E_8(-1) \oplus E_8(-1), \tag{4.39}$$

$$v \in \Gamma^{4,20}, \quad v = (v_A, v_B, v_C), \quad v_A \in \Gamma^{4,4}, \quad v_B, v_C \in E_8(-1),$$

$$g(v_A, v_B, v_C) = (v_A, v_C, v_B). \tag{4.40}$$

Recalling equation (4.13)

$$\epsilon(g(\lambda), g(\mu)) = \epsilon(\lambda, \mu) \frac{\xi_g(\lambda + \mu)}{\xi_g(\lambda)\xi_g(\mu)}, \tag{4.41}$$

we decompose

$$\lambda = (\lambda_A, \lambda_B, \lambda_C), \quad \mu = (\mu_A, \mu_B, \mu_C).$$

For Proposition 4.3.3 we can write the 2-cocycle obtained by the procedure described in Proposition 4.3.1 in the following way:

$$\epsilon(\lambda, \mu) = \epsilon(\lambda_A, \mu_A)\epsilon(\lambda_B, \mu_B)\epsilon(\lambda_C, \mu_C),$$

$$\epsilon(g(\lambda), g(\mu)) = \epsilon((\lambda_A, \lambda_C, \lambda_B), (\mu_A, \mu_C, \mu_B)) = \epsilon(\lambda_A, \mu_A)\epsilon(\lambda_C, \mu_C)\epsilon(\lambda_B, \mu_B) = \epsilon(\lambda, \mu),$$

hence equation (4.13) becomes

$$\xi_g(\lambda + \mu) = \xi_g(\lambda)\xi_g(\mu). \tag{4.42}$$

We can choose $\xi_g(\lambda) = 1$ for every $\lambda \in \Gamma^{4,20}$, and with this simple choice the lift $\hat{g}$ is order 2.
We observe that for this transformation we can easily compute Γ^g, $(1+g)\Gamma$ and Γ_g.

- The condition $(v_A, v_B, v_C) = (v_A, v_C, v_B)$ is equivalent to $v_B = v_C$, so

$$\Gamma^g = \Gamma^{4,4} \oplus E_8(-2), \quad e_i \bullet e_j = \left(\begin{array}{cc|c} 0 & \mathbb{I}_4 & \\ \mathbb{I}_4 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & E_8(-2) \end{array}\right).$$

- From the explicit action of $(1+g)$ on (v_A, v_B, v_C):

$$(1+g)(v_A, v_B, v_C) = (2v_A, v_B + v_C, v_B + v_C),$$

  we conclude

$$(1+g)\Gamma = \Gamma^{4,4}(4) \oplus E_8(-2), \quad e_i \bullet e_j = \left(\begin{array}{cc|c} 0 & 4\mathbb{I}_4 & \\ 4\mathbb{I}_4 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & E_8(-2) \end{array}\right).$$

- Finally, the condition $(v_A, v_B, v_C) = (-v_A, -v_C, -v_B)$ is equivalent to $v_A = 0$, $v_B = -v_C$, so $\Gamma_g = E_8(-2)$.

4.6.2 Order 2: second case

In this case, we do not need an explicit description of the transformation, but we know the fixed lattice Γ^g from [16]:

$$\Gamma^g = \Gamma^{4,4}(2), \quad e_i \bullet e_j = \begin{pmatrix} 0 & 2\mathbb{I}_4 \\ 2\mathbb{I}_4 & 0 \end{pmatrix}. \tag{4.43}$$

The only information we need is that for every $\lambda \in \Gamma^g$ we have that $\lambda \bullet \lambda$ is an integer multiple of 4 (this is a trivial observation, since $\Gamma^{4,4}$ is even and $\Gamma^g = \Gamma^{4,4}(2)$). Note also that if $\lambda \bullet \lambda$ is an integer multiple of 4 for every $\lambda \in \Gamma^g$, then $\lambda_1 \bullet \lambda_2$ is even for every $\lambda_1, \lambda_2 \in \Gamma^g$. Indeed,

$$(\lambda_1 + \lambda_2)^2 - \lambda_1^2 - \lambda_2^2 = 2\lambda_1 \bullet \lambda_2$$

is an integer multiple of 4, and then $\lambda_1 \bullet \lambda_2$ is even. Hence, from Theorem 1 we conclude that there exists a lift of this symmetry that is still of order 2.
Observe also that the condition (4.16)

$$\xi_g(\lambda) = (-1)^{\frac{1}{2}\lambda\bullet\lambda} \quad \text{for every} \quad \lambda \in (1+g)\Gamma$$

becomes simply

$$\xi_g(\lambda) = 1 \quad \text{for every} \quad \lambda \in (1+g)\Gamma.$$

4.6.3 Order 2: third case

As in the previous case, we only need some information from the fixed lattice Γ^g:

$$\Gamma^g = \mathbb{Z}^4(2) \oplus \mathbb{Z}^8(-2). \tag{4.44}$$

Since $\lambda_1 \bullet \lambda_2$ is even for every $\lambda_1, \lambda_2 \in \Gamma^g$, from Theorem 1 we conclude again that there exist a lift of this symmetry that is still of order 2.

Finally, for Theorem 3, we conclude that every cyclic symmetry group of $\Gamma^{4,20}$ admits a lift that preserves its order.

Conclusion and research perspectives

We have presented, with an increasing degree of abstraction, the structure of two-dimensional Conformal Field Theories, highlighting the role palyed by vertex operators. We have reviewed Bosonic String Theory and introduced Superstring Theory, as the main physical motivation of our work comes from these theories. In particular, we have discussed Heterotic String Theory and the phenomenon of enhanced symmetry. We have then described string compactifications, in particular we have focused our attention on both toroidal and orbifold compactifications, and we have gave some elements of more complicated Calabi-Yau manifold compactification. We have faced then the main problem of the Thesis, the study of the lift of symmetry groups that comes from self-dualities of a toroidal two-dimensional Conformal Field Theory. We have found original general results, that culminated in Theorem 3, and we have applied them to the study of cases of physical interest, most notabily the study of the lift of the symmetries of heterotic string theories on $\mathbb{T}^4$ that are dual to K3 sigma models. What we have proved shows that all the cyclic symmetry groups of self-dualities of type IIA superstrings on K3 admit lifts that preserve their order. Outside our general results, crucial to this analysis were the non-perturbative duality between this theory and heterotic strings on the 4-dimensional torus $\mathbb{T}^4$ and the symmetries classification by Gaberdiel, Hohenegger and Volpato presented in [9]. Our results show that, for the models considered and for any cyclic group, the orbifold construction performed in many papers (for example, [16]) is consistent, as there exists a choice of lift that preserves the order of the symmetry on the Conformal Field Theory state space. This however requires a precise choice of lift, and in principle there can be reasons to prefer other permitted lifts that would lead to a symmetry of greater order.

Our analysis opens the door to some interesting problems. Restricting now on the cyclic symmetries, we have seen that the self-duality symmetries of a model admit lifts that preserve their order if and only if this is true for the order two cyclic groups: we have found sufficient conditions to ensure that, e.g. Theorem 1. The most obvious question is whether there exist a complete characterization of order two cyclic groups that admits order-preserving lifts. It would be useful to find and analyze some counterexamples, to find an aswer to that issue.

Another generalization would be the case of arbitrary Abelian, and even non-Abelian symmetry groups. This correspond to a great complexity enhancement of the yet complicated problem, since even the definition of symmetry lift for all the elements of the groups become unhandy. A help may come from group cohomology, as our topic is strictly related to this subject. Indeed, cocycles and symmetry lifts can described in terms of central extensions of groups, that is related with second cohomology. An increase of abstraction may be the price to pay to fly over the technical difficulties we have struggled with in the proofs of our results.

Finally, a rather natural continuation of this work would be a deeper discussion of consistency conditions for orbifold construction. Orbifolds are among the reasons why we are interested in the order of the symmetries. Not always orbifold construction is consistent: there can be problems with the twisted fields OPEs. These problems can manifest as the loss of modular invariance of the partition function of the theory, and with the loss of associativity of twisted fields OPEs. We observe that these two issues are connected: indeed, if the OPE are associative, then the partition function is modular invariant, and the inverse statement should hold for orbifold build with respect to a cyclic symmetry group. A characterization of the conditions that preserve the associativity of the OPEs would be a remarkable result, and our discussion of symmetry lift can be useful to face this problem.

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