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**"Time Series Forecasting for Retail Sales: A Comparative Study of
Traditional Econometric Models and a Machine
Learning Approach"**

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A handwritten signature in black ink, consisting of a stylized, cursive script that is difficult to decipher but appears to be a personal name.

Firma (signature)

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Dedication

"I wholeheartedly dedicate this thesis to my incredible parents, whose unwavering support and relentless commitment to providing me with the finest education have illuminated the path to my radiant and promising future."

Abstract

Aggregate retail sales is one of the important economic indicators which requires precision in forecasting. Most often data scientists and business managers use time series forecasting to predict sales. But it is still a question which time series forecasting method is a better choice. Therefore, this research evaluates and investigates different time series forecasting methods: Box-Jenkins ARIMA, Additive Holt Winters, and the FB Prophet by Facebook on a time series retail sales data. The data used in forecasting was downloaded from the US Census Bureau. Moreover, the forecasting was done over two different time periods split. The rationale was to check the performance of the models in different macroeconomic conditions. The evaluation metric Mean Absolute Percentage Error (MAPE) was used. The Diebold Mariano test was implemented to compare the forecasting accuracy of two competing models. Overall, the results of the research suggest that ARIMA and Winters models outperformed FB Prophet in the first period and ARIMA being the best performing model in first Period having the lowest MAPE ratio and producing significant DM statistics. However, in the second period, the forecasting accuracy of all the models is approximately the same.

1. Introduction

Modern economies have many drivers and private consumption is one of them. This research is related to US aggregate retail sales forecasting. In United States the contribution of retail trade to the GDP is more than 5% retrieved from FRED. This significant contribution to the economy makes retail sales follow similar trends to those followed by the whole economy. Therefore, most of the times, the time series retail sales data observes trend and seasonality. Furthermore, Retail sales forecasting plays a vital role in times of macroeconomic instability, as it assists the policy makers to forecast the shocks on the retail sales and in bigger picture, the economy.

The effect of retail sales forecasting is not only confined to the economy but also, in today's extremely competitive markets, plays a vital role in the performance of any business as well. It can help inventory management by avoiding overproduction and reducing overstocking (Islam & Amin, 2020; Nguyen et al., 2021). In addition, forecasting retail sales at the market level can be crucial for planning and identifying potential growth channels (Fildes et al., 2022). The incorporation of strong seasonality in the retail sales forecasts results in effective predictions of profitability and the allocation of resources. For suppose the forecast done around the fourth quarter (holidays season) marked by events like Christmas and Black Friday can help to do an effective inventory management. Thus, due to the increasing use of big data and data analysis, various functional areas of retail companies can benefit from accurate sales forecasts (Fisher & Raman, 2018) or in contrast no sales forecast results in companies reducing or ceasing to operate (V. Shah, 2019). Moreover, it is also possible to forecast retail sales of any specific company to expand this study.

Time series forecasting has a significant reputation in academia as well. As a matter of fact, for many years, researchers have made great efforts to develop effective models to improve forecast accuracy (Adhikari & Agrawal, 2013). Sometimes using a method that does not fit well to the given time series yields inaccurate results. Therefore, forecasters should be careful to choose a forecast methodology that is appropriate for the situation (Puthran et al., 2014)

In this dissertation we aim to display how different time-series forecasting methods, be it traditional econometrics or machine learning, compare their forecasting accuracy on a univariate

time series data. The data was taken from the US Census Bureau for 28 years from 1992-2019 with monthly frequency. Furthermore, the data was split into two parts to observe how do the models behave in the times of macroeconomic stability and instability. We classify the models used in our analysis into two types. (1) Traditional econometrics models: Box-Jenkins ARIMA (G. E. P., et al Box, 2015) and Holt Winters Exponential Smoothing (Winters, 1960), (2) Machine Learning model: FB Prophet or Prophet (Taylor & Letham, 2018). The models are described below.

ARIMA is the natural extension of Auto Regressive Integrated Moving Average models (G. E. P., et al Box, 2015). Of all the features and advantages, modelling of non-stationary series to a stationary series by taking a sequential difference, makes ARIMA more robust. Furthermore, parsimony is the key principle behind the Box-Jenkins method as it requires that our model should not include any additional coefficients, which may potentially reduce the degrees of freedom in estimation. By integrating AR, differencing, and MA orders, ARIMA becomes more flexible and robust in the prediction of time series data (Hasan et al., 2022).

In our analysis Holt Winters Exponential Smoothing method has also played a significant role. Holt Winters is a popular method to forecast seasonal time series (Rahman et al., 2016). The reason to apply this model is that it is only applicable when there is a presence of trend and seasonality in the time series. So, after the general plotting we came to conclusion that our data has trend and seasonality both. However, among the two available approaches for Holt Winters Exponential Smoothing method (1) Additive, (2) Multiplicative, we chose the former after diagnosing the seasonal decomposition of both the time periods. However, the parameters of α , γ , and δ (level, trend, and seasonality) were found from the Grid Search method, which is more robust as it also considers those hyper parameters for smoothing, which were not previously smoothed (Tran & Van Dai, 2020).

In recent times machine learning models have been in trend for time series forecasting. Therefore, the application of FB Prophet model on our time series data in this study plays a vital role. The prophet model (Taylor & Letham, 2018), built by core data science team of Facebook, belongs to the family of Generalized additive models (GAM). And due to this characteristic FB Prophet model can incorporate different variables with different characteristics like holidays or any other specific event for example, it can consider the data that what is the sales of Christmas trees at Coop Supermercato around the Christmas season? or the sales of hand sanitizers at Lidl

during COVID-19 times. Hence, our research question is: *“does the FB Prophet model surpass the forecasting performance of Box-Jenkins ARIMA & Holt Winters Exponential Smoothing models for predicting aggregate retail sales in the US?”*

Therefore, answers to this question can assist in identifying the best-performing model which can be used to forecast retail sales in specific contexts. The results generated from this forecast can aid future research in selecting the appropriate model for forecasting. In a practical setting, policy makers and business owners can take advantage of the findings and models used in this paper to enhance their forecasts. Moreover, we hope that the findings of this paper will bring positive impact beyond academia.

The research proceeds with holistic literature review incorporating the research done previously on the models used in this paper. Next, the data section displays the plot of entire data with a segregation line pointing two periods having different macroeconomic conditions. Furthermore, the methodology part describes every model in detail and how we have implemented it using different programming languages libraries. The results section discusses the outputs produced by the models and then the research ends with a discussion and conclusion in which the recommendations for possible future work were proposed.

2. Literature Review

Introduction

This chapter reviews the existing literature on the evolution of time series. How traditional and machine learning approaches applied in this field and the significant contributions of the previous researchers. The literature highlighted prominent studies of researchers who contributed to the field of econometrics and machine learning. We also did comparative analysis based upon existing studies and found the gap in the study.

The relevant literature presented in this section has been published in the field of time series forecasting. Several studies suggest that Box-Jenkin's ARIMA is a promising model in the field of sales forecasting with reliable results. The studies presented in this section have also been applied as inspiration in terms of methodology.

Moreover, our research compares the prediction performance of machine learning with traditional methods. To observe, we want to know whether the FB prophet model surpasses the prediction performance of Box-Jenkins, ARIMA, and HW exponential smoothing models to predict aggregate retail sales in the US.

2.1 Time Series Forecasting

A collection of observations done sequentially through time is known as a time series. For example, sales of any specific product in consecutive months, monsoon rain at a particular location in a consecutive month, electricity consumption in specific vicinity for consecutive one-hour periods. The time series forecasting applications include economic planning, sales forecasting, inventory/stock management, production planning, annual budgeting, evaluation of the economic strategies, model evaluation etc. Some of the applications discussed previously provides self-explanation. For example, an accurate forecast for sales at any specific period will assist in efficient production. Furthermore, time series data provides an opportunity to observe a behavior called "out of sample". The time series forecasts can assist in the evaluation of the model, when trying to fit models incorporating time series. The verification of a fitted model is usually carried out by examining the goodness of the connection of the same data used to estimate the model parameters known as the in-sample fit (Chatfield, 2000). The observations provided by the time series model

can be cross checked with the actual observations and based on the results we can declare either in-sample or out of sample a winner.

Time series forecasting is important in every field of research. It is one of the most common and used learning tasks every day, and businesses use it every day to prepare for sales increases and drops, prepare for disasters when observing meteorological data, forecast daily stock prices and foreign exchange rates.

Over the last few decades, time-series modelling has attracted the attention of the research community. The aim of time series modelling is to collect, analyze, and study the past, and to develop appropriate models to describe the nature of the series. Then, the model explains the time series and predicts its future value, i.e., forecasting. Time series forecasting is important in every field of research. It is one of the most common and used learning tasks every day, and businesses use it every day to prepare for sales increases and drops, prepare for disasters when observing meteorological data, forecast daily stock prices and foreign exchange rates.

Various forecasting methods have been developed and proposed in the previous literature, but in this study, only univariate sales data was forecasted.

Computing forecasts from past and present values is called the forecasting method. In broad terms, according to (Chatfield, 2000) forecasts methods are classified into 3 types:

1. Judgment forecasts are based on subjective judgment, intuition, internal knowledge, or other relevant information.
2. Univariate methods where the forecast depends only on the present and past values of a single series that is forecasted, for example, on a time function for a linear trend.
3. Multivariate method depends, in at least some cases, on the value of more than one time series variable called explanatory variable. If a variable depends jointly, it also depends on a multivariate model, including multiple equations.

2.2 Traditional Econometrics Models

Academics, statisticians, and economists have proposed extensive research and models to improve the accuracy of the forecast. Therefore, a variety of time series models are developed and

improved, but this abundance of models does not necessarily mean that these models are universally applicable. The most popular and widely used approaches are still statistical models such as HW and ARIMA. However, in recent years, machine learning models have become increasingly competitive with traditional approaches and provide more expandability to the models mentioned above (Ben Baccar, 2019).

Types of Traditional Econometrics Models

2.2.1 Auto Regressive Integrated Moving Average (ARIMA)

The ARIMA model is widely used in short-term time series with high regularity and periodicity. This model has been extensively studied and applied in forecast studies due to its attractive theoretical properties and various empirical evidence. In addition, the ARIMA model is equivalent to most exponential smoothing models, except for the multiplicative form of Holt-Winters.

ARIMA is a natural extension of Auto Regressive and Moving Average Model (G. E. P. , et al Box, 2015). In addition to all the features and advantages, ARIMA can model nonstationary series into stationary series by taking the sequence of differences. Using these differences, ARIMA can be used more flexibly and robustly for any data set and for comparison of benchmarks with other forecast algorithms.

The Inception of ARIMA Models

The emergence of ARIMA models can be traced back to the novel research work of Box and Jenkins (1970). They combined autoregressive (AR) and moving average (MA) models with the idea of differencing to create a general flexible approach for time series forecasting and modelling and in the result the foundation of Box-Jenkins methodology was laid, which, now, has become standard operating procedure for fitting of ARIMA models and it is widely applied in different fields.

ARIMA Models Theoretical Foundations

There are three key components of ARIMA models: Autoregressive (AR), differencing (I), and Moving Average (MA) processes. A particular aspect of time series modelling is addressed by each component. The components are discussed below:

- **Autoregressive Component:** An autoregressive (AR) component is denoted by P, which in other words called as P^{th} order. It represents the number of lags (past values) that are included in the model. The relationship built between the current value of time series and its past values is modeled by autoregressive component.
- **Differencing Component:** The aim of differencing component is to make the series stationary. The stationarity of the time series can be achieved by the removal of trend and seasonality from the data. The order of differencing d is denoted by ARIMA models that indicates the number of times the, before the modelling, time series is differenced.
- **Moving Average Component:** The moving average (MA) component is denoted by q, which is called as q^{th} order in other words. It represents the number of lags (past errors) that are included in the model. The relationship built between the current value of time series and its past values is modeled by moving average component.
- The ARIMA (p, d, q) model allows possible model specifications with a wide range of its components to fit in different time series patterns.

Box-Jenkins Methodology: The fitting of ARIMA models is a systematic approach. It consists of the following steps:

- **Data Processing:** For an effective and efficient forecasting of the ARIMA model the data needs to be analyzed for the presence of any possible outlier and its removal, missing values, or NAN values.
- **Model Identification:** To identify the appropriate orders of p, d, q the plotting of autocorrelation function (ACF) and partial autocorrelation function (PACF) is done.
- **Parameter Estimation:** The parameters are estimated from various methods like maximum likelihood estimation (MLE), method of conditional least square.
- **Model Diagnostics:** Through Ljung-Box test, autocorrelation residual plots and normality test, the residuals of the fitted models can be analyzed to check model's misspecification.
- **Model Forecasting:** For the desired forecast horizon, the fitted ARIMA model is used to generate forecast also with the prediction intervals to assess the uncertainty associated with the forecasts.

ARIMA Models Application in Time Series Forecasting

For time series forecasting applications ARIMA models are used in many different fields, especially when dealing with time series data. Some of the renowned fields are:

- **Forecasting In Financial Markets:** ARIMA models are widely used to predict financial period sequences, such as stock prices, exchange rates and interest rates, and offer valuable insight into investment decision and risk management (Tsay, 2005).
- **Forecasting Of Macroeconomic Variables:** In the economic field, the ARIMA model is used to predict macroeconomic variables such as GDP growth, inflation and unemployment and contributes to the formulation and evaluation of economic policies (Stock & Watson, 2003).
- **Forecasting Demand:** In industries such as retail, energy, and transport, ARIMA models are used to predict demand, help allocate resources, manage inventory, and plan capacity. (Snyder, 2014)
- **Forecasting For Climate and Environment:** ARIMA models are also used to predict climate and environmental variables such as temperature, rainfall, and air pollution levels, and are essential to environmental management and policy making (G. E. P., et al Box, 2015).

ARIMA Models Limitations and Extensions: Despite being at a broad use, ARIMA models have some limitations. Some of them are discussed below:

- **Assumption For Linearity:** For the underlying time series process, ARIMA models take the assumption that the process is linear. However, this assumption does not apply to all the time series processes to those exhibiting nonlinear or complex dynamics.
- **Assumption For Stationarity:** ARIMA models requires the time series to be stationary, which is not applicable to all the time series. And this results in the loss of information or over-transformation.

- **Uncertainty for Model Selection:** The method for selecting optimal values for p , d , q is subjective and can vary based on the selection of criteria and it is also subjective.

To deal with these limitations for ARIMA models were addressed with several extensions and alternatives.

- **SARIMA or Seasonal ARIMA Models:** The SARIMA or Seasonal ARIMA models contain the seasonal components to capture patterns, both seasonal and regular, in time series data (Box & Jenkins, 1970).
- **ARFIMA or Fractionally Integrated ARIMA Models:** These models allow fractional order differentiation and provide a more flexible approach to modeling long-distance dependency in time series (Granger & Joyeux, 1980).
- **Non-Gaussian and non-linear ARIMA Models:** Many extensions of ARIMA as non-Gaussian and non-linear are proposed such as: (TARIMA) threshold ARIMA models by (Tong, 1983), (GARIMA) generalized ARIMA models by (Davis & Dunsmuir, 1996), and (EARIMA) exponential ARIMA models by (G. E. Box & Tiao, 1975).

For several years ARIMA models have been used as a primary tool for forecasting time series. Several researches have been done to observe the forecasting performance of ARIMA models. The research done by (Chatterjee et al., 2021) for the comparison of different time series models found that ARIMA is the best performing model when forecasting future stock price in shorter period. Furthermore, the work done by (Lolea et al., 2021) observed that the ARIMA (0,1,1) is suitable for stock price prediction, and it yields the least Mean Square Error (MSE). (Choy et al., 2021) observed that ARIMA model of order (4,0,1) is a better performer when forecasting stock prices of any market sector. So, we observed that both the research works have small forecast errors for ARIMA as compared to other time series forecasting models for future stock prices prediction.

The well-established framework and the robustness of ARIMA models to capture and predict linear time series patterns makes it well versed in the research forecasting community. Since, ARIMA can model nonstationary series into stationary series by taking the sequence of differences so using these differences, ARIMA can be used more flexibly and robustly for any data set, and for comparison of benchmarks with other forecast algorithms. Their wide acceptance in various

applications increases the ARIMA model's popularity, despite having some limitations. The proposition of alternatives and further extensions of ARIMA models has expanded their scope. In this study we have compared the forecasting performance of ARIMA VS FB Prophet. Empirical findings suggest that the predicted value of the ARIMA model is more accurate compared to the FB Prophet model.

2.2.2 Holt Winters Models

The renowned Holt Winters models are widely used in time series forecasting because of their characteristics of being simple, and the ability to handle the seasonal data. They are also known as Exponential Smoothing State Space Models. Holt initially introduced an extension, for handling linear trends, of simple exponential smoothing, and later Winters extended it by incorporating seasonal components in it. For Holt Winters Model, we will provide a descriptive overview in this literature including the theoretical background, time series forecasting application, and the limitation and extension.

The Inception of Holt Winters Models

There are several types of exponential smoothing methods (Holt, 2004). Holt-Winters (HW) method originated from 1950s when renowned researchers met to satisfy a request from the Office of Naval Research, Planning, and Control of Industrial Operations in the United States. The main objective was to develop a high accuracy and low-cost forecasting model that could integrate with the existing system. The researcher Charles Holt in 1957 showed that the forecasting method most often used at that time, the method of exponentially weighted moving average could not only be used to smooth the level of a variable but also to smooth the trend, seasonality, and other components of forecasting. According to (V. Shah, 2019), Holt-Winters Exponential Smoothing (HW) methodology is one of the easiest and most widely used in the sales data industry with seasonal patterns and trends.

Holt Winters Theoretical Foundations

Holt Winters consists of the forecast equation and three smoothing equations, one for level, one for trend component, and one for seasonal index, each of them corresponds to smoothing parameters: α, γ, δ respectively. Furthermore, the HW models are classified into two types: (1) Holt Winters Additive (2) Holt Winters Multiplicative models. Each type has a different characteristic.

Holt Winters Multiplicative model is applied when the time series has seasonal fluctuations with a propensity to increase or decrease with respect to time. The three smoothing parameters α, γ, δ represents level, trend and seasonality components in the model.

However, Holt Winters Additive model is applied when the seasonal fluctuations of the time series are constant. The HW-Additive model also has three smoothing parameters α, γ, δ for level, trend, and seasonality. In this study, we have used an additive approach because when we did the seasonal decomposition, we observed that the seasonality amplitude is constant over time and there is an upward trend.

Holt Winters Forecasting and Model Fitting:

The process of forecasting and model fitting with Holt Winters is described below:

- **Data Processing:** For an effective and efficient forecasting the data needs to be analyzed for the presence of any possible outlier and its removal, missing values or Nan values.
- **Model Identification:** Through seasonal decomposition we can identify the model that needed to be implemented. The seasonal decomposition gives a holistic view in terms of the behavior of data at level, the type of trend, seasonality and the remaining is residual.
- **Parameter Estimation:** The parameters α, γ, δ for level, trend and seasonality can be estimated from various methods like maximum likelihood estimation (MLE), method of least square estimation or grid search method.
- **Model Diagnostics:** Through Ljung-Box test, autocorrelation residual plots and normality test, the residuals of the fitted models can be analyzed to check model's misspecification.
- **Model Forecasting:** For the desired forecast horizon, the fitted HW model is used to generate forecast. In the forecast seasonal patterns are considered.

Holt Winters Models Application in Time Series Forecasting:

For time series forecasting applications Holt Winters models are used in many different fields, especially when dealing with time series data. Some of the renowned fields are:

Forecasting For Tourism and Transportation: Holt-Winters models are used to predict arrival and demand for tourist goods, helping to plan and manage resources in the tourism and transportation sectors (Li et al., 2011).

Forecasting For Climate and Environment: The Holt-Winters model also uses the weather and seasonal environmental variables (temperature, precipitation, air quality, etc.) model to provide key information for environmental management and policymaking (Hyndman & Khandakar, 2008).

Forecasting For Retail Sales: In the retail industry, Holt-Winters models are used to forecast product sales based on seasonal demand patterns and to support inventory management and marketing strategies (Kahn et al., 2012).

Forecasting For Energy Demand and Energy Price: Holt-Winters models are widely used to predict energy consumption such as electricity, gas, and other utilities and provide valuable information for resource allocation and infrastructure planning (Taylor, 2010).

Holt Winters Models Limitations and Extensions:

Despite having their wide use, Holt Winters models have some limitations. Some of them are discussed below:

- **Assumption For Linearity:** For the underlying time series process, HW models take the assumption that the process is linear. However, this assumption does not apply to all the time series processes to those exhibiting non-linear or complex dynamics.
- **Assumption For Stationarity:** HW models requires the time series to be stationary, which is not applicable to all the time series. And this results in the loss of information or over-transformation.
- **Parameter Selection Uncertainty:** The method for selecting optimal values for smoothing parameters of level, trend and seasonality (α, γ, δ) can vary based on the selection of criteria and is also subjective.

To deal with these limitations for Holt Winters models were addressed with several extensions and alternatives.

- **Nonlinear, Non-Gaussian, and Multivariate Components with State Space Models:** These models extend Holt-Walter's basic framework by incorporating nonlinear, non-gaussian, and multivariate components and offer a more flexible method for modeling complex time series data (Harvey & Koopman, 2012).
- **Bayesian approaches For HW models:** Bayesian methods are used to adapt and predict Holt-Winters models, incorporating previous information, quantifying parameters, and model uncertainty (Petris et al., 2009).

In recent years, HW models have emerged as an efficient way for forecasting time series that has the incorporation of seasonal patterns. The extension to exponential smoothing framework enables the HW models to observe seasonality and trend both. Even though the HW models have some limitations, their characteristics of being flexible, simple, and intuitive have played a significant role in the model's popularity. The research of (Trull et al., 2020) demonstrated that with seasonal parameters Holt Winters generally does a better prediction rather than without seasonal parameters and incur fewer errors. The work of (Chatterjee et al., 2021) and (Choy et al., 2021) is based on comparison of prediction performance of Holt Winters triple exponential smoothing VS ARIMA models. The results for this study demonstrated the superior performance of ARIMA leaving behind Holt Winters for both small and large data sets. It is worth mentioning that (Mgale et al., 2021) combined both Holt-Winters and ARIMA models to observe seasonal data, which ARIMA was not able to manage well, and it greatly improved the model's forecasting accuracy. This study compares the forecasting performance of Holt Winters with FB Prophet model taking in account the seasonality for both time periods.

2.3 Machine Learning Models

In recent times the machine learning models have earned tremendous success. The machine learning models like Long Short-Term Memory (LSTM) a type of recurrent neural network capable of learning long-term dependencies for prediction (Istiake Sunny et al., 2020) requires a lot of time and various resources for producing efficient results. Despite the recent successes of

machine learning models within classification tasks such as image recognition, the use of machine learning models within the domain of time series forecasting is not considered competitive (Abrahamsson & Ahlqvist, 2022). Therefore today, most of the research has been done using naive classical approaches such as classical Box-Jenkins methods or linear regression models to make forecasts due to their simplicity and ease of use.

2.3.1 Facebook Prophet Model

To make the forecasting process smooth, efficient, and robust many established organizations have also conducted intensive research and introduced their own predictive models, due to the potential of newly discovered time-series forecasts. For example, Facebook, the social networking site launched a new time series prediction model called Prophet. The most attractive feature of the model, especially in the field of weather forecasting, is its ability to adapt to regular changes and to irregularly scheduled holidays. The model can be robust to missing data and easily handle outliers and trend changes, so it predicts with high precision, as indicated in the study (Verma et al., 2021).

The Inception of FB Prophet Model

The members of the core data science team at Facebook Sean J. Taylor and Benjamin Letham designed an open-source forecasting model to address the limitations of doing high quality forecasts. Also, the reason for the development of the Prophet model is to meet business forecasting needs (Kumar Jha & Pande, 2021). It offers semi-automatic forecast models and can be adjusted based on previous user experience. The FB Prophet model combines the strengths of statistical and machine learning approaches and uses flexible trend components and seasonal additive components in a time series model (Taylor & Letham, 2018).

FB Prophet Theoretical Foundations

FB Prophet model uses an additive time series model with three components such as a piece-wise linear or logistic growth function for modelling the trend component, a periodic component, a holiday component which adds also external features (non-only holidays) to the time series that can help the user get better forecasts and properly predictions for rare events. Finally, there can

also be an error component that considers errors in the prediction. In addition, Facebook's tools also manage multiplicative seasonality and sub-daily frequencies, but these two options must be provided manually to the model. In its algorithm the Prophet model applies several curve fitting techniques and frameworks for modelling non-linear trends and the fitting of the model parameters such as logistic growth curve, Bayesian framework, and Fourier series.

FB Prophet Forecasting and Model Fitting:

- **Data Processing:** The prophet model incorporates the data as a two-column data frame, where the first column represents the time variable as 'ds' and the second represents observed time series values as 'y'.
- **Model Initialization:** The initialization of the Prophet model is done by the user for any desired setting of growth model, trend, and seasonal components.
- **Model Fitting:** fit () method is used by FB Prophet on time series data as it deploys the estimation procedure for parameters.
- **Forecasting:** predict () method is called for the prediction of Prophet model, when the desired forecasting horizon is specified.
- **Evaluation and Diagnostics:** Cross validation technique is mostly used for the evaluation of the model.

FB Prophet Model Application in Time Series Forecasting:

Ever since the Prophet model was introduced, it has been applied on a wide array of time series forecasts. Some of them are described below:

- **Forecasting For Financial Markets:** The prophet model was implemented by (Fernández-Rodríguez et al., 2018) to forecast the exchange rates daily for different currencies.
- **Forecasting For Retail Sales:** The research work of (Žunić et al., 2020) did the application of Prophet model on retail sales data. And it displayed good performance to its counterparts such as gradient boosting models.

- **Forecasting For Energy Demand:** (Kang et al., 2019) applied FB Prophet model in their research to forecast short term electricity demand. The model yielded the optimum results and considered the seasonal patterns effectively.

FB Prophet Limitations and Challenges:

- **Assumption Of the Model:** The FB Prophet model assumes through multiplicative or additive seasonality; time series data is generated by a decomposable model. While this is not true for all the time series models.
- **Tuning Of the Parameters:** Even though the entire Prophet model is optimized to apply the algorithm, but still the fine tuning of the parameters is required sometimes by the user to yield optimal results.
- **Irregular or High-Frequency Data:** The data with high frequency or irregular sample may sometimes require additional pre-processing as Prophet is best fit with time series with low frequency and regular intervals. Also, one of the limitations of prophet is that it cannot handle time series without timestamps, as it uses dates to infer frequency and build the model (Taylor & Letham, 2018).

From past research we have observed that the Prophet model is yielding mixed results as it is neither a winner nor a loser all the time. The research work (Garlapati et al., 2021) observed that prediction done by Prophet is adequate and performing well. However, this machine learning model is poor when dealing with small data sets (Choy et al., 2021). Moreover, (Lolea et al., 2021) proved the poor performance of the Prophet model with large data sets and the model lag in terms of forecasting as compared to ARIMA.

2.4 Comparative Analysis of Literature

Many research projects are being carried out for time series forecasting using Facebook Prophet in different business domains. (Kumar Jha & Pande, 2021)'s work has made a significant

contribution to the field by forecasting Google's closing share price during the COVID-19 pandemic and used Automatic Integrated Moving Average (ARIMA) and Facebook Prophet models. They compared the precision of the predictions made by the two models. First, they examined the stationarity of the data set and predicted the stock price during the epidemic using ARIMA (0,1,1), and then trained the Prophet model using the stock price before 1 January 2021 and predicted the stock price after 1 January 2021. The findings indicate that the ARIMA model is more effective in predicting the price of Google shares during the pandemic.

A similar forecast framework proposed by (Gaur, 2020) proposed an ARIMA-based FB-prophet prediction model for the prediction of the trend of COVID-19 cases. The proposed model helps detect COVID-19 outbreaks and irregularities in India and abroad. The proposed method compared with conventional linear regression-based forecasts. According to the proposed research findings, the FB prophet is a better prediction model for low error and better fitting.

The study of (Veiga et al., 2014) aims to compare the performances between ARIMA and Holt-Winters (HW) models for the prediction of a time series based on MAPE and U-Theil accuracy metrics. They applied these methods to forecasting the demand of a time series formed by a group of perishable dairy products with a short life cycle, in a Brazilian retail company. The empirical findings showed that the HW methodology gave better results regarding the performance metrics, having a better adjustment and capturing the linear behavior of the series (Makridakis et al., 1982; Makridakis & Hibon, 2000). Another study by (Yenidogan et al., 2018) compared the prophet model with ARIMA to forecast Bitcoin and showed that the prophet model outperformed the ARIMA model.

(Singh et al., 2020) tried different forecasting approaches to predict Amazon's future quarterly net sales based on its historical quarterly series in 2019. Specifically, they used HW approach, ARIMA, and neural networks autoregression model. After forecasting, they evaluate and justify these approaches by comparing forecast data with actual net sales in 2019. By using accurate metrics such as MAPE and RMSE, the findings showed that seasonal ARIMA gives the most accurate results. These findings can help Amazon to manage its future operations.

2.5 Measures of Forecast Error

One critical factor in sales forecasting is the accuracy of the prediction. Therefore, considerable efforts have been made to make this process more accurate and precise. The work in this field comprises several forecasting accuracy metrics such as RMSE, MAPE, MAE and so on. Comparison of actual sales with predicted results can be used using accuracy metrics. In this study, the Mean Absolute Percentage Error (MAPE) assesses the performance of the model, which is obtained as the Mean Absolute Percentage Error function for the prediction and the eventual outcome. This error measure expresses error as a percentage and can be used in evaluating models for different datasets (Hamzaçebi, 2008).

The Diebold Mariano test (Diebold & Mariano, 2002) was further implemented to compare the forecasting performance of two competing models. DM tests are widely used statistical tests to compare forecasts obtained from different models (I. Shah et al., 2019). The statistics of the modified DM test follow a normal distribution asymptotically (Zhou et al., 2021). The Diebold-Mariano test is still asymptotically valid even for nested models under some regularity assumptions (Giacomini & H, 2003).

2.6 Research Contribution

There has been a lot of work done on ARIMA and Holt Winters models. The ARIMA was introduced in 1970 by Box and Jenkins and the Holt Winters has its traces originating in 1950 so, as compared to its counterparts Prophet is relatively new as it was introduced in 2018 by Taylor and Letham. Every day, there is new research contributing to the literature of FB Prophet such as (Garlapati et al., 2021) predicted stock prices with Prophet, (Choy et al., 2021) did the prediction with Prophet with small data set, (Lolea et al., 2021) did the prediction with large data set etc. However, there is a research gap that this research will fill is that it has compared the performance of ARIMA and Holt Winters model with Prophet for forecasting aggregate retail sales in the US. And the forecasting performance of the Prophet model while predicting retail sales in the times of macroeconomics stability and instability.

Conclusion

Time series forecasts are widely applicable, and studies topics in both academia and industry. Accurate forecasts significantly increase future forecast accuracy, which is particularly important in industries such as retail, where future product sales uncertainty is rapidly shifting.

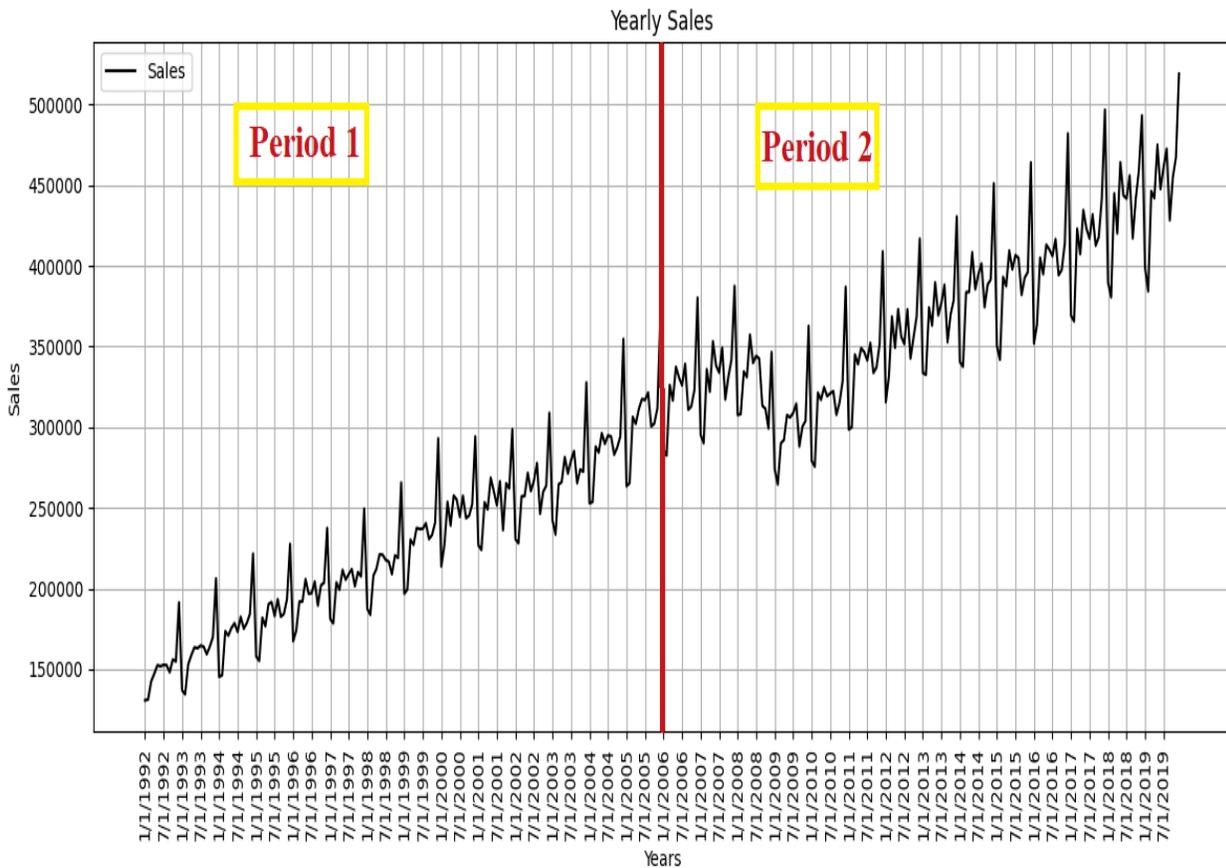
The general approaches described in this study for forecasting are the auto-regressive moving average (ARIMA), Holt-Winters exponential smoothing and machine learning approach, which is the Prophet modelling. The Facebook prophet model has so far attracted the attention of researchers in various fields of science. These methods can be used to forecast retail sales in the United States. Each forecasting approach has its advantages and can be better for specific data sets; that is, a linear model works better for data that are linear. The literature supports and recommends the use of ARIMA models for forecasting a major state-level economic variable, the net retail sales in the United States. In this literature review we have observed the behavior of different time series forecasting models on different time series data types.

3. Data

The data was obtained from United States Census Bureau (United States Census Bureau, 2019) in the section of Historical Data of Monthly Retail Trade. Figure 1 shows that the retail sales have constantly been increasing over time and there is a clear upward trend with seasonality. However, there is a deviation in the positive trend that can be observed at times of financial crisis.

The data for retail sales was analyzed in two time periods. The 1st period from (1992-2005) is the period of relative macroeconomic stability. However, the 2nd period from (2006-2019) has the effect of Global Financial Crisis (2008). The last 12 observations in each period were kept as “out of sample for” multi-step forecasting.

Figure 1: Monthly Aggregate Retail Sales (millions of \$) for the period (1992-2019) in US.



1st period is from (1992-2005), 2nd period is from (2006-2019)

4. Methodology

The forecasting and the evaluation techniques used for the models are explained first in this section. However, the detailed explanation of all the models is discussed later.

4.1 Forecasting

The forecasting technique used for out of sample predictions is multi-step forecasting. The reason for the selection of multi-step forecasting over single step forecasting is that it gives the 12-months projection into the future, and the forecasts are more useful in the practical setting, especially for policy makers.

There are several methods available to measure the forecasting accuracy like Root Mean Square Error (RMSE) etc., But we have used Mean Absolute Percentage Error (MAPE) because it does not only present a better picture of the difference between the actual values and forecast values but also it is not likely to change the magnitude of the time series forecast (Alon et al., 2001).

The Mean Absolute Percentage Error is calculated as follows:

$$MAPE = N^{-1} \sum_{i=1}^N \frac{|y_i - \hat{y}_i|}{y_i} \quad (1)$$

Here, \hat{y}_i is the predicted value and y_i is the actual value.

To compare the forecast between two models, the Diebold Mariano (DM) test was implemented. In DM test forecasts obtained from different models are statistically compared. The implementation of the DM test is based on method from (I. Shah et al., 2019). Let's consider two different forecasts \hat{y}_{1t} and \hat{y}_{2t} , available for time series y_t where $t = 1, \dots, T$. The forecast errors associated with this are $\epsilon_{1t} = y_t - \hat{y}_{1t}$ and $\epsilon_{2t} = y_t - \hat{y}_{2t}$. Since the DM test is based on the loss differential that is the comparison of mean differences among squared errors of two predicted series so, the loss differential between the two forecasts is $\eta_t = L(\epsilon_{1t}) - L(\epsilon_{2t})$. Whereas $E[\eta_t]=0$ is the null hypothesis of same forecast accuracy for two different forecasts. For the loss differential to be covariance stationary, the DM test requires:

$$\begin{aligned} E[\eta_t] &= \mu \\ cov(\eta_t - \eta_{t-\tau}) &= \gamma(\tau) \\ Var(\eta_t) &= \sigma_n, 0 < \sigma_n < \infty \end{aligned}$$

So, Diebold Mariano (DM) test under these assumptions is expressed as:

$$DM = \frac{\bar{\eta}}{\hat{\sigma}_\eta} \rightarrow N(0,1) \quad (2)$$

Here, $\bar{\eta} = \frac{1}{T} \sum_{t=1}^T \eta_t$ is the loss differential sample mean. And $\hat{\sigma}_\eta$ is a consistent standard error estimate of η_t . The null hypothesis of the test is the equal accuracy of two models. The results of the DM test are interpreted by checking the p-values at 5% significance level.

4.2 Econometric models

4.2.1 Auto Regressive Integrated Moving Average (ARIMA)

Among the sophisticated models available for time series forecasting, Autoregressive integrated moving average (ARIMA) is one of them. The three different historical components of the data considered by ARIMA model are autoregressive (AR), differencing (I) and moving average (MA). And quite often these components are mentioned along with model like ARIMA (p, d, q), which means that the model uses p^{th} AR terms, q^{th} MA terms and d differencing to achieve stationarity. The general expression for the ARIMA model is:

$$y'_t = c + \varphi_1 y'_{t-1} + \dots + \varphi_p y'_{t-p} + \theta_1 e_{t-1} + \dots + \theta_q e_{t-q} + \dots + e_t \quad (3)$$

Here, y'_t is representing differenced series at time t and e_t is representing random error at time t. Moreover, φ and θ are the components of ARIMA model.

The Box-Jenkins ARIMA model refers to the method of finding out the best parameters for the ARIMA model to do time series forecasting (G. E. P., et al Box, 2015). Since Classical regression is often not sufficient for exploring all the characteristics of the data, therefore, Box-Jenkins ARIMA is used as it is based on the identification of the auto-correlations function (ACF) structure. The three iterative steps followed by the Box-Jenkins ARIMA model, regarded as a model building process, are: identification of the model, estimation of the parameters, and checking of the diagnostics.

The implementation of Box-Jenkins ARIMA was done on R-programming using forecast library (Hyndman & Khandakar, 2008). For period 1 (1992-2005) the ARIMA model was fitted as:

$$y'_t = c + \varphi_1 y'_{t-1} + \varphi_2 y'_{t-2} + \varphi_{12} y'_{t-12} + \theta_1 e_{t-1} + e_t \quad (4)$$

Here, y'_t is representing differenced series at time t and e_t is representing random error at time t. Moreover, φ and θ are the components of ARIMA model. In the first period some analysis done with the help of R-programming suggested that for AR part, lag 1,2,12 is significant. And for MA part lag 1 was significant, where $\varphi_3 + \dots + \varphi_{11} = 0$ is fitted to y'_{t-p} , using maximum likelihood

method (I. Shah et al., 2019). Therefore, the restricted ARIMA (12,1,1) model was selected for Period 1 (1992-2005) because the maximum likelihood estimates are the representation of “best fit” model parameter values and refrain the model to be overfit. For period 2 (2006-2019) the ARIMA model was fitted as:

$$y'_t = c + \varphi_1 y'_{t-1} + \varphi_2 y'_{t-2} + \varphi_3 y'_{t-3} + \varphi_4 y'_{t-4} + \varphi_{12} y'_{t-12} + \theta_1 e_{t-1} + e_t \quad (5)$$

The model in period 2 (2006-2019) was again implemented in the same manner as restricted ARIMA in period 1. However, the significant lags for this period were lag 1,2,3,4,12 for AR part and for MA part the lag 1 was significant and, therefore, ARIMA (12,1,1) model was again selected for Period 2 (2006-2019). Whereas to avoid overfitting of the model, the parameters $\varphi_5 + \dots + \varphi_{11} = 0$ were fitted to y'_{t-p} , using maximum likelihood method making it restricted.

Figure 2: Flow Chart for Carrying Box-Jenkins ARIMA reproduced from Hao, 2023

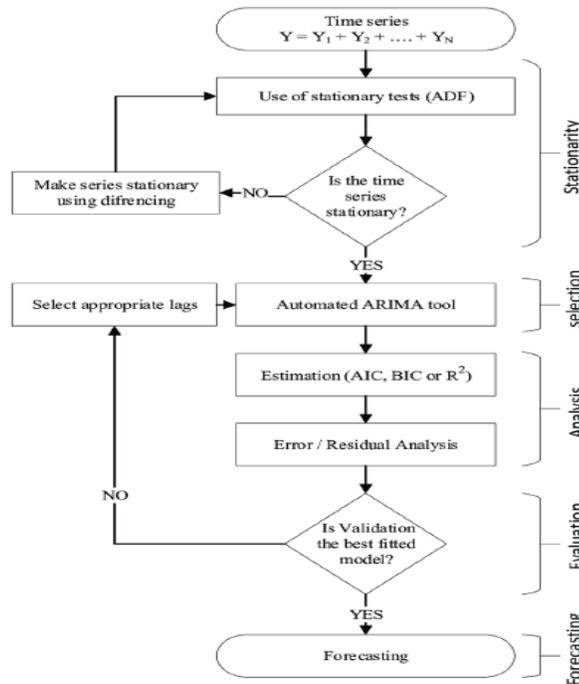


Figure 2 illustrates the process of carrying out Box-Jenkins ARIMA. The step is to make sure that the data for time series forecasting is stationary. The test measures the stationary is the legendary Augmented Dickey-Fuller Test (ADF). Stationarity of the data is achieved through differencing, which can be of first order, second order or the seasonal differencing. The ARIMA will repeat the ADF test until the time series data achieve stationarity. Once, the stationarity is achieved perform

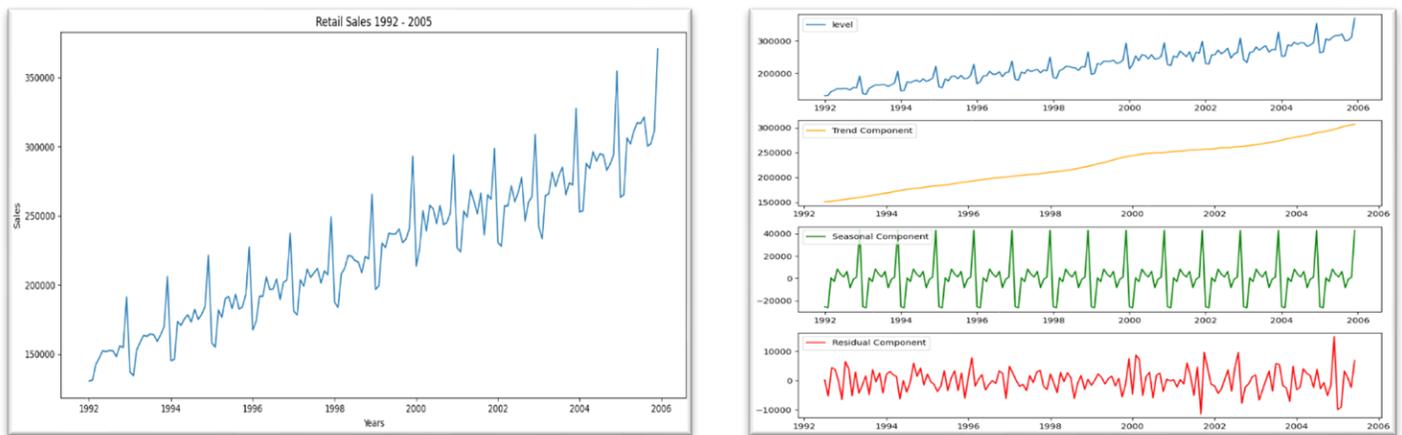
estimation and error test with default lags until the best fit validation of the model is found. Otherwise, repeat the process of error and estimation with different combinations of lags.

4.2.2 Holt Winters Exponential Smoothing

The second model we have used for our forecasting is from (Winters, 1960). The Holt Winters forecasting method is widely used forecasting method that can deal with the data that has trend and seasonality (Chatfield, 1978). The model is based on three smoothing equations and one forecast equation, which are explained in detail below. The Winters model has three smoothing parameters α , γ , and δ which represent level, trend, and seasonality, respectively. The exponential smoothing method gives the most recent observation a higher weight, and the weight decreases exponentially as the observation becomes further away (Rahman et al., 2016).

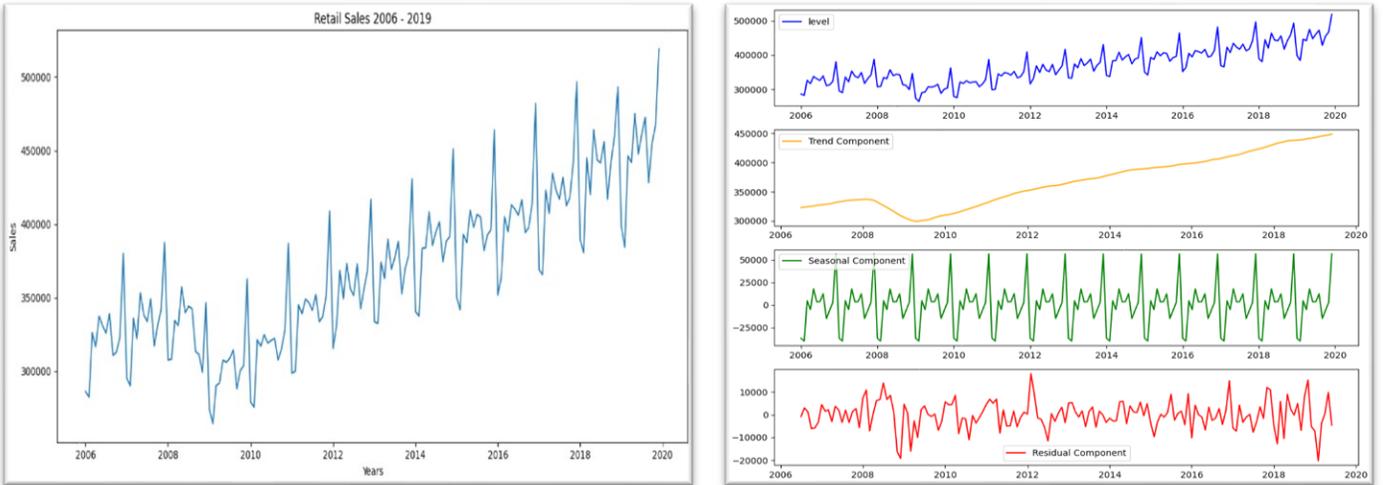
There are two main variations in the seasonal equation of the Winters model. One variation considers additive seasonality called Additive Holt Winters, while the other considers multiplicative seasonality called Multiplicative Holt Winters. The Holt–Winters additive method is used in time series with constant seasonal variations (Rahman et al., 2016). Furthermore, after doing seasonal decomposition, shown in figures (2 and 3) below, the Holt Winters Additive methodology was implemented.

Figure 3: Period 1 Plot with Seasonal Decomposition



From figure 3 we can observe that there is an upward trend, and the seasonality amplitude is constant over time.

Figure 4: Period 2 Plot with Seasonal Decomposition



From figure 4 we can observe again that there is an upward trend, and the seasonality amplitude is constant over time.

We now present the mathematical equations for Holt Winters Additive method that have been fitted on the given time series (Gardner, 2006). Let's consider time-series X_t for observed time series values in period t , with $X = x_1, x_2, x_3, \dots, x_t$. And at the end of period t , S_t is the smoothed level of the series, T_t is the smoothed additive trend, I_t is the smoothed additive seasonal index. Furthermore, p is the number of periods for seasonal cycle, $\hat{X}_t(m)$ represents the time series forecast, where m is the forecast periods. The parameters α , γ , and δ , as discussed above, represent level, trend, and seasonality, respectively. Therefore, the basic equations for Holt Winters Additive method can be written as:

$$\text{Level: } S_t = S_{t-1} + T_{t-1} + \alpha e_t \quad (6)$$

$$\text{Trend: } T_t = T_{t-1} + \gamma e_t \quad (7)$$

$$\text{Seasonal: } I_t = I_{t-p} + \delta(1 - \alpha)e_t \quad (8)$$

$$\text{Forecast: } \hat{X}_t(m) = S_t + mT_t + I_{t-p+m} \quad (9)$$

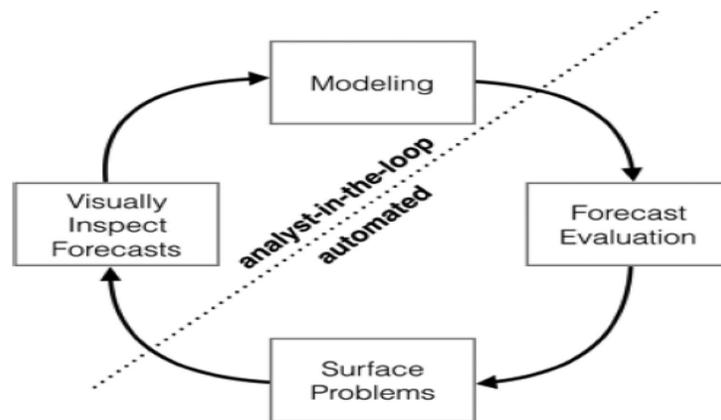
Furthermore, to achieve the accurate smoothed parameters for level, trend and seasonality, Grid Search framework was applied. The significance of Grid Search framework is that it considers those hyper parameters as well, which were not tuned automatically when the model was identified as exponential smoothing model (ES) (Tran & Van Dai, 2020). Thus, ES methods' configuration

parameters include trend type (t), damped type (d), seasonal type (s), seasonal period type (m), box-cox transformer type (b), and removal bias (r) (Brownlee, 2019). We have used Python library Stats model (Seabold & Perktold, 2010) for HW-Additive. The obtained parameters for level, trend, and seasonality were then validated with the cross-validation technique.

4.3 FB Prophet

Facebook Prophet model is one of the powerful methods for time series forecasting as it was introduced by (Taylor & Letham, 2018). The main characteristic of this model is that it uses forecasting method based on generalized additive model (GAM) (Hastie & Tibshirani, 1987). The main advantage of GAM is that it has a decomposition efficiency and can consider new components as necessary when a seasonality is identified from a new source. The Prophet model can also manage the data exhibiting strong seasonality and missing values. It uses the approach called **“Analyst in the loop”** as shown in figure below:

Figure 5: Analyst-in-the-Loop Modeling reproduced from Taylor and Letham (2018)



The **“Analyst in the loop”** approach, to improve the contextual relevance and accuracy of time series forecasting, recognizes the human analyst importance. The task of an analyst is to typically observe the model, identify anomalies, and execute the model based on domain specific insights and observe the model behavior and measure its performance. The advantage of this framework is

that it leads to more accurate actionable insights. The time series data was fitted using the Prophet library in python.

The three main components of the Prophet model are: trend, seasonality, and holidays. The model can be expressed as:

$$Y(t) = s(t) + g(t) + h(t) + e(t) \quad (10)$$

Here, $s(t)$ represents the seasonality (daily, monthly, or yearly), $g(t)$ represents the trend function for modelling nonperiodic changes in time series values, and $h(t)$ is the impact of holidays occur in entire year, but the current implementation of the model is without holidays effect. Moreover, $e(t)$ is an error term under the parametric assumption of normal distribution. The Prophet method uses automatic curve fitting technique where the model is automatically fitted using Stan code (Carpenter et al., 2017) considering seasonality, trend, and holidays.

For trend forecasting Taylor and Letham (2018) have implemented two models: a piecewise linear trend model and a saturated growth model. The growth trend is modeled typically using the logistic growth model, which can be written as:

$$g(t) = \frac{C}{1 + \exp(-k(t - m))} \quad (11)$$

Here, C is the carrying capacity, K is the growth rate, and m is an offsetting parameter. Firstly, the carrying capacity is not constant as the population is increasing every single day therefore, the fixed carrying capacity C is replaced with the time varying capacity $C(t)$. Secondly, the growth rate is also not constant as the new products are introduced to the markets from time to time. So, to fit the historical data a model must be able to incorporate the varying growth rate. FB Prophet model introduces trend changes in the growth model by defining change-points explicitly. S change points are allowed by the Prophet model at times s_j ; $j = 1, 2, \dots, S$ and the growth rate can be readjusted as $\delta \in R^S$, where δ_j is change in rate that occurs at time s_j . Now, at time t the rate consists of base rate K and all the adjustments done till that point of time: $k + \sum_{j:t>s_j} \delta_j$. The introduction of the vector $\mathbf{a}(t) \in [0,1]^S$ will provide the explanation for this:

$$a_i(t) = \begin{cases} 0 & \text{if } t \geq s_j \\ 1, & \text{otherwise.} \end{cases}$$

Therefore, the rate at time t is $k + \mathbf{a}(t)^T \boldsymbol{\delta}$. The offset parameter m must also need to be adjusted when the rate K is adjusted to connect the end points of segments. However, the correct adjustment for change-point j can be expressed as:

$$\gamma_j = (s_j - m - \sum_{l < j} \gamma_l) \left(1 - \frac{k + \sum_{l < j} \delta_l}{k + \sum_{l \leq j} \delta_l} \right)$$

Furthermore, the logistic model is now transformed as piece wise logistic growth model can be written as:

$$g(t) = \frac{c(t)}{1 + \exp(-(k + \mathbf{a}(t)^T \boldsymbol{\delta})(t - (m + \mathbf{a}(t)^T \boldsymbol{\gamma})))} \quad (12)$$

The linear trend model is written as: $g(t) = (k + \mathbf{a}(t)^T \boldsymbol{\delta})t + (m + \mathbf{a}(t)^T \boldsymbol{\gamma})$ where γ_j is set to $-s_j \delta_j$ to make the function continuous. Furthermore, the change-point s_j can automatically be selected by adjusting $\delta_j \sim \text{Laplace}(0, \tau)$ where τ has the characteristic to control rates.

Moreover, seasonality is approximated by standard Fourier Series which can be expressed as:

$$S(t) = \sum_{n=1}^N (a_n \cos\left(\frac{2\pi n t}{P}\right) + b_n \sin\left(\frac{2\pi n t}{P}\right)) \quad (13)$$

Here, $S(t)$ is the representation of seasonality and P being the seasonality period. N is number of Fourier components (harmonics) to be include in the series and a_n and b_n are amplitude representation of n^{th} harmonics.

5. Results

The research results were obtained on the computer system with Intel Core i5-7200 processor on Windows 11. The tables below display the actual and forecasted values for period 1 and period 2, respectively.

Table 1: Observed and predicted values for period 1 (1992-2005)

| Out of Sample Period 1 | Actual | Box Jenkins Forecast | Holt Winters Forecast | FB Prophet Forecast |
|-------------------------------|---------------|-----------------------------|------------------------------|----------------------------|
| 1/1/2005 | 263469 | 270262.6 | 273135.5 | 269442.3 |
| 2/1/2005 | 265320 | 271556.8 | 272392.6 | 275096.0 |
| 3/1/2005 | 306384 | 307429.8 | 304983.3 | 292700.6 |
| 4/1/2005 | 302054 | 303213.6 | 303341.1 | 293415.0 |
| 5/1/2005 | 311292 | 315782.6 | 317950.7 | 304976.2 |
| 6/1/2005 | 317375 | 308287.4 | 311040.6 | 303466.7 |
| 7/1/2005 | 316887 | 312966.8 | 315692.0 | 303853.9 |
| 8/1/2005 | 321409 | 312152.6 | 320661.7 | 311639.3 |
| 9/1/2005 | 300439 | 300669.8 | 303758.8 | 294624.5 |
| 10/1/2005 | 302213 | 305555.9 | 312986.4 | 299764.2 |
| 11/1/2005 | 311715 | 312564.1 | 317431.2 | 304432.7 |
| 12/1/2005 | 370726 | 373435.5 | 365600.2 | 351991.6 |

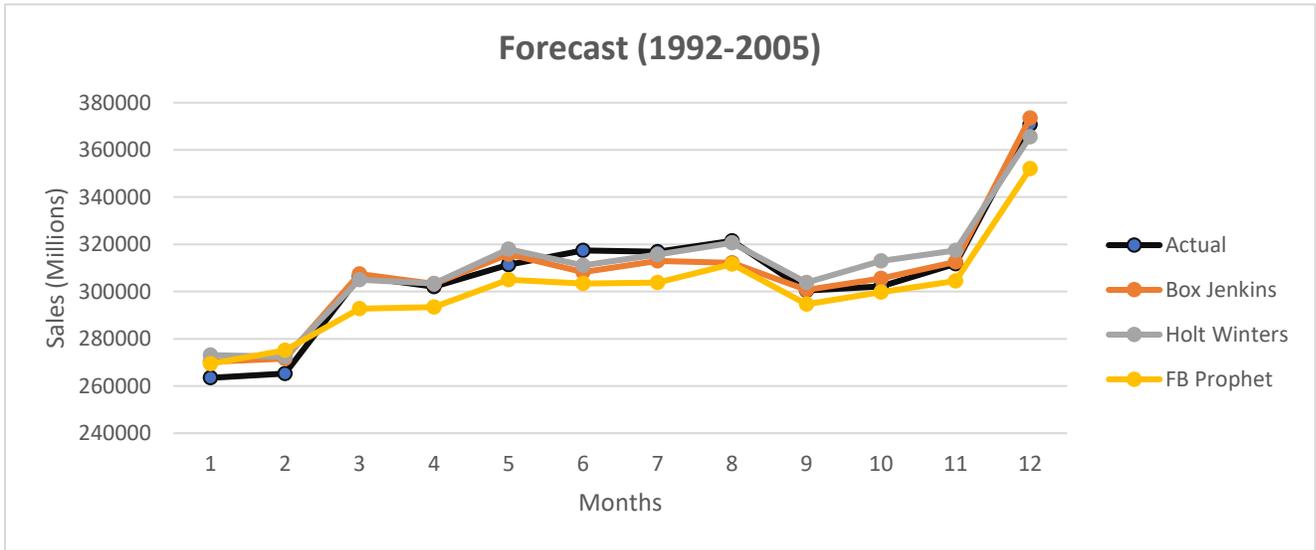
Table 2: Observed and predicted values for period 2 (2006-2019)

| Out of Sample Period 2 | Actual | Box Jenkins Forecast | Holt Winters Forecast | FB Prophet Forecast |
|-------------------------------|---------------|-----------------------------|------------------------------|----------------------------|
| 1/1/2019 | 398783 | 395004.2 | 395286.7 | 402688.3 |
| 2/1/2019 | 384243 | 388870.2 | 392320.3 | 420086.2 |
| 3/1/2019 | 446272 | 453959.1 | 451861.6 | 443561.9 |
| 4/1/2019 | 441878 | 427521.9 | 434294.5 | 442236.9 |
| 5/1/2019 | 474963 | 468910.7 | 469003.5 | 466201.0 |
| 6/1/2019 | 447578 | 448373.0 | 451141.4 | 459004.2 |
| 7/1/2019 | 461010 | 446097.3 | 447831.2 | 460454.4 |
| 8/1/2019 | 472517 | 460481.5 | 459015.4 | 463432.8 |
| 9/1/2019 | 428241 | 422730.1 | 427288.5 | 437936.6 |
| 10/1/2019 | 454941 | 447508.7 | 441609.3 | 451685.2 |
| 11/1/2019 | 467189 | 464275.7 | 454640.8 | 462769.7 |
| 12/1/2019 | 518979 | 495255.2 | 506776.5 | 515404.3 |

***The out of sample observations were for 2005 and 2019 respectively.**

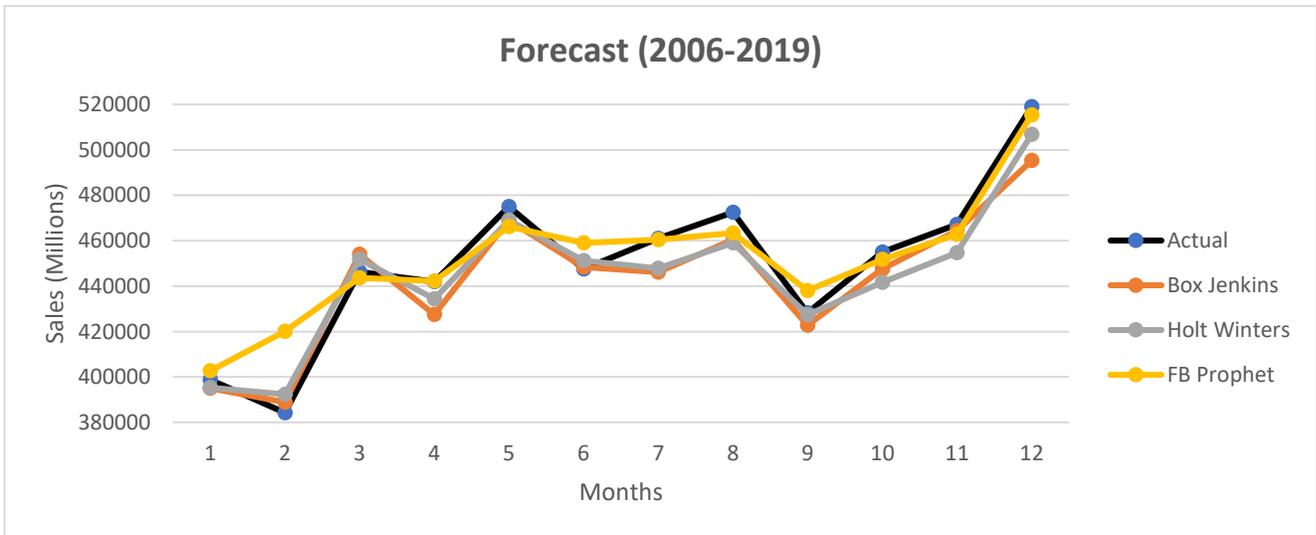
The graphical representation of the values, actual and forecasted, is done below. Now, we can see that the models implemented are predicting approximately accurate in both the time periods. The forecasts for both periods can be seen in figures 6 and 7 below.

Figure 6: Period 1 Actual and Forecasted Plot



From figure 6 it is clearly observed that all three of the model forecasts are exhibiting seasonality and trend. This is evidence that the models have been trained well and their implementation on the test data (out of sample) is exhibiting almost the same patterns as compared to the actual data line.

Figure 7: Period 2 Actual and Forecasted Plot



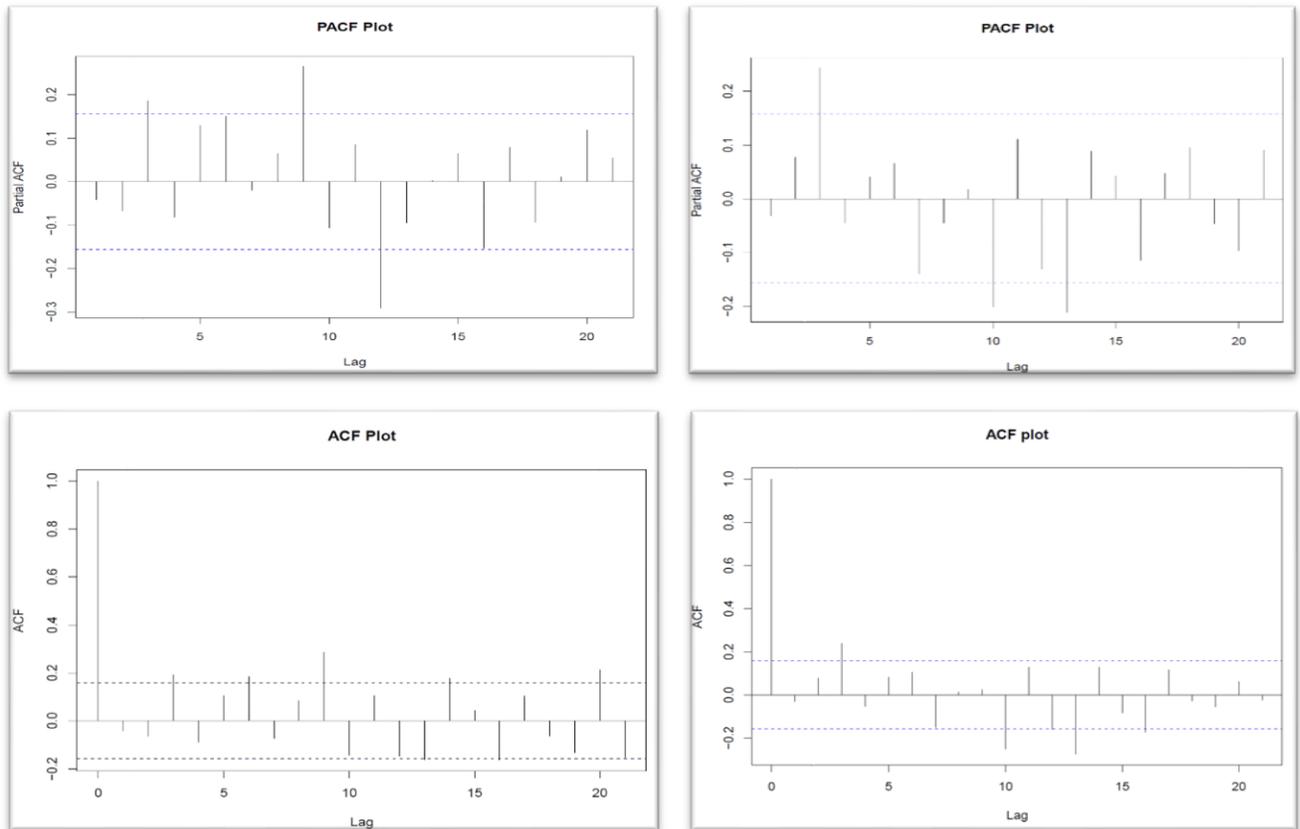
From figure 7, we can observe that for the starting two months FB Prophet is not really observing the trend and fluctuation. We can suspect the behavior of FB Prophet due to the global financial crisis as it did not capture the seasonality and the trend in the initial months. Nevertheless, it is of more importance to compare the test results and Diebold Mariano test statistics to come to any conclusion about the model's performance.

5.1 Models Individual Results

5.1.1 Box-Jenkins ARIMA Residual Plots

After fitting the ARIMA model of order (12,1,1) for both the periods. We plotted the residuals of our restricted ARIMA model which are displayed in the figures below.

Figure 8: Period 1 (left) and Period 2 (right) PACF and ACF plots.

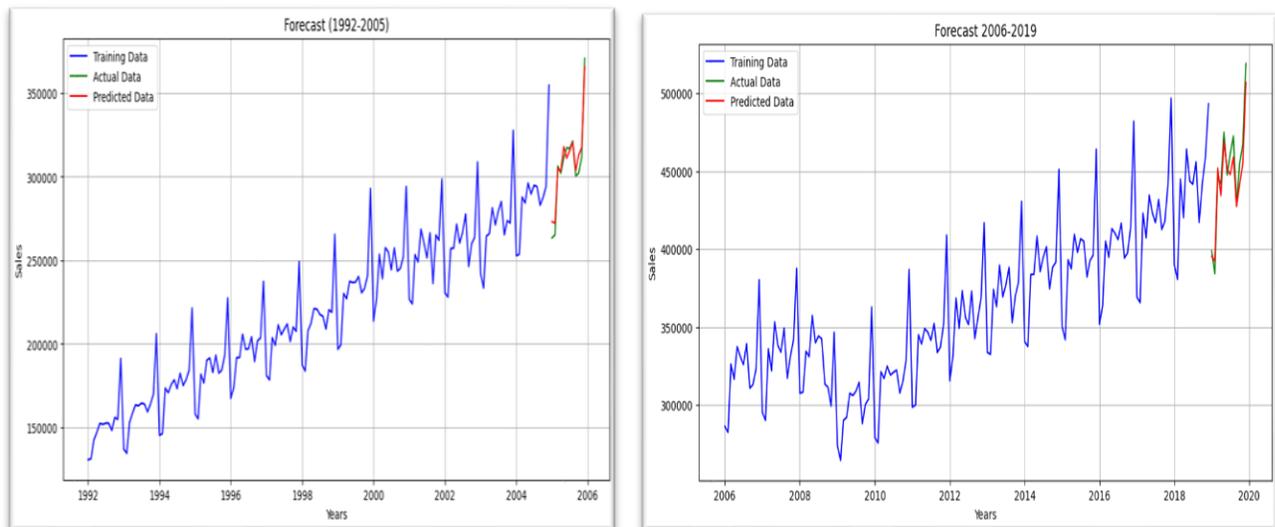


We selected the tentative model based on ACF and PACF plots, but the final model selection was made based on Mean Absolute Percentage Error (MAPE). And then we plotted the final

residuals. However, we can see some significant spikes in both ACF and PACF plots such as in the top right figure of PACF of Period 2 there are significant spikes at lag 3,10, and 13. The absolute value of these lags is -0.2, which is significantly smaller. Because sometimes due to white noise process we often see the significant spikes in the ACF and PACF plots.

5.1.2 Holt Winters Exponential Smoothing Forecast Plot

Figure 9: Holt Winters Forecast Plot

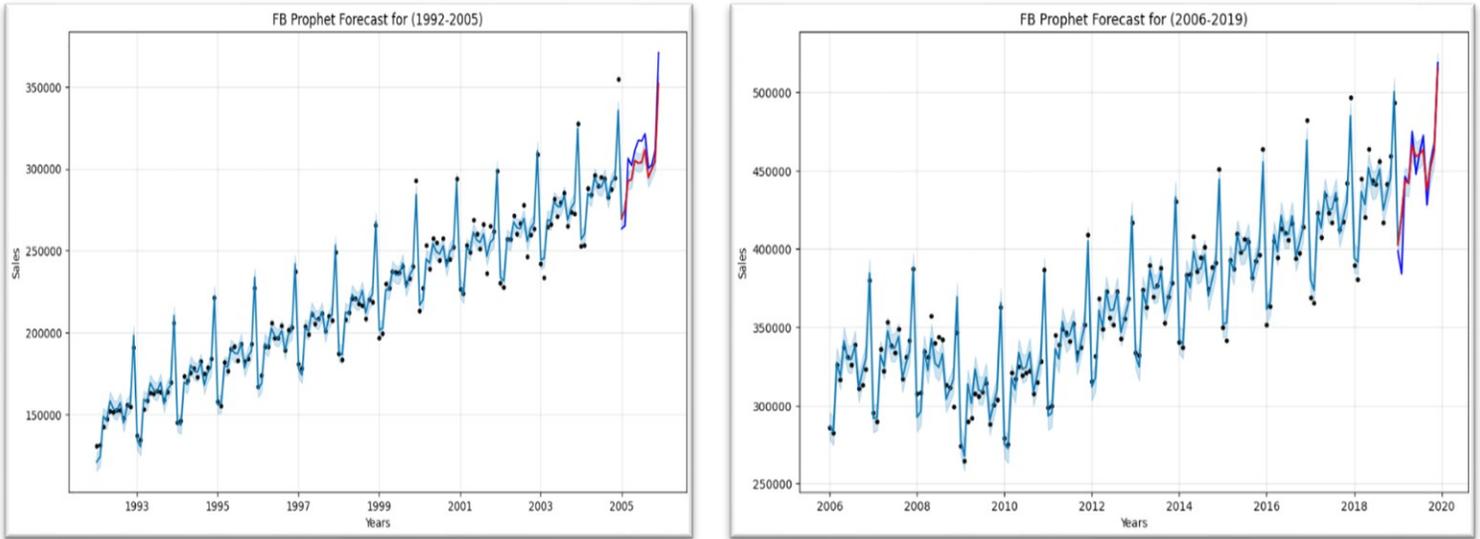


The forecast of both the time periods are almost approximately following the same trend and seasonality pattern as in actual data. The out of sample forecast, as discussed, for period 1 is of 2005 and for period 2 is of 2019. The smoothing parameters (level, trend, and seasonal) were found from Grid Search (Tran & Van Dai, 2020). Complete results of Grid Search is in appendix section.

5.1.3 FB Prophet Forecast Plot

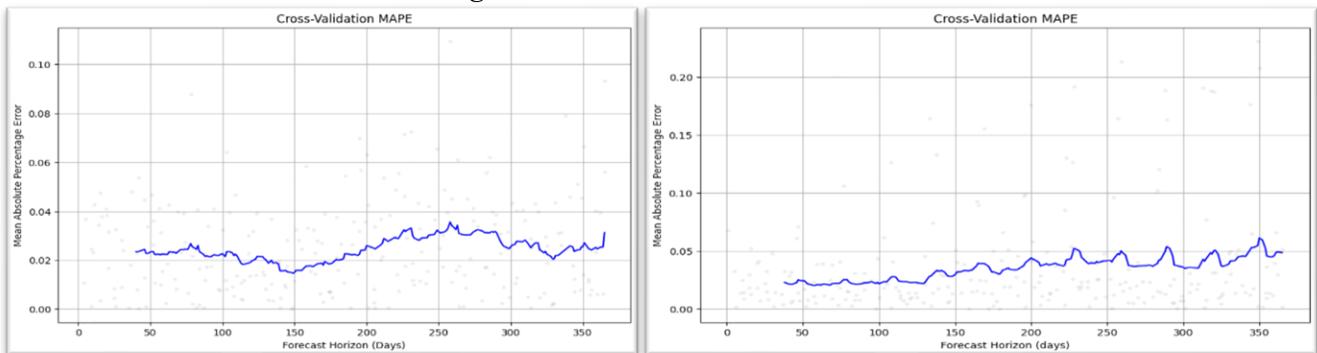
In figure 10 below the black dots are observed data points and the blue line passing the black dots is the forecast line and, in the end, when the training data set is completed the next blue and red line is last 12 months observed and the predicted values.

Figure 10 FB Prophet Forecast Plot



The predicted retail sales (red) in Period 1, which is a period with stable macroeconomic conditions, deviates from observed values of that period (blue). We suspect that this deviation might cause the MAPE values to yield greater error. However, the situation is opposite in period 2, which has the shock effect of Global Financial Crisis (2008), but still the predicted retail sales (red) are almost following the same seasonality and trend pattern as of observed values (blue).

Figure 11: Cross-validate MAPE.



In figure 11 the left plot is of (1992-2005) that has MAPE of 2.47 and the right plot is of (2006-2019) having MAPE 3.52. The cross-validated MAPE represents an average error percentage across cross validation done on entire time periods. The cross validation performed is on training data set using rolling window method. So, now we have a clear picture of how the model will work on our test data set.

5.2 Comparison of Errors

Table 3: Mean Absolute Percentage Error (MAPE)

| <i>MAPE OF THE MODELS</i> | | |
|---------------------------|---------------------|----------------------|
| Models | First Period | Second Period |
| Box Jenkins Arima | 1.35 (1) | 1.87 (3) |
| Holt Winters | 1.65 (2) | 1.82 (1) |
| Fb Prophet | 3.08 (3) | 1.83 (2) |

The **first period** examination starting from 1992 to 2005 has revealed intriguing patterns for models' predictive performance because it is a period of relative macroeconomic stability in the US. The parenthesis along with MAPE values represents the respective positions secured by each model. The Box-Jenkins ARIMA model has reached a notable MAPE i.e., 1.35 by securing first place. This is evidence of the model's remarkable ability to forecast accurate aggregate retail sales in the future. MAPE of 1.35 is evidence that model can inculcate the effect of seasonality and trend well and provide a reasonable forecast. However, Holt Winters Exponential Smoothing model holds the second place in terms of forecasting with an MAPE of 1.65. It can be noted here that the application of additive method of Holt Winters Exponential Smoothing has exhibited the remarkable performance proving its ability to capture the seasonality and trend and producing lower MAPE. Moreover, last place has been held by FB Prophet model with an MAPE of 3.08. The MAPE of FB prophet suggests that the model was not efficient in capturing the data patterns to provide more accurate forecast as compared to Box-Jenkins and Holt Winters Exponential Smoothing models.

For the **second period** from 2006 – 2019 the rankings of the models have changed. This is the period that is marked with macroeconomic volatility capturing the effects of Global Financial Crisis of 2008. From table 3 we can also observe that the MAPE of Box-Jenkins and Holt Winters Exponential Smoothing models have increased. So, in this period the first place has been held by Holt Winters Exponential Smoothing model with an MAPE of 1.82. The best performance of Holt Winters model indicates that the smoothed parameters of model produce more robust results when there is a fluctuation in the data in terms of macroeconomic volatility. Furthermore, the FB Prophet model has improved its rank by holding the second place with an MAPE of 1.82 and we can observe from this result that during the volatility period FB Prophet not only improved its ranking but also enhanced its forecasting performance as it had an MAPE of 3.08 in the first period. So,

we can say that the effect of the economic instability enhanced the model forecasting performance in a way that it captured the data patterns more accurately. Lastly, Box-Jenkins ARIMA holds the last position with an MAPE of 1.87 in contrast to its performance in the first period.

Therefore, we have concluded that the macroeconomic stability and instability do affect the forecasting performance of the model as Box-Jenkins model which has performed well in the first period (1992-2005) by securing the first place, stood last when the macroeconomic conditions were instable in the second period (2006-2019). However, on the other hand we have noted that that MAPE value was reduced for FB Prophet model in the second period, when there was the effect of Global Financial Crisis in the data. Moreover, the first place in the second period which is held by Holt Winters model reveals that the smoothed parameters of level, trend, and seasonality (α, β, γ) produced better results.

5.3 MAPE Ratios and Test Results

Table 4: MAPE Ratios & Diebold Mariano Test Statistics (1992 – 2005)

| Models | Box-Jenkins | Holt Winters | Fb Prophet |
|---------------------|--------------------|---------------------|-------------------|
| Box-Jenkins | - | 1.22 (0.74) | 2.28 (0.99) |
| Holt Winters | 0.82 (0.25) | - | 1.87 (0.97) |
| Fb Prophet | 0.44 (0.00) | 0.53 (0.02) | - |

For **Period 1** (1992-2005) MAPE ratios are displayed in table 4 along with the respective p-values in the brackets. The test was done as implemented in (I. Shah et al., 2019). Where, H_0 : the forecasting accuracy of the model in the column and the model in the row is the same; H_1 : the forecasting accuracy of the model in the column is greater than that of model in the row. So, now we do the comparison between Box-Jenkins Vs FB Prophet, and it is clearly observed that the respective p-value is (0.00) which results in the rejection of H_0 and we can say that the forecasting accuracy of the model in the column (Box-Jenkins) is greater than that of model in the row (FB Prophet). However, for Holt Winter VS FB Prophet the p-value again is (0.02), which means that

we again reject the H_0 and conclude that Holt Winters has a better forecasting accuracy than that of FB Prophet model.

Table 5: MAPE Ratios & Diebold Mariano Test Statistics (2006 – 2019)

| Models | Box-Jenkins | Holt Winters | Fb Prophet |
|---------------------|--------------------|---------------------|-------------------|
| Box-Jenkins | - | 0.97 (0.28) | 1.00 (0.60) |
| Holt Winters | 1.02 (0.72) | - | 1.00 (0.69) |
| Fb Prophet | 1.02 (0.40) | 0.99 (0.31) | - |

The **Period 2** (2006-2019) MAPE ratios are displayed in table 5 along with the respective p-values in the brackets. However, H_0 : the forecasting accuracy of the model in the column and the model in the row is the same; H_1 : the forecasting accuracy of the model in the column is greater than that of model in the row remains the same. So, from this table we will do the comparison between the Box-Jenkins and the FB Prophet and from the respective p-value i.e. (0.40) we do not reject the null hypothesis and therefore, conclude that the forecasting accuracy of Box-Jenkins and FB Prophet is the same. Furthermore, the comparison of forecasting performance between Holt Winters and FB Prophet results in that we do not reject H_0 because of p-value (0.30). Hence the forecasting accuracy of Holt Winters and FB Prophet for the second period is the same.

6. Discussion and Conclusion

This research thesis had the aim of studying whether FB Prophet model surpass the forecasting accuracy of Box-Jenkins ARIMA and Holt Winters Exponential Smoothing models for the prediction of US aggregate retail sales for two different time periods having different macroeconomic conditions.

To answer the research question, we can say that the results obtained from FB Prophet model did not produce significant results in terms of forecasting accuracy. In the first period (1992-2005) FB Prophet model performance had been unsatisfactory, which is the stable period in terms of macroeconomic conditions, however it improved in the second period (2006-2019), but not a clear outperformance by the model. The difference of error from the best model was approximately 2%, but still such difference can have a considerable impact in terms of practical application. For instance, the difference between FB Prophet and Box-Jenkins ARIMA, in the first period is 1.73% which means that \$88 billion was incorrectly predicted during the prediction year. As the aggregate sales increase such figures also tend to increase with respect to time.

The forecasting accuracy of Box-Jenkins ARIMA and Holt Winters Exponential Smoothing models, in the first period, significantly points out the fact that these models are efficient, accurate and robust for time series forecasting as we have already discussed in the literature review section. But between these two AIRMA is the best performing in terms of MAPE ratio and DM statistics in first period. We have also seen in the literature review from different studies that ARIMA outperforms FB Prophet in certain cases. Many nonlinear aspects or extreme volatility are not incorporated in aggregate retail sales like most of the economic variables as is incorporated in time series natural sciences or financial markets. Therefore, the selection of forecasting method must be based on the studied data. Simple models that rely on incorporating linear patterns can manage economic data efficiently, however, there are machine learning methods like Artificial Neural Networks (ANN), Long Short-Term Memory (LSTM) and FB Prophet can analyze more complex processes.

Furthermore, we made the test results more intuitive by incorporating the MAPE's ratios. The rational for incorporating MAPE ratios along with the respective p-values helped in better framing

of the test result picture. And with the help of MAPE ratios we were able to identify the best performing model in the first period i.e., Box-Jenkins ARIMA, because of it being statistically significant and it has the lowest MAPE ratio.

The forecasting accuracy in the second period for all the models is the same, as no model outperforms other models in terms of forecasting accuracy. Since, second period is a period of unstable macroeconomic conditions so, we can say that each model has taken into effect the shock of financial crisis at the same level. However, there is an important note to consider here is the MAPE of the Box-Jenkins ARIMA and Holt Winters Exponential Smoothing models increased, which was obvious, in the second period. But the MAPE of FB Prophet model reduced from 3.08 in first period to 1.83 in the second period. And through this behavior of the Prophet model, we can conclude that this machine learning model improved its forecasting accuracy.

However, this study has a limitation attached to it. Since we have done our analysis on univariate time series data, the effect of holidays was not included in the forecasting of FB Prophet model otherwise the results would have been more optimal. Furthermore, the business users, in the practical setting, require the simplest and fastest short-term forecasting, but it takes time to find the model as a best fit for the required time series because models like Holt Winters take time in finding the suitable parameters. Even though the Box-Jenkins model performs the best in the first period, the need to select the appropriate order of (p, d, q) also plays a pivotal role, otherwise the overfitting of the model can produce biased results.

There are some future research suggestions based on the findings of this thesis. First, building a more consistent and robust model using multivariate time series data. Forecast accuracy can be amplified through data transformation or seasonality removal, so the data seasonally adjusted can be of good use. Furthermore, the forecasting accuracy can be further cross checked on other available tests such as Wilcoxon's signed-rank test (SR) along with Diebold Mariano test (DM). The incorporation of holidays effect in the FB Prophet model can yield domain/field specific insights. For the parameter selection in Holt Winters Exponential Smoothing Method the damping factor can also be considered. Although the sales time series is in demand, for profitability cost is also a factor. And it solely depends on the firm's policy, board, quarter/annual targets and hiring practice. We suspect that the models discussed in this thesis may not give the same performance for cost time series as they did for sales time-series.

7. References

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8. Appendix

Python Code for FB Prophet

```
import pandas as pd
from prophet import Prophet
from prophet.diagnostics import cross_validation
from prophet.plot import plot_cross_validation_metric
from sklearn.metrics import mean_absolute_percentage_error
import matplotlib.pyplot as plt

# Load the data
data = pd.read_csv ("Retail sales ")

# Rename the columns as 'ds' and 'y' for Prophet
data = data.rename(columns={'Years': 'ds', 'sales': 'y'})

# Convert 'ds' column to datetime format
data['ds'] = pd.to_datetime(data['ds'])

# Handle missing values
data = data.dropna()

# Split the data into training and test sets
train_data = data[:-12] # Training data
test_data = data[-12:] # Test data
```

```

# Create a Prophet model
model = Prophet()

# Fit the model
model.fit(train_data)

# Perform cross-validation on training data
df_cv = cross_validation(model, initial='730 days', period='180 days', horizon='365 days')

# Extract the actual values and predicted values from cross-validated data
actual_values_cv = df_cv['y']
predicted_values_cv = df_cv['yhat']

# Calculate MAPE on cross-validated data
mape_cv = mean_absolute_percentage_error(actual_values_cv, predicted_values_cv)

# Print the MAPE
print("Mean Absolute Percentage Error (MAPE) for cross-validated data:", mape_cv)

# Plot cross-validation results
fig = plot_cross_validation_metric(df_cv, metric='mape', rolling_window=0.1, ax=None)

plt.title('Cross-Validation MAPE')
plt.xlabel('Forecast Horizon (Days)')
plt.ylabel('Mean Absolute Percentage Error')

```

```
plt.show()
```

Python Codes for Holt Winters

Parameter Selection

```
import pandas as pd
import numpy as np
from itertools import product
from sklearn.metrics import mean_absolute_percentage_error
from sklearn.model_selection import TimeSeriesSplit

def holt_winters(data, alpha, beta, gamma, forecast_length):
    level = np.zeros(len(data))
    trend = np.zeros(len(data))
    seasonal = np.zeros(len(data))
    forecast = np.zeros(len(data) + forecast_length)

    # Initialize parameter estimates
    level[0] = data[0]
    trend[0] = data[1] - data[0]
    seasonal[0] = data[0]

    for i in range(1, len(data)):
        # Holt-Winters equations
        level[i] = alpha * (data[i] - seasonal[i - 1]) + (1 - alpha) * (level[i - 1] + trend[i - 1])
        trend[i] = beta * (level[i] - level[i - 1]) + (1 - beta) * trend[i - 1]
        seasonal[i] = gamma * (data[i] - level[i]) + (1 - gamma) * seasonal[i - 1]
```

```

forecast[i] = level[i] + trend[i] + seasonal[i]

# Forecasting for future periods
for i in range(len(data), len(data) + forecast_length):
    forecast[i] = level[-1] + (i - len(data) + 1) * trend[-1] + seasonal[-12 + (i - len(data) + 1)
% 12]

return forecast[-forecast_length:]

# Read data from CSV file
data_df = pd.read_csv("Retail Sales")
sales = data_df['sales'].values

# Define the range of values for alpha, beta, and gamma
alpha_values = np.linspace(0, 1, 6) # Reduced to 6 values
beta_values = np.linspace(0, 1, 6)
gamma_values = np.linspace(0, 1, 6)

best_mape = float('inf')
best_params = None

# Cross-validation using TimeSeriesSplit
tscv = TimeSeriesSplit(n_splits=3)
for train_index, test_index in tscv.split(sales):
    train_data, test_data = sales[train_index], sales[test_index]

# Grid search for finding the best combination of alpha, beta, and gamma
for alpha, beta, gamma in product(alpha_values, beta_values, gamma_values):

```

```

forecast_length = len(test_data)

# Generate the entire forecast for the test dataset
forecast = holt_winters(train_data, alpha, beta, gamma, forecast_length)

# Calculate the MAPE for the entire test dataset
mape = mean_absolute_percentage_error(test_data, forecast)

if mape < best_mape:
    best_mape = mape
    best_params = (alpha, beta, gamma)

print("Best MAPE:", best_mape)
print("Best Parameters (alpha, beta, gamma):", best_params)

```

Seasonal Decomposition

```

import pandas as pd
import matplotlib.pyplot as plt
from statsmodels.tsa.seasonal import seasonal_decompose

# Read the time series data from the CSV file
data = pd.read_csv("Retail sales ")

# Convert the 'Years' column to datetime type
data['Years'] = pd.to_datetime(data['Years'])

```

```

# Set the 'Years' column as the index
data.set_index('Years', inplace=True)

# Perform seasonal decomposition
result = seasonal_decompose(data, model='additive') # Use 'additive' for additive
decomposition

# result = seasonal_decompose(data, model='multiplicative') # Use 'multiplicative' for
multiplicative decomposition

# Plot the original data, trend, seasonal, and residual components
plt.figure(figsize=(10, 8))

plt.subplot(411)
plt.plot(data, label='level')
plt.legend()

plt.subplot(412)
plt.plot(result.trend, label='Trend')
plt.legend()

plt.subplot(413)
plt.plot(result.seasonal, label='Seasonal')
plt.legend()

plt.subplot(414)
plt.plot(result.resid, label='Residual')
plt.legend()

plt.tight_layout()

```

```
plt.show()
```

Forecasting

```
import pandas as pd
from statsmodels.tsa.holtwinters import ExponentialSmoothing
from sklearn.metrics import mean_absolute_percentage_error

# Read the time series data from the CSV file
data = pd.read_csv("Retail Sales")

# Convert the 'Years' column to datetime type
data['Years'] = pd.to_datetime(data['Years'])

# Set the 'Years' column as the index
data.set_index('Years', inplace=True)

# Set the frequency as 'MS' (monthly)
data.index.freq = 'MS'

# Split the data into training and test sets
train_data = data[:-12] # Training data
test_data = data[-12:] # Test data (out-of-sample forecasting)

# Apply Holt-Winters Triple Exponential Smoothing on the test data
```

```
model = ExponentialSmoothing(test_data, trend='add', seasonal='add', seasonal_periods=12,
initialization_method="estimated")
```

```
# Use the best parameters (obtained from parameter selection code)
```

```
alpha = **
```

```
beta = **
```

```
gamma = **
```

```
fitted_model = model.fit(smoothing_level=alpha, smoothing_trend=beta,
smoothing_seasonal=gamma)
```

```
# Generate multi-step forecasts
```

```
forecast_steps = 12
```

```
forecasts = fitted_model.forecast(steps=forecast_steps)
```

```
# Calculate MAPE
```

```
mape = mean_absolute_percentage_error(test_data, forecasts)
```

```
print("MAPE:", mape)
```

```
# Print the multi-step forecasts
```

```
print("Multi-step Forecasts:", forecasts)
```

Python Code for Augmented Dickey Fuller (Stationarity Check)

```

import pandas as pd
import matplotlib.pyplot as plt
import statsmodels.api as sm

# Read the data table
file_path = r"(Retail Sales)"
df = pd.read_csv(file_path)

# Convert 'Years' column to datetime
df['Years'] = pd.to_datetime(df['Years'])

# Plotting
plt.figure(figsize=(12, 6))
plt.plot(df['Years'], df['sales'])

# titles for x-axis and y-axis
plt.xlabel('Years', fontsize=12, fontweight='bold', color='black')
plt.ylabel('Sales', fontsize=12, fontweight='bold', color='black')

plt.xticks(rotation=0)
plt.tight_layout()

# ADF test for stationarity
adf_test = sm.tsa.stattools.adfuller(df['sales'])
print("ADF Test Results:")

```

```

print(f"ADF Statistic: {adf_test[0]}")
print(f"p-value: {adf_test[1]}")
print("Critical Values:")
for key, value in adf_test[4].items():
    print(f"\t{key}: {value}")

plt.show()

```

R Programming Code for ARIMA

```

library(forecast)
data <- read.csv("Retail Sales")
sales<- log(data$sales)
plot.ts(sales)
fit<- Arima(sales[1:156],order = c(12,1,1), fixed=c(NA,NA,0,0,0,0,0,0,0,0,NA,NA)) #restricted
pred<- exp(forecast(fit,h=12)$mean)
obs<- exp(sales[157:168])
mean(abs((obs-pred)/ obs))*100
mean(abs((obs-pred)))

```

HOLT WINTERS GRID SEARCH

**The parameters selected for each period are highlighted below*

| GRID SEARCH (1992-2005) | | | | GRID SEARCH (2006-2019) | | | |
|-------------------------|------|-------|---------|-------------------------|------|-------|----------|
| Alpha | Beta | Gamma | MAPE | Alpha | Beta | Gamma | MAPE |
| 0 | 0 | 0 | 0.46221 | 0 | 0 | 0 | 0.183006 |
| 0 | 0 | 0.2 | 0.06212 | 0 | 0 | 0.2 | 0.379782 |
| 0 | 0 | 0.4 | 0.05851 | 0 | 0 | 0.4 | 0.371284 |
| 0 | 0 | 0.6 | 0.06043 | 0 | 0 | 0.6 | 0.370963 |
| 0 | 0 | 0.8 | 0.06878 | 0 | 0 | 0.8 | 0.370953 |
| 0 | 0 | 1 | 0.08033 | 0 | 0 | 1 | 0.370895 |
| 0 | 0.2 | 0 | 0.46221 | 0 | 0.2 | 0 | 0.183006 |
| | | | | 0 | 0.2 | 0.2 | 0.379782 |

| | | | | | | | |
|------------|------------|------------|---------------|-----|-----|-----|----------|
| 0 | 0.2 | 0.2 | 0.06212 | 0 | 0.2 | 0.4 | 0.371284 |
| 0 | 0.2 | 0.4 | 0.05851 | 0 | 0.2 | 0.6 | 0.370963 |
| 0 | 0.2 | 0.6 | 0.06043 | 0 | 0.2 | 0.8 | 0.370953 |
| 0 | 0.2 | 0.8 | 0.06878 | 0 | 0.2 | 1 | 0.370895 |
| 0 | 0.2 | 1 | 0.08033 | 0 | 0.4 | 0 | 0.183006 |
| 0 | 0.4 | 0 | 0.46221 | 0 | 0.4 | 0.2 | 0.379782 |
| 0 | 0.4 | 0.2 | 0.06212 | 0 | 0.4 | 0.4 | 0.371284 |
| 0 | 0.4 | 0.4 | 0.05851 | 0 | 0.4 | 0.6 | 0.370963 |
| 0 | 0.4 | 0.6 | 0.06043 | 0 | 0.4 | 0.8 | 0.370953 |
| 0 | 0.4 | 0.8 | 0.06878 | 0 | 0.4 | 1 | 0.370895 |
| 0 | 0.4 | 1 | 0.08033 | 0 | 0.6 | 0 | 0.183006 |
| 0 | 0.6 | 0 | 0.46221 | 0 | 0.6 | 0.2 | 0.379782 |
| 0 | 0.6 | 0.2 | 0.06212 | 0 | 0.6 | 0.4 | 0.371284 |
| 0 | 0.6 | 0.4 | 0.05851 | 0 | 0.6 | 0.6 | 0.370963 |
| 0 | 0.6 | 0.6 | 0.06043 | 0 | 0.6 | 0.8 | 0.370953 |
| 0 | 0.6 | 0.8 | 0.06878 | 0 | 0.6 | 1 | 0.370895 |
| 0 | 0.6 | 1 | 0.08033 | 0 | 0.8 | 0 | 0.183006 |
| 0 | 0.8 | 0 | 0.46221 | 0 | 0.8 | 0.2 | 0.379782 |
| 0 | 0.8 | 0.2 | 0.06212 | 0 | 0.8 | 0.4 | 0.371284 |
| 0 | 0.8 | 0.4 | 0.05851 | 0 | 0.8 | 0.6 | 0.370963 |
| 0 | 0.8 | 0.6 | 0.06043 | 0 | 0.8 | 0.8 | 0.370953 |
| 0 | 0.8 | 0.8 | 0.06878 | 0 | 0.8 | 0.8 | 0.370953 |
| 0 | 0.8 | 1 | 0.08033 | 0 | 0.8 | 1 | 0.370895 |
| 0 | 1 | 0 | 0.46221 | 0 | 1 | 0 | 0.183006 |
| 0 | 1 | 0.2 | 0.06212 | 0 | 1 | 0.2 | 0.379782 |
| 0 | 1 | 0.4 | 0.05851 | 0 | 1 | 0.4 | 0.371284 |
| 0 | 1 | 0.6 | 0.06043 | 0 | 1 | 0.6 | 0.370963 |
| 0 | 1 | 0.8 | 0.06878 | 0 | 1 | 0.8 | 0.370953 |
| 0 | 1 | 1 | 0.08033 | 0 | 1 | 1 | 0.370895 |
| 0.2 | 0 | 0 | 0.06233 | 0.2 | 0 | 0 | 0.366662 |
| 0.2 | 0 | 0.2 | 0.05607 | 0.2 | 0 | 0.2 | 0.359549 |
| 0.2 | 0 | 0.4 | 0.05436 | 0.2 | 0 | 0.4 | 0.356789 |
| 0.2 | 0 | 0.6 | 0.05584 | 0.2 | 0 | 0.6 | 0.356723 |
| 0.2 | 0 | 0.8 | 0.06165 | 0.2 | 0 | 0.8 | 0.357803 |
| 0.2 | 0 | 1 | 0.06985 | 0.2 | 0 | 1 | 0.359144 |
| 0.2 | 0.2 | 0 | 0.06226 | 0.2 | 0.2 | 0 | 0.409009 |
| 0.2 | 0.2 | 0.2 | 0.05569 | 0.2 | 0.2 | 0.2 | 0.246914 |
| 0.2 | 0.2 | 0.4 | 0.0481 | 0.2 | 0.2 | 0.4 | 0.244975 |
| 0.2 | 0.2 | 0.6 | 0.05494 | 0.2 | 0.2 | 0.6 | 0.274849 |
| 0.2 | 0.2 | 0.8 | 0.07486 | 0.2 | 0.2 | 0.8 | 0.307319 |
| 0.2 | 0.2 | 1 | 0.09661 | 0.2 | 0.2 | 1 | 0.335203 |
| | | | | 0.2 | 0.4 | 0 | 0.277888 |
| | | | | 0.2 | 0.4 | 0.2 | 0.226567 |
| | | | | 0.2 | 0.4 | 0.4 | 0.179671 |
| | | | | 0.2 | 0.4 | 0.6 | 0.174314 |

| | | | | | | | |
|-----|-----|-----|---------|------------|------------|------------|-----------------|
| 0.2 | 0.4 | 0 | 0.04849 | 0.2 | 0.4 | 0.8 | 0.192995 |
| 0.2 | 0.4 | 0.2 | 0.06527 | 0.2 | 0.4 | 1 | 0.220541 |
| 0.2 | 0.4 | 0.4 | 0.09906 | 0.2 | 0.6 | 0 | 0.204516 |
| 0.2 | 0.4 | 0.6 | 0.10431 | 0.2 | 0.6 | 0.2 | 0.098309 |
| 0.2 | 0.4 | 0.8 | 0.09368 | 0.2 | 0.6 | 0.4 | 0.099055 |
| 0.2 | 0.4 | 1 | 0.08073 | 0.2 | 0.6 | 0.6 | 0.115342 |
| 0.2 | 0.6 | 0 | 0.10721 | 0.2 | 0.6 | 0.8 | 0.138921 |
| 0.2 | 0.6 | 0.2 | 0.0963 | 0.2 | 0.6 | 1 | 0.165273 |
| 0.2 | 0.6 | 0.4 | 0.14226 | 0.2 | 0.8 | 0 | 0.189633 |
| 0.2 | 0.6 | 0.6 | 0.15108 | 0.2 | 0.8 | 0.2 | 0.108096 |
| 0.2 | 0.6 | 0.8 | 0.14354 | 0.2 | 0.8 | 0.4 | 0.057267 |
| 0.2 | 0.6 | 1 | 0.13082 | 0.2 | 0.8 | 0.6 | 0.044713 |
| 0.2 | 0.8 | 0 | 0.06164 | 0.2 | 0.8 | 0.8 | 0.087755 |
| 0.2 | 0.8 | 0.2 | 0.16893 | 0.2 | 0.8 | 1 | 0.126192 |
| 0.2 | 0.8 | 0.4 | 0.21375 | 0.2 | 1 | 0 | 0.438732 |
| 0.2 | 0.8 | 0.6 | 0.20668 | 0.2 | 1 | 0.2 | 0.296079 |
| 0.2 | 0.8 | 0.8 | 0.1846 | 0.2 | 1 | 0.4 | 0.169225 |
| 0.2 | 0.8 | 1 | 0.16582 | 0.2 | 1 | 0.6 | 0.082947 |
| 0.2 | 1 | 0 | 0.12336 | 0.2 | 1 | 0.8 | 0.048754 |
| 0.2 | 1 | 0.2 | 0.29853 | 0.2 | 1 | 1 | 0.088493 |
| 0.2 | 1 | 0.4 | 0.31125 | 0.4 | 0 | 0 | 0.345712 |
| 0.2 | 1 | 0.6 | 0.27404 | 0.4 | 0 | 0.2 | 0.342619 |
| 0.2 | 1 | 0.8 | 0.23072 | 0.4 | 0 | 0.4 | 0.342191 |
| 0.2 | 1 | 1 | 0.19676 | 0.4 | 0 | 0.6 | 0.343404 |
| 0.4 | 0 | 0 | 0.05661 | 0.4 | 0 | 0.8 | 0.3454 |
| 0.4 | 0 | 0.2 | 0.05223 | 0.4 | 0 | 1 | 0.34764 |
| 0.4 | 0 | 0.4 | 0.05194 | 0.4 | 0.2 | 0 | 0.200208 |
| 0.4 | 0 | 0.6 | 0.05385 | 0.4 | 0.2 | 0.2 | 0.170351 |
| 0.4 | 0 | 0.8 | 0.05701 | 0.4 | 0.2 | 0.4 | 0.162948 |
| 0.4 | 0 | 1 | 0.06159 | 0.4 | 0.2 | 0.6 | 0.171692 |
| 0.4 | 0.2 | 0 | 0.07934 | 0.4 | 0.2 | 0.8 | 0.189286 |
| 0.4 | 0.2 | 0.2 | 0.10321 | 0.4 | 0.2 | 1 | 0.210542 |
| 0.4 | 0.2 | 0.4 | 0.10969 | 0.4 | 0.4 | 0 | 0.113129 |
| 0.4 | 0.2 | 0.6 | 0.10423 | 0.4 | 0.4 | 0.2 | 0.069196 |
| 0.4 | 0.2 | 0.8 | 0.0931 | 0.4 | 0.4 | 0.4 | 0.042153 |
| 0.4 | 0.2 | 1 | 0.08148 | 0.4 | 0.4 | 0.6 | 0.055218 |
| 0.4 | 0.4 | 0 | 0.19695 | 0.4 | 0.4 | 0.8 | 0.087099 |
| 0.4 | 0.4 | 0.2 | 0.22295 | 0.4 | 0.4 | 1 | 0.11659 |
| 0.4 | 0.4 | 0.4 | 0.21899 | 0.4 | 0.6 | 0 | 0.455289 |
| 0.4 | 0.4 | 0.6 | 0.20211 | 0.4 | 0.6 | 0.2 | 0.314316 |
| 0.4 | 0.4 | 0.8 | 0.18247 | 0.4 | 0.6 | 0.4 | 0.204478 |
| | | | | 0.4 | 0.6 | 0.6 | 0.121167 |
| | | | | 0.4 | 0.6 | 0.8 | 0.065031 |
| | | | | 0.4 | 0.6 | 1 | 0.050426 |



| | | | | | | | |
|-----|-----|-----|---------|-----|-----|-----|----------|
| 0.4 | 0.4 | 1 | 0.16606 | 0.4 | 0.8 | 0 | 0.706752 |
| 0.4 | 0.6 | 0 | 0.4855 | 0.4 | 0.8 | 0.2 | 0.518077 |
| 0.4 | 0.6 | 0.2 | 0.4433 | 0.4 | 0.8 | 0.4 | 0.364409 |
| 0.4 | 0.6 | 0.4 | 0.3833 | 0.4 | 0.8 | 0.6 | 0.239215 |
| 0.4 | 0.6 | 0.6 | 0.32229 | 0.4 | 0.8 | 0.8 | 0.14715 |
| 0.4 | 0.6 | 0.8 | 0.27111 | 0.4 | 0.8 | 1 | 0.085716 |
| 0.4 | 0.6 | 1 | 0.23119 | 0.4 | 1 | 0 | 0.840165 |
| 0.4 | 0.8 | 0 | 0.84936 | 0.4 | 1 | 0.2 | 0.648284 |
| 0.4 | 0.8 | 0.2 | 0.70084 | 0.4 | 1 | 0.4 | 0.467738 |
| 0.4 | 0.8 | 0.4 | 0.56285 | 0.4 | 1 | 0.6 | 0.318642 |
| 0.4 | 0.8 | 0.6 | 0.4477 | 0.4 | 1 | 0.8 | 0.207368 |
| 0.4 | 0.8 | 0.8 | 0.36026 | 0.4 | 1 | 1 | 0.12959 |
| 0.4 | 0.8 | 1 | 0.29633 | 0.6 | 0 | 0 | 0.327989 |
| 0.4 | 1 | 0 | 1.17938 | 0.6 | 0 | 0.2 | 0.328425 |
| 0.4 | 1 | 0.2 | 0.93261 | 0.6 | 0 | 0.4 | 0.329682 |
| 0.4 | 1 | 0.4 | 0.71279 | 0.6 | 0 | 0.6 | 0.331572 |
| 0.4 | 1 | 0.6 | 0.54698 | 0.6 | 0 | 0.8 | 0.333801 |
| 0.4 | 1 | 0.8 | 0.43115 | 0.6 | 0 | 1 | 0.336136 |
| 0.4 | 1 | 1 | 0.34778 | 0.6 | 0.2 | 0 | 0.086061 |
| 0.6 | 0 | 0 | 0.05328 | 0.6 | 0.2 | 0.2 | 0.094767 |
| 0.6 | 0 | 0.2 | 0.05135 | 0.6 | 0.2 | 0.4 | 0.106397 |
| 0.6 | 0 | 0.4 | 0.05119 | 0.6 | 0.2 | 0.6 | 0.119732 |
| 0.6 | 0 | 0.6 | 0.05256 | 0.6 | 0.2 | 0.8 | 0.133465 |
| 0.6 | 0 | 0.8 | 0.05446 | 0.6 | 0.2 | 1 | 0.147443 |
| 0.6 | 0 | 1 | 0.05711 | 0.6 | 0.4 | 0 | 0.257769 |
| 0.6 | 0.2 | 0 | 0.163 | 0.6 | 0.4 | 0.2 | 0.189249 |
| 0.6 | 0.2 | 0.2 | 0.16253 | 0.6 | 0.4 | 0.4 | 0.132864 |
| 0.6 | 0.2 | 0.4 | 0.15692 | 0.6 | 0.4 | 0.6 | 0.088702 |
| 0.6 | 0.2 | 0.6 | 0.14901 | 0.6 | 0.4 | 0.8 | 0.055889 |
| 0.6 | 0.2 | 0.8 | 0.14088 | 0.6 | 0.4 | 1 | 0.04631 |
| 0.6 | 0.2 | 1 | 0.1322 | 0.6 | 0.6 | 0 | 0.497074 |
| 0.6 | 0.4 | 0 | 0.41773 | 0.6 | 0.6 | 0.2 | 0.390719 |
| 0.6 | 0.4 | 0.2 | 0.37407 | 0.6 | 0.6 | 0.4 | 0.298582 |
| 0.6 | 0.4 | 0.4 | 0.33207 | 0.6 | 0.6 | 0.6 | 0.221938 |
| 0.6 | 0.4 | 0.6 | 0.29422 | 0.6 | 0.6 | 0.8 | 0.161148 |
| 0.6 | 0.4 | 0.8 | 0.26164 | 0.6 | 0.6 | 1 | 0.113949 |
| 0.6 | 0.4 | 1 | 0.23429 | 0.6 | 0.8 | 0 | 0.596876 |
| 0.6 | 0.6 | 0 | 0.71764 | 0.6 | 0.8 | 0.2 | 0.466891 |
| 0.6 | 0.6 | 0.2 | 0.60521 | 0.6 | 0.8 | 0.4 | 0.359721 |
| 0.6 | 0.6 | 0.4 | 0.51262 | 0.6 | 0.8 | 0.6 | 0.274645 |
| 0.6 | 0.6 | 0.6 | 0.43802 | 0.6 | 0.8 | 0.8 | 0.208466 |
| | | | | 0.6 | 0.8 | 1 | 0.155678 |
| | | | | 0.6 | 1 | 0 | 0.562209 |
| | | | | 0.6 | 1 | 0.2 | 0.417448 |

| | | | | | | | |
|-----|-----|-----|---------|-----|-----|-----|----------|
| 0.6 | 0.6 | 0.8 | 0.37822 | 0.6 | 1 | 0.4 | 0.312808 |
| 0.6 | 0.6 | 1 | 0.32796 | 0.6 | 1 | 0.6 | 0.23898 |
| 0.6 | 0.8 | 0 | 0.9069 | 0.6 | 1 | 0.8 | 0.186294 |
| 0.6 | 0.8 | 0.2 | 0.73715 | 0.6 | 1 | 1 | 0.142842 |
| 0.6 | 0.8 | 0.4 | 0.60825 | 0.8 | 0 | 0 | 0.317981 |
| 0.6 | 0.8 | 0.6 | 0.51419 | 0.8 | 0 | 0.2 | 0.319245 |
| 0.6 | 0.8 | 0.8 | 0.44285 | 0.8 | 0 | 0.4 | 0.320598 |
| 0.6 | 0.8 | 1 | 0.38236 | 0.8 | 0 | 0.6 | 0.322012 |
| 0.6 | 1 | 0 | 0.89009 | 0.8 | 0 | 0.8 | 0.323465 |
| 0.6 | 1 | 0.2 | 0.70177 | 0.8 | 0 | 1 | 0.324942 |
| 0.6 | 1 | 0.4 | 0.57784 | 0.8 | 0.2 | 0 | 0.058711 |
| 0.6 | 1 | 0.6 | 0.49778 | 0.8 | 0.2 | 0.2 | 0.066312 |
| 0.6 | 1 | 0.8 | 0.43903 | 0.8 | 0.2 | 0.4 | 0.074323 |
| 0.6 | 1 | 1 | 0.38222 | 0.8 | 0.2 | 0.6 | 0.082985 |
| 0.8 | 0 | 0 | 0.05217 | 0.8 | 0.2 | 0.8 | 0.091759 |
| 0.8 | 0 | 0.2 | 0.05172 | 0.8 | 0.2 | 1 | 0.100836 |
| 0.8 | 0 | 0.4 | 0.05147 | 0.8 | 0.4 | 0 | 0.229895 |
| 0.8 | 0 | 0.6 | 0.0517 | 0.8 | 0.4 | 0.2 | 0.19523 |
| 0.8 | 0 | 0.8 | 0.05237 | 0.8 | 0.4 | 0.4 | 0.164785 |
| 0.8 | 0 | 1 | 0.05335 | 0.8 | 0.4 | 0.6 | 0.137692 |
| 0.8 | 0.2 | 0 | 0.20768 | 0.8 | 0.4 | 0.8 | 0.114053 |
| 0.8 | 0.2 | 0.2 | 0.1996 | 0.8 | 0.4 | 1 | 0.093503 |
| 0.8 | 0.2 | 0.4 | 0.19153 | 0.8 | 0.6 | 0 | 0.325706 |
| 0.8 | 0.2 | 0.6 | 0.18352 | 0.8 | 0.6 | 0.2 | 0.28436 |
| 0.8 | 0.2 | 0.8 | 0.1756 | 0.8 | 0.6 | 0.4 | 0.247411 |
| 0.8 | 0.2 | 1 | 0.16802 | 0.8 | 0.6 | 0.6 | 0.214746 |
| 0.8 | 0.4 | 0 | 0.43447 | 0.8 | 0.6 | 0.8 | 0.185944 |
| 0.8 | 0.4 | 0.2 | 0.40348 | 0.8 | 0.6 | 1 | 0.159069 |
| 0.8 | 0.4 | 0.4 | 0.37546 | 0.8 | 0.8 | 0 | 0.246838 |
| 0.8 | 0.4 | 0.6 | 0.34998 | 0.8 | 0.8 | 0.2 | 0.2165 |
| 0.8 | 0.4 | 0.8 | 0.32654 | 0.8 | 0.8 | 0.4 | 0.191596 |
| 0.8 | 0.4 | 1 | 0.30454 | 0.8 | 0.8 | 0.6 | 0.169682 |
| 0.8 | 0.6 | 0 | 0.5657 | 0.8 | 0.8 | 0.8 | 0.149113 |
| 0.8 | 0.6 | 0.2 | 0.52112 | 0.8 | 0.8 | 1 | 0.129174 |
| 0.8 | 0.6 | 0.4 | 0.48255 | 0.8 | 1 | 0 | 0.067685 |
| 0.8 | 0.6 | 0.6 | 0.44826 | 0.8 | 1 | 0.2 | 0.067416 |
| 0.8 | 0.6 | 0.8 | 0.41672 | 0.8 | 1 | 0.4 | 0.066732 |
| 0.8 | 0.6 | 1 | 0.38645 | 0.8 | 1 | 0.6 | 0.063523 |
| 0.8 | 0.8 | 0 | 0.50776 | 0.8 | 1 | 0.8 | 0.058587 |
| 0.8 | 0.8 | 0.2 | 0.47757 | 0.8 | 1 | 1 | 0.053625 |
| 0.8 | 0.8 | 0.4 | 0.45172 | 1 | 0 | 0 | 0.3145 |
| | | | | 1 | 0 | 0.2 | 0.3145 |
| | | | | 1 | 0 | 0.4 | 0.3145 |
| | | | | 1 | 0 | 0.6 | 0.3145 |

| | | | | | | | |
|-----|-----|-----|---------|---|-----|-----|----------|
| 0.8 | 0.8 | 0.6 | 0.42732 | 1 | 0 | 0.8 | 0.3145 |
| 0.8 | 0.8 | 0.8 | 0.40175 | 1 | 0 | 1 | 0.3145 |
| 0.8 | 0.8 | 1 | 0.37287 | 1 | 0.2 | 0 | 0.063596 |
| 0.8 | 1 | 0 | 0.32154 | 1 | 0.2 | 0.2 | 0.063596 |
| 0.8 | 1 | 0.2 | 0.33109 | 1 | 0.2 | 0.4 | 0.063596 |
| 0.8 | 1 | 0.4 | 0.33511 | 1 | 0.2 | 0.6 | 0.063596 |
| 0.8 | 1 | 0.6 | 0.33007 | 1 | 0.2 | 0.8 | 0.063596 |
| 0.8 | 1 | 0.8 | 0.3134 | 1 | 0.2 | 1 | 0.063596 |
| 0.8 | 1 | 1 | 0.28361 | 1 | 0.4 | 0 | 0.143963 |
| 1 | 0 | 0 | 0.05215 | 1 | 0.4 | 0.2 | 0.143963 |
| 1 | 0 | 0.2 | 0.05215 | 1 | 0.4 | 0.4 | 0.143963 |
| 1 | 0 | 0.4 | 0.05215 | 1 | 0.4 | 0.6 | 0.143963 |
| 1 | 0 | 0.6 | 0.05215 | 1 | 0.4 | 0.8 | 0.143963 |
| 1 | 0 | 0.8 | 0.05215 | 1 | 0.4 | 1 | 0.143963 |
| 1 | 0 | 1 | 0.05215 | 1 | 0.6 | 0 | 0.14571 |
| 1 | 0.2 | 0 | 0.20659 | 1 | 0.6 | 0.2 | 0.14571 |
| 1 | 0.2 | 0.2 | 0.20659 | 1 | 0.6 | 0.4 | 0.14571 |
| 1 | 0.2 | 0.4 | 0.20659 | 1 | 0.6 | 0.6 | 0.14571 |
| 1 | 0.2 | 0.6 | 0.20659 | 1 | 0.6 | 0.8 | 0.14571 |
| 1 | 0.2 | 0.8 | 0.20659 | 1 | 0.6 | 1 | 0.14571 |
| 1 | 0.2 | 1 | 0.20659 | 1 | 0.8 | 0 | 0.04858 |
| 1 | 0.4 | 0 | 0.36169 | 1 | 0.8 | 0.2 | 0.04858 |
| 1 | 0.4 | 0.2 | 0.36169 | 1 | 0.8 | 0.4 | 0.04858 |
| 1 | 0.4 | 0.4 | 0.36169 | 1 | 0.8 | 0.6 | 0.04858 |
| 1 | 0.4 | 0.6 | 0.36169 | 1 | 0.8 | 0.8 | 0.04858 |
| 1 | 0.4 | 0.8 | 0.36169 | 1 | 0.8 | 1 | 0.04858 |
| 1 | 0.4 | 1 | 0.36169 | 1 | 1 | 0 | 0.178218 |
| 1 | 0.6 | 0 | 0.38715 | 1 | 1 | 0.2 | 0.178218 |
| 1 | 0.6 | 0.2 | 0.38715 | 1 | 1 | 0.4 | 0.178218 |
| 1 | 0.6 | 0.4 | 0.38715 | 1 | 1 | 0.6 | 0.178218 |
| 1 | 0.6 | 0.6 | 0.38715 | 1 | 1 | 0.8 | 0.178218 |
| 1 | 0.6 | 0.8 | 0.38715 | 1 | 1 | 1 | 0.178218 |
| 1 | 0.6 | 1 | 0.38715 | 0 | 0 | 0 | 0.520611 |
| 1 | 0.8 | 0 | 0.28313 | 0 | 0 | 0.2 | 0.379588 |
| 1 | 0.8 | 0.2 | 0.28313 | 0 | 0 | 0.4 | 0.346652 |
| 1 | 0.8 | 0.4 | 0.28313 | 0 | 0 | 0.6 | 0.336105 |
| 1 | 0.8 | 0.6 | 0.28313 | 0 | 0.2 | 0.8 | 0.330572 |
| 1 | 0.8 | 0.8 | 0.28313 | 0 | 0.2 | 1 | 0.326644 |
| 1 | 0.8 | 1 | 0.28313 | 0 | 0.2 | 0 | 0.520611 |
| 1 | 1 | 0 | 0.10102 | 0 | 0.2 | 0.2 | 0.379588 |
| 1 | 1 | 0.2 | 0.10102 | 0 | 0.2 | 0.4 | 0.346652 |
| | | | | 0 | 0.2 | 0.6 | 0.336105 |
| | | | | 0 | 0.2 | 0.8 | 0.330572 |
| | | | | 0 | 0.2 | 1 | 0.326644 |

| | | | | | | | |
|-----|-----|-----|---------|-----|-----|-----|----------|
| 1 | 1 | 0.4 | 0.10102 | 0 | 0.4 | 0 | 0.520611 |
| 1 | 1 | 0.6 | 0.10102 | 0 | 0.4 | 0.2 | 0.379588 |
| 1 | 1 | 0.8 | 0.10102 | 0 | 0.4 | 0.4 | 0.346652 |
| 1 | 1 | 1 | 0.10102 | 0 | 0.4 | 0.6 | 0.336105 |
| 0 | 0 | 0 | 0.30676 | 0 | 0.4 | 0.8 | 0.330572 |
| 0 | 0 | 0.2 | 0.07772 | 0 | 0.4 | 1 | 0.326644 |
| 0 | 0 | 0.4 | 0.07187 | 0 | 0.6 | 0 | 0.520611 |
| 0 | 0 | 0.6 | 0.0731 | 0 | 0.6 | 0.2 | 0.379588 |
| 0 | 0 | 0.8 | 0.0778 | 0 | 0.6 | 0.4 | 0.346652 |
| 0 | 0 | 1 | 0.08451 | 0 | 0.6 | 0.6 | 0.336105 |
| 0 | 0.2 | 0 | 0.30676 | 0 | 0.6 | 0.8 | 0.330572 |
| 0 | 0.2 | 0.2 | 0.07772 | 0 | 0.6 | 1 | 0.326644 |
| 0 | 0.2 | 0.4 | 0.07187 | 0 | 0.8 | 0 | 0.520611 |
| 0 | 0.2 | 0.6 | 0.0731 | 0 | 0.8 | 0.2 | 0.379588 |
| 0 | 0.2 | 0.8 | 0.0778 | 0 | 0.8 | 0.4 | 0.346652 |
| 0 | 0.2 | 1 | 0.08451 | 0 | 0.8 | 0.6 | 0.336105 |
| 0 | 0.4 | 0 | 0.30676 | 0 | 0.8 | 0.8 | 0.330572 |
| 0 | 0.4 | 0.2 | 0.07772 | 0 | 0.8 | 1 | 0.326644 |
| 0 | 0.4 | 0.4 | 0.07187 | 0 | 1 | 0 | 0.520611 |
| 0 | 0.4 | 0.6 | 0.0731 | 0 | 1 | 0.2 | 0.379588 |
| 0 | 0.4 | 0.8 | 0.0778 | 0 | 1 | 0.4 | 0.346652 |
| 0 | 0.4 | 1 | 0.08451 | 0 | 1 | 0.6 | 0.336105 |
| 0 | 0.6 | 0 | 0.30676 | 0 | 1 | 0.8 | 0.330572 |
| 0 | 0.6 | 0.2 | 0.07772 | 0.2 | 1 | 1 | 0.326644 |
| 0 | 0.6 | 0.4 | 0.07187 | 0.2 | 0 | 0 | 0.284412 |
| 0 | 0.6 | 0.6 | 0.0731 | 0.2 | 0 | 0.2 | 0.284285 |
| 0 | 0.6 | 0.8 | 0.0778 | 0.2 | 0 | 0.4 | 0.284358 |
| 0 | 0.6 | 1 | 0.08451 | 0.2 | 0 | 0.6 | 0.284848 |
| 0 | 0.8 | 0 | 0.30676 | 0.2 | 0 | 0.8 | 0.285558 |
| 0 | 0.8 | 0.2 | 0.07772 | 0.2 | 0 | 1 | 0.286327 |
| 0 | 0.8 | 0.4 | 0.07187 | 0.2 | 0.2 | 0 | 0.137732 |
| 0 | 0.8 | 0.6 | 0.0731 | 0.2 | 0.2 | 0.2 | 0.162412 |
| 0 | 0.8 | 0.8 | 0.0778 | 0.2 | 0.2 | 0.4 | 0.167564 |
| 0 | 0.8 | 1 | 0.08451 | 0.2 | 0.2 | 0.6 | 0.16053 |
| 0 | 1 | 0 | 0.30676 | 0.2 | 0.2 | 0.8 | 0.147416 |
| 0 | 1 | 0.2 | 0.07772 | 0.2 | 0.2 | 1 | 0.135119 |
| 0 | 1 | 0.4 | 0.07187 | 0.2 | 0.4 | 0 | 0.240297 |
| 0 | 1 | 0.6 | 0.0731 | 0.2 | 0.4 | 0.2 | 0.263756 |
| 0 | 1 | 0.8 | 0.0778 | 0.2 | 0.4 | 0.4 | 0.279426 |
| 0 | 1 | 1 | 0.08451 | 0.2 | 0.4 | 0.6 | 0.287435 |
| 0.2 | 0 | 0 | 0.06083 | 0.2 | 0.4 | 0.8 | 0.288441 |
| | | | | 0.2 | 0.4 | 1 | 0.2844 |
| | | | | 0.2 | 0.6 | 0 | 0.378606 |
| | | | | 0.2 | 0.6 | 0.2 | 0.385255 |

| | | | | | | | |
|-----|-----|-----|---------|-----|-----|-----|----------|
| 0.2 | 0 | 0.2 | 0.05527 | 0.2 | 0.6 | 0.4 | 0.398518 |
| 0.2 | 0 | 0.4 | 0.05173 | 0.2 | 0.6 | 0.6 | 0.405621 |
| 0.2 | 0 | 0.6 | 0.05135 | 0.2 | 0.6 | 0.8 | 0.405572 |
| 0.2 | 0 | 0.8 | 0.05464 | 0.2 | 0.6 | 1 | 0.400381 |
| 0.2 | 0 | 1 | 0.06039 | 0.2 | 0.8 | 0 | 0.498192 |
| 0.2 | 0.2 | 0 | 0.17465 | 0.2 | 0.8 | 0.2 | 0.512638 |
| 0.2 | 0.2 | 0.2 | 0.16805 | 0.2 | 0.8 | 0.4 | 0.525793 |
| 0.2 | 0.2 | 0.4 | 0.16589 | 0.2 | 0.8 | 0.6 | 0.529831 |
| 0.2 | 0.2 | 0.6 | 0.16391 | 0.2 | 0.8 | 0.8 | 0.525322 |
| 0.2 | 0.2 | 0.8 | 0.16078 | 0.2 | 0.8 | 1 | 0.515755 |
| 0.2 | 0.2 | 1 | 0.15793 | 0.2 | 1 | 0 | 0.625977 |
| 0.2 | 0.4 | 0 | 0.35049 | 0.2 | 1 | 0.2 | 0.641799 |
| 0.2 | 0.4 | 0.2 | 0.33104 | 0.2 | 1 | 0.4 | 0.658119 |
| 0.2 | 0.4 | 0.4 | 0.32449 | 0.2 | 1 | 0.6 | 0.658673 |
| 0.2 | 0.4 | 0.6 | 0.32364 | 0.2 | 1 | 0.8 | 0.648259 |
| 0.2 | 0.4 | 0.8 | 0.32345 | 0.2 | 1 | 1 | 0.632588 |
| 0.2 | 0.4 | 1 | 0.3237 | 0.4 | 0 | 0 | 0.23095 |
| 0.2 | 0.6 | 0 | 0.52672 | 0.4 | 0 | 0.2 | 0.236778 |
| 0.2 | 0.6 | 0.2 | 0.4876 | 0.4 | 0 | 0.4 | 0.240334 |
| 0.2 | 0.6 | 0.4 | 0.4818 | 0.4 | 0 | 0.6 | 0.242992 |
| 0.2 | 0.6 | 0.6 | 0.48199 | 0.4 | 0 | 0.8 | 0.245329 |
| 0.2 | 0.6 | 0.8 | 0.48076 | 0.4 | 0 | 1 | 0.247558 |
| 0.2 | 0.6 | 1 | 0.48074 | 0.4 | 0.2 | 0 | 0.294916 |
| 0.2 | 0.8 | 0 | 0.68386 | 0.4 | 0.2 | 0.2 | 0.306522 |
| 0.2 | 0.8 | 0.2 | 0.62976 | 0.4 | 0.2 | 0.4 | 0.313596 |
| 0.2 | 0.8 | 0.4 | 0.63593 | 0.4 | 0.2 | 0.6 | 0.315915 |
| 0.2 | 0.8 | 0.6 | 0.6416 | 0.4 | 0.2 | 0.8 | 0.314084 |
| 0.2 | 0.8 | 0.8 | 0.64066 | 0.4 | 0.2 | 1 | 0.309391 |
| 0.2 | 0.8 | 1 | 0.63866 | 0.4 | 0.4 | 0 | 0.545595 |
| 0.2 | 1 | 0 | 0.7706 | 0.4 | 0.4 | 0.2 | 0.555056 |
| 0.2 | 1 | 0.2 | 0.75291 | 0.4 | 0.4 | 0.4 | 0.558855 |
| 0.2 | 1 | 0.4 | 0.78674 | 0.4 | 0.4 | 0.6 | 0.557129 |
| 0.2 | 1 | 0.6 | 0.80044 | 0.4 | 0.4 | 0.8 | 0.551289 |
| 0.2 | 1 | 0.8 | 0.80019 | 0.4 | 0.4 | 1 | 0.54303 |
| 0.2 | 1 | 1 | 0.79774 | 0.4 | 0.6 | 0 | 0.812476 |
| 0.4 | 0 | 0 | 0.05733 | 0.4 | 0.6 | 0.2 | 0.825161 |
| 0.4 | 0 | 0.2 | 0.05525 | 0.4 | 0.6 | 0.4 | 0.823521 |
| 0.4 | 0 | 0.4 | 0.05516 | 0.4 | 0.6 | 0.6 | 0.812595 |
| 0.4 | 0 | 0.6 | 0.05611 | 0.4 | 0.6 | 0.8 | 0.796444 |
| 0.4 | 0 | 0.8 | 0.05805 | 0.4 | 0.6 | 1 | 0.778246 |
| 0.4 | 0 | 1 | 0.06107 | 0.4 | 0.8 | 0 | 1.106251 |
| | | | | 0.4 | 0.8 | 0.2 | 1.113339 |
| | | | | 0.4 | 0.8 | 0.4 | 1.099363 |
| | | | | 0.4 | 0.8 | 0.6 | 1.07452 |

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|-----|-----|-----|---------|-----|-----|-----|----------|
| 0.4 | 0.2 | 0 | 0.36708 | 0.4 | 0.8 | 0.8 | 1.044987 |
| 0.4 | 0.2 | 0.2 | 0.36241 | 0.4 | 0.8 | 1 | 1.014852 |
| 0.4 | 0.2 | 0.4 | 0.3611 | 0.4 | 1 | 0 | 1.434298 |
| 0.4 | 0.2 | 0.6 | 0.36042 | 0.4 | 1 | 0.2 | 1.418464 |
| 0.4 | 0.2 | 0.8 | 0.35979 | 0.4 | 1 | 0.4 | 1.381163 |
| 0.4 | 0.2 | 1 | 0.35957 | 0.4 | 1 | 0.6 | 1.33683 |
| 0.4 | 0.4 | 0 | 0.66808 | 0.4 | 1 | 0.8 | 1.291172 |
| 0.4 | 0.4 | 0.2 | 0.6746 | 0.4 | 1 | 1 | 1.247876 |
| 0.4 | 0.4 | 0.4 | 0.67972 | 0.6 | 0 | 0 | 0.20012 |
| 0.4 | 0.4 | 0.6 | 0.68025 | 0.6 | 0 | 0.2 | 0.202403 |
| 0.4 | 0.4 | 0.8 | 0.67833 | 0.6 | 0 | 0.4 | 0.205237 |
| 0.4 | 0.4 | 1 | 0.67664 | 0.6 | 0 | 0.6 | 0.208063 |
| 0.4 | 0.6 | 0 | 0.9272 | 0.6 | 0 | 0.8 | 0.210755 |
| 0.4 | 0.6 | 0.2 | 0.97682 | 0.6 | 0 | 1 | 0.213331 |
| 0.4 | 0.6 | 0.4 | 0.9963 | 0.6 | 0.2 | 0 | 0.459807 |
| 0.4 | 0.6 | 0.6 | 0.99932 | 0.6 | 0.2 | 0.2 | 0.4622 |
| 0.4 | 0.6 | 0.8 | 0.99643 | 0.6 | 0.2 | 0.4 | 0.462559 |
| 0.4 | 0.6 | 1 | 0.99398 | 0.6 | 0.2 | 0.6 | 0.461137 |
| 0.4 | 0.8 | 0 | 1.23503 | 0.6 | 0.2 | 0.8 | 0.458277 |
| 0.4 | 0.8 | 0.2 | 1.30872 | 0.6 | 0.2 | 1 | 0.454394 |
| 0.4 | 0.8 | 0.4 | 1.32668 | 0.6 | 0.4 | 0 | 0.855617 |
| 0.4 | 0.8 | 0.6 | 1.32269 | 0.6 | 0.4 | 0.2 | 0.850633 |
| 0.4 | 0.8 | 0.8 | 1.31443 | 0.6 | 0.4 | 0.4 | 0.842077 |
| 0.4 | 0.8 | 1 | 1.30991 | 0.6 | 0.4 | 0.6 | 0.831124 |
| 0.4 | 1 | 0 | 1.66371 | 0.6 | 0.4 | 0.8 | 0.818747 |
| 0.4 | 1 | 0.2 | 1.69433 | 0.6 | 0.4 | 1 | 0.805901 |
| 0.4 | 1 | 0.4 | 1.67768 | 0.6 | 0.6 | 0 | 1.281051 |
| 0.4 | 1 | 0.6 | 1.65218 | 0.6 | 0.6 | 0.2 | 1.260105 |
| 0.4 | 1 | 0.8 | 1.63314 | 0.6 | 0.6 | 0.4 | 1.236067 |
| 0.4 | 1 | 1 | 1.62523 | 0.6 | 0.6 | 0.6 | 1.210535 |
| 0.6 | 0 | 0 | 0.0764 | 0.6 | 0.6 | 0.8 | 1.18466 |
| 0.6 | 0 | 0.2 | 0.07589 | 0.6 | 0.6 | 1 | 1.159779 |
| 0.6 | 0 | 0.4 | 0.07591 | 0.6 | 0.8 | 0 | 1.715993 |
| 0.6 | 0 | 0.6 | 0.07646 | 0.6 | 0.8 | 0.2 | 1.668781 |
| 0.6 | 0 | 0.8 | 0.07746 | 0.6 | 0.8 | 0.4 | 1.623571 |
| 0.6 | 0 | 1 | 0.07892 | 0.6 | 0.8 | 0.6 | 1.580114 |
| 0.6 | 0.2 | 0 | 0.56029 | 0.6 | 0.8 | 0.8 | 1.538737 |
| 0.6 | 0.2 | 0.2 | 0.55999 | 0.6 | 0.8 | 1 | 1.501093 |
| 0.6 | 0.2 | 0.4 | 0.55904 | 0.6 | 1 | 0 | 2.120773 |
| 0.6 | 0.2 | 0.6 | 0.55765 | 0.6 | 1 | 0.2 | 2.049036 |
| 0.6 | 0.2 | 0.8 | 0.55624 | 0.6 | 1 | 0.4 | 1.98497 |
| | | | | 0.6 | 1 | 0.6 | 1.924819 |
| | | | | 0.6 | 1 | 0.8 | 1.868799 |
| | | | | 0.6 | 1 | 1 | 1.820182 |

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|-----|-----|-----|---------|-----|-----|-----|----------|
| 0.6 | 0.2 | 1 | 0.55526 | 0.8 | 0 | 0 | 0.180755 |
| 0.6 | 0.4 | 0 | 1.03059 | 0.8 | 0 | 0.2 | 0.181747 |
| 0.6 | 0.4 | 0.2 | 1.03692 | 0.8 | 0 | 0.4 | 0.18273 |
| 0.6 | 0.4 | 0.4 | 1.03761 | 0.8 | 0 | 0.6 | 0.183714 |
| 0.6 | 0.4 | 0.6 | 1.03567 | 0.8 | 0 | 0.8 | 0.184706 |
| 0.6 | 0.4 | 0.8 | 1.03321 | 0.8 | 0 | 1 | 0.185712 |
| 0.6 | 0.4 | 1 | 1.03166 | 0.8 | 0.2 | 0 | 0.612534 |
| 0.6 | 0.6 | 0 | 1.54065 | 0.8 | 0.2 | 0.2 | 0.610133 |
| 0.6 | 0.6 | 0.2 | 1.53499 | 0.8 | 0.2 | 0.4 | 0.607363 |
| 0.6 | 0.6 | 0.4 | 1.52511 | 0.8 | 0.2 | 0.6 | 0.604295 |
| 0.6 | 0.6 | 0.6 | 1.51543 | 0.8 | 0.2 | 0.8 | 0.601006 |
| 0.6 | 0.6 | 0.8 | 1.50828 | 0.8 | 0.2 | 1 | 0.59758 |
| 0.6 | 0.6 | 1 | 1.50494 | 0.8 | 0.4 | 0 | 1.124157 |
| 0.6 | 0.8 | 0 | 2.10001 | 0.8 | 0.4 | 0.2 | 1.113546 |
| 0.6 | 0.8 | 0.2 | 2.05424 | 0.8 | 0.4 | 0.4 | 1.102702 |
| 0.6 | 0.8 | 0.4 | 2.01959 | 0.8 | 0.4 | 0.6 | 1.091773 |
| 0.6 | 0.8 | 0.6 | 1.9957 | 0.8 | 0.4 | 0.8 | 1.080927 |
| 0.6 | 0.8 | 0.8 | 1.98153 | 0.8 | 0.4 | 1 | 1.070354 |
| 0.6 | 0.8 | 0.8 | 1.98153 | 0.8 | 0.6 | 0 | 1.631282 |
| 0.6 | 0.8 | 1 | 1.97647 | 0.8 | 0.6 | 0.2 | 1.609282 |
| 0.6 | 1 | 0 | 2.62253 | 0.8 | 0.6 | 0.4 | 1.587789 |
| 0.6 | 1 | 0.2 | 2.54245 | 0.8 | 0.6 | 0.6 | 1.566969 |
| 0.6 | 1 | 0.4 | 2.4912 | 0.8 | 0.6 | 0.8 | 1.547077 |
| 0.6 | 1 | 0.6 | 2.46023 | 0.8 | 0.6 | 1 | 1.52846 |
| 0.6 | 1 | 0.8 | 2.44488 | 0.8 | 0.8 | 0 | 2.093367 |
| 0.6 | 1 | 1 | 2.44333 | 0.8 | 0.8 | 0.2 | 2.060768 |
| 0.8 | 0 | 0 | 0.10846 | 0.8 | 0.8 | 0.4 | 2.029273 |
| 0.8 | 0 | 0.2 | 0.10842 | 0.8 | 0.8 | 0.6 | 1.99937 |
| 0.8 | 0 | 0.4 | 0.10848 | 0.8 | 0.8 | 0.8 | 1.971703 |
| 0.8 | 0 | 0.6 | 0.10864 | 0.8 | 0.8 | 1 | 1.947046 |
| 0.8 | 0 | 0.8 | 0.10891 | 0.8 | 1 | 0 | 2.490533 |
| 0.8 | 0 | 1 | 0.10931 | 0.8 | 1 | 0.2 | 2.448914 |
| 0.8 | 0.2 | 0 | 0.75677 | 0.8 | 1 | 0.4 | 2.409269 |
| 0.8 | 0.2 | 0.2 | 0.75588 | 0.8 | 1 | 0.6 | 2.373002 |
| 0.8 | 0.2 | 0.4 | 0.75492 | 0.8 | 1 | 0.8 | 2.341569 |
| 0.8 | 0.2 | 0.6 | 0.75398 | 0.8 | 1 | 1 | 2.316477 |
| 0.8 | 0.2 | 0.8 | 0.75318 | 1 | 0 | 0 | 0.167458 |
| 0.8 | 0.2 | 1 | 0.75258 | 1 | 0 | 0.2 | 0.167458 |
| 0.8 | 0.4 | 0 | 1.39597 | 1 | 0 | 0.4 | 0.167458 |
| 0.8 | 0.4 | 0.2 | 1.39282 | 1 | 0 | 0.6 | 0.167458 |
| 0.8 | 0.4 | 0.4 | 1.38993 | 1 | 0 | 0.8 | 0.167458 |
| 0.8 | 0.4 | 0.6 | 1.38751 | 1 | 0 | 1 | 0.167458 |
| | | | | 1 | 0.2 | 0 | 0.742603 |
| | | | | 1 | 0.2 | 0.2 | 0.742603 |

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|-----|-----|-----|---------|---|-----|-----|----------|
| 0.8 | 0.4 | 0.8 | 1.38573 | 1 | 0.2 | 0.4 | 0.742603 |
| 0.8 | 0.4 | 1 | 1.38474 | 1 | 0.2 | 0.6 | 0.742603 |
| 0.8 | 0.6 | 0 | 2.04425 | 1 | 0.2 | 0.8 | 0.742603 |
| 0.8 | 0.6 | 0.2 | 2.03331 | 1 | 0.2 | 1 | 0.742603 |
| 0.8 | 0.6 | 0.4 | 2.02484 | 1 | 0.4 | 0 | 1.330795 |
| 0.8 | 0.6 | 0.6 | 2.01874 | 1 | 0.4 | 0.2 | 1.330795 |
| 0.8 | 0.6 | 0.8 | 2.01495 | 1 | 0.4 | 0.4 | 1.330795 |
| 0.8 | 0.6 | 1 | 2.01345 | 1 | 0.4 | 0.6 | 1.330795 |
| 0.8 | 0.8 | 0 | 2.65931 | 1 | 0.4 | 0.8 | 1.330795 |
| 0.8 | 0.8 | 0.2 | 2.64642 | 1 | 0.4 | 1 | 1.330795 |
| 0.8 | 0.8 | 0.4 | 2.63765 | 1 | 0.6 | 0 | 1.873619 |
| 0.8 | 0.8 | 0.6 | 2.63275 | 1 | 0.6 | 0.2 | 1.873619 |
| 0.8 | 0.8 | 0.8 | 2.6316 | 1 | 0.6 | 0.4 | 1.873619 |
| 0.8 | 0.8 | 1 | 2.63417 | 1 | 0.6 | 0.6 | 1.873619 |
| 0.8 | 1 | 0 | 3.22047 | 1 | 0.6 | 0.8 | 1.873619 |
| 0.8 | 1 | 0.2 | 3.21766 | 1 | 0.6 | 1 | 1.873619 |
| 0.8 | 1 | 0.4 | 3.21879 | 1 | 0.8 | 0 | 2.341682 |
| 0.8 | 1 | 0.6 | 3.22414 | 1 | 0.8 | 0.2 | 2.341682 |
| 0.8 | 1 | 0.8 | 3.23401 | 1 | 0.8 | 0.4 | 2.341682 |
| 0.8 | 1 | 1 | 3.24865 | 1 | 0.8 | 0.6 | 2.341682 |
| 1 | 0 | 0 | 0.14165 | 1 | 0.8 | 0.8 | 2.341682 |
| 1 | 0 | 0.2 | 0.14165 | 1 | 0.8 | 1 | 2.341682 |
| 1 | 0 | 0.4 | 0.14165 | 1 | 1 | 0 | 2.746537 |
| 1 | 0 | 0.6 | 0.14165 | 1 | 1 | 0.2 | 2.746537 |
| 1 | 0 | 0.8 | 0.14165 | 1 | 1 | 0.4 | 2.746537 |
| 1 | 0 | 1 | 0.14165 | 1 | 1 | 0.6 | 2.746537 |
| 1 | 0.2 | 0 | 0.94904 | 1 | 1 | 0.8 | 2.746537 |
| 1 | 0.2 | 0.2 | 0.94904 | 1 | 1 | 1 | 2.746537 |
| 1 | 0.2 | 0.4 | 0.94904 | 0 | 0 | 0 | 0.937234 |
| 1 | 0.2 | 0.6 | 0.94904 | 0 | 0 | 0.2 | 0.347986 |
| 1 | 0.2 | 0.8 | 0.94904 | 0 | 0 | 0.4 | 0.321732 |
| 1 | 0.2 | 1 | 0.94904 | 0 | 0 | 0.6 | 0.313083 |
| 1 | 0.4 | 0 | 1.73681 | 0 | 0 | 0.8 | 0.308711 |
| 1 | 0.4 | 0.2 | 1.73681 | 0 | 0 | 1 | 0.306899 |
| 1 | 0.4 | 0.4 | 1.73681 | 0 | 0.2 | 0 | 0.937234 |
| 1 | 0.4 | 0.6 | 1.73681 | 0 | 0.2 | 0.2 | 0.347986 |
| 1 | 0.4 | 0.8 | 1.73681 | 0 | 0.2 | 0.4 | 0.321732 |
| 1 | 0.4 | 1 | 1.73681 | 0 | 0.2 | 0.6 | 0.313083 |
| 1 | 0.6 | 0 | 2.51661 | 0 | 0.2 | 0.8 | 0.308711 |
| 1 | 0.6 | 0.2 | 2.51661 | 0 | 0.2 | 1 | 0.306899 |
| 1 | 0.6 | 0.4 | 2.51661 | 0 | 0.4 | 0 | 0.937234 |
| 1 | 0.6 | 0.4 | 2.51661 | 0 | 0.4 | 0.2 | 0.347986 |
| 1 | 0.6 | 0.4 | 2.51661 | 0 | 0.4 | 0.4 | 0.321732 |
| 1 | 0.6 | 0.4 | 2.51661 | 0 | 0.4 | 0.6 | 0.313083 |

| | | | | | | | |
|---|-----|-----|---------|-----|-----|-----|----------|
| 1 | 0.6 | 0.6 | 2.51661 | 0 | 0.4 | 0.8 | 0.308711 |
| 1 | 0.6 | 0.8 | 2.51661 | 0 | 0.4 | 1 | 0.306899 |
| 1 | 0.6 | 1 | 2.51661 | 0 | 0.6 | 0 | 0.937234 |
| 1 | 0.8 | 0 | 3.28472 | 0 | 0.6 | 0.2 | 0.347986 |
| 1 | 0.8 | 0.2 | 3.28472 | 0 | 0.6 | 0.4 | 0.321732 |
| 1 | 0.8 | 0.4 | 3.28472 | 0 | 0.6 | 0.6 | 0.313083 |
| 1 | 0.8 | 0.6 | 3.28472 | 0 | 0.6 | 0.8 | 0.308711 |
| 1 | 0.8 | 0.8 | 3.28472 | 0 | 0.6 | 1 | 0.306899 |
| 1 | 0.8 | 1 | 3.28472 | 0 | 0.8 | 0 | 0.937234 |
| 1 | 1 | 0 | 4.07647 | 0 | 0.8 | 0.2 | 0.347986 |
| 1 | 1 | 0.2 | 4.07647 | 0 | 0.8 | 0.4 | 0.321732 |
| 1 | 1 | 0.4 | 4.07647 | 0 | 0.8 | 0.6 | 0.313083 |
| 1 | 1 | 0.6 | 4.07647 | 0 | 0.8 | 0.8 | 0.308711 |
| 1 | 1 | 0.8 | 4.07647 | 0 | 0.8 | 1 | 0.306899 |
| 1 | 1 | 1 | 4.07647 | 0 | 1 | 0 | 0.937234 |
| 0 | 0 | 0 | 0.20995 | 0 | 1 | 0.2 | 0.347986 |
| 0 | 0 | 0.2 | 0.0662 | 0 | 1 | 0.4 | 0.321732 |
| 0 | 0 | 0.4 | 0.06366 | 0 | 1 | 0.6 | 0.313083 |
| 0 | 0 | 0.6 | 0.06583 | 0 | 1 | 0.8 | 0.308711 |
| 0 | 0 | 0.8 | 0.07415 | 0 | 1 | 1 | 0.306899 |
| 0 | 0 | 1 | 0.08477 | 0.2 | 0 | 0 | 0.2882 |
| 0 | 0.2 | 0 | 0.20995 | 0.2 | 0 | 0.2 | 0.288809 |
| 0 | 0.2 | 0.2 | 0.0662 | 0.2 | 0 | 0.4 | 0.288569 |
| 0 | 0.2 | 0.4 | 0.06366 | 0.2 | 0 | 0.6 | 0.288978 |
| 0 | 0.2 | 0.6 | 0.06583 | 0.2 | 0 | 0.8 | 0.289778 |
| 0 | 0.2 | 0.8 | 0.07415 | 0.2 | 0 | 1 | 0.291141 |
| 0 | 0.2 | 1 | 0.08477 | 0.2 | 0.2 | 0 | 0.059918 |
| 0 | 0.4 | 0 | 0.20995 | 0.2 | 0.2 | 0.2 | 0.048478 |
| 0 | 0.4 | 0.2 | 0.0662 | 0.2 | 0.2 | 0.4 | 0.045964 |
| 0 | 0.4 | 0.4 | 0.06366 | 0.2 | 0.2 | 0.6 | 0.049604 |
| 0 | 0.4 | 0.6 | 0.06583 | 0.2 | 0.2 | 0.8 | 0.054506 |
| 0 | 0.4 | 0.8 | 0.07415 | 0.2 | 0.2 | 1 | 0.061461 |
| 0 | 0.4 | 1 | 0.08477 | 0.2 | 0.4 | 0 | 0.054008 |
| 0 | 0.6 | 0 | 0.20995 | 0.2 | 0.4 | 0.2 | 0.048017 |
| 0 | 0.6 | 0.2 | 0.0662 | 0.2 | 0.4 | 0.4 | 0.05545 |
| 0 | 0.6 | 0.4 | 0.06366 | 0.2 | 0.4 | 0.6 | 0.058499 |
| 0 | 0.6 | 0.6 | 0.06583 | 0.2 | 0.4 | 0.8 | 0.061064 |
| 0 | 0.6 | 0.8 | 0.07415 | 0.2 | 0.6 | 1 | 0.06526 |
| 0 | 0.6 | 1 | 0.08477 | 0.2 | 0.6 | 0 | 0.053804 |
| 0 | 0.8 | 0 | 0.20995 | 0.2 | 0.6 | 0.2 | 0.073061 |
| 0 | 0.8 | 0.2 | 0.0662 | 0.2 | 0.6 | 0.4 | 0.084289 |
| 0 | 0.8 | 0.4 | 0.06366 | 0.2 | 0.6 | 0.6 | 0.079109 |
| 0 | 0.8 | 0.6 | 0.06583 | 0.2 | 0.6 | 0.8 | 0.072906 |
| 0 | 0.8 | 0.8 | 0.07415 | 0.2 | 0.6 | 1 | 0.072608 |
| 0 | 0.8 | 1 | 0.08477 | | | | |

| | | | | | | | |
|-----|-----|-----|---------|-----|-----|-----|----------|
| 0 | 0.8 | 0.4 | 0.06366 | 0.2 | 0.8 | 0 | 0.058092 |
| 0 | 0.8 | 0.6 | 0.06583 | 0.2 | 0.8 | 0.2 | 0.134273 |
| 0 | 0.8 | 0.8 | 0.07415 | 0.2 | 0.8 | 0.4 | 0.133109 |
| 0 | 0.8 | 1 | 0.08477 | 0.2 | 0.8 | 0.6 | 0.111135 |
| 0 | 1 | 0 | 0.20995 | 0.2 | 0.8 | 0.8 | 0.092886 |
| 0 | 1 | 0.2 | 0.0662 | 0.2 | 0.8 | 1 | 0.08238 |
| 0 | 1 | 0.4 | 0.06366 | 0.2 | 1 | 0 | 0.210395 |
| 0 | 1 | 0.6 | 0.06583 | 0.2 | 1 | 0.2 | 0.228733 |
| 0 | 1 | 0.8 | 0.07415 | 0.2 | 1 | 0.4 | 0.19364 |
| 0 | 1 | 1 | 0.08477 | 0.2 | 1 | 0.6 | 0.150793 |
| 0.2 | 0 | 0 | 0.07013 | 0.2 | 1 | 0.8 | 0.115849 |
| 0.2 | 0 | 0.2 | 0.06642 | 0.2 | 1 | 1 | 0.095762 |
| 0.2 | 0 | 0.4 | 0.06429 | 0.4 | 0 | 0 | 0.2566 |
| 0.2 | 0 | 0.6 | 0.06506 | 0.4 | 0 | 0.2 | 0.261633 |
| 0.2 | 0 | 0.8 | 0.06973 | 0.4 | 0 | 0.4 | 0.265752 |
| 0.2 | 0 | 1 | 0.07748 | 0.4 | 0 | 0.6 | 0.269441 |
| 0.2 | 0.2 | 0 | 0.08099 | 0.4 | 0 | 0.8 | 0.272864 |
| 0.2 | 0.2 | 0.2 | 0.06524 | 0.4 | 0 | 1 | 0.276018 |
| 0.2 | 0.2 | 0.4 | 0.056 | 0.4 | 0.2 | 0 | 0.052544 |
| 0.2 | 0.2 | 0.6 | 0.05683 | 0.4 | 0.2 | 0.2 | 0.056382 |
| 0.2 | 0.2 | 0.8 | 0.06242 | 0.4 | 0.2 | 0.4 | 0.057841 |
| 0.2 | 0.2 | 1 | 0.07264 | 0.4 | 0.2 | 0.6 | 0.058454 |
| 0.2 | 0.4 | 0 | 0.0878 | 0.4 | 0.2 | 0.8 | 0.059529 |
| 0.2 | 0.4 | 0.2 | 0.05691 | 0.4 | 0.2 | 1 | 0.061621 |
| 0.2 | 0.4 | 0.4 | 0.05155 | 0.4 | 0.4 | 0 | 0.145183 |
| 0.2 | 0.4 | 0.6 | 0.05545 | 0.4 | 0.4 | 0.2 | 0.139632 |
| 0.2 | 0.4 | 0.8 | 0.06342 | 0.4 | 0.4 | 0.4 | 0.122026 |
| 0.2 | 0.4 | 1 | 0.07667 | 0.4 | 0.4 | 0.6 | 0.102948 |
| 0.2 | 0.6 | 0 | 0.08708 | 0.4 | 0.4 | 0.8 | 0.088857 |
| 0.2 | 0.6 | 0.2 | 0.04998 | 0.4 | 0.4 | 1 | 0.079362 |
| 0.2 | 0.6 | 0.4 | 0.05629 | 0.4 | 0.6 | 0 | 0.331689 |
| 0.2 | 0.6 | 0.6 | 0.05699 | 0.4 | 0.6 | 0.2 | 0.26964 |
| 0.2 | 0.6 | 0.8 | 0.06183 | 0.4 | 0.6 | 0.4 | 0.213592 |
| 0.2 | 0.6 | 1 | 0.07901 | 0.4 | 0.6 | 0.6 | 0.167093 |
| 0.2 | 0.8 | 0 | 0.05099 | 0.4 | 0.6 | 0.8 | 0.130801 |
| 0.2 | 0.8 | 0.2 | 0.09057 | 0.4 | 0.6 | 1 | 0.107727 |
| 0.2 | 0.8 | 0.4 | 0.09117 | 0.4 | 0.8 | 0 | 0.507088 |
| 0.2 | 0.8 | 0.6 | 0.06986 | 0.4 | 0.8 | 0.2 | 0.396775 |
| 0.2 | 0.8 | 0.8 | 0.06281 | 0.4 | 0.8 | 0.4 | 0.300006 |
| 0.2 | 0.8 | 1 | 0.0799 | 0.4 | 0.8 | 0.6 | 0.222947 |
| 0.2 | 1 | 0 | 0.13065 | 0.4 | 0.8 | 0.8 | 0.168557 |
| | | | | 0.4 | 0.8 | 1 | 0.131405 |
| | | | | 0.4 | 1 | 0 | 0.629953 |
| | | | | 0.4 | 1 | 0.2 | 0.489635 |

| | | | | | | | |
|-----|-----|-----|---------|-----|-----|-----|----------|
| 0.2 | 1 | 0.2 | 0.17323 | 0.4 | 1 | 0.4 | 0.357349 |
| 0.2 | 1 | 0.4 | 0.14166 | 0.4 | 1 | 0.6 | 0.255475 |
| 0.2 | 1 | 0.6 | 0.09565 | 0.4 | 1 | 0.8 | 0.189278 |
| 0.2 | 1 | 0.8 | 0.06582 | 0.4 | 1 | 1 | 0.143972 |
| 0.2 | 1 | 1 | 0.08062 | 0.6 | 0 | 0 | 0.24263 |
| 0.4 | 0 | 0 | 0.06858 | 0.6 | 0 | 0.2 | 0.246699 |
| 0.4 | 0 | 0.2 | 0.0661 | 0.6 | 0 | 0.4 | 0.250378 |
| 0.4 | 0 | 0.4 | 0.06512 | 0.6 | 0 | 0.6 | 0.25418 |
| 0.4 | 0 | 0.6 | 0.06548 | 0.6 | 0 | 0.8 | 0.257819 |
| 0.4 | 0 | 0.8 | 0.06785 | 0.6 | 0 | 1 | 0.261204 |
| 0.4 | 0 | 1 | 0.07208 | 0.6 | 0.2 | 0 | 0.089329 |
| 0.4 | 0.2 | 0 | 0.05617 | 0.6 | 0.2 | 0.2 | 0.083816 |
| 0.4 | 0.2 | 0.2 | 0.0518 | 0.6 | 0.2 | 0.4 | 0.077561 |
| 0.4 | 0.2 | 0.4 | 0.05287 | 0.6 | 0.2 | 0.6 | 0.071836 |
| 0.4 | 0.2 | 0.6 | 0.05629 | 0.6 | 0.2 | 0.8 | 0.067393 |
| 0.4 | 0.2 | 0.8 | 0.0622 | 0.6 | 0.2 | 1 | 0.065259 |
| 0.4 | 0.2 | 1 | 0.07013 | 0.6 | 0.4 | 0 | 0.240064 |
| 0.4 | 0.4 | 0 | 0.09991 | 0.6 | 0.4 | 0.2 | 0.202756 |
| 0.4 | 0.4 | 0.2 | 0.09489 | 0.6 | 0.4 | 0.4 | 0.171069 |
| 0.4 | 0.4 | 0.4 | 0.07627 | 0.6 | 0.4 | 0.6 | 0.144157 |
| 0.4 | 0.4 | 0.6 | 0.05979 | 0.6 | 0.4 | 0.8 | 0.121898 |
| 0.4 | 0.4 | 0.8 | 0.05987 | 0.6 | 0.4 | 1 | 0.105733 |
| 0.4 | 0.4 | 1 | 0.07298 | 0.6 | 0.6 | 0 | 0.378672 |
| 0.4 | 0.6 | 0 | 0.27781 | 0.6 | 0.6 | 0.2 | 0.307033 |
| 0.4 | 0.6 | 0.2 | 0.21213 | 0.6 | 0.6 | 0.4 | 0.247459 |
| 0.4 | 0.6 | 0.4 | 0.14514 | 0.6 | 0.6 | 0.6 | 0.201509 |
| 0.4 | 0.6 | 0.6 | 0.09125 | 0.6 | 0.6 | 0.8 | 0.166416 |
| 0.4 | 0.6 | 0.8 | 0.06283 | 0.6 | 0.8 | 1 | 0.137502 |
| 0.4 | 0.6 | 1 | 0.07451 | 0.6 | 0.8 | 0 | 0.432522 |
| 0.4 | 0.8 | 0 | 0.48168 | 0.6 | 0.8 | 0.2 | 0.333621 |
| 0.4 | 0.8 | 0.2 | 0.34521 | 0.6 | 0.8 | 0.4 | 0.259487 |
| 0.4 | 0.8 | 0.4 | 0.21594 | 0.6 | 0.8 | 0.6 | 0.207665 |
| 0.4 | 0.8 | 0.6 | 0.11877 | 0.6 | 0.8 | 0.8 | 0.170903 |
| 0.4 | 0.8 | 0.8 | 0.06595 | 0.6 | 0.8 | 1 | 0.139858 |
| 0.4 | 0.8 | 1 | 0.08329 | 0.6 | 1 | 0 | 0.365363 |
| 0.4 | 1 | 0 | 0.65351 | 0.6 | 1 | 0.2 | 0.257448 |
| 0.4 | 1 | 0.2 | 0.44436 | 0.6 | 1 | 0.4 | 0.193111 |
| 0.4 | 1 | 0.4 | 0.25512 | 0.6 | 1 | 0.6 | 0.157721 |
| 0.4 | 1 | 0.6 | 0.12061 | 0.6 | 1 | 0.8 | 0.134125 |
| 0.4 | 1 | 0.8 | 0.06497 | 0.6 | 1 | 1 | 0.112294 |
| 0.4 | 1 | 1 | 0.1058 | 0.8 | 0 | 0 | 0.236107 |
| | | | | 0.8 | 0 | 0.2 | 0.238261 |
| | | | | 0.8 | 0 | 0.4 | 0.240347 |
| | | | | 0.8 | 0 | 0.6 | 0.242543 |

| | | | | | | | |
|-----|-----|-----|---------|-----|-----|-----|----------|
| 0.6 | 0 | 0 | 0.06705 | 0.8 | 0 | 0.8 | 0.244663 |
| 0.6 | 0 | 0.2 | 0.06638 | 0.8 | 0 | 1 | 0.24672 |
| 0.6 | 0 | 0.4 | 0.06638 | 0.8 | 0.2 | 0 | 0.101298 |
| 0.6 | 0 | 0.6 | 0.06697 | 0.8 | 0.2 | 0.2 | 0.094413 |
| 0.6 | 0 | 0.8 | 0.0681 | 0.8 | 0.2 | 0.4 | 0.088696 |
| 0.6 | 0 | 1 | 0.06975 | 0.8 | 0.2 | 0.6 | 0.083618 |
| 0.6 | 0.2 | 0 | 0.05604 | 0.8 | 0.2 | 0.8 | 0.078955 |
| 0.6 | 0.2 | 0.2 | 0.05311 | 0.8 | 0.2 | 1 | 0.074846 |
| 0.6 | 0.2 | 0.4 | 0.05304 | 0.8 | 0.4 | 0 | 0.210148 |
| 0.6 | 0.2 | 0.6 | 0.05588 | 0.8 | 0.4 | 0.2 | 0.189833 |
| 0.6 | 0.2 | 0.8 | 0.06203 | 0.8 | 0.4 | 0.4 | 0.172258 |
| 0.6 | 0.2 | 1 | 0.0696 | 0.8 | 0.4 | 0.6 | 0.156549 |
| 0.6 | 0.4 | 0 | 0.17167 | 0.8 | 0.4 | 0.8 | 0.142236 |
| 0.6 | 0.4 | 0.2 | 0.12703 | 0.8 | 0.4 | 1 | 0.128806 |
| 0.6 | 0.4 | 0.4 | 0.0904 | 0.8 | 0.6 | 0 | 0.234039 |
| 0.6 | 0.4 | 0.6 | 0.06245 | 0.8 | 0.6 | 0.2 | 0.209158 |
| 0.6 | 0.4 | 0.8 | 0.05797 | 0.8 | 0.6 | 0.4 | 0.188449 |
| 0.6 | 0.4 | 1 | 0.07099 | 0.8 | 0.6 | 0.6 | 0.170542 |
| 0.6 | 0.6 | 0 | 0.29838 | 0.8 | 0.6 | 0.8 | 0.15411 |
| 0.6 | 0.6 | 0.2 | 0.19865 | 0.8 | 0.6 | 1 | 0.137946 |
| 0.6 | 0.6 | 0.4 | 0.1224 | 0.8 | 0.8 | 0 | 0.140264 |
| 0.6 | 0.6 | 0.6 | 0.06975 | 0.8 | 0.8 | 0.2 | 0.129335 |
| 0.6 | 0.6 | 0.8 | 0.06187 | 0.8 | 0.8 | 0.4 | 0.120468 |
| 0.6 | 0.6 | 1 | 0.08817 | 0.8 | 0.8 | 0.6 | 0.111845 |
| 0.6 | 0.8 | 0 | 0.31713 | 0.8 | 0.8 | 0.8 | 0.102707 |
| 0.6 | 0.8 | 0.2 | 0.17457 | 0.8 | 0.8 | 1 | 0.091605 |
| 0.6 | 0.8 | 0.4 | 0.08147 | 0.8 | 1 | 0 | 0.064282 |
| 0.6 | 0.8 | 0.6 | 0.06538 | 0.8 | 1 | 0.2 | 0.057365 |
| 0.6 | 0.8 | 0.8 | 0.10485 | 0.8 | 1 | 0.4 | 0.054053 |
| 0.6 | 0.8 | 1 | 0.15047 | 0.8 | 1 | 0.6 | 0.054114 |
| 0.6 | 1 | 0 | 0.16063 | 0.8 | 1 | 0.8 | 0.057067 |
| 0.6 | 1 | 0.2 | 0.05919 | 0.8 | 1 | 1 | 0.064277 |
| 0.6 | 1 | 0.4 | 0.12343 | 1 | 0 | 0 | 0.234098 |
| 0.6 | 1 | 0.6 | 0.18437 | 1 | 0 | 0.2 | 0.234098 |
| 0.6 | 1 | 0.8 | 0.2288 | 1 | 0 | 0.4 | 0.234098 |
| 0.6 | 1 | 1 | 0.26999 | 1 | 0 | 0.6 | 0.234098 |
| 0.8 | 0 | 0 | 0.06744 | 1 | 0 | 0.8 | 0.234098 |
| 0.8 | 0 | 0.2 | 0.06749 | 1 | 0.2 | 1 | 0.234098 |
| 0.8 | 0 | 0.4 | 0.06781 | 1 | 0.2 | 0 | 0.087764 |
| 0.8 | 0 | 0.6 | 0.06825 | 1 | 0.2 | 0.2 | 0.087764 |
| 0.8 | 0 | 0.8 | 0.06882 | 1 | 0.2 | 0.4 | 0.087764 |
| | | | | 1 | 0.2 | 0.6 | 0.087764 |
| | | | | 1 | 0.2 | 0.8 | 0.087764 |
| | | | | 1 | 0.2 | 1 | 0.087764 |

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|-----|-----|-----|---------|---|-----|-----|----------|
| 0.8 | 0 | 1 | 0.06952 | 1 | 0.4 | 0 | 0.141075 |
| 0.8 | 0.2 | 0 | 0.05208 | 1 | 0.4 | 0.2 | 0.141075 |
| 0.8 | 0.2 | 0.2 | 0.05216 | 1 | 0.4 | 0.4 | 0.141075 |
| 0.8 | 0.2 | 0.4 | 0.05413 | 1 | 0.4 | 0.6 | 0.141075 |
| 0.8 | 0.2 | 0.6 | 0.05833 | 1 | 0.4 | 0.8 | 0.141075 |
| 0.8 | 0.2 | 0.8 | 0.06339 | 1 | 0.4 | 1 | 0.141075 |
| 0.8 | 0.2 | 1 | 0.06885 | 1 | 0.6 | 0 | 0.101896 |
| 0.8 | 0.4 | 0 | 0.0909 | 1 | 0.6 | 0.2 | 0.101896 |
| 0.8 | 0.4 | 0.2 | 0.06759 | 1 | 0.6 | 0.4 | 0.101896 |
| 0.8 | 0.4 | 0.4 | 0.05457 | 1 | 0.6 | 0.6 | 0.101896 |
| 0.8 | 0.4 | 0.6 | 0.05496 | 1 | 0.6 | 0.8 | 0.101896 |
| 0.8 | 0.4 | 0.8 | 0.06526 | 1 | 0.6 | 1 | 0.101896 |
| 0.8 | 0.4 | 1 | 0.07698 | 1 | 0.8 | 0 | 0.059452 |
| 0.8 | 0.6 | 0 | 0.05244 | 1 | 0.8 | 0.2 | 0.059452 |
| 0.8 | 0.6 | 0.2 | 0.06321 | 1 | 0.8 | 0.4 | 0.059452 |
| 0.8 | 0.6 | 0.4 | 0.08195 | 1 | 0.8 | 0.6 | 0.059452 |
| 0.8 | 0.6 | 0.6 | 0.10407 | 1 | 0.8 | 0.8 | 0.059452 |
| 0.8 | 0.6 | 0.8 | 0.12712 | 1 | 0.8 | 1 | 0.059452 |
| 0.8 | 0.6 | 1 | 0.15063 | 1 | 1 | 0 | 0.19825 |
| 0.8 | 0.8 | 0 | 0.22671 | 1 | 1 | 0.2 | 0.19825 |
| 0.8 | 0.8 | 0.2 | 0.249 | 1 | 1 | 0.4 | 0.19825 |
| 0.8 | 0.8 | 0.4 | 0.26748 | 1 | 1 | 0.6 | 0.19825 |
| 0.8 | 0.8 | 0.6 | 0.28485 | 1 | 1 | 0.8 | 0.19825 |
| 0.8 | 0.8 | 0.8 | 0.30249 | 1 | 1 | 1 | 0.19825 |
| 0.8 | 0.8 | 1 | 0.32189 | | | | |
| 0.8 | 1 | 0 | 0.56026 | | | | |
| 0.8 | 1 | 0.2 | 0.54736 | | | | |
| 0.8 | 1 | 0.4 | 0.53844 | | | | |
| 0.8 | 1 | 0.6 | 0.53602 | | | | |
| 0.8 | 1 | 0.8 | 0.54186 | | | | |
| 0.8 | 1 | 1 | 0.55688 | | | | |
| 1 | 0 | 0 | 0.06983 | | | | |
| 1 | 0 | 0.2 | 0.06983 | | | | |
| 1 | 0 | 0.4 | 0.06983 | | | | |
| 1 | 0 | 0.6 | 0.06983 | | | | |
| 1 | 0 | 0.8 | 0.06983 | | | | |
| 1 | 0 | 1 | 0.06983 | | | | |
| 1 | 0.2 | 0 | 0.06785 | | | | |
| 1 | 0.2 | 0.2 | 0.06785 | | | | |
| 1 | 0.2 | 0.4 | 0.06785 | | | | |
| 1 | 0.2 | 0.6 | 0.06785 | | | | |

| | | | |
|---|-----|-----|---------|
| 1 | 0.2 | 0.8 | 0.06785 |
| 1 | 0.2 | 1 | 0.06785 |
| 1 | 0.4 | 0 | 0.10073 |
| 1 | 0.4 | 0.2 | 0.10073 |
| 1 | 0.4 | 0.4 | 0.10073 |
| 1 | 0.4 | 0.6 | 0.10073 |
| 1 | 0.4 | 0.8 | 0.10073 |
| 1 | 0.4 | 1 | 0.10073 |
| 1 | 0.6 | 0 | 0.27363 |
| 1 | 0.6 | 0.2 | 0.27363 |
| 1 | 0.6 | 0.4 | 0.27363 |
| 1 | 0.6 | 0.6 | 0.27363 |
| 1 | 0.6 | 0.8 | 0.27363 |
| 1 | 0.6 | 1 | 0.27363 |
| 1 | 0.8 | 0 | 0.56325 |
| 1 | 0.8 | 0.2 | 0.56325 |
| 1 | 0.8 | 0.4 | 0.56325 |
| 1 | 0.8 | 0.6 | 0.56325 |
| 1 | 0.8 | 0.8 | 0.56325 |
| 1 | 0.8 | 1 | 0.56325 |
| 1 | 1 | 0 | 0.90792 |
| 1 | 1 | 0.2 | 0.90792 |
| 1 | 1 | 0.4 | 0.90792 |
| 1 | 1 | 0.6 | 0.90792 |
| 1 | 1 | 0.8 | 0.90792 |
| 1 | 1 | 1 | 0.90792 |

Facebook Prophet Cross-Validation (MAPE)

| (1992-2005) | (2006-2019) |
|---|---|
| Horizon: 1994-02-01 00:00:00, MAPE: 0.0012 | Horizon: 2008-02-01 00:00:00, MAPE: 0.0135 |
| Horizon: 1994-03-01 00:00:00, MAPE: 0.0480 | Horizon: 2008-03-01 00:00:00, MAPE: 0.0325 |
| Horizon: 1994-04-01 00:00:00, MAPE: 0.0081 | Horizon: 2008-04-01 00:00:00, MAPE: 0.0381 |
| Horizon: 1994-05-01 00:00:00, MAPE: 0.0077 | Horizon: 2008-05-01 00:00:00, MAPE: 0.0433 |
| Horizon: 1994-06-01 00:00:00, MAPE: 0.0138 | Horizon: 2008-06-01 00:00:00, MAPE: 0.0454 |

| | |
|---|---|
| Horizon: 1994-07-01 00:00:00, MAPE: 0.0332 | Horizon: 2008-07-01 00:00:00, MAPE: 0.0253 |
| Horizon: 1994-08-01 00:00:00, MAPE: 0.0248 | Horizon: 2008-08-01 00:00:00, MAPE: 0.0271 |
| Horizon: 1994-09-01 00:00:00, MAPE: 0.0110 | Horizon: 2008-09-01 00:00:00, MAPE: 0.0585 |
| Horizon: 1994-10-01 00:00:00, MAPE: 0.0240 | Horizon: 2008-10-01 00:00:00, MAPE: 0.0235 |
| Horizon: 1994-11-01 00:00:00, MAPE: 0.0144 | Horizon: 2008-11-01 00:00:00, MAPE: 0.0820 |
| Horizon: 1994-12-01 00:00:00, MAPE: 0.0043 | Horizon: 2008-12-01 00:00:00, MAPE: 0.1775 |
| Horizon: 1995-01-01 00:00:00, MAPE: 0.0250 | Horizon: 2009-01-01 00:00:00, MAPE: 0.1346 |
| Horizon: 1995-02-01 00:00:00, MAPE: 0.0244 | Horizon: 2009-02-01 00:00:00, MAPE: 0.0577 |
| Horizon: 1995-03-01 00:00:00, MAPE: 0.0545 | Horizon: 2009-03-01 00:00:00, MAPE: 0.0687 |
| Horizon: 1995-04-01 00:00:00, MAPE: 0.0347 | Horizon: 2009-04-01 00:00:00, MAPE: 0.0685 |
| Horizon: 1995-05-01 00:00:00, MAPE: 0.0126 | Horizon: 2009-05-01 00:00:00, MAPE: 0.0719 |
| Horizon: 1995-06-01 00:00:00, MAPE: 0.0053 | Horizon: 2009-06-01 00:00:00, MAPE: 0.0754 |
| Horizon: 1995-07-01 00:00:00, MAPE: 0.0176 | Horizon: 2009-07-01 00:00:00, MAPE: 0.0775 |
| Horizon: 1995-08-01 00:00:00, MAPE: 0.0176 | Horizon: 2009-08-01 00:00:00, MAPE: 0.0754 |
| Horizon: 1995-09-01 00:00:00, MAPE: 0.0295 | Horizon: 2009-09-01 00:00:00, MAPE: 0.0985 |
| Horizon: 1995-10-01 00:00:00, MAPE: 0.0230 | Horizon: 2009-10-01 00:00:00, MAPE: 0.1350 |
| Horizon: 1995-11-01 00:00:00, MAPE: 0.0243 | Horizon: 2009-11-01 00:00:00, MAPE: 0.1575 |
| Horizon: 1995-12-01 00:00:00, MAPE: 0.0307 | Horizon: 2009-12-01 00:00:00, MAPE: 0.1603 |
| Horizon: 1996-01-01 00:00:00, MAPE: 0.0070 | Horizon: 2010-01-01 00:00:00, MAPE: 0.1932 |
| Horizon: 1996-02-01 00:00:00, MAPE: 0.0383 | Horizon: 2010-02-01 00:00:00, MAPE: 0.0972 |
| Horizon: 1996-03-01 00:00:00, MAPE: 0.0312 | Horizon: 2010-03-01 00:00:00, MAPE: 0.0994 |
| Horizon: 1996-04-01 00:00:00, MAPE: 0.0066 | Horizon: 2010-04-01 00:00:00, MAPE: 0.1143 |
| Horizon: 1996-05-01 00:00:00, MAPE: 0.0406 | Horizon: 2010-05-01 00:00:00, MAPE: 0.1019 |

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|---|---|
| Horizon: 1996-06-01 00:00:00, MAPE: 0.0046 | Horizon: 2010-06-01 00:00:00, MAPE: 0.1097 |
| Horizon: 1996-07-01 00:00:00, MAPE: 0.0084 | Horizon: 2010-07-01 00:00:00, MAPE: 0.1159 |
| Horizon: 1996-08-01 00:00:00, MAPE: 0.0369 | Horizon: 2010-08-01 00:00:00, MAPE: 0.0500 |
| Horizon: 1996-09-01 00:00:00, MAPE: 0.0191 | Horizon: 2010-09-01 00:00:00, MAPE: 0.0096 |
| Horizon: 1996-10-01 00:00:00, MAPE: 0.0081 | Horizon: 2010-10-01 00:00:00, MAPE: 0.0131 |
| Horizon: 1996-11-01 00:00:00, MAPE: 0.0135 | Horizon: 2010-11-01 00:00:00, MAPE: 0.0105 |
| Horizon: 1996-12-01 00:00:00, MAPE: 0.0083 | Horizon: 2010-12-01 00:00:00, MAPE: 0.0191 |
| Horizon: 1997-01-01 00:00:00, MAPE: 0.0097 | Horizon: 2011-01-01 00:00:00, MAPE: 0.0134 |
| Horizon: 1997-02-01 00:00:00, MAPE: 0.0082 | Horizon: 2011-02-01 00:00:00, MAPE: 0.0226 |
| Horizon: 1997-03-01 00:00:00, MAPE: 0.0103 | Horizon: 2011-03-01 00:00:00, MAPE: 0.0222 |
| Horizon: 1997-04-01 00:00:00, MAPE: 0.0170 | Horizon: 2011-04-01 00:00:00, MAPE: 0.0324 |
| Horizon: 1997-05-01 00:00:00, MAPE: 0.0030 | Horizon: 2011-05-01 00:00:00, MAPE: 0.0069 |
| Horizon: 1997-06-01 00:00:00, MAPE: 0.0181 | Horizon: 2011-06-01 00:00:00, MAPE: 0.0083 |
| Horizon: 1997-07-01 00:00:00, MAPE: 0.0063 | Horizon: 2011-07-01 00:00:00, MAPE: 0.0060 |
| Horizon: 1997-08-01 00:00:00, MAPE: 0.0007 | Horizon: 2011-08-01 00:00:00, MAPE: 0.0091 |
| Horizon: 1997-09-01 00:00:00, MAPE: 0.0107 | Horizon: 2011-09-01 00:00:00, MAPE: 0.0171 |
| Horizon: 1997-10-01 00:00:00, MAPE: 0.0008 | Horizon: 2011-10-01 00:00:00, MAPE: 0.0063 |
| Horizon: 1997-11-01 00:00:00, MAPE: 0.0308 | Horizon: 2011-11-01 00:00:00, MAPE: 0.0059 |
| Horizon: 1997-12-01 00:00:00, MAPE: 0.0009 | Horizon: 2011-12-01 00:00:00, MAPE: 0.0182 |
| Horizon: 1998-01-01 00:00:00, MAPE: 0.0222 | Horizon: 2012-01-01 00:00:00, MAPE: 0.0060 |
| Horizon: 1998-02-01 00:00:00, MAPE: 0.0324 | Horizon: 2012-02-01 00:00:00, MAPE: 0.0486 |
| Horizon: 1998-03-01 00:00:00, MAPE: 0.0195 | Horizon: 2012-03-01 00:00:00, MAPE: 0.0400 |
| Horizon: 1998-04-01 00:00:00, MAPE: 0.0129 | Horizon: 2012-04-01 00:00:00, MAPE: 0.0132 |

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| Horizon: 1998-05-01 00:00:00, MAPE: 0.0091 | Horizon: 2012-05-01 00:00:00, MAPE: 0.0082 |
| Horizon: 1998-06-01 00:00:00, MAPE: 0.0068 | Horizon: 2012-06-01 00:00:00, MAPE: 0.0304 |
| Horizon: 1998-07-01 00:00:00, MAPE: 0.0109 | Horizon: 2012-07-01 00:00:00, MAPE: 0.0665 |
| Horizon: 1998-08-01 00:00:00, MAPE: 0.0227 | Horizon: 2012-08-01 00:00:00, MAPE: 0.0140 |
| Horizon: 1998-09-01 00:00:00, MAPE: 0.0170 | Horizon: 2012-09-01 00:00:00, MAPE: 0.0391 |
| Horizon: 1998-10-01 00:00:00, MAPE: 0.0189 | Horizon: 2012-10-01 00:00:00, MAPE: 0.0237 |
| Horizon: 1998-11-01 00:00:00, MAPE: 0.0120 | Horizon: 2012-11-01 00:00:00, MAPE: 0.0141 |
| Horizon: 1998-12-01 00:00:00, MAPE: 0.0295 | Horizon: 2012-12-01 00:00:00, MAPE: 0.0137 |
| Horizon: 1999-01-01 00:00:00, MAPE: 0.0092 | Horizon: 2013-01-01 00:00:00, MAPE: 0.0450 |
| Horizon: 1999-02-01 00:00:00, MAPE: 0.0005 | Horizon: 2013-02-01 00:00:00, MAPE: 0.0248 |
| Horizon: 1999-03-01 00:00:00, MAPE: 0.0422 | Horizon: 2013-03-01 00:00:00, MAPE: 0.0187 |
| Horizon: 1999-04-01 00:00:00, MAPE: 0.0410 | Horizon: 2013-04-01 00:00:00, MAPE: 0.0314 |
| Horizon: 1999-05-01 00:00:00, MAPE: 0.0396 | Horizon: 2013-05-01 00:00:00, MAPE: 0.0158 |
| Horizon: 1999-06-01 00:00:00, MAPE: 0.0253 | Horizon: 2013-06-01 00:00:00, MAPE: 0.0450 |
| Horizon: 1999-07-01 00:00:00, MAPE: 0.0491 | Horizon: 2013-07-01 00:00:00, MAPE: 0.0105 |
| Horizon: 1999-08-01 00:00:00, MAPE: 0.0282 | Horizon: 2013-08-01 00:00:00, MAPE: 0.0103 |
| Horizon: 1999-09-01 00:00:00, MAPE: 0.0244 | Horizon: 2013-09-01 00:00:00, MAPE: 0.0362 |
| Horizon: 1999-10-01 00:00:00, MAPE: 0.0210 | Horizon: 2013-10-01 00:00:00, MAPE: 0.0116 |
| Horizon: 1999-11-01 00:00:00, MAPE: 0.0277 | Horizon: 2013-11-01 00:00:00, MAPE: 0.0113 |
| Horizon: 1999-12-01 00:00:00, MAPE: 0.0686 | Horizon: 2013-12-01 00:00:00, MAPE: 0.0103 |
| Horizon: 2000-01-01 00:00:00, MAPE: 0.0124 | Horizon: 2014-01-01 00:00:00, MAPE: 0.0261 |
| Horizon: 2000-02-01 00:00:00, MAPE: 0.0359 | Horizon: 2014-02-01 00:00:00, MAPE: 0.0304 |
| Horizon: 2000-03-01 00:00:00, MAPE: 0.0526 | Horizon: 2014-03-01 00:00:00, MAPE: 0.0096 |

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| Horizon: 2000-04-01 00:00:00, MAPE: 0.0179 | Horizon: 2014-04-01 00:00:00, MAPE: 0.0102 |
| Horizon: 2000-05-01 00:00:00, MAPE: 0.0222 | Horizon: 2014-05-01 00:00:00, MAPE: 0.0210 |
| Horizon: 2000-06-01 00:00:00, MAPE: 0.0307 | Horizon: 2014-06-01 00:00:00, MAPE: 0.0145 |
| Horizon: 2000-07-01 00:00:00, MAPE: 0.0392 | Horizon: 2014-07-01 00:00:00, MAPE: 0.0130 |
| Horizon: 2000-08-01 00:00:00, MAPE: 0.0037 | Horizon: 2014-08-01 00:00:00, MAPE: 0.0044 |
| Horizon: 2000-09-01 00:00:00, MAPE: 0.0212 | Horizon: 2014-09-01 00:00:00, MAPE: 0.0018 |
| Horizon: 2000-10-01 00:00:00, MAPE: 0.0599 | Horizon: 2014-10-01 00:00:00, MAPE: 0.0169 |
| Horizon: 2000-11-01 00:00:00, MAPE: 0.0327 | Horizon: 2014-11-01 00:00:00, MAPE: 0.0018 |
| Horizon: 2000-12-01 00:00:00, MAPE: 0.0107 | Horizon: 2014-12-01 00:00:00, MAPE: 0.0138 |
| Horizon: 2001-01-01 00:00:00, MAPE: 0.0587 | Horizon: 2015-01-01 00:00:00, MAPE: 0.0267 |
| Horizon: 2001-02-01 00:00:00, MAPE: 0.0580 | Horizon: 2015-02-01 00:00:00, MAPE: 0.0574 |
| Horizon: 2001-03-01 00:00:00, MAPE: 0.0381 | Horizon: 2015-03-01 00:00:00, MAPE: 0.0150 |
| Horizon: 2001-04-01 00:00:00, MAPE: 0.0507 | Horizon: 2015-04-01 00:00:00, MAPE: 0.0179 |
| Horizon: 2001-05-01 00:00:00, MAPE: 0.0143 | Horizon: 2015-05-01 00:00:00, MAPE: 0.0109 |
| Horizon: 2001-06-01 00:00:00, MAPE: 0.0395 | Horizon: 2015-06-01 00:00:00, MAPE: 0.0212 |
| Horizon: 2001-07-01 00:00:00, MAPE: 0.0473 | Horizon: 2015-07-01 00:00:00, MAPE: 0.0070 |
| Horizon: 2001-08-01 00:00:00, MAPE: 0.0093 | Horizon: 2015-08-01 00:00:00, MAPE: 0.0199 |
| Horizon: 2001-09-01 00:00:00, MAPE: 0.0987 | Horizon: 2015-09-01 00:00:00, MAPE: 0.0147 |
| Horizon: 2001-10-01 00:00:00, MAPE: 0.0098 | Horizon: 2015-10-01 00:00:00, MAPE: 0.0068 |
| Horizon: 2001-11-01 00:00:00, MAPE: 0.0295 | Horizon: 2015-11-01 00:00:00, MAPE: 0.0244 |
| Horizon: 2001-12-01 00:00:00, MAPE: 0.0361 | Horizon: 2015-12-01 00:00:00, MAPE: 0.0078 |
| Horizon: 2002-01-01 00:00:00, MAPE: 0.0507 | Horizon: 2016-01-01 00:00:00, MAPE: 0.0403 |
| Horizon: 2002-02-01 00:00:00, MAPE: 0.0596 | Horizon: 2016-02-01 00:00:00, MAPE: 0.0182 |

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| Horizon: 2002-03-01 00:00:00, MAPE: 0.0335 | Horizon: 2016-03-01 00:00:00, MAPE: 0.0154 |
| Horizon: 2002-04-01 00:00:00, MAPE: 0.0261 | Horizon: 2016-04-01 00:00:00, MAPE: 0.0194 |
| Horizon: 2002-05-01 00:00:00, MAPE: 0.0124 | Horizon: 2016-05-01 00:00:00, MAPE: 0.0305 |
| Horizon: 2002-06-01 00:00:00, MAPE: 0.0527 | Horizon: 2016-06-01 00:00:00, MAPE: 0.0008 |
| Horizon: 2002-07-01 00:00:00, MAPE: 0.0086 | Horizon: 2016-07-01 00:00:00, MAPE: 0.0130 |
| Horizon: 2002-08-01 00:00:00, MAPE: 0.0274 | Horizon: 2016-08-01 00:00:00, MAPE: 0.0134 |
| Horizon: 2002-09-01 00:00:00, MAPE: 0.0509 | Horizon: 2016-09-01 00:00:00, MAPE: 0.0065 |
| Horizon: 2002-10-01 00:00:00, MAPE: 0.0207 | Horizon: 2016-10-01 00:00:00, MAPE: 0.0185 |
| Horizon: 2002-11-01 00:00:00, MAPE: 0.0192 | Horizon: 2016-11-01 00:00:00, MAPE: 0.0118 |
| Horizon: 2002-12-01 00:00:00, MAPE: 0.0092 | Horizon: 2016-12-01 00:00:00, MAPE: 0.0429 |
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| Horizon: 2003-02-01 00:00:00, MAPE: 0.0422 | Horizon: 2017-02-01 00:00:00, MAPE: 0.0209 |
| Horizon: 2003-03-01 00:00:00, MAPE: 0.0014 | Horizon: 2017-03-01 00:00:00, MAPE: 0.0151 |
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| Horizon: 2003-07-01 00:00:00, MAPE: 0.0332 | Horizon: 2017-07-01 00:00:00, MAPE: 0.0097 |
| Horizon: 2003-08-01 00:00:00, MAPE: 0.0320 | Horizon: 2017-08-01 00:00:00, MAPE: 0.0036 |
| Horizon: 2003-09-01 00:00:00, MAPE: 0.0128 | Horizon: 2017-09-01 00:00:00, MAPE: 0.0210 |
| Horizon: 2003-10-01 00:00:00, MAPE: 0.0236 | Horizon: 2017-10-01 00:00:00, MAPE: 0.0123 |
| Horizon: 2003-11-01 00:00:00, MAPE: 0.0013 | Horizon: 2017-11-01 00:00:00, MAPE: 0.0492 |
| Horizon: 2003-12-01 00:00:00, MAPE: 0.0387 | Horizon: 2017-12-01 00:00:00, MAPE: 0.0405 |
| Horizon: 2004-01-01 00:00:00, MAPE: 0.0145 | Horizon: 2018-01-01 00:00:00, MAPE: 0.0051 |

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| Horizon: 2004-02-01 00:00:00, MAPE: 0.0063 | Horizon: 2018-02-01 00:00:00, MAPE: 0.0148 |
| Horizon: 2004-03-01 00:00:00, MAPE: 0.0395 | Horizon: 2018-03-01 00:00:00, MAPE: 0.0354 |
| Horizon: 2004-04-01 00:00:00, MAPE: 0.0376 | Horizon: 2018-04-01 00:00:00, MAPE: 0.0050 |
| Horizon: 2004-05-01 00:00:00, MAPE: 0.0379 | Horizon: 2018-05-01 00:00:00, MAPE: 0.0452 |
| Horizon: 2004-06-01 00:00:00, MAPE: 0.0330 | Horizon: 2018-06-01 00:00:00, MAPE: 0.0247 |
| Horizon: 2004-07-01 00:00:00, MAPE: 0.0538 | Horizon: 2018-07-01 00:00:00, MAPE: 0.0132 |
| Horizon: 2004-08-01 00:00:00, MAPE: 0.0300 | Horizon: 2018-08-01 00:00:00, MAPE: 0.0253 |
| Horizon: 2004-09-01 00:00:00, MAPE: 0.0361 | Horizon: 2018-09-01 00:00:00, MAPE: 0.0056 |
| Horizon: 2004-10-01 00:00:00, MAPE: 0.0127 | Horizon: 2018-10-01 00:00:00, MAPE: 0.0308 |
| Horizon: 2004-11-01 00:00:00, MAPE: 0.0312 | Horizon: 2018-11-01 00:00:00, MAPE: 0.0474 |
| Horizon: 2004-12-01 00:00:00, MAPE: 0.0933 | Horizon: 2018-12-01 00:00:00, MAPE: 0.0017 |