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On magnetic fields and sub-Riemannian geodesics

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Introduction

In this thesis we consider particular sub-Riemannian structures that are obtained from other sub-Riemannian structures with the presence of a magnetic field. The project was inspired by an article of Richard Montgomery [Mon95], and was structured following the approach and methods of [ABB19].

Sub-Riemannian geometry. A sub-Riemannian structure on a smooth manifold M can be regarded as a generalization of a Riemannian structure in which the degrees of freedom of the velocity of a particle moving in the manifold is limited to a vector sub-bundle of the tangent bundle, which is also called a distribution and denoted with $D \subset TM$. The distribution is endowed with a metric, the so called sub-Riemannian metric g_s, which allows us to measure length of admissible curves, i.e. curves tangent to the distribution.

Sub-Riemannian structures emerge naturally in the study of non-holonomic dynamical systems. For instance, consider a rolling-without-sliding disk on a plane or on a surface (e.g. the wheel of a bike). It is clear that the wheel cannot translate in a direction orthogonal to its plane. The configuration space is four dimensional (more precisely is locally diffeomorphic to $\mathbb{R}^2 \times \mathbb{T}^2$), however, the phase space is not eight but only six dimensional! In fact the velocities are contained in a distribution of rank two. A crucial observation, which can be derived also by practical experimentation, is that even if not all movements are admissible, all configurations can be reached following admissible curves. We refer to this fundamental property by saying that the distribution describing the system is bracket generating. We call step of the distribution the number of successive brackets augmented by one needed in order to span all the tangent space. If our structure is also analytic, this situation is the exact opposite to the one in which distribution is involutive.

In the involutive case, by Frobenius theorem the dynamics is stuck in the integral leaf tangent to the distribution. The opposite result for bracket generating distributions, is called Rashewsky-Chow theorem, and states that we can fill an open neighborhood of any point by moving always tangent to the distribution. The limit case in which the distribution is all the tangent space, the Riemannian case, the bracket generating condition and the involutivity are equivalent and the two theorems coincides; the integral leaf being diffeomorhic to a Riemannian ball.

sub-Riemannian geodesics. Riemannian length minimizers are solutions of the geodesics equation and hence are regular curves. We can see sub-Riemannian length minimizers or sub-Riemannian geodesics, as constrained minima of the length, where the constraint is given by the requirement that the candidate minimizer must be tangent to the distribution. Where the constraint is regular, we find that minimizers are solution of differential equations, hence they are regular curves. We call abnormal curves the ones for which this constraint is not regular, and for them we need to check separately if they are actual minimizers. This is the infinite dimensional version of what happens in finite dimension, when we minimize functions restricted to a sub-manifold of the domain. In that case, where the given constraint is regular, we apply the Lagrange multipliers rule (v.s. the geodesic equation), and where the manifold is not regular, we shall check separately if in such points the function have actually a minimum.

An important, difficult, problem is then to understand when abnormal minimizers are regular curves or not, depending on the properties of the distribution.

Contents and Results. The underlying idea behind the thesis can be summarized in the following general principle.

Principle. *Given a sub-Riemannian manifold of dimension n and step s and a magnetic field on it, we can construct a sub-Riemannian manifold of dimension $n+1$ and step greater or equal to $s+1$. Moreover, the step is equal to $s+1$ exactly where the magnetic field is different from zero.*

We are able to make this principle precise, and actually prove it, in the case in which the starting sub-Riemannian manifold is a Riemannian manifold or a contact sub-Riemannian manifold.

In particular, in the first part of the thesis we show how to pass from a Riemannian surface M (i.e. a sub-Riemannian manifold of step one) to a sub-Riemannian manifold C_M^β of step grater or equal to two, by introducing a magnetic field β in M . This new space is realized as a real line bundle over M . The sub-Riemannian metric is defined as the pull-back of the Riemannian metric by the canonical projection of the bundle. We then prove both in coordinates and using orthonormal frames that

Theorem. *Normal sub-Riemannian length minimizers of C_M^β project into the trajectories of charged particles in M subject to the Lorentz force, i.e. the images of curves $\sigma : [0,1] \to M$ that are solutions of the Newton equation*

$$\nabla_{\dot\sigma}\dot\sigma = \sharp i_{\dot\sigma}\beta \ . \tag{1}$$

This theorem is a particular case of a more general result present in [Mon90], but is here presented from a different viewpoint.

We verify that the step of C_M^β is greater or equal to two, depending on the magnetic field, and that when the magnetic field is nonzero, the step is exactly two. Moreover, we show that when the magnetic field is a nonzero constant, C_M^β naturally carries a Lie group structure, isomorphic to the three dimensional Heisenberg group. In this case the normal sub-Riemannian length minimizers have a clear geometric interpretation as lifts of solutions of an isoperimetric problem.

In the following we generalize the construction of C_M^β with M a Riemannian manifold of any finite dimension. First, using the frames formalism we recover the well known Hamiltonian description of Riemannian geodesics (1) and the minimal coupling principle (2).

Theorem. *1) If $\gamma : [0,1] \to M$ is a geodesic, then it is the projection of a solution $\lambda : [0,1] \to T^*M$ of the Hamiltonian system given by the kinetic energy $H = \sum_i h_i^2$.*
2) If $\gamma : [0,1] \to M$ is the motion of a charged particle subject to a magnetic field $\beta = dA$, then it is the projection of a solution $\lambda : [0,1] \to T^*M$ of the Hamiltonian system given by the shifted kinetic energy $H_A = \sum_i (h_i + A_i)^2$.

Using this result we are able two show that normal sub-Riemannian length minimizers of C_M^β projects into paths of charged particles.

In the second part of the thesis we consider the case in which M is a sub-Riemannian manifold of step two, and we find significant differences with the Riemannian case, proving some original results. Some of these results will be part of a research article that we plan to prepare in collaboration with the thesis supervisors.

In particular, we consider the Heisenberg group, which is a sub-Riemannian manifold of contact type. The contact structure allows us to describe the magnetic fields using the Rumin complex. With an original, explicit calculation, we show that the step of C_M^β is greater or equal to three, and exactly three where the magnetic field is nonzero. We show that the choice of a constant magnetic field leads to a sub-Riemannian structure of Engel type. In this case we cannot find an equation analogous to 1, since the magnetic fields in a contact structure contain second order derivatives of the potential, while in the Riemannian case, where we use the standard exterior differential, we have only derivatives of order one.

The final goal is to give a description of abnormal curves of the sub-Riemannian structures studied. It is known that abnormal curves of C_M^β, where M is a Riemannian surface, are all contained in the zero locus of the magnetic field (see for example [ABB19]). More in general we show that

Theorem. *Given (M, D, g_s) a three dimensional sub-Riemannian structure of rank 2, then all abnormal curves belong to the Martinet set, i.e. the subset of M in which $D^2 \subset TM$. Moreover where such a set is a smooth sub-manifold $\mathcal{S}$ of M the abnormal curves contained in $\mathcal{S}$ are nice if and only if $D|_m$ is transversal to $T_m\mathcal{S}$.*

In other words the Martinet set for C_M^β is given by the zero locus of the magnetic field, and $\mathcal{S}$ by the points of this set in which $\mathrm{d}\beta \neq 0$.

We then consider the sub-Riemannian structure constructed from the three dimensional Heisenberg group using a constant magnetic field and verify a well known fact (see again [ABB19]) that there exists an abnormal curve passing through each point of it, which we shall call Engel abnormal. When considering a generic magnetic field we show that its zero locus is made of equilibrium points of the Engel abnormals. However inside such a locus, generically there are no other abnormal curves. This shows in particular that the relation between the abnormals and the zero locus of the magnetic field encountered when we introduce a magnetic field in a Riemannian surface is lost in the case of a magnetic field in a contact structure.

Chapter 1

Charged particle in a magnetic field

We begin by recalling the description of a charged particle in the three dimensional Euclidean space with the presence of a magnetic field. By analogy we pass to the Euclidean plane and then to a generic Riemannian surface. This not only allows us to fix some notation and terminology, but also helps to build some intuition about the main ideas of the constructions in the other sections. Next we introduce a way to pass from a given Riemannian surface to a three dimensional space using the magnetic interaction. Finally we show the relation between Sub-Riemannian length minimizers in this new space and the trajectories of charged particles with a magnetic field in the starting surface. The chapter ends with a purely geometric description of the minimizers which makes use of the Hamiltonian formalism.

1.1 Euclidean space

Lagrangian formulation. A charged particle in the three dimensional Euclidean space with the presence of a magnetic field is subject to a force known as the Lorentz force. Using vector notation (letters in bold), the dynamic is described by the Newton equation

$$m\ddot{\mathbf{x}}(t) = q\ \dot{\mathbf{x}}(t) \times \mathbf{B}(\mathbf{x}(t)) \ , \tag{1.1}$$

where m is the mass of the particle, q its electric charge, $\mathbf{x}(t) \in \mathbb{R}^3$ is its position at time $t \in \mathbb{R}$, $\dot{\mathbf{x}}(t) \in T_{\mathbf{x}(t)}\mathbb{R}^3$ is its velocity at the same time, $\mathbf{B} \in \mathfrak{X}(\mathbb{R}^3)$ is the magnetic field[1], and $\times$ denotes the standard vector product of $\mathbb{R}^3$. We want to derive a Lagrangian description of this dynamic. In order to do so, we look for a function $\mathcal{L} : T\mathbb{R}^3 \simeq \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R},\ (\mathbf{x}, \mathbf{v}) \mapsto \mathcal{L}(\mathbf{x}, \mathbf{v})$ of the form

$$\mathcal{L}(\mathbf{x}, \mathbf{v}) = \frac{1}{2} m \|\mathbf{v}\|^2 - V(\mathbf{x}, \mathbf{v}) \ , \tag{1.2}$$

where $\|\cdot\|$ is the norm induced by the standard scalar product in $\mathbb{R}^3$ and $V : T\mathbb{R}^3 \to \mathbb{R}$ is called the potential. We use Cartesian coordinates in $\mathbb{R}^3$, $\mathbf{x} \mapsto x = (x^1, x^2, x^3)^T$ and the induced fiber coordinates in $T\mathbb{R}^3$, $\mathbf{v} \mapsto v = (v^1, v^2, v^3)^T$ with $\mathbf{v} = v^i \partial_i$, $i = 1, 2, 3$. For simplicity we denote the representative of the Lagrangian in coordinate and the Lagrangian in the same way (since there is a global chart for $T\mathbb{R}^3$, this distinction is not fundamental). The Lagrangian is then written as[2]

$$\mathcal{L}(x, v) = \frac{1}{2} m v^i v_i - V(x, v) \ , \tag{1.3}$$

and the Lagrange equations are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v^i}(x, \dot{x}) - \frac{\partial \mathcal{L}}{\partial x^i}(x, \dot{x}) = m\ddot{x}_i + \frac{\partial V}{\partial v^j v^i}\ddot{x}^j + \frac{\partial V}{\partial x^j v^i}\dot{x}^j - \frac{\partial V}{\partial x^i} = 0 \quad i = 1, 2, 3 \ .$$

[1]We denote with $\mathfrak{X}(M)$ the space of vector fields over a smooth manifold M.

[2]We are using Einstein notation for the sum, i.e. we understand sum for indices that appear both in an upper and lower position at the same time.

Since we do not want other terms proportional to the acceleration $\ddot{\mathbf{x}}$ apart from the one coming from the kinetic term $\frac{1}{2}m\|\mathbf{v}\|^2 = \frac{1}{2}mv_iv^i$, we make the following ansatz

$$V(x,v) = qA_i(x)v^i \ . \tag{1.4}$$

In other words we want a potential that is linear in the velocities. In this way the Lagrange equations become (we omit explicit dependencies)

$$\frac{d}{dt}(mv_i - qA_i) + q(\partial_i A_j)v^j = m\dot{v}_i - q(\partial_j A_i)\dot{x}^i + q(\partial_i A_j)v^j = 0 \ .$$

Equivalently

$$m\dot{v}_i = q(\partial_j A_i - \partial_i A_j)v^j \ . \tag{1.5}$$

We call $\beta_{ij}(x) := -(\partial_j A_i(x) - \partial_i A_j(x))$ the (components of the) magnetic 2-form. We can also rearrange the RHS to recover the Lorentz force. To do so we simply define $B_i(x) := \frac{1}{2}\varepsilon_i^{jk}\beta_{jk}(x)$ where ε_i^{jk} is the Levi-Civita 3-symbol relative to the standard scalar product, and call it the (components of the) magnetic 1-form. By the properties of this symbol we have also that $\beta_{ij} = \varepsilon_{ij}^k B_k$. Then we get

$$m\dot{v}_i = -q\beta_{ij}v^j = -q\varepsilon_{ij}^k B_k v^j \ . \tag{1.6}$$

Using the duality given by the scalar product we have

$$m\dot{v}^i = -q\varepsilon_{kj}^i B^k v^j = -q(\mathbf{B}\times\mathbf{v})^i = q(\mathbf{v}\times\mathbf{B})^i \ . \tag{1.7}$$

Finally we have fully recovered the Newton equation with the Lorentz force. To summarize the relations introduced so far, we notice that the magnetic field $\mathbf{B} \in \mathfrak{X}(\mathbb{R}^3)$ corresponds to a closed 2-form $\beta \in \Lambda^2(\mathbb{R}^3)$ which is again related to a 1-form $B \in \Lambda^1(\mathbb{R}^3)$ in the following way[3]

$$\begin{cases} \mathbf{B} = \sharp B \\ \beta = \star B \ , \end{cases} \tag{1.8}$$

where $\star : \Lambda^k(\mathbb{R}^3) \to \Lambda^{3-k}(\mathbb{R}^3)$ is the Hodge duality, and $\sharp : \Lambda^1(\mathbb{R}^3) \to \mathfrak{X}(\mathbb{R}^3)$, is the sharp isomorphism relative to the standard scalar product[4]. By definition of external differential we also see that $\beta = \mathrm{d}A$ if we think A_i, $i = 1,2,3$, to be the components of a 1-form $A \in \Lambda^1(\mathbb{R}^3)$, called the magnetic potential.

Hamiltonian formulation. We now compute the Hamiltonian and the Hamilton equations for the previous system. The Hamiltonian is the Legendre transform of $\mathcal{L}$ with respect to v. As usual we denote the momenta with $p \in T_x^*\mathbb{R}^3$. Here $p_i := \frac{\partial\mathcal{L}}{\partial v^i} = mv_i - qA_i$, and hence $v_i(p) = \frac{p_i + qA_i}{m}$, so that

$$\begin{aligned} H(x,p) &= \frac{\partial\mathcal{L}}{\partial v^i}(x,v(p))v^i(p) - \mathcal{L}(x,v(p)) = \\ &= (mv_i - qA_i)v^i - \frac{1}{2}mv_iv^i + qA_iv^i\Big|_{v=v(p)} = \\ &= \frac{1}{2}mv_iv^i\Big|_{v=v(p)} = \sum_{i=1}^{3}\frac{(p_i+qA_i)^2}{2m} \ . \end{aligned}$$

[3] For a more detailed and relativistic treatment of electromagnetism using differential forms see [Fra11] section 7.2.

[4] In a three dimensional Riemannian manifold with metric g, in the case $k = 1$ we define $\star B := i_{\sharp B} vol_g(\mathbb{R}^3)$ with $\sharp$ and $vol_g(\mathbb{R}^3)$ the sharp isomorphism and the volume form relative to the metric g respectively.

The Hamilton equations are

$$\begin{cases} \dot{x}^i = \frac{\partial H}{\partial p_i} = \frac{p^i + qA^i}{m} \\ \dot{p}_i = -\frac{\partial H}{\partial x^i} = -q\frac{(p^j+qA^j)}{m}\frac{\partial A_j}{\partial x^i} \end{cases} . \tag{1.9}$$

Remark. We notice how the gauge invariance of the Lagrange equations is guaranteed by the fact that under a gauge transformation of the potential, $A_i \mapsto A_i + \partial_i f$, with $f : x \mapsto f(x)$, of class $\mathcal{C}^2(\mathbb{R}^3)$, the Lagrangian is simply shifted by a total derivative. Indeed, along a curve $x(t)$ with $v(t) = \dot{x}(t)$

$$\mathcal{L}(x(t), \dot{x}(t)) \mapsto \mathcal{L}(x(t), \dot{x}(t)) + q(\partial_i f(x(t)))\dot{x}^i(t) = \mathcal{L}(x(t), \dot{x}(t)) + q\frac{df}{dt}(x(t)) .$$

In the Hamiltonian description on the contrary, the gauge transformation changes definitely the equations 1.9 . Nevertheless we have the following result.

Proposition. Let H_A be the Hamiltonian of a charged particle in the presence of a magnetic potential A and let $A \mapsto A + \mathrm{d}f$ be a gauge transformation with $f \in \mathcal{C}^2$. Then there exists a canonical transformation[5] such that the Hamilton equations relative to $H_{A+\mathrm{d}f}$ are the same of the ones of H_A.

Proof. We want to find a canonical transformation $(p, q) \mapsto \Psi(p, q)$ such that $H_{A+\mathrm{d}f} \circ \Psi^{-1} = H_A$. It is clear that we shall choose the shift in the momenta $(x^i, p_i) \mapsto (x^i, p_i + q\partial_i f)$. We only need to check if this transformation is canonical, hence we want to verify if the Jacobian of the shift, $M(x, p)$, which is given by

$$M(x,p) = \begin{pmatrix} \mathbb{I}_3 & \mathbb{O}_3 \\ q\mathrm{Hess}_x(f) & \mathbb{I}_3 \end{pmatrix}$$

where $(\mathrm{Hess}_x(f))_{ij} = \partial_i\partial_j f$ is the Hessian of f with respect to x, is symplectic, i.e. if

$$M(x,p)^T \mathbb{J}_6 M(x,p) = \mathbb{J}_6 \quad \forall (x,p) \in \mathbb{R}^3 \times \mathbb{R}^3 ,$$

with $\mathbb{J}_6$ the symplectic unit in six dimensions. Performing the row-by-column multiplication we get the condition

$$\begin{pmatrix} \mathbb{I}_3 & (q\mathrm{Hess}_x(f))^T \\ \mathbb{O}_3 & \mathbb{I}_3 \end{pmatrix} \begin{pmatrix} \mathbb{O}_3 & \mathbb{I}_3 \\ -\mathbb{I}_3 & \mathbb{O}_3 \end{pmatrix} \begin{pmatrix} \mathbb{I}_3 & \mathbb{O}_3 \\ q\mathrm{Hess}_x(f) & \mathbb{I}_3 \end{pmatrix} =$$

$$= \begin{pmatrix} q\left(\mathrm{Hess}_x(f) - (\mathrm{Hess}_x(f))^T\right) & \mathbb{I}_3 \\ -\mathbb{I}_3 & \mathbb{O}_3 \end{pmatrix} = \begin{pmatrix} \mathbb{O}_3 & \mathbb{I}_3 \\ -\mathbb{I}_3 & \mathbb{O}_3 \end{pmatrix} \quad \forall x \in \mathbb{R}^3 , \tag{1.10}$$

in other words $\partial_i\partial_j f - \partial_j\partial_i f = 0$, which is satisfied if $f \in \mathcal{C}^2(\mathbb{R}^3)$.

□

We have hence seen that invariance of Hamilton equations under a gauge transformation is recovered via a canonical transformation involving only the momenta. Moreover with a similar computation as the one done in the proof we see that also the shift $p \mapsto p + \Pi$ where Π is a closed 1-form is canonical. Indeed a slight modification of the computation 1.10 gives the relation $\partial_i \Pi_j - \partial_j \Pi_i = (\mathrm{d}\Pi)_{ij} = 0$.

[5] A canonical transformation for an Hamiltonian system is a coordinate transformation $(\tilde{p}, \tilde{q}) = \Psi(p, q)$ that brings the Hamilton equations for $H(p, q)$ into the ones of $\tilde{H}(\tilde{p}, \tilde{q}) := H(\Psi^{-1}(\tilde{p}, \tilde{q}))$.

1.2 From flat to curved

Charged particle in the Euclidean plane. We now consider a particle in the two dimensional Euclidean plane. We define a Lagrangian of the same type of before (locally, directly in coordinates), where we put $m = 1 = q$ for convenience

$$\mathcal{L}(x, v) := \frac{1}{2} v_i v^i - A_i(x) v^i \ , \qquad (x, v) \in T\mathbb{R}^2 \ . \tag{1.11}$$

The Lagrange equations have the same form of the previous ones

$$\dot{v}_i + \beta_{ij} v^j = 0 \ . \tag{1.12}$$

The only difference is that now we cannot represent the magnetic 2-form as a tangent vector, as it happened in the three dimensional case. We notice however that the vector representation is accidental and do not reflect any fundamental property of the magnetic interaction, which emerges naturally as a 2-form in the theory of electromagnetism considered as a gauge theory (see [Ble13] Chapters 1 and 2, and also [Fra11] section 7.2).
For later convenience we recall that in dimension two there is a correspondence between 2-forms and functions. If $\beta = \beta_{12} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \in \Lambda^2(\mathbb{R}^2)$ we denote with $b := \beta_{12} \in \mathcal{C}^\infty(\mathbb{R}^2)$ the corresponding function.

Curvature and magnetic field. Consider a piece-wise smooth curve in the Euclidean plane $\sigma : [0, 1] \to \mathbb{R}^2$. Whenever σ has nonzero speed we can find a unit tangent vector, namely $\tau(t) := \frac{\dot{\sigma}(t)}{\|\dot{\sigma}(t)\|}$. Moreover it is easily verified that the vector $\dot{\tau}(t)$ is orthogonal to $\tau(t)$ at any time. We define the first principal curvature to be the quantity

$$\kappa(t) := \frac{\|\dot{\tau}(t)\|}{\|\dot{\sigma}(t)\|} \ . \tag{1.13}$$

First of all we notice that this ratio is independent of the reparametrization of the time. If we use the arc parameter s we have $\left\| \frac{d\sigma(t(s))}{ds} \right\| = 1$ and hence $\kappa(s) = \frac{d\tau}{ds}(s)$. We also recall that the curvature has a clear geometric interpretation: it is the inverse of the ray of the circumference that best approximate the curve at the point $\sigma(t)$, called the osculating circle.

Going back to the dynamics of a charged particle in the plane we observe that the modulus of the velocity is a constant of motion (kinetic energy is conserved). Explicitly

$$\frac{d}{dt} v_i v^i = 2\dot{v}_i v^i = -2\beta_{ij} v^j v^i = 0 \ . \tag{1.14}$$

So up to a constant factor the time is the arc parameter, say $t = \lambda s$. After taking the Euclidean norm of both sides of the Lagrange equations 1.12, we finally obtain the relation

$$\kappa(t) = \lambda |b(x(t))| \ . \tag{1.15}$$

This relation tells us that if the magnetic field is constant the solutions are curves with constant curvature i.e. straight lines or arcs of circles. We also notice that the constant λ is related to the kinetic energy $E = \frac{1}{2} v_i v^i$ as $\lambda = \frac{1}{\sqrt{2E}}$. So higher is the energy less is the curvature.

Charged particle in a Riemannian surface. Consider now the case of a Riemannian surface (M, g) with g a Riemannian metric. In local coordinates, the kinetic term in the

Lagrangian (1.20) will now contain the metric coefficients while the magnetic potential should be thought as 1-form which is defined only locally[6]. We have

$$\mathcal{L}(x,v) = \frac{1}{2} g_{ij}(x) v^i v^j - A_i(x) v^i \ . \tag{1.16}$$

Computing the Lagrange equations for the kinetic part we have

$$\frac{1}{2}\left(2g_{ik}\ddot{x}^k + 2\partial_j g_{ik}\dot{x}^j\dot{x}^k - \partial_i g_{jk}\dot{x}^j\dot{x}^k\right) = g_{ik}\ddot{x}^k + \frac{1}{2}\left(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}\right) \ .$$

For the potential, as before, we find $\beta_{ij}\dot{x}^j$. Finally

$$g_{ik}\ddot{x}^k + \frac{1}{2}(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk})\dot{x}^k\dot{x}^j + \beta_{ik}\dot{x}^k = 0 \ .$$

Multiplying by the inverse metric g^{hi} both sides we get

$$\ddot{x}^h + \frac{g^{hi}}{2}(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk})\dot{x}^k\dot{x}^j + g^{hi}\beta_{ik}\dot{x}^k = 0 \ .$$

We recognize the that the quantities in parenthesis are the Christoffel symbols (Γ^h_{kj}) of the Levi-Civita connection of the metric g. Hence we can write

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k + g^{ij}\beta_{jk}\dot{x}^k = 0 \ . \tag{1.17}$$

In terms of covariant derivative and internal multiplication[7] we can write these equations in a purely geometric, hence global, way

$$\nabla_{\dot{x}}\dot{x} = \sharp i_{\dot{x}}\beta \ . \tag{1.18}$$

Since energy is still conserved, we are parametrizing with constant multiples of the arc parameter s, say again $t = \lambda s$, hence, taking the g-norm in both sides, we recover a more general version of our previous result on the first principal curvature 1.15

$$\kappa_g(t) = \lambda|b(x(t))| \ , \tag{1.19}$$

where κ_g is now the geodesic curvature[8], that coincides with the former if the metric is flat.

1.3 A three dimensional space

Three dimensional dynamics. Consider a piece-wise smooth curve $\sigma : [0,1] \to M$ in a Riemannian surface (M,g) with a magnetic 2-form $\beta \in \Lambda^2(M)$, which is contained in a open subset $U \subseteq M$. In U we have a magnetic potential $A \in \Lambda^1(U)$ for β, and for a given point $O = \sigma(0) \in U$ we define for all $t \in [0,1]$ the real quantity

$$z(t) := \int_{\sigma([0,t])} A \ . \tag{1.20}$$

[6] See again [Ble13] paragraph 1.2.7.

[7] Given a vector field $X \in \mathfrak{X}(M)$ we define the internal multiplication or contraction by X, as the map $i_X : \Lambda^k(M) \to \Lambda^{k-1}(M)$, $\omega \mapsto i_X\omega$, by $(i_X\omega)(Y_1,\cdots,Y_{k-1}) := \omega(X,Y_1,\cdots,Y_{k-1})$ for all $Y_1,\cdots,Y_{k-1} \in \mathfrak{X}(M)$.

[8] Given a smooth curve $\sigma : [0,1] \to M$ in a Riemannian manifold (M,g), we define its geodesic curvature with respect to g at the point $\sigma(t)$ as $\kappa_g(t) := \frac{\|\nabla_{\dot\sigma}\dot\sigma(t)\|_g}{\|\dot\sigma(t)\|_g}$ where $\|\cdot\|_g$ is the norm induced by g.

The curve $\gamma := (\sigma, z)$ takes values in the trivial bundle $U \times \mathbb{R}$ which we equip with the canonical projection $\pi : U \times \mathbb{R} \to U$, $(m, z) \mapsto m$. Using an orthonormal frame $e_{1,2}$ in U we can express the velocity of γ as

$$\dot{\gamma}(t) = u^1(t)e_1(\sigma(t)) + u^2(t)e_2(\sigma(t)) + A|_{\sigma(t)}(\dot{\sigma}(t))\partial_z \ , \tag{1.21}$$

where $u^{1,2} : [0, 1] \to \mathbb{R}$ are the components of $\dot{\sigma}$ with respect to the frame $e_{1,2}$ in order, because $\sigma = \pi(\gamma)$ implies $\dot{\sigma} = \pi_* \dot{\gamma}$. We recognize that the curve γ is always tangent to the distribution[9] $D|_{(m,z)} := \text{span}(\{X_1(m, z), X_2(m, z)\})$ with $(m, z) \in U \times \mathbb{R}$ and $X_{1,2} \in \mathfrak{X}(U \times \mathbb{R})$ given by

$$\begin{cases} X_1(m, z) := e_1(m) + A|_m(e_1(m))\partial_z \\ X_2(m, z) := e_2(m) + A|_m(e_2(m))\partial_z \end{cases} . \tag{1.22}$$

The rank of D is always equal to two, so the distribution defines a field of planes in the bundle. We now check if this distribution is bracket generating. If $[e_1, e_2] = c^1 e_1 + c^2 e_2$, with $c^{1,2} \in \mathcal{C}^\infty(M)$, we have

$$[X_1, X_2] = [e_1 + A(e_1)\partial_z, e_2 + A(e_2)\partial_z] = [e_1, e_2] + (e_1(A(e_2)) - e_2(A(e_1)))\, \partial_z \ .$$

By the remarkable formula by which any 1-form τ satisfies $\mathrm{d}\tau(X, Y) = X(\tau(Y)) - Y(\tau(X)) - \tau([X, Y])$, with X, Y vector fields, and recalling that $\beta = \mathrm{d}A$, we can express the preceding Lie bracket as

$$[X_1, X_2] = c^1 e_1 + c^2 e_2 + \mathrm{d}A(e_1, e_2)\partial_z + c^1 A(e_1)\partial_z + c^2 A(e_2)\partial_z = c^1 X_1 + c^2 X_2 + \beta(e_1, e_2)\partial_z \ .$$

Moreover, using the dual frame of $e_{1,2}$, call it $\mu^{1,2}$, we see that $\mathrm{d}A = \beta = b\mu^1 \wedge \mu^2$ for some $b \in \mathcal{C}^\infty(U)$, and we can write

$$[X_1, X_2] = c^1 X_1 + c^2 X_2 + b\partial_z \ . \tag{1.23}$$

We conclude that D is bracket generating at (m, z) whenever $b(m) \neq 0$. We can finally complete the frame $\{X_1, X_2\}$ to a frame in the product using $\partial_z =: X_3$.
It is a remarkable fact that we can see the distribution D as the kernel of a 1-form in the bundle, namely

$$\alpha := \mathrm{d}z - \pi^* A \in \Lambda^1(\pi^{-1}(U)) \ . \tag{1.24}$$

In the following we will see that this 1-form can be defined 'in the large'.

Globalization. In the preceding paragraph we have seen that the dynamic 1.20 brings ('lifts') us into a trivial bundle equipped with a certain distribution D described by 1.22 or 1.24. It is a natural question to ask in what space the former dynamic takes place in the large, i.e. if we escape from U. Guided by the form of 1.24, it is natural to think that this space can be constructed by gluing together the various products $U \times \mathbb{R}$ considered above. To construct such a space we remark that contrary to the magnetic potential A, the magnetic field β is a globally defined closed 2-form on M [10]. If A is a potential in the open subset $U \subseteq M$ and $\tilde{A}$ is another potential in another open subset $V \subseteq M$, in the intersection $U \cap V$ we must have $\mathrm{d}A = \beta = \mathrm{d}\tilde{A}$. Suppose in particular that $A - \tilde{A} = \mathrm{d}S$ with $S \in \mathcal{C}^\infty(U \cap V)$. Then we can

[9] By a rank-k distribution we mean a sub-bundle of the tangent bundle, i.e. a collection $\bigsqcup_{m \in M} D_m$, with D_m a k-dimensional subspace of $T_m M$ for all $m \in M$.

[10] An explanation of this fact, which is valid for abelian gauge theories as electromagnetism, is given in [Ble13] paragraph 1.2.7 and 2.2.16 and [Fra11] section 16.4.

construct the transition functions of the bundle as follows. If $m \in U \cap V$ we say that the two fiber coordinates (m, z) and $(m, \tilde{z})$ correspond to the same point in the bundle if and only if

$$z = \tilde{z} + S(m) \ . \tag{1.25}$$

In this way we can globally define a 1-form in the bundle starting from $\alpha \in \Lambda^1(\pi^{-1}(U \cap V))$ defined as $\alpha := \mathrm{d}z - \pi^* A$. Indeed, in a different trivialization we have $\tilde{\alpha} := \mathrm{d}\tilde{z} - \pi^* \tilde{A}$, but

$$\begin{aligned} \tilde{\alpha} &= \mathrm{d}\tilde{z} - \pi^*\tilde{A} = \mathrm{d}\tilde{z} - \pi^*\tilde{A} \pm \pi^* A = \mathrm{d}\tilde{z} + \pi^*\mathrm{d}S - \pi^* A = \\ &= \mathrm{d}(\tilde{z} + \pi^* S) - \pi^* A = \mathrm{d}z - \pi^* A = \alpha \ . \end{aligned} \tag{1.26}$$

In conclusion, we were able to construct a nontrivial line bundle, that we denote as C_M^β, with $\pi : C_M^\beta \to M$ locally equal to the canonical projection, equipped with a globally defined 1-form $\alpha \in \Lambda^1(C_M^\beta)$. We further point out an interesting fact, namely that even if β may be not globally exact, $\pi^*\beta$ is! Indeed

$$\mathrm{d}\alpha = \mathrm{d}(\mathrm{d}z - \pi^* A) = -\mathrm{d}\pi^* A = -\pi^*\mathrm{d}A = -\pi^*\beta \ . \tag{1.27}$$

Remark. Notice that the globalization procedure needs the difference $A - \tilde{A}$ to be exact, otherwise our construction is only local. A last important remark is about topology. Depending on the cohomology class of the magnetic 2-form, the topology of the fibers of C_M^β changes[11].

1.4 Sub-Riemannian structure

Previously, thanks to the introduction of a magnetic 2-form $\beta \in \Lambda^2(M)$ in a Riemannian surface (M, g), we were able to construct a line bundle over it $\pi : C_M^\beta \to M$ naturally endowed with a 1-form $\alpha \in \Lambda^1(C_M^\beta)$, hence with a rank-2 distribution $D \subset TC_M^\beta$, which was proven to be bracket generating wherever the magnetic 2-form is nonzero. We can give to the bundle C_M^β, together with D, a Sub-Riemannian structure by introducing a metric in the distribution. We denote such a structure with the triplet (C_M^β, D, g_s), where g_s denotes the Sub-Riemannian metric defined via the metric on the base as

$$g_s|_x(u, v) := g|_{\pi(x)}(\pi_* u, \pi_* v) \quad x \in C_M^\beta, \ u, v \in D|_x \subset T_x C_M^\beta \ . \tag{1.28}$$

Notice that $X_{1,2}$ of 1.22 are orthonormal with respect to g_s, i.e. $g_s(X_i, X_j) = \delta_{ij}, \ i = 1, 2$. It is a natural question to ask if there exists length minimizers among curves tangent to the distribution (admissible curves), joining two distinct points $x_1, x_2 \in C_M^\beta$, in the following sense

$$\min\{l_s[\gamma] \ : \ \partial\gamma = \{x_1, x_2\} \subset C_M^\beta, \ \dot{\gamma} \in D|_\gamma\} \ , \tag{1.29}$$

with $\gamma : [0, 1] \to C_M^\beta$ a piece-wise smooth curve and

$$l_s[\gamma] := \int_0^1 \sqrt{g_s|_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} \ \mathrm{d}t \ . \tag{1.30}$$

The quantity $l_s[\gamma]$ is called the Sub-Riemannian length of γ.

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[11] See comment in [VWA13] appendix 4, paragraph L, where the construction of C_M^β is performed using as magnetic 2-form a symplectic form. The construction made in the reference is a special case of the present one. The exact relation between the two will be studied later.

Sub-Riemannian length minimizers in C^β_M. We will develop two strategies to find minimizers. The first will use coordinates and will allow us to make direct connection with the study made in section 1.2. The other strategy will be developed in the next section because of its generality and fundamental geometric meaning.

Coordinate approach. Since our study is local, we use coordinates in M so that we express the metric with its coefficients $g = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$, and also $A = A_i\mathrm{d}x^i$ and $\dot\sigma = \dot\sigma^i\partial_i$, with $i = 1, 2$. By definition we can express the Sub-Riemannian length of $\gamma : [0, 1] \to C^\beta_M$ as (like before $\sigma = \pi(\gamma)$)

$$\begin{aligned} l_s[\gamma] &:= \int_0^1 \sqrt{g_s|_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\ \mathrm{d}t = \int_0^1 \sqrt{g|_{\sigma(t)}(\dot\sigma(t), \dot\sigma(t))}\ \mathrm{d}t = \\ &= \int_0^1 \sqrt{g_{ij}(\sigma(t))\dot\sigma^i(t)\dot\sigma^j(t)}\ \mathrm{d}t =: l[\sigma]\ . \end{aligned} \tag{1.31}$$

Minimizing the Sub-Riemannian length $l_s[\cdot]$ between admissible curves, by 1.20, is equivalent to minimize the Riemannian length $l[\cdot]$ between curves on the base M subject to the integral constraint

$$\int_0^1 A_i(\sigma(t))\dot\sigma^i(t)\ \mathrm{d}t = const. = z(1)\ , \tag{1.32}$$

where $(\sigma(t), z(t)) = \gamma(t)$. We know that free length minimizers in a Riemannian manifold have constant speed[12], moreover the integral constraint is parametrization independent, hence, by Hölder inequality such minimizers are equivalently energy minimizers, i.e. they minimize the functional $\frac{1}{2}\int_0^1 g|_{\sigma(t)}(\dot\sigma(t), \dot\sigma(t))\ \mathrm{d}t$. Our integral constraint is regular whenever $\mathrm{d}A = \beta \neq 0$, indeed, by computing the variational derivative of 1.32 with respect to σ we get the condition

$$\frac{\delta}{\delta\sigma^i}\int_0^1 A_j(\sigma(t))\dot\sigma^j(t)\ \mathrm{d}t = (\partial_i A_j - \partial_j A_i)\dot\sigma^j = \beta_{ij}\dot\sigma^j \neq 0\ .$$

In the regular case, by the Lagrange multipliers rule we finally get the constrained Euler-Lagrange equations for σ.

$$\frac{\delta}{\delta\sigma^i}\frac{1}{2}\int_0^1 g_{ij}(\sigma(t))\dot\sigma^i\dot\sigma^j\ \mathrm{d}t = \lambda\frac{\delta}{\delta\sigma^i}\int_0^1 A_i(\sigma(t))\dot\sigma^i(t)\ \mathrm{d}t\ . \tag{1.33}$$

We further notice that the RHS is parametrization independent while the LHS is homogeneous of degree one in the velocities. This allows us to discard λ. Having said this there is no more work to do in fact, since we have already computed Lagrange equations for both the members and the result is again equation 1.17.
In conclusion the problem of finding Sub-Riemannian length minimizers of (C^β_M, D, g_s) is equivalent to the problem of finding the trajectories of a charged particle with the presence of a magnetic 2-form in the Riemannian surface (M, g) downstairs studied in section 1.2.

1.5 Hamiltonian description of Sub-Riemannian length minimizers

Up to now we have seen that Sub-Riemannian length minimizers in C^β_M project into solutions of constrained Euler-Lagrange equations that are the ones of a charged particle with the presence

[12]The length functional does not depend explicitly on the time, hence, by Noether theorem we have a conserved quantity which is the norm of the velocity, i.e. the Lagrangian itself.

of a magnetic 2-form in M. This description made inevitably use of coordinates in C^β_M. We now want to give a purely geometric description of such minimizers, making use of the natural symplectic structure[13] of $T^*C^\beta_M$. In the following $\pi : T^*C^\beta_M \to C^\beta_M$ is the canonical projection from the cotangent bundle. No confusion should arise since we wont use the bundle structure of C^β_M except at the very end. This entire section retrace slavishly chapter 4 of [ABB19], in particular section 4.4.
From what we have seen C^β_M is naturally endowed with a 1-form $\alpha \in \Lambda^1(C^\beta_M)$. This form defines a distribution $D \subset TC^\beta_M$ given by $D|_x := \ker(\alpha|_x)$, for all $x \in C^\beta_M$. Such a distribution was spanned by the two vector fields $X_{1,2} \in \mathfrak{X}(C^\beta_M)$ described in 1.22 that we completed to a frame in TC^β_M with X_3. We can write the velocity of an admissible curve $\gamma : [0, 1] \to C^\beta_M$, i.e. a curve always tangent to D, with two controls $u^{1,2} : [0, 1] \to \mathbb{R}$ as

$$\dot\gamma(t) = u^i(t)X_i(\gamma(t)) = u^1(t)X_1(\gamma(t)) + u^2(t)X_2(\gamma(t)), \quad t \in [0, 1] \ . \tag{1.34}$$

We now want to define a dynamic in the cotangent bundle. To do so we notice that one can naturally define three functions in $T^*C^\beta_M$, namely $h_i : T^*C^\beta_M \to \mathbb{R}$, $i = 1, 2, 3$, as

$$h_i(\lambda) := \lambda(X_i(\pi(\lambda))) \ , \tag{1.35}$$

where we stress that now $\pi : T^*C^\beta_M \to C^\beta_M$ is the canonical projection. It is important to remark that these are linear function on the fibers in the sense that for $\lambda, \rho \in \pi^{-1}(x)$, $x \in C^\beta_M$, we have that for all $a, b \in \mathbb{R}$

$$h_i(a\lambda + b\rho) = ah_i(\lambda) + bh_i(\rho) \ .$$

The linear independence of $X_{1,2,3}$ implies that $h_{1,2,3}$ define a fiber coordinate system in $T^*C^\beta_M$, i.e. we can represent points as $(x, h_1, h_2, h_3) \in T^*C^\beta_M$, $x \in C^\beta_M$, without using coordinates in the base. In more precise words we are using the trivialization of $T^*C^\beta_M$ which is the dual of the one induced in TC^β_M by the choice of the vector fields X_i's. We now define a new function $H : T^*C^\beta_M \to \mathbb{R}$, called the Sub-Riemannian Hamiltonian, as

$$H(\lambda) = \frac{1}{2}(h_1^2(\lambda) + h_2^2(\lambda)) \ . \tag{1.36}$$

Using the canonical symplectic form on $T^*C^\beta_M$, we can define the Hamiltonian vector field[14] $X_H \in \mathfrak{X}(T^*C^\beta_M)$ associated with H

$$X_H := \sharp \mathrm{d}H \ . \tag{1.37}$$

We warn that here $\sharp : \Lambda^1(T^*C^\beta_M) \to \mathfrak{X}(T^*C^\beta_M)$ is the duality induced by the symplectic form ω, and not by a Riemannian metric. This vector field defines a dynamic in the cotangent bundle that is described by the Hamilton equations

$$\dot\lambda = X_H \ . \tag{1.38}$$

We can easily verify (even easier computation after a quick switch to coordinates) that

$$(\mathrm{d}\pi)|_\lambda (X_H(\lambda)) = h^i(\lambda)X_i(\pi(\lambda)) \in T_{\pi(\lambda)}C^\beta_M \quad i = 1, 2 \ . \tag{1.39}$$

Here it comes the crucial point. We first notice that the projection of the solutions of the Hamilton equations for H are admissible curves. It turns out (see [ABB19] chapter 4, section

[13] In every cotangent bundle T^*M, M a smooth manifold, there exists the so called tautological or Liouville 1-form $\theta \in \Lambda^1(T^*M)$ given by $\theta|_\lambda(\xi) := \lambda(\pi_*\xi)$, for all $\lambda \in T^*M$ and $\xi \in T_\lambda(T^*M)$, with $\pi : T^*M \to M$ the canonical projection. The symplectic form is then given by $\omega := \mathrm{d}\theta$.

[14] Contrary to the Riemannian case there is a sign ambiguity due to the skew symmetry of ω. We make a choice of sign defining $\mathrm{d}H = -i_{X_H}\omega$.

4.3, theorem 4.20) that these projections are also length minimizers. This means that if $\bar{\lambda}$ is a solution of 1.38, and we denote with $\bar{u}^i : [0,1] \to \mathbb{R}$, $i = 1, 2$ the controls of a length minimizer $\bar{\gamma} = \pi(\bar{\lambda})$, we shall have that

$$\bar{u}_i(t) = h_i(\bar{\lambda}(t)) \quad \forall t \in [0,1],\ i = 1, 2\ . \tag{1.40}$$

Consequently, the next step is to look at the dynamic on the fibers to get the evolution of the h_i's. To simplify computations we introduce the Poisson brackets, $\{\cdot,\cdot\} : \mathcal{C}^\infty(T^*C_M^\beta) \times \mathcal{C}^\infty(T^*C_M^\beta) \to \mathcal{C}^\infty(T^*C_M^\beta)$, $(f, g) \mapsto \{f, g\} := \omega(X_f, X_g)$, with $X_{f,g}$ the Hamiltonian vector fields associated with f, g respectively. We recall that the Lie derivative along the flux of an Hamiltonian vector field X_H of a function $a \in \mathcal{C}^\infty(T^*C_M^\beta)$, is given in terms of the Poisson brackets by $\dot{a} = \{H, a\}$. Since the Poisson brackets act like a derivation on both arguments we can easily write the dynamics on the fibers ($i = 1, 2$)

$$\begin{cases} \dot{h}_1 = \{H, h_1\} = \frac{1}{2}\{h_i h^i, h_1\} = \{h_2, h_1\}h_2 \\ \dot{h}_2 = \{H, h_2\} = \frac{1}{2}\{h_i h^i, h_2\} = \{h_1, h_2\}h_1 \\ \dot{h}_3 = \{H, h_3\} = \frac{1}{2}\{h_i h^i, h_3\} = \{h_1, h_3\}h_1 + \{h_2, h_3\}h_2\ . \end{cases} \tag{1.41}$$

We now recall the following nontrivial fact. Given a function of the type 1.35, $a(\lambda) := \lambda(X(\pi(\lambda)))$, for some $X \in \mathfrak{X}(C_M^\beta)$, we have $\pi_* X_a = X$. To show this we suppose to be able to find $\tilde{X} \in \mathfrak{X}(T^*C_M^\beta)$ such that $\pi_*\tilde{X} = X$. As a consequence we can express the function a in terms of the Liouville 1-form θ as

$$a(\lambda) = \lambda(X(\pi(\lambda))) = \theta|_\lambda(\tilde{X}(\lambda))\ .$$

By the homotopy formula $L_{\tilde{X}} = i_{\tilde{X}} \circ \mathrm{d} + \mathrm{d} \circ i_{\tilde{X}}$, we have

$$\mathrm{d}a = \mathrm{d}\,(i_{\tilde{X}}\theta) = L_{\tilde{X}}\theta - i_{\tilde{X}}\omega\ .$$

Consequently, if we can find a $\tilde{X}$ s.t. $L_{\tilde{X}}\theta = 0$ we obtain what we look for

$$\tilde{X} = X_a\ .$$

We finally observe that such a field $\tilde{X}$ always exists. We only need to consider the cotangent lift of X. A classical result[15] tells us that cotangent lifts of flows of vector fields in the base manifold are symplectomorphism in the cotangent bundle. In fact, for these lifts we have that not only the symplectic form is invariant but also the Liouville form is. This implies that the Lie derivative of θ along the flow of the cotangent lift is zero. In particular we have hence shown that $X_i = \pi_* X_{h_i}$ for $i = 1, 2, 3$.
To compute explicitly the equations on the fibers we also need to recall the following Lie algebras homomorphism $(\mathcal{C}^\infty(T^*C_M^\beta), \{\cdot,\cdot\}) \simeq (\mathfrak{X}_\omega(T^*C_M^\beta), [\cdot,\cdot])$, where $\mathfrak{X}_\omega$ denotes the space of Hamiltonian vector fields, given by

$$[X_a, X_b] = X_{\{a,b\}}\ ,$$

In the present case

$$\begin{aligned} \{h_1, h_2\}(\lambda) &= \lambda(\pi_* X_{\{h_1,h_2\}}) = \lambda(\pi_*([X_{h_1}, X_{h_2}])) = \lambda([\pi_* X_{h_1}, \pi_* X_{h_2}]) = \\ &= \lambda([X_1, X_2]) = \lambda(c^1 X_i + c^2 X_2 + b X_3) = c^1 h_1(\lambda) + c^2 h_2(\lambda) + b h_3(\lambda)\ , \end{aligned} \tag{1.42}$$

and

$$\begin{aligned} \{h_i, h_3\}(\lambda) &= \lambda(\pi_* X_{\{h_i,h_3\}}) = \lambda(\pi_*([X_{h_i}, X_{h_3}])) = \lambda([\pi_* X_{h_i}, \pi_* X_{h_3}]) = \\ &= \lambda([X_i, X_3]) = \lambda(0) = 0\ . \end{aligned} \tag{1.43}$$

[15] See for example [AM08] theorem 3.2.12.

Finally, combining 1.41 with 1.42 and 1.43, we have the explicit expression for the equations in the fibers

$$\begin{cases} \dot{h}_1 = -(c^1 h_1 + c^2 h_2 + b h_3) h_2 \\ \dot{h}_2 = (c^1 h_1 + c^2 h_2 + b h_3) h_1 \\ \dot{h}_3 = 0 \ . \end{cases} \tag{1.44}$$

Equivalence between Hamiltonian and coordinate approach. Observe that, since H is conserved, we can restrict to the cylinder $h_1^2 + h_2^2 = 1$, and introduce cylindrical coordinates $(\vartheta, h_3) \in \mathbb{S}^1 \times \mathbb{R}$, such that

$$\begin{cases} h_1 = \cos(\vartheta) \\ h_2 = \sin(\vartheta) \\ h_3 = h_3 \ . \end{cases} \tag{1.45}$$

After changing variables according to 1.45, the system in the fibers 1.44 becomes

$$\begin{cases} \dot{\vartheta} = c^1 \cos(\vartheta) + c^2 \sin(\vartheta) + b h_3 \\ \dot{h}_3 = 0 \ . \end{cases}$$

Consequently, on the base C_M^β we have

$$\dot{\gamma} := \pi_* \dot{\lambda} = \pi_* X_H = h^i X_i = \cos(\vartheta) X_1 + \sin(\vartheta) X_2 \ .$$

To make contact with the coordinate approach we reintroduce the fibered coordinates in C_M^β that we used in the previous section, in which $X_i = e_i + A(e_1)\partial_z$, $i = 1, 2$, and the projection $x = (m, z) \mapsto m$. The projection of γ on the base of C_M^β, that we called σ, solves the equation

$$\dot{\sigma} = \cos(\vartheta) e_1 + \sin(\vartheta) e_2 \ .$$

Notice that $\|\dot{\sigma}\|_g = 1$, so we are parametrizing with arc parameter and hence we can regard the motion $(\sigma, \dot{\sigma})$ as a motion in SM, where SM is the sphere bundle[16] over M. We can define a connection over it with a $\mathfrak{s}^1 \simeq \mathbb{R}$-valued 1-form on SM. Denoting the canonical projection of SM as $\pi_s : SM \to M$ we set

$$\tau := \mathrm{d}\vartheta + \pi_s^*(a_i \mu^i) \in \Lambda^1(SM, \mathfrak{s}^1) \simeq \Lambda^1(SM), \ i = 1, 2 \ ,$$

where ϑ is the coordinate on the fibers, $\mu^{1,2}$ is the dual basis to $e_{1,2}$, and $a_{1,2} \in \mathcal{C}^\infty(M)$. Among all possible connections we choose the Levi-Civita one[17], which corresponds to the choice $a_1 = -c^1$ and $a_2 = -c^2$. We verify that in absence of the magnetic form, i.e. when $b = 0$, we have that σ is a geodesic, indeed

$$\tau|_{(\sigma,\vartheta)^T}((\dot{\sigma}, \dot{\vartheta})^T) = c^1 \cos(\vartheta) + c^2 \sin(\vartheta) + a_1 \cos(\vartheta) + a_2 \sin(\vartheta) = 0 \ .$$

As a consequence, the geodesic curvature with a nonzero b is (set $h_3 = const. = 1$)

$$\kappa_g(t) := \left| \tau|_{(\sigma(t),\vartheta(t))^T}((\dot{\sigma}(t), \dot{\vartheta}(t))^T) \right| = |b(\sigma(t))| \ . \tag{1.46}$$

[16] $SM := \bigsqcup_{m \in M} \{v \in T_m M \ : \ \|v\|_g = 1\}$. When M is 2-dimensional, SM has the structure of a principal bundle with base M and fiber $\mathbb{S}^1$.

[17] See [ST15] chapter 7 section 7.1 or [ABB19] chapter 1 section 1.2. For a brief treatment of these topics see Appendix A.

Chapter 2

The Heisenberg group

In this chapter we apply the constructions made in the previous one to a particular case. More precisely we will introduce a constant magnetic 2-form on the Euclidean plane $(\mathbb{R}^2, \cdot)$ and we will see that the bundle $C^{\beta}_{\mathbb{R}^2}$ will be globally diffeomorphic to $\mathbb{R}^3$. From the 1-form α defined in 1.24, whose kernel constitute a distribution in the new space, we derive a Lie group structure on $C^{\beta}_{\mathbb{R}^2}$. This is recognized to be the three dimensional Heisenberg group $\mathbb{H}_3$. After a simple generalization, we describe the length minimizers of a particular Sub-Riemannian structure on $\mathbb{H}_3$ using the Hamiltonian formalism of section 1.5.

2.1 Euclidean plane with constant magnetic 2-form

The Heisenberg algebra. Consider the Euclidean plane described by Cartesian coordinates $x^{1,2}$ with the magnetic 2-form given by $\beta = \mathrm{d}x^1 \wedge \mathrm{d}x^2$. This form admits an entire class of primitives between which we choose

$$A := \frac{1}{2}(x^1 \mathrm{d}x^2 - x^2 \mathrm{d}x^1) \ . \tag{2.1}$$

$C^{\beta}_{\mathbb{R}^2}$ is now the trivial bundle $\mathbb{R}^2 \times \mathbb{R} \ni (x, z) = (x^1, x^2, z) \in \mathbb{R}^3$, with projection $\pi(x, z) = x$, equipped with the form

$$\alpha := \mathrm{d}z - \pi^* A = \mathrm{d}z - \frac{1}{2}(x^1 \mathrm{d}x^2 - x^2 \mathrm{d}x^1) \ . \tag{2.2}$$

The kernel of α is spanned by the distribution D in 1.22 that now is given by

$$\begin{cases} X_1(x, z) := \partial_1 + A_1(x)\partial_z \\ X_2(x, z) := \partial_2 + A_2(x)\partial_z \end{cases} \tag{2.3}$$

In the present case we have

$$[X_1, X_2] = (\partial_1 A_2 - \partial_2 A_1)\partial_z = \beta_{12}\partial_z = \partial_z =: X_3 \ .$$

In conclusion D is bracket generating and, together with X_3, verifies the Heisenberg algebra

$$\begin{cases} [X_1, X_3] = 0 \\ [X_2, X_3] = 0 \\ [X_1, X_2] = X_3 \ . \end{cases} \tag{2.4}$$

We remark that a different choice of the magnetic potential A would lead to a different α. However, shifting the fiber coordinate z as prescribed by 1.25, we recover the same distribution.

The Heisenberg group. In the previous paragraph we ended up with a bracket generating distribution that solves a particular algebra, the Heisenberg algebra. The Rashewski-Chow theorem (in its global version, see [ABB19] chapter 3 section 3.2, theorem 3.31) tells us that following curves tangent to this distribution we can fill all $\mathbb{R}^3$. Being connected and simply connected, by Lie theorem (see for example [FC30] chapter II section 21, 'Le troisième théorème fondamental de S. Lie'), this $\mathbb{R}^3$ is a Lie group isomorphic to the three dimensional Heisenberg group $\mathbb{H}_3$. From the flows of the vector fields spanning the distribution we can recover the group multiplication law. Denoting the fluxes as $\Phi^{X_i} : \mathbb{R} \times \mathbb{H}_3 \to \mathbb{H}_3$, $(t_i, x) \mapsto \Phi^{X_i}(t_i, x) =: \Phi^{X_i}_{t_i}(x)$, for $i = 1, 2, 3$ we have (setting $b = 1$)

$$\Phi^{X_1}_{t_1}(x) = \left(x^1 + t_1, x^2, x^3 + \frac{x^2 t_1}{2}\right)^T \quad ,$$

$$\Phi^{X_2}_{t_2}(x) = \left(x^1, x^2 + t_2, x^3 - \frac{x^1 t_2}{2}\right)^T \quad ,$$

$$\Phi^{X_3}_{t_3}(x) = \left(x^1, x^2, x^3 + t_3\right)^T \quad .$$

Regarding the t_i's as the coordinates of a point in the group $(t_1, t_2, t_3) \leftrightarrow (y^1, y^2, y^3)^T = y \in \mathbb{H}_3$, we reconstruct the group multiplication as follows. Given $x, y \in \mathbb{H}_3$ the group operation is

$$y.x = \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} . \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} := \begin{pmatrix} x^1 + y^1 \\ x^2 + y^2 \\ x^3 + y^3 + \frac{(x^2 y^1 - x^1 y^2)}{2} \end{pmatrix} \quad .$$

This is actually the operation defining the Lie group structure of $\mathbb{H}_3$. There is also a three dimensional matrix representation of the Heisenberg group obtained via the following identification

$$x = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & x^1 & x^3 + \frac{1}{2} x^1 x^2 \\ 0 & 1 & x^2 \\ 0 & 0 & 1 \end{pmatrix} \quad .$$

The group multiplication then simply corresponds to the raw-by-column matrix multiplication. Higher odd-dimensional Heisenberg groups are define with an analogous multiplication in $\mathbb{R}^{2n+1}$, $n > 0$ natural, and denoted as $\mathbb{H}_{2n+1}$.

2.2 The symplectic $\mathbb{R}^{2n}$ and $\mathbb{H}_{2n+1}$

Contactification. We notice that the procedure of constructing C^β_M from a manifold M performed in section 1.3 contains the case in which we have a symplectic manifold (M, ω), where ω plays the role of a (non degenerate) magnetic field $\beta = -\omega$. Analogously to the magnetic 2-form the symplectic form admits a primitive (locally defined), i.e. the Liouville 1-form θ which is the opposite of the magnetic potential A. The form $\alpha := \mathrm{d}z - \pi^* A = \mathrm{d}z + \pi^* \theta$ will have the property that $\mathrm{d}\alpha$ is non degenerate when restricted to the kernel of α because of the non degeneracy condition on ω due to the symplectic structure (see equation 1.27). We refer to this property by saying that C^β_M admits a contact structure given by the contact form α and denote it as (C^β_M, α). For this reason (C^β_M, α) is called contactification[1] of (M, ω).

[1] See again [VWA13] appendix 4 paragraph L. Now the relation with our construction of C^β_M should be manifest. Here in particular, different potentials, i.e. Liouville 1-forms, always differ by an exact form, that is the differential of the generating function of the canonical transformation that brings the two potentials one into the other.

Remark. From this perspective the construction of section 1.3 can be seen as a generalization of the contactification where we relax the hypothesis on the non degeneracy of ω, using instead of a symplectic form a generic closed 2-form representing the magnetic field, the potentials of which are related by gauge transformations. Note that apart from the symplectic case, the structure on C_M^β is not contact.

The group $\mathbb{H}_{2n+1}$ as contactification. We consider the symplectic $\mathbb{R}^{2n}$. Using Darboux coordinates (p_i, q^i), with $i = 1, \cdots, n$, the contact form can be chosen as

$$\alpha = \mathrm{d}z + \frac{1}{2}(p_i \mathrm{d}q^i - q^i \mathrm{d}p_i) \in \Lambda^1(C_M^\beta \simeq \mathbb{R}^{2n} \times \mathbb{R}) ,$$

where $\theta = \frac{1}{2}(p_i \mathrm{d}q^i - q^i \mathrm{d}p_i)$ is the Liouville 1-form that gives the symplectic form $\omega = \mathrm{d}\theta = \mathrm{d}p_i \wedge \mathrm{d}q^i$. The kernel of α is then spanned by the distribution described by the $2n$ vector fields[2] in C_M^β $(i = 1, \cdots, n)$

$$\begin{cases} X^i := \frac{\partial}{\partial p_i} + \frac{q^i}{2}\frac{\partial}{\partial z} \\ Y_i := \frac{\partial}{\partial q^i} - \frac{p_i}{2}\frac{\partial}{\partial z} \ . \end{cases}$$

After an easy computation we see these vector fields, together with $Z := \frac{\partial}{\partial z}$, solves the $\mathbb{H}_{2n+1}$ algebra

$$\begin{cases} [Y_j, X^i] = \delta^i_j Z \\ [X^i, Z] = 0 \\ [Y_i, Z] = 0 \ . \end{cases}$$

Following the same procedure of section 2.1 we can recover the entire group $\mathbb{H}_{2n+1}$.
In conclusion the $(2n+1)$-dimensional Heisenberg group can be realized as the contactification of the symplectic $\mathbb{R}^{2n}$ for all $n > 0$ natural.

2.3 Sub-Riemannian structure on the Heisenberg group

Up to now we obtained the Heisenberg group $\mathbb{H}_3$ from a constant magnetic 2-form in the Euclidean plane. This group carries naturally a contact structure described by a bracket generating distribution D of rank two spanned by the vector fields defined in equations 2.3. We can give to the Heisenberg group, together with this distribution, a Sub-Riemannian structure by introducing a metric in D. We denote such a structure with the triplet $(\mathbb{H}_3, D, g_s)$, where g_s denotes the Sub-Riemannian metric defined as

$$g_s|_x(u, v) := (\pi_* u) \cdot (\pi_* v) \ , \tag{2.5}$$

with $x \in \mathbb{H}_3$, $\pi : \mathbb{H}_3 \to \mathbb{R}^2$, $u, v \in D|_x \subset T_x\mathbb{H}_3$, and '$\cdot$' the Euclidean scalar product in the plane.

Sub-Riemannian length minimizers. We follow the Hamiltonian approach developed in section 1.5. Using the Hamilton equations in the fibers 1.41 and the Heisenberg algebra 2.4, we compute the Poisson brackets using the same strategy of before (see equations 1.42 and 1.43). We get (keeping track of the magnetic field $b := \beta_{12} = 1$ for clarity)

$$\begin{cases} \dot{h}_1 = -b h_3 h_2 \\ \dot{h}_2 = b h_3 h_1 \\ \dot{h}_3 = 0 \ . \end{cases} \tag{2.6}$$

[2]We simply generalize the definition 2.3 using $A = -\theta$.

Since energy is conserved we again restrict to the cylinder $h_1^2+h_2^2=1$, and introduce cylindrical coordinates as before (see 1.45). The Hamilton equations 2.6 become

$$\begin{cases} \dot\gamma = \cos(\vartheta)X_1 + \sin(\vartheta)X_2 \\ \dot\vartheta = bh_3 \\ \dot h_3 = 0 \ . \end{cases}$$

Integrating the fiber part of the system we get the solutions (choose $h_3 = const. = 1$),

$$\begin{cases} \vartheta(t) = bt + c \\ h_3 = 1 \ , \end{cases}$$

with c an integration constant, that we put equal to zero in the following. In the base we have that

$$\dot\gamma = \cos(bt)X_1(\gamma(t)) + \sin(bt)X_2(\gamma(t)) \ .$$

Using the coordinates in $\mathbb{H}_3$ that we used in the previous paragraphs, in which $X_i = \partial_i + A_i\partial_z$, $i = 1, 2$, and the projection $(x^1, x^2, z) \mapsto (x^1, x^2)$. On the plane (x^1, x^2) for the projection of γ, that we called σ, we get the equations

$$\begin{cases} \dot\sigma^1 = \cos(bt) \\ \dot\sigma^2 = \sin(bt) \ . \end{cases}$$

Notice that $\|\dot\sigma\| = 1$, so we are parametrizing with arc parameter, so, finally

$$\kappa(t) = \|\ddot\sigma(t)\| = |b(\sigma(t))| \ . \tag{2.7}$$

With this equation we recovered the result 1.15, i.e. we have shown that projections of Sub-Riemannian length minimizers are the trajectories of a charged particle in the Euclidean plane subject to a constant magnetic field. These curves are then curves with constant principal curvature, hence straight lines or arcs of circles.
Finally we remark that since in the Heisenberg case the magnetic field is constant and nonzero, there cannot be abnormal minimizers!

Isoperimetric problem. It is noteworthy the fact that in the Heisenberg case the problem of finding Sub-Riemannian length minimizers can be rephrased as an isoperimetric problem. Indeed the constraint 1.20 for the present choice of the magnetic potential 2.1, can be written as

$$\int_{\sigma([0,1])} A + k = \int_\Sigma \mathrm{d}A = \int_\Sigma \frac{1}{2}\mathrm{d}(x^1\mathrm{d}x^2 - x^2\mathrm{d}x^1) = \int_\Sigma \mathrm{d}x^1 \wedge \mathrm{d}x^2 =: \mathrm{Vol}(\Sigma) = const. \ , \tag{2.8}$$

where k is a constant due to the integration of A along the segment $\varsigma : [0,1] \to \mathbb{R}^2$, $t \mapsto \sigma(0) + (t-1)(\sigma(0) - \sigma(1))$, $\partial\Sigma = \sigma([0,1]) + \varsigma([0,1])$, in the sense of simplicial complexes, and $\mathrm{Vol}(\Sigma)$ is the (oriented) area of $\Sigma \subset \mathbb{R}^2$. On the other hand, the functional we want to minimize is the Euclidean length of σ. In conclusion we are looking for planar curves with minimal length, between the ones that with the segment $\varsigma([0,1])$ enclose a fixed area. This problem is in some sense dual to the classical Dido problem where we want to find closed planar curves with fixed length enclosing the maximal area.

Chapter 3

Magnetic forms and Sub-Riemannian manifiolds

We study the effect of introducing a magnetic form in a Sub-Riemannian manifold, starting from the Riemannian case. In this way we generalize what we have seen in the previous chapter for Riemannian surfaces. In the last section we treat the Sub-Riemannian case and we will see how the Sub-Riemannian structure interacts with the magnetic field. In particular, in the contact case this interaction leads to the notion of the Rumin complex.

3.1 Hamiltonian description of geodesics

We start with a brief treatment of some topics in symplectic geometry useful for the next paragraphs, where we prove that the Riemannian geodesics of (M, g) can be seen as the projection on M of the solutions of an Hamiltonian system in T^*M.

Fiber-homogeneous functions on T^*M. Consider the cotangent bundle T^*M of an n-dimensional manifold M, with projection $\pi : T^*M \to M$. T^*M is naturally endowed with the Liouville 1-form $\theta \in \Lambda^1(T^*M)$ whose differential gives the symplectic form $\sigma := \mathrm{d}\theta$. On T^*M operates the group of dilatations, i.e. the multiplicative $\mathbb{R}$, in the following way. $\delta : \mathbb{R} \times T^*M \to T^*M$

$$\delta(a, \lambda) := a\lambda \ ,$$

where $(a\lambda)(v) := a\ \lambda(v)$ for all $v \in T_{\pi(\lambda)}M$, $a \in \mathbb{R}$. We also denote as $\delta_a : T^*M \to T^*M$ the fiber-preserving map $\delta_a(\lambda) := \delta(a, \lambda)$. We are now able to define fiber homogeneous functions on T^*M.

Definition. We say that $H \in \mathcal{C}^\infty(T^*M)$ is homogeneous of degree k in the fibers if and only if

$$\delta_a^* H = a^k H \ .$$

We have the following characterization.

Theorem. A function $H \in \mathcal{C}^\infty(T^*M)$ is homogeneous of degree k if and only if

$$kH = i_{X_H}\theta \ ,$$

where X_H is the Hamiltonian vector field associated with H.

Proof. We prove this using Darboux coordinates. Let (p_i, q^i), $i = 1, \cdots, n$ be Darboux coordinates in M. A point $\lambda \in T^*M$ can then be expressed using the frame $\{\mathrm{d}q^i\}_{i=1,\cdots,n}$ as $\lambda = p_i \mathrm{d}q^i|_{\pi(\lambda)=q}$. We denote H and its representative in coordinates in the same way. Hence

$$i_{X_H}\theta(\lambda) := \lambda(\pi_* X_H(\pi(\lambda))) = p_i \mathrm{d}q^i|_q \left(\frac{\partial H}{\partial p_j}(p,q) \frac{\partial}{\partial q^j}\bigg|_q \right) = p_i \frac{\partial H}{\partial p_j}(p,q)\delta^i_j = p_i \frac{\partial H}{\partial p_i}(p,q) \ .$$

Now, by Euler theorem on homogeneous functions we conclude

$$i_{X_H}\theta(\lambda) = p_i \frac{\partial H}{\partial p_i}(p,q) = kH(p,q) = kH(\lambda) \ .$$

□

We can also refer to fiber-homogeneous functions as tautological functions because of the form of the expression $kH = i_{X_H}\theta$. The following proposition is rather obvious (think in coordinates).

Proposition. Let $H \in \mathcal{C}^\infty(T^*M)$ be a fiber-homogeneous function of degree one. Then the flow of the Hamiltonian vector field X_H preserves the Liouville 1-form.

Proof. By the homotopy formula

$$\mathrm{d}H = \mathrm{d}(i_{X_H}\theta) = L_{X_H}\theta - i_{X_H}\sigma = L_{X_H}\theta + \mathrm{d}H \ ,$$

hence $L_{X_H}\theta = 0$. □

The same computation shows us that for a homogeneous function of degree k we have

$$L_{X_H}\theta = (k-1)\mathrm{d}H \ . \tag{3.1}$$

We also have the following remarkable theorem concerning the Poisson brackets between homogeneous functions.

Theorem. Let $f, g \in \mathcal{C}^\infty(T^*M)$ be fiber-homogeneous functions of degree h, k respectively. Then the Poisson bracket $\{f, g\}$ is a fiber-homogeneous function of degree $h + k - 1$.

Proof. We need to prove that $(h + k - 1)\{f, g\} = i_{X_{\{f,g\}}}\theta$. We have

$$i_{X_{\{f,g\}}}\theta = i_{[X_f, X_g]}\theta = L_{X_f} i_{X_g}\theta - i_{X_g} L_{X_f}\theta = kL_{X_f} g - (h-1) i_{X_g}\mathrm{d}f =$$

$$= -(h+k-1)\mathrm{d}f(X_g) = (h+k-1)\{f,g\} \ .$$

In the second equality we used a standard identity and in the third equation 3.1. For the remaining equalities we simply used the definition of Hamiltonian vector field and of Poisson bracket. □

We will use in particular the case where both the functions are homogeneous of degree one, in which case

$$\{f, g\} = i_{X_{\{f,g\}}}\theta \ . \tag{3.2}$$

Symplectic formalism using frames. Let $\{X_1, \cdots, X_n\}$ be a frame of M, and $\{\mu^1, \cdots, \mu^n\}$ its dual. We define a linear function for each of the X_i's, namely $h_i : T^*M \to \mathbb{R}$ as

$$h_i(\lambda) := \lambda(X_i(\pi(\lambda))), \quad \lambda \in T^*M \ .$$

As already noticed in section 1.5, these functions are coordinates in the fibers of the bundle. We also remark that from the treatment of the previous paragraph, since these functions are fiber-linear, hence fiber-homogeneous of degree one, we can write them in the tautological form

$$h_i = i_{X_{h_i}}\theta \ . \tag{3.3}$$

We also observe that from the definition of the Liouville 1-form, we have

$$\pi_* X_{h_i} = X_i \ . \tag{3.4}$$

The coordinates h_i's locally define a splitting in $T(T^*M)$ described point-wise as

$$T_\lambda(T^*M) \simeq T_{\pi(\lambda)}M \oplus T_\lambda(T^*_{\pi(\lambda)}M), \quad \lambda \in T^*M \ . \tag{3.5}$$

Observe that in this way π_* becomes the canonical projection on the first addend of 3.5. Consequently, a section of $T(T^*M)$ can be described using sections of TM and $T(T^*_{\pi(\lambda)}M) \simeq \mathbb{R}^{2n}$ as

$$V = x^i X_i + v_i \frac{\partial}{\partial h_i} \ ,$$

where $x^i, v_i \in \mathcal{C}^\infty(T^*M)$ and $\{\frac{\partial}{\partial h_i}\}_{i=1,\cdots,n}$ are the coordinate sections of the second addend[1]. Similarly we can represent forms on T^*M using the dual basis $\{\mu^1, \cdots, \mu^n, \mathrm{d}h_1, \cdots, \mathrm{d}h_n\}$. For example we have $\theta = h_j\mu^j$. Indeed, since we can write $\theta = \tilde{h}_j\mu^j$ for some $\tilde{h}_j$, using the tautological formula 3.3, $h_j := i_{X_{h_j}}\theta = \tilde{h}_k\mu^k(X_j) = \tilde{h}_k\delta^k_j = \tilde{h}_j$.
We end this paragraph with a useful expression for the symplectic form and an immediate corollary.

Proposition. Using the dual basis $\{\mu^1, \cdots, \mu^n, \mathrm{d}h_1, \cdots, \mathrm{d}h_n\}$, we can write the symplectic form σ as

$$\sigma = \mathrm{d}h_j \wedge \mu^j - \frac{1}{2}c^k_{ij}h_k \ \mu^i \wedge \mu^j \ , \tag{3.6}$$

where $c^k_{ij} \in \mathcal{C}^\infty(M)$ are defined by $[X_i, X_j] = c^k_{ij}X_k$.

Proof. By definition

$$\sigma = \mathrm{d}\theta = \mathrm{d}(h_j\mu^j) = \mathrm{d}h_j \wedge \mu^j + h_j\mathrm{d}\mu^j \ .$$

Using a remarkable formula

$$\mathrm{d}\mu^j(X_i, X_k) = X_i(\mu^j(X_k)) - X_k(\mu^j(X_i)) - \mu^j([X_i, X_k]) =$$

$$= X_i(\delta^j_k) - X_k(\delta^j_i) - \mu^j([X_i, X_k]) = \mu^j(c^l_{ki}X_l) = c^j_{ki} \ .$$

Finally

$$\sigma = \mathrm{d}h_j \wedge \mu^j - \frac{1}{2}h_j c^j_{ik} \ \mu^i \wedge \mu^k \ .$$

□

[1] Beware that although the X_i's are sections of TM, x^iX_i are not. Similarly for the $\frac{\partial}{\partial h_i}$'s.

In this way we can find the following expression for the X_{h_i}'s.

Proposition. Using the frame of $T(T^*M)$ given by $\{X_1, \cdots, X_n, \frac{\partial}{\partial h_1}, \cdots \frac{\partial}{\partial h_n}\}$ we have

$$X_{h_i} = X_i + c_{ij}^k h_k \frac{\partial}{\partial h_j} \ . \tag{3.7}$$

Proof. We can use the explicit expression of the sharp isomorphism induced by σ, the so called Poisson tensor, and then the result is manifest. Alternatively we notice that we can express the fiber part of the Hamilton equations using the Poisson brackets. We have

$$X_{h_i} = x^j X_j + v_j \frac{\partial}{\partial h_j} \ ,$$

with x^j and v_j to be determined. The x^j are fixed by the condition $\pi_* X_{h_i} = X_i$, so $x^j = \delta_i^j$. For the fiber part we have

$$v_j = \dot{h}_j = \{h_i, h_j\} \ .$$

By equation 3.2, we have

$$\{h_i, h_j\} = i_{X_{\{h_i,h_j\}}}\theta = i_{[X_{h_i}, X_{h_j}]}\theta = h_k \mu^k([X_i, X_j]) = h_k c_{ij}^k \ .$$

□

The Riemannian Hamiltonian. Let g be a Riemannian metric on M and $\{X_i\}_{i=1\cdots,n}$ be an orthonormal frame with dual frame $\{\mu^i\}_{i=1,\cdots,n}$. We want to define a dynamic in the cotangent bundle that will return us the geodesics in the base. To do so we introduce the Riemannian Hamiltonian $H : T^*M \to \mathbb{R}$ as

$$H(\lambda) := \frac{1}{2} g|_{\pi(\lambda)} \left(\sum_{i=1}^n h_i(\lambda) X_i(\pi(\lambda)), \sum_{j=1}^n h_j(\lambda) X_j(\pi(\lambda)) \right) \ . \tag{3.8}$$

Using the orthonormality condition on the X_i's we get the expression

$$H(\lambda) = \frac{1}{2} \sum_{i=1}^n (h_i(\lambda))^2 \ . \tag{3.9}$$

We easily verify that $\mathrm{d}H = \sum_{i=1}^n h_i \mathrm{d}h_i$, hence, by linearity of the sharp isomorphism (the one of σ) we get

$$X_H = \sum_{i=1}^n h_i X_{h_i} \ . \tag{3.10}$$

Moreover using expression 3.7, we can write

$$X_H = \sum_{i=1}^n h_i \left(X_i + c_{ij}^k h_k \frac{\partial}{\partial h_j} \right) \ . \tag{3.11}$$

By equation 3.4, Hamilton equations $\dot{\lambda} = X_H$ project in the base giving[2] (define $\gamma := \pi(\lambda)$)

$$\dot{\gamma} = \pi_* \dot{\lambda} = \pi_* X_H = \sum_{j=1}^n h_j X_j \ .$$

The functions h_i's are hence the components of the velocity of the projections of solutions of the Hamilton equations in T^*M into the base manifold M with respect to the orthonormal frame $\{X_i\}_{i=1\cdots,n}$.

We finally need to verify that γ with $\dot{\gamma} = \pi_* X_H$ is geodetic. Contrary to the two dimensional Riemannian case we cannot exploit the principal bundle structure of the sphere bundle over M. Instead we need to work in the whole TM and use affine connections.

[2]Beware that $\pi_* X_H$ is not a vector field on M.

The geodesics equation. We need to verify if the projection on M of a solution of the equation $\dot{\lambda} = X_H$, denoted with $\dot{\gamma} = \pi_* X_H = \sum_{j=1}^n h_j X_j$, is a geodesic, i.e. if we have $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$, where ∇ is the Levi-Civita connection of g. We obtain

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \sum_{i,j}^{n} \nabla_{h_i X_i} h_j X_j = \sum_{i,j}^{n} \left(\dot{h}_j + \sum_{k=1}^{n} \Gamma_{ik}^{j} h_k h_i \right) X_j \ . \tag{3.12}$$

But $\dot{h}_j = \sum_{i,k=1}^n c_{ij}^k h_k h_i$, hence

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \sum_{i,j,k}^{n} \left(c_{ij}^k + \Gamma_{ik}^j \right) h_k h_i X_j \ . \tag{3.13}$$

Now we use the properties of the Levi-Civita connection coefficients[3] to get the final result.

$$(c_{ij}^k + \Gamma_{ik}^j) h_i h_k = (\Gamma_{ij}^k - \Gamma_{ji}^k + \Gamma_{ik}^j) h_k h_i = (-\Gamma_{ik}^j - \Gamma_{ji}^k + \Gamma_{ik}^j) h_k h_i = -\Gamma_{ji}^k h_k h_i = 0 \ .$$

In conclusion

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0 \ .$$

3.2 Magnetic forms in a Riemannian manifold

We now introduce a magnetic field in a Riemannian manifold (M, g) and compute the Hamilton equations corresponding to the Riemannian Hamiltonian, which now become shifted by the magnetic potential.

The shifted Hamiltonian. Let $\beta \in \Lambda^2(M)$ be a magnetic field, with local potential $A \in \Lambda^1(U)$, $U \subseteq M$ open subset. With orthonormal frames we can write the Riemannian Hamiltonian as $\frac{1}{2}\sum_{j=1}^n h_j^2$. Using the tautological definition of h_j 3.3, we can regard the introduction of A as a shift in θ, namely

$$\theta \mapsto \tilde{\theta} := \theta + \pi^* A \ .$$

In this way we recover the usual shift in the momenta $h_j \mapsto \tilde{h}_j = h_j + A_j$. Indeed

$$\tilde{h}_j = i_{X_{h_j}} \tilde{\theta} = h_j + \pi^*(A(X_{h_j})) = h_j + \pi^*(A(\pi_* X_{h_j})) = h_j + \pi^*(A(X_j)) = h_j + \pi^* A_j \ .$$

Since we always use the splitting 3.5, in the following we simply write A instead of $\pi^* A$.
In conclusion the new shifted Hamiltonian becomes

$$H = \frac{1}{2}\sum_{j=1}^{n} (h_j + A_j)^2 \ . \tag{3.14}$$

The Hamilton equations. We need to compute X_H. Obviously $\mathrm{d}H = \sum_{j=1}^n (h_j + A_j)(\mathrm{d}h_j + \mathrm{d}A_j)$, and by linearity of the sharp isomorphism of σ, that we denote as $\sharp_\sigma$, we have

$$X_H = \sum_{j=1}^{n} (h_j + A_j)(X_{h_j} + \sharp_\sigma \mathrm{d}A_j) \ . \tag{3.15}$$

From a simple computation, making use of the expression of σ found previously, we have that $\sharp_\sigma \mu^j = -\frac{\partial}{\partial h_j}$. Moreover $\mathrm{d}A_j = (X_j(A_i) + c_{ij}^k A_k + \beta_{ij})\mu^i$. So, equation 3.15 becomes

$$X_H = \sum_{j=1}^{n} (h_j + A_j) \left(X_j + \left(c_{ji}^k (h_k + A_k) - X_j(A_i) - \beta_{ij} \right) \frac{\partial}{\partial h_i} \right) \ . \tag{3.16}$$

[3] We have recalled them in Appendix B.

Equations on the base. We now compute $\nabla_{\dot\gamma}\dot\gamma$, where here $\dot\gamma = \pi_* X_H = \sum_{j=1}^n (h_j + A_j)X_j$. Using 3.16, have

$$\sum_{i,j=1}^n \nabla_{(h_i+A_i)X_i}(h_j+A_j)X_j = \sum_{i,j=1}(h_i+A_i)\nabla_{X_i}(h_j+A_j)X_j =$$
$$= \sum_{i,j=1}^n \left((\dot h_j + \dot A_j) + \sum_{k=i}^n (h_i+A_i)(h_k+A_k)\Gamma^j_{ik}\right)X_j =$$
$$= \sum_{i,j=1}^n \left(-\dot A_j + \beta_{ij}(h_i+A_i) + \dot A_j + \sum_{k=1}^n (c^k_{ij} + \Gamma^j_{ik})(h_i+A_i)(h_k+A_k)\right)X_j =$$
$$= \sum_{i,j=1}^n \left(\beta_{ij}(h_i+A_i)\right)X_j \ .$$

In the last passage we used the properties of the Levi-Civita connection coefficients. Finally we recovered the result obtained in the first chapter, where we used coordinates and the variational approach

$$\nabla_{\dot\gamma}\dot\gamma = \sum_{i,j=1}^n (\beta_{ij}\dot\gamma_i)X_j = \sharp_g i_{\dot\gamma}\beta \ . \tag{3.17}$$

In this last equation we used the sharp isomorphism of g, denoted as $\sharp_g$, that for orthonormal frames is simply δ^{ij}.

3.3 Sub-Riemannian Geodesics of C_M^β

We start by observing that our treatment of section 1.3 does not involve the dimension of the Riemannian surface. Therefore we can apply the construction of C_M^β to a Riemannian manifold (M, g) of any finite dimension $n \in \mathbb{N}$. Again, the possibility of globalizing the trivial bundle structure depends on the topology of M. We can forget about this problems if we limit ourselves to study the local properties of the construction. Since this is the case for now, we still denote the bundle we define as C_M^β regardless of its global existence. Let $\beta \in \Lambda^2(M)$ be a magnetic 2-form with potential $A \in \Lambda^1(U)$ in the open subset $U \subseteq M$. We consider a piece-wise smooth curve $\gamma : [0,1] \to U$, and define a dynamic in $U \times \mathbb{R}$ as we did with 1.20, i.e. we set

$$z(t) := \int_{\gamma([0,t])} A \ ,$$

and $\lambda(t) := (\gamma(t), z(t)) \in U \times \mathbb{R}$ for all $t \in [0,1]$. Using the splitting $T(U\times\mathbb{R}) = TU \oplus T\mathbb{R}$ and an orthonormal frame of M, $\{X_1, \cdots, X_n\}$, we have that

$$\dot\lambda(t) = (\dot\gamma(t), \dot z(t)) = u^i(t)X_i(\gamma(t)) + A_{\gamma(t)}(\dot\gamma(t))\partial_z \ ,$$

where $u^i : [0,1] \to \mathbb{R}$, $i = 1, \cdots, n$, are piece-wise smooth functions. The curve $\lambda : [0,1] \to U \times \mathbb{R}$ is always tangent to the distribution $D \subset T(U \times \mathbb{R})$ described by the vector fields

$$T_i := X_i + A(X_i)\partial_z \quad i = 1, \cdots, n \ . \tag{3.18}$$

Clearly the distribution has constant rank equal to n, and we easily verify wherever it is bracket generating. Using the same computations of section 1.3 we obtain

$$[T_i, T_j] = [X_i, X_j] + A([X_i, X_j])\partial_z + \mathrm{d}A(X_i, X_j) = c^k_{ij}T_k + \beta_{ij}\partial_z \ , \tag{3.19}$$

where $c^k_{ij} \in \mathcal{C}^\infty(M)$. This means that if one of the β_{ij} is non zero the distribution is bracket generating. This is the n-dimensional analogue of 1.23. In the following we denote $T_z := \partial_z$.

Sub-Riemannian geodesics. We consider the Sub-Riemannian metric given by

$$g_s|_x(u,v) := g|_{\pi(x)}(\pi_* u, \pi_* v) \quad x \in C_M^\beta,\ u,v \in D|_x \subset T_x C_M^\beta \ . \tag{3.20}$$

which is the essentially 1.28 in the n-dimensional case. Following the approach of section 1.5 we define the Sub-Riemannian Hamiltonian H_s analogously to 1.36, i.e. we set

$$H_s = \frac{1}{2}\sum_{i=1}^n h_i^2 \ ,$$

with $h_i \in \mathcal{C}^\infty(T^* C_M^\beta),\ i = 1, \cdots, n$ the fiber linear function on $T^* C_M^\beta$ corresponding to T_i. We also denote with h_z the fiber linear function relative to T_z. The fiber part of the Hamilton equations is then

$$\begin{cases} \dot{h}_i = \{H_s, h_i\} = \sum_{j=1}^n \{h_j, h_i\} h_j = \sum_{j=1}^n c_{ji}^k h_j h_k + \beta_{ji} h_j h_z \\ \dot{h}_z = \{H_s, h_z\} = \sum_{j=1}^n \{h_j, h_z\} h_j = 0 \ . \end{cases} \tag{3.21}$$

If we consider the projection of the system on U which is given by $\dot\gamma(t) = \sum_{i=1}^n h_i X_i$, and compute the covariant derivative along γ relative to the Levi-Civita connection of g, using 3.12 we have

$$\nabla_{\dot\gamma}\dot\gamma = \sum_{i,j}^n \left(\dot{h}_j + \sum_{k=1}^n \Gamma_{ik}^j h_k h_i \right) X_j \ . \tag{3.22}$$

From this equation, thanks to 3.21 we obtain

$$\nabla_{\dot\gamma}\dot\gamma = \sum_{i,j,k}^n \left(c_{ij}^k + \Gamma_{ik}^j \right) h_k h_i X_j + \sum_{i,j} \beta_{ij} h_i h_z X_j \ . \tag{3.23}$$

Now, as we already know, the parenthesis in the first addend is zero by the properties of the Levi-Civita connection coefficients and h_z is a constant that we can fix to be equal to one, hence

$$\nabla_{\dot\gamma}\dot\gamma = \sum_{i,j} (\beta_{ij} h_i) X_j = \sharp_g i_{\dot\gamma}\beta \ .$$

Finally we were able to show that as in the case of Riemannian surfaces the Sub-Riemannian (normal) geodesics of C_M^β projects into the trajectories of charged particles in M with the presence of a magnetic field.

3.4 Magnetic forms in a Sub-Riemannian manifold

We pass now to the general Sub-Riemannian case, where we are given a distribution $D \subset TM$ and a Sub-Riemannian metric g_s. In the following $k \leq n = \dim(M)$ will be the (constant) rank of D that we consider to be spanned by $\{X_1, \cdots, X_k\} \subset \mathfrak{X}(M)$ orthonormal frame with respect to g_s.

The Sub-Riemannian Hamiltonian. Like in the previous section, we consider tautological functions

$$h_j = i_{X_{h_j}}\theta \quad j = 1, \cdots, k \ .$$

Observe that these are coordinates on the fibers of the dual distribution $D^* \subset T^*M$.
Next we define the Sub-Riemannian Hamiltonian as

$$H_s(\lambda) := \frac{1}{2} g_s|_{\pi(\lambda)} \left(\sum_{j=1}^k h_j(\lambda) X_j(\pi(\lambda)), \sum_{j=1}^k h_j(\lambda) X_j(\pi(\lambda)) \right) \ . \tag{3.24}$$

Using the orthonormality of the X_i's, we recover the expression of the previous chapters

$$H_s(\lambda) = \frac{1}{2}\sum_{j=1}^{k}(h_j(\lambda))^2 \ . \tag{3.25}$$

Equivalent magnetic potentials. We introduce a magnetic form $\beta \in \Lambda^2(M)$, with local magnetic potential $A \in \Lambda^1(U)$, $U \subseteq M$ open subset. Suppose also that the distribution D is described as the kernel of $n-k$ 1-forms $\{\alpha_1, \cdots, \alpha_{n-k}\} \subset \Lambda^1(M)$. We say that D is described by the Pfaffian equations

$$\begin{cases} \alpha_1 = 0 \\ \quad \vdots \\ \alpha_{n-k} = 0 \ . \end{cases} \tag{3.26}$$

As in the Riemannian case, we interpret the introduction of a magnetic form as a shift in the h_j's given by

$$\theta \mapsto \tilde{\theta} = \theta + \pi^* A \ .$$

However now we have a redundance, because if we shift A with any of the α_i's nothing changes, and we are left with the same Hamiltonian. Explicitly, for all $i = 1, \cdots, n-k$ we have

$$i_{X_{h_j}}(\theta + \pi^*(A + \alpha_i)) = h_j + A_j + \pi^*(\alpha_i(X_j)) = h_j + A_j = i_{X_{h_j}}(\theta + \pi^* A) \ .$$

In conclusion we should consider magnetic potentials up to the equivalence relation induced by the shifting with the α_i's

$$A \mapsto A + f^i \alpha_i \quad i = 1, \cdots, n-k \ , \tag{3.27}$$

with $f^i \in \mathcal{C}^\infty(M)$.

The Rumin complex. In the case in which the distribution has rank $n-1$, i.e. we are given a single 1-form α, which we suppose to be also contact, we can construct from the set of equivalent magnetic potentials a differential complex, the Rumin complex, whose cohomology is equal to the De Rham cohomology of the manifold M. Here we present the general construction following [Rum94] and the next chapter we will describe it for the contact structure given by 2.2 in the three dimensional Heisenberg group.

Let (M, α) be a $(2n+1)$-dimensional contact manifold, with contact form α. We denote as $\Lambda^*(M) := \bigoplus_{k=0}^{2n+1} \Lambda^k(M)$ the graded algebra of differential forms on M. We define the ideal (with respect to the exterior multiplication)

$$\mathcal{I}^*(M) := \{\alpha \wedge \beta + \mathrm{d}\alpha \wedge \gamma \ : \ \beta, \gamma \in \Lambda^*(M)\} \ , \tag{3.28}$$

and the annihilator

$$\mathcal{J}^*(M) := \{\omega \in \Lambda^*(M) \ : \ \alpha \wedge \omega = 0 = \mathrm{d}\alpha \wedge \omega\} \ . \tag{3.29}$$

Notice that none of these sets depends on the normalization of the contact form α and that if $\omega \in \mathcal{I}^*$ (respectively $\omega \in \mathcal{J}^*(M)$) then also $\mathrm{d}\omega \in \mathcal{I}^*(M)$ ($\mathrm{d}\omega \in \mathcal{J}^*(M)$). This means that the exterior differential induces an operator $\mathrm{d}_\alpha : \Lambda^*(M)/\mathcal{I}^*(M) \to \Lambda^*(M)/\mathcal{I}^*(M)$, i.e. for every $i = 1, \cdots, 2n+1$, $\mathrm{d}_\alpha : \Lambda^i(M)/\mathcal{I}^i(M) \to \Lambda^{i+1}(M)/\mathcal{I}^{i+1}(M)$. From the symplecticity of $\mathrm{d}\alpha$ a classical result tells us that the exterior multiplication by $\mathrm{d}\alpha$, which we now regard as a mapping of horizontal forms[4] of degree k to horizontal forms of degree $k+2$, is surjective for

[4] We recall that horizontal k-forms are forms that depends only on the contact distribution $D := \ker(\alpha)$. We recognize that these k-forms are the sections of the k^{th} exterior power of the sub-bundle $D^* \subseteq T^*M$. We denote such space of sections as $\Lambda^k_\alpha(M)$.

every $k \geq n-1$ and injective for $k \leq n-1$. From this fact it follows that $\Lambda^k(M)/\mathcal{I}^k(M) = \{0\}$ for $k \geq n+1$ and $\mathcal{J}^k(M) = \{0\}$ for $k \leq n$. We can finally state (without proving) the following result.

Proposition. It exists a differential operator $\mathcal{D} : \Lambda^n(M)/\mathcal{I}^n(M) \to \mathcal{J}^{n+1}$ such that the sequence

$$\{0\} \longrightarrow \mathbb{R} \longrightarrow \mathcal{C}^\infty(M) \xrightarrow{\mathrm{d}_\alpha} \Lambda^1(M)/\mathcal{I}^1(M) \xrightarrow{\mathrm{d}_\alpha} \cdots \xrightarrow{\mathrm{d}_\alpha} \Lambda^n(M)/\mathcal{I}^n(M) \xrightarrow{\mathcal{D}}$$

$$\xrightarrow{\mathcal{D}} \mathcal{J}^{n+1} \xrightarrow{\mathrm{d}_\alpha} \cdots \xrightarrow{\mathrm{d}_\alpha} \mathcal{J}^{2n+1} \xrightarrow{\mathrm{d}_\alpha} \{0\}$$

is exact and its cohomology coincides with the De Rham cohomology of M.

Remark. Notice that $\mathcal{I}^1(M) = \mathrm{span}(\alpha)$, hence we recover the symmetry 3.27 present in the Sub-Riemannian Hamiltonian.

Description of $\mathcal{D}$. We now describe the differential operator $\mathcal{D}$ and verify its cohomological properties, i.e. that $\mathrm{d}_\alpha \circ \mathcal{D} = 0$ and $\mathcal{D} \circ \mathrm{d}_\alpha = 0$. Always following [Rum94], we have

Proposition. Given $\varphi \in \Lambda^n_\alpha(M)$ an horizontal n-form, it exists a unique lift $\tilde{\varphi} \in \Lambda^n(M)$ such that $\mathrm{d}\tilde{\varphi} \in \mathcal{J}^{n+1}(M)$.

Proof. Let $\bar{\varphi} \in \Lambda^n(M)$ be any lift of φ. We then look for $\beta \in \Lambda^{n-1}_\alpha(M)$ such that if $\tilde{\varphi} = \bar{\varphi} + \alpha \wedge \beta$ we have $\mathrm{d}\tilde{\varphi} \in \mathcal{J}^{n+1}(M)$. In particular $\mathrm{d}\tilde{\varphi} = \mathrm{d}\bar{\varphi} + \mathrm{d}\alpha \wedge \beta - \alpha \wedge \mathrm{d}\beta$. Hence $\alpha \wedge \mathrm{d}\tilde{\varphi} = \alpha \wedge (\mathrm{d}\bar{\varphi} + \mathrm{d}\alpha \wedge \beta) = 0$ if and only if the second factor is zero on D, i.e if and only if $(\mathrm{d}\alpha \wedge \beta)|_D = -(\mathrm{d}\bar{\varphi})|_D$. Since for $n-1$ the homomorphism given by exterior multiplication by $(\mathrm{d}\alpha)|_D$ is an isomorphism $\Lambda^{n-1}_\alpha(M) \simeq \Lambda^{n+1}_\alpha(M)$, this last equation has a unique solution. On the other hand $\mathrm{d}\alpha \wedge \mathrm{d}\tilde{\varphi} = \mathrm{d}(\alpha \wedge \mathrm{d}\tilde{\varphi}) = 0$ by what we have just found. In conclusion $\mathrm{d}\tilde{\varphi} \in \mathcal{J}^{n+1}(M)$. ☐

We further notice that since $\Lambda^n_\alpha(M) \simeq \Lambda^n(M)/\{\alpha \wedge \beta \ : \ \beta \in \Lambda^{n-1}(M)\}$, then $\Lambda^n(M)/\mathcal{I}^n(M) \simeq \Lambda^n_\alpha(M)/\{\mathrm{d}\alpha \wedge \beta \ : \ \beta \in \Lambda^{n-2}(M)\}$. We can now state the following proposition.

Proposition. The operator $\tilde{\mathcal{D}} : \Lambda^n_\alpha(M) \to \mathcal{J}^{n+1}(M)$ defined by $\tilde{\mathcal{D}}\varphi := \mathrm{d}\tilde{\varphi}$ pass to the quotient with respect to $\{\mathrm{d}\alpha \wedge \beta \ : \ \beta \in \Lambda^{n-2}(M)\}$. Therefore we set $\mathcal{D}[\varphi] := \tilde{\mathcal{D}}\varphi$, having $\varphi \in \Lambda^n_\alpha(M)$ and $[\varphi] \in \Lambda^n(M)/\mathcal{I}^n(M)$ the corresponding equivalence class.

Proof. Let $(\mathrm{d}\alpha \wedge \beta)^D$ be the horizontal part[5] of $\mathrm{d}\alpha \wedge \beta$. We clearly have $\mathrm{d}(\mathrm{d}\alpha \wedge \beta - \alpha \wedge \mathrm{d}\beta) = 0 \in \mathcal{J}^{n+1}(M)$. By our previous proposition this means that $\mathrm{d}\alpha \wedge \beta - \alpha \wedge \mathrm{d}\beta$ is the unique lift of $(\mathrm{d}\alpha \wedge \beta)^D$ to $\Lambda^n(M)$. Consequently, by definition, $\tilde{\mathcal{D}}((\mathrm{d}\alpha \wedge \beta)^D) = \mathrm{d}(\mathrm{d}\alpha \wedge \beta - \alpha \wedge \mathrm{d}\beta) = 0$. ☐

We now show the local exactness of the Rumin complex for n and $n+1$. We adopt the shorthand notation for the quotients $\Lambda^k(M)/\mathcal{I}^k(M) =: \Omega^k(M)$, with $k = 1, \cdots, 2n+1$.
If $\varphi \in \Omega^n(M)$ is such that $\mathcal{D}\varphi = 0$, then by taking the unique lift $\tilde{\varphi} \in \Lambda^n(M)$, we have $\mathcal{D}\varphi = \mathrm{d}\tilde{\varphi} = 0$. Consequently there exist, locally, $\beta \in \Lambda^{n-1}(M)$ such that $\tilde{\varphi} = \mathrm{d}\beta$. But the exterior differential pass to the quotient with respect to $\mathcal{I}^{n-1}(M)$, hence $\varphi = \mathrm{d}_\alpha[\beta]$ with $[\beta] \in \Omega^{n-1}(M)$, the equivalence class of β.
Let now $\varphi \in \mathcal{J}^{n+1}(M)$ such that $\mathrm{d}_\alpha\varphi = 0$. Taking a representative $\tilde{\varphi} \in \Lambda^{n+1}(M)$, we have $\mathrm{d}\tilde{\varphi} = 0$. Therefore, locally, there exists $\beta \in \Lambda^n(M)$ such that $\tilde{\varphi} = \mathrm{d}\beta$. Passing to the quotient in this last equation we get $\varphi = \mathcal{D}[\beta]$, where again $[\beta] \in \Omega^n(M)$ is the equivalence class of β.

[5] Given a k-form $\varphi \in \Lambda^k(M)$, and $D := \ker(\alpha) \subset TM$, the horizontal (contact) distribution, $\varphi^D(X_1, \cdots, X_k) := \varphi(X_1^D, \cdots, X_k^D)$, where X_i^D is the component of $X_i \in \mathfrak{X}(M)$ along D, is the horizontal part of φ. This means that $\varphi^D \in \Lambda^k_\alpha(M)$.

Chapter 4

Magnetic forms in the Heisenberg group

In this chapter we describe the Rumin complex for the three dimensional Heisenberg group $\mathbb{H}_3$, using the general construction made in the previous chapter. Next we construct the space $C^{\beta}_{\mathbb{H}_3}$ by introducing a magnetic 2-form $\beta \in \Omega^2(\mathbb{H}_3)$. In the following, we study the particular case of a constant magnetic 2-form that will make $C^{\beta}_{\mathbb{H}_3}$ a four dimensional Engel-type group. This new space we endow with a Sub-Riemannian structure and we study its Sub-Riemannian length minimizers, giving a geometric interpretation of them.

4.1 The Rumin complex in $\mathbb{H}_3$

We consider the Heisenberg group $\mathbb{H}_3$ with the contact structure given by

$$\alpha := \mathrm{d}z + \frac{1}{2}(y\mathrm{d}x - x\mathrm{d}y) \ , \tag{4.1}$$

which is the same of 2.2, with the renaming $x^1 = x$ and $x^2 = y$. The horizontal distribution, given by the kernel of α, is spanned by the vector fields 2.3 that we recall here for convenience

$$\begin{cases} X = \partial_x - \frac{y}{2}\partial_z \\ Y = \partial_y + \frac{x}{2}\partial_z \ . \end{cases} \tag{4.2}$$

As already remarked in the Sub-Riemannian case the possible magnetic potentials that we can introduce, that in principle can be any 1-form $A \in \Lambda^1(U)$, $U \subseteq \mathbb{H}_3$ open, are redundant. We shall consider instead

$$A \in \Omega^1(U) := \Lambda^1(U)/\mathrm{span}\{\alpha\} \ , \tag{4.3}$$

in the sense that $A, \tilde{A} \in \Lambda^1(U)$ are equivalent if and only if $A - \tilde{A} = f\alpha$, with $f \in \mathcal{C}^\infty(U)$. Thanks to this symmetry we have

Proposition. Let $\alpha = \mathrm{d}z + \frac{1}{2}(y\mathrm{d}x - x\mathrm{d}y)$. Then

$$\Omega^1(U) = \mathrm{span}\{\mathrm{d}x, \mathrm{d}y\} \ .$$

Proof. The set of 1-forms $\{\mathrm{d}x, \mathrm{d}y, \alpha\} \subset \Lambda^1(U)$ provides a basis for $T^*_m U$ forall $m \in U$, hence a generic form $A \in \Lambda^1(U)$ can be written as

$$A = A_x\mathrm{d}x + A_y\mathrm{d}y + A_\alpha\alpha \ ,$$

for suitable $A_x, A_y, A_\alpha \in \mathcal{C}^\infty(U)$. But by definition of $\Omega^1(U)$, $\tilde{A} := A - A_\alpha\alpha = A_x\mathrm{d}x + A_y\mathrm{d}y$ is equivalent to A. □

In order to pass from functions on U to $\Omega^1(U)$ we can define the following first order differential operator acting on $\mathcal{C}^\infty(U)$

$$d_\alpha(f) := X(f)\mathrm{d}x - Y(f)\mathrm{d}y \ . \tag{4.4}$$

We notice that $\mathrm{d}_\alpha f = \mathrm{d}f \mod(\alpha)$, indeed $\mathrm{d}_\alpha f = \mathrm{d}f - (\partial_z f)\alpha$, i.e. the exterior differential passes to the quotient. The operator d_α is therefore the one defined in the Rumin complex. Since $\{\mathrm{d}x, \mathrm{d}y, \alpha\} \subset \Lambda^1(U)$ gives a frame for T^*U we can use the 2-forms $\{\mathrm{d}x \wedge \alpha, \mathrm{d}y \wedge \alpha, \mathrm{d}x \wedge \mathrm{d}y\} \subset \Lambda^2(U)$ to express a generic 2-form $\beta \in \Lambda^2(U)$ as

$$\beta = \beta_x \mathrm{d}x \wedge \alpha + \beta_y \mathrm{d}y \wedge \alpha + b_\alpha \mathrm{d}x \wedge \mathrm{d}y \ .$$

Following the general construction of the Rumin complex, the non trivial 2-forms that we have to consider are $\Omega^2(\mathbb{H}_3) = \mathrm{span}\{\mathrm{d}x \wedge \alpha, \mathrm{d}y \wedge \alpha\}$. In this way we are able to characterize the magnetic fields in a way that reflects the degeneracy of the potentials, i.e. the underlying contact distribution. Using the construction of $\mathcal{D}$ of the Rumin complex for the contact structure 4.1 in $\mathbb{H}_3$ we have

Proposition. Given $A \in \Omega^1(\mathbb{H}_3)$ the unique lift $\tilde{A} \in \Lambda^1(\mathbb{H}_3)$ such that $\mathrm{d}\tilde{A} \in \mathcal{J}^2(\mathbb{H}_3) = \Omega^2(\mathbb{H}_3) = \mathrm{span}\{\mathrm{d}x \wedge \alpha, \mathrm{d}y \wedge \alpha\}$ is given by $\tilde{A} = A + f\alpha$ with

$$f = X(A_y) - Y(A_x) \ .$$

Furthermore we have

$$\mathcal{D}A = \mathcal{D}(A_x \mathrm{d}x + A_y \mathrm{d}y) := \mathrm{d}\tilde{A} = \beta_x \mathrm{d}x \wedge \alpha + \beta_y \mathrm{d}y \wedge \alpha \ , \tag{4.5}$$

with

$$\begin{cases} \beta_x := X(X(A_y) - Y(A_x)) - Z(A_x) \\ \beta_y := Y(X(A_y) - Y(A_x)) - Z(A_y) \end{cases} \tag{4.6}$$

Proof. We can compute f from the condition $\alpha \wedge \mathrm{d}\tilde{A} = 0$. We have $\mathrm{d}\tilde{A} = \mathrm{d}A + \mathrm{d}f \wedge \alpha + f\mathrm{d}\alpha$, so we consider the equation $\alpha \wedge (\mathrm{d}A + f\mathrm{d}\alpha) = 0$, which is satisfied if and only if $(\mathrm{d}A + f\mathrm{d}\alpha)|_D = 0$. Then we simply need to solve $f(\mathrm{d}\alpha)|_D = -(\mathrm{d}A)|_D$ for $f \in \mathcal{C}^\infty(\mathbb{H}_3)$. Recalling that on D it is $\alpha = 0$, i.e. $\mathrm{d}z = \frac{x}{2}\mathrm{d}y - \frac{y}{2}\mathrm{d}x$ we get

$$\begin{aligned} f\mathrm{d}x \wedge \mathrm{d}y &= (\partial_x A_y - \partial_y A_x)\mathrm{d}x \wedge \mathrm{d}y + \partial_z A_x \mathrm{d}z \wedge \mathrm{d}x + \partial_z A_y \mathrm{d}z \wedge \mathrm{d}y = \\ &= \left(\partial_x A_y - \partial_y A_x - \frac{y}{2}\partial_z A_y - \frac{x}{2}\partial_z A_x\right)\mathrm{d}x \wedge \mathrm{d}y = \\ &= (X(A_y) - Y(A_x))\mathrm{d}x \wedge \mathrm{d}y \ , \end{aligned}$$

which is what we looked for.
Recalling that for any function $g \in \mathcal{C}^\infty(\mathbb{H}_3)$ it is $\mathrm{d}g = X(g)\mathrm{d}x + Y(g)\mathrm{d}y + (\partial_z g)\alpha$, we are now in a position to prove 4.5.

$$\mathcal{D}A := \mathrm{d}\tilde{A} = \mathrm{d}A + \mathrm{d}f \wedge \alpha + f\mathrm{d}\alpha = \mathrm{d}A + X(f)\mathrm{d}x \wedge \alpha + Y(f)\mathrm{d}y \wedge \alpha + f\mathrm{d}\alpha =$$

$$\begin{aligned} &= (\partial_x A_y - \partial_y A_x - f)\mathrm{d}x \wedge \mathrm{d}y + \partial_z A_x \mathrm{d}z \wedge \mathrm{d}x + \partial_z A_y \mathrm{d}z \wedge \mathrm{d}y + X(f)\mathrm{d}x \wedge \alpha + Y(f)\mathrm{d}y \wedge \alpha = \\ &= \left(\frac{x}{2}\partial_z A_x + \frac{y}{2}\partial_z A_y\right)\mathrm{d}x \wedge \mathrm{d}y + \partial_z A_x \mathrm{d}z \wedge \mathrm{d}x + \partial_z A_y \mathrm{d}z \wedge \mathrm{d}y + X(f)\mathrm{d}x \wedge \alpha + Y(f)\mathrm{d}y \wedge \alpha = \\ &= (X(f) - Z(A_x))\mathrm{d}x \wedge \alpha + (Y(f) - Z(A_y))\mathrm{d}y \wedge \alpha \ . \end{aligned}$$

This is exactly our claim.

□

Even if we already proved this in general in the previous chapter, we show that the differential operator $\mathcal{D}$ satisfies the cohomological property

$$\mathcal{D} \circ \mathrm{d}_\alpha = 0 \ . \tag{4.7}$$

A simple computation proves this fact. Indeed, for all $f \in \mathcal{C}^\infty(U)$ we have

$$\mathcal{D}(\mathrm{d}_\alpha f) = \mathcal{D}(X(f)\mathrm{d}x + Y(f)\mathrm{d}y) = (X(X(Y(f)) - Y(X(f))) - Z(X(f)))\mathrm{d}x \wedge \alpha +$$

$$+(Y(X(Y(f)) - Y(X(f))) - Z(Y(f)))\mathrm{d}y \wedge \alpha = (X([X,Y](f)) - Z(X(f)))\mathrm{d}x \wedge \alpha +$$

$$+(Y([X,Y](f)) - Z(Y(f)))\mathrm{d}y \wedge \alpha = ([X,Z](f))\mathrm{d}x \wedge \alpha + ([Y,Z](f))\mathrm{d}y \wedge \alpha = 0 \ .$$

Notice that we used only the definitions of d_α and $\mathcal{D}$ and the Heisenberg algebra 2.4.
Finally the closed three forms are clearly spanned by the volume form $\alpha \wedge \mathrm{d}\alpha$, and are obtained from $\Omega^2(U)$ with the standard exterior differential. Even in this case the cohomological property $\mathrm{d} \circ \mathcal{D} = 0$ is straightforward to prove since $\mathrm{d}(\mathrm{d}x \wedge \alpha) = -\mathrm{d}x \wedge \mathrm{d}\alpha = \mathrm{d}x \wedge \mathrm{d}x \wedge \mathrm{d}y = 0$ and similarly for $\mathrm{d}(\mathrm{d}y \wedge \alpha) = \mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}y = 0$.
In conclusion the Rumin complex for $\mathbb{H}_3$ is then given by the following short exact sequence.

$$\{0\} \longrightarrow \mathcal{C}^\infty(U) \xrightarrow{\mathrm{d}_\alpha} \Omega^1(U) \xrightarrow{\mathcal{D}} \Omega^2(U) \xrightarrow{\mathrm{d}} \Omega^3(U) \longrightarrow \{0\}$$

Equivalent potentials. The cohomological property $\mathcal{D} \circ \mathrm{d}_\alpha = 0$ implies that we can further restrict to potentials of the type $A = a\mathrm{d}y$, with $a \in \mathcal{C}^\infty(\mathbb{H}_3)$. Indeed let us consider $\tilde{A} := A + \mathrm{d}_\alpha g$, $g \in \mathcal{C}^\infty(\mathbb{H}_3)$, which satisfies $\mathcal{D}\tilde{A} = \mathcal{D}(A + \mathrm{d}_\alpha g) = \mathcal{D}A$. We have that

$$\tilde{A} = (A_x + X(g))\mathrm{d}x + (A_y + Y(g))\mathrm{d}y \ ,$$

and consequently we can find g such that

$$A_x + \left(\partial_x - \frac{y}{2}\partial_z\right) g = 0 \ , \tag{4.8}$$

having $a = A_y + Y(g)$.

4.2 A four dimensional space

A four dimensional dynamic. Let $\beta \in \Omega^2(\mathbb{H}_3)$ a magnetic field, and let $A \in \Omega^1(\mathbb{H}_3)$ be a potential for β, i.e. $\beta = \mathcal{D}A$. Following the procedure of section 1.3 we consider a piece-wise smooth curve $\gamma : [0,1] \to \mathbb{H}_3$, with the difference that here we ask γ to be admissible, i.e. $\dot{\gamma}(t) \in \ker(\alpha|_{\gamma(t)})$ for all $t \in [0,1]$. This means that

$$\dot{\gamma}(t) = u_x(t)X(\gamma(t)) + u_y(t)Y(\gamma(t)) \ , \tag{4.9}$$

with $u_x, u_y : [0,1] \to \mathbb{R}$ controls.
We then define a real quantity

$$w(t) := \int_{\gamma([0,t])} A. \tag{4.10}$$

In complete analogy with section 1.3 we study the dynamic in the bundle $\mathbb{H}_3 \times \mathbb{R} \simeq \mathbb{R}^4$ given by $\lambda(t) := (\gamma(t), w(t))$. The velocity of λ is then

$$\dot{\lambda}(t) = u_x(t)X(\gamma(t)) + u_y(t)Y(\gamma(t)) + A|_{\gamma(t)}(\dot{\gamma}(t))\partial_w \ . \tag{4.11}$$

We notice that the curve λ can be seen as a curve tangent to a rank-2 distribution D in $T\mathbb{R}^4$ which is defined by the vector fields

$$\begin{cases} T_1 := X + A(X)\partial_w \\ T_2 := Y + A(Y)\partial_w \ , \end{cases} \tag{4.12}$$

where the sum refers to the canonical splitting $T(\mathbb{H}_3 \times \mathbb{R}) \simeq T\mathbb{H}_3 \oplus T\mathbb{R}$. In this way we can rewrite 4.11 as

$$\dot{\lambda}(t) = u_1(t)T_1(\lambda(t)) + u_2(t)T_2(\lambda(t)) \ , \tag{4.13}$$

where we simply renamed the controls $u_x = u_1$ and $u_y = u_2$. Finally we remark that the distribution $D = \mathrm{span}(\{T_1, T_2\}) \subset T\mathbb{R}^4$, can also be seen as the kernel of the Pfaffian equations

$$\begin{cases} \alpha = 0 \\ \mathrm{d}w - A = 0 \ . \end{cases} \tag{4.14}$$

Bracket generating condition. We want to study wherever D is bracket generating. We start from the first Lie bracket.

$$[T_1, T_2] = [X + A(X)\partial_w, Y + A(Y)\partial_w] = [X, Y] + (X(A(Y)) - Y(A(X)))\,\partial_w \ .$$

Using the Heisenberg algebra 2.4 (renaming $X_3 = \partial_z =: Z$) we get

$$[T_1, T_2] = Z + (X(A(Y)) - Y(A(X)))\,\partial_w \ ,$$

and using a shorthand notation for the coefficient of ∂_w we write

$$[T_1, T_2] = Z + \mathcal{B}\partial_w \ . \tag{4.15}$$

We call $T_3 := [T_1, T_2]$ and compute the next bracket $[T_1, T_3] =: T_4$. Using again 2.4, and 4.15 just found

$$[T_1, T_3] = [X + A(X)\partial_w, Z + \mathcal{B}\partial_w] = [X, Z] + (X(\mathcal{B}) - Z(A(X)))\,\partial_w = (X(\mathcal{B}) - Z(A(X)))\,\partial_w \ .$$

We summarize the four vector fields that we have found so far

$$\begin{cases} T_1 = X + A(X)\partial_w \\ T_2 = Y + A(Y)\partial_w \\ T_3 = Z + \mathcal{B}\partial_w \\ T_4 = (X(\mathcal{B}) - Z(A(X)))\,\partial_w \ . \end{cases} \tag{4.16}$$

From the explicit expressions 4.16 we see that this distribution has at least rank equal to three. However the step can be greater than 3. In fact, using the coordinate frame $\{\partial_x, \partial_y, \partial_z, \partial_w\}$ the condition for the T_i's to be independent comes from the equation

$$\det \begin{pmatrix} 1 & 0 & -\frac{y}{2} & A(X) \\ 0 & 1 & +\frac{x}{2} & A(Y) \\ 0 & 0 & 1 & \mathcal{B} \\ 0 & 0 & 0 & (X(\mathcal{B}) - Z(A(X))) \end{pmatrix} = (X(\mathcal{B}) - Z(A(X))) = \beta_x = 0 \ . \tag{4.17}$$

This condition is also gauge invariant, i.e. it does not depend on the choice of the potential A. We verify this immediately. Given $\tilde{A} := A + \mathrm{d}_\alpha g$ a different potential for β the horizontal vector fields change as

$$\begin{cases} T_1 := X + (A + \mathrm{d}_\alpha g)(X)\partial_w = X + (A_x + X(g))\partial_w \\ T_2 := Y + (A + \mathrm{d}_\alpha g)(Y)\partial_w = Y + (A_y + Y(g))\partial_w , \end{cases}$$

In this way we obtain

$$[T_1, T_2] = Z + (\mathcal{B} + Z(g))\partial_w \ .$$

The second order bracket is therefore

$$[T_1, [T_1, T_2]] = [X + (A_x + X(g))\partial_w, Z + (\mathcal{B} + Z(g))\partial_w] =$$

$$= (X(\mathcal{B}) - Z(A_x) - Z(X(g)) + X(Z(g)))\partial_w = (\beta_x + [X, Z](g))\partial_w = \beta_x \partial_w \ .$$

In this way the condition 4.17 for D to be bracket generating is again $\beta_x \neq 0$. Now, if $\beta_x = 0$ we shall consider instead of T_4, the vector field $T_5 := [T_2, T_3] = [T_2, [T_1, T_2]]$. In this case, repeating the computation 4.17 with T_5 in place of T_4 we obtain that $\{T_1, T_2, T_3, T_5\}$ are independent if and only if

$$Y(\mathcal{B}) - Z(A(Y)) = \beta_y \neq 0 \ .$$

We have hence proved the following.

Proposition. The magnetic 2-form $\beta = \mathcal{D}A \neq 0$, $A \in \Omega^1(\mathbb{H}_3)$, if and only if $D = \mathrm{span}(\{T_1, T_2\})$, given by 4.12, has step equal to three.

Remark. Similarly to the Heisenberg case, in which the Sub-Riemannian structure has step two as long as the magnetic 2-form $\beta \neq 0 \in \Lambda^2(\mathbb{R}^2)$, in the Engel case, to have step three, we obtain the condition $\beta \neq 0 \in \Omega^2(\mathbb{H}_3)$. In this sense we can say that the Rumin complex provides the appropriate description of the magnetic fields in a contact structure, in order to extend its relation with the step of the distribution of C_M^β.

Constant magnetic field. We are now interested in a particular class of magnetic fields, namely the ones of the type

$$\beta = b \ \mathrm{d}x \wedge \alpha \ , \tag{4.18}$$

with $b \in \mathbb{R}$ a constant[1]. We can easily verify that a candidate potential for such a field is given by the form

$$A = \frac{bx^2}{2}\mathrm{d}y \ . \tag{4.19}$$

In this case the vector fields 4.16 become

$$\begin{cases} T_1 = X \\ T_2 = Y + \frac{bx^2}{2}\partial_w \\ T_3 = Z + bx\partial_w \\ T_4 = b\partial_w \ . \end{cases} \tag{4.20}$$

From the previous proposition we know that if $b \neq 0$ the distribution $D = \mathrm{span}(\{T_1, T_2\}) \subset T\mathbb{R}^4$ is bracket generating, hence we can fill all $\mathbb{R}^4$ following admissible curves.

[1]Notice that these type of fields are constant multiples of one of the generators of $\Omega^2(U)$, and hence we shall call them 'constant'. The general linear combination with constant coefficients of the generators can be brought to the form 4.18 with a suitable rotation in the space of the coefficients.

The elements T_i's of the algebra 4.20 satisfy the following property, called the Engel property i.e. that for all $i, j = 1, 2, 3, 4$, there exist $k_{ij} \in \mathbb{N}$ such that

$$\underbrace{(\mathrm{ad}_{T_i} \circ \cdots \circ \mathrm{ad}_{T_i})}_{k_{ij}\text{-times}}(T_j) = \mathrm{ad}_{T_i}^{k_{ij}}(T_j) = 0 \ , \tag{4.21}$$

where $\mathrm{ad}_Y(\cdot) := [Y, \cdot]$ is another notation for the Lie brackets[2]. The Lie group associated with the Lie algebra 4.20 is therefore an Engel group that we denote as $\mathbb{E}_4$. It is also easy to see that $\mathbb{E}_4$ is also a Carnot group.
We finally notice that in this case the space $C^{\beta}_{\mathbb{H}_3}$, which is simply $\mathbb{R}^4$, cannot be endowed with a contact structure[3].

The group operation. Starting from the Engel algebra 4.20 we can recover the group multiplication law as we did in section 2.1 for the Heisenberg group. Computing all possible brackets we find

$$\begin{cases} [T_1, T_2] = T_3 \\ [T_1, T_3] = T_4 \\ [T_1, T_4] = [T_2, T_3] = [T_2, T_4] = [T_3, T_4] = 0 \ . \end{cases} \tag{4.22}$$

We shall exploit the surgectivity of the exponential map $\exp : \mathrm{Lie}(\mathbb{E}_4) \to \mathbb{E}_4$, i.e. we suppose every element $g \in \mathbb{E}_4$ can be written as exponential of a suitable element in the Lie algebra

$$g = \exp(T) = \exp(aT_1 + bT_2 + cT_3 + dT_4) \ ,$$

where a, b, c, d are the coordinates of $T \in \mathrm{Lie}(\mathbb{E}_4)$ with respect to the base given by the T_i's. For another element $\tilde{g} \in \mathbb{E}_4$ we shall write analogously

$$\tilde{g} = \exp(\tilde{T}) = \exp(\tilde{a}T_1 + \tilde{b}T_2 + \tilde{c}T_3 + \tilde{d}T_4) \ .$$

Now, if G is a k-dimensional Lie group, it can be shown that given a basis $\{e_i\}_{i=1,\cdots,k}$ of $\mathfrak{g} := \mathrm{Lie}(G)$, the correspondence $G \to \mathbb{R}^k$ given by $\exp(x^i e_i) \mapsto (x^1, \cdots, x^k)$ is a local diffeomorphism. Using these coordinates we can find the group operation using the Baker-Campbell-Hausdorff formula as follows. This formula says that in a (connected and simply connected) Lie group G we have for all $X, Y \in \mathfrak{g}$ the identity

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}\left([X, [X, Y] - [Y, [X, Y]]\right) + \cdots\right) \ ,$$

where the dots indicates higher brackets that we do not need since for the present algebra they all vanish. Consequently, to obtain the product $g\tilde{g}$ in coordinates, we shall first compute

$$\begin{aligned} &[aT_1 + bT_2 + cT_3 + dT_4, \tilde{a}T_1 + \tilde{b}T_2 + \tilde{c}T_3 + \tilde{d}T_4] = \\ &= [aT_1 + bT_2 + cT_3, \tilde{a}T_1 + \tilde{b}T_2 + \tilde{c}T_3] = \\ &= (a\tilde{b} - \tilde{a}b)[T_1, T_2] + (a\tilde{c} - \tilde{a}c)[T_1, T_3] = \mathcal{A}T_3 + \mathcal{B}T_4 \ , \end{aligned}$$

where we introduced a shorthand notation for the coefficients of T_3 and T_4. Next we consider the double brackets

$$[aT_1 + bT_2 + cT_3 + dT_4, \mathcal{A}T_3 + \mathcal{B}T_4] = [aT_1, \mathcal{A}T_3] = a\mathcal{A}T_4 \ ,$$

[2]The notation comes from the fact that $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}),\ Y \mapsto \mathrm{ad}_Y$, is the differential at the identity of the adjoint action $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g}),\ g \mapsto \mathrm{Ad}_g$. We recall that $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is the differential at the identity of the conjugation by $g \in G$, i.e. of the map $C_g : G \to G,\ h \mapsto R_{g^{-1}}(L_g h) = ghg^{-1}$.

[3]The non degeneracy condition on the exterior differential of the form $\mathrm{d}w - A$ cannot be satisfied when the dimension of the manifold is even.

and

$$[\tilde{a}T_1 + \tilde{b}T_2 + \tilde{c}T_3 + \tilde{d}T_4, \mathcal{A}T_3 + \mathcal{B}T_4] = [\tilde{a}T_1, \mathcal{A}T_3] = \tilde{a}\mathcal{A}T_4 \ .$$

In conclusion

$$g\tilde{g} = \exp(aT_1 + bT_2 + cT_3 + dT_4)\exp(\tilde{a}T_1 + \tilde{b}T_2 + \tilde{c}T_3 + \tilde{d}T_4) =$$

$$= \exp\left((a+\tilde{a})T_1 + (b+\tilde{b})T_2 + \left[(c+\tilde{c}) + \frac{\mathcal{A}}{2}\right]T_3 + \left[(d+\tilde{d}) + \frac{\mathcal{B}}{2} + \frac{\mathcal{A}(a-\tilde{a})}{12}\right]T_4\right) \ .$$

We have finally recovered the group multiplication law in $\mathbb{E}_4$ in local coordinates

$$(a,b,c,d)^T. \, (\tilde{a},\tilde{b},\tilde{c},\tilde{d})^T = \left(a+\tilde{a}, b+\tilde{b}, c+\tilde{c} + \frac{(a\tilde{b}-\tilde{a}b)}{2}, d+\tilde{d} + \frac{a\tilde{c}-\tilde{a}c}{2} + \frac{(a-\tilde{a})(a\tilde{b}-\tilde{a}b)}{12}\right)^T \ ,$$

for $(a,b,c,d)^T, (\tilde{a},\tilde{b},\tilde{c},\tilde{d})^T \in \mathbb{R}^4$.

4.3 Sub-Riemannian structure on $\mathbb{E}_4$

We can endow $\mathbb{E}_4 \simeq \mathbb{R}^4$, together with the distribution $D = \mathrm{span}(\{T_1, T_2\})$, with a Sub-Riemannian structure starting from the the one of $\mathbb{H}_3$ described in section 2.3, i.e. the one defined by the Sub-Riemannian metric 2.5. Let $\pi_w : \mathbb{R}^4 \simeq \mathbb{H}_3 \times \mathbb{R} \to \mathbb{H}_3$ be the canonical projection $(m, w) \mapsto \pi_w(m, w) = m$, and $\pi_z : \mathbb{H}_3 \to \mathbb{R}^2$, $\pi_2(x, y, z) \mapsto (x, y)$ the projection of section 2.3. We consider the composition $\pi := \pi_z \circ \pi_w$ and we define a Sub-Riemannian metric on D as

$$g_s|_\lambda(u, v) := (\pi_* u) \cdot (\pi_* v) \ , \tag{4.23}$$

with $\lambda \in \mathbb{R}^4$, $u, v \in D|_\lambda \subset T_\lambda\mathbb{R}^4$, and '$\cdot$' is again the Euclidean scalar product in the plane. In other words the Sub-Riemannian metric 4.23 just defined is 2.5 pulled back with π_w. As usual we denote the Sub-Riemannian manifold $\mathbb{R}^4$ with the distribution D and the Sub-Riemannian metric g_s with the triplet $(\mathbb{R}^4, D, g_s)$.

Normal length minimizers. We now study the length minimizers of $(\mathbb{R}^4, D, g_s)$ using the Hamiltonian formalism of section 1.5. The phase space is now $T^*\mathbb{R}^4 \simeq \mathbb{R}^8$, with $\bar{\pi} : \mathbb{R}^8 \to \mathbb{R}^4$ the canonical projection, and the Sub-Riemannian Hamiltonian is

$$H_s(\xi) = \frac{1}{2}\left(h_1^2(\xi) + h_2^2(\xi)\right) \ , \tag{4.24}$$

with $h_i(\xi) := \xi(\bar{\pi}_* T_i)$, $i = 1, 2, 3, 4$ and $\xi \in \mathbb{R}^8$, the fiber-linear functions that we use as coordinates in the fibers of the cotangent bundle. We compute the fiber part of the Hamilton equations using the properties of the Poisson brackets

$$\begin{cases} \dot{h}_1 = \{H_s, h_1\} = \{h_2, h_1\}h_2 \\ \dot{h}_2 = \{H_s, h_2\} = \{h_1, h_2\}h_1 \\ \dot{h}_3 = \{H_s, h_3\} = \{h_1, h_3\}h_1 + \{h_2, h_3\}h_2 \\ \dot{h}_4 = \{H_s, h_4\} = \{h_1, h_4\}h_1 + \{h_2, h_4\}h_2 \ . \end{cases} \tag{4.25}$$

Moreover we know from previous computations (see 1.42) that

$$\{h_i, h_j\}(\xi) = \xi([T_i, T_j]) \ ,$$

and from 4.22 we have

$$\begin{cases} \dot{h}_1 = -h_3 h_2 \\ \dot{h}_2 = h_3 h_1 \\ \dot{h}_3 = h_4 \\ \dot{h}_4 = 0 \ . \end{cases} \tag{4.26}$$

Again the Hamiltonian is constant along the solutions of the Hamilton equations. We can then restrict to the cylinder $h_1^2 + h_2^2 = 1$ and use the change of coordinates 1.45 while leaving h_4 unaltered. The system 4.26 reduces to

$$\begin{cases} \dot{\vartheta} = h_3 \\ \dot{h}_3 = h_4 \\ \dot{h}_4 = 0 \ . \end{cases} \tag{4.27}$$

We know that the Hamiltonian vector field of 4.24 project into $\mathbb{R}^4$ producing the dynamic $\dot{\lambda} = \bar{\pi}_* X_{H_s} = h^i X_i$. In our case $\dot{\lambda} = \cos(\vartheta) T_1 + \sin(\vartheta) T_2$. From the structure of T_1 and T_2 we can easily see that the projections of normal minimizers into the plane $\pi(\mathbb{R}^4) = \mathbb{R}^2$ (recall that $\pi(x, y, z, w) = (\pi_z \circ \pi_w)(x, y, z, w) = (x, y)$) are again arcs of circles, i.e. curves with constant principal curvature.

Centroid problem. From the variational viewpoint the problem of finding Sub-Riemannian length minimizers of $(\mathbb{R}^4, D, g_s)$ as above is equivalent to the problem of finding the Euclidean length minimizers in the plane $\pi(\mathbb{R}^4) \simeq \mathbb{R}^2$ with two additional constraints. Indeed by definition, the Sub-Riemannian length l_s of an admissible curve $\lambda : [0, 1] \to \mathbb{R}^4$ is

$$l_s(\lambda) = \int_0^1 \sqrt{g_s|_{\lambda(t)}(\dot{\lambda}(t), \dot{\lambda}(t))} \ \mathrm{d}t = \int_0^1 \sqrt{(\pi_* \dot{\lambda}(t)) \cdot (\pi_* \dot{\lambda}(t))} \ \mathrm{d}t \ . \tag{4.28}$$

Calling $\sigma := \pi(\lambda)$ we have

$$l_s(\lambda) = \int_0^1 \sqrt{\dot{\sigma} \cdot \dot{\sigma}} \ \mathrm{d}t = l(\sigma) \ , \tag{4.29}$$

where l is the Euclidean length of σ. Furthermore, since λ is admissible we also have that its projection $\pi_w(\lambda) =: \gamma$ satisfies the condition 4.10, i.e.

$$\int_0^1 A|_{\gamma(t)}(\dot{\gamma}(t)) \ \mathrm{d}t = w(1) = const.$$

In our case $A = \frac{bx^2}{2}\mathrm{d}y$, so that using 4.9, we find $A(\dot{\gamma}) = A(u_x X + u_y Y) = \frac{bx^2}{2} u_y$. Now, since $\pi_z(\gamma) = \sigma$ we have that $\dot{\sigma}_y = u_y$, and then the constraint on $\dot{\lambda}$ 4.10 is equivalent to the constraint on σ given by

$$\int_0^1 \frac{b\sigma_x^2}{2} \dot{\sigma}_y \ \mathrm{d}t = w(1) \ . \tag{4.30}$$

Moreover, we have already seen that the condition on γ to be admissible is equivalent to the constraint on σ given by 1.32 that we recall here

$$\frac{1}{2} \int_0^1 (\sigma_x(t) \dot{\sigma}_y(t) - \sigma_y(t) \dot{\sigma}_x(t)) \ \mathrm{d}t = \gamma(1) \ . \tag{4.31}$$

Apart from constant integrations along the segment $\sigma(0)\sigma(1)$ we can use Stokes theorem to describe such constraints as double integrals over the bounded subset Σ of the plane, whose boundary is given by σ and the segment $\sigma(0)\sigma(1)$. We finally get the conditions

$$\begin{cases} \int_\Sigma x \mathrm{d}x \wedge \mathrm{d}y = k_1 \\ \int_\Sigma \mathrm{d}x \wedge \mathrm{d}y = k_2 \ , \end{cases} \tag{4.32}$$

with $k_1,\ k_2$ real constants. Geometrically, the first condition says that we are fixing the product of the area of Σ times the coordinate of the centroid along x. But since by the second integral condition we are also fixing the area of Σ, we are restricting to curves that enclose a fixed area equal to k_2, and have the centroid lying on the line $x = \frac{k1}{k2}$.

Chapter 5

Abnormal curves

In this chapter we give a description of abnormal curves arising from the sub-Riemannian structures studied in the previous chapters. After some generalities about abnormals we concentrate on the case of distributions of growth vector $(2,3)$ and $(2,3,4)$, which are the ones that arises from the construction of C_M^β when we use a constant magnetic field. After that we take the magnetic field to be generic and describe the effect on abnormal curves.

5.1 Sub-Riemannian length minimizers

Let (M, D, g_s) be a sub-Riemannian manifold and $k \in \mathbb{N}$ the constant rank of D. The problem of finding a length minimizer between two points $P, Q \in M$, called a sub-Riemannian geodesic connecting P to Q, is expressed as the problem of finding the

$$\min\{l_s[\gamma] \ s.t. \ \gamma \in L^2([0,1], M),\ \gamma(0) = P,\ \gamma(1) = Q,\ \dot\gamma(t) \in D|_{\gamma(t)}\ \forall t \in [0,1]\}\ , \tag{5.1}$$

with

$$l_s[\gamma] := \int_0^1 \sqrt{g_s|_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\ \mathrm{d}t\ . \tag{5.2}$$

Clearly, since $\dot\gamma$ exists almost everywhere in $[0,1]$, where it does not, the condition $\dot\gamma(t) \in D|_{\gamma(t)}$ is considered to be satisfied. We suppose that both the points are contained in an open subset $U \subseteq M$ where there we choose an orthonormal frame for D, $\{X_i\}_{i=1,\cdots,k}$. In this way we can write $\dot\gamma$ using $u^i \in L^2([0,1], \mathbb{R})$, $i = 1, \cdots, k$, controls

$$\dot\gamma(t) = u^i(t) X_i(\gamma(t))\ . \tag{5.3}$$

The length functional then becomes a functional over the space of controls. Denoting comprehensively the controls as $u := (u^1, \cdots, u^k) \in L^2([0,1], \mathbb{R}^k)$, we have

$$l_s[\gamma] = \int_0^1 \sqrt{g_s|_{\gamma(t)}(u^i(t)X_i(\gamma(t)), u^j(t)X_j(\gamma(t)))}\ \mathrm{d}t = \int_0^1 \sqrt{u^i(t)u_i(t)}\ \mathrm{d}t =: \tilde l_s[u]\ . \tag{5.4}$$

Notice that since the constraint on γ and l_s are parametrization independent, (by Hölder inequality) the length minimizers are energy minimizers, i.e. they minimize the energy functional

$$\mathcal{E}_s[\gamma] := \frac{1}{2}\int_0^1 g_s|_{\gamma(t)}\left(\dot\gamma(t), \dot\gamma(t)\right)\ \mathrm{d}t = \frac{1}{2}\int_0^1 u^i(t)u_i(t)\ \mathrm{d}t =: \tilde{\mathcal{E}}_s[u]\ . \tag{5.5}$$

We can readily compute the differential at u of the energy functional

$$(\mathrm{d}\mathcal{E}_s)|_u(v) = \int_0^1 u_i(t)v^i(t)\ \mathrm{d}t\ , \tag{5.6}$$

for all variation (tangent vector) $v \in T_u L^2([0,1], \mathbb{R}^k) \simeq L^2([0,1], \mathbb{R}^k)$.

Endpoint map. We now want to write explicitly the constraint on γ as a constraint on the controls. If $u \in L^2([0,1],\mathbb{R}^k)$ is a control we denote as $\gamma_u \in L^2([0,1],M)$ the solution of $\dot{\gamma} = u^i X_i$ starting from $P \in U$.

Definition. The t-point map based on $P \in U$ is the map $E_P^t : L^2([0,1],\mathbb{R}^k) \to M$, $u \mapsto \gamma_u(t)$, with $t \in [0,1]$. In particular we call the endpoint map based on P, the 1-point map and denote it simply as $E_P := E_P^1$.

Differential of the Endpoint map. The following proposition tells us how to compute the differential of the endpoint map.

Proposition. Let $E_P : L^2([0,1],\mathbb{R}^k) \to M$ be the endpoint map relative to the distribution $D = \mathrm{span}(\{X_1, \cdots, X_k\})$ then its differential at u applied to the 'tangent vector' v at u is equal to

$$(\mathrm{d}E_P)|_u(v) = \int_0^1 \left((\Phi_{t,1}^u)_* X_t^v\right)|_{\gamma_u(1)} \, \mathrm{d}t \ , \tag{5.7}$$

where $\Phi_{t,1}^u$ is the flow of $X^u = u^i X_i$ and $X^v = v^i X_i$.

Proof. See [ABB19], sub-section 8.1.1. □

Lagrange multipliers. We state a fairly general theorem about critical points. Let $F : \Omega \to M$ and $f : \Omega \to \mathbb{R}$ be differentiable functions, Ω and M being manifolds (possibly infinite dimensional).

Theorem. Given $m \in M$, if u is a minimum (or a maximum) for the function $f|_{F^{-1}(m)}$, then there exists a nonzero $(\lambda, \nu) \in T_m^* M \times \mathbb{R}$ such that

$$\langle \lambda | (\mathrm{d}F)|_u(v) \rangle + \nu (\mathrm{d}f)|_u(v) = 0 \ , \tag{5.8}$$

for all $v \in T_u \Omega$.

We distinguish between two different situations.

Definition. We call u a minimum of $f|_{F^{-1}(m)}$ normal iff $\mathrm{rank}\,((\mathrm{d}F)|_u) = \dim(M)$ and $(\mathrm{d}f)|_u \neq 0$. We call a minimum (strictly) abnormal iff $\mathrm{rank}\,((\mathrm{d}F)|_u) < \dim(M)$ and $(\mathrm{d}f)|_u \neq 0$.

The normal case correspond to the usual Lagrange multipliers rule in Analysis. In that case there is a co-vector (λ, ν), with $\nu \neq 0$, such that 5.8 is satisfied. In the abnormal case instead the solution to 5.8 is given by $(\lambda_i, 0)$, $i = 1, \cdots, \mathrm{corank}\,((\mathrm{d}F)|_u) = r$, such that $\mathrm{span}(\{\lambda_1, \cdots, \lambda_r\}) \simeq \mathrm{coker}\,((\mathrm{d}F)|_u)$. We can now prove the following.

Theorem. If γ_u is a sub-Riemannian length minimizer of (M, D, g_s), connecting $\gamma_u(0) = P$ and $\gamma_u(1) = Q$, then one of the following relations holds ((N) stands for normal minimizer and (A) for abnormal minimizer)

$$\text{(N)} \quad h_i(t) := \langle \lambda_u(t) | X_i(\gamma_u(t)) \rangle = u_i(t) \ , \tag{5.9}$$

$$\text{(A)} \quad h_i(t) \equiv 0 \ , \tag{5.10}$$

where $\lambda_u(t) \in T^*_{\gamma_u(t)} M$ for all $t \in [0,1]$.

Proof. We only need to consider the Lagrange multipliers rule 5.8 (i.e. the constrained Euler-Lagrange equations) for $\mathcal{E}_s$ and E_P, which thanks to 5.6 and 5.7, give us for $\lambda_Q \in T^*_Q M$, $\nu \in \mathbb{R}$

$$\langle \lambda_Q | (\Phi^u_{t,1})_* X_i)|_Q \rangle = -\nu u_i(t) \ , \tag{5.11}$$

or equivalently,

$$\langle (\mathrm{d}\Phi^u_{t,1})|^{-T}_{\gamma_u(t)} \lambda_Q | X_i(\gamma_u(t)) \rangle = -\nu u_i(t) \ . \tag{5.12}$$

Setting $\lambda_u(t) := (\mathrm{d}\Phi^u_{t,1})|^{-T}_{\gamma_u(t)} \lambda_Q$, in the normal case we can choose $\nu = -1$ and get the desired result (N). In the abnormal case we have $\nu = 0$, hence also (A) easily follows. Moreover, observe that the transposed differential $(\mathrm{d}\Phi^u_{t,1})|^{-T}$ defines a flow in T^*M, $(\lambda, t) \mapsto (\mathrm{d}\Phi^u_{t,1})|^{-T}_{\pi(\lambda)} \lambda$, which corresponds to the (time dependent) vector field $u^i X_{h_i}$, i.e. the cotangent lift of the (time dependent) vector field $u^i X_i$. □

Normal minimizers. The following theorem tells us that normal minimizers are projection of an Hamiltonian system on T^*M, $\pi : T^*M \to M$.

Theorem. (Normal extremals.) Let $\gamma_u : [0,1] \to M$ be a normal length minimizer of (M, D, g_s), parametrized with constant speed. Then there exists a curve $\lambda_u : [0,1] \to T^*M$, such that $\pi(\lambda_u) = \gamma_u$ and

$$\dot{\lambda}_u = X_{H_s} \ ,$$

with H_s the Sub-Riemannian Hamiltonian of g_s, which is given by .

$$H_s(\lambda) := \frac{1}{2} g_s|_{\pi(\lambda)} \left(\sum_{j=1}^k h_j(\lambda) X_j(\pi(\lambda)), \sum_{j=1}^k h_j(\lambda) X_j(\pi(\lambda)) \right) \ , \tag{5.13}$$

having $h_i(\lambda) = \langle \lambda | X_i(\pi(\lambda)) \rangle$ for all $\lambda \in T^*M$.

Proof. We have already shown that the cotangent lift the flow of a vector field $X_i \in \mathfrak{X}(M)$ is the flow of the Hamiltonian vector field associated with $h_i(\lambda) = \langle \lambda | X_i(\pi(\lambda)) \rangle$. This means that, if $\dot{\gamma}(t)_u = u^i(t) X_i(\gamma(t))$, then $\dot{\lambda}_u(t) = u^i(t) X_{h_i}(\lambda(t))$, but since γ_u is a normal minimizer, $u^i(t) = h_i(\lambda_u(t))$, hence $\dot{\lambda}_u = h^i X_{h_i}$. On the other hand, since the X_i's are orthonormal $H = \frac{1}{2} h^i h_i$ and the Hamilton equations become $\dot{\lambda} = X_H = h^i X_{h_i}$. □

5.1.1 Step of a distribution

We always suppose the distribution $D \subset TM$ to be bracket generating at all points of M. This means that taking a sufficiently high number N of brackets we obtain enough linearly independent vector fields to span TM. Given $\mathcal{F} := \{X_1, \cdots, X_k\} \subset \mathfrak{X}(M)$, $k \geq \mathrm{rank}(D)$, a family of vector fields describing D, we define the following subspace of $T_m M$

$$\mathrm{Lie}^N_m(\mathcal{F}) := \mathrm{span}\left(\{ [X_{i_1}, \cdots, [X_{i_{n-1}}, X_{i_n}]](m) \ : \ m \in M, \ X_{i_j} \in \mathcal{F}, \ n \leq N \} \right) \ , \tag{5.14}$$

and call the step of D at $m \in M$ the minimum integer $\mathfrak{s}(m)$ such that $\mathrm{Lie}^{\mathfrak{s}(m)}_m(\mathcal{F}) = T_m M$. By convention we set $\mathrm{Lie}^1_m(\mathcal{F}) := D|_m$. Clearly the step can vary with m. It is also useful to record the progression in the dimension of $\mathrm{Lie}^N_m(\mathcal{F})$ when N varies and fixing m. We define the growth vector at $m \in M$ as the vector $(k_1(m), k_2(m), \cdots, k_{\mathfrak{s}(m)})$, where $k_j := \dim(\mathrm{Lie}^j_m(\mathcal{F}))$. This means that $k_1 = \mathrm{rank}(D)$ and $k_{\mathfrak{s}(m)} = \dim(M)$.

Proposition. The integers k_j do not depend on the family $\mathcal{F} = \{X_1, \cdots, X_k\} \subset \mathfrak{X}(M)$ describing D.

Proof. Locally, in an open $U \subset M$, $k = \mathrm{rank}(D)$. Start by considering k_2. We have $[X_i, X_j] = c_{ij}^h X_h + s_{ij}^h Y_h$, where $Y_h \in \mathfrak{X}(M)$, $h = 1, \cdots, \mathrm{corank}(D)$, complete $\mathcal{F}$ to a frame for TM. Let $\{\tilde{X}_1, \cdots, \tilde{X}_k\}$ be another frame describing D. Then we have

$$\tilde{X}_i(m) = a_i^j(m) X_j(m) \ ,$$

with $a_i^j \in \mathcal{C}^\infty(M)$ giving the coefficients of a linear invertible map $a(m) \in \mathrm{GL}_k(\mathbb{R})$ at every $m \in M$. Computing the bracket using the new fields we get

$$[\tilde{X}_i, \tilde{X}_j] = [a_i^l X_l, a_j^h X_h] = a_i^l a_j^h [X_l, X_h] \ \mathrm{mod}(D) = a_i^l a_j^h s_{lh}^m Y_m \ \mathrm{mod}(D) = \tilde{s}_{ij}^m Y_m \ \mathrm{mod}(D) \ .$$

Now, since a is invertible, $\tilde{s}_{ij}^m = 0$ if and only if $s_{lh}^m = 0$. This means that there exists l and h such that $s_{lh}^m \neq 0$, i.e. $k_2 = k + 1$ if and only if there exist i and j such that $\tilde{s}_{ij}^m \neq 0$, i.e. $\tilde{k}_2 = k + 1$. We conclude that $k_2 = \tilde{k}_2$. The equivalence of higher order brackets conditions is similar. □

The previous proposition tells us that $\mathrm{Lie}_m^N(\mathcal{F})$ does not depend on the set $\mathcal{F} = \{X_1, \cdots, X_k\}$, and that is equivalent to the following geometric definition. Let $\mathfrak{X}_D(M)$ be the set of sections of TM taking values in D. Then we could set

$$\mathrm{Lie}_m^N(D) := \mathrm{span}\left([X_{i_1}, \cdots, [X_{i_{n-1}}, X_{i_n}]](m) \ : \ m \in M, \ X_{i_j} \in \mathfrak{X}_D(M), \ n \leq N\}\right) \ , \tag{5.15}$$

which we shorten as $\mathrm{Lie}_m^N(D) := \left(\mathrm{ad}_D^{1 \leq n \leq N} D\right)\Big|_m$, without mentioning any frame.

5.2 Abnormal curves

It is easy to show that the zero level set of the sub-Riemannian Hamiltonian $H_s^{-1}(0)$ is a $(2n-k)$-dimensional submanifold of T^*M which can be seen as the sub-bundle given by the annihilator $D^\perp \subset T^*M$ of the distribution $D \subset TM$

$$D^\perp := \bigsqcup_{m \in M} D^\perp|_m \ , \tag{5.16}$$

with

$$D^\perp|_m := \{p \in T_m^*M \ : \ \langle p | X_i(m) \rangle, \ i = 1, \cdots, k\} \ . \tag{5.17}$$

In other words, if the distribution is described by the Pfaffian equations

$$\begin{cases} \alpha^1 = 0 \\ \vdots \\ \alpha^{n-k} = 0 \ , \end{cases} \tag{5.18}$$

with $\alpha^j \in \Lambda^1(M)$, $j = 1, \cdots, n-k$, we have

$$D^\perp = \mathrm{span}(\{\alpha^1, \cdots, \alpha^{n-k}\}) \ . \tag{5.19}$$

The structure of $D^\perp$ is made clear by expressing the symplectic form in terms of the frame $\{\mu^1, \cdots, \mu^k\}$ dual to $\{X_1, \cdots, X_k\}$ and $\{\alpha_1, \cdots, \alpha^{n-k}\}$ dual to $\{Y_1, \cdots, Y_{n-k}\}$. Again denoting as h_i the coordinates induced by the X_i's and k_j the ones induced by the Y_j's from section 3.1 we have

$$\sigma = \mathrm{d}h_i \wedge \mu^i + h_i \mathrm{d}\mu^i + \mathrm{d}k_j \wedge \alpha^j + k_j \mathrm{d}\alpha^j \ , \tag{5.20}$$

hence

$$\sigma|_{D^\perp} = \mathrm{d}k_j \wedge \alpha^j + k_j \mathrm{d}\alpha^j \ . \tag{5.21}$$

From condition 5.10, abnormal curves belong to the subset $D^\perp \subset T^*M$. We can further characterize them as follows.

Theorem. Let $\gamma_u : [0,1] \to M$ be an abnormal curve of (M, D, g_s). Then there exists a Lipschitz curve $\lambda_u : [0,1] \to D^\perp \subseteq T^*M$ such that $\pi(\lambda_u) = \gamma_u$ and

$$\dot{\lambda}_u \in \ker(\sigma|_{D^\perp}) \tag{5.22}$$

Proof. Given an abnormal γ_u with u Lipschitz controls, the previous theorem tells us that there exits a Lipschitz curve $\lambda_u : [0,1] \to D^\perp$ solution of

$$\dot{\lambda}_u = u^i(t) X_{h_i}(\lambda_u(t)) \ .$$

The tangent space to $D^\perp$ at $\lambda \in D^\perp$ is given by

$$T_\lambda D^\perp = \bigcap_{i=1}^{k} \ker\left((\mathrm{d}h_i)|_\lambda\right) = \bigcap_{i=1}^{k} \ker\left((i_{X_{h_i}}\sigma)|_\lambda\right) = \bigcap_{i=1}^{k} \left(\mathrm{span}(X_{h_i}(\lambda))\right)^\S \ ,$$

where the superscript '§' denotes the orthogonal with respect to σ. Consequently we obtain

$$\ker(\sigma|_{D^\perp}) := (TD^\perp)^\S = \left(\bigcap_{i=1}^{k} \left(\mathrm{span}(X_{h_i}(\lambda))\right)^\S\right)^\S = \mathrm{span}(\{X_{h_1}, \cdots, X_{h_k}\}) \ .$$

This allows us to conclude that $\dot{\lambda}_u \in \ker(\sigma|_{D^\perp})$.

□

The previous theorem states that abnormal curves[1] live in the set

$$\mathrm{Char}(D) := \{\lambda \in D^\perp \ : \ \ker\left((\sigma|_{D^\perp})|_\lambda\right) \neq \{0\}\} \ . \tag{5.23}$$

We also observe that being λ an abnormal, from $h_i(\lambda(t)) = 0$ for all $t \in [0,1]$ and $i = 1, \cdots, k$, we obtain

$$0 = \frac{d}{dt} h_i(\lambda(t)) = (\mathrm{d}h_i)|_{\lambda(t)}(\dot{\lambda}(t)) = u^j(t)(\mathrm{d}h_i)|_{\lambda(t)}(X_{h_j}(\lambda(t)) = u^i(t)\{h_j, h_i\}(\lambda(t)) \ . \tag{5.24}$$

Hence defining the map $\mathcal{H} : D^\perp \to \mathrm{Skew}(k, \mathbb{R})$

$$\mathcal{H}_{ij}(\lambda) := \{h_i, h_j\}(\lambda), \tag{5.25}$$

we get that

$$\mathrm{Char}(D) = \{\lambda \in D^\perp \ : \ \det(\mathcal{H}(\lambda)) = 0\} \ . \tag{5.26}$$

We immediately notice that if the rank of the distribution is odd, then $\mathrm{Char}(D) = D^\perp$. In the even case instead, we can write the determinant of $\mathcal{H}$ as the square of its Pfaffian $\mathrm{Pf}(\mathcal{H})$. In this way $\mathrm{Char}(D) = D^\perp \cap \left(\mathrm{Pf}(\mathcal{H})^{-1}\right)(0)$, which whenever $\mathrm{d}(\mathrm{Pf}(\mathcal{H}))|_\lambda$ is nonzero, is a $(2n-k-1)$-dimensional sub-manifold of T^*M.

[1] We should call λ (abnormal) and $\gamma = \pi(\lambda)$ (abnormal curve or abnormal trajectory) differently, however since their meaning is clear we use the same name for both the objects.

Abnormals and corank-1 distributions. Let $D \subset TM$ be a rank-$(n-1)$ distribution. Suppose that $S_2 \subseteq M$ is the set in which the distribution have step greater than two (in other words the points in which the distribution is involutive), i.e.

$$S_2 := \{m \in M \ : \ D^2|_m \subset T_m M\} \ . \tag{5.27}$$

If $D = \mathrm{span}\left(\{X_1, \cdots, X_{n-1}\}\right)$, we can find a frame of TM by adding a suitable vector field $Y \in \mathfrak{X}(M)$. Thanks to the proposition of section 1.1 we can describe S_2 as the zero level set of $\frac{(n-1)(n-2)}{2}$ functions s_{ij}. Indeed if

$$D^2 = \mathrm{span}(\{X_1, \cdots, X_{n-1}, s_{12}Y, \cdots, s_{(n-2)(n-1)}Y\}) \ , \tag{5.28}$$

where the functions s_{ij} are defined by the Lie brackets

$$[X_i, X_j] = c_{ij}^k X_k + s_{ij} Y \ , \tag{5.29}$$

with $1 \le i < j \le n-1$ and $1 \le k \le n-1$, we can write S_2 as

$$S_2 = \{m \in M \ : \ s_{ij}(m) = 0, \ 1 \le i < j \le n-1\} \ . \tag{5.30}$$

Computing the next order brackets we get

$$[X_k, [X_i, X_j]] = [X_k, c_{ij}^l X_l + s_{ij} Y] = X_k(s_{ij}) Y \ \mathrm{mod}(D) \ . \tag{5.31}$$

This allows us to give a geometric description of the points in which the step is greater than three. Indeed setting

$$S_3 := \{m \in M \ : \ D^3|_m \subset T_m M\} \ , \tag{5.32}$$

we have that

$$S_3 = \{m \in M \ : \ (\mathrm{d}s_{ij})|_m (X_k(m)) = 0, \ 1 \le i < j \le n-1, \ 1 \le k \le n-1\} \ , \tag{5.33}$$

in other words

$$S_3 = \{m \in M \ : \ D|_m \subseteq (\mathrm{d}s_{ij})|_m, \ 1 \le i < j \le n-1\} \ . \tag{5.34}$$

Since s_{ij} are smooth functions, apart from a discrete set of points, S_2 is a smooth sub-manifold of co-dimension $\frac{(n-1)(n-2)}{2}$, and

$$T_m S_2 = \bigcap_{1 \le i < j \le n-1} \ker\left((\mathrm{d}s_{ij})|_m\right) \ ,$$

so S_3 is the set of points of M in which D is tangent to S_2. We also see from 5.31 that in general S_3 have co-dimension $\frac{(n-1)^2(n-2)}{2}$, so the co-dimension of $S_2 \cap S_3$ is $\frac{n(n-1)(n-2)}{2}$.
In this case $\mathcal{H}_{ij} = \{h_i, h_j\}|_{D^\perp} = (c_{ij}^k h_k + s_{ij} h_y)|_{D^\perp} = s_{ij} h_y$, where $h_y(\lambda) := \langle \lambda | Y(\pi(\lambda)) \rangle$. From this it follows that admissible curves inside S_2 are abnormals. Moreover it is rather clear that the converse is not always true. More precisely, curves in S_2 belong to the co-isotropic part of $\mathrm{Char}(D)$.

Remark. This type of distribution is the one arising from the construction of C_M^β when M is a n-dimensional Riemannian manifold. The role of the functions s_{ij} is played by the components of the magnetic field β_{ij}. A naive dimensional counting tells us that, since the magnetic field does not depend on the vertical coordinate of C_M^β, the co-dimension of S_2 is equal to $\frac{n(n-1)}{2}$, so the only non trivial dimensions of the manifold are $n = 2, 3$. For a magnetic field in a Riemannian surface we obtain a surface in C_M^β, for a magnetic field in a three dimensional Riemannian manifold we obtain a line in C_M^β. In the first case the distribution is generically transversal to the surface, in the second the distribution is skew with respect to the lines.

Contact distributions. Suppose now that the sub-Riemannian structure is of contact type. This means that $D = \ker(\alpha)$, and $\alpha \in \Lambda^1(M)$ the contact form, is such that $(\mathrm{d}\alpha)|_D$ is non degenerate. From a remarkable formula we also get that

$$\mathrm{d}\alpha(X_i, X_j) = X_i(\langle\alpha|X_j\rangle) - X_j(\langle\alpha|X_i\rangle) - \langle\alpha|[X_i, X_j]\rangle = -s_{ij}\langle\alpha|Y\rangle = -s_{ij} f h_y = f\mathcal{H}_{ji} \ , \quad (5.35)$$

for a non zero $f \in \mathcal{C}^\infty(M)$. In other words $-(\mathrm{d}\alpha)|_D = f\mathcal{H}$. But then $\det(\mathcal{H}) \neq 0$, so $\mathrm{Char}(D) = \emptyset$, and there cannot be abnormals. We summarize what just found into a theorem.

Theorem. Let (M, D, g_s) be a sub-Riemannian structure of contact type. Then there are no abnormal curves.

5.3 Abnormals of rank-2 distributions.

We now address the case in which we are given a rank-2 distribution, that is locally described by two vector fields $X_{1,2} \in \mathfrak{X}(M)$. It will be useful to define a particular type of abnormal extremals.

Definition. An abnormal $\lambda : [0,1] \to D^\perp$ is called a nice abnormal iff

$$\lambda(t) \in \left(D^2|_{\pi(\lambda(t))}\right)^\perp \setminus \left(D^3|_{\pi(\lambda(t))}\right)^\perp \quad \forall t \in [0,1] \ . \quad (5.36)$$

Proposition. Along a nice abnormal of a rank-2 sub-Riemannian structure we have that $h_{12} := \{h_1, h_2\} \equiv 0$ (Goh condition) and $(h_{112}\lambda(t), h_{221}(\lambda(t)) \neq 0$, for all $t \in [0,1]$, where $h_{112} := \{h_1, \{h_1, h_2\}\}$ and $h_{221} := \{h_2, \{h_2, h_1\}\}$.

Proof. For the first part, since along an abnormal λ, for $i = 1, 2$, $h_i \equiv 0$, we have that

$$\frac{d}{dt} h_i(\lambda) = \langle \mathrm{d}h_i | u^j X_{h_j} \rangle = u^j\{h_i, h_j\} = \mathcal{H}_{ji} u^j \equiv 0 \ .$$

Since $u \neq 0$, it must be $\det(\mathcal{H}) = h_{12}^2 = 0$, i.e. $h_{12} = 0$.
Let now $h_{112}(\lambda(t)) = 0$ for some $t \in [0,1]$. But then $0 = \langle\lambda(t)|[X_1, [X_1, X_2]](\pi(\lambda(t)))\rangle$. The same computation goes for h_{221}, and hence at least one of the two quantities must be nonzero, otherwise $\dot\lambda(t) \in (D^3|_{\pi(\lambda(t))})^\perp$. □

The following theorem tells us that nice abnormals are (reparametrizations of) solutions of an Hamiltonian system, hence are regular curves.

Theorem. Let $\lambda : [0,1] \to D^\perp$ be an abnormal of a rank-2 sub-Riemannian structure. Then λ is nice if and only if it is a reperametrization of a solution of the Hamiltonian system

$$\dot\lambda(t) = X_H(\lambda(t)) \ , \quad (5.37)$$

with initial datum $\lambda(0) \in \left(D^2|_{\pi(\lambda(0))}\right)^\perp \setminus \left(D^3|_{\pi(\lambda(0))}\right)^\perp$, and where $H = \{h_2, \{h_2, h_1\}\}h_1 + \{h_1, \{h_1, h_2\}\}h_2 = h_{221}h_1 + h_{112}h_2$.

Proof. See [ABB19], theorem 12.30 page 425. □

5.3.1 Rank-2 distribution in dimension 3.

Let $[X_1, X_2] = sY \mod(D)$, so that $S_2 = \{m \in M \ : \ s(m) = 0\}$. Given λ an abnormal curve, the condition $\{h_1, h_2\}(\lambda) \equiv 0$ leads to

$$\{h_1, h_2\}(\lambda) = \langle \lambda | [X_1, X_2](\pi(\lambda)) \rangle = \langle \lambda | s(\pi(\lambda)) Y(\pi(\lambda)) \rangle = s(\pi(\lambda)) h_3(\lambda) \ , \tag{5.38}$$

where h_3 is the dual coordinate of Y. Since h_3 cannot be zero, it must be $s(\pi(\lambda)) = 0$, hence all abnormals are contained in $S_2 \subset M$, which is called the Martinet set.

Consider the next brackets $[X_k, [X_1, X_2]] = X_k(s)Y \mod(D)$. Always from the condition $\{h_1, h_2\} \equiv 0$ we get

$$\frac{d}{dt} h_{12}(\lambda) = \langle \mathrm{d}h_{12} | \dot{\lambda} \rangle = u^k \{h_k, \{h_1, h_2\}\}(\lambda) = u^k h_{k12}(\lambda) \equiv 0 \ , \tag{5.39}$$

but we have that

$$h_{k12}(\lambda) = \langle \lambda | [X_k, [X_1, X_2]] \rangle = \langle \mathrm{d}s | X_k \rangle (\pi(\lambda)) h_3(\lambda) \ .$$

In conclusion, since $h_3(\lambda) \neq 0$, we have necessarily that

$$\langle \mathrm{d}s | u^k X_k \rangle (\pi(\lambda)) \equiv 0 \ , \tag{5.40}$$

in other words $\dot{\gamma} = \pi_* \dot{\lambda} = u^k X_k \in TS_2$. Condition 5.39 allows us to find the right control by setting

$$\begin{cases} u^1(t) = h_{221}(\lambda(t)) \\ u^2(t) = h_{112}(\lambda(t)) \ . \end{cases} \tag{5.41}$$

On the other hand the Hamiltonian vector field 5.37, restricted to $D^\perp$ is

$$(X_H)|_{D^\perp} = h_{112} X_{h_2} + h_{221} X_{h_1} \ . \tag{5.42}$$

This means that the projection $\gamma = \pi(\lambda)$ on M satisfies

$$\dot{\gamma} = h_{112} X_2 + h_{221} X_1 =: V \ , \tag{5.43}$$

which exactly the vector field that we just found as the intersection $D \cap TS_2$. Observe also that if $\gamma(t) \in S_3$ we have $h_{112} = h_{221} = 0$, hence there would be an equilibrium.
Furthermore we can see that V has zero divergence[2] at points of S_3. Consider $\{\mu^1, \mu^2, \mu^3 := \alpha\} \subset \Lambda^1(M)$ the dual basis of $\{X_1, X_2, X_3 := Y\} \subset \mathfrak{X}(M)$. Since $\mathcal{L}_{X_i} \mu^j = i_{X_i} \mathrm{d}\mu^j$, we see that if $[X_i, X_j] = c_{ij}^k X_k$ then $\mathrm{d}\mu^k = c_{ij}^k \mu^i \wedge \mu^j$. Moreover, since $\mathrm{d}\alpha = b \mu^1 \wedge \mu^2$ for some function $b \in \mathcal{C}^\infty(M)$, we have

$$\mathcal{L}_V(\mu^1 \wedge \mu^2 \wedge \mu^3) = (\mathcal{L}_V \mu^1) \wedge \mu^2 \wedge \mu^3 + \mu^1 \wedge (\mathcal{L}_V \mu^2) \wedge \mu^3 = c_{12}^1 (h_{221} - h_{112})(\mu^1 \wedge \mu^2 \wedge \mu^3)$$

This means that $\mathrm{div}(V) = c_{12}^1(h_{221} - h_{112})$, so in conclusion $(\mathrm{div}(V))|_{S_3} = 0$. The zero divergence condition implies that the trace of the linearization of the system $\dot{\gamma} = V$ around the equilibria is zero. This means that the eigenvalues are either real (hyperbolic equilibrium) or purely imaginary (a center). The non linear contributions then turns centers into focuses, but leave the hyperbolic equilibria qualitatively the same (Grobman-Hartman theorem).

This results are applicable to the sub-Riemannian structure of C_M^β made in Chapter 1, where M was a Riemannian surface and β a magnetic field in it. The Martinet set here corresponds to the zero locus of the magnetic field $\{(m, z) \in C_M^\beta \ : \ \beta(m) = 0\}$. If $b \in \mathcal{C}^\infty(M)$ denotes the component of the magnetic field the nice abnormal extremal contained in the regular part of the Martinet set is solution of the system

$$\dot{\gamma} = X_1(b) X_2 - X_2(b) X_1 \ .$$

[2]Given ω volume form for M the divergence $\mathrm{div}(X) \in \mathcal{C}^\infty(M)$ of a vector field X is defined by the relation $\mathcal{L}_X \omega = (\mathrm{div}(X))\omega$.

5.3.2 Rank-2 distribution in dimension 4.

Let $D = \mathrm{span}(\{X_1, X_2\}) \subset TM$. We can complete $\{X_1, X_2\}$ to a frame for TM by adding two independent vector fields Y_1 and Y_2. The first bracket will be

$$[X_1, X_2] = s_1 Y_1 + s_2 Y_2 \quad \mathrm{mod}(D) \ ,$$

where $s_{1,2} \in \mathcal{C}^\infty(M)$. We denote with $k_{1,2}$ the coordinates induced on the fibers of T^*M by $Y_{1,2}$, i.e. $k_i(\lambda) := \langle \lambda | Y_i(\pi(\lambda)) \rangle$, $i = 1, 2$. Consequently we expect that the points m in which $(D^2)|_m = D|_m$ belong to a sub-manifold of dimension two.

To simplify the notation we write $[X_i, X_j] = c_{ij}^k X_k + s_{ij}^k Y_k$, with $i, j, k = 1, 2$. The next brackets are given by

$$[X_k, [X_i, X_j]] = [X_k, c_{ij}^h X_h + s_{ij}^h Y_h] = \left(X_k(s_{ij}^l) + s_{ij}^h r_{kh}^l \right) Y_l \quad \mathrm{mod}(D) \ , \tag{5.44}$$

where $r_{kh}^l \in \mathcal{C}^\infty(M)$ are defined by $[X_k, Y_h] = r_{kh}^l Y_l$ mod(D).

Growth vector (2, 3, ..., 3, 4). We consider first the equiregular case, in which the growth vector is forced to be $(2, 3, 4)$. This means that $Y(m) := s_1(m) Y_1(m) + s_2(m) Y_2(m) \neq 0$ for all $m \in M$. Up to a change of frame we can always set $Y_1 := Y$ and then we can complete $\{X_1, X_2, Y\}$ to a frame of TM by adding an independent vector field $Z \in \mathfrak{X}(M)$. The second order brackets become

$$[X_k, [X_1, X_2]] = [X_k, Y \ \mathrm{mod}(D)] = r_k Z \quad \mathrm{mod}(D^2) \ , \tag{5.45}$$

where $r_{1,2} \in \mathcal{C}^\infty(M)$. As a consequence, the set of points $m \in M$ at which $(D^3)|_m \subseteq (D^2)|_m$ is again given by a sub-manifold of dimension two. This is exactly the case encountered in Chapter 4 when we introduced a magnetic field in the Heisenberg group. In particular the functions $r_{1,2}$ correspond to the components of the magnetic field $b_{x,y}$. When the magnetic field is a non zero constant, we then obtain the Engel distribution, and the growth vector is $(2, 3, 4)$.

Consider an abnormal curve $\lambda : [0, 1] \to D^\perp$, having $\dot{\lambda} = u^i X_{h_i}$, $i = 1, 2$. The Goh conditions tell us that if $[X_1, X_2] = Y$ mod(D), we have

$$0 \equiv h_{12}(\lambda) = \langle \lambda | [X_1, X_2](\pi(\lambda)) \rangle = \langle \lambda | Y(\pi(\lambda)) \rangle = k_1(\lambda) \ , \tag{5.46}$$

where k_1 is the dual coordinate induced on the fibers of T^*M by Y. Contrary to the co-rank one distribution case, here $k_1(\lambda) \equiv 0$ is a non trivial solution since $k_2(\lambda) := \langle \lambda | Z(\pi(\lambda)) \rangle$ is free. In order to find the controls of the abnormal curve we observe that

$$0 \equiv \dot{k}_1(\lambda) = \langle \mathrm{d}k_1 | u^i X_{h_i} \rangle = u^i \{k_1, h_i\}(\lambda) \ . \tag{5.47}$$

Now, form 5.45 we obtain

$$\{k_1, h_i\}(\lambda) = \langle \lambda | [[X_1, X_2], X_i] \rangle = -r_i(\pi(\lambda)) \langle \lambda | Z \rangle =: -r_i(\pi(\lambda)) k_2(\lambda) \ ,$$

and since $k_2 \neq 0$ we see that 5.47 is satisfied as long as

$$\begin{cases} u^1(t) = r_2(\pi(\lambda(t))) \\ u^2(t) = -r_1(\pi(\lambda(t))) \ . \end{cases} \tag{5.48}$$

The unique abnormal of an Engel type distribution, where at least one of the r_i's is nonzero, is therefore nice. Moreover, since there are no restrictions on the base, there is an Engel abnormal passing through each point of M. Notice in particular that if $r_1 = 0$, and $r_2 \neq 0$, the abnormal trajectory is generated by the flow of X_1 and vice versa if $r_2 = 0$, and $r_1 \neq 0$ the abnormal is generated by the flow of X_2. In a more intrinsic way we see that the abnormal curve is generated by the flow of the vector field $X \in D$ such that $[X, D^2] = 0 \bmod(D^2)$.

In the case in which $r_1 = r_2 = 0$, i.e. in the zero locus of the magnetic field, the Engel abnormal becomes an equilibrium point. As we said previously, apart from a disctete set of points, this set is a sub-manifold of dimension two $\mathcal{S}$, and since the distribution has rank two we expect the intersection $D \cap T\mathcal{S}$ to be generically empty. It follows that the only abnormal curves that project inside $\mathcal{S}$ are the equilibria of the abnormal specified by 5.48.

Abnormal of the Engel distribution. Consider the case of a constant magnetic field studied in Chapter 4 which give rise to the Engel distribution. We recall here the vector fields of the distribution

$$\begin{cases} T_1 = X \\ T_2 = Y + \frac{bx^2}{2}\partial_w \ , \end{cases} \tag{5.49}$$

where X and Y are the vector fields of the Heisenberg structure 2.3

$$\begin{cases} X = \partial_x - \frac{y}{2}\partial_z \\ Y = \partial_y + \frac{x}{2}\partial_z \ . \end{cases} \tag{5.50}$$

The second order bracket gives

$$\begin{cases} [T_1, [T_1, T_2]] = \partial_w \\ [T_2, [T_1, T_2]] = 0 \ , \end{cases} \tag{5.51}$$

which corresponds to a magnetic field $\beta = \mathrm{d}x \wedge \alpha$. The Engel abnormal is then a reparametrization of a solution of $\dot{\gamma} = -T_2 = -\partial_y - \frac{x}{2}\partial_z - \frac{x^2}{2}\partial_w$, i.e. a straight line

$$\begin{cases} x(t) = x_0 \\ y(t) = y_0 - t \\ z(t) = z_0 - \frac{x_0}{2}t \\ w(t) = w_0 - \frac{x_0^2}{2}t \ , \end{cases} \tag{5.52}$$

with $(x_0, y_0, z_0, w_0) \in \mathbb{R}^4$ initial datum.

In the following we consider non trivial magnetic fields coming from physical models, namely the monopole and the dipole.

Magnetic monopole. We now consider the magnetic potential of a magnetic monopole. We start by rephrasing the description of forms using a cylindrical coordinate system in $\mathbb{H}_3$, $(x, y, z) \mapsto (r, \varphi, z)$. In these coordinates we use the frame $\{\mathrm{d}r, r\mathrm{d}\varphi, \alpha\}$, where now the contact form 2.2 is written as $\alpha = \mathrm{d}z - \frac{r^2}{2}\mathrm{d}\varphi$. The dual frame of $\{\mathrm{d}r, r\mathrm{d}\varphi, \alpha\}$ is given by $\{R, \Phi, Z\}$ with

$$\begin{cases} R := \partial_r \\ \Phi := \frac{\partial_\varphi}{r} + \frac{r\partial_z}{2} \\ Z := \partial_z \ . \end{cases} \tag{5.53}$$

Its is straightforward to check that this frame is orthonormal with respect to the flat metric 2.5 of $\mathbb{H}_3$. The Rumin complex can be described as $\Omega^1(\mathbb{H}_3) = \mathrm{span}(\{\mathrm{d}r, r\mathrm{d}\varphi\})$, $\Omega^2(\mathbb{H}_3) =$

span($\{\mathrm{d}r \wedge \alpha, r\mathrm{d}\varphi \wedge \alpha\}$) and $\Omega^3(\mathbb{H}_3) = \mathrm{span}\{\alpha \wedge \mathrm{d}\alpha\}$. We also have an expression for the operator $\mathcal{D}$ in these coordinates.

Proposition. Given $A \in \Omega^1(\mathbb{H}_3)$, with $A = A_r \mathrm{d}r + A_\varphi r\mathrm{d}\varphi$, we have

$$\mathcal{D}A = (R(\gamma) - Z(A_r))\mathrm{d}r \wedge \alpha + (\Phi(\gamma) - Z(A_\varphi))r\mathrm{d}\varphi \wedge \alpha \ , \tag{5.54}$$

where

$$\gamma := \frac{R(rA_\varphi)}{r} - \Phi(A_r) \ . \tag{5.55}$$

Proof. See [Cas+21], proposition 4.1. The proof is completely analogous to the one of the proposition at page 36. □

We further notice that the closure of the magnetic field implies that its components are not independent. We have indeed the following proposition.

Proposition. Given $\beta \in \Omega^2(\mathbb{H}_3)$, with $\beta = \mathcal{D}A$ and $\beta = \beta_r \mathrm{d}r \wedge \alpha + \beta_\varphi r\mathrm{d}\varphi \wedge \alpha$, we have

$$\beta_\varphi(r, \varphi, z) = \int_0^r \Phi(\beta_r)(\tilde{r}, \varphi, z)\tilde{r} \ \mathrm{d}\tilde{r} \ . \tag{5.56}$$

Proof. See again [Cas+21], proposition 4.2. □

Notice that from 5.56 it follows that if the magnetic field depends only on the radial coordinate, we have $\beta_\varphi \equiv 0$. In this case the zero locus of the magnetic field is generically a co-dimension one sub-manifold of $\mathbb{E}_4$.

The horizontal distribution in the Heisenberg group (see 2.3) is spanned by R and Φ. If $\beta = \mathcal{D}A$, and $A = A_r \mathrm{d}r + A_\varphi r\mathrm{d}\varphi$, the Horizontal distribution of $C^\beta_{\mathbb{H}_3}$ is spanned by

$$\begin{cases} T_1 := R + A(R)\partial_w = R + A_r \partial_w \\ T_2 := \Phi + A(\Phi)\partial_w = \Phi + A_\varphi \partial_w \ . \end{cases} \tag{5.57}$$

The first bracket is given by

$$[T_1, T_2] = [R + A_r\partial_w, \Phi + A_\varphi \partial_w] = [R, \Phi] + (R(A_\varphi) - \Phi(A_r))\partial_w = [R, \Phi] + \left(\gamma - \frac{A_\varphi}{r}\right)\partial_w \ . \tag{5.58}$$

As we already proved in general, from this expression we see that the growth vector of $C^\beta_{\mathbb{H}_3}$ is of type $(2, 3, \cdots, 4)$. Observe also that since R and Φ are not L-invariant, $\{R, \Phi, Z\}$ do not solve the Heisenberg algebra. The next bracket is given by the following expression

$$[T_1, [T_1, T_2]] = [R, [R, \Phi]] + \left(R(\gamma) - R\left(\frac{A_\varphi}{r}\right) - [R, \Phi](A_r)\right)\partial_w \ . \tag{5.59}$$

Since we have $[R, \Phi] = -\frac{\partial_\varphi}{r^2} + \frac{\partial_z}{2}$ and $[R, [R, \Phi]] = \frac{2\partial_\varphi}{r^3}$, we can compute the step by looking at the following determinant

$$\det \begin{pmatrix} 1 & 0 & 0 & A_r \\ 0 & \frac{1}{r} & +\frac{r}{2} & A_\varphi \\ 0 & -\frac{1}{r^2} & \frac{1}{2} & \mathcal{B} \\ 0 & \frac{2}{r^3} & 0 & \mathcal{C} \ . \end{pmatrix} = \frac{\mathcal{B}}{r^2} + \frac{\mathcal{C}}{r} - \frac{A_\varphi}{r^3} \ , \tag{5.60}$$

where we used a shorthand notation for the coefficients of ∂_w. With a bit of work we can rewrite this determinant in a more meaningful way.

$$\frac{\mathcal{B}}{r^2}+\frac{\mathcal{C}}{r}-\frac{A_\varphi}{r^3}=\frac{\gamma}{r^2}-\frac{A_\varphi}{r^3}+\frac{R(\gamma)}{r}-\frac{1}{r}R\left(\frac{A_\varphi}{r}\right)+\frac{1}{r}\left(\frac{\partial_\varphi}{r^2}-\frac{\partial_z}{2}\right)(A_r)-\frac{A_\varphi}{r^3}=$$

$$=\frac{1}{r}\left(R(\gamma)-Z(A_r)\right)+\frac{1}{r^2}\left(\left(\frac{\partial_\varphi}{r^2}+\frac{\partial_z}{2}\right)(A_r)-R(A_\varphi)-\frac{A_\varphi}{r}+\gamma\right)=$$

$$=\frac{\beta_r}{r}+\frac{1}{r^2}\left(\Phi(A_r)-R(A_\varphi)+R(A_\varphi)-\Phi(A_r)\right)=\frac{\beta_r}{r}\ .$$

As it happened in the case of L-invariant frames the step depends on the component of the magnetic field. However in this case the second order bracket does not involve directly the magnetic field (confront 5.59 with 4.16). The other second order bracket is given by

$$[T_2,[T_1,T_2]]=\left[\Phi+A_\varphi\partial_w,[R,\Phi]+\left(\gamma-\frac{A_\varphi}{r}\right)\partial_w\right]=$$

$$=[\Phi,[R,\Phi]]+\left(\Phi(\gamma)-\frac{\Phi(A_\varphi)}{r}+\frac{\Phi(A_\varphi)}{r}-Z(A_\varphi)\right)\partial_w=\beta_\varphi\partial_w\ .$$

The the second bracket is independent if and only if the following determinant is non zero.

$$\det\begin{pmatrix}1 & 0 & 0 & A_r\\ 0 & \frac{1}{r} & +\frac{r}{2} & A_\varphi\\ 0 & -\frac{1}{r^2} & \frac{1}{2} & \mathcal{B}\\ 0 & 0 & 0 & \beta_\varphi\ .\end{pmatrix}=\frac{\beta_\varphi}{r}\ . \tag{5.61}$$

We have then recovered our previous result on the step of the distribution $D=\mathrm{span}(\{T_1,T_2\})$ in $C^\beta_{\mathbb{H}_3}$, i.e. that the step is strictly greater than 3 if and only if $\beta\neq 0$.

We now consider the following scalar potential of a magnetic monopole in $\mathbb{H}_3$

$$V(r,z)=\frac{1}{\sqrt{r^2+z^2}}\ , \tag{5.62}$$

the standard magnetic 1-form being $B:=\mathrm{d}V=-\frac{r\mathrm{d}r}{\sqrt{r^2+z^2}^3}-\frac{z\mathrm{d}z}{\sqrt{r^2+z^2}^3}=-\frac{r\mathrm{d}r+z\mathrm{d}z}{\rho^3}$, where $\rho^2=r^2+z^2$. Since we are looking for a magnetic potential we shall compute first the standard magnetic 2-form $\star B$. In cylindrical coordinates we have

$$(\star B)_{ij}=r\varepsilon_{ijk}B^k=r(\varepsilon_{ijr}B^r+\varepsilon_{ijz}B^z)\ ,$$

hence

$$\star B=\frac{r^2}{\rho^3}\mathrm{d}z\wedge\mathrm{d}\varphi-\frac{zr}{\rho^3}\mathrm{d}r\wedge\mathrm{d}\varphi\ .$$

It is easy to see that the magnetic potential A, such that $\mathrm{d}A=\star B$ is given by

$$A=\frac{z\mathrm{d}\varphi}{\rho}\ .$$

We can now express A in terms of the generators of $\Omega^1(\mathbb{H}_3)$ as $A=A_r\mathrm{d}r+A_\varphi r\mathrm{d}\varphi=\frac{z}{r\rho}r\mathrm{d}\varphi$, and compute the magnetic field according to Rumin as $\beta=\mathcal{D}A$. We have $\gamma=-\frac{z}{\rho^3}$, then

$$\beta_r=\partial_r\left(-\frac{z}{\rho^3}\right)=\frac{3rz}{\rho^5}\ ,$$

and

$$\beta_\varphi = \left(\frac{\partial_\varphi}{r} + \frac{r\partial_z}{2}\right)\left(-\frac{z}{\rho^3}\right) - \partial_z\left(\frac{z}{r\rho}\right) = -\frac{r^4 + 2z^4}{2r\rho^5} \ .$$

As a consequence, except for the origin, which is outside the coordinate system, the magnetic field is non zero, hence the growth vector is $(2, 3, 4)$. Moreover, the only abnormal curves are Engel abnormals.

Dipole-like magnetic potential. We consider now the potential of a magnet in $\mathbb{H}_3$ lying in the (x, y) plane. If the dipole moment is $\mathbf{m} \in \mathbb{H}^3$, the vector potential $\mathbf{A} \in \mathfrak{X}(\mathbb{H}_3)$ at $\mathbf{x} \in \mathbb{H}_3$ is given by

$$\mathbf{A}(\mathbf{x}) = \frac{\mathbf{m} \times \mathbf{x}}{\|\mathbf{x}\|^3} \ , \tag{5.63}$$

where the bold letters denotes three dimensional vectors and "$\times$" stands for the standard vector product in $\mathbb{R}^3$. Referring to the Cartesian coordinates of $\mathbb{H}_3$, (x, y, z), we assume that $\mathbf{m} = (1, 0, 0)$, and hence, the potential is

$$A = \frac{y\mathrm{d}z}{r^3} - \frac{z\mathrm{d}y}{r^3} \ , \tag{5.64}$$

where $r := \|\mathbf{x}\|$. This potential is equivalent modulo the contact form α (see 4.1) to the form

$$A = -\frac{y^2\mathrm{d}x}{2r^3} + \frac{(xy - 2z)\mathrm{d}y}{2r^3} \tag{5.65}$$

We can compute the magnetic field according to Rumin $\beta = \beta_x \mathrm{d}x \wedge \alpha + \beta_y \mathrm{d}y \wedge \alpha$. From 4.6 we have

$$\begin{aligned}
\beta_x &= X\left(X\left(\frac{(xy-2z)}{2r^3}\right) - Y\left(-\frac{y^2}{2r^3}\right)\right) - Z\left(-\frac{y^2}{2r^3}\right) = \\
&= X\left(\left((xy-2z)X + y^2Y\right)\left(\frac{1}{2r^3}\right) + \frac{2y}{r^3}\right) + y^2Z\left(\frac{1}{2r^3}\right) = \\
&= (6yX + y^2Z)\left(\frac{1}{2r^3}\right) + \left((xy-2z)X^2 + y^2XY\right)\left(\frac{1}{2r^3}\right) \ .
\end{aligned}$$

For β_y we get

$$\begin{aligned}
\beta_y &= Y\left(X\left(\frac{(xy-2z)}{2r^3}\right) - Y\left(-\frac{y^2}{2r^3}\right)\right) - Z\left(\frac{(xy-2z)}{2r^3}\right) = \\
&= Y\left(\left((xy-2z)X + y^2Y\right)\left(\frac{1}{2r^3}\right) + \frac{2y}{r^3}\right) + (xy-2z)Z\left(\frac{1}{2r^3}\right) + \frac{1}{r^3} = \\
&= \frac{3}{r^3} + ((6yY + (xy-2z)Z))\left(\frac{1}{2r^3}\right) + \left((xy-2z)YX + y^2Y^2\right)\left(\frac{1}{2r^3}\right) \ .
\end{aligned}$$

Notice that by neglecting the inverse cubic term we obtain again the Engel distribution. After a long computation we get

$$\beta_x = \frac{3}{8r^7}(-24x^4y + x^3y^3 - xy^3(4+y^2) + 4xy(-1+y^2)z^2 - 2(-4+y(-1+7y))z(y^2+z^2) -$$

$$-x^2y(24y^2 + x(-16+y^2) - 2(-16+y+3y^2)z + 24yz^2) + xy(y^4 + 20z(-xy+2z) - 4y^2(-5+z^2)))$$

We obtain equally complicated expression for the other component. This makes the computation of the Engel abnormal not very enlightening.

Remark. The expression of the magnetic potential for the monopole and the dipole are in contrast with the metric structure of $\mathbb{H}_3$. We shall instead use for the monopole a potential like

$$V(r,z) = \frac{1}{\sqrt{r^4 + z^2}} \ ,$$

since it has the property of being homogeneous under the natural dilation in $\mathbb{H}_3$, $(x,y,z) \mapsto (\epsilon x, \epsilon y, \epsilon^2 z)$.

Appendix A

Principal connections

Let $\pi : P \to M$ be a principal bundle with base M and characteristic fiber a Lie group G. The projection defines a distribution in TP called the vertical distribution $\mathcal{V}$ as

$$\mathcal{V} = \bigsqcup_{p \in P} \mathcal{V}_p = \bigsqcup_{p \in P} \ker(\pi_*|_p) \subset TP \ .$$

Definition. A connection on P is a distribution $\mathcal{H} \subset TP$, called an horizontal distribution, such that

$$TP = \mathcal{H} \oplus \mathcal{V} \ ,$$

in the sense that $T_pP = H_p \oplus \mathcal{V}_p$, for all $p \in P$ and that is equivariant with respect to the right translations $R_g(p) = pg$ of P, i.e.

$$\mathcal{H}_{pg} = R_{g_*}\mathcal{H}_p \ .$$

We shall give two other equivalent definitions of a principal connection.

Definition. A connection on P is a $\mathfrak{g}$-valued 1-form on P called the connection 1-form, i.e. a certain $\omega \in \Lambda^1(P, \mathfrak{g})$ such that

$$R_g^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega \quad \forall g \in G \ ,$$

and brings the vector fields of the right actions to their generators in the Lie algebra

$$\omega|_p(X_\xi(p)) = \xi \quad \forall p \in P \ ,$$

with $X_\xi(p) := \frac{d}{dt} R_{\exp(t\xi)}(p)|_{t=0} = \frac{d}{dt} p \exp(t\xi)|_{t=0}$.

Definition. A connection on P is a correspondence between the local trivializations $T_U : \pi^{-1}(U) \to U \times G$ of P and $\mathfrak{g}$-valued 1-forms on U open subset of M. If $\omega_U \in \Lambda^1(U, \mathfrak{g})$ corresponds to T_U and $\omega_V \in \Lambda^1(V, \mathfrak{g})$ to T_V, in the intersection $U \cap V$ we have the compatibility relation

$$\omega_V|_p(v) = \left((L_{g_{UV}^{-1}(m)})_* \circ (g_{UV})_* \right)(v) + \left(\mathrm{Ad}_{g_{UV}^{-1}(m)} \circ \omega_U|_p \right)(v) \quad \forall v \in T_m(U \cap V) \ ,$$

where $m = \pi(p) \in U \cap V$, and $g_{UV} \in \mathcal{C}^\infty(U \cap V, G)$ is the transition function between the two trivializations.

Remark. It can be shown that the three definitions just given are in fact equivalent. It is customary in Physics to call the locally defined 1-forms ω_U the potentials. The reason for this is that, at least for G Abelian, the exterior differentials of these forms lead to a globally defined 2-form $F \in \Lambda^2(M, \mathfrak{g})$ called the field strength, which is an observable field. Important is the case of Electromagnetism in which M is the Minkowski 4-dimensional space-time and $G = \mathbb{S}^1$.

In this case F contains the Electric and Magnetic fields.

We notice that it is clear, mostly from the first definition, that there is an isomorphism between $\mathcal{H}$ and TM. This comes from the definition of $\mathcal{H}$ and that π is a submersion. Secondly, if $\omega \in \Lambda^1(P, \mathfrak{g})$, we have that $\ker(\omega) \simeq \mathcal{H} \simeq TM$. We finally observe that the relation between the local potentials ω_U and a connection 1-form ω can be established using the local sections. If $s_U : U \to P$ is a local section we can take $\omega_U := s_U^* \omega$ as local potential.

Curvature of a principal connection. We define the following operation

Definition. Let $\varphi \in \Lambda^i(P, \mathfrak{g})$, $\psi \in \Lambda^j(P, \mathfrak{g})$, we define $[\varphi, \psi] \in \Lambda^1(P, \mathfrak{g})$ through the relation

$$[\varphi, \psi](X_1, \cdots, X_{i+j}) := \frac{1}{i!j!} \sum_{\sigma \in \mathfrak{S}(i+j)} (-1)^{|\sigma|} [\varphi(X_{\sigma(1)}, \cdots, X_{\sigma(i)}), \psi(X_{\sigma(i+1)}, \cdots, X_{\sigma(i+j)})] \ ,$$

where $\mathfrak{S}(i+j)$ is the permutation group of $i+j$ elements, $|\sigma|$ the sign of the permutation σ, $X_1 \cdots, X_{i+j} \in \mathfrak{X}(P)$, and the bracket on the RHS is the Lie bracket of $\mathfrak{g}$.

We turn our attention back to the splitting defined by the choice of a connection $TP = \mathcal{H} \oplus \mathcal{V}$. We can then uniquely write every vector field $X \in \mathfrak{X}(P)$ as the sum of its vertical and horizontal components

$$X(p) = X^{\mathcal{V}}(p) + X^{\mathcal{H}}(p), \quad X^{\mathcal{V}}(p) \in \mathcal{V}_p, \ X^{\mathcal{H}}(p) \in \mathcal{H}_p \ .$$

Consequently we have that $\pi_*(X^{\mathcal{V}}) = 0$ and if ω is the connection 1-form $\omega(X^{\mathcal{H}}) = 0$. We can therefore define horizontal differential forms as follows.

Definition. Let $\varphi \in \Lambda^i(P, \mathfrak{g})$. The horizontal part of φ is the form $\varphi^{\mathcal{H}} \in \Lambda^i(P, \mathfrak{g})$ defined through the relation

$$\varphi^{\mathcal{H}}(X_1, \cdots, X_i) := \varphi(X_1^{\mathcal{H}}, \cdots, X_i^{\mathcal{H}}) \ .$$

Definition. Let $\mathcal{H} \subset TP$ be a connection on P. The exterior covariant derivative of a form $\varphi \in \Lambda^i(P, \mathfrak{g})$ relative to the connection $\mathcal{H}$ is the form $\nabla^{\mathcal{H}} \varphi \in \Lambda^{i+1}(P, \mathfrak{g})$ defined as

$$\nabla^{\mathcal{H}} \varphi := (\mathrm{d}\varphi)^{\mathcal{H}} \ .$$

If the connection $\mathcal{H}$ is specified by the connection 1-form ω, we use also the notation $\nabla^{\omega} := \nabla^{\mathcal{H}}$. In the case only one connection is considered we will omit the superscript.

We are now in the position to define the curvature of a principal connection.

Definition. The curvature of a principal connection 1-form ω, denoted as Ω^{ω} is the exterior covariant derivative relative to the connection itself of ω, i.e.

$$\Omega^{\omega} := \nabla^{\omega} \omega \in \Lambda^2(P, \mathfrak{g}) \ .$$

Theorem. (Structural equation) If ω is a connection 1-form on a principal bundle, we have the following identity

$$\Omega^{\omega} := \nabla^{\omega} \omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega] \ .$$

Notice that in the Abelian case the covariant derivative coincides with the exterior differential, as it happens in Electromagnetism.

The circle bundle on a surface. If we consider a Riemannian surface M, we can construct the circle bundle over it as the collection of tangent vectors with unitary length with respect to the Riemannian metric. We denote this set as SM. This space has the structure of a principal bundle $\pi : SM \to M$ with base M and characteristic fiber the Abelian Lie group $\mathbb{S}^1$. Let $\{e_1, e_2\}$ be an orthonormal frame for TM over the open subset $U \subset M$. The two sections defines a local trivialization of SM as follows. Given $s \in \pi^{-1}(U)$ we can then write it as ($\pi(s) = m$)

$$s = \cos(\vartheta(m))e_1 + \sin(\vartheta(m))e_2 \ ,$$

where $\vartheta : U \to \mathbb{S}^1$ is a uniquely determined function. The local trivialization of TS over U is the map $T_U : \pi^{-1}(U) \to U \times \mathbb{S}^1$, $s \mapsto (m, \vartheta(m))$. The right translations by $\mathbb{S}^1$ are simply rotation of the frame, i.e. $R_\alpha s = s\alpha$ is such that in the trivialization of before $T_U(s\alpha) = (m, \vartheta + \alpha)$.
A connection over SM is a Lie algebra valued one form over SM. Since the Lie algebra of $\mathbb{S}^1$ is diffeomorphic to $\mathbb{R}$, we can consider ordinary 1-forms in $\Lambda^1(SM)$.
We denote with $V \in \mathfrak{X}(SM)$ the (locally defined) generator of the vertical distribution $\ker(\pi_*)$, and consider a connection $\tau \in \Lambda^1(SM)$. There exists a unique horizontal lift of vector fields on the base M, in particular we denote as $\{E_1, E_2\}$ the horizontal lift of the orthonormal frame $\{e_1, e_2\}$. We recall that by this we mean that for $i = 1, 2$ we have

$$\begin{cases} \tau(E_i) = 0 \\ \pi_* E_i = e_i \ . \end{cases} \tag{5.66}$$

We choose a suitable normalization such that $\tau(V) = 1$. This means that in local coordinates $V = \partial_\vartheta$ and $\tau_\vartheta = 1$. Observe that $\{E_1, E_2, V\}$ is now a frame on SM. Next we consider the brackets

$$[E_1, E_2] = \tilde{c}_1 E_1 + \tilde{c}_2 E_2 + vV \ , \tag{5.67}$$

where $\tilde{c}_1, \tilde{c}_2, v \in \mathcal{C}^\infty(SM)$. Using 5.66 we have

$$\pi_*([E_1, E_2]) = [\pi_* E_1, \pi_* E_2] = [e_1, e_2] = c_1 e_i + c_2 e_2 \ ,$$

and on the other hand, by 5.67

$$\pi_*[E_1, E_2] = \tilde{c}_1 \pi_* E_1 + \tilde{c}_2 \pi_* E_2 + v\pi_* V = \tilde{c}_1 e_1 + \tilde{c}_2 e_2 \ .$$

Consequently $\tilde{c}_1 = c_1$ and $\tilde{c}_2 = c_2$. Finally we can write

$$[E_1, E_2] = c_1 E_1 + c_2 E_2 + vV \tag{5.68}$$

We now define for $i = 1, 2$ the following forms[3] $\omega_i \in \Lambda^1(SM)$ through the relation

$$\omega_i(X) := g(\pi_* X, e_i) \ , \tag{5.69}$$

for all $X \in \mathfrak{X}(SM)$. Notice that $\omega_i(X)$ is the component of $\pi_* X$ along the vector field e_i with respect the metric g. One possible way to define the Levi-Civita connection in a circle bundle is the following (see [ST15] chapter 7 section 7.2).

Theorem. There is a unique principal connection $\tau \in \Lambda^1(SM)$, called the Levi-Civita connection relative to the metric g, such that

$$\begin{cases} \mathrm{d}\omega_1 = \tau \wedge \omega_2 \\ \mathrm{d}\omega_2 = -\tau \wedge \omega_1 \ . \end{cases} \tag{5.70}$$

[3] These are horizontal forms in the sense that $\omega_i = \omega_i^{\mathcal{H}}$, with $\mathcal{H} := \ker(\tau)$ the chosen connection. Using the dual frame of $\{e_1, e_2\}$, $\{\mu^1, \mu^2\}$ we can express them as $\omega_i = \pi^* \mu^i$.

From these conditions we can compute τ using the frame of T^*SM given by $\{\omega_1, \omega_2, \mathrm{d}\vartheta\}$. On the one hand we have for $i = 1, 2$

$$\mathrm{d}\omega_i(E_1, E_2) = E_1(\omega_i(E_2)) - E_2(\omega_i(E_1)) - \omega_i([E_1, E_2]) =$$

$$= E_1(\delta_{i2}) - E_2(\delta_{i1}) - \omega_i(c_1 E_1 + c_2 E_2 + vV) = -c_1\delta_{i1} - c_2\delta_{i2} \ .$$

The RHS of equations 5.70 returns for $i = 1, 2$

$$(-1)^i \left(\tau \wedge \omega_i\right)(E_1, E_2) = (-1)^i \left(\tau_1\delta_{i2} - \tau_2\delta_{i1}\right) \ .$$

Equations then 5.70 lead to

$$\begin{cases} \tau_1 = -c_1 \\ \tau_2 = -c_2 \ . \end{cases}$$

In conclusion The Levi-Civita connection can be written as

$$\tau = \mathrm{d}\vartheta - c_1\omega_1 - c_2\omega_2 \ ,$$

or using the dual frame $\{\mu^1, \mu^2\}$ we get

$$\tau = \mathrm{d}\vartheta - c_i\pi^*\mu^i \ , \tag{5.71}$$

which is the form used in the first chapter section 1.5.

Curvature. Since $\mathbb{S}^1$ is abelian, from the structural equation we see that the curvature two form is $\Omega^\tau = \mathrm{d}\tau + \frac{1}{2}[\tau, \tau] = \mathrm{d}\tau = -\mathrm{d}c_i \wedge \pi^*\mu^i - c_i\pi^*\mathrm{d}\mu^i$. But we have just proved that $\mathrm{d}\mu^i = -(c_1\delta_{i1} + c_2\delta_{i2})\mu^1 \wedge \mu^2 = -c_i\mu^1 \wedge \mu^2$. On the other hand the c_i's are functions on M, hence $\mathrm{d}c_i = e_j(c_i)\pi^*\mu^j$ so that at the end we arrive at the expression

$$\Omega^\tau = -\left(e_1(c_2) - e_2(c_1) - c_1^2 - c_2^2\right)\pi^*(\mu^1 \wedge \mu^2) \ . \tag{5.72}$$

In the present case of a Riemannian surface the curvature form has only one component that can be shown to be precisely the opposite of the Gaussian curvature of the surface. The same expression is found for example in [ABB19] chapter 4 section 4.4.

Appendix B

Affine connections on a vector bundle

Let $\pi : E \to M$ be a vector bundle of rank k with base M of dimension n. The projection defines a distribution in TE, called vertical distribution, that is the subset

$$\mathcal{V} := \bigsqcup_{e \in E} \ker\, (\pi_*|_e) \subset TE\ .$$

Definition. A connection in E is a distribution $\mathcal{H} \subset TE$, called the horizontal distribution, such that

$$TE = \mathcal{H} \oplus \mathcal{V}\ .$$

Theorem. For all $X \in \mathfrak{X}(M)$ there exists a unique $\nabla_X \in \mathfrak{X}(E)$ such that $\pi_* \nabla_X = X$ and ∇_X is horizontal, i.e. takes values in $\mathcal{H}$. We call the operator $\nabla : \mathfrak{X}(M) \to \mathfrak{X}(E)$ the horizontal lift.

Proof. Since π is by hypothesis a submersion and $\mathcal{H}$ is a connection, π_* is an isomorphism between $\mathcal{H}$ and TM. □

With a slight abuse of notation we can then regard $\mathcal{H}$ as the image of TM via ∇, i.e. $\mathcal{H} = \nabla(TM)$. The notion of a connection naturally produce the notion of parallel transport of sections of E along curves in the base. Indeed a section of E along a curve $\gamma : [0,1] \to M$, i.e. a map $V : [0,1] \to E$ such that $\pi(V(t)) = \gamma(t)$ for all $t \in [0,1]$, will be said to be parallel transported if and only if $\dot{V}(t) \in \mathcal{H}$ for all $t \in [0,1]$.
A natural question arise about whether the horizontal distribution is involutive or not. To measure how much $\mathcal{H}$ fails to be involutive, we define the following object.

Definition. Let $\nabla : \mathfrak{X}(M) \to \mathfrak{X}(E)$ be a connection. We define the Riemannian curvature tensor via

$$R(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \in \mathfrak{X}(E), \quad X, Y \in \mathfrak{X}(M)\ .$$

Observe that $\nabla_{[X,Y]}$ is the horizontal part of $[\nabla_X, \nabla_Y]$, and hence $R(X,Y)$ is the remaining vertical part. We also remark that $R(\cdot,\cdot)$ is a skew-symmetric covariant tensor taking values in $\mathfrak{X}(E)$ since it is $\mathcal{C}^\infty(M)$-linear and skewsymmetric in the arguments. [4] We end this paragraph stating the following theorem, which is a direct consequence of Frobenius theorem applied to the distribution $\mathcal{H} \subset TE$.

Theorem. The horizontal distribution $\mathcal{H}$ is integrable if and only if the curvature tensor vanish.

[4]Regarding ∇ as a 1-form on M, taking values in $\mathfrak{X}(E)$, we can regard R a 2-form with values in $\mathfrak{X}(E)$.

The cotangent bundle. We consider the case in which $E = T^*M$. In this case we can use the splitting 3.5 and express ∇_{X_i} in the following way

$$\nabla_{X_i} = X_i + \Gamma_{ij}\frac{\partial}{\partial h_j} \ ,$$

where $\Gamma_{ij} \in \mathcal{C}^\infty(T^*M)$ measure how the horizontal subspace is immersed in $T(T^*M)$ with respect the present splitting.
From the linear structure of the bundle we have the following fact.

Proposition. Thanks to the linear structure on the fibers we have

$$\nabla_{X_i} = X_i + \Gamma_{ij}^k h_k \frac{\partial}{\partial h_j} \ ,$$

where $\Gamma_{ij}^k \in \mathcal{C}^\infty(M)$ are called the connection coefficient in the chosen frame.
We are now able to define the covariant differentiation of a section of TM.

Definition. Let $X, Y \in \mathfrak{X}(M)$. We call the covariant derivative of X with respect to Y the unique $Z \in \mathfrak{X}(M)$ such that

$$\lambda(Z(\pi(\lambda))) = \nabla_X \lambda(Y(\pi(\lambda))) \quad \forall\ \lambda \in T^*M \ .$$

Proposition. Using the frame $\{X_i\}_{i=1,\cdots,n}$ we have

$$\nabla_{X_i} X_j = \Gamma_{ij}^k X_k \ .$$

Moreover for a generic vector field $X = x^i X_i$ we have the following Leibniz rule

$$\nabla_{X_i} x^j X_j = X_i(x^j) + x^j \nabla_{X_i} X_j = \left(X_i(x^k) + \Gamma_{ij}^k x^j\right) X_k \ .$$

Proof. By previous results

$$\nabla_{X_i} = X_i + \Gamma_{ij}^k h_k \frac{\partial}{\partial h_j} \ .$$

Moreover we have $\lambda(X_j(\pi(\lambda))) = h_j$. Then the result easily follows. □

Being able to differentiate functions and vector fields we can now extend the covariant derivative to all tensor fields over M. In particular we are interested to derive forms. To do so we use the Leibniz rule.

Definition. Let $X, Y \in \mathfrak{X}(M)$ and $\alpha \in \lambda^1(M)$ we define the covariant derivative along X of α, denoted as $\nabla_X \alpha \in \Lambda^1(M)$, through the following identity (Leibniz rule)

$$\nabla_X(\alpha(Y)) = (\nabla_X \alpha)(Y) + \alpha(\nabla_X Y) \ .$$

Proposition. Using the dual frames $\{X_i\}_{i=1,\cdots,n}$ and $\{\mu^i\}_{i=1,\cdots,n}$ we have

$$\nabla_{X_i} \mu^j = -\Gamma_{ik}^j \mu^k \ .$$

Furthermore for a generic form $\alpha = \alpha_i \mu^i$ we have

$$\nabla_{X_i} \alpha = \left(X_i(\alpha_k) - \Gamma_{ik}^j \alpha_j\right) \mu^k \ .$$

Proof. We use the Leibniz rule

$$(\nabla_{X_i} \mu^j)(X_k) = \nabla_{X_i}(\mu^j(X_k)) - \mu^j(\nabla_{X_i} X_k) = \nabla_{X_i}(\delta_k^j) - \Gamma_{ik}^l \delta_l^j = -\Gamma_{ik}^j \ .$$

Again is trivial to check the result for generic forms. □

The Levi-Civita connection. Between all connections we choose the only one compatible with the Riemannian structure and torsion free, in the sense specified in the following.

Definition. We define the torsion of an affine connection ∇ to be the map $T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by the relation

$$\lambda(T(X,Y)(\pi(\lambda))) = \sigma|_\lambda(\nabla_X(\lambda), \nabla_Y(\lambda)) \quad \forall\ X, Y \in \mathfrak{X}(M),\ \forall \lambda \in T^*M\ ,$$

where σ is the symplectic form of T^*M.

Remark. Observe that T measures whether the horizontal distribution is Lagrangian[5] or not.

Definition. An affine connection ∇ is said to be compatible with the metric g if and only if

$$\nabla_X g = 0 \quad \forall\ X \in \mathfrak{X}(M)\ ,$$

i.e. if the covariant derivatives along all vector fields of the metric tensor vanish.

Remark. Observe that the compatibility with the metric implies that ∇ is also compatible with the duality induced by the metric, i.e. $\nabla_X \sharp\alpha = \sharp\nabla_X\alpha$ for $X \in \mathfrak{X}(M)$ and $\alpha \in \Lambda^1(M)$.

Theorem.(Levi-Civita) Let (M, g) be a Riemannian manifold. There exists a unique affine connection over T^*M that is at the same time compatible with the metric and has zero torsion.

We do not prove the theorem, but recall a corollary to perform computations.

Corollary. If ∇ is the Levi-Civita connection relative to the metric g, and $\{X_i\}_{i=1,\cdots,n}$ is an orthonormal frame with $[X_i, X_j] = c_{ij}^k X_k$, we have.

$$\Gamma^i_{jk} - \Gamma^i_{kj} = c^i_{jk}\ ,$$

and also

$$\Gamma^i_{jk} = -\Gamma^k_{ji}\ .$$

Proof. The first condition comes from the fact that ∇ has zero torsion and the second from the compatibility with the metric. They are in fact simply the conditions above mentioned, expressed with the orthonormal frames. □

[5] A distribution $D \subset T(T^*M)$ of dimension n is said to be Lagrangian iff $\sigma|_D = 0$.

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