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A real variation on the Stationary Phase Lemma

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Introduction

Let $\mathbb{V}$ be a one-dimensional complex vector space with coordinate z and let $\mathbb{V}^*$ be its dual with coordinate w.
The Fourier-Laplace transform, denoted by ${}^{\mathsf{L}}$, is a functor between the bounded derived category of $\mathscr{D}_{\mathbb{V}}$-modules $\mathsf{D}^b(\mathscr{D}_{\mathbb{V}})$ and the category $\mathsf{D}^b(\mathscr{D}_{\mathbb{V}^*})$: it is defined as an analogue of the integral transform with kernel associated to e^{-zw}. In particular it gives an equivalence between the full triangulated subcategory $\mathsf{D}^b_{hol}(\mathscr{D}_{\mathbb{V}})$ of $\mathsf{D}^b(\mathscr{D}_{\mathbb{V}})$ consisting of objects with holonomic cohomologies and $\mathsf{D}^b_{hol}(\mathscr{D}_{\mathbb{V}^*})$.
If a is a singular point of $\mathscr{M} \in \mathsf{D}^b_{hol}(\mathscr{D}_{\mathbb{V}})$ then, after a ramification, $\mathscr{M}$ can be asimptotically written on a sector V_a as a finite direct sum of exponential modules $\mathscr{E}^f := \mathscr{D}_{\mathbb{V}} e^f \otimes^D \mathscr{O}_{\mathbb{V}}(*a)$ where $\mathscr{D}_{\mathbb{V}} e^f := \mathscr{D}_{\mathbb{V}}/\{P \in \mathscr{D}_{\mathbb{V}};\, Pe^f = 0 \text{ on } V_a$ and $f \in \mathscr{O}_{\mathbb{V}}(*a)$ is a meromorphic function with pole at a. The functions f in this decomposition are called *exponential factors* of $\mathscr{M}$. We say that such an f is *admissible* if it is unbounded at a and, if $a = \infty$, if it's not linear at ∞.
In this setting the stationary phase lemma states that the admissible exponential factors of ${}^{\mathsf{L}}\mathscr{M}$ are obtained by applying the Legendre transform to the admissible exponential factors of $\mathscr{M}$.
Classically, the stationary phase formula is stated in terms of the so-called local Fourier-Laplace transform for formal holonomic $\mathscr{D}$-modules. This was introduced in [**3**] (see also [**8**, **2**]), by analogy with the ℓ-adic case treated in [**15**]. An explicit stationary phase formula was obtained in [**16**, **6**] (see also [**9**]) for $\mathscr{D}$-modules, and in [**7** , **1**] for ℓ-adic sheaves.
Consider now $\mathrm{E}^b_+(\mathrm{I}\mathbb{C}_{\mathbb{V}_\infty})$ and $\mathrm{E}^b_+(\mathrm{I}\mathbb{C}_{\mathbb{V}_\infty{}^*})$, the categories of enhanced indsheaves on $\mathbb{V}_\infty$ and on $\mathbb{V}^*_\infty$ ($\mathbb{V}_\infty$ is the bounded compactification of $\mathbb{V}$ and $\mathbb{V}^*_\infty$ of $\mathbb{V}^*$, see Section 1.3). The enhanced Fourier-Sato transform, still denoted by ${}^{\mathsf{L}}$, is a functor from $\mathrm{E}^b_+(\mathrm{I}\mathbb{C}_{\mathbb{V}_\infty})$ into $\mathrm{E}^b_+(\mathrm{I}\mathbb{C}_{\mathbb{V}^*_\infty})$: also this functor is defined as an analogue of the integral transform with kernel associated to e^{-zw}, and it gives an equivalence between the full triangulated subcategory $\mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_{\mathbb{V}_\infty})$ of $\mathrm{E}^b_+(\mathrm{I}\mathbb{C}_{\mathbb{V}_\infty})$ of $\mathbb{R}$-constructible indsheaves on $\mathbb{V}_\infty$ and $\mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_{\mathbb{V}^*_\infty})$.
We say that $K \in \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_{\mathbb{V}_\infty})$ has a *normal form* at $a \in \mathbb{V}_\infty$ if, after a ramification, it can be written on a sector V_a as a finite direct sum of $\mathbb{R}$-constructible exponential indsheaves of the form $\mathbb{E}^{\mathrm{Re} f}$ defined near a where $\mathbb{E}^{\mathrm{Re} f}$ is associated

to the sheaf $\mathbb{C}_{\{(z,t)\in\mathbb{V}\times\mathbb{R};\, t+\mathrm{Re}f(z)\geq 0\}}$ with f a meromorphic function with pole at a. It is possible to define a functor $\mathscr{S}ol^E_{\mathbb{V}_\infty} : \mathsf{D}^b(\mathscr{D}_{\mathbb{V}_\infty})^{op} \longrightarrow \mathrm{E}^b_+(\mathrm{I}\mathbb{C}_{\mathbb{V}_\infty})$ which gives an enhanced version of the Riemann-Hilbert corrispondence, i.e. a fully faithful functor $\mathscr{S}ol^E_{\mathbb{V}_\infty} : \mathsf{D}^b_{hol}(\mathscr{D}_{\mathbb{V}_\infty})^{op} \longrightarrow \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_{\mathbb{V}_\infty})$; in particular it is possible to reconstruct $\mathscr{M}$ from $\mathscr{S}ol^E_{\mathbb{V}_\infty}(\mathscr{M})$ functorially. Moreover we have $\mathscr{S}ol^E_{\mathbb{V}^*_\infty}({}^{\mathsf{L}}\mathscr{M}) \simeq {}^{\mathsf{L}}\mathscr{S}ol^E_{\mathbb{V}_\infty}(\mathscr{M})$.
This correspondence allows us to translate the stationary phase lemma in terms of enhanced indsheaves. In this framework a microlocal proof of the lemma is given in [**4**].
If instead of $\mathbb{V}_\infty$ we decide to consider $\mathbb{R}_\infty$ we have that for each $a \in \mathbb{R}_\infty$ the enhanced indsheaf $K \in \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_{\mathbb{R}_\infty})$ can be written as a finite direct sum of $\mathbb{R}$-constructible exponential indsheaves of the form $\mathbb{E}^f$ *and* $\mathbb{E}^{f^+\triangleright f^-}$ defined near a. The exponential indsheaves $\mathbb{E}^f$ and $\mathbb{E}^{f^+\triangleright f^-}$ are associated respectively to the sheaf $\mathbb{C}_{\{(x,t)\in\mathbb{R}\times\mathbb{R};\, t+f(x)\geq 0\}}$ and to the sheaf $\mathbb{C}_{\{(x,t)\in\mathbb{R}\times\mathbb{R};\, -f^+(x)\leq t<-f^-(x)\}}$ where f, f^+, f^- : $V^u_a \to \mathbb{R}$ are analytic functions "with a good behaviour" (that we'll explain in Section 2.4) defined near a and such that $f^-(x) \leq f^+(x)$ for any $x \in V^u_a$. Also in this case the functions f, f^+, f^- in the decomposition are called *exponential factors* of K at a. In this case the Riemann-Hilbert correspondence is not available, so we'll focus only on the $\mathbb{R}$-constructible exponential indsheaves.
In this setting we can rephrase the stationary phase lemma as follows: f is an admissible exponential factor defined on V^u_a in the decomposition of K at a if and only if g is an admissible exponential factor defined on U^v_b in the decomposition of ${}^{\mathsf{L}}K$ at b, where (b, v, g) is given by the Legendre transform of (a, u, f), for $u, v \in \{+, -\}$. In particular we have that, for $x \in V^u_a$ and $y \in U^v_b$,

$$g(y) - f(x) + xy = 0 \quad \text{for} \quad y = f'(x) \ :$$

this is called the *stationary phase formula.*

In Chapter 1 we recall some basic notions about sheaves of k-modules, then we define the constructible sheaves and in particular the $\mathbb{R}$-constructible sheaves on bordered spaces; then we construct the category of indsheaves, which are ind-objects with values in the category of sheaves with compact support.
In Chapter 2 we introduce the convolution functors: we focus mainly on the convolution product, which is an important functor that allows us to define the category of enhanced sheaves (they're basically sheaves with an extra variable that satisfy some properties, explained in Section 2.1, involving the convolution product). Then we define the six Grothendieck operations for enhanced sheaves and give the analogous notions of constructible enhanced sheaves, in particular on bordered spaces. Later we define the category of enhanced indsheaves and generalize some of the notions given in the chapter.

In Chapter 3 we recall briefly some definitions regarding the $\mathscr{D}$-modules, then we present the solution functors and the enhanced version of the Riemann-Hilbert correspondence: this explains the relation between $\mathscr{D}$-modules and enhanced indsheaves; anyway we don't study in depth the $\mathscr{D}$-modules since we will focus only on the enhanced indsheaves.
In Chapter 4 we define the analogues of the Fourier-Laplace transform for $\mathscr{D}$-modules and for enhanced sheaves and indheaves: the latter one is called enhanced Fourier-Sato transform. Then we describe some properties of the enhanced Fourier-Sato transform and we compute the transform of an exponential enhanced sheaf with explicit computations. In the end we show an interesting link between the Fourier-Sato transform of an enhanced sheaf and its microsupport.
In the first section of Chapter 5 we summarize the notions and results in [**4**] regarding the stationary phase lemma in the one-dimensional complex case. In the second section firstly we translate the notions given previously for $\mathbb{R}$ and then we compute explicitly the enhanced Fourier-Sato transform of another exponential sheaf: the complex behaviour of the resulting enhanced sheaf leads us to focus only on the enhanced indsheaves. After some considerations about enhanced $\mathbb{R}$-constructible indsheaves we give the statement of the stationary phase lemma in the real case, and we prove it via direct computations.

Chapter 1

Sheaves and indsheaves

1.1 Sheaves and operations

Firstly, let us recall the notions that we will need later on, following the notations in [**14**].
We say that a topological space is *good* if it is Hausdorff, locally compact, countable at infinity and has finite flabby dimension.
Let k be a field. If M is a good topological space, we denote by $\mathrm{Mod}(k_M)$ the abelian category of sheaves of k-modules on M and by $\mathsf{D}^b(k_M)$ the bounded derived category of $\mathrm{Mod}(k_M)$. If $A \subset M$ is a locally closed subset we denote by k_A the constant sheaf on A with stalk k extended by 0 on $M \setminus A$, and if $F \in \mathsf{D}^b(k_M)$ we set $F_A := F \otimes k_A$.
We have two internal operations:

$$\cdot \otimes \cdot : \mathsf{D}^b(k_M) \times \mathsf{D}^b(k_M) \to \mathsf{D}^b(k_M),$$

$$R\mathscr{H}om(\cdot\,,\cdot) : \mathsf{D}^b(k_M)^{\mathrm{op}} \times \mathsf{D}^b(k_M) \to \mathsf{D}^b(k_M).$$

Let $f : M \to N$ be a morphism of good topological spaces. We have the following functors:

$$Rf_* : \mathsf{D}^b(k_M) \to \mathsf{D}^b(k_N),$$

$$Rf_! : \mathsf{D}^b(k_M) \to \mathsf{D}^b(k_N),$$

$$f^{-1} : \mathsf{D}^b(k_N) \to \mathsf{D}^b(k_M),$$

$$f^! : \mathsf{D}^b(k_N) \to \mathsf{D}^b(k_M).$$

These functors together with $\cdot \otimes \cdot$ and $R\mathscr{H}om(\cdot\,,\cdot)$ are called *Grothendieck's six operations.*

1.2 Constructible sheaves

Assume now that M is a real analytic manifold.

Definition 1.2.1. Let Z be a subset of M. We say that Z is *subanalytic at* $x \in M$ if there exist an open neighborhood U of x and some compact manifolds X_j^1, X_j^2 with morphisms $f_j^1 : X_j^1 \to M$, $f_j^2 : X_j^2 \to M$ $(1 \leq j \leq N)$, such that:

$$Z \cap U = U \cap \bigcup_{j=1}^{N} (f_j^1(X_j^1) \setminus f_j^2(X_j^2)) .$$

If Z is subanalytic at each $x \in M$ then we say that Z is *subanalytic in* M.

Definition 1.2.2. We say that $F \in \mathsf{D}^b(k_M)$ is $\mathbb{R}$-*constructible* if there exists a locally finite covering $M = \bigcup_{i \in I} M_i$ by subanalytic subsets such that for all $j \in \mathbb{Z}$ and all $i \in I$ both $F|_{M_i}$ and $H^j(F)|_{M_i}$ are locally constant of finite rank. We denote by $\mathsf{D}^b_{\mathbb{R}-c}(k_M)$ the full subcategory of $\mathsf{D}^b(k_M)$ consisting of $\mathbb{R}$-constructible sheaves.

Example 1.2.3. If Z is a locally closed subanalytic subset of M, then the sheaf k_Z is $\mathbb{R}$-constructible.

Proposition 1.2.4. *Let $f : M \to N$ be a morphism of real analytic manifolds and let F, F_1, $F_2 \in \mathsf{D}^b_{\mathbb{R}-c}(k_M)$, $G \in \mathsf{D}^b_{\mathbb{R}-c}(k_N)$. Then:*

i. $F_1 \otimes F_2$, $R\mathscr{H}om(F_1, F_2) \in \mathsf{D}^b_{\mathbb{R}-c}(k_M)$;

ii. $f^{-1}G$, $f^!G \in \mathsf{D}^b_{\mathbb{R}-c}(k_M)$;

iii. $Rf_*F \in \mathsf{D}^b_{\mathbb{R}-c}(k_N)$ *if moreover f is proper on supp(F).*

Consider now a complex analytic manifold X.

Definition 1.2.5. A subset $S \subset X$ is called $\mathbb{C}$-*analytic* if both $\overline{S}$ and $\overline{S} \setminus S$ are complex analytic subsets.

Definition 1.2.6. We say that $F \in \mathsf{D}^b(k_X)$ is $\mathbb{C}$-*constructible* if there exists a locally finite covering $X = \bigcup_{i \in I} X_i$ by $\mathbb{C}$-analytic subsets such that for all $j \in \mathbb{Z}$ and all $i \in I$ both $F|_{X_i}$ and $H^j(F)|_{X_i}$ are locally constant of finite rank. We denote by $\mathsf{D}^b_{\mathbb{C}-c}(k_X)$ the full subcategory of $\mathsf{D}^b(k_X)$ consisting of $\mathbb{C}$-constructible sheaves.

Remark. Notice that $\mathsf{D}^b_{\mathbb{C}-c}(k_X)$ is a subcategory of $\mathsf{D}^b_{\mathbb{R}-c}(k_{X^\mathbb{R}})$ where $X^\mathbb{R}$ the underlying real analytic manifold of X.

Proposition 1.2.7. *Let $f : X \to Y$ be a morphism of complex analytic manifolds and let F, F_1, $F_2 \in \mathsf{D}^b_{\mathbb{C}-c}(k_X)$, $G \in \mathsf{D}^b_{\mathbb{C}-c}(k_Y)$. Then:*

i. $F_1 \otimes F_2$, $R\mathscr{H}om(F_1, F_2) \in \mathsf{D}^b_{\mathbb{C}-c}(k_X)$;

ii. $f^{-1}G$, $f^!G \in \mathsf{D}^b_{\mathbb{C}-c}(k_X)$;

iii. $Rf_*F \in \mathsf{D}^b_{\mathbb{C}-c}(k_Y)$ *if moreover* f *is proper on* $supp(F)$.

1.3 Bordered spaces

Definition 1.3.1. A *bordered space* $M_\infty = (M, \check{M})$ is a pair of a good topological space $\check{M}$ and an open subset $M \subset \check{M}$.

Let $M_\infty = (M, \check{M})$, $N_\infty = (N, \check{N})$ be two bordered spaces. For a continuous map $f : M \to N$ we denote by $\Gamma_f \subset M \times N$ its graph and by $\overline{\Gamma}_f$ the closure of Γ_f in $\check{M} \times \check{N}$. We denote by q_1, q_2 the projections:

$$\check{M} \xleftarrow{q_1} \check{M} \times \check{N} \xrightarrow{q_2} \check{N}\,.$$

Definition 1.3.2. A *morphism of bordered spaces* $f : M_\infty \to N_\infty$ is a continuous map $f : M \to N$ such that $q_1|_{\overline{\Gamma}_f} : \overline{\Gamma}_f \to \check{M}$ is proper. The composition of two morphisms of bordered spaces is given by the composition of the underlying continuous maps. If moreover $q_2|_{\overline{\Gamma}_f} : \overline{\Gamma}_f \to \check{N}$ is proper then we say that f is *semiproper*.

Remark. The bordered spaces together with the morphisms of bordered spaces form a category, in which the identity id_{M_∞} is given by id_M. Moreover the category of good topological spaces embeds into the category of bordered spaces by the identification $M = (M, M)$.

Definition 1.3.3. A *subanalytic bordered space* M_∞ is a bordered space $M_\infty = (M, \check{M})$ such that $\check{M}$ is a subanalytic space and M is an open subanalytic subset of $\check{M}$. A subset $U \subset M$ is *subanalytic in* M_∞ if U is a subanalytic subset of $\check{M}$. Let $N_\infty = (N, \check{N})$ be another subanalytic bordered space. A *morphism of subanalytic bordered spaces* is a morphism $f : M_\infty \to N_\infty$ of bordered spaces whose graph is subanalytic in $\check{M} \times \check{N}$.

Let M_∞ be a subanalytic bordered space.

Definition 1.3.4. We say that a sheaf on M is an $\mathbb{R}$*-constructible sheaf on* M_∞ if it can be extended to an $\mathbb{R}$-constructible sheaf on $\check{M}$. We denote by $\mathrm{Mod}_{\mathbb{R}-c}(k_{M_\infty})$ the full subcategory of $\mathrm{Mod}(k_{M_\infty})$ consisting of $\mathbb{R}$-constructible sheaves on M_∞, and by $\mathsf{D}^b_{\mathbb{R}-c}(k_{M_\infty})$ its bounded derived category.

Definition 1.3.5. A *complex bordered space* $X_\infty = (X, \check{X})$ is a pair of a complex manifold $\check{X}$ and an open subset $X \subset \check{X}$ such that $\check{X} \setminus X$ is a complex analytic subset of $\check{X}$.

Definition 1.3.6. A *morphism of complex bordered spaces* $f : X_\infty \to Y_\infty$ is a complex analytic map $f : X \to Y$ such that $\overline{\Gamma}_f$ is a complex analytic subset of $\check{X} \times \check{Y}$ and $q_1|_{\overline{\Gamma}_f} : \overline{\Gamma}_f \to \check{X}$ is proper. If moreover $q_2|_{\overline{\Gamma}_f} : \overline{\Gamma}_f \to \check{Y}$ is proper then we say that f is *semiproper*.

1.4 Indsheaves

Let $\mathscr{C}$ be a category. We denote by $\mathscr{C}^\vee$ the category of functors from $\mathscr{C}^{\mathrm{op}}$ to **Set** and by h the Yoneda embedding $h : \mathscr{C} \to \mathscr{C}^\vee$ given by $X \mapsto \mathrm{Hom}_{\mathscr{C}}(\,\cdot\,, X)$. We denote by "$\varinjlim$" the inductive limit in $\mathscr{C}^\vee$, i.e. if I is a small filtrant category and $a : I \to \mathscr{C}$ is an inductive system then "$\varinjlim$"$a = \varinjlim(h \circ a)$, and so "$\varinjlim$"$a : \mathscr{C} \ni X \mapsto \varinjlim_{i\in I} \mathrm{Hom}_{\mathscr{C}}(X, a(i)) \in \mathbf{Set}$.

Definition 1.4.1. An object $F \in \mathscr{C}^\vee$ is an *ind-object* if there exists a small filtrant category I and an inductive system $a : I \to \mathscr{C}$ such that $F \simeq$ "$\varinjlim_{i\in I}$"a. We denote by $\mathrm{Ind}(\mathscr{C})$ the full subcategory of $\mathscr{C}^\vee$ consisting of ind-objects.

Consider now a good topological space M and a field k. We denote by $\mathrm{Mod}^c(k_M)$ the full subcategory of $\mathrm{Mod}(k_M)$ consisting of sheaves with compact support.

Definition 1.4.2. An *indsheaf* is an object in $\mathrm{Ind}(\mathrm{Mod}^c(k_M)) =: \mathrm{I}(k_M)$ (or $\mathrm{I}k_M$ if there's no risk of confusion), i.e. is an ind-object in the category of sheaves with compact support.

We have a natural embedding of the category of sheaves into the category of indsheaves:

$$\begin{aligned} \iota : \mathrm{Mod}(k_M) &\longrightarrow \mathrm{I}(k_M) \\ F &\longmapsto \text{“}\varinjlim_{U\subset M}\text{”} F_U \end{aligned}$$

with U relatively compact open subset of M. The functor ι is fully faithful and admits an exact left adjoint:

$$\begin{aligned} \alpha : \mathrm{I}(k_M) &\longrightarrow \mathrm{Mod}(k_M) \\ \text{“}\varinjlim_{i\in I}\text{”} F_i &\longmapsto \varinjlim_{i\in I} F_i \end{aligned} \quad . \tag{1.1}$$

Moreover α admits an exact fully faithful left adjoint, denoted by β.

Remark. Let $F = \text{“}\varinjlim_i\text{”} F_i$, $G = \text{“}\varinjlim_j\text{”} G_j \in \mathrm{I}(k_M)$ with $F_i, G_j \in \mathrm{Mod}^c(k_M)$. In $\mathrm{I}(k_M)$ there is an inner hom-functor $\mathscr{I}hom(F,G) := \varprojlim_i \text{“}\varinjlim_j\text{”} \mathscr{H}om(F_i, G_j)$. We set $\mathscr{H}om := \alpha \circ \mathscr{I}hom$. We have also a tensor product, defined as $F \otimes G := \text{“}\varinjlim_{i,j}\text{”} F_i \otimes G_j$.

Definition 1.4.3. Let $f : M \to N$ be a morphism of good topological spaces, $F = \text{“}\varinjlim_i\text{”} F_i \in \mathrm{I}(k_M)$ and $G = \text{“}\varinjlim_i\text{”} G_i \in \mathrm{I}(k_N)$ with $F_i \in \mathrm{Mod}^c(k_M)$, $G_i \in \mathrm{Mod}^c(k_N)$. We define the functors f^{-1}, f_*, $f_{!!}$ as:

$$\begin{aligned} f^{-1} : \mathrm{I}(k_N) &\longrightarrow \mathrm{I}(k_M) \\ G &\longmapsto f^{-1}G = \text{“}\varinjlim_i\text{”}\text{“}\varinjlim_{U \subset M}\text{”} (f^{-1}G_i)_U \,, \end{aligned}$$

with U relatively compact in M,

$$\begin{aligned} f_* : \mathrm{I}(k_M) &\longrightarrow \mathrm{I}(k_N) \\ F &\longmapsto f_*F = \varprojlim_K \text{“}\varinjlim_i\text{”} f_* F_{iK} \end{aligned}$$

with K compact in M, and

$$\begin{aligned} f_{!!} : \mathrm{I}(k_M) &\longrightarrow \mathrm{I}(k_N) \\ F &\longmapsto f_{!!}F = \text{“}\varinjlim_i\text{”} f_* F_i \,. \end{aligned}$$

Notice that we denote the proper direct image of an indsheaf with $f_{!!}$ because in general $f_{!!} \circ \iota_M \neq \iota_N \circ f_!$.
If we take the bounded derived categories of $\mathrm{I}(k_M)$ and $\mathrm{I}(k_N)$, respectively $\mathsf{D}^b(\mathrm{I}k_M)$ and $\mathsf{D}^b(\mathrm{I}k_N)$, we can define the derived functors $\otimes$, $R\mathscr{I}hom(\cdot\,,\cdot)$, f^{-1}, Rf_*, $Rf_{!!}$. $Rf_{!!}$ admits a right adjoint, denoted by $f^!$: in this way we have obtained the six Grothendieck operations for indsheaves.

Now let $M_\infty = (M, \check{M})$ be a bordered space.

Definition 1.4.4. We define $\mathsf{D}^b(\mathrm{I}k_{M_\infty})$, the bounded derived category of indsheaves on M_∞, as $\mathsf{D}^b(\mathrm{Ind}(\mathrm{Mod}^c(k_{M_\infty})))$ where $\mathrm{Mod}^c(k_{M_\infty})$ denotes the full subcategory of $\mathrm{Mod}(k_{M_\infty})$ consisting of sheaves on M whose support is relatively compact in M_∞ (i. e. such that it is contained in a compact subset of $\check{M}$).

Remark. There is a natural equivalence of categories $\mathsf{D}^b(\mathrm{I}k_{M_\infty}) \simeq \mathsf{D}^b(\mathrm{I}k_{\check{M}})/\mathsf{D}^b(\mathrm{I}k_{\check{M}\setminus M})$ and a quotient functor $\mathsf{q} : \mathsf{D}^b(\mathrm{I}k_{\check{M}}) \to \mathsf{D}^b(\mathrm{I}k_{M_\infty})$. Moreover there is a natural exact embedding $\mathsf{D}^b(k_M) \rightarrowtail \mathsf{D}^b(\mathrm{I}k_{M_\infty})$ wich has an exact left adjoint α.

The functors $\cdot \otimes \cdot$, $R\mathscr{I}hom(\cdot\,,\cdot)$ in $\mathsf{D}^b(\mathrm{I}k_M)$ induce well-defined functors (which we denote in the same way) in $\mathsf{D}^b(\mathrm{I}k_{M_\infty})$.

Definition 1.4.5. Let $f : M_\infty \to N_\infty$ be a morphism of bordered spaces. For $F \in \mathsf{D}^b(\mathrm{I}k_{\check{M}})$, $G \in \mathsf{D}^b(\mathrm{I}k_{\check{N}})$ we define:

$$\begin{aligned} Rf_*F &= Rq_{2*}R\mathscr{I}hom(k_{\Gamma_f},\, q_1^!F), \\ f^{-1}G &= Rq_{1!!}(k_{\Gamma_f} \otimes q_2^{-1}G), \\ Rf_{!!}F &= Rq_{2!!}(k_{\Gamma_f} \otimes q_1^{-1}F), \\ f^!G &= Rq_{1*}R\mathscr{I}hom(k_{\Gamma_f},\, q_2^!G). \end{aligned}$$

These definitions induce well-defined functors $Rf_*, Rf_{!!} : \mathsf{D}^b(\mathrm{I}k_{M_\infty}) \to \mathsf{D}^b(\mathrm{I}k_{N_\infty})$ and $f^{-1}, f^! : \mathsf{D}^b(\mathrm{I}k_{N_\infty}) \to \mathsf{D}^b(\mathrm{I}k_{M_\infty})$ through the quotient functor q. Together with $\cdot \otimes \cdot$, $R\mathscr{I}hom(\cdot\,,\cdot)$ we have constructed the six Grothendieck operations for $\mathsf{D}^b(\mathrm{I}k_{M_\infty})$.

Chapter 2

Enhanced sheaves and indsheaves

Let M be a good topological space and let μ, q_1, $q_2 : M\times\mathbb{R}\times\mathbb{R} \to M\times\mathbb{R}$ be defined as $\mu(x,t_1,t_2) := (x,t_1+t_2)$, $q_1(x,t_1,t_2) := (x,t_1)$, $q_2(x,t_1,t_2) := (x,t_2)$. We'll use the notation $k_{\{t=0\}}$ (resp. $k_{\{t\geq a\}}$, $k_{\{t\leq a\}}$) to indicate $k_{M\times\{0\}}$ (resp. $k_{M\times\{t\in\mathbb{R}:\, t\geq a\}}$, $k_{M\times\{t\in\mathbb{R}:\, t\leq a\}}$) in $\mathsf{D}^b(k_{M\times\mathbb{R}})$.

2.1 Convolution and enhanced sheaves

Definition 2.1.1. We define the *convolution functors* in $\mathsf{D}^b(k_{M\times\mathbb{R}})$ as

$$F_1 \overset{+}{\otimes} F_2 := R\mu_!(q_1^{-1}F_1 \otimes q_2^{-1}F_2)\,,$$

$$R\mathscr{H}om^+(F_1,\, F_2) := Rq_{1*}R\mathscr{H}om(q_2^{-1}F_1,\, \mu^!F_2)\,.$$

The functor $\cdot \overset{+}{\otimes} \cdot$ is called *convolution product* in $\mathsf{D}^b(k_{M\times\mathbb{R}})$.

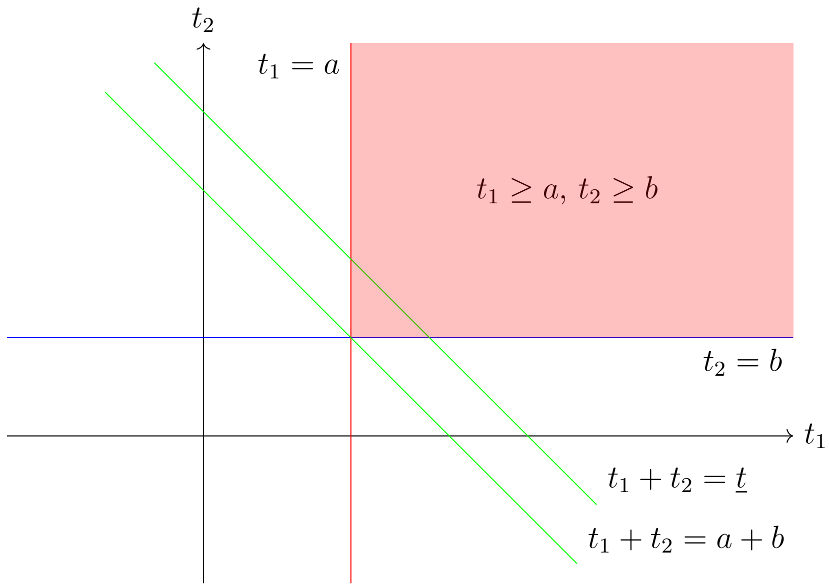
Remark. The convolution product makes $\mathsf{D}^b(k_{M\times\mathbb{R}})$ into a commutative tensor category, with $k_{\{t=0\}}$ as unit object.

Remark. In general $k_{\{t\geq a\}} \overset{+}{\otimes} k_{\{t\geq b\}} \simeq k_{\{t\geq a+b\}}$, in fact we have $k_{\{t\geq a\}} \overset{+}{\otimes} k_{\{t\geq b\}} \simeq R\mu_!(k_{\{(x,t_1,t_2)\in M\times\mathbb{R}\times\mathbb{R};\, t_1\geq a,\, t_2\geq b\}}) \xrightarrow{\varphi} R\mu_!(k_{\{(x,t_1,t_2)\in M\times\mathbb{R}\times\mathbb{R};\, t_1\geq a,\, t_2\geq b,\, t_1+t_2=a+b\}}) \simeq k_{\{t\geq a+b\}}$.
Let's compute the fibers of $k_{\{t\geq a\}} \overset{+}{\otimes} k_{\{t\geq b\}} \simeq k_{\{t\geq a+b\}}$ in order to show that φ is an isomorphism. We can regard M as the single-point space $\{\star\}$; then, fixed

$(\star,\,\underline{t}) \in M \times \mathbb{R}$, we have:

$$
\begin{aligned}
&(k_{\{t\geq a\}} \overset{+}{\otimes} k_{\{t\geq b\}})_{(\star,\underline{t})} = (R\mu_!(q_1^{-1}k_{\{t\geq a\}} \otimes q_2^{-1}k_{\{t\geq b\}}))_{(\star,\underline{t})}\\
&\simeq R\Gamma_c(\mu^{-1}(\star,\,\underline{t});\; q_1^{-1}k_{\{t\geq a\}} \otimes q_2^{-1}k_{\{t\geq b\}}|_{\mu^{-1}(\star,\underline{t})})\\
&= R\Gamma_c(\{(\star,\,t_1,\,t_2) : t_1,t_2 \in \mathbb{R},\, t_1+t_2=\underline{t}\};\; q_1^{-1}k_{\{t\geq a\}} \otimes q_2^{-1}k_{\{t\geq b\}}|_{\mu^{-1}(\star,\underline{t})})\\
&\simeq R\Gamma_c(\{(t_1,\,t_2) : t_1,t_2 \in \mathbb{R},\, t_1+t_2=\underline{t}\};\; k_{\{(t_1,t_2): t_1,t_2\in\mathbb{R},\, t_1\geq a,\, t_2\geq b\}}|_{\{(t_1,t_2):\, t_1,t_2\in\mathbb{R}, t_1+t_2=\underline{t}\}})\\
&= \begin{cases} 0 & \text{if } \underline{t} < a+b \\ k & \text{if } \underline{t} \geq a+b \end{cases},
\end{aligned}
$$

so φ is an isomorphism.



Moreover we have $k_{\{t\geq a\}} \overset{+}{\otimes} k_{\{t\geq 0\}} \simeq k_{\{t\geq a\}}$, $k_{\{t>a\}} \overset{+}{\otimes} k_{\{t\geq 0\}} \simeq 0$, $k_{\{t>a\}} \overset{+}{\otimes} k_{\{t>0\}} \simeq k_{\{t>a\}}[-1]$, $k_{\{t\leq a\}} \overset{+}{\otimes} k_{\{t\geq 0\}} \simeq 0$, $k_{\{t<a\}} \overset{+}{\otimes} k_{\{t\geq 0\}} \simeq k_{\{t\geq a\}}[-1]$ and $k_{\{t<a\}} \overset{+}{\otimes} k_{\{t>0\}} \simeq k_{M\times\mathbb{R}}[-1]$.

Definition 2.1.2. We define the category of *enhanced sheaves* as the quotient category $\mathrm{E}^b_+(k_M) := \mathsf{D}^b(k_{M\times\mathbb{R}})/\mathscr{N}$, where $\mathscr{N}$ is the full subcategory of $\mathsf{D}^b(k_{M\times\mathbb{R}})$ defined as $\{F \in \mathsf{D}^b(k_{M\times\mathbb{R}}) :\ F \overset{+}{\otimes} k_{\{t\geq 0\}} \simeq 0\}$.

Remark. The quotient functor $Q : \mathsf{D}^b(k_{M\times\mathbb{R}}) \to \mathrm{E}^b_+(k_M)$ induces an equivalence of categories: $\{F \in \mathsf{D}^b(k_{M\times\mathbb{R}}) :\ F \overset{+}{\otimes} k_{\{t\geq 0\}} \simeq F\} \xrightarrow{\sim} \mathrm{E}^b_+(k_M)$. Moreover Q admits fully faithful left and right adjoints defined respectively as $L^E(QF) := k_{\{t\geq 0\}} \overset{+}{\otimes} F$

and $R^E(QF) := R\mathscr{H}om^+(k_{\{t\geq 0\}},\, F)$.
There is also a natural embedding $\epsilon : \mathsf{D}^b(k_M) \rightarrowtail \mathrm{E}^b_+(k_M)$, $F \mapsto Q(k_{\{t\geq 0\}} \otimes \pi^{-1}F)$ where $\pi : M \times \mathbb{R} \to M$ is the projection.
The natural t-structure of $\mathsf{D}^b(k_{M\times\mathbb{R}})$ induces by L^E a t-structure for $\mathrm{E}^b_+(k_M)$, and we denote by $\mathrm{E}^0_+(k_M) = \{K \in \mathrm{E}^b_+(k_M); H^jL^E(K) = 0 \text{ for any } j \neq 0\}$ its heart.

2.2 Operations on enhanced sheaves

Consider a morphism of good topological spaces $f : M \to N$ and denote $f_\mathbb{R} := f \times id_\mathbb{R} : M \times \mathbb{R} \to N \times \mathbb{R}$.

Definition 2.2.1. Let $F \in \mathsf{D}^b(k_{M\times\mathbb{R}})$ and $G \in \mathsf{D}^b(k_{N\times\mathbb{R}})$. We define the functors on enhanced sheaves Ef_*, $Ef_!$, Ef^{-1}, $Ef^!$ as $Ef_*(QF) := Q(Rf_{\mathbb{R}*}F)$, $Ef_!(QF) := Q(Rf_{\mathbb{R}!}F)$, $Ef^{-1}(QG) := Q(f_\mathbb{R}^{-1}G)$, $Ef^!(QG) := Q(f_\mathbb{R}^!G)$. Moreover we define $\cdot \overset{+}{\otimes} \cdot$ and $R\mathscr{H}om^+(\cdot,\, \cdot)$ for enhanced sheaves as the functors induced from the ones of $\mathsf{D}^b(k_{M\times\mathbb{R}})$. These functors are the six operations for enhanced sheaves.

There are some useful relations that hold also for the operations of enhanced sheaves, e.g the analogue of the projection formula for usual sheaves. Let's prove some of them:

Proposition 2.2.2. *Let $f : M \to N$ be a morphism of good topological spaces and $K \in \mathsf{E}^b(k_M)$, L, L_1, $L_2 \in \mathsf{E}^b(k_N)$. Then:*

i. $Ef^{-1}L_1 \overset{+}{\otimes} Ef^{-1}L_2 \simeq Ef^{-1}(L_1 \overset{+}{\otimes} L_2)$;

ii. $Ef_!K \overset{+}{\otimes} L \simeq Ef_!(K \overset{+}{\otimes} Ef^{-1}L)$;

iii. $R\mathscr{H}om^+(L,\, Ef_*K) \simeq Ef_*R\mathscr{H}om^+(Ef^{-1}L,\, K)$;

iv. $R\mathscr{H}om^+(Ef_!K,\, L) \simeq Ef_*R\mathscr{H}om^+(K,\, Ef^!L)$;

v. $Ef^!R\mathscr{H}om^+(L_1,\, L_2) \simeq R\mathscr{H}om^+(Ef^{-1}LG_1,\, Ef^!L_2)$.

Proof. Consider the projections $\pi_M : M \times \mathbb{R} \to M$, $\pi_N : N \times \mathbb{R} \to N$ and the maps $\mu, q_1, q_2 : M \times \mathbb{R} \times \mathbb{R} \to M \times \mathbb{R}$ defined above. Define analogously $\mu', q_1', q_2' : N\times\mathbb{R}\times\mathbb{R} \to N\times\mathbb{R}$ and put $f_{\mathbb{R}^2} := f\times id_\mathbb{R}\times id_\mathbb{R} : M\times\mathbb{R}\times\mathbb{R} \to N\times\mathbb{R}\times\mathbb{R}$. We have the following Cartesian squares:

$$\begin{array}{ccc} M\times\mathbb{R}\times\mathbb{R} & \xrightarrow{q_1} & M\times\mathbb{R} \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\pi_M} \\ M\times\mathbb{R} & \xrightarrow{\pi_M} & M \end{array}, \quad \begin{array}{ccc} M\times\mathbb{R}\times\mathbb{R} & \xrightarrow{q_2} & M\times\mathbb{R} \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\pi_M} \\ M\times\mathbb{R} & \xrightarrow{\pi_M} & M \end{array}, \quad \begin{array}{ccc} M\times\mathbb{R}\times\mathbb{R} & \xrightarrow{q_1} & M\times\mathbb{R} \\ \downarrow{\scriptstyle q_2} & & \downarrow{\scriptstyle\pi_M} \\ M\times\mathbb{R} & \xrightarrow{\pi_M} & M \end{array},$$

and the same with N, μ', q_1', q_2', π_N instead of M, μ, q_1, q_2, π_M, and:

$$\begin{array}{ccc}
M\times\mathbb{R}\times\mathbb{R} & \xrightarrow{\mu} & M\times\mathbb{R} \\
\downarrow{\scriptstyle f_{\mathbb{R}^2}} & & \downarrow{\scriptstyle f_{\mathbb{R}}} \\
N\times\mathbb{R}\times\mathbb{R} & \xrightarrow{\mu'} & N\times\mathbb{R}
\end{array}\ ,\quad
\begin{array}{ccc}
M\times\mathbb{R}\times\mathbb{R} & \xrightarrow{q_1} & M\times\mathbb{R} \\
\downarrow{\scriptstyle f_{\mathbb{R}^2}} & & \downarrow{\scriptstyle f_{\mathbb{R}}} \\
N\times\mathbb{R}\times\mathbb{R} & \xrightarrow{q_1'} & N\times\mathbb{R}
\end{array}\ ,\quad
\begin{array}{ccc}
M\times\mathbb{R}\times\mathbb{R} & \xrightarrow{q_2} & M\times\mathbb{R} \\
\downarrow{\scriptstyle f_{\mathbb{R}^2}} & & \downarrow{\scriptstyle f_{\mathbb{R}}} \\
N\times\mathbb{R}\times\mathbb{R} & \xrightarrow{q_2'} & N\times\mathbb{R}
\end{array}\ .$$

Remember that in general for a Cartesian square of locally compact spaces with continuous maps

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\
X' & \xrightarrow{f'} & Y'
\end{array}$$

it holds $Rf_! \circ g^{-1} \simeq g'^{-1} \circ Rf'_!$ and $Rg_* \circ f^! \simeq f'^! \circ Rg'_*$.
We will use also some known isomorphisms of sheaf functors (proofs can be found in **[14]**).
Let $F \in \mathsf{D}^b(k_{M\times\mathbb{R}})$, $G, G_1, G_2 \in \mathsf{D}^b(k_{N\times\mathbb{R}})$ be such that $K = QF$ and $L = QG$, $L_1 = QG_1$, $L_2 = QG_2$.
i.

$$\begin{aligned}
Ef^{-1}(QG_1) \overset{+}{\otimes} Ef^{-1}(QG_2) &= Qf_{\mathbb{R}}^{-1}G_1 \overset{+}{\otimes} Qf_{\mathbb{R}}^{-1}G_2 \\
&= QR\mu_!(q_1^{-1}f_{\mathbb{R}}^{-1}G_1 \otimes q_2^{-1}f_{\mathbb{R}}^{-1}G_2) \\
&\simeq QR\mu_!(f_{\mathbb{R}^2}^{-1}q_1'^{-1}G_1 \otimes f_{\mathbb{R}^2}^{-1}q_2'^{-1}G_2) \\
&\simeq QR\mu_! f_{\mathbb{R}^2}^{-1}(q_1'^{-1}G_1 \otimes q_2'^{-1}G_2)) \\
&\simeq Qf_{\mathbb{R}}^{-1}R\mu'_!(q_1'^{-1}G_1 \otimes q_2'^{-1}G_2) \\
&= Ef^{-1}(QR\mu'_!(q_1'^{-1}G_1 \otimes q_2'^{-1}G_2)) \\
&= Ef^{-1}(QG_1 \overset{+}{\otimes} QG_2)\,;
\end{aligned}$$

ii.

$$\begin{aligned}
Ef_!(QF) \overset{+}{\otimes} QG &= QRf_{\mathbb{R}!}F \overset{+}{\otimes} QG \\
&= QR\mu'_!(q_1'^{-1}Rf_{\mathbb{R}!}F \otimes q_2'^{-1}G_2) \\
&\simeq QR\mu'_!(Rf_{\mathbb{R}^2!}q_1^{-1}F \otimes q_2'^{-1}G_2) \\
&\simeq QR\mu'_!Rf_{\mathbb{R}^2!}(q_1^{-1}F \otimes Rf_{\mathbb{R}^2}^{-1}q_2'^{-1}G_2) \\
&\simeq QR\mu'_!Rf_{\mathbb{R}^2!}(q_1^{-1}F \otimes q_2^{-1}Rf_{\mathbb{R}}^{-1}G_2) \\
&\simeq QRf_{\mathbb{R}!}R\mu_!(q_1^{-1}F \otimes q_2^{-1}Rf_{\mathbb{R}}^{-1}G_2) \\
&= Ef_!(QR\mu_!(q_1^{-1}F \otimes Rf_{\mathbb{R}^2}^{-1}q_2'^{-1}G_2)) \\
&= Ef_!(QF \overset{+}{\otimes} QRf_{\mathbb{R}}^{-1}G_2)) \\
&= Ef_!(QF \overset{+}{\otimes} Ef^{-1}(QG))\,;
\end{aligned}$$

iii.

$$
\begin{aligned}
R\mathscr{H}om^{+}(QG,\, Ef_*(QF)) &= R\mathscr{H}om^{+}(QG,\, QRf_{\mathbb{R}*}F)\\
&= QRq'_{1*}(R\mathscr{H}om(q_2'^{-1}G,\, \mu'^{!}Rf_{\mathbb{R}*}F))\\
&\simeq QRq'_{1*}(R\mathscr{H}om(q_2'^{-1}G,\, Rf_{\mathbb{R}^2*}\mu^{!}F))\\
&\simeq QRq'_{1*}(Rf_{\mathbb{R}^2*}R\mathscr{H}om(Rf_{\mathbb{R}^2}^{-1}q_2'^{-1}G,\, \mu^{!}F))\\
&\simeq QRq'_{1*}Rf_{\mathbb{R}^2*}R\mathscr{H}om(q_2^{-1}Rf_{\mathbb{R}}^{-1}G,\, \mu^{!}F)\\
&\simeq QRf_{\mathbb{R}*}Rq_{1*}R\mathscr{H}om(q_2^{-1}Rf_{\mathbb{R}}^{-1}G,\, \mu^{!}F)\\
&= Ef_*(QRq_{1*}R\mathscr{H}om(q_2^{-1}Rf_{\mathbb{R}}^{-1}G,\, \mu^{!}F))\\
&= Ef_*(Rq_{1*}R\mathscr{H}om(QRf_{\mathbb{R}}^{-1}G,\, QF))\\
&= Ef_*R\mathscr{H}om^{+}(Ef^{-1}(QG),\, QF)\,;
\end{aligned}
$$

iv.

$$
\begin{aligned}
R\mathscr{H}om^{+}(Ef_!(QF),\, QG) &= R\mathscr{H}om^{+}(QRf_{\mathbb{R}!}F,\, QG))\\
&= QRq'_{1*}(R\mathscr{H}om(q_2'^{-1}Rf_{\mathbb{R}!}F,\, \mu'^{!}G))\\
&\simeq QRq'_{1*}(R\mathscr{H}om(Rf_{\mathbb{R}^2!}q_2^{-1}F,\, \mu'^{!}G))\\
&\simeq QRq'_{1*}(Rf_{\mathbb{R}^2*}R\mathscr{H}om(q_2^{-1}F,\, f_{\mathbb{R}^2}^{!}\mu'^{!}G))\\
&\simeq QRq'_{1*}Rf_{\mathbb{R}^2*}R\mathscr{H}om(q_2^{-1}F,\, \mu^{!}f_{\mathbb{R}}^{!}G)\\
&\simeq QRf_{\mathbb{R}*}Rq_{1*}R\mathscr{H}om(q_2^{-1}F,\, \mu^{!}f_{\mathbb{R}}^{!}G)\\
&= Ef_*(QRq_{1*}R\mathscr{H}om(q_2^{-1}F,\, \mu^{!}f_{\mathbb{R}}^{!}G))\\
&= Ef_*R\mathscr{H}om^{+}(QF,\, Qf_{\mathbb{R}}^{!}G)\\
&= Ef_*R\mathscr{H}om^{+}(QF,\, Ef^{!}(QG))\,;
\end{aligned}
$$

v.

$$
\begin{aligned}
Ef^{!}R\mathscr{H}om^{+}(QG_1,\, QG_2) &= Ef^{!}(QRq'_{1*}R\mathscr{H}om(q_2'^{-1}G_1,\, \mu'^{!}G_2))\\
&= Qf_{\mathbb{R}}^{!}Rq'_{1*}R\mathscr{H}om(q_2'^{-1}G_1,\, \mu'^{!}G_2)\\
&\simeq QRq_{1*}f_{\mathbb{R}^2}^{!}R\mathscr{H}om(q_2'^{-1}G_1,\, \mu'^{!}G_2)\\
&\simeq QRq_{1*}R\mathscr{H}om(f_{\mathbb{R}^2}^{-1}q_2'^{-1}G_1,\, f_{\mathbb{R}^2}^{!}\mu'^{!}G_2)\\
&\simeq QRq_{1*}R\mathscr{H}om(q_2^{-1}f_{\mathbb{R}}^{-1}G_1,\, \mu^{!}f_{\mathbb{R}}^{!}G_2)\\
&= R\mathscr{H}om^{+}(Q(f_{\mathbb{R}}^{-1}G_1),\, Q(f_{\mathbb{R}}^{!}G_2))\\
&= R\mathscr{H}om^{+}(Ef^{-1}(QG_1),\, Ef^{!}(QG_2))\,.
\end{aligned}
$$

$\square$

2.3 $\mathbb{R}$-constructible enhanced sheaves

Denote by $\mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}})$ the bordered space in which $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$ is the two-point compactification of the real line. Notice that $\mathbb{R}_\infty$ is isomorphic to $(\mathbb{R}, \mathbb{P}^1(\mathbb{R}))$ where $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ is the real projective line.
Consider a subanalytic space M and a subanalytic bordered space M_∞: then also $M_\infty \times \mathbb{R}_\infty$ is a subanalytic bordered space.

Definition 2.3.1. We define the category $\mathrm{E}^b_{\mathbb{R}-c}(k_M)$ (resp. $\mathrm{E}^b_{\mathbb{R}-c}(k_{M_\infty})$) of $\mathbb{R}$-constructible enhanced sheaves on M (resp. on M_∞) as the full triangulated subcategory of $\mathsf{D}^b_{\mathbb{R}-c}(k_{M\times\mathbb{R}_\infty})$ (resp. of $\mathsf{D}^b_{\mathbb{R}-c}(k_{M_\infty\times\mathbb{R}_\infty})$) whose objects K satisfy the condition $K \overset{+}{\otimes} k_{\{t\geq 0\}} \xrightarrow{\sim} K$. The heart of the t-structure of $\mathrm{E}^b_{\mathbb{R}-c}(k_M)$ is denoted by $\mathrm{E}^0_{\mathbb{R}-c}(k_M)$.

Remark. If $f : M \to N$ is a semiproper morphism of real analytic manifolds then the six operations send $\mathbb{R}$-constructible enhanced sheaves into $\mathbb{R}$-constructible enhanced sheaves; in particular the convolution functors send $\mathbb{R}$-constructible enhanced sheaves into $\mathbb{R}$-constructible enhanced sheaves.

Remark. If X is a complex manifold then an $\mathbb{R}$-constructible sheaf on X is defined as an $\mathbb{R}$-constructible sheaf on the underlying real analytic manifold $X^{\mathbb{R}}$.

2.4 Exponential enhanced sheaves

Let M be a good topological space.

Definition 2.4.1. Let $U \subset M$ be an open subset and let $\varphi, \varphi^+, \varphi^- : U \to \mathbb{R}$ be continuous functions with $\varphi^-(x) \leq \varphi^+(x)$ for any $x \in U$. The associated *exponential enhanced sheaves* are defined by, respectively, $\mathsf{E}^{\varphi}_{U|M} := Qk_{\{t+\varphi\geq 0\}}$ and $\mathsf{E}^{\varphi^+\triangleright\varphi^-}_{U|M} := Qk_{\{-\varphi^+\leq t<-\varphi^-\}}$, where $\{t+\varphi \geq 0\}$ denotes $\{(x,t) \in U\times\mathbb{R}; t+\varphi(x) \geq 0\}$ and similarly for $\{-\varphi^+ \leq t < -\varphi^-\}$. If $U = M$ we write E^{φ} and $\mathsf{E}^{\varphi^+\triangleright\varphi^-}$.

Remark. Notice that $L^E(\mathsf{E}^{\varphi}_{U|M}) \simeq k_{\{t+\varphi\geq 0\}}$ and $L^E(\mathsf{E}^{\varphi^+\triangleright\varphi^-}_{U|M}) \simeq k_{\{-\varphi^+\leq t<-\varphi^-\}}$, and so $\mathsf{E}^{\varphi}_{U|M}, \mathsf{E}^{\varphi^+\triangleright\varphi^-}_{U|M} \in \mathrm{E}^0_+(k_M)$. Moreover the exact sequence in $\mathsf{D}^b(k_{M\times\mathbb{R}})$

$$0 \to k_{\{-\varphi^+\leq t<-\varphi^-\}} \to k_{\{t+\varphi^+\geq 0\}} \to k_{\{t+\varphi^-\geq 0\}} \to 0$$

induces the exact sequence in $\mathrm{E}^0_+(k_M)$

$$0 \to \mathsf{E}^{\varphi^+\triangleright\varphi^-}_{U|M} \to \mathsf{E}^{\varphi^+}_{U|M} \to \mathsf{E}^{\varphi^-}_{U|M} \to 0\,.$$

Remark. If we have $\varphi, \psi : U \to \mathbb{R}$ then $\mathsf{E}^{\varphi}_{U|M} \overset{+}{\otimes} \mathsf{E}^{\psi}_{U|M} \simeq \mathsf{E}^{\varphi+\psi}_{U|M}$. It can be proven analogously to $k_{\{t\geq a\}} \overset{+}{\otimes} k_{\{t\geq b\}} \simeq k_{\{t\geq a+b\}}$.

Let $M_\infty = (M, \check{M})$ be a subanalytic bordered space.

Definition 2.4.2. Let U be an open subanalytic subset of M_∞. A function $\varphi : U \to \mathbb{R}$ is *globally subanalytic* if its graph is subanalytic in $M_\infty \times \mathbb{R}_\infty$.

Remark. If φ, φ^+, $\varphi^- : U \to \mathbb{R}$ are continuous globally subanalytic functions with $\varphi^-(x) \leq \varphi^+(x)$ for any $x \in U$ then $\mathsf{E}^{\varphi}_{U|M_\infty}$, $\mathsf{E}^{\varphi^+ \rhd \varphi^-}_{U|M_\infty} \in \mathrm{E}^0_{\mathbb{R}-c}(k_{M_\infty})$.

2.5 Enhanced indsheaves

Let M be a good topological space and let $M_\infty = (M, \check{M})$ be a bordered space; consider the morphisms μ, q_1, q_2 : $M \times \mathbb{R}_\infty \times \mathbb{R}_\infty \to M \times \mathbb{R}_\infty$ and μ, q_1, q_2 : $M_\infty \times \mathbb{R}_\infty \times \mathbb{R}_\infty \to M_\infty \times \mathbb{R}_\infty$ induced by the ones defined above.

Definition 2.5.1. We define the *convolution functors* in $\mathsf{D}^b(\mathrm{I}k_{M_\infty \times \mathbb{R}_\infty})$ as $F_1 \overset{+}{\otimes} F_2 := R\mu_{!!}(q_1^{-1}F_1 \otimes q_2^{-1}F_2)$ and $\mathscr{I}hom^+(F_1, F_2) := Rq_{1*}R\mathscr{I}hom(q_2^{-1}F_1, \mu^! F_2)$.

We will keep the notations $k_{\{t=0\}}$, $k_{\{t\geq a\}}$, $k_{\{t\leq a\}}$ as above, with $M_\infty \times \mathbb{R}_\infty$ instead of $M \times \mathbb{R}$ where $k_{\{t=0\}}$, $k_{\{t\geq a\}}$, $k_{\{t\leq a\}}$ are regarded as objects of $\mathsf{D}^b(\mathrm{I}k_{M_\infty \times \mathbb{R}_\infty})$.

Remark. The convolution product makes $\mathsf{D}^b(\mathrm{I}k_{M_\infty \times \mathbb{R}_\infty})$ into a commutative tensor category, with $k_{\{t=0\}}$ as unit object.

Definition 2.5.2. We define the category of *enhanced indsheaves* as the quotient category $\mathrm{E}^b_+(\mathrm{I}k_M) := \mathsf{D}^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})/\mathscr{N}$ (or $\mathrm{E}^b_+(\mathrm{I}k_{M_\infty}) := \mathsf{D}^b(\mathrm{I}k_{M_\infty\times\mathbb{R}_\infty})/\mathscr{N}$), where $\mathscr{N}$ is the full subcategory of $\mathsf{D}^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$ defined as $\{F \in \mathsf{D}^b(\mathrm{I}k_{M\times\mathbb{R}_\infty}) : F \overset{+}{\otimes} k_{\{t\geq 0\}} \simeq 0\}$ (or the full subcategory of $\mathsf{D}^b(\mathrm{I}k_{M_\infty\times\mathbb{R}_\infty})$ defined as $\{F \in \mathsf{D}^b(\mathrm{I}k_{M_\infty\times\mathbb{R}_\infty}) : F \overset{+}{\otimes} k_{\{t\geq 0\}} \simeq 0\}$).

Remark. The quotient functor $Q : \mathsf{D}^b(\mathrm{I}k_{M\times\mathbb{R}_\infty}) \to \mathrm{E}^b_+(\mathrm{I}k_M)$ induces an equivalence of categories as for the enhanced sheaves:

$$\{F \in \mathsf{D}^b(\mathrm{I}k_{M\times\mathbb{R}_\infty}) :\ F \overset{+}{\otimes} k_{\{t\geq 0\}} \simeq F\} \xrightarrow{\sim} \mathrm{E}^b_+(\mathrm{I}k_{M_\infty}).$$

Moreover Q admits fully faithful left and right adjoints L^E and R^E defined as for the enhanced sheaves. The same holds with M_∞ instead of M.

We have the natural embeddings $\mathrm{E}^b_+(k_M) \rightarrowtail \mathrm{E}^b_+(\mathrm{I}k_{M_\infty})$ and $\epsilon : \mathsf{D}^b(\mathrm{I}k_{M_\infty}) \rightarrowtail \mathrm{E}^b_+(\mathrm{I}k_{M_\infty})$ where ϵ is defined as for the enhanced sheaves.

We denote by $\mathrm{E}^0_+(\mathrm{I}k_M)$ (or $\mathrm{E}^0_+(\mathrm{I}k_{M_\infty})$) the heart of the natural t-structure of $\mathrm{E}^b_+(\mathrm{I}k_M)$ (or of $\mathrm{E}^b_+(\mathrm{I}k_{M_\infty})$).

Definition 2.5.3. Let $f : M \to N$ be a morphism of good topological spaces (or let $f : M_\infty \to N_\infty$ be a morphism of bordered spaces). We define the *six operations for enhanced indsheaves* as the functors $\cdot \overset{+}{\otimes} \cdot$, $\mathscr{I}hom^+(\cdot, \cdot)$, Ef_*, $Ef_{!!}$, Ef^{-1}, $Ef^!$ induced by the functors $\cdot \overset{+}{\otimes} \cdot$, $\mathscr{I}hom^+(\cdot, \cdot)$, $Rf_{\mathbb{R}_\infty *}$, $Rf_{\mathbb{R}_\infty !!}$, $f^{-1}_{\mathbb{R}_\infty}$, $f^!_{\mathbb{R}_\infty}$ for $\mathsf{D}^b(\mathrm{I}k_{M\times\mathbb{R}_\infty})$ (or for $\mathsf{D}^b(\mathrm{I}k_{M_\infty\times\mathbb{R}_\infty})$).

Remark. If $f : M \to N$ is a morphism of good topological spaces and $K \in \mathrm{E}^b(\mathrm{I}k_M)$, L, L_1, $L_2 \in \mathrm{E}^b(\mathrm{I}k_N)$ then the same isomorphisms as the ones in Proposition 2.2.2 hold by changing $Ef_!$ and $R\mathscr{H}om^+(\cdot, \cdot)$ with $Ef_{!!}$ and $\mathscr{I}hom^+(\cdot, \cdot)$.

Consider the projection $\pi : M_\infty \times \mathbb{R}_\infty \to M_\infty$. We define the *outer hom functors* with values respectively in $\mathsf{D}^b(\mathrm{I}k_{M_\infty})$ and in $\mathsf{D}^b(k_{M_\infty})$ as respectively:

$$\mathscr{I}hom^E(K_1, K_2) := R\pi_* R\mathscr{I}hom(L^E K_1, R^E K_2)\,,$$

$$\mathscr{H}om^E(K_1, K_2) := \alpha R\mathscr{I}hom^E(K_1, K_2)$$

where α is induced by the functor (1.1).
We define also $RHom^E(K_1, K_2) := R\Gamma(M; \mathscr{H}om^E(K_1, K_2)) \in \mathsf{D}^b(k)$.
Consider the projections $p_1 : M_\infty \times N_\infty \to M_\infty$, $p_2 : M_\infty \times N_\infty \to N_\infty$ and let $K \in \mathrm{E}^b_+(\mathrm{I}k_{M_\infty})$, $L \in \mathrm{E}^b_+(\mathrm{I}k_{N_\infty})$. We define their *external tensor product* as $K \overset{+}{\boxtimes} L := Ep_1^{-1} K \overset{+}{\otimes} Ep_2^{-1} L$.

We denote by $k^E_M := Q(\text{“}\varinjlim_{c\to+\infty}\text{”} k_{\{t\geq c\}}) \in \mathrm{E}^b_+(\mathrm{I}k_M)$ and by $k^E_{M_\infty} := Ej^{-1}(k^E_{\check{M}}) \in \mathrm{E}^b_+(\mathrm{I}k_{M_\infty})$ where $j : M_\infty \to \check{M}$ is the natural morphism.

Lemma 2.5.4. *The functor $k^E_{M_\infty} \overset{+}{\otimes} \cdot$ is an exact functor.*

Definition 2.5.5. A *stable object* is an object $K \in \mathrm{E}^b_+(\mathrm{I}k_M)$ such that

$$K \xleftarrow{\simeq} k_{\{t\geq 0\}} \overset{+}{\otimes} K \xrightarrow{\simeq} k_{\{t\geq a\}} \overset{+}{\otimes} K$$

for any $a \geq 0$ or, equivalently, such that

$$k_{\{t\geq 0\}} \overset{+}{\otimes} K \simeq k^E_M \overset{+}{\otimes} K\,.$$

Proposition 2.5.6. *Let $f : M \to N$ be a continuous map of good topological spaces and let $K \in \mathrm{E}^b_+(\mathrm{I}k_M)$, $L \in \mathrm{E}^b_+(\mathrm{I}k_N)$. Then:*

i. $Ef_{!!}(k^E_M \overset{+}{\otimes} K) \simeq k^E_N \overset{+}{\otimes} Ef_{!!} K$;

ii. $Ef^{-1}(k_N^E \overset{+}{\otimes} L) \simeq k_M^E \overset{+}{\otimes} Ef^{-1}L$;

iii. $Ef^!(k_N^E \overset{+}{\otimes} L) \simeq k_M^E \overset{+}{\otimes} Ef^!L$.

Thus $Ef_{!!}$, Ef^{-1} and $Ef^!$ send stable objects into stable objects.

Assume now that M_∞ is a subanalytic bordered space and consider the projection $\pi : M_\infty \times \mathbb{R}_\infty \to M_\infty$.

Definition 2.5.7. We say that an object $K \in \mathrm{E}^b_+(\mathrm{I}k_M)$ (or $K \in \mathrm{E}^b_+(\mathrm{I}k_{M_\infty})$) is $\mathbb{R}$-*constructible* if for any relatively compact subanalytic open subset $U \subset M$ (or for any subanalytic open subset $U \subset M$ relatively compact in $\check{M}$) there exists $F \in \mathsf{D}^b_{\mathbb{R}-c}(k_{M\times\mathbb{R}_\infty})$ (or $F \in \mathsf{D}^b_{\mathbb{R}-c}(k_{M_\infty\times\mathbb{R}_\infty})$) such that $\pi^{-1}k_U \otimes K \simeq k_M^E \overset{+}{\otimes} QF$ (or $\pi^{-1}k_U \otimes K \simeq k_{M_\infty}^E \overset{+}{\otimes} QF$). We denote by $\mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}k_M)$ (or $\mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}k_{M_\infty})$) the full subcategory of $\mathrm{E}^b_+(\mathrm{I}k_M)$ (or of $\mathrm{E}^b_+(\mathrm{I}k_{M_\infty})$) consisting of $\mathbb{R}$-constructible objects.

Remark. There is another natural embedding $e : \mathsf{D}^b_{\mathbb{R}-c}(k_{M_\infty}) \rightarrowtail \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}k_{M_\infty})$, $F \mapsto k_{M_\infty}^E \overset{+}{\otimes} \epsilon(F)$, and a canonical functor $\mathrm{E}^b_{\mathbb{R}-c}(k_{M_\infty}) \to \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}k_{M_\infty})$, $K \mapsto k_{M_\infty}^E \overset{+}{\otimes} K$; the latter is essentially surjective but not fully faithful.

Remark. Note that $\mathbb{R}$-constructible objects in $\mathrm{E}^b_+(\mathrm{I}k_M)$ are stable. Moreover if $f : M \to N$ is a semiproper morphism of real analytic manifolds then the six operations send $\mathbb{R}$-constructible enhanced indsheaves into $\mathbb{R}$-constructible enhanced indsheaves.

Definition 2.5.8. Let $U \subset M$ be an open subset and let φ, φ^+, $\varphi^- : U \to \mathbb{R}$ be continuous functions with $\varphi^-(x) \leq \varphi^+(x)$ for any $x \in U$. The associated *exponential enhanced indsheaves* are defined by, respectively, $\mathbb{E}^{\varphi}_{U|M_\infty} := k_{M_\infty}^E \overset{+}{\otimes} \mathsf{E}^{\varphi}_{U|M}$ and $\mathbb{E}^{\varphi^+\rhd\varphi^-}_{U|M_\infty} := k_{M_\infty}^E \overset{+}{\otimes} \mathsf{E}^{\varphi^+\rhd\varphi^-}_{U|M}$, where $\mathsf{E}^{\varphi}_{U|M}$, $\mathsf{E}^{\varphi^+\rhd\varphi^-}_{U|M}$ are regarded as objects of $\mathrm{E}^b_+(\mathrm{I}k_{M_\infty})$.

Lemma 2.5.9. *Let φ^+, $\varphi^- : U \to \mathbb{R}$ be continuous functions with $\varphi^-(x) \leq \varphi^+(x)$ for any $x \in U$. Then $\mathbb{E}^{\varphi^+\rhd\varphi^-}_{U|M_\infty} \simeq 0$ if and only if $\varphi^+ - \varphi^-$ is bounded on $K \cap U$ for any relatively compact subset K of M_∞.*

Remark. Since the functor $k_{M_\infty}^E \overset{+}{\otimes} \cdot$ is exact, we have that $\mathbb{E}^{\varphi}_{U|M_\infty}, \mathbb{E}^{\varphi^+\rhd\varphi^-}_{U|M_\infty} \in \mathrm{E}^0_+(\mathrm{I}k_M)$. Moreover we have the short exact sequence in $\mathrm{E}^0_+(\mathrm{I}k_M)$:

$$0 \to \mathbb{E}^{\varphi^+\rhd\varphi^-}_{U|M_\infty} \to \mathbb{E}^{\varphi^+}_{U|M_\infty} \to \mathbb{E}^{\varphi^-}_{U|M_\infty} \to 0\,.$$

In particular if $\varphi : U \to \mathbb{R}$ is bounded with $m = \inf_{x\in U} \varphi(x)$ then we have the short exact sequence:

$$0 \to \mathbb{E}^{\varphi\rhd m}_{U|M_\infty} \to \mathbb{E}^{\varphi}_{U|M_\infty} \to \mathbb{E}^{m}_{U|M_\infty} \to 0\,.$$

By using the lemma above we find $\mathbb{E}^{\varphi \rhd m}_{U|M_\infty} \simeq 0$, hence $\mathbb{E}^{\varphi}_{U|M_\infty} \simeq \mathbb{E}^{m}_{U|M_\infty} \simeq \mathbb{E}^{0}_{U|M_\infty}$, and $\mathbb{E}^{0}_{U|M_\infty} \simeq K^E_U := K^E_{M_\infty} \overset{+}{\otimes} Q(\pi^{-1}k_U)$.

Remark. If φ, φ^+, $\varphi^- : U \to \mathbb{R}$ are continuous globally subanalytic functions with $\varphi^-(x) \leq \varphi^+(x)$ for any $x \in U$ then $\mathbb{E}^{\varphi}_{U|M_\infty}, \mathbb{E}^{\varphi^+ \rhd \varphi^-}_{U|M_\infty} \in \mathrm{E}^0_{\mathbb{R}-c}(\mathrm{I}k_{M_\infty})$.

Chapter 3

Riemann-Hilbert correspondence

3.1 $\mathscr{D}$-modules

Let X be a complex manifold. We denote by:

- d_X the complex dimension of X,
- $\mathscr{O}_X$ the sheaf of holomorphic functions on X,
- Θ_X the sheaf of vector fields on X,
- $\mathscr{D}_X$ the sheaf of differential operators on X,
- Ω_X the invertible $\mathscr{O}_X$-module of differential forms of degree d_X,
- $\mathrm{Mod}(\mathscr{D}_X)$ and $\mathrm{Mod}(\mathscr{D}_X^{op})$ respectively the abelian category of left $\mathscr{D}_X$-modules and the one of right $\mathscr{D}_X$-modules,
- $\mathsf{D}^b(\mathscr{D}_X)$ the bounded derived category of $\mathrm{Mod}(\mathscr{D}_X)$,
- $\otimes^D$, $R\mathscr{H}om_{\mathscr{D}_X}(\cdot\,,\cdot)$ Df^*, Df_*, $Df_!$ the operations $\mathsf{D}^b(\mathscr{D}_X)$, given a morphism of complex manifolds $f : X \to Y$.

Remark. There is an equivalence of categories

$$\begin{aligned} r : \mathrm{Mod}(\mathscr{D}_X) &\longrightarrow \mathrm{Mod}(\mathscr{D}_X^{op}) \\ \mathscr{M} &\longmapsto \mathscr{M}^r := \Omega_X \underset{\mathscr{O}_X}{\otimes} \mathscr{M}\,, \end{aligned}$$

so it is enough to study left $\mathscr{D}_X$-modules.

A $\mathscr{D}_X$-module $\mathscr{M}$ is *coherent* if it is locally finitely generated (i. e. locally there exists $n \in \mathbb{N}$ such that there is an exact sequence $\mathscr{D}_X^n \to \mathscr{M} \to 0$) and for every open subset $U \subset X$ all its locally finitely generated $\mathscr{D}_X|_U$-submodules are locally finitely presented (i.e. locally there exist n_1, $n_2 \in \mathbb{N}$ such that there is an exact sequence $\mathscr{D}_X^{n_1} \to \mathscr{D}_X^{n_2} \to \mathscr{M} \to 0$). We denote by $\mathsf{D}^b_{coh}(\mathscr{D}_X)$ the full triangulated subcategory of $\mathsf{D}^b(\mathscr{D}_X)$ consisting of objects with coherent cohomologies. It is possible to associate to a coherent $\mathscr{D}_X$-module $\mathscr{M}$ its *characteristic variety* $char(\mathscr{M})$, which is a closed conic involutive (in particular such that $dim_{\mathbb{C}}(char(\mathscr{M})) \geq d_X$) subset of the cotangent bundle T^*X. If moreover $dim_{\mathbb{C}}(char(\mathscr{M})) = d_X$ we say that $\mathscr{M}$ is *holonomic*.
We denote by $\mathsf{D}^b_{hol}(\mathscr{D}_X)$ the full triangulated subcategory of $\mathsf{D}^b_{coh}(\mathscr{D}_X)$ consisting of objects with holonomic cohomologies. We denote by $\mathsf{D}^b_{rh}(\mathscr{D}_X)$ the full triangulated subcategory of $\mathsf{D}^b_{hol}(\mathscr{D}_X)$ consisting of objects with regular holonomic cohomologies; if X is one-dimensional then an object $\mathscr{M} \in \mathsf{D}^b_{hol}(\mathscr{D}_X)$ has regular cohomologies if they consist on Fuchsian differential operators, i.e. differential operators in which every singular point (including the point at infinity) is a regular singularity.

3.2 Solution functors

Definition 3.2.1. Let X be a complex analytic manifold and $Y \subset X$ be a complex analytic hypersurface. We denote by $\mathscr{O}_X(*Y)$ the sheaf of meromorphic functions with poles at Y. For $\mathscr{M} \in \mathsf{D}^b(\mathscr{D}_X)$ we define $\mathscr{M}(*Y) := \mathscr{M} \otimes^D \mathscr{O}_X(*Y)$. Let $U = X \setminus Y$; for $f \in \mathscr{O}_X(*Y)$ we set $\mathscr{D}_X e^f := \mathscr{D}_X / \{P \in \mathscr{D}_X;\, Pe^f = 0 \text{ on } U\}$ and $\mathscr{E}^f_{U|X} := \mathscr{D}_X e^f(*Y)$; $\mathscr{E}^f_{U|X}$ is called *exponential module* with exponent f. These are holonomic $\mathscr{D}_X$-modules.

Definition 3.2.2. Let $X_\infty = (X, \check{X})$ be a complex bordered space and let $Z = \check{X} \setminus X$. We define the triangolated category $\mathsf{D}^b_{hol}(\mathscr{D}_{X_\infty})$ as the full triangulated subcategory of $\mathsf{D}^b_{hol}(\mathscr{D}_{\check{X}})$ consisting of objects $\mathscr{M}$ such that $\mathscr{M}(*Z) \simeq \mathscr{M}$.

Remark. The operations for $\mathscr{D}_X$-modules can be extended for $\mathscr{D}_{X_\infty}$-modules. If $f : X_\infty \to Y_\infty$ is a semiproper morphism of complex bordered spaces then the operations send holonomic $\mathscr{D}$-modules into holonomic $\mathscr{D}$-modules.

Definition 3.2.3. The *solution functor* is defined as

$$\begin{aligned} \mathscr{S}ol_X : \mathsf{D}^b(\mathscr{D}_X)^{op} &\longrightarrow \mathsf{D}^b(\mathbb{C}_X) \\ \mathscr{M} &\longmapsto R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\, \mathscr{O}_X)\,. \end{aligned}$$

Remark. Notice that if $\mathscr{M} = \frac{\mathscr{D}_X}{\mathscr{D}_X P}$ with $P \in \mathscr{D}_X$ then in $\mathsf{D}^b(\mathbb{C}_X)$ we have the distinguished triangle $\mathscr{S}ol_X(\mathscr{M}) \longrightarrow \mathscr{O}_X \xrightarrow{P\cdot} \mathscr{O}_X \xrightarrow{+1}$. In particular $H^0\mathscr{S}ol_X(\mathscr{M}) \simeq \{u \in \mathscr{O}_X; Pu = 0\}$ and $H^1\mathscr{S}ol_X(\mathscr{M}) \simeq \frac{\mathscr{O}_X}{P\mathscr{O}_X}$.

Definition 3.2.4. The *enhanced solution functor* is defined as

$$\begin{aligned} \mathscr{S}ol^E_X : \mathsf{D}^b(\mathscr{D}_X)^{op} &\longrightarrow \mathrm{E}^b_+(\mathrm{I}\mathbb{C}_X) \\ \mathscr{M} &\longmapsto R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\, \mathscr{O}^E_X)\,, \end{aligned}$$

(here we don't recall the definition of $\mathscr{O}^E_X$).

Theorem 3.2.5. *Let $Y \subset X$ be a complex analytic hypersurface, $U = X \setminus Y$ and $f \in \mathscr{O}_X(*Y)$. Then*

$$\mathscr{S}ol^E_X(\mathscr{E}^f_{U|X}) \simeq \mathbb{E}^{\mathrm{Re} f}_{U|X}\,.$$

3.3 Riemann-Hilbert correspondence

Theorem 3.3.1 (Classical Riemann-Hilbert correspondence)**.** *The solution functor gives an equivalence of categories:*

$$\mathscr{S}ol_X : \mathsf{D}^b_{rh}(\mathscr{D}_X)^{op} \xrightarrow{\sim} \mathsf{D}^b_{\mathbb{C}-c}(\mathbb{C}_X)\,.$$

Theorem 3.3.2 (Enhanced Riemann-Hilbert correspondence)**.** *The enhanced solution functor gives a fully faithful functor:*

$$\mathscr{S}ol^E_X : \mathsf{D}^b_{hol}(\mathscr{D}_X)^{op} \longrightarrow \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_X)\,;$$

in particular it is possible to reconstruct $\mathscr{M}$ from $\mathscr{S}ol^E_X(\mathscr{M})$ functorially.

The two correspondences are compatible, in fact we have the following quasi-commutative diagram:

$$\begin{tikzcd}
\mathsf{D}^b_{rh}(\mathscr{D}_X)^{op} \arrow[r, "\sim", "\mathscr{S}ol_X"'] \arrow[d, hook] \arrow[rr, bend left, "\mathrm{id}"] & \mathsf{D}^b_{\mathbb{C}-c}(\mathbb{C}_X) \arrow[r, "\sim"] \arrow[d, hook, "e"] & \mathsf{D}^b_{rh}(\mathscr{D}_X)^{op} \arrow[d, hook] \\
\mathsf{D}^b_{hol}(\mathscr{D}_X)^{op} \arrow[r, "\mathscr{S}ol^E_X"'] \arrow[rr, bend right, "\text{canonical embedding}"'] & \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_X) \arrow[r] & \mathsf{D}^b(\mathscr{D}_X)^{op}
\end{tikzcd}\,.$$

Chapter 4

Fourier transforms

4.1 Integral transforms

Consider the following morphisms of complex manifolds:

$$\begin{tikzcd} & S \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y\,. \end{tikzcd}$$

Definition 4.1.1. Let $\mathscr{L} \in \mathsf{D}^b(\mathscr{D}_S)$. The *integral transform with kernel* $\mathscr{L}$ for $\mathscr{D}_X$-modules is the functor

$$\begin{aligned} \cdot \overset{D}{\circ} \mathscr{L} : \mathsf{D}^b(\mathscr{D}_X) &\longrightarrow \mathsf{D}^b(\mathscr{D}_Y) \\ \mathscr{M} &\longmapsto \mathscr{M} \overset{D}{\circ} \mathscr{L} := Dq_*(Dp^*\mathscr{M} \overset{D}{\otimes} \mathscr{L})\,. \end{aligned}$$

Definition 4.1.2. Let $L \in \mathrm{E}^b_+(\mathrm{I}k_S)$. The *integral transform with kernel* L for enhanced indsheaves is the functor

$$\begin{aligned} \cdot \overset{E}{\circ} L : \mathrm{E}^b_+(\mathrm{I}k_X) &\longrightarrow \mathrm{E}^b_+(\mathrm{I}k_Y) \\ K &\longmapsto F \overset{E}{\circ} L := Eq_{!!}(Ep^{-1}K \overset{+}{\otimes} L)\,. \end{aligned}$$

Notice that we can define the functor in the above definition analogously for the enhanced sheaves by changing $L \in \mathrm{E}^b_+(\mathrm{I}k_S)$ with $L \in \mathrm{E}^b_+(k_S)$ and $Eq_{!!}$ with $Eq_!$.

Consider the commutative diagram of complex manifolds

$$\begin{tikzcd}
 & S \arrow[d, "r"] \arrow[ddl, "p"'] \arrow[ddr, "q"] & \\
 & X \times Y \arrow[dl, "p'"] \arrow[dr, "q'"'] & \\
X & & Y
\end{tikzcd}$$

where $r := (p, q)$.

Proposition 4.1.3. *If $L \in \mathrm{E}^b_+(\mathrm{I}k_S)$ and $K \in \mathrm{E}^b_+(\mathrm{I}k_X)$ then $K \overset{E}{\circ} L \simeq K \overset{E}{\circ} Er_{!!}L$.*

Proof. We have $K \overset{E}{\circ} L = Eq_{!!}(Ep^{-1}K \overset{+}{\otimes} L) \simeq Eq'_{!!}Er_{!!}(Er^{-1}Ep'^{-1}K \overset{+}{\otimes} L) \simeq Eq'_{!!}(Ep'^{-1}K \overset{+}{\otimes} Er_{!!}L) = K \overset{E}{\circ} Er_{!!}L$. □

Let Z be another complex manifold. Consider the following diagram with cartesian square:

$$\begin{tikzcd}
 & & X \times Y \times Z \arrow[dl, "r'"'] \arrow[dr, "q'"] & & \\
 & X \times Y \arrow[dl, "p"] \arrow[dr, "q"'] \arrow[rr, phantom, "\square"] & & Y \times Z \arrow[dl, "r"] \arrow[dr, "s"'] & \\
X & & Y & & Z\,.
\end{tikzcd}$$

Proposition 4.1.4. *Let $L \in \mathrm{E}^b_+(\mathrm{I}k_{X\times Y})$, $\widetilde{L} \in \mathrm{E}^b_+(\mathrm{I}k_{Y\times Z})$ and set $L \overset{+}{\circ} \widetilde{L} := Er'^{-1} \overset{+}{\otimes} Eq'^{-1}\widetilde{L} \in \mathrm{E}^b_+(\mathrm{I}k_{X\times Y\times Z})$. If $K \in \mathrm{E}^b_+(\mathrm{I}k_X)$ then $(K \overset{E}{\circ} L) \overset{E}{\circ} \widetilde{L} \simeq K \overset{E}{\circ} (L \overset{+}{\circ} \widetilde{L})$.*

Proof. We have:

$$\begin{aligned}
(K \overset{E}{\circ} L) \overset{E}{\circ} \widetilde{L} &= Eq_{!!}(Ep^{-1}K \overset{+}{\otimes} L) \overset{E}{\circ} \widetilde{L} = Es_{!!}(Er^{-1}Eq_{!!}(Ep^{-1}K \overset{+}{\otimes} L) \overset{+}{\otimes} \widetilde{L}) \\
&\simeq Es_{!!}(Eq'_{!!}Er'^{-1}(Ep^{-1}K \overset{+}{\otimes} L) \overset{+}{\otimes} \widetilde{L}) \\
&\simeq Es_{!!}(Eq'_{!!}(Er'^{-1}Ep^{-1}K \overset{+}{\otimes} Er'^{-1}L) \overset{+}{\otimes} \widetilde{L}) \\
&\simeq Es_{!!}Eq'_{!!}(Er'^{-1}Ep^{-1}K \overset{+}{\otimes} Er'^{-1}L \overset{+}{\otimes} Eq'^{-1}\widetilde{L}) \\
&\simeq E(s \circ q')_{!!}(E(p \circ r')^{-1}K \overset{+}{\otimes} (L \overset{+}{\circ} \widetilde{L})) = K \overset{E}{\circ} (L \overset{+}{\circ} \widetilde{L})\,.
\end{aligned}$$

□

4.2 Fourier-Laplace transform

Let $\mathbb{V}$ be a one-dimensional complex vector space with coordinate z and let $\mathbb{V}^*$ be its dual with coordinate w. Let $\mathbb{P} := \mathbb{V} \cup \{\infty\}$ and $\mathbb{P}^* := \mathbb{V}^* \cup \{\infty\}$ be their associated projective lines: then we have the bordered spaces $\mathbb{V}_\infty := (\mathbb{V}, \mathbb{P})$ and $\mathbb{V}^*_\infty := (\mathbb{V}^*, \mathbb{P}^*)$. Consider the morphisms

$$\begin{tikzcd} & \mathbb{V}_\infty \times \mathbb{V}^*_\infty \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{V}_\infty & & \mathbb{V}^*_\infty \end{tikzcd}$$

induced by the projections $(z, w) \mapsto z$ and $(z, w) \mapsto w$.

Definition 4.2.1. Let $\mathscr{L} := \mathscr{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}$, $\mathscr{L}^a := \mathscr{E}^{zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty} \in \mathsf{D}^b(\mathscr{D}_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty})$. The *Fourier-Laplace transform* for $\mathscr{D}$-modules is the functor

$$\begin{aligned} {}^{\mathsf{L}} : \mathsf{D}^b(\mathscr{D}_{\mathbb{V}_\infty}) &\longrightarrow \mathsf{D}^b(\mathscr{D}_{\mathbb{V}^*_\infty}) \\ \mathscr{M} &\longmapsto \mathscr{M} \overset{D}{\circ} \mathscr{L} = Dq_*(\mathscr{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty} \overset{D}{\otimes} Dp^*\mathscr{M})\,. \end{aligned}$$

It admits a quasi-inverse, defined as:

$$\begin{aligned} {}^{\lrcorner} : \mathsf{D}^b(\mathscr{D}_{\mathbb{V}^*_\infty}) &\longrightarrow \mathsf{D}^b(\mathscr{D}_{\mathbb{V}_\infty}) \\ \mathscr{N} &\longmapsto \mathscr{L}^a \overset{D}{\circ} \mathscr{N} = Dp_*(\mathscr{E}^{zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty} \overset{D}{\otimes} Dq^*\mathscr{N})\,. \end{aligned}$$

Remark. The Fourier-Laplace transform and its quasi-inverse interchange $\mathsf{D}^b_{hol}(\mathscr{D}_{\mathbb{V}_\infty})$ and $\mathsf{D}^b_{hol}(\mathscr{D}_{\mathbb{V}^*_\infty})$.

4.3 Enhanced Fourier-Sato transform

Consider again the two bordered spaces $\mathbb{V}_\infty := (\mathbb{V}, \mathbb{P})$ and $\mathbb{V}^*_\infty := (\mathbb{V}^*, \mathbb{P}^*)$ defined before and the morphisms

$$\begin{tikzcd} & \mathbb{V}_\infty \times \mathbb{V}^*_\infty \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{V}_\infty & & \mathbb{V}^*_\infty \end{tikzcd}$$

induced by the projections $(z, w) \mapsto z$ and $(z, w) \mapsto w$.

Definition 4.3.1. Let $L := \mathsf{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1]$, $L^a := \mathsf{E}^{zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \in \mathrm{E}^b_+(k_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty})$. The *enhanced Fourier-Sato transform* for enhanced sheaves is the functor

$$\begin{aligned} {}^{\llcorner} : \mathrm{E}^b_+(k_{\mathbb{V}_\infty}) &\longrightarrow \mathrm{E}^b_+(k_{\mathbb{V}^*_\infty}) \\ K &\longmapsto K \overset{E}{\circ} L = Eq_!(\mathsf{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \overset{+}{\otimes} Ep^{-1}K)\,. \end{aligned}$$

It admits a quasi-inverse, defined as:

$$\begin{aligned} {}^{\lrcorner} : \mathrm{E}^b_+(k_{\mathbb{V}^*_\infty}) &\longrightarrow \mathrm{E}^b_+(k_{\mathbb{V}_\infty}) \\ P &\longmapsto L^a \overset{E}{\circ} P = Ep_!(\mathsf{E}^{zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \overset{+}{\otimes} Eq^{-1}P)\,. \end{aligned}$$

If $L := \mathbb{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1]$, $L^a := \mathbb{E}^{zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \in \mathrm{E}^b_+(\mathrm{I}k_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty})$ then we define analogously the enhanced Fourier-Sato transform with kernel L for enhanced ind-sheaves and its quasi-inverse by replacing $Ep_!$ and $Eq_!$ with $Ep_{!!}$ and $Eq_{!!}$.

Let's show that ${}^{\llcorner}$ and ${}^{\lrcorner}$ are quasi-inverse of each other. Recall that that $\mathbb{V}^{**}_\infty \simeq \mathbb{V}_\infty$ and let $\tilde{z}$ be its coordinate. Consider the diagram with cartesian square

$$\begin{tikzcd} & & \mathbb{V}_\infty\times\mathbb{V}^*_\infty\times\mathbb{V}_\infty \arrow[dl, "q_{12}"'] \arrow[dr, "q_{23}"] & & \\ & \mathbb{V}_\infty\times\mathbb{V}^*_\infty \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \square & \mathbb{V}^*_\infty\times\mathbb{V}_\infty \arrow[dl, "q_2"'] \arrow[dr, "q_3"] & \\ \mathbb{V}_\infty & & \mathbb{V}^*_\infty & & \mathbb{V}_\infty\,. \end{tikzcd}$$

where the maps are induced by the projections $(z,w,\tilde{z}) \overset{q_{12}}{\longmapsto} (z,w)$, $(z,w,\tilde{z}) \overset{q_{23}}{\longmapsto} (w,\tilde{z})$, $(z,w) \overset{q_1}{\longmapsto} z$, $(w,\tilde{z}) \overset{q_2}{\longmapsto} w$ and $(w,\tilde{z}) \overset{q_3}{\longmapsto} \tilde{z}$.
Let $\mathbb{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \in \mathrm{E}^b_+(\mathrm{I}k_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty})$, $\mathbb{E}^{w\tilde{z}}_{\mathbb{V}^*\times\mathbb{V}|\mathbb{V}^*_\infty\times\mathbb{V}_\infty}[1] \in \mathrm{E}^b_+(\mathrm{I}k_{\mathbb{V}^*_\infty\times\mathbb{V}_\infty})$. If $K \in \mathrm{E}^b_+(\mathrm{I}k_{\mathbb{V}_\infty})$ then $(K \overset{E}{\circ} \mathbb{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1]) \overset{E}{\circ} \mathbb{E}^{w\tilde{z}}_{\mathbb{V}^*\times\mathbb{V}|\mathbb{V}^*_\infty\times\mathbb{V}_\infty}[1] \simeq K \overset{E}{\circ} (\mathbb{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \overset{+}{\circ} \mathbb{E}^{w\tilde{z}}_{\mathbb{V}^*\times\mathbb{V}|\mathbb{V}^*_\infty\times\mathbb{V}_\infty}[1])$, thanks to Proposition 4.1.4.
Consider now the commutative diagram

$$\begin{tikzcd} & \mathbb{V}_\infty\times\mathbb{V}^*_\infty\times\mathbb{V}_\infty \arrow[d, "q_{13}"] \arrow[ddl, "q_1\circ q_{12}"'] \arrow[ddr, "q_2\circ q_{23}"] & \\ & \mathbb{V}_\infty\times\mathbb{V}_\infty \arrow[dl, "q_1"] \arrow[dr, "q_3"'] & \\ \mathbb{V}_\infty & & \mathbb{V}_\infty \end{tikzcd}$$

where q_{13} is induced by the projection $(z,w,\tilde{z}) \mapsto (z,\tilde{z})$. Then $K \overset{E}{\circ} (\mathbb{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \overset{+}{\circ} \mathbb{E}^{w\tilde{z}}_{\mathbb{V}^*\times\mathbb{V}|\mathbb{V}^*_\infty\times\mathbb{V}_\infty}[1]) \simeq K \overset{E}{\circ} Eq_{13!!}(\mathbb{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \overset{+}{\circ} \mathbb{E}^{w\tilde{z}}_{\mathbb{V}^*\times\mathbb{V}|\mathbb{V}^*_\infty\times\mathbb{V}_\infty}[1])$, thanks to Proposition 4.1.3.
Now consider the following:

Proposition 4.3.2. *We have:*

$$Eq_{13!!}(\mathbb{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \overset{+}{\circ} \mathbb{E}^{w\tilde{z}}_{\mathbb{V}^*\times\mathbb{V}|\mathbb{V}^*_\infty\times\mathbb{V}_\infty}[1]) \simeq k^E_{\mathbb{V}_\infty\times\mathbb{V}_\infty} \overset{+}{\otimes} Qk_{\{(z,\tilde{z},t)\in\mathbb{V}\times\mathbb{V}\times\mathbb{R};\, z=\tilde{z},\, t\geq 0\}}\,.$$

Proof. We have:

$$\begin{aligned}
&Eq_{13!!}(\mathbb{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \overset{+}{\circ} \mathbb{E}^{w\tilde{z}}_{\mathbb{V}^*\times\mathbb{V}|\mathbb{V}^*_\infty\times\mathbb{V}_\infty}[1]) = \\
&= Eq_{13!!}((k^E_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty} \overset{+}{\otimes} \mathsf{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty})[1] \overset{+}{\circ} (k^E_{\mathbb{V}^*_\infty\times\mathbb{V}_\infty} \overset{+}{\otimes} \mathsf{E}^{w\tilde{z}}_{\mathbb{V}^*\times\mathbb{V}|\mathbb{V}^*_\infty\times\mathbb{V}_\infty})[1]) \\
&= Eq_{13!!}(Eq_{12}^{-1}(k^E_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty} \overset{+}{\otimes} \mathsf{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty})[1] \overset{+}{\otimes} Eq_{23}^{-1}(k^E_{\mathbb{V}^*_\infty\times\mathbb{V}_\infty} \overset{+}{\otimes} \mathsf{E}^{w\tilde{z}}_{\mathbb{V}^*\times\mathbb{V}|\mathbb{V}^*_\infty\times\mathbb{V}_\infty})[1]) \\
&\simeq Eq_{13!!}(k^E_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty\times\mathbb{V}_\infty} \overset{+}{\otimes} (Eq_{12}^{-1}\mathsf{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \overset{+}{\otimes} Eq_{23}^{-1}\mathsf{E}^{w\tilde{z}}_{\mathbb{V}^*\times\mathbb{V}|\mathbb{V}^*_\infty\times\mathbb{V}_\infty}[1])) \\
&\simeq k^E_{\mathbb{V}_\infty\times\mathbb{V}_\infty} \overset{+}{\otimes} Eq_{13!}(Eq_{12}^{-1}\mathsf{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \overset{+}{\otimes} Eq_{23}^{-1}\mathsf{E}^{w\tilde{z}}_{\mathbb{V}^*\times\mathbb{V}|\mathbb{V}^*_\infty\times\mathbb{V}_\infty}[1])\,,
\end{aligned}$$

so let's study $Eq_{13!}(Eq_{12}^{-1}\mathsf{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \overset{+}{\otimes} Eq_{23}^{-1}\mathsf{E}^{w\tilde{z}}_{\mathbb{V}^*\times\mathbb{V}|\mathbb{V}^*_\infty\times\mathbb{V}_\infty}[1])$.
We have:

$$\begin{aligned}
&Eq_{13!}(Eq_{12}^{-1}\mathsf{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \overset{+}{\otimes} Eq_{23}^{-1}\mathsf{E}^{w\tilde{z}}_{\mathbb{V}^*\times\mathbb{V}|\mathbb{V}^*_\infty\times\mathbb{V}_\infty}[1]) \\
&\simeq Eq_{13!}(\mathsf{E}^{(\tilde{z}-z)w}_{\mathbb{V}\times\mathbb{V}^*\times\mathbb{V}|\mathbb{V}_\infty\times\mathbb{V}^*_\infty\times\mathbb{V}_\infty}[1]) \\
&\simeq QRq_{13!}(k_{\{t+(\tilde{z}-z)w\geq 0\}})
\end{aligned}$$

and $QRq_{13!}(k_{\{t+(\tilde{z}-z)w\geq 0\}}) \to Qk_{\{(z,\tilde{z},t)\in\mathbb{V}\times\mathbb{V}\times\mathbb{R};\, z=\tilde{z},\, t\geq 0\}}$ which is induced by the projection $q_{13} : \mathbb{V}\times\mathbb{V}^*\times\mathbb{V} \to \mathbb{V}\times\mathbb{V}$.
Fix $(\underline{z},\underline{\tilde{z}},\underline{t}) \in \mathbb{V}\times\mathbb{V}\times\mathbb{R}$: $(Rq_{13!}(k_{\{t+(\tilde{z}-z)w\geq 0\}}))_{(\underline{z},\underline{\tilde{z}},\underline{t})} \simeq R\Gamma_c(w\in\mathbb{V}^*;\, k_{\{\underline{t}+(\underline{\tilde{z}}-\underline{z})w\geq 0\}})$ which is isomorphic to 0 if $\tilde{z}\neq z$, and, if $\tilde{z}=z$, is isomorphic to $k_{\{(z,t)\in\mathbb{V}\times\mathbb{R};\, t\geq 0\}} \simeq k_{\{(z,\tilde{z},t)\in\mathbb{V}\times\mathbb{V}\times\mathbb{R};\, z=\tilde{z},\, t\geq 0\}}$. □

Finally let $\Delta_{\mathbb{V}_\infty} := \{(z,\tilde{z})\in\mathbb{V}_\infty\times\mathbb{V}_\infty;\, z=\tilde{z}\}$ and consider the commutative diagram

$$\begin{tikzcd}
 & \Delta_{\mathbb{V}_\infty} \arrow[ddl, "\tilde{q}"'] \arrow[d, "\tilde{q}"] \arrow[ddr, "\tilde{q}"] & \\
 & \mathbb{V}_\infty \arrow[dl, "\mathrm{id}_{\mathbb{V}_\infty}"] \arrow[dr, "\mathrm{id}_{\mathbb{V}_\infty}"'] & \\
\mathbb{V}_\infty & & \mathbb{V}_\infty
\end{tikzcd}$$

where $\tilde{q}$ is the projection $(z, z) \mapsto z$. Then:

$$\begin{aligned}
&K \overset{E}{\circ} (k^E_{\mathbb{V}_\infty\times\mathbb{V}_\infty} \overset{+}{\otimes} Qk_{\{(z,\tilde{z},t)\in\mathbb{V}\times\mathbb{V}\times\mathbb{R};\, z=\tilde{z},\, t\geq 0\}}) \\
&\simeq K \overset{E}{\circ} E\tilde{q}_{!!}(k^E_{\mathbb{V}_\infty\times\mathbb{V}_\infty} \overset{+}{\otimes} Qk_{\{(z,\tilde{z},t)\in\mathbb{V}\times\mathbb{V}\times\mathbb{R};\, z=\tilde{z},\, t\geq 0\}}) \\
&\simeq K \overset{E}{\circ} (k^E_{\mathbb{V}_\infty} \overset{+}{\otimes} E\tilde{q}_{!}(Qk_{\{(z,\tilde{z},t)\in\mathbb{V}\times\mathbb{V}\times\mathbb{R};\, z=\tilde{z},\, t\geq 0\}})) \\
&\simeq K \overset{E}{\circ} (k^E_{\mathbb{V}_\infty} \overset{+}{\otimes} Qk_{\{(z,t)\in\mathbb{V}\times\mathbb{R};\, t\geq 0\}}) \\
&\simeq K \overset{E}{\circ} \mathbb{E}^0_{\mathbb{V}|\mathbb{V}_\infty} \simeq K\,.
\end{aligned}$$

Hence ${}^{\lrcorner}({}^{\llcorner}K) \simeq K$.

Remark. The maps p and q are semiproper and L, L^a are $\mathbb{R}$-constructible (ind)sheaves, hence the functors ${}^{\llcorner}$ and ${}^{\lrcorner}$ send enhanced $\mathbb{R}$-constructible (ind)sheaves into enhanced $\mathbb{R}$-constructible (ind)sheaves.

Remark. Let $K \in \mathrm{E}^b_+(\mathrm{I}k_{\mathbb{V}_\infty})$ and let $F \in \mathrm{E}^b_+(k_{\mathbb{V}_\infty})$ be such that $K \simeq k^E_{\mathbb{V}_\infty} \overset{+}{\otimes} F$. We have

$$\begin{aligned}
{}^{\llcorner}K \simeq {}^{\llcorner}(k^E_{\mathbb{V}_\infty} \overset{+}{\otimes} F) &= Eq_{!!}(\mathbb{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \overset{+}{\otimes} Ep^{-1}(k^E_{\mathbb{V}_\infty} \overset{+}{\otimes} F)) \\
&\simeq Eq_{!!}((k^E_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty} \overset{+}{\otimes} \mathsf{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1]) \overset{+}{\otimes} (Ep^{-1}k^E_{\mathbb{V}_\infty} \overset{+}{\otimes} Ep^{-1}F)) \\
&\simeq Eq_{!!}((k^E_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty} \overset{+}{\otimes} (\mathsf{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \overset{+}{\otimes} Ep^{-1}F)) \\
&\simeq k^E_{\mathbb{V}^*_\infty} \overset{+}{\otimes} Eq_{!}(\mathsf{E}^{-zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}[1] \overset{+}{\otimes} Ep^{-1}F) = k^E_{\mathbb{V}^*_\infty} \overset{+}{\otimes} {}^{\llcorner}F\,.
\end{aligned}$$

Let $a \in \mathbb{V}$ and let $\tau_a : \mathbb{V}_\infty \to \mathbb{V}_\infty$ be the morphism induced by the translation $\tau_a(z) = z + a$.

Lemma 4.3.3. *If $K \in \mathrm{E}^b_+(\mathrm{I}k_{\mathbb{V}_\infty})$ then ${}^{\llcorner}(E\tau_a^{-1}K) \simeq \mathbb{E}^{\mathrm{Re}aw}_{\mathbb{V}^*|\mathbb{V}^*_\infty} \overset{+}{\otimes} {}^{\llcorner}K$.*

Recall that $\mathscr{S}\!ol^E_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty}$ is a fully faithful functor and $\mathscr{S}\!ol^E_{\mathbb{V}_\infty\times\mathbb{V}^*_\infty}(\mathscr{E}^{\pm zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}) \simeq \mathbb{E}^{\pm zw}_{\mathbb{V}\times\mathbb{V}^*|\mathbb{V}_\infty\times\mathbb{V}^*_\infty}$. If $k = \mathbb{C}$ we have the following:

Proposition 4.3.4. *Let $\mathscr{M} \in \mathsf{D}^b_{\mathrm{hol}}(\mathscr{D}_{\mathbb{V}_\infty})$, $\mathscr{N} \in \mathsf{D}^b_{\mathrm{hol}}(\mathscr{D}_{\mathbb{V}^*_\infty})$. Then*

$$\mathscr{S}\!ol^E_{\mathbb{V}^*_\infty}({}^{\llcorner}\mathscr{M}) \simeq {}^{\llcorner}\mathscr{S}\!ol^E_{\mathbb{V}_\infty}(\mathscr{M})\,, \quad \mathscr{S}\!ol^E_{\mathbb{V}_\infty}({}^{\lrcorner}\mathscr{N}) \simeq {}^{\lrcorner}\mathscr{S}\!ol^E_{\mathbb{V}^*_\infty}(\mathscr{N})\,.$$

Remark. If we consider $\mathbb{R}_\infty$, with coordinate x, instead of $\mathbb{V}_\infty$ then $\mathbb{R}^*_\infty$, with coordinate y, is isomorphic to $\mathbb{R}_\infty$. If we take $L := \mathsf{E}^{-xy}_{\mathbb{R}\times\mathbb{R}^*|\mathbb{R}_\infty\times\mathbb{R}^*_\infty}[1]$ or $L := \mathbb{E}^{-xy}_{\mathbb{R}\times\mathbb{R}^*|\mathbb{R}_\infty\times\mathbb{R}^*_\infty}[1]$ and $L^a := \mathsf{E}^{xy}_{\mathbb{R}\times\mathbb{R}^*|\mathbb{R}_\infty\times\mathbb{R}^*_\infty}[1]$ or $L^a := \mathbb{E}^{xy}_{\mathbb{R}\times\mathbb{R}^*|\mathbb{R}_\infty\times\mathbb{R}^*_\infty}[1]$ then the definitions and results concerning only enhanced sheaves and indsheaves given above are still valid.

Example 4.3.5. Consider $\mathsf{E}^f_{\mathbb{R}|\mathbb{R}_\infty} \in \mathrm{E}^b_+(k_{\mathbb{R}_\infty})$ with $f(x) = \frac{x^3}{3}$, that we'll denote with E^f.
Let's compute the enhanced Fourier-Sato transform of E^f:

$$\begin{aligned} {}^{\mathsf{L}}\mathsf{E}^f &= Eq_!(\mathsf{E}^{-xy}[1] \overset{+}{\otimes} Ep^{-1}\mathsf{E}^f) \simeq Eq_!(\mathsf{E}^{x^3/3-xy}) \\ &\simeq Eq_!(Qk_{\{(x,\,y,\,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R};\, t+x^3/3-xy\geq 0\}}) \,. \end{aligned}$$

Fix $(\underline{y},\, \underline{t}) \in \mathbb{R}\times\mathbb{R}$; then:

$$\begin{aligned} ({}^{\mathsf{L}}\mathsf{E}^f)_{(\underline{y},\,\underline{t})} &\simeq R\Gamma_c(q_{\mathbb{R}}^{-1}(\underline{y},\, \underline{t});\, k_{\{(x,\,y,\,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R};\, t+x^3/3-xy\geq 0\}}|_{q_{\mathbb{R}}^{-1}(\underline{y},\,\underline{t})}) \\ &\simeq R\Gamma_c(\{x\in\mathbb{R}\};\, k_{\{x\in\mathbb{R};\, \underline{t}+x^3/3-x\underline{y}\geq 0\}}) \,. \end{aligned}$$

If $x^3/3 - x\underline{y}$ hasn't any local maxima and minima then $({}^{\mathsf{L}}\mathsf{E}^f)_{(\underline{y},\,\underline{t})} = 0$, and this happens when $x^2 - \underline{y} \geq 0$ for every $x \in \mathbb{R}$, i.e. for $\underline{y} \leq 0$ (see Figure 4.1).

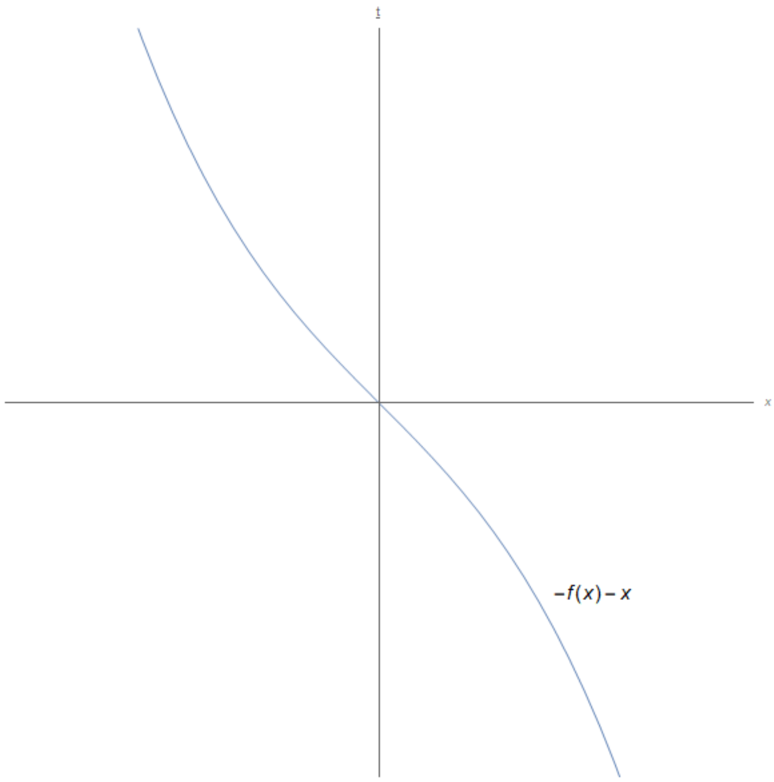


Figure 4.1: Example of $\underline{t} = -f(x) + x\underline{y}$ for $\underline{y} = -1$

Assume $\underline{y} > 0$: in this case there is one local maximum M and one local minimum m respectively at $x_M = \widetilde{g_M}'(\underline{y})$ and at $x_m = \widetilde{g_m}'(\underline{y})$ where $\widetilde{g_M}'$ and $\widetilde{g_m}'$ are the inverse functions of $f'(x)$ respectively for x near x_M and for x near x_m (see Figure

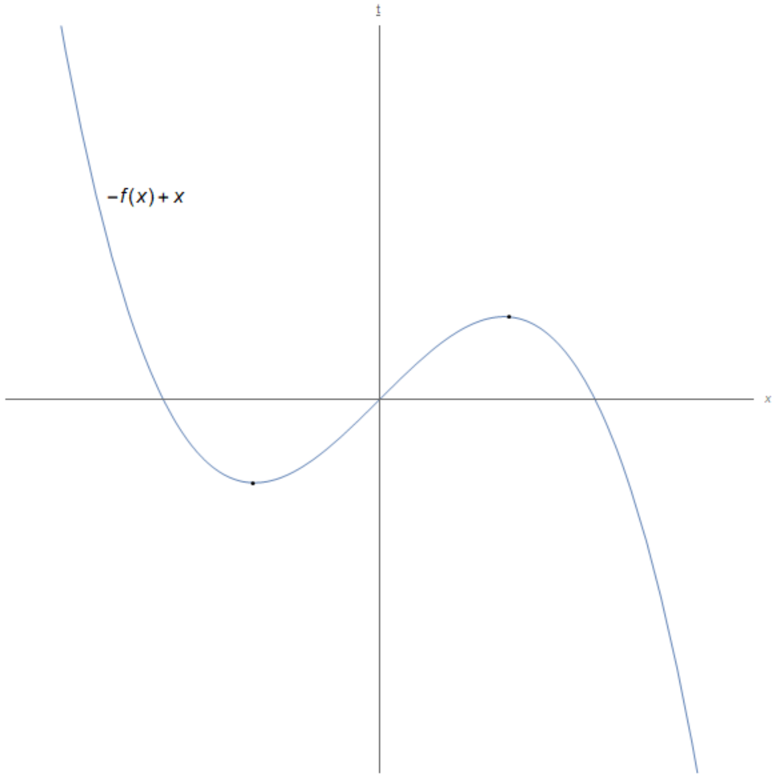


Figure 4.2: Example of $\underline{t} = -f(x) + x\underline{y}$ for $\underline{y} = 1$

4.2).
So:

$$({}^{\mathsf{L}}\mathsf{E}^{f})_{(\underline{y},\underline{t})} = \begin{cases} 0 & \text{if } \underline{t} < -(f(\widetilde{g_M}'(\underline{y})) - \widetilde{g_M}'(\underline{y})\underline{y}) \\ k & \text{if } -(f(\widetilde{g_M}'(\underline{y})) - \widetilde{g_M}'(\underline{y})\underline{y}) \leq \underline{t} < -(f(\widetilde{g_m}'(\underline{y})) - \widetilde{g_m}'(\underline{y})\underline{y}) \\ 0 & \text{if } \underline{t} \geq -(f(\widetilde{g_m}'(\underline{y})) - \widetilde{g_m}'(\underline{y})\underline{y}) \end{cases}.$$

Notice that $(f(\widetilde{g_*}'(\underline{y})) - \widetilde{g_*}'(\underline{y})\underline{y})' = -\widetilde{g_*}'$, hence we can take the primitive g_* of $-\widetilde{g_*}'$ that satisfies $f(\widetilde{g_*}'(\underline{y})) - \widetilde{g_*}'(\underline{y})\underline{y} = g_*$ for $* = M, m$; let's compute g_*.
The derivative of $f(x)$ is $f'(x) = x^2$: for $x \geq 0$ it has inverse $x = \widetilde{g}'_m(y) = y^{1/2}$ and for $x \leq 0$ it has inverse $x = \widetilde{g}'_M(y) = -y^{1/2}$, so, for $x \geq 0$, we find $x = g_m(y) = -\frac{3}{2}y^{3/2}$ and, for $x \leq 0$, $x = g_M(y) = \frac{3}{2}y^{3/2}$. Hence ${}^{\mathsf{L}}\mathsf{E}^{f} \simeq \mathsf{E}^{g_M \triangleright g_m}$ (see Figure 4.3).

Remark. Consider $\mathsf{E}^{f}_{\mathbb{R}|\mathbb{R}_\infty} \in \mathrm{E}^b_+(k_{\mathbb{R}_\infty})$ with f smooth. If we apply two times the enhanced Fourier-Sato transform to $\mathsf{E}^{f}_{\mathbb{R}|\mathbb{R}_\infty}$ we find ${}^{\mathsf{L}}({}^{\mathsf{L}}\mathsf{E}^{f}_{\mathbb{R}|\mathbb{R}_\infty}) \simeq \mathsf{E}^{f^a}_{\mathbb{R}|\mathbb{R}_\infty} \in \mathrm{E}^b_+(k_{\mathbb{R}_\infty})$ where $a : \mathbb{R} \to \mathbb{R}$ is the antipodal map (in this case it's $x \mapsto -x$) and $f^a := f \circ a$. In fact we have just seen that the functions defining ${}^{\mathsf{L}}\mathsf{E}^{f}_{\mathbb{R}|\mathbb{R}_\infty}$ are obtained

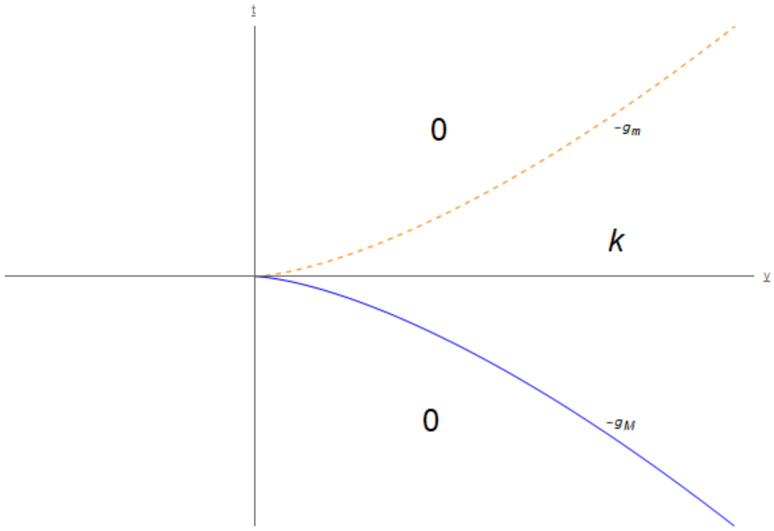


Figure 4.3: Fibers of $^{\mathsf{L}}\mathsf{E}^f \simeq \mathsf{E}^{g_M \triangleright g_m}$

by integrating $-\tilde{g}'$ where $x = \tilde{g}'(y)$ is the inverse of $y = f'(x)$. Let's use the same procedure to find the functions defining $^{\mathsf{L}}(^{\mathsf{L}}\mathsf{E}^f_{\mathbb{R}|\mathbb{R}_\infty})$: the derivative of $g(y)$ is $-\tilde{g}'(y)$, hence the inverse of $x = -\tilde{g}'(y)$ is $y = f'(-x)$ and so the primitive $h(x)$ of $-f'(-x)$ such that $h(x) = g(f'(-x)) - xf'(-x)$ for $y = f'(-x)$ is $h(x) = f(-x) = f^a$.

4.4 Microsupport and enhanced Fourier-Sato transform

Let X be a manifold and let $F \in \mathsf{D}^b(k_X)$. Assume that X is open in a vector space E and let $p = (x_0, \xi_0) \in T^*X$ and let $F \in \mathsf{D}^b(k_X)$.

Definition 4.4.1. The *microsupport* of F, denoted by $SS(F)$, is the subset of T^*X defined in this way: $p \notin SS(F)$ if and only if there exists an open neighborhood U of p such that for any $x_1 \in X$ and any real function φ of class $\mathscr{C}^1$ defined in a neighborhood of x_1 with $\varphi(x_1) = 0$, $\mathrm{d}\varphi(x_1) \in U$, we have $(R\Gamma_{\{x;\varphi(x)\geq 0\}}(F))_{x_1} = 0$.

Proposition 4.4.2. *Let $\varphi : X \to \mathbb{R}$ a function of class C^1 such that $\mathrm{d}\varphi \neq 0$ on the set $\{x;\ \varphi(x) = 0\}$. Then:*

$$SS(k_{\{x\in X;\, \varphi(x)\geq 0\}}) = \{(x;\ \lambda \mathrm{d}\varphi(x));\ \lambda\varphi(x) = 0,\ \lambda \geq 0,\ \varphi(x) \geq 0\}\,.$$

Assume now that M is a real analytic manifold. Denote by $(x, t; x^*, t^*) \in T^*(M \times \mathbb{R})$ the homogeneous symplectic coordinates of the cotangent bundle of

$M \times \mathbb{R}$. Consider the map

$$\begin{aligned} T^*(M\times\mathbb{R}) \supset \{t^*>0\} &\xrightarrow{\gamma} T^*M \\ (x,t;x^*,t^*) &\longmapsto (x;x^*/t^*) \end{aligned}.$$

Definition 4.4.3. Let $K \in \mathrm{E}^b_+(k_M)$. We define $SS^E(K) := \overline{\gamma(SS(F)\cap\{t^*>0\})} \subset T^*M$ where $F \in \mathsf{D}^b(k_{M\times\mathbb{R}})$ is such that $Q(F) \simeq K$. The definition of $SS^E(K)$ does not depend on the choice of F. We call $SS^E(K)$ the *enhanced microsupport* of K.

Notice that $SS^E(\epsilon(F)) = SS(F)$ for $F \in \mathsf{D}^b(k_M)$.
If instead of M we consider a complex manifold X and $K \in \mathrm{E}^b_+(k_X)$ then $SS^E(K)$ is defined as above as a subset of $T^*(X^{\mathbb{R}})$ where $X^{\mathbb{R}}$ denotes the underlying real analytic manifold of X.
Consider now a one-dimensional vector space $\mathbb{V}$ and let $(z,t,w,s;x^*,t^*,w^*,s^*)$ be the coordinates of $T^*(\mathbb{V}\times\mathbb{R}\times\mathbb{V}^*\times\mathbb{R})$. Consider the subsets of $T^*(\mathbb{V}\times\mathbb{R}\times\mathbb{V}^*\times\mathbb{R})$

$$\begin{aligned} \Lambda_{\mathsf{L}} &:= \{s^*>0\}\cap SS(k_{\{s-t-\mathrm{Re}(zw)\geq 0\}})\,, \\ \Lambda_{\lrcorner} &:= \{t^*>0\}\cap SS(k_{\{t-s+\mathrm{Re}(zw)\geq 0\}})\,. \end{aligned}$$

Notice that

$$\Lambda_{\mathsf{L}} = \{s-t-\mathrm{Re}(zw)=0, z^*=wt^*, w^*=zt^*, s^*=-t^*, s^*>0\}.$$

Let Λ^a_{L} be the image of Λ_{L} by the map

$$\begin{aligned} a: T^*(\mathbb{V}\times\mathbb{R}\times\mathbb{V}^*\times\mathbb{R}) &\longrightarrow T^*(\mathbb{V}\times\mathbb{R}\times\mathbb{V}^*\times\mathbb{R}) \\ (z,t,w,s;x^*,t^*,w^*,s^*) &\longmapsto (z,t,w,s;-x^*,-t^*,w^*,s^*)\,; \end{aligned}$$

we have that Λ^a_{L} is the graph of the map

$$\begin{aligned} \widetilde{\chi}: T^*(\mathbb{V}\times\mathbb{R})\cap\{t^*>0\} &\longrightarrow T^*(\mathbb{V}^*\times\mathbb{R})\cap\{s^*>0\} \\ (z,t;x^*,t^*) &\longmapsto (z^*/t^*, t+\mathrm{Re}(zz^*/t^*); -zt^*, t^*)\,. \end{aligned}$$

The map $\widetilde{\chi}$ induces a morphism $\chi : T^*\mathbb{V} \to T^*\mathbb{V}^*$ defined as the composition $\gamma\circ\widetilde{\chi}\circ\gamma^{-1}$:

$$\begin{aligned} T^*\mathbb{V} \xrightarrow{\gamma^{-1}} T^*(\mathbb{V}\times\mathbb{R})\cap\{t^*>0\} &\xrightarrow{\widetilde{\chi}} T^*(\mathbb{V}^*\times\mathbb{R})\cap\{s^*>0\} \xrightarrow{\gamma} T^*\mathbb{V}^* \\ (z,z^*) \longmapsto (z,t,z^*t^*,t^*) &\longmapsto (z^*/t^*, t+\mathrm{Re}(zz^*/t^*); -zt^*, t^*) \longmapsto (z^*,-z)\,. \end{aligned}$$

With analogous considerations for $\Lambda_{\lrcorner}$ we can define another morphism χ^{-1} : $T^*\mathbb{V}^* \to T^*\mathbb{V}$ given by $(w,w^*) \mapsto (-w^*,w)$.
There is an important link between the two morphisms χ, χ^{-1} and the enhanced Fourier-Sato transform $^{\mathsf{L}}$ and its quasi-inverse $^{\lrcorner}$:

Theorem 4.4.4. *Let $K \in \mathrm{E}^b_+(k_{\mathbb{V}})$ and $P \in \mathrm{E}^b_+(k_{\mathbb{V}^*})$. Then:*

$$SS^E({}^{\mathsf{L}}K) = \chi(SS^E(K)), \quad SS^E({}^{\lrcorner}P) = \chi^{-1}(SS^E(P))\,.$$

Chapter 5

Stationary phase lemma

5.1 Stationary phase lemma in the complex case

Let M be a smooth manifold of dimension $n \geq 1$, and let $a \in M$. The *total real blow-up of M along a* is the map of smooth manifolds $\varpi_a^{\text{tot}} : \widetilde{M}_a^{\text{tot}} \to M$ defined in local coordinates $(x_1, \dots, x_n)$ with $a = (0, \dots, 0)$ as follows:

$$\begin{aligned}
&\widetilde{M}_a^{\text{tot}} := \{(\rho, \xi) \in \mathbb{R} \times \mathbb{R}^n;\ |\xi| = 1\},\\
&\varpi_a^{\text{tot}} : \widetilde{M}_a^{\text{tot}} \to M, \quad (\rho, \xi) \mapsto \rho\xi.
\end{aligned}$$

The *real blow-up of M at a* is the closed subset $\widetilde{M}_a := \{(\rho, \xi) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n;\ |\xi| = 1\}$ of $\widetilde{M}_a^{\text{tot}}$. Set $\varpi_a := \varpi_a^{\text{tot}}|_{\widetilde{M}_a}$ and $S_aM := \varpi_a^{-1}(a) \simeq S^{n-1}$, the sphere of tangent directions at a; we have the commutative diagram:

$$\begin{tikzcd}
S_aM \arrow[r, hook, "\tilde{\imath}_a"] & \widetilde{M}_a \arrow[dd, "\varpi_a"'] & \\
& & M \setminus \{a\} \arrow[ul, hook', "\tilde{\jmath}_a"'] \arrow[dl, hook, "j_a"] \\
& M &
\end{tikzcd}$$

Let $\theta \in S_aM$ and $V \subset M$. We say that V is a *sectorial neighborhood* of θ if $V \subset M \setminus \{a\}$ and $S_aM \cup \tilde{\jmath}_a(V)$ is a neighborhood of θ in $\widetilde{M}_a$ (this is equivalent to ask $V = \tilde{\jmath}_a^{-1}(U)$ for some neighborhood U of θ in $\widetilde{M}_a$). If V is a sectorial neighborhood of θ we write $\theta \dot{\in} V$.
We say that a statement $P(\theta)$ on $\theta \in S_aM$ holds for *generic* θ if it holds for θ outside a finite subset of S_aM.

Lemma 5.1.1. *Let M be a real analytic smooth surface and let $K \in \mathrm{E}^b_{\mathbb{R}-c}(k_M)$. Then, for generic $\theta \in S_aM$, there exists a subanalytic open subset $V \subset M$ such that $\theta \dot{\in} V$ and*

$$\pi^{-1}k_V \otimes K \simeq \bigoplus_{i\in I} \mathsf{E}^{f_i}_{V|M}[d_i] \oplus \bigoplus_{j\in J} \mathsf{E}^{f_j^+ \triangleright f_j^-}_{V|M}[d_j]$$

with I, J finite sets, d_i, $d_j \in \mathbb{Z}$ and f_i, f_j^+, f_j^- $: V \to \mathbb{R}$ analytic and globally subanalytic functions such that $f_j^-(x) > f_j^+(x)$ for any $x \in V$.

A similar statement holds for $K \in \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}k_M)$ by replacing E with $\mathbb{E}$.

Now let X be a smooth complex analytic curve and let $a \in X$. Consider

$$\begin{tikzcd}
S_aX \arrow[r, hook, "\tilde{\imath}_a"] & \widetilde{X}_a \arrow[dd, "\varpi_a"'] & \\
& & X \setminus \{a\} \arrow[ul, hook', "\tilde{\jmath}_a"'] \arrow[dl, hook', "j_a"] \\
& X &
\end{tikzcd}$$

where $\widetilde{X}_a$ denotes the real blow-up of the smooth real analytic surface underlying X. In this case $S_aX \simeq S^1$, the circle of tangent directions at a; a local coordinate z_a at a is a holomorphic function defined on a neighborhood of a such that $z_a(a) = 0$ and $(dz_a)(a) \neq 0$.

Definition 5.1.2. Let $\theta \in S_aX$ and $U \dot{\ni} \theta$. We say that $f \in \mathscr{O}_X(U)$ *admits a Puiseux expansion* at θ if there exist $p \in \mathbb{Z}_{>0}$, a local coordinate z_a at a, an open subset $V \subset U$ with $\theta \dot{\in} V$ and a determination of $z_a^{1/p}$ on V such that $f(x) = h(z_a^{1/p}(x))$ for $x \in V$ for some section $h \in \mathscr{O}_{\mathbb{C}}(*0)$ in a neighborhood of 0.
We denote by $\mathscr{P}_{\widetilde{X}_a}$ the subsheaf of $\tilde{\jmath}_{a*}\tilde{\jmath}_a^{-1}\mathscr{O}_X$ whose sections on $\Omega \subset \widetilde{X}_a$ are holomorphic functions on $\tilde{\jmath}_a^{-1}\Omega$ admitting a Puiseux expansion for each point of $\Omega \cap S_aX$. The sheaf $\mathscr{P}_{S_aX} := \tilde{\imath}_a^{-1}\mathscr{P}_{\widetilde{X}_a}$ is called the *sheaf of Puiseux germs* on S_aX; if we need more precision we will write (a, θ, f) instead of $f \in \mathscr{P}_{S_aX}$.
Let $\lambda \in \mathbb{Q}$; we denote by $\mathscr{P}^{\leq\lambda}_{S_aX}$ the subsheaf of $\mathscr{P}_{S_aX}$ whose sections locally belong to $\bigcup_{p\in\mathbb{Z}_{\geq 1}} z_a^{-\lambda}\mathbb{C}\{z_a^{1/p}\}$ for a local coordinate z_a at a and a determination of $z_a^{1/p}$ at θ.
We set $\overline{\mathscr{P}}_{S_aX} := \mathscr{P}_{S_aX}/\mathscr{P}^{\leq 0}_{S_aX}$ and we denote by $[f]$ the image of $f \in \mathscr{P}_{S_aX}$ in $\overline{\mathscr{P}}_{S_aX}$.

Definition 5.1.3. Let $\theta \in S_aX$ and $\Phi \in \mathscr{P}_{S_aX,\theta}$. We say that Φ is *well separated* if for any f, $h \in \Phi$:

i. $[f] = 0$ implies $f = 0$;

ii. $[f] = [h]$ implies $f = h$.

Definition 5.1.4. A *multiplicity* at $a \in X$ is a morphism of sheaves of sets $N : \mathscr{P}_{S_aX} \to (\mathbb{Z}_{\geq 0})_{S_aX}$ such that $N_\theta^{>0} := N_\theta^{-1}(\mathbb{Z}_{>0}) \subset \mathscr{P}_{S_aX,\theta}$ is well separated and finite for some $\theta \in S_aX$.
A *multiplicity class* at $a \in X$ is a morphism of sheaves of sets $\overline{N} : \overline{\mathscr{P}}_{S_aX} \to (\mathbb{Z}_{\geq 0})_{S_aX}$ such that $\overline{N}_\theta^{>0} := \overline{N}_\theta^{-1}(\mathbb{Z}_{>0}) \in \overline{\mathscr{P}}_{S_aX,\theta}$ is finite for some $\theta \in S_aX$.
A Puiseux germ $f \in N_\theta^{>0}$ is called an *exponential factor* of N at θ and the positive integer $N(f)$ is called *multiplicity* of f. Moreover for $f \in \mathscr{P}_{S_aX}$ we set $\overline{N}(f) := \overline{N}([f])$.
If N is a multiplicity then we denote by $\overline{N}$ its class, defined by setting $\overline{N}(f) = N(h)$ if there exists $h \in N_\theta^{>0}$ such that $[f] = [h]$, and $\overline{N}(f) = 0$ otherwise.

Definition 5.1.5. Let $K \in \mathrm{E}^b_{\mathbb{R}-c}(k_X)$. We say that K has a *normal form* at $a \in X$ if there exists a multiplicity at a, $N : \mathscr{P}_{S_aX} \to (\mathbb{Z}_{\geq 0})_{S_aX}$, such that for any $\theta \in S_aX$ there exists an open sectorial neighborhood $V_\theta \dot{\ni} \theta$ such that

$$\pi^{-1}k_{V_\theta} \otimes K \simeq \bigoplus_{f \in N_\theta^{>0}} (\mathsf{E}^{\mathrm{Re} f}_{V_\theta|X})^{N(f)} .$$

Let $K \in \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}k_X)$. We say that K has a *normal form* at $a \in X$ if there exists a multiplicity at a, $N : \mathscr{P}_{S_aX} \to (\mathbb{Z}_{\geq 0})_{S_aX}$, such that for any $\theta \in S_aX$ there exists an open sectorial neighborhood $V_\theta \dot{\ni} \theta$ such that

$$\pi^{-1}k_{V_\theta} \otimes K \simeq \bigoplus_{f \in N_\theta^{>0}} (\mathbb{E}^{\mathrm{Re} f}_{V_\theta|X})^{N(f)} .$$

The multiplicity N and its class $\overline{N}$ are uniquely determined by K. We call $\overline{N}$ the *multiplicity class of* K.

Remark. Let $X = \mathbb{V}_\infty = (\mathbb{V}, \mathbb{P})$ where $\mathbb{V}$ is an one-dimensional complex vector space, with coordinate z, and $\mathbb{P} = \mathbb{V} \cup \{\infty\}$, and consider $\mathscr{M} \in \mathsf{D}^b_{hol}(\mathscr{D}_{\mathbb{V}_\infty})$. Let $a \in \mathbb{P}$ be a singular point of $\mathscr{M}$: if $a \in \mathbb{V}$ then take as a local coordinate $z_a = z - a$ and if $a = \infty$ then take $z_\infty = z^{-1}$. Then, after a ramification, $\mathscr{M}$ decomposes on a sector V_a as a finite direct sum of exponential modules $\mathscr{E}^f_{V_a|\mathbb{V}_\infty}$ where f admits a Puiseux expansion at $\theta \in S_a\mathbb{P}$. We call (a, θ, f) an *exponential factor* of $\mathscr{M}$. If $k = \mathbb{C}$ then the enhanced solution functor $\mathscr{S}ol^E_{\mathbb{V}_\infty}$ gives an important link between the exponential factors of $\mathscr{M}$ and exponential factors in the normal form of $\mathscr{S}ol^E_{\mathbb{V}_\infty}(\mathscr{M}) = K \in \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_{\mathbb{V}_\infty})$, since $\mathscr{S}ol^E_{\mathbb{V}_\infty}(\mathscr{E}^f_{V_a|\mathbb{V}_\infty}) \simeq \mathbb{E}^{\mathrm{Re} f}_{V_a|\mathbb{V}_\infty}$.

Definition 5.1.6. Let (a,θ,f) be a Puiseux germ on X. The *multiplicity test functor* at (a,θ,f) is defined as

$$\begin{aligned} G_{(a,\theta,f)} : \mathrm{E}^b_+(\mathrm{I}k_X) &\longrightarrow \mathsf{D}^b(k) \\ K &\longmapsto \varinjlim_{V,c,\delta,\varepsilon} RHom^E(\mathsf{E}_{V|X}^{(\mathrm{Re}f+c)\triangleright(\mathrm{Re}f-\delta|z_a|^{-\varepsilon})}, K) \end{aligned}$$

where z_a is a local coordinate at a, V runs over the open sectorial neighborhoods of θ, $c \to +\infty$ and δ, $\varepsilon \to 0^+$.

Proposition 5.1.7. *Let (a,θ,f) be a Puiseux germ on X. Let $K \in \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}k_X)$ have normal form at a with multiplicity class $\overline{N}$. Then $G_{(a,\theta,f)}K \simeq k^{\overline{N}(f)}$.*

Consider now $\mathbb{V}_\infty = (\mathbb{V},\mathbb{P})$ where $\mathbb{V}$ is an one-dimensional complex vector space, with coordinate z, and $\mathbb{P} = \mathbb{V} \cup \{\infty\}$.

Definition 5.1.8. Let (a,θ,f) be a Puiseux germ in $\mathscr{P}_{S_a\mathbb{P}}$. We say that it is:

i. *unbounded* if $\mathrm{ord}_a(f) > 0$;

ii. *linear* if $a = \infty$ and $f(z) - bz \in \mathscr{P}^{\leq 0}_{S_\infty\mathbb{P}}$ for some $b \in \mathbb{V}$, $b \neq 0$;

iii. *admissible* if it is unbounded and not linear.

Definition 5.1.9. Let (a,θ,f) be an admissible Puiseux germ on $\mathbb{P}$. We define the *Legendre transform* $\mathsf{L}(a,\theta,f) := (b,\eta,g)$ of (a,θ,f), which is an admissible Puiseux germ on $\mathbb{P}^*$, in this way:
1) derive $w = f(z)$ with z near θ;
2) take $b \in \mathbb{P}^*$ and $\eta \in S_b\mathbb{P}^*$ such that $w = f'(z) \to \eta$ for $z \to \theta$;
3) take the inverse $z = \varphi(w)$ of $w = f'(z)$ for z near θ;
4) take the primitive $g(w)$ of $-\varphi(w)$ which satisfies

$$zw - f(z) + g(w) = 0 \quad \text{for} \quad w = f'(z)\,. \tag{5.1}$$

This procedure gives $(b,\eta,g) = \mathsf{L}(a,\theta,f)$. The equation (5.1) is called *stationary phase formula.*

Theorem 5.1.10 (Stationary phase lemma). *Let (a,θ,f) be an admissible Puiseux germ on $\mathbb{P}$ and let $(b,\eta,g) = \mathsf{L}(a,\theta,f)$. Let $K \in \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}k_{\mathbb{V}_\infty})$ have normal form at a. Then, for generic η, we have*

$$G_{(b,\eta,g)}({}^{\mathsf{L}}K) \simeq G_{(a,\theta,f)}(K)\,.$$

Let $k = \mathbb{C}$ and $\mathscr{S}\!ol^E_{\mathbb{V}_\infty}(\mathscr{M}) \simeq K$ for $\mathscr{M} \in \mathsf{D}^b_{hol}(\mathscr{D}_{\mathbb{V}_\infty})$. Then $\mathscr{S}\!ol^E_{\mathbb{V}^*_\infty}({}^{\mathsf{L}}\mathscr{M}) \simeq {}^{\mathsf{L}}K$, and in particular the stationary phase lemma stated for $\mathscr{D}$-modules becomes a corollary of the theorem above:

Corollary 5.1.11. *Let (a,θ,f) be an admissible Puiseux germ on $\mathbb{P}$ and let $\mathscr{M} \in \mathsf{D}^b_{hol}(\mathscr{D}_{\mathbb{V}_\infty})$. Then (a,θ,f) is an exponential factor of $\mathscr{M}$ if and only if $\mathsf{L}(a,\theta,f)$ is an exponential factor of ${}^{\mathsf{L}}\mathscr{M}$.*

5.2 Stationary phase lemma in the real case

Let $\mathbb{V}_\infty = \mathbb{R}_\infty = (\mathbb{R}, \overline{\mathbb{R}})$ where $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$: in this case $\mathbb{V}_\infty \times \mathbb{V}_\infty^* \simeq \mathbb{R}_\infty \times \mathbb{R}_\infty \ni (x, y)$; in this section we will focus only on the study of enhanced (ind)sheaves, since the Riemann-Hilbert correspondence is not available.
Notice that since for each $a \in \mathbb{R}$ we have $S_a\overline{\mathbb{R}} \simeq \{+, -\}$ (and $S_{+\infty}\overline{\mathbb{R}} \simeq \{-\}$, $S_{-\infty}\overline{\mathbb{R}} \simeq \{+\}$) a sectorial neighborhood of $a \in \mathbb{R}$ is simply the union of the two disjoint open subsets $V_a^+ := ((a, a+\varepsilon), [a, a+\varepsilon))$ and $V_a^- := ((a-\varepsilon, a), (a-\varepsilon, a])$ for $\varepsilon > 0$, and a sectorial neighborhood of $\pm\infty$ is either $V_{+\infty} := ((M, +\infty), (M, +\infty])$ or $V_{-\infty} := ((-\infty, -M), [-\infty, -M))$ for some $M \in \mathbb{R}$, $M \gg 1$.
Recall also that in this case it is possible to obtain exponential sheaves of the form $\mathsf{E}^{g_1 \rhd g_2}$ after applying the enhanced Fourier-Sato transform to exponential sheaves of the form E^f (see Example 4.3.5).
Let $K \in \mathrm{E}^0_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{R}_\infty})$: Lemma 5.1.1 holds also for $\mathbb{R}_\infty$, hence for $a \in \mathbb{R}$, we have the decomposition

$$\pi^{-1} k_{V_a^\pm} \otimes K \simeq \bigoplus_{i \in I_\pm} \mathsf{E}^{f_i}_{V_a^\pm|\mathbb{R}_\infty} \oplus \bigoplus_{j \in J_\pm} \mathsf{E}^{f_j^+ \rhd f_j^-}_{V_a^\pm|\mathbb{R}_\infty}$$

with $I_\pm$, $J_\pm$ finite sets and $f_i, f_j^+, f_j^- : V_a^\pm \to \mathbb{R}$ analytic and globally subanalytic functions such that $f_j^-(x) < f_j^+(x)$ for any $x \in V_a^\pm$.
For $a = \pm\infty$, we have the decomposition

$$\pi^{-1} k_{V_{\pm\infty}} \otimes K \simeq \bigoplus_{i \in I_\pm} \mathsf{E}^{f_i}_{V_{\pm\infty}|\mathbb{R}_\infty} \oplus \bigoplus_{j \in J_\pm} \mathsf{E}^{f_j^+ \rhd f_j^-}_{V_{\pm\infty}|\mathbb{R}_\infty}$$

with $I_\pm$, $J_\pm$ finite sets and $f_i, f_j^+, f_j^- : V_{\pm\infty} \to \mathbb{R}$ analytic and globally subanalytic functions such that $f_j^-(x) > f_j^+(x)$ for any $x \in V_{\pm\infty}$. If $K \in \mathrm{E}^0_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_{\mathbb{R}_\infty})$ it holds an analogous result with $\mathbb{E}$ instead of E. We call the functions in these decompositions *exponential factors* of K at a.
Notice that in this setting the notion of admissible Puiseux germ can be translated in this way: assume that the exponential sheaf (or indsheaf) $\mathsf{E}^f_{V_a|\mathbb{R}_\infty}$ (or $\mathbb{E}^f_{V_a|\mathbb{R}_\infty}$) appears in the decomposition of $K \in \mathrm{E}^b_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{R}_\infty})$ (or of $K \in \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_{\mathbb{R}_\infty})$) at a. We say that the exponential factor f is *admissible* if:

- $\mathrm{ord}_a f > 0$;
- $f(x) \neq bx + c$ for every $b, c \in \mathbb{R}$ if $a = \infty$.

Assume now that the exponential sheaf (or indsheaf) $\mathsf{E}^{f_1 \rhd f_2}_{V_a|\mathbb{R}_\infty}$ (or $\mathbb{E}^{f_1 \rhd f_2}_{V_a|\mathbb{R}_\infty}$) appears in the decomposition of $K \in \mathrm{E}^b_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{R}_\infty})$ (or of $K \in \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_{\mathbb{R}_\infty})$) at a. We say that the exponential factors f_1, f_2 are *admissible* if:

- $\mathrm{ord}_a f_1,\ \mathrm{ord}_a f_2 > 0$;
- $f_1(x),\ f_2(x) \neq bx + c$ for every $b,\ c \in \mathbb{R}$ if $a = \infty$;
- $f_1 - f_2$ is unbounded on V_a.

Definition 5.2.1. Let $K \in \mathrm{E}^0_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{R}_\infty})$ (or $K \in \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_{\mathbb{R}_\infty})$), let f be an admissible exponential factor defined on V_a^u where $u \in S_a\overline{\mathbb{R}} = \{+,-\}$ in the decomposition of K at a and consider the triplet (a,u,f). The *Legendre transform of* (a,u,f), denoted by $\mathsf{L}(a,u,f)$, is the triplet (b,v,g) where $v \in S_b\overline{\mathbb{R}}^* \simeq S_b\overline{\mathbb{R}} = \{+,-\}$, obtained in this way:
1) derive $y = f(x)$ with x in V_a^u;
2) take $b \in \overline{\mathbb{R}}$ and $v \in S_b\overline{\mathbb{R}}$ such that $y = f'(x) \to b$ in U_b^v for $x \to a$ in V_a^u;
3) take the inverse $x = \varphi(y)$ of $y = f'(x)$ for x in V_a^u;
4) take the primitive $g(y)$ of $-\varphi(y)$ which satisfies, for $x \in V_a^u$ and $y \in U_b^v$,

$$g(y) - f(x) + xy = 0 \quad \text{for} \quad y = f'(x)\,. \tag{5.2}$$

This procedure gives $(b,v,g) = \mathsf{L}(a,\theta,f)$ where g is admissible. The equation (5.2) is the stationary phase formula in the real case.

Remark. Notice that if f is not admissible then we can't apply the Legendre transform to (a,u,f) since $y = f'(x)$ is not invertible.
Moreover the Legendre transform admits an inverse obtained by changing x with y and y with $-x$.

Let's start with the following explicit example, recalling that we will use

$$\begin{tikzcd} & \mathbb{R}_\infty \times \mathbb{R}_\infty \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{R}_\infty & & \mathbb{R}_\infty \end{tikzcd}$$

where $p,\ q$ are induced by the projections (that we denote in the same way) $(x,y) \mapsto x$ and $(x,y) \mapsto y$.

Example 5.2.2. Let's describe in detail the case of E^f with $f(x) = \frac{x^4}{4} - \frac{x^2}{2}$.
The Fourier transform of E^f is ${}^{\mathsf{L}}\mathsf{E}^f = Eq_!(\mathsf{E}^{-xy}[1]\overset{+}{\otimes} Ep^{-1}\mathsf{E}^f) \simeq Eq_!(\mathsf{E}^{x^4/4-x^2/2-xy}) \simeq Eq_!(Q\mathbb{C}_{\{(x,\,y,\,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R};\, t+x^4/4-x^2/2-xy\geq 0\}})$. It has fiber at $(\underline{y},\ \underline{t})$ given by

$$({}^{\mathsf{L}}\mathsf{E}^f)_{(\underline{y},\,\underline{t})} \simeq R\Gamma_c(q_{\mathbb{R}}^{-1}(\underline{y},\ \underline{t});\ \mathbb{C}_{\{(x,\,y,\,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R};\, t+x^4/4-x^2/2-xy\geq 0\}}|_{q_{\mathbb{R}}^{-1}(\underline{y},\,\underline{t})}) \simeq$$
$$R\Gamma_c(\{x \in \mathbb{R}\};\ \mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}+x^4/4-x^2/2-x\underline{y}\geq 0\}})\,.$$

Recall that $f(x)-xy=g(y)$ with $x=\tilde{g}'(y)$, where $\tilde{g}'(y)$ is the inverse of $y=f'(x)$ near x and $g(y)$ is the function obtained by integrating $-\tilde{g}'(y)$.
Let's denote by $g_1'(y)$, $g_2'(y)$, $g_3'(y)$ the functions obtained by changing the sign of the inverses of $y=x^3+x$ respectively in $x<-1/\sqrt{3}$, $-1/\sqrt{3}<x<1/\sqrt{3}$ and $x>1/\sqrt{3}$ (see Figure 5.1).

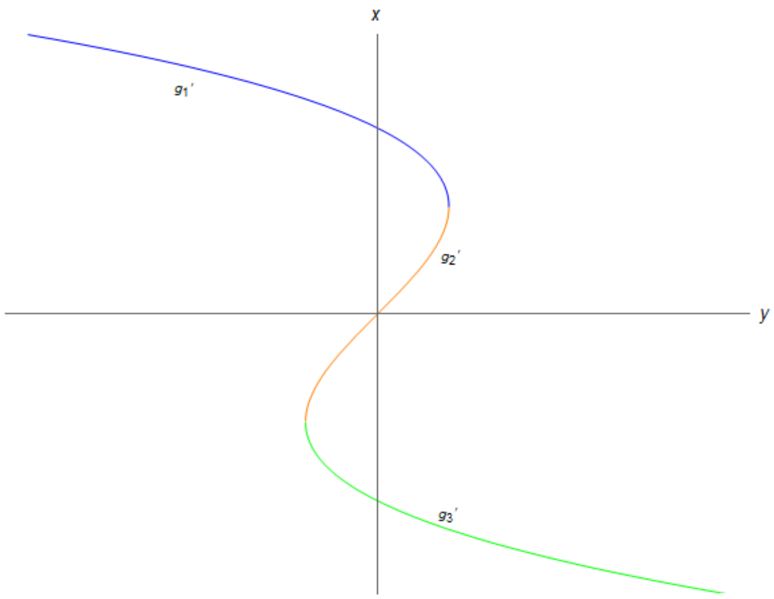


Figure 5.1: Functions g_1', g_2', g_3'

Let's integrate $g_2'(y)$ in order to get a function $g_2(y)$ which passes by $(0,0)$ and consequently integrate $g_1'(y)$ and $g_3'(y)$ such as they connect to $g_2(y)$: with this procedure we obtain the functions defining ${}^{\mathsf{L}}\mathsf{E}^f$ such that $g_*(y)=f(x)-xy$ where $x=-g_*'(y)$ for $*=1,2,3$ (see Figure 5.2).
If $\underline{y}\leq-\frac{2}{3\sqrt{3}}$ or $\underline{y}\geq\frac{2}{3\sqrt{3}}$ then $h_{\underline{y}}'(x)=x^3-x-\underline{y}$ has only one zero (respectively in $x=-g_1'(\underline{y})$ or in $x=-g_3'(\underline{y})$), and in particular $h_{\underline{y}}(x)$ has only one stationary point which is a global minimum, so (respectively with $*=1$ or with $*=3$) we have

$$({}^{\mathsf{L}}\mathsf{E}^f)_{(\underline{y},\underline{t})}=\begin{cases}0 & \text{if } \underline{t}<-g_*(\underline{y})\\ \mathbb{C}[-1] & \text{if } \underline{t}\geq-g_*(\underline{y})\end{cases}.$$

If instead $-\frac{2}{3\sqrt{3}}<\underline{y}<\frac{2}{3\sqrt{3}}$ then $h_{\underline{y}}(x)$ has three stationary points, two minima and a maximum (the two minima are in $x_1=-g_1'(\underline{y})$ and in $x_3=-g_3'(\underline{y})$, and the maximum is in $x_2=-g_2'(\underline{y})$; for $\underline{y}\lesseqqgtr 0$ we have $h_{\underline{y}}(x_1)\lesseqqgtr h_{\underline{y}}(x_3)$).

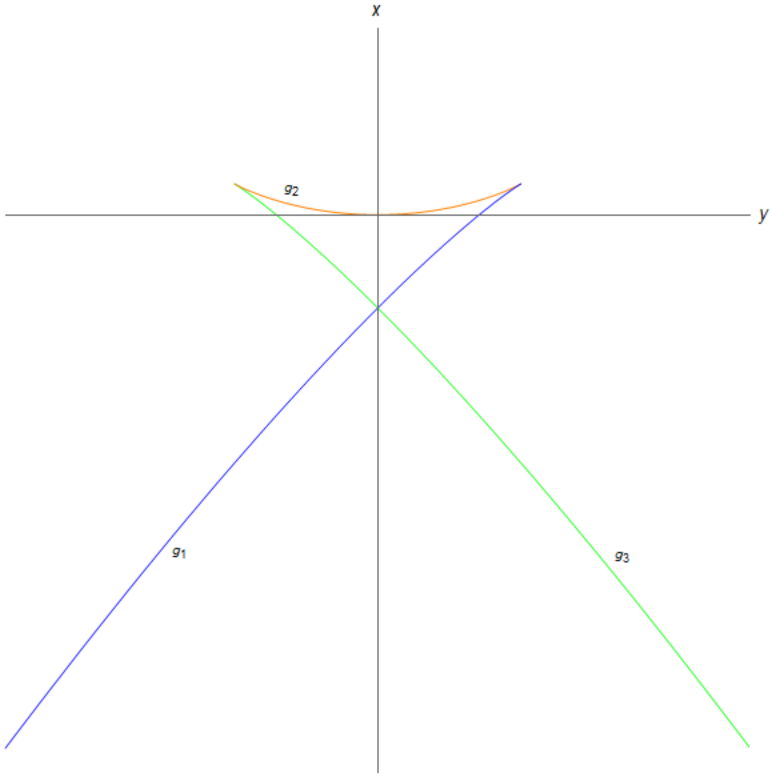


Figure 5.2: Functions g_1, g_2, g_3

If $\underline{y} \leq 0$

$$({}^{\mathsf{L}}\mathsf{E}^f)_{(\underline{y},\underline{t})} = \begin{cases} 0 & \text{if } \underline{t} < -g_2(\underline{y}) \\ \mathbb{C} & \text{if } -g_2(\underline{y}) \leq \underline{t} < -g_3(\underline{y}) \\ 0 & \text{if } -g_3(\underline{y}) \leq \underline{t} < -g_1(\underline{y}) \\ \mathbb{C}[-1] & \text{if } \underline{t} \geq -g_1(\underline{y}) \end{cases}$$

and if $\underline{y} \geq 0$

$$({}^{\mathsf{L}}\mathsf{E}^f)_{(\underline{y},\underline{t})} = \begin{cases} 0 & \text{if } \underline{t} < -g_2(\underline{y}) \\ \mathbb{C} & \text{if } -g_2(\underline{y}) \leq \underline{t} < -g_1(\underline{y}) \\ 0 & \text{if } -g_1(\underline{y}) \leq \underline{t} < -g_3(\underline{y}) \\ \mathbb{C}[-1] & \text{if } \underline{t} \geq -g_3(\underline{y}) \end{cases}$$

(see figure 5.3).

Notice that ${}^{\mathsf{L}}\mathsf{E}^f$ has a complex behaviour for $-\frac{2}{3\sqrt{3}} < y < \frac{2}{3\sqrt{3}}$: by focusing on $\mathbb{R}$-constructible enhanced indsheaves we will need only to study their singular points, which are finite in number.

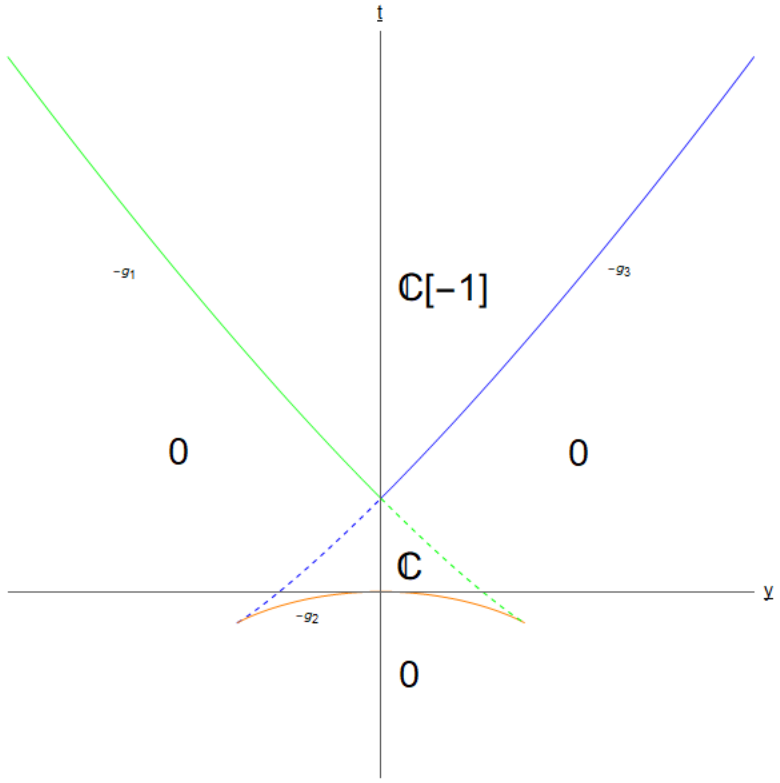


Figure 5.3: Fibers of ${}^{\mathsf{L}}\mathsf{E}^f$

We say that a point $a \in \mathbb{R}_\infty$ is a *regular point* of $K \in \mathrm{E}^b_{\mathbb{R}-c}(\mathrm{I}k_{\mathbb{R}_\infty})$ if there exists an open neighborhood U of a such that $K|_U$ is isomorphic to a finite direct sum of constant enhanced indseaves. We say that K is *regular* on U if every $a \in U$ is a regular point of K. A point $a \in \mathbb{R}_\infty$ is a *singular point* of K if there exists an open neighborhood U of a such that K is regular on $U \setminus \{a\}$ and not on U.
Let $K \in \mathrm{E}^0_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_{\mathbb{R}_\infty})$ have singular points only at $\pm\infty$ and consider its decomposition at $\pm\infty$; we have the following short exact sequence:

$$0 \longrightarrow K|_{V_{-\infty}\cup V_{+\infty}} \longrightarrow K \longrightarrow K|_{\mathbb{R}_\infty\setminus(V_{-\infty}\cup V_{+\infty})} \longrightarrow 0\,. \tag{5.3}$$

Since $K|_{\mathbb{R}} \simeq (\mathbb{C}^E_{\mathbb{R}})^N$ with $N \in \mathbb{N}$ then $K|_{\mathbb{R}_\infty\setminus(V_{-\infty}\cup V_{-\infty})} \simeq e(\mathbb{C}_{[a,b]})^N$ for some $a,\, b \in \mathbb{R}$, $a < 0$, $b > 0$, and $K|_{V_{\pm\infty}} = \bigoplus_{i\in I_\pm} \mathbb{E}^{f_i}_{V_{\pm\infty}|\mathbb{R}_\infty} \oplus \bigoplus_{j\in J_\pm} \mathbb{E}^{f_j^+\triangleright f_j^-}_{V_{\pm\infty}|\mathbb{R}_\infty}$ with $N = |I_+| + |J_+| = |I_-| + |J_-|$. If we apply the enhanced Fourier-Sato transform to this short exact sequence we get

$$\begin{aligned} 0 \longrightarrow \bigoplus_{i\in I_-}({}^{\mathsf{L}}\mathbb{E}^{f_i}_{V_{-\infty}|\mathbb{R}_\infty}) \oplus \bigoplus_{j\in J_-}({}^{\mathsf{L}}\mathbb{E}^{f_j^+\triangleright f_j^-}_{V_{-\infty}|\mathbb{R}_\infty}) \oplus \bigoplus_{i\in I_+}({}^{\mathsf{L}}\mathbb{E}^{f_i}_{V_{+\infty}|\mathbb{R}_\infty}) \oplus \bigoplus_{j\in J_+}({}^{\mathsf{L}}\mathbb{E}^{f_j^+\triangleright f_j^-}_{V_{+\infty}|\mathbb{R}_\infty}) \longrightarrow \\ \longrightarrow {}^{\mathsf{L}}K \longrightarrow ({}^{\mathsf{L}}e(\mathbb{C}_{[a,b]}))^N \longrightarrow 0\,, \end{aligned}$$

thus ${}^{\mathsf{L}}K$ is a combinations of the ${}^{\mathsf{L}}\mathbb{E}^{f_i}_{V_{\pm\infty}|\mathbb{R}_\infty}$, ${}^{\mathsf{L}}\mathbb{E}^{f_j^+ \rhd f_j^-}_{V_{\pm\infty}|\mathbb{R}_\infty}$ and ${}^{\mathsf{L}}e(\mathbb{C}_{[a,b]})$.
We have ${}^{\mathsf{L}}e(\mathbb{C}_{[a,b]}) = {}^{\mathsf{L}}(\mathbb{C}^E_{\mathbb{R}_\infty} \overset{+}{\otimes} Q(\mathbb{C}_{\{t\geq 0\}} \otimes \pi^{-1}\mathbb{C}_{[a,b]}) \simeq \mathbb{C}^E_{\mathbb{R}_\infty} \overset{+}{\otimes} {}^{\mathsf{L}}Q(\mathbb{C}_{\{t\geq 0\}} \otimes \pi^{-1}\mathbb{C}_{[a,b]})$ $\simeq \mathbb{C}^E_{\mathbb{R}_\infty} \overset{+}{\otimes} {}^{\mathsf{L}}Q\mathbb{C}_{\{(x,t)\in\mathbb{R}\times\mathbb{R}:\, t\geq 0,\, a\leq x\leq b\}}$. We have

$$\begin{aligned}{}^{\mathsf{L}}Q\mathbb{C}_{\{(x,t)\in\mathbb{R}\times\mathbb{R}:\, t\geq 0,\, a\leq x\leq b\}} &= Eq_!(\mathsf{E}^{-xy}[1] \overset{+}{\otimes} Q\mathbb{C}_{\{(x,t)\in\mathbb{R}\times\mathbb{R}:\, t\geq 0,\, a\leq x\leq b\}}) \\ &\simeq Eq_!(Q\mathbb{C}_{\{(x,y,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}:\, t-xy\geq 0,\, a\leq x\leq b\}}).\end{aligned}$$

The projection $(x,y,t)\mapsto(y,t)$ induces a morphism

$$\mathbb{C}_{\{(x,y,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}:\, t-xy\geq 0,\, a\leq x\leq b\}} \to \mathbb{C}_{\{(y,t)\in\mathbb{R}\times\mathbb{R}:\, t-ay\geq 0 \text{ for } y\geq 0,\, t-by\geq 0 \text{ for } y\leq 0\}}$$

hence $Eq_!(Q\mathbb{C}_{\{(x,y,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}:\, t-xy\geq 0,\, a\leq x\leq b\}}) \to \mathsf{E}^g$ where $g:\mathbb{R}\to\mathbb{R}$ is defined as

$$g(y) = \begin{cases} -ay & \text{if } y\geq 0 \\ -by & \text{if } y\leq 0 \end{cases}.$$

Let $(\underline{y},\underline{t})\in\mathbb{R}\times\mathbb{R}$ be fixed: $(Eq_!(Q\mathbb{C}_{\{(x,y,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}:\, t-xy\geq 0,\, a\leq x\leq b\}}))_{(\underline{y},\underline{t})} \simeq$ $R\Gamma_c(\{x\in\mathbb{R}\};\, \mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}-x\underline{y}\geq 0,\, a\leq x\leq b\}})$; if $\underline{y}\geq 0$ then

$$R\Gamma_c(\{x\in\mathbb{R}\};\, \mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}-x\underline{y}\geq 0,\, a\leq x\leq b\}}) = \begin{cases} 0 & \text{if } \underline{t} < a\underline{y} \\ \mathbb{C} & \text{if } \underline{t}\geq a\underline{y} \end{cases}$$

and if $\underline{y}\leq 0$ then

$$R\Gamma_c(\{x\in\mathbb{R}\};\, \mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}-x\underline{y}\geq 0,\, a\leq x\leq b\}}) = \begin{cases} 0 & \text{if } \underline{t} < b\underline{y} \\ \mathbb{C} & \text{if } \underline{t}\geq b\underline{y} \end{cases}$$

so $Eq_!(Q\mathbb{C}_{\{(x,y,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}:\, t-xy\geq 0,\, a\leq x\leq b\}}) \simeq \mathsf{E}^g$.
Let $f_1,\, f_2 : (b,+\infty)\to\mathbb{R}$ be the two exponential factors of $\mathbb{E}^{f_1\rhd f_2}_{V_{+\infty}|\mathbb{R}_\infty}$ in the decomposition at $+\infty$ of K. Recall that we have the following short exact sequence:

$$0 \longrightarrow \mathbb{E}^{f_1\rhd f_2}_{V_{+\infty}|\mathbb{R}_\infty} \longrightarrow \mathbb{E}^{f_1}_{V_{+\infty}|\mathbb{R}_\infty} \longrightarrow \mathbb{E}^{f_2}_{V_{+\infty}|\mathbb{R}_\infty} \longrightarrow 0 \tag{5.4}$$

Since the enhanced Fourier-Sato transform is an exact functor, we get the short exact sequence

$$0 \longrightarrow {}^{\mathsf{L}}\mathbb{E}^{f_1\rhd f_2}_{V_{+\infty}|\mathbb{R}_\infty} \longrightarrow {}^{\mathsf{L}}\mathbb{E}^{f_1}_{V_{+\infty}|\mathbb{R}_\infty} \longrightarrow {}^{\mathsf{L}}\mathbb{E}^{f_2}_{V_{+\infty}|\mathbb{R}_\infty} \longrightarrow 0$$

hence we can study f_1 and f_2 separately.
In conclusion in order to compute the admissible exponential factors in ${}^{\mathsf{L}}K$ we can focus only the exponential sheaves of the form ${}^{\mathsf{L}}\mathsf{E}^{f}_{V_{\pm\infty}|\mathbb{R}_\infty}$, ${}^{\mathsf{L}}\mathsf{E}^{f_1}_{V_{\pm\infty}|\mathbb{R}_\infty}$ and ${}^{\mathsf{L}}\mathsf{E}^{f_2}_{V_{\pm\infty}|\mathbb{R}_\infty}$

given by the admissible exponential factors f and f_1, f_2 with $f_1 \geq f_2$ that appear in the decomposition of K, thanks to the short exact sequences (5.3) and (5.4) and the fact that ${}^{\mathsf{L}}K \simeq \mathbb{C}^E_{\mathbb{R}_\infty} \overset{+}{\otimes} {}^{\mathsf{L}}F$ for $F \in \mathrm{E}^0_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{R}_\infty})$ such that $K \simeq \mathbb{C}^E_{\mathbb{R}_\infty} \overset{+}{\otimes} F$; anyway we have to keep in mind that in the decomposition of ${}^{\mathsf{L}}K$ there will be also some exponential indsheaves given by non admissible exponential factors.

If now we assume that $K \in \mathrm{E}^0_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_{\mathbb{R}_\infty})$ has only one singular point at $a \in \mathbb{R}$ then with the same considerations as above we can prove that in order to compute the exponential factors in ${}^{\mathsf{L}}K$ we can study only the exponential sheaves of the form ${}^{\mathsf{L}}\mathsf{E}^f_{V^\pm_a|\mathbb{R}_\infty}$, ${}^{\mathsf{L}}\mathsf{E}^{f_1}_{V^\pm_a|\mathbb{R}_\infty}$ and ${}^{\mathsf{L}}\mathsf{E}^{f_2}_{V^\pm_a|\mathbb{R}_\infty}$ given by the exponential factors f and f_1, f_2 with $f_1 \geq f_2$ that appear in the decomposition of K. Again in the decomposition of ${}^{\mathsf{L}}K$ we will find also some exponential indsheaves given by non admissible exponential factors.

Theorem 5.2.3. *Consider the decomposition at a of $K \in \mathrm{E}^0_{\mathbb{R}-c}(\mathrm{I}\mathbb{C}_{\mathbb{R}_\infty})$. The Legendre transform establishes a 1-1 correspondence from the admissible exponential factors of K at a defined on V^u_a to the admissible exponential factors of ${}^{\mathsf{L}}K$ at b defined on U^v_b, where $\mathsf{L}(a,u,f) = (b,v,g)$.*

Proof. We'll study only the behaviour in $V_{+\infty}$ and in V^-_a with $a \in \mathbb{R}$ since ${}^{\mathsf{L}}({}^{\mathsf{L}}\mathsf{E}^f_{\mathbb{R}|\mathbb{R}_\infty}) \simeq \mathsf{E}^{f^a}_{\mathbb{R}|\mathbb{R}_\infty} \in \mathrm{E}^0_{\mathbb{R}-c}(k_{\mathbb{R}_\infty})$. We will consider the exponential sheaves $\mathsf{E}^f_{V_{+\infty}|\mathbb{R}_\infty}$, $\mathsf{E}^{f_1}_{V_{+\infty}|\mathbb{R}_\infty}$, $\mathsf{E}^{f_2}_{V_{+\infty}|\mathbb{R}_\infty}$ and $\mathsf{E}^f_{V^-_a|\mathbb{R}_\infty}$, $\mathsf{E}^{f_1}_{V^-_a|\mathbb{R}_\infty}$, $\mathsf{E}^{f_2}_{V^-_a|\mathbb{R}_\infty}$ with $a \in \mathbb{R}$, as explained before.

Let $a = +\infty$ and assume that $\mathrm{ord}_{+\infty} f > 1$, $\mathrm{ord}_{+\infty} f_1 > 1$, $\mathrm{ord}_{+\infty} f_2 > 1$ and $V_{+\infty} = ((c,+\infty), (c,+\infty])$ where c is chosen in order to have $f''(x)$, $f''_1(x)$, $f''_2(x) \neq 0$ for any $x \in (c,+\infty)$.
i) Let $\mathbb{E}^f_{V_{+\infty}|\mathbb{R}_\infty}$ be in the decomposition of K at $+\infty$ with $\lim\limits_{x\to+\infty} f(x) = +\infty$. We have ${}^{\mathsf{L}}\mathsf{E}^f_{V_{+\infty}|\mathbb{R}_\infty} \simeq Eq_!(Q(\mathbb{C}_{\{(x,y,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}:\, t+f(x)-xy\geq 0,\, c<x\}}))$; if we fix $(\underline{y},\underline{t}) \in \mathbb{R}\times\mathbb{R}$ then $({}^{\mathsf{L}}\mathsf{E}^f_{V_{+\infty}|\mathbb{R}_\infty})_{(\underline{y},\underline{t})} \simeq R\Gamma_c(\{x \in \mathbb{R}\}; \mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}+f(x)-x\underline{y}\geq 0,\, c<x\}})$. If $-f(x) + x\underline{y}$ hasn't any stationary points for every $x > c$ then

$$R\Gamma_c(\{x \in \mathbb{R}\}; \mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}+f(x)-x\underline{y}\geq 0,\, c<x\}}) = \begin{cases} 0 & \text{if } \underline{t} < -f(c) + c\underline{y} \\ \mathbb{C}[-1] & \text{if } \underline{t} \geq -f(c) + c\underline{y} \end{cases};$$

we are in this situation when $f'(x) - \underline{y} > 0$ for every $x > c$ since $f(x)$ is increasing in $(c,+\infty)$, so for $\underline{y} \leq f'(c)$.
Assume now that $\underline{y} > f'(c)$: in this case $f(x) - x\underline{y}$ has global minimum at $x = \tilde{g}'(\underline{y})$ where $\tilde{g}'(\underline{y})$ is the inverse of $\underline{y} = f'(x)$, and so

$$R\Gamma_c(\{x \in \mathbb{R}\}; \mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}+f(x)-x\underline{y}\geq 0,\, c<x\}}) = \begin{cases} 0 & \text{if } \underline{t} < -f(\tilde{g}'(\underline{y})) + \tilde{g}'(\underline{y})\underline{y} \\ \mathbb{C}[-1] & \text{if } \underline{t} \geq -f(\tilde{g}'(\underline{y})) + \tilde{g}'(\underline{y})\underline{y} \end{cases}.$$

Recall that $f(\tilde{g}'(\underline{y})) - \tilde{g}'(\underline{y})\underline{y} = g(\underline{y})$ where $g(\underline{y})$ is the integral of $-\tilde{g}'(\underline{y})$: let's compute it.
Notice that f is increasing and convex in $(c, +\infty)$ since $\lim_{x\to+\infty} f(x) = +\infty$ and $\mathrm{ord}_{+\infty} f > 1$, hence f' is positive and increasing in $(c, +\infty)$ with $\lim_{x\to+\infty} f'(x) = +\infty$ and $\mathrm{ord}_{+\infty} f' > 0$. Let $x = \tilde{g}'(y)$ be the inverse of $y = f'(x)$ for $x > c$: then $\lim_{y\to+\infty} \tilde{g}'(y) = +\infty$ and $\mathrm{ord}_{+\infty}\tilde{g}' > 0$. In particular $\tilde{g}'$ is increasing and positive in $(f'(c), +\infty)$, so $-\tilde{g}'$ is decreasing and negative in $(f'(c), +\infty)$, and $\lim_{y\to+\infty} -\tilde{g}'(y) = -\infty$.
Let $g(y)$ be a primitive of $-\tilde{g}'(y)$ in $(f'(c), +\infty)$: then g is decreasing with $\lim_{y\to+\infty} g(y) = -\infty$ and $\mathrm{ord}_{+\infty} g > 1$. Notice that these computations are exactly what one needs to do to find the Legendre transform of $(+\infty, -, f)$, and so $\mathsf{L}(+\infty, -, f) = (+\infty, -, g)$.
In this way we have found that ${}^{\mathsf{L}}\mathsf{E}^f_{V_{+\infty}|\mathbb{R}_\infty} \simeq \mathsf{E}^h[-1]$ with $h : \mathbb{R} \to \mathbb{R}$ defined as

$$h(y) = \begin{cases} f(c) - cy & \text{if } y \leq f'(c) \\ g(y) & \text{if } y > f'(c) \end{cases} .$$

Notice that $f(c) - cy$ is not admissible, hence we'll consider only g in the decomposition of ${}^{\mathsf{L}}K$: in fact g is admissible and moreover it has $\mathrm{ord}_{+\infty} g > 1$ and $\lim_{y\to+\infty} g(y) = -\infty$.
ii) Let $\mathbb{E}^f_{V_{+\infty}|\mathbb{R}_\infty}$ be in the decomposition of K at $+\infty$ with $\lim_{x\to+\infty} f(x) = -\infty$. We have again ${}^{\mathsf{L}}\mathsf{E}^f_{V_{+\infty}|\mathbb{R}_\infty} \simeq Eq_!(Q(\mathbb{C}_{\{(x,y,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}:\, t+f(x)-xy\geq 0,\, c<x\}}))$; if we fix $(\underline{y}, \underline{t}) \in \mathbb{R}\times\mathbb{R}$ then $({}^{\mathsf{L}}\mathsf{E}^f_{V_{+\infty}|\mathbb{R}_\infty})_{(\underline{y},\underline{t})} \simeq R\Gamma_c(\{x \in \mathbb{R}\}; \mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}+f(x)-x\underline{y}\geq 0,\, c<x\}})$. If $-f(x) + x\underline{y}$ hasn't any stationary points for every $x > c$ this time we have $R\Gamma_c(\{x \in \mathbb{R}\}; \mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}+f(x)-x\underline{y}\geq 0,\, c<x\}}) = 0$ for every $\underline{t}$. We are in this situation when $f'(x) - \underline{y} < 0$ for every $x > c$ since $f(x)$ is decreasing in $(c, +\infty)$, so for $\underline{y} \leq f'(c)$.
Assume now that $\underline{y} > f'(c)$: in this case $f(x) - x\underline{y}$ has global minimum at $x = \tilde{g}'(\underline{y})$ where $\tilde{g}'(\underline{y})$ is the inverse of $\underline{y} = f'(x)$, and so

$$R\Gamma_c(\{x \in \mathbb{R}\}; \mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}+f(x)-x\underline{y}\geq 0,\, c<x\}}) = \begin{cases} 0 & \text{if } \underline{t} < -g(\underline{y}) \\ \mathbb{C} & \text{if } -g(\underline{y}) \leq \underline{t} < -f(c) + c\underline{y} \\ 0 & \text{if } \underline{t} \geq -f(c) + c\underline{y} \end{cases} .$$

Let's compute $g(y)$.
Notice that f is decreasing and concave in $(c, +\infty)$ since $\lim_{x\to+\infty} f(x) = -\infty$ and $\mathrm{ord}_{+\infty} f > 1$, hence f' is negative and decreasing in $(c, +\infty)$ with $\lim_{x\to+\infty} f'(x) = -\infty$ and $\mathrm{ord}_{+\infty} f' > 0$. Let $x = \tilde{g}'(y)$ be the inverse of $y = f'(x)$ for $x > c$:

then $\lim_{y\to-\infty} \tilde{g}'(y) = +\infty$ and $\mathrm{ord}_{+\infty}\tilde{g}' > 0$. In particular $\tilde{g}'$ is decreasing and positive in $(-\infty, f'(c))$, so $-\tilde{g}'$ is increasing and negative in $(-\infty, f'(c))$, and $\lim_{y\to-\infty} -\tilde{g}'(y) = -\infty$.
Let $g(y)$ be a primitive of $-\tilde{g}'(y)$ in $(-\infty, f'(c))$: then g is decreasing with $\lim_{y\to-\infty} g(y) = +\infty$ and $\mathrm{ord}_{+\infty}g > 1$.
In this way we have found that ${}^{\mathsf{L}}\mathsf{E}^{f}_{V_{+\infty}|\mathbb{R}_\infty} \simeq \mathsf{E}^{g\rhd h}_{(-\infty,f'(c))|\mathbb{R}_\infty}$ with $h : (-\infty, f'(c)) \to \mathbb{R}$ defined as $h(y) = f(c)-cy$. Notice that $h(y)$ is not admissible, hence we'll consider again only g in the decomposition of ${}^{\mathsf{L}}K$: g is admissible and it has $\mathrm{ord}_{+\infty}g > 1$ and $\lim_{y\to-\infty} g(y) = +\infty$.
iii) Let $\mathbb{E}^{f_1\rhd f_2}_{V_{+\infty}|\mathbb{R}_\infty}$ be in the decomposition of K at $+\infty$ with $\lim_{x\to+\infty} f_1(x) = +\infty$ and $\lim_{x\to+\infty} f_2(x) = -\infty$. In this case we'll study separately $\mathsf{E}^{f_1}_{V_{+\infty}|\mathbb{R}_\infty}$ and $\mathsf{E}^{f_2}_{V_{+\infty}|\mathbb{R}_\infty}$.
Thanks to i) we know that in ${}^{\mathsf{L}}\mathsf{E}^{f_1}_{V_{+\infty}|\mathbb{R}_\infty}$ we have the function $g_1(y) : (f_1'(c), +\infty) \to \mathbb{R}$ which is admissible and has $\mathrm{ord}_{+\infty}g_1 > 1$ and $\lim_{y\to+\infty} g_1(y) = -\infty$. By applying ii) to $\mathsf{E}^{f_2}_{V_{+\infty}|\mathbb{R}_\infty}$ we find in ${}^{\mathsf{L}}\mathsf{E}^{f_2}_{V_{+\infty}|\mathbb{R}_\infty}$ the function $g_2(y) : (-\infty, f_2'(c)) \to \mathbb{R}$, admissible, with $\mathrm{ord}_{-\infty}g_2 > 1$ and $\lim_{y\to-\infty} g_2(y) = +\infty$.
Hence in ${}^{\mathsf{L}}K$ there are the two admissible exponential factors g_1 and g_2, respectively at $+\infty$ and at $-\infty$.
iv) Let $\mathbb{E}^{f_1\rhd f_2}_{V_{+\infty}|\mathbb{R}_\infty}$ be in the decomposition of K at $+\infty$ with $\lim_{x\to+\infty} f_1(x) = +\infty$ and $\lim_{x\to+\infty} f_2(x) = +\infty$. Again we'll study separately $\mathsf{E}^{f_1}_{V_{+\infty}|\mathbb{R}_\infty}$ and $\mathsf{E}^{f_2}_{V_{+\infty}|\mathbb{R}_\infty}$.
Using the results in i) for both $\mathsf{E}^{f_1}_{V_{+\infty}|\mathbb{R}_\infty}$ and $\mathsf{E}^{f_2}_{V_{+\infty}|\mathbb{R}_\infty}$ we find that in ${}^{\mathsf{L}}\mathsf{E}^{f_1}_{V_{+\infty}|\mathbb{R}_\infty}$ there's the function $g_1(y) : (f_1'(c), +\infty) \to \mathbb{R}$, admissible, with $\mathrm{ord}_{+\infty}g_1 > 1$ and $\lim_{y\to+\infty} g_1(y) = -\infty$, and in ${}^{\mathsf{L}}\mathsf{E}^{f_2}_{V_{+\infty}|\mathbb{R}_\infty}$ there's the function $g_2(y) : (f_2'(c), +\infty) \to \mathbb{R}$, admissible, with $\mathrm{ord}_{+\infty}g_2 > 1$ and $\lim_{y\to+\infty} g_2(y) = -\infty$. In particular $\mathrm{ord}_{+\infty}f_1 \geq \mathrm{ord}_{+\infty}f_2$ because $f_1 \geq f_2$ with f_1, f_2 both positive and $f_1 - f_2$ is unbounded at $+\infty$, so $\mathrm{ord}_{+\infty}g_1 \leq \mathrm{ord}_{+\infty}g_2$ hence $g_1 \geq g_2$ since they're both negative. Moreover $g_1 - g_2$ is unbounded at $+\infty$, thus (g_1, g_2) is also admissible.
Recall that in ${}^{\mathsf{L}}K$ there are also some exponential sheaves given by non admissible functions which may interfere with g_1 and g_2, so we can't assume the presence of $\mathsf{E}^{g_1\rhd g_2}_{U_{+\infty}|\mathbb{R}_\infty}$ in the decomposition of ${}^{\mathsf{L}}K$ at $+\infty$, therefore we have to consider g_1 and g_2 separately.
v) Let $\mathbb{E}^{f_1\rhd f_2}_{V_{+\infty}|\mathbb{R}_\infty}$ be in the decomposition of K at $+\infty$ with $\lim_{x\to+\infty} f_1(x) = -\infty$ and $\lim_{x\to+\infty} f_2(x) = -\infty$. Also here we'll study separately $\mathsf{E}^{f_1}_{V_{+\infty}|\mathbb{R}_\infty}$ and $\mathsf{E}^{f_2}_{V_{+\infty}|\mathbb{R}_\infty}$.
Using the results in ii) for $\mathsf{E}^{f_1}_{V_{+\infty}|\mathbb{R}_\infty}$ and $\mathsf{E}^{f_2}_{V_{+\infty}|\mathbb{R}_\infty}$ we find that in ${}^{\mathsf{L}}\mathsf{E}^{f_1}_{V_{+\infty}|\mathbb{R}_\infty}$ there's the function $g_1(y) : (-\infty, f_1'(c)) \to \mathbb{R}$, admissible, with $\mathrm{ord}_{-\infty}g_1 > 1$ and $\lim_{y\to-\infty} g_1(y) =$

$+\infty$, and in ${}^{\mathsf{L}}\mathsf{E}^{f_2}_{V_{+\infty}|\mathbb{R}_\infty}$ there's the function $g_2(y) : (-\infty, f_2'(c)) \to \mathbb{R}$, admissible, with $\mathrm{ord}_{+\infty} g_2 > 1$ and $\lim_{y\to-\infty} g_2(y) = +\infty$. This time $\mathrm{ord}_{+\infty} f_1 \leq \mathrm{ord}_{+\infty} f_2$ because $f_1 \geq f_2$ with f_1, f_2 both negative and $f_1 - f_2$ is unbounded at $+\infty$, so $\mathrm{ord}_{-\infty} g_1 \geq \mathrm{ord}_{-\infty} g_2$ hence $g_1 \geq g_2$ since they're both positive. Moreover $g_1 - g_2$ is unbounded at $-\infty$, thus (g_1, g_2) is also admissible.
Hence in ${}^{\mathsf{L}}\mathsf{E}^{f_1 \triangleright f_2}_{V_{+\infty}|\mathbb{R}_\infty}$ there is the pair of admissible exponential factors (g_1, g_2). Also in this case we can't assume to have $\mathsf{E}^{g_1 \triangleright g_2}_{U_{-\infty}|\mathbb{R}_\infty}$ in the decomposition of ${}^{\mathsf{L}}K$ at $-\infty$, so we have to consider g_1 and g_2 separately.

Assume now that $a = 0$.
Assume that $V_0^- = ((c, 0), (c, 0])$ where c is chosen in order to have $f''(x)$, $f_1''(x)$, $f_2''(x) \neq 0$ for any $x \in (c, 0)$.
i) Let $\mathbb{E}^f_{V_0^-|\mathbb{R}_\infty}$ be in the decomposition of K at 0 with $\lim_{x\to 0^-} f(x) = +\infty$. We have ${}^{\mathsf{L}}\mathsf{E}^f_{V_0^-|\mathbb{R}_\infty} \simeq Eq_!(Q(\mathbb{C}_{\{(x,y,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}:\, t+f(x)-xy\geq 0,\, c<x<0\}}))$; if we fix $(\underline{y}, \underline{t}) \in \mathbb{R}\times\mathbb{R}$ then $({}^{\mathsf{L}}\mathsf{E}^f_{V_0^-|\mathbb{R}_\infty})_{(\underline{y},\underline{t})} \simeq R\Gamma_c(\{x \in \mathbb{R}\}; \mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}+f(x)-x\underline{y}\geq 0,\, c<x<0\}})$. If $-f(x) + x\underline{y}$ hasn't any stationary points for every $c < x < 0$ then

$$R\Gamma_c(\{x \in \mathbb{R}\}; \mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}+f(x)-x\underline{y}\geq 0,\, c<x<0\}}) = \begin{cases} 0 & \text{if } \underline{t} < -f(c) + c\underline{y} \\ \mathbb{C}[-1] & \text{if } \underline{t} \geq -f(c) + c\underline{y} \end{cases};$$

we are in this situation when $f'(x) - \underline{y} > 0$ for every $c < x < 0$ since $f(x)$ is increasing in $(c, 0)$, so for $\underline{y} \leq f'(c)$.
Assume now that $\underline{y} > f'(c)$: in this case $f(x) - x\underline{y}$ has global minimum at $x = \tilde{g}'(\underline{y})$ where $\tilde{g}'(\underline{y})$ is the inverse of $\underline{y} = f'(x)$, and so

$$R\Gamma_c(\{x \in \mathbb{R}\}; \mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}+f(x)-x\underline{y}\geq 0,\, c<x<0\}}) = \begin{cases} 0 & \text{if } \underline{t} < -f(\tilde{g}'(\underline{y})) + \tilde{g}'(\underline{y})\underline{y} \\ \mathbb{C}[-1] & \text{if } \underline{t} \geq -f(\tilde{g}'(\underline{y})) + \tilde{g}'(\underline{y})\underline{y} \end{cases}.$$

Let's compute $g(\underline{y}) = f(\tilde{g}'(\underline{y})) - \tilde{g}'(\underline{y})\underline{y}$.
Notice that f is increasing and convex in $(c, 0)$ since $\lim_{x\to 0^-} f(x) = +\infty$, hence f' is positive and increasing in $(c, 0)$ with $\lim_{x\to 0^-} f'(x) = +\infty$ and $\mathrm{ord}_0 f' \geq 1$. Let $x = \tilde{g}'(y)$ be the inverse of $y = f'(x)$ for $c < x < 0$: then $\lim_{y\to+\infty} \tilde{g}'(y) = 0^-$. In particular $\tilde{g}'$ is increasing and negative in $(f'(c), +\infty)$, so $-\tilde{g}'$ is decreasing and positive in $(f'(c), +\infty)$, and $\lim_{y\to+\infty} -\tilde{g}'(y) = 0^+$.
Let $g(y)$ be a primitive of $-\tilde{g}'(y)$ in $(f'(c), +\infty)$: g is increasing with $\lim_{y\to+\infty} g(y) = +\infty$ and $0 < \mathrm{ord}_{+\infty} g < 1$.

Hence ${}^{\mathsf{L}}\mathsf{E}^{f}_{V_0^-|\mathbb{R}_\infty} \simeq \mathsf{E}^{h}[-1]$ with $h:\mathbb{R}\to\mathbb{R}$ defined as

$$h(y)=\begin{cases} f(c)-cy & \text{if } y\leq f'(c)\\ g(y) & \text{if } y>f'(c)\end{cases}.$$

Notice that $f(c)-cy$ is not admissible, hence we'll consider only g in the decomposition of ${}^{\mathsf{L}}K$: in fact g is admissible with $0<\mathrm{ord}_{+\infty}g<1$ and $\lim\limits_{y\to+\infty} g(y)=+\infty$.

ii) Let $\mathbb{E}^{f}_{V_0^-|\mathbb{R}_\infty}$ be in the decomposition of K at 0 with $\lim\limits_{x\to 0^-} f(x)=-\infty$. We have again ${}^{\mathsf{L}}\mathsf{E}^{f}_{V_0^-|\mathbb{R}_\infty} \simeq Eq_!(Q(\mathbb{C}_{\{(x,y,t)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}:\, t+f(x)-xy\geq 0,\, c<x<0\}}))$; if we fix $(\underline{y},\underline{t})\in\mathbb{R}\times\mathbb{R}$ then $({}^{\mathsf{L}}\mathsf{E}^{f}_{V_0^-|\mathbb{R}_\infty})_{(\underline{y},\underline{t})} \simeq R\Gamma_c(\{x\in\mathbb{R}\};\mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}+f(x)-x\underline{y}\geq 0,\, c<x<0\}})$. If $-f(x)+x\underline{y}$ hasn't any stationary points for every $c<x<0$ this time we have $R\Gamma_c(\{x\in\mathbb{R}\};\mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}+f(x)-x\underline{y}\geq 0,\, c<x<0\}})=0$ for every $\underline{t}$. We are in this situation when $f'(x)-\underline{y}<0$ for every $c<x<0$ since $f(x)$ is decreasing in $(c,0)$, so for $\underline{y}\leq f'(c)$. Assume now that $\underline{y}>f'(c)$: in this case $f(x)-x\underline{y}$ has global minimum at $x=\tilde{g}'(\underline{y})$ where $\tilde{g}'(\underline{y})$ is the inverse of $\underline{y}=f'(x)$, and so

$$R\Gamma_c(\{x\in\mathbb{R}\};\mathbb{C}_{\{x\in\mathbb{R};\, \underline{t}+f(x)-x\underline{y}\geq 0,\, c<x<0\}})=\begin{cases} 0 & \text{if } \underline{t}<-g(\underline{y})\\ \mathbb{C} & \text{if } -g(\underline{y})\leq\underline{t}<-f(c)+c\underline{y}\\ 0 & \text{if } \underline{t}\geq -f(c)+c\underline{y}\end{cases}.$$

Let's compute $g(y)$.

Notice that f is decreasing and concave in $(c,0)$ since $\lim\limits_{x\to 0^-} f(x)=-\infty$, hence f' is negative and decreasing in $(c,0)$ with $\lim\limits_{x\to 0^-} f'(x)=-\infty$ and $\mathrm{ord}_0 f'\geq 1$. Let $x=\tilde{g}'(y)$ be the inverse of $y=f'(x)$ for $c<x<0$: then $\lim\limits_{y\to-\infty}\tilde{g}'(y)=0^-$. In particular $\tilde{g}'$ is decreasing and negative in $(-\infty,f'(c))$, so $-\tilde{g}'$ is increasing and positive in $(-\infty,f'(c))$, and $\lim\limits_{y\to-\infty} -\tilde{g}'(y)=0^+$.

Let $g(y)$ be a primitive of $-\tilde{g}'(y)$ in $(-\infty,f'(c))$: then g is increasing with $\lim\limits_{y\to-\infty} g(y)=-\infty$ and $0<\mathrm{ord}_{-\infty}g<1$.

So ${}^{\mathsf{L}}\mathsf{E}^{f}_{V_0^-|\mathbb{R}_\infty}\simeq \mathsf{E}^{g\rhd h}_{(-\infty,f'(c))|\mathbb{R}_\infty}$ with $h:(-\infty,f'(c))\to\mathbb{R}$ defined as $h(y)=f(c)-cy$. Notice that $h(y)$ is not admissible, hence we'll consider again only g in the decomposition of ${}^{\mathsf{L}}K$: g is admissible and it has $0<\mathrm{ord}_{-\infty}g<1$ and $\lim\limits_{y\to-\infty} g(y)=-\infty$.

iii) Let $\mathbb{E}^{f_1\rhd f_2}_{V_0^-|\mathbb{R}_\infty}$ be in the decomposition of K at 0. With the same considerations made in the case $a=+\infty$ and using the results i) and ii) of the case $a=0$ it is possible to prove that in the decomposition of ${}^{\mathsf{L}}K$ at $\pm\infty$ there are the admissible exponential factors g_1, g_2 where g_1, g_2 are given by the Legendre transform of $(0,-,f_1)$ and $(0,-,f_2)$.

Assume that $a \in \mathbb{R}$, $a > 0$.
Assume that $V_a^- = ((c,a),\ (c,a])$ where c is chosen in order to have $f''(x),\ f_1''(x),\ f_2''(x) \neq 0$ for any $x \in (c,a)$.
Let $\tau_{-a} : \mathbb{R}_\infty \to \mathbb{R}_\infty$ be the morphism induced by the translation $\tau_{-a}(x) = x - a$. Then we have $\mathsf{E}^{f}_{V_a^-|\mathbb{R}_\infty} \simeq E\tau_{-a}^{-1}(\mathsf{E}^{\hat{f}}_{V_0^-|\mathbb{R}_\infty})$ where $\hat{f} := f \circ \tau_{-a}$. Notice that $\hat{f}$ has the same limit for $x \to 0^-$ as the one of f for $x \to a^-$. Then, thanks to Lemma 4.3.3, we have ${}^{\mathsf{L}}\mathsf{E}^{f}_{V_a^-|\mathbb{R}_\infty} \simeq \mathsf{E}^{-ay}_{\mathbb{R}|\mathbb{R}_\infty} \overset{+}{\otimes} {}^{\mathsf{L}}\mathsf{E}^{\hat{f}}_{V_0^-|\mathbb{R}_\infty}$.
i) Assume that $\mathbb{E}^{f}_{V_a^-|\mathbb{R}_\infty}$ is in the decomposition of K at a with $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to 0^-} \hat{f}(x) = +\infty$. Then in the decomposition at $+\infty$ of ${}^{\mathsf{L}}\mathsf{E}^{\hat{f}}_{V_0^-|\mathbb{R}_\infty}$ we obtain the admissible exponential factor $\hat{g}(y) : (\hat{f}'(c), +\infty) \to \mathbb{R}$ with $\lim\limits_{y\to+\infty} \hat{g}(y) = +\infty$ and $0 < \mathrm{ord}_{+\infty}\hat{g} < 1$; hence in the decomposition at $+\infty$ of ${}^{\mathsf{L}}K$ there is the exponential factor $g(y) = \hat{g}(y) - ay$: it has $\lim\limits_{x\to+\infty} g(y) = -\infty$ and $\mathrm{ord}_{+\infty} g = 1$, and it's admissible because $\hat{g}(y)$ is not constant.
ii) Assume that $\mathbb{E}^{f}_{V_a^-|\mathbb{R}_\infty}$ is in the decomposition of K at a with $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to 0^-} \hat{f}(x) = -\infty$. Analogously we find that in the decomposition at $-\infty$ of ${}^{\mathsf{L}}K$ there is the exponential factor $g(y) = \hat{g}(y) - ay$ with $\lim\limits_{x\to-\infty} g(y) = +\infty$ and $\mathrm{ord}_{-\infty} g = 1$, which is admissible because $\hat{g}(y)$ is not constant.
iii) Let $\mathbb{E}^{f_1 \triangleright f_2}_{V_a^-|\mathbb{R}_\infty}$ be in the decomposition of K at a. Then in the decomposition of ${}^{\mathsf{L}}K$ at $\pm\infty$ there are the admissible exponential factors g_1, g_2 obtained applying i) or ii) or both i) and ii) to f_1 and f_2.

Assume that $a \in \mathbb{R}$, $a < 0$.
Assume that $V_a^- = ((c,a),\ (c,a])$ where c is chosen in order to have $f''(x)$, $f_1''(x)$, $f_2''(x) \neq 0$ for any $x \in (c,a)$.
Let $\tau_a : \mathbb{R}_\infty \to \mathbb{R}_\infty$ be the morphism induced by the translation $\tau_a(x) = x + a$. Then again ${}^{\mathsf{L}}\mathsf{E}^{f}_{V_a^-|\mathbb{R}_\infty} \simeq \mathsf{E}^{ay}_{\mathbb{R}|\mathbb{R}_\infty} \overset{+}{\otimes} {}^{\mathsf{L}}\mathsf{E}^{\hat{f}}_{V_0^-|\mathbb{R}_\infty}$, where $\hat{f} := f \circ \tau_a$.
With the same considerations as above we find that the admissible exponential factor f with $\lim\limits_{x\to a^-} f(x) = +\infty$ corresponds to the admissible exponential factor g in the decomposition of ${}^{\mathsf{L}}K$ at $+\infty$ with $\lim\limits_{y\to+\infty} g(y) = +\infty$ and $\mathrm{ord}_{+\infty} g = 1$, which is admissible, and that the admissible exponential factor f with $\lim\limits_{x\to a^-} f(x) = -\infty$ corresponds to the admissible exponential factor g in the decomposition of ${}^{\mathsf{L}}K$ at $-\infty$ with $\lim\limits_{y\to-\infty} g(y) = -\infty$ and $\mathrm{ord}_{-\infty} g = 1$, admissible.

Consider now ${}^{\mathsf{L}}K$ instead of K and assume that in the decomposition of ${}^{\mathsf{L}}K$ at b there is the exponential indsheaf $\mathbb{E}^{g}_{U^v_b|\mathbb{R}_\infty}$ given by the admissible exponential factor g where $v \in \{+,-\}$. Notice that ${}^{\lrcorner}({}^{\mathsf{L}}K) \simeq K$ and $\mathsf{L}(\mathsf{J}(b,v,g)) \simeq (b,v,g)$. Hence, by the same computations as above, $(a,u,f) = \mathsf{J}(b,v,g)$ where f is an admissible exponential factor of K.
Similar considerations hold if we assume that in the decomposition of ${}^{\mathsf{L}}K$ at b there is the exponential indsheaf $\mathbb{E}^{g_1 \triangleright g_2}_{U^v_b|\mathbb{R}_\infty}$ given by the admissible exponential factors g_1, g_2.
Hence we have a 1-1 correspondence between the admissible exponential factors of K at a and the admissible exponential factors of ${}^{\mathsf{L}}K$ at b, where $\mathsf{L}(a,u,f) = (b,v,g)$. □

Let's summarize the correspondence between $(a,-,f)$ of K and $\mathsf{L}(a,-,f) = (b,v,g)$ of ${}^{\mathsf{L}}K$ in the following table:

K			${}^{\mathsf{L}}K$		
a	u	f	b	v	g
$+\infty$	$-$	$f(x) \underset{x\to+\infty}{\longrightarrow} +\infty$, $\mathrm{ord}_{+\infty} f > 1$	$+\infty$	$-$	$g(y) \underset{y\to+\infty}{\longrightarrow} -\infty$, $\mathrm{ord}_{+\infty} g > 1$
$+\infty$	$-$	$f(x) \underset{x\to+\infty}{\longrightarrow} -\infty$, $\mathrm{ord}_{+\infty} f > 1$	$-\infty$	$+$	$g(y) \underset{y\to-\infty}{\longrightarrow} +\infty$, $\mathrm{ord}_{-\infty} g > 1$
0	$-$	$f(x) \underset{x\to 0^-}{\longrightarrow} +\infty$	$+\infty$	$-$	$g(y) \underset{y\to+\infty}{\longrightarrow} +\infty$, $0 < \mathrm{ord}_{+\infty} g < 1$
0	$-$	$f(x) \underset{x\to 0^-}{\longrightarrow} -\infty$	$-\infty$	$+$	$g(y) \underset{y\to-\infty}{\longrightarrow} -\infty$, $0 < \mathrm{ord}_{-\infty} g < 1$

In particular if $a \in \mathbb{R}$, $a > 0$ then $g(y) = \hat{g}(y) - ay$ where $\hat{g}$ is given by the Legendre transform of $(0,u,\hat{f})$ with $\hat{f} = f \circ \tau_{-a}$ and if $a \in \mathbb{R}$, $a < 0$ then $g(y) = \hat{g}(y) + ay$ where $\hat{g}$ is given by the Legendre transform of $(0,u,\hat{f})$ with $\hat{f} = f \circ \tau_a$.

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