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Inflationary Flow Equations and their implications

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1 Introduction

Inflation is the most accredited theory of the early universe. Introduced by Starobinsky, Guth and Linde in the early '80s, after the detection of the Cosmic Microwave Background radiation and the definitive validation of the Big Bang model, it solves several of the problems which arose in the framework of a standard Big Bang evolution of the universe. This theory makes a number of strong predictions: first of all, if inflation occurred, the universe would have undergone an extremely brief period of exponential expansion, growing of a factor of at least 10^{26} (usually quoted as "60 e-folds of expansion", since $10^{26} \approx e^{60}$) in the arc of 10^{-33} seconds. Secondly, inflation describes a number of important features of the primordial universe, such as the near scale-independence of its perturbation and the generation of primordial gravitational waves; the latter are of particular importance, since no other model predicts their presence in the early stages of our universe. This is why the detection of such primordial waves is considered the "smoking gun" probe of inflation.
As observations of CMB anisotropies, Large Scale Structure and gravitational waves have become more and more detailed and precise, various model of inflation have been put to the test: in the simplest scenario, the inflationary expansion is driven by a quantum scalar field, known as the inflaton ϕ, whose potential $V(\phi)$ is unknown. Different forms of the potential yield different predictions on a number of early-universe observables, to be confronted with the observed data. In this context, an interesting approach is then to use a stochastic process to generate models of inflation, looking for those whose predictions match the data: this is the aim of the Flow Equation approach in inflationary cosmology. This method, first proposed by Hoffman and Turner [8] and later generalized by Kinney [9] to arbitrary order, consists in the definition of a hierarchy of parameters, called *slow-roll parameters*, which depend on the Hubble parameter and its derivative with respect to the inflaton ϕ. The slow-roll parameters are related to each other by a set of differential equations, the flow equations, and can be used to predict the values of cosmological observables given a specific potential $V(\phi)$. Numerical integration of the flow equations yield a trajectory in the slow-roll parameters space, which completely specify the form of the inflaton potential and then of the observables.
In this thesis we present the basic principles of cosmological inflation, and use them to define the early-universe observables of main interest and the slow-roll parameters. We then present the flow equations approach to the inflationary problem, discussing how it can be used to make predictions on said observables and how these prediction are confronted with the most recent data gatherings.

2 Inflationary dynamics [3]

2.1 Equations of motion

The Friedmann-Robertson-Walker metric of an evolving universe with curvature k is (we will always use $c = \hbar = k_B = 1$ units)

$$g_{\mu\nu}dx^\mu dx^\nu = ds^2 = dt^2 - a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) = a(t)^2\left(d\tau^2 - \frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{1}$$

where we used, through definition of the time-dependent scale factor $a(t)$, comoving spatial coordinates $\mathbf{x} = (r, \Omega)$ and conformal time τ, which are related to physical ("standard") spacetime coordinates by $\mathbf{x}_{phys} = a(t)\mathbf{x}$ and $dt = a(t)d\tau$. Isotropy and homogeneity reasons require the stress-energy tensor of the matter sources of the universe to be that of a perfect fluid

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu - P g_{\mu\nu} \tag{2}$$

where ρ and P are the energy density and pressure of the fluid and U^μ is its 4-velocity. In fact we can decompose the stress-energy tensor into the scalar T_{00}, the 3-vectors T_{0i} and T_{i0} and the 3-tensor T_{ij}, and argue that the *mean* values of the 3-vectors components, which represent the flux of energy in the direction x^i, should vanish for isotropy, and for the same reason the 3-tensor should be proportional to δ_{ij} and then to $g_{ij}(\mathbf{x}=0)$. Moreover, homogeneity requires T_{00} and the diagonal components of T_{ij} to depend only on time. Notice these considerations are valid *in a comoving frame*:

$$T_{00} = \rho(t); \qquad T_{i0} = T_{0i} = 0; \qquad T_{ij} = -P(t)g_{ij}(t, \mathbf{x}) \tag{3}$$

This expressions correspond to (2) if we are in a comoving frame, where $U_\mu = (1, 0, 0, 0)$; then (2) is the generalization of the above expressions to a non-comoving frame.
Applying the Einstein equations, $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ to the perfect fluid stress energy tensor, we obtain the equations of motion for the metric, which relates the evolution of the scale factor to the matter-energy content of the universe. Remembering the definitions [1]

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g^{\rho\sigma} R_{\rho\sigma} g_{\mu\nu}, \tag{4}$$

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho}, \tag{5}$$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}), \tag{6}$$

with some gruesome calculations we find that the only non-zero components of the Einstein tensor are

$$G^0{}_0 = 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] \tag{7}$$

$$G^i{}_j = \left[2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right]\delta^i_j. \tag{8}$$

Substituting this and (2) into the Einstein equations gives the *Friedmann equations*:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{9}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \tag{10}$$

where the *Hubble parameter* $H \equiv \dot{a}/a$ has been defined. We will later limit ourselves to the case of a flat universe ($k = 0$). Another fundamental equation is the continuity equation, $\nabla_\mu T^\mu{}_\nu = 0$. This implies

$$\nabla_\mu T^\mu{}_\nu = \partial_\mu T^\mu{}_\nu + \Gamma^\mu_{\mu\lambda} T^\lambda{}_\nu - \Gamma^\lambda_{\mu\nu} T^\mu{}_\lambda = 0. \tag{11}$$

Using the definition of the Christoffel symbols one can recast this as

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0 \tag{12}$$

which is known as the *fluid equation*.
The Friedmann equations can also be (naively) derived in the context of Newtonian gravity [2], by applying energy conservation: let's consider the idealized case of a perfectly uniform expanding medium of mass density ρ. If the universe is homogeneous and isotropic, we can consider any point to be is center, and compute the gravitational attraction felt by a body of mass m at distance r from it as

$$\mathbf{F} = -\frac{GMm\mathbf{r}}{r^3} = -\frac{4\pi G\rho\mathbf{r}m}{3} \tag{13}$$

where, using a famous argument due to Newton, we only considered the mass within the sphere of radius r to be the source of the force. The potential energy associated with this configuration is

$$V = -\frac{GMm}{r} = -\frac{4\pi G\rho r^2 m}{3} \tag{14}$$

for a total energy of

$$U = \frac{1}{2}m\dot{r}^2 - \frac{4\pi G\rho r^2 m}{3} \tag{15}$$

which can be written using comoving coordinates ($r = ax$) as

$$U = \frac{1}{2}m\dot{a}^2 x^2 - \frac{4\pi G\rho a^2 x^2 m}{3} \tag{16}$$

(notice we are considering objects fixed in comoving coordinates, $\dot{x} = 0$). Multiplying by $2/ma^2x^2$ gives exactly equation (9), where $k \equiv -2U/mx^2$ is a constant. We can derive the second equation using the perfect fluid assumption: if we only consider reversible adiabatic expansions, where $dS = 0$, the first law of thermodynamics reduces to

$$dE + pdV = 0. \tag{17}$$

Differentiating this in time, and considering that $E = m = (4\pi/3)\rho(ax)^3$ and that $V = (4\pi/3)(ax)^3$, we get

$$0 = \frac{dE}{dt} + p\frac{dV}{dt} = 4\pi\rho a^2 x^3 \frac{da}{dt} + \frac{4\pi}{3}a^3 x^3 \frac{d\rho}{dt} + 4\pi p a^2 x^3 \frac{da}{dt} \implies \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p\right) = 0. \tag{18}$$

Which is equation (12). Differentiating the first Friedmann equation and substituting in for $\dot{\rho}$ from the fluid equation yields

$$2\frac{\dot{a}}{a}\frac{a\ddot{a} - \dot{a}^2}{a^2} = \frac{8\pi G}{3}\left(-3\frac{\dot{a}}{a}(\rho + p)\right) + 2\frac{k\dot{a}}{a^3} \tag{19}$$

and then dividing by $2\dot{a}/a$ and using once again the first Friedmann equation to express the factor $(\dot{a}/a)^2$ on the l.h.s. we get the second Friedmann equation, (10).
These equations completely specify the dynamics of the scale parameter (and consequently of the spacetime metric) and can be solved if a specific equation of state, i.e. a relation between the energy density and pressure of the cosmic fluid, $P = w(\rho)\rho$, is assigned. The energy sources of main interest in cosmology are radiation ($w \equiv 1/3$), non-relativistic matter ($w \approx 0$) and dark energy ($w \equiv -1$). Standard cosmology predicts that, for "conventional" sources as radiation and matter, $a(t) \underset{t,\tau\to 0}{\longrightarrow} 0$, which is famously known as the Big Bang singularity. This can be straightforwardly derived from the equations of motion: since the fluid equation (12) implies

$$\rho \propto a^{-3(1+w)} \implies \rho \propto \begin{cases} a^{-3} & \text{matter} \\ a^{-4} & \text{radiation} \\ (a^0 & \text{dark energy}) \end{cases} \tag{20}$$

(radiation energy density scales with an additional a^{-1} factor because of the contribution of gravitational redshift), in the case of a flat, single component universe we can recast the Friedmann equation as

$$H(a) = \frac{\dot{a}}{a} \propto a^{-\frac{3}{2}(1+w)} \implies \dot{a} \propto a^{-\frac{1}{2}(1+3w)} \tag{21}$$

which is solved by

$$a(t) \propto t^{\frac{2}{3(1+w)}} = \begin{cases} t^{\frac{2}{3}} & \text{matter} \\ t^{\frac{1}{2}} & \text{radiation} \end{cases} \tag{22}$$

$$(\ \ a(t) \propto e^{Ht},\ H = \text{const.} \quad \text{dark energy}\ \ )$$

2.2 The horizon problem and the inflationary solution

The theory of the Big Bang and standard cosmology generate a number of problems, the most critical of which is the *horizon problem.* To formalise it, let us consider a photon travelling in a FRW metric: it holds that $ds^2 = a(\tau)(d\tau^2 - d\mathbf{x} \cdot d\mathbf{x}) = 0 \implies d|\mathbf{x}| = d\tau$, so the maximum comoving distance the photon can travel from the beginning of time to present is

$$|\Delta \mathbf{x}| = \int_{\tau_i}^{\tau} d\tau = \int_{t_i}^{t} \frac{dt}{a} = \int_{a_i}^{a} \frac{1}{a\dot{a}} da = \int_{\ln a_i}^{\ln a} (aH)^{-1} d\ln a \equiv \int_{N_i}^{N} (aH)^{-1} dN \tag{23}$$

where we have defined the *number of e-folds of expansion* as $dN = d\ln a = Hdt$. The integrand $(aH)^{-1}$ is known as *comoving Hubble radius.* It represents the maximum comoving distance a particle can travel during an expansion time (which the time it takes $a(t)$ to increase by a factor of e), therefore it measures causal connection: if two points in spacetime are separated more than the Hubble radius at that moment, they will not be able to communicate within the next expansion time (while if they are separated more than their particle horizon they *never were* able to communicate). As we have seen in (21), in a universe dominated by an energy source of constant equation of state $P = w\rho$, we can write

$$(aH)^{-1} = (\dot{a})^{-1} \propto a^{\frac{1}{2}(1+3w)} \tag{24}$$

and since every convential energy source satisfies the so called Strong Energy Condition (SEC), $1+3w > 0$ (as is the case for radiation and matter), the Hubble radius should be a constantly increasing quantity as the universe expands. It follows that the integral above is convergent, and this comoving distance (called the *particle horizon*) is finite: we conclude that the Big Bang theory predicts the early universe to be composed of many casually disconnected patches of spacetime, failing to explain, for instance, why the Cosmic Microwave Background temperature is observed to be homogeneously $\sim 2.7K$ in every direction, up to perturbations of order 10^{-5}.

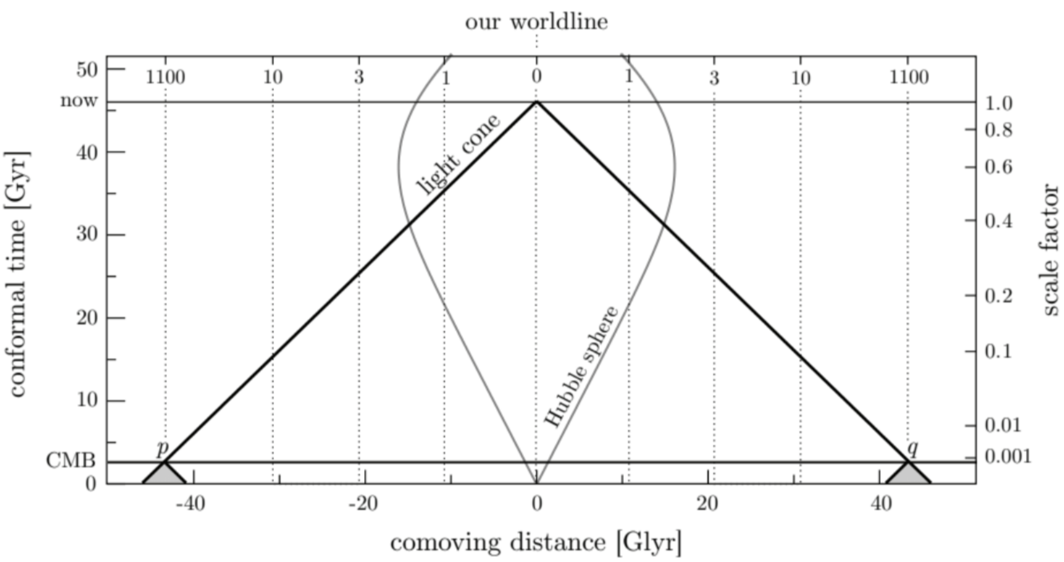

Figure 1: Image taken from [3]. Our past light cone intersect the CMB in two antipodal points (being this a 1D projection), but the past light cones of these two points don't intersect before they hit the Big Bang singularity, so those two points can never have been in causal contact. Notice the light cones are actually cones (like in the Minkowskian case) only if we use conformal time and comoving distances; in physical coordinates, their shape would be deformed by the evolution of the scale factor over time [1]

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A solution is offered by inflationary cosmology, which postulates an early period of *decreasing*

Hubble radius or, equivalently, of *accelerated expansion*, since

$$\frac{d}{dt}(aH)^{-1} = \frac{d}{dt}(\dot{a})^{-1} = -\frac{\ddot{a}}{(\dot{a})^2} < 0 \implies \ddot{a} > 0. \tag{25}$$

Notice this can be achieved only with the introduction of a SEC-violating fluid, with $1 + 3w < 0$. Postulating a decreasing Hubble radius at the beginning of time solves the horizon problem, because now the integral (23) is divergent:

$$|\Delta\mathbf{x}| = \tau - \tau_i \propto \frac{2}{1+3w}(a^{\frac{1}{2}(1+3w)} - a_i^{\frac{1}{2}(1+3w)}) \underset{a_i \to 0}{\longrightarrow} +\infty \quad \text{if } 1+3w < 0. \tag{26}$$

Having a potentially infinite particle horizon, every point in the universe is now in causal contact with every other point if we go sufficiently back in (conformal) time. Notice how the inflationary solution corresponds to pushing the Big Bang singularity ($a_i = 0$), to *infinitely negative* conformal time

$$\tau_i \propto \frac{2}{(1+3w)} a_i^{\frac{1}{2}(1+3w)} \underset{a_i \to 0}{\longrightarrow} -\infty \quad \text{if } 1+3w < 0. \tag{27}$$

In inflationary cosmology, the moment $\tau = 0$ is no longer the time of the singularity, *but the transition point between inflation and standard Big Bang evolution.*

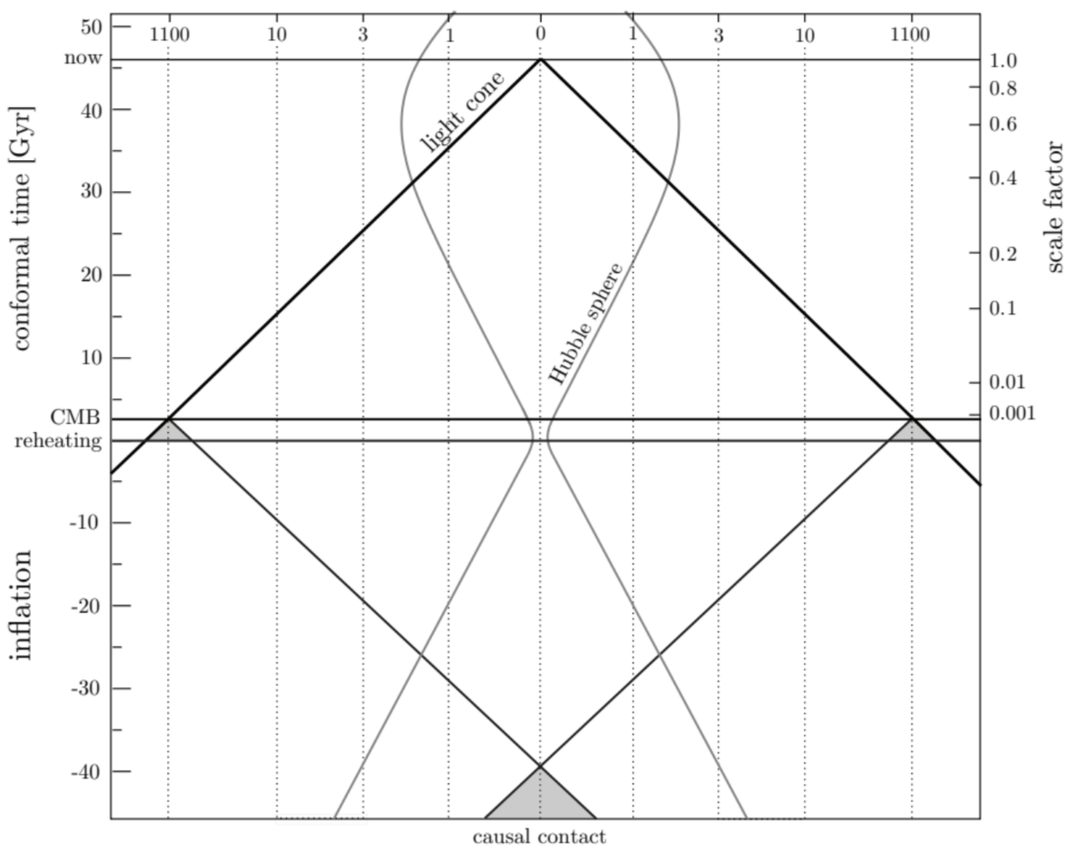

Figure 2: Image taken from [3]. The hypothesis of a shrinking Hubble radius during inflation allows the conformal time to run backwards from $\tau = 0$ to $\tau = -\infty$, expanding the particle horizon of each CMB event until they all overlap. Notice we also have a period of shrinking Hubble radius after inflation has ended, which corresponds to a Dark Energy dominated regime, where $w = -1 < -1/3$.

Another important condition for inflation is that it lasts for a sufficiently long period of time, so that the particle horizon of every point in space-time increases until it includes the whole universe. At the very minimum, we have to ask that the observable universe *today* - which we *know*, from CMB surveys, has been in causal contact in the past - fits entirely into the Hubble radius at the beginning of time, so that within the following expansion time every point within that radius could have been in causal contact with every other point. This implies

$$(a_0 H_0)^{-1} < (a_i H_i)^{-1} \tag{28}$$

where the 0 subscript indicates quantities evaluated at present time. Now, by assuming the universe was radiation dominated after the inflation (which, as clear from (20), was the case of the early phases of the Big Bang expansion), and remembering (21), we have $H \propto a^{-2}$, so we can relate the Hubble radius now to its value at the end of inflation:

$$\frac{a_0 H_0}{a_e H_e} \approx \frac{a_0}{a_e}\left(\frac{a_e}{a_0}\right)^2 = \frac{a_e}{a_0} \approx \frac{T_0}{T_e} \tag{29}$$

where we have related the scale factor to the temperature of the CMB using the fact that for a perfect black-body radiation

$$\rho \propto T^4 \overset{(20)}{\Longrightarrow} T \propto a^{-1}. \tag{30}$$

We can estimate the CMB temperature at inflation end using its dependence on a and solving the Friedmann equations for the scale factor: this gives $T_e \sim 10^{15} GeV$, and we know $T_0 = 10^{-4} eV (\sim 2.7K$ in $k_B = 1$ units). Then we can make the rough estimate

$$(a_i H_i)^{-1} > (a_0 H_0)^{-1} \approx \frac{T_e}{T_0}(a_e H_e)^{-1} \approx 10^{28}(a_e H_e)^{-1} \tag{31}$$

and since during inflation $H \sim$ constant (as will be clear later, in (39)), this means

$$\frac{a_e}{a_i} > 10^{28} \implies \ln\left(\frac{a_e}{a_i}\right) > 64 \implies \Delta N = N_e - N_i > 64 \tag{32}$$

So we will say that about 60 e-folds of inflation are required to solve the horizon problem.

2.3 Inflaton dynamics and slow-roll inflation

To achieve the conditions for inflation, we introduce a scalar field, the inflaton $\phi(t, \mathbf{x})$. The Lagrangian density associated to a scalar field like the inflaton is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} - V(\phi) \tag{33}$$

and, through Noether's theorem, the stress-energy tensor associated to it is

$$T_{\mu\nu} = \partial_\nu\phi\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)} - g_{\mu\nu}\mathcal{L} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right). \tag{34}$$

Confronting this with (3), we find

$$T^0{}_0 = \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi); \qquad T^i{}_j = -P_\phi\delta^i_j = -\left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right)\delta^i{}_j \tag{35}$$

and therefore the Friedmann equations, (9) and (10), become

$$\begin{cases} H^2 = \dfrac{8\pi G}{3}\left(\dfrac{1}{2}\dot{\phi}^2 + V(\phi)\right) \\ \dot{H} = -\dfrac{4\pi G}{3}(\rho + 3P) - H^2 = -4\pi G\dot{\phi}^2. \end{cases} \tag{36}$$

Derivating the first one with respect to time we get

$$2H\dot{H} = \frac{8\pi G}{3}(\dot{\phi}\ddot{\phi} + V'(\phi)\dot{\phi}), \tag{37}$$

where V' is derivative of the potential with respect to ϕ. Combining this with the second equation we find the *Klein-Gordon equation*

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{38}$$

which sums up the evolution equations of the system in the case of scalar field dynamics.
Now we can make the above conditions for inflation mathematically precise: the requirement $\frac{d}{dt}(aH)^{-1} < 0$ becomes

$$\frac{d}{dt}(aH)^{-1} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}\left(1 + \frac{\dot{H}}{H^2}\right) \equiv -\frac{1}{a}(1-\epsilon) < 0 \implies \epsilon < 1 \tag{39}$$

and we can compute ϵ substituting the second Friedmann equation from (36) into $\dot{H}$:

$$\epsilon = -\frac{\dot{H}}{H^2}\left(= -\frac{d\ln H}{dN}\right) = 8\pi G\frac{\frac{1}{2}\dot{\phi}^2}{H^2} \tag{40}$$

and therefore we see that the inflation condition reduces to the requirement of the kinetic energy associated to the field being negligible with respect to the total energy (remembering that $H^2 \propto \rho_\phi$). In order to have a long enough period of inflation, we also want ϵ to stay small for long enough. This is measured by a second parameter

$$\eta \equiv \frac{d\ln\epsilon}{dN} = \frac{\dot{\epsilon}}{H\epsilon} = 2\frac{\ddot{\phi}}{H\dot{\phi}} - 2\frac{\dot{H}}{H^2} \equiv 2(\epsilon - \delta) \quad \text{where } \delta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{41}$$

A regime in which the conditions $\epsilon, |\delta| \ll 1$ (and then $\epsilon, |\eta| \ll 1$) are respected is called *slow-roll inflation*; ϵ and η are also called Hubble slow-roll parameters. Since $\epsilon \ll 1 \implies \dot{\phi}^2 \ll V$ and $\delta \ll 1 \implies \ddot{\phi} \ll 1$, in a slow-roll regime the following approximations of equations (9) and (38) hold:

$$H^2 \approx \frac{8\pi GV}{3}; \qquad 3H\dot{\phi} \approx -V'. \tag{42}$$

Therefore the Hubble parameters defined above can be substituted by the *potential* slow-roll parameters

$$\epsilon = 8\pi G\frac{\frac{1}{2}\dot{\phi}^2}{H^2} \approx \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \equiv \epsilon_V; \tag{43}$$

$$\delta + \epsilon = -\frac{\ddot{\phi}}{H\dot{\phi}} - \frac{\dot{H}}{H^2} \approx \frac{1}{8\pi G}\frac{V''}{V} \equiv \eta_V. \tag{44}$$

Notice however that the condition for inflation $\epsilon_V < 1$ is only an approximated one, while $\epsilon < 1$ is the precise one.

2.4 Conserved curvature perturbation

We now want to state an important conservation law, derivable in the context of relativistic perturbation theory: although we won't go through the details, the final result has an important application in the study of primordial perturbations, which we will later discuss.
Considering infinitesimal perturbations of the metric (1), one could derive a set of equations of motion

linearised in said perturbations, equivalent to the Einstein and Klein-Gordon equations. More precisely, the perturbed metric is

$$\begin{aligned} ds^2 = a(t)^2 \Big[&(1+2A)d\tau^2 + -2(\partial_i B + \hat{B}_i)dx^i d\tau \\ &- \left((1+2C)\delta_{ij} + 2\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E + 2\partial_{(i}\hat{E}_{j)} + 2\hat{E}_{ij}\right)dx^i dx^j \end{aligned} \tag{45}$$

where the g_{00} component contains scalar perturbations, the g_{0i} components scalar and vectorial perturbations, the g_{ij} components scalar, vectorial and tensorial perturbations. To obtain the equations of motions for said perturbations, we have to compute the corresponding perturbed Einstein tensor, and apply to it the perturbed Einstein equation

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \tag{46}$$

where the perturbation in the stress-energy tensor is obtained perturbing the expression (2) (barred variables represent the unperturbed value of each quantity):

$$\delta T^\mu_\nu = (\delta\rho + \delta P)\bar{U}^\mu \bar{U}_\nu + (\bar{\rho} + \bar{P})(\delta U^\mu \bar{U}_\nu + \bar{U}^\mu \delta U_\nu) - \delta P \delta^\mu_\nu - \Pi^\mu_\nu. \tag{47}$$

The freedom in the choice of the reference frame also introduces a gauge freedom in these equations. It can be shown that the Ricci scalar curvature of a surface at constant time with the perturbed metric has the gauge-invariant expression

$$\mathcal{R} = C - \frac{1}{3}\nabla^2 E + \mathcal{H}(B + v) \tag{48}$$

where $\mathcal{H}$ is the conformal Hubble parameter, v is the fluid velocity and C, E and B are scalar coefficients describing the perturbation of the metric. Using the so called Newtonian gauge, where $B = E = 0$ and $C = -\Phi$, this reduces to

$$\mathcal{R} = -\Phi - \frac{\mathcal{H}(\Phi' + \mathcal{H}\Phi)}{4\pi G a^2(\bar{P} + \bar{\rho})} \tag{49}$$

where we used the Einstein perturbed equations, in which $\bar{p}$ and $\bar{\rho}$ are the unperturbed, background values of pressure and energy density, and Φ and Φ' are the gravitational potential and its (conformal) time derivative. Taking a time derivative and applying once again the perturbed Einstein equations, one finds

$$-4\pi G a^2(\bar{P} + \bar{\rho})\mathcal{R}' = 4\pi G a^2 \mathcal{H}\left(\delta P - \frac{\bar{P}'}{\bar{\rho}'}\delta\rho\right) + \mathcal{H}\frac{\bar{P}'}{\bar{\rho}'}\nabla^2\Phi. \tag{50}$$

For fluids with constant equation of state, $P = w\rho$ (but also more generally for barotropic fluids, $P = P(\rho)$) the first term on the r.h.s. vanishes, leaving us with

$$\frac{d\mathcal{R}}{d\tau} \propto \mathcal{H}k^2\Phi \sim \mathcal{H}k^2\mathcal{R} \implies \frac{d\ln\mathcal{R}}{d\ln a} \propto \left(\frac{k}{\mathcal{H}}\right)^2. \tag{51}$$

Notice how we substituted the term $\nabla^2\Phi$ with its Fourier counterpart $k^2\Phi$, making the implicit assumption that Φ is Fourier expandable and rewriting equation (50) for each of its Fourier components. Here, the wave number $|\mathbf{k}| = k$ (or more precisely, its inverse) represents the typical scale on which the potential fluctuates. This scale is to be confronted with the Hubble radius: equation (51) implies that, on superhorizon scales $(k)^{-1} \gg (aH)^{-1}$, the scalar curvature has the property of being conserved along the expansion.

3 Quantum Inflationary Fluctuations [3]

A remarkable feature of inflation is that primordial fluctuations in the value of the inflaton field can explain the large scale structure we observe today. As we have shown in the previous section, the inflaton dynamics essentially governs the energy density and pressure of the cosmic fluid, and hence the expansion of the universe; therefore, we can qualitatively think of the inflaton as a kind of "internal clock" of inflation. Since, on quantum scales, arbitrarily precise clocks do not exist by the uncertainty principle, the inflaton will necessarily fluctuate, making different regions of spacetime "more inflated" than others. At the end of inflation, these differences will have produced regions of different energy content, explaining why today we observe local differences in matter content (Large Scale Structure) and temperature (CMB anisotropies).

3.1 Mukhanov-Sasaki equation

Starting from the inflaton action

$$S = \int d\tau\, d^3x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi)\right] \tag{52}$$

one can show that perturbations of the inflaton field, of the form $\phi(\tau, \mathbf{x}) = \bar{\phi}(\tau) + f(\tau, \mathbf{x})/a(\tau)$, satisfy the equation of motion of a harmonic oscillator; however, it is important to notice that this statement is only true, as will be clear in a moment, in a slow-roll regime, and only if we linearize the equations of motion by considering contributions to the action up to second order in inflaton perturbations.
In spatially flat gauge ($C = E = 0$ in (45)), metric perturbations are negligible relative to inflaton ones, and (52) can be evaluated for an unperturbed (flat) FRW metric:

$$S = \int d\tau\, d^3x \left[\frac{1}{2} a^2((\phi')^2 - (\nabla\phi)^2) - a^4 V(\phi)\right] \tag{53}$$

(here primes denote derivatives with respect to conformal time, $\tau = t/a(t)$); applying the variation principle one gets

$$\begin{aligned}\delta S &= \delta S^{(1)} + \delta S^{(2)} \\ &= \int d\tau\, d^3x \left[a\bar{\phi}' f' - a'\bar{\phi}' f - a^3 V_{,\phi} f\right] + \frac{1}{2}\int d\tau\, d^3x \left[(f')^2 - (\nabla f)^2 - 2\mathcal{H} f f' + (\mathcal{H}^2 - a^2 V_{,\phi\phi}) f^2\right],\end{aligned} \tag{54}$$

where contributions of first and second order in f have been separated (here $\mathcal{H}$ is the conformal Hubble parameter). Integrating by parts and dropping boundary terms

$$\begin{aligned} &= -\int d\tau\, d^3x \left[\partial_\tau(a\bar{\phi}') + a'\bar{\phi}' + a^3 V_{,\phi}\right] f + \frac{1}{2}\int d\tau\, d^3x \left[(f')^2 - (\nabla f)^2 + (\mathcal{H}' + \mathcal{H}^2 - a^2 V, \phi\phi) f^2\right] \\ &= -\int d\tau\, d^3x a \left[\bar{\phi}'' + 2\mathcal{H}\bar{\phi}' + a^2 V_{,\phi}\right] f + \frac{1}{2}\int d\tau\, d^3x \left[(f')^2 - (\nabla f)^2 + \left(\frac{a''}{a} - a^2 V, \phi\phi\right) f^2\right].\end{aligned} \tag{55}$$

The last $V, \phi\phi$ term in $\delta S^{(2)}$ can be neglected during slow-roll inflation, since

$$\frac{V_{,\phi\phi}}{H^2} \approx 3\eta_V \ll 1 \underset{a''/a \approx 2a^2H^2}{\Longrightarrow} \frac{a''}{a} \gg a^2 V_{,\phi\phi}. \tag{56}$$

Finally, requiring $\delta S^{(1)} = 0$ *for all* f leads to the equation of motion

$$\bar{\phi}'' + 2\mathcal{H}\bar{\phi}' + a^2 V_{,\phi} = 0 \tag{57}$$

while applying the Euler-Lagrange equation to the integrand in $\delta S^{(2)}$ (which can be written as $\partial_\mu f \partial^\mu f + (a''/a) f^2$), one gets

$$f'' - \nabla^2 f - \frac{a''}{a} f = 0. \tag{58}$$

The first one is just the Klein Gordon equation for the background field, while the second one, known as *Mukhanov-Sasaki equation*, can be recast in Fourier space as

$$f''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right) f_{\mathbf{k}} = 0 \tag{59}$$

which holds for every oscillation mode of wave vector $\mathbf{k}$. On subhorizon (inflationary) scales, $k^2 \gg a''/a \approx 2\mathcal{H}^2$, this reduces to a harmonic oscillator equation, $f''_{\mathbf{k}} + k^2 f_{\mathbf{k}} = 0$.

3.2 Canonical quantisation of fluctuations

Since inflaton deviations behave harmonically, they can be completely solved. This is done properly in a quantum field theory formalism, but it is possible to understand the main results in analogy with the quantum mechanics of a 1D harmonic oscillator. Namely, considering the Lagrangian density of the inflaton (the integrand of $\delta S^{(2)}$ in (55)) we can define the momentum conjugate to f

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial f'} = f' \tag{60}$$

and promote f and π to quantum operators. These satisfy the canonical commutation relation

$$[\hat{f}(\tau, \mathbf{x}), \hat{\pi}(\tau, \mathbf{x'})] = i\delta(\mathbf{x} - \mathbf{x'}) \qquad \text{(the equivalent of } [\hat{x}, \hat{p}] = i) \tag{61}$$

which can be recast in Fourier space as

$$\begin{aligned}[\hat{f}_{\mathbf{k}}(\tau), \hat{\pi}_{\mathbf{k}'}(\tau)] &= \int \frac{d^3x}{(2\pi)^{3/2}} \int \frac{d^3x'}{(2\pi)^{3/2}} [\hat{f}(\tau, \mathbf{x}), \hat{\pi}(\tau, \mathbf{x})] e^{-i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}'\cdot\mathbf{x}'} \overset{(61)}{=} \\ &= i \int \frac{d^3x}{(2\pi)^{3/2}} e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} = i\delta(\mathbf{k} + \mathbf{k}').\end{aligned} \tag{62}$$

Each Fourier mode can be written in terms of raising and lowering operators as

$$\hat{f}_{\mathbf{k}}(\tau) = f_k(\tau)\hat{a}_{\mathbf{k}} + f_k^*(\tau)\hat{a}^\dagger_{\mathbf{k}} \implies \hat{\pi}_{\mathbf{k}}(\tau) = \pi_k(\tau)\hat{a}_{\mathbf{k}} + \pi_k^*(\tau)\hat{a}^\dagger_{\mathbf{k}} \tag{63}$$

where $f_k(\tau)$ is a solution of wavenumber k to the Mukhanov-Sasaki equation (59), and f_k is its complex conjugate. Then, substituting (63) into (62), the CCR becomes

$$-i(f_k f_k^{*\prime} - f_k^* f_k')[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] \equiv W[f_k, f_k^*][\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta(\mathbf{k} + \mathbf{k}'). \tag{64}$$

Here, the canonical normalization is $W[f_k, f_k^*] = 1 \implies [\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta(\mathbf{k} + \mathbf{k}')$. The vacuum or fundamental state is annihilated by the lowering operator

$$\hat{a}_{\mathbf{k}} |0\rangle = 0 \tag{65}$$

while excited states are obtained by applying the raising operators to the vacuum state (and each Fourier mode can be excited to a specific level) as

$$|n_{\mathbf{k}_1}, n_{\mathbf{k}_2}, ...\rangle = \frac{1}{\sqrt{n_{\mathbf{k}_1}! n_{\mathbf{k}_2}! ...}} [(\hat{a}^\dagger_{\mathbf{k}_1})^{n_{\mathbf{k}_1}} (\hat{a}^\dagger_{\mathbf{k}_2})^{n_{\mathbf{k}_2}} ...] |0\rangle \tag{66}$$

Through these definitions, we can compute the amplitude of inflaton fluctuations. Given the operator

$$\hat{f} = \int \frac{d^3k}{(2\pi)^{3/2}} [f_k(\tau)\hat{a}_{\mathbf{k}} + f_k^*(\tau)\hat{a}^\dagger_{\mathbf{k}}] e^{i\mathbf{k}\cdot\mathbf{x}} \tag{67}$$

it is straightforward to see that its average value in the vacuum state vanishes, $\hat{f}_{|0\rangle} = \langle 0| \hat{f} |0\rangle = 0$; on the other hand, its variance is

$$\begin{aligned} \left\langle \left| \hat{f} \right|^2 \right\rangle_{|0\rangle} &= \langle 0| \hat{f}^\dagger(\tau, \mathbf{0}) \hat{f}(\tau, \mathbf{0}) |0\rangle \\ &= \int \frac{d^3k}{(2\pi)^{3/2}} \int \frac{d^3k'}{(2\pi)^{3/2}} \langle 0| (f_k^*(\tau)\hat{a}_{\mathbf{k}}^\dagger + f_k(\tau)\hat{a}_{\mathbf{k}})(f_{k'}(\tau)\hat{a}_{\mathbf{k}'} + f_{k'}^*(\tau)\hat{a}_{\mathbf{k}'}^\dagger) |0\rangle \\ &= \int \frac{d^3k}{(2\pi)^{3/2}} \int \frac{d^3k'}{(2\pi)^{3/2}} f_k(\tau) f_{k'}^*(\tau) \langle 0| [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] |0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3} |f_k(\tau)|^2 = \int dk 4\pi k^2 \frac{|f_k(\tau)|^2}{(2\pi)^3} = \int d\ln k \frac{k^3}{2\pi^2} |f_k(\tau)|^2. \end{aligned} \tag{68}$$

Therefore, the square of the solution to the Muhkanov-Sasaki equation determines the variance of quantum fluctuations. Here we can define the power spectrum

$$\Delta_f^2(k, \tau) \equiv \frac{k^3}{2\pi^2} |f_k(\tau)|^2 \tag{69}$$

which is the central probing quantity of the inflationary period. To write it explicitly, we need to find a solution $f_k(\tau)$ of the Mukhanov-Sasaki equation in Fourier space. For slow-roll inflation, we can approximate that equation in the limit of a perfect de Sitter space, namely $a(t) = e^{Ht}$, which implies $a(\tau) = -\tau^{-1}$; then, equation (59) reduces to

$$f_{\mathbf{k}}'' + \left(k^2 - \frac{2}{\tau^2}\right) f_{\mathbf{k}} = 0. \tag{70}$$

This has an exact solution

$$\alpha \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) + \beta \frac{e^{ik\tau}}{\sqrt{2k}} \left(1 + \frac{i}{k\tau}\right) \tag{71}$$

where α and β are fixed by the boundary conditions. In this case we have to impose that, at early times, the Mukhanov-Sasaki equation reduces to a harmonic oscillator one

$$k^2 - \frac{2}{\tau^2} \overset{\tau \to -\infty}{\longrightarrow} k^2 \implies f_k'' + k^2 f_k \approx 0 \implies \underset{\tau \to -\infty}{lim} f_k(\tau) = \frac{1}{\sqrt{2k}} e^{\pm ik\tau} \tag{72}$$

However, only the negative sign solution satisfies the canonical condition $W[f_k, f_k^*] = 1$, thus imposing $\alpha = 1$, $\beta = 0$, or

$$f_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right) \tag{73}$$

Substituting (73) into (69), and switching to the $\delta\phi$ power spectrum, we get

$$\Delta_{\delta\phi}^2(k, \tau) = a^{-2} \Delta_f^2(k, \tau) = \left(\frac{H}{2\pi}\right)^2 \left(1 + \left(\frac{k}{aH}\right)^2\right). \tag{74}$$

Since we want to to use the power spectrum to relate quantities at late times (after horizon re-entry) to the inflationary ones, as better stated in the next section, we can approximate the above expression for superhorizon scales, $k \ll aH$. The power spectrum of the inflaton deviations at horizon crossing is then

$$\Delta_{\delta\phi}^2(k) \approx \left(\frac{H}{2\pi}\right)^2 \Big|_{k=aH}. \tag{75}$$

3.3 Curvature Perturbations

Since the curvature perturbation is conserved on superhorizon scales, as stated by (51), it would be convenient to relate the power spectrum of fluctuations (equation (74)), to the power spectrum of $\mathcal{R}$. In this way, measurements of $\Delta^2_{\mathcal{R}}$ conducted at late times would offer a direct probe of the inflationary perturbations at very early times, before horizon exit.

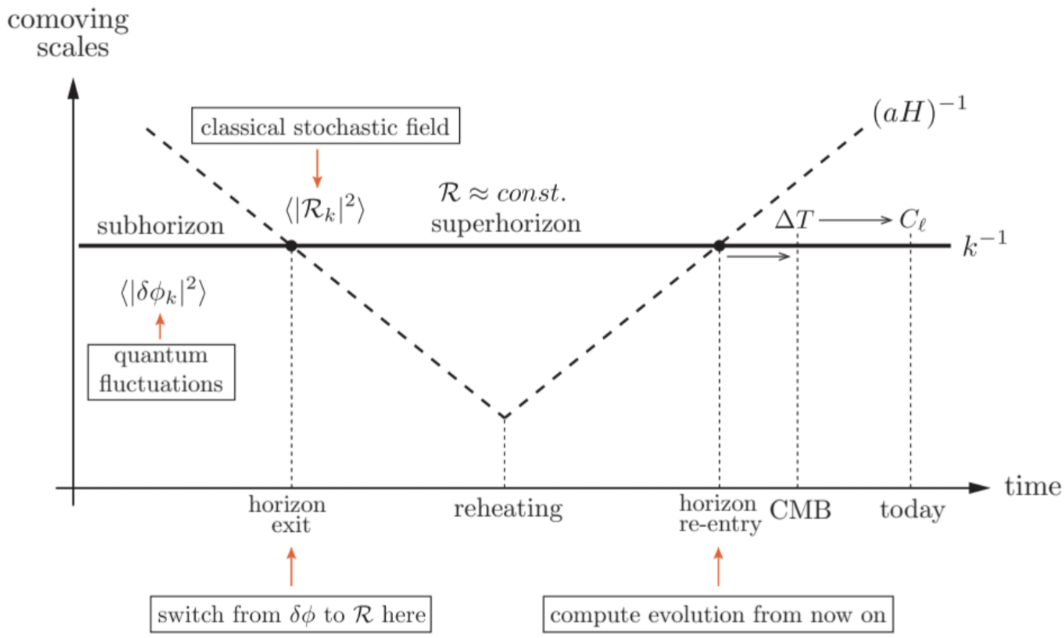

Figure 3: Image taken from [3], showing how comoving scales compare to the comoving horizon (Hubble radius) during and after inflation. The comoving horizon $(aH)^{-1}$ shrinks during inflation, causing all comoving scales to exit the horizon ($k^{-1} \gg (aH)^{-1}$), and then grows back when inflation has ended, causing them to re-entry the horizon. It is then convenient to compute the scale-dependent inflaton fluctuation $|\delta\phi_k|^2$ before horizon exit and the conserved curvature perturbation $|\mathcal{R}_k|^2$ after horizon exit, since this is constant and is related directly to its value at late times, after horizon re-entry.

From the definition (48) we can get the expression of the curvature perturbation in spatially flat gauge

$$\mathcal{R} = C - \frac{1}{3}\nabla^2 E + \mathcal{H}(B+v) \overset{C=E=0}{\longrightarrow} \mathcal{H}(B+v). \tag{76}$$

It can be shown that the quantity $B+v$ is related to the expression of the perturbed stress-energy, $\delta T^0_j = -(\bar{\rho} + \bar{P})\partial_j(B+v)$; comparing this with the expression of the stress-energy tensor of a scalar field (34),

$$\delta T^0{}_j = g^{0\mu}\partial_\mu\phi\partial_j\delta\phi = \bar{g}^{00}\partial_0\phi\partial_j\delta\phi = \frac{\bar{\phi}'}{a^2}\partial_j\delta\phi \tag{77}$$

we get (remembering that $\bar{\rho} + \bar{P} = \dot{\bar{\phi}}^2$ from (35))

$$B + v = -\frac{\delta\phi}{\bar{\phi}'} \implies \mathcal{R} = -\frac{\mathcal{H}}{\bar{\phi}'}\delta\phi = -\frac{H}{\dot{\bar{\phi}}}\delta\phi. \tag{78}$$

Therefore the relations between power spectra is

$$\Delta^2_{\mathcal{R}} = \left(\frac{H}{\dot{\bar{\phi}}}\right)^2 \Delta^2_{\delta\phi} = \frac{4\pi G}{\epsilon}\Delta^2_{\delta\phi} \tag{79}$$

where $\epsilon = 4\pi G\dot{\phi}/H^2$ is the first slow-roll parameter. Substituting (74) into (79) we finally get

$$\Delta^2_{\mathcal{R}}(k) = \frac{GH^2}{\pi\epsilon}\Big|_{k=aH}. \tag{80}$$

In this expression the dependence from the scale mode k is hidden inside H^2 and ϵ (because at horizon crossing $k = aH$). Since, for slow-roll inflation, these are very slowly varying functions of time, we expect the power spectrum to be nearly independent of scale. The deviations from scale-invariance can be expressed in a power law form

$$\Delta^2_{\mathcal{R}}(k) = A_s \left(\frac{k}{k_\star}\right)^{n_s-1} \tag{81}$$

where $k_\star$ is some reference scale and we have defined a scalar spectral amplitude A_s and a scalar spectral index

$$n_s - 1 \equiv \frac{d\ln \Delta^2_{\mathcal{R}}}{d\ln k}. \tag{82}$$

This quantity is an important probe of inflation, since it is connected to the slow-roll parameters: in fact

$$n_s - 1 \equiv \frac{d\ln \Delta^2_{\mathcal{R}}}{d\ln k} = \frac{d\ln \Delta^2_{\mathcal{R}}}{dN}\frac{dN}{d\ln k} \overset{(80)}{=} \left(2\frac{d\ln H}{dN} - \frac{d\ln \epsilon}{dN}\right)\frac{dN}{d\ln k} \tag{83}$$

and the term in parentheses is just $-2\epsilon - \eta$, according to equations (40) and (41). The second term is evaluated at horizon crossing: therefore $\ln k = N + \ln H$ and we get, via a Taylor expansion in the first slow-roll parameter,

$$\frac{dN}{d\ln k} = \left(\frac{d\ln k}{dN}\right)^{-1} = \left(1 + \frac{d\ln H}{dN}\right)^{-1} \approx 1 + \epsilon. \tag{84}$$

Then, to first order in slow-roll parameters

$$n_s - 1 = -2\epsilon - \eta \tag{85}$$

3.4 Tensorial perturbations

This power spectrum formalism can be extended to tensorial perturbations in the metric, aka gravitational waves. These are

$$ds^2 = a(\tau)^2(d\tau^2 - (\delta_{ij} + 2\hat{E}_{ij})dx^i dx^j); \tag{86}$$

substituting them into the Einstein-Hilbert action, and computing second order perturbations of said action, one gets

$$S_{EH} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \implies \delta S^{(2)} = \frac{1}{64\pi G}\int d^4x a^2((\hat{E}'_{ij})^2 - (\nabla \hat{E}_{ij})^2) \tag{87}$$

Gravitational waves have two polarization modes [1], so these perturbations can be parametrized as

$$\frac{1}{4\sqrt{2\pi G}} a\hat{E}_{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix} f_+ & f_\times & 0 \\ f_\times & -f_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{88}$$

giving

$$S^{(2)} = \frac{1}{2}\sum_{I=+,\times}\int d^4x \left((f'_I)^2 - (\nabla f_I)^2 + \frac{a''}{a}f_I^2\right) \tag{89}$$

Since this is just two copies of the Mukhanov-Sasaki action in (55), we can immediately conclude that

$$\Delta^2_t(k) = 2\left(\frac{4\sqrt{2\pi G}}{a}\right)^2 \Delta^2_f(k) \overset{(75)}{=} \frac{16GH^2}{\pi}\Big|_{k=aH} \tag{90}$$

and, in analogy to (81)

$$\Delta^2_t(k) = A_t\left(\frac{k}{k_\star}\right)^{n_t}. \tag{91}$$

Another important probing quantity is given by

$$r \equiv \frac{A_t}{A_s} \tag{92}$$

known as *tensor-to-scalar ratio*. With calculations similar to (82) - (85) we can show n_t and r's relation to the slow-roll parameters:

$$n_t \equiv \frac{d\ln\Delta_t^2}{d\ln k} = \frac{d\ln\Delta_t^2}{dN}\frac{dN}{d\ln k} \overset{(90)}{=} 2\frac{d\ln H}{dN}\left(1+\frac{d\ln H}{dN}\right)^{-1} \approx -2\epsilon(1+\epsilon) \approx -2\epsilon \tag{93}$$

$$r \equiv \frac{A_t}{A_s} = \frac{\Delta_t^2/\left(\frac{k}{k_\star}\right)^{n_t}}{\Delta_{\mathcal{R}}^2/\left(\frac{k}{k_\star}\right)^{n_s-1}} \overset{(80),(90)}{=} 16\epsilon\left(\frac{k}{k_\star}\right)^{n_t-n_s+1} \overset{(85),(93)}{\approx} 16\epsilon. \tag{94}$$

This also implies the *consistency relation*, $n_t = -r/8$. This relation shows how the shape of the tensor power spectrum does not provide an additional independent observable.
The two quantities n_s and r are the main cosmological observables that a given theory of inflation should be able to reproduce. They are connected, as we have shown, to the slow-roll parameters, and then in turn to the flow equation formalism we are going to develop in the next section.

4 The flow equations approach [9]

Since the theory of inflationary cosmology has been developed, numerous models for the inflaton field potential, $V(\phi)$, have been proposed. All these models are distinguishable by their different predictions on the values of the probing quantities introduced above, the spectral scalar index, n_s, and the tensor-to-scalar ratio, r. Thanks to many years of data gatherings, one can now proceed to exclude these models one by one; however, it would be desirable to have at our disposal a method to make *generic* predictions, without having to work within a specific model. This is provided by the flow equation approach, first introduced by Hoffman and Turner [8] in 2001. This method aims at describing the inflationary dynamics through a hierarchy of generalized slow-roll parameters, in which the time derivative of each parameter is written in terms of some higher order parameter; this set of equations can be truncated at any arbitrarily high order, and numerically integrated to obtain a prediction of the values of slow-roll parameters, and thus of our probing variables. Of course, initial conditions in this integration are set by the specific model assumed. This approach shows how the majority of these flows tend to cluster in certain regions of the parameter space, thus imposing constrictions on the possible values of the cosmological observables.

4.1 Hamilton-Jacobi formalism

A convenient way to describe the slow-roll hierarchy is to rewrite the equation of motion (38) of the inflaton directly in terms of ϕ rather than t. This is possible as long as the sign of $\dot\phi$ is preserved. The Klein-Gordon equation is equivalent to (here primes denotes derivation with respect to ϕ)

$$\begin{cases} \dot\phi = -\dfrac{1}{4\pi G}H'(\phi) \\ H'(\phi)^2 - 12\pi G H(\phi)^2 = -32\pi^2 G^2 V(\phi) \end{cases} \tag{95}$$

since, differentiating the second equation with respect to ϕ and the first one with respect to t

$$\begin{cases} H'(\phi) = -4\pi G\dot\phi; \quad H''(\phi) = -4\pi G\dfrac{\ddot\phi}{\dot\phi} \\ \dfrac{1}{(4\pi G)^2}H''(\phi)H'(\phi) - \dfrac{3}{4\pi G}H'(\phi)H(\phi) + V'(\phi) = 0 \end{cases} \tag{96}$$

and substituting the first two equations into the third reduces it to equation (38). So (95) implies (38); the vice versa is true because the first equation in (95) follows straightforwardly from (36) if we consider $H = H(\phi)$, and substituting it into the Klein-Gordon equation gives exactly the second equation in (96).
The second equation of (95) is usually called the Hamilton-Jacobi equation; it can be recast in the useful form

$$H(\phi)^2\left(1-\frac{1}{3}\epsilon(\phi)\right)=\frac{8\pi G}{3}V(\phi) \tag{97}$$

where we have defined

$$\epsilon(\phi)\equiv\frac{1}{4\pi G}\left(\frac{H'(\phi)}{H(\phi)}\right)^2. \tag{98}$$

This new definition of the first slow-roll parameter, because of the first equation in (95), coincides with (40), while also underlining more strongly its relation with the first derivative of the Hubble parameter. This also allows us to relate nicely the inflaton to the number of e-folds of inflation, since

$$dN=d\ln a=Hdt=\frac{H}{\dot\phi}d\phi\overset{(95),(98)}{=}2\sqrt{\pi G}\frac{d\phi}{\sqrt{\epsilon(\phi)}} \tag{99}$$

4.2 Slow-roll hierarchy and flow equations

Differentiating the newly found expression for the first slow-roll parameter, one gets

$$\frac{d\epsilon}{d\phi}=2\frac{H'}{H}\left(\frac{1}{4\pi G}\frac{H''}{H}-\frac{1}{4\pi G}\left(\frac{H'}{H}\right)^2\right)=4\sqrt{\pi G}\sqrt{\epsilon}(\delta-\epsilon) \tag{100}$$

where we used the definition of δ given at (41), and used once again the Hamilton-Jacobi formalism (the second equation in (96)) to re-define it:

$$\frac{1}{4\pi G}\left(\frac{H''}{H}\right)=-\frac{\ddot\phi}{\dot\phi H}\equiv\delta. \tag{101}$$

Therefore the derivative of the first slow-roll parameter is related to the second. Further derivation of δ yields

$$\begin{aligned}\frac{d\delta}{d\phi}&=\frac{1}{4\pi G}\frac{H'''H-H''H'}{H^2}=\frac{1}{4\pi G}\frac{H}{H'}\left(\frac{H'''H'}{H^2}-\frac{H''}{H}\left(\frac{H'}{H}\right)^2\right)\\&=2\sqrt{\pi G}\frac{1}{\sqrt{\epsilon}}\left(\frac{1}{(4\pi G)^2}\frac{H'''H'}{H^2}-\delta\epsilon\right)\equiv2\sqrt{\pi G}\frac{1}{\sqrt{\epsilon}}(\xi^2-\delta\epsilon)\end{aligned} \tag{102}$$

where

$$\xi^2\equiv\frac{1}{(4\pi G)^2}\frac{H'''H'}{H^2} \tag{103}$$

can be regarded as a new, third order slow-roll parameter, related to the third order derivative of H (just like ϵ and δ are related to first and second order derivatives). Using the relation (99) we can recast equations (100) and (102) in an enlightening way:

$$\frac{d\epsilon}{dN}=2\epsilon(\delta-\epsilon) \tag{104}$$

$$\frac{d\delta}{dN}=\xi^2-\epsilon\delta. \tag{105}$$

These equations seem to suggest a pattern: the derivative of each slow-roll parameter is itself higher order in slow-roll parameters. This statement can be in fact generalized by introducing a whole *hierarchy* of Hubble slow-roll parameters

$$ {}^{l}\lambda_H \equiv \frac{1}{(4\pi G)^l}\frac{(H')^{l-1}}{H^l}\frac{d^{l+1}H}{d\phi^{l+1}}. \tag{106} $$

So, for instance, for $l=1$ we get δ, for $l=2$ we get ξ^2, and so on. Derivating this expression with respect to ϕ and using again (99), we find the generic, l-th order flow equation

$$ \begin{aligned} \frac{d({}^{l}\lambda_H)}{d\phi} &= \frac{1}{(4\pi G)^l}\left(\frac{(l-1)(H')^{l-2}H''}{H^l}\frac{d^{l+1}H}{d\phi^{l+1}} + \frac{(H')^{l-1}}{H^l}\frac{d^{l+2}H}{d\phi^{l+2}} - \frac{l(H')^l}{H^{l+1}}\frac{d^{l+1}H}{d\phi^{l+1}}\right) \\ &= 4\pi G\frac{H}{H'}\left((l-1)\delta - l\epsilon\right)({}^{l}\lambda_H) + \frac{1}{(4\pi G)^l}\frac{(H')^{l-1}}{H^l}\frac{d^{l+2}H}{d\phi^{l+2}} \end{aligned} \tag{107} $$

$$ \frac{d({}^{l}\lambda_H)}{dN} = \frac{1}{4\pi G}\frac{H'}{H}\frac{d({}^{l}\lambda_H)}{d\phi} = ((l-1)\delta - l\epsilon)({}^{l}\lambda_H) + ({}^{l+1}\lambda_H) \tag{108} $$

This equations, along with equation (104), form a system of *exact* differential equation (notice how, despite the use of the term "slow-roll parameters", no approximation was made) which can be truncated at a certain order and numerically integrated to obtain the slow-roll parameters. Since these are related to cosmological observables (as stated by (85) and (94), and it is *there* we make the slow-roll approximation) this provides a method to predict their values. In fact, one could also write flow equations directly in terms of r and n_s (as was first done by Hoffmann and Turner): we can define a new parameter

$$ \sigma \equiv 2\delta - 4\epsilon = -2\epsilon - \eta \tag{109} $$

which coincides, to first order in slow-roll, with the scalar spectral index, $n_s - 1$ (equation (85)). Equations (104) and (105) are re-written in terms of σ as

$$ \begin{cases} \dfrac{d\epsilon}{dN} = \epsilon(\sigma + 2\epsilon) \\ \dfrac{d\sigma}{dN} = 2\dfrac{d\delta}{dN} - 4\dfrac{d\epsilon}{dN} = 2\xi^2 - 2\epsilon\delta - 4\epsilon(\sigma+2\epsilon) = 2\xi^2 - 5\epsilon\sigma - 12\epsilon^2. \end{cases} \tag{110} $$

Now, remembering the slow-roll approximations (94) and (109), the first equation simply becomes

$$ \frac{dr}{dN} = (n_s - 1)r + \frac{1}{8}r^2 \tag{111} $$

while for the second equation a bit of work is required to express ξ^2 in terms of the observables: first we can say

$$ 2\xi^2 = \frac{1}{8\pi^2 G^2}\frac{H'H'''}{H^2} = \frac{1}{4(\pi G)^{3/2}}\sqrt{\epsilon}\frac{H'''}{H}; \tag{112} $$

then, defining

$$ x(\phi) \equiv \frac{V'(\phi)}{V(\phi)} \tag{113} $$

and differentiating it with respect to ϕ

$$ x'' = \frac{V'''}{V} - 3\frac{V'V''}{V^2} + 2\left(\frac{V'}{V}\right)^3 \tag{114} $$

we can use the *potential* expressions of ϵ and η, equations (43) and (44) (which are only valid in slow-roll regime) to write

$$ = \frac{V'''}{V} + 16(\pi G)^{3/2}\sqrt{\epsilon_V}(8\epsilon_V - 6\eta_V). \tag{115} $$

In a slow-roll regime and in first order to slow-roll parameters, we can neglect the second term of this expression, since it is order 3/2 in slow-roll parameters. It can be proven that the first term is proportional to H'''/H in the Hamilton-Jacobi formalism,

$$\frac{V'''}{V} = A\frac{H'''}{H} \tag{116}$$

Then

$$2\xi^2 \approx \frac{1}{4(\pi G)^{3/2}A}\sqrt{\epsilon}x'' \tag{117}$$

and it holds that

$$\frac{d(n_s-1)}{dN} = -\frac{5}{16}r(n_s-1) - \frac{3}{64}r^2 + \frac{1}{16(\pi G)^{3/2}A}\sqrt{r}x''. \tag{118}$$

Thus, equations (111) and (118) show how the flow equations method is applied directly to cosmological observables. Hoffman and Turner solved them numerically assuming x'' to be small and constant, practically truncating the flow equations to the first order.

4.3 Fixed points in the parameter space

We obtained the hierarchy of equations

$$\begin{cases} \dfrac{d\epsilon}{dN} = \epsilon(\sigma + 2\epsilon) \\ \dfrac{d\sigma}{dN} = 2\xi^2 - 5\epsilon\sigma - 12\epsilon^2. \\ \dfrac{d({}^l\lambda_H)}{dN} = ((l-1)\delta - l\epsilon)({}^l\lambda_H) + ({}^{l+1}\lambda_H) \end{cases} \tag{119}$$

which describe exactly the flow of a given inflaton model in the slow-roll parameter space. In the study of such a flow is of primary importance the identification of fixed points, where all derivatives in (119) vanish. We can easily find two classes of fixed points. First is the case with vanishing tensor-to-scalar ratio

$$\begin{cases} r = 16\epsilon = 0 \\ {}^l\lambda_H = 0 \quad \text{for } \ l \geq 2 \\ \sigma = 2\delta = -\eta = \text{const.} \end{cases} \tag{120}$$

Since the flow equations imply

$$\frac{d^2\epsilon}{dN d\epsilon} = \sigma \tag{121}$$

and

$$\frac{d^2\epsilon}{dN d\sigma} = 0 \tag{122}$$

this class of fixed points is stable (for perturbations of ϵ and σ around equilibrium) if $\sigma > 0$ or $n_s > 1$, unstable if $\sigma < 0$ or $n_s < 1$. This means that inflationary evolution will flow away from $r = 0$ if $n_s < 1$ and towards $r = 0$ if $n_s > 1$. This behaviour can be explained in terms of the inflaton potential: in slow-roll approximation, and with $\epsilon = 0$, we have

$$n_s - 1 = -\eta = 2\delta \approx \frac{1}{4\pi G}\frac{V''(\phi)}{V(\phi)} \tag{123}$$

so $n_s < 1$ implies the field is sitting atop a maximum of the potential (remembering that $\epsilon = 0 \implies V'(\phi) = 0$), while $n_s > 1$ implies it is resting at a minimum of the potential. It is important to notice then, that if we want inflation to end (i.e. to reach a value $\epsilon = 1$) it must be $n_s < 1$ (at least for points in parameter space with $r \approx 0$), and this seems indeed to be the case for our early universe. A second class of known fixed points is that with $\epsilon = \delta = \xi = \text{const}$, or more specifically

$$^{l+1}\lambda_H = \epsilon(^l\lambda_H) = \epsilon^l = \text{const} \implies \begin{cases} \delta = \epsilon = \text{const} \\ \sigma = 2\delta - 4\epsilon = -2\epsilon = \text{const} \\ \xi^2 = \epsilon^2 = \text{const} \\ \ldots \end{cases} \tag{124}$$

In fact these conditions imply that every derivative in (119) vanishes. This class of fixed points can make inflation potentially infinite; luckily, it is possible to proof that they are not attractors for long time evolution, a conclusion supported by the results of numerical integration of the flow equations. It is however interesting to notice that these points correspond to the case of *perfect power-law inflation* [6].

4.4 Numerical integration

With the flow equations method described, it is possible to find the *generic* predictions of inflation. A certain inflaton model can be completely specified by a point in the slow-roll parameter space. Given this point, we can define a simple procedure to determine the predicted values for the cosmological observables after a fixed amount of inflation (a fixed number of e-folds, N):

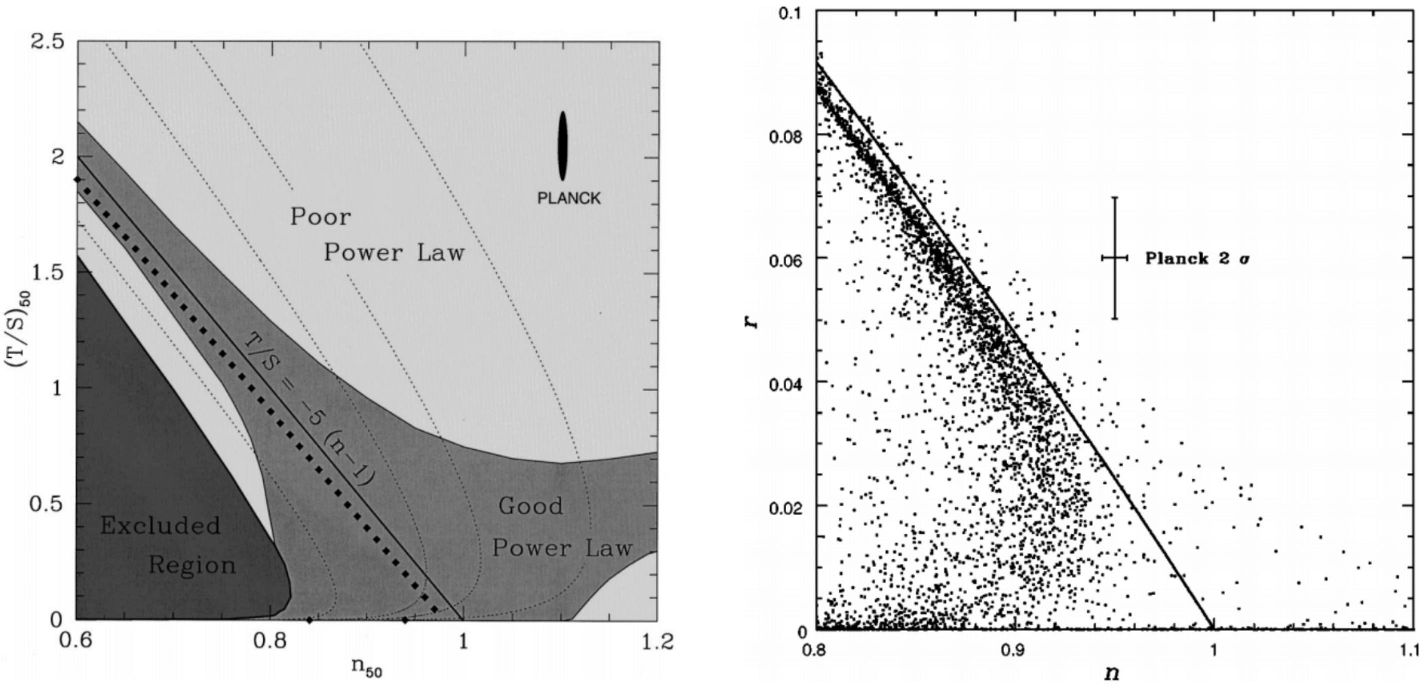

Figure 4: A couple examples of numerical integration of the flow equations. The image on the left is taken from [8]: as mentioned, Hoffman and Turner performed an integration truncating the hierarchy to the first order. The dotted lines represent solutions for different values of x'' (see equation (118)), while the diagonal straight line is the case of perfect power law, where various constant solutions (diamonds) are found, as mentioned in the previous section. Notice a different definition of the tensor-to-scalar ratio (here called T/S) has been used. The image on the right, taken from [9], shows instead an integration for $M = 5$, as well as the power law solution. Both pictures depict for comparison a 2σ uncertainty region for the Planck experiment.

1. Select a point in parameter space, $(\epsilon, ^l\lambda_H)$;
2. Evaluate forward in time the flow equations (119), until inflation ends ($\epsilon = 1$) or a fixed point is reached;

3. If a fixed point is reached, evaluate the slow-roll parameters, and the observables in turn, at that point. If inflation ends after N e-folds, evaluate *back* the flow equations for $N^* = 50 \sim 60$ e-folds from the end of inflation, and compute the observables at the point obtained. As we have seen in (32), this is roughly the amount of inflation required to solve the horizon problem; values of N^* in the range $50 \sim 60$ will yield a range of possible values for the cosmological variables (a segment in the $n_s - r$ plane), to be compared with the observational results.

4. In the latter case there is a third possibility: that the inflation also ends while evolving back in time; this means that the selected model is incapable of sustaining N^* e-folds of inflation.

In principle this method produces exact results. In practice, the flow equations must be truncated at a certain order and integrated numerically. The truncation is simply achieved by setting ${}^l\lambda_H = 0$ for $l > M$, and integrating the first $M + 1$ equations.

4.5 Dynamical interpretation

It is important to understand how exactly the flow equations are related to the inflationary dynamics. As noted by Liddle in this paper [10], they are, in fact, not related at all: to obtain the system (119), we only had to define the slow-roll parameters and their relations in term of H and its derivatives, through equations (106) and (108); this was carried out without ever mentioning the dynamical equations of the system, which is the Hamilton-Jacobi version of the Friedmann equation,

$$H'(\phi)^2 - 12\pi G H(\phi)^2 = -32\pi^2 G^2 V(\phi). \tag{125}$$

This equation tells us how H and its derivative are related to the inflaton potential, which is the object that really contains the dynamical description of the system. The flow equations only describe the evolution of a solution along a trajectory in terms of some parameter, which can be ϕ or N, but do nothing to determine what this trajectory in parameter space is. The reason flow equations are related to the dynamics is that their ultimate output is a function $\epsilon(\phi)$ (from ϵ every other slow-roll parameter can be determined) which specifies a certain inflation model, in contrast to the more traditional views that use $V(\phi)$ or $H(\phi)$ for such specification. The different perspectives are in fact correlated: from (98) it follows

$$H(\phi) = H_i e^{\int_{\phi_i}^{\phi} \sqrt{4\pi G\epsilon(\phi)} d\phi} \tag{126}$$

and from (125) it follows

$$V(\phi) = \frac{3}{8\pi G} H^2(\phi) \left(1 - \frac{1}{3}\epsilon(\phi)\right). \tag{127}$$

Therefore, the flow equations should be regarded as an algorithm for generating inflationary models, as they do not incorporate themselves the inflationary dynamics.
An interesting application of these relations between $\epsilon(\phi)$ and $H(\phi)$ and $V(\phi)$ is the description of a class of exact analytic solutions to the flow equations: we start by noting that the truncation required to solve the equations, meaning ${}^{M+1}\lambda_H = 0$, can be obtained automatically if the solution satisfies

$$\frac{d^{M+2}H}{d\phi^{M+2}} \equiv 0 \tag{128}$$

as follows straightforwardly from (106). This corresponds to a class of polynomial solutions for $H(\phi)$:

$$H(\phi) = H_0(1 + A_1\phi + ... + A_{M+1}\phi^{M+1}) \tag{129}$$

where the coefficients A_i are related to the initial values of the slow-roll parameters through their definitions. For instance

$$\epsilon(\phi) = \frac{1}{4\pi G} \left(\frac{A_1 + ... + (M+1)A_{M+1}\phi^M}{1 + A_1\phi + ... + A_{M+1}\phi^{M+1}} \right)^2 \tag{130}$$

and then

$$\epsilon(\phi = 0) = \frac{A_1^2}{4\pi G} \tag{131}$$

and, analogously, $A_l + 1$ will be related to the initial value of ${}^l\lambda_H$. We can also write explicitly the potential for this model using (127):

$$V(\phi) = \frac{3}{8\pi G} H_0^2 (1 + A_1\phi + ... + A_{M+1}\phi^{M+1})^2 \left[1 - \frac{1}{3}\frac{1}{4\pi G}\left(\frac{A_1 + ... + (M+1)A_{M+1}\phi^M}{1 + A_1\phi + ... + A_{M+1}\phi^{M+1}}\right)^2\right]. \tag{132}$$

This expression clarifies how the specific inflation models depends on the starting point of inflation in the slow-roll parameter space. In particular, in order to have initial conditions which produce inflation, we have to require $|A_1| < \sqrt{4\pi G}$, and we can impose a slow-roll regime by requiring $|{}^l\lambda_H| \ll 1$. This polynomial solutions are capable of reproducing the two main behaviours discussed in the previous section: if the potential can take negative values, at some point in the evolution we will have $\epsilon = 1$, leading to the end of inflation; if the potential has a minimum at a positive value, it reaches a fixed point of the kind $\epsilon = 0$, potentially driving infinite inflation.

5 Confronting with the observations [5]

The theory of inflation elegantly solves many problems of the standard hot Big Bang model, but is also in good agreement with observational results. The main sources of experimental data to confront with theoretical predictions are the studies on CMB anisotropies and the Large Scale Structure surveys (LSS). The most recent results come from the Planck satellite experiment:

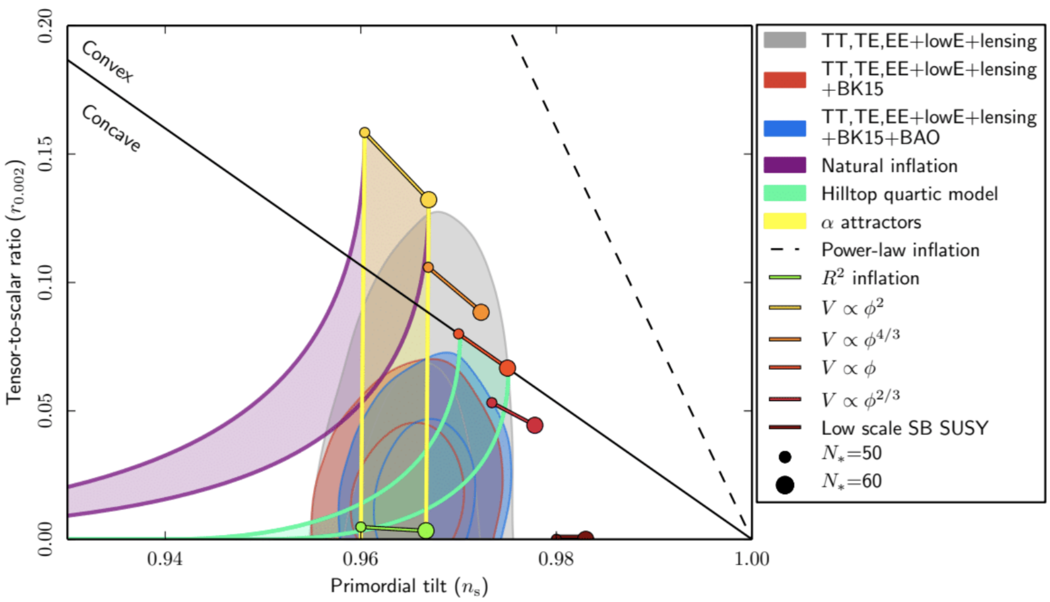

Figure 5: Image from [5], comparing Planck data, in combinations with other datasets, and different theoretical models. The highlighted regions represent 68% and 95% CL. The power law solution is depicted as a dashed line.

$$\begin{gathered} n_s = 0.9649 \pm 0.0042 \\ r < 0.064 \end{gathered} \tag{133}$$

As clear from the image above, many models for the inflaton potential predict values of the cosmological variables in good agreement with the observations. Notice that the theoretical values vary along a segment, due to the different value of $N*$ used in the flow equations integration, as discussed above. The tensor-to-scalar ratio is very small, but possibly non-zero: since they are not predicted by any other, non-inflationary theory of the early universe, the detection of gravitational waves is often referred as the *smoking-gun probe* of inflation.

6 Conclusions

In this thesis we have summarised the fundamental ideas regarding the theory of inflation in modern cosmology, focusing on how it solved the horizon problem (among many others) which arose in standard Big Bang cosmology. We have then studied inflationary dynamics in the limited case of a scalar field model, discussing how such dynamics can be described through the definition of slow-roll parameters, and how these parameters are related to some key early-universe observables. Finally, we have shown how the flow equations method can be used to predict the values of these observables for a generic inflationary model. Despite not being strictly related to the dynamics of the system, the flow equations provide a stochastic approach to compute the predictions of many different models, allowing for a broader, more general view of the problem, rather a case-by-case study of each model. In future years this method is sure to be used time and time again, in attempt to match predictions and data, and of course even more general approaches are being implemented, for instance considering multi-fields models or introducing non-standard terms to the Lagrangian of a given model. A major breakthrough would be the detection of primordial gravitational waves: when such a discovery will be made, the flow equation method could be used to distinguish between models which predict $r \neq 0$, allowing for the presence of gravitational waves, and models which predict $r = 0$.

References

[1] Sean M. Carroll, *Spacetime and Geometry*, Cambridge University Press, 2019.

[2] Andrew R. Liddle, *An Introduction to Modern Cosmology*, Wiley, 2015.

[3] Daniel Baumann, *Lectures on Cosmology*, 2019.

[4] Daniel Baumann, TASI Lectures on Inflation. arXiv:0907.5424, 2012.

[5] Planck Collaboration: Y. Akrami et al., Constraints on Inflation. Astronomy and Astrophysics 641, A10, 2020.

[6] F. Lucchin and S. Matarrese. Power law inflation. *Phys. Rev.*, D.**32**.1316, 1985.

[7] Andrew R. Liddle, Paul Parsons and John D. Barrow. Formalizing the slow roll approximation in inflation. *Phys. Rev.*, D.**50**.7222, 1994.

[8] Mark B. Hoffman and Micheal S. Turner. Kinematic constraints to the key inflationary observables. *Phys. Rev.*, D.**64**.023506, 2001.

[9] William H. Kinney. Inflation: Flow, fixed points and observables to arbitrary order in slow roll. *Phys. Rev.*, D.**66**.083508, 2002.

[10] Andrew R. Liddle. Inflationary flow equations. *Phys. Rev.*, D.**68**.103504, 2003.