UNIVERSITÀ DEGLI STUDI DI PADOVA

DIPARTIMENTO DI SCIENZE ECONOMICHE ED AZIENDALI "M. FANNO"

DIPARTIMENTO DI MATEMATICA "T. LEVI-CIVITA"

CORSO DI LAUREA MAGISTRALE IN ECONOMICS AND FINANCE

TESI DI LAUREA

LIBOR TRANSITION: NEW RISK-FREE RATES MODELS AND THEIR USE FOR DERIVATIVE PRICING

Relatore:
Ch.mo Prof. CLAUDIO FONTANA

Laureanda: CLAUDIA REDI
Matricola: 2013365

Anno accademico 2021/2022

Dichiaro di aver preso visione del "Regolamento antiplagio" approvato dal Consiglio del Dipartimento di Scienze Economiche e Aziendali e, consapevole delle conseguenze derivanti da dichiarazioni mendaci, dichiaro che il presente lavoro non è già stato sottoposto, in tutto o in parte, per il conseguimento di un titolo accademico in altre Università italiane o straniere. Dichiaro inoltre che tutte le fonti utilizzate per la realizzazione del presente lavoro, inclusi i materiali digitali, sono state correttamente citate nel corpo del testo e nella sezione 'Riferimenti bibliografici'.

I hereby declare that I have read and understood the "Anti-plagiarism rules and regulations" approved by the Council of the Department of Economics and Management and I am aware of the consequences of making false statements. I declare that this piece of work has not been previously submitted – either fully or partially – for fulfilling the requirements of an academic degree, whether in Italy or abroad. Furthermore, I declare that the references used for this work – including the digital materials – have been appropriately cited and acknowledged in the text and in the section 'References'.

Firma dello studente:

Claudia Redi

Abstract

In 2017, following manipulation episodes and post-crisis issues, the Financial Conduct Authority has announced that markets will be transitioning away from LIBOR starting 2021. This has led jurisdictions to the selection of alternative new risk-free rates (RFRs), which should be more reliable since they are anchored to effective market transactions and do not derive from the quotes of a panel of banks (as their predecessor). This thesis analyses the characteristics of these new rates and proposes some possible solutions for modeling them. It also suggests a way to use one of this models (Hull-White model) to obtain pricing formulas for a particular type of derivatives, namely options on RFRs futures. Moreover, it provides a numerical sensitivity analysis studying how option prices vary with respect to some Hull-White model's parameters.

Contents

Introduction **1**

1 LIBOR reform: towards new reference rates **4**
- 1.1 LIBOR: an overview . . . 4
  - 1.1.1 LIBOR issues: what brought to the reform . . . 8
- 1.2 New Risk-Free Rates (RFRs) . . . 13
  - 1.2.1 SOFR . . . 14
  - 1.2.2 ESTR . . . 16
  - 1.2.3 SONIA . . . 18
  - 1.2.4 RFRs drawbacks and challenges . . . 19

2 An overview of the primary RFRs models **22**
- 2.1 RFRs models' generalities . . . 23
  - 2.1.1 The extended zero-coupon bond price formula . . . 25
  - 2.1.2 Backward-looking in arrears rates . . . 26
  - 2.1.3 Forward-looking rates . . . 27
- 2.2 The general term structure equation . . . 28
- 2.3 The Hull-White Model . . . 34
  - 2.3.1 The Vasicek Model . . . 34
  - 2.3.2 The Hull-White Model (extended-Vasicek) . . . 35
  - 2.3.3 The pricing formula for a European call option . . . 37
- 2.4 The Forward Market Model . . . 37
  - 2.4.1 The generalised FMM . . . 38
  - 2.4.2 Model characteristics . . . 41

3 Derivative pricing using the Hull-White model **43**
- 3.1 Options on RFRs futures . . . 44
- 3.2 Pricing of the future contract . . . 45
- 3.3 Pricing of the option contract . . . 49
  - 3.3.1 Option on 3M future . . . 51
  - 3.3.2 Option on 1M future . . . 56

4 Numerical study of options on RFRs futures **59**
- 4.1 Generalities . . . 59

4.1.1 Hull-White model parameters 59
4.1.2 Additional inputs 61
4.2 Option on 3M future's numerical analysis 63
4.2.1 Change in the strike price 65
4.2.2 Change in the maturity 67
4.3 Option on 1M future's numerical analysis 69
4.3.1 Change in the strike price 71
4.3.2 Change in the maturity 73

Conclusion **76**

A 3M option code **79**

B 1M option code **106**

C Strike price code **130**

Introduction

For decades, **Interbank Offered Rates** (commonly known as **IBOR rates**) have served as widely accepted reference rates for numerous financial instruments. Of these, the most widespread is the USD **London Interbank Offered Rate (LIBOR)**, which is estimated to be used as a reference rate for USD 350 trillion of outstanding contracts, in maturities ranging from overnight to more than 30 years, in both financial markets and commercial fields (see [21]).

For many years, LIBOR has been treated as a proxy for the risk-free interest rate. In fact, it referred to AA-rated banks, therefore to transactions that were perceived to have credit risk almost equal to zero. This is why financial markets considered it suitable to reference financial instruments.

Nonetheless, after the 2008 global financial crisis, LIBOR started to face some issues. Firstly, since LIBOR is constructed from a survey of a small panel of banks reporting non-binding quotes rather than actual transactions, some banks started to manipulate their quotes, by understating the borrowing costs they reported for LIBOR. At the same time, post-crisis interbank trading dropped, especially in the unsecured segment. In fact, banks suffered substantial losses and higher balance sheet costs, which made them overall reluctant to lend. Moreover, Central Banks undertook unconventional policies, that created a copious supply of reserve balances, making banks less in need to trade with each other.

As LIBOR faced all these issues, it became clearer and clearer that it was no longer suitable as a reference rate in financial markets. This is why in 2017, the Financial Conduct Authority, the institution supervising LIBOR production, announced that, starting 2021, banks will no longer be asked to communicate their quotes for LIBOR computation.

LIBOR termination brought to the identification of **alternative risk-free rates (RFRs)**, to serve as its adequate substitutes in financial markets. Particularly, jurisdictions have focused on selecting rates that are anchored to *effective market transactions*, in order to fix LIBOR's manipulation problems. All these rates are *overnight*, that is based on one-day transactions, while LIBOR was produced for 7 different maturities.

LIBOR transition and the introduction of new RFRs brings into markets two challenges. The first one is how to address changes in evaluation and discounting of already existing derivatives. In fact, all instruments anchored to LIBOR which have not expired yet would face an abrupt shift of benchmark, were additional measures not be taken. Additionally, markets are already trading innovative instruments anchored to the new RFRs, raising the issue of their evaluation.
The main objective of this thesis is to find a way to price these new RFR derivatives, assuming that the interest rate follows a stochastic model. To accomplish this, first we will illustrate some stochastic models that can be found in the literature to model interest rates. Afterwards, supposing that the interest rate follows one of these models, we will focus on the evaluation of a particular type of RFR derivative.
Specifically, the structure of the thesis is as follows:

- In *Chapter 1*, we present a brief overview of the main LIBOR characteristics and we study in detail the reasons that brought to the reform. Consequently, we analyse the main features of the new RFRs, focusing specifically on the Secured overnight financing rate (SOFR), the Euro short-term rate (ESTR) and the Sterling overnight index average (SONIA). We also discuss some drawbacks and challenges that the introduction of these new rates brings into markets.

- In *Chapter 2*, since the new reference rates are all overnight rates, we study the associated term structure stochastic models, in order for them to be used in the evaluation of financial instruments. We identify two main approaches for the term structure evaluation: either build a backward-looking rate, based on past realisation of overnight rates and known at the end of the application period, or a forward-looking rate, which reflects the expectations of future realisations of the former and is known at the beginning of the application period. Afterwards, we list some solutions for modelling RFRs proposed by different authors. In particular, first we present the Hull-White model as in [7], which seems to be the standard choice for RFRs evaluation in current markets. Next, we introduce the Forward Market Model as in [28], which is an extension to the less complete LIBOR Market Model.

- In *Chapter 3*, we try to obtain an evaluation formula for a particular type of derivatives referencing RFRs, namely options on RFRs futures. We decided to analyse this type of derivatives because of the increasing popularity they are gaining in capital markets. These instruments are composed by a future contract (that is, a derivative according to which two counterparties agree to exchange a

RFR interest-bearing instrument at a pre-specified price some time in the future), that constitutes the underlying of an option (which gives the right, but not the obligation, to buy or sell it at a strike price at a certain date). In current financial markets, there exist two types of futures depending on their maturity, namely 1-month (1M) and 3-months (3M) futures. Firstly, we derive two pricing formulas for these futures. Afterwards, we use our results to price the options. We obtain two different evaluation equations, one for options on 3M future and one for options on 1M futures. Notice that the option on 1M future's evaluation equation constitutes an original result. In fact, in the current literature, we cannot find a comparable explicit evaluation formula to price this kind of instruments.
Observe that, since it represents one of the most popular models in use in the financial industry, we assume that our interest rate follows the Hull-White dynamics.

- In *Chapter* 4, we perform a numerical analysis using the platform Matlab. Particularly, we perform a sensitivity analysis, studying how changes in the Hull-White model's parameters affect the options on RFRs future' prices. We also analyse how our results behave in relation to changes in the strike price and maturity of the option.

Chapter 1

LIBOR reform: towards new reference rates

1.1. LIBOR: an overview

Interbank rates (commonly labelled as **IBOR**, short for *Interbank Offered Rates*) represent the cost at which primary financial institutions can borrow money from each other and they serve as widely accepted reference rates for numerous transactions. In financial markets, the most widespread interbank rate is LIBOR (short for *London Interbank Offered Rate*), even if there exist other similar rates, such as EURIBOR.[1] **LIBOR** started to be used in the 1970s as a benchmark rate for offshore Eurodollar transactions. Particularly, LIBOR origin is sometimes credited to Minos Zombanakis of Manufacturers Hanover Trust, who seemingly arranged an USD 80 million loan for the Shah of Iran with a rate based on a set of reported interbank funding ask. The loan's rate was made of a spread over the said interbank funding rate, hence it could be syndicated or readjusted over time depending on the interbank lending conditions. In the following years, it became clear that an increasing number of banks were actively trading in a variety of relatively new market instruments, such as interest rate swaps, foreign currency options and forward rate agreements.[2] While recognizing that such instruments increased the business and brought greater depth to the London Interbank

[1]The **Euro Interbank Offered Rate (Euribor)** is a daily reference rate, published by the European Money Markets Institute, based on the averaged interest rates at which Eurozone banks offer to lend unsecured funds to other banks in the euro wholesale money market (or interbank market).

[2]**Interest rate swaps** are forward contracts where one stream of future interest payments is exchanged for another based on a specified principal amount. A **foreign currency option** is a contract giving the option purchaser (the buyer) the right, but not the obligation, to buy or sell a fixed amount of foreign exchange at a fixed price per unit for a specified time period. A **forward rate agreement (FRA)** is an agreement between two parties who agree on a fixed rate of interest to be paid/received at a fixed date in the future.

market, the BBA (*British Banker's Association*)[3] felt the need to provide some measures of uniformity. In October 1984, the Association — together with other bodies, such as the Bank of England — established various working parties, which eventually produced the *BBA standard for interest rate swaps*, or "BBA IRS" terms: BBA IRS was the precursor of LIBOR, hence these standards constitute the first form of regulation of the rate. In January 1986, the *British Bankers' Association* published LIBOR - initially in US Dollars, Japanese Yen and Sterling (and later in 10 currencies with fifteen maturities calculated for each) – as the average of each submitting bank's estimate of the rate at which panel banks could borrow from each other (see [22]).

In 1998, the BBA published the LIBOR's definition, which reads "*the rate at which an individual contributor panel bank could borrow funds, were it to do so, by asking for and then accepting interbank offers in reasonable market size just prior to 11:00 London time*". This definition in still in force today. Specifically, every day the BBA asks a panel of AA-rated banks the question "*At what rate could you borrow funds, were you to do so, by asking for and then accepting interbank offers in a reasonable market size just prior to 11 am?*". It then eliminates the highest and lowest 25% of the quotes reported and computes the trimmed average of the remaining. LIBOR continued to be prefixed and administrated by BBA (and therefore known as **BBA LIBOR**) until February 2014, when the *Intercontinental Exchange Group* (ICE) took over its administration, changing it to the **ICE LIBOR** (see [22]).

Today, LIBOR is produced for **5 currencies** (USD, EUR, GBP, JPY, CHF) and for **7 maturities**, denoted as *tenors* (1 day, 1 week, 1, 2, 3, 6 and 12 months).

LIBOR is an **unsecured rate**, that is based on unsecured loans and, since it is derived from the quotes of a panel of banks, it is not anchored to actual transactions in active and liquid markets.

Among the LIBOR rates produced for the five currencies, the most popular and active in financial markets is currently USD LIBOR. In fact, it is broadly used as a reference rate for USD 350 trillion of outstanding contracts in maturities ranging from overnight to more than 30 years, in both financial markets and commercial fields (see [21]). The most important ones are summarised in Table 1.1. The panel for USD LIBOR is composed by 15 major banks, including Bank of America, Barclays, Citibank, Deutsche Bank, JPMorgan Chase, and UBS. These banks are selected according to the USD LIBOR Contributor Bank Criteria (see [24]), which are designed so that the contributed input data is able to produce a rate that is representative of the economic reality.

In order to correctly price derivatives, it is crucial that the applied interbank interest

[3] The British Banker's Association (BBA) was a trade association for the UK banking and financial services sector. From 1 July 2017, it was merged into UK Finance.

Interbank derivative products	Commercial field products	Hybrid products
Forward rate agreements	Floating rate notes	Range accrual notes
Interest rate futures	Floating rate certificates of deposit	Step up callable notes
Interest rate swaps	Syndicated loans	Target redemption notes
Swaptions	Variable rate mortgages	Hybrid perpetual notes
Overnight indexed swaps	Term loans	Collateralized mortgage obligations
Interest rates options, caps and floors		Collateralized debt obligations

Table 1.1: A summary of the financial contracts in which LIBOR is adopted.

rate is **risk-free**. For many years, LIBOR has been considered a proxy for the risk-free interest rate. In fact, it referred to AA-rated banks, therefore to transactions that were perceived to have credit risk almost equal to zero. This is why, mathematically, LIBOR has been determined as a **forward simply compounded rate** assuming absence of risk, as follows (from [7]).

Suppose that we are standing at time t, and we fix two points in time S and T, with $t < S < T$. We set up the following construction:

1. At time t we sell one S-bond. This will earn us $P(t,S)$ dollars.
2. We use this income to buy $\frac{P(t,S)}{P(t,T)}$ T-bonds. Therefore, our net investment at time t is worth $P(t,S) - \frac{P(t,S)}{P(t,T)} \cdot P(t,T) = 0$.
3. At time S the S-bond matures, so we need to pay one dollar.
4. At time T the T-bonds mature at one dollar each, so we will receive the amount $\frac{P(t,S)}{P(t,T)}$ dollars.
5. Thus, net effect overall obtained is that, based on a contract at t, an investment of one dollar at time S has brought $\frac{P(t,S)}{P(t,T)}$ dollars at time T.
6. This means that, at time t, we have set up a contract guaranteeing a riskless rate of interest over the future interval $[S,T]$. Such an interest rate is called **forward rate**, or **LIBOR rate**.

Now, we compute the relevant interest rate implied by the construction above.

Definition 1.1.1. *The **simple forward rate L**, is the solution to the equation*

$$1 + (T-S)L = \frac{P(t,S)}{P(t,T)}$$

Therefore,

$$L(t,S,T) = -\frac{P(t,T) - P(t,S)}{(T-S)P(t,T)}.$$

Nonetheless, after the global financial crisis, it is no longer reasonable to assume that LIBOR is a risk-free rate. To investigate the phenomenon, we compare LIBOR with some commonly accepted proxy for the risk-free rate. Figure 1.1 shows the spread of three-months USD LIBOR (and other relevant rates) over the three-month USD OIS rate, which can be considered free of risk since it refers to a short time horizon (24 hours). As noticeable, LIBOR performed well solely until the global financial crisis, as the LIBOR-OIS spread was always close to 0. However, after 2008, the spread became large and positive, meaning that some risk components were embedded in LIBOR.
This is why the mathematical construction above can no longer be applied. Indeed, the pre-crisis LIBOR rates associated to different tenors could simply be determined by the no-arbitrage condition, resulting in an interest rate market characterised by a single yield curve. On the contrary, after 2008 the market became segmented, in the sense that different yield curves arose from market instruments that depended on a specific tenor, thus leading to multiple yield curves. One of the most general approaches to model these multiple curves is based on affine processes, as carried out in [10].

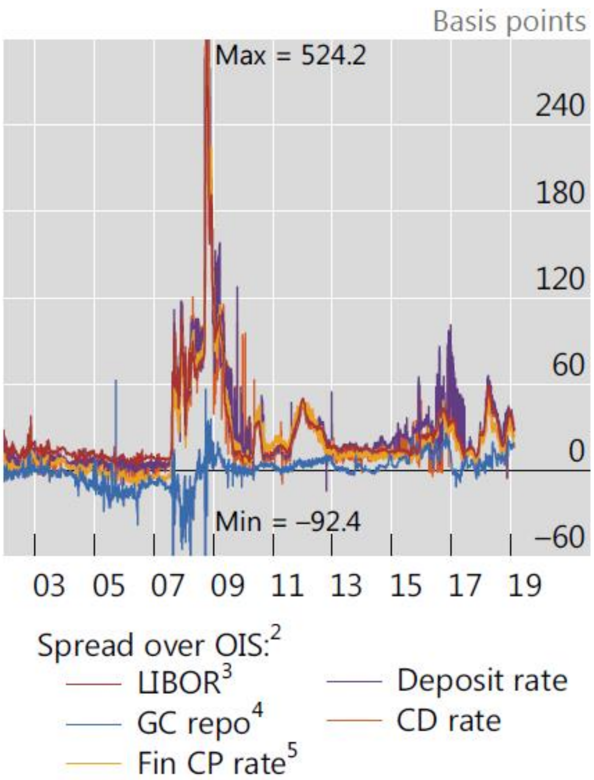

Figure 1.1: USD three-months LIBOR spread over three-months USD OIS rate. *Notice that the peak in 2007-2009 was related to the global financial crisis. Source: [34]*

1.1.1 LIBOR issues: what brought to the reform

After the 2008 financial crisis, it became clear that LIBOR could no longer be used as a benchmark rate, as it was not an appropriate proxy for the risk-free rate. This paragraph summarises some bank practises and crisis fallbacks which brought to this outcome.

Firstly, since LIBOR is constructed from a survey of a small set of banks reporting non-binding quotes rather than actual transactions, said banks could easily **manipulate** their quotes, by understating the borrowing costs they reported for LIBOR. For example, if among the panel banks' assets is a derivative whose payoff is positively related to LIBOR, it is tempting for them to report a higher quote and encourage the others to do the same. This practise generates the impression that banks could borrow from other banks more cheaply than they can in reality. Moreover, it makes the banking system seem healthier than it actually is.

On 29 May 2008, The Wall Street Journal (WSJ) published a study suggesting that banks might have been manipulating their submissions during the 2008 credit crunch (see [31]). In order to assess the borrowing rates reported by the US panel of banks, the Journal gathered numbers from the default-insurance market, which helped in assessing the financial health of the banks. In fact, before the global financial crisis, the cost of insuring against banks defaulting on their debts moved together with LIBOR in the same direction: both rose when the market thought banks were in difficulty. However, after the crisis outburst, as investors worried about possible bank failures, the two measures began to diverge, with reported LIBOR rates failing to reflect rising default-insurance costs. Figure 1.2 reports the spread between the cost of default insurance and the borrowing rates reported by the panel banks. As noticeable, the gap between the two measures was wider for Citigroup, Germany's WestLB, the United Kingdom's HBOS, J.P. Morgan Chase & Co. and Switzerland's UBS.

In response to the study released by the WSJ, some authorities stated that LIBOR continued to be reliable even during the financial crisis. For example, in its March 2008 Quarterly Review, the Bank for International Settlements declared that "*available data do not support the hypothesis that contributor banks manipulated their quotes to profit from positions based on fixings* (source: [5])". Furthermore, the International Monetary Fund, in its October 2008 Global Financial Stability Review, affirmed that "*although the integrity of the USD LIBOR-fixing process has been questioned by some market participants and the financial press, it appears that USD LIBOR remains an accurate measure of a typical creditworthy bank's marginal cost of unsecured U.S. dollar term funding* (source: [25])".

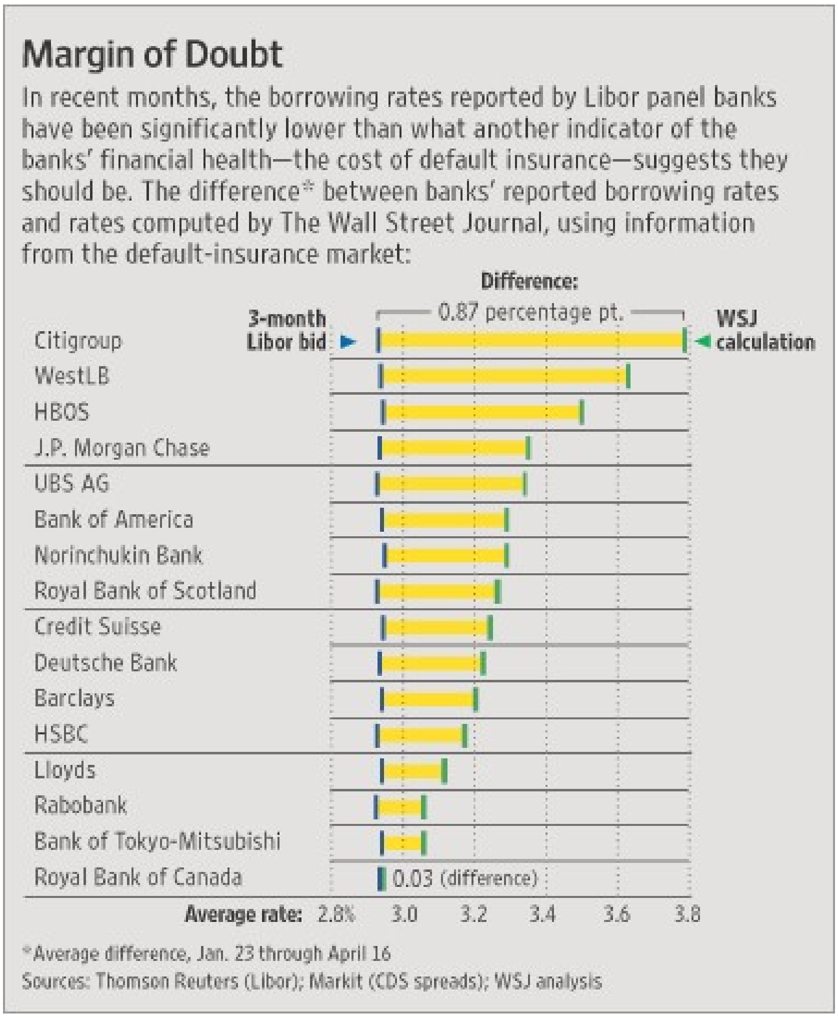

Figure 1.2: Spread between the cost of default insurance and the borrowing rates reported by the 2008 panel banks. Source: [31]

Nonetheless, in March 2011, the WSJ published another article reporting that regulators were focusing on Bank of America Corp., Citigroup Inc. and UBS about LIBOR rate manipulation, thus strengthening their first study's hypothesis (see [32]).
In 2012, the US Department of Justice initiated a criminal investigation concerning LIBOR abuse and manipulation. Barclays Bank was the first bank to be fined on 27 June 2012, for an amount of USD 200 million by the Commodity Futures Trading Commission[4], USD 160 million by the United States Department of Justice[5], and £59.5 million by the Financial Services Authority[6], for attempted manipulation of the LIBOR

[4]Commodity Future Trading Commission, Press Release, 2012. URL: https://www.cftc.gov/PressRoom/PressReleases/6289-12

[5]United States Department of Justice, Justice News, 2012. URL: https://www.justice.gov/opa/pr/barclays-bank-plc-admits-misconduct-related-submissions-london-interbank-offered-rate-and

[6]Financial Services Authority, Press Release, 2012. URL: http://www.fsa.gov.uk/library/communication/pr/2012/070.shtml

rate. Later, in December 2013, the European Commission announced fines for other six banks, which had taken part to one or more bilateral cartels for LIBOR submission of Japanese yen from 2007 to 2010. Of these, the Royal Bank of Scotland was fined €260 million, Deutsche Bank €259 million and JPMorgan €80 million. Citigroup received a smaller fine for about €70 million, thanks to an immunity for one of the infringements to which it took part.[7]

Already by mid-2012, the issue started to be discussed by the news and financial programs, and made the front page of several newspapers. The media defined the manipulation-practise as **LIBOR scandal**.

Although the scandal came into light only after the financial crisis, there is evidence that it had been on-going for a long time: in an article of July 2012, the Financial Times stated that LIBOR manipulation had been in use since at least 1991 (see [26]).

Consequently LIBOR manipulation, some reforms were taken. On July 2013, the administration of the rate started to be regulated and supervised by the UK's Financial Conduct Authority (FCA). Moreover, as mentioned in section 1.1, in early 2014 LIBOR's administration passed from BBA to ICE (see [21]). Furthermore, knowingly or deliberately making false or misleading statements in relation to benchmark-setting became a criminal offence in the UK law, under the Financial Services Act of 2012. The Danish, Swedish, Canadian, Australian and New Zealand LIBOR rates were terminated: only the five rates still produced today remained.[8] In addition, on July 2013, the BBA established the Interim LIBOR Oversight Committee (ILOC), which must follow an interim code, concerning how banks must behave in relation to LIBOR. For example, each bank must indicate a named person responsible for LIBOR, chargeable in case of wrongdoings. The banks must also keep records to be audited by the regulators if necessary.[9]

Apart from manipulation practises, LIBOR faced other important drawbacks as a consequence of the global financial crisis.

For starters, post-crisis **interbank trading** dropped, especially in the unsecured segment. This was driven by the Central Banks (CBs) doings, which characterised the years after 2008. In particular, CBs lowered interest rates, keeping them close or below zero, in order to increase money supply and boost economic activity. However, they kept them low for a long time. Therefore, to manage the resulting decrease of inflation

[7] European Commission, Press Release, 2013. URL: https://ec.europa.eu/commission/presscorner/detail/en/IP_13_1208

[8] The British Bankers' Association, LIBOR becomes a regulated activity, Press Release, 2013. URL: http://www.bbalibor.com/news/libor-becomes-a-regulated-activity

[9] The British Bankers' Association, BBA Libor Limited has established the Interim LIBOR Oversight Committee (ILOC), Press Release, 2013. URL: http://www.bbalibor.com/news/bba-libor-limited-has-established-the-interim-libor-oversight-committee-ilo

– which negatively impacts economies and their ability to grow in a healthy way - they were forced to use unconventional policies, such as asset purchase programs (APPs). AAPs are monetary policy instruments through which CBs purchase certain amounts of government bonds or other financial assets, to inject money into the economy and expand economic activity. In the US, the Federal Reserve conducted large-scale asset purchases between 2008 and 2014.[10] The European Central Bank implemented the "quantitative easing" starting 2015.[11] This copious supply of reserve balances created by APPs made banks less in need to trade with each other.
Moreover, consequently the Great Financial Crisis, banks suffered substantial losses. At the same time, they faced higher balance sheet costs due to tighter risk management and new regulatory standards – typically, they established minimum capital requirements, caps for the risk-weighted assets, ranges for the leverage ratio and supervision by institutional bodies, to ensure the soundness of the bank. The general instability in their bank accounts increased adverse selection among banks. As banks began to fear that their counterparty was not trustworthy, they became reluctant to lend to one another and started to hoard liquidity. For these reasons, the interbank market was no longer sufficiently liquid to produce a reasonable reference rate.
Furthermore, in order to reduce the counterparty credit risk in interbank exposures, banks started to gather funds from non-banks sources. Nevertheless, when they did trade with other banks, they used only less risky wholesale instruments (such as repurchase agreements or repos, instruments used to raise short-term capital by selling and then repurchasing government securities at a slightly higher price). This resulted in the interbank market shrinking, contributing to making LIBOR no more appropriate as a benchmark rate. In fact, banks became hesitant themselves in providing submissions: indeed, when interbank lending and borrowing activity is so scarce, there is no adequate way to validate the judgements upon which the banks' quotes are based.

As LIBOR faced all these issues, it became clearer and clearer that it was no longer suitable as a reference rate in financial markets. In 2017, the FCA and the Bank of England's Financial Policy Committee (FPC) noticed that the absence of active underlying markets and the scarcity of term unsecured deposit transactions had raised serious questions about the future of LIBOR benchmarks. In fact, while the precise volume of transactions in markets underlying LIBOR is unknown, estimates show that, on a typical day, the volume of three-month wholesale funding transactions by major

[10] Federal Reserve Bank of New York, Large-Scale Asset Purchases, 2008. URL: https://www.newyorkfed.org/markets/programs-archive/large-scale-asset-purchases

[11] European Central Bank, ECB announces expanded asset purchase program, 2015. URL: https://www.ecb.europa.eu/press/pr/date/2015/html/pr150122_1.en.html

global banks was about USD 500 million. This is a very low number compared to the trillions of financial contracts referencing USD LIBOR. Eventually, they reached the conclusion that LIBOR had become unsustainable and unsuitable for the widespread reliance that had been placed upon it. In particular, in a speech delivered on 27 July 2017, FCA Chief Executive Andrew Bailey stated that "*data from IBA and from central banks indicate that there are relatively few eligible term borrowing transactions by any large banks – i.e. these banks receive few loans or deposits of a twelve, six or even three month term from other banks or eligible corporate depositors.* [. . .] *On the basis of what we can currently observe, activity in these markets is limited, and there seems little prospect of these markets becoming substantially more active in the near future. The absence of active underlying markets raises a serious question about the sustainability of the LIBOR benchmarks that are based upon these markets. If an active market does not exist, how can even the best run benchmark measure it? Moreover, panel banks feel understandable discomfort about providing submissions based on judgements with so little actual borrowing activity against which to validate those judgements.* [. . .] *In our view it is not only potentially unsustainable, but also undesirable, for market participants to rely indefinitely on reference rates that do not have active underlying markets to support them. As well as an inherently greater vulnerability to manipulation when rates are based on judgements rather than the real price of term funding, there are a host of questions about whether and how such reference rates can respond to stressed market conditions.*". For these reasons, he anticipated that from 2021 banks will be no longer obliged to report rates for LIBOR computation. The early announcement was made to ensure enough time for the market to move away from LIBOR, in order to have a planned and orderly transition, thus less risky and less expensive.[12]
In March 2021, the FCA and ICE Benchmark Administration announced that sterling, euro, Swiss franc and Japanese yen LIBOR panels, as well as panels for 1-week and 2-month US dollar LIBOR, would cease at the end of 2021. Only 1-month, 3-month, 6-month and 12-month US dollar LIBOR were extended to June 2023, given their importance in terms of volume of derivatives referencing to them in financial markets.[13] Based on the undertakings received from the panel banks, the FCA does not anticipate that any LIBOR settings will become unrepresentative before these relevant dates. However, representative LIBOR rates will not be available past these dates: the publication of most of the LIBOR settings will stop immediately after this time. Accordingly, both FCA and the Bank of England's FPC have worked together with

[12] A. Bailey, The future of LIBOR, Bloomberg London event, UK, 2017. URL: https://www.fca.org.uk/news/speeches/the-future-of-libor

[13] Financial Conduct Authority, Further arrangements for the orderly wind-down of LIBOR at end-2021, Press Release, 2021. URL: https://www.fca.org.uk/news/press-releases/further-arrangements-orderly-wind-down-libor-end-2021

market participants and the other regulatory authorities from all over the world, to ensure that robust alternatives to LIBOR are available and that existing contracts can be transitioned onto these alternatives to ensure financial stability and market integrity.

1.2. New Risk-Free Rates (RFRs)

LIBOR termination raised the need for the identification of one or more interest rates to serve as its adequate substitutes in financial markets. Ideally, in order to appropriately work as reference rates, the new interest rates should:

- Accurately represent interest rates in core money markets, in a way that is not open to manipulation. Benchmarks anchored to **actual transactions in active and liquid markets** and not derived from a poll of selected banks fulfil this feature;
- Provide a reference rate for **discounting and pricing** financial instruments such as derivatives without difficulty;
- Function as a benchmark for **term lending and funding**. In fact, financial intermediaries are simultaneously lenders and borrowers, hence they require a lending benchmark that behaves not so differently from the borrowing one.

Given its issues (discussed in section 1.1.1), it is evident that LIBOR fails to meet the first criterion out of the three. This explains why the reforms have mostly focused in selecting rates linked with actual transactions in the most liquid market segments. These rates are commonly identified as **RFRs** (that is, **risk-free rates**) and should present the following attributes, in order to try fixing LIBOR's main problems:

- They should have a **shorter tenor** than LIBOR. In fact, they generally refer to overnight (O/N) markets, where traded volumes are larger than longer-dated tenors;
- They should reflect borrowing costs from **wholesale non-bank counterparties**, and not exclusively of the interbank market;
- They should be based on **collateralised (secured) transactions**, which include banks' repurchase agreements (repos) with non-bank counterparties.

Taking this into account, the authorities of each jurisdiction have selected alternative RFRs benchmarks. Although pursuing similar schemes, each country took different steps in identifying a new rate. This resulted in different currency areas identifying

Currency area	**United States**	**United Kingdom**	**Euro area**	**Switzerland**	**Japan**
Alternative rate	**SOFR** (Secured overnight financing rate)	**SONIA** (Sterling overnight index average)	**ESTR** (Euro short-term rate)	**SARON** (Swiss average overnight rate)	**TONA** (Tokyo overnight average rate)
Administrator	Federal Reserve Bank of New York	Bank of England	European Central Bank	SIX Swiss Exchange	Bank of Japan
Wholesale non-bank counterparties	Yes	Yes	Yes	No	Yes
Secured	Yes	No	No	Yes	No
Overnight rate	Yes	Yes	Yes	Yes	Yes

Table 1.2: Identified alternative RFRs in different currency areas. Source: https://www.fca.org.uk/markets/libor-transition

different benchmark rates for their financial instruments and no longer in a uniformly-calculated widespread one. The most important ones and their characteristics are summarised in Table 1.2. As noticeable, the alternative benchmark rates in the US, UK, Europe, Switzerland and Japan are all O/N rates. In the US and Switzerland, the underlying transactions are collateralised (with US and Swiss Treasuries), while they are not in the UK, Japan and Euro area.

In the next paragraphs, SOFR, ESTR and SONIA's main characteristics will be presented. The reason behind this choice of selection is related to the relevance these three rates have in current financial markets, both in terms of volume of underlying transactions and types of referencing derivatives.

1.2.1 SOFR

The *Secured overnight financing rate* (SOFR) is a reference rate based on the US Treasury repurchase agreement (repo) market, specifically on tri-party repo, General Collateral Finance (GCF) repo and bilateral repo transactions cleared through the Fixed Income Clearing Corporation (FICC). [14] The New York Fed is its administrator and produces the rate in cooperation with the Treasury Department's Office of Financial Research

[14] In **broad repos**, also **called tri-party repos** (because they are cleared through a third party which is either Bank of New York Mellon or JP Morgan Chase), the typical lenders are MMFs and other non-banks. **General collateral financing** (GCF) repos are inter-dealer repos. **Bilateral repo transactions** are typically between dealers and non-banks.

(OFR). The New York Fed publishes SOFR on a daily basis at approximately 8:00 AM. It also publishes 30, 90 and 180-day SOFR Averages and a SOFR Index, to better support a successful transition away from USD LIBOR. The rate production is periodically reviewed by an internal New York Fed Oversight Committee – which consists of members from across the New York Fed's organizational structure who are not involved in the daily production of SOFR.

In 2014, the Federal Reserve Board and the New York Fed jointly convened the *Alternative Reference Rates Committee* (ARRC), to identify risk-free alternative reference rates for USD LIBOR.[15] ARRC is composed of a diverse set of private-sector entities, each with an important presence in markets affected by USD LIBOR, and a wide array of official-sector entities, including banking and financial sector regulators. The ARRC identified a first set of criteria that the new RFRs must respect, which contains *benchmark quality*, *methodological quality*, *accountability* (that is satisfaction of the four IOSCO principles) and *ease of interpretation.* At the same time, the *Bank of International Settlements* (BIS) - an international financial institution owned by central banks that fosters international monetary and financial cooperation and serves as a bank for central banks - selected another set of standards, namely *reliability*, *robustness*, *frequency*, *ready availability* and *representativeness*. As noticeable, these standards overlap for certain aspects but disagree for others, creating some confusion. The issue is studied deeply in [4], and a solution proposed.

In 2017, the ARRC identified SOFR as the most appropriate reference rate for USD derivatives and other financial contracts (see [2]). The ARRC considered a list of potential alternatives, including term unsecured rates, overnight unsecured rates like the Overnight Bank Funding Rate (OBFR), term secured rates, overnight secured rates like SOFR and treasury bill and bond rates. Eventually, the ARRC selected SOFR because it is a fully transaction-based, overnight nearly risk-free reference rate and a good representation of the general funding conditions in the US money markets. Additionally, it is suitable to be used across a broad range of financial products, including derivatives and many variable-rate cash products that have historically referenced USD LIBOR. Moreover, it is based on transactions that take place in the Treasury repo market, which is characterised by considerable depth and breadth. Indeed, the transaction volume underlying SOFR is far larger than volumes in other US money markets: since SOFR was first published in April 2018, the daily transaction volume underlying SOFR has been on average more than USD 980 billion (Figure 1.3). At the same time, SOFR reflects activity undertaken by different types of institutions, including asset managers, banks, corporate treasurers, insurance companies, money

[15] From URL: `https://www.newyorkfed.org/arrc/about`

market funds, pension funds, and more.
Already in 2017, the ARRC had published a Paced Transition Plan, with specific steps and timelines designed to encourage adoption of SOFR (see [1]). In 2020, the commission issued the Recommended Best Practices, which provide timelines and interim milestones for transitioning away from USD LIBOR in a way that will minimize market disruption and support a smooth transition (see [3]).

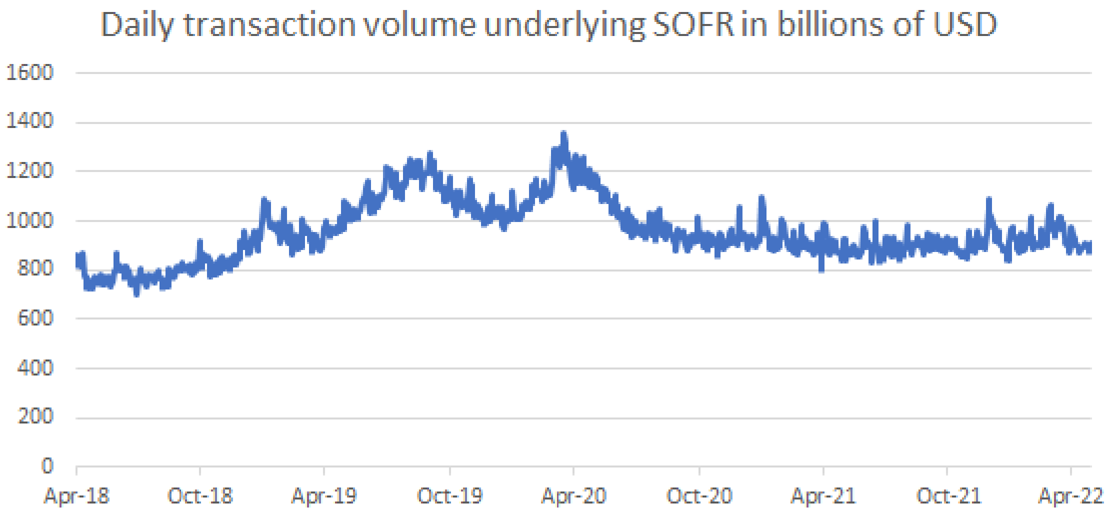

Figure 1.3: Daily transaction volume underlying SOFR. *Data retrieved from source:* *https://www.newyorkfed.org/markets/reference-rates/sofr*

1.2.2 ESTR

The *Euro short-term rate* (ESTR) reflects the wholesale euro unsecured overnight borrowing costs of banks located in the euro area. ESTR is published at 08:00 AM on each TARGET2 business day.[16] It is based on transactions conducted and settled on the previous TARGET2 business day (the reporting date t) with a maturity date of $t+1$, which are considered to have been executed at arm's length and thus reflect market rates in an unbiased way.
ESTR is based entirely on daily confidential statistical information relating to money market transactions, collected with the assistance of Deutsche Bundesbank, Banco de España, Banque de France and Banca d'Italia. ESTR is administered and overseen by the ECB, with the ESTR Oversight Committee reviewing all aspects of the rate determination process.
In 2017, the European Central Bank (ECB), the European Securities and Markets Authority, the European Commission and the Belgian Financial Services and Markets

[16] TARGET2 is open every day, with the exception of: Saturdays, Sundays, New Year's Day, Good Friday and Easter Monday, 1 May (Labour Day), Christmas Day and 26 December.

Authority established the *working group on euro risk-free rates* - a private sector group which the public institutions attended as observers - to identify and recommend risk-free rates that could serve as an alternative to current benchmarks used in a variety of financial instruments and contracts in the euro area (see [11]). The *working group* selected a set of macro criteria that the new rates should respect, namely *benchmark quality*, *methodological quality*, *governance* and *accountability*. In 2018, the group chose ESTR to be used as the risk-free rate for the euro area.[17] ESTR was published for the first time by the ECB in October 2019.[18]
The working group identified ESTR as the substitute for LIBOR because of the full transaction-based, overnight risk-free nature of the rate. Moreover, as noticeable in figure 1.4, the transactions underlying ESTR stand on average 45 billions EUR, evidence of the considerable depth of the market.
In 2019, the ECB published a preliminary rate, called pre-ESTR, which follows the same calculation methodology as ESTR, but was based on final data. Pre-ESTR was only intended as a set of indicators for ESTR: its publication was solely for information purposes, to help market participants in understanding its nature, but the data were not meant to be used as a reference rate in any market transaction (see [12]).

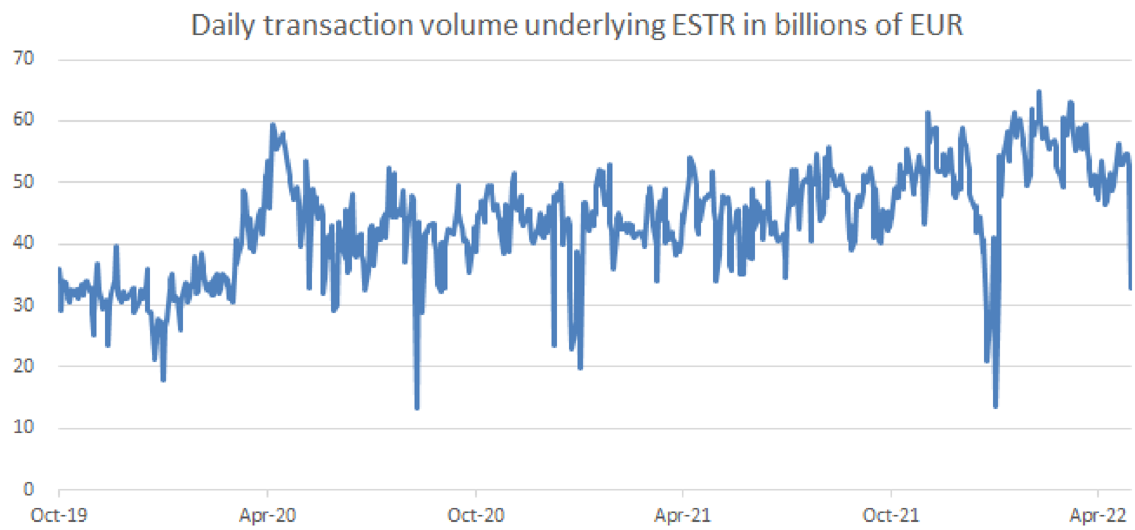

Figure 1.4: Daily transaction volume underlying ESTR. *Data retrieved from source:* *https://sdw.ecb.europa.eu/browse.do?node=9698150*

[17]European working group, Private sector working group on euro risk-free rates recommends ESTER as euro risk-free rate, Press Release, 2018. URL: https://www.ecb.europa.eu/press/pr/date/2018/html/ecb.pr180913.en.html

[18]European Central Bank, ECB announces start date for euro short-term rate (ESTR), Press Release, 2019. URL: https://www.ecb.europa.eu/press/pr/date/2019/html/ecb.pr190314~28790a71ef.en.html

1.2.3 SONIA

The *Sterling overnight index average* (SONIA) is the rate at which interests are paid on sterling short-term wholesale funds when credit, liquidity and other risks are minimal. The Bank of England administrates the rate and takes responsibility for its governance and publication every London business day at 9:00 AM. SONIA is measured as the trimmed mean, rounded to four decimal places, of interest rates paid on eligible sterling denominated deposit transactions. The SONIA Oversight Committee reviews all aspects of the benchmark determination process and provides scrutiny of the administration of the rate.
SONIA was first introduced in March 1997.[19] In 2015, the Bank of England set up a *working group* to choose risk-free rates (RFRs) to provide an alternative to LIBOR.[20] In 2016, the Bank of England took responsibility for SONIA.[21] The following year, the *working group* published a paper in which SONIA was identified as the preferred alternative rate for sterling markets (see [6]). In fact, it is based on actual transactions and reflects the average of the interest rates that banks pay to borrow sterling overnight from other financial institutions and other institutional investors. Moreover, its market depth is considerable, as the daily transaction volume underlying SONIA is on average £40 billion. (Figure 1.5).

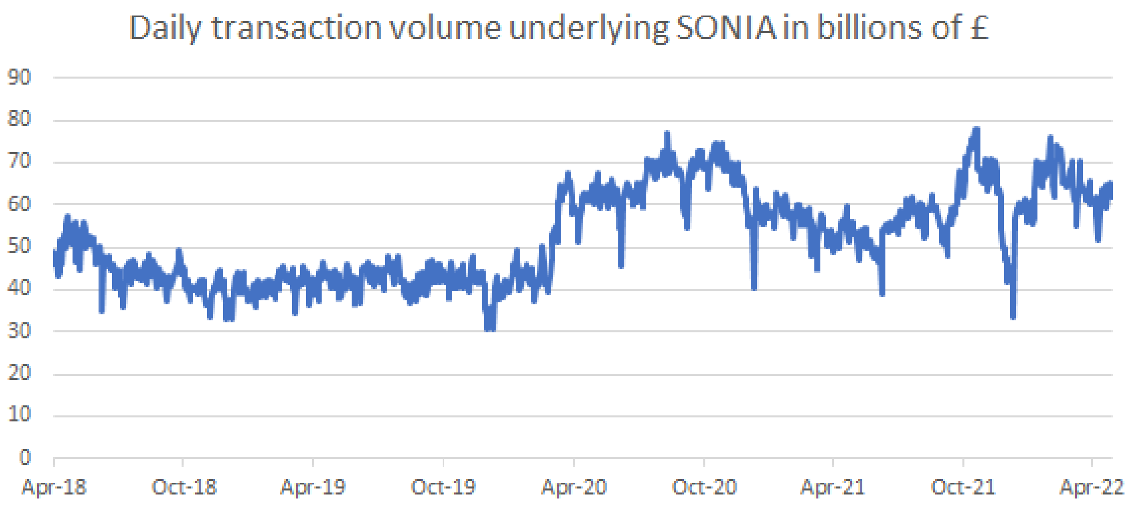

Figure 1.5: Daily transaction volume underlying SONIA. Data retrieved from source: *https://www.bankofengland.co.uk/boeapps/database/fromshowcolumns.asp?ShowData.x=51&ShowData.y=30&Travel=NIx&ShadowPage=1&SearchText=sonia&SearchExclude=&SearchTextFields=TC&Thes=&SearchType=&Cats=&ActualResNumPerPage=&TotalNumResults=12&XNotes2=Y&C=5JK&C=UH6*

[19] From URL: https://www.bankofengland.co.uk/markets/sonia-benchmark

[20] From URL: https://www.bankofengland.co.uk/markets/transition-to-sterling-risk-free-rates-from-libor/working-group-on-sterling-risk-free-reference-rates

[21] From URL: https://www.bankofengland.co.uk/markets/sonia-benchmark

Generally speaking, the underlying transactions of these reference rates can be grouped into three different types: non-bank to bank lending (*Type 1*), bank to bank lending (*Type 2*) and bank to non-bank lending (*Type 3*). *Type 1* transactions are typically made by cash-rich companies or money market mutual funds (MMFs). In *Type 2* transactions, banks gather funds from each other; while in *Type 3* they raise cash from hedge funds or investment managers. SOFR incorporates transactions of all three types, as it is based on broad repos (similar to *Type 1*), inter-dealer repos (similar to *Type 2*) and bilateral repos (similar to *Type 3*). On the contrary, both SONIA and ESTR comprise only *Type 1* and *2* transactions (Table 1.3).

RFR	**Type 1 transactions**	**Type 2 transactions**	**Type 3 transactions**
SOFR	Yes	Yes	Yes
ESTR	Yes	Yes	No
SONIA	Yes	Yes	No

Table 1.3: Main RFRs underlying transaction types.

1.2.4 RFRs drawbacks and challenges

Although successfully fixing some LIBOR issues, new RFRs are not immune to some drawbacks.

Firstly, since the new reference rates are overnight rates, they theoretically should be virtually risk-free. However, in practise, they present some **disadvantages**, as RFRs are generally more volatile than LIBOR. In fact, as empirically demonstrated in [27], they:

- are prone to upward or downward spikes due to **regulatory constraints**. Specifically, it is proved that tighter regulatory constraints (such as minimum risk-weighted capital and leverage ratio requirements) slightly decrease SONIA and ESTR but increase SOFR. In fact, more binding regulations reduce the banks' ability to lend to other banks and borrow from non-banks, increasing interbank rates and decreasing *Type 1* rates. Given that *Type 1* transactions dominate *Type 2* in terms of volume for the European and British rate, the increase in interbank rates is offset by a higher decrease in non-bank to bank lending rates. On the contrary, SOFR is mainly based on repo rates. Since these are instruments that rely on large banks intermediation, if said banks are subject to stricter constraints, bank to non-bank lending rates will increase;

- increase as the **Treasury debt outstanding** becomes larger. This happens because an increase in the supply of safe assets increases *Type 1* transaction rates, because non-bank lenders can invest more in government debt. The impact is even greater for SOFR, as an increase in the Treasury supply increases the demand for repos and thus the repo-rate;

- increase if the amount of **central bank's reserves** decreases. In fact, more central bank reserves lower banks' demand for overnight borrowing, reducing interest rates.

Nonetheless, the main concern for financial markets is another. In fact, the reform brings into markets two challenges. The first one is how to address changes in evaluation and discounting of **already existent derivatives**. As a matter of a fact, all instruments anchored to LIBOR which have not expired yet would face an abrupt shift of benchmark, were additional measures not be taken. This would possibly result in a part of the contract favoured by the sudden change of value and the other experiencing a potential loss. Ideally, the problem should be solved by giving a compensation to the disadvantaged part, that should account for both the value change and the new sensitivity (delta) to the new discounting rate. The first one is likely to be addressed via a cash payment, while the second with an exchange of basis swap. A basis swap is an interest rate swap - a type of a derivative contract through which two counterparties agree to exchange one stream of future interest payments for another, referenced against an interest rate index - which involves the exchange of two floating rate financial instruments.
On the other hand, the second issue is related to the evaluation and pricing of the **new kinds of derivatives** that will be born after the adoption of the new benchmark rates. In fact, we can imagine several innovative instruments anchored to the new RFRs. An example is given by options on SOFR futures, a type of derivative launched by CME in 2020. The option gives the right, but not the obligation, to exercise sometime before maturity the SOFR future contract, in order to protect the holder from unfavourable variations of its price. These derivative instruments are gaining increasing popularity in the last weeks, hence it is crucial to understand how they work and the interest rate they reference to.
In order to address these issues, in October 2020, the Financial Stability Board[22] published a Global Transition Roadmap for LIBOR, to inform those with exposure to LIBOR benchmarks of some of the steps they should have been taking over the remaining period until end-2021 to successfully mitigate these risks (see [14]). Particularly, firms should identify and assess all existing LIBOR exposures and other dependencies, to implement a plan for potential fallbacks with end-users of LIBOR

[22]The **Financial Stability Board** (FSB) is an international body that monitors and makes recommendations about the global financial system.

referencing products maturing beyond end-2021. In doing this, firms should adhere to the 2020 ISDA Fallbacks Protocol, according to which counterparties are encouraged to agree to contractual fallback provisions that would provide for adjusted versions of the RFRs as replacement rates.[23] At the same time, by the end of 2020, lenders should be in a position to offer non-LIBOR linked loan products to their customers. This could be done by giving borrowers a choice in terms of the reference rate underlying their loans.

Finding a way to correctly price and evaluate existing and new derivatives instruments will be the main focus of the following chapters. Particularly, some interest rates models will be presented and then applied to RFRs derivatives (specifically to options on futures).

[23]See URL: `https://www.isda.org/protocol/isda-2020-ibor-fallbacks-protocol/`

Chapter 2

An overview of the primary RFRs models

Since all selected RFRs are overnight rates, in order for them to be used as a replacement of LIBOR in both new and already existing contracts, they first need to be converted into term rates. The **term structure** of interest rates is the relationship between interest rates or bond yields and different terms or maturities. If plotted, the term structure of interest rates is called **yield curve**, and it useful to identify the current state of an economy. The term structure of interest rates reflects the expectations of market participants about future changes in interest rates and their assessment of monetary policy conditions.

In this chapter, after some generalities, we will present different solutions to model interest rates, that can be found in the literature, and we will obtain different term structures, depending on the RFRs model presented. Particularly, first we will present the **Hull-White (extended-Vasicek) model**, basing our analysis on [7]. This model seems to be the standard choice for interest rate modeling in current markets. In fact, in [30], which is one of the first papers about SOFR modeling, a Gaussian Hull-White short rate model is adopted. The Hull-White model remains dominant also in other recent short rate approaches to RFRs modeling, such as [17], [18], [36] and [37].

Given that the Hull-White model constitutes one of its extensions, we will first introduce also the **Vasicek model** as done in [7], for pedagogical purposes.

Lastly, we will present the **Forward Market Model** as done in [28], which constitutes another solution for modeling risk-free rates.

The main difference between these models is that the Hull-White model (together with the Vasicek model) models short rates of interest, while the Forward Market Model models forward rates. A **forward rate** is the yield of a security that will not be traded until a predetermined date in the future. On the contrary, the **short rate** is the interest rate at which an entity can borrow money for an infinitesimally short period of time.

2.1. RFRs models' generalities

In this section, we will introduce some generalities, that later will be useful to present and understand RFRs models.
Assume the existence of a continuous-time financial market with an arbitrage-free family of zero-coupon bonds, whose price process is $\{P(\cdot, T); T \geq 0\}$. Therefore, $P(t, T)$ denotes the price at time t of a risk-free zero-coupon bond with maturity T. Intuitively, the price $P(t, T)$ should depend upon the behaviour of the short rate of interest over the interval $[t, T]$. A short rate is an interest rate at which someone can borrow money for an infinitesimally short period of time from time t. Let us denote the time-t value of this instantaneous rate by $r(t)$. Assume that $r(t)$ is the overnight interest associated with funding/remunerating cash collateral posted as variation margin. At the same time, $r(t)$ should represent the discounting rate. [1] This is consistent with the overall direction of the LIBOR reform and transition to the new rate benchmarks. In fact, as explained in chapter 1, both rates should have been moved to the new RFRs by the second quarter of 2021. If the same RFRs are used for discounting and for deriving term rates, this implies a return to the classic single-curve modelling environment, rather than a multiple-curves one.
A natural starting point for modeling interest rates is to give an *a priori* specification of the dynamics of the short rate of interest. Suppose that, under the objective probability $\mathbb{P}$, the short rate $r(t)$ is the solution to the following SDE

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))d\bar{W}(t) \tag{2.1}$$

where μ and σ are functions representing respectively the drift and volatility of the process and $\bar{W}$ is a Brownian motion under the objective probability $\mathbb{P}$. The short rate of interest is the only object given *a priori*, thus the only exogenously given asset is the money account, with price process B defined by the dynamics

$$\begin{cases} dB(t) = r(t)B(t)dt \\ B(0) = 1 \end{cases}$$

[1] An interest rate that is both the overnight interest associated with funding/remunerating cash collateral posted as variation margin and the discounting rate is known as **Price Alignement Interest (PAI)**.

whose solution is given by

$$B(t) = e^{\int_0^t r(u)\,du}.$$

This is to be interpreted as the model of a bank with a stochastic short rate of interest r, hence the dynamics of B can be interpreted as the ones of a bank account.
Assume also the existence of a risk-neutral measure $\mathbb{Q}$, whose associated numeraire is $B(t)$. A numeraire, or base value, is an economic term that represents a unit of measure. Having a numeraire allows for the comparison of values against one another. Recall also that a risk-neutral measure is a probability measure such that the discounted price of every traded asset (or portfolio) is a martingale under the risk-neutral measure $\mathbb{Q}$.
Notice that our market contains all possible types of bonds, among which the only exogenously given one is the risk-free asset. Nonetheless, the price of a particular bond will not be completely determined by the specification of the (2.1) r-dynamics and the assumption that the bond market is arbitrage-free. In fact, arbitrage pricing is always a case of pricing a derivative in terms of the price of some underlying assets, however we do not have a sufficient number in our market. Even so, if we take the price of one particular "benchmark" bond, then the prices of all other bonds can be uniquely determined in terms of the price of the benchmark. This is true because bonds of different maturities satisfy certain internal relations to ensure absence of arbitrage. This fact will be demonstrated in the next section.
Before that, however, for later use, we show that it is possible to obtain an expression for the zero-coupon bond price for both the times before and after maturity T. Moreover, we will also introduce two other concepts that will be of great importance both for the Forward Market Model and for derivative pricing: backward-looking and forward-looking rates. In fact, ISDA and other regulators identified two main approaches that can be used for the computation of the rates' term structure. The first one is "**in-arrears**", that is the calculation of interest using daily rates published during the relevant application period (and not over a period of time prior to the start of the application period). In the SOFR "In Arrears" Conventions for Syndicated Business Loans (see [13]), the Federal Reserve recommends two structures for this kind of rates:

- **daily simple in arrears rate**: the rate is sourced daily and multiplied by the outstanding principal of the loan;
- **daily compounded in arrears rate**: the rate is obtained through a methodology that compounds daily values of the overnight rate, throughout the relevant term period.

Although the compounded interest rate more accurately reflects the time value of money, implementing the simple interest rate is more straightforward. Nonetheless, the rates resulting from the two methods differ by a few basis points. In the case of either compounded or simple interest in arrears, the rate for the entire application period would not be known at the beginning of this period. Instead, the overnight rate would be pulled daily (and compounded based on a previous day's rate in the case of daily compounded rate). This means that these structures allow for interest accruals to be calculated daily and they are not set in advance but fixed during each application period. For this reason, "in arrears" rates are **backward-looking** and known at the end of the corresponding application period.

The second approach for the identification of RFRs' term structures is through a market implied prediction of the compounded setting-in-arrears rate. In this case, the rate is calculated over a period of time prior to the start of the application periods (and not using daily rates published during this time). In fact, it represents the market's predictions for the interest rate rather than the prior day's overnight performance. For this reason, it is **forward-looking** in nature and known at the beginning of the application period.

Notice that, according to its definition (see 1.1), LIBOR is a forward-looking rate, since it is calculated over a period of time prior to the start of the application period and not using daily rates published during this time.

2.1.1 The extended zero-coupon bond price formula

Define as $\mathcal{F}_t$ the "information" available in the market at time t, that is the sigma-algebra generated by the model risk factors up to time t.

Because of risk-neutral methodology, we have that

$$\frac{P(t,T)}{B(t)} = \mathbb{E}^{\mathbb{Q}}\left[\frac{P(T,T)}{B(T)}\bigg|\mathcal{F}_t\right]$$

where $\mathbb{E}^{\mathbb{Q}}$ stands for the expected value under the risk-neutral measure $\mathbb{Q}$. From this, we obtain:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\frac{B(t)}{B(T)}\bigg|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\,du}\Big|\mathcal{F}_t\right].$$

Since it constitutes the value of a contract expiring at time T, the previous equation is valid for every $t \leq T$.

Nonetheless, we show that it is possible to extend the definition of the zero-coupon bond also for the times after maturity, as carried out in [28]. To do so, we use the previous equation and the definition of $B(t)$, hence for $t > T$ we have:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{\int_T^t r(u)\,du}\middle|\mathcal{F}_t\right] = e^{\int_T^t r(u)\,du} = \frac{B(t)}{B(T)} \tag{2.2}$$

where the second equality derives from the fact that $e^{\int_T^t r(u)\,du}$ is $\mathcal{F}_t$-measurable.
We can consider a self-financing strategy Y_T, that consists of buying the zero-coupon bond with maturity T and reinvesting the bond's unit notional received at T at the risk-free rate $r(t)$ fom T onwards. If we denote $Y_T(t)$ as the time t value of the strategy, we have:

$$Y_T(t) = \begin{cases} P(t,T) & \text{for } t \leq T \\ e^{\int_T^t r(u)\,du} & \text{for } t > T. \end{cases}$$

Notice that for $t > T$, $Y_T(t)$ is exactly the value of the extended bond price defined by (2.2). Because of this, we can conclude that for each given T, $Y_T(t) = P(t,T)$ for all times t. Therefore, the strategy $Y_T(t)$ is the extended zero-coupon bond with maturity T.

2.1.2 Backward-looking in arrears rates

Let us consider $M+1$ dates $T_0, T_1, \ldots, T_M$ and denote with τ_j the year fraction for the time interval $[T_{j-1}, T_j]$. Assume that from now on bond prices $P(t,T)$ are meant in the extended sense. The daily compounded setting-in-arrears rate for the interval $[T_{j-1}, T_j)$, which we denote by $R(T_{j-1}, T_j)$, is

$$R(T_{j-1}, T_j) = \frac{1}{\tau_j}\left[\prod_{i=1}^{n}(1 + r_i\delta_i) - 1\right]$$

where the product is over the business days in $[T_{j-1}, T_j)$ and r_i is the RFR fixing on date i with associated day-count fraction δ_i. For each $j = 1, \ldots, M$, we apply the same approximation used in [28] of the rate for the interval $[T_{j-1}, T_j)$[2] and use the extended definition of zero-coupon bond

$$R(T_{j-1}, T_j) = \frac{1}{\tau_j}\left[e^{\int_{T_{j-1}}^{T_j} r(u)\,du} - 1\right] = \frac{1}{\tau_j}\left[\frac{B(T_j)}{B(T_{j-1})} - 1\right] = \frac{1}{\tau_j}\left[P(T_{j-1}, T_j) - 1\right]. \tag{2.3}$$

Now, we can define the backward-looking forward rate $R_j(t)$ at time t as the expected value of $R(T_{j-1}, T_j)$ conditioned to the extended T_j-forward measure. A T_j-forward measure is a pricing measure with respect to a risk-neutral measure $\mathbb{Q}^{T_j}$, which, rather

[2] The approximation is obtained taking the limit for the mesh of $\{\delta_1, ..., \delta_n\}$ for the daily compounded setting-in-arrears rate, going to zero in the daily-compounded setting-in-arrears rate formula.

than using the money market as numeraire, uses a bond with maturity T_j

$$R_j(t) = \mathbb{E}^{T_j}\left[R(T_{j-1}, T_j)\middle|\mathcal{F}_t\right] \tag{2.4}$$

where $\mathbb{E}^{T_j}$ is the expected value with respect to the risk-neutral probability measure $\mathbb{Q}^{T_j}$. In other terms, we can see $R_j(t)$ as the value of the fixed rate K in the swaplet paying $\tau_j[R(T_{j-1}, T_j) - K]$ at time T_j such that the swaplet has zero value at time t. Recall that a swaplet is an interest rate swap that has a single payment. From (2.3) and (2.4), and changing the measure from $\mathbb{Q}^{T_j}$ to $\mathbb{Q}$ - and therefore returning to use the bank account as a numeraire rather than the bond - we obtain

$$1 + \tau_j R_j(t) = \mathbb{E}^{T_j}\left[e^{\int_{T_{j-1}}^{T_j} r(u)\,du}\middle|\mathcal{F}_t\right] = \frac{1}{P(t, T_j)}\mathbb{E}\left[e^{-\int_t^{T_j} r(u)\,du}e^{\int_{T_{j-1}}^{T_j} r(u)\,du}\middle|\mathcal{F}_t\right] =$$

$$= \frac{1}{P(t, T_j)}\mathbb{E}\left[e^{-\int_t^{T_{j-1}} r(u)\,du}\middle|\mathcal{F}_t\right] = \frac{P(t, T_{j-1})}{P(t, T_j)}.$$

Hence,

$$R_j(t) = \frac{1}{\tau_j}\left[\frac{P(t, T_{j-1})}{P(t, T_j)} - 1\right]. \tag{2.5}$$

Notice that this is the classic, simply-compunded, forward-rate formula (see Definition 1.1.1) which, thanks to the extended definition of the bond price, is true for every t, even those after maturity T_j.

2.1.3 Forward-looking rates

By definition, the forward-looking spot rate is the market-implied prediction of the daily compounded setting-in arrears rate. In other words, it is the expected value conditioned to the extended T_j-forward measure of $R(T_{j-1}, T_j)$, taking into consideration all the information available in the previous period (that is the sigma-algebra $\mathcal{F}_{T_{j-1}}$)

$$F(T_{j-1}, T_j) = \mathbb{E}^{T_j}\left[R(T_{j-1}, T_j)\middle|\mathcal{F}_{T_{j-1}}\right].$$

The forward-looking forward rate $F_j(t)$ at time t is defined as as the expected value of $F(T_{j-1}, T_j)$ conditioned to the extended T_j-forward measure. In other terms, it is the value of the fixed rate K in the swaplet that pays $\tau_j[F(T_{j-1}, T_j) - K]$ at time T_j, such that the swaplet has zero value at time t, hence

$$F_j(t) = \mathbb{E}^{T_j}\left[F(T_{j-1}, T_j)\middle|\mathcal{F}_t\right]$$

Thus, by no arbitrage, for $t \leq T_{j-1}$:

$$F_j(t) = \mathbb{E}^{T_j}\left[\mathbb{E}^{T_j}\left[R(T_{j-1}, T_j)\middle|\mathcal{F}_{T_{j-1}}\right]\middle|\mathcal{F}_t\right] =$$

$$= \mathbb{E}^{T_j}\left[R(T_{j-1}, T_j)\middle|\mathcal{F}_t\right] = R_j(t)$$

On the contrary, for $t > T_{j-1}$, since $F(T_{j-1}, T_j)$ is known at T_{j-1},

$$F_j(t) = F(T_{j-1}, T_j)$$

hence the value is fixed and constant.

2.2. The general term structure equation

As anticipated in the previous section, now we will demonstrate that if we take the price of one particular bond as a benchmark, then it is possible to uniquely determine the prices of all other bonds in terms of the price of the benchmark.
To show this, we assume that the price of a T-bond has the form

$$P(t, T) = F(t, r(t), T)$$

where F is a smooth function of three real variables. At the time of maturity T, the bond is worth 1 dollar

$$F(T, r, T) = 1$$

for all r.[3] Let us build a portfolio with bonds of two different maturities T and S. From (2.1) and the Ito's formula, we get the following dynamics for the T-bond

$$dF^T = F^T \alpha_T dt + F^T \sigma_T d\bar{W} \tag{2.6}$$

where

$$\alpha_T = \frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T}{F^T} \tag{2.7}$$

[3]Notice that in the equation r denotes a real variable, but at the same time it is used for the stochastic process for the short rate. For a better understanding, the stochastic process should henceforth be denoted differently. However, with some abuse of notation, r will continue to be used for both.

$$\sigma_T = \frac{\sigma F_r^T}{F^T} \tag{2.8}$$

with subindices r and t denoting partial derivatives. The S-bond has analogous corresponding equations.
Denoting the relative portfolio by (u_s, u_T) we have the following value dynamics for our portfolio

$$dV = V\left[u_T\frac{dF^T}{F^T} + u_S\frac{dF^s}{F^S}\right].$$

Using (2.6) and the corresponding equation for the S-bond, we obtain

$$dV = V\left[u_T\alpha_T + u_S\alpha_S\right]dt + V\left[u_T\sigma_T + u_S\sigma_S\right]d\bar{W}. \tag{2.9}$$

Since the sum of portfolio weights must be 1 and assuming that the portfolio is well diversified, we have that

$$\begin{cases} u_T + u_S = 1 \\ u_T\sigma_T + u_S\sigma_S = 0. \end{cases} \tag{2.10}$$

Given this, the $d\bar{W}$-term of (2.9) disappears and the value reduces to

$$dV = V\left[u_T\alpha_T + u_S\alpha_S\right]dt. \tag{2.11}$$

The system (2.10) has solutions

$$u_T = -\tfrac{\sigma_S}{\sigma_T - \sigma_S}$$

$$u_S = \frac{\sigma_T}{\sigma_T - \sigma_S}.$$

Substituting this into (2.11), we obtain

$$dV = V\left[\frac{\alpha_S\sigma_T - \alpha_T\sigma_S}{\sigma_T - \sigma_S}\right]dt.$$

Given the no-arbitrage assumption, if the portfolio is self-financed, its rate of return must be equal to the short rate of interest (from [7]). Hence, the following condition must hold

$$\frac{\alpha_S\sigma_T - \alpha_T\sigma_S}{\sigma_T - \sigma_S} = r(t) \tag{2.12}$$

for all t. Written differently, this is

$$\frac{\alpha_S(t) - r(t)}{\sigma_S(t)} = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)}.$$

Notice that the left-hand side stochastic process is independent of the choice of T, and the right-hand side process is independent of S. Thus, we can define an universal process λ, such that

$$\frac{\alpha_T(t) - r(t)}{\sigma_T(t)} = \lambda(t) \tag{2.13}$$

for all t and T. $\alpha_T(t)$ is the local rate of return on the T-bond, r is the rate of return on the risk-free asset and $\sigma_T(t)$ is the local volatility of the T-bond. The term $\alpha_T(t) - r(t)$ is the risk premium of the T-bond and measures the excess rate of return for the risky T-bond over the riskless rate of return which is required by the market to avoid arbitrage possibilities. Therefore, the process λ represents the risk premium per unit of volatility, that is the market price of risk. By substituting previous formulas (2.7) and (2.8) into (2.13), we can obtain the **term structure equation** for F^T.

Definition 2.2.1. *The **general term structure equation** for F^T is*

$$\begin{cases} F_t^T + \{\mu - \lambda\sigma\}F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0 \\ F^T(T, r) = 1. \end{cases} \tag{2.14}$$

Using the Feyman-Kac theorem (see [7]), we can obtain the following explicit formula for $F(t, r, T)$

$$F(t, r, T) = \mathbb{E}_{t,r}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)\,ds}\right]. \tag{2.15}$$

The risk-neutral probability measure $\mathbb{Q}$ and the subscrits t and r inside equation (2.15) denote that the expectation shall be taken given the following dynamics of the short rate

$$dr(s) = \{\mu - \lambda\sigma\}ds + \sigma dW(s)$$

$$r(t) = r$$

where W is the Brownian motion under the risk neutral measure $\mathbb{Q}$.
The term structure will be determined as soon as the drift term μ, the diffusion term (volatility) σ and the market price of risk λ will be specified. Suppose for a moment that σ is given a priori. Then, it is irrelevant exactly how μ and λ are specified *per se*. In fact, the object that, apart from σ, really determines the term structure is the term $\mu - \lambda\sigma$. From (2.12), we notice that this is exactly the drift term of the short rate of interest under the martingale measure $\mathbb{Q}$. Hence, instead of specifying μ and λ under the objective probability measure $\mathbb{P}$, we will specify the dynamics of the short rate

r directly under the risk-neutral measure $\mathbb{Q}$. This procedure is known as martingale modeling and the typical assumption will thus be that r under $\mathbb{Q}$ has dynamics given by

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t) \tag{2.16}$$

where μ and σ are given functions and W is a Brownian motion under the measure $\mathbb{Q}$.[4] In the literature, there are a large number of proposals on how to specify the $\mathbb{Q}$-dynamics for r. Some of the most popular models are:

1. Vasicek:
$$dr = (b - ar)dt + \sigma dW$$
with $a > 0$;

2. Cox-Ingersoll-Ross (CIR):
$$dr = a(b - r)dt + \sigma\sqrt{r}dW;$$

3. Dothan:
$$dr = ardt + \sigma rdW;$$

4. Black-Derman-Toy:
$$dr = \Theta(t)rdt + \sigma(t)rdW;$$

5. Ho-Lee:
$$dr = \Theta(t)dt + \sigma dW;$$

6. Hull-White (extended-Vasicek):
$$dr = (\Theta(t) - a(t)r)dt + \sigma(t)dW$$
with $a(t) > 0$;[5]

Remark 1 (***models' parameters estimation***)**.** The main concern about these specifications is how to estimate the various parameters in the models above. In fact, all processes follow $\mathbb{Q}$-dynamics, hence all parameters hold under the martingale measure

[4] Notice that from now on the letter μ will always denote the drift term of the short rate under the martingale measure $\mathbb{Q}$.

[5] Notice that there exists another short rate dynamics specification derived by J. Hull and A. White, which is an extension of the Cox-Ingersoll-Ross model, that however will not be analysed in this dissertation. Specifically, under this model, r has $\mathbb{Q}$-dynamics $dr = (\Theta(t) - a(t)r)dt + \sigma(t)\sqrt{r}dW$, $a(t) > 0$. For more details, see [19].

$\mathbb{Q}$. However, when we make observations in the real world, we are observing r under the objective probability $\mathbb{P}$ and not $\mathbb{Q}$. This is why standard statistical procedures cannot be applied. Nonetheless, it is possible to show that the diffusion term is the same under $\mathbb{P}$ and under $\mathbb{Q}$ (from [7]). Specifically, parameters can be estimated by inverting the yield curve. Suppose that we want to estimate a parameter vector α that specifies both μ and σ. To do that, first we need to solve the term structure equation (2.14) and compute the theoretical term structure $P(t,T,\alpha) = F^T(t,r,\alpha)$. Afterwards, we have to collect prices from the market and obtain the empirical structure $\{P^*(0,T), T \geq 0\}$. At this point, we can estimate the parameter α^* by fitting the theoretical curve $\{P(0,T,\alpha), T \geq 0\}$ into the empirical curve $\{P^*(0,T), T \geq 0\}$. For a more detailed explanation, see [7].
This process involves some PDEs, which are sometimes difficult to resolve. Nevertheless, there exists an easy way to overcome the problem, which is by using **affine term structures**.

Definition 2.2.2. *The model is said to possess an* ***affine term structure (ATS)*** *if the term structure* $\{P(t,T); 0 \leq t \leq T, T > 0\}$ *has the form* $P(t,T) = F(t,r(t),T)$*, with*

$$F(t,r,T) = e^{A(t,T)-B(t,T)r} \tag{2.17}$$

where A *and* B *are deterministic functions.*

Now, suppose that we have a family of T-bonds, whose price is $P(t,T) = F(t,r(t),T)$ and that are worth 1 dollar at maturity $F(T,r,T) = 1$. Assume the usual $\mathbb{Q}$-dynamics for the short rate (2.16) and that our model possesses an ATS. Using (2.17), we can easily compute the partial derivatives of F and substitute them into the term structure equation (2.14), and we obtain

$$A_t(t,T) - \{1 + B_t(t,T)\}r - \mu(t,r)B(t,T) + \frac{1}{2}\sigma^2(t,r)B^2(t,T) = 0. \tag{2.18}$$

The boundary value $F(T,r,T) = 1$ implies

$$\begin{cases} A(T,T) = 0 \\ B(T,T) = 0. \end{cases}$$

Equation (2.18) gives us the relations which must hold between A, B, μ and σ in order for the ATS to exist. Notice that for certain choices of μ and σ there may or may not exist functions A and B that satisfies (2.18). Henceforth, we must give conditions on μ and σ that fulfill this requirement. We observe that if μ and σ are both affine (i.e. linear plus a constant) functions of r, with possibly time-dependent coefficients, (2.18) becomes a separable differential equation for the unknown functions A and B. Particularly, suppose that μ and σ have the form

$$\begin{cases} \mu(t,r) = \alpha(t)r + \beta(t) \\ \sigma(t,r) = \sqrt{\gamma(t)r + \delta(t)}. \end{cases} \tag{2.19}$$

Then, after collecting terms, (2.18) becomes

$$A_t(t,T) - \beta(t)B(t,T) + \frac{1}{2}\delta(t)B^2(t,T) - \{1 + B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T)\}r = 0. \tag{2.20}$$

This equation holds for all t, T and r, so let us consider it for a fixed choice of T and t. Since the equation holds for all values of r, its coefficient must be equal to zero. Thus we have the equation

$$B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = -1. \tag{2.21}$$

Also, given (2.21), equation (2.20) becomes

$$A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T). \tag{2.22}$$

Hence, the model admits an ATS of the form (2.17) when A and B satisfy both (2.21) and (2.22) taking into account the boundary conditions, that is

$$\begin{cases} B_t(t,T) + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = -1 \\ B(T,T) = 0 \end{cases} \tag{2.23}$$

and

$$\begin{cases} A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T) \\ A(T,T) = 0. \end{cases} \tag{2.24}$$

In the next section, we will derive the term structure equation for the Hull-White model's specification of the short rate, which seems to be the standard choice for interest rate modeling in current markets. However, since the Hull-White model constitutes its extension, we will first present the Vasicek model, for pedagogical purposes.

2.3. The Hull-White Model

This section will present the **Hull-White model (extended-Vasicek)** as done in [7]. The Hull-White model is a short rate model, therefore it models instantaneous short rates of interest. Since the Hull-White model constitutes its extension, first we will introduce the **Vasicek model** as in [7], for a better understanding.

2.3.1 The Vasicek Model

Suppose that the model possesses an affine term structure of the same form as the one presented in the previous section. Recall that, under the Vasicek model, the $\mathbb{Q}$-dynamics of the short rate r are

$$dr(t) = (b - ar(t))dt + \sigma dW$$

with $a > 0$. Given this specification, we can conclude that in (2.19), $\alpha(t) = -a$, $\beta(t) = b$, $\gamma(t) = 0$ and $\delta(t) = \sigma$. The two equations systems (2.23) and (2.24) then become

$$\begin{cases} B_t(t,T) - aB(t,T) = -1 \\ B(T,T) = 0 \end{cases} \tag{2.25}$$

and

$$\begin{cases} A_t(t,T) = bB(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) \\ A(T,T) = 0. \end{cases} \tag{2.26}$$

Equation (2.25) is a simple ODE in the t-variable for each fixed T and can be solved as

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right].$$

Contrarily, integrating (2.26) we obtain

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T B^2(s,T)\,ds - b\int_t^T B(s,T)\,ds.$$

Substituting the expression for B above, we obtain that, in the Vasicek model, the bond prices are given by the usual ATS formula (2.17), that is

$$P(t,T) = e^{A(t,T) - B(t,T)r(t)}$$

where

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]$$

and

$$A(t,T) = \frac{(B(t,T) - T + t)(ab - \frac{1}{2}\sigma^2)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a}.$$

2.3.2 The Hull-White Model (extended-Vasicek)

J. Hull and A. White (see [19]) extended the Vasicek model by adding a time-dependent drift $\Theta(t)$ to the process for r and allowing both the coefficient a and the volatility factor σ to be functions of the time t. This leads to the following $\mathbb{Q}$-dynamics of the short rate r

$$dr = (\Theta(t) - a(t)r)dt + \sigma(t)dW$$

with $a(t) > 0$. Notice that, by using an appropriate time-dependent function $\Theta(t)$, the Hull-White model is able to perfectly fit the initially observed term structure of interest rates. This feature will be crucial for the determination of the term structure equation. In this section, we will present a simplified version of the Hull-White extension, where the coefficient a and the volatility σ are constants while Θ is a deterministic function of time (see [7]). The $\mathbb{Q}$-dynamics of the short rate becomes

$$dr(t) = (\Theta(t) - ar(t))dt + \sigma dW(t) \tag{2.27}$$

where a and σ are typically chosen to obtain a nice volatility structure, whereas Θ is chosen in order to fit the theoretical bond prices $\{P(0,T), T > 0\}$ to the observed curve $\{P^*(0,T), T > 0\}$.

Suppose that we have an affine term structure so bond prices are given by equation (2.17). Given the short rate dynamics, we can conclude that in (2.19) $a(t) = -a$, $\beta(t) = \Theta(t)$, $\gamma(t) = 0$ and $\delta(t) = \sigma$. Hence, the two equation systems (2.23) and (2.24) become

$$\begin{cases} B_t(t,T) = aB(t,T) - 1 \\ B(T,T) = 0 \end{cases} \tag{2.28}$$

and

$$\begin{cases} A_t(t,T) = \Theta(t)B(t,T) - \frac{1}{2}\sigma^2 B^2(t,T) \\ A(T,T) = 0. \end{cases} \tag{2.29}$$

Equation (2.28) is a simple ODE in the t-variable for each fixed T and can be solved as

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right]. \tag{2.30}$$

On the contrary, integrating (2.29), we obtain

$$A(t,T) = \int_t^T \left[\frac{1}{2}\sigma^2 B^2(s,T) - \Theta(s)B(s,T)\right] ds. \tag{2.31}$$

Now, we want to fit the theoretical prices above to the observed prices and it is convenient to do so by using the forward rates. Since there is a one-to-one correspondence between forward rates and bond prices[6], we may as well fit the theoretical forward curve $\{f(0,T), T > 0\}$ to the observed curve $\{f^*(0,T), T > 0\}$, where f stands for the forward rate, with $f(t,T) = -\frac{d \ln p(t,T)}{dT}$ and $f^*(t,T) = -\frac{d \ln p^*(t,T)}{dT}$.[7] Since the model is assumed to possess an affine term structure, the forward rate at time $t = 0$ is given by (from [7])

$$f(0,T) = B_T(0,T)r(0) - A_T(0,T)$$

which, after inserting the expressions for $A(t,T)$ and $B(t,T)$ becomes

$$f(0,T) = e^{-aT}r(0) + \int_0^T e^{-a(T-s)}\Theta(s)\, ds - \frac{\sigma^2}{2a^2}(1-e^{-aT})^2.$$

Given an observed forward rate structure f^*, our problem is to find a function Θ that solves the equation

$$f^*(0,T) = e^{-aT}r(0) + \int_0^T e^{-a(T-s)}\Theta(s)\, ds - \frac{\sigma^2}{2a^2}(1-e^{-aT})^2 \tag{2.32}$$

for every $T > 0$. One way to solve it is by writing

$$f^*(0,T) = x(T) - g(T)$$

where the two functions x and g are defined as

$$\begin{cases} \dot{x} = -ax(t) + \Theta(t) \\ x(0) = r(0) \end{cases}$$

and

$$g(t) = \frac{\sigma^2}{2a^2}(1-e^{-at})^2 = \frac{\sigma^2}{2}B^2(0,t).$$

The solution to equation (2.32) then is

[6] For $t \leq s \leq T$ we have that $p(t,T) = p(t,s)e^{-\int_s^T f(t,u)\,du}$ and in particular $p(t,T) = e^{-\int_t^T f(t,s)\,ds}$, where f stands for the forward rate (see [7]).

[7] Notice that this is the usual definition of an instantaneous forward rate with maturity T contracted at T. For a more specific definition and derivation, see [7].

$$\Theta(T) = \dot{x}(T) + ax(T) = f_T^*(0,T) + \dot{g}(T) + ax(T) = \\ = f_T^*(0,T) + \dot{g}(T) + a\left[f^*(0,T) + g(T)\right]. \tag{2.33}$$

By choosing Θ according to (2.33), we have determined our martingale measure for a fixed choice of a and σ. Substituting this expression for Θ into (2.31), performing the integration and inserting the result, as well as (2.30), into (2.17), we obtain the Hull-White term structure, that is

$$P(t,T) = \frac{P^*(0,T)}{P^*(0,t)} e^{B(t,T)f^*(0,t) - \frac{\sigma^2}{4a}B^2(t,T)(1-e^{-2at}) - B(t,T)r(t)}$$

where B is given by (2.30).

2.3.3 The pricing formula for a European call option

As reported in [7], both the Hull-White model and the Vasicek model have the same pricing formula for a European call option.[8] Particularly, assume the existence of a European call option on an S-bond with maturity T and strike price K, where $T < S$. Its pricing formula is

$$c(t,T,K,S) = P(t,S)N(d) - P(t,T)KN(d - \sigma_p)$$

with

$$d = \frac{1}{\sigma_p} \ln\left[\frac{P(t,S)}{P(t,T)K}\right] + \frac{1}{2}\sigma_p$$

$$\sigma_p = \frac{1}{a}\left[1 - e^{-a(S-T)}\right]\sqrt{\frac{\sigma^2}{2a}\left[1 - e^{-2a(T-t)}\right]}$$

and where $P(t,T)$ and $P(t,S)$ do not have to be computed since they can directly be observed on the market.[9]

2.4. The Forward Market Model

The **Forward Market Model (FMM)**, developed by A. Lyashenko and F. Mercurio (see [28]), models both backward-looking daily-compounded in arrears and forward-looking term rates, using a single stochastic process. It constitutes a natural extension of the classic single-curve LIBOR Market Model (LMM), which models LIBOR forward

[8] Recall that an European call option is a call option that can be exercised only at maturity.
[9] For complete derivation of the formula, see [7].

rates - thus exclusively forward-looking rates. As the LMM, the FMM models a set of forward rates, and not short rates. Recall that a **forward rate** is the yield of a security that will not be traded until a predetermined date in the future. Nonetheless, the FMM constitutes a more complete model than LMM because, while preserving the dynamics of the forward-looking (LIBOR-like) rates, it also provides a model for interests that use daily rates published during the relevant application periods (and therefore not exclusively over a period of time prior to the their start).

2.4.1 The generalised FMM

As demonstrated in sections 2.1.2 and 2.1.3, we can conclude that, for each $j = 1, ..., M$, the backward-looking forward rate $R_j(t)$ and the forward-looking forward rate $F_j(t)$ can be expressed by a single rate. We will use the same notation of [28], and denote these rates as $R_j(t)$. In fact, both are described by a single common value, $R_j(t)$, when $t \leq T_{j-1}$. At time $t = T_{j-1}$, the forward-looking forward rate fixes at $R_j(T_{j-1}) = F(T_{j-1}, T_j)$ and stops evolving. On the contrary, the backward-looking forward rate continues its journey until it fixes at time T_j: for $t \geq T_j$, $R_j(t) = R_j(T_j)$.
Because of its own definition (2.4), the forward rate $R_j(t)$ is a martingale under the corresponding T_j-forward measure. The $\mathbb{Q}^{T_j}$-dynamics of $R_j(t)$ can be defined for every t, including $t \geq T_j$. As in [28], we assume that $R_j(t)$ has the following $\mathbb{Q}^{T_j}$-dynamics:

$$dR_j(t) = \sigma_j(t)\mathbb{1}_{\{t \leq T_j\}} dW_j(t)$$

where, for each $j = 1, ..., M$, $\sigma_j(t)$ is an adapted process representing the volatility of the forward rate and $W_j(t)$ is a standard Brownian motion such that $dW_i(t)dW_j(t) = \rho_{i,j}dt$, with $\rho_{i,j}$ being the correlation between the two processes. The indicator function $\mathbb{1}_{\{t \leq T_j\}}$ is introduced to ensure that the process is well defined and constant for times greater than (or equal to) T_j.
In order to properly define the forward rate dynamics, it is crucial to model the behaviour of its volatility in the accrual period $[T_{j-1}, T_j]$. In [28], the authors choose a differentiable function g_j such that: $g_j(t) = 1$ for $t \leq T_{j-1}$, $g_j(t)$ is monotonically decreasing in $[T_{j-1}, T_j]$ and $g_j(t) = 0$ for $t \geq T_j$. An example for the function $g_j(t)$, assuming a linear decay, is

$$g_j(t) = \min\left[\frac{(T_j - t)^+}{(T_j - T_{j-1})}, 1\right].$$

The dynamics of $R_j(t)$ then becomes:

$$dR_j(t) = \sigma_j(t) g_j(t) dW_j(t). \tag{2.34}$$

Equation (2.34) defines the dynamics of each forward rate $R_j(t)$ under the corresponding T_j-forward measure. By deriving the dynamics of each forward under a common probability measure, we can define a market model where all forward rates are modeled jointly, for $j = 1, ..., M$. To do this, we apply the change-of-numeraire formula relating the drifts of a given process under two measures with known numeraires. Specifically, we know the dynamics of $R_j(t)$ under the T_j-forward measure and we want to derive its dynamics under the measure $\mathbb{Q}^N$, that is associated to the generic numeraire $N(t)$. To compute the drift, we use the same solution adopted in [28][10]: assuming continuous dynamics, the drift of R_j under $\mathbb{Q}^N$, as a function of time t is

$$\textbf{Drift}(R_j; \mathbb{Q}^N)(t) = \frac{dR_j(t) d \ln \left[\frac{N(t)}{P(t,T_j)} \right]}{dt}. \tag{2.35}$$

Let us consider a specific case for the value of the generic numeraire $N(t)$, that is $N(t) = B(t)$. If this is true, the probability measure $\mathbb{Q}^N$ is the risk-neutral probability measure $\mathbb{Q}$.

Equation (2.35) becomes

$$\textbf{Drift}(R_j; \mathbb{Q})(t) = \frac{dR_j(t) d \ln \left[\frac{B(t)}{P(t,T_j)} \right]}{dt}.$$

Let us focus first on the logarithmic part of the drift equation. Using the definition of extended bond prices (see (2.2)), we can write

$$\ln \left[\frac{B(t)}{P(t,T_j)} \right] = \ln \left[\frac{P(t,0)}{P(t,T_j)} \right]$$

if we take $T = 0$.[11]

Expanding the formula by filling all the intermediate points between 0 and t, we can write

$$\ln \left[\frac{P(t,0)}{P(t,T_j)} \right] = \ln \left[\prod_{i=1}^{j} \left(\frac{P(t,T_{i-1})}{P(t,T_i)} \right) \right].$$

[10] Specifically, Lyashenko and Mercurio use Brigo and Mercurio's change-of-numeraire formula (2006).

[11] Notice that $P(t,0) = \frac{B(t)}{B(0)}$, where $B(0) = 1$ by definition.

Using (2.5),

$$\ln\left[\prod_{i=1}^{j}\left(\frac{P(t,T_{i-1})}{P(t,T_i)}\right)\right]=\ln\left[\prod_{i=1}^{j}(1+\tau_i R_i(t))\right]=\sum_{i=1}^{j}\ln\left[1+\tau_i R_i(t)\right].$$

Therefore,

$$\mathbf{Drift}(R_j;\mathbb{Q})(t)=\frac{dR_j(t)d\sum_{i=1}^{j}\ln\left[1+\tau_i R_i(t)\right]}{dt}=$$

$$=\sum_{i=1}^{j}\left[\frac{dR_j(t)d\ln\left[1+\tau_i R_i(t)\right]}{dt}\right]=$$

$$=\sum_{i=1}^{j}\left[\frac{\tau_i}{1+\tau_i R_i(t)}\frac{dR_j(t)dR_i(t)}{dt}\right]=$$

$$=\sigma_j(t)g_j(t)\sum_{i=1}^{j}\left[\rho_{i,j}\frac{\tau_i\sigma_i(t)g_i(t)}{1+\tau_i R_i(t)}\right] \tag{2.36}$$

where the last equality is obtained substituting (2.34) and $dW_i(t)dW_j(t)=\rho_{i,j}dt$. The $\mathbb{Q}$-dynamics of R_j then becomes:

$$dR_j(t)=\sigma_j(t)g_j(t)\sum_{i=1}^{j}\left[\rho_{i,j}\frac{\tau_i\sigma_i(t)g_i(t)}{1+\tau_i R_i(t)}\right]+\sigma_j(t)g_j(t)dW_j^{\mathbb{Q}}(t)$$

where $W_j^{\mathbb{Q}}(t)$ is a $\mathbb{Q}$-Brownian motion.

A. Lyashenko and F. Mercurio consider also two additional cases for the value of the generic numeraire $N(t)$ that will not be presented here. Specifically, they study the drift nature when $N(t)=B_d(t)$, where $B_d(t)$ is the time-t value of a particular discrete bank account (hence, the probability measure $\mathbb{Q}^N$ becomes the classic spot-LIBOR probability measure $\mathbb{Q}^d$, which corresponds to using the discretely-compounded money market account as numeraire within the LIBOR market model). Moreover, they analyse the case in which $N(t)=P(t,T_k)$, where k is a generic scalar (hence, the probability measure $\mathbb{Q}^N$ becomes the T_k-forward measure, that is a pricing measure that uses a bond with maturity T_k as a numeraire). For the complete mathematical derivation of the $\mathbb{Q}$-dynamics in these two cases, see [28].

2.4.2 Model characteristics

As already stressed, the FMM is an extension of the classic single-curve LMM, that models jointly the dynamics of both forward-looking forward rates $F_j(t)$ and backward-looking forward rates $R_j(t)$, since $F_j(t) = R_j(t)$ for all times t before the expiry time T_{j-1} of $F_j(t)$. Additionally, the FMM has other properties:

- The generalised forward rates $R_j(t)$ are more complete than the than forward-looking LIBOR rates in terms of spanning the periods defined by the time grid $T_0, ..., T_M$. In fact, for any index $j = 1, ..., M$ and for any time t, we can express the price of a zero-coupon bond with maturity T_j in terms of the bank account $B(t)$ and forward rates $R_i(t)$ as follows

$$P(t, T_j) = B(t) \prod_{i=1}^{j} \frac{1}{1 + \tau_i R_i(t)}$$

  with the equality holding for all t, including $t > T_j$. This means that

$$\frac{dP(t, T_j)}{P(t, T_j)} = r(t)dt - \sum_{i=1}^{j} \frac{\tau_i}{1 + \tau_i R_i(t)} \sigma_i(t) g_i(t) dW_i^{\mathbb{Q}}(t)$$

  so the volatility of all bonds $P(t, T_j)$ and their instantaneous covariance structure are known and a function of rates $R_i(t)$.
  An analogous representation cannot be found under the LMM.

- Under the FMM, it is possible to price future contracts more precisely than under the LMM. In fact, generally the time-t future price of a contract that pays out H_T at time $T > t$ can be computed as done by Hunt and Kennedy (cited in [28]):

$$f(t) = \mathbb{E}\left[H_T | \mathcal{F}_t\right].$$

  In the classic LMM model, $\mathbb{Q}$-dynamics are not directly available, hence $\mathbb{Q}$ is typically approximated with $\mathbb{Q}^d$ to explicitely compute the future price $f(t)$:

$$f(t) \approx \mathbb{E}^d\left[H_T | \mathcal{F}_t\right]$$

  where $\mathbb{E}^d$ denotes expectation under $\mathbb{Q}^d$. Such an approximation is no longer needed in the FMM, as the forward rate dynamics are perfectly known under $\mathbb{Q}$, hence the first formula can be used without issues.

- The FMM also provides an easier extension to a cross-currency interest-rate model than the LMM. In a two-currency economy, where domestic and foreign rates are driven by the corresponding FMMs, the dynamics of the foreign FMM under the domestic measure $\mathbb{Q}$ and the dynamics of the domestic FMM under the foreign money-market risk-neutral measure $\mathbb{Q}^f$ can be easily derived. Contrarily, this is not possible under the classic LMM. For more details on the derivation of these dynamics, see [28].

Chapter 3

Derivative pricing using the Hull-White model

As previously stressed in section 1.2.4, LIBOR transition and the introduction of new RFRs rose the issue of how to price new kinds of derivatives, that were born after the adoption of the new benchmark rates. An example of these new instruments is given by options on three months SOFR futures, a type of derivative launched by CME on January 2020. These derivative instruments are gaining increasing popularity in the last weeks: in July 2022, the daily traded volume has been on average around 600 millions, with peaks even reaching 800 millions.[1] Following the successful launches of options on three months SOFR futures, CME Group launched also options on one month SOFR futures in May 2020. Indeed, recall that a SOFR future contract is a derivative whose underlying is an interest-bearing instrument referencing SOFR and that in financial markets there exists two types of future contracts depending on their maturity (three months or one month). Also, remember that the option gives the right, but not the obligation, to exercise sometime before maturity the SOFR future contract, in order to protect the holder from unfavourable variations of its price.

At the same time, the ICE has launched options on three months SONIA index futures in December 2020. These are derivatives very similar to options on SOFR future, that deliver into the nearest three month SONIA index future contract, and that are also experiencing a considerable diffusion in financial markets.[2]

The increasing spread and market popularity of these instruments attest why it is crucial to understand how they work and can be priced.

[1] See URL: https://www.cmegroup.com/markets/interest-rates/stirs/three-month-sofr.volume.options.html#optionProductId=8849

[2] See URL: https://www.theice.com/products/79341513/Options-on-Three-Month-SONIA-Index-Future

In this chapter, we will focus on the evaluation of the general category of **options on RFRs future**, with the future having a generic RFR as an underlying, considering both three months and one month future contracts. Notice that we will assume that the RFRs dynamics are described by the Hull-White model, since it is currently the most popular for interest rate modeling.

3.1. Options on RFRs futures

Options on RFRs futures are particular derivative instruments composed of two parts. The first one is a future contract on a RFR, that is a derivative according to which two counterparties agree to exchange a RFR interest-bearing instrument at a pre-specified price some time in the future. Notice that, in current financial markets, there exist two types of future contracts depending on their maturity, namely 1-month (1M) and 3-months (3M) future contracts. As explained in [30], [28] and [15], 1M and 3M future contracts are characterized by different settlement specifications:

- a 3M future contract settles at T_j at the backward-looking rate $R(T_{j-1}, T_j)$ (representing the geometric average of overnight rates over the period $[T_{j-1}, T_j]$, with $\tau_j = T_j - T_{j-1}$ being equal to three months);
- a 1M future contract settles at the rate representing the arithmetic average of overnight rates over the period $[T_{j-1}, T_j]$, with $\tau_j = T_j - T_{j-1}$ being equal to one month.

Let us suppose that the interest rate underlying the future contract has the Hull-White $\mathbb{Q}$-dynamics specified in chapter 2, with parameters a and σ constant and positive, that is

$$dr(t) = (\Theta(t) - ar(t))dt + \sigma dW(t) \tag{3.1}$$

where $W(t)$ indicates the Brownian motion with respect to the risk neutral measure. The reason behind this assumption is the popularity of the Hull-White model for interest rate modeling in current financial markets.
The second derivative composing the option on RFR future is an option contract. Specifically, the future contract constitutes the underlying instrument of the option, which gives the holder the right, but not the obligation, to buy or sell it at a strike price on or before the option's expiration date.
Since there exists two types of future contracts, we can imagine two types of options that have each of them as an underlying instrument, namely an option on a 3M future and an option on a 1M future.

In the next sections, we will try to obtain a mathematical formula that describes the price of options on RFRs futures. In particular, first we will start by evaluating the future contracts (both 3M and 1M) and then we will use the results to price the options (on both 3M and 1M futures).

3.2. Pricing of the future contract

Let us first consider a **3M future contract**, which we know settles at T_j at the backward-looking rate $R(T_{j-1}, T_j)$. Denoting as $f_j^{3M}(t)$ the 3M future rate at time t, it holds that

$$\begin{aligned} f_j^{3M}(t) &= \mathbb{E}^{\mathbb{Q}}\left[R(T_{j-1}, T_j) \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\tau_j}(e^{\int_{T_{j-1}}^{T_j} r(u)\,du} - 1) \middle| \mathcal{F}_t\right] \\ &= \frac{1}{\tau_j}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{\int_{T_{j-1}}^{T_j} r(u)\,du} \middle| \mathcal{F}_t\right] - 1\right] \end{aligned} \tag{3.2}$$

where the second equality derives from equation (2.3). Assume that $r(t)$ satisfies the Hull-White stochastic differential equation (3.1).
From equation (3.2), we notice that the future price determination can be traced back to the computation of the expected value $\mathbb{E}^{\mathbb{Q}}\left[e^{\int_{T_{j-1}}^{T_j} r(u)\,du} \middle| \mathcal{F}_t\right]$. The starting point for the resolution of this expected value is the computation of the integral $\int_{T_{j-1}}^{T_j} r(u)\,du$. Integrating equation (3.1), we obtain

$$r(u) = r(t)e^{-a(u-t)} + \int_t^u e^{-a(u-s)}\Theta(s)\,ds + \sigma \int_t^u e^{-a(u-s)}\,dW(s). \tag{3.3}$$

For $t \leq T_{j-1}$, by substituting (3.3) into the integral we want to compute, we obtain

$$\begin{aligned} \int_{T_{j-1}}^{T_j} r(u)\,du = \int_{T_{j-1}}^{T_j} r(t)e^{-a(u-t)}\,du + \int_{T_{j-1}}^{T_j} \left(\int_t^u e^{-a(u-s)}\Theta(s)\,ds\right) du \\ + \int_{T_{j-1}}^{T_j} \sigma \int_t^u e^{-a(u-s)}\,dW(s)\,du. \end{aligned} \tag{3.4}$$

Using Fubini's theorem (see Theorem A.48 of [7]), the previous equation becomes

$$\int_{T_{j-1}}^{T_j} r(u)\,du = \int_{T_{j-1}}^{T_j} r(t)e^{-a(u-t)}\,du + \int_t^{T_{j-1}} \left(\int_{T_{j-1}}^{T_j} e^{-a(u-s)}\Theta(s)\,du \right) ds$$

$$+ \int_{T_{j-1}}^{T_j} \left(\int_s^{T_j} e^{-a(u-s)}\Theta(s)\,du \right) ds + \sigma \int_t^{T_{j-1}} \left(\int_{T_{j-1}}^{T_j} e^{-a(u-s)}\,du \right) dW(s)$$

$$+\sigma \int_{T_{j-1}}^{T_j} \left(\int_s^{T_j} e^{-a(u-s)}\,du \right) dW(s).$$

Solving the integrals

$$\int_{T_{j-1}}^{T_j} r(u)\,du = \left[-\frac{r(t)}{a} e^{-a(u-t)} \right]_{u=T_{j-1}}^{u=T_j} + \int_t^{T_{j-1}} \left[-\frac{1}{a} e^{-a(u-s)}\Theta(s) \right]_{u=T_{j-1}}^{u=T_j} ds$$

$$+ \int_{T_{j-1}}^{T_j} \left[-\frac{1}{a} e^{-a(u-s)}\Theta(s) \right]_{u=s}^{u=T_j} ds - \frac{\sigma}{a} \int_t^{T_{j-1}} \left[e^{-a(u-s)} \right]_{u=T_{j-1}}^{u=T_j} dW(s)$$

$$-\frac{\sigma}{a} \int_{T_{j-1}}^{T_j} \left[e^{-a(u-s)} \right]_{u=s}^{u=T_j} dW(s) =$$

$$= \frac{r(t)}{a} e^{-a(T_{j-1}-t)} - \frac{r(t)}{a} e^{-a(T_j-t)} + \frac{1}{a} \int_t^{T_{j-1}} (e^{-a(T_{j-1}-s)} - e^{-a(T_j-s)})\Theta(s)\,ds$$

$$+\frac{1}{a} \int_{T_{j-1}}^{T_j} (1 - e^{-a(T_j-s)})\Theta(s)\,ds - \frac{\sigma}{a} \int_t^{T_{j-1}} (e^{-a(T_j-s)} - e^{-a(T_{j-1}-s)})\,dW(s)$$

$$-\frac{\sigma}{a} \int_{T_{j-1}}^{T_j} (e^{-a(T_j-s)} - 1)\,dW(s). \tag{3.5}$$

For notation simplicity, let us denote

$$\eta_{t,j} = \frac{1}{a} \int_t^{T_{j-1}} (e^{-a(T_{j-1}-s)} - e^{-a(T_j-s)})\Theta(s)\,ds + \frac{1}{a} \int_{T_{j-1}}^{T_j} (1 - e^{-a(T_j-s)})\Theta(s)\,ds. \tag{3.6}$$

We can now notice that $\int_{T_{j-1}}^{T_j} r(u)\,du$, conditioned to $\mathcal{F}_t$, is distributed with respect to the risk neutral measure as a normal. We can compute the mean and variance using their general definitions as

$$\mu_{t,j} = \mathbb{E}^{\mathbb{Q}} \left[\int_{T_{j-1}}^{T_j} r(u)\,du \middle| \mathcal{F}_t \right] = \frac{r(t)}{a} (e^{-a(T_{j-1}-t)} - e^{-a(T_j-t)}) + \eta_{t,j} \tag{3.7}$$

and

$$\Sigma_{t,j}^2 = Var^{\mathbb{Q}} \left[\int_{T_{j-1}}^{T_j} r(u)\,du \middle| \mathcal{F}_t \right] =$$

$$= \frac{\sigma^2}{a^2}\mathbb{E}^{\mathbb{Q}}\left[\left(\int_t^{T_{j-1}}(e^{-a(T_j-s)}-e^{-a(T_{j-1}-s)})\,dW(s)+\int_{T_{j-1}}^{T_j}(e^{-a(T_j-s)}-1)\,dW(s)\right)^2\middle|\mathcal{F}_t\right]=$$

$$= \frac{\sigma^2}{a^2}\mathbb{E}^{\mathbb{Q}}\left[\left(\int_t^{T_{j-1}}(e^{-a(T_j-s)}-e^{-a(T_{j-1}-s)})\,dW(s)\right)^2+\left(\int_{T_{j-1}}^{T_j}(e^{-a(T_j-s)}-1)\,dW(s)\right)^2\right.$$

$$\left.+2\left(\int_t^{T_{j-1}}(e^{-a(T_j-s)}-e^{-a(T_{j-1}-s)})\,dW(s)\right)\cdot\left(\int_{T_{j-1}}^{T_j}(e^{-a(T_j-s)}-1)\,dW(s)\right)\middle|\mathcal{F}_t\right].$$

Using the expected value linearity property and the law of iterated expectations, we have

$$\Sigma_{t,j}^2 = \frac{\sigma^2}{a^2}\left[\mathbb{E}^{\mathbb{Q}}\left[\left(\int_t^{T_{j-1}}(e^{-a(T_j-s)}-e^{-a(T_{j-1}-s)})\,dW(s)\right)^2\middle|\mathcal{F}_t\right]\right.$$

$$+\mathbb{E}^{\mathbb{Q}}\left[\left(\int_{T_{j-1}}^{T_j}(e^{-a(T_j-s)}-1)\,dW(s)\right)^2\middle|\mathcal{F}_t\right]$$

$$+2\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\left(\int_t^{T_{j-1}}(e^{-a(T_j-s)}-e^{-a(T_{j-1}-s)})\,dW(s)\right)\cdot\right.\right.$$

$$\left.\left.\left.\left(\int_{T_{j-1}}^{T_j}(e^{-a(T_j-s)}-1)\,dW(s)\right)\middle|\mathcal{F}_{T_{j-1}}\right]\middle|\mathcal{F}_t\right]\right].$$

Notice that the third addend of the previous equation disappears, since its second factor turns equal to 0 when solving the integral. Therefore, by using Ito isometry and taking into account that our integrals become $\mathcal{F}_t$-measurable, we can rewrite our equation as

$$\Sigma_{t,j}^2 = \frac{\sigma^2}{a^2}\left[\int_t^{T_{j-1}}(e^{-a(T_j-s)}-e^{-a(T_{j-1}-s)})^2\,ds+\int_{T_{j-1}}^{T_j}(e^{-a(T_j-s)}-1)^2\,ds\right].$$

Solving the integrals, we obtain

$$\Sigma_{t,j}^2 = \frac{\sigma^2}{a^2}\left[\int_t^{T_{j-1}}(e^{-2a(T_j-s)}+e^{-2a(T_{j-1}-s)}-2e^{-a(T_{j-1}+T_j-2s)}\,ds\right.$$

$$\left.+\int_{T_{j-1}}^{T_j}(e^{-2a(T_j-s)}+1-2e^{-a(T_j-s)})\,ds\right]=\frac{\sigma^2}{a^2}\left[\left[\frac{1}{2a}e^{-2a(T_j-s)}\right]_{s=t}^{s=T_{j-1}}\right.$$

$$+\left[\frac{1}{2a}e^{-2a(T_{j-1}-s)}\right]_{s=t}^{s=T_{j-1}}-\left[\frac{1}{a}e^{-a(T_j+T_{j-1}-2s)}\right]_{s=t}^{s=T_{j-1}}+\left[\frac{1}{2a}e^{-2a(T_j-s)}\right]_{s=T_{j-1}}^{s=T_j}$$

$$+(T_{j-1}-T_j)-\left[\left[\frac{2}{a}e^{-a(T_j-s)}\right]_{s=T_{j-1}}^{s=T_j}\right]=\frac{\sigma^2}{a^2}\left[\frac{1}{2a}e^{-2a(T_j-T_{j-1})}-\frac{1}{2a}e^{-2a(T_j-t)}\right.$$
$$+\frac{1}{2a}-\frac{1}{2a}e^{-2a(T_{j-1}-t)}-\frac{1}{a}e^{-a(T_j-T_{j-1})}+\frac{1}{a}e^{-a(T_j+T_{j-1}-2t)}+\frac{1}{2a}$$
$$\left.-\frac{1}{2a}e^{-2a(T_j-T_{j-1})}+(T_{j-1}-T_j)-\frac{2}{a}+\frac{2}{a}e^{-a(T_j-T_{j-1})}\right].$$

Hence, the variance is

$$\Sigma_{t,j}^2=\frac{\sigma^2}{a^2}\left[\frac{1}{a}(e^{-a(T_j-T_{j-1})}+e^{-a(T_j+T_{j-1}-2t)}-1)-\frac{1}{2a}(e^{-2a(T_j-t)}+e^{-2a(T_{j-1}-t)})\right.$$
$$\left.+(T_j-T_{j-1})\right]. \tag{3.8}$$

We can now find the 3M future price. Starting from (3.2), given the normal distribution of the integral, we can write

$$f_j^{3M}(t)=\frac{1}{\tau_j}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{\int_{T_{j-1}}^{T_j}r(u)\,du}\middle|\mathcal{F}_t\right]-1\right]=\frac{1}{\tau_j}(e^{\mu_{t,j}+\frac{1}{2}\Sigma_{t,j}^2}-1)$$

and get the following result, taking into account that $\eta_{t,j}$ and $\Sigma_{t,j}^2$ are given by equations (3.6) and (3.8).

Proposition 1. *Given the hypothesis made, the* ***3M future price*** $f_j^{3M}(t)$ *at time* $t\le T_{j-1}$ *is given by*

$$f_j^{3M}(t)=\frac{1}{\tau_j}(e^{\frac{r(t)}{a}(e^{-a(T_{j-1}-t)}-e^{-a(T_j-t)})+\eta_{t,j}+\frac{1}{2}\Sigma_{t,j}^2}-1)$$

where

$$\eta_{t,j}=\frac{1}{a}\int_t^{T_{j-1}}(e^{-a(T_{j-1}-s)}-e^{-a(T_j-s)})\Theta(s)\,ds+\frac{1}{a}\int_{T_{j-1}}^{T_j}(1-e^{-a(T_j-s)})\Theta(s)\,ds$$

and

$$\Sigma_{t,j}^2=\frac{\sigma^2}{a^2}\left[\frac{1}{a}(e^{-a(T_j-T_{j-1})}+e^{-a(T_j+T_{j-1}-2t)}-1)-\frac{1}{2a}(e^{-2a(T_j-t)}+e^{-2a(T_{j-1}-t)})\right.$$
$$\left.+(T_j-T_{j-1})\right].$$

Let us now consider the **1M future**, that we know settles at the rate representing the arithmetic average of overnight rates over the period $[T_{j-1},T_j]$, with τ_j being equal to one month. The 1M future rate $f_j^{1M}(t)$ is

$$f_j^{1M}(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{\tau_j}\int_{T_{j-1}}^{T_j} r(u)\,du \middle| \mathcal{F}_t\right].$$

Using equations (3.6) and (3.7), we obtain the following result for $t \leq T_{j-1}$.

Proposition 2. *Given the hypothesis made, the* ***1M future price*** $f_j^{1M}(t)$ *at time* $t \leq T_{j-1}$ *is given by*

$$f_j^{1M}(t) = \frac{1}{\tau_j}\left(\frac{r(t)}{a}(e^{-a(T_{j-1}-t)} - e^{-a(T_j-t)}) + \eta_{t,j}\right)$$

where

$$\eta_{t,j} = \frac{1}{a}\int_t^{T_{j-1}}(e^{-a(T_{j-1}-s)} - e^{-a(T_j-s)})\Theta(s)\,ds + \frac{1}{a}\int_{T_{j-1}}^{T_j}(1 - e^{-a(T_j-s)})\Theta(s)\,ds.$$

3.3. Pricing of the option contract

Let us now focus on the option part of our derivative. For simplicity, we will consider an European call option. Our assumption is consistent with the options on futures that we find in current financial markets, which are usually of European type. Therefore, our results can be applied to both options on SONIA index futures and options on SOFR futures. Moreover, remember that by the call-put parity, once we have computed the European call prices for some expiry dates and strikes, we can easily obtain the European put prices for those expiry dates and strikes.

Let us consider a European call option on $f_j^{iM}(t)$, with $i = 3$ in the case of a 3M future contract and $i = 1$ in the case of a 1M future contract. Assume that the option has date of maturity $T \leq T_j$[3] and strike price K. The payoff at time T associated with this claim is

$$\mathcal{X} = \max[f_j^{iM}(T) - K, 0].$$

We now want to compute the option price at the time when we decide whether to buy the option, which we assume happens at a certain time t. We can write the option as

$$\mathcal{X} = [f_j^{iM}(T) - K] \cdot I\{f_j^{iM}(T) \geq K\}$$

where I is an indicator function of the form

[3]Notice that when the option expires at the same date of the future contract, that is at T_j, it expires into cash. However, this is not always the case: in fact, the two dates may not always coincide and the derivative would expire into the future contract. This is why we leave T as generic.

$$I\{f_j^{iM}(T) \geq K\} = \begin{cases} 1 & \text{if } f_j^{iM}(T) \geq K \\ 0 & \text{if } f_j^{iM}(T) < K. \end{cases}$$

Denoting the price of the option at time t as $\Pi^{iM}(t,T)$, we have that

$$\Pi^{iM}(t,T) = \mathbb{E}^{\mathbb{Q}}[B^{-1}(T)[f_j^{iM}(T) - K]I\{f_j^{iM}(T) \geq K\}|\mathcal{F}_t] =$$

$$= \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r(s)\,ds} f_j^{iM}(T) \cdot I\{f_j^{iM}(T) \geq K\}|\mathcal{F}_t] - K\mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r(s)\,ds} \cdot I\{f_j^{iM}(T) \geq K\}|\mathcal{F}_t]$$

where $\mathbb{Q}$ is the usual risk neutral martingale measure. The price can be rewritten as

$$\Pi^{iM}(t,T) = P(t,T)\mathbb{E}^T\left[f_j^{iM}(T) \cdot I\{f_j^{iM}(T) \geq K\}\middle|\mathcal{F}_t\right] - KP(t,T)\mathbb{E}^T\left[I\{f_j^{iM}(T) \geq K\}\middle|\mathcal{F}_t\right] \tag{3.9}$$

where $\mathbb{E}^T$ denotes the expectation under the T-forward measure $\mathbb{Q}^T$ and $P(t,T)$ is the price of a zero-coupon bond at time t.[4] In order to solve the two expectations in the previous equation, we first need to derive the $r(t)$ dynamics with respect to the T-forward measure. Let us suppose that this dynamics depends on the same parameters of the Hull-White model a and σ, as we defined them in chapter 2. Then, the following result is true.

Proposition 3. *The dynamics of the short rate $r(t)$ with respect to the T-forward measure is given by the following stochastic differential equation*

$$dr(t) = (\Theta(t) - ar(t) - \sigma^2 B(t,T))dt + \sigma dW^T(t) \tag{3.10}$$

where $B(t,T)$ is given by equation (2.25) and W^T is a Brownian motion with respect to the T-forward measure.

Proof. As demonstrated in section 2.3.2, the Hull-White model is characterised by an affine term structure, that is

$$P(t,T) = e^{A(t,T) - B(t,T)r(t)} \tag{3.11}$$

where $A(t,T)$ and $B(t,T)$ are given respectively by equations (2.26) and (2.25). Using Ito's formula, from equation (3.11) we obtain that

$$dP(t,T) = r(t)P(t,T)dt - \sigma B(t,T)P(t,T)dW(t).$$

[4] The result derives from the fact that, for a general T-claim X, we have that $\Pi(t,X) = P(t,T)\mathbb{E}^T[X|\mathcal{F}_t]$. For more details, see [7].

Recalling that the bank account has dynamics

$$dB(t) = r(t)B(t)dt$$

and that the Girsanov kernel $\varphi^T(t)$ for the transition from the risk neutral measure to the T-forward measure $\mathbb{Q}^T$ is given by the difference between the volatilities of the two numeraires (as demonstrated in [7]); since the bank account $B(t)$ has volatility equal to zero, we have that

$$\varphi^T(t) = -\sigma B(t, T).$$

Therefore,

$$dW^T(t) = dW(t) + \sigma B(t, T)dt$$

from which we obtain the following stochastic differential equation for the dynamics of $r(t)$ with respect to the T-forward measure

$$dr(t) = (\Theta(t) - ar(t) - \sigma^2 B(t, T))dt + \sigma dW^T(t)$$

where W^T is a Brownian motion with respect to the T-forward measure. $\square$

Now, we are ready to solve the two expectations in equation (3.9). Particularly, in the next two paragraphs, we will derive the option price first when the underlying is a 3M future ($i = 3$) and then when it is a 1M future ($i = 1$). The reason why we need to distinguish between the two cases is related to the different nature of the two future prices $f_j^{3M}(t)$ and $f_j^{1M}(t)$, as we derived them respectively in Proposition 1 and Proposition 2. In fact, $f_j^{3M}(t)$ is described by an exponential function of $r(t)$, allowing us to follow a procedure similar to the one used for the Black and Scholes formula derivation. On the contrary, in $f_j^{1M}(t)$, we do not find a comparable exponential function of $r(t)$. This is why we need to follow a different procedure.

3.3.1 Option on 3M future

In this paragraph, we will study the case in which the underlying of the option contract is a **3M future contract**, that is when $i = 3$. Specifically, let us focus first on the expected value of the second term of equation (3.9), which can be rewritten as

$$\mathbb{E}^T\left[I\{f_j^{3M}(T) \geq K\}\middle|\mathcal{F}_t\right] = \mathbb{Q}^T(f_j^{3M}(T) \geq K\,|\mathcal{F}_t) =$$

$$= \mathbb{Q}^T \left(\frac{1}{\tau_j} (e^{\frac{r(T)}{a}(e^{-a(T_{j-1}-T)} - e^{-a(T_j-T)}) + \eta_{T,j} + \frac{1}{2}\Sigma^2_{T,j}} - 1) \geq K \,\middle|\, \mathcal{F}_t \right) \tag{3.12}$$

where $\mathbb{Q}^T$ stands for the probability under the T-forward measure, the third equality is obtained substituting the result of Proposition 1, and

$$\eta_{T,j} = \frac{1}{a}\int_T^{T_{j-1}} (e^{-a(T_{j-1}-s)} - e^{-a(T_j-s)})\Theta(s)\,ds + \frac{1}{a}\int_{T_{j-1}}^{T_j} (1 - e^{-a(T_j-s)})\Theta(s)\,ds, \tag{3.13}$$

$$\Sigma^2_{T,j} = \frac{\sigma^2}{a^2}\left[\frac{1}{a}(e^{-a(T_j-T_{j-1})} + e^{-a(T_j+T_{j-1}-2T)} - 1) - \frac{1}{2a}(e^{-2a(T_j-T)} + e^{-2a(T_{j-1}-T)}) \right.$$
$$\left. + (T_j - T_{j-1})\right].$$

In order to compute this probability measure, we first need to find the distribution of $r(T)$ with respect to the T-forward measure conditional to $\mathcal{F}_t$. To do so, we integrate equation (3.10) and obtain the solution

$$r(T) = r(t)e^{-a(T-t)} + \int_t^T e^{-a(T-s)}\left[\Theta(s) - \sigma^2 B(s,T)\right]\,ds + \sigma\int_t^T e^{-a(T-s)}\,dW^T(s) =$$

$$= r(t)e^{-a(T-t)} + \int_t^T e^{-a(T-s)}\Theta(s)\,ds - \frac{\sigma^2}{a}\int_t^T e^{-a(T-s)}\,ds + \frac{\sigma^2}{a}\int_t^T e^{-a(2T-2s)}\,ds$$
$$+\sigma\int_t^T e^{-a(T-s)}\,dW^T(s)$$

$$= r(t)e^{-a(T-t)} + \int_t^T e^{-a(T-s)}\Theta(s)\,ds - \frac{\sigma^2}{a^2}\left[e^{-a(T-s)}\right]_{s=t}^{s=T} + \frac{\sigma^2}{2a^2}\left[e^{-a(2T-2s)}\right]_{s=t}^{s=T}$$
$$+\sigma\int_t^T e^{-a(T-s)}\,dW^T(s)$$

$$= r(t)e^{-a(T-t)} + \int_t^T e^{-a(T-s)}\Theta(s)\,ds - \frac{\sigma^2}{a^2} + \frac{\sigma^2}{a^2}e^{-a(T-t)} + \frac{\sigma^2}{2a^2}$$

$$-\frac{\sigma^2}{2a^2}e^{-a(2T-2t)} + \sigma\int_t^T e^{-a(T-s)}\,dW^T(s) \tag{3.14}$$

where the second equality is obtained substituting the expression for $B(t,T)$ (see equation (2.25)). Thus, we notice that $r(T)$ is distributed as a normal with respect to the T-forward measure. The mean can be computed as

$$\alpha(t,T) = \mathbb{E}^T\left[r(T)|\mathcal{F}_t\right] = r(t)e^{-a(T-t)} + \int_t^T e^{-a(T-s)}\Theta(s)\,ds - \frac{\sigma^2}{a^2}$$

$$+\frac{\sigma^2}{a^2}e^{-a(T-t)} + \frac{\sigma^2}{2a^2} - \frac{\sigma^2}{2a^2}e^{-a(2T-2t)}. \tag{3.15}$$

The variance of $r(T)$ is

$$\beta^2(t,T) = Var^T\left[r(T)|\mathcal{F}_t\right] = \sigma^2\mathbb{E}^T\left[\left(\int_t^T e^{-a(T-s)}\,dW^T(s)\right)^2\middle|\mathcal{F}_t\right].$$

Using Ito's isometry, the variance becomes

$$\beta^2(t,T) = \sigma^2\left[\int_t^T (e^{-a(T-s)})^2\,ds\right] = \sigma^2\left[\int_t^T e^{-2a(T-s)}\,ds\right] = \sigma^2\left[\frac{1}{2a}e^{-2a(T-s)}\right]_{s=t}^{s=T} =$$

$$= \frac{\sigma^2}{2a} - \frac{\sigma^2}{2a}e^{-2a(T-t)}. \tag{3.16}$$

Now, for simplicity, let us rewrite equation (3.12) as

$$\mathbb{Q}^T\left(\frac{1}{\tau_j}(e^{\gamma(T)r(T)+\delta(T)}-1) \geq K \middle| \mathcal{F}_t\right) \tag{3.17}$$

where

$$\gamma(T) = \frac{e^{-a(T_{j-1}-T)} - e^{-a(T_j-T)}}{a}$$

and

$$\delta(T) = \eta_{T,j} + \frac{1}{2}\Sigma^2_{T,j}.$$

We notice that the exponent inside equation (3.17) is a linear transformation of $r(T)$, which is therefore distributed as a normal

$$(\gamma(T)r(T)+\delta(T)) \sim \mathcal{N}(\gamma(T)\alpha(t,T)+\delta(T);\gamma^2(T)\beta^2(t,T)).$$

For better comprehensibility purposes, let us now define

$$Y(T) = e^{\gamma(T)r(T)+\delta(T)} = e^{\gamma(T)\left(\alpha(t,T)+\sigma\int_t^T e^{-a(T-s)}\,dW^T(s)\right)+\delta(T)}$$

and

$$Y(t) = \mathbb{E}^T\left[e^{\gamma(T)r(T)+\delta(T)}\middle|\mathcal{F}_t\right] = e^{\delta(T)+\gamma(T)\alpha(t,T)+\frac{1}{2}\gamma^2(T)\beta^2(t,T)}.$$

From these two equations we observe that

$$Y(T) = Y(t)e^{-\gamma^2(T)\frac{\beta^2(t,T)}{2}+\gamma(T)\sigma\int_t^T e^{-a(T-s)}\,dW^T(s)}.$$

We notice that the term $\gamma(T)\sigma \int_t^T e^{-a(T-s)}\, dW^T(s)$ is distributed as a normal with zero mean and variance $\gamma^2(T)\beta^2(t,T)$, under the T-forward measure conditional on $\mathcal{F}_t$. Therefore, by means of standardisation, we can rewrite term as

$$\sigma \int_t^T e^{-a(T-s)}\, dW^T(s) = \gamma(T)\beta(t,T)y$$

where y is a standardised variable $\sim \mathcal{N}(0,1)$.
We now move back to equation (3.17) and make use of these results.

$$\mathbb{Q}^T\left(\frac{1}{\tau_j}(e^{\gamma(T)r(T)+\delta(T)}-1) \geq K \,\middle|\, \mathcal{F}_t\right) = \mathbb{Q}^T\left(\frac{1}{\tau_j}(Y(T)-1) \geq K \,\middle|\, \mathcal{F}_t\right) =$$

$$= \mathbb{Q}^T\left(Y(T) \geq \tau_j K + 1 \,\middle|\, \mathcal{F}_t\right) = \mathbb{Q}^T\left(Y(t)e^{-\gamma^2(T)\frac{\beta^2(t,T)}{2}+\gamma(T)\beta(t,T)y} \geq \tau_j K + 1 \,\middle|\, \mathcal{F}_t\right) =$$

$$= \mathbb{Q}^T\left(e^{-\gamma^2(T)\frac{\beta^2(t,T)}{2}+\gamma(T)\beta(t,T)y} \geq \frac{\tau_j K + 1}{Y(t)} \,\middle|\, \mathcal{F}_t\right) =$$

$$= \mathbb{Q}^T\left(-\gamma^2(T)\frac{\beta^2(t,T)}{2} + \gamma(T)\beta(t,T)y \geq \ln\left[\frac{\tau_j K + 1}{Y(t)}\right] \,\middle|\, \mathcal{F}_t\right) =$$

$$= \mathbb{Q}^T\left(y \geq \frac{\ln\left[\frac{\tau_j K+1}{Y(t)}\right] + \gamma^2(T)\frac{\beta^2(t,T)}{2}}{\beta(t,T)\gamma(T)} \,\middle|\, \mathcal{F}_t\right) =$$

$$= \mathbb{Q}^T\left(y \geq -d_2^{3M} \,\middle|\, \mathcal{F}_t\right) = \mathbb{Q}^T\left(y \leq d_2^{3M} \,\middle|\, \mathcal{F}_t\right) = N[d_2^{3M}]$$

where

$$N[d_2^{3M}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2^{3M}} e^{-\frac{y^2}{2}}\, dy.$$

Let us now focus on the expected value of the first term of equation (3.9), which can be rewritten as

$$\mathbb{E}^T\left[f_j^{3M}(T) \cdot I\{f_j^{3M}(T) \geq K\} \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}^T\left[\left(\frac{1}{\tau_j}\left(e^{\gamma(T)r(T)+\delta(t)}-1\right)\right) \cdot \mathbb{1}_{\{y \geq -d_2^{3M}\}} \,\middle|\, \mathcal{F}_t\right] =$$

$$= \frac{1}{\tau_j}\left[\mathbb{E}^T\left[e^{\gamma(T)+r(T)\delta(T)} \,\middle|\, \mathcal{F}_t\right] \cdot \mathbb{1}_{\{y \geq -d_2^{3M}\}} - \mathbb{1}_{\{y \geq -d_2^{3M}\}}\right] =$$

$$= \frac{1}{\tau_j}\left[\int_{-d_2^{3M}}^{+\infty} Y(t)e^{-\gamma^2(T)\frac{\beta^2(t,T)}{2}+\gamma(T)\beta(t,T)y-\frac{y^2}{2}} \frac{1}{\sqrt{2\pi}}\, dy - N[d_2^{3M}]\right] =$$

$$= \frac{1}{\tau_j}\left[Y(t)\int_{-d_2^{3M}}^{+\infty} e^{-\frac{1}{2}(y-\gamma(T)\beta(t,T))^2} \frac{1}{\sqrt{2\pi}}\, dy - N[d_2^{3M}]\right].$$

Denoting $x = y - \gamma(T)\beta(t,T)$, the integral in the previous equation becomes

$$\int_{-d_2^{3M}-\gamma(T)\beta(t,T)}^{+\infty} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}}\, dx = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}}\, dx - \int_{-\infty}^{-d_2^{3M}-\gamma(T)\beta(t,T)} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}}\, dx =$$

$$= 1 - N[-d_2^{3M} - \gamma(T)\beta(t,T)] = N[d_2^{3M} + \gamma(T)\beta(t,T)] = N[d_1^{3M}]$$

where

$$N[d_1^{3M}] = \frac{1}{2\pi} \int_{-\infty}^{d_1^{3M}} e^{-\frac{y^2}{2}}\, dy.$$

Thus, the expected value of the first addend of equation (3.9) is

$$\frac{1}{\tau_j} \left[Y(t)N[d_1^{3M}] - N[d_2^{3M}] \right].$$

By substituting the expressions of the two expected values that we just obtained above into equation (3.9), we get that

$$\Pi^{3M}(t,T) = \frac{P(t,T)}{\tau_j} \left(Y(t)N[d_1^{3M}] - N[d_2^{3M}] \right) - KP(t,T)N[d_2^{3M}].$$

By multiplying and dividing the previous equation by τ_j, we get

$$\left(P(t,T)Y(t)N[d_1^{3M}] - P(t,T)N[d_2^{3M}] - \tau_j KP(t,T)N[d_2^{3M}] \right) \frac{1}{\tau_j} =$$

$$= \frac{P(t,T)}{\tau_j} \left(Y(t)N[d_1^{3M}] - (1+\tau_j K)N[d_2^{3M}] \right).$$

Therefore, we obtain the following result.

Proposition 4. *Under the given assumptions, the price of an European call option at time 0 having as an underling a* ***3M future contract*** *with future price* $f_j^{3M}(T)$ *is*

$$\Pi^{3M}(t,T) = \frac{P(t,T)}{\tau_j} \left(Y(t)N[d_1^{3M}] - (1+\tau_j K)N[d_2^{3M}] \right)$$

where

$$d_2^{3M} = \frac{\ln \frac{Y(t)}{\tau_j K+1} - \gamma^2(T)\frac{\beta^2(t,T)}{2}}{\gamma(T)\beta(t,T)}$$

and

$$d_1^{3M} = d_2^{3M} + \gamma(T)\beta(t,T).$$

3.3.2 Option on 1M future

In this paragraph, we will study the case in which the underlying of the option is a **1M future contract**, that is when $i = 1$. As before, we need to solve the two expected values that we find inside equation (3.9), which depend on $f_j^{1M}(T)$. From proposition 2, we know that

$$f_j^{1M}(T) = \frac{1}{\tau_j}\left(\frac{r(T)}{a}(e^{-a(T_{j-1}-T)} - e^{-a(T_j-T)}) + \eta_{T,j}\right) \tag{3.18}$$

where $\eta_{T,j}$ is equal to equation (3.13). For simplicity, let us rewrite equation (3.18) as

$$f_j^{1M}(T) = \epsilon(T)r(T) + \omega(T) \tag{3.19}$$

with

$$\epsilon(T) = \frac{e^{-a(T_{j-1}-T)} - e^{-a(T_j-T)}}{a\tau_j}$$

and

$$\omega(T) = \frac{\eta_{T,j}}{\tau_j}.$$

From the previous paragraph, we know that $r(T)$ is described by equation (3.14) and it is therefore distributed as a normal with respect to the T-forward measure conditional to $\mathcal{F}_t$

$$r(T) \sim \mathcal{N}(\alpha(t,T); \beta^2(t,T))$$

with $\alpha(t,T)$ and $\beta^2(t,T)$ defined by equations (3.15) and (3.16) respectively.
Looking at equation (3.19), we notice that $f_j^{1M}(T)$ is a linear transformation of $r(T)$ and thus it is distributed as a normal with respect to the T-forward measure conditional to $\mathcal{F}_t$

$$f_j^{1M}(T) \sim \mathcal{N}(\epsilon(T)\alpha(t,T) + \omega(T); \epsilon^2(T)\beta^2(t,T)). \tag{3.20}$$

Now, we move back to the determination of the two expected values in the option price. Specifically, let us start considering the expected value that we find in the second term of equation (3.9), which we can rewrite as

$$\mathbb{E}^T\left[I\{f_j^{1M}(T) \geq K\}\Big|\mathcal{F}_t\right] = \mathbb{Q}^T\left(f_j^{1M}(T) \geq K\,\Big|\mathcal{F}_t\right)$$

where $\mathbb{Q}^T$ indicates the probability under the T-forward measure. Since we know that $f_j^{1M}(T)$ is distributed as a normal, it can be standardised by writing

$$\mathbb{Q}^T\left(\frac{f_j^{1M}(T)-\epsilon(T)\alpha(t,T)-\omega(T)}{\epsilon(T)\beta(t,T)} \geq \frac{K-\epsilon(T)\alpha(t,T)-\omega(T)}{\epsilon(T)\beta(t,T)}\,\middle|\mathcal{F}_t\right).$$

Let us call the standardised variable $Z \sim \mathcal{N}(0;1)$. Thus, the previous equation becomes

$$\mathbb{Q}^T\left(Z \geq -d^{1M}\,\middle|\mathcal{F}_t\right) = \mathbb{Q}^T\left(Z \leq d^{1M}\,\middle|\mathcal{F}_t\right) = N[d^{1M}]$$

where

$$N[d^{1M}] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d^{1M}} e^{-\frac{z^2}{2}}\,dz.$$

Let us now focus instead on the expected value of the first term of equation (3.9). Since we know that $f_j^{1M}(T)$ is distributed as a normal (from equation (3.20)), the expected value of the first addend of equation (3.9) becomes

$$\mathbb{E}^T\left[f_j^{1M}(T)\cdot I\{f_j^{1M}(T)\geq K\}\middle|\mathcal{F}_t\right] = (\epsilon(T)\alpha(t,T)+\omega(T))\,\mathbb{Q}^T\left(f_j^{1M}(T)\geq K\,\middle|\mathcal{F}_t\right)$$

$$+\epsilon(T)\beta(t,T)\mathbb{E}^T\left[Y\cdot\mathbb{1}_{\left\{Y\geq\frac{K-\epsilon(T)\alpha(T)-\omega(T)}{\epsilon(T)\beta(t,T)}\right\}}\middle|\mathcal{F}_t\right].$$

Consider the expected value of the second addend in the previous equation. This is equal to

$$\mathbb{E}^T\left[Y\cdot\mathbb{1}_{\{Y\geq -d^{1M}\}}\middle|\mathcal{F}_t\right] = \frac{1}{\sqrt{2\pi}}\int_{-d^{1M}}^{+\infty} ye^{-\frac{y^2}{2}}\,dy =$$

$$= \frac{1}{\sqrt{2\pi}}\left[-e^{-\frac{y^2}{2}}\right]_{y=-d^{1M}}^{y=+\infty} = \frac{1}{\sqrt{2\pi}}e^{-\frac{(-d^{1M})^2}{2}} = \varphi(d^{1M})$$

where $\varphi(\cdot)$ is the density function of the standard variable Y.
The expected value of the first addend of equation (3.9) becomes

$$\mathbb{E}^T\left[f_j^{1M}(T)\cdot I\{f_j^{1M}(T)\geq K\}\middle|\mathcal{F}_t\right] = (\epsilon(T)\alpha(T)+\omega(T))\,N[d^{1M}]+\epsilon(T)\beta(t,T)\varphi(d^{1M}).$$

By substituting the expressions of the two expected values that we just obtained above into equation (3.9), we get that

$$\Pi^{1M}(t,T) = P(t,T)\left[(\epsilon(T)\alpha(t,T)+\omega(T))\,N[d^{1M}]+\epsilon(T)\beta(t,T)\varphi(d^{1M})\right]$$

$$-KP(t,T)N[d^{1M}] = P(t,T)\left[(\epsilon(T)\alpha(t,T)+\omega(T)-K)\,N[d^{1M}]+\epsilon(T)\beta(t,T)\varphi(d^{1M})\right].$$

Therefore, we obtain the following result.

Proposition 5. *Under the given assumptions, the price of an European call option at time 0* *having as an underling a **1M future contract** with future price* $f_j^{1M}(T)$ *is*

$$\Pi^{1M}(t,T) = P(t,T)\left[\left(\epsilon(T)\alpha(t,T) + \omega(T) - K\right) N[d^{1M}] + \epsilon(T)\beta(t,T)\varphi(d^{1M})\right]$$

with

$$d^{1M} = \frac{\epsilon(T)\alpha(t,T) + \omega(T) - K}{\epsilon(T)\beta(t,T)}.$$

Chapter 4

Numerical study of options on RFRs futures

In this chapter, we will perform a numerical study on options on 3M and 1M futures' prices, using the mathematical results that we obtained in the previous chapter, specifically Proposition 4 and 5. Particularly, using the programming and computing platform Matlab, we will perform a **sensitivity analysis**, studying how option prices vary across a reasonable range of values of the Hull-White model's parameters a and σ. Additionally, we will show how changes of the strike price K and the maturity T of the option affect our results.

4.1. Generalities

4.1.1 Hull-White model parameters

From equation (2.27), we know that the $\mathbb{Q}$-dynamics of the short rate $r(t)$ depends on two parameters, namely the drift term a and the volatility σ, and a time-dependent function $\Theta(t)$. We want to perform a sensitivity analysis using the software Matlab, to assess how changes in both a and σ impact on options on 3M and 1M futures' prices. To this effect, we first need to obtain an explicit form for the function $\Theta(t)$ and set reasonable ranges of values for a and σ, across which the price will vary.
The function $\Theta(t)$ was determined in chapter 2 by fitting the initially observed term structure of interest rates to the market prices, a procedure that resulted in equation (2.33). For more clarity, we can rewrite this equation as

$$\Theta(T) = \frac{df^*(0,T)}{dT} + af^*(0,T) + \frac{\sigma^2}{2a}(1 - e^{-2aT}) \tag{4.1}$$

which is obtained by substituting the relative expressions for $g(t)$ and $\dot{g}(t)$ into equation (2.33) and where we wrote the derivative $f^*_T(0,T)$ explicitly. As we can notice, this formula is determined by the forward rates $f^*(0,T)$ observed on the market at the initial time $t=0$. From chapter 2, we know that the forward rates $f^*(0,T)$ are related to the zero coupon bond prices $P^*(0,T)$ observed on the market at time $t=0$ through the formula

$$f^*(0,T) = -\frac{d \ln P^*(0,T)}{dT}.$$

Therefore, the function $\Theta(t)$ could be traced back to the yield curve, which we recall is the relationship between the zero-coupon bond yields and their maturity T.
In the previous chapter, we stressed that the most widespread options on RFR futures in current financial markets are the ones launched by the CME, whose reference rate is SOFR. Hence, it would be natural to use the SOFR yield curve to derive our function $\Theta(t)$. Unfortunately, this curve is not publicly available. In fact, since SOFR has been introduced recently, the SOFR yield curve is currently being constructed with different *ad hoc* methodologies (which generally involve observable market data, including futures contracts, market swap rates and outstanding government debt instruments) and there is no unique referenceable procedure. This is why, for simplicity, we decided to make use of the ECB yield curve in our analysis, that is published daily at the URL: `https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/index.en.html`, and thus is uniquely determined for the market. We are aware of the limitations of this choice, given the options on SOFR futures significant diffusion in current financial markets.
The ECB estimates a parametric functional form for the forward curve using the Svensson model[1], that is

$$f^*(0,T) = \beta_0 + \beta_1 e^{-\frac{T}{\tau_1}} + \beta_2 \frac{T}{\tau_1} e^{-\frac{T}{\tau_1}} + \beta_3 \frac{T}{\tau_2} e^{-\frac{T}{\tau_2}}$$

where β_0, β_1, β_2, β_3, τ_1 and τ_2 are parameters estimated daily by the ECB. Taking the derivative with respect to T of the previous equation, we get

$$\frac{df^*(0,T)}{dT} = -\frac{\beta_1}{\tau_1} e^{-\frac{T}{\tau_1}} + \beta_2 \left(\frac{1}{\tau_1} - \frac{T}{\tau_1^2} \right) e^{-\frac{T}{\tau_1}} + \beta_3 \left(\frac{1}{\tau_2} - \frac{T}{\tau_2^2} \right) e^{-\frac{T}{\tau_2}}.$$

We now have all the elements of equation (4.1), which becomes

[1]Svensson, L. E., Estimating and Interpreting Forward Interest Rates: Sweden 1992-1994, *Centre for Economic Policy Research*, 1994; cited in European Central Bank, *Technical Notes*, URL: `https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/technical_notes.pdf`.

$$\Theta(T) = -\frac{\beta_1}{\tau_1}e^{-\frac{T}{\tau_1}} + \beta_2\left(\frac{1}{\tau_1} - \frac{T}{\tau_1^2}\right)e^{-\frac{T}{\tau_1}} + \beta_3\left(\frac{1}{\tau_2} - \frac{T}{\tau_2^2}\right)e^{-\frac{T}{\tau_2}}$$

$$+a\left(\beta_0 + \beta_1 e^{-\frac{T}{\tau_1}} + \beta_2\frac{T}{\tau_1}e^{-\frac{T}{\tau_1}} + \beta_3\frac{T}{\tau_2}e^{-\frac{T}{\tau_2}}\right) + \frac{\sigma^2}{2a}(1 - e^{-2aT}). \tag{4.2}$$

On the day we performed our analysis, that is on the 24th August 2022, the ECB estimated the following values for the forward rate equation's parameters[2]

- $\beta_0 = 1.641204$;
- $\beta_1 = -1.766233$;
- $\beta_2 = 25.191842$;
- $\beta_3 = -25.607108$;
- $\tau_1 = 1.344229$;
- $\tau_2 = 1.421672$.

Contrarily to the function $\Theta(t)$, it is not possible to deduce the parameters a and σ directly from the market prices of zero-coupon bonds. This is why we decided to assume they vary across a range of plausible values. Specifically, we supposed that the parameter a varies across the interval

$$5\% \leq a \leq 15\%$$

while σ varies across the range

$$3\% \leq \sigma \leq 10\%.$$

In Matlab, these parameters were made vary within these ranges with the function *linspace*, through which we generated linearly spaced vectors of 100 points.

4.1.2 Additional inputs

Let us now define some other additional inputs that we need for our analysis. First of all, we assume that the starting time t, that is the time at which we compute the prices,

[2]The parameters were found at the same website of the ECB yield curve, URL: `https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/index.en.html`

is equal to $t = 0$.
Moreover, for the value of $r(0)$, which is the value of the short rate at $t = 0$, we use the value of ESTR on the 24^{th} August 2022, that is $r(0) = -0.082\%$. This choice is made to be as coherent as possible with the other parameters's assumptions, since we adopted the ECB's Svensson function for the yield curve.
Afterwards, we decide the characteristics of the future contracts that will constitute the underlying of the options in our analysis. We consider the following futures

	3M FUTURE ($\tau_j = 0.25$)	
	T_{j-1}	T_j
1.	1 Year and 9 Months	2 Years
2.	2 Years and 9 Months	3 Years

	1M FUTURE ($\tau_j = 0.091$)	
	T_{j-1}	T_j
3.	1 Year and 11 Months	2 Years
4.	2 Years and 11 Months	3 Years

Initially, we decide to set the maturities of the options on each of the underlying futures equal to T_j. Recall that, from the previous chapter, we know that the maturity T of the option cannot be greater than T_j. Thus, we have that

- Option on future *1* has maturity 2 years;
- Option on future *2* has maturity 3 years;
- Option on future *3* has maturity 2 years;
- Option on future *4* has maturity 3 years.

Remark 2 (***maturity of the option***)**.** Notice that, when choosing a maturity $T = T_j$, the underlying of the option simply becomes the RFR (which is computed in a geometric or arithmetic way for a 3M or 1M future respectively, as explained in section 3.1). On the contrary, when $T < T_j$, the option expires at maturity into the 3M or 1M future contract (described respectively by Proposition 4 and Proposition 5). This latter case will be studied in sections 4.2.2 and 4.3.2, where we analyse how our results vary when the maturity of the 3M or 1M option is reduced.

Additionally, we observe that, in both Proposition 4 and 5, the price of options on 3M and 1M futures depends on $P(t, T)$, which is the market price of zero-coupon bonds at time t. To compute these prices at time $t = 0$ we use the formula

$$P(0,T) = e^{-yT}$$

where y is the yield from the yield curve, as usual retrieved from the ECB website[3]. For our analysis, we obtain

T	$P(0,T)$
2 Years	0.9837
3 Years	0.9723

Lastly, we need to set the strike price K. We will start by assuming that the investor opts for a call option whose strike price is at the money (which means equal to the price of the underlying instrument). Thus, we compute the future prices at time $t = 0$ of the contracts set before, making use of Proposition 1 for the 3M futures and Proposition 2 for the 1M future. Notice that, for simplicity, we set a and σ equal to their medium values for our computation, that is 10% and 6,5% respectively. We obtain that

- Option on future *1* should have a strike price $K = 1.1248$;
- Option on future *2* should have a strike price $K = 1.1798$;
- Option on future *3* should have a strike price $K = 0.9923$;
- Option on future *4* should have a strike price $K = 1.0392$.

The detailed computations for these strike prices can be found in Appendix C.

4.2. Option on 3M future's numerical analysis

We start by performing the analysis on options on 3M futures, specifically on future *1* and *2*. Particularly, making use of the assumptions made in the previous section, we implement our pricing formula (Proposition 4) in Matlab and see how it varies across the two ranges of values for the parameters a and σ. In this section we summarise and comment the results of our analysis. The detailed code can be found in Appendix A. In Figure 4.1, we plotted in blue the price of the option on future *1* and in red the price of the option on future *2*, with respect to both a and σ. As we can notice, both prices

[3] As before, the curve is found at URL: `https://www.ecb.europa.eu/stats/financial_markets_and_interest_rates/euro_area_yield_curves/html/index.en.html`.

display an increasing pattern. To better investigate their relationship, in Figure 4.2 we represented the option prices with respect to the two parameters separately.

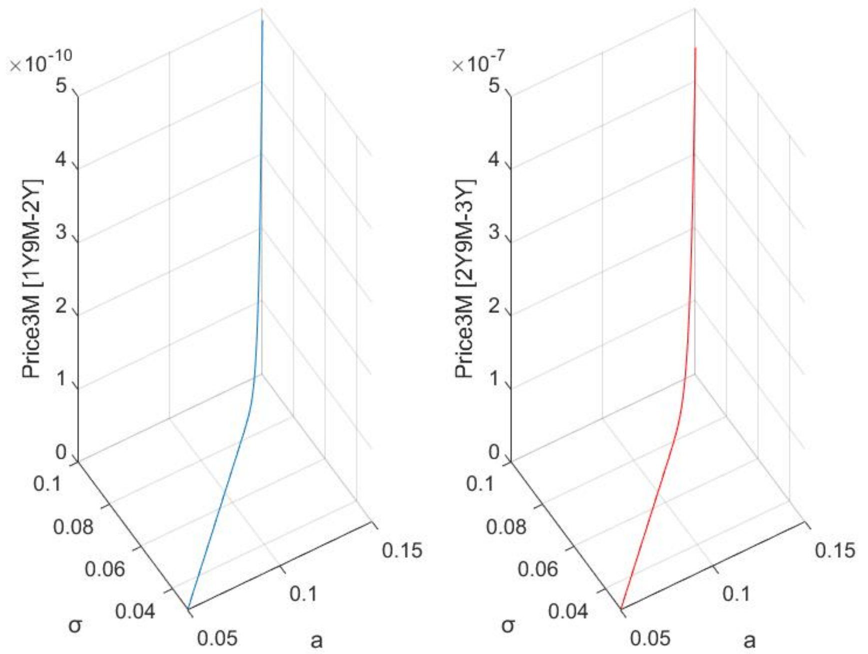

Figure 4.1: Option on 3M future's prices with respect to a and σ simultaneously.

As we can see, the price of the options increases as the parameter a increases. As shown in chapter 3, a influences various measures relevant for the price determination. For example, it affects the variance of $r(t)$, reducing it when the parameter increases. At the same time, it influences the expected value of the integral of $r(t)$. The positive or negative correlation of a with the latter is however relatively difficult to assess, as a appears also inside the explicit function of $\Theta(T)$ (see (4.2)). Thus, there exists a trade-off between various effects to take into account when studying the relationship of the option price with the parameter a. From our analysis, we can conclude that its net balance is positive.

At the same time, we witness that the price increases as the volatility (risk) of the short rate σ increases. An option is an instrument used by investors to protect themselves from fluctuations in the price, thus they will be prone to pay more for a financial tool that is granting them insurance against risk as the volatility increases.

The last thing that we can notice from these graphs is that for higher T_{j-1} and T_j (recall that option on future *2* has both T_{j-1} and T_j greater than option on future *1*) the price is higher. This results from the fact that the more the investment is further in time, the more it is uncertain. For this reason, investors will want to pay more today to get a form of protection in the future.

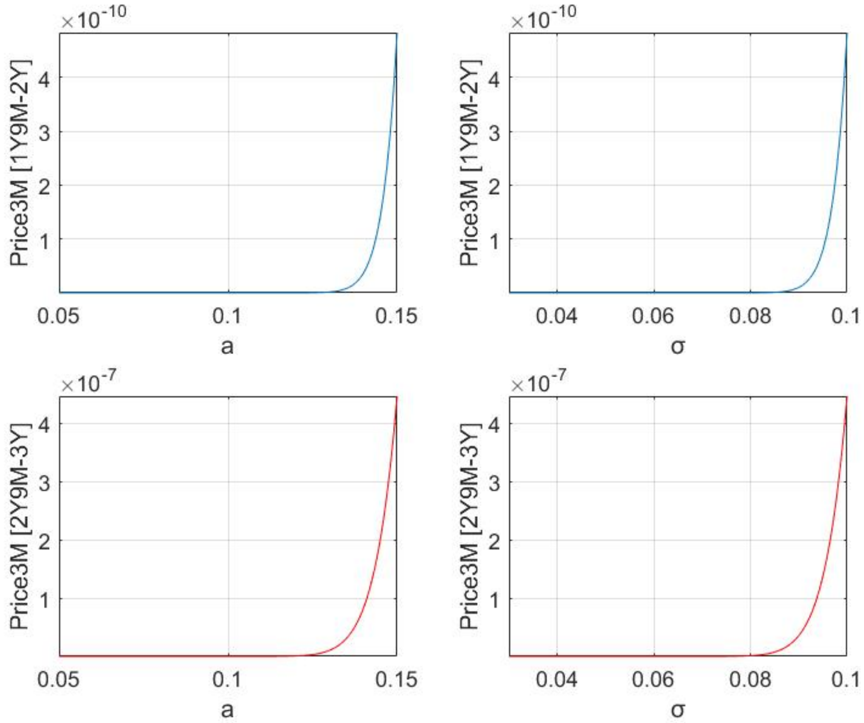

Figure 4.2: Option on 3M future's prices with respect to a and σ separately.

Remark 3 (***option price curve***)**.** In the graphs above (as well as in the following ones), it seems that the price curve is equal to zero until a and σ reach relatively high values, in correspondence of which the curve becomes exponential. In truth, this phenomenon is solely related to the scale of the graphs. In fact, even for small values of the parameters, the price is never equal to zero and follows an exponential trend. The issue derives from the fact that, for the lowest values of a and σ, the 3M option price has an order of magnitude of 10^{-60}. Contrarily, for the highest values of the parameters, the price's magnitude increases up to 10^{-10}. The relatively small orders of magnitude derive from the implicit assumption of a notional value of 1, while in reality it is usually a significant number. The consistent difference between the highest and lowest values of the price makes the price curve appear flattened toward the horizontal axis and the exponential trend not noticeable for low values of a and σ.

4.2.1 Change in the strike price

We now want to study how a change in the strike price K affects our results. Particularly, we analyse how the price curve moves in response to an increase or a decrease of 0.05 points of K, that is the following cases:

- Option on future *1* has strike $K = 1.0748$;
- Option on future *1* has strike $K = 1.1748$;
- Option on future *2* has strike $K = 1.1298$;
- Option on future *2* has strike $K = 1.2298$.

Notice that, when the the strike price is reduced by 0.05, the option becomes in the money (which means that the strike price is below the underlying price). On the contrary, when the strike price is increased by 0.05, the option becomes out of the money (that is, the strike price is above the underlying price).

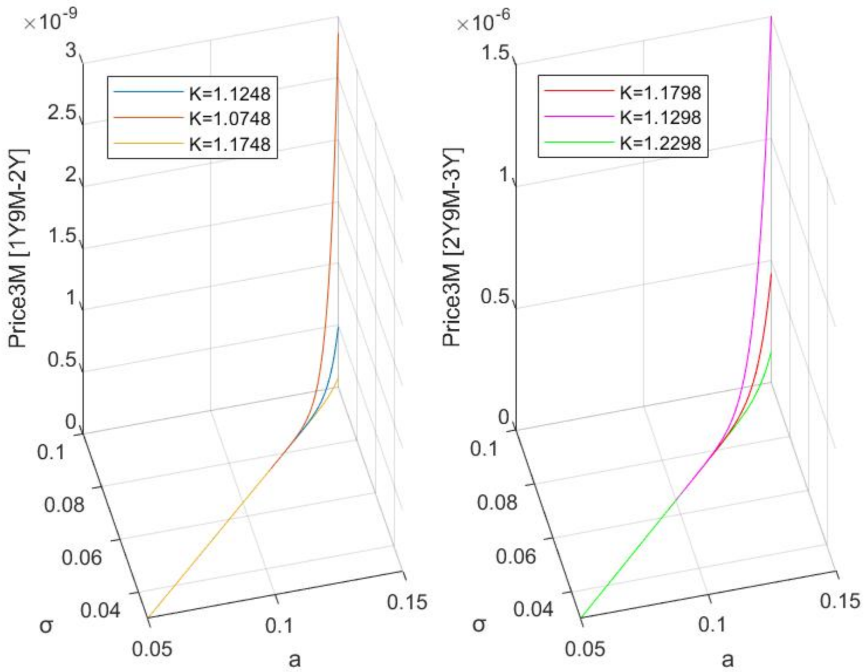

Figure 4.3: Option on 3M future's prices with different strike prices K with respect to a and σ simultaneously.

Figures 4.3 and 4.4 summarise our results. As before, the first figure shows how option prices with different strikes simultaneously move with respect to a and σ, while the latter investigates the relationships separately.
As we can notice, even if the strike price changes, the price curve remains increasing with respect to the parameters a and σ.
Nonetheless, we witness that as the strike price increases, the option price decreases. Indeed, for the option on future *1*, the orange line (that is, when K is decreased by 0.05) sits above the blue line (which corresponds to the initial K), which is in turn over

the yellow line (that is, the case in which K is increased by 0.05). Similarly, for the option on future *2*, the pink line (that is, when K is decreased by 0.05) stands above the red line (which corresponds to the initially set K), which is in turn over the green line (that is, the case in which K is increased by 0.05).
These findings are related to the intrinsic nature of a call option. Recall that a call option gives the right (but not the obligation) to buy at maturity the underlying instrument at the strike price K. Thus, the higher is the price that the investor might potentially want to pay in the future if faced by averse changes in the price of the underlying, the lower will be the one that they are willing to pay today.

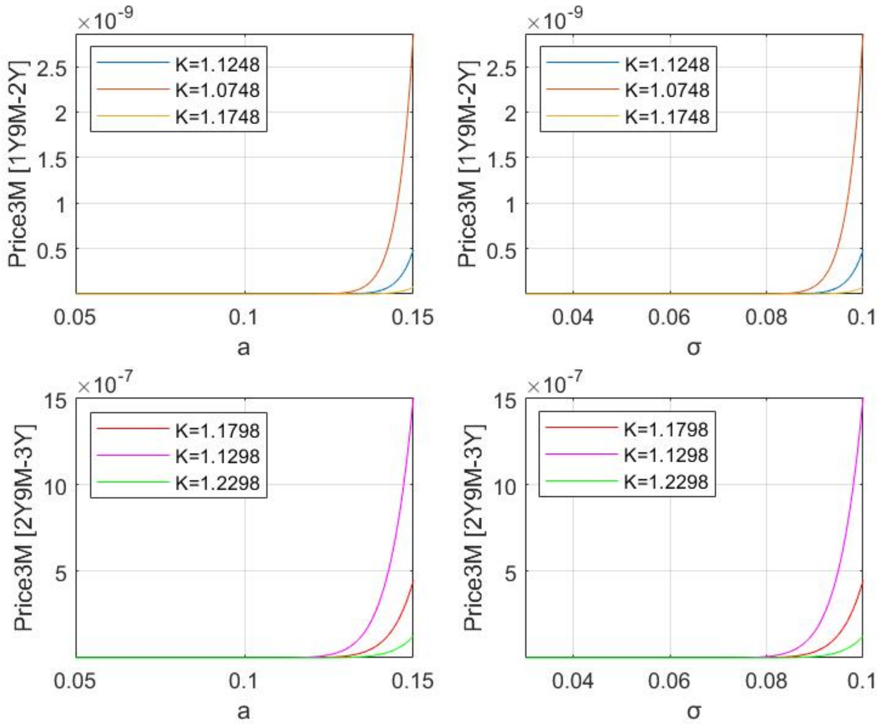

Figure 4.4: Option on 3M future's prices with different strike prices K with respect to a and σ separately.

4.2.2 Change in the maturity

Lastly, we also want to study how a change in the maturity T affects our results. Particularly, we analyse how the price curve changes in response to a decrease of 3 and 4 months of T, that is the following cases:

- Option on future *1* has maturity 1 year and 9 months;
- Option on future *1* has maturity 1 year and 8 months;

- Option on future 2 has maturity 2 years and 9 months;
- Option on future 2 has maturity 2 years and 8 months.

Notice that, by reducing the maturity to a certain $T < T_j$, the underlying at maturity is no longer the RFR (as in the case of $T = T_j$), but the option expires into a 3M future contract.
As before, knowing that the price formula depends on $P(t, T)$, we need to compute the market prices of the zero-coupon bonds at time $t = 0$ for the new maturities, using the usual formula and retrieving the yields from the ECB website. We obtain

T	$P(0, T)$
1 Year and 9 Months	0.9864
1 Year and 8 Months	0.9873
2 Years and 9 Months	0.9753
2 Years and 8 Months	0.9762

Figures 4.5 and 4.6 summarise our results. As before, the first figure shows how option prices with different maturities simultaneously move with respect to a and σ, while the latter investigates the relationships separately.

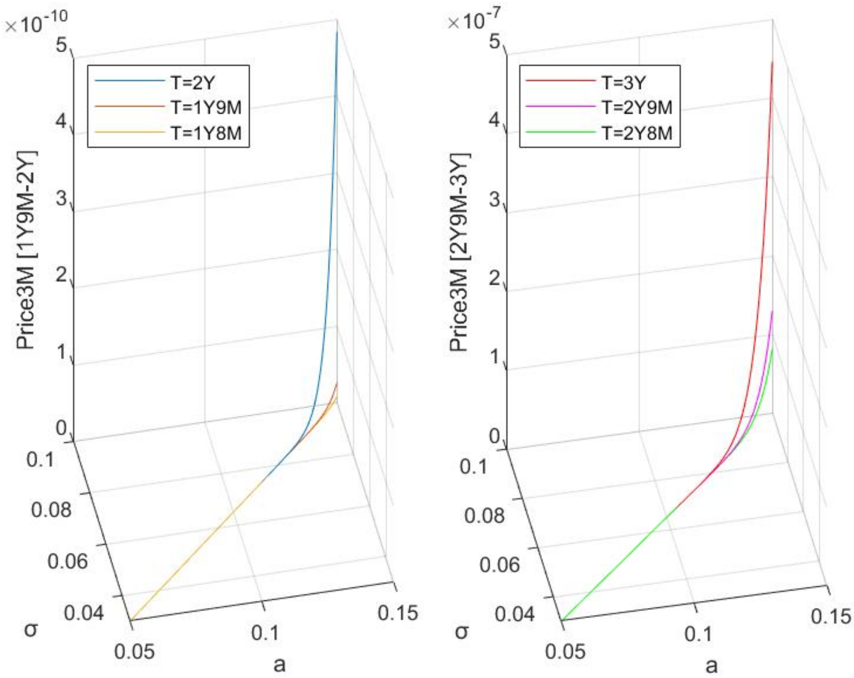

Figure 4.5: Option on 3M future's prices with different maturities T with respect to a and σ simultaneously.

As noticeable, even if the maturity changes, the price curve remains increasing with respect to the parameters a and σ.
Nonetheless, we witness that as the maturity decreases, the option price decreases as well. Indeed, for the option on future *1*, the yellow line (that is, the case in which T is decreased by 4 months) sits below the orange line (that is, when T is decreased by 3 months) which is in turn under the blue line (which corresponds to the initial T).
Similarly, for the option on future *2*, the green line (that is, the case in which T is reduced by 4 months) stands below the pink line (that is, when T is decreased by 3 months) which is in turn under the red line (which corresponds to the initially set T).
These findings are related to the intrinsic nature of an option. In fact, the lower the maturity, the lower will be the time span across which the price of the underlying can fluctuate adversely. For this reason, as T decreases, investors will need less protection against prices' risky movements, hence they will be willing to pay less for the option.

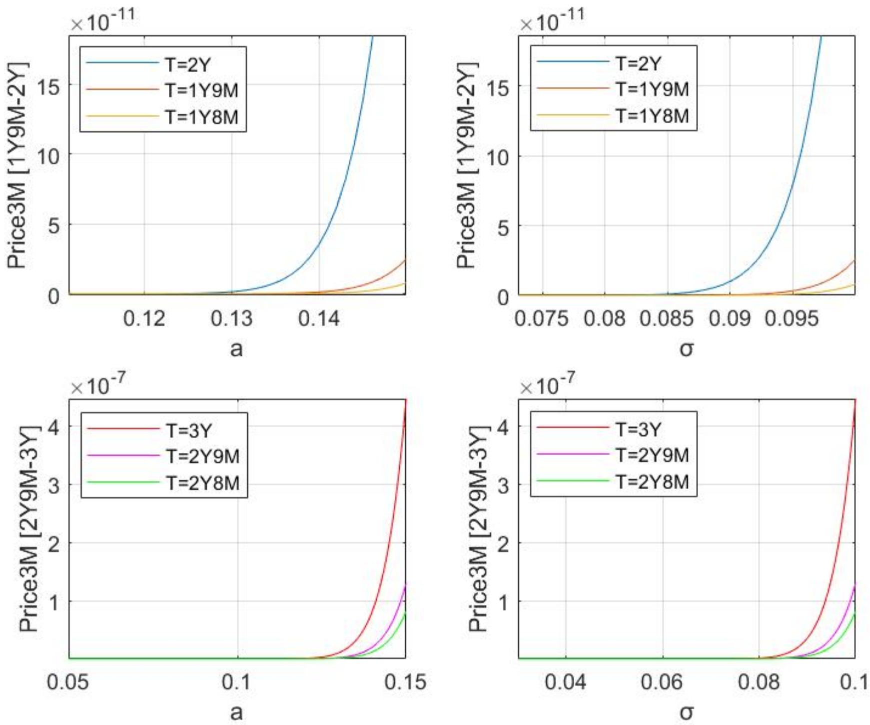

Figure 4.6: Option on 3M future's prices with different maturities T with respect to a and σ separately.

4.3. Option on 1M future's numerical analysis

We now perform the analysis on options on 1M futures, specifically on future *3* and *4*. Particularly, making use of the assumptions made in the first section, we implement the pricing formula of Proposition 5 in Matlab and see how it varies across the two decided ranges of values for the parameters a and σ. In this section we summarise and comment the results of our analysis. The detailed code can be found in Appendix B. In Figure 4.7, we plotted in blue the price of the option on future *3* and in red the price of the option on future *4*, with respect to both a and σ. As in the case of an option on 3M future, we notice that the prices display an increasing pattern. Moreover, as before, to better investigate their relationship, in Figure 4.8 we represented the option prices with respect to the two parameters separately.

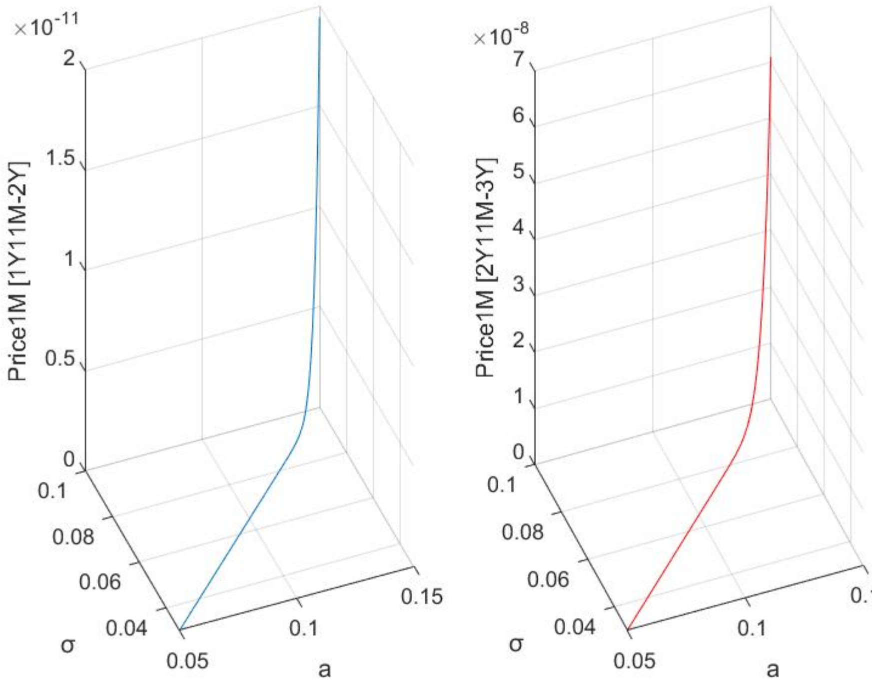

Figure 4.7: Option on 1M future's prices with respect to a and σ simultaneously.

As before, we witness that the net effect of the parameter a on the option price is positive.
The price still has a positive relationship with the volatility of the short rate σ, since an option is an instrument granting investors a form of insurance against the risk associated to fluctuations in the price.
Lastly, as in the previous case, for higher T_{j-1} and T_j (recall that option on future *4* has both T_{j-1} and T_j greater than option on future *3*) the price is greater, a result of the larger uncertainty of further in time investments.

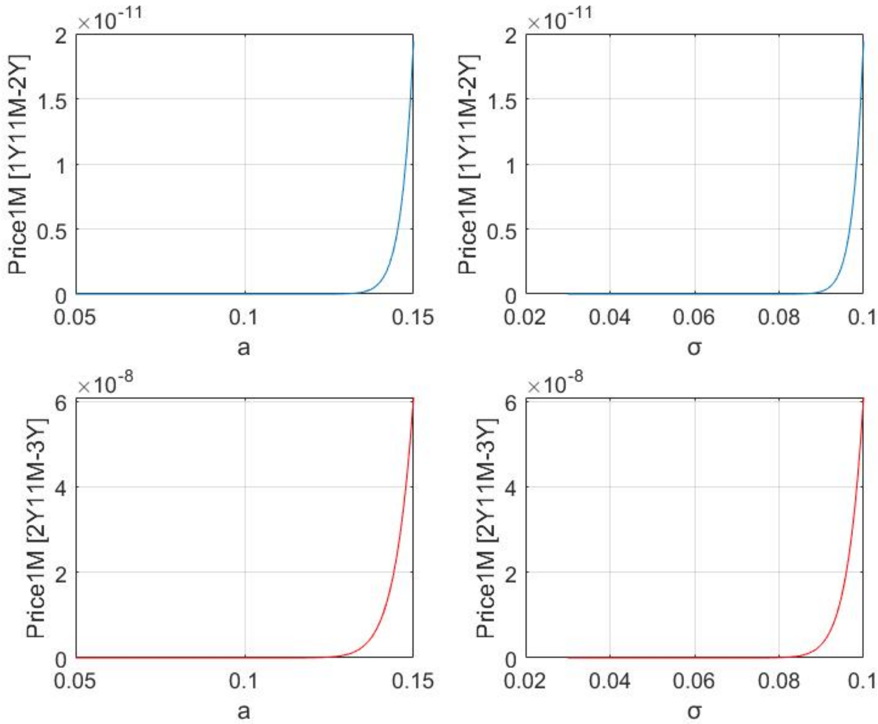

Figure 4.8: Option on 1M future's prices with respect to a and σ separately.

Remark 4 (***option price curve***)**.** Similar considerations regarding the price curve can be made for the option on 1M future. In fact, it seems equal to zero until a and σ reach relatively high values, in correspondence of which the curve becomes exponential. However this phenomenon is only related to the scale of the graphs. In fact, the 1M price's highest value is in the order of 10^{-10} and lowest of 10^{-70}. Thus, given the scale chosen, the price curve appears flattened toward the horizontal axis, but in reality there is still an exponential trend even when a and σ are low.

4.3.1 Change in the strike price

Now, we study how a change in the strike price K affects our findings. Particularly, as in the case of an option on 3M future, we analyse how the price curve moves when K increases or decreases by 0.05 points, that is the following cases:

- Option on future *3* has strike $K = 0.9423$;
- Option on future *3* has strike $K = 1.0423$;
- Option on future *4* has strike $K = 0.9892$;
- Option on future *4* has strike $K = 1.0892$.

Figures 4.9 and 4.10 summarise our results. As always, the first figure shows how option prices with different strikes simultaneously move with respect to a and σ, while the latter investigates the relationships separately.

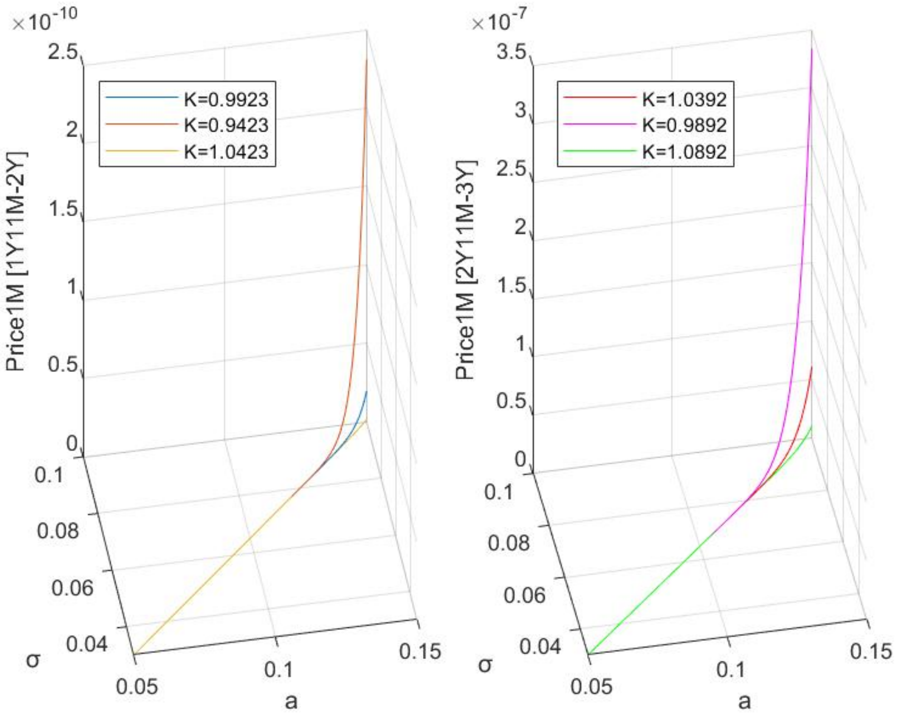

Figure 4.9: Option on 1M future's prices with different strike prices K with respect to a and σ simultaneously.

As before, even if the strike price changes, the price curve stays increasing with respect to the parameters a and σ.
Also, as the strike price increases, the call option price decreases, a result coming from the reluctance of investors to pay more today if they might have to pay a high strike price in the future. In fact, for the option on future 3, the orange line (that is, when K is decreased by 0.05) is above the blue line (which corresponds to the initial K), which in turn stands over the yellow line (that is, the case in which K is increased by 0.05). Similarly, for the option on future 4, the pink line (that is, when K is decreased by 0.05) sits above the red line (which corresponds to the initially set K), which is in turn over the green line (that is, the case in which K is increased by 0.05).

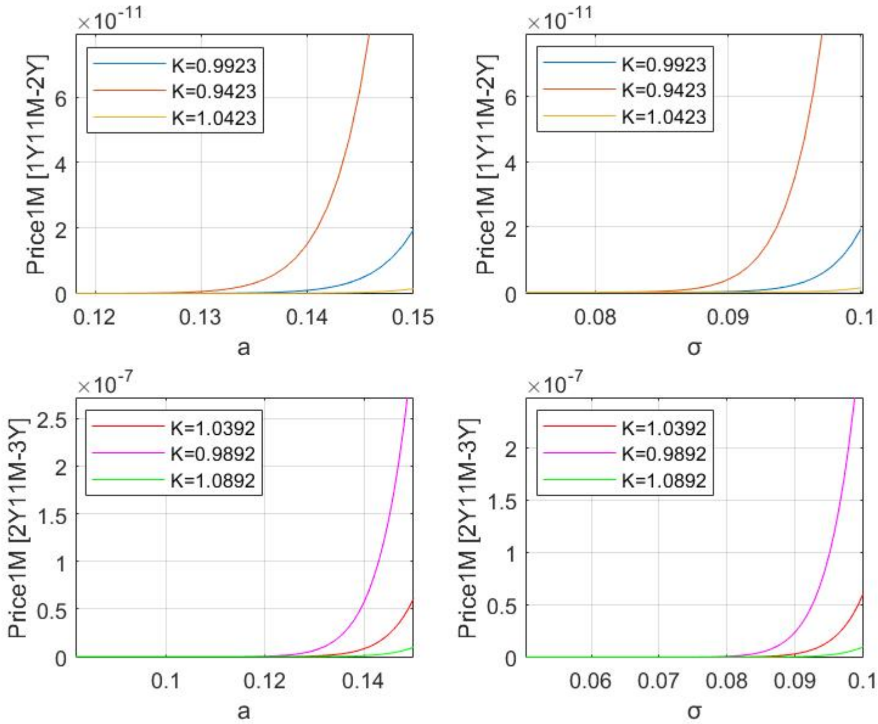

Figure 4.10: Option on 1M future's prices with different strike prices K with respect to a and σ separately.

4.3.2 Change in the maturity

Finally, as in the case of an option on 3M future, we study how a change in the maturity T affects our results. Particularly, we analyse how the price curve moves in response to a decrease of 1 and 2 months of T, that is the following cases:

- Option on future *3* has maturity 1 year 11 months;
- Option on future *3* has maturity 1 year 10 months;
- Option on future *4* has maturity 2 years 11 months;
- Option on future *4* has maturity 2 years 10 months.

Notice that, by reducing the maturity to a certain $T < T_j$, the underlying is no longer the RFR (as in the case of $T = T_j$), but the option expires into a 1M future contract. Since the price formula depends on $P(t, T)$, we need to compute the market price of the zero-coupon bonds at time $t = 0$ for the new maturities, using the usual formula and retrieving the yields from the ECB website. We get

T	$P(0,T)$
1 Year and 11 Months	0.9846
1 Year and 10 Months	0.9855
2 Years and 11 Months	0.9733
2 Years and 10 Months	0.9743

Figures 4.11 and 4.12 summarise our findings. Again, recall that the first figure shows how option prices with different maturities simultaneously move with respect to a and σ, while the latter investigates the relationships separately.

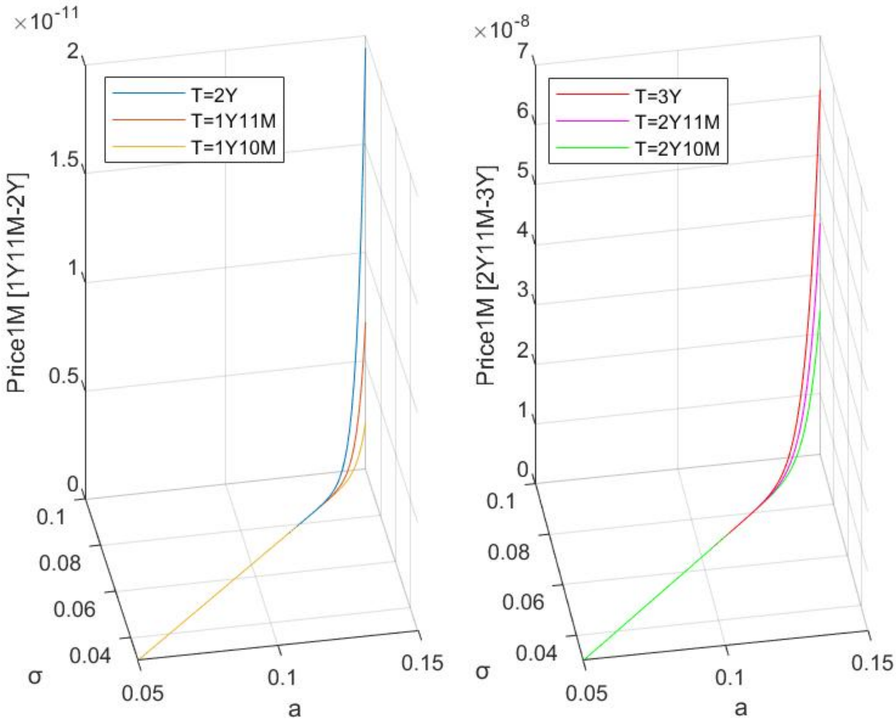

Figure 4.11: Option on 1M future's prices with different maturities T with respect to a and σ simultaneously.

As in the case of an option on 3M future, even if the maturity changes, the price curve stays increasing with respect to the parameters a and σ.
Also, as the maturity decreases, the option price decreases as well, since the time span across which the investors can encounter averse price movements decreases, thus they will be reluctant to pay high prices today. Indeed, for the option on future 3, the yellow line (that is, the case in which T is decreased by 2 months) is below the orange line (that is, when T is decreased by 1 months), which in turn sits under the blue line (which corresponds to the initial T). Similarly, for the option on future 4, the green line

(that is, the case in which T is reduced by 2 months) stays below the pink line (that is, when T is decreased by 1 months), which in turn stands under the red line (which corresponds to the initially set T).

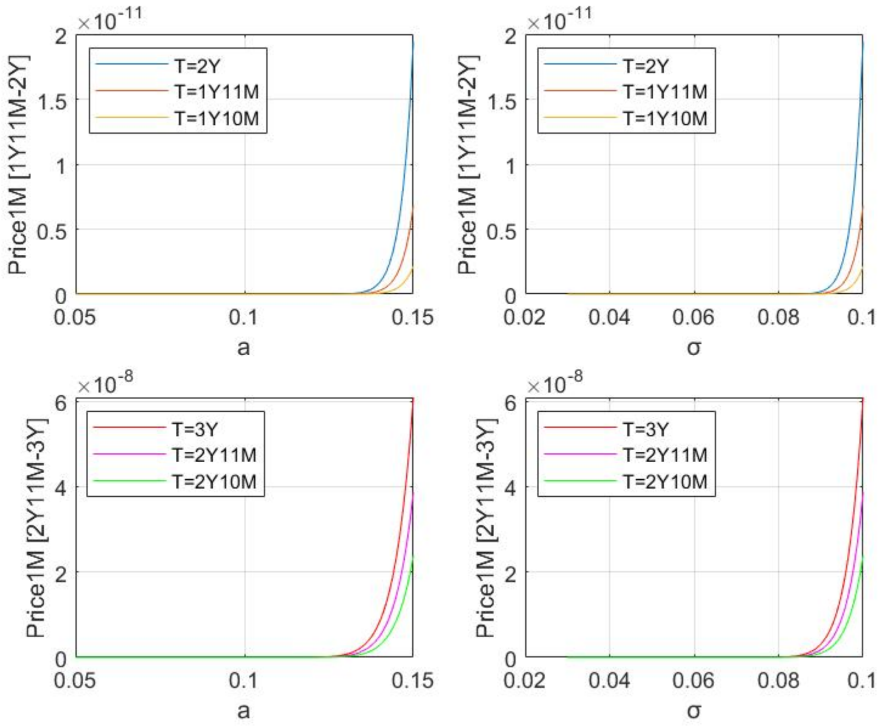

Figure 4.12: Option on 1M future's prices with different maturities T with respect to a and σ separately.

Conclusion

After the 2008 Financial Crisis, during which the rate experienced manipulation practises and interbank-market related issues, it became clear that LIBOR could no longer be used as a reference rate for market transactions. This is why, in 2017, the Financial Conduct Authority announced that markets will be transitioning away from LIBOR starting 2021. For this reason, jurisdictions have focused on selecting alternative new risk-free rates. These should be more reliable since they are anchored to effective market transactions and do not derive from the quotes of a panel of banks (as their predecessor). The main focus of this thesis has been analysing the characteristics of these new rates, studying possible solutions for modeling them and for pricing new kinds of derivatives. Particularly:

- In *Chapter 1*, we presented an overview of the characteristics of LIBOR. After that, we identified as the main reasons that brought to the reform:

    ➢ The *manipulation practices* of the rate, carried out by the panel of banks in charge of submitting the quotes of its estimation;

    ➢ The *Great Financial Crisis*, that made banks suffer substantial losses and general instability, which resulted into adverse selection in interbank lending;

    ➢ The *unconventional policies* enforced by Central Banks, which injected supply of reserve balances into markets and made banks less in need to trade with each other.

  Consequently, we analysed the main features of the new RFRs, which should be anchored to actual market transactions in active and liquid markets, in order to avoid the risk of manipulation. Afterwards, we presented specifically the characteristics of the Secured overnight financing rate (SOFR), the Euro short-term rate (ESTR) and the Sterling overnight index average (SONIA). Lastly, we discussed some drawbacks and challenges that the introduction of these new rates brings into markets. Particularly, the primary concern for financial markets is how to evaluate already existent and new kinds of derivatives.

- In *Chapter 2*, since the new reference rates are all overnight rates, we derived their term structure, so that they can be used in the evaluation of financial instruments. We identified two main approaches to obtain the term structure: either construct a *backward-looking rate*, based on past realisation of overnight rates and known at the end of the application period, or a *forward-looking rate*, which reflects the expectations of future realisations of the backward-looking rate and is known at the beginning of the application period.
  Afterwards, we presented some solutions for modelling RFRs proposed by different authors. In particular,
  - ➢ The Hull-White model as done in [7], in which the $\mathbb{Q}$-dynamics of the short rate of interest are characterised by a drift term, that depends on a time-dependent function $\Theta(t)$ and a constant a, and by a constant volatility term σ. This model seems to be the most popular choice for RFRs evaluation in current markets;
  - ➢ The Forward Market Model as in [28], which models both forward-looking (LIBOR-like) forward rates and backward-looking forward rates, providing a more complete extension to the LIBOR Market Model.
- In *Chapter 3*, we obtained an evaluation formula for a particular type of derivative referencing RFRs, called option on RFRs future. These are instruments composed by a future contract, which in turn constitutes the underlying of an option. Since it is currently the most popular in financial markets, we assumed that our interest rate follows the Hull-White dynamics.
  In current financial markets, there exist two types of futures depending on their maturity, namely 1-month (1M) and 3-months (3M) futures. Firstly, we derived two pricing formulas for these futures. Afterwards, we used our results to price the options. Notice that we obtained two different evaluation equations, one for options on 3M future and one for options on 1M futures. The option on 1M future's evaluation equation constitutes an original result. In fact, in the literature, there does not exist a comparable explicit evaluation formula to price this kind of instruments.
- In *Chapter 4*, we performed a numerical sensitivity analysis using the platform Matlab. Particularly, we studied how changes in the Hull-White model parameters a and σ across reasonable ranges of values affect the options on RFRs futures' prices. We also analysed how our results adjust in relation to changes in the strike price K and the maturity T of the option. We observed that

- ➢ Both options on 3M and 1M futures' prices follow an increasing pattern. They increase both with respect to the parameter a and the volatility σ (in fact, as risk increases, investors will be prone to pay more for a form of insurance);
- ➢ Both call options on 3M and 1M futures' prices reduce as the strike price K increases. In fact, investors will be reluctant to pay more today if they might pay a high strike price tomorrow;
- ➢ Both options on 3M and 1M futures' prices decrease if the maturity T is reduced. In fact, the lower the maturity, the lower is the time span across which the underlying can fluctuate adversely.

Concluding, in this thesis we were able to better understand the characteristics of the new RFRs and what are the possible solutions to model them that can be found in the literature. Also, we were able to obtain evaluation formulas for options on RFRs futures, that could be used after LIBOR's reform. Of course, our study was limited to a particular type of derivative referencing the new RFRs, that we chose since they are gaining a considerable popularity in current financial markets. Certainly, it would be interesting to find possible solutions to price other kinds of derivatives in future studies.

Appendix A

3M option code

Here is reported the code used for the analysis of the option on 3M future.

```
%HW MODEL PARAMETERS%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
a=linspace(0.05,0.15,100); %a varies from 5% to 15%
sigma=linspace(0.03,0.10,100); %sigma varies from 3% to 10%

%Find the function Theta(t) - function parameters at 23/08/2022
beta_0=1.641204;
beta_1=-1.766233;
beta_2=25.191842;
beta_3=-25.607108;
tau_1=1.344229;
tau_2=1.421672;
%Theta=@(a,x,sigma) a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1)
% .*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*%exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.%*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(%-2.*a.*x);

%Find prices of zero-coupon bonds for different maturities (euro-area yield
%curve at 23/08/2022)
AllBond=[-0.00113136 0.00195457 0.00408054 0.00554793 0.00767 0.00783 0.00823246 0.00902 0.00910 0.00934709 0.01027293 0.01115277 0.01194251 0.01261471 0.01317121 0.01362780 0.01400331 0.01431471 0.01457576 0.01479712 0.01498690 0.01515127 0.01529493 0.01542154 0.01553392
```

```
    0.01563434 0.01572458 0.01580612 0.01588014 0.01594762 0.01600938
    0.01606612 0.01611841 0.01616675 0.01621157 0.01625322 0.01629204];
T = [0.25 0.5 0.75 1 5/3 1.75 2 8/3 2.75 3 4 5 6 7 8 9 10 11 12 13 14 15
    16 17 18 19 20 21 22 23 24 25 26 27 28 29 30];
pT=zeros(1,37);
for i=1:37
pT(1,i)=exp(-(T(i)*AllBond(i)));
end

%OPTION ON 3M FUTURE%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%1st case: [1Year+9months 2 Years] future contract, option has maturity 2
    years, strike 1.124.
tau_j=0.25;
T_j=2;
T_j1=1.75;
T=2;
t=0;
r_0=-0.00082; %ESTR value on 23/08/2022
K=1.124;
p2Y=pT(1,7);

%find delta
fun = @(x,a,sigma) (1-exp(-a.*(T_j-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1
    )+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2)
    )-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp
    (-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma
    .^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int1=@(a,sigma)integral(@(x)fun(x,a,sigma),T_j1,T_j);

int_v=zeros(1,100);
for i=1:100
int_v(1,i)=int1(a(i),sigma(i));
end

fun2 = @(x,a,sigma) (exp(-a.*(T_j1-x))-exp(-a.*(T_j-x))).*(a.*(beta_0+
    beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./
```

```
    tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./
    tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*
    exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int2=@(a,sigma)integral(@(x)fun2(x,a,sigma),T,T_j1);

int_v2=zeros(1,100);
for i=1:100
int_v2(1,i)=int2(a(i),sigma(i));
end

delta1=zeros(1,100);
for i=1:100
delta1(1,i)=(1/a(i))*int_v2(i)+(1/a(i))*int_v(i);
end

delta2=zeros(1,100);
for i=1:100
delta2(1,i)=(1/2)*((sigma(i)^2)/(a(i)^2))*((1/a(i))*(exp(-a(i)*(T_j-T_j1))
    +exp(-a(i)*(T_j+T_j1-2*T))-1)-(1/(2*a(i)))*(exp(-2*a(i)*(T_j-T))+exp
    (-2*a(i)*(T_j1-T)))+(T_j-T_j1));
end

delta=zeros(1,100);
for i=1:100
delta(1,i)=delta1(i)+delta2(i);
end

%find gamma
gamma=zeros(1,100);
for i=1:100
gamma(1,i)=(exp(-a(i)*(T_j1-T))-exp(-a(i)*(T_j-T)))/(a(i));
end

%find alpha
fun3 = @(x,a,sigma) (exp(-a.*(T-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+
    beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))
    -(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-
```

```
    x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)
    ./(2.*a)).*(1-exp(-2.*a.*x)));
int3=@(a,sigma)integral(@(x)fun3(x,a,sigma),t,T);

int_v3=zeros(1,100);
for i=1:100
int_v3(1,i)=int3(a(i),sigma(i));
end

alpha1=zeros(1,100);
for i=1:100
alpha1(1,i)=r_0*exp(-a(i)*(T-t))-((sigma(i)^2)/(a(i)^2))-((sigma(i)^2)/(a(
    i)^2))*exp(-a(i)*(T-t))+((sigma(i)^2)/(2*a(i)^2))-((sigma(i)^2)/(2*a(i)
    ^2))*exp(-a(i)*(2*T-2*t));
end

alpha=zeros(1,100);
for i=1:100
alpha(1,i)=alpha1(i)+int_v3(i);
end

%find beta
beta=zeros(1,100);
for i=1:100
beta(1,i)=((sigma(i)^2)/(2*a(i)))-((sigma(i)^2)/(2*a(i)))*exp(-2*a(i)*(T-t
    ));
end

%find Y_t, N(d_1), N(d_2)
Y_t=zeros(1,100);
for i=1:100
Y_t(1,i)=exp(delta(i)+gamma(i)*alpha(i)+(1/2)*(gamma(i).^2)*beta(i));
end

d2_3=zeros(1,100);
for i=1:100
```

```
d2_3(1,i)=(log(Y_t(i)/(tau_j*K+1))-(gamma(i)^2)*(beta(i)/2))/(gamma(i)*
    sqrt(beta(i)));
end

N_d2_3=zeros(1,100);
for i=1:100
N_d2_3(1,i)=normcdf(d2_3(i));
end

d1_3=zeros(1,100);
for i=1:100
    d1_3(1,i)=d2_3(i)+gamma(i)*sqrt(beta(i));
end

N_d1_3=zeros(1,100);
for i=1:100
N_d1_3(1,i)=normcdf(d1_3(i));
end

Price3M=zeros(1,100);
for i=1:100
Price3M(1,i)=(p2Y/tau_j)*(Y_t(i)*N_d1_3(i)-(1+tau_j*K)*N_d2_3(i));
end

%2nd case: [2Years+9Months 3Years] future contract, option has maturity
% 3 years, strike 1.1798.
K2=1.1798;
T_j_2=3;
T_j1_2=2.75;
T_2=3;
p3Y=pT(1,10);

%find delta
fun_2 = @(x,a,sigma) (1-exp(-a.*(T_j_2-x))).*(a.*(beta_0+beta_1.*exp(-x./
    tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./
    tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2)
    .*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((
```

```
sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int1_2=@(a,sigma)integral(@(x)fun_2(x,a,sigma),T_j1_2,T_j_2);

int_v_2=zeros(1,100);
for i=1:100
    int_v_2(1,i)=int1_2(a(i),sigma(i));
end

fun2_2 = @(x,a,sigma) (exp(-a.*(T_j1_2-x))-exp(-a.*(T_j_2-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int2_2=@(a,sigma)integral(@(x)fun2_2(x,a,sigma),T_2,T_j1_2);

int_v2_2=zeros(1,100);
for i=1:100
int_v2_2(1,i)=int2_2(a(i),sigma(i));
end

delta1_2=zeros(1,100);
for i=1:100
delta1_2(1,i)=(1/a(i))*int_v2_2(i)+(1/a(i))*int_v_2(i);
end

delta2_2=zeros(1,100);
for i=1:100
delta2_2(1,i)=(1/2)*((sigma(i)^2)/(a(i)^2))*((1/a(i))*(exp(-a(i)*(T_j_2-T_j1_2))+exp(-a(i)*(T_j_2+T_j1_2-2*T_2))-1)-(1/(2*a(i)))*(exp(-2*a(i)*(T_j_2-T_2))+exp(-2*a(i)*(T_j1_2-T_2)))+(T_j_2-T_j1_2));
end

delta_2=zeros(1,100);
for i=1:100
delta_2(1,i)=delta1_2(i)+delta2_2(i);
end

```

```
%find gamma
gamma_2=zeros(1,100);
for i=1:100
gamma_2(1,i)=(exp(-a(i)*(T_j1_2-T_2))-exp(-a(i)*(T_j_2-T_2)))/(a(i));
end

%find alpha
fun3_2 = @(x,a,sigma) (exp(-a.*(T_2-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int3_2=@(a,sigma)integral(@(x)fun3_2(x,a,sigma),t,T_2);

int_v3_2=zeros(1,100);
for i=1:100
int_v3_2(1,i)=int3_2(a(i),sigma(i));
end

alpha1_2=zeros(1,100);
for i=1:100
alpha1_2(1,i)=r_0*exp(-a(i)*(T_2-t))-((sigma(i)^2)/(a(i)^2))-((sigma(i)^2)/(a(i)^2))*exp(-a(i)*(T_2-t))+((sigma(i)^2)/(2*a(i)^2))-((sigma(i)^2)/(2*a(i)^2))*exp(-a(i)*(2*T_2-2*t));
end

alpha_2=zeros(1,100);
for i=1:100
alpha_2(1,i)=alpha1_2(i)+int_v3_2(i);
end

%find beta
beta2=zeros(1,100);
for i=1:100
beta2(1,i)=((sigma(i)^2)/(2*a(i)))-((sigma(i)^2)/(2*a(i)))*exp(-2*a(i)*(T_2-t));
end
```

```
%find Y_t, N(d_1), N(d_2)
Y_t_2=zeros(1,100);
for i=1:100
Y_t_2(1,i)=exp(delta_2(i)+gamma_2(i)*alpha_2(i)+(1/2)*(gamma_2(i).^2)*
    beta2(i));
end

d2_3_2=zeros(1,100);
for i=1:100
d2_3_2(1,i)=(log(Y_t_2(i)/(tau_j*K2+1))-(gamma_2(i)^2)*(beta2(i)/2))/(
    gamma_2(i)*sqrt(beta2(i)));
end

N_d2_3_2=zeros(1,100);
for i=1:100
N_d2_3_2(1,i)=normcdf(d2_3_2(i));
end

d1_3_2=zeros(1,100);
for i=1:100
d1_3_2(1,i)=d2_3_2(i)+gamma_2(i)*sqrt(beta2(i));
end

N_d1_3_2=zeros(1,100);
for i=1:100
N_d1_3_2(1,i)=normcdf(d1_3_2(i));
end

Price3M_2=zeros(1,100);
for i=1:100
Price3M_2(1,i)=(p3Y/tau_j)*(Y_t_2(i)*N_d1_3_2(i)-(1+tau_j*K2)*N_d2_3_2(i))
    ;
end

figure
subplot(1,2,1)
```

```
plot3(a, sigma, Price3M)
xlabel('a')
ylabel('sigma')
zlabel('Price3M [1Y9M-2Y]')
grid on
subplot(1,2,2)
plot3( a, sigma, Price3M_2, 'r')
xlabel('a')
ylabel('sigma')
zlabel('Price3M [2Y9M-3Y]')
grid on

figure
subplot(2,2,1)
plot(a, Price3M)
xlabel('a')
ylabel('Price3M [1Y9M-2Y]')
axis tight
grid on
subplot(2,2,2)
plot(sigma, Price3M)
xlabel('sigma')
ylabel('Price3M [1Y9M-2Y]')
axis tight
grid on
subplot(2,2,3)
plot( a, Price3M_2, 'r')
xlabel('a')
ylabel('Price3M [2Y9M-3Y]')
axis tight
grid on
subplot(2,2,4)
plot(sigma, Price3M_2, 'r')
xlabel('sigma')
ylabel('Price3M [2Y9M-3Y]')
grid on
axis tight
```

```
%reduce the strike by 0.05

%1st case: [1Year+9months 2 Years] future contract, option has maturity 2
    years, strike 1.074.
K_2=1.074;

d2_3_K2=zeros(1,100);
for i=1:100
d2_3_K2(1,i)=(log(Y_t(i)/(tau_j*K_2+1))-(gamma(i)^2)*(beta(i)/2))/(gamma(i
    )*sqrt(beta(i)));
end

N_d2_3_K2=zeros(1,100);
for i=1:100
N_d2_3_K2(1,i)=normcdf(d2_3_K2(i));
end

d1_3_K2=zeros(1,100);
for i=1:100
    d1_3_K2(1,i)=d2_3_K2(i)+gamma(i)*sqrt(beta(i));
end

N_d1_3_K2=zeros(1,100);
for i=1:100
N_d1_3_K2(1,i)=normcdf(d1_3_K2(i));
end

Price3M_K2=zeros(1,100);
for i=1:100
Price3M_K2(1,i)=(p2Y/tau_j)*(Y_t(i)*N_d1_3_K2(i)-(1+tau_j*K_2)*N_d2_3_K2(i
    ));
end

%2nd case: [2Years+9Months 3Years] future contract, option has maturity
% 3 years, strike 1.1298.
K2_2=1.1298;
```

```
d2_3_2_K2=zeros(1,100);
for i=1:100
d2_3_2_K2(1,i)=(log(Y_t_2(i)/(tau_j*K2_2+1))-(gamma_2(i)^2)*(beta2(i)/2))
    /(gamma_2(i)*sqrt(beta2(i)));
end

N_d2_3_2_K2=zeros(1,100);
for i=1:100
N_d2_3_2_K2(1,i)=normcdf(d2_3_2_K2(i));
end

d1_3_2_K2=zeros(1,100);
for i=1:100
    d1_3_2_K2(1,i)=d2_3_2_K2(i)+gamma_2(i)*sqrt(beta2(i));
end

N_d1_3_2_K2=zeros(1,100);
for i=1:100
N_d1_3_2_K2(1,i)=normcdf(d1_3_2_K2(i));
end

Price3M_2_K2=zeros(1,100);
for i=1:100
Price3M_2_K2(1,i)=(p3Y/tau_j)*(Y_t_2(i)*N_d1_3_2_K2(i)-(1+tau_j*K2_2)*
    N_d2_3_2_K2(i));
end

%increase the strike by 0.05

%1st case: [1Year+9months 2 Years] future contract, option has maturity 2
    years, strike 1.174.
K_3=1.174;

d2_3_K3=zeros(1,100);
for i=1:100
d2_3_K3(1,i)=(log(Y_t(i)/(tau_j*K_3+1))-(gamma(i)^2)*(beta(i)/2))/(gamma(i
    )*sqrt(beta(i)));
```

```
end

N_d2_3_K3=zeros(1,100);
for i=1:100
N_d2_3_K3(1,i)=normcdf(d2_3_K3(i));
end

d1_3_K3=zeros(1,100);
for i=1:100
    d1_3_K3(1,i)=d2_3_K3(i)+gamma(i)*sqrt(beta(i));
end

N_d1_3_K3=zeros(1,100);
for i=1:100
N_d1_3_K3(1,i)=normcdf(d1_3_K3(i));
end

Price3M_K3=zeros(1,100);
for i=1:100
Price3M_K3(1,i)=(p2Y/tau_j)*(Y_t(i)*N_d1_3_K3(i)-(1+tau_j*K_3)*N_d2_3_K3(i));
end

%2nd case: [2Years+9Months 3Years] future contract, option has maturity
% 3 years, strike 1.2298.
K2_3=1.2298;

d2_3_2_K3=zeros(1,100);
for i=1:100
d2_3_2_K3(1,i)=(log(Y_t_2(i)/(tau_j*K2_3+1))-(gamma_2(i)^2)*(beta2(i)/2))/(gamma_2(i)*sqrt(beta2(i)));
end

N_d2_3_2_K3=zeros(1,100);
for i=1:100
N_d2_3_2_K3(1,i)=normcdf(d2_3_2_K3(i));
end
```

```
d1_3_2_K3=zeros(1,100);
for i=1:100
    d1_3_2_K3(1,i)=d2_3_2_K3(i)+gamma_2(i)*sqrt(beta2(i));
end

N_d1_3_2_K3=zeros(1,100);
for i=1:100
N_d1_3_2_K3(1,i)=normcdf(d1_3_2_K3(i));
end

Price3M_2_K3=zeros(1,100);
for i=1:100
Price3M_2_K3(1,i)=(p3Y/tau_j)*(Y_t_2(i)*N_d1_3_2_K3(i)-(1+tau_j*K2_3)*
    N_d2_3_2_K3(i));
end

figure
subplot(1,2,1)
plot3(a, sigma, Price3M, a, sigma, Price3M_K2, a, sigma, Price3M_K3)
xlabel('a')
ylabel('sigma')
zlabel('Price3M [1Y9M-2Y]')
grid on
legend('K=1.1248','K=1.0748','K=1.1748', 'Orientation','vertical')
subplot(1,2,2)
plot3( a, sigma, Price3M_2, 'r', a, sigma, Price3M_2_K2, 'm', a, sigma,
    Price3M_2_K3, 'g')
xlabel('a')
ylabel('sigma')
zlabel('Price3M [2Y9M-3Y]')
grid on
legend('K=1.1798','K=1.1298','K=1.2298', 'Orientation','vertical')

figure
subplot(2,2,1)
plot(a, Price3M, a, Price3M_K2, a, Price3M_K3)
```

```
xlabel('a')
ylabel('Price3M [1Y9M—2Y]')
axis tight
grid on
legend('K=1.1248','K=1.0748','K=1.1748', 'Orientation','vertical')
subplot(2,2,2)
plot(sigma, Price3M, sigma, Price3M_K2, sigma, Price3M_K3)
xlabel('sigma')
ylabel('Price3M [1Y9M—2Y]')
axis tight
grid on
legend('K=1.1248','K=1.0748','K=1.1748', 'Orientation','vertical')
subplot(2,2,3)
plot( a, Price3M_2, 'r', a, Price3M_2_K2, 'm', a, Price3M_2_K3, 'g')
xlabel('a')
ylabel('Price3M [2Y9M—3Y]')
axis tight
grid on
legend('K=1.1798','K=1.1298','K=1.2298', 'Orientation','vertical')
subplot(2,2,4)
plot(sigma, Price3M_2, 'r', sigma, Price3M_2_K2, 'm', sigma, Price3M_2_K3,
    'g')
xlabel('sigma')
ylabel('Price3M [2Y9M—3Y]')
grid on
axis tight
legend('K=1.1798','K=1.1298','K=1.2298', 'Orientation','vertical')

%reduce the maturity by 3 months

%1st case: [1Year+9months 2 Years] future contract, option has maturity 1
    year 9 months, strike 1.1248.
T2=1.75;
p1Y9M=pT(1,6);

%find delta
```

```
fun2_T2 = @(x,a,sigma) (exp(-a.*(T_j1-x))-exp(-a.*(T_j-x))).*(a.*(beta_0+
    beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./
    tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./
    tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*
    exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int2_T2=@(a,sigma)integral(@(x)fun2_T2(x,a,sigma),T2,T_j1);

int_v2_T2=zeros(1,100);
for i=1:100
int_v2_T2(1,i)=int2_T2(a(i),sigma(i));
end

delta1_T2=zeros(1,100);
for i=1:100
delta1_T2(1,i)=(1/a(i))*int_v2_T2(i)+(1/a(i))*int_v(i);
end

delta2_T2=zeros(1,100);
for i=1:100
delta2_T2(1,i)=(1/2)*((sigma(i)^2)/(a(i)^2))*((1/a(i))*(exp(-a(i)*(T_j-
    T_j1))+exp(-a(i)*(T_j+T_j1-2*T2))-1)-(1/(2*a(i)))*(exp(-2*a(i)*(T_j-T2)
    )+exp(-2*a(i)*(T_j1-T2)))+(T_j-T_j1));
end

delta_T2=zeros(1,100);
for i=1:100
delta_T2(1,i)=delta1_T2(i)+delta2_T2(i);
end

%find gamma
gamma_T2=zeros(1,100);
for i=1:100
gamma_T2(1,i)=(exp(-a(i)*(T_j1-T2))-exp(-a(i)*(T_j-T2)))/(a(i));
end

%find alpha
```

```
fun3_T2 = @(x,a,sigma) (exp(-a.*(T2-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int3_T2=@(a,sigma)integral(@(x)fun3_T2(x,a,sigma),t,T2);

int_v3_T2=zeros(1,100);
for i=1:100
int_v3_T2(1,i)=int3_T2(a(i),sigma(i));
end

alpha1_T2=zeros(1,100);
for i=1:100
alpha1_T2(1,i)=r_0*exp(-a(i)*(T2-t))-((sigma(i)^2)/(a(i)^2))-((sigma(i)^2)/(a(i)^2))*exp(-a(i)*(T2-t))+((sigma(i)^2)/(2*a(i)^2))-((sigma(i)^2)/(2*a(i)^2))*exp(-a(i)*(2*T2-2*t));
end

alpha_T2=zeros(1,100);
for i=1:100
alpha_T2(1,i)=alpha1_T2(i)+int_v3_T2(i);
end

%find beta
beta_T2=zeros(1,100);
for i=1:100
beta_T2(1,i)=((sigma(i)^2)/(2*a(i)))-((sigma(i)^2)/(2*a(i)))*exp(-2*a(i)*(T2-t));
end

%find Y_t, N(d_1), N(d_2)
Y_t_T2=zeros(1,100);
for i=1:100
Y_t_T2(1,i)=exp(delta_T2(i)+gamma_T2(i)*alpha_T2(i)+(1/2)*(gamma_T2(i).^2)*beta_T2(i));
end
```

```
d2_3_T2=zeros(1,100);
for i=1:100
d2_3_T2(1,i)=(log(Y_t_T2(i)/(tau_j*K+1))-(gamma_T2(i)^2)*(beta_T2(i)/2))/(
    gamma_T2(i)*sqrt(beta_T2(i)));
end

N_d2_3_T2=zeros(1,100);
for i=1:100
N_d2_3_T2(1,i)=normcdf(d2_3_T2(i));
end

d1_3_T2=zeros(1,100);
for i=1:100
    d1_3_T2(1,i)=d2_3_T2(i)+gamma_T2(i)*sqrt(beta_T2(i));
end

N_d1_3_T2=zeros(1,100);
for i=1:100
N_d1_3_T2(1,i)=normcdf(d1_3_T2(i));
end

Price3M_T2=zeros(1,100);
for i=1:100
Price3M_T2(1,i)=(p1Y9M/tau_j)*(Y_t_T2(i)*N_d1_3_T2(i)-(1+tau_j*K)*
    N_d2_3_T2(i));
end

%2nd case: [2Years+9Months 3Years] future contract, option has maturity
% 2Yers 9 Months, strike 1.1798.
T_2_2=2.75;
p2Y9M=pT(1,9);

%find delta
fun2_2_T2 = @(x,a,sigma) (exp(-a.*(T_j1_2-x))-exp(-a.*(T_j_2-x))).*(a.*(
    beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3
    .*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2
```

```
.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int2_2_T2=@(a,sigma)integral(@(x)fun2_2_T2(x,a,sigma),T_2_2,T_j1_2);

int_v2_2_T2=zeros(1,100);
for i=1:100
int_v2_2_T2(1,i)=int2_2_T2(a(i),sigma(i));
end

delta1_2_T2=zeros(1,100);
for i=1:100
delta1_2_T2(1,i)=(1/a(i))*int_v2_2_T2(i)+(1/a(i))*int_v_2(i);
end

delta2_2_T2=zeros(1,100);
for i=1:100
delta2_2_T2(1,i)=(1/2)*((sigma(i)^2)/(a(i)^2))*((1/a(i))*(exp(-a(i)*(T_j_2-T_j1_2))+exp(-a(i)*(T_j_2+T_j1_2-2*T_2_2))-1)-(1/(2*a(i)))*(exp(-2*a(i)*(T_j_2-T_2_2))+exp(-2*a(i)*(T_j1_2-T_2_2)))+(T_j_2-T_j1_2));
end

delta_2_T2=zeros(1,100);
for i=1:100
delta_2_T2(1,i)=delta1_2_T2(i)+delta2_2_T2(i);
end

%find gamma
gamma_2_T2=zeros(1,100);
for i=1:100
gamma_2_T2(1,i)=(exp(-a(i)*(T_j1_2-T_2_2))-exp(-a(i)*(T_j_2-T_2_2)))/(a(i));
end

%find alpha
fun3_2_T2 = @(x,a,sigma)(exp(-a.*(T_2_2-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2)
```

```
.*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int3_2_T2=@(a,sigma)integral(@(x)fun3_2_T2(x,a,sigma),t,T_2_2);

int_v3_2_T2=zeros(1,100);
for i=1:100
int_v3_2_T2(1,i)=int3_2_T2(a(i),sigma(i));
end

alpha1_2_T2=zeros(1,100);
for i=1:100
alpha1_2_T2(1,i)=r_0*exp(-a(i)*(T_2_2-t))-((sigma(i)^2)/(a(i)^2))-((sigma(i)^2)/(a(i)^2))*exp(-a(i)*(T_2_2-t))+((sigma(i)^2)/(2*a(i)^2))-((sigma(i)^2)/(2*a(i)^2))*exp(-a(i)*(2*T_2_2-2*t));
end

alpha_2_T2=zeros(1,100);
for i=1:100
alpha_2_T2(1,i)=alpha1_2_T2(i)+int_v3_2_T2(i);
end

%find beta
beta2_T2=zeros(1,100);
for i=1:100
beta2_T2(1,i)=((sigma(i)^2)/(2*a(i)))-((sigma(i)^2)/(2*a(i)))*exp(-2*a(i)*(T_2_2-t));
end

%find Y_t, N(d_1), N(d_2)
Y_t_2_T2=zeros(1,100);
for i=1:100
Y_t_2_T2(1,i)=exp(delta_2_T2(i)+gamma_2_T2(i)*alpha_2_T2(i)+(1/2)*(gamma_2_T2(i).^2)*beta2_T2(i));
end

d2_3_2_T2=zeros(1,100);
for i=1:100
```

```
d2_3_2_T2(1,i)=(log(Y_t_2_T2(i)/(tau_j*K2+1))-(gamma_2_T2(i)^2)*(beta2_T2(
    i)/2))/(gamma_2_T2(i)*sqrt(beta2_T2(i)));
end

N_d2_3_2_T2=zeros(1,100);
for i=1:100
N_d2_3_2_T2(1,i)=normcdf(d2_3_2_T2(i));
end

d1_3_2_T2=zeros(1,100);
for i=1:100
    d1_3_2_T2(1,i)=d2_3_2_T2(i)+gamma_2_T2(i)*sqrt(beta2_T2(i));
end

N_d1_3_2_T2=zeros(1,100);
for i=1:100
N_d1_3_2_T2(1,i)=normcdf(d1_3_2_T2(i));
end

Price3M_2_T2=zeros(1,100);
for i=1:100
Price3M_2_T2(1,i)=(p2Y9M/tau_j)*(Y_t_2_T2(i)*N_d1_3_2_T2(i)-(1+tau_j*K2)*
    N_d2_3_2_T2(i));
end

%reduce the maturity by 4 months

%1st case: [1Year+9months 2 Years] future contract, option has maturity 1
    year 8 months, strike 1.1248.
T3=5/3;
p1Y8M=pT(1,5);

%find delta
fun2_T3 = @(x,a,sigma) (exp(-a.*(T_j1-x))-exp(-a.*(T_j-x))).*(a.*(beta_0+
    beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./
    tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./
    tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*
```

```
exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int2_T3=@(a,sigma)integral(@(x)fun2_T3(x,a,sigma),T3,T_j1);

int_v2_T3=zeros(1,100);
for i=1:100
int_v2_T3(1,i)=int2_T3(a(i),sigma(i));
end

delta1_T3=zeros(1,100);
for i=1:100
delta1_T3(1,i)=(1/a(i))*int_v2_T3(i)+(1/a(i))*int_v(i);
end

delta2_T3=zeros(1,100);
for i=1:100
delta2_T3(1,i)=(1/2)*((sigma(i)^2)/(a(i)^2))*((1/a(i))*(exp(-a(i)*(T_j-T_j1))+exp(-a(i)*(T_j+T_j1-2*T3))-1)-(1/(2*a(i)))*(exp(-2*a(i)*(T_j-T3))+exp(-2*a(i)*(T_j1-T3)))+(T_j-T_j1));
end

delta_T3=zeros(1,100);
for i=1:100
delta_T3(1,i)=delta1_T3(i)+delta2_T3(i);
end

%find gamma
gamma_T3=zeros(1,100);
for i=1:100
gamma_T3(1,i)=(exp(-a(i)*(T_j1-T3))-exp(-a(i)*(T_j-T3)))/(a(i));
end

%find alpha
fun3_T3 = @(x,a,sigma) (exp(-a.*(T3-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
```

```
int3_T3=@(a,sigma)integral(@(x)fun3_T3(x,a,sigma),t,T3);

int_v3_T3=zeros(1,100);
for i=1:100
int_v3_T3(1,i)=int3_T3(a(i),sigma(i));
end

alpha1_T3=zeros(1,100);
for i=1:100
alpha1_T3(1,i)=r_0*exp(-a(i)*(T3-t))-((sigma(i)^2)/(a(i)^2))-((sigma(i)^2)/(a(i)^2))*exp(-a(i)*(T3-t))+((sigma(i)^2)/(2*a(i)^2))-((sigma(i)^2)/(2*a(i)^2))*exp(-a(i)*(2*T3-2*t));
end

alpha_T3=zeros(1,100);
for i=1:100
alpha_T3(1,i)=alpha1_T3(i)+int_v3_T3(i);
end

%find beta
beta_T3=zeros(1,100);
for i=1:100
beta_T3(1,i)=((sigma(i)^2)/(2*a(i)))-((sigma(i)^2)/(2*a(i)))*exp(-2*a(i)*(T3-t));
end

%find Y_t, N(d_1), N(d_2)
Y_t_T3=zeros(1,100);
for i=1:100
Y_t_T3(1,i)=exp(delta_T3(i)+gamma_T3(i)*alpha_T3(i)+(1/2)*(gamma_T3(i).^2)*beta_T3(i));
end

d2_3_T3=zeros(1,100);
for i=1:100
d2_3_T3(1,i)=(log(Y_t_T3(i)/(tau_j*K+1))-(gamma_T3(i)^2)*(beta_T3(i)/2))/(gamma_T3(i)*sqrt(beta_T3(i)));
```

```
end

N_d2_3_T3=zeros(1,100);
for i=1:100
N_d2_3_T3(1,i)=normcdf(d2_3_T3(i));
end

d1_3_T3=zeros(1,100);
for i=1:100
d1_3_T3(1,i)=d2_3_T3(i)+gamma_T3(i)*sqrt(beta_T3(i));
end


N_d1_3_T3=zeros(1,100);
for i=1:100
N_d1_3_T3(1,i)=normcdf(d1_3_T3(i));
end


Price3M_T3=zeros(1,100);
for i=1:100
Price3M_T3(1,i)=(p1Y8M/tau_j)*(Y_t_T3(i)*N_d1_3_T3(i)-(1+tau_j*K)*
    N_d2_3_T3(i));
end


%2nd case: [2Years+9Months 3Years] future contract, option has maturity
% 2Yers 8 Months, strike 1.1798.
T_2_3=8/3;
p2Y8M=pT(1,8);

%find delta
fun2_2_T3 = @(x,a,sigma) (exp(-a.*(T_j1_2-x))-exp(-a.*(T_j_2-x))).*(a.*(
    beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3
    .*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2
    .*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2
    .^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
```

```
int2_2_T3=@(a,sigma)integral(@(x)fun2_2_T3(x,a,sigma),T_2_3,T_j1_2);

int_v2_2_T3=zeros(1,100);
for i=1:100
int_v2_2_T3(1,i)=int2_2_T3(a(i),sigma(i));
end

delta1_2_T3=zeros(1,100);
for i=1:100
delta1_2_T3(1,i)=(1/a(i))*int_v2_2_T3(i)+(1/a(i))*int_v_2(i);
end

delta2_2_T3=zeros(1,100);
for i=1:100
delta2_2_T3(1,i)=(1/2)*((sigma(i)^2)/(a(i)^2))*((1/a(i))*(exp(-a(i)*(T_j_2-T_j1_2))+exp(-a(i)*(T_j_2+T_j1_2-2*T_2_3))-1)-(1/(2*a(i)))*(exp(-2*a(i)*(T_j_2-T_2_3))+exp(-2*a(i)*(T_j1_2-T_2_3)))+(T_j_2-T_j1_2));
end

delta_2_T3=zeros(1,100);
for i=1:100
delta_2_T3(1,i)=delta1_2_T3(i)+delta2_2_T3(i);
end

%find gamma
gamma_2_T3=zeros(1,100);
for i=1:100
gamma_2_T3(1,i)=(exp(-a(i)*(T_j1_2-T_2_3))-exp(-a(i)*(T_j_2-T_2_3)))/(a(i));
end

%find alpha
fun3_2_T3 = @(x,a,sigma) (exp(-a.*(T_2_3-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
```

```
int3_2_T3=@(a,sigma)integral(@(x)fun3_2_T3(x,a,sigma),t,T_2_3);

int_v3_2_T3=zeros(1,100);
for i=1:100
int_v3_2_T3(1,i)=int3_2_T3(a(i),sigma(i));
end

alpha1_2_T3=zeros(1,100);
for i=1:100
alpha1_2_T3(1,i)=r_0*exp(-a(i)*(T_2_3-t))-((sigma(i)^2)/(a(i)^2))-((sigma(i)^2)/(a(i)^2))*exp(-a(i)*(T_2_3-t))+((sigma(i)^2)/(2*a(i)^2))-((sigma(i)^2)/(2*a(i)^2))*exp(-a(i)*(2*T_2_3-2*t));
end

alpha_2_T3=zeros(1,100);
for i=1:100
alpha_2_T3(1,i)=alpha1_2_T3(i)+int_v3_2_T3(i);
end

%find beta
beta2_T3=zeros(1,100);
for i=1:100
beta2_T3(1,i)=((sigma(i)^2)/(2*a(i)))-((sigma(i)^2)/(2*a(i)))*exp(-2*a(i)*(T_2_3-t));
end

%find Y_t, N(d_1), N(d_2)
Y_t_2_T3=zeros(1,100);
for i=1:100
Y_t_2_T3(1,i)=exp(delta_2_T3(i)+gamma_2_T3(i)*alpha_2_T3(i)+(1/2)*(gamma_2_T3(i).^2)*beta2_T3(i));
end

d2_3_2_T3=zeros(1,100);
for i=1:100
d2_3_2_T3(1,i)=(log(Y_t_2_T3(i)/(tau_j*K2+1))-(gamma_2_T3(i)^2)*(beta2_T3(i)/2))/(gamma_2_T3(i)*sqrt(beta2_T3(i)));
```

```
end

N_d2_3_2_T3=zeros(1,100);
for i=1:100
N_d2_3_2_T3(1,i)=normcdf(d2_3_2_T3(i));
end

d1_3_2_T3=zeros(1,100);
for i=1:100
d1_3_2_T3(1,i)=d2_3_2_T3(i)+gamma_2_T3(i)*sqrt(beta2_T3(i));
end

N_d1_3_2_T2=zeros(1,100);
for i=1:100
N_d1_3_2_T3(1,i)=normcdf(d1_3_2_T3(i));
end

Price3M_2_T3=zeros(1,100);
for i=1:100
Price3M_2_T3(1,i)=(p2Y8M/tau_j)*(Y_t_2_T3(i)*N_d1_3_2_T3(i)-(1+tau_j*K2)*
    N_d2_3_2_T3(i));
end

figure
subplot(1,2,1)
plot3(a, sigma, Price3M, a, sigma, Price3M_T2, a, sigma, Price3M_T3)
xlabel('a')
ylabel('sigma')
zlabel('Price3M [1Y9M-2Y]')
grid on
legend('T=2Y','T=1Y9M','T=1Y8M', 'Orientation','vertical')
subplot(1,2,2)
plot3( a, sigma, Price3M_2, 'r', a, sigma, Price3M_2_T2, 'm', a, sigma,
    Price3M_2_T3, 'g')
xlabel('a')
ylabel('sigma')
zlabel('Price3M [2Y9M-3Y]')
```

```
grid on
legend('T=3Y','T=2Y9M','T=2Y8M', 'Orientation','vertical')

figure
subplot(2,2,1)
plot(a, Price3M, a, Price3M_T2, a, Price3M_T3)
xlabel('a')
ylabel('Price3M [1Y9M-2Y]')
axis tight
grid on
legend('T=2Y','T=1Y9M','T=1Y8M', 'Orientation','vertical')
subplot(2,2,2)
plot(sigma, Price3M, sigma, Price3M_T2, sigma, Price3M_T3)
xlabel('sigma')
ylabel('Price3M [1Y9M-2Y]')
axis tight
grid on
legend('T=2Y','T=1Y9M','T=1Y8M', 'Orientation','vertical')
subplot(2,2,3)
plot( a, Price3M_2, 'r', a, Price3M_2_T2, 'm', a, Price3M_2_T3, 'g')
xlabel('a')
ylabel('Price3M [2Y9M-3Y]')
axis tight
grid on
legend('T=3Y','T=2Y9M','T=2Y8M', 'Orientation','vertical')
subplot(2,2,4)
plot(sigma, Price3M_2, 'r', sigma, Price3M_2_T2, 'm', sigma, Price3M_2_T3,
    'g')
xlabel('sigma')
ylabel('Price3M [2Y9M-3Y]')
grid on
axis tight
legend('T=3Y','T=2Y9M','T=2Y8M', 'Orientation','vertical')
```

Appendix B

1M option code

Here is reported the code used for the analysis of the option on 1M future.

```
%HW MODEL PARAMETERS%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
a=linspace(0.05,0.15,100); %a varies from 5% to 15%
sigma=linspace(0.03,0.10,100); %sigma varies from 3% to 10%

%Find the function Theta(t) - function parameters at 23/08/2022
beta_0=1.641204;
beta_1=-1.766233;
beta_2=25.191842;
beta_3=-25.607108;
tau_1=1.344229;
tau_2=1.421672;
%Theta=@(a,x,sigma) a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1)
% .*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1)
% .*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta%
   _3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-e
   %xp(-2.*a.*x);

%Find prices of zero-coupon bonds for different maturities(euro-area yield
     curve at 23/08/2022)
AllBond=[-0.00113136 0.00195457 0.00408054 0.00554793 0.00797 0.00811
   0.00823246 0.00919 0.00927 0.00934709 0.01027293 0.01115277 0.01194251
   0.01261471 0.01317121 0.01362780 0.01400331 0.01431471 0.01457576
   0.01479712 0.01498690 0.01515127 0.01529493 0.01542154 0.01553392
   0.01563434 0.01572458 0.01580612 0.01588014 0.01594762 0.01600938
```

```
    0.01606612 0.01611841 0.01616675 0.01621157 0.01625322 0.01629204];
T = [0.25 0.5 0.75 1 11/6 23/12 2 17/6 35/12 3 4 5 6 7 8 9 10 11 12 13 14
    15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30];
pT=zeros(1,37);
for i=1:37
pT(1,i)=exp(-(T(i)*AllBond(i)));
end

%OPTION ON 1M FUTURE%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%1st case: [1Year+11Months  2Years] future contract, option has maturity 2
     Years, strike 0.9923.
tau_j=1/12;
T_j1=2-1/12;
T_j=2;
T=2;
t=0;
r_0=-0.082; %ESTR value on 23/08/2022
K=0.9923;
p2Y=pT(1,7);

%find omega
fun = @(x,a,sigma) (1-exp(-a.*(T_j-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1
    )+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2)
    )-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp
    (-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma
    .^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int1=@(a,sigma)integral(@(x)fun(x,a,sigma),T_j1,T_j);

int_v=zeros(1,100);
for i=1:100
int_v(1,i)=int1(a(i),sigma(i));
end

fun2 = @(x,a,sigma) (exp(-a.*(T_j1-x))-exp(-a.*(T_j-x))).*(a.*(beta_0+
    beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./
    tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./
    tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*
```

```
    exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int2=@(a,sigma)integral(@(x)fun2(x,a,sigma),T,T_j1);

int_v2=zeros(1,100);
for i=1:100
int_v2(1,i)=int2(a(i),sigma(i));
end

delta1=zeros(1,100);
for i=1:100
delta1(1,i)=1/a(i)*int_v2(i)+1/a(i)*int_v(i);
end

omega=zeros(1,100);
for i=1:100
omega(1,i)=(1/tau_j)*delta1(i);
end

%find alpha
fun3 = @(x,a,sigma)(exp(-a.*(T-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+
    beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))
    -(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-
    x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)
    ./(2.*a)).*(1-exp(-2.*a.*x)));
int3=@(a,sigma)integral(@(x)fun3(x,a,sigma),t,T);

int_v3=zeros(1,100);
for i=1:100
int_v3(1,i)=int3(a(i),sigma(i));
end

alpha1=zeros(1,100);
for i=1:100
alpha1(1,i)=r_0*exp(-a(i)*(T-t))-((sigma(i)^2)/(a(i)^2))-((sigma(i)^2)/(a(
    i)^2))*exp(-a(i)*(T-t))+((sigma(i)^2)/(2*a(i)^2))-((sigma(i)^2)/(2*a(i)
    ^2))*exp(-a(i)*(2*T-2*t));
end
```

```
alpha=zeros(1,100);
for i=1:100
alpha(1,i)=alpha1(i)+int_v3(i);
end

%find beta
beta=zeros(1,100);
for i=1:100
beta(1,i)=((sigma(i)^2)/(2*a(i)))-((sigma(i)^2)/(2*a(i)))*exp(-2*a(i)*(T-t
    ));
end

%find epsilon
epsilon=zeros(1,100);
for i=1:100
epsilon(1,i)=(1/(a(i)*tau_j))*(exp(-a(i)*(T_j1-T))-exp(-a(i)*(T_j-T)));
end

%find N(d_1) and phi(d_1)
d_1=zeros(1,100);
for i=1:100
d_1(1,i)=(-K+epsilon(i)*alpha(i)+omega(i))/(epsilon(i)*sqrt(beta(i)));
end

N_d_1=zeros(1,100);
for i=1:100
N_d_1(1,i)=normcdf(d_1(i));
end

D_d_1=zeros(1,100);
for i=1:100
D_d_1(1,i)=normpdf(d_1(i));
end

Price1M=zeros(1,100);
for i=1:100
```

```
Price1M(1,i)=p2Y*((epsilon(i)*alpha(i)+omega(i)-K)*N_d_1(i)+epsilon(i)*sqrt(beta(i))*D_d_1(i));
end

%2nd case: [2Years+11Months 3 years] future contract, option has maturity 3 years, strike 1.0392.
T_j_2=3;
T_j1_2=3-1/12;
T_2=3;
K2=1.0392;
p3Y=pT(1,10);

%find omega
fun_2 = @(x,a,sigma) (1-exp(-a.*(T_j_2-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int1_2=@(a,sigma)integral(@(x)fun_2(x,a,sigma),T_j1_2,T_j_2);

int_v_2=zeros(1,100);
for i=1:100
int_v_2(1,i)=int1_2(a(i),sigma(i));
end

fun2_2 = @(x,a,sigma) (exp(-a.*(T_j1_2-x))-exp(-a.*(T_j_2-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int2_2=@(a,sigma)integral(@(x)fun2_2(x,a,sigma),T_2,T_j1_2);

int_v2_2=zeros(1,100);
for i=1:100
int_v2_2(1,i)=int2_2(a(i),sigma(i));
end

```

```
delta1_2=zeros(1,100);
for i=1:100
delta1_2(1,i)=1/a(i)*int_v2_2(i)+1/a(i)*int_v_2(i);
end

omega_2=zeros(1,100);
for i=1:100
omega_2(1,i)=(1/tau_j)*delta1_2(i);
end

%find alpha
fun3_2 = @(x,a,sigma) (exp(-a.*(T_2-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int3_2=@(a,sigma)integral(@(x)fun3_2(x,a,sigma),t,T_2);

int_v3_2=zeros(1,100);
for i=1:100
int_v3_2(1,i)=int3_2(a(i),sigma(i));
end

alpha1_2=zeros(1,100);
for i=1:100
alpha1_2(1,i)=r_0*exp(-a(i)*(T_2-t))-((sigma(i)^2)/(a(i)^2))-((sigma(i)^2)/(a(i)^2))*exp(-a(i)*(T_2-t))+((sigma(i)^2)/(2*a(i)^2))-((sigma(i)^2)/(2*a(i)^2))*exp(-a(i)*(2*T_2-2*t));
end

alpha_2=zeros(1,100);
for i=1:100
alpha_2(1,i)=alpha1_2(i)+int_v3_2(i);
end

%find beta
beta2=zeros(1,100);
```

```
for i=1:100
beta2(1,i)=((sigma(i)^2)/(2*a(i)))-((sigma(i)^2)/(2*a(i)))*exp(-2*a(i)*(
    T_2-t));
end

%find epsilon
epsilon_2=zeros(1,100);
for i=1:100
epsilon_2(1,i)=(1/(a(i)*tau_j))*(exp(-a(i)*(T_j1_2-T_2))-exp(-a(i)*(T_j_2-
    T_2)));
end

%find N(d_1) and phi(d_1)
d_1_2=zeros(1,100);
for i=1:100
d_1_2(1,i)=(-K2+epsilon_2(i)*alpha_2(i)+omega_2(i))/(epsilon_2(i)*sqrt(
    beta2(i)));
end

N_d_1_2=zeros(1,100);
for i=1:100
N_d_1_2(1,i)=normcdf(d_1_2(i));
end

D_d_1_2=zeros(1,100);
for i=1:100
D_d_1_2(1,i)=normpdf(d_1_2(i));
end

Price1M_2=zeros(1,100);
for i=1:100
Price1M_2(1,i)=p3Y*((epsilon_2(i)*alpha_2(i)+omega_2(i)-K2)*N_d_1_2(i)+
    epsilon_2(i)*sqrt(beta2(i))*D_d_1_2(i));
end

figure
subplot(1,2,1)
```

```
plot3(a, sigma, Price1M)
xlabel('a')
ylabel('sigma')
zlabel('Price1M [1Y11M–2Y]')
grid on
subplot(1,2,2)
plot3(a, sigma, Price1M_2, 'r')
xlabel('a')
ylabel('sigma')
zlabel('Price1M [2Y11M–3Y]')
grid on

figure
subplot(2,2,1)
plot(a, Price1M)
xlabel('a')
ylabel('Price1M [1Y11M–2Y]')
grid on
subplot(2,2,2)
plot(sigma, Price1M)
xlabel('sigma')
ylabel('Price1M [1Y11M–2Y]')
grid on
subplot(2,2,3)
plot( a, Price1M_2, 'r')
xlabel('a')
ylabel('Price1M [2Y11M–3Y]')
grid on
subplot(2,2,4)
plot(sigma, Price1M_2, 'r')
xlabel('sigma')
ylabel('Price1M [2Y11M–3Y]')
grid on

%reduce the strike by 0.05

%1st case: %1st case: [1Year+11Months  2Years] future contract, option
```

```
% has maturity 2 Years, strike 0.9423.
K_2=0.9423;

d_1_K2=zeros(1,100);
for i=1:100
d_1_K2(1,i)=(-K_2+epsilon(i)*alpha(i)+omega(i))/(epsilon(i)*sqrt(beta(i)));
end

N_d_1_K2=zeros(1,100);
for i=1:100
N_d_1_K2(1,i)=normcdf(d_1_K2(i));
end

D_d_1_K2=zeros(1,100);
for i=1:100
D_d_1_K2(1,i)=normpdf(d_1_K2(i));
end

Price1M_K2=zeros(1,100);
for i=1:100
Price1M_K2(1,i)=p2Y*((epsilon(i)*alpha(i)+omega(i)-K_2)*N_d_1_K2(i)+epsilon(i)*sqrt(beta(i))*D_d_1_K2(i));
end

%2nd case: [2Years+11Months 3 years] future contract, option has maturity 3 years, strike 0.600.
K2_2=0.9892;

d_1_2_K2=zeros(1,100);
for i=1:100
d_1_2_K2(1,i)=(-K2_2+epsilon_2(i)*alpha_2(i)+omega_2(i))/(epsilon_2(i)*sqrt(beta2(i)));
end

N_d_1_2_K2=zeros(1,100);
for i=1:100
```

```
N_d_1_2_K2(1,i)=normcdf(d_1_2_K2(i));
end

D_d_1_2_K2=zeros(1,100);
for i=1:100
D_d_1_2_K2(1,i)=normpdf(d_1_2_K2(i));
end

Price1M_2_K2=zeros(1,100);
for i=1:100
Price1M_2_K2(1,i)=p3Y*((epsilon_2(i)*alpha_2(i)+omega_2(i)-K2_2)*N_d_1_2_K2(i)+epsilon_2(i)*sqrt(beta2(i))*D_d_1_2_K2(i));
end

%increase the strike by 0.05

%1st case: [1Year+11Months  2Years] future contract, option has maturity 2 Years, strike 1.0423.
K_3=1.0423;

d_1_K3=zeros(1,100);
for i=1:100
d_1_K3(1,i)=(-K_3+epsilon(i)*alpha(i)+omega(i))/(epsilon(i)*sqrt(beta(i)));
end

N_d_1_K3=zeros(1,100);
for i=1:100
N_d_1_K3(1,i)=normcdf(d_1_K3(i));
end

D_d_1_K3=zeros(1,100);
for i=1:100
D_d_1_K3(1,i)=normpdf(d_1_K3(i));
end

Price1M_K3=zeros(1,100);
```

```
for i=1:100
Price1M_K3(1,i)=p2Y*((epsilon(i)*alpha(i)+omega(i)-K_3)*N_d_1_K3(i)+
    epsilon(i)*sqrt(beta(i))*D_d_1_K3(i));
end

%2nd case: [2Years+11Months 3 years] future contract, option has maturity
    3 years, strike 1.0892.
K2_3=1.0892;

d_1_2_K3=zeros(1,100);
for i=1:100
d_1_2_K3(1,i)=(-K2_3+epsilon_2(i)*alpha_2(i)+omega_2(i))/(epsilon_2(i)*
    sqrt(beta2(i)));
end

N_d_1_2_K3=zeros(1,100);
for i=1:100
N_d_1_2_K3(1,i)=normcdf(d_1_2_K3(i));
end

D_d_1_2_K3=zeros(1,100);
for i=1:100
D_d_1_2_K3(1,i)=normpdf(d_1_2_K3(i));
end

Price1M_2_K3=zeros(1,100);
for i=1:100
Price1M_2_K3(1,i)=p3Y*((epsilon_2(i)*alpha_2(i)+omega_2(i)-K2_3)*
    N_d_1_2_K3(i)+epsilon_2(i)*sqrt(beta2(i))*D_d_1_2_K3(i));
end

figure
subplot(1,2,1)
plot3(a, sigma, Price1M, a, sigma, Price1M_K2, a, sigma, Price1M_K3)
xlabel('a')
ylabel('sigma')
zlabel('Price1M [1Y11M-2Y]')
```

```
grid on
legend('K=0.9923','K=0.9423','K=1.0423', 'Orientation', 'vertical')
subplot(1,2,2)
plot3( a, sigma, Price1M_2, 'r', a, sigma, Price1M_2_K2, 'm', a, sigma, Price1M_2_K3, 'g')
xlabel('a')
ylabel('sigma')
zlabel('Price1M [2Y11M-3Y]')
grid on
legend('K=1.0392','K=0.9892','K=1.0892', 'Orientation', 'vertical')

figure
subplot(2,2,1)
plot(a, Price1M, a, Price1M_K2, a, Price1M_K3)
xlabel('a')
ylabel('Price1M [1Y11M-2Y]')
grid on
legend('K=0.9923','K=0.9423','K=1.0423', 'Orientation', 'vertical')
subplot(2,2,2)
plot(sigma, Price1M, sigma, Price1M_K2, sigma, Price1M_K3)
xlabel('sigma')
ylabel('Price1M [1Y11M-2Y]')
grid on
legend('K=0.9923','K=0.9423','K=1.0423', 'Orientation', 'vertical')
subplot(2,2,3)
plot( a, Price1M_2, 'r', a, Price1M_2_K2, 'm', a, Price1M_2_K3,'g')
xlabel('a')
ylabel('Price1M [2Y11M-3Y]')
grid on
legend('K=1.0392','K=0.9892','K=1.0892', 'Orientation', 'vertical')
subplot(2,2,4)
plot(sigma, Price1M_2, 'r', sigma, Price1M_2_K2, 'm', sigma, Price1M_2_K3, 'g')
xlabel('sigma')
ylabel('Price1M [2Y11M-3Y]')
grid on
legend('K=1.0392','K=0.9892','K=1.0892', 'Orientation', 'vertical')
```

```
%reduce the maturity by 1 month

%1st case: [1Year+11Months  2Years] future contract, option has maturity 1 year 11 months, strike 0.9923.
T2=2-1/12;
p1Y11M=pT(1,6);

%find omega
fun2_T2 = @(x,a,sigma) (exp(-a.*(T_j1-x))-exp(-a.*(T_j-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int2_T2=@(a,sigma)integral(@(x)fun2_T2(x,a,sigma),T2,T_j1);

int_v2_T2=zeros(1,100);
for i=1:100
int_v2_T2(1,i)=int2_T2(a(i),sigma(i));
end

delta1_T2=zeros(1,100);
for i=1:100
delta1_T2(1,i)=1/a(i)*int_v2_T2(i)+1/a(i)*int_v(i);
end

omega_T2=zeros(1,100);
for i=1:100
omega_T2(1,i)=(1/tau_j)*delta1_T2(i);
end

%find alpha
fun3_T2 = @(x,a,sigma) (exp(-a.*(T2-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
```

```
int3_T2=@(a,sigma)integral(@(x)fun3_T2(x,a,sigma),t,T2);

int_v3_T2=zeros(1,100);
for i=1:100
int_v3_T2(1,i)=int3_T2(a(i),sigma(i));
end

alpha1_T2=zeros(1,100);
for i=1:100
alpha1_T2(1,i)=r_0*exp(-a(i)*(T2-t))-((sigma(i)^2)/(a(i)^2))-((sigma(i)^2)
    /(a(i)^2))*exp(-a(i)*(T2-t))+((sigma(i)^2)/(2*a(i)^2))-((sigma(i)^2)
    /(2*a(i)^2))*exp(-a(i)*(2*T2-2*t));
end

alpha_T2=zeros(1,100);
for i=1:100
alpha_T2(1,i)=alpha1_T2(i)+int_v3_T2(i);
end

%find beta
beta_T2=zeros(1,100);
for i=1:100
beta_T2(1,i)=((sigma(i)^2)/(2*a(i)))-((sigma(i)^2)/(2*a(i)))*exp(-2*a(i)*(
    T2-t));
end

%find epsilon
epsilon_T2=zeros(1,100);
for i=1:100
epsilon_T2(1,i)=(1/(a(i)*tau_j))*(exp(-a(i)*(T_j1-T2))-exp(-a(i)*(T_j-T2))
    );
end

%find N(d_1) and phi(d_1)
d_1_T2=zeros(1,100);
for i=1:100
```

```
d_1_T2(1,i)=(-K+epsilon_T2(i)*alpha_T2(i)+omega_T2(i))/(epsilon_T2(i)*sqrt(beta_T2(i)));
end

N_d_1_T2=zeros(1,100);
for i=1:100
N_d_1_T2(1,i)=normcdf(d_1_T2(i));
end

D_d_1_T2=zeros(1,100);
for i=1:100
D_d_1_T2(1,i)=normpdf(d_1_T2(i));
end

Price1M_T2=zeros(1,100);
for i=1:100
Price1M_T2(1,i)=p1Y11M*((epsilon_T2(i)*alpha_T2(i)+omega_T2(i)-K)*N_d_1_T2(i)+epsilon_T2(i)*sqrt(beta_T2(i))*D_d_1_T2(i));
end

%2nd case: [2Years+11Months 3 years] future contract, option has maturity 2 years 11 months years, strike 1.0392.
T_2_2=3-1/12;
p2Y11M=pT(1,9);

%find omega
fun2_2_T2 = @(x,a,sigma) (exp(-a.*(T_j1_2-x))-exp(-a.*(T_j_2-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int2_2_T2=@(a,sigma)integral(@(x)fun2_2_T2(x,a,sigma),T_2_2,T_j1_2);

int_v2_2_T2=zeros(1,100);
for i=1:100
int_v2_2_T2(1,i)=int2_2_T2(a(i),sigma(i));
end
```

```
delta1_2_T2=zeros(1,100);
for i=1:100
delta1_2_T2(1,i)=1/a(i)*int_v2_2_T2(i)+1/a(i)*int_v_2(i);
end

omega_2_T2=zeros(1,100);
for i=1:100
omega_2_T2(1,i)=(1/tau_j)*delta1_2_T2(i);
end

%find alpha
fun3_2_T2 = @(x,a,sigma) (exp(-a.*(T_2_2-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int3_2_T2=@(a,sigma)integral(@(x)fun3_2_T2(x,a,sigma),t,T_2_2);

int_v3_2_T2=zeros(1,100);
for i=1:100
int_v3_2_T2(1,i)=int3_2_T2(a(i),sigma(i));
end

alpha1_2_T2=zeros(1,100);
for i=1:100
alpha1_2_T2(1,i)=r_0*exp(-a(i)*(T_2_2-t))-((sigma(i)^2)/(a(i)^2))-((sigma(i)^2)/(a(i)^2))*exp(-a(i)*(T_2_2-t))+((sigma(i)^2)/(2*a(i)^2))-((sigma(i)^2)/(2*a(i)^2))*exp(-a(i)*(2*T_2_2-2*t));
end

alpha_2_T2=zeros(1,100);
for i=1:100
alpha_2_T2(1,i)=alpha1_2_T2(i)+int_v3_2_T2(i);
end

%find beta
```

```
beta2_T2=zeros(1,100);
for i=1:100
beta2_T2(1,i)=((sigma(i)^2)/(2*a(i)))-((sigma(i)^2)/(2*a(i)))*exp(-2*a(i)*(T_2_2-t));
end

%find epsilon
epsilon_2_T2=zeros(1,100);
for i=1:100
epsilon_2_T2(1,i)=(1/(a(i)*tau_j))*(exp(-a(i)*(T_j1_2-T_2_2))-exp(-a(i)*(T_j_2-T_2_2)));
end

%find N(d_1) and phi(d_1)
d_1_2_T2=zeros(1,100);
for i=1:100
d_1_2_T2(1,i)=(-K2+epsilon_2_T2(i)*alpha_2_T2(i)+omega_2_T2(i))/(epsilon_2_T2(i)*sqrt(beta2_T2(i)));
end

N_d_1_2_T2=zeros(1,100);
for i=1:100
N_d_1_2_T2(1,i)=normcdf(d_1_2_T2(i));
end

D_d_1_2_T2=zeros(1,100);
for i=1:100
D_d_1_2_T2(1,i)=normpdf(d_1_2_T2(i));
end

Price1M_2_T2=zeros(1,100);
for i=1:100
Price1M_2_T2(1,i)=p2Y11M*((epsilon_2_T2(i)*alpha_2_T2(i)+omega_2_T2(i)-K2)*N_d_1_2_T2(i)+epsilon_2_T2(i)*sqrt(beta2_T2(i))*D_d_1_2_T2(i));
end

%reduce the maturity by 2 months
```

```
%1st case: [1Year+11Months  2Years] future contract, option has maturity 1 Year 10 months, strike 0.9923.
T3=2-2/12;
p1Y10M=pT(1,5);

%find omega
fun2_T3 = @(x,a,sigma)(exp(-a.*(T_j1-x))-exp(-a.*(T_j-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int2_T3=@(a,sigma)integral(@(x)fun2_T3(x,a,sigma),T3,T_j1);

int_v2_T3=zeros(1,100);
for i=1:100
int_v2_T3(1,i)=int2_T3(a(i),sigma(i));
end

delta1_T3=zeros(1,100);
for i=1:100
delta1_T3(1,i)=1/a(i)*int_v2_T3(i)+1/a(i)*int_v(i);
end

omega_T3=zeros(1,100);
for i=1:100
omega_T3(1,i)=(1/tau_j)*delta1_T3(i);
end

%find alpha
fun3_T3 = @(x,a,sigma)(exp(-a.*(T3-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int3_T3=@(a,sigma)integral(@(x)fun3_T3(x,a,sigma),t,T3);

```

```
int_v3_T3=zeros(1,100);
for i=1:100
int_v3_T3(1,i)=int3_T3(a(i),sigma(i));
end

alpha1_T3=zeros(1,100);
for i=1:100
alpha1_T3(1,i)=r_0*exp(-a(i)*(T3-t))-((sigma(i)^2)/(a(i)^2))-((sigma(i)^2)/(a(i)^2))*exp(-a(i)*(T3-t))+((sigma(i)^2)/(2*a(i)^2))-((sigma(i)^2)/(2*a(i)^2))*exp(-a(i)*(2*T3-2*t));
end

alpha_T3=zeros(1,100);
for i=1:100
alpha_T3(1,i)=alpha1_T3(i)+int_v3_T3(i);
end

%find beta
beta_T3=zeros(1,100);
for i=1:100
beta_T3(1,i)=((sigma(i)^2)/(2*a(i)))-((sigma(i)^2)/(2*a(i)))*exp(-2*a(i)*(T3-t));
end

%find epsilon
epsilon_T3=zeros(1,100);
for i=1:100
epsilon_T3(1,i)=(1/(a(i)*tau_j))*(exp(-a(i)*(T_j1-T3))-exp(-a(i)*(T_j-T3)));
end

%find N(d_1) and phi(d_1)
d_1_T3=zeros(1,100);
for i=1:100
d_1_T3(1,i)=(-K+epsilon_T3(i)*alpha_T3(i)+omega_T3(i))/(epsilon_T3(i)*sqrt(beta_T3(i)));
end
```

```
N_d_1_T3=zeros(1,100);
for i=1:100
N_d_1_T3(1,i)=normcdf(d_1_T3(i));
end

D_d_1_T3=zeros(1,100);
for i=1:100
D_d_1_T3(1,i)=normpdf(d_1_T3(i));
end

Price1M_T3=zeros(1,100);
for i=1:100
Price1M_T3(1,i)=p1Y10M*((epsilon_T3(i)*alpha_T3(i)+omega_T3(i)-K)*N_d_1_T3(i)+epsilon_T3(i)*sqrt(beta_T3(i))*D_d_1_T3(i));
end

%2nd case: [2Years+11Months 3 years] future contract, option has maturity 2 years 10 months, strike 1.0392.
T_2_3=3-2/12;
p2Y10M=pT(1,8);

%find omega
fun2_2_T3 = @(x,a,sigma) (exp(-a.*(T_j1_2-x))-exp(-a.*(T_j_2-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int2_2_T3=@(a,sigma)integral(@(x)fun2_2_T3(x,a,sigma),T_2_3,T_j1_2);

int_v2_2_T2=zeros(1,100);
for i=1:100
int_v2_2_T3(1,i)=int2_2_T3(a(i),sigma(i));
end

delta1_2_T3=zeros(1,100);
for i=1:100
```

```
delta1_2_T3(1,i)=1/a(i)*int_v2_2_T3(i)+1/a(i)*int_v_2(i);
end

omega_2_T3=zeros(1,100);
for i=1:100
omega_2_T3(1,i)=(1/tau_j)*delta1_2_T3(i);
end

%find alpha
fun3_2_T3 = @(x,a,sigma) (exp(-a.*(T_2_3-x))).*(a.*(beta_0+beta_1.*exp(-x
    ./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x
    ./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1
    .^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))
    +((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
int3_2_T3=@(a,sigma)integral(@(x)fun3_2_T3(x,a,sigma),t,T_2_3);

int_v3_2_T3=zeros(1,100);
for i=1:100
    int_v3_2_T3(1,i)=int3_2_T3(a(i),sigma(i));
end

alpha1_2_T3=zeros(1,100);
for i=1:100
alpha1_2_T3(1,i)=r_0*exp(-a(i)*(T_2_3-t))-((sigma(i)^2)/(a(i)^2))-((sigma(
    i)^2)/(a(i)^2))*exp(-a(i)*(T_2_3-t))+((sigma(i)^2)/(2*a(i)^2))-((sigma(
    i)^2)/(2*a(i)^2))*exp(-a(i)*(2*T_2_3-2*t));
end

alpha_2_T3=zeros(1,100);
for i=1:100
alpha_2_T3(1,i)=alpha1_2_T3(i)+int_v3_2_T3(i);
end

%find beta
beta2_T3=zeros(1,100);
for i=1:100
```

```
beta2_T3(1,i)=((sigma(i)^2)/(2*a(i)))-((sigma(i)^2)/(2*a(i)))*exp(-2*a(i)*(T_2_3-t));
end

%find epsilon
epsilon_2_T3=zeros(1,100);
for i=1:100
epsilon_2_T3(1,i)=(1/(a(i)*tau_j))*(exp(-a(i)*(T_j1_2-T_2_3))-exp(-a(i)*(T_j_2-T_2_3)));
end

%find N(d_1) and phi(d_1)
d_1_2_T3=zeros(1,100);
for i=1:100
d_1_2_T3(1,i)=(-K2+epsilon_2_T3(i)*alpha_2_T3(i)+omega_2_T3(i))/(epsilon_2_T3(i)*sqrt(beta2_T3(i)));
end

N_d_1_2_T3=zeros(1,100);
for i=1:100
N_d_1_2_T3(1,i)=normcdf(d_1_2_T3(i));
end

D_d_1_2_T3=zeros(1,100);
for i=1:100
D_d_1_2_T3(1,i)=normpdf(d_1_2_T3(i));
end

Price1M_2_T3=zeros(1,100);
for i=1:100
Price1M_2_T3(1,i)=p2Y10M*((epsilon_2_T3(i)*alpha_2_T3(i)+omega_2_T3(i)-K2)*N_d_1_2_T3(i)+epsilon_2_T3(i)*sqrt(beta2_T3(i))*D_d_1_2_T3(i));
end

figure
subplot(1,2,1)
plot3(a, sigma, Price1M, a, sigma, Price1M_T2, a, sigma, Price1M_T3)
```

```
xlabel('a')
ylabel('sigma')
zlabel('Price1M [1Y11M-2Y]')
grid on
legend('T=2Y','T=1Y11M','T=1Y10M', 'Orientation', 'vertical')
subplot(1,2,2)
plot3( a, sigma, Price1M_2, 'r', a, sigma, Price1M_2_T2, 'm', a, sigma,
    Price1M_2_T3, 'g')
xlabel('a')
ylabel('sigma')
zlabel('Price1M [2Y11M-3Y]')
grid on
legend('T=3Y','T=2Y11M','T=2Y10M', 'Orientation', 'vertical')

figure
subplot(2,2,1)
plot(a, Price1M, a, Price1M_T2, a, Price1M_T3)
xlabel('a')
ylabel('Price1M [1Y11M-2Y]')
grid on
legend('T=2Y','T=1Y11M','T=1Y10M', 'Orientation', 'vertical')
subplot(2,2,2)
plot(sigma, Price1M, sigma, Price1M_T2, sigma, Price1M_T3)
xlabel('sigma')
ylabel('Price1M [1Y11M-2Y]')
grid on
legend('T=2Y','T=1Y11M','T=1Y10M', 'Orientation', 'vertical')
subplot(2,2,3)
plot( a, Price1M_2, 'r', a, Price1M_2_T2, 'm', a, Price1M_2_T3, 'g')
xlabel('a')
ylabel('Price1M [2Y11M-3Y]')
grid on
legend('T=3Y','T=2Y11M','T=2Y10M', 'Orientation', 'vertical')
subplot(2,2,4)
plot(sigma, Price1M_2, 'r', sigma, Price1M_2_T2, 'm', sigma, Price1M_2_T3,
    'g')
xlabel('sigma')
```

```
ylabel('Price1M [2Y11M—3Y]')
grid on
legend('T=3Y','T=2Y11M','T=2Y10M', 'Orientation', 'vertical')
```

Appendix C

Strike price code

Here we report the code used to obtain reasonable values for the strike prices.

- **Option on 3M future**

```
%1st case: [1Year+9months 2 Years] future contract.
a=0.1;
sigma=0.065;
tau_j=0.25;
T_j=2;
T_j1=1.75;
t=0;
beta_0=1.641204;
beta_1=-1.766233;
beta_2=25.191842;
beta_3=-25.607108;
tau_1=1.344229;
tau_2=1.421672;
r_0=-0.00082;

%find eta
funk = @(x)(1-exp(-a.*(T_j-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
intk1=integral(funk,T_j1,T_j);

```

```
funk2 = @(x) (exp(-a.*(T_j1-x))-exp(-a.*(T_j-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
intk2=integral(funk2,t,T_j1);

eta=(1/a)*intk1+(1/a)*intk2;

%find Sigma
Sigma=(1/2)*((sigma^2)/(a^2))*((1/a)*(exp(-a*(T_j-T_j1))+exp(-a*(T_j+T_j1-2*t))-1)-(1/(2*a))*(exp(-2*a*(T_j-t))+exp(-2*a*(T_j1-t)))+(T_j-T_j1));

%future price
f_0=(1/tau_j)*(exp((r_0/a)*(exp(-a*T_j1)-exp(-a*T_j))+eta+Sigma)-1);

%2nd case: [2Years+9Months 3Years] future contract.
T_j_2=3;
T_j1_2=2.75;

%find eta
funk_2 = @(x)(1-exp(-a.*(T_j_2-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
intk1_2=integral(funk_2,T_j1_2,T_j_2);

funk2_2 = @(x) (exp(-a.*(T_j1_2-x))-exp(-a.*(T_j_2-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
intk2_2=integral(funk2_2,t,T_j1_2);
```

```
eta_2=(1/a)*intk1_2+(1/a)*intk2_2;

%find Sigma
Sigma_2=(1/2)*((sigma^2)/(a^2))*((1/a)*(exp(-a*(T_j_2-T_j1_2))+exp(-a
    *(T_j_2+T_j1_2-2*t))-1)-(1/(2*a))*(exp(-2*a*(T_j_2-t))+exp(-2*a*(
    T_j1_2-t)))+(T_j_2-T_j1_2));

%future price
f_0_2=(1/tau_j)*(exp((r_0/a)*(exp(-a*T_j1_2)-exp(-a*T_j_2))+eta_2+
    Sigma_2)-1);
```

- **Option on 1M future**

```
%1st case: [1Year+11months 2 Years] future contract.
a=0.1;
sigma=0.065;
tau_j=1/12;
T_j=2;
T_j1=2-1/12;
t=0;
beta_0=1.641204;
beta_1=-1.766233;
beta_2=25.191842;
beta_3=-25.607108;
tau_1=1.344229;
tau_2=1.421672;
r_0=-0.00082;

%find eta
funk = @(x)(1-exp(-a.*(T_j-x))).*(a.*(beta_0+beta_1.*exp(-x./tau_1)+
    beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp(-x./
    tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x./
    tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp(-x
    ./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
intk1=integral(funk,T_j1,T_j);

funk2 = @(x) (exp(-a.*(T_j1-x))-exp(-a.*(T_j-x))).*(a.*(beta_0+beta_1
```

```
.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./
    tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2
    .*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./
    tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x))
    );
intk2=integral(funk2,t,T_j1);

eta=(1/a)*intk1+(1/a)*intk2;

%future price
f_0=(1/tau_j)*((r_0/a)*(exp(-a*(T_j1-t))-exp(-a*(T_j-t)))+eta);

%2nd case: [2Years+11Months 3Years] future contract.
T_j_2=3;
T_j1_2=3-1/12;

funk_2 = @(x)(1-exp(-a.*(T_j_2-x))).*(a.*(beta_0+beta_1.*exp(-x./
    tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3.*(x./tau_2).*exp
    (-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+beta_2.*((1./tau_1-x
    ./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./tau_2-x./tau_2.^2).*exp
    (-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp(-2.*a.*x)));
intk1_2=integral(funk_2,T_j1_2,T_j_2);

funk2_2 = @(x) (exp(-a.*(T_j1_2-x))-exp(-a.*(T_j_2-x))).*(a.*(beta_0+
    beta_1.*exp(-x./tau_1)+beta_2.*(x./tau_1).*exp(-x./tau_1)+beta_3
    .*(x./tau_2).*exp(-x./tau_2))-(beta_1./tau_1).*exp(-x./tau_1)+
    beta_2.*((1./tau_1-x./tau_1.^2).*exp(-x./tau_1))+beta_3.*((1./
    tau_2-x./tau_2.^2).*exp(-x./tau_2))+((sigma.^2)./(2.*a)).*(1-exp
    (-2.*a.*x)));
intk2_2=integral(funk2_2,t,T_j1_2);

eta_2=(1/a)*intk1_2+(1/a)*intk2_2;

%future price
f_0_2=(1/tau_j)*((r_0/a)*(exp(-a*(T_j1_2-t))-exp(-a*(T_j_2-t)))+eta_2
    );
```

Bibliography

[1] Alternative Reference Rates Committee. The ARRC's Paced Transition Plan for Developing SOFR Markets. *Institutional Report*. URL: https://www.newyorkfed.org/medialibrary/microsites/arrc/files/paced-timeline-plan.pdf, 2017.

[2] Alternative Reference rates Committee. The ARRC Second Report. *Institutional Report*. URL: https://www.newyorkfed.org/medialibrary/Microsites/arrc/files/2018/ARRC-Second-report, 2018.

[3] Alternative Reference Rates Committee. The ARRC Best Practice Recommendations Related to Scope of Use of the Term Rate. *Institutional Report*. URL: https://www.newyorkfed.org/medialibrary/Microsites/arrc/files/2021/ARRC_Scope_of_Use.pdf, 2020.

[4] A. Baig and D. B. Winters. The search for a new reference rate. *Review of Quantitative Finance and Accounting*, 58(3):939–976, 2022.

[5] Bank For International Settlements. BIS quarterly review. *Institutional Report*. URL: https://www.bis.org/publ/qtrpdf/r_qt0803.pdf, 2008.

[6] Bank of England. SONIA as the RFR and approaches to adoption. *Institutional Report*. URL: https://www.bankofengland.co.uk/-/media/boe/files/markets/benchmarks/sonia-as-the-risk-free-reference-rate-and-approaches-to-adoption.pdf, 2017.

[7] T. Björk. Arbitrage theory in continuous time. *Oxford university press*, 2009.

[8] H. Braml. The manipulation of LIBOR and related interest rates. *Studies in Economics and Finance*, 2016.

[9] D. Brigo, F. Mercurio, et al. Interest rate models: theory and practice. *Springer*, 2001.

[10] C. Cuchiero, C. Fontana, and A. Gnoatto. Affine multiple yield curve models. *Mathematical Finance*, 29(2):568–611, 2019.

[11] European Central Bank. Establishment of the working group on euro risk-free rates. *Institutional Report*. URL: https://www.ecb.europa.eu/paym/interest_rate_benchmarks/WG_euro_risk-free_rates/shared/pdf/2017_11_29_terms_of_reference.pdf, 2018.

[12] European Central Bank. Pre-ESTR. *Institutional Report*. URL: https://www.ecb.europa.eu/paym/interest_rate_benchmarks/WG_euro_risk-free_rates/shared/pdf/ecb.Pre-ESTER.en.pdf, 2019.

[13] Federal Reserve Bank of New York. SOFR "In Arrears" Conventions for Syndicated Business Loans. *Institutional Report*. URL: https://www.newyorkfed.org/medialibrary/Microsites/arrc/files/2020/ARRC_SOFR_Synd_Loan_Conventions.pdf, 2020.

[14] Financial Stability Board. Global Transition Roadmap for LIBOR. *Institutional Report*. URL: https://www.fsb.org/wp-content/uploads/P020621-1.pdf, 2021.

[15] C. Fontana. Caplet pricing in affine models for risk-free rates. *arXiv preprint arXiv:2202.09116*, 2022.

[16] J. Fouquau and P. K. Spieser. Statistical evidence about LIBOR manipulation: A "Sherlock Holmes" investigation. *Journal of Banking & Finance*, 50:632–643, 2015.

[17] T. Hasegawa. Caplet formulae for backward-looking term rates with Hull-White model. *Available at SSRN 3909949*, 2021.

[18] K. F. Hofmann. Implied volatilities for options on backward-looking term rates. *Available at SSRN 3593284*, 2020.

[19] J. Hull and A. White. Pricing interest-rate-derivative securities. *The review of financial studies*, 3(4):573–592, 1990.

[20] J. C. Hull. Options futures and other derivatives. *Pearson Education India*, 2003.

[21] ICE Benchmark Administration. Summary of ICE LIBOR evolution. *Institutional Report*.

[22] ICE Benchmark Administration. Second position paper on the evolution of ICE LIBOR. *Institutional Report*. URL: https://www.theice.com/publicdocs/ICE_LIBOR_Second_Position_Paper.pdf, 2015.

[23] ICE Benchmark Administration. Roadmap for ICE LIBOR. *Institutional Report*. URL: https://www.theice.com/publicdocs/ICE_LIBOR_Roadmap0316.pdf, 2016.

[24] ICE Benchmark Administration. USD LIBOR panel bank criteria. *Institutional Report*. URL: https://www.theice.com/publicdocs/USD_LIBOR_Panel_Bank_Criteria.pdf, 2022.

[25] International Monetary Fund. Global Financial Stability Report. *Institutional Report*. URL: https://www.imf.org/en/Publications/GFSR/Issues/2016/12/31/Global-Financial-Stability-Report-October-2008-Financial-Stress-and-Deleveraging-Macrofi-22027, 2008.

[26] D. Keenan. My thwarted attempt to tell of LIBOR shenanigans. *Financial Times*, 2012.

[27] S. Klingler and O. Syrstad. Life after LIBOR. *Journal of Financial Economics*, 141(2):783–801, 2021.

[28] A. Lyashenko and F. Mercurio. Looking forward to backward-looking rates: a modeling framework for term rates replacing LIBOR. *Available at SSRN 3330240*, 2019.

[29] A. Lyashenko and F. Mercurio. Looking Forward to Backward-Looking Rates: Completing the Generalized Forward Market Model. *Available at SSRN*, 2019.

[30] F. Mercurio. A simple multi-curve model for pricing SOFR futures and other derivatives. *Available at SSRN 3225872*, 2018.

[31] C. Mollenkamp and M. Whitehouse. Study casts doubt on key rate. *Wall Street Journal*, 29(5), 2008.

[32] E. Mollenkamp and J. Eaglesham. US LIBOR Probe includes BofA, Citi, UBS. *Wall Street Journal*, 2011.

[33] V. Piterbarg. Interest rates benchmark reform and options markets. *Available at SSRN 3537925*, 2020.

[34] A. Schrimpf and V. Sushko. Beyond LIBOR: a primer on the new benchmark rates. *BIS Quarterly Review March*, 2019.

[35] F. Siboulet, R. Kumar, R. Douady, and S. Crepey. LIBOR Inside Out Transition and Challenges. *Wilmott*, 2019(100):12–29, 2019.

[36] C. Turfus. Caplet pricing with backward-looking rates. *Available at SSRN 3527091*, 2020.

[37] M. Xu. SOFR Derivative Pricing Using a Short Rate Model. *Available at SSRN 4007604*, 2021.