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Noise-induced periodicity in networks of interacting diffusions

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Preface

In this thesis we investigate the emergence of collective periodic behaviors in a frustrated network of stochastic interacting diffusions. The study proposed revisits the work on [12]. We provide a model of noisy interacting particles, arranged in two communities of units, which depend on their mutual coupling interactions. Motivated by insights on numerical simulations, we show that this model features the phenomenon of noise-induced periodicity: when the number of particles goes to infinity, in a certain range of interaction strengths, although the system has no periodic behavior in the zero-noise limit, a moderate amount of noise may generate attractive periodic rules.

Contents

Introduction

Living systems are characterized by the emergence of recurrent dynamical patterns, that arise from the aggregate of a very large number of interacting units. Such patterns are due to self-organization and are observed at all scales of magnitude. Self-organized behaviors are noted both in large communities of microscopic components - like neural oscillations (alpha and beta waves), gene network activity and chemical reactions - as well as on larger levels - as predator-prey equilibria and applauding audiences to name few. In particular, *collective periodic behaviors* of many elements systems are among the most commonly observed ways of self-organization in biology, ecology and socio-economics. The attempt of modeling such complex systems leads naturally to consider large families of microscopic identical units (particles). Self-organized oscillations, then, arise on a macroscopic scale from the dynamic of these minimal components that evolve coupled by interaction terms.

In this framework lies the topic of this thesis: the phenomenon of *noise-induced periodicity*. Indeed, large-volume natural systems of *noisy* interacting particles often exhibit robust collective periodic behaviors [17]. Frequently are encountered systems, with this property, in which the particles neither have tendency to behave periodically on their own nor are subject to a periodic external forcing; nevertheless, they organize to produce a regular motion perceived only macroscopically: a collective self-sustained rhythm.

The presence, within a system, of *noise*, meant as an intrinsic, unpredictable, disturbance element in the particle's interactions, may stimulate the emergence of such collective periodic motions. However, how such families of "non-periodic" particles can generate macroscopic oscillations, is, still, poorly understood from a theoretical standpoint.

The difficulty of treating theoretically large noisy interacting components leads to give great attention to *mean-field* theory and mean-field interacting particles systems, due to their more analytical properties. In this context, the attempt of explaining rigorously possible origins of self-organized rhythms identified various essential aspects that enhance the emergence of such coherent and structured dynamics. Since, rhythmic behaviors are intrinsically non-equilibrium phenomena, a breaking mechanism needs to enter the microscopic design of the models.

Quite a number of such mechanisms, lately, have been taken into account: for example, the addition of delay in the information's transmission [1, 9], and/or frustration in the interaction's network in multi-population discrete particle systems [2]. However, several works have highlighted the importance of the interplay between the reciprocal interaction of the units and noise. Examples of *nonlinear diffusion processes*, for which periodic behaviors are caused by the presence of noise, meaning that no periodicity occurs in the system when noise is turned off, where given long time ago [15]. Recent works have stressed the specific importance of noise, as a equilibrium-breaking element. For instance, the phenomenon, known as excitability by noise, is widely observed [10]. From this point of view, the role of noise is believed to be twofold. On one hand, noise can lead to oscillatory states in systems whose deterministic counterparts do not display any periodic behavior; on the other, it can facilitate the transition from incoherence to macroscopic pulsing [17].

In this thesis we investigate the emergence of a collective periodic behavior in a *frustrated network* of interacting diffusions. We retrace the work made on [12], conducting a similar study. In the model we provide, particles are divided into two populations and they interact only via the respective empirical means. The frustration of the network is designed as follows. On one hand, both intra-population interaction parameters are positive: each particle wants to conform to the average position of the particles in its own community. On the other hand, inter-population couplings have opposite signs: the particles of one population want to conform to the average position of the particles of the other community, while the particles in the latter want to move away from the empirical mean of the first community. We show that this system features the phenomenon of noise-induced periodicity: in the *infinite volume limit*, that is when the number of particles goes to infinity, in a certain range of interaction strengths, although the system has no periodic behavior in the zero-noise limit, a moderate amount of noise may generate an attractive periodic law.

The thesis is divided into the following chapters.

In Chapter 1 we introduce the model described above as a system of N diffusive particles on $\mathbb{R}$, divided in two populations of N_1 and N_2 elements. Mean-field theory is introduced as, by design of our model, the interaction between particles is managed, only, via the respective means of the populations. Then we give a description of the study, as we analyzed the occurring of noise-induced periodicity in the model. We summarize it here.

We observed, on numerical simulations of the model, periodic oscillations in the trajectories of the empirical means of the two populations. This motivated the investigation of the *thermodynamic/macroscopic limit* of the system (when N goes to infinity). Through a *propagation of chaos* statement we prove that, in the macroscopic limit, two generic particles of the system, one for each population, follow *non-linear diffusion equations*. As non-linear diffusion processes can have time-periodic law, we argue that, for specific choices of the parameters, the presence of a noise component may generate a robust, self-sustained, rhythmic behaviour in the mean-field trajectories. While, when the noise is turned off the system moves towards stable equilibria.
Therefore, we further analyze the time-evolution of the macroscopic limit and prove that, the noisy limiting dynamic can evolve as a pair of *Gaussian processes*. This reduces the study of our problem to a finite dimensional one, since we derive the explicit (deterministic) equations for the mean and variance of these processes. Finally, we show that the dynamical system describing the time evolution of these means and variances undergo a *phase transition* via a *Hopf bifurcation* at an equilibrium point. As a consequence, in a certain range of the noise intensity and parameter's choice, the system has

a stable limit cycle as a long-time attractor, implying that the laws of the previously mentioned Gaussian processes are periodic.

In Chapter 2 we give some general background and mathematical tools to the study of *diffusions*. In particular, after recalling some basics in stochastic calculus and stochastic differential equations, we give a short introduction to some modern concepts about *propagation of chaos* for interacting diffusions. We state the general theorem that provides the convergence of a wide class of mean-field interacting dynamics (*microscopic models*) to a *macroscopic limit*, as the number of particles goes to infinity. This can be useful to describe whenever stochastic independence of two random particles in a many-particle system, persists in time, as the number of particles tends to infinity. Originally designed for Statistical Mechanics, the emergence of application of stochastic mean-field dynamics includes life and social sciences. We illustrate, without technical details, a famous model of *interacting Fitzhugh-Nagumo neurons*.
We include also in this chapter the relevant result about Hopf bifurcations, which provides the technical conditions to be checked when searching for limit cycles around equilibrium points in dynamical systems.

Finally, in Chapter 3 we provide all the results in the study. In order, in Sec. 3.1, after a well posedness statement for the finite-size system which describes the model, we give the results and the analysis of the numerical simulations we ran. Sec. 3.2 contains the well-posedness proof of the macroscopic limit, while in Sec. 3.3 we state the propagation of chaos theorem for our model. After that, we argue, in Sec. 3.4, that no periodic behaviour are present for the macroscopic limit when the noise is absent. In Sec. 3.5 we prove the existence of a pair of independent Gaussian processes which approximate the macroscopic dynamic and, finally, in Sec. 3.6 we show that the system which describes the means and variances of these processes displays a Hopf bifurcation.

Chapter 1

Description of the model and outline of the results

In the first section, we introduce the (microscopic) model as a diffusion process. We define the dynamics of the system with N Itô stochastic differential equations on $\mathbb{R}$. The analytical expression of the equation is inspired by a model of interacting FitzHugh-Nagumo neurons, of which we give a brief description in Sec. 2.4; here we divide the units into two populations.
Then, in latter section, we give a walk through of the study and a brief summary of the results: starting from the numerical simulations, trough the theoretical work, finishing with the main topic, the noise-induced periodicity.

1.1. The model as a finite-size system of interacting diffusions

So to introduce the subject let us consider a system of N diffusive particles on $\mathbb{R}$ and divide the N particles into two disjoint communities of sizes N_1 and N_2 respectively. Denote by I_1 (resp. I_2) the set of sites belonging to the first (resp. second) community. In this setting, we indicate with $\left(x_j^{(N)}(t)\right)_{j=1,\dots,N_1}$ the "positions" at time t of the particles of population I_1 and with $\left(y_j^{(N)}(t)\right)_{j=1,\dots,N_2}$ the "positions" at time t of the particles of population I_2, so that

$$\mathbf{x}^{(N)}(t) = \left(x_1^{(N)}(t), x_2^{(N)}(t), \dots, x_{N_1}^{(N)}(t), y_1^{(N)}(t), y_2^{(N)}(t), \dots, y_{N_2}^{(N)}(t)\right)$$

represents the state of the whole system at time t.

The basic feature we want to introduce in the following model is that the strength of the interaction between particles depends on the community they belong to: θ_{11} and θ_{22} tune the interaction between sites of the same community, whereas θ_{12} and θ_{21} control the coupling strength between particles of different groups.

More specifically, each population, taken alone, is a mean field system with interaction strength θ_{11} (resp. θ_{22}). When we couple the two communities, the population I_1 (resp. I_2) influences the population I_2 (resp. I_1) through the average position of its particles with strength θ_{21} (resp. θ_{12}).
A crucial feature for the system to show periodic behavior is frustration of the network, i.e. the inter-community interactions must have opposite signs.

Now we introduce the microscopic dynamics we are interested in. Let

$$m_1^{(N)}(t) := \frac{1}{N_1}\sum_{j=1}^{N_1} x_j^{(N)}(t) \qquad \text{and} \qquad m_2^{(N)}(t) := \frac{1}{N_2}\sum_{j=1}^{N_2} y_j^{(N)}(t)$$

be the empirical means of the positions of the particles in populations I_1 and I_2, respectively, at time t. Moreover, denote by $\alpha := \frac{N_1}{N}$ the fraction of sites belonging to the first group. Then, omitting time dependence for notations convenience, the interacting particle's system we are going to study reads:

$$\begin{aligned} dx_j^{(N)} &= \left(-\left(x_j^{(N)}\right)^3 + x_j^{(N)}\right)dt - \alpha\theta_{11}\left(x_j^{(N)} - m_1^{(N)}\right)dt \\ &\quad - (1-\alpha)\theta_{12}\left(x_j^{(N)} - m_2^{(N)}\right)dt + \sigma dW^j \qquad \text{for } j = 1,\dots,N_1, \\ dy_j^{(N)} &= \left(-\left(y_j^{(N)}\right)^3 + y_j^{(N)}\right)dt - \alpha\theta_{11}\left(y_j^{(N)} - m_1^{(N)}\right)dt \\ &\quad - (1-\alpha)\theta_{12}\left(y_j^{(N)} - m_2^{(N)}\right)dt + \sigma dW^{N_1+j} \qquad \text{for } j = 1,\dots,N_1, \end{aligned} \tag{1.1}$$

where $\left(W_t^j\right)_{j=1,\dots,N}$ are N independent copies of a standard Brownian motion. Here $\sigma \geq 0$ is the parameter that tunes the amount of noise in the system, since the diffusion coefficient is the same for each coordinate.
Note that in (1.1), the two populations interact only via their empirical means. This feature puts our model into the mean-field theory.

When $\theta_{11} = \theta_{22} = \theta_{12} = \theta_{21} = \theta > 0$ the model reduces to the mean field interacting diffusions considered in [5]. In particular, it describes a dynamical model of a collection of an-harmonic oscillators, which can be proved to experience a phase transition as the number of oscillators goes to infinity.

In a general setting, all the coupling constants could be either positive or negative, allowing both cooperative and uncooperative interactions. Nevertheless, in the present work, we focus on the case $\theta_{11}, \theta_{22} > 0$ and $\theta_{12}\theta_{21} < 0$. In particular, we make the specific choice: $\theta_{12} > 0$ and $\theta_{21} < 0$. This means that a generic particle from I_1 tends to conform to the average particle position of community I_2, whereas particles in I_2 are incline to differ from the average particle position of community I_1.

1.2. Noise-induced periodicity: synopsis of the study

The numerical simulations of the model (1.1), that we have computed in Section 3.1.1, show that the empirical means $\left(m_1^{(N)}(t) \text{ and } m_1^{(N)}(t)\right)$ display an oscillatory behavior for appropriate choices of the parameters. This motivated the investigation of thermodynamic limit of (1.1), i.e when the number of particles N goes to infinity.
Indeed, it is known that diffusion processes, described by SDEs as (1.1), cannot have a time periodic law.

Remark. Solutions to stochastic differential equation of the form:

$$d\mathbf{X}(t) = \mathbf{b}(\mathbf{X}(t))dt + \sigma d\mathbf{W}(t), \quad t \geq 0, \quad \sigma > 0$$

with $\mathbf{b} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ Lipschitz continuous, $\{\mathbf{W}(t); t \geq 0\}$ a d-dimensional Brownian motion cannot have a time periodic law. This follows from the fact that solutions to such equations either have an invariant probability measure $\boldsymbol{\pi}$, which is globally asymptotically stable $\big($the $Law\{\mathbf{X}(t)\}$ converges to $\boldsymbol{\pi}$ weakly as $t \to \infty$, for every initial $Law\{\mathbf{X}(0)\}\big)$ or $\mathbb{P}\left(\mathbf{X}(t) \in K \,|\, \mathbf{X}(0) = \mathbf{x}\right) \xrightarrow{t\to\infty} 0$, for every compact subset $K \subseteq \mathbb{R}^d$ and every initial condition $\mathbf{x} \in \mathcal{P}(\mathbb{R}^d)$, the space of probability measures on $\mathbb{R}^d$ [15, 8].

While, this need not to be true for *nonlinear* diffusion processes, i.e. solutions of stochastic differential equations of the form:

$$d\mathbf{X}(t) = \mathbf{b}\left(\mathbf{X}(t), Law\{\mathbf{X}(t)\}\right)dt + \sigma d\mathbf{W}(t);$$

where $\mathbf{b} : \mathbb{R}^d \times \mathcal{M}(\mathbb{R}^d) \longrightarrow \mathbb{R}^d$. They are called "nonlinear" because the associated *Fokker-Plank* equation is a nonlinear partial (integro-)differential equation.
In [16] can be found a proof of this fact, whereas [15] even provides examples of a nonlinear diffusions for which periodic behaviours are created by the noise, i.e. no periodicity occurs when the noise is turned off ($\sigma = 0$).

In our model (1.1) the mean-field interaction has a peculiar feature: when the number of particles N goes to infinity, the empirical averages $m_1^{(N)}(t)$ and $m_2^{(N)}(t)$ are expected to converge to a limit given by the solution of a nonlinear stochastic differential equation, which, can have a time periodic law. So the oscillatory behaviour, observed in the numerical simulations, could be theoretically explained trough the macroscopic limit. In the following we outline the study in more details.

First, we prove that, starting form i.i.d. initial conditions, if we let the number of particles N grow large, independence propagates in time. As a result, time evolution of a pair of representative particles, one for each population, is described by the stochastic differential system of equations:

$$\begin{aligned} dX_t &= \left[-X_t^3 + X_t - \alpha\theta_{11}(X_t - \mathbb{E}[X_t]) - (1-\alpha)\theta_{12}(X_t - \mathbb{E}[Y_t])\right]dt + \sigma dW_t^1, \\ dY_t &= \left[-Y_t^3 + Y_t - \alpha\theta_{21}(Y_t - \mathbb{E}[X_t]) - (1-\alpha)\theta_{22}(Y_t - \mathbb{E}[Y_t])\right]dt + \sigma dW_t^2; \end{aligned} \tag{1.2}$$

where $\{W_t^i; t \geq 0\}_{i=1,2}$ are two independent Brownian motions and $\mathbb{E}[\cdot]$ is the expectation respect of the joint probability measure $Law\{X(t), Y(t)\}$, given the initial conditions. The system (1.2) is well posed and a proof of the existence and uniqueness of a strong solution is given.

To be more precise, we show that, for all $0 \leq k_1, k_2 \in \mathbb{N}$, for all $T > 0$ and for all $t \in [0, T]$, as N goes to infinity, the random vector $\left(x_1^{(N)}(t), \ldots, x_{k_1}^{(N)}(t), y_1^{(N)}(t), \ldots, y_{k_2}^{(N)}(t)\right)$ converges in distribution to the vector $\left(X^1(t), \ldots, X^{k_1}(t), Y^1(t), \ldots, Y^{k_2}(t)\right)$, whose entries are independent random variables such that $\left(X^i(t)\right)_{i=1,\ldots,k_1}$ are copies of the solution of the first equation in (1.2) and $\left(Y^i(t)\right)_{i=1,\ldots,k_2}$ are copies of the solution to the second equation of (1.2). This is referred as the phenomenon of *propagation of chaos*.

Moreover, if we introduce the transition densities of the laws of X_t and Y_t as $q_t^X(x,y)$ and $q_t^Y(x,y)$ and denote their respective means $\mathbb{E}^X[\cdot]$ and $\mathbb{E}^Y[\cdot]$, the Fokker-Plank equation, associated to (1.2), is nonlinear:

$$\begin{aligned}
\frac{\partial q_t^X(x,z)}{\partial t} &= \frac{\sigma^2}{2}\frac{\partial^2 q_t^X(x,z)}{\partial z^2} - \frac{\partial}{\partial z}\left\{\left[(1-\alpha\theta_{11} - (1-\alpha)\theta_{12})z - z^3\right] q_t^X(x,z)\right\} \\
&\quad - \left[\alpha\theta_{11}\,\mathbb{E}^X[z] - (1-\alpha)\theta_{12}\,\mathbb{E}^Y[z]\right]\frac{\partial q_t^X(x,z)}{\partial z}, \\
\frac{\partial q_t^Y(y,z)}{\partial t} &= \frac{\sigma^2}{2}\frac{\partial^2 q_t^Y(y,z)}{\partial z^2} - \frac{\partial}{\partial z}\left\{\left[(1-\alpha\theta_{21} - (1-\alpha)\theta_{22})z - z^3\right] q_t^Y(y,z)\right\} \\
&\quad - \left[\alpha\theta_{21}\,\mathbb{E}^X[z] - (1-\alpha)\theta_{22}\,\mathbb{E}^Y[z]\right]\frac{\partial q_t^Y(y,z)}{\partial z};
\end{aligned} \tag{1.3}$$

Therefore system (1.2) is a good candidate for having a solution $\{(X_t, Y_t); 0 \leq t \leq T\}$ with time periodic law, in view of the previous remark. However, it is very hard to have insight into the long-time behaviours or to find periodic solutions, since the presence of non-linearity and noise makes the problem infinite dimensional, as we'll find out later. One could also perform numerical simulations of system (1.3) using the *F.E.M. method* with point-wise initial conditions $\left(q_0^X(x,z), q_0^Y(y,z)\right) = \left(\delta_x(z), \delta_y(z)\right)$, $x, y \in \mathbb{R}$. Nevertheless, we proceeded with a different approach.

First, in Section 3.4, we analyze the system (1.2) in the absence of noise and prove that oscillatory/periodic behaviour are not observed when $\sigma = 0$. We fully analyze the equilibrium points of the system and, doing so, we provide a scheme for the parameters space. Then, in the main part of the study, we investigate our limiting model, with $\sigma > 0$.

We show, in Sec. 3.5, that in presence of an appropriate amount of noise, the positions, at the thermodynamical limit, of the two representative particles of system (1.2), evolve as a pair of two independent Gaussian processes; we refer to this as *small-noise Gaussian approximation*. More precisely, we construct two independent Gaussian processes so that they solve the first two moments equations of (1.2), and, starting from same i.i.d. initial conditions, their evolution in time is close to the one of the macroscopic limit.

Consequently, we reduce the study to the means and variances of the Gaussian processes, for which we provide an explicit and deterministic system of differential equations. The dynamical system, which describes the time evolution of these, is parameterized by the noise (σ) and displays, in different parameters regions, a *Hopf bifurcation* for some critical values σ_c. In Section 3.6, we prove that, as the noise-parameter is decreased to cross the thresholds σ_c, a stable limit cycle appears around an equilibrium point. Therefore, within a certain range of the noise intensity ($0 < \sigma < \sigma_c$) the two Gaussian processes have a limit cycle, as long-time atractor of their evolution.

This, in particular, denotes that the laws of the Gaussian processes are periodic and, therefore, the small-noise approximation gives a good qualitative description of the emergence of the self-sustained oscillations observed in the numerical simulations of the system (1.1).

Chapter 2

Mathematical background

In this chapter we introduce a basic background for diffusion processes. In view of provide a general context for diffusions and give some useful notions to the study of them, we present some famous results in a discursive manner, without giving any rigorous demonstration. One can find extensive description about diffusion processes in [7], while we refer to [8, 6] for the study of stochastic differential equation.
In Sec. 2.2 we introduce some basic facts about propagation of chaos in a stochastic mean-field dynamic. In Sec. 2.4 we use them in an application of a Fitzhugh-Nagumo model for interacting neurons. For these we used [4] as a reference.
Sec. 2.3 reports the condition, that we used in our study, to check when showing the occurrence of a Hopf bifurcation within a dynamical system.

2.1. Basics on diffusion processes

Before diving into the subject, we bring in the common notions of basic probability theory on Euclidean n-spaces, as we will use most of them and their properties on defining diffusion processes.

We denote with the triple $(\Omega, \mathcal{U}, \mathbb{P})$ a general probability space, provided: Ω is a non-empty set, $\mathcal{U}$ is a σ-algebra of subsets of Ω and $\mathbb{P} : \mathcal{U} \to [0,1]$ is a probability measure. The smallest σ-algebra of subsets of $\mathbb{R}^n$, containing all the open sets, is called the *Borel* subsets of $\mathbb{R}^n$ and it is indicated with $\mathcal{B}$. In the following, the presence of a probability space will be implicit.
The elements $A \in \mathcal{U}$ are called events, and $\mathbb{P}(A)$ is the probability of the event A. Whenever a property is true except for an event of probability zero, we say that it holds *almost surely/almost everywhere* (usually abbreviated with "a.s.").

A $\mathcal{U}$-measurable function $\mathbf{X} : \Omega \to \mathbb{R}^n$, which is almost everywhere finite, is a generic random variable. Boldface characters denote vector valued quantities.
Usually, for a random variable, we omit the ω dependence and write "$\mathbf{X}$" instead of "$\mathbf{X}(\omega)$". Moreover, if $B \in \mathcal{B}$, $\mathbb{P}(\mathbf{X} \in B)$ indicates $\mathbb{P}(\mathbf{X}^{-1}(B))$, the probability that $\mathbf{X}$ takes values within B.
If $\mathbf{X}$ a random variable, $\mathcal{U}(\mathbf{X}) := \{\mathbf{X}^{-1}(B) \,|\, B \in \mathcal{B}\}$ is a σ-algebra, generated by $\mathbf{X}$.

The expectation/mean of a random variable is defined to be the integral

$$\mathbb{E}[\mathbf{X}] := \int_{\Omega} \mathbf{X} d\mathbb{P} \left(= \int_{\Omega} \mathbf{X}(\omega)\mathbb{P}(d\omega) \right),$$

provided $\left|\mathbf{X}(\omega)\right|$ is integrable; $|\cdot|$ denotes the Euclidean norm. So we can set the covariance function of two random variables $\mathbf{X}, \mathbf{Y}$: $\mathrm{Cov}(\mathbf{X}, \mathbf{Y}) := \mathbb{E}[(\mathbf{X} - \mathbb{E}[\mathbf{X}])(\mathbf{Y} - \mathbb{E}[\mathbf{Y}])]$; and the variance $\mathrm{Var}(\mathbf{X}) := \mathrm{Cov}(\mathbf{X}, \mathbf{X}) = \mathbb{E}[|\mathbf{X}|^2] - \left|\mathbb{E}[\mathbf{X}]\right|^2$.

Additionally, a random variable $\mathbf{X}$, can be defined in terms of the joint distribution function $F_{\mathbf{X}}(x_1, ..., x_n)$, that is, by specifying the probability of the event $\{\mathbf{X}(\omega)_1 < x_1, ..., \mathbf{X}(\omega)_n < x_n\}$, for $\omega \in \Omega$ and $(x_1, ..., x_n) \in \mathbb{R}^n$. In particular, if there exists a nonnegative, integrable function $f : \mathbb{R}^n \to \mathbb{R}$ such that $F_{\mathbf{X}}(x_1, ..., x_n) = \int_{-\infty}^{x_1} ... \int_{-\infty}^{x_n} f(y_1, ..., y_n)\, dy_n ... dy_1$, we say that $\mathbf{X}$ has probability density f ($\mathbf{X} \sim ...$).
In this case, it follows that, for any event $B \in \mathcal{B}$:

$$\mathbb{P}(\mathbf{X} \in B) = \int_B f(\mathbf{x}) d\mathbf{x}.$$

The expectation of a random variable can be extended and written using its density function.

For any pair of events A, B with $\mathbb{P}(B) > 0$, the conditional probability of A, given B, is $\mathbb{P}(A\,|\,B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$. Thus, A and B are said to be independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. This definition is naturally extended to sequence of events and σ-algebras. In particular, $\{\mathbf{X}_i : \Omega \to \mathbb{R}^n\}_{i=1,...}$ is a collection of independent random variables, if for all $k \geq 2$ and choices of $B_1, ..., B_k \in \mathcal{B}$,

$$\mathbb{P}(\mathbf{X}_1 \in B_1, \mathbf{X}_2 \in B_2, ..., \mathbf{X}_k \in B_k) = \mathbb{P}(\mathbf{X}_1 \in B_1)\mathbb{P}(\mathbf{X}_2 \in B_2) \cdots \mathbb{P}(\mathbf{X}_k \in B_k).$$

Upon this, one defines the conditional expectation of a random variable $\mathbf{X}$, given the event B ($\mathbb{P}(B) > 0$) as the mean over the probability measure $\mathbb{P}(\cdot|B)$:

$$\mathbb{E}[\mathbf{X}|B] := \frac{1}{\mathbb{P}(B)} \int_{\Omega} X d\mathbb{P};$$

and, more in general, given the σ-algebra $\mathcal{V} \subseteq \mathcal{U}$, $\mathbb{E}[\mathbf{X}|\mathcal{V}]$ is defined to be the unique $\mathcal{V}$-measurable random variable such that $\int_A \mathbf{X} d\mathbb{P} = \int_A \mathbb{E}[\mathbf{X}|\mathcal{V}]\, d\mathbb{P}$ for every $A \in \mathcal{V}$.

The properties that follow from these notions will be used without giving too much details and considerations.

2.1.1 Diffusions as Markov processes

For simplicity of notation, in the following, most of the definitions are given on the one dimensional case, but the concepts we present are easily generalized.

The concept of a *stochastic process* is used to describe, in a loose sense, systems which evolve probabilistically in time, or, more precisely, systems in which a certain time-dependent random variable $\mathbf{X}(t)$ exists.

We can measure values $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \dots$, etc., of $\mathbf{X}(t)$ at times $t_1, t_2, t_3, \dots$ and assume that a set of joint probability densities exists $p(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2; \mathbf{x}_3, t_3; \dots)$, which describe the system completely. In terms of these probability densities, one can define the conditional probabilities

$$p(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2; \dots | \mathbf{y}_1, \tau_1; \mathbf{y}_2, \tau_2; \dots) = \frac{p(\mathbf{x}_1, t_1; \mathbf{x}_2, t_2; \dots; \mathbf{y}_1, \tau_1; \mathbf{y}_2, \tau_2; \dots)}{p(\mathbf{y}_1, \tau_1; \mathbf{y}_2, \tau_2; \dots)};$$

where the time-ordering is: $t_1 \geq t_2 \geq t_3 \geq \dots \geq \tau_1 \geq \tau_2 \geq \dots$. The concept of an "evolution" leads to consider the conditional probabilities as the predictions of the future values of $\mathbf{X}(t)$ (i.e. $\mathbf{x}_1, \mathbf{x}_2, \dots$ at times $t_1, t_2, \dots$) given the knowledge of the past (values $\mathbf{y}_1, \mathbf{y}_2, \dots$ at times $\tau_1, \tau_2, \dots$).

Definition 1 (Stochastic process). Let T be a set, denoting the time, and $(\Omega, \mathcal{U}, \mathbb{P})$ a probability space. A *stochastic process* is a collection of real random variables $\mathbf{X} = \{\mathbf{X}_t \,|\, t \in T\}$ that can be expressed as a function of two variables $\mathbf{X} : T \times \Omega \to \mathbb{R}^n$, where:

- $\mathbf{X}_t := \mathbf{X}(t, \cdot) : \Omega \to \mathbb{R}^n$ is a random variable for every $t \in T$.
- $\mathbf{X}(\cdot, \omega) : T \to \mathbb{R}^n$ is the realization or *sample path* for each $\omega \in \Omega$.

Depending on T being a discrete or a continuous time set, we call the stochastic process a *discrete* or a *continuous* time process. Most of the times, we will denote a stochastic process by time dependence $\{\mathbf{X}_t\}_{t \geq 0}$ or simply by $\mathbf{X}_t$.

The time variability of a stochastic process, technically, is described by all its conditional probabilities. Though, substantial information can be gained studying the quantities:

- Mean: $\mathbb{E}[\mathbf{X}_t] = \mu(t)$ for each $t \in T$
- Variance: $\mathrm{Var}(\mathbf{X}_t) = \nu^2(t)$ for each $t \in T$.
- (two-time) Co-variance: $\mathrm{Cov}(\mathbf{X}_t, \mathbf{X}_s)$ for distinct time instants $s, t \in T$.

A stochastic process $\mathbf{X}_t$, for which the random variables $\mathbf{X}_{t_j+1} - \mathbf{X}_{t_j}$, $j = 1, \dots, n-1$, are independent for any finite combinations of time instants $t_1 < \dots < t_n$ in T is a stochastic process with *independent increments*.

Example 1. A stochastic process X_t, such that any joined distribution

$$F_{X_{t_{i_1}}, \dots, X_{t_{i_n}}}(x_{i_1}, \dots, x_{i_n})$$

is normal, i.e. has density of the form

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{|x-\mu|^2}{2\sigma^2}} \qquad \text{for some } \mu, \sigma \in \mathbb{R},$$

for every choices of $t_{i_j} \in T$, is called *Gaussian*.

Example 2. A Poisson process is a continuous time stochastic process $X = \{X_t \,|\, t \geq 0\}$ with (non-overlapping) independent increments for which

$$X_0 = 0 \text{ a.s.},$$
$$\mathbb{E}[X_t] = 0,$$
$$X_t - X_s \sim \mathcal{P}(\lambda(t-s))$$

for all $0 \leq s \leq t$; λ is called *intensity parameter*.

We call a stochastic process *strictly stationary* if all its joint distributions are invariant under time displacement, that is

$$F_{X_{t_{i_1+h}},\dots,X_{t_{i_n+h}}}(\cdot) = F_{X_{t_{i_1}},\dots,X_{t_{i_n}}}(\cdot)$$

for every $t_{i_j}, t_{i_{j+1}} \in T$ and $h \geq 0$. While a stochastic process X_t is *wide-sense stationary* if there exists a constant $m \in \mathbb{R}$ and a function $c : T \to \mathbb{R}$, such that

$$\mathbb{E}[X_t] = m, \quad \mathrm{Var}[X_t] = c(0) \quad \text{and } \mathrm{Cov}[X_t, X_s] = c(t-s)$$

for all $s, t \in T$.

The setting of a generic stochastic process is very loose, so, next, we focus on the concept of *Markov process*, which rises from the simple idea that only the knowledge of the present determines the future. We start from discrete time and discrete spaces and the definition of *Markov chain*; this helps to better understand *diffusion* processes.

Definition 2 (Markov chain). Let $\mathcal{X} = \{x_1, \dots, x_N\}$ be the set of a finite number of discrete states. The discrete time stochastic process $X = \{X_n : \Omega \to \mathcal{X} \,|\, n = 1, 2, \dots\}$ is a discrete time *Markov chain* if it satisfies the *Markov property*, that is

$$\mathbb{P}(X_{n+1} = x_j \,|\, X_n = x_{i_n}) = \mathbb{P}(X_{n+1} = x_j \,|\, X_1 = x_{i_1}, \dots, X_n = x_{i_n})$$

for all possible $x_j, x_{i_1}, \dots, x_{i_n} \in \mathcal{X}$ with $n = 1, 2, \dots$.

Heuristically, the Markov property tells that the future depends on the past only through the present, or, in other words, that only the present state is needed to determine the future ones.

For a Markov chain we can define the *transition matrix* $\mathcal{P}_n \in \mathbb{R}^{N\times N}$, its entries are given by

$$p_n^{(i,j)} = \mathbb{P}(X_{n+1} = x_j \,|\, X_n = x_{i_n})$$

for $i, j = 1, \dots, N$. We call them the *transition probabilities* and they satisfy $\sum_j^N p_n^{(i,j)} = 1$ for each i, as X_{n+1} can only attain state in $\mathcal{X}$.

If we call $\mathbf{p}_n$ the column vector of the marginal probabilities $(\mathbb{P}(X_n = x_1), \dots, \mathbb{P}(X_n = x_N))$, then the probability vector $\mathbf{p}_{n+1}$ is given by $\mathbf{p}_{n+1} = \mathcal{P}_n^T \mathbf{p}_n$.

A discrete time Markov chain is called *homogeneous* if $\mathcal{P}_n = \overline{\mathcal{P}}$ for all $n = 1, 2, \dots$. Therefore the probability vector of a homogeneous Markov chain satisfies

$$\mathbf{p}_{n+k} = \left(\overline{\mathcal{P}}^k\right)^T \mathbf{p}_n$$

for every $k \geq 1$, and the probability distributions depend only on the time that has elapsed.
This does not mean that the Markov is strictly stationary. In other to be so, it is also required that $\mathbf{p}_n = \overline{\mathbf{p}}$ for each $n = 1, 2, ...$, which implies that the probability distributions are equal for all times such that $\overline{\mathbf{p}} = \overline{\mathcal{P}}^T \overline{\mathbf{p}}$.

Remark. It can be shown that a homogeneous Markov chain has at least one stationary probability vector solution. Therefore, it is sufficient that the initial random variable X_0 is distributed according to one of its stationary probability vectors for the Markov chain to be stationary.

Definition 3. Let $\mathcal{X} = \{x_1, ..., x_N\}$ the set of a finite number of discrete states. The stochastic process $X = \{X_t : \Omega \to \mathcal{X} | t \in \mathbb{R}^+\}$ is a *continuous time Markov chain* if it satisfies the following property:

$$\mathbb{P}(X_t = x_j | X_s = x_i) = \mathbb{P}(X_t = x_j | X_{r_1} = x_{i_1}, ..., X_{r_n} = x_{i_n}, X_s = x_i)$$

for $0 \leq r_1 \leq ... \leq r_n < s < t$ and all $x_{i_1}, ..., x_{i_n}, x_i, x_j \in \mathcal{X}$.

The entries of the *transition matrix* $\mathcal{P}_{s,t} \in \mathbb{R}^{N \times N}$ and the probability vectors are now respectively:

$$p_{s,t}^{(i,j)} = \mathbb{P}(X_t = x_j | X_s = x_i) \qquad \mathbf{p}_t = \mathcal{P}_{s,t}^T \mathbf{p}_s$$

for any $0 \leq s \leq t$; and the transition matrices satisfy the relationship $\mathcal{P}_{r,t} = \mathcal{P}_{r,s}\mathcal{P}_{s,t}$, for any $0 \leq r \leq s \leq t$.
If all the transition matrices depend only on the time differences, the we say that the continuous time Markov chain is *homogeneous* and we write $\mathcal{P}_{s,t} = \mathcal{P}_{0,t-s} \equiv \mathcal{P}_{t-s}$ for any $0 \leq s \leq t$. So we have $\mathcal{P}_{t+s} = \mathcal{P}_t\mathcal{P}_s = \mathcal{P}_s\mathcal{P}_t$ for all $s, t \geq 0$.

Moreover, for an homogeneous continuous time Markov chain, we define the *infinitesimal generator* or *intensity matrix* $\mathcal{G} \in \mathbb{R}^{N \times N}$, as follows:

$$g^{(i,j)} = \begin{cases} \lim_{t\to\infty} \frac{p_t^{(i,j)}}{t} & \text{if } i \neq j, \\ \lim_{t\to\infty} \frac{p_t^{(i,j)} - 1}{t} & \text{if } i = j. \end{cases}$$

Theorem 1. *A homogeneous continuous time Markov chain X_t is completely characterized by the initial probability vector $\mathbf{p}_0 = \mathbb{P}(X_0 = x_i)_i$ and its intensity matrix $\mathcal{G}$. Moreover, if all the diagonal elements of $\mathcal{G}$ are finite, then, the transition probabilities satisfy the Kolmogorov forward and backward equation, respectively:*

$$\frac{d}{dt}\mathcal{P}_t - \mathcal{P}_t\mathcal{G} = 0$$
$$\frac{d}{dt}\mathcal{P}_t - \mathcal{G}^T\mathcal{P}_t = 0$$

We move on now to define *continuous time continuous state* Markov processes, when the state space $\mathcal{X} \subseteq \mathbb{R}$.

Definition 4 (Markov process). Let $\mathcal{X} \subseteq \mathbb{R}$ be the state space, the stochastic process $X = \{X_t : \Omega \to \mathcal{X} | t \in \mathbb{R}^+\}$ is a continuous time continuous state Markov process if it satisfies the following Markov property:

$$\mathbb{P}(X_t \in B | X_s = x) = \mathbb{P}(X_t \in B | X_{r_1} = x_1, ..., X_{r_n} = x_n, X_s = x)$$

for all Borel subsets $B \subseteq \mathbb{R}$, time instants $0 \le r_1 \le ... \le r_n \le s \le t$ and all $x_1, ..., x_n, x \in \mathbb{R}$ for which the conditional probabilities are defined.

Fore fixed s, x and t the transition probability $\mathbb{P}(X_t \in B | X_s = x)$ is a probability measure on the σ-algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$ such that

$$\mathbb{P}(X_t \in B | X_s = x) = \int_B p(s, x; t, y)\, dy$$

for all $B \in \mathcal{B}$. The quantity $p(s, x; t, \cdot)$ is called *transition density*, it generalize the role of the transition matrix of Markov chains. It follows form the Markov property that

$$p(s, x; t, y) = \int_{-\infty}^{\infty} p(s, x; \tau, z) p(\tau, z; t, y)\, dz$$

for all $0 \le s \le \tau \le t$ and $x, y \in \mathbb{R}$. This equation is known as the *Chapman-Kolmogorov equation*.

Remark. From this point of view, one can construct a complete Markov process given a transition probability function $p(s, x; t, y)$ and starting with an arbitrary initial distribution X_0.

If all the transition density of a Markov process X_t depend only on the time differences $t - s$, then it is called *homogeneous* and we write $p(s, x; t, y) = p(0, x; t - s, y) \equiv p_{t-s}(x, y)$ for any $0 \le s \le t$. It is called *periodic* if the function $p(s, x; t + s, y)$ is periodic in s.

Now we have all the tools to give the definition of a diffusion process.

Definition 5 (Diffusion process). A continuous time continuous state Markov process $X = \{X_t | t \in \mathbb{R}^+\}$ is a *diffusion process* if the following limits, involving the transition densities $p(s, x; t, \cdot)$, exists for all $\epsilon > 0$, $s \ge 0$ and $x \in \mathbb{R}$:

$$\lim_{t \downarrow s} \frac{1}{t - s} \int_{|y - x| > \epsilon} p(s, x; t, y)\, dy = 0, \tag{2.1}$$

$$\lim_{t \downarrow s} \frac{1}{t - s} \int_{|y - x| < \epsilon} (y - x) p(s, x; t, y)\, dy = b(s, x), \tag{2.2}$$

$$\lim_{t \downarrow s} \frac{1}{t - s} \int_{|y - x| < \epsilon} (y - x)^2 p(s, x; t, y)\, dy = \sigma^2(s, x). \tag{2.3}$$

The functions $\alpha(s, x)$ and $\beta(s, x)$ are called respectively the *drift* and *diffusion* coefficient at time s and position x. Usually it is assumed that these limit relations are uniform with respect to t in each finite interval $t_0 \le t \le t_1$ and with respect to x in $-\infty < x < \infty$.

The first condition (2.1) prevents the diffusion process from having instantaneous jumps, assuring the "continuity". It means that the probability for the final position $\mathbf{y}$ to be finitely different from $\mathbf{x}$ goes to zero *faster* than $t-s$, as $t-s$ goes to zero. Diffusion processes are *almost surely continuous* functions of time, i.e. the map $X(\cdot,\omega): \mathbb{R}^+ \to \mathbb{R}$ is continuous function in t, for $\mathbb{P}$-almost all $\omega \in \Omega$, this is also called *sample continuity*. However they not need to be differentiable.

Remark. The continuity of a stochastic process, can be defined in several ways (continuity with probability one, mean square continuity, continuity in probability, etc.) An interesting result is the *Kolmogorov's continuity criterion*, which states that if a continuous time continuous states stochastic process X_t satisfy

$$\mathbb{E}\left[|X_t - X_s|^a\right] \leq c|t-s|^{1+b}$$

for all $s,t \geq 0$ and $|t-s| \leq h$, for some $a,b,c,h > 0$, then there exists a modification $\widetilde{X}_t$ such that $\widetilde{X}_t$ is sample continuous and for all $t \geq 0$ $\mathbb{P}(X_t = \widetilde{X}_t) = 1$.

Conditions (2.2) and (2.3), on the other hand, shows that $b(s,x)$ and $\sigma^2(s,x)$ represent, respectively, the instantaneous rate of change of the mean and the instantaneous rate of change of the squared fluctuations of the process, given $X_s = x$:

$$b(s,x) = \lim_{t \downarrow s} \frac{\mathbb{E}[W_t - W_s \,|\, W_s = x]}{t-s}$$

$$\sigma^2(s,x) = \lim_{t \downarrow s} \frac{\mathbb{E}[(W_t - W_s)^2 \,|\, W_s = x]}{t-s}$$

Remark. If X_t is a homogeneous diffusion process, i.e. its probability densities depend only on the time differences, it follows that the drift and the diffusion coefficients are independent of time, that is $b(s,x) \equiv b(x)$ and $\sigma(s,x) \equiv \sigma(x)$ for every $s,x \in \mathbb{R}$.

We give now two important notions that describe the evolution of a diffusion process in time. For simplicity, we focus here on *homogeneous diffusion processes*, since our model falls into this class, but the concepts are generic.

A Markov process is well portrayed by its "infinitesimal behaviour", i.e. the evolution for small time increments.
Suppose $\mathbf{X}_t$ is an homogeneous Markov process that takes values in $\mathbb{R}^n$, we use the abbreviations $\mathbb{E}_\mathbf{x}[\cdot] := \mathbb{E}[\cdot|\mathbf{X}_0 = \mathbf{x}]$ and $\mathbb{P}_\mathbf{x}(\cdot) := \mathbb{P}(\cdot|\mathbf{X}_0 = \mathbf{x})$, so we can set the *transition kernel*:

$$Q_t(\mathbf{x};A) := \mathbb{P}_\mathbf{x}(\mathbf{X}_t \in A) = \int_A p_t(\mathbf{x},\mathbf{y})d\mathbf{y}.$$

We denote with $\mathcal{C}_0(\mathbb{R}^n,\mathbb{R})$ the space of continuous functions $f: \mathbb{R}^n \to \mathbb{R}$ such that $\lim_{|x|\to\infty} f(\mathbf{x}) = 0$.

Definition 6 (Infinitesimal generator). We define the *infinitesimal generator* $\mathcal{L}$ of a homogeneous Markov process $\mathbf{X}_t$:

$$\mathcal{L}f(\mathbf{x}) := \lim_{t\to 0} \frac{\mathbb{E}_\mathbf{x}\left[f(\mathbf{X}_t)\right] - f(\mathbf{x})}{t} \tag{2.4}$$

for every function $f \in \mathcal{C}_0(\mathbb{R}^n,\mathbb{R})$ for which such limit exists. The set of such functions is called the *domain* of the generator.

It can be shown that the generator completely describes the law of the process.

Note. This relation is valid and can be extended also to the class of functions twice continuously differentiable with compact support $\mathcal{C}_0^2(\mathbb{R}^n, \mathbb{R})$.

If $\mathbf{X}_t$ is a diffusion process that takes values in $\mathbb{R}^n$, its diffusion and drift functions are respectively $\boldsymbol{\sigma} : \mathbb{R} \times \mathbb{R}^n \to M(n \times d, \mathbb{R})$ and $\mathbf{b} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Moreover, if $\mathbf{X}_t$ is homogeneous $\mathbf{b}$ and $\boldsymbol{\sigma}$ are time independent and it can be proved that the infinitesimal generator acts:

$$\mathcal{L}f(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{b}_i(\mathbf{x}) \frac{\partial}{\partial x_i} f(\mathbf{x}) + \frac{1}{2} \sum_{i,j=1}^{n} (\boldsymbol{\sigma}\boldsymbol{\sigma}^T)_{i,j}(\mathbf{x}) \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}),$$

for every $f \in \mathcal{C}_0(\mathbb{R}^n, \mathbb{R})$.

Furthermore, we can define a family of operators $\{\mathcal{P}_t \,|\, t \geq 0\}$, called *semigroup* of the process, that acts on the functions $f \in \mathcal{C}_0(\mathbb{R}^n, \mathbb{R})$ in the following way:

$$\mathcal{P}_t f(\mathbf{x}) := \mathbb{E}_{\mathbf{x}}[f(\mathbf{X}_t)] = \int_{\mathbb{R}^n} f(\mathbf{y}) Q_t(\mathbf{x}; d\mathbf{y});$$

It is possible to show that $\mathcal{P}_t f \in \mathcal{C}_0(\mathbb{R}^n, \mathbb{R})$ for every f.

The name "semigroup" comes from the fact that $\mathcal{P}_t \circ \mathcal{P}_s = \mathcal{P}_{t+s}$, i.e. $\mathcal{P}_t(\mathcal{P}_s f) = \mathcal{P}_{t+s} f$ for every $t, s \geq 0$ and $f \in \mathcal{C}_0(\mathbb{R}^n, \mathbb{R})$, as it follows from the Chapman-Kolmogorv equation. Note that

$$\mathcal{L}f = \lim_{t \to 0} \frac{1}{t}(\mathcal{P}_t f - f) = \frac{d}{dt} \mathcal{P}_t(f)|_{t=0},$$

for every f in the domain of $\mathcal{L}$. More generally, for every f, $\mathcal{P}_t(f)$ is still in the domain of $\mathcal{L}$ and the following relation holds

$$\frac{d}{dt} \mathcal{P}_t f = \mathcal{L}(\mathcal{P}_t) = \mathcal{P}_t(\mathcal{L}f), \qquad \forall t \geq 0.$$

Therefore, the generator $\mathcal{L}$ determines the semigroup.

We shall see later an useful criterion on stochastic differential equations, that employ the infinitesimal generator.

For a diffusion process, another way to gain sight into its behaviour is to look at the backward-forward evolution of its transition density $p(s, x; t, y)$.

Let $\mathbf{X}_t$ be a homogeneous diffusion process such that its transition distribution $Q_t(\mathbf{x}; d\mathbf{y}) = \mathbb{P}_{\mathbf{x}}\left(\mathbf{X}_t \in d\mathbf{y}\right)$ is absolutely continuous for every $t \geq 0$, i.e. suppose that there exists a measurable function $q_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$Q_t(\mathbf{x}; d\mathbf{y}) = q_t(\mathbf{x}, \mathbf{y}) d\mathbf{y}, \qquad \forall t > 0, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Giving some regularity assumptions, it can be shown that the density $q_t(\mathbf{x}, \mathbf{y})$ satisfy the *backward Kolmogorov equation*, that is, for every fixed $\mathbf{y} \in \mathbb{R}^n$,

$$\frac{\partial}{\partial t} q_t(\mathbf{x}, \mathbf{y}) = \mathcal{L}_{\mathbf{x}} q_t(\mathbf{x}, \mathbf{y}), \qquad \forall t > 0, \quad \forall \mathbf{x} \in \mathbb{R}^n;$$

here $\mathcal{L}_{\mathbf{X}}$ means that the infinitesimal generator is acting on the variable $\mathbf{x}$ of $q_t(\mathbf{x},\mathbf{y})$. Furthermore, if we denote with $\mathcal{L}^*$ the *Hermitian adjoint* operator of $\mathcal{L}$, it holds that, for every fixed $\mathbf{x} \in \mathbb{R}^n$,

$$\frac{\partial}{\partial t} q_t(\mathbf{x},\mathbf{y}) = \mathcal{L}^*_{\mathbf{y}} q_t(\mathbf{x},\mathbf{y}), \qquad \forall t > 0, \quad \forall \mathbf{y} \in \mathbb{R}^n;$$

This is the so called *Kolmogorov forward equations*, which is also know as the *Fokker-Plank* equation. It can be proved that for a generic homogeneous diffusion process $\mathbf{X}_t$ the equation is:

$$\frac{\partial}{\partial t} q_t(\mathbf{x},\mathbf{y}) = -\sum_{i=1}^{n} \frac{\partial}{\partial y_i}\Big\{b_i(\mathbf{y}) q_t(\mathbf{x},\mathbf{y})\Big\} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial y_i \partial y_j}\Big\{(\sigma\sigma^T)_{i,j}(\mathbf{y}) q_t(\mathbf{x},\mathbf{y})\Big\};$$

where the initial conditions are given by $q_{t_0}(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$. The study of such equations can give useful insight of the long time behaviour of a diffusion process.

2.1.2 The Wiener process

The Wiener process was proposed by Wiener as mathematical description of the *Brownian motion*. It is the fundamental building block of the theory of stochastic differential equations, as we will show. The physical phenomenon of the Brownian motion was investigated by the famous botanist *Robert Brown* in the nineteenth century, when observing the motion of pollen grains suspended in water. Essentially, it characterizes the erratic motion (i.e. diffusion) of a grain pollen on a water surface due to the fact that is continually bombarded by water molecules. The modern formulation is quite straight forward and we shall use both terms (Brownian motion and Wiener process) to indicate it.

Definition 7 (1-dim Brownian motion). A standard 1-dim *Wiener process* is a continuous time continuous states Gaussian Markov process $W = \{W_t | t \geq 0\}$, with (non-overlapping) independent increments for which

- $\mathbb{P}(W_0 = 0) = 1$
- $\mathbb{E}[W_t] = 0$ for all $t \geq 0$
- $W_t - W_s \sim \mathcal{N}(0, t-s)$ for all $0 \leq s \leq t$

The covariance is $\mathrm{Cov}[W_t, W_s] = \min\{t,s\}$. Indeed, if $0 \leq s \leq t$, then

$$\begin{aligned}
\mathrm{Cov}[W_t, W_s] &= \mathbb{E}[(W_t - \mathbb{E}[W_t])(W_s - \mathbb{E}[W_s])] \\
&= \mathbb{E}[W_t W_s] \\
&= \mathbb{E}[(W_t - W_s + W_s) W_s] \\
&= \mathbb{E}[W_t - W_s]\,\mathbb{E}[W_s] + \mathbb{E}[W_s^2] = 0 \cdot 0 + s
\end{aligned}$$

Hence, it is not a wide-sense stationary process. However, it is homogeneous since its transition probability is given by

$$p(s,x;t,y) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left\{-\frac{(y-x)^2}{2(t-s)}\right\}.$$

Although the sample paths of Wiener processes are almost surely continuous functions of time (the Kolmogorov continuity criterion is satisfied for $a = 4$, $b = 1$ and $c = 3$), they are almost surely nowhere differentiable. The following lines give a proof of this fact.

Consider the partition of a bounded time interval $[s,t]$ into sub-intervals $[\tau_k^{(n)}, \tau_{k+1}^{(n)}]$ of equal length $\frac{(t-s)}{2^n}$, where $\tau_k(n) = s + \frac{k(t-s)}{2^n}$ for $k = 0,1,...,2^n - 1$. It can be shown that

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left(W_{\tau_{k+1}^{(n)}}(\omega) - W_{\tau_k^{(n)}}(\omega) \right)^2 = t - s, \quad \text{a.s.};$$

$W_\tau(\omega)$ is a realization of the standard Wiener process $W = \{W_\tau \,|\, \tau \in [s,t]\}$ for any $\omega \in \Omega$. Hence,

$$\begin{aligned} t - s \leq & \limsup_{n\to\infty} \max_{0\leq k\leq 2^{(n)}-1} \left| W_{\tau_{k+1}^{(n)}}(\omega) - W_{\tau_k^{(n)}}(\omega) \right| \\ & \times \sum_{k=0}^{2^n-1} \left| W_{\tau_{k+1}^{(n)}}(\omega) - W_{\tau_k^{(n)}}(\omega) \right|. \end{aligned}$$

From the sample path continuity, we have that

$$\max_{0\leq k\leq 2^{(n)}-1} \left| W_{\tau_{k+1}^{(n)}}(\omega) - W_{\tau_k^{(n)}}(\omega) \right| \to 0, \quad \text{a.s. , as } n \to \infty,$$

and therefore, we must have that:

$$\sum_{k=0}^{2^n-1} \left| W_{\tau_{k+1}^{(n)}}(\omega) - W_{\tau_k^{(n)}}(\omega) \right| \to \infty, \quad \text{a.s. , as } n \to \infty$$

As a consequence, the sample paths do, almost surely, not have bounded variation on $[s,t]$ and cannot be differentiated.

The standard Wiener process is a diffusion process with drift and diffusion coefficient $b(s,x) = 0$ and $\sigma(s,x) = 1$:

$$\begin{aligned} b(s,x) &= \lim_{t\downarrow s} \frac{\mathbb{E}[y] - x}{t - s} = 0 \\ \sigma^2(s,x) &= \lim_{t\downarrow s} \frac{\mathbb{E}[y^2] - 2\,\mathbb{E}[y]x + x^2}{t - s} = \lim_{t\downarrow s} \left\{ \frac{t-s}{t-s} + 0 \right\} = 1 \end{aligned}$$

Closely related to the Brownian motion is the *white noise*, as we show in the following.

Let X_t be a (wide-sense) stationary process, i.e. $\mathbb{E}[X_t] = m \in \mathbb{R}$, $\mathrm{Var}[X_t] = c(0)$ and $\mathrm{Cov}[X_t, X_s] = c(t-s)$ for some function $c : \mathbb{R}^+ \to \mathbb{R}$.
We define for X_t the *power spectral density* as the *Fourier transform* of its covariance, that is:

$$\tilde{c}(\xi) = \int_{-\infty}^{\infty} c(t) e^{-i\xi t}\, dt,$$

where $\xi = 2\pi f$ and $c(t) \equiv c(t-0)$, f indicates the frequency.

We can recover the covariance of X_t as the inverse transform of the spectral density: $c(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{c}(\xi) e^{-i\xi t}\, d\xi$. So, the variance of the process can be interpreted as the average power (or energy):

$$\mathrm{Var}[X_t] = c(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{c}(\xi)\, d\xi$$

Definition 8 (White noise). A *white noise* w_t is a zero-mean wide-sense stationary process with constant non-zero spectral density $\tilde{c}(\xi) = \tilde{c}(0)$ for all $\xi \in \mathbb{R}$.

The white noise has flat spectral density, all "frequencies" contribute equally in the correlation function. The covariance of the white noise $\mathrm{Cov}[w_t, x_s] = \mathbb{E}[w_t w_s] = c(t-s)$ satisfy $c(t) = \tilde{c}(0)\delta(t)$ for all $t \in \mathbb{R}^+$.
So, without loss of generality, if we assume that $\tilde{c}(0) = 1$, it can be shown that Gaussian white noise correspond to the following limit process

$$w_t = \lim_{h\to 0} \frac{W_{t+h} - W_t}{h},$$

where W_t is a standard Wiener process. To see this, suppose $h > 0$, fix $t > 0$, and set

$$\begin{aligned}
\phi_h(s) := \mathbb{E}&\left[\left(\frac{W_{t+h} - W_t}{h}\right)\left(\frac{W_{s+h} - W_s}{h}\right)\right] \\
&= \frac{1}{h^2}\left(\mathbb{E}[W_{t+h}W_{s+h}] - \mathbb{E}[W_{t+h}W_s] - \mathbb{E}[W_t W_{s+h}] + \mathbb{E}[W_t W_s]\right) \\
&= \frac{1}{h^2}\left(\min\{t+h, s+h\} - \min\{t+h, s\} - \min\{t, s+h\} + \min\{t, s\}\right)
\end{aligned}$$

Then, $\phi_h(s) \to 0$ as $h \to 0$ for each $s \neq t$. But $\phi_h \geq 0$ and $\int \phi_h(s)\, ds = 1$; and so presumably $\phi_h(s) \to \delta_0(s-t)$ in some sense, as $h \to 0$. In addition, we expect that $\phi_h(s) \to \mathbb{E}[w_t w_s]$. These heuristic considerations, suggest the definition of the white noise.

Hence the white noise can be seen as the "derivative" of a Wiener process. However, the sample paths of a Wiener process are not differentiable anywhere. We will see how to interpret this in the next section.

2.1.3 Diffusion processes as solution to stochastic differential equations

Diffusion processes can be described by solutions of *Itô stochastic differential equations* (**SDE**). Generally speaking, stochastic differential equations describe processes for which a variable x following the rule $a(t, x)$ might be subject to some random environmental effect, i.e. *noise*.

The solution to an ordinary differential equation (**ODE**) of the form

$$\frac{d}{dt}x(t) = a(t, x),$$

can be written in its symbolic differential form or as an integral equation, which are respectively:

$$dx = a(t,x)dt$$
$$x(t) = x(t_0) + \int_{t_0}^{t} a(s, x(s))\, ds;$$

$x(t;t_0,x_0)$ denotes the solution satisfying the initial condition $x(t_0) = x_0$. For some regularity conditions on $a(t,x)$, this solution is unique, which means that the future is completely defined by the present given the initial condition.

In a similar manner, we want to extend this to deal with the presence of an intrinsic random component ξ_t, which represent the environmental noise, in such a way that we can find solutions to equations of the form:

$$\frac{d}{dt}x(t) = a(t,x) + \text{``}b(t,x)\xi_t\text{''};$$

for suitable functions $a(t,x)$ and $b(t,x)$.
We want to associate the noise ξ_t to some random process and expect it has the properties:

- zero mean: $\mathbb{E}[\xi_t] = 0$ for all t;
- uncorrelation: $\mathbb{E}[\xi_t \xi_s] = 0$ for every $t \neq s$;
- (wide-sense) stationarity.

One can easily check that the white noise w_t satisfies all these.
Therefore, the symbolic form of a stochastic differential equation can be written as follows:

$$dX_t = \alpha(t,x)dt + \beta(t,x)w_t dt \tag{2.5}$$

for suitable functions $\alpha(t,x), \beta(t,x)$. It defines a continuous time continuous space Markov process X_t, as one can prove.

The next step is to connect this to the standard Wiener process W_t, and read $w_t dt$ as the increment dW_t. Roughly speaking, we want the process X_t to be connected with W_t in such a way that if $X_t = x$, then the increment $dX_t = X_{t+dt} - X_t$ during the next period of time dt is

$$dX_t \sim \alpha(t,x)dt + \beta(t,x)dW_t.$$

Indeed, if this relation holds, one can prove that

$$\mathbb{E}[dX_t - (\alpha(t,x)dt + \beta(t,x)dW_t) \,|\, X_t = x] = o(dt),$$
$$\mathbb{E}[dX_t - (\alpha(t,x)dt + \beta(t,x)dW_t)^2 \,|\, X_t = x] = o(dt);$$

where $o(dt)$ approach zero as dt, and therefore, that X_t defines a diffusion process in the sense of the definitions (2.1) to (2.3).

The interpretation of the symbolic expressions (2.5) is the integral equation along a sample path

$$X_t(\omega) = X_{t_0}(\omega) + \int_{t_0}^{t} \alpha(s, X_s(\omega))\,ds + \int_{t_0}^{t} \beta(s, X_s(\omega))dW_s(\omega), \tag{2.6}$$

given some probability space $(\Omega, \mathcal{U}, \mathbb{P})$; where we interpreted $w_t dt$ as dW_t

However, the Wiener process W_t is (almost surely) nowhere differentiable such that the white noise process w_t does not really exist as a conventional function of t. As a result, the second integral in (2.6) cannot be understood as an ordinary (Riemann or Lebesgue) integral. Worse, it is not a Riemann-Stieltjes integral since the continuous sample paths of a Wiener process are not of bounded variation for each sample path. A way to read (2.6) is given by the *Itô's stochastic integral*, which we will explain in the next section.

2.1.4 Itô Stochastic Calculus

Consider a probability space $(\Omega, \mathcal{U}, \mathbb{P})$, a Wiener process $W = \{W_t \mid t \geq 0\}$ and an increasing family $\{\mathcal{U}_t, t \geq 0\}$ of sub-σ-algebras of $\mathcal{U}$ such that W_t is a $\mathcal{U}_t$-measurable for each $t \geq 0$ and with

$$\mathbb{E}[W_t | \mathcal{U}_0] = 0 \quad \text{and} \quad \mathbb{E}[W_t - W_s | \mathcal{U}_s] = 0 \quad \text{a.s. ,}$$

for $0 \leq s \leq t$. Consider the integral expression of the random function $f : T \times \Omega \to \mathbb{R}$ on the unit time interval:

$$I[f](\omega) := \int_0^1 f(s, w) dW_s(\omega). \tag{2.7}$$

Step 1. If the function f is a nonrandom step function, i.e. $f(t, \omega) = f_j$ on $t_j \leq t < t_{j+1}$ for $j = 1, 2, ..., n-1$ with $0 = t_1 < t_2 < ... < t_n = 1$, then

$$I[f](\omega) = \sum_{j=1}^{n-1} f_j \left(W_{t_{j+1}}(\omega) - W_{t_j}\right), \quad \text{a.s. .} \tag{2.8}$$

Remark. This integral is a random variable with zero mean as it is a sum of random variables with zero mean.

Furthermore, we have that:

$$\mathbb{E}[I[f](\omega)] = \sum_{j=1}^{n-1} f_j^2 (t_{j+1} - t_j), \tag{2.9}$$

by the properties of the Brownian motion.

Step 2. If the function f is a random step function, that is $f(t, \omega) = f_j(\omega)$ on $t_j \leq t < t_j$ for $j = 1, 2, ..., n-1$ with $t_1 < t_2 < ... < t_n$ is $\mathcal{U}_t$-measurable and mean square integrable over Ω, that is $\mathbb{E}[f_j^2] < \infty$ for $j = 1, 2, ..., n$. The stochastic integral is defined as follows

$$I[f](\omega) = \sum_{j=1}^{n-1} f_j(\omega) \left(W_{t_{j+1}}(\omega) - W_{t_j}\right), \quad \text{w.p. 1.} \tag{2.10}$$

Then, for any $a, b \in \mathbb{R}$ and any random step functions f, g such that f_j, g_j on $t_j \leq t < t_{j+1}$ for $j = 1, 2, ..., n-1$ with $0 = t_1 < t_2 < ... < t_n = 1$ is $\mathcal{U}_{t_j}$-measurable and mean square integrable, the stochastic integral (2.10) satisfies the properties a.s.:

$$I[f] \text{ is } \mathcal{U}_1\text{-measurable,} \tag{2.11}$$

$$\mathbb{E}[I[f]] = 0, \tag{2.12}$$

$$\mathbb{E}[I^2[f]] = \sum_j \mathbb{E}[f_j^2](t_{j+1} - t_j) \tag{2.13}$$

$$I[af + bg] = aI[f] + bI[g], \tag{2.14}$$

Step 3. If f is a continuous function such that $f(t, \cdot)$ is $\mathcal{U}_t$-measurable and mean square integrable, then we define the stochastic integral $I[f]$ as the limit of integrals $I[f^{(n)}]$ of random step functions $f^{(n)}$ converging to f, with the following convergence description. We characterize the limit of the following finite sums:

$$I[f^{(n)}](\omega) = \sum_{j=1}^{n-1} f\left(t_j^{(n)}, \omega\right)\left[W_{t_{j+1}}(\omega) - W_{t_j}(\omega)\right], \quad \text{a.s. ;}$$

where $f^{(n)}(t, \omega) = f\left(t_j^{(n)}, \omega\right)$ on $t_j \leq t \leq t_{j+1}$ for $j = 1, 2, ..., n-1$ with $t_1 < t_2 < ... < t_n$. From the property (2.13) it follows:

$$\mathbb{E}\left[I^2\left[f^{(n)}\right]\right] = \sum_{j=1}^{n-1} \mathbb{E}\left[f^2\left(t_j^{(n)}, \cdot\right)\right](t_{j+1} - t_j).$$

This quantity converges to the Riemann integral $\int_0^1 \mathbb{E}[f^2(s, \cdot)]\,ds$ for $n \to \infty$. This result, along with the well-behaved mean square property of the Wiener process, i.e. $\mathbb{E}[(W_t - W_s)^2] = t - s$, suggests defining the stochastic integral in terms of mean square convergence.

Theorem 2 (Itô stochastic integral). *The Itô (stochastic) integral $I[f]$ of a function $f : T \times \Omega \to \mathbb{R}$, is the (unique) mean square limit of sequences $I[f^{(n)}]$ for any sequence of random step functions $f^{(n)}$ converging to f (in mean square):*

$$I[f](\omega) = m.s. \lim_{n \to \infty} \sum_{j=1}^{n-1} f\left(t_j^{(n)}, \omega\right)\left[W_{t_{j+1}}(\omega) - W_{t_j}(\omega)\right], \quad a.s. \text{ ;}$$

$f^{(n)}$ is converges to f in mean square if $\mathbb{E}\left[\int_s^t \left(f^{(n)}(\tau, \omega) - f(\tau, \omega))^2\right) d\tau\right] \to 0$, as $n \to \infty$.

The time-dependent Itô integral

$$X_t(\omega) = \int_{t_0}^t f(s, \omega)\,dW_s(\omega)$$

is a random variable defined on any interval $[t_0, t]$, and it $\mathcal{U}_t$-measurable and mean square integrable for every $t \geq 0$. Moreover, properties (2.11) to (2.14) still hold.

Remark. Since we chose f to be mean square integrable, one can prove that $I[f]$ has a version with continuous samples path almost surely, which leads us to consider $I[f]$ satisfying sample continuity.

As the Riemann and the Riemann-Stieltjes integrals, the Itô integral satisfies conventional properties such as the linearity property and the additivity property. However, it has also the following unusual property.

Example 3. Let W_t be the standard Brownian motion, then

$$\int_0^t W_s(\omega)dW_s(\omega) = \frac{1}{2}W_t^2(\omega) - \frac{1}{2}t, \quad \text{a.s.} \; .$$

This follows from the fact that

$$\sum_j W_{t_j}(W_{t_{j+1}} - W_{t_j}) = \frac{1}{2}W_t^2 - \frac{1}{2}\sum_j (W_{t_{j+1}} - W_{t_j})^2,$$

for every choice of $t_j < t$.

Indeed, another famous interpretation of $\int f(s,w)dW_s(\omega)$ is the *Stratonovich integral*. In Itô, the integrand f is approximated in the left end point t_j in every interval $[t_j, t_{j+1}]$. An alternative option is to choose the middle point $t_j = \frac{t_j + t_{j+1}}{2}$, which characterizes the Stratonovich integral. The two definitions lead to different results in various cases and are useful in diverse contexts. The Stratonovich integral, for example, obeys the usual chain rule when performing change of variables, so can be easier to use to perform some calculations, while the Itô integral does not, as we shall se in the next section.

Stochastic differentials that are interpreted as Itô stochastic integrals do not follow the chain rule of classical calculus. Roughly speaking, when differentiating, an additional term is appearing due to the fact that dW_t^2 is equal to dt in the mean square sense.

Let X_t be a general diffusion process and consider $Y = \{Y_t = U(t, X_t) | t \geq 0\}$ with $U(t,x)$ having continuous second order partial derivatives. If X_t were continuously differentiable, the chain rule of classical calculus would give the following expression:

$$dY_t = \frac{\partial U}{\partial t}dt + \frac{\partial U}{\partial x}dX_t$$

This follows from a Taylor expansion of U. If X_t is a process such that $dX_t = f(t, X_t)dW_t$ for some function f, we get,

$$dY_t = \left\{ \frac{\partial U}{\partial t} + \frac{1}{2}f^2\frac{\partial^2 U}{\partial x^2} \right\} dt + \frac{\partial U}{\partial x}dX_t;$$

where the equality is interpreted in the mean square sense. The additional term is due to the fact that $\mathbb{E}[dX_t^2] = \mathbb{E}[f^2]dt$ gives rise to an additional term of the order in Δt of the Taylor expansion for U:

$$\Delta Y_t = \left\{ \frac{\partial U}{\partial t}\Delta t + \frac{\partial U}{\partial x}\Delta x \right\} + \left\{ \frac{\partial^2 U}{\partial t^2}\Delta t^2 + 2\frac{\partial^2 U}{\partial t \partial x}\Delta t \Delta x + \frac{\partial^2 U}{\partial x^2}\Delta x^2 \right\} + \dots$$

Theorem 3 (Itô Formula). *Consider the following general stochastic differential:*

$$X_t(\omega) - X_s(\omega) = \int_s^t \alpha(u, X_u)\, du + \int_s^t \beta(u, X_u)\, dW_u(\omega).$$

Let $Y_t = U(t, X_t)$, with $U(t,x)$ having continuous partial derivatives in t and up to second order in x. Then the following stochastic chain rule (Itô formula) holds :

$$Y_t - Y_s = \int_s^t \left\{ \frac{\partial U}{\partial t}\Big|_{(u,X_u)} + \frac{\partial \alpha}{\partial u} \frac{\partial U}{\partial x}\Big|_{(u,X_u)} + \frac{1}{2} \frac{\partial^2 U}{\partial x^2}\Big|_{(u,X_u)} \right\} du + \int_s^t \frac{\partial U}{\partial x}\Big|_{(u,X_u)} dX_u$$

Now that we gave an precise interpretation to the expression (2.6), we we give the standard definition of stochastic differential equation.

Let $\mathbf{W}_t$ be a standard d-dimensional Brownian motion and $\mathbf{Z}$ a random variable independent of the process $\mathbf{W}_t$. Define $\mathcal{F}_t := \mathcal{U}(\{\mathbf{W}_s | 0 \leq s \leq t\}, \mathbf{Z})$ with $t \geq 0$ the σ-algebra generated by $\mathbf{Z}$ and all the history of the Wiener process up to (and including) time t.

Definition 9 (Itô Stochastic differential equations). Assume $T > 0$ and

$$\mathbf{b} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n, \quad \boldsymbol{\sigma} : [0,T] \times \mathbb{R}^n \to M(n \times d, \mathbb{R})$$

are given functions. We say that a stochastic process $\mathbf{X}_t$ is a *strong* solution of the Itô stochastic differential equation

$$\begin{cases} d\mathbf{X}_t = \mathbf{b}(t, \mathbf{X}_t)dt + \boldsymbol{\sigma}(t, \mathbf{X}_t) d\mathbf{W}_t \\ \mathbf{X}_0 = \mathbf{z} \end{cases} \tag{2.15}$$

for $0 \leq t \leq T$ provided for all $t \geq 0$:

1. $\mathbf{X}_t$ is $\mathcal{F}_t$-measurable,
2. $\mathbb{E}\left[\int_0^T \left| b_i(s, \mathbf{X}_s) \right| ds\right] < \infty$, for all $i = 1, \ldots, n$;
3. $\mathbb{E}\left[\int_0^T \sigma_{i,j}^2(s, \mathbf{X}_s)\, ds\right] < \infty$, for all $i = 1, \ldots, n$ and $j = 1, \ldots, d$;
4. for all times $0 \leq t \leq T$,

$$\mathbf{X}_t = \mathbf{X}_0 + \int_0^t \mathbf{b}(s, \mathbf{X}_s)\, ds + \int_0^t \boldsymbol{\sigma}(s, \mathbf{X}_s)\, d\mathbf{W}_s$$

Remark. In view of previous assertions, $X_t(\omega)$ can be always assumed to have sample continuous paths almost surely.

In the following we recall two famous results about stochastic differential equations we use in the study of our models.

Theorem 4. *Let $T > 0$, $\{\mathbf{W}_t \,|\, 0 \le t \le T\}$ a standard Wiener process and let $\mathbf{b} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $\boldsymbol{\sigma} : [0,T] \times \mathbb{R}^n \to M(n \times d, \mathbb{R})$ be measurable functions such that:*

$$\left|\mathbf{b}(t,\mathbf{x})\right|^2 + \sum_{j=1}^{d} \left|\sigma_j(t,\mathbf{x})\right|^2 \le L(1 + |\mathbf{x}|^2); \qquad x \in \mathbb{R},\, t \in [0,T]$$

$$\left|\mathbf{b}(t,\mathbf{x}) - \mathbf{b}(t,\mathbf{y})\right| + \sum_{j=1}^{d} \left|\sigma_j(t,\mathbf{x}) - \sigma_j(t,\mathbf{y})\right| \le D\left|\mathbf{x} - \mathbf{y}\right| \qquad x,y \in \mathbb{R},\, t \in [0,T]$$

for some constants L, D. Then,

1. *For every σ-algebra $\{\mathcal{F}_t \,|\, 0 \le t \le T\}$ generated by the history of the Brownian motion and fixed (square-integrable) random variable $\mathbf{Z}$ independent of the processes $\mathbf{W}_t - \mathbf{W}_{t_0}$, there exists a solution $\mathbf{X}_t$ of the stochastic differential equation*

$$d\mathbf{X}_t = \mathbf{b}(t,\mathbf{X}_t)dt + \boldsymbol{\sigma}(t,\mathbf{X}_t)d\mathbf{W}_t; \qquad 0 \le t \le T, \qquad \mathbf{X}_0 = \mathbf{Z}$$

   which is an almost surely continuous stochastic process and is unique w.p. 1.

2. *This solution is a Markov process whose transition probability density $p(s,x;t,y)$ is defined for $t > s$ by the relation $p(s,x;t,y) = \mathbb{P}(\mathbf{X}_t = \mathbf{y} \,|\, \mathbf{X}_s = \mathbf{x})$;*

Moreover,

3. *The transition probability density $p(s,x;t,y)$ satisfies a generalized version of limits (2.1) to (2.3), and define a diffusion process with drift and diffusion constants respectively $\mathbf{b}(t,\mathbf{x})$ and $\boldsymbol{\sigma}^2(t,\mathbf{x})$.*

The following result is useful to prove the existence of solution to stochastic differential equations. In the study of our model we use the following theorem to prove the well posedness of system (1.1).

Theorem 5. *Suppose that conditions of the previous theorem are valid in every cylinder $I \times U_R$ and, moreover, that there exists a non-negative function $V \in C_0^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ on the domain of the infinitesimal generator $\mathcal{L}$ such that for some constant $c > 0$*

$$\mathcal{L}V(t,\mathbf{x}) \le cV(t,\mathbf{x}), \qquad \text{for every } t \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^n,$$

$$V_R = \inf_{|\mathbf{x}|>R} V(t,\mathbf{x}) \to \infty, \quad \text{as } R \to \infty$$

then part 1, 2 and 3 of the previous theorem hold true.

2.2. Stochastic mean-field dynamics and propagation of chaos

We give in this section a brief view on the stochastic mean-field dynamics perceived as the random evolution of a system comprised by N interacting components, invariant in law for permutation of components and such that the contribution of each component to the evolution of any other is of order $\frac{1}{N}$. The permutation invariance clearly does not allow any freedom in the choice of the geometry of the interactions, however this is the feature that makes these models analytically treatable. Applications of such theory span in many field including social and life sciences.

2.2.1 A first example

First we give a result to better understand the general setting.

Consider a system of N interacting diffusions on $\mathbb{R}^d$ solving the following system of SDEs:

$$dX_t^{i,N} = \frac{1}{N}\sum_{j=1}^{N} b\left(X_t^{i,N}, X^{j,N}\right)dt + dW_t^i \qquad i = 1,...,N$$

where $b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a Lipschitz function, $\left(W_t^i\right)_{i\geq 1}$ are independent Brownian motions, and we assumed that $\left(X_0^{i,N}\right)_{i=1,...,N}$ is a family of i.i.d. square integrable random variables; so that that the above stochastic differential equation is well posed, by the fundamental theorem [13].
Consider the single component $X^{i,N}$, assume $X_0^{i,N} = X_0^i$ does not depend on the specific number of particles N and "believe in laws of large numbers" as $N \to \infty$. It is natural to guess that $X^{i,N}$ converges, in some sense, to a limit process $\overline{X}^i$ solving:

$$\begin{cases} d\overline{X}_t^i = \left\{\int b\left(\overline{X}_t^i, y\right) q_t(dy)\right\} dt + dW_t^i \\ \overline{X}_0^i = X_0^i \end{cases} \tag{2.16}$$

where $q_t(\cdot)$ denotes $Law(\overline{X}_t^i)$.
Once the nontrivial problem of well-posedness of (2.16) is settled, one aims at showing that, for any given $T > 0$ and for any given $m \geq 1$:

$$\left(X_{[0,T]}^{1,N}, X_{[0,T]}^{2,N}, ..., X_{[0,T]}^{m,N}\right) \longrightarrow \left(\overline{X}_{[0,T]}^1, \overline{X}_{[0,T]}^2, ..., \overline{X}_{[0,T]}^m\right)$$

in distribution, as $N \to \infty$; indicating with $X_{[0,T]} \in \mathcal{C}([0,T])$, the whole trajectory up to time T. Note that the components of the process $(\overline{X}_{[0,T]}^1, \overline{X}_{[0,T]}^2, ..., \overline{X}_{[0,T]}^m)$ are independent. This means that independence at time 0 propagates in time, at least in the macroscopic limit $N \to \infty$. This property is referred to as *propagation of chaos*.

Propagation of chaos can be actually rephrased as a *Law of Large numbers*. To this aim, given a generic vector $\mathbf{x} = (x_1, x_2, ..., x_N)$, denote with $\rho(\mathbf{x}; dy) := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}(dy)$ the related empirical measure. The propagation of chaos property given above, is equivalent to the fact that the sequence of empirical measures $\rho(\mathbf{X}_{[0,T]}^N)$ converge in distribution to $Q \in \mathcal{P}(\mathcal{C}([0,T]))$; where $\mathcal{P}(\mathcal{C}([0,T]))$ denotes the set of probabilities on $\mathcal{C}([0,T])$, provided with the weak topology of weak convergence and Q is the law of the solution of (2.16). This is established in the following result.

Proposition 6. *Let $(X^{i,N} : N \geq 1, 1 \leq i \leq N)$ be a triangular array of random variables taking values in a topological space E, such that for each N the law of $(X^{i,N})_{1\leq i\leq N}$ is symmetric (i.e. invariant by permutation of components). Moreover let $(\overline{X}^i)_{i\geq 1}$ be a i.i.d. sequence of E-valued random variables. Then the following statements are equivalent:*

i) for every $m \leq 1$

$$(X^{1,N}, X^{2,N}, ..., X^{m,N}) \longrightarrow (\overline{X}^1, \overline{X}^2, ..., \overline{X}^m)$$

in distribution as $N \to \infty$

ii) the sequence of empirical measures $\rho(\mathbf{X}^N_{[0,T]})$ converges in distribution to $Q := Law(\overline{X}^1)$ as $N \to \infty$.

2.2.2 Propagation of chaos for interacting systems

Now we extend this result to a wide class of $\mathbb{R}^d$-valued interacting dynamics, which includes the relaxed model above. Systems given by (1.1) and (1.2), fall into this very general setting, however here the main aim is to introduce *quenched disorder*, which accounts for inhomogeneities in the system and jumps in the dynamics, which allows to include process with discrete state space. The dynamics is determined by the following characteristics.

- "Local" parameters $(h_i)_{i=1}^N$, drawn independently from a distribution $\boldsymbol{\mu}$ on $\mathbb{R}^{d'}$ with compact support.
- A *drift* coefficient
$$b(x_i,h_i,\rho_N(\mathbf{x},\mathbf{h})) : \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}) \longrightarrow \mathbb{R}^d,$$
where
$$\rho_N(\mathbf{x},\mathbf{h}) = \frac{1}{N}\sum_{i=1}^N \delta_{(x_i,h_i)}.$$
- A *diffusion* coefficient
$$\sigma(x_i,h_i,\rho_N(\mathbf{x},\mathbf{h})) : \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}) \longrightarrow \mathbb{R}^{d\times n}$$
where n is the dimension of the Brownian motion.
- A *jump rate*
$$\lambda(x_i,h_i,\rho_N(\mathbf{x},\mathbf{h})) : \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}) \longrightarrow [0,+\infty),$$
together with
- a distribution $f(x_i,h_i,\rho_N(\mathbf{x},\mathbf{h});v)\alpha(dv)$ with
$$f : \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}) \times [0,1] \longrightarrow \mathbb{R}^d;$$
$\alpha(dv)$ is a probability measure on $[0,1]$.

The microscopic model. The dynamics could be introduce also via the infinitesimal generator and the semigroup, but it is convenient to use the language of stochastic differential equations. Let $(W^i)_{i\geq 1}$ be a i.i.d. sequence of the n-dimensional Brownian motions; $(N^i(dt,du,dv))_{i\geq 1}$ be i.i.d. Poisson random measures on $[0,+\infty)\times[0,+\infty)\times[0,1]$ with characteristic measure $dt\otimes du\otimes\alpha(dv)$. The microscopic model is given as solution of the SDEs:

$$\begin{aligned} X_t^{i,N} =& X_0^i + \int_0^t b(X_s^{i,N},h_i,\rho(\mathbf{X}_s^N,\mathbf{h}))\,ds + \int_0^t \sigma(X_s^{i,N},h_i;\rho(\mathbf{X}_s^N,\mathbf{h}))\,dW_s^i \\ &+ \int_{[0,t)\times[0,+\infty)\times[0,1]} f(X_{s^-}^{i,N},h_i;\rho_N(\mathbf{X}_{s^-}^{i,N},\mathbf{h}),\alpha)\mathbf{1}_{[0,\lambda(X_{s^-}^{i,N},h_i;\rho_N(\mathbf{X}_{s^-}^N,\mathbf{h}))]}(u)N^i(ds,du,dv); \end{aligned} \tag{2.17}$$

it will be assumed that the inital conditions X_0^i are i.i.d., square integrable, independent of both the local parameters (h_i) and of the driving noises (W^i,N^i).

The macroscopic limit. At heurisitc level it is not hard to identify the limit of a give component $X^{i,N}$ of (2.17) subject to a local field h. We omit the apex i on the process and of the driving noises

$$\begin{aligned}\overline{X}_t(h) = \overline{X}_0^i &+ \int_0^t b(\overline{X}_s(h), h, r_s)\, ds + \int_0^t \sigma(\overline{X}_s(h), h, r_s)\, dW_s^i \\ &+ \int_{[0,t)\times[0,+\infty)\times[0,1]} f(\overline{X}_{s^-}(h), h, r_s, \alpha) \mathbf{1}_{[0,\lambda(\overline{X}_{s^-}(h),h,r_s)]}(u) N^i(ds, du, dv);\end{aligned} \tag{2.18}$$

where $r_s = \mathrm{Law}(\overline{X}_s(h)) \otimes \boldsymbol{\mu}(dh)$. We indicate by $\overline{X}^i$ the solution to (2.18) with $\overline{X}_0 = \overline{X}_0^i$ and driving noises W^i, N^i.

Well posedness conditions We now give conditions that guarantee well posedness of (2.17) and (2.18). Weaker conditions exists but the two *Lipschitz* conditions we provide allow a reasonable economy of notations. It is useful to work with the following probability measure:

$$\mathcal{P}^1(\mathbb{R}^d) := \left\{ \nu \in \mathcal{P}(\mathbb{R}^d) \mid \int |x|\, \nu(dx) < \infty \right\}$$

which is provide with the *Wasserstein metric*:

$$d(\nu, \nu') := \inf \left\{ \int \left| x - y \right| \Pi(dx, dy) \,|\, \Pi \text{ has marginals } \nu \text{ and } \nu' \right\}.$$

- **L1**: The function $b(x,h,r)$ and $\sigma(x,h,r)$ defined in $R^d \times \mathbb{R}^{d'} \times \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'})$ are continuous, and globally Lipschitz in (x,r) uniformly in h
- **L2**: We assume $f : \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}) \times [0,1] \longrightarrow \mathbb{R}^d$ and $\lambda : \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^{d'}) \longrightarrow [0, +\infty)$ are continuous and obey the following condition:

$$\int \left| f(\mathbf{x}, \mathbf{h}; r; v) \mathbf{1}_{[0,\lambda(\mathbf{x},\mathbf{h};r)]}(u) - f(\mathbf{y}, \mathbf{h}; r'; v) \mathbf{1}_{[0,\lambda(\mathbf{x},\mathbf{h};r')]}(u) \right| du\, \alpha(dv) \le L\Big(\left| \mathbf{x} - \mathbf{y} \right| + d(r, r') \Big)$$

for all $\mathbf{x}, \mathbf{y}, r, r', \mathbf{h}$.

Remark. The above assumptions imply that when one replaces r by the empirical measure $\rho_N(\mathbf{x}, \mathbf{h})$, recovers a Lipschitz condition in x. For instance, the function $b(x_i, h_i, \rho_N(\mathbf{x}, \mathbf{h}))$ is globally Lipschitz in $\mathbf{x}$ uniformly in $\mathbf{h}$.

Theorem 7 (Propagation of chaos). *Under conditions **L1** and **L2** both the system* (2.17) *and* (2.18) *admit a unique strong solution. Moreover, for $i \ge 1$ denote by $\overline{X}^i(h_i)$ the solution of (2.18) with the local parameter h_i and the same initial condition X_0^i of (2.17). Then for each i and $T > 0$:*

$$\lim_{N\to\infty} \int \mathbb{E}\left[\sup_{t\in[0,T]} \left| X_t^{i,N} - \overline{X}_t^i(h_i) \right| \right] \boldsymbol{\mu}^{\otimes N}(d\mathbf{h}) = 0$$

where $\boldsymbol{\mu}^{\otimes N}$ is the N-fold product of $\boldsymbol{\mu}$.

We have seen in proposition (6), that propagation of chaos is equivalent to a Law of Large Numbers:

$$\rho(\mathbf{X}^N) = \frac{1}{N}\sum_{i=1}^{N}\delta_{X^{i,N}} \longrightarrow Q$$

as $N \to \infty$, where Q is the law of the macroscopic dynamics. It is therefore natural to consider a corresponding Central Limit Theorem, which describes the *fluctuations* around the limit. In particular, one can consider the normalized distribution-valued process

$$\Phi_t^N := \sqrt{N}\left[\frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^{i,N}} - q_t\right];$$

where q_t is the marginal probability of Q at time t and prove, with remarkable generality, that for any bounded time-interval $[0,T]$, the process Φ^N converges quickly to a distribution valued Gaussian process.

2.3. Limit cycle from a Hopf bifurcation

Here we recall a famous result in dynamical system which it is useful in the study of our model. The aim is to provide some tools that allow us to justify the presence of periodic behaviours for a diffusion process.

Consider a generic non-linear dynamical system of differential equations, depending on a parameter $k \in \mathbb{R}$:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, k),$$

with $f \in \mathcal{C}^1(E)$, where E is an open subset of $\mathbb{R}^n$. We focus on changes of the parameter k, if the qualitative behavior of the system remains the same, we say that there is *structurally stability*. A vector field which is not structurally stable, it belongs to the *bifurcation* set in $\mathcal{C}^1(E)$. A bifurcation occurs when a small smooth change made to the parameter values causes a sudden "qualitative" or topological change in the behavior of the system. A value $\widetilde{k}$ of the parameter k for which the vector field $\mathbf{f}(\mathbf{x}, \widetilde{k})$ is not structurally stable is called a bifurcation value. We are interested in a particular type of these phenomena.

The analysis around non-hyperbolic critical points of a vector field is relevant to the notion of *Hopf bifurcations*. These occur when a periodic solution or a limit cycle branches-out around the equilibrium point $\mathbf{x}_0$, as the parameter k changes to cross the threshold $\widetilde{k}$.

Note. When a equilibrium point locally changes its stability, as the parameter varies, from stable to unstable or vice-versa, the phase portrait far from it will be qualitatively unaffected: if the non-linearity makes the far flow contracting, then orbits will still be coming in (or out) and we expect a periodic orbit to appear where the near and far flow find a balance.

One can detect Hopf bifurcations simply by looking whether a pair of complex eigenvalues of the linearized system, given by the Jacobian matrix $D\mathbf{f}(\mathbf{x_0}, \widetilde{k})$, around the equilibrium point $\mathbf{x}_0$, crosses the imaginary axis at $k = \widetilde{k}$.

Theorem 8 (Hopf). *Suppose that $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, k)$ defines a $\mathcal{C}^4$-system, where $\mathbf{x} \in \mathbb{R}^n$ and $k \in \mathbb{R}$, with a critical point $\mathbf{x}_0$ for $k = \widetilde{k}$. Moreover, suppose that $\mathrm{D}\mathbf{f}\left(\mathbf{x_0}, \widetilde{k}\right)$ has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real part. Then there is a smooth curve of equilibrium points $\mathbf{x}(k)$ with $\mathbf{x}(\widetilde{k}) = \mathbf{x}_0$ and the eigenvalues $\lambda(k)$ and $\overline{\lambda(k)}$, which are pure imaginary at $k = \widetilde{k}$, vary smoothly with k. Furthermore, if*

$$\frac{d}{dk}\left[\mathrm{Re}\left(\lambda(k)\right)\right]_{k=\widetilde{k}} \neq 0, \tag{2.19}$$

then there is a unique two-dimensional center manifold passing through the point $\left(\mathbf{x}_0, \widetilde{k}\right)$ and a smooth transformation of coordinates such that the system on the center manifold is transformed into the normal form

$$\begin{aligned} \dot{x} &= -y + ax(x^2 + y^2) - by(x^2 + y^2) + O(|\mathbf{x}|^4) \\ \dot{x} &= x + bx(x^2 + y^2) + ay(x^2 + y^2) + O(|\mathbf{x}|^4) \end{aligned} \tag{2.20}$$

in a neighborhood of the origin which, for $a \neq 0$, has a weak focus of multiplicity one.

For a function $\mathbf{f} \in \mathcal{C}^k(E)$, where E is an open subset of $\mathbb{R}^n$, we define

$$\|\mathbf{f}\|_k = \sup_{\mathbf{x}\in E}\left|\mathbf{f}(\mathbf{x})\right| + \sup_{\mathbf{x}\in E}\left\|\mathrm{D}\mathbf{f}(\mathbf{x})\right\| + \ldots + \sup_{\mathbf{x}\in E}\left\|\mathrm{D}^k\mathbf{f}(\mathbf{x})\right\|$$

where for the norms $\|\cdot\|$ on the right-hand side of this equation we use

$$\left\|\mathrm{D}^k\mathbf{f}(\mathbf{x})\right\| = \max\left|\frac{\partial^k\mathbf{f}(\mathbf{x})}{\partial x_{j_1}\ldots\partial x_{j_k}}\right|$$

with $j_1, \ldots, j_k = 1, \ldots, n$.

Theorem 9. *If the origin is a multiple focus of multiplicity m of the analytic system (2.20) then for $k \leq 2m+1$*

1. *there is a $\delta > 0$ and an $\epsilon > 0$ such that any system ϵ-close to (2.20) in the $\mathcal{C}^k$-norm, defined above, has at most m limit cycles in a neighbourhood $N_\delta(0)$ and*

2. *for any $\delta > 0$ and an $\epsilon > 0$ there is an analytic system which is ϵ-close to (2.20) in the $\mathcal{C}^k$-norm and has exactly m simple limit cycles in $N_\delta(0)$.*

2.4. An application: The Fitzhugh-Nagumo model for neurons

The FitzHugh-Nagumo model describes (qualitatively rather than quantitatively) the response of an excitable neuron membrane to external current stimuli. It was designed to isolate conceptually the essentially mathematical properties of excitation and propagation from the electrochemical properties of sodium and potassium ion flow, but it has many more apllications in various fields of science.

The FitzHugh-Nagumo model comes from a reduction of more realistic and complex models (e.g. the Hodgkin-Huxley model) and describes the evolution of the membrane potential x_t through the following differential equation

$$\begin{cases} \dot{x}_t = x_t - \frac{1}{3}x_t^3 + y_t + I_t^{ext} \\ \dot{y}_t = \epsilon(a + bx_t - \gamma y_t) \end{cases} \tag{2.21}$$

where:

- y_t is a *recovery variable* obtained by reduction of other variables
- I_t^{ext} is the input current, assumed to be random and stationary. Without loss of generality, choosing a properly, we can assume I_t^{ext} has zero mean.
- b is the interaction strength between x and y, $\gamma \geq 0$ is a dissipation parameter and a is a kinetic parameter related with the input current and synaptic conductance.

The parameter ϵ can be used to separate the time scales of the evolution of the two variables. We assume the input current takes the form a scaled Brownian motion: $dI_t^{ext} = \sigma dW_t$.

If we analyze the system in absence of randomness, i.e. consider in the equations no input current ($\sigma = 0$) and set $b = -1, \gamma = 0$ to make the analysis simpler, then (2.21) has a unique equilibrium point in $(a, -a + \frac{a^3}{3})$, which is globally stable for $|a| < 1$, resulting in small amplitude trajectory dynamics.
The system display a Hopf bifurcation at $|a| = 1$ and a stable periodic orbit emerges for $|a| > 1$. This means that the system can be excited by the input current, producing, at least for appropriate choice of the parameters, rapid variations of the potential (*spikes*) which occur periodically.

There are many ways to make several neurons interact with each others in a network, even within the mean-field scheme, depending of how we model synaptic connections. The simplest that correspond to electrical synapses, leads to the following system.

We denote with $X^{i,N}$ the membrane potential of the i-th neuron. Assigning the local parameter h_i, that can be interpreted as the *macroscopic location* of the neuron, or its *type*, we derive the following stochastic differential system of equations:

$$\begin{aligned} dX_t^{i,N} &= \left(X_t^{i,N} - \frac{1}{3}\left(X_t^{i,N}\right)^3 + Y_t^{i,N}\right)dt \\ &\quad + \frac{1}{N}\sum_{j=1}^{N} J(h_i, h_j)\left(X_t^{i,N} - X_t^{j,N}\right)dt + \sigma dW_t^i \\ dY_t^{i,N} &= \epsilon(h_i)\left[a(h_i) + b(h_i)X_t^{i,N} - \gamma(h_i)Y_t^{i,N}\right] \end{aligned} \tag{2.22}$$

where the coupling parameters $J(h_i, h_j)$ are introduced. These quantities tune the interaction between neurons. One can introduce a delay dynamics to enririch the dynamic. So, denoting the delay with τ in the transmission of information between different neurons, we have:

$$\begin{aligned} dX_t^{i,N} &= \left(X_t^{i,N} - \frac{1}{3}\left(X_t^{i,N}\right)^3 + Y_t^{i,N}\right)dt \\ &+ \frac{1}{N}\sum_{j=1}^{N} J(h_i,h_j)\left(X_t^{i,N} - X_{t-\tau(h_i,h_j)}^{j,N}\right)dt + \sigma dW_t^i \\ dY_t^{i,N} &= \epsilon(h_i)\left[a(h_i) + b(h_i)X_t^{i,N} - \gamma(h_i)Y_t^{i,N}\right] \end{aligned} \tag{2.23}$$

However the delay makes a bit more difficult to prove the well posedness of the system, for both the model and its macroscopic limit, nevertheless one can prove that, as $N \to \infty$, the propagation of chaos carries through, giving the following macroscopic description:

$$\begin{aligned} d\overline{X}_t(h) &= \left(\overline{X}_t(h) - \frac{1}{3}\overline{X}_t^3(h) + \overline{Y}_t(h)\right)dt \\ &+ \left(\int J(h,h')(\overline{X}_t(h) - y)q_{t-\tau(h,h')}(dy;h')\mu(dh')\right)dt + \sigma dW_t, \\ d\overline{Y}_t(h) &= \epsilon(h)\left(a(h) + b(h)\overline{X}_t(h) - \gamma(h)\overline{Y}_t(h)\right)dt. \end{aligned} \tag{2.24}$$

Here $q_t(dx;h)$ denotes the law of $\overline{X}_t(h)$.

Not much is known at this level of generality, so we consider the simplest, homogeneous case in which h is constant, $\gamma = 0$ and $b = -1$. The following SDEs describe the evolution of the state of a generic particle of the system, when $N \to \infty$:

$$\begin{aligned} d\overline{X}_t &= \left[\overline{X}_t - \frac{1}{3}\overline{X}_t^3 + \overline{Y}_t + J\left(\overline{X}_t - \mathbb{E}[\overline{X}_{t-\tau}]\right)\right]dt + \sigma dW_t \\ d\overline{Y}_t &= \epsilon\left(a + \overline{X}_t\right)dt \end{aligned} \tag{2.25}$$

here $\mathbb{E}[\,\cdot\,]$ is the conditional expectation of the process, given the initial conditions.

As a further simplification, we let the noise go to zero, in both the equation and the initial condition. We obtain the deterministic system with delay

$$\begin{cases} \dot{x}_t = x_t - \frac{1}{3}x_t^3 + y_t + J(x_t - x_{t-\tau}) \\ \dot{y}_t = \epsilon(a + x_t) \end{cases} \tag{2.26}$$

This system has been extensively studied in [9]. Here we assume $J \geq 0$.

- For every fixed $a \in \mathbb{R}$, the point $(a, -a + \frac{a^3}{3})$ is still the unique fixed point of system (2.26). It is stable for $|a| > \sqrt{1+2J}$ and unstable for $|a| < 1$, no matter what τ is.

- For $1 < |a| < \sqrt{1+2J}$ loss of stability via a Hopf bifurcation can be obtained by increasing τ: *interaction and transmission delay may produce oscillations even if single neurons are in the stability region.*

In this setting the noise may play a role in exciting the neuronal network. Consider the simplified system (2.25) and remove the delay. We obtain:

$$\begin{aligned} dX_t &= \left[X_t - \frac{1}{3}X_t^3 + Y_t + J\left(X_t - \mathbb{E}[X_t]\right) \right] dt + \sigma dW_t \\ dY_t &= \epsilon(a - X_t)dt \end{aligned} \tag{2.27}$$

Some indications on the behavior of this system, confirmed by numerical simulations, are obtained via the following heuristic argument. A similar approach can be found also in [3].

Using the *Ito formula*, write the moments equations of (2.27):

$$\begin{aligned} dX_t^p &= \left\{ \left[X_t - \frac{1}{3}X_t^3 + Y_t + J\left(X_t - \mathbb{E}[X_t]\right) \right] pX_t^{p-1} + \sigma^2 p(p-1)X_t^{p-2} \right\} dt + \sigma p X_t^{p-1} dW_t \\ dY_t^p &= \epsilon(a - X_t)pY_t^{p-1}dt \end{aligned}$$

If we pretend the two variables (X_t, Y_t) describe Gaussian processes, one can derive a closed system for the mean functions $\left(\mu_X(t), \mu_Y(t)\right)$ and the covariance matrix $\left(\mathrm{Cov}(X_t, X_t), \mathrm{Cov}(X_t, Y_t), \mathrm{Cov}(Y_t, Y_t)\right)$. The solutions to this system fully describe two Gaussian processes $(\widetilde{X}_t, \widetilde{Y}_t)$ which can be shown to be a good approximation of the solution to (2.27) for σ small.
Therefore, one can focus on the study of the evolution of the law of $(\widetilde{X}_t, \widetilde{Y}_t)$ and gain, at least locally around the fixed point, a sufficient approximation of (X_t, Y_t), the solution to our model.

It can be shown that for $|a| > 1$ but sufficiently close to 1, periodic solutions for the law of $(\widetilde{X}_t, \widetilde{Y}_t)$ emerge for moderate values of σ, i.e. within some interval $0 < \sigma_0 < \sigma < \sigma_1$: we therefore obtain noise-induced oscillations. On can find in [11] a considerable analysis of (2.22) and its parameters.

This is an example of a model that undergo a *phase transition*: in the macroscopic dynamics, the stationary solution that is unique for small interaction, loses its stability as the interaction strength crosses a threshold, and is subject to bifurcation, which generates periodic solutions.

Chapter 3

Noise-induced periodicity: illustration of the results

This chapter illustrates all the results, previously mentioned, and presents the study as we demonstrate the occurring of the phenomenon of noise-induced periodicity in the model.

3.1. Preliminary study of the model

We prove, firstly, the existence and uniqueness of a strong solution to (1.1):

$$\left\{\mathbf{x}^{(N)}(t); 0 \le t \le T\right\} = \left\{\left(x_1^{(N)}(t), \dots, x_{N_1}^{(N)}(t), y_1^{(N)}(t), \dots, y_{N_2}^{(N)}(t)\right); 0 \le t \le T\right\}$$

using the *Khasminskii criterion* [8].

Consider the auxiliary norm-like function $V : \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \longrightarrow \mathbb{R}$

$$V\left(\mathbf{x}^{(N)}\right) = \frac{1}{N_1}\sum_{i=1}^{N_1}\left[\frac{\left(x_i^{(N)}\right)^4}{4} + \frac{\left(x_i^{(N)}\right)^2}{2}\right] + \frac{1}{N_2}\sum_{i=1}^{N_2}\left[\frac{\left(y_i^{(N)}\right)^4}{4} + \frac{\left(y_i^{(N)}\right)^2}{2}\right]$$

and the infinitesimal generator $\mathcal{L}$ of (1.1).

Remark. We recall that for a general Markov process defined by the stochastic differential equation $d\mathbf{X}_t = b(\mathbf{X}_t)dt + \sigma d\mathbf{W}_t$ (the prototype equation of our model), the infinitesimal generator acts like:

$$\mathcal{L}F(\mathbf{x}) = \sum_{i=1} b_i(\mathbf{x})\frac{\partial}{\partial x_i}F(\mathbf{x}) + \sigma^2 \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}F(\mathbf{x}),$$

for every, at least $\mathcal{C}^2$, time independent real function F.

It is easy to prove that there exists some real constant $k > 0$ such that the inequality

$$\mathcal{L}V\left(\mathbf{x}^{(N)}\right) \leq k\left(1 + V\left(\mathbf{x}^{(N)}\right)\right)$$

holds true in every bounded domain of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. This fact ensures the existence and uniqueness of a strong solution to (1.1) (Khasminskii criterion).
In particular, for every initial condition $\mathbf{x}^{(N)}(t_0) = \mathbf{z}$, independent of the processes $\left(W_t^i - W_0^i\right)_{i=1,\dots,N}$, there exist a solution $\mathbf{x}_{\mathbf{z}}^{(N)}(t)$ to (1.1) which is an almost surely continuous stochastic process in some probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

3.1.1 Numerical simulations of the finite-size system

In this section we present the results of numerical simulations of the finite-size system (1.1) with the aim of giving evidence of the phenomenon of noise-induced periodicity. In the view of the model, all the simulations were ran with different choices of σ and several values of the interaction strengths. We made use of the classical Euler method to perform simulations of 10^6 iterations with time-step $dt = 0.005$ for a system of 1000 particles equally divided between the two populations, i.e. we set $\alpha = \frac{1}{2}$. We computed these with the help of the software *Mathematica*: the appendix A contains the code.

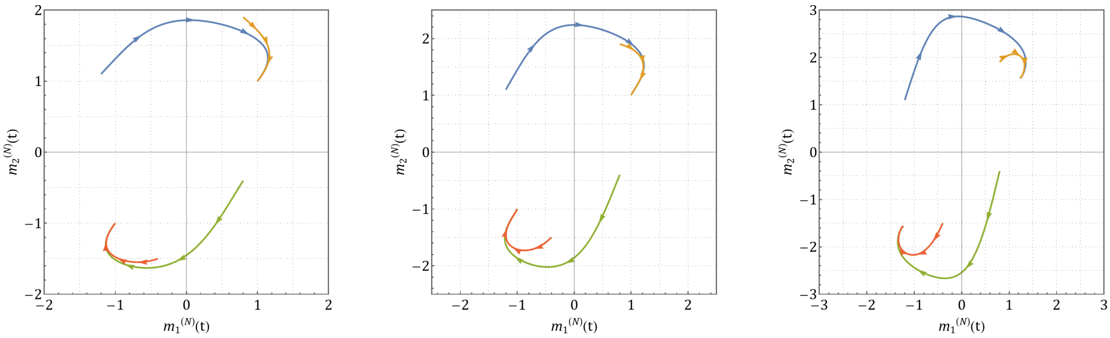
All the particles in the same population were given identical initial conditions. For the analysis we fixed $\theta_{11} = \theta_{22} = 8$ and let $A := (1-\alpha)\theta_{12} > 0$ and $B := -\alpha\theta_{21} > 0$ vary. The parameter's space of the simulations in this work can be summarized in the following scheme:

1. Noiseless dynamics ($\sigma = 0$):
   (a) $A = 2, B = 2.5$; $(A - 1 < B < A + 2)$
   (b) $A = 2, B = 4$; $(B = A + 2)$
   (c) $A = 2, B = 7$; $(B > A + 2)$
2. Intermediate noise ($\sigma > 0$)
   (a) $\sigma = 0.5$; $A = 2, B = 2.5$; $(A - 1 < B < A + 2)$
   (b) $\sigma = 0.1$; $A = 2, B = 4$; $(B = A + 2)$
   (c) $\sigma = 0.6$; $A = 2, B = 7$; $(B > A + 2)$
3. Large noise value ($\sigma \gg 1$)
   (a) $\sigma = 5$; $A = 2, B = 2.5$; $(A - 1 < B < A + 2)$
   (b) $\sigma = 5$; $A = 2, B = 4$; $(B = A + 2)$
   (c) $\sigma = 5$; $A = 2, B = 7$; $(B > A + 2)$

Remark. This particular structure was motivated by the investigation conducted on the deterministic dynamical system that drives (1.2). The equilibrium point analysis of the related vector field is found in Section 3.4.

Ours examinations establish the following.

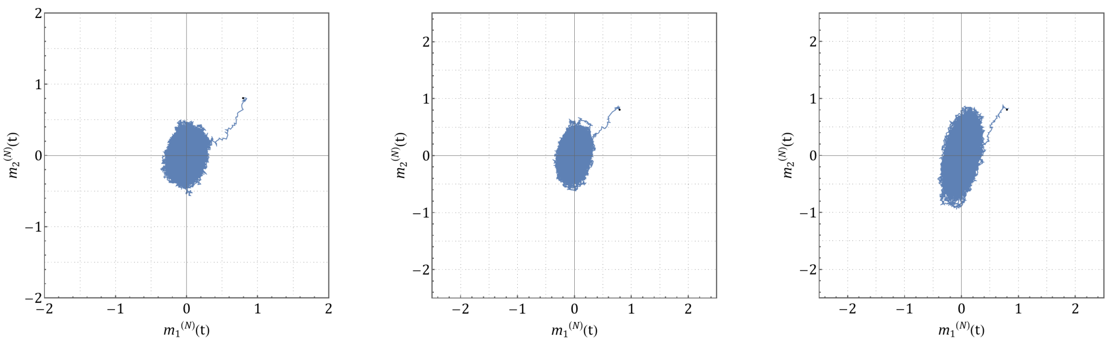
1. Deterministic behaviour If $\sigma = 0$ the system is attracted to a fixed point (Fig. 3.1). During a time interval of $T = 0.005 \cdot 10^6$, for different initial conditions, no periodic behaviors arise in any of the three considered cases and the system ends up in one of the stable equilibria. In section 3.4 we will explore better the nature of the dynamics in this case. Numerical evidences support the idea that this behavior persists also in the parameter range $A - 1 < B < A + 2$ and $B > A + 2$ for small values of σ.



(a) (b) (c)

Figure 3.1: Each graph represent the trajectory $t \mapsto \left(m_1^{(N)}(t), m_2^{(N)}(t)\right)$ obtained with numerical simulations of system (1.1) starting from the same 4 random initial points. In (a) $A = 2, B = 2.5$, (b) $A = 2, B = 4$, (c) $A = 2, B = 7$.

2. Large noise values If σ is increased to larger values ($\gg 1$), the dynamics are completely altered, the zero-mean Brownian disturbance dominates and the trajectories exhibit random excursions around the origin. The system essentially becomes a Brownian motion: see Fig. 3.2



(a) (b) (c)

Figure 3.2: In these simulations $\sigma = 5$ and the coupling constants are respectively: (a) $A = 2, B = 2.5$; (b) $A = 2, B = 4$; (c) $A = 2, B = 7$. It is clear from the graphics that the stochastic component dominates the dynamics.

3. intermediate noise range We find that if the intensity of the noise is tuned to an intermediate range of values, that are different for each case, we observe robust oscilla-

Noise	Coupling constants	Poincaré return time	P.r.t projection
$\sigma = 0.5$	$A = 2, B = 2.5$	19.30 ± 0.16	19.23 ± 0.45
$\sigma = 0.1$	$A = 2, B = 4$	29.27 ± 0.26	29.24 ± 0.61
$\sigma = 0.6$	$A = 2, B = 7$	6.45 ± 0.01	6.46 ± 0.03

Table 3.3: First and second column display the different choices for the constants. Third column: period of $t \longmapsto \left(m_1^{(N)}(t), m_2^{(N)}(t)\right)$ obtained by computing the average passing time from positive to negative values of $m_2^{(N)}$ for each simulations. In the fourth column the period of every simulation is recovered from the power spectrum of the Fourier transform, considering a sampling period of $dt = 0.005$.

tory behaviors on the $\left\{m_1^{(N)}, m_2^{(N)}\right\}$ plane. This suggests the presence of a time periodic law. Therefore this model seems to exhibit the phenomenon of the noise-induced periodicity.

A simple explanation could be the following. An intermediate amount of noise may create/stabilize some attractors and destabilize others. In this setting, seems that the noise destabilizes (some of the) fixed points and generates a stable rhythmic behavior on the empirical averages of the particle positions of the two communities.

The oscillatory behavior emerging from the simulations of system (1.1) is analyzed in Fig. 3.4 and Table 3.3. In particular, in the analysis we computed the average return time of the system to the Poincaré section $\left\{m_2^{(N)} = 0, m_1^{(N)} > 0\right\}$ and its standard deviation, in the various cases. These are reported in the third column of Table (3.3). On top of this we computed the discrete Fourier transform (DFT), averaged over $n = 50$ simulations, of the mean particle position $m_2^{(N)}$. Using the peak frequency of the Fourier transform, we computed a projection of the period of the trajectory $t \mapsto m_2^{(N)}(t)$. The average of the periods and their standard deviations are reported in the fourth column of Table 3.3, for the different values of the constants, as the Poincarè return time (P.r.t.) projection. In this setting, in every simulation we assigned different initial points, identical for particles of the same population.

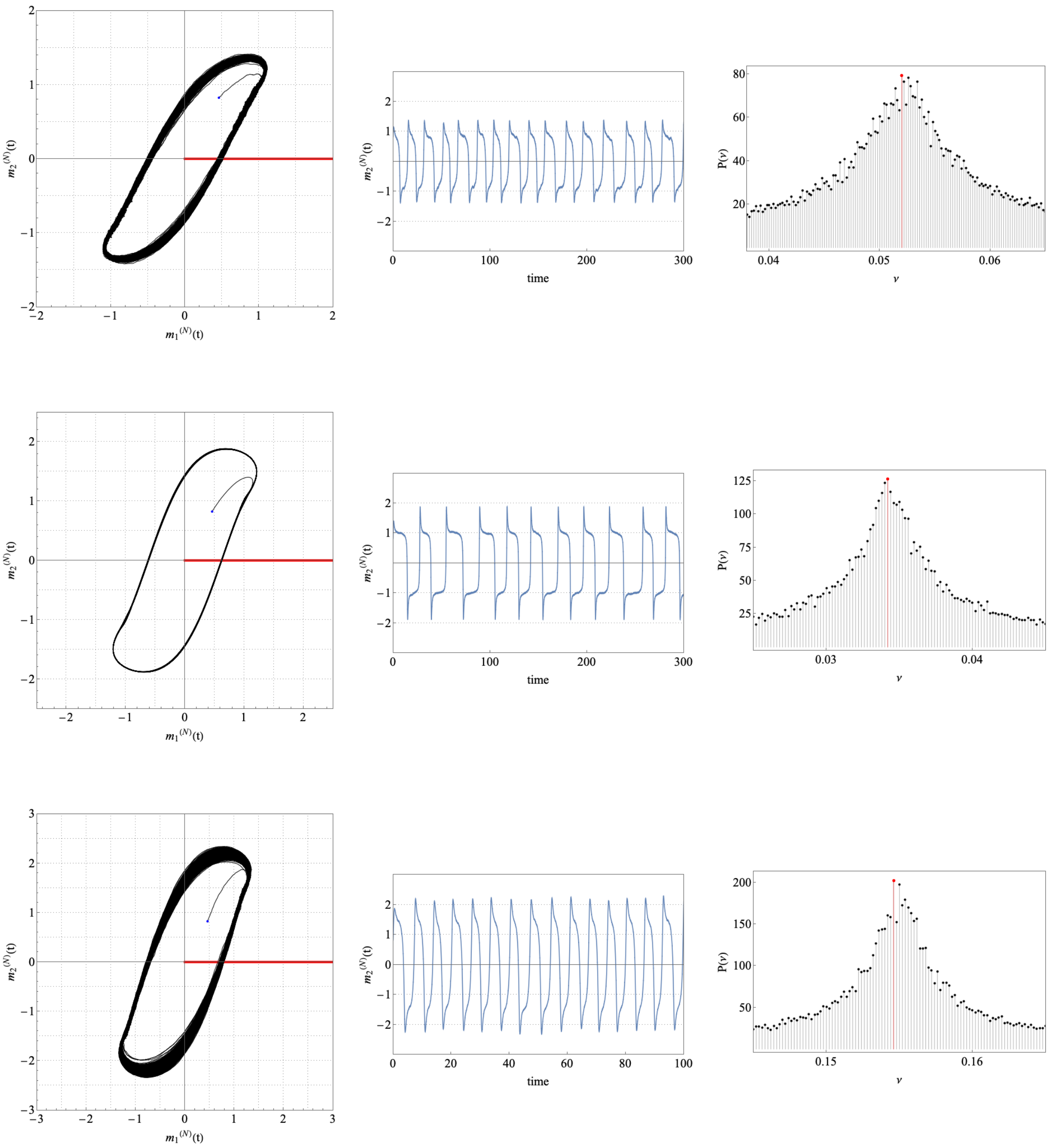


Figure 3.4: Analysis of the trajectories $t \longmapsto \left(m_1^{(N)}(t), m_2^{(N)}(t)\right)$ and $t \longmapsto \left(m_2^{(N)}(t)\right)$ from the numerical simulations of system (1.1), when the noise is tuned to an intermediate range of values. From top to bottom: (a) $A = 2, B = 2.5, \sigma = 0.5$ $(A - 1 < B < A + 2)$; (b) $A = 2, B = 4, \sigma = 0.1$ $(B = A + 2)$; (c) $A = 2, B = 7, \sigma = 0.6$ $(A + 2 < B)$. In the first column: trajectory $\left(m_1^{(N)}(t), m_2^{(N)}(t)\right)$ of sample simulation, with the Poincaré section. The second column display the time evolution of $m_2^{(N)}(t)$. In the third column: we plotted the modulus of the discrete Fourier transform (averaged over the simulations), i.e. the power spectrum $P(\nu)$ against the frequencies ν, in the relevant spectral region. To produce these figures we used the Fourier built-in function of *Mathematica* applied to the trajectory $t \longmapsto m_2^{(N)}(t)$ over 10^6 steps and averaged the obtained spectrum over the 50 simulations. In the three cases the projection of the period was obtained as the reciprocal of the frequency highlighted by the red peak lines in the graphs.

3.2. Well-posedness of the macroscopic limit

As a first step to the study of (1.2), we prove that the system (1.2) is well posed, i.e. for every suitable initial condition (X^*, Y^*) the system (1.2) exhibits a *unique strong solution*.

Theorem A (Existence and uniqueness of a strong solution of system (1.2)). *Let $T > 0$ and any initial condition $(X_0, Y_0) = (X^*, Y^*)$, having finite first moment and being independent of the Brownian motions $\{W^i(t); 0 \le t \le T\}_{i=1,2}$. Then the system (1.2)*

$$dX_t = \Big[-X_t^3 + X_t - \alpha\theta_{11}(X_t - \mathbb{E}[X_t]) - (1-\alpha)\theta_{12}(X_t - \mathbb{E}[Y_t])\Big]dt + \sigma dW_t^1,$$

$$dY_t = \Big[-Y_t^3 + Y_t - \alpha\theta_{21}(Y_t - \mathbb{E}[X_t]) - (1-\alpha)\theta_{22}(Y_t - \mathbb{E}[Y_t])\Big]dt + \sigma dW_t^2;$$

has a unique strong solution.

Proof. The argument follows a Picard iteration. A similar approach is used, for example, in [5].

Step 1: Picard iteration.
Consider the two sequence of stochastic processes $\{X_n(t); 0 \le t \le T\}$ and $\{Y_n(t); 0 \le t \le T\}$, indexed by $n \ge 1$, defined by the following integral equations and iteration scheme: for $n \ge 1$

$$\begin{aligned} X_n(t) - X_n(0) &= \int_0^t -X_n(s)^3 + X_n(s) - \alpha\theta_{11}X_n(s) - (1-\alpha)\theta_{12}X_n(s)\,ds \\ &\quad + \int_0^t \alpha\theta_{11}\,\mathbb{E}[X_{n-1}(s)] + (1-\alpha)\theta_{12}\,\mathbb{E}[Y_{n-1}(s)]\,ds + \sigma W^1(t) \\ &= \int_0^t -X_n(s)^3 + X_n(s)(1 - \alpha\theta_{11} - (1-\alpha)\theta_{12})\,ds \\ &\quad + \int_0^t \alpha\theta_{11}\,\mathbb{E}[X_{n-1}(s)] + (1-\alpha)\theta_{12}\,\mathbb{E}[Y_{n-1}(s)]\,ds + \sigma W^1(t) \end{aligned}$$

$$\begin{aligned} Y_n(t) - Y_n(0) = \ldots &= \int_0^t -Y_n(s)^3 + Y_n(s)(1 - \alpha\theta_{21} - (1-\alpha)\theta_{22})\,ds \\ &\quad + \int_0^t \alpha\theta_{21}\,\mathbb{E}[X_{n-1}(s)] + (1-\alpha)\theta_{22}\,\mathbb{E}[Y_{n-1}(s)]\,ds + \sigma W^2(t) \end{aligned}$$

$(0 \le t \le T)$; all with the same initial conditions $\big(X_n(0), Y_n(0)\big) = (X^*, Y^*)$, together with $X_0(t) = X^*$ and $Y_0(t) = Y^*$. Now, by subtracting two subsequent approximations we get:

$$\begin{aligned} X_{n+1}(t) - X_n(t) &= \int_0^t -[X_{n+1}(s)^3 - X_n(s)^3] + [X_{n+1}(s) - X_n(s)](1 - \alpha\theta_{11} - (1-\alpha)\theta_{12})\,ds \\ &\quad + \int_0^t \alpha\theta_{11}\,\mathbb{E}[X_n(s) - X_{n-1}(s)] + (1-\alpha)\theta_{12}\,\mathbb{E}[Y_n(s) - Y_{n-1}(s)]\,ds \end{aligned}$$

$$Y_{n+1}(t) - Y_n(t) = \int_0^t -[Y_{n+1}(s)^3 - Y_n(s)^3] + [Y_{n+1}(s) - Y_n(s)](1 - \alpha\theta_{21} - (1-\alpha)\theta_{22})\,ds$$
$$+ \int_0^t \alpha\theta_{21}\,\mathbb{E}[X_n(s) - X_{n-1}(s)] + (1-\alpha)\theta_{22}\,\mathbb{E}[Y_n(s) - Y_{n-1}(s)]\,ds$$

Note that the Brownian motions cancel out. Now, using the identity $(a^3 - b^3) = (a-b)(a^2 + ab + b^2)$, we can re-write the system as:

$$\begin{aligned} X_{n+1}(t) - X_n(t) = &\int_0^t [X_{n+1}(s) - X_n(s)](1 - f_n(s) - \alpha\theta_{11} - (1-\alpha)\theta_{12})\,ds \\ &+ \int_0^t \alpha\theta_{11}\,\mathbb{E}[X_n(s) - X_{n-1}(s)] + (1-\alpha)\theta_{12}\,\mathbb{E}[Y_n(s) - Y_{n-1}(s)]\,ds, \end{aligned} \tag{3.1}$$

$$\begin{aligned} Y_{n+1}(t) - Y_n(t) = &\int_0^t [Y_{n+1}(s) - Y_n(s)](1 - g_n(s) - \alpha\theta_{21} - (1-\alpha)\theta_{22})\,ds \\ &+ \int_0^t \alpha\theta_{21}\,\mathbb{E}[X_n(s) - X_{n-1}(s)] + (1-\alpha)\theta_{22}\,\mathbb{E}[Y_n(s) - Y_{n-1}(s)]\,ds; \end{aligned} \tag{3.2}$$

where, we put:

$$\begin{aligned} f_n(t) :=& X_{n+1}(t)^2 + X_{n+1}(t)X_n(t) + X_n(t)^2, \\ g_n(t) :=& Y_{n+1}(t)^2 + Y_{n+1}(t)Y_n(t) + Y_n(t)^2; \end{aligned}$$

note that $f_n(t), g_n(t) \geq 0$ for all $t \in [0, T]$.
Equations (3.1) and (3.2) have the form:

$$\varphi(t) = \int_0^t \varphi(s)H(s)\,ds + \int_0^t Q(s)\,ds; \tag{3.3}$$

where $\varphi(t)$ is given by $X_{n+1}(t) - X_n(t)$ and $Y_{n+1}(t) - Y_n(t)$ respectively.
Solutions to this kind of equations can be written as

$$\varphi(t) = \varphi(0) + \int_0^t Q(s)e^{\int_s^t H(r)\,dr}\,ds.$$

Therefore, applying this to Eq. (3.1) and (3.2) we have that for $t \in [0, T]$ and $n \geq 1$:

$$\begin{aligned} X_{n+1}(t) - X_n(t) = &\int_0^t \Big\{\alpha\theta_{11}\,\mathbb{E}[X_n(s) - X_{n-1}(s)] + (1-\alpha)\theta_{12}\,\mathbb{E}[Y_n(s) - Y_{n-1}(s)]\Big\} \\ &\cdot e^{\int_s^t 1 - f_n(r) - \alpha\theta_{11} - (1-\alpha)\theta_{12}\,dr}\,ds, \end{aligned} \tag{3.4}$$

$$\phi(0) = X_{n+1}(0) - X_n(0) = X^* - X^* = 0;$$

and

$$Y_{n+1}(t) - Y_n(t) = \int_0^t \Big\{\alpha\theta_{21}\,\mathbb{E}[X_n(s) - X_{n-1}(s)] + (1-\alpha)\theta_{22}\,\mathbb{E}[Y_n(s) - Y_{n-1}(s)]\Big\} \cdot e^{\int_s^t 1 - g_n(r) - \alpha\theta_{21} - (1-\alpha)\theta_{22}\,dr}\,ds, \tag{3.5}$$

$$\phi(0) = Y_{n+1}(0) - Y_n(0) = Y^* - Y^* = 0.$$

Above we used respectively $1-f_n(r)-\alpha\theta_{11}-(1-\alpha)\theta_{12}$ and $1-g_n(r)-\alpha\theta_{21}-(1-\alpha)\theta_{22}$ as $H(r)$, and it is easy to notice that $H(r) \le 1$ in both cases. In the next lines we use these facts to bound ours solutions.

Step 2: Convergence's property.
We want to show now that for every $T > 0$, $\{\mathbb{E}[X_n(t)]; 0 \le t \le T\}_{n\ge1}$ and $\{\mathbb{E}[Y_n(t)]; 0 \le t \le T\}_{n\ge1}$ are Cauchy sequences in $\mathcal{C}([0,T])$, equipped with the sup-norm:

$$d(f,g) := \sup_{t\in[0,T]} \big|f(t) - g(t)\big| \qquad \forall f,g \in \mathcal{C}([0,T])$$

Hence, since $(\mathcal{C}([0,T]), d)$ is a complete metric space, we expect that the two sequences converge to some elements $\{M_X(t); 0 \le t \le T\}$, $\{M_Y(t); 0 \le t \le T\} \in \mathcal{C}([0,T])$.

Consider Eq.(3.4) and take the Expectation of the absolute value on both side. Define $\phi_n(t) := \sup_{s\in[0,t]} \mathbb{E}\Big[\big|X_{n+1}(s) - X_n(s)\big|\Big]$ and $\psi_n(t) := \sup_{s\in[0,t]} \mathbb{E}\Big[\big|Y_{n+1}(s) - Y_n(s)\big|\Big]$. Then, there exists some positive constants $\widetilde{C}_t$ and $\widetilde{D}_t$ such that:

$$\phi_n(t) \le \widetilde{C}_t \int_0^t \phi_{n-1}(s)\,ds + \widetilde{D}_t \int_0^t \psi_{n-1}(s)\,ds; \tag{3.6}$$

From Eq. (3.5) we get an identical inequality for $\psi_n(t)$. Hereby, applying iteratively the above inequalities in $[0,T]$, we prove that:

$$\phi_n(T) \le C_T \phi_1(T) \frac{T^{n-1}}{(n-1)!} + D_T \psi_1(T) \frac{T^{n-1}}{(n-1)!}$$

for some real constants C_T and D_T; and the same for $\psi_n(T)$.
This proves that, for every $T > 0$:

$$\lim_{n\to\infty} \phi_n(T) = 0 \quad \text{and} \quad \lim_{n\to\infty} \psi_n(T) = 0$$

It follows that $\{\mathbb{E}[X_n(t)]; 0 \le t \le T\}_{n\ge1}$ and $\{\mathbb{E}[Y_n(t)]; 0 \le t \le T\}_{n\ge1}$ are two Cauchy sequences, so they converge to some continuous limits $\{M_X(t); 0 \le t \le T\}$ and $\{M_Y(t); 0 \le t \le T\}$ in $\mathcal{C}([0,T])$.

Step 3: Existence.
Consider now the following stochastic differential equations

$$
\begin{aligned}
dX_t &= \Big[-X_t^3 + X_t - \alpha\theta_{11}(X_t - M_X(t)) - (1-\alpha)\theta_{12}(X_t - M_Y(t))\Big]dt + \sigma dW_t^1 \\
dY_t &= \Big[-Y_t^3 + Y_t - \alpha\theta_{21}(Y_t - M_X(t)) - (1-\alpha)\theta_{22}(Y_t - M_Y(t))\Big]dt + \sigma dW_t^2
\end{aligned}
\tag{3.7}
$$

with initial conditions $(X_0, Y_0) = (X^*, Y^*)$. Now, since the terms $M_X(t)$ and $M_Y(t)$ are bounded for every $t \in [0,T]$, existence and uniqueness of a strong solution to system (3.7), can be proven with the *Khasminskii*'s test, see [8].
So, let $\Big\{\Big(\widetilde{X}_t, \widetilde{Y}_t\Big); 0 \le t \le T\Big\}$ be the solution of (3.7). We can repeat the scheme of (3.1) and (3.2) and construct the two approximations:

$$
\begin{aligned}
X_{n+1}(t) - \widetilde{X}_t = &\int_0^t [X_{n+1}(s) - \widetilde{X}(s)](1 - \widetilde{f}_n(s) - \alpha\theta_{11} - (1-\alpha)\theta_{12})\, ds \\
&+ \int_0^t \alpha\theta_{11}\,\mathbb{E}[X_n(s) - M_X(s)] + (1-\alpha)\theta_{12}\,\mathbb{E}[Y_n(s) - M_Y(s)]\, ds
\end{aligned}
$$

$$
\begin{aligned}
X_{n+1}(t) - \widetilde{Y}_t = &\int_0^t [Y_{n+1}(s) - \widetilde{Y}(s)](1 - \widetilde{g}_n(s) - \alpha\theta_{21} - (1-\alpha)\theta_{22})\, ds \\
&+ \int_0^t \alpha\theta_{21}\,\mathbb{E}[X_n(s) - M_X(s)] + (1-\alpha)\theta_{22}\,\mathbb{E}[Y_n(s) - M_Y(s)]\, ds
\end{aligned}
$$

where $\widetilde{f}_n(s)$ comes from replacing the terms $X_n(s)$ with $\widetilde{X}_s$ and analogously $\widetilde{g}_n(s)$ with $\widetilde{Y}_s$. Now, applying the same arguments as before, we have:

$$
\begin{aligned}
\sup_{s\in[0,t]}\Big|X_{n+1}(t) - \widetilde{X}_t\Big| \le & k_1 \int_0^t \sup_{r\in[0,s]}\Big|\mathbb{E}[X_n(r) - M_X(r)]\Big|\, ds \\
&+ k_2 \int_0^t \sup_{r\in[0,s]}\Big|\mathbb{E}[Y_n(r) - M_Y(r)]\Big|\, ds \quad \text{for every } t \in [0,T];
\end{aligned}
$$

for some positive constants k_1, k_2.
Hereby, we obtain that:

$$
\Big\{\mathbb{E}[X_n(t)]; 0 \le t \le T\Big\} \xrightarrow{n\to\infty} \Big\{\mathbb{E}[\widetilde{X}_t]; 0 \le t \le T\Big\} \quad \text{in } \mathcal{C}([0,T]).
$$

The same result can be shown for $\widetilde{Y}_t$.
This proves that $\mathbb{E}[\widetilde{X}_t] = M_X(t)$, $\mathbb{E}[\widetilde{Y}_t] = M_Y(t)$ and, therefore, (3.7) provides a solution for the system (1.2). It remains only to prove uniqueness.

Step 4: Uniqueness.
Let $\Big\{\Big(U(t), V(t)\Big); 0 \le t \le T\Big\}$ be another solution of (1.2). Consider the integral expressions for $X(t) - U(t)$ and $Y(t) - V(t)$ as in (3.4). We can use them to estimate

the quantities:

$$\Phi(t) = \left|\mathbb{E}[X(t) - U(t)]\right|,$$
$$\Psi(t) = \left|\mathbb{E}[Y(t) - V(t)]\right|.$$

Applying an identical argument, we easily obtain the following inequalities:

$$\Phi(t) \leq \widetilde{K}_T \int_0^t \Phi(s) + \Psi(s)\, ds \qquad \text{and} \qquad \Psi(t) \leq \widetilde{H}_T \int_0^t \Phi(s) + \Psi(s)\, ds,$$

for some real constants $\widetilde{K}_T, \widetilde{H}_T$. Summing up them we get that

$$\Phi(t) + \Psi(t) \leq (\widetilde{K}_T + \widetilde{H}_T) \int_0^t \Phi(s) + \Psi(s)\, ds.$$

Note that, in this particular expression, the *Gronwall's Lemma* can be applied, and it can be used to prove that $\Phi(t) + \Psi(t) \leq 0$ for every $t \in [0, T]$.
This means that $\mathbb{E}[X(t)] = \mathbb{E}[U(t)]$ and $\mathbb{E}[Y(t)] = \mathbb{E}[V(t)]$ for every $t \in [0, T]$, and so $\{(U_t, V_t); 0 \leq t \leq T\}$ and $\{(X_t, Y_t); 0 \leq t \leq T\}$ are both solutions to (3.7) with same pair $\left(M_X(t), M_Y(t)\right)$ of moments and initial conditions (X^*, Y^*). By the uniqueness, it follows that:

$$\mathbb{P}\left\{\left(X(t), Y(t)\right) = \left(U(t), V(t)\right), \text{ for all } t \in [0, T]\right\} = 1.$$

This concludes the proof.

□

3.3. Propagation of chaos in the model

Follows, in this section, a propagation of chaos statement, in which we prove that the macroscopic description of system (1.1) is exactly the system (1.2). We claim that, as $N \to \infty$, the evolution of each particle remains independent of the evolution of any finite subset of others. Indeed, one key feature of our model, is that individual particles interact only via the empirical means of the two populations. Consequently, when taking the infinite volume limit, the influence of a finite number of particles becomes negligible. In our case, the macroscopic evolution of a pair of representative particles, one for each population, is the process $\{(X(t), Y(t)); t \leq T\}$ described by the system (1.2).

Theorem B (Propagation of chaos). *Fix $T > 0$. Let*

$$\left\{\left(x_1^{(N)}(t), \ldots, x_{N_1}^{(N)}(t), y_1^{(N)}(t), \ldots, y_{N_2}^{(N)}(t)\right); 0 \leq t \leq T\right\}$$

be the solution to (1.1), with the initial condition satisfying the following properties:

i) the vector $\left(x_1^{(N)}(0), \ldots, x_{N_1}^{(N)}(0), y_1^{(N)}(0), \ldots, y_{N_2}^{(N)}(0)\right)$ is a family of independent random variables, such that each component is also independent of the Brownian motions $\left(W_t^j\right)_{j=1,\ldots,N}$.

ii) the random variables $\left(x_1^{(N)}(0),\dots,x_{N_1}^{(N)}(0)\right)$ *are identically distributed with the law* $\mathcal{Q}_X$*, respectively* $\left(y_1^{(N)}(0),\dots,y_{N_2}^{(N)}(0)\right)$ *with* $\mathcal{Q}_Y$*; and such that* $\mathcal{Q}_X$ *and* $\mathcal{Q}_Y$ *have finite second moment.*

Let

$$\left\{\left(x_1(t),\dots,x_{N_1}(t),y_1(t),\dots,y_{N_2}(t)\right);0\le t\le T\right\}$$

be the random processes whose components $\{x_i(t);0\le t\le T\}_{i=1,\dots,N_1}$ *are copies of the solution to the first equation of (1.2) and* $\{y_j(t);0\le t\le T\}_{j=1,\dots,N_2}$ *copies of the solution of the second equation of (1.2); with both sharing the same initial conditions (component-wise equal) and Brownian motions that define system (1.1). Define the index sets* $\mathcal{I}=\{i_1,\dots,i_{k_1}\}\subseteq\{1,\dots,N_1\}$ *and* $\mathcal{J}=\{j_1,\dots,j_{k_2}\}\subseteq\{1,\dots,N_2\}$*, so that* $|\mathcal{I}|=k_1$ *and* $|\mathcal{J}|=k_2$*.*
Then, we have

$$\lim_{N\to\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\left|\mathbf{x}_{k_1,k_2}^{(N)}(t)-\mathbf{x}_{k_1,k_2}(t)\right|\right]=0; \tag{3.8}$$

here $|\cdot|$ *denotes the* ℓ_1*-norm and*

$$\mathbf{x}_{k_1,k_2}^{(N)}(t)=\left(x_{i_1}^{(N)}(t),\dots,x_{i_{k_1}}^{(N)}(t),y_{j_1}^{(N)}(t),\dots,y_{j_{k_2}}^{(N)}(t)\right),$$

$$\mathbf{x}_{k_1,k_2}(t)=\left(x_1(t),\dots,x_{k_1}(t),y_1(t),\dots,y_{k_2}(t)\right).$$

Proof. The proof relies on a coupling method. Similar approaches were used, for example, in [3]. The goal here is to prove (3.8).

To begin with, without loss of generality, we can take $\mathcal{I}=\{1,\dots,k_1\}$ and $\mathcal{J}=\{1,\dots,k_2\}$. Next, we can note immediately that:

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\mathbf{x}_{k_1,k_2}^{(N)}(t)-\mathbf{x}_{k_1,k_2}(t)\right|\right]\le \sum_{i=1}^{k_1}\mathbb{E}\left[\sup_{t\in[0,T]}\left|x_i^{(N)}(t)-x_i(t)\right|\right]+\sum_{j=1}^{k_2}\mathbb{E}\left[\sup_{t\in[0,T]}\left|y_j^{(N)}(t)-y_j(t)\right|\right];$$

so, to conclude it suffices to show that each of the k_1+k_2 terms goes to zero when $N\to\infty$.
We will show this holds for $i=1$, i.e. $\mathbb{E}\left[\sup_{t\in[0,T]}\left|x_1^{(N)}(t)-x_1(t)\right|\right]$, since identical arguments can be carried out for the others terms.
As we did in the proof of existence e uniqueness of system (1.2), we consider the integral equations for $\left\{x_1^{(N)}(t);\,0\le t\le T\right\}$ and $\left\{x_1(t);\,0\le t\le T\right\}$.

Therefore we can write:

$$x_1^{(N)}(t) - x_1(t) = \int_0^t [x_1^{(N)}(s) - x_1(s)](1 - f(s) - \alpha\theta_{11} - (1-\alpha)\theta_{12})\,ds$$
$$+ \int_0^t \alpha\theta_{11}\left(m_1^{(N)}(s) - \mathbb{E}[X_1(s)]\right) + (1-\alpha)\theta_{12}\left(m_2^{(N)}(s) - \mathbb{E}[y_1(s)]\right)ds;$$

here, as we did before, we used the identity $a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$ and put $f(s) := \left(x_1^{(N)}(s)\right)^2 + x_1(s)^2 + x_1^{(N)}(s)x_1(s)$. Moreover we can set:

$$\mu(s) := \alpha\theta_{11}\left(m_1^{(N)}(s) - \mathbb{E}[X_1(s)]\right) + (1-\alpha)\theta_{12}\left(m_2^{(N)}(s) - \mathbb{E}[y_1(s)]\right),$$

so

$$x_1^{(N)}(t) - x_1(t) = \int_0^t \left[\left(x_1^{(N)}(s) - x_1(s)\right)\left(1 - f(s) - \alpha\theta_{11} - (1-\alpha)\theta_{12}\right) + \mu(s)\right] ds \quad (3.9)$$

We can see that equation (3.9) is of the type $\varphi(t) = \int_0^t \varphi(s)H(s) + Q(s)\,ds$, where $\varphi(t) = x_1^{(N)}(t) - x_1(t)$. We already saw previously, that a solution to this kind of equation can be written as $\varphi(t) = \varphi(0) + \int_0^t Q(s)e^{\int_s^t H(r)\,dr}\,ds$; this fact will be useful to estimate $\left|x_1^{(N)}(t) - x_1(t)\right|$ with $\left|\mu(s)\right|$.
Since by assumption $x_1^{(N)}(0) = x_1(0)$, we can have that, for every $t \in [0,T]$:

$$\left|x_1^{(N)}(t) - x_1(t)\right| \le \int_0^t \left|\mu(s)\right| e^{\int_s^t 1 - f_n(r) - \alpha\theta_{11} - (1-\alpha)\theta_{12}\,dr}\,ds \le C_T \int_0^t \left|\mu(s)\right| ds \quad (3.10)$$

for some real positive constantC_T (we used the above fact). Moreover we have that:

$$\begin{aligned}\mathbb{E}\left[\left|\mu(s)\right|\right] &\le \mathbb{E}\left[\alpha\theta_{11}\left|m_1^{(N)}(s) - \mathbb{E}[X_1(s)]\right| + (1-\alpha)\theta_{12}\left|m_2^{(N)}(s) - \mathbb{E}[y_1(s)]\right|\right] \\ &\le \mathbb{E}\left[\alpha\theta_{11}\left|m_1^{(N)}(s) + \frac{1}{N_1}\sum_{i=1}^{N_1} x_i(s) - \frac{1}{N_1}\sum_{i=1}^{N_1} x_i(s) - \mathbb{E}[X_1(s)]\right|\right. \\ &\quad \left. + (1-\alpha)\theta_{12}\left|m_2^{(N)}(s) + \frac{1}{N_2}\sum_{j=1}^{N_2} y_i(s) - \frac{1}{N_2}\sum_{j=1}^{N_2} y_i(s) - \mathbb{E}[y_1(s)]\right|\right].\end{aligned} \quad (3.11)$$

We added and subtracted the quantities $\frac{1}{N_1}\sum_{i=1}^{N_1} x_i(s)$ and $\frac{1}{N_2}\sum_{j=1}^{N_2} y_i(s)$, respectively in the first and second absolute values of the first line.

Continuing we have:

$$\mathbb{E}\Big[\big|\mu(s)\big|\Big] \leq \frac{\alpha\theta_{11}}{N_1}\sum_{i=1}^{N_1}\mathbb{E}\left[\left|x_i^{(N)}(s)-x_i(s)\right|\right]+\alpha\theta_{11}\,\mathbb{E}\left[\left|\frac{1}{N_1}\sum_{i=1}^{N_1}x_i(s)-\mathbb{E}[x_1(s)]\right|\right]$$
$$+\frac{(1-\alpha)\theta_{12}}{N_2}\sum_{j=1}^{N_2}\mathbb{E}\left[\left|y_j^{(N)}(s)-x_j(s)\right|\right]+(1-\alpha)\theta_{12}\,\mathbb{E}\left[\left|\frac{1}{N_2}\sum_{j=1}^{N_2}y_j(s)-\mathbb{E}[y_1(s)]\right|\right] \tag{3.12}$$

Now we want to bound the four relevant terms:

$$\mathbb{E}\left[\left|x_i^{(N)}(s)-x_i(s)\right|\right],\quad \mathbb{E}\left[\left|y_j^{(N)(s)-x_j(s)}\right|\right];$$
$$\mathbb{E}\left[\left|\frac{1}{N_1}\sum_{i=1}^{N_1}x_i(s)-\mathbb{E}[x_1(s)]\right|\right],\quad \mathbb{E}\left[\left|\frac{1}{N_2}\sum_{j=1}^{N_2}y_j(s)-\mathbb{E}[y_1(s)]\right|\right].$$

The first two are easy, since:

$$\mathbb{E}\left[\left|x_i^{(N)}(s)-x_i(s)\right|\right] \leq \mathbb{E}\left[\sup_{r\in[0,s]}\left|x_i^{(N)}(r)-x_i(r)\right|\right]$$
$$\mathbb{E}\left[\left|y_j^{(N)}(s)-y_j(s)\right|\right] \leq \mathbb{E}\left[\sup_{r\in[0,s]}\left|y_j^{(N)}(r)-y_j(r)\right|\right]$$

Note, in addition, that these terms are independent of the indexes i, j, due to the symmetry of our system, which, on the other hand, depends on the initial conditions and the choice of the constants. So we can chose for example $i = 1$ and $j = 1$. Regarding the other two terms, we can employ the *central limit theorem*. The two limiting processes $\{x_i(t)\}_{i\in N_1}$ and $\{x_j(t)\}_{j\in N_2}$ are a family with independent identical distributions and have uniformly bounded second moments (since the system (1.2) is well posed). As a consequence the standard limit theorem assures that there exists a positive constant K_T, such that, uniformly for all $s \in [0,T]$, it holds:

$$\mathbb{E}\left[\left|\frac{1}{N_1}\sum_{i=1}^{N_1}x_i(s)-\mathbb{E}[x_1(s)]\right|\right] \leq \frac{K_T}{\sqrt{N_1}} \quad\text{and}\quad \mathbb{E}\left[\left|\frac{1}{N_2}\sum_{j=1}^{N_1}y_j(s)-\mathbb{E}[y_1(s)]\right|\right] \leq \frac{K_T}{\sqrt{N_2}}.$$

Now, if we take the supremum and the expectation on both sides of (3.9), for every $\tilde{t} \in [0,T]$ for which:

$$\mathbb{E}\left[\sup_{t\in[0,\tilde{t}]}\left|x_1^{(N)}(t)-x_1(t)\right|\right] \leq C_T\int_0^{\tilde{t}}\mathbb{E}\Big[\big|\mu(s)\big|\Big]ds$$

We now put everything together, recall that $\alpha = \frac{N_1}{N}$. We obtain:

$$\begin{aligned}\mathbb{E}\left[\left|\mu(s)\right|\right] \leq &\alpha\theta_{11}\,\mathbb{E}\left[\sup_{r\in[0,s]}\left|x_1^{(N)}(r)-x_1(r)\right|\right]+\frac{\sqrt{\alpha}\theta_{11}K_T}{\sqrt{N}}\\ &+(1-\alpha)\theta_{12}\,\mathbb{E}\left[\sup_{r\in[0,s]}\left|y_1^{(N)}(r)-y_1(r)\right|\right]+\frac{\sqrt{(1-\alpha)}\theta_{12}K_T}{\sqrt{N}};\end{aligned} \tag{3.13}$$

and so, surely there exists some real positive arbitrary constant $\mathcal{D}$, depending on T and on all the parameters ($\alpha,\theta_{11},\theta_{12},\theta_{21}$ and θ_{22}), such that:

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,\tilde{t}]}\left|x_1^{(N)}(t)-x_1(t)\right|\right] \leq &\mathcal{D}\int_0^{\tilde{t}}\mathbb{E}\left[\sup_{r\in[0,s]}\left|x_1^{(N)}(r)-x_1(r)\right|\right]ds\\ &+\mathcal{D}\int_0^{\tilde{t}}\mathbb{E}\left[\sup_{r\in[0,s]}\left|y_1^{(N)}(r)-y_1(r)\right|\right]ds\\ &+\frac{\mathcal{D}}{\sqrt{N}};\end{aligned} \tag{3.14}$$

In the same way, we can achieve:

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,\tilde{t}]}\left|y_1^{(N)}(t)-y_1(t)\right|\right] \leq &\mathcal{D}\int_0^{\tilde{t}}\mathbb{E}\left[\sup_{r\in[0,s]}\left|x_1^{(N)}(r)-x_1(r)\right|\right]ds\\ &+\mathcal{D}\int_0^{\tilde{t}}\mathbb{E}\left[\sup_{r\in[0,s]}\left|y_1^{(N)}(r)-y_1(r)\right|\right]ds\\ &+\frac{\mathcal{D}}{\sqrt{N}}\end{aligned} \tag{3.15}$$

Therefore, if we define the function

$$g(\tilde{t}) := \mathbb{E}\left[\sup_{t\in[0,\tilde{t}]}\left|x_1^{(N)}(t)-x_1(t)\right|\right]+\mathbb{E}\left[\sup_{t\in[0,\tilde{t}]}\left|y_1^{(N)}(t)-y_1(t)\right|\right].$$

Summing up the inequalities (3.14) and (3.15), we get:

$$g(\tilde{t}) \leq 2\mathcal{D}\int_0^{\tilde{t}} g(s)\,ds + 2\frac{\mathcal{D}}{\sqrt{N}}.$$

An easy application of the *Gronwall's lemma* leads, in particular, to:

$$g(T) \leq \frac{2\mathcal{D}e^{2\mathcal{D}T}}{\sqrt{N}}$$

that proves $\lim_{N\to\infty} g(T) = 0$. This concludes the proof.

□

3.4. Noiseless dynamic of the macroscopic limit

Before giving the main result of the thesis, we argue here that, in absence of a noise component, the system (1.2) does not exhibit oscillatory behaviours. Thus, it behave, essentially, in the same way as the finite-size system.

If $\sigma = 0$, the system (1.2)

$$dX_t = \Big[-X_t^3 + X_t - \alpha\theta_{11}(X_t - \mathbb{E}[X_t]) - (1-\alpha)\theta_{12}(X_t - \mathbb{E}[Y_t])\Big]dt + \sigma dW_t^1$$

$$dY_t = \Big[-Y_t^3 + Y_t - \alpha\theta_{21}(Y_t - \mathbb{E}[X_t]) - (1-\alpha)\theta_{22}(Y_t - \mathbb{E}[Y_t])\Big]dt + \sigma dW_t^2,$$

reduces to a system of differential equation, in which the variables X_t and Y_t are deterministic functions of time. Thus, in the equations, both terms $\alpha\theta_{11}(X_t - \mathbb{E}[X_t])$ and $(1-\alpha)\theta_{22}(Y_t - \mathbb{E}[Y_T])$ are equal to zero. Setting $A := (1-\alpha)\theta_{12}(> 0), B := -\alpha\theta_{21}(> 0)$ and $X_t = x, Y_t = y$ the system can be re-written as:

$$\begin{cases} \dot{x} = -x^3 + x - A(x-y) \\ \dot{y} = -y^3 + y - B(x-y) \end{cases} \tag{3.16}$$

where time dependence is implicit and $\dot{z} := \frac{\partial}{\partial t} z(t)$.

Note. We assume that $A > 1$ and $B > 0$, unless otherwise specified.

In the following we summarize the equilibria analysis of the vector field of (3.16). Many results about dynamical system were used, we refer to [14] for complete explanations.

1. The fixed points $(0,0)$ and $\pm(1,1)$ are present for any values of A and B, in particular:

   - The linearization of (3.16) around the origin has the eigenvalues:

     $$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 1 - A + B = 1 - \gamma,$$

     setting $\gamma := A - B$. Thus $(0,0)$ is a saddle if $\gamma > 1$, it has a unstable and a neutral direction for $\gamma = 1$ and it is a unstable node in the other cases.

   - The points $\pm(1,1)$ have identical properties since the eigenvalues for the linearized system around them are:

     $$\lambda_1 = -2 \quad \text{and} \quad \lambda_2 = -2 - A + B = -2 - \gamma$$

     Therefore $\pm(1,1)$ are stable nodes for $\gamma > -2$, they have a neutral and a stable direction when $\gamma = -2$ and they are saddle points otherwise.

   So, summarising we have: *i)* $\gamma < -2$, $(0,0)$ is unstable and $\pm(1,1)$ are saddle points; *ii)* when $-2 < \gamma < 1$, $(0,0)$ is unstable and $\pm(1,1)$ are stable nodes; *iii)* for $\gamma > 1$, $(0,0)$ is a saddle point and $\pm(1,1)$ are stable nodes.

2. Two additional equilibria points might be present, depending on the values of the parameter space. To find them we seek through the lines $y = \beta x$ ($\beta \neq 0$). Substituting on the first equation of the system

$$\begin{cases} -x^3 + x - A(x-y) = 0 \\ -y^3 + y - B(x-y) = 0 \end{cases} \tag{3.17}$$

gives the points:

$$\tilde{x}_\beta = \pm(\sqrt{1 - A(1-\beta)}, \beta\sqrt{1 - A(1-\beta)}) \tag{3.18}$$

where β is subject to the condition: $\beta > \frac{A-1}{A}$.
Since we assumed $A > 1$, we must have $\beta > 0$. Therefore fixed points of the form $(x, \beta x)$ can only appear in the first and third quadrant. Meanwhile the second equation of (3.17) gives the extra condition on β:

$$\beta = f(\beta) \quad \text{with} \quad f(\beta) := \sqrt{\frac{1 - B\frac{1-\beta}{\beta}}{1 - A(1-\beta)}} \tag{3.19}$$

Note that $\beta = 1$ is a solution, for which we find again the equilibria $\pm(1,1)$.
Fixed points $\tilde{x}_\beta$ may exist only if the argument inside the square root in (3.19) is strictly positive, that is if:

$$\beta > \max\left\{\frac{A-1}{A}, \frac{B}{1+B}\right\}, \tag{3.20}$$

which brings to the following cases, since both terms are positive.

$$\begin{cases} \beta > \frac{A-1}{A} = \frac{B}{1+B} & \text{iff} \quad B = A-1 \\ \beta > \frac{A-1}{A} & \text{iff} \quad B < A-1 \\ \beta > \frac{B}{1+B} & \text{iff} \quad B > A-1 \end{cases} \tag{3.21}$$

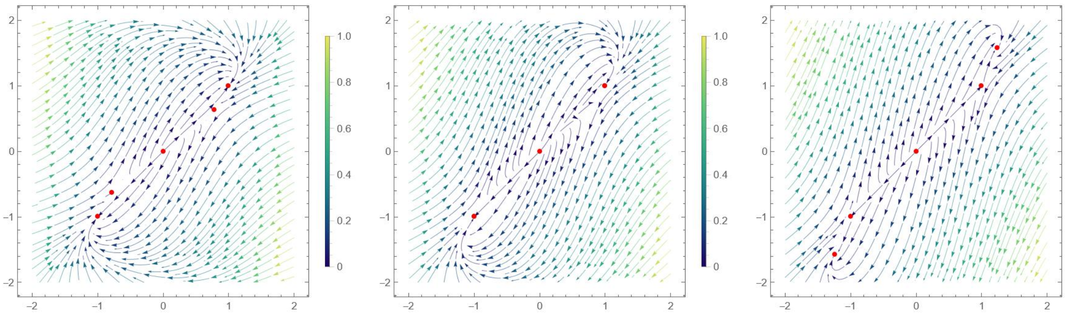
However,it is possible to prove that two additional fixed points only appear in the last case, i.e. when $B > A-1$; we postpone the details. The others either have no solution or the solutions are again $(0,0)$ and $\pm(1,1)$.

Moreover, in this range of values ($B > A-1$), three situations arise:

- $A-1 < B < A+2$: there exist $\beta > 0$ such that $\pm x_\beta$ are fixed points for system (3.16), with $|x_\beta| < 1$ and $\beta < 1$. In this case $\pm x_\beta$ are saddle points.
- $B = A+2$: no other fixed points are present apart from $(0,0)$ and $\pm(1,1)$.
- $B > A+2$: there exist $\beta > 0$ such that $\pm x_\beta$ are fixed points for system (3.16), with $|x_\beta| > 1$ and $\beta > 1$. In this situation $\pm x_\beta$ are stable nodes.

The table (3.6) summarizes the study, combining points 1 and 2. The parameters space of interest is $A > 1$, $B > A-1$.

The scenarios summarized in Table 3.6 have inspired the numerical study carried out on the model. In addition, we have examined in more details the behaviour of



(a) (b) (c)

Figure 3.5: Phase portrait of system (3.16) for various values of A and B. (a) Case $A-1 < B < A+2$ with $A = 2$ and $B = 2.5$. Fixed points: $(0,0)$ is an unstable node, $\pm(1,1)$ are stable nodes and $\pm(0.78', 0.63')$, numerically computed, are saddle points. (b) Case $B = A+2$ with $A = 2$ and $B = 4$. Fixed points: $(0,0)$ is an unstable node $\pm(1,1)$ have a negative and a zero eigenvalue. (c) Case $B > A+2$ with $A = 2$ and $B = 7$. Fixed points: $(0,0)$ is an unstable node, $\pm(1,1)$ are saddle points and $\pm(1.24', 1.58')$, numerically computed, are stable spirals. Red dots mark the equilibria. Streamline colors corresponds to the magnitude of vector field scaled to $[0,1]$ (relative magnitude).

(3.16) in the three different cases, providing some examples. In Fig. 3.5, numerically obtained phase portraits are displayed for specific values of the parameters in the cases $A-1 < B < A+2$, $B = A+2$ and $B > A+2$.

- $(B < A+2)$. If $A = 2, B = 2.5$ the Eq. $\beta = f(\beta)$ has the solution $\beta = 1$ and $\beta = \beta_x < 1$, numerically obtained. Thus we obtain respectively the fixed points $\pm\tilde{x}_1 = \pm(1,1)$ and $\pm\tilde{x}_{\beta_x} \approx \pm(0.78, 0.63)$. The eigenvalues of the linearized system around $\pm\tilde{x}_1$ are both real and negative, therefore they are stable nodes. The fixed points $\pm\tilde{x}_{\beta_x}$ are saddle points. The phase portrait numerically obtained is shown in Fig. 3.5 (a).
- $(B = A+2)$. If $A = 2, B = 4$ the only solution for β is 1, so the only fixed points, apart from $(0,0)$, are $\pm\tilde{x}_1$. The analysis above, for this particular case, gives a good explanation of the dynamics. Also the Fig. 3.5 (b) displays the phase portrait for this choice of constants.
- $(B > A+2)$. If $A = 2, B = 7$, there are two solutions for β: $\beta = 1$ and $\beta = \beta_x > 1$. In this case the fixed points $\pm\tilde{x}_1$ are saddle points, while the fixed points $\pm\tilde{x}_{\beta_x} \approx \pm(1.24, 1.58)$ have complexes eigenvalues with negative real part, thus they are stable spirals. The phase portrait is shown in Fig. 3.5 (c).

Remark. In the 3 cases, $(0,0)$ is a unstable fixed point.

3.4.1 Equilibrium points of the system

We include here the case study of (3.21) and hence conclude the analysis of the equilibria points of the system.

1. If $B = A-1$, $f(\beta) = \frac{1}{\sqrt{\beta}}$. So the unique solution to the equation (3.19) is $\beta = 1$ and $\tilde{x}_\beta = \pm(1,1)$. In this case $\gamma = 1$ so $\pm(1,1)$ are stable nodes and $(0,0)$ has a zero

	$(0,0)$	$\pm(1,1)$	$\pm(x,\beta x)$
$A-1<B<A+2$	unstable node	stable nodes	$0<x<1, 0<\beta<1$, saddle points
$B=A+2$	unstable node	one negative and one null eigenvalue	–
$B>A+2$	unstable node	saddle points	$x>1, \beta>1$, stable nodes

Table 3.6: Parameters space is $A>1, \quad B>A-1$. The table outline the nature of the fixed points of system (3.16) for different values of the parameters A,B. It defines also the scheme used to arrange the numerical simulations conducted on (1.1).

eigenvalue, thus the linearization cannot give information about the behavior in the phase space close to it. Nevertheless, the system can be re-written as

$$\begin{cases} \dot{x} = -x^3 - x(A-1) + Ay \\ \dot{y} = -y^3 - x(A-1) + Ay. \end{cases}$$

We observe that above the line $t : y = \frac{A-1}{A}x$, that is the eigen-direction of the zero eigenvalue, the linear component of the vector field is positive and negative below it. So close to $(0,0)$ we can neglect the third-order terms and the get a good approximation for the local dynamics. Furthermore along the line t, the linear component is equal to zero and so only the third-order terms count. We get that $\dot{x}<0, \dot{y}<0$ in the first quadrant and $\dot{x}>0, \dot{y}>0$ in the third one, which give a good understanding of the flow.

2. If $B<A-1$, the only solution to Eq. (3.19) is again $\beta=1$.
First of all, observe that $f(\beta)$ has a vertical asymptote to positive infinity as β approaches $\frac{A-1}{A}$ and it has a horizontal asymptote to zero as β grows to infinity. Moreover, $\frac{\partial}{\partial\beta}f(\beta)=0$ in:

$$\beta_\pm = \frac{AB \pm \sqrt{AB(B-(A-1))}}{A(1+B)},$$

that are not real for $B<A-1$. Now, in this case, we are searching for solutions $\beta>\frac{A-1}{A}$. In this region $f(\beta)$ is strictly decreasing and its graph cannot have more than one intersection with the line $y=\beta$, so the only solution to the Eq. (3.19) is $\beta=1$. This means that the only fixed points of the system are $(0,0)$ and $\pm(1,1)$ and we already portrait this behavior in the previous paragraph.

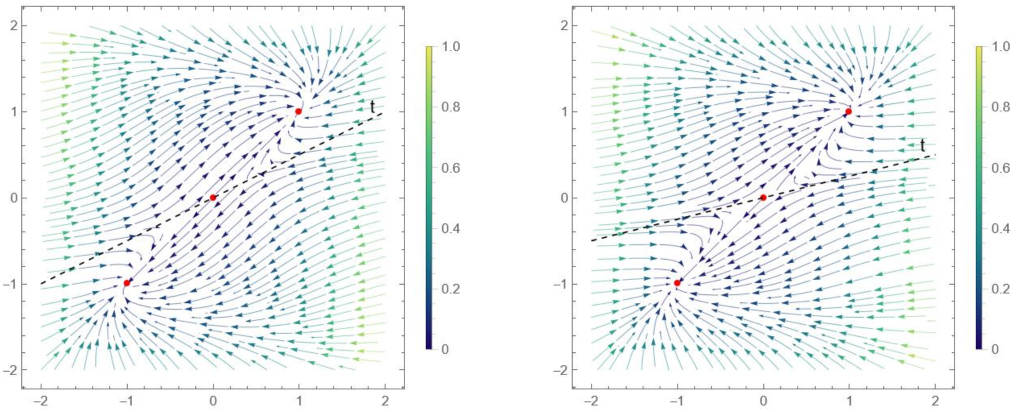


Figure 3.7: This figures represent the phase portrait of the system (3.16) when $B < A-1$. In red are displayed the nodes, and the dashed lines t show the neutral eigen-directions of $(0,0)$: $y = \frac{A-1}{A}x$ in the first graph and $y = \frac{B}{A}x$ in the second. In order, the choice for the constants was $A = 2, B = 1$ and $A = 2, B = 0.5$.

3. When $B > A-1$, i.e. $\beta > \frac{B}{B+1}$, $(0,0)$ is unstable and the nature of $\pm(1,1)$ changes according to γ being greater, less or equal to -2.
   Observe that $f(\frac{B}{B+1}) = 0$, $\lim_{\beta\to\infty} f(\beta) = 0$ and in this case $\frac{\partial}{\partial\beta} f(\beta) = 0$ gives two distinct real points $\beta_\pm$, with $\beta_- < \frac{B}{B+1}$. So the function $f(\beta)$ has only one critical point (maximum) $\beta = \beta_x > \frac{B}{B+1}$, therefore it may cross the line $y = \beta$ once, twice or never. But since we know that $\beta = 1$ is a solution of $\beta = f(\beta)$, there are 3 possibilities: $\beta = \beta_x = 1$ is the only solution [*i)*], or there is another intersection β_x that could be greater [*ii)*] or less [*iii)*] than 1. In the following lines we analyze the 3 sub-cases.

   i) $\beta = \beta_x = 1$ is the only solution to Eq. (3.19).
   This happens when the line $y = \beta$ is tangent to $f(\beta)$ in $\beta = 1$, i.e. $\frac{\partial f}{\partial\beta}(1) = 1$. That is if $B = A+2$ ($\gamma = -2$). In this case $\pm(1,1)$ have a negative and a zero eigenvalue, so to check stability one has to take into account higher-order terms. We study only $(1,1)$, since $-(1,1)$ is similar.
   To ease the computations we make the change of variables: $\hat{x} = x-1$, $\hat{y} = y-1$; so $(1,1)$ is shifted to $(0,0)$. The system (3.16) becomes

$$\begin{cases} \dot{\hat{x}} = -(A+2)\hat{x} + A\hat{y} - 3\hat{x}^2 - \hat{x}^3 \\ \dot{\hat{y}} = -(A+2)\hat{x} + A\hat{y} - 3\hat{y}^2 - \hat{y}^3 \end{cases} \tag{3.22}$$

   Along the eigen-direction of the zero eigenvalue, represented by the line $r : \hat{y} = \frac{A+2}{A}\hat{x}$, the first-order terms of the system above vanish, and the line r always lies above $\hat{y} = \hat{x}$, that is the eigen-direction of the non-zero eigenvalue. Also the first-order terms of the system are positive above r and negative below it. So, out of this line, higher-order terms can be neglected close to the origin, while the second-order terms give a good approximation along the line r, where is immediate to see that the vector field points downward-left.

   ii) If $0 < \frac{\partial f}{\partial\beta}(1) < 1$, i.e. $B < A+2$, there are two intersections, one at $\beta = 1$ and one at $\beta = \beta_x(A,B) < 1$. This means that there are two extra fixed point of type $\pm(x, \beta_x x)$, with absolute value less than 1.

iii) If $\frac{\partial f}{\partial \beta}(1) > 1$, i.e. $B > A + 2$, in addition to the intersection $\beta = 1$ there is a second intersection at $\beta = \beta_x > 1$, so there are two extra fixed points of type $\pm(x, \beta x)$, with absolute value greater than 1.

3.5. Small noise approximations of the macroscopic limit

We derive now a *small-noise approximation* of the system (1.2). In particular, we aim to create a pair of independent Gaussian processes $\left\{\left(\widetilde{X}_t, \widetilde{Y}_t\right); 0 \le t \le T\right\}$ that closely follows in time $\{(X_t, Y_t); 0 \le t \le T\}$, the solution of the system (1.2). Although we prove that this approximation, holds rigorously true when $\sigma \to 0$, numerical simulations show that it remains valid also beyond the assumption of $\sigma \ll 1$, which explains the observations made on the numerical study.

Note that the specific form of the system allows to consider the two processes $\widetilde{X}_t$ and $\widetilde{Y}_t$ to be independent. Indeed, in the equations

$$dX_t = \left[-X_t^3 + X_t - \alpha\theta_{11}(X_t - \mathbb{E}[X_t]) - (1-\alpha)\theta_{12}(X_t - \mathbb{E}[Y_t])\right]dt + \sigma dW_t^1,$$

$$dY_t = \left[-Y_t^3 + Y_t - \alpha\theta_{21}(Y_t - \mathbb{E}[X_t]) - (1-\alpha)\theta_{22}(Y_t - \mathbb{E}[Y_t])\right]dt + \sigma dW_t^2,$$

the two variables interact only through their expected mean; if we had mixed terms of the type $X^n Y^m$, such approach would not be possible.

The following theorem gives the result.

Theorem C (Small noise approximations). *Let $T > 0$ and $\{(X_t, Y_t); t \le T\}$ be the solution of (1.2) with initial conditions X^* and Y^*. There exist two Gauss-Markov processes $\{\widetilde{X}(t); 0 \le t \le T\}$ and $\{\widetilde{Y}(t); 0 \le t \le T\}$, with $\widetilde{X}(0) = X^*$ and $\widetilde{Y}(0) = Y^*$, satisfying the following properties.*

1. *$\widetilde{X}_t$ and $\widetilde{Y}_t$ solve the first two moments equations of the system (1.2)*
2. *$\widetilde{X}_t$ and $\widetilde{Y}_t$ are simultaneously σ-closed to the solution of (1.2), that is, for every $T > 0$ there exists some real constant $C_T > 0$ such that for every $\sigma > 0$:*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\{\left|X(t) - \widetilde{X}(t)\right| + \left|Y(t) - \widetilde{Y}(t)\right|\right\}\right] \le C_T \sigma^2$$

Remark. $\widetilde{X}_t$ and $\widetilde{Y}_t$ are Gaussian processes, so, among other properties, we use here the fact that their higher-order moments are polynomial functions of the first two moments. In particular, this means that their behaviour is completely described by the mean and variance functions. As a result, rather than studying some infinite dimensional system, given by the moments equations of the solution of (1.2), we can reduce our analysis to a 4-dim system of differential equations, which portrays the behaviour of the means and variances of the Gaussian-approximation processes. We shall see how to do this in the proof of the theorem.

Proof. We divide the proof in several steps and begin by deriving the equations of the moments for (1.2).

Step 1: Moments equations.
Applying *Itô's formula* to the equations of the system (1.2) we obtain the following SDEs, which are solved by all $X(t)^p$ and $Y(t)^p$, with $p \geq 1$:

$$\begin{aligned} dX_t^p =& \Big[-pX_t^{p+2} + pX_t^p - [\alpha\theta_{11}(X_t - \mathbb{E}[X_t]) + (1-\alpha)\theta_{12}(X_t - \mathbb{E}[Y_t])]pX_t^{p-1} \\ & + \frac{\sigma^2}{2}p(p-1)X_t^{p-2}\Big]dt + \sigma p X_t^{p-1}dW_t \\ dY_t^p =& \Big[-pY_t^{p+2} + pY_t^p - [\alpha\theta_{21}(Y_t - \mathbb{E}[X_t]) + (1-\alpha)\theta_{22}(Y_t - \mathbb{E}[Y_t])]pY_t^{p-1} \\ & + \frac{\sigma^2}{2}p(p-1)Y_t^{p-2}\Big]dt + \sigma p Y_t^{p-1}dW_t^Y \end{aligned} \tag{3.23}$$

Now, set $m_p^X(t) := \mathbb{E}[X^p(t)]$ and $m_p^Y(t) := \mathbb{E}[Y^p(t)]$ the p-th moments of X_t and Y_t. Taking the expectation $\mathbb{E}[\cdot]$ on both side of the above equations we obtain the system:

$$\begin{aligned} \frac{d}{dt}m_p^X(t) =& -pm_{p+2}^X(t) + pm_p^X(t) + \frac{\sigma^2}{2}p(p-1)m_{p-2}^X \\ & - p\alpha\theta_{11}\Big(m_p^X - m_1^X(t)m_{p-1}^X(t)\Big) - p(1-\alpha)\theta_{12}\Big(m_p^X(t) - m_1^Y(t)m_{p-1}^X(t)\Big) \\ \frac{d}{dt}m_p^Y(t) =& -pm_{p+2}^Y(t) + pm_p^Y(t) + \frac{\sigma^2}{2}p(p-1)m_{p-2}^Y \\ & - p\alpha\theta_{21}\Big(m_p^Y - m_1^X(t)m_{p-1}^Y(t)\Big) - p(1-\alpha)\theta_{22}\Big(m_p^Y(t) - m_1^Y(t)m_{p-1}^Y(t))\Big); \end{aligned} \tag{3.24}$$

(the Brownian motion has zero mean: $\mathbb{E}[W_t] = 0$). We formally divided by dt on both sides of the equations.
Note that in (3.24) the p-th moments depend on the $(p+2)$-th moments, this makes the system infinite dimensional, unless some higher order term depends on the first two moments of X_t and Y_t.

Nevertheless, consider the first two moments equations, so take $p = 1, 2$ in (3.24):

$$\begin{aligned} \frac{d}{dt}m_1^X(t) = -m_3^X(t) + m_1^X(t) - \alpha\theta_{11}m_1^X - (1-\alpha)\theta_{12}m_1^X(t), \\ \frac{d}{dt}m_1^Y(t) = -m_3^Y(t) + m_1^Y(t) - \alpha\theta_{21}m_1^Y - (1-\alpha)\theta_{22}m_1^Y(t); \end{aligned} \tag{3.25}$$

$$\begin{aligned}\frac{d}{dt}m_2^X(t) &= -2m_4^X(t) + 2m_2^X(t) - 2\alpha\theta_{11}\Big(m_2^X - m_1^X(t)^2\Big)\\ &\qquad - 2A\Big(m_2^X(t) - m_1^Y(t)m_1^X(t)\Big) + \sigma^2\\ \frac{d}{dt}m_2^Y(t) &= -2m_4^Y(t) + 2m_2^Y(t) + 2B\Big(m_2^Y - m_1^X(t)m_1^Y(t)\Big)\\ &\qquad - 2(1-\alpha)\theta_{22}\Big(m_2^Y(t) - m_1^Y(t)^2\Big) + \sigma^2;\end{aligned} \tag{3.26}$$

here, we called $A := (1-\alpha)\theta_{12}$ and $B := -\alpha\theta_{21}$.

Recall that if Z is a Gauss random variable with normal distribution $\mathcal{N}(\mu,\nu)$, where μ and ν denote the mean and variance respectively, then we have the identities:

$$\mathbb{E}[Z^3] = \mu^3 + 3\mu\nu \quad \text{and} \quad \mathbb{E}[Z^4] = \mu^4 + 6\mu^2\nu + 3\nu^2$$

Therefore, if we suppose that a couple of Gauss-Markov random processes with mean and variance $\mu_X(t), \nu_X(t)$ and $\mu_Y(t), \nu_Y(t)$ respectively, solve equations (3.25) and (3.26), then, plugging the identities into the equations, we obtain the following system of differential equations, to which the mean and variance functions must obey:

$$\begin{aligned}\dot{\mu}_X &= -\mu_X^3 + \mu_X(1-3\nu_X) - A(\mu_X - \mu_Y)\\ \dot{\mu}_Y &= -\mu_Y^3 + \mu_Y(1-3\nu_Y) + B(\mu_Y - \mu_X)\\ \dot{\nu}_X &= -6\nu_X^2 - 6\mu_X^2\nu_X + 2\nu_X - 2\alpha\theta_{11}\nu_X + 2A\nu_X + \sigma^2\\ \dot{\nu}_Y &= -6\nu_Y^2 - 6\mu_X^2\nu_Y + 2\nu_Y + 2B\nu_Y - 2(1-\alpha)\theta_{22}\nu_Y + \sigma^2\end{aligned} \tag{3.27}$$

In particular, (3.27) has a four-dimensional vector field which is continuous in each variable and has continuous partial derivatives. Hereby the system has a unique global solution $\Big\{\Big(\overline{\mu}_X(t), \overline{\mu}_Y(t), \overline{\nu}_X(t), \overline{\nu}_Y(t)\Big); t \geq 0\Big\}$ with the initial conditions $\overline{\mu}_X(0) = X^*$, $\overline{\nu}_Y(0) = Y^*$ and $\overline{\nu}_X(0) = \overline{\nu}_Y(0) = 0$.

Step 2: approximations processes' definition.
Now let $T > 0$, and set $V_X(t) := \sigma^{-2}\overline{\nu}_X(t)$ and $V_Y(t) := \sigma^{-2}\overline{\nu}_Y(t)$. We want to define two centered (with zero mean) Gaussian processes $\{Z_X(t)\}_{t\leq T}$ and $\{Z_Y(t)\}_{t\leq T}$, so that

$$\mathbb{E}\Big[Z_X(t)^2\Big] = V_X(t) \quad \text{and} \quad \mathbb{E}\Big[Z_Y(t)^2\Big] = V_Y(t) \quad \text{for all } t \in [0,T]; \tag{3.28}$$

and obtain the differential characterization of these, that will be useful later to prove the σ-closedness to the solution.
Let's consider $\{Z_X(t)\}_{t\leq T}$ first and its differential as a generic Ito's process:

$$dZ_X(t) = a(t)dt + b(t)dW_1(t);$$

with suitable functions $a(t), b(t)$ and $W_1(\cdot)$ a standard Brownian motion. Using Ito's formula we derive:

$$dZ_X(t)^2 = (2Z_X(t)a(t) + b(t)^2)dt + 2Z_X(t)b(t)dW_1(t);$$

and taking the expectation $\mathbb{E}[\cdot]$ and dividing by dt on both side of the equation we otain:

$$\frac{d}{dt}\mathbb{E}[Z_X(t)^2] = 2\mathbb{E}[Z_X(t)a(t)] + \mathbb{E}[b(t)^2]. \tag{3.29}$$

Now we have to give $b(t)$ and $a(t)$ such that (3.29) respects condition (3.28), that is $\mathbb{E}[Z_X(t)^2]$ satisfies the third equation of (3.27):

$$\dot{\nu}_X = -6{\nu_X}^2 - 6{\mu_X}^2\nu_X + 2\nu_X - 2\alpha\theta_{11}\nu_X + 2A\nu_X + \sigma^2,$$

which, dividing both side by σ^2, becomes:

$$\dot{V}_X(t) = -2\Big(V_X(t)(3\sigma^2 V_X(t) - 3\mu_X(t)^2 + 1 - \alpha\theta_{11} + A)\Big) + 1; \tag{3.30}$$

where we used the definition of V_X and made the time-dependency explicit.
Now we can set $b(t) = 1$ and define $a(t)$ in the following way.
Take the deterministic function

$$\tilde{a}(t) = 3\sigma^2 V_X(t) - 3\mu_X(t)^2 + 1 - \alpha\theta_{11} + A.$$

We can define $a(t)$ such that $Z_X(t)a(t) = Z_X^2(t)\tilde{a}(t)$. Then we have a process $Z_X(t)$, whose second moment function $\mathbb{E}Z_X(t)^2$ satisfies 3.30), for which $V_X(t)$ is the solution. In the same way, with straight-forward modifications, we construct $\{Z_Y(t)\}_{t\le T}$.

Hence, putting together the results, we have constructed two processes $\{Z_X(t)\}_{t\le T}$ and $\{Z_Y(t)\}_{t\le T}$ satisfying the stochastic equations:

$$\begin{aligned} dZ_X(t) &= \Big(-3\sigma^2 V_X(t) - 3\mu_X(t)^2 + 1 - \alpha\theta_{11} - (1-\alpha)\theta_{12}\Big)Z_X(t)dt + dW_1(t) \\ dZ_Y(t) &= \Big(-3\sigma^2 V_Y(t) - 3\mu_Y(t)^2 + 1 - \alpha\theta_{21} - (1-\alpha)\theta_{22}\Big)Z_Y(t)dt + dW_2(t) \\ Z_X(0) &= Z_Y(0) = 0 \end{aligned} \tag{3.31}$$

They are both Gauss Markov with zero mean and such that $\mathrm{Var}[Z_X(t)] = V_X(t)$ and $\mathrm{Var}[Z_Y(t)] = V_Y(t)$ for all $t \in [0,T]$; they are also well-defined, since the solution of (3.27) is unique.

Part 3: σ-closedness.
Define the two processes:

$$\widetilde{X}(t) := \mu_X(t) + \sigma Z_X(t) \qquad \widetilde{Y}(t) := \mu_Y(t) + \sigma Z_Y(t) \qquad \text{for all } t \in [0,T];$$

They both are Gaussian Markov processes, with the respective means $\mu_X(t), \mu_Y(t)$ and variances $\nu_X = \sigma^2 V_X, \nu_Y = \sigma^2 V_Y$ that satisfy system (3.30). This also imply they solve the first two moments equations: (3.25) and (3.26). So we proved the

first statement of the theorem.
Now, we consider the following trivial inequality:

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\{\left|X(t)-\widetilde{X}(t)\right|+\left|Y(t)-\widetilde{Y}(t)\right|\right\}\right]\leq\mathbb{E}\left[\sup_{t\in[0,T]}\left|X(t)-\widetilde{X}(t)\right|\right] + \mathbb{E}\left[\sup_{t\in[0,T]}\left|Y(t)-\widetilde{Y}(t)\right|\right]. \tag{3.32}$$

To conclude the proof of the theorem, it is sufficient to find an upper bound for the right-hand side terms.
So, by construction we have that:

$$\begin{aligned} d\widetilde{X}_t &= d\mu_X(t)+\sigma dZ_X(t)\\ &=\Big[-\mu_X(t)^3+\mu_X(t)\Big(1-3\sigma^2V_X(t)\Big)-(1-\alpha)\theta_{12}\Big(\mu_X(t)-\mu_Y(t)\Big)\Big]dt\\ &\quad+\sigma Z_X(t)\Big[-3\sigma^2V_X(t)-3\mu_X(t)^2+1-\alpha\theta_{11}-(1-\alpha)\theta_{12}\Big]dt+\sigma dW_1(t)\\ &=\Big[1-3\sigma^2V_X(t)-(1-\alpha)\theta_{12}\Big]\Big(\mu_X(t)+\sigma Z_X(t)\Big)dt+\Big[-\mu_X(t)^3-3\sigma Z_X(t)\mu_X(t)^2\Big]dt\\ &\quad+\Big[(1-\alpha)\theta_{12}\mu_Y(t)-\alpha\theta_{11}\sigma Z_X(t)\Big]dt+\sigma dW_1(t)\\ &=\Big[\Big(1-3\sigma^2V_X(t)-(1-\alpha)\theta_{12}\Big)\widetilde{X}_t-\widetilde{X}_t^3+\sigma^3Z_X(t)^3+3\sigma^2Z_X(t)^2\mu_X(t)\Big]dt\\ &\quad+\Big[(1-\alpha)\theta_{12}\mu_Y(t)-\alpha\theta_{11}\sigma Z_X(t)\Big]dt+\sigma dW_1(t); \end{aligned} \tag{3.33}$$

we used the identity $(a+b)^3=a^3+3a^2b+3ab^2+b^3$. Following the same computations we obtain also:

$$\begin{aligned} d\widetilde{Y}_t=\ldots&=\Big[\Big(1-3\sigma^2V_Y(t)-\alpha\theta_{21}\Big)\widetilde{Y}_t-\widetilde{Y}_t^3+\sigma^3Z_Y(t)^3+3\sigma^2Z_Y(t)^2\mu_Y(t)\Big]dt\\ &\quad+\Big[\alpha\theta_{21}\mu_Y(t)-(1-\alpha)\theta_{22}\sigma Z_Y(t)\Big]dt+\sigma dW_2(t). \end{aligned} \tag{3.34}$$

Then, we can look at their integral equations. For $\widetilde{X}_t$, we have:

$$\begin{aligned} X(t)-\widetilde{X}(t)&=\int_0^t\Big(X(s)-\widetilde{X}(s)\Big)\Big[1-f(s)-\alpha\theta_{11}-(1-\alpha)\theta_{12}\Big]ds\\ &\quad-\sigma^2\int_0^t\Big(\sigma Z_X(s)^3+3\mu_X(s)Z_X(s)^2-3\mu_X(s)V_X(s)-3\sigma V_X(s)Z_X(s)\Big)ds; \end{aligned}$$

recall that $\mathbb{E}[X_t]=\mu_X(t)=\mathbb{E}[\widetilde{X}_t]$, since they solve the same equations. We used $f(t)=X(t)^2+X(t)\widetilde{X}(t)+\widetilde{X}(t)^2$.
Again, the above equation is of the form $\phi(t)=\int_0^t\phi(s)H(s)ds+\int_0^t Q(s)ds$, with $\phi(t)=$

$X(t)-\widetilde{X}(t)$. So, as $\phi(0)=0$, the solution is $\phi(t)=\int_0^t Q(s)e^{\int_s^t H(r)dr}ds$, with:

$$H(t) = 1 - f(t) - \alpha\theta_{11} - (1-\alpha)\theta_{12}$$
$$Q(t) = -\sigma^2\Big(\sigma Z_X(t)^3 + 3\mu_X(t)Z_X(t)^2 - 3\mu_X(t)V_X(t) - 3\sigma V_X(t)Z_X(t)\Big)$$

So, we have the inequality:

$$\Big|X(t)-\widetilde{X}(t)\Big| \le \int_0^t \Big|Q(s)\Big| e^{\int_s^t 1-f(r)-\alpha\theta_{11}-(1-\alpha)\theta_{12}\,dr}\, ds$$

and, therefore,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\Big|X(t)-\widetilde{X}(t)\Big|\right] \le \mathbb{E}\left[\sup_{t\in[0,T]}\int_0^t \Big|Q(s)\Big| e^{\int_s^t 1-f(r)-\alpha\theta_{11}-(1-\alpha)\theta_{12}\,dr}\, ds\right]$$
$$\le \mathbb{E}\left[\int_0^T \Big|Q(s)\Big| \sup_{t\in[0,T]} e^{\int_s^t 1-f(r)-\alpha\theta_{11}-(1-\alpha)\theta_{12}\,dr}\, ds\right] \le C_T\int_0^T \mathbb{E}\Big[\Big|Q(s)\Big|\Big]ds$$

Now, since $Q(s)$ is polynomial function of a Gauss Markov process, it has a time-local finite L^1-norm. Furthermore we can define $\widetilde{Q}(s) := \sigma^2 Q(s)$, and the last term of the above inequality can be bounded by the constant $\widetilde{C}_T\sigma^2$. Everything said can be applied in the same way to prove it holds that:

$$\mathbb{E}\left[\sup_{t\in[0,T]}\Big|Y(t)-\widetilde{Y}(t)\Big|\right] \le \widetilde{D}_T\sigma^2$$

for some positive constant $\widetilde{D}_T$. All of this proves the second statement of the theorem and concludes the proof.

□

3.6. Occuring of a Hopf bifurcation

This section concludes our study of the model. Here we reduce our analysis to a four dimension dynamical system. With the help of bifurcation theory, we provide the results that motivates the observations made in the numerical simulations.

In the proof of the theorem C we constructed a pair of Gauss-Markov processes, $\Big\{\Big(\widetilde{X}_t,\widetilde{Y}_t\Big); 0\le t\le T\Big\}$, that closely follows the solution of the system (1.2). These processes are σ-closed to the solution of (1.2) and their respective mean and variance functions $\Big(\mu_X(t),\mu_Y(t),\nu_X(t),\nu_Y(t)\Big)$ solve the system of differential equations (3.27):

$$\begin{cases} \dot{\mu}_X = -\mu_X^3 + \mu_X(1-3\nu_X) - A(\mu_X - \mu_X) \\ \dot{\mu}_Y = -\mu_Y^3 + \mu_Y(1-3\nu_Y) + B(\mu_Y - \mu_X) \\ \dot{\nu}_X = -6\nu_X^2 - 6\mu_X^2\nu_X + 2\nu_X - 2\alpha\theta_{11}\nu_X + 2A\nu_X + \sigma^2 \\ \dot{\nu}_Y = -6\nu_Y^2 - 6\mu_X^2\nu_Y + 2\nu_Y + 2B\nu_Y - 2(1-\alpha)\theta_{22}\nu_Y + \sigma^2 \end{cases} \tag{3.27}$$

Since the equations above completely describe the processes $\widetilde{X}_t$ and $\widetilde{Y}_t$, we can focus on the study of the system (3.27) to have a sufficiently good approximation of the solution to the model (1.2), which is the macroscopic limit of our model.

Result For fixed values $\theta_{11}, \theta_{22}, A, B$, considered in this work, the dynamical system (3.27) displays a *Hopf bifurcation* at the equilibrium point

$$\left(\mu_X, \mu_X, \nu_X, \nu_X\right) = \left(0, 0, \nu_1^{A,B}, \nu_2^{A,B}\right)$$

with a critical value $\widetilde{\sigma}_{A,B}$ of the noise parameter.
We recall that a Hopf bifurcation occurs whenever a fixed point of a dynamical system loses stability, as a pair of complex conjugate eigenvalues (of the linearization around the fixed point) crosses the imaginary axis of the complex plane.

Consider the vector field of the system (3.27):

$$\mathbf{F}(\mu_X, \mu_Y, \nu_X, \nu_Y) = \begin{pmatrix} -\mu_X^3 + \mu_X(1-3\nu_X) - A(\mu_X - \mu_X) \\ -\mu_Y^3 + \mu_Y(1-3\nu_Y) + B(\mu_Y - \mu_X) \\ -6\nu_X^2 - 6\mu_X^2\nu_X + 2\nu_X - 2\alpha\theta_{11}\nu_X + 2A\nu_X + \sigma^2 \\ -6\nu_Y^2 - 6\mu_X^2\nu_Y + 2\nu_Y + 2B\nu_Y - 2(1-\alpha)\theta_{22}\nu_Y + \sigma^2 \end{pmatrix}$$

with the noise-parameter σ. For $\mathbf{F}(\,\cdot\,, \sigma)$ we find the equilibrium point:

$$\left(0, 0, \nu_1^{A,B}, \nu_2^{A,B}\right) = \left(0, 0, \frac{1}{6}\left(\sqrt{(A+3)^2 + 6\sigma^2} - A - 3\right), \frac{1}{6}\left(\sqrt{(B-3)^2 + 6\sigma^2} + B - 3\right)\right);$$

and the linearized system $\mathrm{D}\mathbf{F}\left(\left(0, 0, \nu_1^{A,B}, \nu_2^{A,B}\right), \sigma\right)$, given by the Jacobian matrix:

$$J_{A,B}(\sigma) = \begin{bmatrix} -A - 3\nu_1^{A,B} + 1 & A & 0 & 0 \\ -B & B - 3\nu_2^{A,B} & 0 & 0 \\ 0 & 0 & -2A - 12\nu_1^{A,B} - 6 & 0 \\ 0 & 0 & 0 & 2B - 12\nu_2^{A,B} - 6 \end{bmatrix}$$

The matrix $J_{A,B}(\sigma)$ has the eigenvalues:

$$\lambda_1^{A,B}(\sigma) = -2\sqrt{A^2 + 6A + 6\sigma^2 + 9}$$
$$\lambda_2^{A,B}(\sigma) = -2\sqrt{B^2 - 6B + 6\sigma^2 + 9},$$

and

$$\begin{aligned} \lambda_3^{A,B}(\sigma) = &\frac{1}{4}\Bigg\{ - \sqrt{(A+3)^2 + 6\sigma^2} - A - \sqrt{(B-3)^2 + 6\sigma^2} + B + 10 \\ &- \sqrt{2}\Bigg[A^2 + A\left(\sqrt{(A+3)^2 + 6\sigma^2} - \sqrt{(B-3)^2 + 6\sigma^2} - 7B + 3\right) \\ &- \left(\sqrt{(B-3)^2 + 6\sigma^2} - B\right)\left(\sqrt{(A+3)^2 + 6\sigma^2} + B\right) - 3B + 6\sigma^2 + 9\Bigg]^{\frac{1}{2}}\Bigg\} \end{aligned}$$

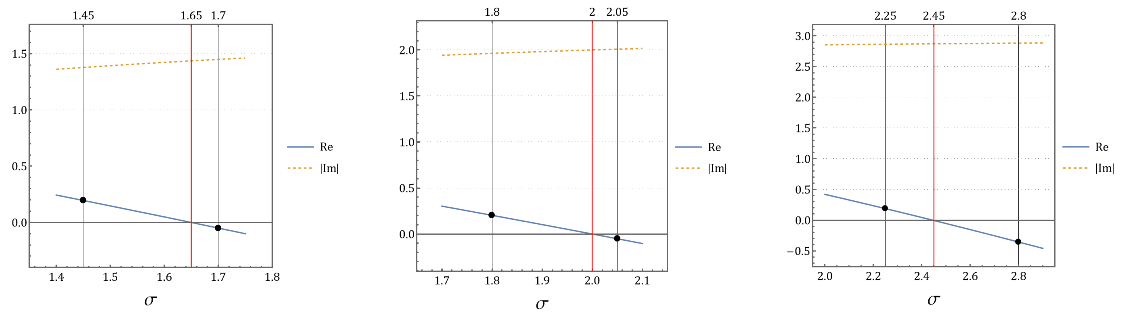


Figure 3.8: The figures displays the behaviour of the eigenvalues $\lambda_3^{A,B}$ and $\lambda_4^{A,B}$ (the real part in blue and in orange the absolute value of the imaginary part) as the values of the parameter σ change, in the three cases: *i*) $A = 2, B = 2.5$, *ii*) $A = 2, B = 4$ and *iii*) $A = 2, B = 7$. The red line represent, in each case, the threshold, i.e. the value of σ at which the eigenvalues cross the imaginary axis: $\widetilde{\sigma}_{2,2.5} \approx 1.65$, $\widetilde{\sigma}_{2,4} \approx 2$ and $\widetilde{\sigma}_{2,7} \approx 2.45$; while the thin black ones represent, in each case, the values for which we computed the simulations in Fig. 3.11. All the results were obtained numerically using the software *Mathematica*

$$\begin{aligned}\lambda_4^{A,B}(\sigma) =& \frac{1}{4}\Bigg\{-\sqrt{(A+3)^2+6\sigma^2}-A-\sqrt{(B-3)^2+6\sigma^2}+B+10\\ &+\sqrt{2}\Bigg[A^2+A\left(\sqrt{(A+3)^2+6\sigma^2}-\sqrt{(B-3)^2+6\sigma^2}-7B+3\right)\\ &-\left(\sqrt{(B-3)^2+6\sigma^2}-B\right)\left(\sqrt{(A+3)^2+6\sigma^2}+B\right)-3B+6\sigma^2+9\Bigg]^{\frac{1}{2}}\Bigg\}\end{aligned}$$

In the three cases considered in this study $\big($*i*) $A = 2, B = 2.5$, *ii*) $A = 2, B = 4$ and *iii*) $A = 2, B = 7$; $\theta_{11} = \theta_{22} = 8\big)$, $\lambda_1^{A,B}, \lambda_2^{A,B}$ are non-zero for $\sigma > 0$. Moreover, $\lambda_3^{A,B}$ and $\lambda_4^{A,B}$ are complexes conjugate numbers. Fig. 3.8 represents the behaviour of them in the three different choices of A and B, respect to the noise's parameter changes. We can see that, in each case, the eigenvalues have a threshold $\widetilde{\sigma}_{A,B}$ for which the system stability changes as we cross it. We have $\widetilde{\sigma}_{2,2.5} \approx 1.65$, $\widetilde{\sigma}_{2,4} \approx 2$ and $\widetilde{\sigma}_{2,7} \approx 2.45$. Therefore the conditions for the Hopf bifurcation are satisfied, which proves the presence of a limit cycle.

The Fig. 3.8 shows that in all the three cases, the equilibrium point $\left(0, 0, \nu_1^{A,B}, \nu_2^{A,B}\right)$ changes its nature from stable to unstable, as the noise decrease and the real part of $\lambda_3^{A,B}, \lambda_4^{A,B}$ becomes positive ($\lambda_1^{A,B}, \lambda_2^{A,B}$ are always negative for every $\sigma > 0$). This suggest the presence of a *stable* limit cycle. We computed numerical simulations of system (3.27) that prove the occurring of a Hopf bifurcation in all the cases considered and the presence of a stable limit cycle for an intermediate range of values of the noise. In Fig. 3.11 we analyze the results. For completeness we also show in Fig. 3.9 the cases $\sigma = 0$ and $\sigma = 5$.

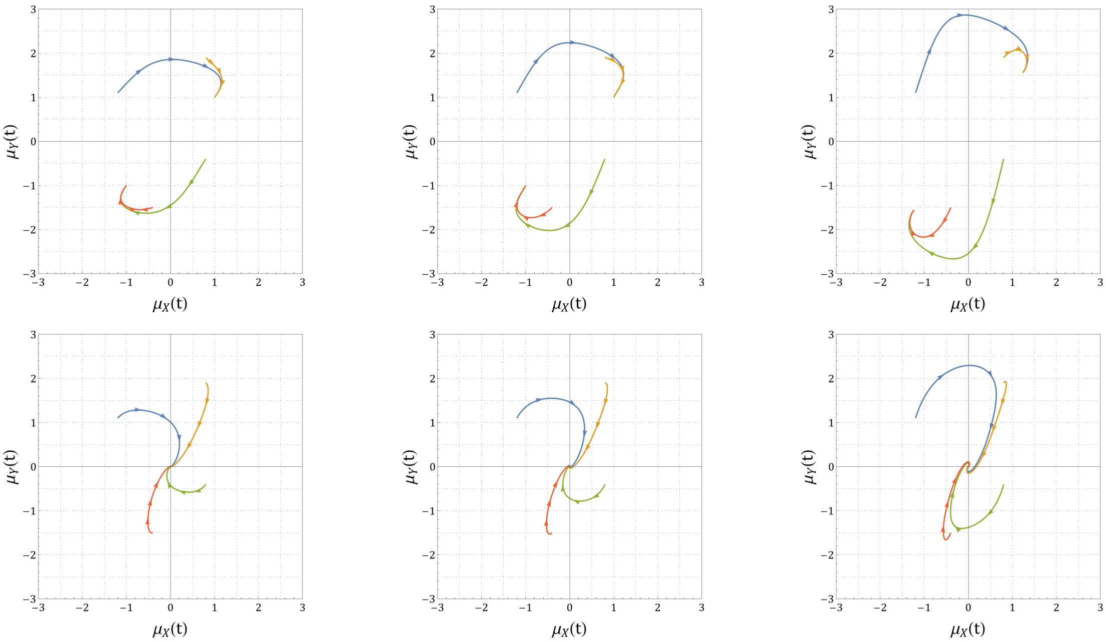


Figure 3.9: Dynamics of system (3.27) when $\sigma = 0$ and $\sigma = 5$. The figure display trajectories of $\mu_X(t)$ and $\mu_Y(t)$ during the numerical simulations of the system (3.27). For these we used the Euler method. In the first row we plotted the simulations for $\sigma = 0$ and $\sigma = 5$ for the second row. Each column represent respectively: first $A = 2, B = 2.5$, second $A = 2, B = 4$ and third $A = 2, B = 7$.

To complete the study, in Fig. 3.10 we simulated the behaviour of the system (3.27) with the same choices of the parameters A, B, σ as Fig. 3.4. Our analysis shows that the behaviour of our model 1.1 is qualitatively well described by the the Gaussian approximation given by (3.27), which drives its robust, self-sustained, periodic rhythm.

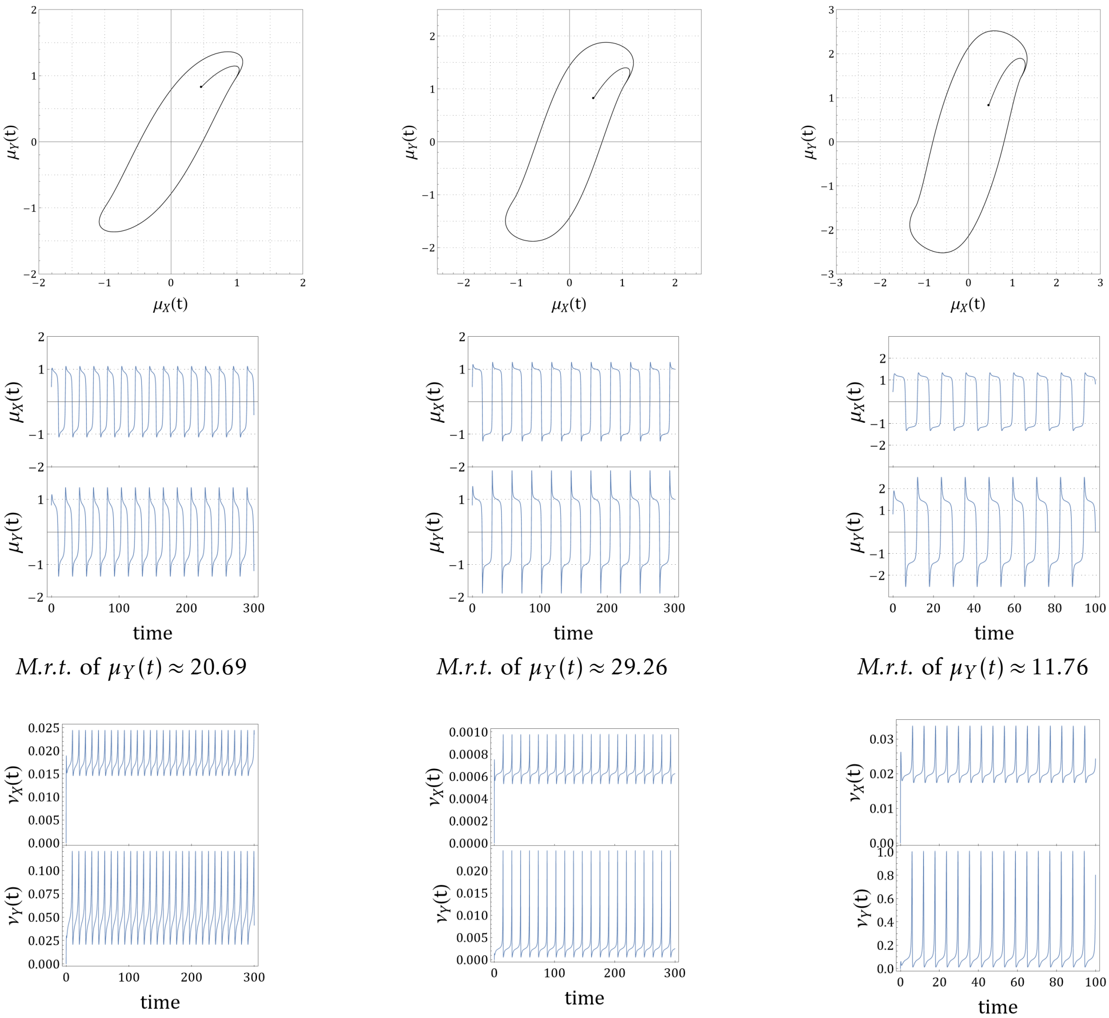


Figure 3.10: The figure shows numerical simulations of the system (3.27), in three parameters cases: $A = 2$, $B = 2.5$, $\sigma = 0.5$ first column, $A = 2$, $B = 4$, $\sigma = 0.1$ for the second and $A = 2$, $B = 4$, $\sigma = 0.6$ for the third. On the top we plotted the path of $\left(\mu_X(t), \mu_Y(t)\right)$, while under the trajectory of each variable. We ran the simulations using the Euler method, with the same choice for the constants as in Fig. 3.4, with the same time step $dt = 0.005$, for 10^6 iterations. We computed also the mean return time *(M.r.t.)*to the Poincaré section $\left\{\mu_X(t) > 0, \mu_Y(t) = 0\right\}$ for the trajectory $t \mapsto \mu_Y(t)$.

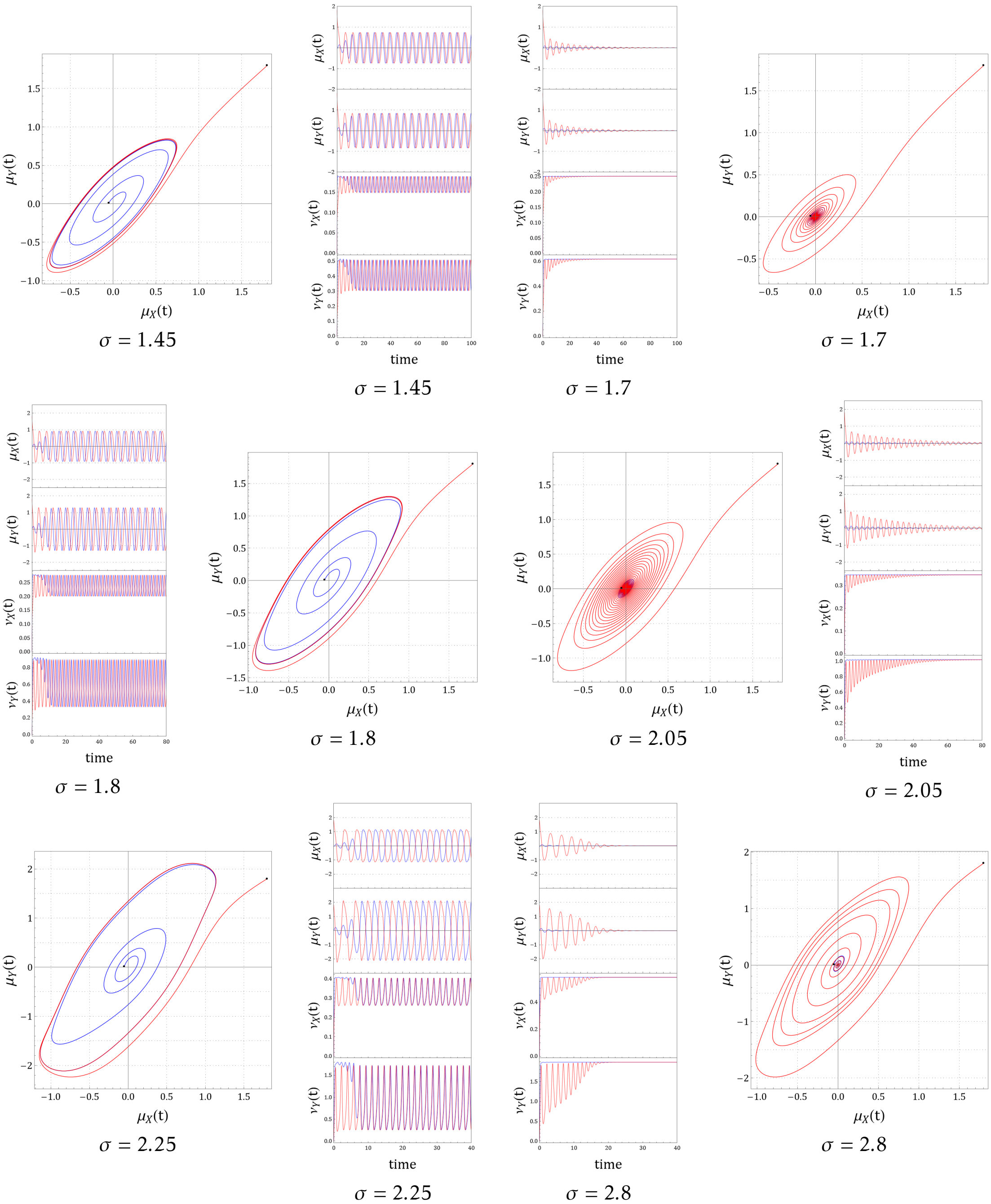


Figure 3.11: We ran numerical simulations of the system (3.27), to prove the occurring of an Hopf bifurcation at the equilibrium point $\left(0, 0, \nu_1^{A,B}, \nu_2^{A,B}\right)$, and the rise of a stable limit cycle. From top to bottom in each row we have the three cases: *i*) $A = 2, B = 2.5$, *ii*) $A = 2, B = 4$ and *iii*) $A = 2, B = 7$. In each row we plotted the trajectory $t \mapsto (\mu_X(t), \mu_Y(t))$ as well as the time evolution of the 4 dimensional vector $\left(\mu_X(t), \mu_y(t), \nu_X(t), \nu_Y(t)\right)$. We choose in each case two values of σ, that are from opposite sides of the threshold $\widetilde{\sigma}_{A,B}$. We recall that: $\widetilde{\sigma}_{2,2.5} \approx 1.65$, $\widetilde{\sigma}_{2,4} \approx 2$ and $\widetilde{\sigma}_{2,7} \approx 2.45$. We can clearly see the presence of stable limit cycles in each case. The simulations were run with the Euler method, with a time step $dt = 0.005$ for 10^6 steps, using he software *Mathematica*.

Conclusions

In this thesis we investigated the emergence of collective periodic behaviours in a frustrated network of interacting diffusion particles. In particular, we were interested in the role of the noise as equilibrium-breaking element and, thus, as an essential component, for the model considered in this thesis, to develop self-sustained periodicities.

In our view, the emergence of periodic motions in the model can be explained as follows. If we imagine to start with two independent communities, that is, particles evolve according to system (1.1) with $\theta_{12} = \theta_{21} = 0$. When the interaction constants θ_{11} and θ_{22} are positive and large enough, each population tends to its own equilibrium. In this case a well description is found in [5]. However, as soon as the two populations are linked together within a interaction network, with $\theta_{12}\theta_{21} < 0$, dynamical frustration is generated between the two populations. If $\sigma > 0$ and large enough, diffusion in enhanced and the interaction terms start to play a significant role. Indeed the rest state of the first community is not compatible with the rest position of the second. Thus, as a consequence, the dynamics does not settle down to a fixed equilibrium and keeps oscillating. While, when $\sigma = 0$ and all the particles in a same population share the same initial condition (as in the simulations), the system is attracted to a critical point where $x_i^{(N)} = y_j^{(N)} = m_1^{(N)} = m_2^{(N)}$ for all $i = 1, \dots, N_1$ and $j = 1, \dots, N_2$ and, thus, the interaction's terms in (1.1) vanish. This makes noise responsible for the emergence of a stable periodic rhythm and, hence, the occurring of the phenomenon of noise-induced periodicity.

In particular we argued that oscillations appear for an intermediate amount of noise. Our analysis goes as follows. First, we derive the large volume limit of the system and show that it has no periodic behaviour in absence of noise. Second, we increase noise and study our system trough numerical simulations. Such simulations shows clearly that self-organized periodic behaviour appear for an intermediate size of the noise to disappear when the noise is too large. Finally, to explain rigorously observed behaviour we prove a small noise approximation. We reduce the system of interest to a system of ODEs for the time evolution of the means and the variances of the two communities. For this system we are able to prove there is a Hopf bifurcation, which explain at least qualitatively the result of numerical simulations.

Appendix A

Code

A.1. Euler's algorithm for SDEs

Consider the one-dimensional ordinary differential equation $\dot{\mathrm{x}} = \mathrm{A}(\mathrm{x}, t)$ with initial conditions $\mathrm{x}(0) = \mathrm{x}_0$ in the time interval $[0, 1]$. Let's choose our step size h and the number of iteration we want to perform N , which define the grid points $t_n^h := nh, 0 \leq n \leq N$. Then a numerical solution x^h can be found with the following inductive scheme:

$$\mathrm{x}^h(0) = \mathrm{x}_0 \qquad \mathrm{x}_{n+1}^h = \mathrm{x}_n^h + \mathrm{A}(\mathrm{x}_n^h, t_n^h)h$$

Let's consider now the stochastic differential equation

$$dX(t) = \mathrm{A}(X(t), t)dt + \mathrm{B}(X(t), t)dW(t) \tag{A.1}$$

with initial conditions $X(0) = X_0$, where $\{W(t)\}_{t\geq 0}$ is the standard Brownian motion. The meaning of (A.1) is

$$X(t) = X_0 + \int_0^t \mathrm{A}(X(s), s)ds + \int_0^t \mathrm{B}(X(s), s)dW(s) \qquad t \geq 0 \tag{A.2}$$

where the second integral has to be interpreted in the Itô sense, while the first one is a standard integral. Like before, with the step-size h and the number of iterations we construct the grid points $t_n^h := nh$. We recall that, since $\{W(t)\}_{t\geq 0}$ is a Brownian motion, $W(t) - W(s)$ are independent normal distribution for every $t \geq s \geq 0$: $W(t) - W(s) \sim \mathcal{N}(0, t-s)$. Therefore, $\Delta_n^h W := W(t_n^h) - W(t_{n-1}^h)$ are i.i.d. $\mathcal{N}(0, h)$. So, we can implement a numerical solution X^h to (A.1):

$$X^h(0) = X_0 \qquad X_{n+1}^h = X_n^h + \mathrm{A}(X_n^h, t_n^h)h + \mathrm{B}(X_n^h, t_n^h)\Delta_n^h W$$

The following Mathematica script uses this method to find a numerical solution to the system (1.1).

Mathematica Session

```
In[1]:= sdesim[n1_,n2_,α_,θ_,dt_,iter_,σ_,z0_] := Module[{m1m2k,cix,ciy,avg,
          ci,sim},

        cix = Table[z0[[1]],{i,1,n1}];
        ciy = Table[z0[[2]],{i,1,n2}];
        ci = {cix,ciy}; avg = {}; m1m2k = {};

        fx[x_,m1_,m2_] := -x^3 + x - αθ[[1,1]](x - m1) - (1 - α)θ[[1,2]](x - m2);
        fy[y_,m1_,m2_] := -y^3 + y - αθ[[2,1]](y - m1) - (1 - α)θ[[2,2]](y - m2);
        F[z_] := Module[{m1,m2},
          m1 = Mean[z[[1]]];m2 = Mean[z[[1]]];
          AppendTo[m1m2k,{m1,m2}];
          {fx[z[[1]],m1,m2],fy[z[[2]],m1,m2]}
          ];

        AlgEulero[h_][z_] := z + hF[z] +
          σ {RandomVariate[NormalDistribution[0,√h],n1],
             RandomVariate[NormalDistribution[0,√h],n2]};

        Monitor[Do[
          Clear[sim];
          sim = NestList[AlgEulero[dt],ci,iter/100];
          AppendTo[avg,m1m2k];
          m1m2k = {}; ci = sim[[-1]],
          {n,100}],ProgressIndicator[n,{1,100}]
        ];
        Flatten[avg,1]
        ]
```

List of Figures and Tables

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