Universitá degli Studi di Paova

Dipartimento di Matematica "Tullio Levi-Civita"

Corso di Laurea Magistrale in Matematica

Combinatorical Algebraic Topology: dualities for Manifolds with boundary

Relatore

Prof. Bruno Chiarellotto

Correlatore

Prof. Federico Moro

Candidato

Daniele Tramontano

Mat. 1236328

Anno Accademico 2019/2020

11 Dicembre 2020

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Introduction

During the last years techniques from Algebraic Topology have been applied to a variety of fields ranging from data analysis **(30)** to the construction of numerical methods for PDE's **(28)**.
In this thesis we will focus on this second kind of application. In particular we will deal with spaces (manifolds) which admits a "triangularization" and then we will use the combinatoric of the data to deal with the problems. In fact we will show how to create a model that solves a Poisson problem in dimension 2: we will translate the smooth geometric techniques of Differential Geometry in discrete form via Combinatorical Algebraic Topology .
The methods that we study go under the name of "cell methods", and have been used mostly in computational electromagnetism **(32; 33)**. In this framework one uses the identification between de Rham and simplicial cohomology and then in this way interpretates differential forms in terms of co-chains in the simplicial setting. As a dividend of such a construction it is possible to write a set of algebraic equations that representes the laws of nature (this construction is explained in the third chapter of **(32)** for the case of Maxwell's equations). One of the paradigms of this method is the necessity of introducing two different meshes that are oriented in a different way in order to discretize "natural " and "twisted" differential forms (the difference between these two kinds of differential forms consists of the fact that the latter changes sign when the orientation of the space changes while the former does not, motivations for the introduction of the "twisted" differential forms can be found in **(27; 32; 33; 34)**). This approach although gives a precise description of the physical quantities has a drawback from the computational point of view. This is due to the fact that the dual mesh gives not rise to a simplicial complex and so it is not possible to build the Whitney forms on it.
What we propose in this thesis is an approach to the cell method based on a single mesh. We show that it is possible to find a discrete counter part of all the smooth operators working just on one mesh. We find converge results for the discrete operators although we notice that the definition of the Hodge star operator is problematic from a computational point of view since it involves a non perfect pairing **(3.2.0.3)**.
The thesis is divided in three chapters:
•) In the first chapter we recall the mathematical background needed in the rest of the thesis. We introduce the concepts of (co)homology and duality, and in the last paragraph we define some concept of Riemannian Geometry that we

will need later on.

•) In the second chapter we will propose an extension of Wilson's work **(23)** to manifold with boundary. Wilson has introduced a discrete interpretation of the differential constructions as Hodge operator and codifferential which we have introduced in chapter 1 once the manifold has a simplicial structure. We will extend his work to the manifold with boundary and we prove that the operators that we define in the discrete setting converges to the smooth ones when mesh goes to zero.

•) In the last chapter we describe a model to solve a Poisson problem in dimension 2 (which later we have found to be equivalent to that one described in **(31)**). We test it on a problem of which we know the exact solution and we prove that the approximate solution converges to the analytic solution when the mesh goes to 0. Moreover we compare our result with the one given by the Matlab function *"AssemPde"* and we see that the convergence rates of the two methods are the same.

Acknowledgements

I would like to express my gratitude to my advisor *Prof. Bruno Chiarellotto*, who guided me throughout this thesis.
I would also like to thank my co-advisor *Prof. Federico Moro*, for the helpful discussions and the remarkable personal involvement he put in this work.

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1 Mathematical Background

In this chapter, I will briefly present all the basic concepts that we will need during the rest of the thesis.
I will often skip the proofs, but in those cases, I will indicate some reference where it is possible to find them.

1.1 A Glimpse on Category Theory and Homological algebra

We can say (co)homology theory is a way to associate to some topological space a sequence of abelian grups, in such a way that these associations respects a "natural" set of rules.
In order to make this definition formal we need a framework in which we can work, for this reason in this paragraph we will briefly introduce the basic notions of Category theory and Homological Algebra.
Main references for this paragraph are **(9; 13)**.

Definition 1.1.0.1. *A category $\mathcal{C}$ is made by A class of objects $Ob(\mathcal{C})$ and for every $X, Y \in Ob(\mathcal{C})$ a set of morphisms that satisfies the following composition law " $\circ$ ":*
$\forall f \in Hom_{\mathcal{C}}(X,Y)$ and $\forall g \in Hom_{\mathcal{C}}(Y,Z)$ $\exists!$ $g \circ f \in Hom_{\mathcal{C}}(X,Z)$ such that the composition operation is associative and $\forall X \in Ob(\mathcal{C})$, $\exists! 1_X \in Hom_{\mathcal{C}}(X,X)$ such that $f \circ 1_X = f$ $\forall f : X \to Y$ and $1_X \circ g = g$ $\forall g : Y \to X$.

Example 1.1.0.2. *The simpler example of category is <u>Set</u> where the elements are sets and morphisms are morphisms between sets.*
Other useful categories are:
1) *<u>Top</u> where the objects are topological spaces and the morphisms are continuous maps.*
2) *<u>$R - Mod$</u> where R is a ring, the elements are left R modules and the morphisms are R-linear maps.*
3) *<u>Ab</u> where, the elements are abelian groups and the morphisms are group morphisms.*

Remark 1.1.0.3. *Most of the notion that we will define are not defined in every category, so during the whole section we suppose that all the categories are Abelian.*
*The definition of Abelian Category can be found in **(13)**, **Definition A.4.2**.*

Definition 1.1.0.4. *A covariant functor F between two categories $\mathcal{C}$ and $\mathcal{D}$ associate to each $C \in Ob(\mathcal{C})$ a unique element $F(C) \in Ob(\mathcal{D})$ and $\forall f \in Hom_{\mathcal{C}}(X,Y)$ a unique element $F(f) \in Hom_{\mathcal{D}}(F(X), F(Y)$(a functor is said to be contravariant if $F(f) \in Hom_{\mathcal{D}}(F(Y), F(X))$.*

Definition 1.1.0.5. *A congruence in a category $\mathcal{C}$ is an equivalence relation on $Hom_{\mathcal{C}}(X,Y)$ $\forall X,Y \in Ob(\mathcal{C})$ such that if $f \sim f^{'}$ in $Hom_{\mathcal{C}}(X,Y)$ and $g \sim g^{'}$ in $Hom_{\mathcal{C}}(Y,Z)$ then $g \circ f \sim g^{'} \circ f^{'}$ in $Hom_{\mathcal{C}}(X,Z)$.*

Example 1.1.0.6. *The most important example of congruence in a category is the homotopy relation in $\underline{Top}$.*
Two maps $f, g : X \to Y$ are homotopic if exists $F : X \times I \to Y$ such that $F(-0) = f$ and $F(-,1) = g$, where $I = [0,1]$. Moreover if there are two maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim Id_Y$ and $g \circ f \sim Id_X$ then the spaces X and Y are said to be homotopic. $\underline{hTop}$ is the category that has topological spaces as objects and $Hom_{\underline{hTop}}(X,Y) = Hom_{\underline{Top}}(X,Y)/\sim$ where $\sim$ is the homotopy relation.

Definition 1.1.0.7. *An exact sequence in a category $\mathcal{C}$ is composed by a set of objects $\{C_i\}_{i\in\mathbb{Z}}$ and a set of morphisms $\{f_i : C_i \to C_{i+1}\}_{i\in\mathbb{Z}}$ such that $Im(f_i) = Ker(f_{i+1})$.*
A short exact sequence is an exact sequence of the shape:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{1}$$

Definition 1.1.0.8. *A chain complex over a category $\mathcal{C}$ is a pair $(X_i, d_i)_{i\in\mathbb{Z}}$ where $X_i \in Ob(\mathcal{C})$ and $d_i : X_i \to X_{i-1}$ such that $d_{i-1} \circ d_i = 0$.*
A morphism between two complexes $(X_i, d_i^X)_{i\in\mathbb{Z}}$ and $(Y_i, d_i^Y)_{i\in\mathbb{Z}}$ is given by a sequence of morphisms $\phi_i : X_i \to Y_i$ such that $\phi_{i-1} \circ d_i^X = d_i^y \circ \phi_i$ i.e the following diagram is commutative:

$$\begin{array}{ccccccc} \dots & \xrightarrow{d_{i+1}^X} & X_i & \xrightarrow{d_i^X} & X_{i-1} & \xrightarrow{d_{i-1}^X} & \dots \\ & & \downarrow{\phi_i} & & \downarrow{\phi_{i-1}} & & \\ \dots & \xrightarrow{d_{i+1}^Y} & Y_i & \xrightarrow{d_i^Y} & Y_{i-1} & \xrightarrow{d_{i-1}^Y} & \dots \end{array} \tag{2}$$

Definition 1.1.0.9. *Given a chain complex (X_i, d_i) over a category $\mathcal{C}$ the n^{th} homology of the complex is $H_n(X_*) = \frac{Ker(d_n)}{Im(d_{n+1})} \in \mathcal{C}$.*

Lemma 1.1.0.10. *Any morphism of complexes induces naturally a morphisms between the homology groups of the complexes.*

Proof. Let $\phi_* : X_* \to Y_*$ be a morphism of complexes it's enough to prove that:
1) Let $a \in ker(d_n^X)$ then $\phi_n(a) \in ker(d_n^Y)$.
2) Let $b \in Im(d_{n+1}^X)$ then $\phi_n(b) \in Im(d_{n+1}^Y)$.
The commutativity of the diagram **(2)** implies that $d_n^Y(\phi_n(a)) = \phi_n(d_n^X(a)) = 0$ and this proves **1**.
Take $c \in X_{n+1}$ such that $d_{n+1}^X(c) = b$ then $d_{n+1}^Y(\phi_{n+1}(c)) = \phi_{n+1}(d_{n+1}^X(c)) =$ $= \phi(b)$ and this proves **2**. □

Theorem 1.1.0.11 (Zig-Zag lemma)**.** *Any time we have a short exact sequence of morphism:*

$$0 \to X_* \to Y_* \to Z_* \to 0 \tag{3}$$

we will have a long exact sequnce in homology of the shape:

$$.. \to H_n(X_*) \to H_n(Y_*) \to H_n(Z_*) \xrightarrow{\delta} H_{n-1}(X_*) \to ... \tag{4}$$

and the morphisms δ are called connecting morphisms.

Proof. Theorem 1.3.1 in **(13)**. □

Proposition 1.1.0.12. *A morphism between two short exact sequences of chain complexes:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{\alpha} & B_* & \xrightarrow{\beta} & C_* & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & X_* & \xrightarrow{\alpha'} & Y_* & \xrightarrow{\beta'} & Z_* & \longrightarrow & 0 \end{array} \tag{5}$$

induces a commutative diagram on the homology groups of the shape:

$$\begin{array}{ccccccccccc} ... & \longrightarrow & H_n(A_*) & \longrightarrow & H_n(B_*) & \longrightarrow & H_n(C_*) & \longrightarrow & H_{n-1}(A_*) & \longrightarrow & ... \\ & & \downarrow H_n(f) & & \downarrow H_n(g) & & \downarrow H_n(h) & & \downarrow H_n(f) & & \\ ... & \longrightarrow & H_n(X_*) & \longrightarrow & H_n(Y_*) & \longrightarrow & H_n(Z_*) & \longrightarrow & H_{n-1}(X_*) & \longrightarrow & ... \end{array} \tag{6}$$

Proof. Proposition 1.3.4 in **(13)**. □

Remark 1.1.0.13. *The notion of chain complex have a natural dual that is the notion of cochain complex, the latter is defined in the same of the former with the only difference that the arrows are reversed i.e $\delta^n : X^n \to X^{n+1}$, in this case we define the cohomology groups of the cochain complex as $H^n(X^*) = \frac{Ker(\delta^n)}{Im(\delta^{n+1})}$, of course the same theorems above hold with the only caution of reversing the arrows.*

1.2 (Co)Homology Theories

Now that we have a formal framework in which we can we work, we can define the (co)homology theories that we will need during the thesis, these theories will be defined as sequences of (contra)covariant functors from some category of topological spaces to $\underline{Ab}$.
Main references for this paragraph are **(5; 10; 6)**.

1.2.1 Simplicial Complexes

Simplexes and simplicial complexes are crucial tools both for Simplicial and Singular (co)homology theories, for this reason this section will be used to introduce these objects and to explain their fundamental properties.

Definition 1.2.1.1 (Convex envelope)**.** *The convex envelope of the points* $p_0, ..., p_k$ *in* $\mathbb{R}^N$ *with* $k \leq N$ *is the set of points* $\{\sum_{i=0}^{k} \lambda_i p_i | \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geq 0\}$ *and it is denoted by* $[p_0, ..., p_k]$

Definition 1.2.1.2. *The points* $\{v_0, .., v_k\}$ *in* $\mathbb{R}^N$ *are affine independent if* $\{v_1 - v_0, .., v_k - v_0\}$ *are linearly independent, in this case they span a k-simplex* $s = [v_0, ..., v_k]$*, and the* $\{v_1 - v_0, .., v_k - v_0\}$ *is the set of vertices of* s *and it is denoted by* $Vert(s)$*.*

Remark 1.2.1.3. *The concept of simplex is easy to visualize in fact for* $k = 0$ *it is a point, for* $k = 1$ *it is a line segment, for* $k = 2$ *it is a triangle, for* $k = 3$ *it is a tetrahedron and so on..*

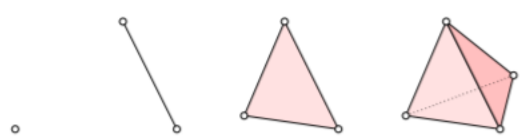

Figure 1: 0,1,2,3-simplexes.

Notation 1.2.1.4. *If* $s = [v_0, .., v_k]$ *is a simplex :*
k *is the dimensions of* s*.*
A face of s *is a simplex* s' *such that* $Vert(s') \subset Vert(s)$ *in this case we write* $s' \leq s$*, if* $Vert(s') \subsetneq Vert(s)$ s' *is a proper face of* s *and we write* $s' \lneq s$*.*
The union of the proper faces of s *is the boundary of* s *and it is denoted by*

Bd(s).
The interior of s is denoted by Int(s) and it is the complement is s of the boundary, sometimes it is called an open simplex.

Definition 1.2.1.5 (Simplicial Complex)**.** *A simplicial complex K is a collection of simplexes in some euclidean space such that:*
1*) if $s \in K$, then every face of s also belongs to K.*
2*) if $s, t \in K$ then $s \cap t$ is either empty or a common face of s and t.*
We write $Vert(K)$ to denote the vertices set of K and $|K|$ is the union of all the simplexes in K and it is called "polytope pf K".

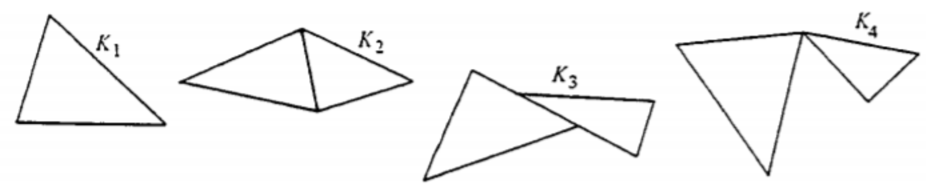

Figure 2: K_1, K_2, K_4 are simplicial complexes while K_3 is not.

Definition 1.2.1.6 (Subcomplex)**.** *If L is a subcollection of K that contains all faces of its elements, then L is a simplicial complex in its own right; it is called a subcomplex of K. One subcomplex of K is the collection of all simplexes of K of dimension at most p; it is called the p-skeleton of K and is denoted $K^{(p)}$. Clearly $K^{(0)} = Vert(K)$.*

Definition 1.2.1.7 (Trinagulation)**.** *A topological space X is a polyhedron if there exists a simplicial complex K and a homemorphism $h : |K| \to X$. The ordered pair (K, h) is called a triangulation of X.*

Definition 1.2.1.8 (Star of a simplex)**.** *If σ is a simplex of K, the star of σ in K, denoted by $St(\sigma, K)$, is the union of the interiors of those simplexes of K that have σ as a face. Its closure, denoted $\overline{St(\sigma, K)}$, is called the closed star of σ in K. It is the union of all simplexes of K having σ as a vertex, and is the polytope of a subcomplex of K. The set $\overline{St(\sigma, K)} \setminus St(\sigma, K)$ is called the link of σ in K and is denoted $Lk(\sigma, K)$ (When the simplicial complex is clear from the context the K is removed from the notation).*

Definition 1.2.1.9 (Simplicial map)**.** *Let K and L be simplicial complexes. A simplicial map $\phi : k \to L$ is function $\phi : Vert(K) \to Vert(L)$ such that any time $\{p_0, .., p_k\}$ spans a simplex in K then $\{\phi(p_0), .., \phi(p_k)\}$ spans a simplex in L.*

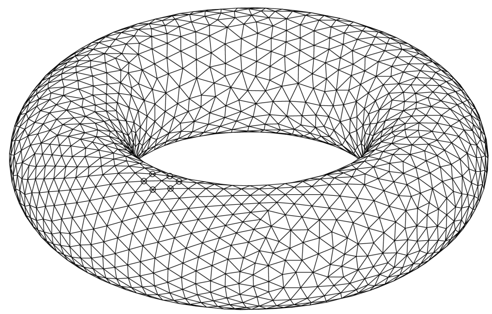

Figure 3: Triangulation of a torus

Theorem 1.2.1.10. *If $\mathcal{K}$ consists of all simplicial complexes and all simplicial maps (with usual composition), then it is a category, and underlying defines a functor $|\ \ | : \mathcal{K} \to \underline{Top}$.*

Proof. Theorem 7.2 in **(10)**. ☐

Definition 1.2.1.11 (Orientation)**.** *An oriented simplicial complex K is a simplicial complex with a partial order on $Vert(K)$ whose restriction to the vertices of any simplex in K is a total order.*

Definition 1.2.1.12 (Subdivision)**.** *Let K be a simplicial complex. A simplicial complex K' is said to be a subdivision if :*
1) Each simplex of K' is contained in a simplex of K.
2) Each simplex of K equals the union of finitely many simplexes of K'.
These conditions trivially imply that $|K| = |K'|$.

Definition 1.2.1.13 (Cone)**.** *Suppose that K is a simplicial complex in $\mathbb{R}^l$, and w is a point of $\mathbb{R}^l$ such that each ray emanating from w intersects $|K|$ in at most one point. We define the cone on K with vertex w to be the collection of all simplexes of the form $[w, a_0, .., a_p]$, where $[a_0, ..., a_p]$ is a simplex of K, along with all faces of such simplexes. We denote this collection $w * K$.*

Definition 1.2.1.14. *Let K be a complex; suppose that L_p, is a subdivision of the p-skeleton of K. Let σ be a $p+1$-simplex of K. The set $Bd(\sigma)$ is the polytope of a subcomplex of the p-skeleton of K, and hence of a subcomplex of L_p; we denote the latter subcomplex by L_σ. If w_σ, is an interior point of σ, then the cone $w_\sigma * L_\sigma$ is a simplicial complex whose underlying space is σ. We define L_{p+1} , to be the union of L_p, and the complexes $w_\sigma * L_\sigma$, as σ ranges over all*

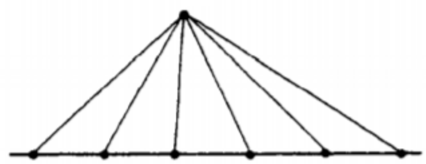

Figure 4: Cone of a vertex along an edge.

$p+1$-simplexes of K. Can be shown that L_{p+1}, is a complex; it is said to be the subdivision of K^{p+1} obtained by starring L_p from the points w_σ.

Definition 1.2.1.15 (Barycenter)**.** *If $\sigma = [v_0, .., v_p]$, the barycenter of σ is $\hat{\sigma} = \sum_{i=0}^{p} \frac{1}{p+1} v_i$.*

Lemma 1.2.1.16. *If K is a complex, then the intersection of any collection of subcomplexes of K is a subcomplex of K. Conversely, if K_α is a collection of complexes in $\mathbb{R}^l$, and if the intersection of every pair $|K_\alpha| \cap |K_\beta|$ is the polytope of a simplicial complex that is a subcomplex of both K_α and K_β, then the union $\bigcup_\alpha K_\alpha$ is a complex.*

Proof. If $\{K_\alpha\}$ is a collection of subcomplexes of K then for every $s, t \in \bigcap K_\alpha$ then all their faces are in $\bigcap K_\alpha$ and since $s \cap t$ is a common face in every K_α it is a common face in $\bigcap K_\alpha$ and this proves the first part of the lemma.
If $s \in \bigcup_\alpha K_\alpha$ then in particular $s \in K_\alpha$ for some α so all the faces of s are in $\bigcup_\alpha K_\alpha$, moreover if $s, t \in \bigcup_\alpha K_\alpha$ then if the they belong to the same K_α trivially they intersect in a common face, while if $s \in K_\alpha$ and $t \in K_\alpha$ their intersection is in $|K_\alpha| \cap |K_\beta|$ so it is a subcomplex of both K_α and K_β so in particular is a common face in $\bigcup_\alpha K_\alpha$. □

Definition 1.2.1.17 (Barycentric Subdivision)**.** *We define a sequence of subdivisions of the skeleton of K in the following way:*
1) *$L_0 = K^{(0)}$.*
2) *L_{p+1} is the be the subdivision of the $p+1$-skeleton obtained by starring L_p from the barycenters of the $p+1$-simplexes of K.*
The union of the L_p is called first barycentric subdivision of K and it is denoted by $sd(K)$.

Lemma 1.2.1.18. *The simplicial complex $sd(K)$ equals the collections of all simplexes of the form*
$[\hat{\sigma}_1, .., \hat{\sigma}_p]$ such that $\sigma_1 > \sigma_2 > .. > \sigma_p$.

Proof. The proof proceeds by induction on p.
If $p = 0$ it is trivial since $v = \hat{v}$.
Suppose that each simplex of $sd(K)$ lying in $|K^{(p)}|$ is of this form. Let τ be a simplex of $sd(K)$ lying in $|K^{(p+1)}|$ and not in $|K^{(p}|$. Then τ belongs to some $\hat{\sigma} * L_\sigma$ with σ being a $p+1$ simplex of K and L_σ the first barycentric subdivision of the complex consisting of the proper faces of σ, so using the induction hypotesis and the definition of $*$ we are done. □

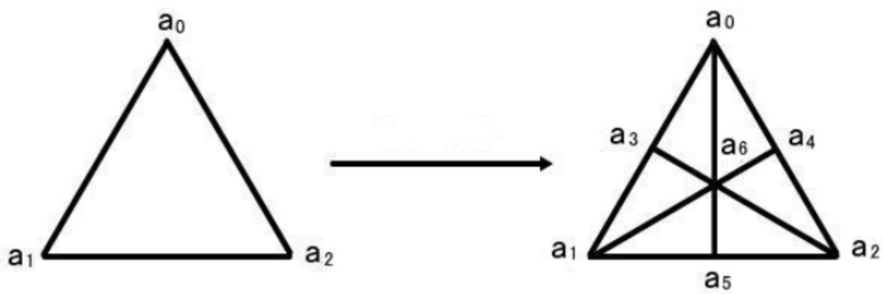

Figure 5: Barycentric subdivision of a 2-simplex.

1.2.2 Simplicial (co)Homology

In this section we define the first of our (co)homology theories.
As we will see all the groups that are needed in order to define this theory are finite dimensional.
In **Chapter 3** we will use this fact to find a method of solving a Poisson' system.

Definition 1.2.2.1 (Simplicial Chains)**.** *If K is an oriented simplicial complex and $q \geq 0$ let $C_q(K)$ be the abelian group with the following presentation.*
Generators: *the $q+1$-tuples $(p_0, ..., p_q)$ of vertices of K such that $\{p_0, ..., p_q\}$ spans a q-simplex in K.*
Relations: $(p_0, ..., p_q) = 0$ *if some vertex is repeated and*
$(p_0, ..., p_q) = (sgn\sigma)(p_{\sigma(0)}, .., p_{\sigma(q)})$ *for any σ permutation of* $\{0, .., q\}$.

Notation 1.2.2.2. *The element of $C_q(K)$ corresponding to $(p_0, .., p_q)$ is denoted by $< p_0, .., p_q >$.*

Lemma 1.2.2.3. *Let K be an oriented simplicial complex of dimension m.*
1) *$C_q(K)$ is a free abelian group with basis all symbols $< p_0, ..., p_q >$ where $\{p_0, .., p_q\}$ span a q-simplex in K with $p_0 < p_1 < .. < p_q$.*
2) *$C_q(K) = 0$ if $q > m$.*

Proof. Lemma 7.10 in **(10)**. □

Definition 1.2.2.4 (Simplicial Boundary Operator)**.** *We define the boundary morphisms in the following way*

$$\partial_n : C_q(X) \to C_{q-1}(X) \qquad < p_0, .., p_q > \mapsto \sum_{i=0}^{q} (-1)^i < p_0, .., \hat{p}_i, ..p_q > \tag{7}$$

Of course can be easily proved that $(C_*(K), \partial_*)$ *is a chain complex and the homology groups of this chain complexes will be called simplicial homology groups of* K *and denoted by* $H_q(K, \mathbb{Z})$*.*

Remark 1.2.2.5. *It is possible to show that once we have a simplicial map* $\phi : K \to L$ *this induces a morphisms of chain complexes* $\phi_* : C_*(K) \to C_*(L)$*, this is defined in a natural way, in fact sends the element* $< p_0, .., p_q >$ *of* $C_q(K)$ *to the element* $< \phi(p_0), .., \phi(p_q) >$ *of* $C_q(L)$*.*
As a consequence of this fact we have that the H_n *are covariant functors from* $\mathcal{K}$ *to* $\underline{Ab}$ *(this fact is proved in Theorem 7.13 of* ***(10)****).*

Definition 1.2.2.6 (Reduced Simplicial Complex)**.** *Define* $C_1(K)$ *as the free abelian group generated by the symbol* $<>$ *and* $\partial_0 : C_0(K) \to C_{-1}(K)$ *such that* $\tilde{\partial}_0(\sum m_p < p >) = (\sum m_p) <>$*, and define the augmented simplicial complex:*

$$\tilde{C}_*(K) = 0 \to C_m(K) \xrightarrow{\partial_m} .... \xrightarrow{\partial_1} C_0(K) \xrightarrow{\tilde{\partial}_0} C_{-1}(K) \to 0 \tag{8}$$

We define the reduced simplicial homology groups by $\tilde{H}_q(K) = H_q(\tilde{C}_*(K))$

Definition 1.2.2.7 (Relative Homology)**.** *If* K_0 *is a subcomplex of* K*, the quotient group* $\frac{C_q(K)}{C_q(K_0)}$ *is called the group of relative chains of* K *modulo* K_0*, and is denoted by* $C_q(K, K_0)$*, clearly it is free abelian and it is generated by the equivalnce classes of the q-simplexes of* K *that are not in* K_0*. Obviouvsly the boundary operator* ∂ *is well defined on the relative chain and so gives rise to a chain complex* $(C_*(K, K_0), \partial_*)$ *and the homology of this chain is called relative homology and it is denoted by* $H_q(K, K_0)$*.*

Proposition 1.2.2.8 (Long exact sequence)**.** *If* K_0 *is a subcomplex of* K *we have along exact sequence in homology:*

$$.. \to H_n(K_0) \to H_n(K) \to H_n(K, K_0) \xrightarrow{\delta} H_{n-1}(K_0) \to ... \tag{9}$$

Proof. Clearly the inclusion map $i : K_0 \to K$ is a simplicial map and so induces a short exact sequnce:

$$0 \to C_*(K_0) \xrightarrow{i_*} C_*(K) \xrightarrow{\pi_*} C_*(K, K_0) \to 0 \tag{10}$$

where $\pi_* : C_*(K) \to \frac{C_*(K)}{C_*(K_0)}$ is the natural projection morphism, using the Zig-Zag lemma this induces the long exact sequence that we were looking for. $\square$

Theorem 1.2.2.9 (Exact sequence of triple)**.** *If K_1 is a subcomplex of K_0 that is a subcomlex of K we have a long exact sequence:*

$$... \to H_n(K_0, K_1) \to H_n(K, K_1) \to H_n(K, K_0) \xrightarrow{\partial} H_{n-1}(K_0, K_1) \to ... \tag{11}$$

Proof. Theorem 5.9 in **(10)**. $\square$

Theorem 1.2.2.10 (Excision)**.** *Let K be a complex; let K_0 be a subcomplex. Let U be an open set contained in $|K_0|$ such that $|K| \setminus U$ is the polytope of a subcomplex L of K. Let L_0 the subcomplex of $|K_0|$ whose polytope is $|K_0| \setminus U$ then the inclusion induces an isomorphism:*

$$H_q(L, L_0) \cong H_q(K, K_0) \tag{12}$$

Proof. Theorem 9.1 in **(5)**. $\square$

Theorem 1.2.2.11 (Mayer-Vietoris)**.** *Let K be a complex; let K_0 and K_1 be subcomplexes such that $K = K_0 \cup K_1$. Let $A = K_0 \cap K_1$. Then there is an exact sequence*

$$... \to H_p(A) \to H_p(K_0) \oplus H_p(K_1) \to H_p(K) \to H_{p-1}(A) \to ... \tag{13}$$

Proof. Theorem 25.1 in **(5)**. $\square$

Definition 1.2.2.12 (Simplicial cochains)**.** *Let K be simplicial complex. The group of p-dimensional cochains of K, with coefficients in G, is the group*
$C^p(K, G) = Hom(C_p(K), G)$.
We can define a coboundary opeator $\delta : C^p(K, G) \to C^{p+1}(K, G)$ defined in such a way that if c^p is p-cochain and c_p a p-chain and $< c^p, c_p >$ the evaluation of c^p at c_p then $< \delta c^p, c_{p+1} >=< c^p, \partial c_{p+1} >$, clearly the coboundary operator inherits from the boundary operators the fact that $\delta \circ \delta = 0$, so (C^*, δ^*) is a cochain complex and the cohomology of the complex is the simplicial cohomology of K with coefficients in G and it is denoted by $H^p(K, G)$.

Remark 1.2.2.13. *If we have a simplicial map $\phi : K \to L$ this clearly define a map $\phi^* : C^*(L) \to C^*(K)$ defined in such a way that*
$< \phi^p(c^p), c_p >=< c^p, \phi_p(c_p) >$ where $c^p \in C^p(L)$ and $C_p \in C_p(K)$, this proves that H^* are contravariant functors from $\mathcal{K}$ to $\underline{Ab}$.
With the same computation we did for Homology we can build relative cohomology groups and we have also the long exact sequence, excision property and

Mayer-Vietoris (but with the arrows reversed). (the details are in Chapter 5 of (4)).

1.2.3 Singular Homology

In this section we define Singular (co)homology theory, this is the most general (co)homology theory that we have, indeed it is defined on every topological space X and turn out to be equivalent to the Simplicial (co)homology theory when the space X is triangulable.

Definition 1.2.3.1 (Standard n-simplex)**.** *A standard n-simplex is the set $\Delta_n = \{(x_0, ..., x_n)| \sum_{i=0}^{k} x_i = 1, x_i \geq 0\}$, this can be seen as the convex envelope $[e_0, ..., e_k]$ of the points $e_0 = (1, 0, ..., 0), .., e_n = (0, ..., 0, 1)$ in $\mathbb{R}^{n+1}$, in particular for $n = 0$ is a point, for $n = 1$ is a segment, for $n = 2$ is a triangle, and for $n = 3$ is a tetrahedron.*

Definition 1.2.3.2 (Face)**.** *Let $\Delta_n = [e_0, ..., e_n]$ a face of Δ_n is $[e_0, .., e_{i-1}, \hat{e}_i, e_{i+1}, .., e_n]$ (where the hat means that the element is deleted.).*

We can embed Δ_{n-1} into Δ_n using the morphism ϵ_i^{n-1} that sends $(x_0, .., x_{n-1}) \to (x_0, ., x_{i-1}, 0, x_{i+1}, .., x_n)$

Definition 1.2.3.3 (Orientation)**.** *An orientation of $\Delta_n = [e_0, ..., e_n]$ is a linear ordering of its vertices, two orientations are the same if they have the same parity as permutations of $\{e_0, .., e_n\}$ otherwise the orientations are opposite.*
An orientation of Δ_n induces an orientation on the faces in the sense $(-1)^i[e_0, ..., \hat{e}_i, ..., e_n]$,

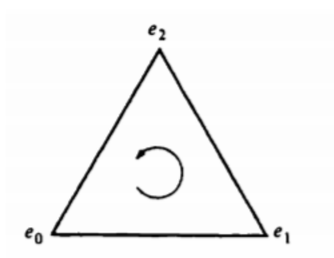

Figure 6: 2simplex with orientation $e_0 < e_1 < e_2$

Definition 1.2.3.4 (Singular complex)**.** *Let X be a topological space. A (singular) n-simplex in X is a continuous map $\sigma : \Delta_n \to X$ where Δ_n is the standard n-simplex.*

Definition 1.2.3.5 (n-chains)**.** *Let X be a topological space. For each $n \geq 0$, define $S_n(X)$ as the free abelian group with basis all singular n-simplexes in X; define $S_{-1}(X) = 0$. The elements of $S_n(X)$ are called (singular) n-chains in X.*

Now we have our groups $S_n(X)$ so in order to build a chain complex we need the morphisms $\partial_n : S_n(X) \to S_{n-1}(X)$, we will call them boundary morphisms.

Definition 1.2.3.6 (Singular Boundary morphisms)**.** *We define the boundary morphisms in the following way:*

$$\partial_n : S_n(X) \to S_{n-1}(X) \qquad \sigma \mapsto \sum_{i=0}^{n} (-1)^i \sigma \circ \epsilon_i^{n-1} \tag{14}$$

∂_n is defined explicitly only on the singular n-simplexes and clearly is extended by linearity on the whole $S_n(X)$.

Proposition 1.2.3.7. *For all $n \in \mathbb{Z}$ we have $\partial_n \circ \partial_{n+1} = 0$.*

Proof. Theorem 4.6 in **(10)**. □

Remark 1.2.3.8. *Thanks to this proposition we know that $(S_*(X), \partial_*)$ is a chain complex so we can build homology groups from this.*

Definition 1.2.3.9 (Singular Homology Group)**.** $H_n^{Sing}(X, \mathbb{Z}) = \frac{ker(\partial_n)}{Im(\partial_{n+1})}$.
We will name the elements of $Z_n(X, \mathbb{Z}) = ker(\partial_n)$ n-cycles and the elements of $B_n(X, \mathbb{Z}) = Im(\partial_{n+1})$ n-boundaries.

Remark 1.2.3.10 (Reduced Singular Homology)**.** *With the same procedure of the **Definition 2.2.2.6** but using the symbol [ ] as generator of $S_{-1}(X)$ we can define the reduced singular complex $\tilde{S}_*(X)$ (this construction is formally explained at Chapter 5 of **(10)**).*

Remark 1.2.3.11. *The Theorem 4.23 of **(10)** proves that H_n is really a covariant functor from hTop to Ab, this is made in two steps, the first is to show that any time we have a map f from two topological spaces X and Y there is a natural chain map from $S_*(X)$ to $S_*(Y)$ induced by f and it is defined sending any generator σ of $S_q(X)$ to $f \circ \sigma$ in $S_q(Y)$, and the second step consist in showing that the induced map preserve the homotopy class, i.e if $f \sim g$ then $H_n(f) = H_n(g)$. Of course thanks to the result that we have seen in the previuos section this induces a map between the homology groups of X and Y.*
Notice that as a direct consequence of the fact that two homotopic maps induce the same morphism in homology we also have that two homotopic spaces have the same homology groups.

Remark 1.2.3.12. *All the concepts that we have defined for simplicial homology like relative homology, long exact sequences, Mayer-Vietoris etc.. are defined in the same way also in the singular case. (look at Paragraph 10.7 of **(2)** for the construction)*

So now if we have an ordered simplicial complex K we can associate to it both $H_*(K, \mathbb{Z})$ and $H_*^{Sing}(|K|, \mathbb{Z})$ and those two classes of functors are *a propri* different, but it is not difficoult to prove that those groups are isomporphic and the isomorphism is induced by the map j_* between the two reduced complexes $\tilde{C}_*(K)$ and $\tilde{S}_*(K)$ defined in the following way:
1) j_{-1} sends $<>$ to $[\quad]$ and is extended by linearity.
2) if $q \geq 0$ $j_q(< p_0, .., p_q >) = \sigma$ where $\sigma : \Delta_q \to |K|$ is the affine map that sends $e_i \to q_i$ and is extended by linearity.
All the details of the isomorphism are given in **(10)**, Theorem 7.22.

Remark 1.2.3.13 (Homology of a contractible spaces)**.** *A space is said to be contractible if it is homotopy equivalent to a point, so in particular if X is contractible space $H_n(X) = H_n(\{pt\}), \quad \forall n \in \mathbb{Z}$.*
*Now using **(1.2.2.3)** we have that $C_q(\{pt\}) = 0, \quad \forall q \gneq 0$ so also $H_q(\{pt\}) = 0, \quad \forall q \gneq 0$, moreover can be proved that for every space X, $H_0(X) = \mathbb{Z}^\alpha$ where α is the cardinality of the set of the path connected components of X (for the proof look at Theorem 4.14 in **(10)**), so in particular $H_0(\{pt\}) = \mathbb{Z}$. So the homology of every contractible space equal to 0 for every $n \gneq 0$ and it is equal to $\mathbb{Z}$ for $n = 0$, it is possible to prove that every convex space is contractible (Theorem 1.7 of **(10)**) so in particular $\mathbb{R}^n$ is so for every n.*

Proposition 1.2.3.14 (Homology of the sphere)**.** *Let $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ be the n dimensional sphere, i.e $\mathbb{S}^n = \{(x_0, .., x_n) \in \mathbb{R}^{n+1} | \sum_i x_i^2 = 1\}$ then:*

$$\begin{cases} H_k(\mathbb{S}^n) = \mathbb{Z}, & k = 0, n \\ H_k(\mathbb{S}^n) = 0, & k \neq 0, n \end{cases} \tag{15}$$

if $n \neq 0$ while :

$$\begin{cases} H_k(\mathbb{S}^0) = \mathbb{Z}^2, & k = 0 \\ H_k(\mathbb{S}^0) = 0, & k \neq 0 \end{cases} \tag{16}$$

Proof. Theorem 6.5 in **(10)**. □

We can notice that there is still some important difference between singular and simplicial homology, the most obvious is that singular homology is defined on every topological space while the simplicial homology is defined just on the spaces that have a triangulation, (there are topological spaces that doesn't admit a triangulation), so singular homology gives rise to a more general theory, but on the other hand from a computational viewpoint simplicial homology is much more useful, this is due mainly to the fact that while the groups $S_q(|K|)$ are infinitely generated, when K is finite (iff $|K|$ is compact) the groups $C_q(K)$ are finitely generated and so the morphisms ∂_q can be represented by matrices and this is clearly an advantage.

1.2.4 Universal Coefficient Theorem

In this section we are going to see how to change the coefficient group of the (co)homology and how to pass from homology to cohomology.

Definition 1.2.4.1 (Tensor product)**.** *Let A and B be abelian groups. Their tensor product, denoted by $A \otimes B$, is the abelian group having the following presentation:*
Generators: $A \times B$, that is, all ordered pairs (a, b).
Relations: : $(a + a', b) = (a, b) + (a', b)$ and $(a, b + b') = (a, b) + (a, b')$ for all $a, a' \in A$ and all $b, b' \in B$.
*The equivalence classes of the pair (a, b) in $A \otimes B$ is denoted by $a \otimes b$. The tensor product has a lot of good properties (that are proved in Chapter 9 of **(10)**) the two that we will need later are:*
***1)** $F \otimes G \cong G$ for every free abelian group F and every abelian group G.*
***2)** Any time that we have two group homomorphisms $f : A \to A'$ and $g : B \to B'$ we can induce a unique group homomorphism $f \otimes g : A \otimes B \to A' \otimes B'$ such that $(f \otimes g)(a \otimes b) = (f(a) \otimes g(b))$.*
***3)** If $B' \xrightarrow{i} B \xrightarrow{p} B" \to 0$ is an exact sequnce of abelian groups then $B' \otimes G \xrightarrow{i \otimes Id_G} B \otimes G \xrightarrow{p \otimes Id_G} B" \otimes G \to 0$ is exact too (the same holds also tensoring on the left).*

Definition 1.2.4.2. *Let (X, A) be a pair of spaces and let G be an abelian group. If $(S_*(X, A), \partial)$ is the singular chain complex of (X, A), then the singular complex with coefliicients G is the complex:*

$$... \to S_{n+1}(X, A) \otimes G \xrightarrow{\partial \otimes Id_g} S_n(X, A) \otimes G \xrightarrow{\partial \otimes Id_g} S_{n-1}(X, A) \otimes G \xrightarrow{\partial \otimes Id_g} ... \tag{17}$$

The n^{th} homology group of (X, A) with coefficients G is the n^{th} homology group of this chain complex.

Remark 1.2.4.3. *Notice that*
$S_n(X) \otimes G = \{\sum g_\sigma \sigma | \sigma : \Delta_n \to X, g_\sigma \in G, g_\sigma = 0 \quad a.e \quad \sigma\}$ *(where a.e means all but finitely many of them).*

Definition 1.2.4.4 (Tor)**.** *For any abelian group A it is possible to build a short exact sequence:*

$$0 \to R \xrightarrow{i} F \to A \to 0 \tag{18}$$

where F is a free abelian group.
For any abelian group B we define $Tor(A, B) = ker(i \otimes 1_B)$. $Tor(-, B)$ is a covariant functor form $\underline{Ab}$ to himself and has a lot of good properties in particular we will use the fact that $Tor(A, B) = 0 \quad \forall B$ if A is torsion free. (Corollary 3.1.5 9 in ***(13)****).*

Theorem 1.2.4.5 (Universal coefficient Theorem for Homology)**.** ***1****) For every space X and every abelian group G, there are exact sequences for all $n \geq 0$:*

$$0 \to H_n(X) \otimes G \xrightarrow{\alpha} H_n(X, G) \to Tor(H_{n-1}(X), G)) \to 0 \tag{19}$$

where $\alpha([(\sigma)] \otimes g) = [\sigma \otimes g]$.
2*) This sequence splits, i.e:*

$$H_n(X, G) \cong (H_n(X) \otimes G) \oplus Tor(H_{n-1}(X), G). \tag{20}$$

Proof. Theorem 9.21 in **(10)**. □

Corollary 1.2.4.6. *If G is the additive group of $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ then:*

$$H_n(X, G) = H_n(X) \otimes G \tag{21}$$

Proof. All those groups are torsion free. □

So now we have a formula that allow us to pass from the homology with coefficients in $\mathbb{Z}$ to the homology with coefficients in any group G, in particular we will use it to find the homology coefficints in $\mathbb{R}$.
Now with a similar argument we are going to find a formula that allows us to pass from homology to cohomology.

Definition 1.2.4.7 (Ext)**.** *For each abelian group A, choose an exact sequence $0 \to R \xrightarrow{i} F \to A \to 0$ with F free abelian, then for every abelian group G we* *can induce an exact sequence:*

$$0 \to Hom(A,G) \to Hom(R,G) \xrightarrow{i^*} Hom(F,G) \tag{22}$$

then we define $Ext(A,G) = cocker(i^*) = \frac{Hom(R,G)}{i^*(Hom(F,G))}$.

Definition 1.2.4.8. *An abelian group G is divisible if, for every $x \in G$ and every integer $n > 0$, there exists $y \in G$ with $ny = x$.*

Remark 1.2.4.9. *The groups $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are divisible, furthermore the functor Ext enjoys the property that $Ext(A,D) = 0$ for every divisible group D and every abelian group A.(for a proof look at Chapter 12 of* ***(10)****)*

Theorem 1.2.4.10 (Dual Universal Coefficient)**.** ***1****) For every space X and every abelian group G, there are exact sequences for all $n \geq 0$:*

$$0 \to Ext(H_{n-1}(X), G) \to H^n(X,G) \xrightarrow{\beta} Hom(H_n(X), G)) \to 0 \tag{23}$$

where $(\beta([\phi]))(z_n + B_n) = \phi(z_n)$.
2*) This sequence splits, i.e:*

$$H^n(X,G) \cong Ext(H_{n-1}(X), G) \oplus Hom(H_n(X), G)). \tag{24}$$

Proof. Theorem 12.11 in **(10)**. □

Corollary 1.2.4.11. *If F is a field of characteristic zero (e.g. $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$), then, for all $n \geq 0$ $H^n(X,F) \cong Hom(H_n(X), F)$*

Proof. Any field of characteristic 0 has a divisible additive group. □

Theorem 1.2.4.12 (Universal coefficient theorem for cohomology)**.** ***1****) For every space X and every abelian group G, there are exact sequences for all $n \geq 0$:*

$$0 \to H^n(X) \otimes G \xrightarrow{\alpha} H^n(X,G) \to Tor(H^{n+1}(X), G)) \to 0 \tag{25}$$

where $\alpha([z] \otimes g) = [zg]$ and $zg : \sigma \mapsto z(\sigma)g$ for any n-simplex in X
2*) This sequence splits, i.e:*

$$H_n(X,G) \cong (H_n(X) \otimes G) \oplus Tor(H_{n-1}(X), G). \tag{26}$$

Proof. Theorem 12.15 in **(10)**. □

1.2.5 De Rham Cohomology

In this section we are going to present a cohomology theory that works on smooth manifolds, this cohomology will turn out to be isomorphic to the singular and simplical cohomology with coefficients in $\mathbb{R}$ and the isomorphism will be given by the integration.
I'm not going to present all the basic notions of differential geometry that can be found in **(11)**.

Definition 1.2.5.1 (Manifold)**.** *An n manifold with boundary is a second countable, Hausdorff topological space M such that $\forall x \in M$ exists U open neighborhood of x, and an homeomorphism ϕ_U from U to an open set of $\mathbb{R}^n$ or $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$, such that $\phi_U \circ \phi_V^{-1} : \phi_V(U \cap V) \to \phi_U(U \cap V)$ is a $\mathcal{C}^\infty$ diffemorphism.*
A pair (U, ϕ_U) is called a chart, while $\{(U_\alpha, \phi_{U_\alpha})\}_{\alpha \in \Delta}$ where $\{U_\alpha\}_{\alpha \in \Delta}$ is an open covering of M is called an atlas.
The boundary of M is the preimage of the points $(0, x_1, ..x_{n-1}) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$, and it is denoted by ∂M, the interior M is $M \setminus \partial M$.
M is manifold without boundary if $\partial M = \emptyset$.
A closed manifold is a compact manifold without boundary.

Definition 1.2.5.2 (Alternate form)**.** *Given a k-vector space V an alternating multilinear n-form is a multilinear map $\alpha : V^n \to k$ such that*
$\alpha(v_{\sigma(1)}, .., v_{\sigma(n)}) = sgn(\sigma)\alpha(v_1, .., v_n)$ for every σ permutation of $\{1, .., n\}$ and every $\{v_1, .., v_n\} \subset V$.
As a consequence of the definition we have that if the characteristic of the field is different from 2, $\alpha(v_1, .., v_n) = 0$ whenever $v_i = v_j$ for $i \neq j$.
The space of alternating multilinear k-forms is a vector space over k and it is denoted as $\mathcal{A}^k(V)$.
We can also define an exterior product $\wedge : \mathcal{A}^r(V) \otimes \mathcal{A}^s(V) \to \mathcal{A}^{r+s}(V)$ such that:

$$(f \wedge g)(v_1, .., v_{r+s}) = \frac{1}{r!s!} \sum_{\sigma \in S_{r+s}} (sgn(\sigma)) f(v_{\sigma(1)}, .., v_{\sigma(r)}) g(v_{\sigma(r+1)}, .., v_{\sigma(r+s)}) \tag{27}$$

for all $f \in \mathcal{A}^r(V)$ and $g \in \mathcal{A}^s(V)$ where S_{r+s} is the set of the permutations of $\{1, .., r+s\}$. The wedge product is bilinear, associative and graded alternating, i.e $f \wedge g = (-1)^{rs} g \wedge f$.
Given $\{e_1, .., e_n\}$ a basis of V and $\{e_1^*, .., e_n^*\}$ the corresponding dual basis of V^*, then a basis of $\mathcal{A}^k(V)$ is the set $\{e_{i_1}^* \wedge .., \wedge e_{i_k}^* | i_1 \lneq .. \lneq i_k\}$ so the dimension of $\mathcal{A}^k(V)$ is $\binom{n}{x}$.

Definition 1.2.5.3 (Differential forms)**.** *If we have a differential smooth n-manifold M, then for every $x \in M$ the tangent space T_xM has the structure of vector space so we can define $\mathcal{A}^k(T_xM)$ and we can define also*
$\mathcal{A}^k M = \coprod_{x\in M} \mathcal{A}^k(T_xM)$*, a smooth section of $\mathcal{A}^k M$ is a differential k form, and the set of differential k-form is denoted by $\Omega^k(M)$.*
Locally any differential k-form can be represented as
$\omega = \omega_{i_1,..,i_k} dx^{i_i} \wedge .. \wedge dx^{i_k}$ *(using Eistein convention) where $\omega_{i_1,..,i_k}$ are $\mathcal{C}^\infty$ maps. (details on this construction are in Chapter 9 of **(11)**).*

Definition 1.2.5.4 (Exterior derivative)**.** *We define the exterior derivative $d^0 : \Omega^k(M) \to \Omega^{k+1}(M)$ inductively:*
1) *If k=0, $\omega \in \Omega^k(M) = \mathcal{C}^\infty(M)$ and $d^0\omega = \frac{\partial \omega}{\partial x_j} dx^j$.*
2) *If $k > 0$, and $\omega = \omega_{i_1,..,i_k} dx^{i_i} \wedge .. \wedge dx^{i_k}$ then*
$d^k(\omega) = d^0(\omega_{i_1,..,i_k}) \wedge dx^{i_i} \wedge .. \wedge dx^{i_k}$

Definition 1.2.5.5 (De Rham Cohomology)**.** *Can be proved that $d^k \circ d^{k+1} = 0$ (Proposition 1.4 of **(4)**) so we can define the De Rham cochain complex:*

$$0 \to \Omega^0(M) \xrightarrow{d^0} \Omega^1(M) \to ... \xrightarrow{d^{k-1}} \Omega^k(M) \to 0 \tag{28}$$

The cohomology groups of the complex are the De Rham cohomology groups of M, and it are denoted $H^k_{dR}(M)$.

Now we want to compare De Rham and simplicial cohomology of a smooth manifold M.
Notice that if we have a p-simplex $\sigma : \Delta^p \to M$ and a p-differential form ω then if σ is a $\mathcal{C}^\infty$ map we can integrate ω over σ in the following way:

$$\int_\sigma \omega = \int_{(\Delta^p)} \sigma^* \omega \tag{29}$$

so if we define $S^\infty_p(M)$ to be the groups of $\mathcal{C}^\infty$ chains , those are subgroups of $S_p(M)$ and are preserved by ∂_* so from this chain complex we can build a C^∞ singular homology $H^\infty_n(M)$.

Theorem 1.2.5.6 (Stokes' Theorem)**.** *For every $\sigma \in S^\infty_k(M, \mathbb{R})$ and $\omega \in \Omega^{k-1}(M)$ we have:*

$$\int_{\partial\sigma} \omega = \int_\sigma d\omega. \tag{30}$$

Thanks to the Stokes' Theorem, we have a morphism of cochain $\int : \Omega^*(M) \to Hom(S^\infty_*(M), \mathbb{R})$, and it is proved in **(12)** (Theorem 3.3) that it

induces an isomorphism in cohomology. Now we want to show that the singular homology groups and the C^∞ singular homology groups are isomorphic.
The second statement is completely proved in **(12)**(Theorem 2.4) and the proof is based on the following lemma.

Lemma 1.2.5.7. *For every k-simplex $\sigma : \Delta_k \to M$ there exists a continuous map $H_\sigma : \Delta_k \times I \to M$ satisfying the following:*
1) H_σ is an homotopy from σ to a smooth k-simplex $\tilde{\sigma}$.
2) For each boundary face inclusion $F_{i,k} : \Delta_{k-1} \hookrightarrow \Delta_k$:

$$H_{\sigma \circ F_{i,k}}(x,t) = H_\sigma(F_{i,k}, t) \tag{31}$$

for $(x,t) \in \Delta k - 1 \times I$
2) If σ is smooth then H_σ is the constant homotopy.

Proof. Lemma 2.3 in **(12)**. □

Theorem 1.2.5.8. *The inclusion $i : S_*^\infty(M, \mathbb{R}) \hookrightarrow S_*(M, \mathbb{R})$ induces an isomorphism in homology.*

Proof. Theorem 2.4 in **(12)**. □

Both the proofs are quite long and technical so I will not provide them but should be clear that thanks to the first statement of the lemma every k-chain lies in the equivalence class of a smooth k-chain so it is reasonable to have the isomorphism.
Now recall that thanks to the Universal Coefficient theorem $Hom(H_k(M), \mathbb{R}) \cong H^k(M, \mathbb{R})$ so using the previous theorem we have:

$$H^k_{dR}(M) \cong Hom(H_k^\infty(M), \mathbb{R}) \cong H^k_\infty(M, \mathbb{R}) \cong H^k(M, \mathbb{R}). \tag{32}$$

Definition 1.2.5.9 (Smooth triangulation)**.** *Let M be a compact manifold then a triangulation $f : |K| \to M$, is said to be smooth if for every n simplex in K exists an open set $\Delta^n \subset U \subset \mathbb{R}^n$ (or $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$) and an extension F of $f_{|\Delta^n}$ to U that is a smooth embedding of U in M.*

Theorem 1.2.5.10 (de-Rham Theorem)**.** *Let M be a compact manifold, possibly with boundary, with a smooth triangulation on M.*
Consider the map $R^k : \Omega^k(M) \to C^k(M, \mathbb{R})$, called de-Rham map, such that $< R(\omega), \Delta^k >= \int_{\Delta^k} \omega$, then according to the Stokes theorem $Rd(\omega) = dR(\omega)$ so R induced a map in cohomology $R^* : H^*_{dR}(M) \to H^*(M)$.
The de-Rham map induces an isomorphism in cohomology.

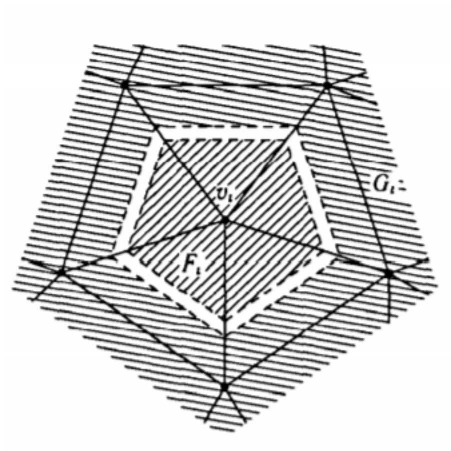

Figure 7: The sets F_i and G_i.

Proof. (Sketch) The full proof is at Section 5.3 of **(17)** in the case of a closed manifold, but applies with no restrictions to the case of a compact manifold with boundary.
We will build an inverse of R^*. Let $Vert(K) = \{v_1, .., v_n\}$ and x_i the barycentric coordinate associated to the vertex v_i (with $x_i = 0$ for any point outside $St(v, K)$. Define $F_i = \{x \in M | x_i \geq \frac{1}{n+1}\}$ and $G_i = \{x \in M | x_i \leq \frac{1}{n+2}\}$, let λ_i a non negative smooth function which is positive on F_i and vanishes on G_i.
The function $\lambda(x) = \lambda_1(x) + .. + \lambda_n(x)$ is non negative and the F_i's cover M since the sum of the barycentric coordinates at any point equals 1, so at least one of the is at least $\frac{1}{n+1}$. Now define $\mu_i(x) = \frac{\lambda_i(x)}{\lambda(x)}$ and associate to each $\sigma^* \in C^k(K)$ dual to the simplex $\sigma = [v_{i_0}, .., v_{i_k}]$:

$$W^k(\hat{\sigma}^k) = k! \sum_{j=0}^{k} (-1)^j \mu_{i_j} d\mu_{i_0} \wedge .. \wedge d\hat{\mu}_{i_j} \wedge .. \wedge d\mu_{i_k} \tag{33}$$

extending W^k by linearity gives a morphism $W^k : C^k(K) \to \Omega^k(M)$, we now want to show that W^* induces a morphism in cohomology and that it is in fact an inverse of R^*, this will be done in many steps.
1) W^* is a cochain morphism.
We have to show that $dW\sigma^* = Wd\sigma^*$ for all $\sigma = [v_{i_0}, .., v_{i_k}]$.
$dW\sigma^* = (k+1)! d\mu_{i_0} \wedge .. \wedge d\mu_{i_k}$ since $(-1)^j d(\mu_{i_j} d\mu_{i_0} \wedge .. \wedge d\hat{\mu}_{i_j} \wedge .. \wedge d\mu_{i_k}) =$
$= (-1)^j d\mu_{i_j} \wedge d\mu_{i_0} .. \wedge d\hat{\mu}_{i_j} \wedge .. \wedge d\mu_{i_k} = d\mu_{i_0} \wedge .. \wedge d\mu_{i_k}$ for every j.
We know from the definition of d that $d(\sigma^*)$ is the sum of the dual of simplexes

of the form $[v_p, v_{i_0}, .., v_{i_k}]$ so:

$$W(d(\sigma^*)) = (k+1)!(\sum_p \mu_p d\mu_{i_0} \wedge .. \wedge d\mu_{i_k} - \sum_{j=0}^{k} (-1)^j \mu_{i_j} d\mu_p d\mu_{i_0} \wedge .. \wedge d\hat{\mu_{i_j}} \wedge .. \wedge d\mu_{i_k}) \tag{34}$$

where the sum is extended to all the p such that $[v_p, v_{i_0}, .., v_{i_k}] \in K^{(k+1)}$, using $\sum \mu_i = 1$ after some straightforward computation it's possible to show that the term on the right hand side is equal to $(k+1)! d\mu_{i_0} \wedge .. \wedge d\mu_{i_k}$.
2) If $I = \sum_{p \in k^{(0)}} p^*$ then $W(I) \equiv 1$.
From the definition $W(I) = \sum_p (W(p^*)) = \sum_p \mu_p = 1$.
3) If σ^* is the dual of $\sigma = [v_{i_0}, .., v_{i_k}]$ then $W(\sigma^*)$ vanishes identically on a neighborhood of $M \setminus St(\sigma)$.
Both μ_i and $d\mu_i$ vanish identically on G_i so $W(\sigma^*)$ vanishes identically on $G_{i_0} \cup .. \cup G_{i_k}$ and this contains $M \setminus Star(\sigma)$.
4 $R \circ W = Id$.
This can be proved by induction on k, if $k = 0$ then $R(W(v_i^*))(v_j) = \int_{v_j} W(v_i^*) =$
$= W(v_i^*)(v_j) = \mu_i(v_j) = \delta_i^j$ where we used that if $i \neq j$ then $v_j \notin Star(v_j)$.
Now let $k \geq 1$ and the hypothesis holds for $k-1$ and σ^* be the dual of $\sigma = [v_{i_0}, .., v_{i_k}]$, by **3** we know that $R(W(\sigma^*)) = 0$ on any k simplex different from σ, it remains to show that $\int_\sigma (W(\sigma^*)) = 1$, this can be done using Stockes' formula and the induction hypothesis.
5) $W^* \circ R^* = Id$, this part of the proof differs from the one that you find in **(17)**.
We want to prove that $[\omega] = [W^*(R^*(\omega))]$ for every $\omega \in H^i(M)$, using the isomorphism between simplicial and singular homology and **(1.2.4.11)** it's enough to show that $\int_{\sigma_i} \omega = \int_{\sigma_i} (W^*(R^*(\omega)))$ where $\{\sigma_1, ..\sigma_l\}$ are generators of $H_i(K)$.
In order to do so notice that $[R(\omega)] = [\sum_i (\int_{\sigma_i} \omega) \sigma_i^*]$ and so $\int_{\sigma_i} (W^*(R^*(\omega))) =$
$= \sum_j (\int_{\sigma_j} \omega \int_{\sigma_i} W(\sigma_j^*)) = \int_{\sigma_i} \omega$, where we have used $\int_{\sigma_j} W(\sigma_i^*) = \delta_j^i$ that we proved in **4**. □

1.2.6 Fundamental Class

In this section we introduce the fundamental class for an n-manifold with boundary M, this will be an element $[M] \in H_n^{sing}(M, \partial M)$.
The fundamental class will be used to prove the Lefschetz Duality **(1.3.2.1)**

and its counterpart in simplicial homology will be used in **(2.1.0.7)** to define the pairing that we need for the definition of the Hodge star operator.

Remark 1.2.6.1. *Using excision it is easy to prove that for every point in the interior of M, $H_n(M, M \setminus \{x\}; \mathbb{Z}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}) \cong \mathbb{Z}$, for the details refers to Chapter 3.3 of **(6)**.*

Notation 1.2.6.2. *For every $A \subset M$, $H_n(M, M \setminus A; \mathbb{Z})$ will be denoted as $H_n(M|A; \mathbb{Z})$.*

Definition 1.2.6.3 (Local Orientation)**.** *Let $x \in M$, with M a smooth manifold without boundary, a local orientation of M at x is a choice of a generator μ_x of $H_n(M|x; \mathbb{Z})$.*

Definition 1.2.6.4 (Orientation)**.** *An orientation for a manifold M, is a function $x \mapsto \mu_x$ assigning to every point in x a local orientation, such that each $x \in M$ has a neighborhood $U \cong \mathbb{R}^n$ in M, containing an open ball B around x, such that for every $y \in B$ μ_y is the image of $\mu_B \in H_n(M|B; \mathbb{Z})$ under the natural map $H_n(M|B; \mathbb{Z}) \to H_n(M|x; \mathbb{Z})$,*

Definition 1.2.6.5 (R-orientation)**.** *The same definition can be generalized to the homology with coefficients in any ring R, in this case a local orientation will be a generator of $H_n(M|x; R)$ and an orientation of M is defined in the same way as before.*

Definition 1.2.6.6 (Covering space)**.** *For any ring R, the covering space M_R is defined as:*

$$M_R = \{\alpha_x | x \in M, \alpha_x \in H_n(M|x; R)\} \tag{35}$$

It is possible to give a topological structure to M_R, a section of the covering space is a continuous map $M \to M_R$ such that any point $x \in M$ in sent to an element of $H_n(M|x; R)$, an R orientation of M is a section that sends each x to a local orientation.

Lemma 1.2.6.7. *Let M be a manifold of dimension n and let $A \subset M$ be a compact subset. Then if $x \mapsto \mu_x$ is a section of the covering space M_R then there is a unique class $\alpha_A \in H_n(M|A; R)$ whose image in $H_n(M|x; R)$ is α_x for all $x \in A$.*

Proof. Lemma 3.27 in **(6)**. □

Definition 1.2.6.8 (Fundamental class)**.** *An R-fundamental class of M is an element $[M] \in H_n(M; R)$ whose image in $H_n(M|x; R)$ is a generator for every x.*

Definition 1.2.6.9 (Collar neighborhood)**.** *Let M be a compact manifold with boundary ∂M then a collar neighborhood of ∂M in M open neighborhood homeomorphic to $\partial M \times [0, 1)$ by a homeomorphism taking ∂M to $\partial M \times \{0\}$.*

Proposition 1.2.6.10. *If M is a compact manifold with boundary, then ∂M has a collar neighborhood.*

Proof. Proposition 3.42 in **(6)**. □

Definition 1.2.6.11 (R-Orientation in Manifold with boundary)**.** *A manifold M with boundary ∂M is R-oriented if $M \setminus \partial M$ is R-oriented as a manifold without boundary.*

Definition 1.2.6.12 (Fundamental Class for manifold with boundary)**.** *Using excision we have $H_i(M, \partial M; R) \cong H_i(M \setminus \partial M, \partial M \times (0, \epsilon); R) = H_i(M \setminus \partial M | M \setminus (\partial M \times (0, \epsilon)))$, if M is R orientable then $M \setminus \partial M$ is R orietable and we can use **(1.2.6.7)** to define an element in $H_n(M, \partial M; R)$ such that the image in $H_n(M|x; R)$ is a generator for every $x \notin \partial M \times (0, \epsilon)$, since this can be done for every ϵ, we can define the fundamental class of M as the element of $H_n(M, \partial M; R)$ whose image in $H_n(M|x; R)$ is a generator for every $x \in M \setminus \partial M$.*

1.2.7 Cap/Cup product

In this section we define both a cap and a cup product, these will be fundamental tools in the proof of the Lefschetz Duality **(1.3.2.1)**.

Definition 1.2.7.1. *For an arbitrary space X and coefficient ring R, define an R bilinear cap product:*

$$\begin{gathered} \cap : C_k(X;R) \times C^l(X;R) \to C_{k-l}(X;R) \quad k \geq l \\ (\sigma, \phi) \mapsto \phi(\sigma_{|[v_0,..,v_l]})\sigma_{|[v_l,..,v_k]} \end{gathered} \tag{36}$$

Proposition 1.2.7.2. *The cap product induces a product in (co)homology $H_k(X; R) \times H^l(X; R) \to H_{k-l}(X; R)$.*

Proof. Is a consequence of the formula:

$$\partial(\sigma \cap \phi) = (-1)^l((\partial\sigma) \cap \phi - \sigma \cap \delta\sigma) \tag{37}$$

the complete proof is at Pag. 240 of **(6)**. □

Remark 1.2.7.3 (Relative case)**.** *The same procedure can be applied to the relative case and gives rise to the products:*

$$\begin{aligned} H_k(X, A; R) \times H^l(K; R) \to H_{k-l}(X, A; R) \\ H_k(X, A; R) \times H^l(K, A; R) \to H_{k-l}(X; R) \end{aligned} \tag{38}$$

moreover a similar cap prudcut can be defined also for the simplicial (co)homology, when the space is triangulated.

Definition 1.2.7.4 (Cup product)**.** *For an arbitrary space X and coefficient ring R, define an R bilinear cup product, from $C^k(X;R) \times C^l(X;R) \to C^{k+l}(X;R)$ such that for any $\sigma : \Delta^{k+l} \to X$:*

$$(\phi \cup \psi)(\sigma) = \phi(\sigma_{|[v_0,..,v_k]})\phi(\sigma_{|[v_k,..,v_{k+l}]}) \tag{39}$$

Proposition 1.2.7.5. *The cup product induces a product in (co)homology $H^k(X;R) \times H^l(X;R) \to H^{k+l}(X;R)$.*

Proof. Is a consequence of the formula:

$$\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^k \phi \cup \delta\psi \tag{40}$$

. the complete proof is at lemma 3.6 in **(6)**. □

Remark 1.2.7.6 (Relative case)**.** *The same procedure can be applied to the relative case and gives rise to the products:*

$$\begin{aligned} H^k(X; R) \times H^l(K, A; R) \to H^{k+l}(X, A; R) \\ H^k(X, A; R) \times H^l(K; R) \to H^{k+l}(X, A; R) \\ H_k(X, A; R) \times H^l(K, A; R) \to H^{k+l}(X, A; R) \end{aligned} \tag{41}$$

moreover a similar cup prudcut can be defined also for the simplicial (co)homology, when the space is triangulated.

Proposition 1.2.7.7 (Connection between cap and cup product)**.** *Let $\phi \in C^k(X;R)$, $\psi \in C^l(X,A;R)$ and $\alpha \in C_{k+l}(K,A;R)$ then:*

$$\psi(\alpha \cap \phi) = (\phi \cup \psi)(\alpha) \tag{42}$$

Proof. Section 3.3 of **(6)**. □

1.3 Dualities

In this paragraph we will study Lefschetz and Alexander duality, the first will give us a way to connect homology and cohomology of complementary dimensions while the second will be useful to connect the homology of a region in the space with the cohomology of its complement in the space.
These dualities are the corner stone for the applications of (co)homological methods in Electromagnetic Theory, for details look at **(1)**.

1.3.1 Lefschetz Duality with Triangulation

Definition 1.3.1.1 (Relative n-manifold)**.** *A topological pair* (X, A) *is called a relative homology n-manifold if for each point* x *of* X *not in* A*, the local homology group* $H_i(X, X\setminus\{x\})$ *is equal to* 0 *if* $i \neq n$ *and it is infinte cyclic if* $i = n$*.*

Definition 1.3.1.2 (Orientation)**.** *Let* (X, A) *be a compact triangulated relative homology nmanifold. We say that* (X, A) *is orientable if it is possible to orient all the n-simplices* σ_i*, of* X *not in* A *so that their sum* $\gamma = \sum_i \sigma_i$ *is a cycle of* (X, A)*. Such a cycle* γ *will be called an orientation cycle for* (X, A)*.*

Notation 1.3.1.3. *A simplex is said to be locally finite if each vertex belongs to finitely many simplexes of* K*.*

Definition 1.3.1.4 (Dual block decomposition)**.** *Let* X *be a locally finite simplicial complex, and* $Sd(X)$ *the first barycentric subdivision. The simplexes of* $sd(X)$ *are of the form:* $\hat{\sigma}_{i_1}...\hat{\sigma}_{i_n}$ *where* $\sigma_{i_i} > .. > \sigma_{i_n}$ *(where* $\hat{\sigma}$ *is the barycenter of* σ *and* $\sigma_i > \sigma_j$ *iff* σ_j *is a proper face of* σ_i*). We shall partially order the vertices of* $sd(X)$ *by decreasing dimension of the simplexes of* X *of which they are the barycenters; this ordering induces a linear ordering on the vertices of each simplex of* $sd(X)$*. Given a simplex* σ *a of* X*, the union of all open simplexes of* $sd(X)$ *of which* $\hat{\sigma}$ *is the initial vertex is just* $Int(\sigma)$*, i.e* $\sigma \setminus \partial\sigma$*.*
Define $D(\sigma)$ *to be the union of all open simplexes of* $sd(X)$ *of which* $\hat{\sigma}$ *is the final vertex; this set is called the block dual to* σ*.*
We call $\overline{D(\sigma)}$ *the closed block dual to* σ*. It equals the union of all simplexes of* $sd(X)$ *of which* $\hat{\sigma}$ *is the final vertex. We let* $\dot{D}(\sigma) = \overline{D(\sigma)} \setminus D(\sigma)$*.*

Lemma 1.3.1.5. *If* σ, τ *are k-simplexes then* $\overline{D(\sigma)} \cap \tau = \emptyset$ *if* $\sigma \neq \tau$ *and is equal to* $\{\hat{\sigma}\}$ *if* $\sigma = \tau$

Proof. Lemma 1.6.12 in **(7)**. □

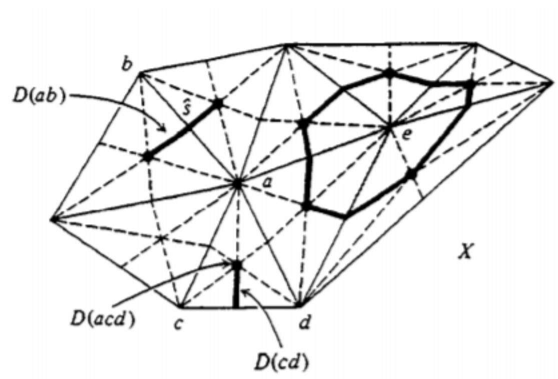

Figure 8: Dual block decomposition of a simplicial complex X, $sd(X)$ is indicated with dotted lines.

Theorem 1.3.1.6. *Let X be a locally finite simplicial complex that consists entirely of n-simplexes and their faces. Let σ be a k-simplex of X. Then:*
a) *The dual blocks are disjoint and their union is $|X|$.*
b) *$\overline{D(\sigma)}$is the polytope of a subcomplex of $sd(X)$ of dimension $n-k$.*
c) *$\dot{D}(\sigma)$ is the union of all blocks $D(\tau)$ for whix τ has σ as a proper face. These blocks have dimensions less then $n-k$.*
d) *$\overline{D(\sigma)}$ equals the cone $|\dot{D}(\sigma) \star \hat{\sigma}|$.*
e) *If $H_i(X, X \setminus \hat{\sigma}) \cong \mathbb{Z}$ if $i = n$ and it is 0 otherwise then $(\overline{D(\sigma)}, \dot{D}(\sigma))$ has the homology of an $n-k$ cell modulo its boundary.*

Proof. Theorem 64.1 in **(5)**. □

Definition 1.3.1.7. *Let X be a locally finite simplicial complex that is a homology n-manifold. Then the preceding theorem applies to each simplex a of X. The collection of dual blocks $D(\sigma)$ will be called the dual block decomposition of X. The union of the blocks of dimension at most p will be denoted by X_p, and called the dual p-skeleton of X. The dual chain complex $\mathcal{D}(X)$ of X is defined by letting its chain group in dimension p be the group $D_p(X) = H_p(X_p, X_{p-1})$. Its boundary operator is the homomorphism ∂_* in the exact sequence of the triple (X_p, X_{p-1}, X_{p-2}).*

Theorem 1.3.1.8. *Let X be a locally finite simplicial complex that is a homology n-manifold. Let X_p be the dual p-skeleton of X. Let $\mathcal{D}(X)$ be the dual chain complex of X.*
a) *The group $H_i(X_p, X_{p-1})$ vanishes for $i \neq p$ and is a free abelian group for $i = p$. A basis when $i = p$ is obtained by choosing generators for the groups*

$H_p(\overline{D(\sigma)}, \dot{D}(\sigma))$, as $D(\sigma)$ ranges over all p-blocks of X, and taking their images, under the homomorphisms induced by inclusion, in $H_p(X_p, X_{p-1})$.
***b**) The dual chain complex $\mathcal{D}(X)$ can be used to compute the homology of X. Indeed, $D_p(X)$ equals the subgroup of $C_p(sd(X))$ consisting of those chains carried by X_p whose boundaries are carried by X_{p-1}. And the inclusion map $D_p(X) \to C_p(sd(X))$ induces a homology isomorphism; therefore, it also induces homology and cohomology isomorphisms with arbitrary coefficients.*

Proof. Theorem 64.2 in **(5)**. □

Definition 1.3.1.9. *Let A be subcomplex of X, it is said to be a full subcomplex of the complex X if every simplex of X whose vertices are in A is itself in A.*

Definition 1.3.1.10 (Deformation retract)**.** *A space $A \subseteq X$ is said to be a deformation retract of X is exists a map $r : X \to A$ such that $r \circ i \sim Id_A$ and $i \circ r \sim Id_X$ where $i \hookrightarrow X$ is the natural inclusion.*
Naturally if A is a deformation rectract of X they have the same homology groups.

Lemma 1.3.1.11. *Let A be a full subcomplex of the finite simplicial complex X. Let C consist of all simplexes of X that are disjoint from $|A|$. Then $|A|$ is a deformation retract of $|X| \setminus |C|$, and $|C|$ is a deformation retract of $|X| \setminus |A|$.*

Proof. Lemma 70.1 in **(5)**. □

Theorem 1.3.1.12 (Lefschetz duality)**.** *Let (X, A) be a compact triangulated relative homology n-manifold. If (X, A) is orientable, there are isomorphisms :*

$$H^k(X, A, \mathbb{Z}) \cong H_{n-k}(|X| \setminus |A|, \mathbb{Z}) \tag{43}$$

$$H_k(X, A, \mathbb{Z}) \cong H^{n-k}(|X| \setminus |A|, \mathbb{Z}) \tag{44}$$

Proof. (Sketch) The complete proof is at Theorem 70.2 in **(5)**.
Let X^* be denote the subcomplex of the first barycentric subdivision of X consisting of all simplexes of $sd(X)$ that are disjoint from $|A|$. Now $|A|$ is the polytope of a full subcomplex of $sd(X)$. By the preceding lemma $|X^*|$ is a deformation retract of $|X| \setminus |A|$. Therefore, we may replace $|X| \setminus |A|$ by X^* in the statement of the theorem. Consider the collection of blocks $D(\sigma)$ dual to the simplexes of X. It is possible to prove that:
The space $|X^*|$ equals the union of all those blocks $D(\sigma)$ dual to simplexes a of X that are not in A. Now from the above theorem we have that the inclusion $\mathcal{D}(X^*) \to \mathcal{C}(X^*)$ induces an isomorphism in the homology and cohomology.

Recall that $C^k(X,A)$ can be considered as the subgroup of $C^k(X)$ consisting of all cochains of X that vanish on simplexes of A so it is free abelian and it is generated by σ^* such that σ is not in A, can be proved that the map $\phi : C^k(X,A) \to D_{n-k}(X^*)$ that sends σ^* to a generator of $H_{n-k}(\overline{D(\sigma)}, \dot{D}(\sigma))$ and is extended by linearity induces the isomorphism in homology that we were looking for. The other isomorphism is given by the dual of the map ϕ. □

Theorem 1.3.1.13. *If $h : |K| \to M$ is a triangulation of M then, $h^{-1}(\partial M)$ is the polytope of a subcomplex of K.*

Proof. Theorem 35.3 of **(5)**. □

Definition 1.3.1.14. *Let σ be a p simplex in X and τ an $n-p$ dual simplex then the intersection number $I(\sigma,\tau)$ is defined as :*

$$I(\sigma,\tau) = \begin{cases} 1 & if \quad \tau = \phi(\sigma^*) \\ -1 & if \quad \tau = -\phi(\sigma^*) \\ 0 & otherwise \end{cases} \tag{45}$$

where ϕ is the morphism defined in the proof of the Lefschetz duality, it can be extended by linearity to a map $I : C_p(X) \otimes D_{n-p}(X) \to \mathbb{Z}$, moreover if A is full subsimplex of X, I can be restricted to a pairing
*$I : C_p(X,A) \otimes D_{n-p}(X^*) \to \mathbb{Z}$ where X^* is defined as in **(1.3.1.12)**.*

Remark 1.3.1.15 (Case n=3)**.** *The isomorphism ϕ is built recursively starting from $k = n$, when $n = 3$ we define $\phi(\sigma^*) = \hat{\sigma}$, for $k = 2$ $\phi(\sigma^*) = [\hat{\sigma}, \hat{\tau_0}] + [\hat{\sigma}, \hat{\tau_1}]$ where τ_0, τ_1 are the 3-simplexes of which σ is a face, (notice that $\overline{D(\sigma)} = [\hat{\sigma}, \hat{\tau_1}] \cup [\hat{\sigma}, \hat{\tau_1}]$), $k = 1$ if σ is a face of $\{\tau_i\}_{i=0\ldots r}$ can be proved (Theorem 70.2 in **(5)**) that $\phi(\delta\sigma^*)$ is a generator of $H_1(\dot{D}(\sigma))$ and using the exact sequence*

$$0 \to H_2(\overline{D(\sigma)}, \dot{D}(\sigma)) \xrightarrow{\partial_*} H_1(\dot{D}(\sigma)) \to 0 \tag{46}$$

*define $\phi(\sigma^*)$ such that $\partial_*(\phi(\sigma^*)) = \phi(\delta\sigma)$, so using **(1.3.1.8)** we have that $|\phi(\sigma^*)| = D(\sigma)$.*
*Now let $\overline{\Omega}$ be a compact three dimensional manifold with boundary, thanks to **(1.3.1.13)** if we give a triangulation to $\overline{\Omega}$ we can use the above theorem with $X = \overline{\Omega}$ and $A = \partial\Omega$, we have a pairing : $I : C_2(\overline{\Omega}, \partial\Omega) \otimes D_1(\Omega^*) \to \mathbb{Z}$, where Ω^* is defined as in the proof of **(1.3.1.12)** such that :*

$$(\sigma,\tau) \to I(\sigma,\tau) \tag{47}$$

We want to prove that the pairing induced in homology is perfect.
Using that the homology groups are torsion free as it is proved in ***(1.3.3.7)****, we know that the paring is perfect if and only if the induced map* $H_2(\overline{\Omega}, \partial\Omega) \to H_1(\Omega^*)^*$ *is an isomorphism and this is true since,* $D_1(\Omega^*)$ *is generated by* $\phi(\sigma_i^*)$ *for where* σ_i *are 1 simplexes in* $C_1(\overline{\Omega}, \partial\Omega)$ *and* $I(\sigma_1, \phi(\sigma_j^*)) = \delta_i^j$ *implies that* $I(-, \phi(\sigma_j^*)) = \sigma_j^*$ *so the map induced in homology is the inverse of the map* ϕ *defined in the proof of the Lefschetz Duality.*
In summary we have proved that there is a perfect pairing $H_1(\Omega^*) \otimes H_2(\overline{\Omega}, \partial\Omega) \to \mathbb{Z}$ *such that if* $\{\gamma_1, .., \gamma_n\}$ *is basis of* $H_1(\Omega^*)$*, and* $\{\Sigma_1, .., \Sigma_n\}$ *is a basis of* $H_2(\overline{\Omega}, \partial\Omega)$ *the matrix* $A = (a_{i,j})_{1 \le i,j \le n}$ *that represents the pairing is defined in such a way that* $a_{i,j} = I(\gamma_i, \Sigma_j)$*. This pairing have a geometrical interpretation, indeed thanks to* ***(1.3.1.5)****, the pairing really counts the "oriented" number of intersection between 1 and 2 simplexes.*

Theorem 1.3.1.16 (Poincare' Duality)**.** *Let* X *be a compact orientable triangulated homology* n*-manifold, there are isomorphisms:*

$$H^k(X, \mathbb{Z}) \cong H_{n-k}(|X|, \mathbb{Z}) \tag{48}$$

Proof. The previous result with $A = \emptyset$. □

1.3.2 Lefschetz Duality without triangulation

Theorem 1.3.2.1 (Lefschetz Duality)**.** *Suppose* M *is a compact* R *orientable* n *manifold with boundary* ∂M*,* $[M] \in H_n(M, \partial M; R)$ *a fundamental class and* $D_M : H^k(M, \partial M; R) \to H_{n-k}(M; R)$ *given by* $D_M(\phi) = [M] \cap \phi$*, then* D_M *is an isomorphism for every* k*.*

Proof. Theorem 3.43 in **(6)**. □

Corollary 1.3.2.2 (Perfect Pairing)**.** *The pairing* $H^k(M; R) \times H^{n-k}(M, \partial M; R) \to R$ *given by* $(\phi, \psi) \mapsto (\phi \cup \psi)([M])$ *is a perfect pairing, when* R *is a field or* $R = \mathbb{Z}$ *and the torsion part of the groups is factored out.*

Proof. Using **(1.2.7.7)** we have $(\phi \cup \psi)([M]) = \psi([M] \cap \phi) = \psi(D_M(\phi))$.
Consider the composition of maps:

$$H^k(M; R) \xrightarrow{h} Hom(H_k(M; R), R) \xrightarrow{D_M^*} Hom(H^{n-k}(M, \partial M; R), R) \tag{49}$$

is the composition of D_M^* is the map dual to D_M and h is the map of the Universal coefficient theorem, then an element $\psi \in H^k(M; R)$ is sent to the morphism $\phi \mapsto \psi([M] \cap \phi) = \psi(D_M(\phi))$, now D_M^* is always an isomorphism, while h is an isomorphism when R is a field or $R = \mathbb{Z}$ and the torsion is factored out, so we are done. □

1.3.3 Alexander Duality

In the application sometimes we need a way a to connect the (co)homology of $\mathbb{R}^3 \setminus \Omega$ and the (co)homology of Ω where Ω is an open bounded 3-manifold inside $\mathbb{R}^3$, (for instance look at **(24)**), it is possible to do so thanks to the so called Alexander Duality theorem, in this section we are going to prove it and we are going to see how to use it in our situation.
The main reference for this duality is **(3)**.

Notation 1.3.3.1. $\Omega_e = \mathbb{R}^3 \setminus \overline{\Omega}$

Claim 1.3.3.2. *$H^1(\Omega_e) \cong H_1(\Omega)$, the proof of the claim will be given at **(1.3.3.8)**.*

Theorem 1.3.3.3 (Alexander Duality, original version)**.** *Let M be a compact orientable n-manifold, A a closed subset of M, and $U = M \setminus A$ the complementary set. Then the relative homology group $H_{n-q}(M, U; G)$ is isomorphic to the Cech-Alexander-Spanier cohomology group $\overline{H}^q(A; G)$.*

Remark 1.3.3.4. *The Cech-Alexander-Spanier cohomology group $\overline{H}^q(A; G)$ is defined as $\varinjlim_{K \subsetneq U, K closed} H^q(M \setminus K)$, in general this is not equal to $H^q(A)$ when A is just a closed subset of M but they are equal when A is a compact submanifold of M (in particular the duality works in our case, a proof can be found in Proposition 18.4.9 in **(2)**).*

Proof. (Sketch) The complete proof can be found at Proposition 6.3 of **(3)** .
I will just explain how the isomorphism is made. If K is a closed subset of U we have an in inclusion of pairs $l : (M \setminus K, U \setminus K) \to (M, U)$ which induce isomorphisms in the relative homology groups by the excision property. Moreover we have a cap product $H^q(M \setminus K) \otimes H_n(M \setminus K, U \setminus K) \xrightarrow{\cap} H_{n-q}(M \setminus K, U \setminus K)$. We denote with μ the orientation on M and with μ_A the image of μ in $H_n(M, U)$. So we can go from $H^q(M \setminus K)$ to $H_{n-q}(M, U)$ in the following way

$$H^q(M \setminus K) \xrightarrow[-\otimes l_*^{-1}(\mu_A)]{} H_{n-q}(M \setminus K, U \setminus K) \xrightarrow[l_*]{\cong} H_{n-q}(M, U) \qquad (50)$$

Now passing to the limit over K we get the isomorphism from $\overline{H}^q(A)$ to $H_{n-q}(M, U)$. □

Remark 1.3.3.5. *In our situation $\overline{\Omega}$ is contained in $\mathbb{R}^3$ so we cannot use directly the above theorem, since $\mathbb{R}^3$ is not compact, but we can always embed $\mathbb{R}^3$ in $\mathbb{S}^3$ using the stereographic projection so we can think to $\overline{\Omega}$ as embedded in $\mathbb{S}^3$, that is compact.*

Lemma 1.3.3.6. *If A is a closed subset of $\mathbb{S}^3$, $H_2(\mathbb{S}^3, \mathbb{S}^3 \setminus A) \cong H_1(\mathbb{S}^3 \setminus A)$.*

Proof. Since $\mathbb{S}^3 \setminus A \hookrightarrow \mathbb{S}^3$ we have the long exact sequence

$$... \to H_2(\mathbb{S}^3) \to H_2(\mathbb{S}^3, \mathbb{S}^3 \setminus A) \to H_1(\mathbb{S}^3 \setminus A) \to H_1(\mathbb{S}^3) \to .... \tag{51}$$

Clearly $H_2(\mathbb{S}^3) = H_1(\mathbb{S}^3) = 0$, so we have the claim. □

Lemma 1.3.3.7. *The integral (co)homology groups of Ω are torsion free.*

Proof. (Sketch) Call Ω_c the image of Ω in $\mathbb{S}^3$ under the stereographic projection and $\Omega_c^e = \mathbb{S}^3 \setminus \mathring{\Omega}_c$.
Clearly Ω_c^e is a closed subset in $\mathbb{S}^3$ so we can use the cohomological version of the previuos lemma and the Alexander duality theorem we have:

$$\tilde{H}^{2-q}(\Omega_c^e, \mathbb{Z}) \cong \tilde{H}_q(\mathring{\Omega}_c, \mathbb{Z}) \quad q = 0, 1, 2 \tag{52}$$

So if we call $T(\cdot)$ the torsion part of the (co)homology groups we have:

$$T^{2-q}(\Omega_c^e) \cong T_q(\mathring{\Omega}_c) \quad q = 0, 1, 2 \tag{53}$$

since the 0-(co)homology groups are always torsion free we have $T_2(\mathring{\Omega}_c) = 0$. Thanks to the Universal Coefficient Theorem we have $T^p(\cdot) = T_{p-1}(\cdot)$ so we have:

$$T_0(\Omega_c^e) = 0 \implies T^1(\Omega_c^e) = 0 = T_1(\mathring{\Omega}_c) \tag{54}$$

So we have:

$$T_0(\mathring{\Omega}_c) = T_1(\mathring{\Omega}_c) = T_2(\mathring{\Omega}_c) = 0 \tag{55}$$

and using the Universal Coefficient Theorem:

$$T^0(\mathring{\Omega}_c) = T^1(\mathring{\Omega}_c) = T^2(\mathring{\Omega}_c) = T^3(\mathring{\Omega}_c) \tag{56}$$

Now because the image of Ω, under stereographic projection, can be contained in a neighborhood of $\mathring{\Omega}_c$ of which he is a deformation retract, one may substitute Ω for $\mathring{\Omega}_c$ Eq. (43), (44) and (45).
The last thing to show is that $T^3(\Omega) = 0$, this can be done using similar arguments, for the proof look at Section 2 of **(8)**. □

Corollary 1.3.3.8. *If $\overline{\Omega}$ is a compact submanifold of $\mathbb{R}^3$, $H^1(\Omega_e) \cong H_1(\Omega)$.*

Proof. Using the stereographic projection we see $\overline{\Omega}$ as a compact submanifold of $\mathbb{S}^3$. Alexander-Duality together with the above lemma give $H^1(\overline{\Omega}) \cong H_1(\mathbb{S}^3 \setminus \overline{\Omega})$. Now $\Omega_e = \mathbb{R}^3 \cap (\mathbb{S}^3 \setminus \overline{\Omega})$ and $\mathbb{S}^3 = \mathbb{R}^3 \cup (\mathbb{S}^3 \setminus \overline{\Omega})$ we have the long exact sequence

$$... \to H_2(\mathbb{S}^3) \to H_1(\Omega_e) \to H_1(\mathbb{R}^3) \oplus H_1(\mathbb{S}^3 \setminus \overline{\Omega}) \to H_1(\mathbb{S}^3) \to ... \tag{57}$$

Since $H_2(\mathbb{S}^3) = H_1(\mathbb{R}^3) = H_1(\mathbb{S}^3) = 0$ we have $H_1(\Omega_e) \cong H_1(\mathbb{S}^3 \setminus \overline{\Omega})$.
So up to now we have proved that $H_1(\Omega_e) \cong H^1(\overline{\Omega})$. Since we can always find an open neighborhood of $\overline{\Omega}$ that is a deformation retract of Ω, $H^1(\Omega) = H^1(\overline{\Omega})$, so we have $H_1(\Omega_e) \cong H^1(\Omega)$. Now using universal coefficient theorem we have :

$$H^1(\Omega) = Hom(H_1(\Omega), \mathbb{Z}) \oplus Ext(H_0(\Omega), \mathbb{Z}) \tag{58}$$

since $H_1(\Omega)$ and $H_0(\Omega)$ are torsion free we have $H^1(\Omega) \cong H_1(\Omega)$. With a similar argument can be proved that $H^1(\Omega_e) \cong H_1(\Omega_e)$ and this concludes the proof. □

Proposition 1.3.3.9. $H_1(\partial\Omega) \cong H_1(\mathbb{R}^3 \setminus \Omega) \oplus H_1(\overline{\Omega})$.

Proof. From $\partial\Omega = (\mathbb{R}^3 \setminus \Omega) \cap \overline{\Omega}$ and $\mathbb{R}^3 = (\mathbb{R}^3 \setminus \Omega) \cup \overline{\Omega}$ we have a long exact sequence:

$$... \to H_2(\mathbb{R}^3) \to H_1(\partial\Omega) \to H_1(\mathbb{R}^3 \setminus \Omega) \oplus H_1(\overline{\Omega}) \to H_1(\mathbb{R}^3) \to ... \tag{59}$$

Since $H_2(\mathbb{R}^3) = H_1(\mathbb{R}^3) = 0$ we have the result.

□

1.4 Riemannian Structure and Hodge Decomposition

In this paragraph we will define, on any Riemannian manifold, an operator $\star$ that in the three dimensional case transform 2-forms into 1-forms and vice-versa, this operator is often used in the discretization process of the Maxwell's equations (for a reference look at **(15; 16)**).
Moreover in the second section we prove that, if the manifold is compact, the $\star$ operator provides also a decomposition of $\Omega^k(M)$, using this decomposition we will find another version of the Lefschetz duality for Riemannian manifolds.
Main references for this paragraph are **(14; 19)**.

1.4.1 Hodge star operator

Definition 1.4.1.1 (Riemannian Metric)**.** *Given a smooth manifold with boundary M, a Riemannian metric on M is a family of positive definite inner products:*

$$g_p : T_pM \times T_pM \to \mathbb{R}, \quad p \in M \tag{60}$$

such that for all smooth vector fields V, W the map $p \mapsto g_p((V(p), W(p))$ is smooth. A Riemannian Manifold (M, g) is a smooth manifold equipped with a Riemannian metric, where smooth means $\mathbb{C}^\infty$.

Definition 1.4.1.2 (Orthonormal Frame)**.** *Given a Riemannian manifold (M, g) and an open set $U \subset M$ a local orthonormal frame on U is a set of (not necessarily smooth) vector fields $\{E_1, .., E_n\}$ on U that are orthonormal with respect to the Riemannian metric at each point $p \in U$, that is, $g_p(E_i(p), E_j(p)) = \delta_i^j$.*
Can be proved that for every $p \in M$ there exists a local orthonormal frame on an open set containing p.

Definition 1.4.1.3 (Inner Product on $\mathcal{A}^k(T_p^*M)$)**.** *Let (M, g) be an oriented Riemannian manifold, let $p \in M$ be a point and let $\{E_1, ..., E_n\}$ be an orthonormal frame at p and $\{e_1, .., e_n\}$ the corresponding dual frame. A basis for $\mathcal{A}^k(T_p^*M)$ is then given by the set: $B = \{e_{i_1} \wedge .. \wedge e_{i_k} | i_i \lneq .. \lneq i_k\}$. We define an inner product $< \cdot, \cdot >_g$ on $\mathcal{A}^k(T_p^*M)$ such that:*

$$< \omega, \eta >_g = \frac{1}{k!} \sum_{1 \le i_1 \lneq .. \lneq i_k \le n} \omega(E_{i_1}, .., E_{i_k}) \eta(E_{i_1}, .., E_{i_k}) \tag{61}$$

This inner product is independent of the choice of orthonormal frame and is hence well defined.

Proposition 1.4.1.4 (Riemannian volume form)**.** *There exists a unique orientation form called the Riemannian volume form, on M, which we denote by ω_g, and which has the defining property that $\omega_g(E_1, .., E_n) = 1$ for every local oriented orthonormal frame $\{E_1, .., E_n\}$, where oriented frame means to be pointwise positively oriented w.r.t the orientation induced by the orientation of M to T_pM, i.e if η is an orientation of M given by a non vanishing smooth n differential form then $\{E_{1,p}, .., E_{n,p}\}$ is positively oriented if $\eta_p(E_{1,p}, .., E_{n,p}) > 0$, otherwise it is negatively oriented.*
Moreover given (U, x) an oriented local chart, then $\omega_g = \sqrt{det(G)} dx^1 \wedge .. \wedge dx^n$ where G is the matrix that represent g in this chart.

Proof. Let $\{e_{1,|p}, .., e_{n,|p}\}$ an oriented orthonormal basis of $T_p(M)$ and define $\{e^1_{|p}, .., e^n_{|p}\}$ as its dual in $T_p^*(M)$, and $\omega_{|p} = e^1_{|p} \wedge .. \wedge e^n_{|p}$, this form is well defined

since or any other positively oriented basis of $T_p(M)$, $\{\tilde{e}_{1,|p}, .., \tilde{e}_{n,|p}\}$ with B transition matrix, $\omega_{|p} = e^1_{|p} \wedge .. \wedge e^n_{|p} = det(B)\tilde{e}^1_{|p} \wedge .. \wedge \tilde{e}^n_{|p}$, but in this case the determinant of B is 1 since both the basis are positively oriented.
Now let x oriented local coordinate chart near p then and A_p the transition matrix from $\{e_{1,|p}, .., e_{n,|p}\}$ to $\{\frac{\partial}{\partial x^1}, .., \frac{\partial}{\partial x^n}\}_{|p}$ we have
$\omega_p(\frac{\partial}{\partial x^1}, .., \frac{\partial}{\partial x^n}) = det(A_p)\omega_p(e_{1,|p}, .., e_{n,|p}) = det(A_p) = \sqrt{det(A_p^T A_p)}$, now by the definition of G we have that $(G) = (A_p^T A_p)$ so we are done. ☐

Remark 1.4.1.5. *Applying $< \cdot, \cdot >_g$ pointwise to differential forms yields the map:*

$$< \cdot, \cdot >_g : \Omega^k(M) \times \Omega^k(M) \to \mathcal{C}^\infty(M) \tag{62}$$

and if M is compact, integrating $< .,. >_g$ over M get an inner product of $\Omega^k(M)$, in the following way.

$$< \eta, \omega > = (\int_M < \eta, \omega >_p \omega_g) \tag{63}$$

Definition 1.4.1.6 (Hodge star operator)**.** *We define the Hodge operator $\star : \Omega^*(M) \to \Omega^*(M)$ such that for a k-form η the identity $\zeta \wedge \eta = < \zeta, \star^{-1}\eta > \omega_g$ holds for all $\zeta \in \Omega^{n-k}(M)$.*

Lemma 1.4.1.7 (Finite dimensional Riesz representation theorem)**.** *Let V be a finite-dimensional vector space endowed with a nondegenerate inner product g, and let f be a linear functional on V. Then there exists a unique vector $v \in V$ such that:*

$$f(w) = g(v, w) \quad \forall w \in V \tag{64}$$

Proof. Let $\{u_1, .., u_n\}$ be an orthonormal basis then define $v = \sum f(u_i)u_i$ then if $w = \sum_i w_i u_i$ we have:

$$f(w) = \sum_i w_i f(u_i) = \sum_{i,j} w_i f(u_j) g(u_i, u_j) = g(w, v) \tag{65}$$

For the unicity suppose v' that satisfies the same property then
$g(v - v', w) = g(v, w) - g(v', w) = f(w) - f(w) = 0$ for all $w \in V$ so $v - v' = 0$. ☐

Proposition 1.4.1.8. *Let (M, g) be a Riemannian manifold with boundary. The Hodge star operator is the unique automorphism on $\Omega^*(M)$ that maps the k-form η to the $(n-k)$-form $\star\eta$. Moreover, for each $k \in \{0, .., n\}$, the map $\star_k$ is an isomorphism from the space of k-forms to the space of $n-k$-forms on M.*

Proof. Every top differential form on M can be written as $f\omega_g$ for some smooth function f on M. Fix $\eta \in \Omega^k(M)$ then $\zeta \wedge \eta \in \Omega^n(M)$ for all $\zeta \in \Omega^{n-k}(M)$, and thus:

$$\zeta \wedge \eta = f_\eta(\zeta)\omega_g \tag{66}$$

where $f_\eta(\zeta)$ is smooth and $f_\eta : \Omega^{n-k} \to \mathcal{C}^\infty(M)$ is linear over $\mathcal{C}^\infty(M)$. Moreover f_η is uniquely determined and at each $p \in M$ the map:

$$f_{\eta|\mathcal{A}^{n-k}(T_p^*M)} : \mathcal{A}^{n-k}(T_p^*M) \to \mathbb{R} \tag{67}$$

is a linear functional, so using the previous lemma we obtain $\theta \in \Omega^{n-k}(M)$ such that $f_\eta(\zeta) =< \zeta, \theta >_{\mathcal{A}^k}$, now call $\star^{-1}\eta = \theta$.
$\star$ is $\mathcal{C}^\infty(M)$ linear since f_η is so, moreover if $\zeta \wedge \eta = 0$ for every $\zeta \in \Omega^{n-k}(M)$ then $\eta = 0$ so $\star$ is injective so using that $dim(\Omega^k(M)) = dim(\Omega^{n-k}(M))$ we are done. □

Proposition 1.4.1.9. *Let (M, g) be a Riemannian manifold with boundary. The following definitions of $\star$ are equivalent:*
1) *Let $\eta \in \Omega^k(M)$, then $\star^{-1}\eta \in \Omega^{n-k}(M)$ such that:*

$$\zeta \wedge \eta =< \zeta, \star^{-1}\eta > \omega_g \quad \forall\zeta \in \Omega^{n-k}(M). \tag{68}$$

2) *Let $\eta \in \Omega^k(M)$, then $\star\eta \in \Omega^{n-k}(M)$ such that:*

$$\zeta \wedge \star\eta =< \zeta, \eta > \omega_g \quad \forall\zeta \in \Omega^k(M). \tag{69}$$

3) *Let $\{e_1, .., e_n\}$ be an orthonormal coframe defined on some open subset $U \subset M$ and $\sigma \in S_n$:*

$$\star\,(e_{\sigma(1)\wedge..\wedge e_{\sigma(n)}}) := sgn(\sigma)e_{\sigma(k+1)} \wedge .. \wedge e_{\sigma(n)} \tag{70}$$

4) *Let $\{E_1, .., E_n\}$ be an orthonormal frame defined on some open subset $U \subset M$ and let $\eta \in \Omega^k(M)$. Then $\star\eta$ is defined on U to be the $(n-k)$-form for which:*

$$(\star\eta)(E_{\sigma(k+1)}, .., E_{\sigma(n)}) := sgn(\sigma)\eta(E_{\sigma(1)}, .., E_{\sigma(n)}) \quad \forall\sigma \in S_n \tag{71}$$

Proof. The first two are clearly equivalent using $\eta = \star \circ \star^{-1}\eta$, while the last two come the definition of ω_g given in **(1.4.1.4)**. □

Corollary 1.4.1.10. *Let (M, g) be a Riemannian manifold with boundary. Then $\star 1 = \omega_g$.*

Proof. From the point **4**) of the previous proposition. □

Corollary 1.4.1.11.

$$\star \circ \star \, \eta = (-1)^{n(n-k)} \eta \tag{72}$$

Proof. Corollary 2.19 in **(19)**. □

Corollary 1.4.1.12. *The inverse map $\star^{-1} : \Omega^*(M) \to \Omega^*(M)$ is equal to $\star$ when n is odd and it is $(-1)^k$ when n is even.*

Proof. Corollary 2.21 in **(19)**. □

Proposition 1.4.1.13. *If M is compact the same inner product defined in the **(1.4.1.5)** can be defined as $< \zeta, \eta >= \int_M \zeta \wedge \star\eta$, moreover this inner product is preserved by the star operator.*

Proof. From the point **2** of **(1.4.1.9)** we know that $\zeta \wedge \star\eta =< \zeta, \eta > \omega_g$ so the first statement is trivial.
The second statement comes from:

$$\begin{aligned} < \star\zeta, \star\eta >= \int_M \star\zeta \wedge \star\star\, \eta \overset{(1.4.1.11)}{=} (-1)^{k(n-k)} \int_M \star\zeta \wedge \eta = \\ = (-1)^{2k(n-k)} \int_M \eta \wedge \star\zeta =< \eta, \zeta >=< \zeta, \eta > . \end{aligned} \tag{73}$$

□

1.4.2 Boundary Conditions

Definition 1.4.2.1 (Codifferential)**.** *Define $\delta^k : \Omega^k(M) \to \Omega^{k-1}(M)$ as $\delta^k = (-1)^{n(k+1)+1} \star d^{n-k} \star$.*

Lemma 1.4.2.2. *Let (M, g) be a Riemannian manifold with boundary. An alternative expression for the codifferential is given by:*

$$\delta\eta = (-1)^k \star^{-1} d \star \eta \tag{74}$$

Proof. Proposition 2.24 in **(19)**. □

Proposition 1.4.2.3. *Let (M, g) be a Riemannian manifold with boundary. Then for any $\zeta \in \Omega^k(M)$ and $\eta \in \Omega^{k+1}(M)$:*

$$< d\zeta, \eta >=< \zeta, \delta\eta > + \int_{\partial M} i^*(\zeta) \wedge i^*(\star\eta) \tag{75}$$

where $i : \partial M \hookrightarrow M$.

Proof. Recall that $d(\omega_1 \wedge \omega_2) = d(\omega_1) \wedge \omega_2 + (-1)^{k_1} \omega_1 \wedge d(\omega_2)$ for all $\omega_i \in \Omega^{k_i}(M)$.

$$< d\zeta, \eta >= \int_M d(\zeta) \wedge \star\eta = \int_M d(\zeta \wedge \star\eta) - (-1)^k \int_M \zeta \wedge d(\star\eta) \tag{76}$$

now the first term on the right hand side of the equation using the Stokes' theorem became $\int_{\partial M} i^*(\zeta) \wedge i^*(\star\eta)$, while :

$$-(-1)^k \int_M \zeta \wedge d(\star\eta) = (-1)^{k+1} \int_M \zeta \wedge d(\star\eta) = \int_M \zeta \wedge \star(-1)^{k+1} \star^{-1} d(\star\eta) \tag{77}$$

and using the previous lemma the term on the right hand side of the equation became:

$$\int_M \zeta \wedge \star\delta\eta =< \zeta, \delta\eta > . \tag{78}$$

□

Remark 1.4.2.4 (Boundary conditions)**.** *From the previous proposition we see that if the manifold is without boundary the differential and the co-differential are adjoint w.r.t the inner product induced by the metric, i.e* $< d\zeta, \eta >=< \zeta, \delta\eta >$. *This is not always true when the manifold has a boundary, but clearly it is true when* $i^*(\zeta) = 0$ *or* $i^*(\star\eta) = 0$ *these two conditions will be called Dirichlet and Neumann boundary conditions.*

Lemma 1.4.2.5. *If* (M, g) *is an oriented Riemannian manifold, exists a unique outward pointing vector field* ν *such that* $g_p(\nu_p, \nu_p) = 1$ *and* $g_p(\nu_p, X_p) = 0, \forall X_p \in T_p\partial M$.

Proof. Lemma 5 in **(26)**. □

Definition 1.4.2.6 (Vector Field Decomposition)**.** *Given a vector field* X *on* M *its normal component* $\boldsymbol{n}(X_{|\partial M}) := g(\nu, X_{|\partial M}) \cdot \nu$ *and its tangential component* $\boldsymbol{t}(X_{|\partial M}) = X_{|\partial M} - \boldsymbol{n}(X_{|\partial M})$. *This allows to define the tangential and normal component of a differential form* $\omega \in \Omega^k(M)$ *over the boundary by* $\boldsymbol{t}(\omega_{|\partial M})(X_1, .., X_k) := \omega_{|\partial M}(\boldsymbol{t}(X_1), .., \boldsymbol{t}(X_k))$ *and* $\boldsymbol{n}(\omega_{|\partial M}) := \omega_{\partial M} - \boldsymbol{t}(\omega_{\partial M})$.

Proposition 1.4.2.7. *Let* (M, g) *a Riemannian manifold with boundary and* $i : \partial M \hookrightarrow M$ *be the inclusion map of the boundary, and* $\eta \in \Omega^k(M)$. *Then:*

$$\boldsymbol{t}\eta = 0 \quad iff \quad i^*(\eta) = 0 \tag{79}$$

Proof. Proposition 5.1 in **(19)**. □

Proposition 1.4.2.8. *Let (M, g) a Riemannian manifold with boundary and $\eta \in \Omega^k(M)$. Then:*

$$\star(\boldsymbol{n}\eta) = \boldsymbol{t}(\star\eta) \qquad \star(\boldsymbol{t}\eta) = \boldsymbol{n}(\star\eta) \tag{80}$$

Proof. Proposition 5.2 in **(19)**. □

Definition 1.4.2.9 (Laplace de-Rham operator)**.** *Let (M, g) a Riemannian manifold with boundary. The Laplace de-Rham operator $\Delta^r : \Omega^r(M) \to \Omega^r(M)$ is defined as $\Delta^r = \delta^{r+1}d^r + d^{r-1}\delta^r$*

Proposition 1.4.2.10. *The Laplace-de Rham operator commutes with $\star$, δ and d.*

Proof. Proposition 3.1 in (19). □

Definition 1.4.2.11. *1) Dirichlet boundary condition*
$\Omega^k_D(M) = \{\omega \in \Omega^k(M) | \boldsymbol{t}\omega = 0\}$.
2) Neumann boundary condition $\Omega^k_N(M) = \{\omega \in \Omega^k(M) | \boldsymbol{n}\omega = 0\}$.
3) Harmonic forms $\mathcal{H}^k(M) = \{\omega \in \Omega^k(M) | d^k(\omega) = 0, \delta^k(\omega) = 0\}$, following the points 1) and 2) we define also $\mathcal{H}^k_N(M)$ and $\mathcal{H}^k_D(M)$.
4) $C^k(M)$ and $E^k(M)$ are the closed and exact k forms, while $cC^k(M)$ and $cE^k(M)$ are coclosed and coexact.

Proposition 1.4.2.12. *Let (M, g)a Riemannian manifold with boundary, then:*

$$\Omega^k_N(M) = \{\omega \in \Omega^k(M) | \boldsymbol{t}(\star\omega) = 0\} \tag{81}$$

Proof. By definition $\omega \in \Omega^k_N(M)$ iff $\mathbf{n}\omega = 0$, since $\star$ is an isomorphism, this happens iff $\star\mathbf{n}\omega = 0$, using **(1.4.2.8)**, $\star\mathbf{n}\omega = \mathbf{t}(\star\omega)$. □

Proposition 1.4.2.13. *Let (M, g) a Riemannian manifold with boundary, the differential preserves Dirichlet boundary condition while the codifferential preserves Neumann boundary condition.*

Proof. Let $\omega \in \Omega^k_D(M)$ then $\mathbf{t}\omega = 0$ so by **(1.4.2.8)** $i^*(\omega) = 0$ and since the differential commutes with the pull back, $i^*(d\omega) = d(i^*(\omega)) = 0$ and so $\mathbf{t}(d\omega) = 0$.
The second statement is made in a similar way using the previous porposition.
□

Definition 1.4.2.14. *In the previous proposition we proved that $(\Omega^*_D(M), d^*)$ is a cochain complex, so we can define $H^k(M, \partial M)$ as the cohomology groups of this complex.*

Proposition 1.4.2.15. *Let (M, g) a Riemannian manifold with boundary then the $\star$ operator provide the following isomorphisms:*
1) $C^k(M) \cong cC^{n-k}(M)$.
2) $E^k(M) \cong cE^{n-k}(M)$.
3) $\mathcal{H}^k(M) \cong \mathcal{H}^{n-k}(M)$.
4) $E^k_D(M) \cong E^{n-k}_N(M)$.
5) $\mathcal{H}^k_D(M) \cong \mathcal{H}^{n-k}_N(M)$.

Proof. Let $\omega \in \Omega^k(M)$ using **(1.4.1.11)** together with the definition of δ, $d\omega = 0$ iff $\delta \star \omega = \star d \star \star \omega = \pm \star d\omega = 0$, this gives the first isomorphism.
In a similar way $\omega = d\eta$ iff $\star\omega = \star d\eta = \pm\delta \star \eta$ and so we have also the second isomorphism.
The other isomorphisms are proved in the same way, the complete proof is at Proposition 5.4 in **(19)**. □

Theorem 1.4.2.16 (Hodge-Morrey-Friedrichs Decomposition)**.** *Let (M, g) a compact oriented, smooth Riemannian n-manifold with boundary. Then $\Omega^k(M)$ decomposes as the orthogonal sum:*

$$\Omega^k(M) = cE^k_N(M) \oplus \mathcal{H}^k(M) \oplus E^k_D(M) \tag{82}$$

moreover, the spaces of harmonic k forms decomposes as:

$$\mathcal{H}^k(M) = \mathcal{H}^k_N(M) \oplus EcC^k(M) = \mathcal{H}^k_D(M) \oplus CcE^k(M). \tag{83}$$

where $EcC^k(M) = \{\omega \in \Omega^k(M) | \delta\omega = 0, \omega = d\eta\}$ and
$CcE^k(M) = \{\omega \in \Omega^k(M) | d\omega = 0, \omega = \delta\eta\}$.

Proof. Corollary 2.4.9 in **(20)** □

Theorem 1.4.2.17. *Let (M, g) a compact oriented, smooth Riemannian n-manifold with boundary, then:*
1) $H^k(M) \cong \mathcal{H}^k_N(M)$.
2) $H^k(M, \partial M) \cong \mathcal{H}^k_D(M)$.

Proof. **1**) We want to show that for every $\omega \in H^k(M)$ exists a unique $\omega_N \in \mathcal{H}^k_N(M)$ such that $[\omega] = [\omega_N]$. Using the Hodge-Morrey-Friedrichs decomposition we have that for every $\omega \in \Omega^k(M)$ we can write, $\omega = \delta\alpha + \beta + d\gamma$, the condition $d\omega = 0$ implies $d\delta\alpha = 0$. Then we have:

$$0 =< d\delta\alpha, \alpha >=< \delta\alpha, \delta\alpha > \tag{84}$$

so $\delta\alpha = 0$. Now using the decomposition $\mathcal{H}^k(M) = \mathcal{H}^k_N(M) \oplus EcC^k(M)$, we can write $\beta \in \mathcal{H}^k(M) = \omega_N + d\eta$, this implies that $\omega = \omega_N + d(\gamma + \eta)$ and so $[\omega] = [\omega_N]$.
2) We want to show that for every $\omega \in H^k(M, \partial M)$ exists a unique $\omega_D \in \mathcal{H}^k_D(M)$ such that $[\omega] = [\omega_D]$. Using the Hodge-Morrey-Friedrichs decomposition we can write $\Omega^k_D(M) = (cE^k_N(M) \cap \Omega^k_D(M)) \oplus \mathcal{H}^k_D(M) \oplus E^k_D(M)$, so write $\omega = \delta\alpha + \omega_D + d\gamma$, as before the condition $d\omega = 0$ implies $\delta\alpha = 0$ so $\omega = \omega_D + d\gamma$ and $[\omega] = [\omega_D]$. □

Corollary 1.4.2.18 (Lefschetz Duality for Riemannian manifolds)**.** *Let (M, g) a compact oriented, smooth Riemannian n-manifold with boundary, then:*

$$H^k(M) \cong H^{n-k}(M, \partial M) \tag{85}$$

Proof. Using the previous theorem we have $H^k(M) \cong \mathcal{H}^k_N(M)$ and $H^{n-k}(M, \partial M) \cong \mathcal{H}^{n-k}_D(M)$ and using **(1.4.2.15)** we know that $\star$ defines and isomorphism between $\mathcal{H}^k_N(M)$ and $\mathcal{H}^{n-k}_D(M)$. □

2 Discrete Hodge star Operator

In this chapter we consider a smooth compact Riemannian manifold with boundary (M, g) with a triangulation K. Our objective is to find a discrete counterpart in the simplicial cohomology of K for the smooth operator $\star$.
We will extend the construction made by Wilson in **(23)** to the case of a compact manifold with boundary, in order to manage boundary conditions we need to define two $\star$ operators, $\bigstar_a : C^k(K) \to C^{n-k}(K, \partial K)$ and $\bigstar_r : C^k(K, \partial K) \to C^{n-k}(K)$, where ∂K is the triangulation induced by K on the boundary of M (that exists because of **(1.3.1.13)**).
In order to do so, we need a pairing $(\cdot, \cdot) : C^k(K) \otimes C^{n-k}(K, \partial K) \to \mathbb{R}$ and two inner products $< \cdot, \cdot >_a$: $C^k(K) \times C^k(K) \to \mathbb{R}$ and $< \cdot, \cdot >_r$: $C^k(K, \partial K) \times C^k(K, \partial K) \to \mathbb{R}$.
Once we have both the inner products and the pairing we can define $\bigstar_a(\sigma)$, for every $\sigma \in C^k(K)$, such that $< \bigstar_a \sigma, \tau >_a = (\sigma, \tau)$ for every $\tau \in C^{n-k}(K, \partial K)$ and viceversa for $\bigstar_r$.

2.1 Pairing

In order to build our pairing we will use a cup product
$\cup : C^k(K) \otimes C^l(K, \partial K) \to C^{k+l}(K, \partial K)$ that mimics the work of the wedge product in the smooth setting, this cup product will be different from the one that we defined in **(1.2.7)** but using the Theorem 4 in **(20)**, the product induced in cohomology will be equal and so also the pairing induced in cohomology will be the same of the one defined in **(1.2.7)** .

Lemma 2.1.0.1. *Let $\phi \in C^k(K, \partial K)$ then $\mathbf{t}(W(\phi)) = 0$, where W is the Whitney map defined in **(1.2.5.10)**.*

Proof. We know from **(1.4.2.7)** that $\mathbf{t}(W(\phi)) = 0$ iff $i^*(W(\phi)) = 0$, moreover we can suppose $\phi = \sigma^*$ with $\sigma = [p_0, .., p_k]$ with $p_0 \notin \partial M$, then using the **step 3** of the proof of **(1.2.5.10)**, the barycentric coordinate μ_{p_0} vanishes on the boundary and so by **(33)**, $i^*(W(\sigma^*)) = 0$ too. □

Remark 2.1.0.2 (Boundary conditions)**.** *From this lemma we see that $C^k(K, \partial K)$ is a good discrete analogue for $\Omega^k_D(M)$, unfortunately we don't have a good counterpart for $\Omega^k_N(M)$ too.*

Definition 2.1.0.3 (Discrete Cup product)**.** *Let $a \in C^k(K)$ and $b \in C^l(K, \partial K)$, then we define $a \cup b \in C^{k+l}(K, \partial K)$ as:*

$$a \cup b = R(W(a) \wedge W(b)) \tag{86}$$

Theorem 2.1.0.4. *Let* $\sigma = [p_{\alpha_0} .., p_{\alpha_j}] \in C_j(K)$ *and* $\tau = [p_{\beta_0}, .., p_{\beta_k}] \in C_k(K, \partial K)$ *and* $\sigma^* \in C^j(K)$, $\tau^* \in C^k(K, \partial K)$ *their dual, then* $\sigma^* \cup \tau^*$ *is zero unless* σ *and* τ *intersect in exactly one vertex and span a* $(j+k)$*-simplex* v*, in which case, for* $\tau = [p_{\alpha_j}, p_{\alpha_{j+1}}, .., p_{\alpha_{j+k}}]$ *we have:*

$$\sigma^* \cup \tau^* = \epsilon(\sigma, \tau) \frac{j!k!}{(j+k+1)!} v^* \tag{87}$$

where $\epsilon(\sigma, \tau)$ *is defined in such a way that*
$orientation(\sigma) \cdot orientation(\tau) = \epsilon(\sigma, \tau) \cdot orientation(v)$*, and* $v = [p_{\alpha_0}, .., p_{\alpha_{j+k}}]$.

Proof. For any simplex α, $W(\alpha^*) = 0$ in $M \setminus \overline{St(\alpha)}$, $\sigma^* \cup \tau^* = R(W(\sigma^*) \wedge W(\tau^*)) = 0$ if $Vert(\sigma) \cap Vert(\tau) = \emptyset$ since this would imply $\overline{St(\sigma)} \cap \overline{St(\tau)} = \emptyset$. If they intersect in more than one vertex $W(\sigma^*) \wedge W(\tau^*) = 0$ since it would be a sum of terms containing $d\mu_{\alpha_i} \wedge d\mu_{\alpha_i}$ for some i. Thus, up to a reordering the vertices of K, it suffices to show that for $\sigma = [p_0, .., p_j]$, $\tau = [p_j, .., p_{j+k}]$ and $v = [p_0, .., p_{j+k}]$ we have $\sigma^* \cup \tau^*(v) = \epsilon(\sigma, \tau) \frac{j!k!}{(j+k1)!}$.
By the definition of W and R we have:
$\sigma^* \cup \tau^*(v) = \int_v W(\sigma^*) \wedge W(\tau^*) = j!k! \int_v \sum_{i=0}^{j+k} (-1)^i \mu_i \mu_j d\mu_0 \wedge .. \wedge \hat{d\mu_i} \wedge .. \wedge d\mu_{j+k}$
Now, $\sum_i \mu_i = 1$ so $d\mu_0 = -\sum_{i=1}^{j+k} d\mu_i$, so the last expression in equal to
$j!k! \int_v \sum_{i=0}^{j+k} (-1)^i \mu_i \mu_j (-d\mu_i) \wedge d\mu_1 \wedge .. \wedge \hat{d\mu_i} \wedge .. \wedge d\mu_{j+k} =$
$= j!k! \int_v \mu_j \sum_{i=0}^{j+k} \mu_i d\mu_1 \wedge .. \wedge d\mu_k = j!k! \int_v \mu_j d\mu_1 \wedge .. \wedge d\mu_{j+k}$, where in the last equation we have used $\sum_i \mu_i = 1$.
Now call $A = \int_v \mu_j d\mu_1 \wedge .. \wedge d\mu_{j+k}$ clearly $A = \int_v \mu_s d\mu_1 \wedge .. \wedge d\mu_{j+k}$ for any s, using $\int_v d\mu_1 \wedge .. \wedge d\mu_{j+k} = \pm \frac{1}{(j+k)!}$ (it is the volume of the standard $(j+k)$-simplex), we get $(j+k+1)A = \int_v d\mu_1 \wedge .. \wedge d\mu_{j+k} = \pm \frac{1}{(j+k)!}$ and so
$A = \int_v \mu_s d\mu_1 \wedge .. \wedge d\mu_{j+k} = \pm \frac{1}{(j+k+1)!}$ where the sign is defined by $\epsilon(\sigma, \tau)$.
□

Remark 2.1.0.5. *Notice that this cup product can be seen as the restriction to* $C^k(K) \times C^j(K, \partial K)$ *of a cup product* $C^k(K) \times C^j(K) \to C^{k+j}(K)$ *defined in* *the same way, and thanks to the previous theorem the latter cup product satisfies $I \cup \phi = \phi \cup I = \phi$, for every $\phi \in C^k(K)$ where $I \in C^0(K)$ is equal to $\sum_{p \in Vert(K)} p^*$.*

Theorem 2.1.0.6. *The element of $H_n(K, \partial K)$ that correspond to the fundamental class $[M] \in H_n(M, \partial M)$ is the orientation cycle for the pair $(K, \partial K)$ defined in **(1.3.1.2)**.*

Proof. The isomorphism between the simplicial homology of K and the singular homology of M is given by the map that associates to each n simplex $\sigma = [v_0, .., v_n]$ in K, the map $\tilde{\sigma} : \Delta^n \to M$ that sends $e_i \to v_i$ and is extended by linearity.
The fundamental class of M is characterized by the fact that for every x in $M \setminus \partial M$, the map $H_n(M) \to H_n(M|x)$ sends $[M]$ into a generator of $H_n(M|x)$. So to prove the theorem we have to show that for any σ, n simplex of K, the map $\tilde{\sigma}$ generates $H_n(M|x)$ for every x interior point of σ. Using excision and a chart of M it's enough to prove it for $\mathbb{R}^n$, and using again excision reduces to show that the identity map on Δ^n generates $H_n(\Delta^n, \Delta^n \setminus \{x\}) \cong H_n(\Delta^n, \partial \Delta^n)$.
This will be done by induction on n. If $n = 0$ $H_0(\Delta^0, \partial \Delta^0) = H_0(\{pt\})$ so the result is trivial.
If $n = 1$ consider the short exact sequence:

$$0 \to H_1(\Delta^1, \partial \Delta^1) \xrightarrow{\delta} H_0(\partial \Delta^1) \xrightarrow{i_*} H_0(\Delta^1) \to 0 \tag{88}$$

with $\Delta^1 = [e_0, e_1]$ and $\partial \Delta^1 = \{e_0, e_1\}$, here $H_0(\partial \Delta^1) = \mathbb{Z} \oplus \mathbb{Z}$ and it is generated by $[e_0]$ and $[e_1]$, the kernel of i_* is given by $K = \{a[e_0] + b[e_1] | a + b = 0\}$. Since the sequence is exact $Id : \Delta^1 \to \Delta^1$ generates $H_1(\Delta^1, \partial \Delta^1)$ iff $\delta(Id)$ generates K. Looking at the construction of the connecting morphism δ, it's clear that $\delta(Id) = [e_0] - [e_1]$ and so it's a generator of K.
Now if $n \geq 2$, define Λ^n as the set of all but one boundary simplexes in Δ^n, this is contractible to the vertex opposite to the face that is not contained in Λ^n. Now we have:

$$H_n(\Delta^n, \partial \Delta^n) \xrightarrow{\delta} H_{n-1}(\partial \Delta^n) = H_{n-1}(\partial \Delta^n, \Lambda^n) \overset{exc}{\cong} H_{n-1}(\Delta^{n-1}, \partial^{n-1}) \tag{89}$$

the identity in Δ^n is sent by δ to the alternated sum of the identities of the simplexes in the boundary, and so by excision this is sent to the identity in Δ^{n-1}.
So the result for Δ^{n-1} implies the one for Δ^n.

□

Definition 2.1.0.7 (Pairing)**.** *Let $[M] \in H_n(K, \partial K)$ be the fundamental class of M, then we define the pairing:*

$$\begin{aligned} (\cdot,\cdot): C^k(K) \times C^{n-k}(K, \partial K) \to \mathbb{R} \\ (\phi, \psi) \mapsto (\phi \cup \psi)([M]) \end{aligned} \tag{90}$$

Remark 2.1.0.8. *As we pointed out in the introduction to this section this pairing induces in cohomology induces in cohomology the same non degenerate pairing defined in **(1.2.7)**, but at level of chains we have no hope of building a non degenerate pairing since the dimension of $C^k(K)$ and $C^{n-k}(K, \partial K)$ are different, in chapter 3 we will see that this does not allow to use the discrete Hodge star operator that we define in this section to solve a Poisson system.*

Proposition 2.1.0.9. *Let $\sigma = [p_{\alpha_0}, .., p_{\alpha_k}] \in C_k(K)$ with $p_{\alpha_0} < .. < p_{\alpha_k}$ and $\tau = [p_{\beta_0}, .., p_{\beta_{n-k}}]$ with $p_{\beta_0} < .. < p_{\beta_k}$, then:*

$$(\sigma^*, \tau^*) = \begin{cases} \pm \frac{k!(n-k)!}{(n+1)!} & \textit{if } \sigma \textit{ and } \tau \textit{ share only 1 vertex and span an } n \textit{ simplex} \\ 0 & \textit{otherwise} \end{cases} \tag{91}$$

Proof. If σ and τ share more than one vertex or they do not span an n simplex then from **(2.1.0.4)** their cup product vanishes.
In this case $\sigma^* \cup \tau^* = \epsilon(\sigma, \tau)\frac{k!(n-k)!}{(n+1)!}v^*$ with $v = [p_0, .., p_n]$ is spanned by σ and τ, since $[M] = \sum_{w \in K^{(n)}} w$ then $\sigma^* \cup \tau^*([M]) = \alpha_w \frac{k!(n-k)!}{(n+1)!}$, with $\alpha_w = \epsilon(\sigma, \tau) \cdot (v^*, w) = \pm 1$ where (v^*, w) is equal to 1 if v and w have the same orientation and it is equal to -1 otherwise. □

Lemma 2.1.0.10. *Let $a \in C^k(K)$ and $b \in C^{n-k}(K, \partial K)$ then:*

$$(a, b) = \int_M W(a) \wedge W(b). \tag{92}$$

Proof. From the definition $(a, b) = (a \cup b)([M]) = R(W(a) \wedge W(b))[M] =$
$= \int_{[M]} W(a) \wedge W(b)$, from **(2.1.0.6)** we know that $[M]$ is equal to the sum of the n simplexes of K, oriented in such a way that if τ is an $n-1$ simplex of K and it is a face of σ_1 and σ_2 then the orientations of τ in σ_1 and σ_2 are opposite, so $\int_{[M]} W(a) \wedge W(b) = \int_M W(a) \wedge W(b)$. □

2.2 Inner products

In this section we are going to define the inner product that we need to build the Hodge star operators, thanks to this inner product we will also able to build a discrete counterpart of the harmonic forms that we defined in **(1.4.2.11)** and also an orthogonal decomposition that is equivalent to the one defined in **(1.4.2.16)** .
The inner product that we use was firstly defined by Eckemann in **(18)** and then used by Dodziuk in **(25; 21)**.

Definition 2.2.0.1 (Inner products)**.** *Let* $\sigma, \tau \in C^k(K)$ *then we define* $< \sigma, \tau >_a = < W(\sigma), W(\tau) >$ *where the inner product of the Whitney forms is the one defined in the **(1.4.1.13)**.*
In the relative case we see $C^k(K, \partial K)$ *as a subsapce of* $C^k(K)$ *and define* $< \cdot, \cdot >_r = < \cdot, \cdot >_{a|C^k(K,\partial K)}$.
The fact that these two products are non degenerate depends on the fact that the Whitney map is injective so if $< \sigma, \sigma > = < W(\sigma), W(\sigma) > = 0$ *then* $W(\sigma) = 0$ *from the non degeneracy of the inner product in* $\Omega^k(M)$ *and so* $\sigma = 0$ *from the injectivity of* W*.*
Since the two inner products are defined in the same way we will refer to them just as $< \cdot, \cdot >$*.*

As we pointed out in the introduction of the **paragraph 1.3.3** the Hodge star operator should model the material properties of the medium, so we expect also the discrete Hodge star operator to do so.
The idea behind the choice of this inner product is that this characteristic of the operator depends on the choice of the inner product and not on the choice of the pairing and so the inner product is built using the continuous inner product (and so the continuous Hodge operator).

Definition 2.2.0.2 (Codifferential)**.** *We define* $\delta : C^k(K) \to C^{k-1}(K)$ *such that:*

$$< d\sigma, \tau > = < \sigma, \delta\tau >, \quad \forall \sigma \in C^{k-1}(K), \tau \in C^k(K) \tag{93}$$

the same can be done for the relative case.

Definition 2.2.0.3 (Discret Harmonic Forms)**.** *Define* $\mathcal{H}^k(K) = \{\sigma \in C^k(K) | d\sigma = 0, \delta\sigma = 0\}$*, and in the same way* $\mathcal{H}^k(K, \partial K)$*.*

Theorem 2.2.0.4 (Discrete Hodge Decomposition)**.** *The following two orthogonal decompositions hold:*
1) $C^k(K) = d(C^{k-1}(K)) \oplus \mathcal{H}^k(K) \oplus \delta(C^{k+1}(K))$.
2) $C^k(K, \partial K) = d(C^{k-1}(K, \partial K)) \oplus \mathcal{H}^k(K, \partial K) \oplus \delta(C^{k+1}(K, \partial K))$.

Proof. **1**) First we prove the orthogonality, let $d\sigma \in d(C^{k-1}(K))$, $\tau \in \mathcal{H}^k(K)$ and $\delta\phi \in \delta(C^{k+1}(K))$, we have:

$$\begin{aligned} &< d\sigma, \tau >=< \sigma, \delta\tau >= 0 \\ &< d\sigma, \delta\phi >=< \sigma, \delta\delta\phi >= 0 \\ &< \tau, \delta\phi >=< d\tau, \phi >= 0. \end{aligned} \tag{94}$$

so it's enough to prove that $dim(C^k(K)) = dim(d(C^{k-1}(K))) + dim(\mathcal{H}^k(K)) +$ $+dim(\delta(C^{k+1}(K)))$.
Let $\delta^k : C^k(K) \to C^{k-1}(K)$, then we will show that
$Ker(\delta^k) = \mathcal{H}^k(K) \oplus \delta^{k+1}(C^{k+1}(K))$. The orthogonality of the two subspaces follows from the previous computations, so we just have to show that for every $\sigma \in Ker(\delta^k)$, if we denote $\pi\sigma$ the orthogonal projection of σ in $\delta^{k+1}(C^{k+1}(K))$ then $d(\sigma - \pi\sigma) = 0$, this is true because $< d(\sigma - \pi\sigma), d(\sigma - \pi\sigma) >=$ $=< \delta d(\sigma - \pi\sigma), \sigma - \pi\sigma >= 0$, where we have used that $\sigma - \pi\sigma$ is orthogonal to $\delta^{k+1}(C^{k+1}(K)$.
Now we have $dim(C^k(K) = dim(Ker(\delta^k)) + dim(\delta^k(C^k(K))) = dim(\mathcal{H}^k(K)) +$ $+dim(\delta^{k+1}(C^{k+1}(K))) + dim(\delta^k(C^k(K)))$.
The last thing to show is that $dim(\delta^k(C^k(K))) = dim(d^{k-1}(C^{k-1}(K))$, but this is true since both $d^{k-1} : \delta^k(C^k(K)) \to d^{k-1}C^{k-1}(K)$ and $\delta^k : d^{k-1}C^{k-1}(K) \to \delta^k(C^k(K))$ are injective.
2) As the proof of **1**). □

Corollary 2.2.0.5. *1)* $H^k(K) \cong R(\mathcal{H}^k_N(M))$.
2) $R(\mathcal{H}^k_D(M)) \cong H^k(K, \partial K) \cong \mathcal{H}^k(K, \partial K)$.

Proof. **1**) Using the point 1 of **(1.4.2.17)** for every $[\sigma] \in H^k(K)$ there is $\omega \in \mathcal{H}^k_N(M)$ such that $[\omega] = [W(\sigma)]$ and so $[R(\omega)] = [\sigma]$.
2) The first isomorphism is proved in the same way of the one in point 1 with the support of **(2.1.0.1)**, while the second isomorphism can be proved in the same way of the point 2 of **(1.4.2.17)**, using the discrete Hodge decomposition of $C^k(K, \partial K)$. □

Remark 2.2.0.6. *The result contained in the last corollary can be viewed as another form of the Leschetz Duality in a triangulated Riemannian Manifold.*

2.3 Hodge star

Definition 2.3.0.1 (Hodge star operator)**.** *Let* $\sigma \in C^k(K)$ *then* $\star_a\sigma \in C^{n-k}(K, \partial K)$ *is defined in such a way that:*

$$< \bigstar_a\sigma, \tau >= (\sigma \cup \tau)([M]), \quad \forall \tau \in C^{n-k}(K, \partial K) \tag{95}$$

While for $\tau \in C^k(K, \partial K)$ then $\star_r \tau \in C^{n-k}(K)$ is defined in such a way that:

$$< \sigma, \bigstar_r \tau >= (\sigma \cup \tau)([M]), \quad \forall \sigma \in C^{n-k}(K). \tag{96}$$

Remark 2.3.0.2 (Matricial form)**.** *Since $\bigstar_a$ is a linear operator from $C^k(K)$ to $C^{n-k}(K, \partial K)$, we can write his associated matrix.*
*Let P^k to be the matrix associated to the pairing $C^k(K) \times C^{n-k}(K, \partial K) \to \mathbb{R}$, i.e $P^k(i,j) = (\sigma_i{}^*, \tau_j{}^*)$, where $\{\sigma_i\}_{i=0,..,N^k}$ are the k simplexes in K and $\{\tau\}_{j=0,..,N_r^{n-k}}$ are the $(n-k)$-simplexes in K that are not in the boundary, then we have $(\sigma^*, \tau^*) = \sigma^{*T} P^k \tau^*$. Moreover if M_r^{n-k} is the matrix associated to the inner product in $C^{n-k}(K, \partial K)$, we have $< \bigstar_a \sigma^*, \tau^* >= \bigstar_a \sigma^{*T} M_r^{n-k} \tau^*$, so the equality **(95)** became $\bigstar_a \sigma^{*T} = \sigma^{*T} P^k \cdot (M_r^{n-k})^{-1}$.*
The same argument applies to prove that the matrix associated to $\bigstar_r$ is $(M^{n-k})^{-1} \cdot P^k$.

Using **(1.4.2.2)** it's easy to show that, in $\Omega^*(M)$, $\star d = (-1)^{k+1} \delta \star$, in the following lemma we prove that this property holds also in the discrete case.

Proposition 2.3.0.3. *The following holds:*
1) For every $\sigma \in C^k(K)$ then $\bigstar_a d\sigma = (-1)^{k+1} \delta \bigstar_a$.
2) The same result for $\bigstar_r$.

Proof. **1**) Let $\tau \in C^{n-k-1}(K, \partial K)$ then:

$$< \bigstar_a d\sigma, \tau >= d\sigma \cup \tau([M]) \tag{97}$$

using the properties of the wedge product $d\sigma \cup \tau = R(W(d\sigma) \wedge W(\tau)) =$
$= R(d(W\sigma) \wedge W\tau) = R(d(W\sigma \wedge \tau) + (-1)^{k+1} W\sigma \wedge W(d\tau))$ and
$R(d(W\sigma \wedge \tau))([M]) = 0$ since it is the evaluation of an exact cochain on a closed chain, so the right hand side of **(97)** became:

$$(-1)^{k+1} \sigma \cup d\tau([M]) = (-1)^{k+1} < \bigstar_a \sigma, d\tau >= (-1)^{k+1} < \delta \bigstar_a \sigma, \tau > \tag{98}$$

since this holds for every τ we have the result.
2) The proof is the same.

□

Proposition 2.3.0.4. *Let $a \in C^k(K)$ and $b \in C^{n-k}(K, \partial K)$ then:*

$$< \bigstar_a a, b >=< a, \bigstar_r b >= \int_M W(a) \wedge W(b). \tag{99}$$

Proof. Is a direct consequence of **(2.1.0.10)** □

A property that we would like to have is that $W\bigstar = \star W$, but since in general $\star W$ is not a Whitney form we cannot expect to have such result, so in the following proposition we prove a weaker results, but it will be enough to prove to convergence of the discrete Hodge star operator in the next section.

Proposition 2.3.0.5. *Let $\pi_r^j : \Omega^j(M) \to W(C^j(K, \partial K))$ and $\pi_a^j : \Omega^j(M) \to W(C^{n-k}(K))$ be the orthogonal projections of $\Omega^j(M)$ onto $W(C^j(K\partial K))$ and $W(C^{n-k}(K))$, then:*
1) *$\forall a \in C^k(K)$, $W\bigstar_a a = \pi_r^{n-k} \star Wa$.*
2) *$\forall a \in C^k(K, \partial K)$, $W\bigstar_r a = \pi_a^{n-k} \star W$.*

Proof. **1**) We have to show that $< W\bigstar_a a, Wb >=< \star Wa, Wb >, \forall b \in C^{n-k}(K, \partial K)$.

$$< W\bigstar_a a, Wb >=< \bigstar_a a, b > \overset{\mathbf{(2.3.0.4)}}{=} \int_M Wa \wedge Wb =< \star Wa, Wb > . \quad (100)$$

2) As the point **1**). □

2.4 Convergence Results

In this section we use the techniques developed by Dodziuk in **(24)** to show that $\bigstar$ converges to $\star$ where the triangulation gets denser.

Definition 2.4.0.1 (Standard Subdivision of a Complex)**.** *Let $\sigma = [p_0, .., p_m]$ be a simplex in $\mathbb{R}^k$, $k \geq m$, the vertices of $S\sigma$ are the points, $p_{i,j} = \frac{1}{2}(p_i + p_j)$, $i \leq j$. We define a partial order of the vertices of $S\sigma$ by setting $p_{i,j} \leq p_{k.l}$ if $i \geq k$ and $j \leq l$.*
The simplexes of $S\sigma$ are increasing sequences of vertices w.r.t the above ordering. If τ is face of σ $S\tau$ equals the subdivision made by the simplexes of $S\sigma$ contained in τ. This allows to define the standard subdivision of an ordered simplicial complex L in a natural way.
Moreover we define inductively $S_0(L) = L$ and $S_{n+1}L = S(S_nL)$.

Definition 2.4.0.2. *Let $\sigma = [p_0, .., p_m]$ and $\sigma' = [q_0, .., q_m]$ be two simplexes in $\mathbb{R}^m$, we say that they are strongly similar if exists $\lambda > 0$ such that:*

$$\lambda(\sigma - p_0) = \sigma' - q_0 \quad (101)$$

where $\sigma - p_0$ is the rigid translation of σ that takes p_0 to the origin, trivially we can see that this defines an equivalent relation of m simplexes in $\mathbb{R}^m$.
Moreover we say that σ is well placed if it's strongly similar to $[0, e_1, .., e_m]$ where $\{e_1, .., e_m\}$ is a standard basis of $\mathbb{R}^m$.

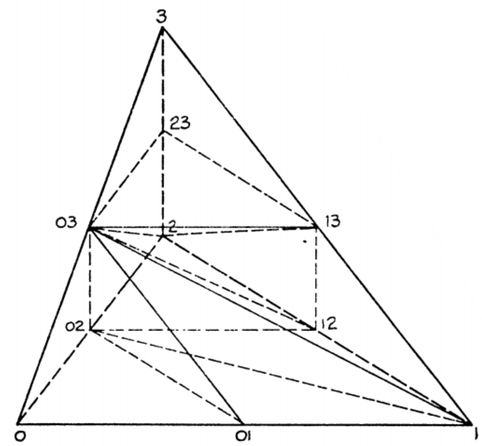

Figure 9: Standard Subdivision of a tetrahedron.

Definition 2.4.0.3. *We say that an n simplex σ of a smooth triangulation of M is well places in a coordinate chart (U, ϕ) if :*
1) $\overline{\sigma} \subset U$.
2) $\phi_{|\overline{\sigma}} : \overline{\sigma} \to \mathbb{R}^n$ *is linear.*
3) $\phi(\sigma)$ *is well places in* $\mathbb{R}^n$

Lemma 2.4.0.4. *There exists a finite set $\mathcal{U}$ of coordinate charts of M with the following property. For every integer $k \geq 0$ and every n-dimensional simplex τ of $S_k K$ there exist a coordinate chart $(U, \phi) \in \mathcal{U}$ and an n-simplex σ of K such that:*
1) τ *is well placed in* (U, ϕ).
2) $\tau \subset \overline{\sigma} \subset U$.

Proof. Lemma 3.4 in **(25)**. □

Definition 2.4.0.5 (Mesh). *Let $\eta_k = \eta_k(K) = \sup_{\sigma \in S_k K} diam(\sigma)$, where $diam(\sigma)$ is measured in metric induced by the euclidean distance in a coordinate neighborhood in which σ is well placed, we call η_k the mesh of $S_k K$.*

Lemma 2.4.0.6. $\lim_{k \to \infty} \eta_k = 0$.

Proof. Lemma 3.6 in **(25)**. □

Notation 2.4.0.7. *We define W_k and R_k the Whitney and de Rham maps defined on the triangulation $S_k K$ of M.*

Lemma 2.4.0.8. *Let $\sigma = [p_0, .., p_n]$, $N = \{1, .., n\}$, $I = \{i_1, .., i_m\} \subset N$ and $\sigma_I^\tau = [p_\tau, p_{i_1}, .., p_{i_m}]$. Then*

$$W(\sigma_I^{0^*}) = m! d\mu_{i_1} \wedge .. \wedge d\mu_{i_m} - \sum_{\tau \in N \setminus I} W(\sigma_I^{\tau^*}) \tag{102}$$

Proof. Let $d\mu_I = d\mu_{i_1} \wedge .. \wedge d\mu_{i_m}$ and $d\mu_I^s = d\mu_{i_1} \wedge .. \wedge d\hat{\mu}_{i_s} \wedge .. d\mu_{i_m}$.
By definition $\frac{1}{m!} W(\sigma_I^{0^*}) = \mu_0 d\mu_I + \sum_{s=1}^{m} (-1)^s \mu_{i_s} d\mu_0 \wedge d\mu_I^s$, now using $\sum \mu_i = 1$ as we did in the proof of **(2.1.0.4)**, the term on the right hand side became:

$$\begin{aligned}
&(1 - \sum_{r=1}^{n} \mu_r) d\mu_I + \sum_{s=1}^{m} (-1)^s \mu_{i_s} (-\sum_{r=1}^{n} d\mu_r) \wedge d\mu_I^s = \\
&= d\mu_I - \sum_{r=1}^{n} \mu_r d\mu_I - \sum_{s=1}^{m} \mu_{i_s} (d\mu_{i_s} + \sum_{r \in N \setminus I} d\mu_r) \wedge d\mu_I^s = \\
&= d\mu_I - \sum_{\tau \in N \setminus I} \mu_r d\mu_I - \sum_{s=1}^{m} \mu_{i_s} (\sum_{r \in N \setminus I} d\mu_r) \wedge d\mu_I^s = \\
&= d\mu_I - \sum_{r \in N \setminus I} (\mu_r d\mu_I + \sum_{s=1}^{m} (-1)^s \mu_{i_s} d\mu_r \wedge d\mu_I^s) = \\
&= d\mu_I - \frac{1}{m!} \sum_{r \in N \setminus I} W(\sigma_I^{r^*}).
\end{aligned} \tag{103}$$

□

Theorem 2.4.0.9. *Let $\omega_1, \in \Omega^j(M)$ and $\omega_2 \in \Omega^k(M)$, then exists a constant $C(\omega_1, \omega_2)$ independent of k such that:*

$$|W_k(R_k(\omega_1) \cup R_k(\omega_2))(p) - \omega_1 \wedge \omega_2(p)|_p \leq C^\sigma(\omega_1, \omega_2) \eta_k \tag{104}$$

almost everywhere on M.

Proof. The proof is based on the techniques developed in **(24)** and **(22)**.
Fix k, the $n-1$ skeleton of $S_k K$ has measure 0 so we can suppose that $p \in \mathring{\sigma}$ for a unique n simplex σ in $S_k K$. Let (U, ϕ) be a coordinate chart in which σ is well placed, we can identify U as a subset of $\mathbb{R}^n$, and since σ is well placed we can suppose that $\sigma = [0, he_1, .., he_n]$ for some $h > 0$, here we can suppose that $\omega_1 = f dx_1 \wedge .. \wedge dx_j$ and $\omega_2 = g dx_{\alpha_1} \wedge .. \wedge dx_{\alpha_k}$, and the baycentric coordinates corresponding to $0, he_1, .., he_n$ are :

$$\begin{aligned}
&\mu_0 = 1 - \frac{1}{h} \sum_{i=1}^{n} x_i \\
&\mu_i = \frac{x_i}{h}, \quad i \neq 0
\end{aligned} \tag{105}$$

Let $N = \{0, .., n\}$, $J = \{1, 2, .., j\}$ and $K = \{\alpha_1, .., \alpha_k\}$, then:

$$R\omega_1 = \sum_{\beta \in N \setminus J} (\int_{[p_\beta, p_1, .., p_j]} \omega_1)[p_\beta, p_1, .., p_j]$$
$$R\omega_2 = \sum_{\gamma \in N \setminus K} (\int_{[p_\gamma, p_{\alpha_1}, .., p_{\alpha_k}]} \omega_2)[p_\gamma, p_{\alpha_1}, .., p_{\alpha_j}] \tag{106}$$

Now using **(2.1.0.4)**, $R\omega_1 \cup R\omega_2 = 0$ if J and K intersect in more then one element, so we have to study the cases in which the intersection is empty or consists of 1 element, we will study the two cases seprately.

1) $J \cap K = \{\alpha_1 = j\}$, we can suppose $K = \{j, j+1, .., j+k-1\}$

Using the notation $\sigma_J^\beta = [p_\beta, p_1, .., p_j]$, $\sigma_K^\gamma = [p_\gamma, p_{\alpha_1}, ..p_{\alpha_k}]$ the cup product ${\sigma_J^\beta}^* \cup {\sigma_K^\gamma}^*$ is non zero iff $\beta \neq \gamma \in Q = N \setminus (J \bigcup K)$. We define:

$$[p_s, p_J, p_K] = [p_s, p_1, .., p_j, p_{\alpha_1}, .., p_{\alpha_k}]$$
$$\int_{[s]} \omega_1 = \int_{[p_s, p_1, .., p_j]} \omega_1 \tag{107}$$
$$\int_{[s]} \omega_2 = \int_{[p_s, p_{\alpha_1}, .., p_{\alpha_k}]} \omega_2$$

and compute

$$R_k\omega_1 \cup R_k\omega_2 = \frac{j!k!}{(j+k+1)!} \sum_{\beta, \gamma \in Q, \beta \neq \gamma} (\int_{[\beta]} \omega_1)(\int_{[\gamma]} \omega_2)[p_\beta, p_\gamma, p_J, p_K] \tag{108}$$

from which we have:

$$W_k(R_k\omega_1 \cup R_k\omega_2) = \frac{j!k!}{(j+k+1)!} \sum_{\beta, \gamma \in Q, \beta < \gamma} (A_{\beta,\gamma}) W(\sigma^{\beta,\gamma}) \tag{109}$$

where $\sigma^{\beta,\gamma}$ is the dual of $[p_\beta, p_\gamma, p_J, p_K]$ and $A_{\beta,\gamma} = (\int_{[\beta]} \omega_1)(\int_{[\gamma]} \omega_2) - (\int_{[\gamma]} \omega_1)(\int_{[\beta]} \omega_2)$, using the definition of W and **(105)** we can prove that:

$$W_k(\sigma^{0,\beta}) = \frac{(j+k)!}{h^{j+k}}(dX^\beta - \frac{1}{h}(\sum_{i=j+k}^{n} x_i dX^\beta + \sum_{i=1}^{j+k-1} \sum_{s=j+k, s\neq\beta}^{n} (-1)^i x_i dX_i^{s,\beta}))$$
$$W_k(\sigma^{\beta,\gamma}) = \frac{(j+k)!}{h^{j+k+1}}(x_\beta dX^\gamma - x_\gamma dX^\beta + \sum_{i=1}^{j+k-1} (-1)^{i+1} x_i dX_i^{\beta,\gamma}) \tag{110}$$

where $dX_l^{i,j} = dx_i \wedge dx_j \wedge dx_1 \wedge .. \wedge dx_{l-1} \wedge dx_{l+1} \wedge .. \wedge dx_{j+k-1}$, notice that $|dx_i|_p \leq 1$ for every i and every $p \in \overset{\circ}{\sigma}$ so it's using triangle inequality it's enough

to give an estimate to to the coefficients of $W_k(R_k\omega_1 \cup R_k\omega_2)$, those are of two kind:
I) $\frac{j!k!}{h^{j+k}}A_{0\beta}$.
II) $\frac{j!k!}{h^{j+k+1}}A_{\beta,\gamma}x_s$.
In order to bound **I**) notice that $S_j = \frac{h^j}{j!}$ is the volume of $[i, 1, .., j]$ and $S_k = \frac{h^k}{k!}$ is the volume of $[l, j, j+1, .., j+k-1]$ for every $i, l \notin Q$, so using the mean value theorem $\frac{j!}{h^j}\int_{[0]}\omega_1 = \frac{\int_{[0]} f(x_1, .., x_j, 0, .., 0)dx_1..dx_j}{S_j} = f(p)$ for some $p \in \sigma$, repeating the same argument for the other integrals we have that:

$$\begin{aligned}
&|\frac{j!k!}{h^{j+k}}A_{0\beta}| = |f(p)g(q) - f(p')g(p')| \leq \\
&\leq |f(p)g(q) - f(p)g(q')| + |f(p)g(q') - f(p')g(q')| = \\
&= |f(p)||g(q) - g(q')| + |g(q')||f(p) - f(p')| \leq \\
&\leq \sup_\sigma |f| \sup_\sigma |\nabla g|\eta_k + \sup_\sigma |g| \sup_\sigma |\nabla f|\eta_k = C\eta_k
\end{aligned} \tag{111}$$

Now for **II**) notice that $\int_{[\beta]}\omega_1 = \int_{[0]} f(x_1, ..x_j, 0.., h(1 - \frac{1}{h}\sum_{i=1}^{j} x_i), 0..0)dx_1..dx_j$ and the same clearly holds for $\int_{[\beta]}\omega_2$ too.
So we have:

$$\begin{aligned}
&\frac{\int_{[0]}\omega_1 - \int_{[\beta]}\omega_1}{S_j} = \frac{\int_{[0]}(f(x_1, .., x_j, 0..0) - f(x_1, ..x_j, 0.., h(1 - \frac{1}{h}\sum_{i=1}^{j} x_i), 0..0)dx_1..dx_j}{S_j} = \\
&= (f(\overline{x}_1, .., \overline{x}_j, 0..0) - f(\overline{x}_1, ..\overline{x}_j, 0.., h(1 - \frac{1}{h}\sum_{i=1}^{j} \overline{x}_i), 0..0)
\end{aligned} \tag{112}$$

moreover

$$|\frac{(f(\overline{x}_1, .., \overline{x}_j, 0..0) - f(\overline{x}_1, ..\overline{x}_j, 0.., h(1 - \frac{1}{h}\sum_{i=1}^{j} \overline{x}_i), 0..0)}{h}| \leq \sup |\frac{\partial f}{\partial x^\beta}| \leq C \tag{113}$$

Finally we can write:

$$
\begin{aligned}
&|\frac{j!k!}{h^{j+k+1}}A_{\beta,\gamma}x_s| = \frac{j!k!}{h^{j+k+1}}|\int_{[\gamma]}\omega_1\int_{[\beta]}\omega_2 - \int_{[\beta]}\omega_1\int_{[\gamma]}\omega_2||x_s| = \\
&= \frac{j!k!}{h^{j+k+1}}|\int_{[\gamma]}\omega_1\int_{[\beta]}\omega_2 + \int_{[0]}\omega_1\int_{[\beta]}\omega_2 - \int_{[0]}\omega_1\int_{[\beta]}\omega_2 + \\
&+ \int_{[\beta]}\omega_1\int_{[0]}\omega_2 - \int_{[\beta]}\omega_1\int_{[0]}\omega_2 + \int_{[0]}\omega_1\int_{[0]}\omega_2 - \int_{[0]}\omega_1\int_{[0]}\omega_2 - \int_{[\beta]}\omega_1\int_{[\gamma]}\omega_2||x_s| \leq \\
&\leq (\frac{j!}{h^j}|\int_{[0]}\omega_1|)\cdot(\frac{k!}{h^{k+1}}\cdot|\int_{[0]}\omega_2 - \int_{[\beta]}\omega_2|)\cdot\eta_k + \text{other terms of the same kind}
\end{aligned}
\tag{114}
$$

where we used $|x_s| \leq \eta_k$.
Now the second term in the product was in bounded in **(113)**, while the first term in the product is bounded by $\sup_\sigma |f|$, and this ends the first part of the proof.
2)$J \cap K = \emptyset$, we can suppose $K = \{j+1, .., j+k\}$. Notice that there are exactly $j+k+1$ products of the form:

$$
[p_\beta, p_J] \cup [p_\gamma, p_K] \tag{115}
$$

that equals a non vanishing multiple of $[p_\tau, p_J, p_K]$, for every $\tau \in Q$ and them are given by:

$$
\begin{aligned}
\beta = \tau, &\quad \gamma \in J \\
\gamma = \tau, &\quad \beta \in K \\
\beta = \gamma &= \tau
\end{aligned}
\tag{116}
$$

so we have:

$$
\begin{aligned}
R(\omega_1) \cup R(\omega_2) = \frac{j!k!}{j+k+1!}(&\sum_{|0|}(\int_{[\beta]}\omega_1)(\int_{[\gamma]}\omega_2)[p_0, p_J, p_K] + \\
&+ \sum_{\tau \in Q\setminus\{0\}}\sum_{|\tau|}(\int_{[\beta]}\omega_1)(\int_{[\gamma]}\omega_2)[p_\tau, p_J, p_K])
\end{aligned}
\tag{117}
$$

where $\sum_{|s|}$ is the sum over all β and γ in **(116)**.
now using 2.4.0.8 we know that:

$$
W(\sigma^0) = (j+k)!d\mu_J \wedge d\mu_K - \sum_{\tau \in Q\setminus\{0\}} W(\sigma^\tau) \tag{118}
$$

So we can write :

$$|W(R\omega_1 \cup R\omega_2)|_p = |\frac{j!k!}{(j+k+1)} \sum_{|0|}(\int_{[\beta]} \omega_1)(\int_{[\gamma]} \omega_2) d\mu_J \wedge d\mu_K - \omega_1(p)\omega_2(p)|_p + \\ + \frac{j!k!}{(j+k+1)!}| \sum_{\tau \in Q\setminus\{0\}} (\sum_{|\tau|}(\int_{[\beta]} \omega_1)(\int_{[\gamma]} \omega_2) - \sum_{|0|}(\int_{[\beta]} \omega_1)(\int_{[\gamma]} \omega_2)) W(\sigma^\tau)|_p \tag{119}$$

Now the second term in the sum can be bounded in the same way of **II** in the first part of the proof, while for the first term notice that $d\mu_J \wedge d\mu_K = \frac{1}{h^{j+k}} dx_J \wedge dx_K$, so we can write:

$$\begin{aligned} &|\frac{j!k!}{(j+k+1)} \sum_{|0|}(\int_{[\beta]} \omega_1)(\int_{[\gamma]} \omega_2) d\mu_J \wedge d\mu_K - \omega_1(p)\omega_2(p)|_p = \\ &= |\frac{j!k!}{h^{j+k}(j+k+1)} \sum_{|0|}(\int_{[\beta]} \omega_1)(\int_{[\gamma]} \omega_2) - f(p)g(p)||dx_J \wedge dx_K|_p \leq \\ &\leq |\frac{j!k!}{h^{j+k}(j+k+1)} \sum_{|0|}((\int_{[\beta]} \omega_1)(\int_{[\gamma]} \omega_2) - \int_{[0]} \omega_1)(\int_{[0]} \omega_2))| + \\ &+ |\frac{j!k!}{h^{j+k}} \int_{[0]} \omega_1 \int_{[0]} \omega_2 - f(p)g(p)| \end{aligned} \tag{120}$$

where we used that $\sum_{[0]}$ contains exactly $j+k+1$ summands, moreover each of these summands have β or γ equal to 0, if $\beta = 0$ we can write:

$$|\frac{j!k!}{h^{j+k}}(\int_{[0]} \omega_1 \int_{[\gamma]} \omega_2 - \int_{[0]} \omega_1 \int_{[0]} \omega_2)| = |\frac{j!}{h^j} \int_{[0]} \omega_1||\frac{k!}{h^{k+1}} \int_{[0]} \omega_2 - \int_{[\gamma]} \omega_2||h| \tag{121}$$

and here the first term is bounded by $\sup|f|$ the second term is bounded by a constant as in **(113)**, and $|h| \leq \eta_k$, so we just have to bound $|\frac{j!k!}{h^{j+k}} \int_{[0]} \omega_1 \int_{[0]} \omega_2 - f(p)g(p)|$ and this can be done using mean value theorem as in **(111)**, and this concludes the proof. □

Corollary 2.4.0.10. *Let $\omega_1, \in \Omega^j(M)$ and $\omega_2 \in \Omega^k(M)$, then exists a constant $C(\omega_1,\omega_2)$ independent of n such that:*

$$||W_k(R_k(\omega_1) \cup R_k(\omega_2)) - \omega_1 \wedge \omega_2|| \leq C(\omega_1,\omega_2)\eta_k \tag{122}$$

Proof. By the definition of this norm we have:

$$\begin{aligned} &||W_k(R_k(\omega_1) \cup R_k(\omega_2)) - \omega_1 \wedge \omega_2||^2 = \\ &= \int_M |W_k(R_k(\omega_1) \cup R_k(\omega_2)) - \omega_1 \wedge \omega_2(p)|_p^2 dVol_M \leq \\ &\leq C^2(\omega_1,\omega_2)\eta_k^2 vol(M)^2 \end{aligned} \tag{123}$$

where $C(\omega_1, \omega_2) = \max_\sigma C^\sigma(\omega_1, \omega_2)$ and $vol(M)$ is finite since M is compact. $\square$

Corollary 2.4.0.11. *Let $\omega \in \Omega^j(M)$ then:*

$$||W_k(R_k(\omega_1)) - \omega_1|| \leq C(\omega_1)\eta_k \tag{124}$$

Proof. Use $\omega_2 \equiv 1 \in \Omega^0(M)$, then $R_k(\omega_2) = I$ defined in the **(2.1.0.5)**, and so $R_k(\omega_1) \cup R_k(\omega_2) = R_k(\omega_1)$. $\square$

Theorem 2.4.0.12. *Let $\omega_1 \in \Omega^j_N(M)$ and $\omega_2 \in \Omega^j_D(M)$ then:*
1) $|| \star \omega_1 - W_k(\bigstar_a(R_k(\omega_1)))|| \leq C(\omega_1)\eta_k$.
2) $|| \star \omega_2 - W_k(\bigstar_r(R_k(\omega_2)))|| \leq C(\omega_2)\eta_k$

Proof. **1**) using **(2.3.0.5)** we have

$$\begin{aligned}
&|| \star \omega_1 - W_k(\bigstar_a(R_k(\omega_1)))|| = || \star \omega_1 - \pi_r^{n-j} \star (W_k(R_k(\omega_1)))|| \leq \\
&\leq || \star \omega_1 - \star(W_k(R_k(\omega_1)))|| + || \star (W_k(R_k(\omega_1))) - \pi_r^{n-j} \star (W_k(R_k(\omega_1)))|| \leq \\
&\leq || \star |||| \omega_1 - (W_k(R_k(\omega_1)))|| + || \star (W_k(R_k(\omega_1))) - W_k(R_k(\star\omega_1))|| \leq \\
&\leq || \star ||C'(\omega_1)\eta_k + || \star (W_k(R_k(\omega_1))) - \star\omega_1|| + || \star \omega_1 - W_k(R_k(\star\omega_1))|| \leq \\
&\leq 2|| \star ||C'(\omega_1)\eta_k + C(\star\omega_1)\eta_k = C(\omega_1)\eta_k
\end{aligned} \tag{125}$$

where in the second inequality we used that $\pi_r^{n-j} \star (W_k(R_k(\omega_1)))$ is the element of $W_k(C^{n-j}(K, \partial K))$ of minimum distance from $\star(W_k(R_k(\omega_1)))$ and that $\star\omega_1 \in \Omega^{n-j}_D(M)$ since the Hodge operator sends $\Omega_N(M)$ in $\Omega_D(M)$ and viceversa **(1.4.2.8)**, moreover we used that the Hodge star operator preserves the metric as it is remarked in **(1.4.1.13)**.
2) as the first part. $\square$

3 Poisson Problem

In this chapter we will use the framework defined in the chapter 2, to find a discretization of the Laplacian problem in dimension 2, the same construction is carried out in **(31)** for the n dimensional case.

3.1 The Model

The problem is defined on a triangulated compact manifold with boundary M, embedded in $\mathbb{R}^2$, and the task is to find a smooth function u on M that satisfies:

$$\begin{aligned} -\Delta u &= f \\ u_{\partial M} &= 0 \end{aligned} \tag{126}$$

where f is a smooth function over M, and $\Delta = div \circ \nabla$.
So we have to find a the right analogues for f, u, div, ∇ in the simplicial complex of K, where K is the triangulation of the M.
Clearly f, u can be considered as elements of $\Omega^0(M)$, so we start writing explicitely the complex $\Omega^*(M)$:

$$0 \to \Omega^0(M) \xrightarrow{\nabla} \Omega^1(M) \xrightarrow{curl} \Omega^2(M) \to 0 \tag{127}$$

where the 2 dimensional curl is defined, locally, as the map that sends $\omega = a(x, y)dx + b(x, y)dy$ to $(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y})dx \wedge dy$.
As we can see the divergence operator does not appear in $\Omega^*(M)$, so what we want to prove now is that $-div : \Omega^1(M) \to \Omega^0(M)$ is the the coadjoint of ∇. There are two equivalent ways to prove this fact, indeed we can use both the definition of δ given in **(1.4.2.1)** and the characterization given in **(1.4.2.3)**, in both cases we have to choose a metric on M, this will be Euclidean metric induced in $\mathbb{R}^2$, in this case using the 2nd characterization of $\star$ defined in **(1.4.1.9)** and $\omega_g = dx \wedge dy$, it's possible to show that:

$$\begin{aligned} \star 1 &= dx \wedge dy \\ \star dx &= dy \\ \star dy &= -dx \\ \star dx \wedge dy &= 1 \end{aligned} \tag{128}$$

so we can use the first definition of δ.
Take $\alpha \in \Omega^1(M)$ then locally we can write $\alpha = a(x, y)dx + b(x, y)dy$, in this

case, we have:

$$\delta\alpha = - \star d^1 \star (\alpha) = - \star d^1 (a(x,y)dy - b(x,y)dx) = - \star ((\frac{\partial b}{\partial x} + \frac{\partial a}{\partial y})dx \wedge dy) =$$
$$= -(\frac{\partial b}{\partial x} + \frac{\partial a}{\partial y}) = (-\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}) \tag{129}$$

so the equation $-\Delta(u) = f$ translates into the equation, $\delta^1 d^0(u) = f$.
The discrete counterparts of u and f are clearly given by $R(u), R(f) \in C^0(K)$ and, so we just need to find the matricial form of δ^1 and d^0.
1) Call D^0 the matrix associated to d^0, then D^0 is matrix of dimension $N_e \times N_v$ where N_v and N_e are the number of vertices in K, and :

$$D^0_{i,j} = \begin{cases} 1, & \textit{if} \quad e_i = [v, v_j] \\ -1, & \textit{if} \quad e_i = [v_j, v] \\ 0, & \textit{otherwise} \end{cases} \tag{130}$$

this is a direct consequence of the definition of d^0 **(1.2.2.12)**.
2) Now we have to find the matrix associated to δ^1, we will call it $\tilde{D}^1$, this is a matrix of dimension $N_v \times N_e$, and it is defined starting from the definition of the discrete codifferental **(2.2.0.2)**, so we have:

$$a^T \cdot D_0^T \cdot M_1 \cdot b = a^t \cdot M_0 \cdot \tilde{D}^1 \cdot b \tag{131}$$

for all $a \in C^0(K)$ and $b \in C^1(K)$, this led to, $\tilde{D}^1 = M_0^{-1} \cdot D_0^T \cdot M_1$.
So the linear system of equation that we have to solve is:

$$(D_0^T \cdot M_1 \cdot D_0)R(u) = M_0 \cdot R(f) \tag{132}$$

3.2 Numerical Experiment

We have implemented in Matlab the discretization model defined above on a Poisson Problem defined on $M = [0,1] \times [0,1]$ and with
$f(x,y) = 2(x + y - x^2 - y^2)$, we know that the solution to this problem is given by $u(x,y) = x(1-x)y(1-y)$.
We have computed the L^2 and the H^1 errors from the exact solution, then we have compared the error made by our model with the one obtained computing the solution with the Matlab function *"AssemPde"*, that implements a classical FEM method.

As you can see from the figure, as the mesh goes to 0 the solution to our system converges to the analytic solution, and the convergence rate is the same of the FEM method.

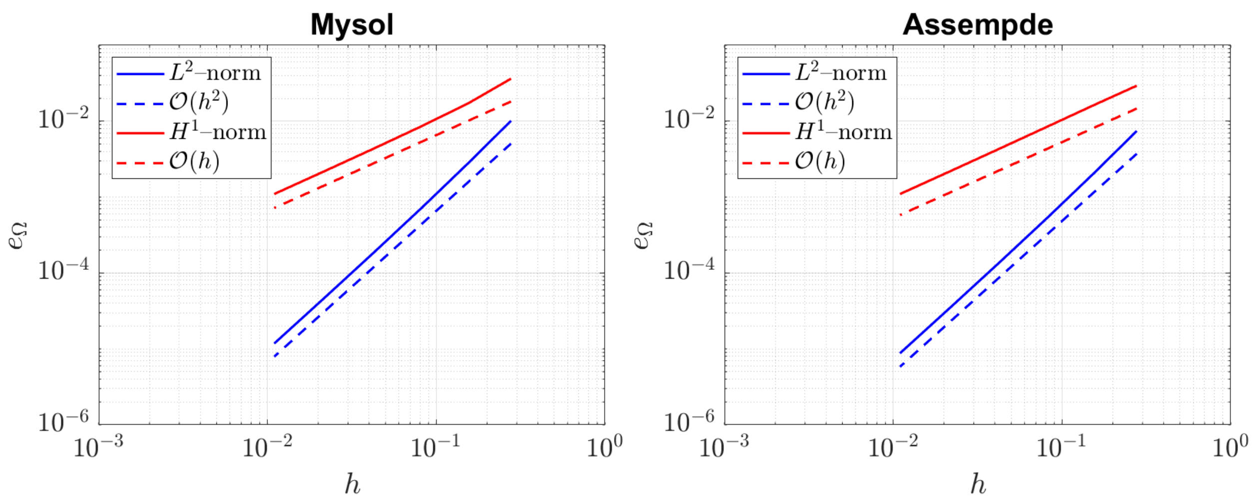

Figure 10: Both the L^2 and the H^1 errors for both the methods

Remark 3.2.0.1. *For the sake of the numerical implementation the fact that* $M = [0,1] \times [0,1]$*, is not a manifold is not a problem, since it is a Lipschitz bounded domain and so we can define on it an* L^2 *theory of differential forms that extends our theory on smooth manifold, both the definition of Lipschitz boundary domain and the definition of the theory can be found at chapter 3 of* ***(28)****.*

Remark 3.2.0.2 (Physical System)**.** *This kind of Poisson problem appears in many Physical models, for example in electromagnetism, where we can interpret the function* f *as the electric charge density, and* u *as the electric potential, so from the Maxwell's equations* ***(29)*** *we have:*

$$\begin{aligned} -\nabla u &= E \\ \epsilon E &= D \\ div(D) &= f \end{aligned} \tag{133}$$

where E *is the electric field,* D *the electric displacement and* ϵ *is the permittivity, the last one is usually represented as a type 2 tensor and usually represents a property of the material.*
When ϵ *is a constant we have the same kind of Poisson problem discussed in the previous section, but there are cases in which this is not true, in these situations we can modify our model in order to take the property of the material into account, in this case it's enough to consider a new inner product between one*

form defined as :

$$< \omega, \eta >_\epsilon = \int_M \epsilon \cdot \omega \wedge \star\eta \tag{134}$$

so the only modification to the model defined in the above section is a slight modification of the matrix M_1.

Remark 3.2.0.3 (Disretization of the Codifferential)**.** *Looking at the definition of the smooth codifferential **(1.4.2.1)** one could ask why we haven't defined the discrete codifferential as $(-1)^{n(k+1)+1}\bigstar \circ d^{n-k} \circ \bigstar$ and actually this is the way in which is defined the discrete codifferential in the numerical methods that uses a secondary mesh (for instance look at **(27)**). Although our discrete Hodge star converges to the smooth one as we proved in **(2.4.0.12)**, it cannot be used in a numerical method since, as we noticed in **(2.1.0.8)**, the pairing involved in the construction of the Hodge operator is not perfect and this causes an ill conditioning of the linear system.*

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