Università degli Studi di Padova

Dipartimento di Matematica "Tullio Levi-Civita"

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LOGARITHMIC CONFORMAL FIELD THEORIES OF TYPE B_n, $p = 2$
AND SYMPLECTIC FERMIONS

CANDIDATO:
Ilaria Flandoli
1129301

RELATORI:
Giovanna Carnovale
Simon Lentner

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Introduction

A Vertex Operator Algebra (VOA) is a complex graded vector space with a vacuum element, a derivation and a vertex operator satisfying three given axioms (Definition 1.13). For any given even integer lattice Λ it is possible to construct an associated VOA $\mathcal{V}_\Lambda$, called lattice VOA, labelling its elements by the ones of Λ. From a physical prospective this is the space of states of a free boson on a torus.

In literature many results are known about this structure and its representation theory. In particular the latter is semisimple and known to be equivalent as abelian category to the category of the Λ^*/Λ-graded vector spaces, where Λ^* is the dual lattice (see Theorem 2.7).

Less known is the case of VOAs with non-semisimple representation theory, the so-called *logarithmic conformal field theories* (LCFT). Those are indeed vertex algebras with some finiteness conditions but with non-semisimple representation theory. Their name comes from Physics where they are associated to correlation functions and these have logarithmic singularities.

Only few examples of this structure are known, as the algebra of n pairs of symplectic fermions $\mathcal{V}_{\mathcal{SF}_n^{even}}$ (see [DR16]) and the triplet algebra $\mathcal{W}_p$ (see [AM08], [FFHST02]).

In particular the latter is constructed through *free field realization*, i.e. as subVOA of a lattice VOA, in the case where the lattice is a rescaled Lie algebra root lattice of type A_1. There are conjectures formulated successively by several authors [FF88], [FFHST02], [TF09], [FGST06a], [AM08], that say that using this procedure it is possible to construct LCFT for every type of root lattice.

The main aim of this work is to prove these conjectures in the case of a root lattice of type B_n rescaled by a parameter $\ell = 4$.

What we do is indeed to construct such a subspace $\mathcal{W}_{B_n, \ell=4}$ and to prove that this is a LCFT. The latter result follows from Corollary 6.3 of section 6 that says:

$$\mathcal{W}_{B_n, \ell=4} \cong \mathcal{V}_{\mathcal{SF}_n^{even}}$$

i.e. it is proven that the new defined structure $\mathcal{W}_{B_n, \ell=4}$ is isomorphic to the well-known LCFT of n pairs of symplectic fermions $\mathcal{V}_{\mathcal{SF}_n^{even}}$.

Let us list in details the contents of this work:

In Chapter 1 we give the preliminary definitions of Hopf algebra, vertex operator algebra and super vector space.

In Chapter 2 we introduce the general setting: we consider a lattice and we define a lattice VOA. Then we give known results about these algebras and about their representation theory, we describe their conformal structure and we define some maps $\mathfrak{Z}_\alpha$ on the lattice VOA $\mathcal{V}_\Lambda$ and on its modules, the *screening charge operators*.

In order to construct a non-semisimple VOA $\mathcal{W}$ through free field realization, we are then interested in the particular case when the lattice is a root lattice Λ_R of $\mathfrak{g}$ complex finite-dimensional semisimple Lie algebra. Moreover,

we want to rescale Λ_R. Hence we consider $\ell = 2p$ even natural number with some divisibility conditions (defined in section 2.5) and we define three lattices rescaled by ℓ, related to the root, coroot and weight lattice: the *short screening* $\Lambda^\ominus$, the *long screening* $\Lambda^\oplus$ and the dual of the long screening $(\Lambda^\oplus)^*$ lattice.

In Chapter 3 we use the screening operators associated to the lattice $\Lambda^\ominus$ to define the subspace $\mathcal{W}$ as their kernel in the lattice VOA $\mathcal{V}_{\Lambda^\oplus}$
The subject of study is $\mathcal{W} := \mathcal{V}_{\Lambda^\oplus} \cap \bigcap_i ker \mathfrak{Z}_{\alpha_i^\ominus}$.

On this space, as we said, a conjecture has been formulated:

Conjecture 0.1. $\mathcal{W}$ is a vertex subalgebra and a logarithmic conformal field theory with the same representation theory as a small quantum group $u_q(\mathfrak{g})$.

This work is focused on two examples: in Chapters 2 and 3 we treat the known one of $\mathfrak{g} = A_1, \ell = 4$ where $\mathcal{W}$ happens to be equal to the triplet algebra $\mathcal{W}_2$ (see [FFHST02]). In Chapter 4, using the same approach, we study the new case $\mathfrak{g} = B_n, \ell = 4$.

More precisely, in each example we describe the involved lattices, the kernel of the short screenings $\mathcal{W}$, the representations $\mathcal{V}_{[\lambda]}$ of the lattice vertex algebra $\mathcal{V}_\Lambda$ and their decomposition when we restrict to $\mathcal{W}$.

We highlight that the second example, $\mathfrak{g} = B_n, \ell = 4$, is special because presents *degeneracy* (4.1.3): singularly the rescaled long roots have still even integer norm. This implies that the short screenings associated to those roots in B_n are still *bosonic* and do not satisfy the Nichols Algebra relations of theorem 2.14. Thus we only work with short screenings associated to short roots of B_n which is a subsystem of type A_1^n. This make the treatise much easier and make possible to deduce the final result by first approaching the easy case A_1.

In Chapter 5 we introduce the VOA of n pairs of Symplectic Fermions $\mathcal{V}_{\mathcal{SF}_n^{even}}$ and we describe it, together with its 4 irreducible modules. We then give the definition of graded dimension of a VOA module and we compute those of the four $\mathcal{W}_{B_n, \ell=4}$ modules $\chi^{\Lambda(1)}$, $\chi^{\Lambda(2)}$, $\chi^{\Pi(1)}$, $\chi^{\Pi(2)}$. Finally we compare these graded dimensions to the ones of the symplectic fermions modules $\chi_i^{\mathcal{SF}}$, $i = 1, \ldots, 4$ defined in [DR16], to verify that they really match:

$$\chi_1^{\mathcal{SF}} + \chi_2^{\mathcal{SF}} = \chi^{\Lambda(1)} + \chi^{\Pi(1)}$$
$$\chi_3^{\mathcal{SF}} = \chi^{\Lambda(2)}$$
$$\chi_4^{\mathcal{SF}} = \chi^{\Pi(2)}$$

This comparison is one of the tools used in Chapter 6 to prove the VOA isomorphism between our $\mathcal{W}_{B_n, \ell=4}$ and the n pairs of Symplectic Fermions $\mathcal{V}_{\mathcal{SF}_n^{even}}$.

Chapter 1

Preliminary definitions

The structure that we are going to study in the next chapters is a commutative cocommutative infinite-dimensional Hopf algebra associated to a given lattice. In this chapter we thus give the definitions of a Hopf algebra building it step by step from the one of algebra following [Kassel95].

1.0.1 Hopf algebra

Definition 1.1. An *algebra* is a ring A together with a ring map $\eta_A : \mathbb{K} \to A$ whose image is contained in the centre of A.
Hence A is a $\mathbb{K}$-vector space and the multiplication map $\mu_A : A \times A \to A$ is $\mathbb{K}$-bilinear

In order to define the concept of a coalgebra, we can paraphrase the previous definition and the dualise it:

Definition 1.2. An *algebra* can be equivalently defined ad a triple (A, μ, η) where A is a vector space, $\mu : A \otimes A \to A$ and $\eta : \mathbb{K} \to A$ are linear maps satisfying the following axioms (Ass) and (Un):
(Ass): The following square commutes

$$
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{\mu \otimes id} & A \otimes A \\
{\scriptstyle id \otimes \mu} \downarrow & & \downarrow {\scriptstyle \mu} \\
A \otimes A & \xrightarrow{\mu} & A
\end{array}
$$

(Un): The following diagram commutes:

$$
\begin{array}{ccccc}
\mathbb{K} \otimes A & \xrightarrow{\eta \otimes id} & A \otimes A & \xleftarrow{id \otimes \eta} & A \otimes \mathbb{K} \\
& {\scriptstyle \cong} \searrow & \downarrow {\scriptstyle \mu} & \swarrow {\scriptstyle \cong} & \\
& & A & &
\end{array}
$$

Remark 1.3. The first axiom express the requirement that the multiplication μ is associative, the latter means that the element $\eta(1)$ of A is a left and right unit for μ.

Definition 1.4. The algebra A is *commutative* if it satisfies a third axiom (Comm): The triangle

$$A \otimes A \xrightarrow{\tau_{A,A}} A \otimes A$$
$$\mu \searrow \qquad \swarrow \mu$$
$$A$$

commutes, where $\tau_{A,A}$ is the flip switching the factors $\tau_{A,A}(a \otimes a') = (a' \otimes a)$.

Dualizing the previous definition, i.e. reversing all arrows of the previous diagrams we have the definition of a coalgebra:

Definition 1.5. A *coalgebra* is a triple (C, Δ, ϵ) where C is a vector space and $\Delta : C \to C \otimes C$ and $\epsilon : C \to \mathbb{K}$ are linear maps satisfying the following axioms (Coass) and (Coun).
(Coass): The following square commutes

$$
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\Delta \downarrow & & \downarrow id \otimes \Delta \\
C \otimes C & \xrightarrow{\Delta \otimes id} & C \otimes C \otimes C
\end{array}
$$

(Coun): The following diagram commutes

$$\mathbb{K} \otimes C \xleftarrow{\epsilon \otimes id} C \otimes C \xrightarrow{id \otimes \epsilon} C \otimes \mathbb{K}$$
$$\cong \nwarrow \quad \uparrow \Delta \quad \nearrow \cong$$
$$C$$

The map Δ is called the *coproduct* or *comultiplication* while ϵ is called the *counit* of the coalgebra. The previous diagrams show that the coproduct is coassociative and counital.

Analogously:

Definition 1.6. The coalgebra C is *cocommutative* if it satisfies a third axiom (Cocomm): The triangle

$$C$$
$$\Delta \swarrow \qquad \searrow \Delta$$
$$C \otimes C \xrightarrow{\tau_{C,C}} C \otimes C$$

commutes, where $\tau_{C,C}$ is the flip.

We now consider a vector space equipped simultaneously with an algebra and a coalgebra structure. If there are some compatibility conditions it is called bialgebra:

Definition 1.7. A *bialgebra* is a quintuple $(H, \mu, \eta, \Delta, \epsilon)$ where (H, μ, η) is an algebra and (H, Δ, ϵ) is a coalgebra verifying the following compatibility conditions:
the following two squares commute

$$
\begin{array}{ccc}
H \otimes H & \xrightarrow{\ \mu\ } & H \\
{\scriptstyle (id \otimes \tau \otimes id)(\Delta \otimes \Delta)} \downarrow & & \downarrow {\scriptstyle \Delta} \\
(H \otimes H) \otimes (H \otimes H) & \xrightarrow[\ \mu \otimes \mu\ ]{} & H \otimes H
\end{array}
\qquad
\begin{array}{ccc}
H \otimes H & \xrightarrow{\ \epsilon \otimes \epsilon\ } & \mathbb{K} \otimes \mathbb{K} \\
{\scriptstyle \mu} \downarrow & & \downarrow {\scriptstyle id} \\
H & \xrightarrow[\ \epsilon\ ]{} & \mathbb{K}
\end{array}
$$

the following two diagrams commute

$$
\begin{array}{ccc}
\mathbb{K} & \xrightarrow{\ \eta\ } & H \\
{\scriptstyle id} \downarrow & & \downarrow {\scriptstyle \Delta} \\
\mathbb{K} \otimes \mathbb{K} & \xrightarrow[\ \eta \otimes \eta\ ]{} & H \otimes H
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{K} & \xrightarrow{\ \eta\ } & H \\
& {\scriptstyle id} \searrow \quad \swarrow {\scriptstyle \epsilon} & \\
& \mathbb{K} &
\end{array}
$$

We finally define the convolution of two linear maps:

Definition 1.8. Given (A, μ, η) algebra and (C, Δ, ϵ) coalgebra, given f and g in $Hom(C, A)$, we define the *convolution* $f \star g$ as the composition of the following maps:

$$
C \xrightarrow{\ \Delta\ } C \otimes C \xrightarrow{\ f \otimes g\ } A \otimes A \xrightarrow{\ \mu\ } A
$$

We are now ready for the wanted definition:

Definition 1.9. A *Hopf algebra* is a bialgebra $(H, \mu, \eta, \Delta, \epsilon)$ together with a $\mathbb{K}$-linear map $S : H \to H$ such that

$$
S \star id = id \star S = \eta \circ \epsilon
$$

The endomorphism S is called *antipode*.

Remark 1.10. If the antipode exists, it is unique by definition.

Example 1.11. Given G group we can construct the group algebra $\mathbb{K}[G]$ as the formal linear combinations of elements of G with coefficients in $\mathbb{K}$.
This happens to be a Hopf algebra $(\mathbb{K}[G], \mu, \eta, \Delta, \epsilon, S)$.
Let us define the related maps:

$$
\begin{aligned}
\Delta : \quad & \mathbb{K}[G] \longrightarrow \mathbb{K}[G] \otimes \mathbb{K}[G] \\
& g \longmapsto g \otimes g \\
\epsilon : \quad & \mathbb{K}[G] \longrightarrow \mathbb{K} \\
& g \longmapsto 1 \quad \forall g \in G \\
\eta : \quad & \mathbb{K} \longrightarrow \mathbb{K}[G] \\
& 1 \longmapsto e_G \\
\mu : \quad & \mathbb{K}[G] \otimes \mathbb{K}[G] \longrightarrow \mathbb{K}[G] \\
& g_1 \otimes g_2 \longmapsto g_1 g_2 \\
S : \quad & \mathbb{K}[G] \longrightarrow \mathbb{K}[G] \\
& g \longmapsto g^{-1}
\end{aligned}
$$

where g, g_1, $g_2 \in G$. These maps can be linearly extended to every element of G. We can verify that $S \star id = \eta \circ \epsilon$:

$$\begin{aligned}
S \star id &= \mu \circ S \otimes id \circ \Delta(g) \\
&= \mu \circ S \otimes id \circ (g \otimes g) \\
&= \mu(g^{-1} \otimes g) = g^{-1}g \\
&= e_G = \eta(1) = \eta \circ \epsilon(g)
\end{aligned}$$

Example 1.12. Analogously for every finite dimensional semisimple complex Lie algebra $\mathfrak{g}$ we can construct a Hopf algebra considering the Universal enveloping algebra $U(\mathfrak{g})$. The maps associated to the Hopf algebra structure are defined $\forall x \in \mathfrak{g}$ as follows

$$\Delta(x) = x \otimes 1 + 1 \otimes x \qquad \epsilon(x) = 0 \qquad S(x) = -x$$

1.0.2 Vertex operator algebra

As we mentioned the first structure that we are going to study is an Hopf algebra associated to a given lattice. In the case in which the lattice is even integer, i.e. when the norm of every vector of the lattice is an even integer, this algebra happens to be also a vertex operator algebra. This is a mathematical object associated to the physical notion of two-dimensional conformal field theory.
We give the definition of it and of the surrounding notions, following [FB68].

Definition 1.13. A *Vertex operator algebra* (VOA) consists of the following data

- a complex graded vector space $\mathcal{V} = \bigoplus_{n=0}^{\infty} \mathcal{V}_n$

- a *Vacuum* element $\mathbf{1} \in \mathcal{V}_0$

- a linear operator called *derivation* $\partial : \mathcal{V} \to \mathcal{V}$

- a linear operator called *vertex operator* $Y = Y(-, z) : \mathcal{V} \to End\mathcal{V}[[z, z^{-1}]]$ defined on every $A \in \mathcal{V}$ as the formal power series

$$Y(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)} z^{-n-1}$$

These data satisfy three axioms:

1. the vacuum axiom: $\forall A \in \mathcal{V}$

$$Y(1, z)A = A = Az^0$$
$$Y(A, z)1 \in A + z\mathcal{V}[z]$$

2. the translation axiom: $\partial(1) = 0$ and $\forall A, B \in \mathcal{V}$

$$[\partial, Y(A, z)]B = \partial Y(A, z)B - Y(A, z)\partial B = \frac{\partial}{\partial z} Y(A, z)B$$

3. the locality axiom: $\forall A, B \in \mathcal{V} \ \exists N > 0, \ N \in \mathbb{Z}$ such that as formal power series
$$(z - x)^N Y(A, z)Y(B, x) = (z - x)^N Y(B, x)Y(A, z)$$

Remark 1.14. The locality axiom says that $\forall A, B \in \mathcal{V}$, the formal power series in two variables obtained by composing $Y(A, z)$ and $Y(B, x)$ in two possible ways are equal to each other, possibily after multiplying them with a large enough power of $(z - x)$.

Definition 1.15. A *vertex algebra homomorphism* ρ between vertex algebras

$$\rho : (\mathcal{V}, 1, \partial, Y) \to (\mathcal{V}', 1', \partial', Y')$$

is a linear map $\mathcal{V} \to \mathcal{V}'$ mapping 1 to $1'$, intertwining the translation operators and satisfying

$$\rho(Y(A, z)B) = Y(\rho(A), z)\rho(B).$$

Definition 1.16. A *vertex subalgebra* $\mathcal{V}' \subset \mathcal{V}$ is a ∂-invariant subspace containing the vacuum vector, and satisfying $Y(A, z)B \in \mathcal{V}'((z))$ for all $A, B \in \mathcal{V}'$.

For completeness' sake, we here introduce the technical definition of a $\mathcal{V}$-*module*. However we will just treat *lattice* vertex algebras: their representation theory is fully known (described in 2.7) and we will just use the results on it.

Definition 1.17. Let $(\mathcal{V}, 1, \partial, Y)$ be a vertex algebra. A vector space M is called a $\mathcal{V}$-*module* if it id equipped with an operation $Y_M : \mathcal{V} \to End M[[z^{\pm 1}]]$ which assigns to each $A \in \mathcal{V}$ a field

$$Y_M(A, z) = \sum_{n \in \mathbb{Z}} A_{(n)}^M z^{-n-1}$$

on M subject to the following axioms:

- $Y_M(1, z) = Id_M$

- for all $A, B \in \mathcal{V}$, $C \in M$ the three expressions

$$Y_M(A, z)Y_M(B, w)C \in M((z))((w)),$$
$$Y_M(B, w)Y_M(A, z)C \in M((w))((z)),$$
$$Y_M(Y_M(A, z - w)B, w)C \in M((w))((z - w))$$

are the expansions, in their respective domains, of the same element of

$$M[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}].$$

1.0.3 Super vector space

Lastly we give a third important definition that will be useful in Chapter 5 and 6 when we will speak about the Symplectic Fermions.

Definition 1.18. A super vector space is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space

$$V = V_0 \oplus V_1$$

The elements of V_0 are called *even* and the elements of V_1 are called *odd*.
The super dimension of a super vector space V is the pair (p, q) where $dim(V_0) = p$ and $dim(V_1) = q$ as ordinary vector spaces. We write $dim V = p|q$.

The super vector spaces form a braided tensor category where the morphisms preserve the $\mathbb{Z}/2\mathbb{Z}$ grading. A braided tensor category is a tensor category equipped with a natural isomorphism called *braiding*, which switch two objects in a tensor product. In this case for example the braiding is given by

$$x \otimes y \to (-1)^{|x|+|y|} y \otimes x$$

We can then define a super algebra as an algebra in the category of super vector spaces; explicitly:

Definition 1.19. A super algebra is a super vector space $\mathcal{A}$ together with a bilinear multiplication morphism $\eta_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ such that

$$A_i A_j \subseteq A_{i+j}$$

where the indexes are read modulo 2.

Analogously it is possible to define a vertex operator superalgebra (VOSA).

Chapter 2

Setting

2.1 First steps in the VOA world

Let us consider a commutative cocommutative Hopf algebra $\mathcal{V}$ and let us construct on it some additional structure, defining:

- an *Hopf pairing* $\langle \, , \, \rangle : \mathcal{V} \otimes \mathcal{V} \longrightarrow \mathbb{C}[z, z^{-1}]$ i.e. a z-dependent map such that the following relations hold:

$$\langle ab, c \rangle = \left\langle a, c^{(1)} \right\rangle \left\langle b, c^{(2)} \right\rangle$$

$$\langle a, bc \rangle = \left\langle a^{(1)}, b \right\rangle \left\langle a^{(2)}, c \right\rangle$$

$$\langle a, 1 \rangle = \epsilon(a)$$

$$\langle 1, b \rangle = \epsilon(b)$$

where $a, b, c \in \mathcal{V}$, ϵ is the Hopf algebra counit and $a^{(1)}, a^{(2)}, c^{(1)}, c^{(2)}$ are the target terms of the Hopf algebra comultiplication Δ where a sum is implicit.

- a *derivation* or translator operator, $\partial : \mathcal{V} \to \mathcal{V}$ such that:

$$\partial.(ab) = (\partial.a)b + a(\partial.b)$$

compatible with the Hopf pairing:

$$\langle a, \partial.b \rangle = -\frac{\partial}{\partial z} \langle a, b \rangle$$

$$\langle \partial.a, b \rangle = \frac{\partial}{\partial z} \langle a, b \rangle$$

From these data, under certain conditions (see [Lent17], [Lent07]), we can define a map, the vertex operator

$$Y : \mathcal{V}_\Lambda \to End\mathcal{V}_\Lambda[[z, z^{-1}]],$$

$$a \longmapsto \left(b \mapsto \sum_{k \geq 0} \left\langle a^{(1)}, b^{(1)} \right\rangle \cdot b^{(2)} \cdot \frac{z^k}{k!} \partial^k.a^{(2)} \right)$$

In turn it defines on $\mathcal{V}$ the structure of a local Vertex Algebra as proven in the quoted references.

Free particle in one dimension

As first example of such a Hopf algebra $\mathcal{V}$ we consider the free commutative algebra generated by the formal symbols

$$\partial\phi, \partial^2\phi, \partial^3\phi, \ldots$$

On $\mathcal{V}$ is naturally defined a $\mathbb{N}_0$-grading such that the $\mathbb{N}_0$-degree of $\partial^k\phi$ is k. Every element is primitive as coalgebra element i.e.

$$\Delta\partial^{1+k}\phi = 1 \otimes \partial^{1+k}\phi + \partial^{1+k}\phi \otimes 1$$

For every $c \in \mathbb{C}^*$ we can construct a derivation and a Hopf pairing as follows:

$$\partial.\partial^k\phi = \partial^{k+1}\phi$$

$$\langle\partial\phi, \partial\phi\rangle = \frac{c}{z^2}$$

The following are some examples of the vertex operator Y applied to generic elements of $\mathcal{V}$:

$$Y(1)\partial^h\phi = \partial^h\phi$$

$$Y(\partial^h\phi)1 = \sum_k \frac{z^k\partial^k}{k!}\partial^h\phi = \sum_k \frac{z^k}{k!}\partial^{h+k}\phi$$

$$Y(\partial\phi)\partial\phi = \frac{c}{z^2} + \partial\phi Y(\partial\phi)1$$

This $\mathcal{V}$ is $\forall c \in \mathbb{C}^*$ the *Heisenberg VOA* defined in Chapter 2.1 of [FB68]. From a physics perspective, $\mathcal{V}$ is as a vector space the space of states of a free particle in one dimension and the map Y represents an interaction between two states to a third one.

Free particle in n-dimensional space or n free particles in one dimension

We extend now the previous construction, considering the free commutative cocommutative Hopf algebra generated $\forall\alpha \in \mathbb{R}^n$ by the formal symbols:

$$\partial\phi_\alpha, \partial^2\phi_\alpha, \partial^3\phi_\alpha, \ldots$$

with the following relation:

$$\partial^{1+k}\phi_{a\alpha+b\beta} = a\partial^{1+k}\phi_\alpha + b\partial^{1+k}\phi_\beta$$

where $a, b \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}^n$. Because of this linear combination it actually suffices to take as generators the elements

$$\partial\phi_{\alpha_1}, \ldots, \partial\phi_{\alpha_n}$$

for any fixed basis $\{\alpha_1, \ldots \alpha_n\}$ of $\mathbb{R}^n$. This Hopf algebra can now be endowed as before with a derivation and an Hopf pairing. In particular we have:

$$\langle\partial\phi_\alpha, \partial\phi_\beta\rangle = \frac{(\alpha, \beta)}{z^2}$$

Physically the system describes a free particle in a n-dimensional space, or equivalently, n particles in one dimension.

2.2 The lattice VOA

In this section we will generalize the previous idea and consider the example in which we are interested: *a free particle on a torus*.

Let $\Lambda \subset \mathbb{C}^n$ be a lattice with basis $\{\alpha_1, \ldots, \alpha_n\}$ and inner product $(\, , \,) : \Lambda \times \Lambda \to \frac{1}{N}\mathbb{Z}$ for some $N \in \mathbb{N}$.

Definition 2.1. We define $\mathcal{V}_\Lambda$ as the commutative, cocommutative, infinite-dimensional $\mathbb{N}_0$-graded Hopf algebra $\mathcal{V}_\Lambda$ generated by the formal symbols

$$e^{\phi_\beta}, \qquad \partial^{1+k}\phi_\alpha$$

where the e^{ϕ_β} has $\mathbb{N}_0$-degree 0 and the $\partial^{1+k}\phi_\alpha$ has $\mathbb{N}_0$-degree $1+k$, parametrized by some elements α and β of Λ.
They fulfill the algebra relations:

$$e^{\phi_\alpha}e^{\phi_\beta} = e^{\phi_{\alpha+\beta}} \qquad\qquad \alpha, \beta \in \Lambda$$

$$\partial^{1+k}\phi_{a\alpha+b\beta} = a\partial^{1+k}\phi_\alpha + b\partial^{1+k}\phi_\beta \qquad\qquad a, b \in \mathbb{Z}$$

In the coalgebra all the symbols $\partial^{1+k}\phi_\alpha$ are primitive, namely

$$\Delta\partial^{1+k}\phi_\alpha = 1 \otimes \partial^{1+k}\phi_\alpha + \partial^{1+k}\phi_\alpha \otimes 1$$

and all the *momentum* elements e^{ϕ_β} are grouplike, namely

$$\Delta e^{\phi_\beta} = e^{\phi_\beta} \otimes e^{\phi_\beta}.$$

An arbitrary element is then a linear combination of elements of the form ue^{ϕ_β} with $\beta \in \Lambda$ and u a differential polynomial, i.e. a poynomial in the elements $\partial^{1+k}\phi_\alpha$ with $k \in \mathbb{N}_0$, $\alpha \in \Lambda$.
We will call $|u|$ the $\mathbb{N}_0$-degree of u.

A derivation $\partial : \mathcal{V}_\Lambda \to \mathcal{V}_\Lambda$ and a Hopf pairing $\langle\, , \,\rangle : \mathcal{V}_\Lambda \otimes \mathcal{V}_\Lambda \to \mathbb{C}[z^{\frac{1}{N}}, z^{-\frac{1}{N}}]$ can be defined as before on the first kind of elements and as follows on the momentum elements:

$$\partial.e^{\phi_\beta} = \partial\phi_\beta e^{\phi_\beta}$$

$$\partial^k.e^{\phi_\beta} = P_{\beta,k}e^{\phi_\beta}$$

$$\langle e^{\phi_\alpha}, e^{\phi_\beta}\rangle = z^{(\alpha,\beta)}$$

$$-\langle e^{\phi_\alpha}, \partial\phi_\beta\rangle = \langle \partial\phi_\beta, e^{\phi_\alpha}\rangle = \frac{(\alpha,\beta)}{z}$$

where $P_{\beta,k}$ is a differential polynomial of degree k depending on the parameter β.

On $\mathcal{V}_\Lambda$, in addition to the $\mathbb{N}_0$-grading, there is a natural Λ-grading

$$\mathcal{V}_\Lambda = \bigoplus_{\lambda \in \Lambda} \mathcal{V}_\lambda$$

where $\mathcal{V}_\lambda = \{ue^{\phi_\lambda}\}$ and u is again a differential polynomial.

Definition 2.2. The *Vertex algebra* operator Y is defined as

$$Y : \mathcal{V}_\Lambda \to End\mathcal{V}_\Lambda[[z^{\frac{1}{N}}, z^{-\frac{1}{N}}]],$$

$$a \longmapsto \left(b \mapsto \sum_{k \geq 0} \left\langle a^{(1)}, b^{(1)} \right\rangle \cdot b^{(2)} \cdot \frac{z^k}{k!} \partial^k . a^{(2)} \right)$$

Depending on the lattice Λ, this algebra $\mathcal{V}_\Lambda$ happens to be a more complex algebraic object.

Definition 2.3. A lattice Λ is called *even integer* lattice if the norm of every vector is an even integer. In particular this implies that the inner product has always integer values.
A lattice Λ is called an *odd integer* lattice if the norm of every vector is an integer but not necessarily even.
A lattice Λ is called a *not integer* or *fractional* lattice if there exist non integer norms.

Theorem 2.4.
- *[FB68] If Λ is an even integer lattice then $(\mathcal{V}_\Lambda, Y)$ is a lattice vertex algebra (VOA), i.e. a vertex algebra whose elements are parametrized by the one of the lattice Λ.*

- *[FB68] If Λ is an odd integer lattice then $(\mathcal{V}_\Lambda, Y)$ is a lattice super vertex algebra (VOSA), i.e. a super vertex algebra whose elements are parametrized by the one of the lattice Λ.*

Remark 2.5. If Λ is not integer, i.e. is fractional, then the locality in $\mathcal{V}_\Lambda$ is replaced by other relations (see [Lent17]) and we can still speak about it as some generalized lattice VOA.

We now consider the dual lattice $\Lambda^* = \{\lambda \in \mathbb{C} | (\lambda, \alpha) \in \mathbb{Z} \, \forall \alpha \in \Lambda\}$. The following result, shown in Chapter 5 of [FB68], gives us informations about the representation category $\mathcal{V}_\Lambda$-Rep of a lattice VOA $\mathcal{V}_\Lambda$.

Remark 2.6. It is possible to have an action of $\mathcal{V}_\Lambda$ on $\mathcal{V}_{\Lambda^*}$ through the vertex operator Y because the inclusion $\Lambda \subset \Lambda^*$ implies

$$Y : \mathcal{V}_\Lambda \times \mathcal{V}_{\Lambda^*} \to \mathcal{V}_{\Lambda^*}$$

and from the definition of the dual lattice we have only integer z-powers. Explicitly, if $\alpha \in \Lambda$ and $\lambda \in \Lambda^*$ then $\alpha + \lambda \in \Lambda^*$ and $(\alpha, \lambda) \in \mathbb{Z}$. Thus

$$Y(e^{\phi_\alpha})e^{\phi_\lambda} = \langle z, \, z^{-1}, \, \phi^{1+k} \rangle \, e^{\phi_{\alpha+\lambda}} \in \mathcal{V}_{\Lambda^*}$$

Theorem 2.7. *Let Λ be a even integer lattice, then $\mathcal{V}_\Lambda$-Rep is a category equivalent to the category of (Λ^*/Λ)-Vect i.e. the vector spaces graded by the abelian group (Λ^*/Λ). We can decompose the module*

$$\mathcal{V}_{\Lambda^*} = \bigoplus_{[\lambda] \in \Lambda^*/\Lambda} \mathcal{V}_{[\lambda]}$$

i.e. as direct sum of simple modules. We call $\mathcal{V}_\Lambda = \mathcal{V}_{[0]}$ the vacuum module.

Remark 2.8. The equivalence of Theorem 2.7 is as abelian categories. Moreover, asking some finiteness conditions on $\mathcal{V}_\Lambda$, its representation category becomes a modular tensor category; then adding a braiding of the form $e^{i\pi(\lambda,\lambda')}$ on the right side the equivalence is also as modular tensor categories.

Remark 2.9. The equivalence as abelian categories implies, in particular, that the simple objects of the two categories corresponds and thus their number coincides. Therefore we have as many $\mathcal{V}_\Lambda$-simple modules in $\mathcal{V}_\Lambda$-Rep as one dimensional vector spaces in the category of (Λ^*/Λ)-graded vector spaces. Since in the latter, we have one dimensional vector spaces for every layer, we have exactly $|\Lambda^*/\Lambda|$ one dimensional vector spaces and thus, irreducible representations of $\mathcal{V}_\Lambda$.

2.3   Screening operators

Once the vertex operator Y is defined, we can produce linear endomorphisms of $\mathcal{V}_\Lambda$ as follows:

Definition 2.10. For a given element $a \in \mathcal{V}_\Lambda$ and $m \in \frac{1}{N}\mathbb{Z}$ we can consider the z^m-term of

$$Y(a)b = \sum_m Y(a)_m z^m b$$

with $b \in \mathcal{V}_\Lambda$, and thus produce an endomorphism: the *mode operator*

$$Y(a)_m : \mathcal{V}_\Lambda \to \mathcal{V}_\Lambda \qquad b \longmapsto \sum_{k \geq 0} \left\langle a^{(1)}, b^{(1)} \right\rangle_{-k+m} b^{(2)} \frac{1}{k!} \partial^k . a^{(2)}$$

where $\langle a,b\rangle_n$ denotes the z^n-coefficient.

From this we can define also the so-called *ResY*-operator, an operator that for every b give us an endomorphism of $\mathcal{V}_\Lambda$:

Definition 2.11. Given $a \in \mathcal{V}_\alpha$, $b \in \mathcal{V}_\beta$ and $m := (\alpha, \beta)$

$$ResY(a) : b \longmapsto Res(Y(a)b)$$

$$Res(Y(a)b) := \begin{cases} Y(a)_{-1} & \text{if } m \in \mathbb{Z} \\ \frac{e^{2\pi i m}-1}{2\pi i} \sum_{k \in \mathbb{Z}} \frac{1}{m+k+1} Y(a)_{m+k} & \text{if } m \notin \mathbb{Z} \end{cases}$$

where the residue is defined as a formal residue of fractional polynomials and geometrically is the integral along the unique lift of a circle with given radius to the multivalued covering on which the polynomial with fractional exponents is defined.

In the integer case this endomorphism has a derivational property thanks to the OPE associativity [Thiel94], [FB68]:

$$Y(a)_{-1}\left(Y(b)_m c\right) = Y(Y(a)_{-1}b)_m c \pm Y(b)_m\left(Y(a)_{-1}c\right) \tag{2.1}$$

Using the residue we arrive to define some interesting maps on the lattice VOA $\mathcal{V}_\Lambda$ and its modules i.e. the *screening charge operators*.

Definition 2.12. For $\alpha \in \Lambda^*$ we define the *screening charge operator* $\mathfrak{Z}_\alpha$ as

$$\mathfrak{Z}_\alpha v := ResY(e^{\phi_\alpha})v$$

By the given definition, if $(\alpha, \beta) \in \mathbb{Z}$ it simplifies to

$$\mathfrak{Z}_\alpha u e^{\phi_\beta} = \sum_{k \geq 0} \langle e^{\phi_\alpha}, u^{(1)} \rangle_{-k-1-(\alpha,\beta)} u^{(2)} e^{\phi_\beta} \frac{1}{k!} \partial^k . e^{\phi_\alpha} \tag{2.2}$$

Remark 2.13. We notice that the operators shift the Λ-grading

$$\mathfrak{Z}_\alpha : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda+\alpha}$$

Moreover in the case of $\alpha \in \Lambda$ they fix the $\mathcal{V}_\Lambda$-modules i.e.

$$\mathfrak{Z}_\alpha : \mathcal{V}_{[\lambda]} \to \mathcal{V}_{[\lambda]}$$

where $[\lambda] \in \Lambda^*/\Lambda$.

The screening operators are an old construction in CFT. In the case of non-integral lattice we have however a new result by [Lent17] that gives us relations of screenings. This result follows in the same paper from the study of the complicated structures of Nichols algebra.

For our purposes it is not necessarily to define such structures but it is interesting to state the Theorem because its consequences (displayed in section 2.6) are one of the motivations of this work.

Theorem 2.14. *[Lentner17] Let Λ be a positive-definite lattice and $\{\alpha_1, \ldots, \alpha_n\}$ be a fixed basis that fulfils $|\alpha_i| \leq 1$. Then the endomorphisms $\mathfrak{Z}_{\alpha_i} := ResY(e^{\phi_{\alpha_i}})$ on the fractional lattice VOA $\mathcal{V}_\Lambda$ constitute an action of the diagonal Nichols algebra generated by the $\mathfrak{Z}_{\alpha_i}$ with braiding matrix*

$$q_{ij} = e^{\pi i (\alpha_i, \alpha_j)}.$$

The theorem proves that the screening operators associated to a small enough lattice obey Nichols algebra relations.

Corollary 2.15. *As consequences of the theorem we have:*

- *if (α, α) is an odd integer $\Rightarrow (\mathfrak{Z}_\alpha)^2 = 0$*

- *if (α, β) is an even integer $\Rightarrow \mathfrak{Z}_\alpha$ and $\mathfrak{Z}_\beta$ commute.*

What we will do next is to find Λ root lattice of some Lie algebra such that q_{ij} is the braiding of a quantum group. But first we want to introduce the notion of Virasoro Algebra and understand how does it act on our VOA.

2.4 Virasoro action

Definition 2.16. The Witt algebra is a Lie algebra generated by the vector fields

$$L_n := -z^{n+1} \frac{\partial}{\partial z}$$

with $n \in \mathbb{Z}$. The Lie bracket of two vector fields is given by

$$[L_m, L_n] = (m - n) L_{m+n}$$

This is the Lie algebra of the group of diffeomorphisms of the circle.

Definition 2.17. The Virasoro Algebra Vir_c is the non-trivial central extension of the Witt Lie algebra. It is generated by the central charge $C = c1$, $c \in \mathbb{C}$, and the operators L_n indexed by an integer $n \in \mathbb{Z}$ and that fulfil the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0}$$

$$[L_n, C] = 0 \qquad \forall n \in \mathbb{Z}.$$

Definition 2.18. A vertex algebra $\mathcal{V}$ is called *conformal*, of central charge $c \in \mathbb{C}$, if there is a non-zero *conformal vector* $\omega \in \mathcal{V}_2$ such that the $Y(\omega)_{-2-n}$ of the corresponding vertex operator satisfy the defining relations of the Virasoro algebra with central charge c and in addition we have $Y(\omega)_{-1} = \partial$, $Y(\omega)_0|_{V_n} = nId$.

On our lattice VOA $\mathcal{V}_\Lambda$ we have the following result by [FF88] which in particular tell us that $\mathcal{V}_\Lambda$ is conformal:

Theorem 2.19. *If we choose an* Energy-Stress tensor *in* $\mathcal{V}_\Lambda$

$$T = \frac{1}{2}\sum_i \partial\phi_{\alpha_i}\partial\phi_{\alpha_i*} + \sum_i Q_i \partial^2\phi_{\alpha_i},$$

depending on an arbitrary vector $Q = (Q_1, \ldots, Q_n) \in span_\mathbb{C}\langle\alpha_1, \ldots, \alpha_n\rangle$ *and independent of the choice of a dual basis* $(\alpha_i, \alpha_j{}^*) = \delta_{ij}$, *then* $L_n := Y(T)_{-2-n}$ *constitute an action of the Virasoro algebra* Vir_c *with central charge* $c = rank - 12(Q,Q)$ *on* $\mathcal{V}_\Lambda$.

Remark 2.20. Since $T \in \mathcal{V}_\Lambda$, in this way we automatically obtain a Virasoro action also on all the vertex modules of $\mathcal{V}_\Lambda$.

Remark 2.21. The action of Vir_c on $\mathcal{V}_\Lambda$ is an action on the vector space and on every module that is compatible with the VOA structure (defined in Chapter 1).

Lemma 2.22. *For the defined* T, *fixed a value of* Q, *the action of the* L_0 *and* L_{-1} *elements of* Vir_c *on a generic element* ue^{ϕ_β} *of the VOA* $\mathcal{V}_\Lambda$ *is given by:*

$$L_{-1}ue^{\phi_\beta} = \partial.(ue^{\phi_\beta})$$

$$L_0 ue^{\phi_\beta} = \left(\frac{(\beta,\beta)}{2} - (\beta, Q) + |u|\right)ue^{\phi_\beta}$$

The L_0 *eigenvalues are called* conformal dimensions.

Proof. Here we just verify the action of L_{-1}.

$$L_{-1}ue^{\phi_\beta} = Y\left(\frac{1}{2}\sum_i \partial\phi_{\alpha_i}\partial\phi_{\alpha_i*} + \sum_i Q_i\partial^2\phi_{\alpha_i}\right)_{-1} ue^{\phi_\beta} =$$

$$= \frac{1}{2}\sum_i Y(\partial\phi_{\alpha_i}\partial\phi_{\alpha_i*})_{-1}ue^{\phi_\beta} + \sum_i Q_i Y(\partial^2\phi_{\alpha_i})_{-1}ue^{\phi_\beta}$$

The co-multiplications of the T summands are:

$$\Delta(\partial\phi_{\alpha_i}\partial\phi_{\alpha_i*}) = \Delta(\partial\phi_{\alpha_i})\Delta(\partial\phi_{\alpha_i*})$$
$$= 1 \otimes \partial\phi_{\alpha_i}\partial\phi_{\alpha_i*} + 2\partial\phi_{\alpha_i} \otimes \partial\phi_{\alpha_i*} + \partial\phi_{\alpha_i}\partial\phi_{\alpha_i*} \otimes 1$$
$$\Delta(\partial^2\phi_{\alpha_i}) = 1 \otimes \partial^2\phi_{\alpha_i} + \partial^2\phi_{\alpha_i} \otimes 1$$

Now we write temporarily $v := ue^{\phi_\beta}$ and define

$$A := Y(\partial\phi_{\alpha_i}\partial\phi_{\alpha_i*})v$$
$$B := Y(\partial^2\phi_{\alpha_i})v$$

and we explicitly compute them:

$$A = \langle 1, v^{(1)}\rangle v^{(2)} \sum_k z^k \frac{\partial^k}{k!}(\partial\phi_{\alpha_i}\partial\phi_{\alpha_i*}) + 2\langle\partial\phi_{\alpha_i}, v^{(1)}\rangle v^{(2)} \sum_k z^k \frac{\partial^k}{k!}\partial\phi_{\alpha_i*}$$

$$+ \langle\partial\phi_{\alpha_i}\partial\phi_{\alpha_i*}, v^{(1)}\rangle v^{(2)} \sum_k z^k \frac{\partial^k}{k!}1$$

$$= 0 + 2\sum_k \langle\partial\phi_{\alpha_i}, v^{(1)}\rangle_n v^{(2)} z^{k+n} \frac{\partial^{k+1}\phi_{\alpha_i*}}{k!} + \langle\partial\phi_{\alpha_i}\partial\phi_{\alpha_i*}, v^{(1)}\rangle_n v^{(2)} z^n$$

$$B = \langle 1, v^{(1)}\rangle v^{(2)} \sum_k z^k \frac{\partial^k}{k!}\partial^2\phi_{\alpha_i} + \langle\partial^2\phi_{\alpha_i}, v^{(1)}\rangle v^{(2)} \sum_k z^k \frac{\partial^k}{k!}1$$

$$= v\sum_k \frac{z^k}{k!}\partial^{k+2}\phi_{\alpha_i} + \langle\partial^2\phi_{\alpha_i}, v^{(1)}\rangle_n v^{(2)} z^n.$$

Going back to the first equation and noticing that the term B vanishes, we obtain:

$$L_{-1}ue^{\phi_\beta} = \frac{1}{2}\sum_i \left(2\sum_k \langle\partial\phi_{\alpha_i}, u^{(1)}e^{\phi_\beta}\rangle_{-k-1} u^{(2)} e^{\phi_\beta} \frac{\partial^{k+1}\phi_{\alpha_i*}}{k!} + \langle\partial\phi_{\alpha_i}\partial\phi_{\alpha_i*}, u^{(1)}e^{\phi_\beta}\rangle_{-1} u^{(2)} e^{\phi_\beta}\right)$$
$$+ \sum_i Q_i \langle\partial^2\phi_{\alpha_i}, u^{(1)}e^{\phi_\beta}\rangle_{-1} u^{(2)} e^{\phi_\beta}$$

The last two terms vanish:

1.
$$\langle\partial\phi_{\alpha_i}\partial\phi_{\alpha_i*}, u^{(1)}e^{\phi_\beta}\rangle = \langle 1, e^{\phi_\beta}\rangle\langle\partial\phi_{\alpha_i}\partial\phi_{\alpha_i*}, u^{(1)}\rangle + 2\langle\partial\phi_{\alpha_i}, e^{\phi_\beta}\rangle\langle\partial\phi_{\alpha_i*}, u^{(1)}\rangle$$
$$+ \langle\partial\phi_{\alpha_i}\partial\phi_{\alpha_i*}, e^{\phi_\beta}\rangle\langle 1, u^{(1)}\rangle$$
$$= \begin{cases} \langle\partial\phi_{\alpha_i}\partial\phi_{\alpha_i*}, e^{\phi_\beta}\rangle & \text{if } u^{(1)} = 1 \\ 2\langle\partial\phi_{\alpha_i}, e^{\phi_\beta}\rangle\langle\partial\phi_{\alpha_i*}, u^{(1)}\rangle & \text{if } u^{(1)} \neq 1 \end{cases}$$

   and both cases give some power of z with exponential lower than -1.

2.
$$\langle\partial^2\phi_{\alpha_i}, u^{(1)}e^{\phi_\beta}\rangle = \partial\langle\partial\phi_{\alpha_i}, u^{(1)}e^{\phi_\beta}\rangle = \partial\left[\langle\partial\phi_{\alpha_i}, u^{(1)}\rangle\langle 1, e^{\phi_\beta}\rangle + \langle 1, u^{(1)}\rangle\langle\partial\phi_{\alpha_i}, e^{\phi_\beta}\rangle\right]$$
$$= \begin{cases} \partial\langle\partial\phi_{\alpha_i}, e^{\phi_\beta}\rangle & \text{if } u^{(1)} = 1 \\ \partial\langle\partial\phi_{\alpha_i}, u^{(1)}\rangle & \text{if } u^{(1)} \neq 1 \end{cases}$$

   and again both cases give some power of z lower than -1.

Hence we obtain

$$L_{-1}ue^{\phi_\beta} = \sum_i \sum_k \langle \partial\phi_{\alpha_i}, u^{(1)}e^{\phi_\beta}\rangle_{-k-1} u^{(2)}e^{\phi_\beta} \frac{\partial^{k+1}\phi_{\alpha_i*}}{k!}$$

$$= \sum_i \sum_{k_1+k_2=-k-1} \left(\langle \partial\phi_{\alpha_i}, u^{(1)}\rangle_{k_1} \langle 1, e^{\phi_\beta}\rangle_{k_2} + \langle 1, u^{(1)}\rangle_{k_1} \langle \partial\phi_{\alpha_i}, e^{\phi_\beta}\rangle_{k_2} \right) u^{(2)}e^{\phi_\beta} \frac{\partial^{k+1}\phi_{\alpha_i*}}{k!}$$

Now we split it again in two cases

- If $u^{(1)} = 1$ then we just have the second summand

$$\sum_i \sum_k \langle \partial\phi_{\alpha_i}, e^{\phi_\beta}\rangle_{-k-1} ue^{\phi_\beta} \frac{\partial^{k+1}\phi_{\alpha_i*}}{k!} = \sum_i (\alpha_i, \beta)ue^{\phi_\beta}\partial\phi_{\alpha_i*}$$

$$= \sum_i (\alpha_i, \beta)ue^{\phi_\beta}\vec{\alpha_i^*} \cdot \partial\vec{\phi} = \sum_i ue^{\phi_\beta}\vec{\alpha_i} \cdot \vec{\beta} \cdot \vec{\alpha_i^*} \cdot \partial\vec{\phi}$$

$$= ue^{\phi_\beta}\vec{\beta} \cdot \partial\vec{\phi} = ue^{\phi_\beta}\partial\phi_\beta = u(\partial.e^{\phi_\beta})$$

  where we had highlighted the vector nature of $\partial\phi$ and the elements α of the n-dimensional lattice Λ.

- If $u^{(1)} \neq 1$ then we just have the first summand

$$\sum_i \sum_k \langle \partial\phi_{\alpha_i}, u^{(1)}\rangle_{-k-1} u^{(2)}e^{\phi_\beta} \frac{\partial^{k+1}\phi_{\alpha_i*}}{k!}$$

  In this case the only possible $u^{(1)}$ such that the Hopf pairing is non-zero is $u^{(1)} = \partial^{n+1}\phi_\gamma$. In particular $u^{(2)} = 1$. We have now

$$\langle \partial\phi_{\alpha_i}, \partial^{n+1}\phi_\gamma\rangle = -\partial.\langle \partial\phi_{\alpha_i}, \partial^n\phi_\gamma\rangle$$

$$= (-1)^2\partial^2.\langle \partial\phi_{\alpha_i}, \partial^{n-1}\phi_\gamma\rangle$$

$$= (-1)^n\partial^n.\langle \partial\phi_{\alpha_i}, \partial\phi_\gamma\rangle$$

$$= (-1)^n\partial^n(\alpha_i, \gamma)z^{-2}$$

$$= (-1)^{2n}(n+1)!z^{-(n+2)}(\alpha_i, \gamma)$$

  Thus, the only term not equal to zero is for $k = n+1$. We obtain:

$$\sum_i \frac{1}{(n+1)!}(n+1)!(\alpha_i, \gamma)e^{\phi_\beta}\partial^{n+2}\phi_{\alpha_i*} = \partial^{n+2}\phi_\gamma e^{\phi_\beta} = \partial.(\partial^{n+1}\phi_\gamma)e^{\phi_\beta}$$

Putting together these two cases we have:

$$L_{-1}ue^{\phi_\beta} = u(\partial.e^{\phi_\beta}) + (\partial.u)e^{\phi_\beta}$$

$$\Rightarrow L_{-1}ue^{\phi_\beta} = \partial.(ue^{\phi_\beta})$$

$\square$

We now qualitatively describe the modules of Vir_c and its action on them.

- *The modules of Vir_c are given by the Λ-grading layers $\mathcal{V}_\lambda$.*
  Indeed we have that every operator L_n acts on an element ue^{ϕ_β} increasing (if $n < 0$) or decreasing (if $n > 0$) its $\mathbb{N}_0$-degree of n, but doesn't shift the Λ-grading.
  For example the L_{-1} operator acts as a derivation, i.e. as follows:

$$\partial.e^{\phi_\lambda} := \partial\phi_\lambda e^{\phi_\lambda}$$
$$\partial.\partial^k\phi_\alpha := \partial^{1+k}\phi_\alpha$$

  and more generally it increases the degree of ue^{ϕ_λ} by 1, but maintains the structure $Polynomial \cdot e^{\phi_\lambda}$.

- *On these modules $\mathcal{V}_\lambda$ is naturally defined another grading by the conformal dimension.*
  For example, for $rank(\Lambda) = 1$ we can represent the $\mathcal{V}_\lambda$ as triangles where on top we have the lowest conformal dimension element e^{ϕ_λ} and below an infinite graded succession of differential elements ue^{ϕ_β} that have higher conformal dimension. As the degree increases, the elements tend towards the triangle's base (see picture).
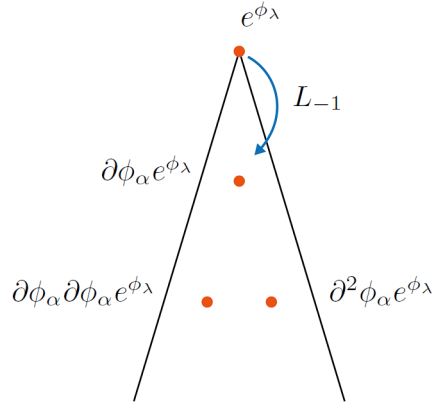

Figure 2.1: Virasoro module, $rank(\Lambda) = 1$

The Vir_c action in not irreducible nor semisimple: for example from $L_{-1}.1 = 0$ we see that $\partial\phi_\alpha$ together with all the elements of higher conformal dimension generate a submodule of the *Vacuum* module $\mathcal{V}_0$. This implies that the module is not irreducible.
But we also have $L_1.\partial\phi_\alpha = 1$ that implies that we can go back from this submodule to the *Vacuum* element 1 and so the module is also not semisimple.

Remark 2.23. The conformal dimension depends on Q. In particular, once Q is fixed, we have a paraboloid structure among the modules $\mathcal{V}_\lambda$: the conformal dimension of a top element e^{ϕ_λ} can be written as follow

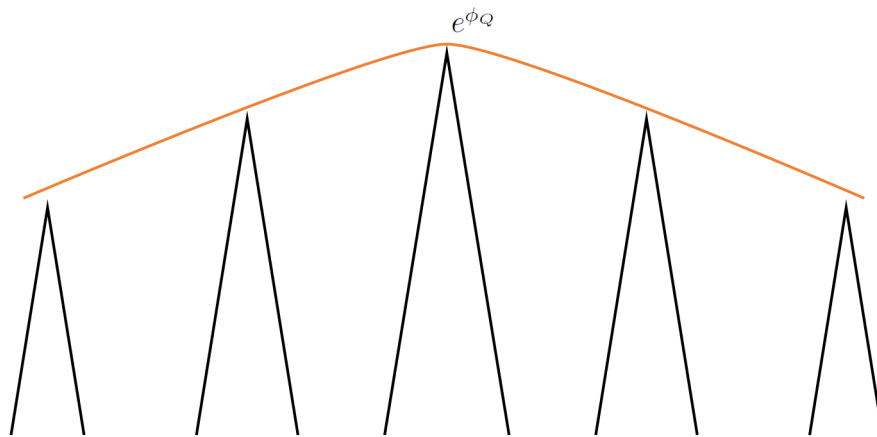$$\frac{1}{2}(\lambda, \lambda) - (\lambda, Q) = \frac{1}{2}\|Q - \lambda\|^2 - \frac{1}{2}(Q, Q)$$

Figure 2.2: Paraboloid structure among Virasoro modules, $rank(\Lambda) = 1$

so for $\lambda \in \Lambda$, the e^{ϕ_λ} elements trace a paraboloid with maximum in $\lambda = Q$.

2.5 Rescaled root lattices

Let $\mathfrak{g}$ be a complex finite dimensional semisimple Lie algebra. Denote by:

- Λ_R its root lattice with basis $\{\alpha_1, \ldots, \alpha_{rank}\}$

- Λ_W its weight lattice i.e. the lattice spanned by the fundamental weights $\{\lambda_1, \ldots, \lambda_{rank}\}$ such that $(\lambda_i, \alpha_j) = d_j \delta_{ij} = \frac{1}{2}(\alpha_j, \alpha_j)\delta_{ij}$

- $\Lambda_R{}^\vee$ its coroot lattice with basis $\{\alpha_1{}^\vee, \ldots, \alpha_{rank}{}^\vee\}$ where $\alpha_i{}^\vee := \alpha_i \frac{2}{(\alpha_i, \alpha_i)}$

Let $\ell = 2p$ be an even natural number divisible by all (α_j, α_j).

We rescale the root lattice Λ_R.

Definition 2.24. We call *short screening* lattice $\Lambda^\ominus := \frac{1}{\sqrt{p}}\Lambda_R$ with basis the *short screening momenta*

$$\left\{ \alpha_1{}^\ominus := -\frac{\alpha_1}{\sqrt{p}}, \ \ldots, \ \alpha_{rank}{}^\ominus := -\frac{\alpha_{rank}}{\sqrt{p}} \right\}.$$

We now consider the (generalized) lattice VOA $\mathcal{V}_{\Lambda^\ominus}$. By Lemma 2.22 we know that for every value of Q there is a Virasoro action on the algebra. Thus we look for the unique value of Q such that the conformal dimension of the elements $e^{\phi_{\alpha_i^\ominus}}$, that we will denote by $h^Q(\alpha_i{}^\ominus)$, is equal to 1. We will understand later why this is convenient.

Lemma 2.25. *The unique Q such that $h^Q(\alpha_i{}^\ominus) = 1$ is given by $Q = (p \cdot \rho_{\mathfrak{g}}{}^\vee - \rho_{\mathfrak{g}})/\sqrt{p}$, where we define $\rho_{\mathfrak{g}}$ as the half sum of all positive roots, and $\rho_{\mathfrak{g}}{}^\vee$ the analogous for the dual root system.*

Proof. To prove it we have to show that

$$\frac{1}{2}(\alpha_i{}^\ominus, \alpha_i{}^\ominus) - (\alpha_i{}^\ominus, Q) = 1$$

So, let us compute the first term:

$$\frac{1}{2}\left(-\frac{\alpha_i}{\sqrt{p}}, -\frac{\alpha_i}{\sqrt{p}}\right) - \left(-\frac{\alpha_i}{\sqrt{p}}, (p \cdot \rho_{\mathfrak{g}}{}^\vee - \rho_{\mathfrak{g}})\frac{1}{\sqrt{p}}\right)$$

$$= \frac{1}{2p}(\alpha_i, \alpha_i) + \frac{1}{p}(\alpha_i, p \cdot \rho_{\mathfrak{g}}{}^\vee) - \frac{1}{p}(\alpha_i, \rho_{\mathfrak{g}})$$

$$= \frac{1}{2p}(\alpha_i, \alpha_i) + (\alpha_i, \rho_{\mathfrak{g}}{}^\vee) - \frac{1}{p}\frac{(\alpha_i, \alpha_i)}{2}$$

$$= (\alpha_i, \rho_{\mathfrak{g}}{}^\vee) = \frac{(\alpha_i, \alpha_i)}{2}(\alpha_i{}^\vee, \rho_{\mathfrak{g}}{}^\vee)$$

$$= \frac{(\alpha_i, \alpha_i)}{2}\frac{(\alpha_i{}^\vee, \alpha_i{}^\vee)}{2} = 1$$

where we used that $\alpha_i{}^\vee = \alpha_i \frac{2}{(\alpha_i, \alpha_i)}$ and $(\alpha_i, \rho_{\mathfrak{g}}) = \frac{(\alpha_i, \alpha_i)}{2}$. $\square$

Lemma 2.26. *Fixed $Q = (p \cdot \rho_{\mathfrak{g}}{}^\vee - \rho_{\mathfrak{g}})/\sqrt{p}$ as above, another combination of elements of Λ_R, β, such that $h^Q(\beta) = 1$ is given by $\beta := \alpha_i{}^\vee\sqrt{p}$*

Proof. Let's compute $h^Q(\beta)$

$$\frac{1}{2}(\alpha_i{}^\vee\sqrt{p}, \alpha_i{}^\vee\sqrt{p}) - (\alpha_i{}^\vee\sqrt{p}, (p\cdot\rho_{\mathfrak{g}}{}^\vee - \rho_{\mathfrak{g}})/\sqrt{p})$$

$$= \frac{p}{2}(\alpha_i{}^\vee, \alpha_i{}^\vee) - (\alpha_i{}^\vee, p\cdot\rho_{\mathfrak{g}}{}^\vee) + (\alpha_i{}^\vee, \rho_{\mathfrak{g}})$$

$$= \frac{p}{2}(\alpha_i{}^\vee, \alpha_i{}^\vee) - \frac{p}{2}(\alpha_i{}^\vee, \alpha_i{}^\vee) + \frac{2}{(\alpha_i, \alpha_i)}(\alpha_i, \rho_{\mathfrak{g}}) = 1$$

$\square$

Hence, once the lattice Λ_R is rescaled to $\Lambda^\ominus$, the choice of such a Q gives directly a second lattice spanned by roots β such that $h^Q(\beta) = 1$.

Definition 2.27. The *long screening* lattice is $\Lambda^\oplus := \sqrt{p}\Lambda_R{}^\vee$ with basis the *long screening momenta*

$$\{\alpha_1{}^\oplus := \alpha_1^\vee\sqrt{p}, \ \ldots, \ \alpha_{rank}{}^\oplus := \alpha_{rank}^\vee\sqrt{p}\}.$$

We consider lastly a third bigger lattice $(\Lambda^\oplus)^*$ defined as the dual lattice of $\Lambda^\oplus$ namely as the one spanned by some $\lambda_1{}^*, \ldots \lambda_{rank}{}^*$ such that $(\lambda_i{}^*, \alpha_j{}^\oplus) = \delta_{ij}$.

Lemma 2.28. *This lattice happens to be equal to the rescaled weight lattice:*

$$(\Lambda^\oplus)^* = \frac{1}{\sqrt{p}}\Lambda_W$$

Proof. The equality can be easily proved:

$$(\lambda_i{}^*, \alpha_j{}^\oplus) = (\frac{1}{\sqrt{p}}\lambda_i, \sqrt{p}\alpha_j{}^\vee) = (\frac{1}{\sqrt{p}}\lambda_i, \sqrt{p}\alpha_j\frac{2}{(\alpha_j, \alpha_j)}) = \frac{2}{(\alpha_j, \alpha_j)}(\lambda_i, \alpha_j) = \delta_{ij}$$

$\square$

The inclusion relationships among these three lattices is as follows:

$$\Lambda^\oplus \subset \Lambda^\ominus \subset (\Lambda^\oplus)^*$$

Remark 2.29. The choice of $l = 2p$ to be divisible by all (α_j, α_j) implies that $\Lambda^\oplus$ is integral. This in turn implies that $\mathcal{V}_{\Lambda^\oplus}$ is a lattice VOA.
Instead we will talk about generalized lattice VOA in the case of $\mathcal{V}_{\Lambda^\ominus}$ and $\mathcal{V}_{(\Lambda^\oplus)^*}$

Corollary 2.30. *The number of irreducible representations of $\mathcal{V}_{\Lambda^\oplus}$ is, by Theorem 2.7 equal to $|(\Lambda^\oplus)^*/\Lambda^\oplus|$. To compute it, it is in general helpful to use the intermediate lattice $\Lambda^\ominus$ as follows*

$$|(\Lambda^\oplus)^*/\Lambda^\oplus| = |(\Lambda^\oplus)^*/\Lambda^\ominus| \cdot |(\Lambda^\ominus)/\Lambda^\oplus|$$

$$= |\frac{1}{\sqrt{p}}\Lambda_W/\frac{1}{\sqrt{p}}\Lambda_R| \cdot |\frac{1}{\sqrt{p}}\Lambda_R/\sqrt{p}\Lambda_R^\vee|$$

$$= |\Lambda_W/\Lambda_R| \cdot \left|\bigoplus_{\alpha_i}\left(\frac{1}{\sqrt{p}}\alpha_i\mathbb{Z}/\sqrt{p}\frac{2}{(\alpha_i, \alpha_i)}\alpha_i\mathbb{Z}\right)\right|$$

$$= |\Lambda_W/\Lambda_R| \cdot \prod_{i=1}^{rank}\frac{\ell}{(\alpha_i, \alpha_i)}$$

where the order of Λ_W/Λ_R is the determinant of the Cartan matrix.

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2.6 Application of results

We will now apply the results displayed in section 2.2, 2.3, 2.4 to the lattice VOAs associated to the three lattices $\Lambda^\ominus, \Lambda^\oplus, (\Lambda^\oplus)^*$.

Let $\mathcal{V}_{\Lambda^\oplus}$ be the lattice VOA associated to the lattice $\Lambda^\oplus$. We know that its simple vertex algebra modules are $\mathcal{V}_{[\lambda]}$ parametrized by $(\Lambda^\oplus)^*/\Lambda^\oplus$ and we can decompose

$$\mathcal{V}_{(\Lambda^\oplus)^*} = \bigoplus_{[\lambda] \in (\Lambda^\oplus)^*/(\Lambda^\oplus)} \mathcal{V}_{[\lambda]}.$$

Moreover, as said in Remark 2.20, the action of Vir_c on $\mathcal{V}_{\Lambda^\oplus}$ given by the vertex operator passes automatically on all the $\mathcal{V}_{\Lambda^\oplus}$-modules $\mathcal{V}_{[\lambda]}$. Any $\mathcal{V}_{\Lambda^\oplus}$-module $\mathcal{V}_{[\lambda]}$ decompose as Virasoro module into its Λ-grading layers:

$$\mathcal{V}_{[\lambda]} = \bigoplus_{\lambda \in [\lambda]} \mathcal{V}_\lambda \tag{2.3}$$

Theorem 2.14 applied to the lattice $\Lambda = \Lambda^\ominus$ implies:

Corollary 2.31. *For those roots $\alpha_i{}^\ominus$ such that $(\alpha_i{}^\ominus, \alpha_i{}^\ominus) \leq 1$, i.e. $(\alpha_i, \alpha_i) \leq p$, the* short screening operators $\mathfrak{Z}_{\alpha_i\ominus}$ *constitute a representation on $\mathcal{V}_{(\Lambda^\oplus)^*}$ of the Nichols algebra for braiding*

$$q_{ij} = e^{\pi i(-\frac{\alpha_i}{\sqrt{p}}, -\frac{\alpha_j}{\sqrt{p}})} = e^{\frac{\pi i}{p}(\alpha_i, \alpha_j)} = q^{(\alpha_i, \alpha_j)}$$

with $q := e^{\frac{2\pi i}{\ell}}$ a ℓ-th root of unity. This is precisely the representation of the postive part of the small quantum group $u_q(\mathfrak{g})^+$.

On the other hand a classical result in [TF09] for $\mathfrak{g}$ simply-laced says:

Corollary 2.32. *The long screening operators $\mathfrak{Z}_{\alpha_i\oplus}$ constitute a representation of the negative part of the enveloping algebra of $\mathfrak{g}^\vee$: $U(\mathfrak{g}^\vee)^-$*

Remark 2.33. A conjecture in [Lent17] affirms it for the general non simply-laced case.

We chose the Virasoro action parametrized by Q such that the following conformal dimensions are

$$h^Q(\alpha_i{}^\ominus) = h^Q(\alpha_i{}^\oplus) = 1$$

Thanks to the usual OPE associativity (equation 2.1) for the integer operators $\mathfrak{Z}_{\alpha_i\oplus}$ this implies:

Corollary 2.34. *The long screening operators $\mathfrak{Z}_{\alpha_i\oplus}$ commute with the Virasoro action, i.e. that they are Virasoro homomorphisms.*

In [Lent17] it is proven that this commutativity relation holds in a weaker form also for the *short screening* case thanks to Nichols algebra relations:

Corollary 2.35. *There exist a $h \in \mathbb{N}$ depending on the modules $\mathcal{V}_{[\lambda]}$ such that the* Weyl *reflections $(\mathfrak{Z}_{\alpha_i \ominus})^h$ around the vector Q, called Steinberg point, commute with the Virasoro action, i.e. are Virasoro homomorphisms.*

For example, in the vacuum module $\mathcal{V}_{[0]} = \mathcal{V}_{\Lambda^\oplus}$ we have $h = 1$. So on this module all $(\mathfrak{Z}_{\alpha_i \ominus})$ are Virasoro homomorphisms. Finally, from the definition of screening operators we have

Corollary 2.36. *The $\mathfrak{Z}_{\alpha_i \oplus}$'s preserve the $\mathcal{V}_{\Lambda^\oplus}$-modules and the $\mathfrak{Z}_{\alpha_i \ominus}$'s preserve the $\Lambda^\ominus$ cosets.*

2.7 Example A_1

Let us now focus on the easiest example of a Lie algebra $\mathfrak{g}$ of type A_1 already treated in [FGST06a], [TW13], [NT11].
We will compute the three lattices $\Lambda^\ominus$, $\Lambda^\oplus$ and $(\Lambda^\oplus)^*$ as defined in section 2.5. We will then consider the lattice VOA $\mathcal{V}_{\Lambda^\oplus}$ and obtain its simple modules.

Consider thus $\mathfrak{g} = \mathfrak{sl}_2$ and its root lattice $\Lambda_R = \alpha\mathbb{Z}$ with α simple root such that $(\alpha, \alpha) = 2$. For the moment we assume $\ell = 2p$ an arbitrary even number.

- We start as before by rescaling the root lattice to the *short screening lattice*
$$\Lambda^\ominus = \alpha^\ominus \mathbb{Z} = -\frac{\alpha}{\sqrt{p}}\mathbb{Z}.$$
  with short screening *momentum* $\alpha^\ominus = -\frac{\alpha}{\sqrt{p}}$.

- To compute Q as in Lemma 2.25, we first notice that in this case $\alpha^\vee = \alpha\frac{2}{(\alpha,\alpha)} = \alpha$ and
$$\rho_\mathfrak{g} = \rho_\mathfrak{g}{}^\vee = \frac{1}{2}\alpha$$
  so we easily arrive to
$$Q = \frac{1}{\sqrt{p}}(p\rho_\mathfrak{g}{}^\vee - \rho_\mathfrak{g}) = \frac{1}{2\sqrt{p}}(p \cdot \alpha - \alpha) = \frac{1}{2}\frac{(p-1)}{\sqrt{p}}\alpha = -\frac{p-1}{2}\alpha^\ominus$$

  Remark 2.37. In this case, since there is just one simple root we can compute Q also directly. From the general ansatz we know indeed that $Q = k\alpha$ with $k \in \mathbb{C}$, is the only one such that the property $h^Q(\alpha_i^\ominus) = 1$ holds. This implies that it has to be:
$$\frac{1}{2}(-\frac{\alpha}{\sqrt{p}}, -\frac{\alpha}{\sqrt{p}}) - (-\frac{\alpha}{\sqrt{p}}, k\alpha) = 1$$
$$\frac{1}{2p}(\alpha, \alpha) + \frac{k}{\sqrt{p}}(\alpha, \alpha) = 1$$
$$\Rightarrow \qquad k = \frac{1}{2}\frac{(p-1)}{\sqrt{p}}$$

  and then $Q = \frac{1}{2}\frac{(p-1)}{\sqrt{p}}\alpha$, as claimed.

- Consider then the *long screening* lattice $\Lambda^\oplus = \mathbb{Z}\alpha^\oplus$ found as before requiring $h^Q(\alpha^\oplus) = 1$. From Lemma 2.26 we obtain:

$$\alpha^\oplus = \alpha\sqrt{p} \qquad \Lambda^\oplus = \alpha\sqrt{p}\,\mathbb{Z}$$

again because $\alpha^\vee = \alpha\frac{2}{(\alpha,\alpha)} = \alpha$.

- To have the third lattice we have to compute the fundamental weight i.e. the $\lambda = k\alpha$ such that $(\lambda, \alpha) = \frac{1}{2}(\alpha, \alpha) = 1$. Immediately we arrive to $k = \frac{1}{2}$ and so:

$$(\Lambda^\oplus)^* = \frac{1}{\sqrt{p}}\Lambda_W = \frac{\alpha}{2\sqrt{p}}\mathbb{Z}$$

- Now we can compute the number of representations of $\mathcal{V}_{\Lambda^\oplus}$:

$$\left|(\Lambda^\oplus)^*/\Lambda^\oplus\right| = \left|(\Lambda^\oplus)^*/\Lambda^\ominus\right| \cdot \left|\Lambda^\ominus/\Lambda^\oplus\right| =$$

$$= \left|\frac{1}{\sqrt{p}}\Lambda_W / \frac{1}{\sqrt{p}}\Lambda_R\right| \cdot \left|\frac{1}{\sqrt{p}}\Lambda_R/\sqrt{p}\Lambda_R{}^\vee\right| =$$

$$= \left|\Lambda_W/\Lambda_R\right| \cdot \left|\frac{1}{\sqrt{p}}\Lambda_R/\sqrt{p}\Lambda_R{}^\vee\right| =$$

$$= 2\left|-\frac{1}{\sqrt{p}}\alpha\mathbb{Z}/\sqrt{p}\alpha\mathbb{Z}\right| = 2\left|\mathbb{Z}_p\right| = 2p$$

- Explicitly these $2p$ representations are given by the cosets with representatives:

$$(\Lambda^\oplus)^*/\Lambda^\oplus = \left\{[\frac{k\alpha}{2\sqrt{p}}] : k = 0, 1, \ldots, 2p-1\right\}$$

that fall into two $\Lambda^\ominus$-cosets

$$0 + \Lambda^\ominus/\Lambda^\oplus = \{\frac{k\alpha}{2\sqrt{p}}, \text{k even}\} \qquad \frac{\alpha}{2\sqrt{p}} + \Lambda^\ominus/\Lambda^\oplus = \{\frac{k\alpha}{2\sqrt{p}}, \text{k odd}\}.$$

The representations are thus $\mathcal{V}_{[\frac{k\alpha}{2\sqrt{p}}]}$, with $k = 0, 1, \ldots, 2p-1$.

Chapter 3

The $\mathcal{W}$ subspace

We now want to use the short screenings to construct a subspace $\mathcal{W}$ of the lattice VOA $\mathcal{V}_{\Lambda\oplus}$ following the so called *free-field realization*.

The subject of study is

$$\mathcal{W} := \mathcal{V}_{\Lambda\oplus} \cap \bigcap_i ker\mathfrak{Z}_{\alpha_i\ominus}.$$

Remark 3.1. The space $\mathcal{W}$ is a sub vertex algebra of $\mathcal{V}_{\Lambda\oplus}$ due to OPE associativity (equation 2.1).

On this space a conjecture has been formulated successively by several authors [FF88], [FFHST02], [TF09], [FGST06a], [AM08]:

Conjecture 3.2.   1. $\mathcal{W}$ is a vertex subalgebra with an action of the Lie algebra $\mathfrak{g}^\vee$ via the long screenings.

  2. It is a Logarithmic conformal field theory (LCFT) i.e. a VOA with finite nonsemisimple representation theory.

  3. Its Representation category is equivalent to the one of the respective small quantum group $u_q(\mathfrak{g})$ at a root of unity q of order $\ell = 2p$.

Remark 3.3. Saying that a LCFT is a VOA with (nonsemisimple) *finite* representation theory means that the Rep category is a finite tensor category in the sense of [EO03].

The construction of such a $\mathcal{W}$ is interesting because of the third point of the conjecture and in its own right, since few logarithmic CFTs are known: most importantly the triplet algebra $\mathcal{W}_p$, which is exactly the one we will construct through free field realization in the A_1 case, and the even part of n pairs of symplectic fermions that we will study later on.

In the next section we will go deeper in the study of the $\mathcal{V}_{\Lambda\oplus}$-modules and we will look at this new space $\mathcal{W}$ and at its representations in the particular case A_1, $\ell = 4$.

3.1 A_1, $\ell = 4$

3.1.1 Lattices

Let us consider again the previous example, specialized to $\ell = 2p = 4$. For this value of ℓ we have:

- $Q = \frac{\alpha}{2\sqrt{2}}$

- $\Lambda^\ominus = \mathbb{Z}\frac{\alpha}{\sqrt{2}}, \qquad \alpha^\ominus = -\frac{\alpha}{\sqrt{2}}$

- $\Lambda^\oplus = \mathbb{Z}\alpha\sqrt{2} \qquad \alpha^\oplus = \sqrt{2}\alpha,$

- $(\Lambda^\oplus)^* = \mathbb{Z}\frac{\alpha}{2\sqrt{2}} \qquad \lambda^* = \frac{\alpha}{2\sqrt{2}}$

- the number of simple representations of $\mathcal{V}_\Lambda^\oplus$ is now 4 and they are given by the classes of $(\Lambda^\oplus)^*/\Lambda^\oplus$, $[\frac{k\alpha}{2\sqrt{2}}]$ for $k = 0, 1, 2, 3$; explicitly:

$$\mathcal{V}_{[0]}, \ \mathcal{V}_{[\frac{\alpha}{2\sqrt{2}}]}, \ \mathcal{V}_{[\frac{2\alpha}{2\sqrt{2}}]}, \ \mathcal{V}_{[\frac{3\alpha}{2\sqrt{2}}]}$$

To simplify what follows we give informal names to them:
Blue (or *Vacuum*), *Steinberg*, *Green* and *Facet* module.

3.1.2 Groundstates

We now introduce the notion of groundstates elements. These give us information about the modules and are the elements on which we will apply the screening operators in the next section.

Definition 3.4. The space of groundstates is the L_0-eigenspace associated with the minimal eigenvalue, i.e. the minimal conformal dimension.

Remark 3.5. For each module $\mathcal{V}_{[\frac{k\alpha}{2\sqrt{2}}]}$ of $\mathcal{V}_{\Lambda^\oplus}$, a preferred basis for this space are the elements e^{ϕ_λ} with $\lambda \in [\frac{k\alpha}{2\sqrt{2}}]$ vector with minimal distance to Q.

Definition 3.6. In particular in this work we will understand under the name of *groundstates elements* exactly the elements e^{ϕ_λ} with λ of minimal distance to Q.

In the following table we will write for each module $\mathcal{V}_{[k]} := \mathcal{V}_{[\frac{k\alpha}{2\sqrt{2}}]}$ the dimension of groundstates space, the respective conformal dimension and explicitly the elements of the groundstates.

	#Groundstates	Conformal dimension	Groundstates elements e^{ϕ_λ}
$\mathcal{V}_{[0]}$	1	0	0
$\mathcal{V}_{[1]}$	1	$-1/8$	$\alpha/2\sqrt{2} = Q$
$\mathcal{V}_{[2]}$	1	0	$\alpha/\sqrt{2}$
$\mathcal{V}_{[3]}$	2	$3/8$	$-\alpha/2\sqrt{2}, \ 3\alpha/2\sqrt{2}$

Table 3.1: Groundstates informations for each module, A_1 case.

3.1.3 Screening operators

We now focus on the screening operators and on their action. From the Corollary 2.34 and 2.35 of Section 2.6 we know that the *long screening* operators and some power of the *short screening* operators are *Vir*-homomorphisms. In particular on the *Blue* and *Green* modules this power is equal to 1, i.e. on them the *short screening* operators are really *Vir*-homomorphisms.

We will now explicitly apply the screening operators short $\mathfrak{Z}_{-\alpha/\sqrt{2}}$ and long $\mathfrak{Z}_{\alpha\sqrt{2}}$ to our modules to see how they act. Since $\forall \beta \in [0], [\frac{2\alpha}{2\sqrt{2}}]$ we have $(-\frac{\alpha}{\sqrt{2}}, \beta) \in \mathbb{Z}$ we can use formula 2.2 of Definition 2.12.

Remark 3.7. We notice that, from the consequences of Theorem 2.14, since $(-\alpha/\sqrt{2}, -\alpha/\sqrt{2}) = 1$ is an odd integer, then $(\mathfrak{Z}_{-\alpha/\sqrt{2}})^2 = 0$ holds.

We start applying the *short screening* operator to the first two layers of *Blue* and *Green* modules:

1. The short screening operator applied to the *Blue* groundstates element (i.e. with conformal dimension equal to 0) give us the result:
$$\mathfrak{Z}_{-\alpha/\sqrt{2}}(e^0) = \mathfrak{Z}_{-\alpha/\sqrt{2}}(1) = 0$$

2. The short screening operator applied to the *second layer* (i.e. with conformal dimension equal to 1) of the *Blue* module gives us the results:

$$\mathfrak{Z}_{-\alpha/\sqrt{2}}(\partial\phi_{\alpha/\sqrt{2}}) =$$
$$= \sum_k \left\langle e^{\phi - \frac{\alpha}{\sqrt{2}}}, \partial\phi_{\alpha/\sqrt{2}} \right\rangle_{-k-1} \frac{1}{k!} \partial^k e^{\phi - \frac{\alpha}{\sqrt{2}}} + \sum_k \left\langle e^{\phi - \frac{\alpha}{\sqrt{2}}}, 1 \right\rangle_{-k-1} \partial\phi_{\alpha/\sqrt{2}} \frac{1}{k!} \partial^k e^{\phi - \frac{\alpha}{\sqrt{2}}}$$
$$= \left\langle e^{\phi - \frac{\alpha}{\sqrt{2}}}, \partial\phi_{\alpha/\sqrt{2}} \right\rangle_{-1} e^{\phi - \frac{\alpha}{\sqrt{2}}} + 0$$
$$= -(\frac{\alpha}{\sqrt{2}}, -\frac{\alpha}{\sqrt{2}}) e^{\phi - \frac{\alpha}{\sqrt{2}}} = e^{\phi - \frac{\alpha}{\sqrt{2}}}$$
$$\mathfrak{Z}_{-\alpha/\sqrt{2}}(e^{\phi_\alpha\sqrt{2}}) =$$
$$= \sum_k \left\langle e^{\phi - \frac{\alpha}{\sqrt{2}}}, 1 \right\rangle_{-k-1+2} e^{\phi_\alpha\sqrt{2}} \frac{1}{k!} \partial^k e^{\phi - \frac{\alpha}{\sqrt{2}}} = \partial\phi_{-\alpha/\sqrt{2}} e^{\phi \frac{\alpha}{\sqrt{2}}}$$

   In the first equation the first term gives a nonzero result if and only if $k = 0$; the second term vanishes because it could be non zero just for $k = -1$ but $k > 0$.
   In the second equation k must be equal to 1.

3. The short screening operator applied to the *Green* groundstates element (i.e. with conformal dimension equal to 0) gives us the result:

$$\mathfrak{Z}_{-\alpha/\sqrt{2}}(e^{\phi \frac{\alpha}{\sqrt{2}}}) = \sum_k \left\langle e^{\phi - \frac{\alpha}{\sqrt{2}}}, 1 \right\rangle_{-k-1+1} e^{\phi \frac{\alpha}{\sqrt{2}}} \frac{1}{k!} \partial^k e^{\phi - \frac{\alpha}{\sqrt{2}}} =$$
$$= \left\langle e^{\phi - \frac{\alpha}{\sqrt{2}}}, 1 \right\rangle_0 e^{\phi \frac{\alpha}{\sqrt{2}}} = e^0 = 1$$

   since the only term non equal to zero is for $k = 0$.

4. The short screening operator applied to the *second layer* (i.e. with conformal dimension equal to 1) of the *Green* module give us the results:

$$3_{-\alpha/\sqrt{2}}(\partial\phi_{\alpha/\sqrt{2}}e^{\phi\frac{\alpha}{\sqrt{2}}}) = 3_{-\alpha/\sqrt{2}}(L_{-1}(e^{\phi\frac{\alpha}{\sqrt{2}}})) = L_{-1}(3_{-\alpha/\sqrt{2}}(e^{\phi\frac{\alpha}{\sqrt{2}}})) = L_{-1}(1) = 0$$

$$3_{-\alpha/\sqrt{2}}(e^{\phi-\frac{\alpha}{\sqrt{2}}}) = \sum_k \left\langle e^{\phi-\frac{\alpha}{\sqrt{2}}}, 1 \right\rangle_{-k-1-1} e^{\phi-\frac{\alpha}{\sqrt{2}}} \frac{1}{k!}\partial^k e^{\phi-\frac{\alpha}{\sqrt{2}}} = 0$$

In the first equation we used the commutativity with the Virasoro action. In the second equation the term vanishes since k should be equal to -2 and this is not possible.

We will now apply the *long screening* operator to these elements:

1. *Blue, first layer*
$$3_{\alpha\sqrt{2}}(e^0) = 3_{\alpha\sqrt{2}}(1) = 0$$

2. *Blue, second layer*
$$3_{\alpha\sqrt{2}}(\partial\phi_{\alpha/\sqrt{2}}) = -2e^{\phi_{\alpha\sqrt{2}}}$$
$$3_{\alpha\sqrt{2}}(e^{\phi_{\alpha\sqrt{2}}}) = 0$$

3. *Green, first layer*
$$3_{\alpha\sqrt{2}}(e^{\phi\frac{\alpha}{\sqrt{2}}}) = 0$$

4. *Green, second layer*
$$3_{\alpha\sqrt{2}}(\partial\phi_{\alpha/\sqrt{2}}e^{\phi\frac{\alpha}{\sqrt{2}}}) = 0$$
$$3_{\alpha\sqrt{2}}(e^{\phi-\frac{\alpha}{\sqrt{2}}}) = \partial\phi_{\alpha\sqrt{2}}e^{\phi\frac{\alpha}{\sqrt{2}}}$$

Finally, we show these and more results in the following picture.
The $(\Lambda^\oplus)^*$-grading $k\alpha/2\sqrt{2}$ is given by k on the x-axis. The conformal dimension is on the y-axis.
In the figure are shown nine Virasoro modules; among them we have one representative of the *Steinberg* module (purple), two of the *Facet* module (orange) and three of the others.
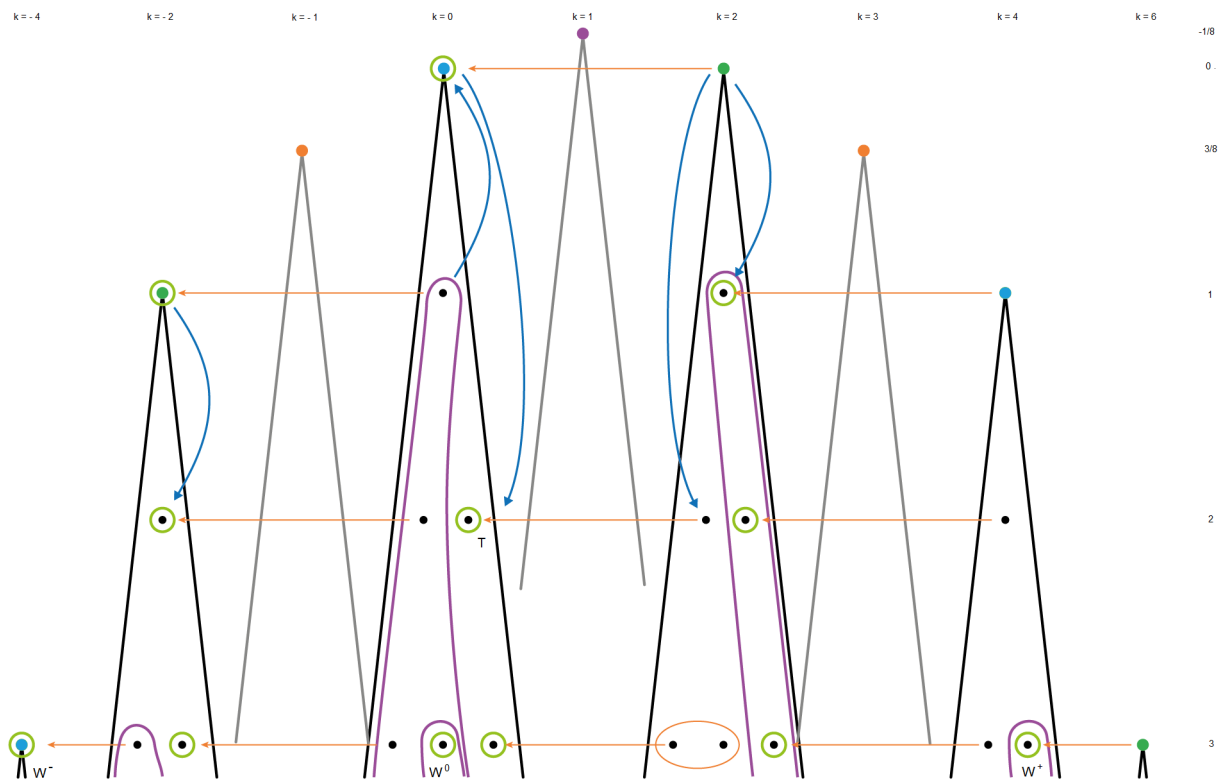
Figure 3.1: The A_1 case

The orange arrows, going from the *Green* to the *Blue* modules and vice versa, represent the short screening operator and the elements in its kernel are encircled with green colour.

We notice that all the elements in the *Blue* or *Green* modules on the left border of this constellation belong to the kernel. Moreover, if an element is in the kernel then it also belongs to the image.

The blue arrows represent some examples of the Virasoro action. In particular it is shown the following result:

Proposition 3.8. *Defined* $T = \frac{1}{2}\sum_i \partial\phi_{\alpha_i}\partial\phi_{\alpha_i*} + \sum_i Q_i \partial^2\phi_{\alpha_i}$ *Energy-Stress tensor as in Theorem 2.19, we have:*

$$L_{-2}1 = T$$

Proof.

$$L_{-2}1 = Y(T)_0 1 = Y\left(\frac{1}{2}\sum_i \partial\phi_{\alpha_i}\partial\phi_{\alpha_i*} + \sum_i Q_i \partial^2\phi_{\alpha_i}\right)_0 1$$

$$= \frac{1}{2}\sum_i Y(\partial\phi_{\alpha_i}\partial\phi_{\alpha_i*}) + \sum_i Q_i Y(\partial^2\phi_{\alpha_i})_0 1)$$

$$= \frac{1}{2}\sum_i \left(\sum_{k\geq 0}\langle 1,1\rangle_{-k}\frac{1}{k!}\partial^k.\partial\phi_{\alpha_i}\partial\phi_{\alpha_i*} + \sum_{k\geq 0}2\langle\partial\phi_{\alpha_i},1\rangle_{-k}\frac{1}{k!}\partial^{k+1}\phi_{\alpha_i*} + \sum_{k\geq 0}\langle\partial\phi_{\alpha_i}\partial\phi_{\alpha_i*},1\rangle_{-k}\frac{1}{k!}\partial^k 1\right) +$$

$$+ \sum_i Q_i \left(\sum_{k\geq 0}\langle 1,1\rangle_{-k}\frac{1}{k!}\partial^k.\partial^2\phi_{\alpha_i}\right)$$

$$= \frac{1}{2}\sum_i \partial\phi_{\alpha_i}\partial\phi_{\alpha_i*} + \sum_i Q_i \partial^2\phi_{\alpha_i} = T$$

in the first and last term of the third line we have $k = 0$ whereas the second and third terms vanish. $\qquad\square$

3.1.4 LCFT $\mathcal{W}$

We now look at the subspace $\mathcal{W}$. In this case, the A_1 case, it is known to be equal to the $\mathcal{W}_2$-triplet algebra. The elements $\mathcal{W}^-, \mathcal{W}^0, \mathcal{W}^+$ showed in picture 3.1 and studied in [FFHST02], form the adjoint 3-dimensional representation of $\mathfrak{g}^\vee = \mathfrak{sl}_2$ acting by long screenings.

Since the *rank* is equal to 1, we have only one short screening operator $\mathfrak{Z}_{-\alpha/\sqrt{2}}$ and so the definition becomes:

$$\mathcal{W} = \mathcal{V}_{\Lambda\oplus} \cap \ker \mathfrak{Z}_{-\alpha/\sqrt{2}}$$
$$= \mathcal{V}_{\sqrt{2}\alpha\mathbb{Z}} \cap \ker \mathfrak{Z}_{-\alpha/\sqrt{2}}$$
$$= \mathcal{V}_{[0]} \cap \ker \mathfrak{Z}_{-\alpha/\sqrt{2}}.$$

Since $\mathcal{W}$ is a subalgebra of $\mathcal{V}_{\Lambda\oplus}$, we can now restrict the modules of $\mathcal{V}_{\Lambda\oplus}$ to $\mathcal{W}$-modules and describe their (non-semisimple) decomposition behaviour into irreducibles.

As shown in Corollary 2.35, the Weyl reflections $(\mathfrak{Z}_{-\alpha/\sqrt{2}})^h$ are Virasoro homomorphisms. Moreover they commute with the $\mathcal{W}$-action by OPE associativity

(equation 2.1). Thus they are vertex module homomorphisms for the restriction to $\mathcal{W}$ and so their kernel and image are $\mathcal{W}$-submodules. In the following table we see which values of the power h are associated to the four modules $\mathcal{V}_{[k]} := \mathcal{V}_{[\frac{k\alpha}{2\sqrt{2}}]}$:

	h
$\mathcal{V}_{[0]}$	1
$\mathcal{V}_{[1]}$	0
$\mathcal{V}_{[2]}$	1
$\mathcal{V}_{[3]}$	2

- For the Steinberg module $\mathcal{V}_{[1]}$ the commuting map is just the identity so the kernel is trivial and the image is all. We get then that the module stays irreducible over $\mathcal{W}$-action and we call it $\Lambda(2)$.

- For the Facet module $\mathcal{V}_{[3]}$, since $(\mathfrak{Z}_{-\frac{\alpha}{\sqrt{2}}})^2 = 0$, the image is trivial and the kernel is all. Then again the module stays irreducible and we call it $\Pi(2)$.

- For the Blue and Green modules, respectively $\mathcal{V}_{[0]}$ and $\mathcal{V}_{[2]}$, we have instead that $(\mathfrak{Z}_{-\frac{\alpha}{\sqrt{2}}})^1$ decomposes them into non-trivial kernel and image. We will call them $\Lambda(1)$ and $\Pi(1)$ and explicitly we have:

$$\Lambda(1) \rightarrow \mathcal{V}_{[0]} \rightarrow \Pi(1)$$
$$\Pi(1) \rightarrow \mathcal{V}_{[2]} \rightarrow \Lambda(1)$$

To summarize we have the following decompositions:

$$\Lambda(1) \rightarrow \mathcal{V}_{[0]} \rightarrow \Pi(1)$$
$$\Pi(1) \rightarrow \mathcal{V}_{[2]} \rightarrow \Lambda(1)$$
$$\mathcal{V}_{[1]} \cong \Lambda(2)$$
$$\mathcal{V}_{[3]} \cong \Pi(2)$$

Remark 3.9. The letters were chosen by Semikhatov to suggest that $\Lambda(1)$ and $\Lambda(2)$ have a 1-dimensional groundstates space and $\Pi(1)$ and $\Pi(2)$ a 2-dimensional groundstates space.

Remark 3.10. We remark that what we said neither proves that $\Lambda(i), \Pi(i)$ are irreducible representations nor that these are all the irreducible representations of $\mathcal{W}$. However for this particular case, A_1, it is proven e.g. in [AM08].

Chapter 4

Case B_n, $\ell = 4$

In the previous chapters we introduced a setting, we defined through free-field realization a subspace $\mathcal{W}$, conjecturally a logarithmic CFT, and then we applied the theory to the example of a Lie algebra $\mathfrak{g}$ with root system A_1.

In this chapter we will follow again this outline applying it to a new example: the B_n case, $\ell = 4$. This is particularly interesting because it is *degenerate* in a sense that will be explained later on. To understand the general instance we will start with the B_2 case.

4.1 $n = 2$

4.1.1 Lattices

Let $\mathfrak{g}$ be a Lie algebra with root system B_2, i.e. $\mathfrak{g} = \mathfrak{so}(5)$. Let ℓ as before be an even integer $\ell = 2p$.

Consider as basis of the root lattice Λ_R the set $\{\alpha_1, \alpha_2\}$ with Killing form:

$$(\alpha_1, \alpha_1) = 4 \qquad (\alpha_2, \alpha_2) = 2 \qquad (\alpha_1, \alpha_2) = -2$$

that means α_1 is the long root and α_2 the short root.

The coroots are then $\alpha_1{}^\vee = \alpha_1/2$, and $\alpha_2{}^\vee = \alpha_2$ and so the sets of positive roots and positive coroots are

$$\Phi^+(B_2) = \{\alpha_1, \ \alpha_2, \ \alpha_1 + \alpha_2, \ \alpha_1 + 2\alpha_2\}$$
$$\Phi^+(B_2{}^\vee) = \{\alpha_1{}^\vee, \ \alpha_2{}^\vee, \ \alpha_1{}^\vee + \alpha_2{}^\vee, \ 2\alpha_1{}^\vee + \alpha_2{}^\vee\}$$

Therefore

$$\rho_{\mathfrak{g}} = 3/2 \ \alpha_1 + 2\alpha_2$$
$$\rho_{\mathfrak{g}}{}^\vee = 2\alpha_1{}^\vee + 3/2 \ \alpha_2{}^\vee = \alpha_1 + 3/2 \ \alpha_2$$

As in the example A_1 we now compute the three lattices $\Lambda^\ominus$, $\Lambda^\oplus$, $(\Lambda^\oplus)^*$; then we focus on the lattice VOA $\mathcal{V}_{\Lambda^\oplus}$ and on its simple modules.

- So, using the result of the second chapter, we fix as in 2.25 the value of Q

$$Q = \frac{1}{\sqrt{p}}(p\rho_{\mathfrak{g}}{}^\vee - \rho_{\mathfrak{g}}) = \frac{1}{\sqrt{2}}\left(2\alpha_1 + 2\frac{3}{2}\alpha_2 - \frac{3}{2}\alpha_1 - 2\alpha_2\right) = \frac{\alpha_1}{2\sqrt{2}} + \frac{\alpha_2}{\sqrt{2}}$$

Remark 4.1. In this case, since the dimension is still small, it is possible to compute Q explicitly (see Appendix).

- The *short screening* lattice with its basis is in this special case an odd integral lattice

$$\Lambda^\ominus = \frac{1}{\sqrt{p}}\Lambda_R, \qquad \{-\frac{1}{\sqrt{p}}\alpha_1, -\frac{1}{\sqrt{p}}\alpha_2\}$$

- The *long screening* lattice with its basis is the even integral lattice

$$\Lambda^\oplus = \sqrt{p}\Lambda_R{}^\vee, \qquad \{\sqrt{p}\alpha_1{}^\vee, \sqrt{p}\alpha_2{}^\vee\}$$

Remark 4.2. We recall that if Λ_R is the root lattice in the case B_2, then $\Lambda_R{}^\vee$ is the root lattice of type C_2.

- As last lattice we consider as before the dual of the *long screening* lattice $(\Lambda^\oplus)^*$ with its basis found by computing the fundamental weights (see Appendix)

$$(\Lambda^\oplus)^* = \frac{1}{\sqrt{p}}\Lambda_W \qquad \{\lambda_1{}^\oplus, \lambda_2{}^\oplus\} = \{\frac{\alpha_1}{\sqrt{p}} + \frac{\alpha_2}{\sqrt{p}}, \frac{\alpha_1}{2\sqrt{p}} + \frac{\alpha_2}{\sqrt{p}}\}$$

Remark 4.3. $\lambda_2{}^\oplus = Q$ and will be the groundstates representative of one of the modules $\mathcal{V}_{[\lambda_2{}^\oplus]}$.

- Now we can compute the number of representations of $\mathcal{V}_\Lambda^\oplus$:

$$\left|(\Lambda^\oplus)^*/\Lambda^\oplus\right| = \left|(\Lambda^\oplus)^*/\Lambda^\ominus\right|\left|\Lambda^\ominus/\Lambda^\oplus\right| = \left|\frac{1}{\sqrt{p}}\Lambda_W / \frac{1}{\sqrt{p}}\Lambda_R\right|\left|\frac{1}{\sqrt{p}}\Lambda_R/\sqrt{p}\Lambda_R{}^\vee\right|$$

$$= |\Lambda_W/\Lambda_R|\left|\frac{1}{\sqrt{p}}\Lambda_R/\sqrt{p}\Lambda_R{}^\vee\right| = 2\left|\bigoplus_{\alpha_i}\left(-\frac{1}{\sqrt{p}}\alpha_i\mathbb{Z}/\sqrt{p}(\alpha_i)^\vee\mathbb{Z}\right)\right| =$$

$$= 2\left|\bigoplus_{\alpha_i}\left(\frac{1}{\sqrt{p}}\alpha_i\mathbb{Z}/\sqrt{p}\frac{2}{(\alpha_i,\alpha_i)}\alpha_i\mathbb{Z}\right)\right| = 2\left|\bigoplus_{\alpha_i}\mathbb{Z}_{\frac{2p}{(\alpha_i,\alpha_i)}}\right|$$

where $|\Lambda_W/\Lambda_R|$ is obtained by looking explicitly to the cosets or computing the determinant of the Cartan matrix. In our $p = 2$ case, remembering the scalar products (α_i, α_i), we get that the number of modules is $2(\frac{p}{2} \cdot p) = 2 \cdot (1 \cdot 2) = 4$.

Specializing to $p = 2$ above and summarising, our lattices are:

$$\Lambda^\ominus = span\{-\frac{\alpha_1}{\sqrt{2}}, -\frac{\alpha_2}{\sqrt{2}}\}$$

$$\Lambda^\oplus = span\{\frac{\alpha_1}{\sqrt{2}}, \sqrt{2}\alpha_2\}$$

$$(\Lambda^\oplus)^* = span\{\frac{\alpha_1}{\sqrt{2}} + \frac{\alpha_2}{\sqrt{2}}, \frac{\alpha_1}{2\sqrt{2}} + \frac{\alpha_2}{\sqrt{2}}\}$$

The modules can now be explicitly determined by looking at the cosets in the quotient $(\Lambda^\oplus)^*/\Lambda^\oplus$:

$$[0] \qquad [\lambda_2{}^\oplus] \qquad [\lambda_1{}^\oplus] \qquad [\lambda_1{}^\oplus + \lambda_2{}^\oplus]$$

To make the further description simpler, we give again the previous pictorial names, i.e. *Blue* $\mathcal{V}_{[0]}$, *Steinberg* $\mathcal{V}_{[\lambda_2{}^\oplus]}$, *Green* $\mathcal{V}_{[\lambda_1{}^\oplus]}$ and *Facet* $\mathcal{V}_{[\lambda_1{}^\oplus + \lambda_2{}^\oplus]}$ module respectively.
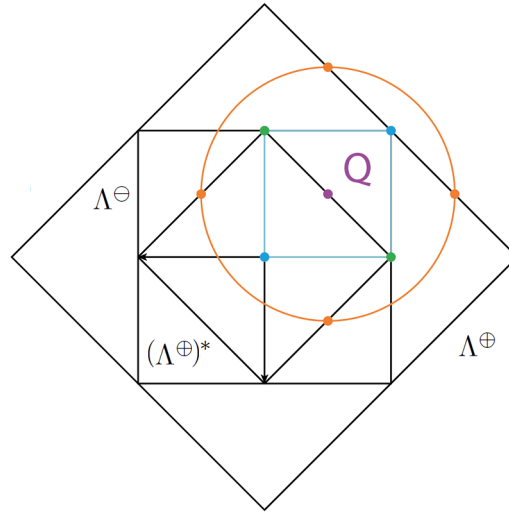

Figure 4.1: The three lattices in the B_2 case

4.1.2 Groundstates

Our next aim is to find out which are the groundstates elements for each module $\mathcal{V}_{[\lambda]}$; we recall that from Definition 3.4 these are the elements e^{ϕ_λ} with $\lambda \in [\lambda]$ of minimal distance to Q.

This is theoretically a hard problem, but in this case the dimension is small and thus it can be concretely solved drawing the three lattices and tracing circles of center Q: the nearest crossing elements with the $(\Lambda^\oplus)^*$ lattice are the groundstates elements.

In picture 4.1 the blue, purple, green and orange points represent precisely the *Blue*, *Steinberg*, *Green* and *Facet* groundstates elements respectively.

In the following table we write next to each module the dimension of the groundstates space, the conformal dimension of its elements and explicitly the groundstates elements:

	#Groundstates	Conformal Dim	Groundstates elements e^{ϕ_λ}
$\mathcal{V}_{[0]}$	2	0	$0\left(e^0 = 1\right),\ \alpha_1/\sqrt{2} + \sqrt{2}\alpha_2$
$\mathcal{V}_{[\lambda_2^{\oplus}]}$	1	$-1/4$	$Q = \alpha_1/2\sqrt{2} + \alpha_2/\sqrt{2}$
$\mathcal{V}_{[\lambda_1^{\oplus}]}$	2	0	$\alpha_1/\sqrt{2} + \alpha_2/\sqrt{2},\ \alpha_2/\sqrt{2}$
$\mathcal{V}_{[\lambda_1^{\oplus}+\lambda_2^{\oplus}]}$	4	$1/4$	$3\alpha_1/2\sqrt{2} + \sqrt{2}\alpha_2,\ \alpha_1/2\sqrt{2},$
			$-\alpha_1/2\sqrt{2},\ \alpha_1/2\sqrt{2} + \sqrt{2}\alpha_2$

Table 4.1: Groundstates informations for each module, B_2 case.

4.1.3 Degeneracy

As we said before B_n, $\ell = 4$ is an interesting case because it is degenerate. Let us see in details for $n = 2$ why it is so.

The important point is that the norm of the short screening momentum $|\alpha_1^{\ominus}| = 2$ is an even integer and thus $e^{i\pi(\alpha_1^{\ominus}, \alpha_1^{\ominus})} = e^{2\pi i} = +1$, i.e., by Corollary 2.15 $(\mathfrak{Z}_{\alpha_1^{\ominus}})^2 \neq 0$ (the short screening operator is said still *bosonic*). Actually in this case we can't apply the Corollary 2.15 at all, since this root is not small enough, i.e. $|\alpha_1^{\ominus}| \nleq 1$.

This happens because the long root in $\Lambda^{\ominus}$, i.e. $\alpha_1^{\ominus}$, is already in $\Lambda^{\oplus}$:

$$\alpha_1^{\oplus} = \sqrt{2}\alpha_1^{\vee} = \sqrt{2}\alpha_1 \frac{2}{(\alpha_1, \alpha_1)} = \frac{\alpha_1}{\sqrt{2}} = -\alpha_1^{\ominus}.$$

Thus, instead of using the two real short screening operators, in order to apply Theorem 2.14, we have to consider those screening operators associated with the vectors α_i such that $|\alpha_i| \leq p$, i.e. $|\alpha_i^{\ominus}| \leq 1$. So we consider:

$$\mathfrak{Z}_{-\frac{\alpha_2}{\sqrt{2}}} \qquad \mathfrak{Z}_{-\frac{\alpha_1+\alpha_2}{\sqrt{2}}}$$

i.e. the short screening operators associated with the short roots $\alpha_2^{\ominus}$ and $\alpha_{12}^{\ominus} := \alpha_1^{\ominus} + \alpha_2^{\ominus}$. These simple roots span a subsystem of type $A_1 \times A_1$.

Remark 4.4. It is now possible to apply again the Corollary 2.15 and since $\alpha_2^{\ominus}$ and $\alpha_{12}^{\ominus}$ have both odd integer norm (they are *fermionic*) and integer inner product, the associated screening operators satisfy:

$$(\mathfrak{Z}_{\alpha_2^{\ominus}})^2 = (\mathfrak{Z}_{\alpha_{12}^{\ominus}})^2 = 0 \qquad [\mathfrak{Z}_{\alpha_2^{\ominus}}, \mathfrak{Z}_{\alpha_{12}^{\ominus}}] = 0$$

Remark 4.5. We can generalize this argument to the B_n case that behaves analogously with $n - 1$ long roots instead of just 1: we obtain n commuting screening operators corresponding to a subset of the B_n lattice generating a A_1^n lattice.

Remark 4.6. This degeneracy behaviour takes also place on the side of the Lusztig quantum group of divided powers (see [Lusz90a]). In particular in [Lent14] it is studied that adding the divisibility condition on ℓ (defined in section 2.5) we have a new short exact sequence of Hopf algebras that in the B_n, $\ell = 4$ case shows degeneracy: the positive part of the small quantum group is no more associated with B_n but to A_1^n, exactly as above (for details see [Lent14]).

4.1.4 Decomposition behaviour of B_2

At this point it is interesting to briefly focus our attention on the decomposition behaviour of B_2 into $A_1 \times A_1$ classes.

The relations between the lattices of B_2 and $A_1 \times A_1$ are the following:

$$\Lambda^{\ominus}{}_{B_2} \supset \Lambda^{\ominus}{}_{(A_1 \times A_1)}$$

$$\Lambda^{\oplus}{}_{B_2} \supset \Lambda^{\oplus}{}_{(A_1 \times A_1)}$$

$$(\Lambda^{\oplus})^*{}_{B_2} \subset (\Lambda^{\oplus})^*{}_{(A_1 \times A_1)}$$

On the level of the lattice VOAs we have that $\mathcal{V}_{\Lambda^{\oplus}}(B_2)$ has got 4 modules, $\mathcal{V}_{\Lambda^{\oplus}}(A_1 \times A_1)$ has got 16 instead.
In particular, every $\mathcal{V}_{\Lambda^{\oplus}}(B_2)$-module can be restricted to $\mathcal{V}_{\Lambda^{\oplus}}(A_1 \times A_1)$-module and so decomposes in direct sum of irreducible $\mathcal{V}_{\Lambda^{\oplus}}(A_1 \times A_1)$-modules. Explicitly the cosets decompose:

$$[Blue] = [0]_{B_n} = \left([0] \oplus [\alpha_2 + \frac{\alpha_{12}}{\sqrt{2}}] \right)_{A_1 \times A_1}$$

$$[Steinberg] = [\lambda_2^{\oplus}]_{B_n} = \left([\lambda_2^{\oplus}] \oplus [\lambda_2^{\oplus} + \frac{\alpha_2 + \alpha_{12}}{\sqrt{2}}] \right)_{A_1 \times A_1}$$

$$[Green] = [\lambda_1^{\oplus}]_{B_n} \left([\frac{\alpha_2}{\sqrt{2}}] \oplus [\frac{\alpha_{12}}{\sqrt{2}}] \right)_{A_1 \times A_1}$$

$$[Facet] = [\lambda_1^{\oplus} + \lambda_2^{\oplus}]_{B_n} = \left([\lambda_2^{\oplus} + \frac{\alpha_2}{\sqrt{2}}] \oplus [\lambda_2^{\oplus} + \frac{\alpha_{12}}{\sqrt{2}}] \right)_{A_1 \times A_1}$$

4.1.5 Short screening operators

Let us now go back to the B_2 case and look at how the short screening operators act in the *Blue* $\mathcal{V}_{[0]}$ and *Green* $\mathcal{V}_{[\lambda_1^{\oplus}]}$ module on the groundstates elements (i.e. conformal dimension equal to 0) and on the higher conformal dimension level elements (i.e. conformal dimension equal to 1). We will list the results and then look at which elements are in the kernel of one or both of the screenings. To simplify the notation we will denote

$$\mathfrak{Z}_1 := \mathfrak{Z}_{-\frac{\alpha_2}{\sqrt{2}}} \text{ and } \mathfrak{Z}_2 := \mathfrak{Z}_{-\frac{\alpha_1 + \alpha_2}{\sqrt{2}}}.$$

Among the elements of conformal dimension 1 we will call *inside* elements the one below the groundstates elements and *outside* those which are on top of more external module elements.

1. The screening operators applied to the groundstates elements of the *Blue* module give as results:

   - $\mathfrak{Z}_1(1) = 0$
   - $\mathfrak{Z}_2(1) = 0$
   - $\mathfrak{Z}_1(e^{\phi \frac{\alpha_1}{\sqrt{2}} + \sqrt{2}\alpha_2}) = e^{\phi \frac{(\alpha_1 + \alpha_2)}{\sqrt{2}}}$
   - $\mathfrak{Z}_2(e^{\phi \frac{\alpha_1}{\sqrt{2}} + \sqrt{2}\alpha_2}) = e^{\phi \frac{\alpha_2}{\sqrt{2}}}$

2. The screening operators applied to the *inside* elements of the upper conformal dimension level of the *Blue* give as results:

- $\mathfrak{Z}_1(\partial\phi_{-\alpha_2/\sqrt{2}}) = e^{\phi_{-\frac{\alpha_2}{\sqrt{2}}}}$

- $\mathfrak{Z}_2(\partial\phi_{-\alpha_2/\sqrt{2}}) = 0$

- $\mathfrak{Z}_1(\partial\phi_{-(\alpha_1+\alpha_2)/\sqrt{2}}) = 0$

- $\mathfrak{Z}_2(\partial\phi_{-(\alpha_1+\alpha_2)/\sqrt{2}}) = e^{\phi_{-\frac{(\alpha_1+\alpha_2)}{\sqrt{2}}}}$

- $\mathfrak{Z}_1(\partial\phi_{(\alpha_1+\alpha_2)/\sqrt{2}}e^{\phi_{\sqrt{2}\alpha_1+\frac{\alpha_2}{\sqrt{2}}}}) = \partial\phi_{(\alpha_1+\alpha_2)/\sqrt{2}}e^{\phi_{\frac{(\alpha_1+\alpha_2)}{\sqrt{2}}}}$

- $\mathfrak{Z}_2(\partial\phi_{(\alpha_1+\alpha_2)/\sqrt{2}}e^{\phi_{\sqrt{2}\alpha_1+\frac{\alpha_2}{\sqrt{2}}}}) = 0$

- $\mathfrak{Z}_1(\partial\phi_{-\alpha_2/\sqrt{2}}e^{\phi_{\sqrt{2}\alpha_1+\frac{\alpha_2}{\sqrt{2}}}}) = 0$

- $\mathfrak{Z}_2(\partial\phi_{-\alpha_2/\sqrt{2}}e^{\phi_{\sqrt{2}\alpha_1+\frac{\alpha_2}{\sqrt{2}}}}) = \partial\phi_{\alpha_2/\sqrt{2}}e^{\phi_{\frac{\alpha_2}{\sqrt{2}}}}$

3. The screening operators applied to the *outside* elements of the upper conformal dimension level of the *Blue* give as results:

- $\mathfrak{Z}_1(e^{\phi_{-\frac{\alpha_1}{\sqrt{2}}}}) = e^{\phi_{-\frac{(\alpha_1+\alpha_2)}{\sqrt{2}}}}$

- $\mathfrak{Z}_2(e^{\phi_{-\frac{\alpha_1}{\sqrt{2}}}}) = 0$

- $\mathfrak{Z}_1(e^{\phi_{\frac{\alpha_1}{\sqrt{2}}}}) = 0$

- $\mathfrak{Z}_2(e^{\phi_{\frac{\alpha_1}{\sqrt{2}}}}) = e^{\phi_{-\frac{\alpha_2}{\sqrt{2}}}}$

- $\mathfrak{Z}_1(e^{\phi_{\sqrt{2}(\alpha_1+\alpha_2)}}) = 0$

- $\mathfrak{Z}_2(e^{\phi_{\sqrt{2}(\alpha_1+\alpha_2)}}) = e^{\phi_{\frac{(\alpha_1+\alpha_2)}{\sqrt{2}}}}$

- $\mathfrak{Z}_1(e^{\phi_{\sqrt{2}\alpha_2}})) = e^{\phi_{\frac{\alpha_2}{\sqrt{2}}}}$

- $\mathfrak{Z}_2(e^{\phi_{\sqrt{2}\alpha_2}})) = 0$

4. The screening operators applied to the groundstates elements of the *Green* module give as results:

- $\mathfrak{Z}_1(e^{\phi_{\frac{(\alpha_1+\alpha_2)}{\sqrt{2}}}}) = 0$

- $\mathfrak{Z}_2(e^{\phi_{\frac{(\alpha_1+\alpha_2)}{\sqrt{2}}}}) = 1$

- $\mathfrak{Z}_1(e^{\phi_{\frac{\alpha_2}{\sqrt{2}}}}) = 1$

- $\mathfrak{Z}_2(e^{\phi_{\frac{\alpha_2}{\sqrt{2}}}}) = 0$

5. The screening operators applied to the *inside* elements of the upper conformal dimension level of the *Green* give as results:

- $\mathfrak{Z}_1((\partial\phi_{\alpha_1/\sqrt{2}} + \partial\phi_{\alpha_2/\sqrt{2}})e^{\phi_{\frac{(\alpha_1+\alpha_2)}{\sqrt{2}}}}) = 0$

- $\mathfrak{Z}_2((\partial\phi_{\alpha_1/\sqrt{2}} + \partial\phi_{\alpha_2/\sqrt{2}})e^{\phi_{\frac{(\alpha_1+\alpha_2)}{\sqrt{2}}}}) = 0$

- $\mathfrak{Z}_1(\partial\phi_{\alpha_2/\sqrt{2}}e^{\phi\frac{\alpha_2}{\sqrt{2}}}) = 0$

- $\mathfrak{Z}_2(\partial\phi_{\alpha_2/\sqrt{2}}e^{\phi\frac{\alpha_2}{\sqrt{2}}}) = 0$

- $\mathfrak{Z}_1(\partial\phi_{\alpha_2/\sqrt{2}}e^{\phi\frac{(\alpha_1+\alpha_2)}{\sqrt{2}}}) = e^{\phi\frac{\alpha_1}{\sqrt{2}}}$

- $\mathfrak{Z}_2(\partial\phi_{\alpha_2/\sqrt{2}}e^{\phi\frac{(\alpha_1+\alpha_2)}{\sqrt{2}}}) = \partial\phi_{\alpha_1/\sqrt{2}}$

- $\mathfrak{Z}_1(\partial\phi_{-(\alpha_1+\alpha_2)/\sqrt{2}}e^{\phi\frac{\alpha_2}{\sqrt{2}}}) = \partial\phi_{-(\alpha_1+\alpha_2)/\sqrt{2}}$

- $\mathfrak{Z}_2(\partial\phi_{-(\alpha_1+\alpha_2)/\sqrt{2}}e^{\phi\frac{\alpha_2}{\sqrt{2}}}) = e^{\phi-\frac{\alpha_1}{\sqrt{2}}}$

6. The screening operators applied to the *outside* elements of the upper conformal dimension level of the *Green* give as results:

- $\mathfrak{Z}_1(e^{\phi-\frac{(\alpha_1+\alpha_2)}{\sqrt{2}}}) = 0$

- $\mathfrak{Z}_2(e^{\phi\frac{(\alpha_1+\alpha_2)}{\sqrt{2}}}) = 0$

- $\mathfrak{Z}_1(e^{\phi-\frac{\alpha_2}{\sqrt{2}}}) = 0$

- $\mathfrak{Z}_2(e^{\phi-\frac{\alpha_2}{\sqrt{2}}}) = 0$

- $\mathfrak{Z}_1(e^{\phi\sqrt{2}\alpha_1+\frac{3}{\sqrt{2}}\alpha_2}) = e^{\phi\sqrt{2}(\alpha_1+\alpha_2)}$

- $\mathfrak{Z}_2(e^{\phi\sqrt{2}\alpha_1+\frac{3}{\sqrt{2}}\alpha_2}) = e^{\phi\frac{\alpha_1}{\sqrt{2}}+\sqrt{2}\alpha_2}$

- $\mathfrak{Z}_1(e^{\phi\frac{\alpha_1}{\sqrt{2}}+\frac{3}{\sqrt{2}}\alpha_2})) = e^{\phi\frac{\alpha_1}{\sqrt{2}}+\sqrt{2}\alpha_2}$

- $\mathfrak{Z}_2(e^{\phi\frac{\alpha_1}{\sqrt{2}}+\frac{3}{\sqrt{2}}\alpha_2})) = e^{\phi\sqrt{2}\alpha_2}$

In the following picture are shown the kernels of the two short screenings in the Blue (left) and Green (right) modules. The blue arrows represent $\mathfrak{Z}_1$, the orange ones $\mathfrak{Z}_2$.

Remark 4.7. Since $(\mathfrak{Z}_i)^2 = 0, i = 1, 2$ then applying $\mathfrak{Z}_i$ we always land in $Ker\mathfrak{Z}_i$.
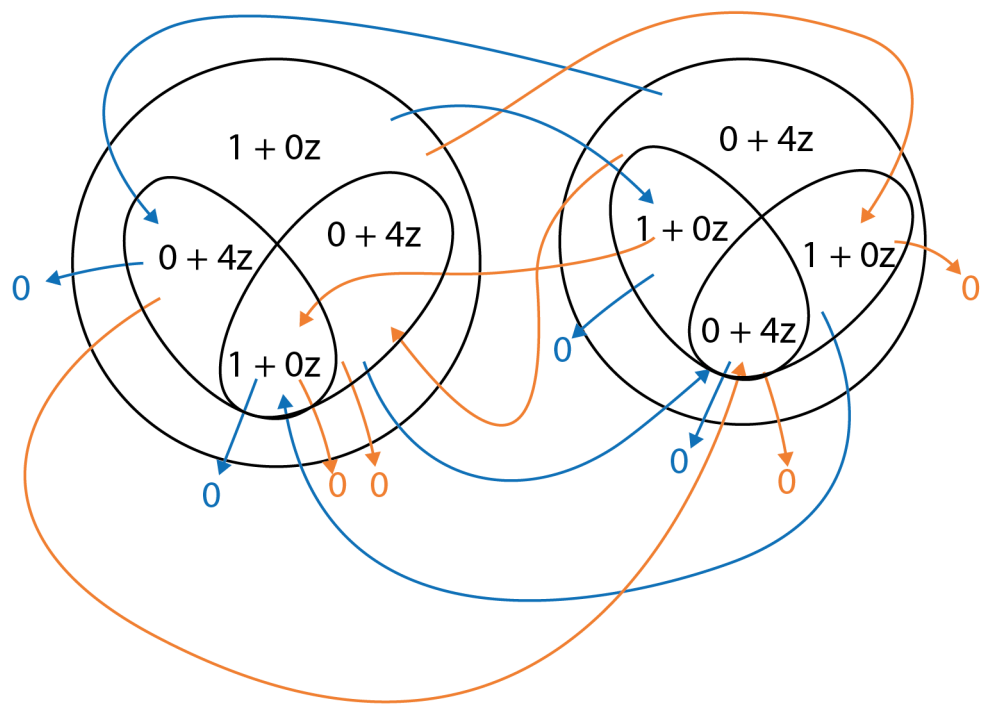
Figure 4.2: The left circle represent the Blue module, the right one represent the Green module containing the intersecting kernels of the screenings. Inside every space we have inserted the corresponding graded dimension. The blue arrows represent $\mathfrak{z}_1$, the orange ones $\mathfrak{z}_2$.

The following table summarizes the computations results about the kernel of the short screenings: for the Blue and Green modules we list the number of elements in each layer and then how many of them are in the intersection of the two kernels, how many are in just one of the two kernels and how many are outside both kernels.

Module		$\sum dim$	$Ker\mathfrak{z}_1 \cap Ker\mathfrak{z}_2$	$Ker\mathfrak{z}_1$	$Ker\mathfrak{z}_2$	In any kernel
Blue	First layer	2	1	0	0	1
	Second layer, inside	4	0	2	2	0
	Second layer, outside	4	0	2	2	0
Green	First layer	2	0	1	1	0
	Second layer, inside	4	2	0	0	2
	Second layer, outside	4	2	0	0	2

Table 4.2

Remark 4.8. For the Steinberg and Facet modules it makes no sense to look at the kernels since on them the short screening operators are not good map namely they are not Vir-homomorphisms. We could look at the kernels of $(\mathfrak{z}_i)^k$ with $k = 0$ and $k = 2$ respectively, i.e. $(\mathfrak{z}_i)^0 = id$, $(\mathfrak{z}_i)^2 = 0$, that are then obviously trivial and full kernels. In particular we would have a similar table:

Module		$\sum dim$	$\bigcap_{i=1,2} Ker(\mathfrak{z}_i)^k$	In any kernel
Steinberg	First layer	1	0	1
	Second layer	6	0	6
Facet	First layer	4	4	0
	Second layer	8	8	0

The following picture shows nine Virasoro modules; among them we have one representative of the *Steinberg* module (purple on the top), four of the *Facet* module (orange on the top) and two of the *Blue* and *Green* modules each.
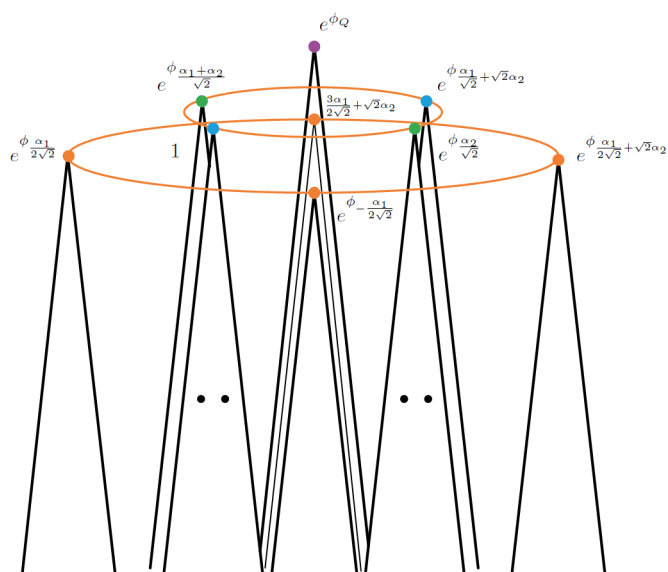
Figure 4.3: The B_2 case.

4.1.6 The $\mathcal{W}$ subspace

We again define as before the $\mathcal{W}$ subalgebra, using the chosen short screening operators, namely

$$\mathcal{W} := \mathcal{V}_{[0]} \cap Ker\mathfrak{Z}_{-\frac{\alpha_2}{\sqrt{2}}} \cap Ker\mathfrak{Z}_{-\frac{\alpha_{12}}{\sqrt{2}}}$$

and study how the action on the 4 modules $\mathcal{V}_{[0]}, \mathcal{V}_{[\lambda_2 \oplus]}, \mathcal{V}_{[\lambda_1 \oplus]}, \mathcal{V}_{[\lambda_1 \oplus + \lambda_2 \oplus]}$ restricted to $\mathcal{W}$ decompose them into kernels and cokernels of the short screening operators.

For the Blue module and Green module we have submodules

$$
\begin{array}{ccc}
 & Ker\mathfrak{Z}_1 & \\
\curvearrowright & & \curvearrowleft \\
\mathcal{V}_{[0]} & & \bigcap_{i=1,2} Ker_{\mathcal{V}_{[0]}}\mathfrak{Z}_i = \mathcal{W} =: \Lambda(1) \\
\curvearrowleft & & \curvearrowright \\
 & Ker\mathfrak{Z}_2 &
\end{array}
$$

$$
\begin{array}{ccc}
 & Ker\mathfrak{Z}_1 & \\
\curvearrowright & & \curvearrowleft \\
\mathcal{V}_{[\lambda_2 \oplus]} & & \bigcap_{i=1,2} Ker_{\mathcal{V}_{[\lambda_2 \oplus]}}\mathfrak{Z}_i =: \Pi(1) \\
\curvearrowleft & & \curvearrowright \\
 & Ker\mathfrak{Z}_2 &
\end{array}
$$

Since $(\alpha_2{}^\ominus, \alpha_{12}{}^\ominus) = 0 \in 2\mathbb{Z}$ these two short screenings commute by Corollary 2.15 and thus we know the isomorphisms between kernels and cokernels.

Lemma 4.9. *We have the following composition series' (for i arbitrary) for the $\mathcal{W}$ action with irreducible quotients indicated by underbraces:*

$$\mathcal{V}_{[0]} \supset \underbrace{\sum_{i=1,2} Ker\,\mathfrak{Z}_i}_{\Lambda(1)} \supset \underbrace{Ker\,\mathfrak{Z}_i}_{\Pi(1)} \supset \underbrace{\bigcap_{i=1,2} Ker\,\mathfrak{Z}_i}_{\Lambda(1)} \supset \underbrace{\{0\}}_{\Pi(1)}$$

$$\mathcal{V}_{[\lambda_2 \oplus]} \supset \underbrace{\sum_{i=1,2} Ker\,\mathfrak{Z}_i}_{\Pi(1)} \supset \underbrace{Ker\,\mathfrak{Z}_i}_{\Lambda(1)} \supset \underbrace{\bigcap_{i=1,2} Ker\,\mathfrak{Z}_i}_{\Pi(1)} \supset \underbrace{\{0\}}_{\Lambda(1)}$$

On the other hand the Steinberg module and Facet module stay irreducible

$$\Lambda(2) := \mathcal{V}_{[\lambda_1 \oplus]} \qquad \Pi(2) := \mathcal{V}_{[\lambda_1 \oplus + \lambda_2 \oplus]}$$

4.2 n arbitrary

4.2.1 Positive roots and lattices

Let us now consider the general case. Let then $\mathfrak{g}$ be a Lie algebra with root system B_n, i.e. $\mathfrak{g} = \mathfrak{so}(2n+1)$. Let $\ell = 2p = 4$.

Consider as basis of the root lattice Λ_R the set of n simple roots $\{\alpha_1, \ldots, \alpha_n\}$

with Killing form matrix:

$$
\begin{pmatrix}
4 & -2 & 0 & \ldots & & 0 \\
-2 & 4 & & & & \vdots \\
0 & & \ddots & & & 0 \\
\vdots & & & & 4 & -2 \\
0 & \ldots & & 0 & -2 & 2
\end{pmatrix}
$$

thus $\alpha_1, \ldots \alpha_{n-1}$ are the long roots and α_n is the short root.

The coroots are then $\alpha_i^\vee = \alpha_i/2$ for $i = 1 \ldots n-1$ and $\alpha_n^\vee = \alpha_n$. In order to compute the value of Q as in Lemma 2.25 we want to compute first the sum of all the positive roots.

In the B_n case the positive roots are n^2 and they are of two forms:

1. Sum of single, pairs, triple, ... , n-ple neighbouring (i.e. with contiguous indices) simple roots.

    Remark 4.10. There are $\binom{n+1}{2}$ positive roots of this form: n of them, exactly the ones where also α_n is involved, are short; the others are long.

2. Sum of twice the short root α_n and then $0, 1$ or 2 times the long roots following this pattern:

$$
\sum_k \alpha_k + \sum_l 2\alpha_l \qquad \text{with } 1 \le k < l \le n
$$

    Remark 4.11. There are $\binom{n}{2}$ positive roots of this form and they are all long roots.

The sum of all the positive roots of the first form is $\sum_j j(n+1-j)\alpha_j$, of the second form is $\sum_j j(n-1)\alpha_j$. We get then:

$$
\rho_\mathfrak{g} = \frac{1}{2}\left(\sum_{j=1}^{n} j(2n-j)\alpha_j \right).
$$

An analogous result can be obtained studying the positive roots of $B_n{}^\vee = C_n$:

$$
\rho_\mathfrak{g}{}^\vee = \frac{1}{2}\left(\sum_{j=1}^{n-1} \frac{j(2n-j+1)}{2}\alpha_j + \frac{n(n+1)}{2}\alpha_n \right).
$$

- It is now possible to compute the value of Q

$$
Q = \frac{1}{\sqrt{p}}(p\rho_\mathfrak{g}{}^\vee - \rho_\mathfrak{g}) = \frac{1}{\sqrt{2}}\left[2\frac{1}{2}\left(\sum_{j=1}^{n-1} \frac{j(2n-j+1)}{2}\alpha_j + \frac{n(n+1)}{2}\alpha_n \right) - \frac{1}{2}\left(\sum_{j=1}^{n} j(2n-j)\alpha_j \right) \right] =
$$

$$
= \frac{1}{2\sqrt{2}}\left(\sum_{j=1}^{n-1} j\alpha_j + n\alpha_n \right) = \frac{1}{2\sqrt{2}} \sum_{j=1}^{n} j\alpha_j
$$

- The *short screening* lattice with its basis is in this special case an odd integral lattice

$$\Lambda^{\ominus} = \frac{1}{\sqrt{2}}\Lambda_R, \qquad \{-\frac{1}{\sqrt{2}}\alpha_1, \ldots, -\frac{1}{\sqrt{2}}\alpha_n\}$$

- The *long screening* lattice with its basis is an even integral lattice

$$\Lambda^{\oplus} = \sqrt{2}\Lambda_R{}^{\vee}, \qquad \left\{\sqrt{2}\alpha_1{}^{\vee}, \ldots, \sqrt{2}\alpha_n{}^{\vee}\right\} = \left\{\frac{\alpha_1}{\sqrt{2}}, \ldots, \frac{\alpha_{n-1}}{\sqrt{2}}, \sqrt{2}\alpha_n\right\}$$

- And finally the dual of the *long screening* lattice with its basis, found through the known fundamental weights, is

$$(\Lambda^{\oplus})^* = \frac{1}{\sqrt{2}}\Lambda_W\{\lambda_1{}^{\oplus}, \ldots, \lambda_n{}^{\oplus}\} = \{\frac{\alpha_1}{\sqrt{2}} + \ldots + \frac{\alpha_n}{\sqrt{2}}, \; \frac{\alpha_1}{\sqrt{2}} + \frac{2}{\sqrt{2}}(\alpha_2 + \ldots + \alpha_n), \ldots$$
$$\ldots, \frac{\alpha_1}{\sqrt{2}} + \frac{2}{\sqrt{2}}\alpha_2 + \ldots + \frac{i-1}{\sqrt{2}}\alpha_{i-1} + \frac{i}{\sqrt{2}}(\alpha_i + \ldots + \alpha_n), \; \frac{1}{2\sqrt{2}}(\alpha_1 + 2\alpha_2 + \ldots + n\alpha_n)\}$$

Remark 4.12. As in the B_2 case 4.1 we obtain the equality $\lambda_n{}^{\oplus} = Q$.

- Let us compute the number of representations of $\mathcal{V}_{\Lambda}^{\oplus}$:

$$\left|(\Lambda^{\oplus})^*/\Lambda^{\oplus}\right| = \left|(\Lambda^{\oplus})^*/\Lambda^{\ominus}\right|\left|\Lambda^{\ominus}/\Lambda^{\oplus}\right| = 2\left|\bigoplus_{\alpha_i} \mathbb{Z}_{\frac{2p}{(\alpha_i,\alpha_i)}}\right|$$
$$= 2\left|\mathbb{Z}_p \times \mathbb{Z}_{p/2} \times \mathbb{Z}_{p/2} \times \cdots \times \mathbb{Z}_{p/2}\right| =$$
$$= 2\left|\mathbb{Z}_2 \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}\right| = 4$$

where again we obtained $\left|(\Lambda^{\oplus})^*/\Lambda^{\ominus}\right| = 2$ by looking at the determinant of the Cartan matrix

$$|\langle\alpha_i, \alpha_j\rangle| = 2\left|\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}\right| = \left|\begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}\right| = 2$$

or explicitly looking at the cosets with respect to $\Lambda^{\ominus}$:

$$(\Lambda^{\oplus})^*/\Lambda^{\ominus} = \left\{0 + \Lambda^{\ominus}, Q + \Lambda^{\ominus}\right\}$$

- Thus, since $\Lambda^{\ominus}/\Lambda^{\oplus} = \left\{0 + \Lambda^{\oplus}, \frac{\alpha_n}{\sqrt{2}} + \Lambda^{\oplus}\right\}$, our 4 modules are given by the cosets

$$[0], \qquad [\frac{\alpha_n}{\sqrt{2}}], \qquad [Q], \qquad [Q + \frac{\alpha_n}{\sqrt{2}}]$$

respectively the *Blue*, *Green*, *Steinberg* and *Facet* module.

4.2.2 Groundstates

Now, as in the A_1 and B_2 case we would like to determine the groundstates elements. In the lower dimensional case this was a drawing or easy-computational problem; in the n dimension case we can proceed in two ways: the more heuristic one, that generalizes those results, and the sharper one, that follows the *wrong walks* approach.

Theorem 4.13. *The groundstates elements of each module in the B_n case are the ones listed in the following table together with the dimension of the associated groundstates space:*

Blue	$Q + \frac{1}{2}\sum_{k=1}^{n} \epsilon_k \alpha_{k...n}{}^{\ominus}$, *such that* $\prod_i \epsilon_i = 1$	2^{n-1}
Steinberg	Q	1
Green	$Q + \frac{1}{2}\sum_{k=1}^{n} \epsilon_k \alpha_{k...n}{}^{\ominus}$, *such that* $\prod_i \epsilon_i = -1$	2^{n-1}
Facet	$Q \pm \alpha_{k...n}{}^{\ominus}$	$2n$

Proof. **Heuristic approach**

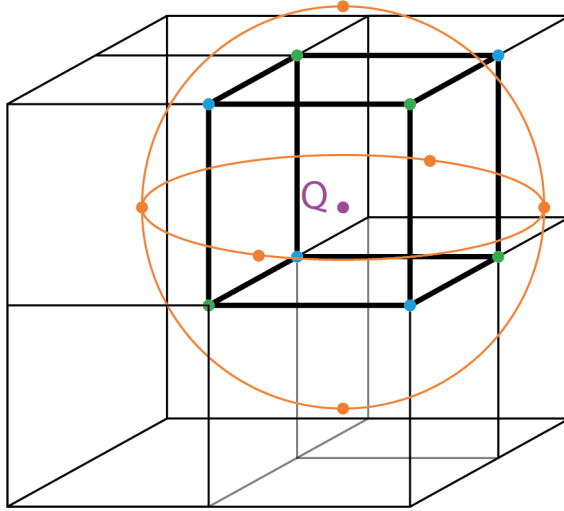We suggest to look at the figure 4.4 that shows the groundstates elements as coloured points in the B_3 case.

Figure 4.4: The crossing points of the cube give us the $\Lambda^{\ominus}$ lattice in the B_3 case. The coloured dots represent the Groundstates elements.

We can think at the groundstates elements constellation as located on a n-dimensional *hypercube* and on a $(n-1)$-dimensional *sphere* (for $n = 1$ a line and a pair of points, for $n = 2$ a square and a circle, for $n = 3$ a 3-dimensional cube and a 2-dimensional sphere in the 3-dimensional space). The hypercube edge and the sphere radius are equal to the short screenings length; the center of both is in the Steinberg point Q.

With this picture in mind we can now place the *Blue* and *Green* representative groundstates elements alternating on the vertices of the hypercube. The *Facet* representative groundstates elements are instead on those sphere points such that tracing a line from them to Q, we cross the hyperplane in the center of the $(n-1)$-faces.

So we obtain that the *Steinberg* module (purple in the figure) has always just one dimensional groundstates space. The dimension of the groundstates space of the *Blue* and *Green* modules together is equal to the number of vertices i.e. 2^n, so they have a 2^{n-1} dimensional groundstates space each. The dimension of the groundstates space of the *Facet* module (orange in the figure) is instead equal to the number of $(n-1)$-faces i.e. $2n$.

Sharper approach

We now try to define these groundstates elements in a clearer, less heuristic way. We consider, as in the case $n = 2$, the very short roots of $\Lambda^{\ominus}$ i.e.

$$\alpha_{k\ldots n}{}^{\ominus} := \alpha_k{}^{\ominus} + \ldots + \alpha_n{}^{\ominus}, \qquad k = 1, \ldots, n$$

with $\alpha_{nn}{}^{\ominus} := \alpha_n{}^{\ominus}$. We notice that if we sum $\alpha_{k\ldots n}{}^{\ominus}$ to a vector of the lattice it just moves one component of it, namely the k-th component.

We then consider the set of vectors $\Gamma = \left\{ \frac{1}{2} \alpha_{k\ldots n}{}^{\ominus} \text{ with } k \in \{1, \ldots, n\} \right\}$. They are a basis of $(\Lambda^{\oplus})^*{}_{A_1{}^n}$ and since they are all orthogonal and of the same length, they form a $\mathbb{Z}^n$-lattice.

What we will do is moving from the starting point Q to the surroundings through these vectors. In this way we will cross the nearest points to Q of $(\Lambda^{\oplus})^*{}_{B_n}$: those will be the wanted groundstates elements.

- Q is obviously the only groundstates element of the *Steinberg* module.

- Summing just one vector of Γ we don't land in $(\Lambda^{\oplus})^*{}_{B_n}$, so the vectors of the form $Q \pm \frac{1}{2} \alpha_{k\ldots n}{}^{\ominus}$ are not groundstates elements.

- Analogously summing several vectors of Γ but not all of them, we again don't reach any $(\Lambda^{\oplus})^*{}_{B_n}$ element.

- To get a $(\Lambda^{\oplus})^*{}_{B_n}$ element we have to sum all of them, i.e. $\left\{ Q + \frac{1}{2} \sum_{k=1}^{n} \epsilon_k \alpha_{k\ldots n}{}^{\ominus} \right\}$ with $\epsilon_k \in \{\pm 1\}$. In this case:

  – we reach an elements of the *Blue* module $\longleftrightarrow \prod_k \epsilon_k = 1$

  – we reach an elements of the *Green* module $\longleftrightarrow \prod_k \epsilon_k = -1$

  This result give us again the dimension of the groundstates space: the number of combinations $\left\{ Q + \frac{1}{2} \sum_{k=1}^{n} \epsilon_k \alpha_{k\ldots n}{}^{\ominus} \right\}$ with $\epsilon_k \in \{\pm 1\}$ are 2^n.

The ones corresponding to the Blue case, i.e. $\prod_k \epsilon_k = 1$ and to the case Green case i.e. $\prod_k \epsilon_k = -1$ are half of them namely 2^{n-1}.

- Finally if we sum the double of a vector of Γ we get $Q \pm \alpha_{k...n}{}^{\ominus}$ and this is in $(\Lambda^{\oplus})^*{}_{B_n}$, in particular it is in the *Facet* module. So these are all the *Facet* groundstates element.
  The dimension of the groundstates space of the *Facet* module results again equal to $2n$ since we can reach them summing or subtracting n vectors $\Rightarrow (n+n) = 2n$.

$\square$

4.3 Conformal dimension of the grounstates

We now want to compute the conformal dimension.

Lemma 4.14. *The conformal dimension of the* Steinberg, Blue, Green *and* Facet *modules are* $-\frac{n}{8}$, 0, 0 *and* $\frac{4-n}{8}$ *respectively.*

Proof.   • To calculate the conformal dimension of the *Steinberg* module it is then sufficient to compute:

$$\frac{1}{2}(Q,Q) - (Q,Q) = -\frac{1}{2}(Q,Q)$$

where the value of Q is the one found in 4.2, namely $Q = \frac{1}{2\sqrt{2}} \sum_{j=1}^{n} j\alpha_j$.

$$
\begin{aligned}
(Q,Q) &= \left(\frac{1}{2\sqrt{2}} \sum_{j=1}^{n} j\alpha_j, \frac{1}{2\sqrt{2}} \sum_{j=1}^{n} j\alpha_j \right) \\
&= \frac{1}{8} \left(\sum_{j=1}^{n} j\alpha_j, \sum_{j=1}^{n} j\alpha_j \right) \\
&= \frac{1}{8} \left[\sum_{j=1}^{n} j^2(\alpha_j, \alpha_j) + 2 \sum_{j=1}^{n-1} (j\alpha_j, (j+1)\alpha_{j+1}) \right] \\
&= \frac{1}{8} \left[\sum_{j=1}^{n-1} j^2 \cdot 4 + n^2 \cdot 2 + 2 \sum_{j=1}^{n-1} j(j+1)(-2) \right] \\
&= \frac{1}{8} \left[-4 \sum_{j=1}^{n-1} j + 2n^2 \right] \\
&= \frac{1}{8} \left[-4 \frac{(n-1)n}{2} + 2n^2 \right] = \frac{1}{8} \cdot 2n
\end{aligned}
$$

Then the conformal dimension of the *Steinberg* module is equal to $-\frac{n}{8}$

- For the *Blue* and *Green* module the calculation is similar. We choose as representative the groundstates element $Q + \frac{1}{2} \sum_{k=1}^{n} \alpha_{k...n}{}^{\ominus}$, i.e., the one

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with $\epsilon_k = 1 \ \forall k$. The computation becomes:

$$\frac{1}{2}(Q + \frac{1}{2}\sum_{k=1}^{n}\alpha_{k...n}^{\ominus}, Q + \frac{1}{2}\sum_{k=1}^{n}\alpha_{k...n}^{\ominus}) - (Q + \frac{1}{2}\sum_{k=1}^{n}\alpha_{k...n}^{\ominus}, Q)$$

$$= -\frac{1}{2}(Q, Q) + \frac{1}{2}(\frac{1}{2}\sum_{k=1}^{n}\alpha_{k...n}^{\ominus}, \frac{1}{2}\sum_{k=1}^{n}\alpha_{k...n}^{\ominus})$$

$$= -\frac{n}{8} + \frac{1}{8}(\sum_{k=1}^{n}\alpha_{k...n}^{\ominus}, \sum_{k=1}^{n}\alpha_{k...n}^{\ominus})$$

$$= -\frac{n}{8} + \frac{1}{8}(\sum_{j=1}^{n}j\alpha_{j}^{\ominus}, \sum_{j=1}^{n}j\alpha_{j}^{\ominus})$$

$$= -\frac{n}{8} + \frac{1}{8}(\sum_{j=1}^{n}j(-\frac{\alpha_j}{\sqrt{2}}), \sum_{j=1}^{n}(-\frac{\alpha_j}{\sqrt{2}}))$$

$$= -\frac{n}{8} + \frac{1}{8} \cdot \frac{1}{2}(\sum_{j=1}^{n}j\alpha_{j}, \sum_{j=1}^{n}j\alpha_{j})$$

$$= -\frac{n}{8} + \frac{1}{8} \cdot \frac{1}{2}2n = 0$$

where the last equality follows from the computation in the previous case. Since the *Blue* and *Green* module are symmetric with respect to the Steinberg point, their groundstates elements have the same conformal dimension.

- Analogously we proceed for the *Facet* module. Here we choose as representative element $Q + \alpha_n^{\ominus}$. The computation in this case is:

$$\frac{1}{2}(Q + \alpha_n^{\ominus}, Q + \alpha_n^{\ominus}) - (Q + \alpha_n^{\ominus}, Q)$$

$$= -\frac{1}{2}(Q, Q) + \frac{1}{2}(\alpha_n^{\ominus}, \alpha_n^{\ominus})$$

$$= -\frac{n}{8} + \frac{1}{2} \cdot \frac{1}{2}(\alpha_n, \alpha_n)$$

$$= -\frac{n}{8} + \frac{1}{2} = \frac{4-n}{8}$$

$\square$

To summarize we write all the results in the following table:

	#Groundstates	Conformal Dim	Groundstates elements
Blue	2^{n-1}	0	$Q + \frac{1}{2}\sum_{k=1}^{n}\epsilon_k\alpha_{k...n}^{\ominus}$, such that $\prod_i \epsilon_i = 1$
Steinberg	1	$-\frac{n}{8}$	Q
Green	2^{n-1}	0	$Q + \frac{1}{2}\sum_{k=1}^{n}\epsilon_k\alpha_{k...n}^{\ominus}$, such that $\prod_i \epsilon_i = -1$
Facet	$2n$	$\frac{4-n}{8}$	$Q \pm \alpha_{k...n}^{\ominus}$

Table 4.3: Groundstates informations for each module, B_n case.

4.4 Screening operators, restriced modules

In the general case $B_n, \ell = 4$ we have degeneracies similar to $B_2, \ell = 4$, because short screening operators of long roots are equal to long screening operators, and have even integral norm (bosonic). So again we define our relevant short screening operators from the short roots in B_n, which form an A_1^n lattice with orthogonal basis $\alpha_{i...n}^{\ominus}$ introduced in the previous section. Their norm is an odd integer (fermionic) and Corollary 2.15 reads

$$(\mathfrak{Z}_{\alpha_{i...n}^{\ominus}})^2 = 0 \qquad [\mathfrak{Z}_{\alpha_{i...n}^{\ominus}}, \mathfrak{Z}_{\alpha_{j...n}^{\ominus}}] = 0.$$

Generalizing the B_2 case, we can prove for induction the following:

Proposition 4.15. *In the B_n case the four $\mathcal{V}_{\Lambda\oplus}$ modules restricted to the kernel of the commuting short screenings*

$$\mathcal{W} = \mathcal{V}_{[0]} \cap \bigcap_{i=1,...n} Ker\mathfrak{Z}_{\alpha_{i...n}}$$

decompose as follows:

- *The Blue module (vacuum module) $\mathcal{V}_{[0]} = \mathcal{V}_{\Lambda\oplus}$ has a decomposition series of 2^{n-1} modules $\Lambda(1)$ and 2^{n-1} modules $\Pi(1)$ and*

$$\Lambda(1) = \bigcap_{i=1,...n} Ker_{\mathcal{V}_{[0]}}\mathfrak{Z}_{\alpha_{i...n}} = \mathcal{W}$$

- *The Green module has a decomposition series of 2^{n-1} modules $\Pi(1)$ and 2^{n-1} modules $\Lambda(1)$ and*

$$\Pi(1) = \bigcap_{i=1,...n} Ker_{\mathcal{V}_{[\frac{\alpha_n}{\sqrt{2}}]}}\mathfrak{Z}_{\alpha_{i...n}}$$

- *The Steinberg module $\mathcal{V}_{[Q]}$ stays irreducible, call it $\Lambda(2)$.*

- *The Facet module $\mathcal{V}_{[Q+\frac{\alpha_n}{\sqrt{2}}]}$ stays irreducible, call it $\Pi(2)$.*

Chapter 5

Comparison with the Symplectic Fermions

In Chapter 4 we constructed the VOA $\mathcal{W}$ in the case $\mathfrak{g} = B_n$, $\ell = 4$. We will call it $\mathcal{W}_{B_n, \ell=4}$.

While studying this structure we conjectured that *it has the same representation category of a known VOA, the Symplectic Fermions VOA*.

We will now introduce the Symplectic Fermions VOA and present some results that convinced us of the plausibility of this conjecture.

We want anyway to highlight that we later proved the conjecture in a stronger form, showing that these VOAs are even isomorphic. The proof is displayed in Chapter 6.

5.1 The Symplectic Fermions VOA

The vertex algebra of a *single pair of symplectic fermions* $\mathcal{V}_{\mathcal{SF}} = \mathcal{V}_{\mathcal{SF}_1}$ is a super-vertex algebra (VOSA) with central charge -2 introduced by [Kausch00]. Our exposition follows [DR16]: in vertex algebra language, the symplectic fermions are generated by two fermionic fields ψ, ψ^*, explicitly

$$\psi(z) = \sum_{n \in \mathbb{Z}} \psi_{-n-1} z^n \qquad \psi^*(z) = \sum_{n \in \mathbb{Z}} \psi^*_{-n-1} z^n$$

fulfilling the following anticommutators

$$[\psi_n, \psi^*_m]_+ = n \delta_{n+m=0} \text{ id}$$

Remark 5.1. This super-vertex algebra can equivalently be described with the following construction: we can first consider a purely odd super-vector space $\mathfrak{h}$ over $\mathbb{C}$ of dimension 2, $\mathfrak{h} = \langle x, y \rangle \cong \mathbb{C}^{0|2}$: from it we can construct a structure, called the affine Lie super-algebra $\hat{\mathfrak{h}}$ of free super-bosons. This has a module, the vacuum $\hat{\mathfrak{h}}$-module, that carries the structure of a vertex operator super-algebra $\mathcal{V}_{\mathcal{SF}}$, the symplectic fermion VOSA.

Similarly one defines the super-vertex algebra of *n-pairs of symplectic fermions*

$$\mathcal{V}_{\mathcal{SF}_n} = (\mathcal{V}_{\mathcal{SF}_1})^n$$

as n copies of the single pair one. This has central charge $-2n$.

To arrive to the definition of the Symplectic Fermions VOA, we now decompose this super-vertex algebra in even and odd part:

$$\mathcal{V}_{\mathcal{SF}_n} = \mathcal{V}_{\mathcal{SF}_n^{even}} \oplus \mathcal{V}_{\mathcal{SF}_n^{odd}}.$$

Theorem 5.2. *The even part $\mathcal{V}_{\mathcal{SF}_n^{even}}$ is a (non-super) VOA, called Symplectic Fermions VOA, and holds what follows:*

$$\mathcal{V}_{\mathcal{SF}_n^{even}} = \left(\mathcal{V}_{\mathcal{SF}_1}\right)^{n,\ even} \supsetneq \left(\mathcal{V}_{\mathcal{SF}_1^{even}}\right)^n$$

where $\mathcal{V}_{\mathcal{SF}_1^{even}}$ is the even part of $\mathcal{V}_{\mathcal{SF}_1}$.

The odd part $\mathcal{V}_{\mathcal{SF}_n^{odd}}$ is an irreducible module of the even part.

Moreover, again in [DR16] we have the following known result about this VOA:

Theorem 5.3. *The Symplectic Fermions VOA $\mathcal{V}_{\mathcal{SF}_n^{even}}$ is a logarithmic CFT with four irreducible representations. It has a conformal structure and the groundstates have conformal dimensions $0, 1, -\frac{n}{8}, -\frac{n}{8} + \frac{1}{2}$.*

Other results and conjectures on this structure can be found in [DR16].

5.2 Three supporting results

1. From Theorem 5.3 we can see that the number of Symplectic Fermions irreducible representations is the same of the $\mathcal{W}_{B_n,\ell=4}$ ones: $\Lambda(1), \Lambda(2), \Pi(1), \Pi(2)$.

2. The central charge of $\mathcal{W}_{B_n,\ell=4}$

$$c = rank - 12(Q,Q) = n - 12\frac{n}{4} = -2n$$

coincide with the Symplectic Fermions one.

3. The graded dimensions, defined in the next section, of the $\mathcal{W}_{B_n,\ell=4}$ modules $\Lambda(i), \Pi(i)$ $i = 1,2$ coincide with the graded dimensions $\chi_j^{\mathcal{SF}}$ $j = 1,\dots,4$ (studied in [DR16]) of the modules of the Symplectic Fermions:

$$\chi_1^{\mathcal{SF}} = \chi^{\Lambda(1)}$$
$$\chi_2^{\mathcal{SF}} = \chi^{\Pi(1)}$$
$$\chi_3^{\mathcal{SF}} = \chi^{\Lambda(2)}$$
$$\chi_4^{\mathcal{SF}} = \chi^{\Pi(2)}$$

In the rest of the Chapter we will prove this third result, after giving the definition of graded dimension of a vertex operator module as in [FB68], section 5.5.

5.3 Graded dimension

Definition 5.4. Let $\mathcal{V}$ be a conformal vertex algebra of central charge c. Let $M = \bigoplus_n M_n$ be a graded module of $\mathcal{V}$, with M_n finite dimensional $\forall n$. We define the *graded dimension*, or *character*, of M, as the formal power series

$$dim M = q^{-\frac{c}{24}} \sum_n dim M_n q^n.$$

We compute now the graded dimension of each Virasoro module and from that the one of the $\mathcal{V}_{\Lambda\oplus}$-modules.

Proposition 5.5. *The graded dimension of a Virasoro module $\mathcal{V}_\lambda$ is*

$$dim \mathcal{V}_\lambda = \frac{q^{\frac{1}{2}(Q-\lambda,Q-\lambda)}}{\eta(q)^n}$$

Proof. The Virasoro modules $\mathcal{V}_\lambda = \left\{ u e^{\phi_\lambda} \text{ with } u \text{ differential polynomial} \right\}$ are graded by the conformal dimension. We will call $E_0 := \frac{1}{2}(\lambda,\lambda) - (\lambda,Q)$ so that the degrees are of the form $E_0 + k$ with $k \in \mathbb{N}_0$.
For $rank = 1$ the graded dimension of a general Virasoro module $\mathcal{V}_\lambda$ with top element e^{ϕ_λ} will then be:

$$dim \mathcal{V}_\lambda = q^{-\frac{c}{24}} \left(q^{E_0} + q^{E_0+1} + 2q^{E_0+2} + 3q^{E_0+3} + 5q^{E_0+4} + \ldots \right) \qquad (5.1)$$

$$= q^{-\frac{c}{24}} \sum_{k \in \mathbb{N}_0} p_k q^{E_0+k} \qquad (5.2)$$

where p_k denotes the number of possible combinations u of differential elements with $|u|$ equal to k.
In our case c is the Virasoro central charge $c = rank - 12(Q,Q) = 1 - 12(Q,Q)$, so Formula 5.1 becomes:

$$dim \mathcal{V}_\lambda = q^{[-\frac{1}{24}(1-12(Q,Q))+\frac{1}{2}(\lambda,\lambda)-(\lambda,Q)]} \sum_{k \in \mathbb{N}_0} p_k q^k$$

$$= q^{\frac{1}{2}(Q-\lambda,Q-\lambda)} q^{-\frac{1}{24}} \sum_{k \in \mathbb{N}_0} p_k q^k$$

Now $\sum_{k \in \mathbb{N}_0} p_k q^k$ is again the dimension of a graded space i.e. the space

$$N := \bigotimes_{k \in \mathbb{N}_0} \mathbb{C}[\partial^k \phi]$$

We compute it:

$$\sum_{k \in \mathbb{N}_0} p_k q^k = dim N = \prod_{k \in \mathbb{N}} dim \mathbb{C}[\partial^k \phi] = \prod_{k \in \mathbb{N}} \sum_{h \in \mathbb{N}_0} q^{hk} = \prod_{k \in \mathbb{N}} \frac{1}{1-q^k}$$

Now we recognise that $q^{-\frac{1}{24}} \prod\limits_{k \in \mathbb{N}} \frac{1}{1-q^k}$ is the inverse of the Dedekind eta function $\eta(q)$ (defined for istance in [Kob93]).

Thus the dimension of the Virasoro module M_λ is equal to:

$$dim\mathcal{V}_\lambda = \frac{q^{\frac{1}{2}(Q-\lambda, Q-\lambda)}}{\eta(q)}$$

Similarly in arbitrary $rank = n$:

$$dim\mathcal{V}_\lambda = \frac{q^{\frac{1}{2}(Q-\lambda, Q-\lambda)}}{\eta(q)^n}$$

$\square$

Proposition 5.6. *The graded dimension of a $\mathcal{V}_{\Lambda^\oplus}$ modules $\mathcal{V}_{[\mu]}$ is:*

$$dim(\mathcal{V}_{[\mu]}) = \chi_{[\mu]} = \frac{\Theta_{[Q-\mu]}}{\eta(q)^n} = \frac{\sum\limits_{\lambda \in [\mu]} q^{\frac{1}{2}(Q-\lambda, Q-\lambda)}}{\left[q^{\frac{1}{24}} \prod\limits_{k \in \mathbb{N}} (1 - q^k)\right]^n}$$

Proof. To compute the graded dimension of a module $\mathcal{V}_{[\mu]}$ of $\mathcal{V}_{\Lambda^\oplus}$, since for 2.3 $\mathcal{V}_{[\mu]} = \bigoplus_{\lambda \in [\mu]} \mathcal{V}_\lambda$, we have now to sum among the graded dimensions of the Virasoro modules V_λ with $\lambda \in [\mu]$ coset of $(\Lambda^\oplus)^*/\Lambda^\oplus$.

We consider the general case $rank = n$ arbitrary and the Jacobi theta function (defined for istance in [HW59])

$$\Theta_{[Q-\mu]}(q) := \sum\limits_{\lambda \in [\mu]} q^{\frac{1}{2}(Q-\lambda, Q-\lambda)}$$

to get finally the graded dimension

$$dim(\mathcal{V}_{[\mu]}) = \chi_{[\mu]} = \frac{\Theta_{[Q-\mu]}}{\eta(q)^n} = \frac{\sum\limits_{\lambda \in [\mu]} q^{\frac{1}{2}(Q-\lambda, Q-\lambda)}}{\left[q^{\frac{1}{24}} \prod\limits_{k \in \mathbb{N}} (1 - q^k)\right]^n}$$

$\square$

5.4 Symplectic Fermions characters

We now copy from [DR16] first the functions:

$$\chi_{ns,+} = (q^{\frac{1}{24}} \prod\limits_{m=1}^\infty (1 + q^m))^{2n} \qquad \chi_{ns,-} = (q^{\frac{1}{24}} \prod\limits_{m=1}^\infty (1 - q^m))^{2n}$$

$$\chi_{r,+} = (q^{-\frac{1}{48}} \prod\limits_{m=1}^\infty (1 + q^{m-\frac{1}{2}}))^{2n} \qquad \chi_{r,-} = (q^{-\frac{1}{48}} \prod\limits_{m=1}^\infty (1 - q^{m-\frac{1}{2}}))^{2n}$$

used to define the characters of the modules of the Symplectic Fermions:

$$\chi_1^{\mathcal{SF}} = \frac{1}{2}(\chi_{ns,+} + \chi_{ns,-}) \qquad \chi_2^{\mathcal{SF}} = \frac{1}{2}(\chi_{ns,+} - \chi_{ns,-})$$

$$\chi_3^{\mathcal{SF}} = \frac{1}{2}(\chi_{r,+} + \chi_{r,-}) \qquad \chi_4^{\mathcal{SF}} = \frac{1}{2}(\chi_{r,+} - \chi_{r,-})$$

5.5 Comparison

The claim is that the graded dimensions computation support the conjecture of the category equivalence between $\mathcal{W}(B_n)$-Rep and $\mathcal{SF}_n - Rep$ by showing

$$\chi_3{}^{\mathcal{SF}} = \chi^{\Lambda(2)} = \chi^{\mathcal{V}_{[1]}}$$

$$\chi_4{}^{\mathcal{SF}} = \chi^{\Pi(2)} = \chi^{\mathcal{V}_{[3]}}$$

$$2^{n-1}(\chi_1{}^{\mathcal{SF}} + \chi_2{}^{\mathcal{SF}}) = 2^{n-1}(\chi^{\Lambda(1)} + \chi^{\Pi(1)}) = \chi^{\mathcal{V}_{[0]}} = \chi^{\mathcal{V}_{[2]}}$$

In particular, $\chi_1{}^{\mathcal{SF}}$ and $\chi_2{}^{\mathcal{SF}}$ should give us the graded dimension of the kernel of the screening in the *Blue* and *Green* modules respectively. The factor 2^{n-1} counts all the possible combinations of kernel of short screenings.

We will now prove this claim building the proof from the A_1 case and arriving to the arbitrary n case. But first, we show it in the $n = 2$ case as a consequence of the results of section 4.1.5.

5.5.1 First approximate comparison for $n = 2$

Going briefly back to section 4.1.5, we can notice that, for $n = 2$, table 4.2 tells us the first terms of the graded dimensions of $\Lambda(1)$ and $\Pi(1)$ since these modules are given by the intersection of the kernels in the Blue and Green module respectively. In this case $c = -2n = -4$; they are

$$\chi^{\Lambda(1)} \approx q^{-\frac{c}{24}}(1 + 0q) \approx q^{\frac{1}{6}}(1 + 0q)$$

$$\chi^{\Pi(1)} \approx q^{-\frac{c}{24}}(0 + 4q) \approx q^{\frac{1}{6}}(0 + 4q)$$

We now make a first approximate comparison, computing the characters $\chi_1{}^{\mathcal{SF}}$ and $\chi_2{}^{\mathcal{SF}}$ of the modules of the symplectic fermions VOA for $n = 2$ up to the second power. We ignore the multiplied term $q^{\frac{1}{24}}$.

$$\chi_{ns,+} \approx q^{\frac{1}{6}}(1 + q)^4(1 + q^2)^4 \ldots \approx q^{\frac{1}{6}}(1 + 4q + 6q^2 + \ldots)(1 + 4q^2 + \ldots) \ldots \approx q^{\frac{1}{6}}(1 + 4q + 10q^2 + \ldots)$$

$$\chi_{ns,-} \approx q^{\frac{1}{6}}(1 - q)^4(1 - q^2)^4 \ldots \approx q^{\frac{1}{6}}(1 - 4q + 6q^2 + \ldots)(1 - 4q^2 + \ldots) \ldots \approx q^{\frac{1}{6}}(1 - 4q + 2q^2 + \ldots)$$

$$\Rightarrow \chi_1{}^{\mathcal{SF}} = \frac{1}{2}(\chi_{ns,+} + \chi_{ns,-}) \approx \frac{1}{2}q^{\frac{1}{6}}(2 + 12q^2) \approx q^{\frac{1}{6}}(1 + 0q + 6q^2)$$

$$\Rightarrow \chi_2{}^{\mathcal{SF}} = \frac{1}{2}(\chi_{ns,+} - \chi_{ns,-}) \approx \frac{1}{2}q^{\frac{1}{6}}(8q + 8q^2) \approx q^{\frac{1}{6}}(0 + 4q + 4q^2)$$

so the results match.

5.5.2 Proofs for $n = 1$

The next results are true for the A_1 case, i.e. for $n = 1$.

Lemma 5.7.

$$\chi_{ns,+} = \chi_1{}^{\mathcal{SF}} + \chi_2{}^{\mathcal{SF}} = \chi^{\mathcal{V}_{[0]}} = \chi^{\mathcal{V}_{[2]}}$$

Proof. Let us compute the graded dimension of the *Blue* module $\mathcal{V}_{[0]}$ and compare it to $\chi_{ns,+}$. We said that for $[\mu] \in (\Lambda^\oplus)^*/\Lambda^\oplus$, the graded dimension is:

$$\chi_{[\mu]} = \frac{\Theta_{[Q-\mu]}(q)}{\eta(q)} = \frac{\sum\limits_{\lambda \in [\mu]} q^{\frac{1}{2}(Q-\lambda, Q-\lambda)}}{q^{\frac{1}{24}} \prod\limits_{m=1}^{\infty}(1 - q^m)}$$

In the case of $[\mu]$ equal to the *Blue module* $\mathcal{V}_{[0]}$ namely in the case of elements of the form $\frac{k\alpha}{2\sqrt{2}}$ with $k \equiv 0$ (4), we have:

$$\frac{1}{2}(Q-\lambda, Q-\lambda) - \frac{1}{2}(Q,Q) = \frac{1}{2}\left(\frac{k\alpha}{2\sqrt{2}}, \frac{k\alpha}{2\sqrt{2}}\right) - \left(\frac{\alpha}{2\sqrt{2}}, \frac{k\alpha}{2\sqrt{2}}\right) = \frac{1}{8}(k^2 - 2k)$$

$$\Rightarrow \frac{1}{2}(Q-\lambda, Q-\lambda) = \frac{1}{8}(k^2 - 2k + 1)$$

and so the dimension turns out to be:

$$\chi^{\mathcal{V}_{[0]}} = q^{\frac{1}{8}} q^{-\frac{1}{24}} \frac{\sum\limits_{k \equiv 0(4)} q^{\frac{1}{8}(k^2 - 2k)}}{\prod\limits_{m=1}^{\infty}(1-q^m)} = q^{\frac{2}{24}} \frac{\sum\limits_{r} q^{2(r^2 - \frac{r}{2})}}{\prod\limits_{m=1}^{\infty}(1-q^m)}$$

where $r := k/4$.

Now, to compare it to $\chi_{ns,+}$ we use the *Jacobi triple product* ([HW59]), namely:

$$\sum_{m=1}^{\infty} w^{2m} \hat{q}^{m^2} = \prod_{m=1}^{\infty}(1 - \hat{q}^{2m})(1 + w^2 \hat{q}^{2m-1})(1 + w^{-2}\hat{q}^{2m-1})$$

Substituting $\hat{q} = q^2$ and $w = q^{\frac{1}{2}}$ we can rewrite the sum above as follows:

$$\chi^{\mathcal{V}_{[0]}} = q^{\frac{2}{24}} \frac{\prod\limits_{m=1}^{\infty}(1 - q^{4m})(1 + q^{-1}q^{4m-2})(1 + qq^{4m-2})}{\prod\limits_{m=1}^{\infty}(1-q^m)}$$

$$= q^{\frac{2}{24}} \frac{\prod\limits_{m=1}^{\infty}(1 - q^{4m})(1 + q^{4m-3})(1 + q^{4m-1})}{\prod\limits_{m=1}^{\infty}(1-q^m)}$$

$$= q^{\frac{2}{24}} \frac{\prod\limits_{m=1}^{\infty}(1 - q^m)(1 + q^m)(1 + q^{2m})(1 + q^{4m-3})(1 + q^{4m-1})}{\prod\limits_{m=1}^{\infty}(1-q^m)}$$

$$= q^{\frac{2}{24}} \prod\limits_{m=1}^{\infty}(1 + q^m)(1 + q^{2m})(1 + q^{4m-3})(1 + q^{4m-1})$$

This result is exactly the expected

$$\chi_{ns,+} = q^{\frac{2}{24}} \prod_{m=1}^{\infty}(1 + q^m)^2$$

since $(1 + q^{2m})$ give us all the even terms and $(1 + q^{4m-3})(1 + q^{4m-1})$ the odd ones of $(1 + q^m)$.

For the *Green* module $\mathcal{V}_{[2]}$ the computation is the same. The Green elements are of the form $\frac{k\alpha}{2\sqrt{2}}$ with $k \equiv 2\,(4)$ that brings us to:

$$\chi^{\mathcal{V}_{[2]}} = q^{\frac{2}{24}} \frac{\sum\limits_{r} q^{2(r^2+\frac{r}{2})}}{\prod\limits_{m=1}^{\infty}(1-q^m)}$$

So the only thing that is changing is in the nominator the plus in the power of q that implies just a switch between the last to terms of the final product:

$$\prod_{m=1}^{\infty}(1+q^m)(1+q^{2m})(1+q^{4m-1})(1+q^{4m-3})$$

The result holds the same. So we have, as claimed:

$$\chi^{\mathcal{V}_{[0]}} = \chi^{\mathcal{V}_{[2]}} = \chi_{ns,+}$$

$\square$

In the same way we can say something about the *Steinberg* $\mathcal{V}_{[1]}$ and the *Facet* $V_{[3]}$ modules. In this case $\chi_3{}^{\mathcal{SF}}$ and $\chi_4{}^{\mathcal{SF}}$ should give us directly their graded dimensions.

Lemma 5.8.
$$\chi_3{}^{\mathcal{SF}} = \chi^{\mathcal{V}_{[1]}} \qquad \chi_4{}^{\mathcal{SF}} = \chi^{\mathcal{V}_{[3]}}$$

Proof. Here the computation is slightly harder. The reason is that in this case we have to deal with $\chi_3{}^{\mathcal{SF}}$ and $\chi_4{}^{\mathcal{SF}}$ that are sums of two products. Thus, it is not convenient to proceed as before. We follow an other strategy.

The trick consist in looking at the two modules together: we can namely consider the sum of their graded dimensions and then split them using the fact that for the *Steinberg* module we have integral powers such as q^m and for the *Facet* module fractional ones $q^{\frac{1}{2}+m}$.

$$\sum_{k\equiv 1(4)} q^{\frac{1}{8}(k^2-2k+1)} = \frac{1}{2}\Big(\sum_{k\equiv 1(2)} q^{\frac{1}{8}(k-1)^2} + \sum_{k\equiv 1(2)} \bar{q}^{\frac{1}{8}(k-1)^2}\Big)$$

where we define $\bar{q}$ such that: $\bar{q}^{\frac{1}{2}} = -q^{\frac{1}{2}}$. In this way we have that if $k \equiv 3(4)$ we catch a minus sign and the terms vanish. If instead $k \equiv 1(4)$ the terms get summed and multiplying them by $\frac{1}{2}$ we get the equality. Now we can indicate $k = 2r+1$

$$\frac{1}{2}\left(\sum_{k\equiv 1(2)} (q^{\frac{1}{2}})^{\frac{1}{4}(k-1)^2} + \sum_{k\equiv 1(2)} (-q^{\frac{1}{2}})^{\frac{1}{4}(k-1)^2}\right) = \frac{1}{2}\left(\sum_{r}(q^{\frac{1}{2}})^{r^2} + \sum_{r}(-q^{\frac{1}{2}})^{r^2}\right)$$

Proceeding as before, we use the *Jacobi triple product* to write the graded di-

mension as sum of products:

$$\frac{1}{2}q^{-\frac{1}{24}}\left(\frac{\prod\limits_{m=1}^{\infty}(1-q^m)(1+q^{m-\frac{1}{2}})(1+q^{m-\frac{1}{2}})}{\prod\limits_{m=1}^{\infty}(1-q^m)}+\frac{\prod\limits_{m=1}^{\infty}(1-q^m)(1-q^{m-\frac{1}{2}})(1-q^{m-\frac{1}{2}})}{\prod\limits_{m=1}^{\infty}(1-q^m)}\right)$$

$$=\frac{1}{2}q^{-\frac{1}{24}}\left(\prod\limits_{m=1}^{\infty}(1+q^{m-\frac{1}{2}})(1+q^{m-\frac{1}{2}})+\prod\limits_{m=1}^{\infty}(1-q^{m-\frac{1}{2}})(1-q^{m-\frac{1}{2}})\right)$$

$$=\frac{1}{2}\left(\chi_{r,+}+\chi_{r,-}\right)=\chi_3{}^{\mathcal{SF}}$$

The *Facet* case is complementary: if $k \equiv 3\ (4)$ the terms have to be summed and if $k \equiv 1\ (4)$, the terms have to be erased. In order to get that we put a minus in the expression above:

$$\sum_{k\equiv 3(4)} q^{\frac{1}{8}(k^2-2k+1)} = \frac{1}{2}\Big(\sum_{k\equiv 1(2)} q^{\frac{1}{8}(k-1)^2} - \sum_{k\equiv 1(2)} \bar{q}^{\frac{1}{8}(k-1)^2}\Big)$$

that proceeding as before turns out to be exactly:

$$\frac{1}{2}(\chi_{r,+}-\chi_{r,-})=\chi_4{}^{\mathcal{SF}}$$

So, as we claimed we reached the following:

$$\chi_3{}^{\mathcal{SF}}=\chi^{\mathcal{V}_{[1]}}\qquad\chi_4{}^{\mathcal{SF}}=\chi^{\mathcal{V}_{[3]}}$$

$\square$

5.5.3 Proofs for arbitrary n

We now generalize these proofs to the n dimensional case. We start with a preparatory result that for simplicity we prove for $n = 2$, but holds for arbitrary n:

Lemma 5.9. *Consider* $[\lambda] \in (\Lambda^{\oplus}{}_{A_1{}^2})^*/\Lambda^{\oplus}{}_{A_1{}^2}$, *which in the* $A_1{}^2$ *coordinates with basis* $\{\alpha_1{}^{\oplus}, \alpha_{12}{}^{\oplus} = \alpha_1{}^{\oplus}+\alpha_2{}^{\oplus}\}$ *has the form:* $\lambda = \lambda_1\alpha_1{}^{\oplus}+\lambda_2\alpha_{12}{}^{\oplus}$.
Then we have the following relation between the associated Jacobi theta functions:

$$\Theta_{[\lambda]}=\Theta_{[\lambda_1]}\cdot\Theta_{[\lambda_2]}$$

Proof. We write $\lambda = x + k$ where $x \in (\Lambda^{\oplus}{}_{A_1{}^2})^*$ and $k \in \Lambda^{\oplus}{}_{A_1{}^2}$ and we write also x and k in coordinates:

$$x = x_1\alpha_1{}^{\oplus}+x_2\alpha_{12}{}^{\oplus}\qquad k = k_1\alpha_1{}^{\oplus}+k_2\alpha_{12}{}^{\oplus}$$

where $k_1, k_2 \in \mathbb{Z}$ and x_1, x_2 can be also fractional.
Then, decomposing also $Q = Q_1\alpha_1{}^{\oplus}+Q_2\alpha_{12}{}^{\oplus}$ with Q_1, Q_2 fractional, we get:

$$\|\lambda - Q\|^2 = \|x_1\alpha_1{}^{\oplus}+x_2\alpha_{12}{}^{\oplus}+k_1\alpha_1{}^{\oplus}+k_2\alpha_{12}{}^{\oplus}-Q_1\alpha_1{}^{\oplus}-Q_2\alpha_{12}{}^{\oplus}\|^2$$

$$= \|(x_1+k_1-Q_1)\alpha_1{}^{\oplus}+(x_2+k_2-Q_2)\alpha_{12}{}^{\oplus}\|^2$$

$$= \|(x_1+k_1-Q_1)\alpha_1{}^{\oplus}\|^2+\|(x_2+k_2-Q_2)\alpha_{12}{}^{\oplus}\|^2$$

So the Theta function, using the associativity of the product, becomes:

$$\Theta_{[\lambda - Q]} = \sum_{\lambda = x + k} q^{\frac{1}{2}(\lambda - Q, \lambda - Q)} = \sum_{k_1, k_2} q^{\frac{1}{2}\|(x_1 + k_1 - Q_1)\alpha_1^{\oplus}\|^2 + \frac{1}{2}\|(x_2 + k_2 - Q_2)\alpha_{12}^{\oplus}\|^2}$$

$$= \left(\sum_{k_1} q^{\frac{1}{2}\|(x_1 + k_1 - Q_1)\alpha_1^{\oplus}\|^2} \right) \cdot \left(\sum_{k_2} q^{\frac{1}{2}\|(x_2 + k_2 - Q_2)\alpha_{12}^{\oplus}\|^2} \right)$$

$$= \left(\sum_{\lambda_1 = x_1 + k_1} q^{\frac{1}{2}(\lambda_1 - Q_1, \lambda_1 - Q_1)} \right) \left(\sum_{\lambda_2 = x_2 + k_2} q^{\frac{1}{2}(\lambda_2 - Q_2, \lambda_2 - Q_2)} \right) = \Theta_{[\lambda_1 - Q]} \cdot \Theta_{[\lambda_2 - Q]}$$

$$\square$$

We will use now this result to compute the graded dimension of the modules for arbitrary n. We start from the Blue $\mathcal{V}_{[0]}$ and Green $\mathcal{V}_{[\frac{\alpha_n}{\sqrt{2}}]}$ modules.

Lemma 5.10.

$$\chi^{\mathcal{V}_{[0]}} = \chi^{\mathcal{V}_{[\frac{\alpha_n}{\sqrt{2}}]}} = 2^{n-1}(\chi_1{}^{\mathcal{SF}} + \chi_2{}^{\mathcal{SF}}) = 2^{n-1}\chi_{ns,+}$$

Proof. To prove this result we will use

1. the fact that these two modules have the same structure and thus they have the same graded dimension

2. the fact that the sum of them gives the $\Lambda^{\ominus}{}_{A_1{}^n}$ lattice

3. Lemma 5.9.

$$\Theta_{\mathcal{V}_{[0]}} = \sum_{\lambda \in 0 + \Lambda^{\oplus}{}_{B_n}} q^{\frac{1}{2}(Q - \lambda, Q - \lambda)}$$

$$\underset{(1)}{=} \frac{1}{2} \left(\sum_{\lambda \in 0 + \Lambda^{\oplus}{}_{B_n}} q^{\frac{1}{2}(Q - \lambda, Q - \lambda)} + \sum_{\lambda \in \alpha_n{}^{\ominus} + \Lambda^{\oplus}{}_{B_n}} q^{\frac{1}{2}(Q - \lambda, Q - \lambda)} \right)$$

$$\underset{(2)}{=} \frac{1}{2} \left(\sum_{\lambda \in \Lambda^{\ominus}{}_{A_1{}^n}} q^{\frac{1}{2}(Q - \lambda, Q - \lambda)} \right)$$

$$\underset{(3)}{=} \frac{1}{2} \left(\sum_{\lambda \in \Lambda^{\ominus}{}_{A_1}} q^{\frac{1}{2}(Q - \lambda, Q - \lambda)} \right)^n$$

Now we proceed as in the case $n = 1$:

$$\lambda \in \Lambda^{\ominus}{}_{A_1} \qquad \Longleftrightarrow \qquad \lambda = -\frac{m\alpha}{\sqrt{2}}, \quad m \in \mathbb{N}$$

Then the graded dimension can be written:

$$\chi^{\mathcal{V}_{[0]}} = \frac{\frac{1}{2} \left(\sum_{\lambda \in \Lambda^{\ominus}{}_{A_1}} q^{\frac{1}{2}(Q - \lambda, Q - \lambda)} \right)^n}{\eta^n} = \frac{1}{2} \left(\frac{\sum_{m=1}^{\infty} q^{\frac{1}{2}(m^2 + m + \frac{1}{4})}}{q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)} \right)^n$$

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Finally we use again the Jacobi triple product to conclude:

$$= \frac{1}{2}\left(q^{\frac{1}{8}}\cdot q^{-\frac{1}{24}}\prod_{m=1}^{\infty}\frac{(1-q^m)(1+q^{\frac{1}{2}}q^{m-\frac{1}{2}})(1+q^{-\frac{1}{2}}q^{m-\frac{1}{2}})}{(1-q^m)}\right)^n$$

$$= \frac{1}{2}\left(q^{\frac{2}{24}}\prod_{m=1}^{\infty}(1+q^m)(1+q^{m-1})\right)^n$$

$$= \frac{1}{2}q^{\frac{2n}{24}}\left((1+q^{1-1})\prod_{m=1}^{\infty}(1+q^m)^2\right)^n$$

$$= \frac{1}{2}q^{\frac{2n}{24}}2^n\left(\prod_{m=1}^{\infty}(1+q^m)\right)^{2n} = 2^{n-1}\chi_{ns,+}$$

$\square$

We now want this matching result also for the Steinberg $\mathcal{V}_{[Q]}$ and Facet $\mathcal{V}_{[Q+\alpha_n/\sqrt{2}]}$ modules. We could proceed as for $n=1$, but it is interesting to follow another path which uses the decomposition behaviour of B_n into $A_1 \times \ldots \times A_1$ coordinates. To make it understandable we first present the result in the B_2 case.

Notation 5.11. χ^i will denote the character associated with the ith dimension i.e. to A_1^i.
χ_j will denote the characters of Symplectic Fermions (instead of $\chi_j^{\mathcal{SF}}$).

Lemma 5.12. *In the B_2 case we have:*

$$\chi^{\mathcal{V}_{[Q]}} = {\chi_3}^2$$

Proof. We consider the class of $(\Lambda^{\oplus}{}_{B_2})^*/\Lambda^{\oplus}{}_{B_2}$ and we want to write it as sum of classes of $(\Lambda^{\oplus}{}_{A_1^2})^*/\Lambda^{\oplus}{}_{A_1^2}$ as in 4.1.4. We have again:

$$[Q]_{B_2} = \left([Q]\oplus[Q+\frac{\alpha_2+\alpha_{12}}{\sqrt{2}}]\right)_{A_1^2} = ([Q]\oplus[-Q])_{A_1^2}$$

Hence in the character we can decompose the sum:

$$\chi^{\mathcal{V}_{[Q]}} = \frac{\sum\limits_{\lambda\in Q+\Lambda^{\oplus}{}_{B_2}}q^{\frac{1}{2}(Q-\lambda,Q-\lambda)}}{\eta^2}$$

$$= \frac{\sum\limits_{\lambda\in Q+\Lambda^{\oplus}{}_{A_1^2}}q^{\frac{1}{2}(Q-\lambda,Q-\lambda)} + \sum\limits_{\lambda\in -Q+\Lambda^{\oplus}{}_{A_1^2}}q^{\frac{1}{2}(Q-\lambda,Q-\lambda)}}{\eta^2}$$

$$= \frac{\Theta^2_{[Q]}}{\eta^2} + \frac{\Theta^2_{[-Q]}}{\eta^2}$$

Now the class $[-Q]$ in A_1^2 can be written in $A_1 \times A_1$ coordinates as (Facet module × Facet module) and thus the second term is the square of the graded dimension of the *Facet* module for $n=1$. The first term is instead simply the

square of the graded dimension of the *Steinberg* module for $n = 1$ (see picture). We obtain:

$$= (\chi_3{}^1)^2 + (\chi_4{}^1)^2 = \frac{1}{4}(\chi_{r,+}{}^1 + \chi_{r,-}{}^1)^2 + \frac{1}{4}(\chi_{r,+}{}^1 - \chi_{r,-}{}^1)^2$$

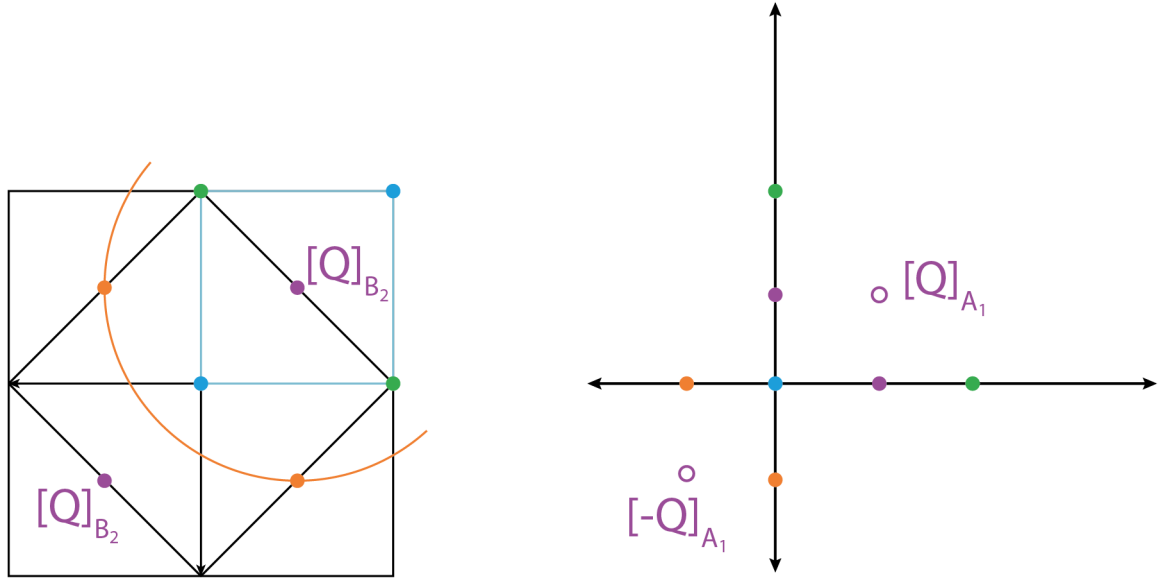$$= \frac{1}{2}(\chi_{r,+}{}^1)^2 + \frac{1}{2}(\chi_{r,-}{}^1)^2 = \frac{1}{2}(\chi_{r,+}{}^2 + \chi_{r,-}{}^2) = \chi_3{}^2$$

Figure 5.1: The groundstates elements on the left in the B_2 case, on the right in the $A_1 \times A_1$ case. In particular two representatives of the Steinberg module case B_2 correspond to two elements of different modules in the case $A_1 \times A_1$.

$\square$

Analogously one can prove that the character of the *Facet* module is equal to $\chi_4{}^2$.

Let us now generalize the decomposition behaviour of the *Steinberg* module to an arbitrary n. Studying the $n = 2$, $n = 3$ cases we found a pattern that brought us to these general results:

- Every time we increase the dimension by 1, the class $[Q]$ of B_n decomposes in the double of the cosets in $A_1 \times \ldots \times A_1$. We will thus have that $[Q]$ of B_n decomposes in the direct sum of exactly 2^{n-1} classes of $A_1 \times \ldots \times A_1$.

- In particular it decomposes in combination of classes of A_1 associated to the *Steinberg* and the *Facet* modules.

- The combination follows the Pascal's triangle; more precisely the coefficient of the combination of the *Steinberg* and the *Facet* classes are the even powers coefficients of the expansion of the n-th power of a binomial.

From these three results we have that the graded dimension of the Steinberg module is given by the graded dimensions of the one dimensional Steinberg and Facet modules combined in a binomial form:

$$\chi_{[Q]} = \sum_{k=0}^{n/2} \binom{n}{2k} (\chi_3{}^1)^{n-2k} (\chi_4{}^1)^{2k}$$

if n even, otherwise the sum arrives to $\frac{n-1}{2}$.

Analogously for the *Facet* module we obtain the coefficients related to the odd powers:

$$\chi_{[Q+\alpha_n/\sqrt{2}]} = \sum_{k=0}^{(n-2)/2} \binom{n}{2k+1} (\chi_3{}^1)^{n-2k-1} (\chi_4{}^1)^{2k+1}$$

if n even, otherwise the sum arrives to $\frac{n-1}{2}$.
This result, as we can see, matches with our claim:

$$\frac{1}{2}(\chi_{[Q]} + \chi_{[Q+\alpha_n/\sqrt{2}]}) =$$

$$= \frac{1}{2}\left[\sum_{k=0}^{n/2} \binom{n}{2k}(\chi_3{}^1)^{n-2k}(\chi_4{}^1)^{2k} + \sum_{k=0}^{(n-2)/2} \binom{n}{2k+1}(\chi_3{}^1)^{n-2k-1}(\chi_4{}^1)^{2k+1} \right]$$

$$= \frac{1}{2}\left[\sum_{k=0}^{n} \binom{n}{k}(\chi_3{}^1)^{n-k}(\chi_4{}^1)^k) \right]$$

$$= \frac{1}{2}(\chi_3{}^1 + \chi_4{}^1)^n = (\chi_{ns,+}{}^1)^n$$

$$= \chi_{ns,+}{}^n = \frac{1}{2}(\chi_3{}^n + \chi_4{}^n)$$

Chapter 6

Isomorphism of VOAs

As we mentioned in Chapter 5, at the end of this project we proved a result stronger than the one conjectured: *our VOA $\mathcal{W}_{B_n, \ell=4}$ and the Symplectic Fermions VOA $\mathcal{V}_{\mathcal{SF}_n^{even}}$ are isomorphic.*
We will now show this proof step by step: we first prove it for $A_1 = B_1$ and then we generalize it to the n dimensional case using that the super-vertex algebra just consists of n copies of the one-dimensional one.

Lemma 6.1. *For the datum $\mathfrak{g} = A_1, \ell = 4$ the lattice $\Lambda^{\ominus} = \frac{1}{\sqrt{2}}\Lambda_R$ is an odd integral lattice. We have the following isomorphisms of super-vertex algebras and vertex algebras*

$$\mathcal{V}_{\mathcal{SF}_1} = Ker_{\mathcal{V}_{\Lambda^{\ominus}}} \mathfrak{Z}_{-\alpha/\sqrt{2}}$$
$$\mathcal{V}_{\mathcal{SF}_1^{even}} \cong \mathcal{W}_{A_1, \ell=4}$$

Proof. This seems to be common knowledge (see e.g. [FGST06a]), but let us draw a quick proof: we define an isomorphisms from $\mathcal{V}_{\mathcal{SF}_1}$ to $\mathcal{V}_{\Lambda^{\ominus}}$ that sends the states (defined in 5.1) as follows

$$\psi \longmapsto e^{\phi_{-\alpha/\sqrt{2}}}$$
$$\psi^* \longmapsto \partial e^{\phi_{+\alpha/\sqrt{2}}}$$

This maps lands clearly in $Ker_{\mathfrak{Z}_{-\alpha/\sqrt{2}}}$ since

$$\mathfrak{Z}_{-\alpha/\sqrt{2}}(e^{\phi_{-\alpha/\sqrt{2}}}) = 0 \qquad \mathfrak{Z}_{-\alpha/\sqrt{2}}(\partial e^{\phi_{+\alpha/\sqrt{2}}}) = \partial \mathfrak{Z}_{-\alpha/\sqrt{2}}(e^{\phi_{+\alpha/\sqrt{2}}}) = \partial.1 = 0$$

and the OPE between the images is the same as the one of the sources in symplectic fermions, in our language:

$$Y(e^{\phi_{-\alpha/\sqrt{2}}})\partial e^{\phi_{+\alpha/\sqrt{2}}} = \sum_{k\geq 0} \left\langle e^{\phi_{-\alpha/\sqrt{2}}}, \partial\phi_{\alpha/\sqrt{2}} \right\rangle \left\langle e^{\phi_{-\alpha/\sqrt{2}}}, e^{\phi_{\alpha/\sqrt{2}}} \right\rangle e^{\phi_{\alpha/\sqrt{2}}} z^k \frac{\partial^k}{k!} e^{\phi_{-\alpha/\sqrt{2}}}$$

$$+ \sum_{k\geq 0} \left\langle e^{\phi_{-\alpha/\sqrt{2}}}, e^{\phi_{\alpha/\sqrt{2}}} \right\rangle \partial\phi_{\alpha/\sqrt{2}} e^{\phi_{\alpha/\sqrt{2}}} z^k \frac{\partial^k}{k!} e^{\phi_{-\alpha/\sqrt{2}}}$$

$$= \sum_{k\geq 0} z^{-1} z^k \partial e^{\phi_{+\alpha/\sqrt{2}}} \frac{\partial^k}{k!} e^{\phi_{-\alpha/\sqrt{2}}} + \sum_{k\geq 0} (+z^{-2}) z^k e^{\phi_{+\alpha/\sqrt{2}}} \frac{\partial^k}{k!} e^{\phi_{-\alpha/\sqrt{2}}}$$

$$= e^0 z^{-2} + 0 \ z^{-1} + \cdots$$

If we want to explicitly prove the defining relations of the mode operators

$$Y(\psi, z) = \psi(z) = \sum_{n \in \mathbb{Z}} Y(\psi)_n z^n$$

$$Y(\psi^*, z) = \psi^*(z) = \sum_{n \in \mathbb{Z}} Y(\psi^*)_n z^n$$

$$[Y(\psi)_{-1-n}, Y(\psi^*)_{-1-m}]_+ = m\delta_{m+n=0}$$

for the images $Y(a)_{-1-n}, Y(b)_{-1-m}$ from the previous calculation of $Y(a)b$, we invoke the associativity formula for integer OPE's (as in [Thiel94], here of a super-vertex algebra)

$$[Y(a)_{-1-n}, Y(b)_{-1-m}]_+ = \sum_{l \geq 0} \binom{n}{l} Y(Y(a)_{-l-1}b)_{l-(1+n+m)}$$

$$= \binom{n}{0} Y(0)_{-(1+n+m)} + \binom{n}{1} Y(e^0)_{1-(1+n+m)}$$

$$= n\delta_{n+m=0} \; id$$

arriving to the wanted result.

Now, we can think about the VOAs as modules over themselves, i.e. as the two vacuum representations, and use the found results. In particular, since the vacuum representation of $\mathcal{V}_{\mathcal{SF}_1}$ is irreducible, the vertex algebra homomorphism is injective: the Kernel must be a trivial subrepresentation and can not be all, since the image is non-zero.

Analogously, surjectivity follows from the matching of the graded dimensions calculated above: the dimensions of the two vacuum modules match, indeed.

So we found a bijective map

$$\mathcal{V}_{\mathcal{SF}_1} \longleftrightarrow Ker_{\mathcal{V}_{\Lambda \ominus}} \mathfrak{Z}_{-\alpha/\sqrt{2}}.$$

It is easy to see that the even subspace $\mathcal{V}_{\mathcal{SF}_1^{even}}$, i.e. differential polynomials in ψ, ψ^* with an even number of factors in each monomial, maps precisely to the subspace with Λ-degrees in $\Lambda^{\oplus} = 2\Lambda^{\ominus}$ hence

$$\mathcal{V}_{\mathcal{SF}_1^{even}} \cong \ker_{\mathcal{V}_{\Lambda \oplus}} \mathfrak{Z}_{-\alpha/\sqrt{2}} = \mathcal{W}_{A_1, \ell=4}$$

$\square$

Corollary 6.2. *By tensoring n copies we get*

$$\mathcal{V}_{\mathcal{SF}_n} = \left(\ker_{\mathcal{V}_{\frac{1}{\sqrt{2}}\Lambda_{R, A_1}}} \mathfrak{Z}_{-\alpha/\sqrt{2}} \right)^n = \bigcap_{i=1,\dots n} \ker_{\mathcal{V}_{\frac{1}{\sqrt{2}}\Lambda_{R, A_1^n}}} \mathfrak{Z}_{-\alpha_i^{A_1^n}/\sqrt{2}}$$

Now we want to prove that the result of Corallary 6.2 holds taking the even part on both sides. The even part of $\mathcal{V}_{\mathcal{SF}_n}$ is again completely characterized

by its lattice-degrees: They are the sublattice $\Lambda \subset \frac{1}{\sqrt{2}}\Lambda_{R,A_1^n} = \mathbb{Z}^n$ consisting of even sums of the basis elements, i.e.

$$\Lambda = \left\{ \sum_i x_i(\alpha_i^{A_1^n}/\sqrt{2}) \mid x_i \in \mathbb{Z}, \ \sum_i x_i \in 2\mathbb{Z} \right\} \subset \mathbb{Z}^n$$

But this Λ is precisely the D_n-root lattice, which is equal to the rescaled root lattice $\frac{1}{\sqrt{2}}C_n$, which is in turn the rescaled coroot lattice $\sqrt{2}B_n^\vee$. And this is precisely the lattice $\Lambda^\oplus$ for $B_n, \ell = 4$ and the images of the orthogonal basis are the dual short roots:

$$\Lambda = \sqrt{2}\Lambda_{R,B_n}^\vee = \Lambda_{B_n,\ell=4}^\oplus \qquad \alpha_i^{A_1^n}/\sqrt{2} = \alpha_{i\ldots n}^{B_n}/\sqrt{2}$$

which we prescribed in 4.2.2 as short screening momenta in the degenerate case $B_n, \ell = 4$. Altogether this gives our final result

Corollary 6.3.

$$\mathcal{V}_{\mathcal{SF}_n^{even}} = \bigcap_{i=1,\ldots n} \ker \mathcal{V}_{\sqrt{2}\Lambda_{R,B_n}^\vee} \, \mathfrak{Z}_{-\alpha_{i\ldots n}^{B_n}} \cong \mathcal{W}_{B_n,\ell=4}$$

Chapter 7

Appendix

7.1 Fundamental weights, B_2

To compute the fundamental weights λ_1 and λ_2 we write them as combination of the simple roots:

$$\lambda_1 = a\alpha_1 + b\alpha_2 \qquad \lambda_2 = c\alpha_1 + d\alpha_2$$

and then we ask $(\lambda_i, \alpha_j) = \delta_{i,j} (\alpha_j, \alpha_j)/2$:

$$a(\alpha_1, \alpha_1) + b(\alpha_2, \alpha_1) = 2 \qquad\qquad c(\alpha_1, \alpha_1) + d(\alpha_2, \alpha_1) = 0$$
$$a(\alpha_1, \alpha_2) + b(\alpha_2, \alpha_2) = 0 \qquad\qquad c(\alpha_1, \alpha_2) + d(\alpha_2, \alpha_2) = 1$$

and so:

$$4a - 2b = 2 \qquad\qquad 4c - 2d = 0$$
$$-2a + 2b = 0 \qquad\qquad -2c + 2d = 1$$

therefore

$$\lambda_1 = \alpha_1 + \alpha_2 \qquad\qquad \lambda_2 = 1/2\alpha_1 + \alpha_2$$

7.2 Explicit computation of Q, $\Lambda^{\oplus}$ in the case B_2

In the B_2 case, since there are just two simple roots α_1, α_2 (long and short respectively) we can compute Q also directly. From the general ansatz we know indeed that $Q = k_1\alpha_1 + k_2\alpha_2$ with $k_1, k_2 \in \mathbb{C}$, is the only vector in $\Lambda^{\oplus*}$ such that the property $h^Q(\alpha_i{}^{\ominus}) = 1$ holds. This implies that it has to be:

1.

$$\frac{1}{2}(-\frac{\alpha_1}{\sqrt{p}}, -\frac{\alpha_1}{\sqrt{p}}) - (-\frac{\alpha_1}{\sqrt{p}}, k_1\alpha_1 + k_2\alpha_2) = 1$$
$$\frac{4}{2p} + \frac{k_1}{\sqrt{p}}(\alpha_1, \alpha_1) + \frac{k_2}{\sqrt{p}}(\alpha_1, \alpha_2) = 1$$
$$2 + 4k_1\sqrt{p} - 2k_2\sqrt{p} = p$$

2.

$$\frac{1}{2}(-\frac{\alpha_2}{\sqrt{p}}, -\frac{\alpha_2}{\sqrt{p}}) - (-\frac{\alpha_2}{\sqrt{p}}, k_1\alpha_1 + k_2\alpha_2) = 1$$

$$\frac{2}{2p} + \frac{k_1}{\sqrt{p}}(\alpha_2, \alpha_1) + \frac{k_2}{\sqrt{p}}(\alpha_2, \alpha_2) = 1$$

$$1 - 2k_1\sqrt{p} + 2k_2\sqrt{p} = p$$

Now we put together the results to find k_1 and k_2:

$$2 + 4k_1\sqrt{p} - 2k_2\sqrt{p} = 1 - 2k_1\sqrt{p} + 2k_2\sqrt{p}$$

$$\Rightarrow k_2 = \frac{3p - 4}{2\sqrt{p}}$$

$$\Rightarrow k_1 = \frac{2p - 3}{2\sqrt{p}}$$

Then we find as value of Q the one of chapter 4.1:

$$Q = \frac{1}{2\sqrt{p}}\left[(2p - 3)\alpha_1 + (3p - 4)\alpha_2\right]$$

If now we want to find the combinations of $x\alpha_1$ and $x\alpha_2$ such that $h^Q(x\alpha_i) = 1$ with that fixed value of Q:

1.

$$\frac{1}{2}(x\alpha_1, x\alpha_1) - (x\alpha_1, \frac{1}{2\sqrt{p}}(2p - 3)\alpha_1 + \frac{1}{2\sqrt{p}}(3p - 4)\alpha_2) = 1$$

$$\frac{x^2}{2}4 - x\frac{1}{2\sqrt{p}}(2p - 3)4 - x\frac{1}{2\sqrt{p}}(3p - 4)(-2) = 1$$

$$2x^2 + \frac{2 - p}{\sqrt{p}}x - 1 = 0$$

Thus we find two solutions:

$$x_1 = \frac{\sqrt{p}}{2}, \quad x_2 = -\frac{1}{\sqrt{p}}$$

2.

$$\frac{1}{2}(x\alpha_2, x\alpha_2) - (x\alpha_2, \frac{1}{2\sqrt{p}}(2p - 3)\alpha_1 + \frac{1}{2\sqrt{p}}(3p - 4)\alpha_2) = 1$$

$$\frac{x^2}{2}2 - x\frac{1}{2\sqrt{p}}(2p - 3)(-2) - x\frac{1}{2\sqrt{p}}(3p - 4) \cdot 2 = 1$$

$$x^2 - \frac{p - 1}{\sqrt{p}}x - 1 = 0$$

Thus we find two solutions:

$$x_1 = \sqrt{p}, \quad x_2 = -\frac{1}{\sqrt{p}}$$

So we found again our four solutions:

$$\alpha_1{}^{\oplus} := \frac{\sqrt{p}}{2}\alpha_1 \qquad\qquad \alpha_1{}^{\ominus} := -\frac{1}{\sqrt{p}}\alpha_1$$

$$\alpha_2{}^{\oplus} := \sqrt{p}\alpha_2 \qquad\qquad \alpha_2{}^{\ominus} := -\frac{1}{\sqrt{p}}\alpha_2$$

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