UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Dipartimento di Matematica "Tullio Levi-Civita"

Master Degree in Physics

Final Dissertation

Convergence to Hartree dynamics for interacting bosons

Thesis supervisor

Prof. Antonio Ponno

Thesis co-supervisor

Prof. Lorenzo Zanelli

Candidate

Pietro Panazzolo

Academic Year 2022/2023

If I have seen further it is by standing
on the shoulders of Giants.

Isaac Newton, letter to Robert Hooke

To Mom and Dad, my Giants.

Abstract

The main goal of the present thesis is to derive a rigorous estimate for the convergence to Hartree dynamics for interacting bosons in the low temperature limit. The 2-body interaction potential is chosen in the Hardy class and we will allow our potential to have negative values, in order to model also attractive forces between particles. The estimate will be obtained working on proper functional norms on the space of square-integrable functions, endowed with a thermal Gaussian measure μ that concentrates around the ground state in the low temperature limit. We will write the normal mode decomposition of the quantum field operators, and then truncate it by introducing an UV cutoff Λ. The cutoff is introduced to switch from the infinitely-many coupled ODEs describing the evolution of the annihilation operator to a finite ODEs system. Dynamics of the system with the cutoff is studied on the Bargmann-Fock space, a subspace of the Fock space of second quantization. We will use coherent state expectation value to obtain functional equations starting from operatorial ones. More precisely, coherent states are introduced algebraically through the action of the Weyl-Heisenberg translation operator; then the Bargmann transform and the corresponding Bargmann representation are introduced. We will use Bargmann representation of the canonical coherent states to compute the Wick symbol of the operators (i.e. the coherent expectation value). Wick symbols will be also used to define the μ-norm for operators. Finally, a bound on the distance (through the above mentioned norms) between the regularized and the full quantum dynamics is provided, and bounding term dependence on cutoff, temperature and time is explicitly shown. Remarkably, we find linear time dependence for the bounding term.

Contents

Introduction

The study of interacting bosons is an active field of research that transversely attracts the interest of both theoretical and experimental communities.

The birth of the study many-bosons dynamics is marked by the 1925 article [19], in which A. Einstein predicted Bose-Einstein condensation in the non-interacting case based upon a previous article by S. N. Bose [15]. The presence of interactions between the bosons represents a major difficulty for a rigorous derivation of results regarding this phenomenon. A key contribution to the theory of weakly interacting bosons is the 1947 paper [13] by N. Bogoliubov, in which the author introduced the eponymous approximation. Even though the study of condensation phenomena related to bosons dates back to the 1920's, the first experimental observation of a Bose-Einstein condensate was realized in the mid 1990's [5]. Since then, the mathematical physics literature has produced many papers in the subject.

A rigorous derivation of the Gross-Pitaevskii equation starting from first principles of many-body quantum dynamics can be found in the paper [9].

In the review [49] a rigorous description of the Bogoliubov theory of superfluids (whose study started in the paper [13]) is developed. Moreover, the same paper [49] illustrates a list of open problems in the field of many-boson systems and Bogoliubov approximation.

An analysis of the scalings of the Hamiltonian to obtain asymptotic results in the limit of infinite number of particles, i.e. the Gross-Pitaevskii limit and the related Thomas-Fermi limit, can be found in the paper [40].

The evolution of the dynamics towards the Hartree dynamics and the study of the rate of convergence can be found in [4], and also in the more recent [47].

An approach similar to the one employed in the present thesis is presented in the works [1, 2, 3], in which the authors recovers the Hartree equation as a mean field limit using the phase space analysis of Wick operators in the infinite dimensional Fock space. We remark however that in these works there is no regularization, meaning that no cutoff is introduced. Another paper using similar techniques (phase space analysis of Wick operators) to the ones used in the present thesis is [42], however the focus on that work is the treatment of Bose-Hubbard models for the derivation of the discrete non-linear Schrödinger flow in the mean field regime.

The first use of coherent states in the topic of Bose-Einstein condensation and superfluidity can be found in the papers [34, 35].

In view of the vastness of the literature regarding the many-body dynamics for bosons and the Bose-Einstein condensate we refer to the reviews [43, 37, 10, 57]. For a historical perspective of advances in the field, see the thesis [39] or the paper [50], in which the development of Bogoliubov's theory of superfluidity (along with many other physical systems) is analyzed in the bigger framework of spontaneous symmetry breaking.

In the present work the main goal is to derive a rigorous estimate for the convergence to Hartree dynamics for interacting spinless bosons in the low temperature limit, following the approach of [44]. We will present a bound on the distance between the quantum many body dynamics and the effective one, after having regularized the fields in a way that will be described shortly.

The interaction potential V is chosen in the Hardy class, and it is allowed to have both negative and positive values, in order to model also attractive phenomena. The estimate will be obtained working on proper functional norms on $L^2(\mathrm{d}\mu)$, where $\mathrm{d}\mu$ is a Gaussian thermal measure that concentrates around the ground state of the system as the temperature goes to zero: for this reason we will regard T as our convergence parameter. Bounding term for the norm will exhibit a dependence on temperature, time and a cutoff that will be introduced to regularize the system: we remark that the dependence on time is linear and that the bounding term is vanishing in the $T \to 0^+$ limit. The coefficient of the interaction term is taken constant and not scaling as $\frac{1}{N}$, with N being the number of bosons. An elliptic property on the Hamiltonian ensures that the Gaussian thermal norm can be used to control from above the norm induced by the standard Gibbs measure.

The thesis is structured over four chapters. **Chapter 1** is a discussion about the physical model: the formalism of second quantization is reviewed and creation operators, destruction operators, Fock space and field operators are introduced. The Hamiltonian in second quantization for identical massive spinless bosons in d dimensions is analyzed, and some hypotheses are made about the interaction potential. In particular, we request that the interaction potential belongs to the Hardy class, a particular class of functions which contains, for instance, the Coulomb potential and all limited and compactly supported potentials. Regarding the confinement potential instead, we choose the harmonic one, and we take as orthonormal basis $\{\varphi_j\}_j$, the set of eigenfunctions of the harmonic oscillator in d dimension. Such choice is motivated by the knowledge of the analytic expression for such functions, together with the existence of estimates, but of course is not restrictive and one could in principle choice any other confinement potential (such as the infinite well).

In **chapter 2** coherent states formalism is developed. Coherent states are crucial in the following chapter, as the concept of Wick symbol relies upon them. First, the Weyl-Heisenberg group is introduced, together with the Weyl translation operator. In this setting, a coherent state is given by the action of the translation operator on an arbitrary function in the Schwartz space, that "generates" the coherent state. We emphasize the case of canonical coherent states, which are the coherent states related to the harmonic oscillator and obtained by choosing the ground state φ_0 as generating function. We prove some relations for such canonical coherent states, such as the product formula or the relation with the basis functions φ_j. Moreover, we check that the canonical coherent states obtained through the Weyl-Heisenberg group are eigenstates of the annihilation operator, thus they satisfy the fundamental property of coherent states. In the following, we define Bargmann representation of coherent states through the Bargmann transform. This representation acts on the Bargmann-Fock space, a particular subspace of the bosonic Fock space: the relation between the two is better explained in the next chapter. We also show that Bargmann representation of canonical coherent states, or simply Bargmann coherent states, still obey the same properties of "standard" canonical coherent states.

In **chapter 3** the regularized theory is presented. The idea behind regularization is to introduce an integer cutoff Λ to switch from the infinitely-many evolution equations for the annihilation operator a_k to a finite system of ODE by selecting only a finite number of summands, summing only on indices that are smaller than Λ (a proper norm on multiindices will be introduced). The cutoff allows us also to define regularized operators, and in particular a regularized version of the field operator in which the normal mode decomposition is truncated, including only Λ^d terms. This regularized theory gives rise to a reduced quantum dynamics, that can be studied in the Bargmann-Fock space built over the coherent phase space $\mathbb{C}^{\Lambda^d}$. In the same chapter we also introduce some important tools for computation, such as the star products and the Wick symbols of operators. The Wick symbol at time $t = 0$ of the regularized Hamiltonian H_Λ is computed, and through this quantity an effective field is defined. The Wick symbols of operators that appears in the (finite) normal mode decomposition of the effective field solve the scalar Hartree equation. We also define the aforementioned Gaussian thermal measure $\mathrm{d}\mu$ and the corresponding norm, that will appear in the main result (proposition 4.8). The behavior of the measure in the $T \to 0$ limit is studied, and we also provide a comparison between $\mathrm{d}\mu$ and the Gibbs measure: in particular the norm induced by the Gibbs measure is controlled from above by the $L^2(\mathrm{d}\mu)$ norm, guaranteeing that the estimates of the main theorem (proposition 4.8) can be rewritten in terms of the Gibbs measure.

In the last chapter, **chapter 4**, the result about convergence (proposition 4.8) is stated and proved. However, the proof is preceded by the statement and proof of several propositions that enters in the proof of proposition 4.8. In particular, the dynamics of we the deviation term between the Wick symbol of the regularized and the effective fields and an we estimate explicitly the remainder of the approximated dynamics with respect to the full Fock dynamics. The proof follows the one presented in [44], presenting however some enhanced estimates. We also present a brief comparison between the methods used in some of the existing literature (e.g. [9, 36, 21]), namely the study of one particle density operators, and the techniques used in [44] and in the present thesis, trying to see how the two approaches are related to each other.

Notations and conventions

We present a brief list of the main notations and conventions that will be used.

L^2 space. We will assume that the vector space $\mathbb{R}^d$ is endowed with the standard Lebesgue measure $\mathrm{d}x$. Given any measure space X with measure μ we will denote as $L^2(X, \mathrm{d}\mu)$ the space of all functions $f : X \to \mathbb{C}$ that are square integrable, namely

$$L^2(X, \mathrm{d}\mu) = \left\{ f : X \to \mathbb{C} \text{ such that } \int_X |f(x)|^2 \, \mathrm{d}\mu \right\}$$

In the case in which $X = \mathbb{R}^d$ with the standard Lebesgue measure $\mathrm{d}x$ we will denote simply $L^2(\mathbb{R}^d, \mathrm{d}x) = L^2(\mathbb{R}^d)$. Furthermore, when the context is clear we will omit to write the vector space, writing for instance $L^2(\mathrm{d}\mu)$.

Operators. Operators will be denoted in sans-serif font, such as H, P, $\mathsf{\Psi}$.

Vectors. Vectorial variables will not have any special notation. For complex vectors $a \in \mathbb{C}^d$ we adopt the following notation

$$a^2 = a \cdot a \qquad \bar{a}^2 = \bar{a} \cdot \bar{a} \qquad |a|^2 = a \cdot \bar{a}$$

Complex measure. Given a complex variable $\alpha \in \mathbb{C}^n$ that can be written $\alpha = q + ip$ where $q, p \in \mathbb{R}^n$ we denote $\mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha} = \frac{1}{\pi^n} \, \mathrm{d}q \, \mathrm{d}p$.

Gaussian integral. We will make use several times of the following n dimensional Gaussian integral

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \; e^{-cx^2 + 2a \cdot x} = \left(\frac{\pi}{c}\right)^{\frac{n}{2}} e^{\frac{1}{c}a^2}$$

where $x \in \mathbb{R}^n$ is the variable over which we are integrating, $a \in \mathbb{C}^n$ is a coefficient vector and $c \in \mathbb{R}^+$ is a positive constant. To prove this expression is sufficient to put $x = (x_1, \ldots, x_n)$, $a = (a_1, \ldots, a_n)$ and write the above integral as

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \; e^{-cx^2 + 2a \cdot x} = \prod_{j=1}^{n} \int e^{-cx_j^2 + 2a_j x_j} \, \mathrm{d}x_j = \left(\frac{\pi}{c}\right)^n \prod_{j=1}^{n} e^{\frac{a_j^2}{c}} = \left(\frac{\pi}{c}\right)^{\frac{n}{2}} e^{\frac{a^2}{c}}$$

Multiindex notation. We will employ multiindex notation, for instance to address the basis elements in arbitrary dimension. A d-dimensional multiindex $k \in \mathbb{N}^d$ is a d-tuple $(k_1, \ldots, k_d) \in \mathbb{N}^d$. We will adopt the standard notation for operations with multiindices, namely

$$|k|_{\text{m.i.}} = k_1 + \cdots + k_d$$

$$k! = k_1! k_2! \ldots k_d!$$

$$k^a = k_1^{a_1} k_2^{a_2} \dots k_d^{a_d}$$

We will also introduce a different norm on multiindices, namely $|k| = \max i\{k_i\}$.

Permutation group. We will denote the permutation group over N elements, or N symmetric group, as S_N. There are $N!$ permutations in S_N, and the composition of two permutations σ_1 and σ_2 will be denoted as $\sigma_1 \circ \sigma_2$.

CHAPTER 1

The physical model

Overview

In this introductory chapter a first overview of the physical model and of the used formalism is provided. A review of the most relevant aspects of the second quantization is provided in a rigorous way: Fock spaces, bosonic/fermionic projectors and field operators are introduced. An analysis of the secondly-quantized Hamiltonian operator with both a confinement (one-body) and a interaction potential (two-body) is provided: regarding the interaction potential, the Hardy class is introduced and a discussion about the potentials is potentials is initiated, exploring their possible features.

1.1 Second quantization

Second quantization was first introduced in 1927 by P. Dirac ([18]), then developed further in the next years by E. Wigner ([55]) and V. Fock ([24]).

A key ingredient of second quantization formalism is the use of Fock space, an Hilbert space that allows the representation of a variable number of particles. To "jump" from a given number of particle to another one the so-called ladder operators, i.e. creation and annihilation operators, have to be used. We give a review of the most relevant features of second quantization in a rigorous fashion, following mainly [51, 17]. A more physical-centered discussion can be found in [41, 23], while for an advanced mathematical treatment see [46]. An interesting paper on the history of second quantization with also a perspective on some advanced mathematical techniques is [52].

We denote a generic vector in the N-particles space as

$$\Psi = \psi_1 \otimes \cdots \otimes \psi_N \in \mathcal{H}_N$$

In braket notation such state is often denoted as

$$\Psi = |\psi_1 \dots \psi_N\rangle$$

We can obtain the so-called *position representation* by computing the braket with the coordinate vectors, namely

$$\Psi(r_1, \dots, r_N) = \langle r_1, \dots, r_N | \psi_1 \dots \psi_N \rangle = \psi_1(r_1)\psi_2(r_2)\dots\psi_N(r_N) \tag{1.1}$$

We know that in nature the observed particles can be divided in two sectors: fermions and bosons[1]. The wavefunctions describing this particles have different symmetry properties:

- bosonic wavefunction is *totally symmetric* under the exchange of particles: let $\sigma \in S_N$ be a permutation of N objects, then
$$\Psi(r_{\sigma(1)}, \cdots, r_{\sigma(N)}) = \Psi(r_1, \cdots, r_N)$$
- on the contrary, fermionic wavefunction is *totally anti-symmetric* under the exchange of any pair of particles[2]:
$$\Psi(r_{\sigma(1)}, \cdots, r_{\sigma(N)}) = (-1)^\sigma \Psi(r_1, \cdots, r_N)$$

Remark. In quantum field theory, Spin-statistic theorem ensures a link between the symmetry properties of the wavefunction and the spin of the particles, so that is equivalent to define bosonic fields as the ones with integer spin and fermionic fields as the ones with half-integer spin (see [22, 53] for a discussion).

If we introduce the parameter $\xi = \pm 1$, where the $+$ is chosen in case of bosons and the $-$ in case of fermions, we can give a unique description of the symmetry properties as
$$\Psi(r_{\sigma(1)}, \ldots, r_{\sigma(N)}) = \xi^\sigma \Psi(r_1, \ldots, r_N)$$

Remark. The anti-symmetry of the many-body wavefunction in the case of fermions has a very important phenomenological implication, namely the Pauli exclusion principle. Indeed, if we exchange any coordinates inside the fermionic wavefunction we get
$$\Psi(r_1, \ldots, r_i, \ldots, r_j, \ldots, r_N) = -\Psi(r_1, \ldots, r_j, \ldots, r_i, \ldots, r_N)$$
and if $r_i = r_j$ we deduce
$$\Psi(r_1, \ldots, r_i, \ldots, r_i, \ldots, r_N) = 0$$

It is clear that the N-particles state that we introduced in equation (1.1) is not forced to obey any of the above symmetries. Therefore, we shall find a way to symmetrize or antisymmetrize the wavefunction Ψ depending on the nature of the particles that describes. For this purpose, we define the *bosonic* or *fermionic projector* as
$$\Pi_\xi(\psi_1 \otimes \cdots \otimes \psi_N) = \frac{1}{N!} \sum_{\sigma \in S_N} \xi^{|\sigma|} \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(N)}$$

It is easy to see that Π is actually a projector, namely that is idempotent: by direct computation we have
$$\begin{aligned}
\Pi_\xi \Pi_\xi(\psi_1 \otimes \cdots \otimes \psi_N) &= \frac{1}{N!} \Pi_\xi \sum_{\sigma \in S_N} \xi^{|\sigma|} \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(N)} \\
&= \frac{1}{N!} \sum_{\sigma \in S_N} \xi^{|\sigma|} \Pi_\xi(\psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(N)}) \\
&= \frac{1}{N!} \frac{1}{N!} \sum_{\sigma, \tau \in S_N} \xi^{|\sigma|+|\tau|} \psi_{\tau \circ \sigma(1)} \otimes \cdots \otimes \psi_{\tau \circ \sigma(N)}
\end{aligned}$$

[1] This is true at least in dimension $d = 3$. For instance, in dimension $d = 2$ also another sector of particles that are not bosons nor fermions can rise, called anyons. In present thesis we will deal with bosons only.

[2] Here the factor $(-1)^\sigma = \pm 1$ represents the parity (or sign) of the permutation, defined as the number of transpositions of two elements to get $(1, \cdots, N)$ starting from $(\sigma(1), \cdots, \sigma(N))$. For instance, if $N = 3$ the permutation $(\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1)$ is even, then $(-1)^\sigma = 1$. On the other hand, the permutation $(\sigma'(1) = 2, \sigma'(2) = 1, \sigma'(3) = 3)$ is odd, $(-1)^{\sigma'} = -1$ (only one exchange is necessary to map $(2, 1, 3)$ into $(1, 2, 3)$).

and putting $\nu = \tau \circ \sigma$ we get[3]

$$\begin{aligned}
\dots &= \frac{1}{N!}\frac{1}{N!}\sum_{\sigma\in S_N}\sum_{\nu\in S_N}\xi^{|\nu|}\psi_{\nu(1)}\otimes\dots\otimes\psi_{\nu(N)} \\
&= \frac{1}{N!}\sum_{\nu\in S_N}\xi^{|\nu|}\psi_{\nu(1)}\otimes\dots\otimes\psi_{\nu(N)} \\
&= \Pi_\xi(\psi_1\otimes\dots\otimes\psi_N)
\end{aligned}$$

The projector acts as

$$\Pi_+ : \mathcal{H}_N \to \Pi_+(\mathcal{H}_N) = \mathcal{H}_N^+ \qquad \Pi_- : \mathcal{H}_N \to \Pi_-(\mathcal{H}_N) = \mathcal{H}_N^-$$

where $\Pi_\pm(\mathcal{H}_N) = \mathcal{H}_N^\pm$ are respectively the *bosonic* and the *fermionic N particles Hilbert spaces*.

To have lighter notation we denote

$$\Pi_+(\psi_1\otimes\dots\otimes\psi_N) = \psi_1\vee\dots\vee\psi_N$$

$$\Pi_-(\psi_1\otimes\dots\otimes\psi_N) = \psi_1\wedge\dots\wedge\psi_N$$

Application of the projector Π_ξ on a multi-particle state gives us the desired symmetrization or anti-symmetrization of the state, depending on the bosonic or fermionic nature of the particle. For instance, if $N = 3$ then the symmetrization (projection on bosonic space) of the state $\mathit{\Psi} = \psi_1\otimes\psi_2\otimes\psi_3$ will be

$$\begin{aligned}
\psi_1\vee\psi_2\vee\psi_3 = \frac{1}{6}\Big[&\psi_1\otimes\psi_2\otimes\psi_3 + \psi_3\otimes\psi_1\otimes\psi_2 + \psi_2\otimes\psi_3\otimes\psi_1 + \\
&+ \psi_3\otimes\psi_2\otimes\psi_1 + \psi_1\otimes\psi_3\otimes\psi_2 + \psi_2\otimes\psi_1\otimes\psi_3\Big]
\end{aligned}$$

while the anti-symmetrization

$$\begin{aligned}
\psi_1\wedge\psi_2\wedge\psi_3 = \frac{1}{6}\Big[&\psi_1\otimes\psi_2\otimes\psi_3 + \psi_3\otimes\psi_1\otimes\psi_2 + \psi_2\otimes\psi_3\otimes\psi_1 + \\
&- \psi_3\otimes\psi_2\otimes\psi_1 - \psi_1\otimes\psi_3\otimes\psi_2 - \psi_2\otimes\psi_1\otimes\psi_3\Big]
\end{aligned}$$

We can define the *bosonic* or *fermionic N-particles state* as

$$\Psi = \sqrt{N!}\;\Pi_\xi(\psi_1\otimes\dots\otimes\psi_N)$$

The symmetrization of a bosonic (or fermionic) state in coordinate representation can be expressed in a much more compact form: indeed, in general we have

$$\Psi(r_1,\dots,r_N) = \frac{1}{\sqrt{N!}}\sum_\sigma \xi^\sigma\psi_{\sigma(1)}(r_1)\dots\psi_{\sigma(N)}(r_N) \tag{1.2}$$

and if $\xi = -1$ we have that equation (1.2) is the determinant of the matrix $A_{ij} = \psi_i(r_j)$, and therefore

$$\Psi(r_1,\dots,r_N) = \frac{1}{\sqrt{N!}}\det\left[\psi_i(r_j)\right]$$

[3] We have $\sum_{\sigma\in S_N} = N!$

this determinant in the context of fermions takes the name of *Slater determinant.*

Also for $\xi = +1$ we can describe the wavefunction as a function of the entries of the matrix: we can write

$$\Psi(r_1, \dots, r_N) = \frac{1}{\sqrt{N!}} \operatorname{per}\left[\psi_i(r_j)\right]$$

where $\operatorname{per}(\cdot)$ denotes the permanent[4] of the matrix $A_{ij} = \psi_i(r_j)$. Computing the permanent of a matrix is basically like computing the determinant, just ignoring all the $-$ signs between the factors.

As an example, let us compute the 2-body wavefunction for both bosons and fermions. Employing the matrix notation we have just introduced we have

$$\Psi^{\text{Bos.}}(r_1, r_2) = \frac{1}{\sqrt{2}} \operatorname{Per}\begin{pmatrix} \psi_1(r_1) & \psi_1(r_2) \\ \psi_2(r_1) & \psi_2(r_2) \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\psi_1(r_1)\psi_2(r_2) + \phi_2(r_1)\phi_1(r_2)\right)$$

and for fermions

$$\Psi^{\text{Ferm.}}(r_1, r_2) = \frac{1}{\sqrt{2}} \operatorname{Det}\begin{pmatrix} \phi_{\alpha_1}(r_1) & \phi_{\alpha_1}(r_2) \\ \phi_{\alpha_2}(r_1) & \phi_{\alpha_2}(r_2) \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\phi_{\alpha_1}(r_1)\phi_{\alpha_2}(r_2) - \phi_{\alpha_2}(r_1)\phi_{\alpha_1}(r_2)\right)$$

Notice that the requested symmetry properties are satisfied:

$$\Psi^{\text{Bos.}}(r_1, r_2) = \Psi^{\text{Bos.}}(r_2, r_1) \qquad \Psi^{\text{Ferm.}}(r_1, r_2) = -\Psi^{\text{Ferm.}}(r_2, r_1)$$

Remark. The state for the N-particles system is not merely the straightforward tensor product of single-particle wavefunctions. Instead we have just seen how the multi-particles state is given by linear combinations of such products, which indicates that single particles are in an *entangled state.* This is the mathematical consequence of the physical loss of individuality for quantum particles: since it does not make sense to label single particles the only possible way to describe a composite system is treating it globally.

We define the *bosonic and fermionic Fock spaces* as the following infinite direct sum of the N-particles spaces[5]:

$$\mathcal{F}^{\pm} = \mathcal{H}_0^{\pm} \oplus \mathcal{H}_1^{\pm} \oplus \mathcal{H}_2^{\pm} \oplus \cdots = \bigoplus_{i=0}^{\infty} \mathcal{H}_i^{\pm}$$

Conventionally, $\mathcal{H}_0^{\pm} = \mathbb{C}$ is the *zero-particle space.*

On Fock space it is natural to define two operators that allow to jump from a certain number of particles to another one, namely the creation and annihilation operators. Let Ψ be our N particles state, then we define the *creation operator* as

$$(\mathsf{a}^{\dagger}(f)(\Psi))(r_0, r_1, \dots, r_N) = \frac{1}{\sqrt{N+1}} \sum_{j=0}^{N} \xi^{j-1} f(r_j) \Psi(r_1, \dots, r_{j-1}, r_{j+1}, r_N) \tag{1.3}$$

[4]The permanent of a $N \times N$ matrix A_{ij} is defined as

$$\operatorname{Per}(A) = \sum_{\sigma \in \operatorname{Sym}(n)} \prod_{i=1}^{N} A_{i\sigma(i)}$$

[5]More formally, given any separable Hilbert space $\mathcal{H}_1$, the Fock space over $\mathcal{H}_1$ is the complete tensor algebra over $\mathcal{H}_1$ (see [25]).

Creation operator adds a particle to the N-particles wavefunction, transforming it into a $N+1$-particles wavefunction

$$\mathsf{a}^\dagger : \mathcal{H}_N \to \mathcal{H}_{N+1}$$

New particle has wavefunction f, and the position of f is averaged by taking the sum of all possible coordinates appearing into f.

Directly from equation (1.3) we can see that the creation operator is linear in f, namely

$$\mathsf{a}^\dagger(\alpha f_1 + \beta f_2) = \alpha \mathsf{a}^\dagger(f_1) + \beta \mathsf{a}^\dagger(f_2)$$

The *annihilation operator* is defined as the hermitian adjoint of the creation operator, namely

$$\mathsf{a} = \left(\mathsf{a}^\dagger\right)^\dagger$$

We want to find the analogous of equation (1.3) for a, namely an explicit expression for the action of the annihilation operator on a multi-particle state. We shall begin with the identity

$$\langle \mathsf{a}^\dagger(f)\Xi, \Psi\rangle = \langle \Xi, \mathsf{a}(f)\Psi\rangle \tag{1.4}$$

The left hand side of this relation can be computed as

$$\begin{aligned}
\langle \mathsf{a}^\dagger(f)\Xi, \Psi\rangle &= \int \mathrm{d}r_1 \dots \mathrm{d}r_N \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \xi^{j-1} \left(f(r_j)\Xi(r_1, \dots, r_{j-1}, r_{j+1}, \dots, r_N)\right)^* \Psi(r_1, \dots r_N) \\
&\overset{\diamondsuit}{=} \int \mathrm{d}s_1 \dots \mathrm{d}s_{N-1} \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int \mathrm{d}r_j \left(f(r_j)\Xi(s_1, \dots, s_{N-1})\right)^* \Psi(r_j, s_1, \dots, s_{N-1}) \\
&= \int \mathrm{d}s_1 \dots \mathrm{d}s_{N-1} \frac{N}{\sqrt{N}} \int \mathrm{d}q f^*(q) \Psi(q, s_1, \dots, s_{N-1})
\end{aligned}$$

where in the passage marked with $\diamondsuit$ we renamed the variables as

$$s_i = \begin{cases} r_i & \text{for } i < j \\ r_{i+1} & \text{for } i > j \end{cases}$$

We found that

$$\langle \mathsf{a}^\dagger(f)\Xi, \Psi\rangle = \sqrt{N} \int \mathrm{d}s_1 \dots \mathrm{d}s_{N-1} \, \mathrm{d}q f^*(q) \Psi(q, s_1, \dots, s_{N-1})$$

and by comparing this expression with the right-hand side of equation (1.4) we get that the action of the annihilation operator on any N-particles state Ψ is

$$(\mathsf{a}(f)(\Psi))(r_1, \dots, r_{N-1}) = \sqrt{N} \int \mathrm{d}q f^*(q) \Psi(q, r_1, \dots, r_{N-1})$$

Similarly to the creation case, here the f is the wavefunction of the destroyed particle, and the projection along each existing wavefunction composing the multi-particle state is taken. We can write in a different fashion the annihilation operator, namely as

$$(\mathsf{a}(f)(\Psi))(r_1, \dots, r_{N-1}) = \sum_{\sigma \in S_N} \langle f, \psi_{\sigma(1)}\rangle \psi_{\sigma(2)}(r_1) \vee \dots \vee \psi_{\sigma(N)}(r_{N-1}) \tag{1.5}$$

From equation (1.5) it is easy to see that the annihilation operator is antilinear[6] in f, namely

$$\mathsf{a}(\alpha f_1 + \beta f_2) = \alpha^* \mathsf{a}(f_1) + \beta^* \mathsf{a}(f_2)$$

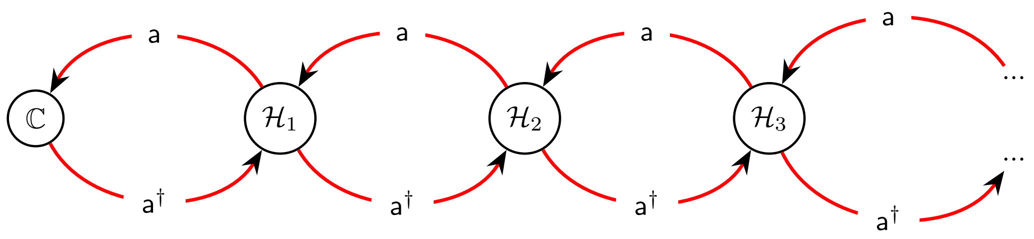


Figure 1.1: Pictorial representation of the action of $\mathsf{a}, \mathsf{a}^\dagger$ operators. The leftmost space is $\mathbb{C} = \mathcal{H}_0$, the zero particle space.

The two operators we just introduced satisfy some interesting commutation relations: let us define the operatorial commutator/anticommutator as $[\mathsf{A}, \mathsf{B}]_\xi = \mathsf{AB} - \xi \mathsf{BA}$, then for $f, g \in \mathcal{H}_1$ we have the following relations

$$[\mathsf{a}(f), \mathsf{a}^\dagger(g)]_\xi = \langle f, g \rangle \mathbf{1} \qquad [\mathsf{a}(f), \mathsf{a}(g)]_\xi = 0 \qquad [\mathsf{a}^\dagger(f), \mathsf{a}^\dagger(g)]_\xi = 0 \tag{1.6}$$

Equations (1.6) in the bosonic case are called *canonical commutation relations*, or CCR, and in the fermionic case *canonical anticommutation relations*, or CAR. Such relations are of great importance, since the different behaviour between fermions and bosons is due to the commuting or anticommuing nature of the respective annihilation or creation operators.

For instance, for fermions the CAR imply the Pauli exclusion principle, since by taking $f = g$ in equation (1.6) we get

$$[\mathsf{a}(f), \mathsf{a}(f)]_- = 2\mathsf{a}(f)^2 = 0 \qquad [\mathsf{a}^\dagger(f), \mathsf{a}^\dagger(f)]_- = 2\mathsf{a}^\dagger(f)^2 = 0$$

and so we conclude that a and $\mathsf{a}^\dagger$ in the fermionic case are nilpotent operators.

The most common realization of creation and annihilation operators is by using an orthonormal basis: let $\{\varphi_k\}_k$ be an orthonormal basis of $\mathcal{H}_1$, then we denote

$$\mathsf{a}^\dagger(\varphi_k) = \mathsf{a}_k^\dagger \qquad \mathsf{a}(\varphi_k) = \mathsf{a}_k$$

Notice that if $\mathcal{H}_1 = L^2(\mathbb{R}^d)$ then k is a d-dimensional multiindex.

Let us focus on bosons: thanks to the orthonormality of the basis the CCR becomes

$$[\mathsf{a}_k, \mathsf{a}_l^\dagger] = \delta_{kl} \tag{1.7}$$

while all other commutators are vanishing.

Let $\Omega \in \mathcal{H}_0$ be the *zero-particles state*, also called the *vacuum state*. Application of the destruction operator to the vacuum state gives identically zero

$$\mathsf{a}(f)\Omega = 0$$

Conversely, we can create any other state in Fock space by repeated action of the creation operator on the vacuum state. Let us consider the first iteration, that creates from the vacuum a single particle state of wavefunction f:

$$\mathsf{a}^\dagger(f)\Omega = f$$

[6]Indeed, recall that in general the scalar product in a Hilbert space is *sesquilinear* (literally "one and a half linear"), meaning linear in one component and antilinear on the other.

Then, by applying another creation operator we get the two-particle state

$$\mathsf{a}^\dagger(g)\mathsf{a}^\dagger(f)\Omega = \mathsf{a}^\dagger(g)f = \frac{1}{\sqrt{2}}\left(g\otimes f + f\otimes g\right)$$

and so on. In general, for the bosonic N-particles state we can write

$$\Psi = \sqrt{N!}\psi_1 \vee \dots \vee \psi_N = \mathsf{a}^\dagger(\psi_1)\dots\mathsf{a}^\dagger(\psi_N)\Omega$$

This characterization of Fock states allows us to define the scalar product between two bosonic Fock states Ψ and Φ as

$$\langle\Psi,\Phi\rangle = \mathrm{per}\left(\langle\psi_i,\phi_j\rangle\right)$$

A pictorial representation of the action of creation and annihilation operators is showed in figure 1.1.

1.2 Field operators

Let us consider for an arbitrary point in space $x\in\mathbb{R}^d$ the following state

$$\delta_x(r) = \delta(r-x)$$

The scalar product of two states is

$$\langle\delta_x,\delta_y\rangle = \int_{\mathbb{R}^d}\mathrm{d}r\delta(r-x)\delta(r-y) = \delta(x-y)$$

We denote the creation and annihilation operators associated to δ_x and δ_y as

$$\mathsf{\Psi}(x) = \mathsf{a}(\delta_x) \qquad \mathsf{\Psi}^\dagger(x) = \mathsf{a}^\dagger(\delta_x)$$

We have the following CCR

$$[\mathsf{\Psi}(x),\mathsf{\Psi}^\dagger(y)] = \delta(x-y) \text{ for all } x,y\in\mathbb{R}^3$$

How can we connect the operators $\mathsf{\Psi}^\dagger$ and $\mathsf{\Psi}$ to the standard (and well-defined) creation and annihilation operators? Let $f\in\mathcal{H}_1$ be an arbitrary single particle wavefunction, then

$$f(r) = \int_{\mathbb{R}^d}\mathrm{d}x f(x)\delta_r(x)$$

or, choosing not to use a coordinate representation

$$f = \int_{\mathbb{R}^d}\mathrm{d}x f(x)\delta_x$$

Then creation and annihilation operators can be expressed as

$$\mathsf{a}^\dagger(f) = \int_{\mathbb{R}^d}\mathrm{d}x f(x)\mathsf{\Psi}^\dagger(x)$$

$$\mathsf{a}(f) = \int_{\mathbb{R}^d}\mathrm{d}x f^*(x)\mathsf{\Psi}(x)$$

Now, instead of choosing an arbitrary $f\in\mathcal{H}_1$ let us consider the orthonormal basis $\{\varphi_k\}_k$, then

$$\delta_x(r) = \sum_k \varphi_k(x)^*\varphi_k(r)$$

and by linearity

$$\Psi(x) = \sum_k \varphi_k(x)\mathsf{a}_k \qquad \Psi^\dagger(x) = \sum_k \varphi_k^*(x)\mathsf{a}_k^\dagger$$

We can add a dependence of field operators from time by considering the time evolution of the annihilation and creation operators. We define the *quantum field operators* as

$$\Psi(t,x) = \sum_k \mathsf{a}_k(t)\varphi_k(x) \qquad \Psi^\dagger(t,x) = \sum_k \mathsf{a}_k^\dagger(t)\varphi_k^*(x) \tag{1.8}$$

where the time evolution is ruled by the Heisenberg equation $i\Psi(t,x) = [\Psi(t,x), \mathsf{H}]$, that admits $\Psi(t,x) = \mathsf{U}(t)\Psi(x)\mathsf{U}^\dagger(t)$ as a solution[7]. Equation (1.8) is also called the *normal mode decomposition* of the field operators. Correspondingly we have

$$\mathsf{a}_k(t) = \langle \varphi_k, \Psi(t,\cdot)\rangle = \int_{\mathbb{R}^d} \varphi_k^*(x)\Psi(t,x)\,\mathrm{d}x$$

$$\mathsf{a}_k^\dagger(t) = \langle \varphi_k^*, \Psi^\dagger(t,\cdot)\rangle = \int_{\mathbb{R}^d} \varphi_k(x)\Psi^\dagger(t,x)\,\mathrm{d}x$$

CCR of the destruction and creation operators is inherited by the field operators, indeed using the linearity of the scalar product we have

$$[\mathsf{a}_k(t), \mathsf{a}_l^\dagger(t)] = [\langle \phi_k, \Psi(t,\cdot)\rangle, \langle \phi_l^*, \Psi^\dagger(t,\cdot)\rangle] = \int_{\mathbb{R}^{2d}} [\Psi(t,x), \Psi^\dagger(t,y)]\phi_k^*(x)\phi_l(y)\,\mathrm{d}^d x\,\mathrm{d}^d y \overset{(!)}{=} \delta_{kl}$$

where the equality marked with (!) holds if and only if

$$[\Psi(t,x), \Psi^\dagger(t,y)] = \delta(x-y) \tag{1.9}$$

Remark. Notice that we are stating the CCR for two fields operators in different spatial points, but at the same time: indeed, equation (1.9) is also commonly named *equal time commutation relation*, or ETCR. Computing the commutator of two field operators at arbitrary space-time points is more complex and involves the notion of *field propagator* (for a relativistic treatment, see chapter 6 of [53] or also chapter 4 of [27]).

1.3 The Hamiltonian

We consider the Hamiltonian in second quantization for identical bosons of spin 0 in $\mathbb{R}^d$

$$\begin{aligned}\mathsf{H} &= \int_{\mathbb{R}^d} \Psi^\dagger(x)\mathsf{h}(x)\Psi(x)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^{2d}} \Psi^\dagger(x)\Psi^\dagger(y)V(|x-y|)\Psi(y)\Psi(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \mathsf{H}^{(0)} + \mathsf{H}^{(\mathrm{int})}\end{aligned} \tag{1.10}$$

where $\mathsf{h}(x)$ is the single particle Hamiltonian, given by

$$\mathsf{h}(x) = -\frac{1}{2m}\nabla^2 + u(x)$$

[7] Here $\mathsf{U}(t) = \exp(-i\mathsf{H}t)$, where H is the Hamiltonian that will be introduced shortly. The map $\mathsf{U} : \mathcal{F}_+ \to \mathcal{F}_+$ is well defined, see [17].

and $u(x)$, $V(|x-y|)$ are respectively the one- and two- body interaction potentials. Notice that for spatial homogeneity the potential V depends on the difference $|x-y|$ rather than from x and y separately. We will denote with $\{\varphi_k\}_k$, with $k \in \mathbb{N}^d$, the set of orthonormal eigenfunctions that satisfy

$$\mathsf{h}\varphi_k = \varepsilon_k \varphi_k$$

Heisenberg equation rules the time evolution of the system, namely

$$\begin{aligned} i\frac{\partial \Psi(t,x)}{\partial t} &= [\Psi(t,x), \mathsf{H}] \\ &= \int_{\mathbb{R}^d} [\Psi(t,x), \Psi^\dagger(t,y)\mathsf{h}(y)\Psi(t,y)]\,\mathrm{d}y + \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^{2d}} [\Psi(t,x), \Psi^\dagger(t,y)\Psi^\dagger(t,z)V(|y-z|)\Psi(t,z)\Psi(t,y)]\,\mathrm{d}y\,\mathrm{d}z \end{aligned} \tag{1.11}$$

Now, focusing on the first term we have

$$\begin{aligned} [\Psi(t,x), \Psi^\dagger(t,y)\mathsf{h}(y)\Psi(t,y)] &= [\Psi(t,x), \Psi^\dagger(t,y)]\mathsf{h}(y)\Psi(t,y) + \Psi^\dagger(t,y)[\Psi(t,x), \mathsf{h}(y)\Psi(t,y)] \\ &= [\Psi(t,x), \Psi^\dagger(t,y)]\mathsf{h}(y)\Psi(t,y) + \Psi^\dagger(t,y)\mathsf{h}(y)\overbrace{[\Psi(t,x), \Psi(t,y)]}^{=0} \\ &= \delta(x-y)\mathsf{h}(y)\Psi(t,y) \end{aligned}$$

and analogously focusing on the second[8]

$$\begin{aligned} [\Psi(x), \Psi^\dagger(y)\Psi^\dagger(z)V(|y-z|)\Psi(z)\Psi(y)] &= [\Psi(x), \Psi^\dagger(y)]\Psi^\dagger(z)V(|y-z|)\Psi(z)\Psi(y) \\ &\quad + \Psi^\dagger(y)[\Psi(x), \Psi^\dagger(z)V(|y-z|)\Psi(z)\Psi(y)] \\ &= \delta(x-y)\Psi^\dagger(z)V(|y-z|)\Psi(z)\Psi(y) \\ &\quad + \Psi^\dagger(y)\Big([\Psi(x), \Psi^\dagger(z)]V(|y-z|)\Psi(z)\Psi(y) \\ &\qquad + \Psi^\dagger(z)\overbrace{[\Psi(x), V(|y-z|)\Psi(z)\Psi(y)]}^{=0}\Big) \\ &= \delta(x-y)\Psi^\dagger(z)V(|y-z|)\Psi(z)\Psi(y) \\ &\quad + \delta(x-z)\Psi^\dagger(y)V(|y-z|)\Psi(z)\Psi(y) \end{aligned}$$

Putting together these two results we can rewrite equation (1.11) as

$$\begin{aligned} i\frac{\partial \Psi(t,x)}{\partial t} &= [\Psi(t,x), \mathsf{H}] \\ &= \int_{\mathbb{R}^d} \delta(x-y)\mathsf{h}(y)\Psi(t,y)\,\mathrm{d}y + \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^{2d}} \delta(x-y)\Psi^\dagger(t,z)V(|y-z|)\Psi(t,z)\Psi(t,y)\,\mathrm{d}y\,\mathrm{d}z \\ &\qquad + \frac{1}{2}\int_{\mathbb{R}^{2d}} \delta(x-z)\Psi^\dagger(t,y)V(|y-z|)\Psi(t,z)\Psi(t,y)\,\mathrm{d}y\,\mathrm{d}z \end{aligned}$$

[8]Here we omit the time dependence of the field operators for better readability.

and if we perform the integrals

$$\begin{aligned} i\frac{\partial \Psi(t,x)}{\partial t} = [\Psi(t,x),\mathsf{H}] &= \mathsf{h}(x)\Psi(t,x) + \\ &+\frac{1}{2}\int_{\mathbb{R}^d}\Psi^\dagger(t,z)V(|x-z|)\Psi(t,z)\Psi(t,x)\,\mathrm{d}z \\ &+\frac{1}{2}\int_{\mathbb{R}^d}\Psi^\dagger(t,y)V(|y-x|)\Psi(t,x)\Psi(t,y)\,\mathrm{d}y \\ &= \mathsf{h}(x)\Psi(t,x) + \left(\int_{\mathbb{R}^d}V(|y-x|)\Psi^\dagger(t,y)\Psi(t,y)\,\mathrm{d}y\right)\Psi(t,x) \\ &= \left(\mathsf{h}(x) + V*\Psi^\dagger\Psi\right)\Psi(t,x) \end{aligned}$$

where $*$ denotes the convolution product[9]. Therefore, the evolution of the field operator Ψ and of its hermitian conjugate $\Psi^\dagger$ are ruled by the following equations

$$i\frac{\partial \Psi(t,x)}{\partial t} = \left(\mathsf{h}(x) + V*\Psi^\dagger\Psi\right)\Psi(t,x) \tag{1.12}$$

$$-i\frac{\partial \Psi^\dagger(t,x)}{\partial t} = \left(\mathsf{h}(x) + V*\Psi^\dagger\Psi\right)\Psi^\dagger(t,x) \tag{1.13}$$

Both equations (1.12) and (1.13) can be seen as a modified Schrödinger equation in second quantization, in which linearity is lost due to the presence of the interaction potential V.

The *Number operator* is defined as

$$\mathsf{N}(t) = \int_{\mathbb{R}^d}\Psi^\dagger(t,x)\Psi(t,x)\,\mathrm{d}^d x$$

We can easily compute the equal time commutation relations of $\mathsf{N}(t)$ with the field operator as

$$[\mathsf{N}(t),\Psi^\dagger(t,x)] = \int_{\mathbb{R}^d}[\Psi^\dagger(t,y)\Psi(t,y),\Psi^\dagger(t,x)]\,\mathrm{d}^d y = \int_{\mathbb{R}^d}\Psi^\dagger(t,y)[\Psi(t,y),\Psi^\dagger(t,x)]\,\mathrm{d}^d y = \Psi^\dagger(t,x)$$

$$[\mathsf{N}(t),\Psi(t,x)] = \int_{\mathbb{R}^d}[\Psi^\dagger(t,y)\Psi(t,y),\Psi(t,x)]\,\mathrm{d}^d y = \int_{\mathbb{R}^d}[\Psi^\dagger(t,y),\Psi(t,x)]\Psi^\dagger(t,y)\,\mathrm{d}^d y = -\Psi(t,x)$$

Using the above relations it is simple to show that the number operator is preserved by the dynamics: indeed[10]

$$[\mathsf{N},\Psi^\dagger(x)\mathsf{h}(x)\Psi(x)] = \Psi^\dagger(x)\mathsf{h}(x)[\mathsf{N},\Psi(x)] + [\mathsf{N},\Psi^\dagger(x)]\mathsf{h}(x)\Psi(x) = 0$$

$$\begin{aligned} [\mathsf{N},\Psi^\dagger(x)\Psi^\dagger(y)V(|x-y|)\Psi(y)\Psi(x)] &= [\mathsf{N},\Psi^\dagger(x)]\Psi^\dagger(y)V(|x-y|)\Psi(y)\Psi(x) + \\ &+\Psi^\dagger(x)[\mathsf{N},\Psi^\dagger(y)]V(|x-y|)\Psi(y)\Psi(x) + \\ &+\Psi^\dagger(x)\Psi^\dagger(y)[\mathsf{N},V(|x-y|)\Psi(y)]\Psi(x) + \\ &+\Psi^\dagger(x)\Psi^\dagger(y)V(|x-y|)\Psi(y)[\mathsf{N},\Psi(x)] = 0 \end{aligned}$$

and therefore

$$i\frac{\partial \mathsf{N}}{\partial t} = [\mathsf{N},\mathsf{H}] = 0$$

[9] Given two operators A and B their *convolution product* is defined as

$$\mathsf{A}*\mathsf{B}(x) = \int \mathsf{A}(x-y)\mathsf{B}(y)\,\mathrm{d}^d y$$

[10] We omit the time dependence for simplicity.

The conservation of the N operator can be seen also as a consequence of Noether's theorem: indeed, the Hamiltonian (1.10) is invariant under the following global $U(1)$ transformation:

$$\Psi \mapsto e^{i\alpha}\Psi \qquad \Psi^\dagger \mapsto e^{-i\alpha}\Psi^\dagger \qquad \alpha \in \mathbb{R}$$

and the number operator is the conserved quantity provided by the theorem.

Now, it is convenient to rewrite the Hamiltonian in an alternative form, using the creation and annihilation operators. Let us start from the first piece of equation (1.10)

$$\int_{\mathbb{R}^d} \Psi^\dagger(x)\mathsf{h}(x)\Psi(x)\,\mathrm{d}x = \sum_{kl} \mathsf{a}_k^\dagger \mathsf{a}_l \int_{\mathbb{R}^d} \varphi_k^*(x)\mathsf{h}(x)\varphi_l(x)\,\mathrm{d}x = \sum_{kl} \mathsf{a}_k^\dagger \mathsf{a}_l \langle \varphi_k, \mathsf{h}\varphi_l\rangle = \sum_k \varepsilon_k \mathsf{a}_k^\dagger \mathsf{a}_k \tag{1.14}$$

then move on to the second

$$\frac{1}{2}\int_{\mathbb{R}^{2d}} \Psi^\dagger(x)\Psi^\dagger(y)V(|x-y|)\Psi(y)\Psi(x)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\sum_{klmn} \mathsf{a}_k^\dagger(t)\mathsf{a}_l^\dagger(t)\mathsf{a}_m(t)\mathsf{a}_n(t)V_{klmn} \tag{1.15}$$

where

$$V_{klmn} = \int_{\mathbb{R}^{2d}} \varphi_k^*(x)\varphi_l^*(y)\varphi_m(x)\varphi_n(y)V(|x-y|)\,\mathrm{d}^d x\,\mathrm{d}^d y$$

are the *interaction coefficients* of the potential. We will discuss more in depth some properties of this object later on. Combining equations (1.14) and (1.15) we can write

$$\mathsf{H} = \sum_k \varepsilon_k \mathsf{a}_k^\dagger \mathsf{a}_k + \frac{1}{2}\sum_{klmn} \mathsf{a}_k^\dagger(t)\mathsf{a}_l^\dagger(t)\mathsf{a}_m(t)\mathsf{a}_n(t)V_{klmn}$$

Time evolution of the destruction operator satisfies the Heisenberg equation, namely

$$i\frac{\mathrm{d}\mathsf{a}_k}{\mathrm{d}t} = [\mathsf{a}_k, \mathsf{H}] = \sum_l \varepsilon_l[\mathsf{a}_k, \mathsf{a}_l^\dagger \mathsf{a}_l] + \frac{1}{2}\sum_{lmnr}[\mathsf{a}_k, \mathsf{a}_l^\dagger \mathsf{a}_m^\dagger \mathsf{a}_n \mathsf{a}_r]V_{lmnr} \tag{1.16}$$

Evaluation of the commutators is easily done through the CCR:

$$[\mathsf{a}_k, \mathsf{a}_l^\dagger \mathsf{a}_l] = [\mathsf{a}_k, \mathsf{a}_l^\dagger]\mathsf{a}_l = \mathsf{a}_k \delta_{kl}$$

$$\begin{aligned}
[\mathsf{a}_k, \mathsf{a}_l^\dagger \mathsf{a}_m^\dagger \mathsf{a}_n \mathsf{a}_r] &= [\mathsf{a}_k, \mathsf{a}_l^\dagger]\mathsf{a}_m^\dagger \mathsf{a}_n \mathsf{a}_r + \mathsf{a}_l^\dagger[\mathsf{a}_k, \mathsf{a}_m^\dagger \mathsf{a}_n \mathsf{a}_r] = \delta_{lk}\mathsf{a}_m^\dagger \mathsf{a}_n \mathsf{a}_r + \mathsf{a}_l^\dagger[\mathsf{a}_k, \mathsf{a}_m^\dagger]\mathsf{a}_n \mathsf{a}_r \\
&= \delta_{lk}\mathsf{a}_m^\dagger \mathsf{a}_n \mathsf{a}_r + \delta_{mk}\mathsf{a}_l^\dagger \mathsf{a}_n \mathsf{a}_r
\end{aligned}$$

and so one gets

$$\begin{aligned}
i\frac{\mathrm{d}\mathsf{a}_k}{\mathrm{d}t} &= \epsilon_k \mathsf{a}_k + \frac{1}{2}\sum_{lmnr}(\delta_{lk}\mathsf{a}_m^\dagger \mathsf{a}_n \mathsf{a}_r + \delta_{mk}\mathsf{a}_l^\dagger \mathsf{a}_n \mathsf{a}_r)V_{lmnr} \\
&= \epsilon_k \mathsf{a}_k + \frac{1}{2}\sum_{mnr}\left(\mathsf{a}_m^\dagger \mathsf{a}_n \mathsf{a}_r V_{kmnr} + \mathsf{a}_m^\dagger \mathsf{a}_n \mathsf{a}_r V_{mknr}\right) \\
&= \epsilon_k \mathsf{a}_k + \sum_{mnl}\mathsf{a}_m^\dagger \mathsf{a}_n \mathsf{a}_l V_{(km)nl}
\end{aligned}$$

where $V_{(km)nl} = \frac{1}{2}(V_{kmnl} + V_{mknl})$ is the *symmetrized interaction coefficient.*

1.4 Potentials

Confinement potential

We stated above that

$$\mathsf{h}(x) = -\frac{1}{2}\nabla^2 + u(x)$$

is our single particle Hamiltonian. We take the confinement potential to be the harmonic one, namely

$$u(x) = \frac{1}{2}|x|^2$$

where again $m = 1$ and $\omega = 1$. The single particle eigenfunctions φ_j are therefore given by the solutions of the d dimensional harmonic oscillator.

Interaction potential and the Hardy class

Let us define the H^1-*Sobolev space* as

$$H^1(\mathbb{R}^d) = \{\psi : \mathbb{R}^d \to \mathbb{R} \text{ such that } \|\psi\|_{L^2(\mathbb{R}^d)} + \|\nabla\psi\|_{L^2(\mathbb{R}^d)} < +\infty\}$$

we call the quantity $\|\psi\|_{H^1(\mathbb{R}^d)} := \|\psi\|_{L^2(\mathbb{R}^d)} + \|\nabla\psi\|_{L^2(\mathbb{R}^d)}$ the H^1-*Sobolev norm* of ψ.

Given a potential $V : \mathbb{R}^d \to \mathbb{R}$ we say it belongs to the *Hardy class* if

$$\|V\psi\|_{L^2(\mathbb{R}^d)} \leq C_V \, \|\psi\|_{H^1(\mathbb{R}^d)}$$

where the constant C_V is the *Hardy constant* related to the potential V.

Potentials in Hardy class include the familiar Coulomb-type potential, indeed we have the following proposition:

Proposition 1.1. The Coulomb potential $V(x) = \frac{1}{|x|}$ obeys the *Hardy inequality*, namely

$$\left(\frac{d-2}{2}\right)^2 \int_{\mathbb{R}^d} \frac{|\psi|^2}{|x|^2} \, \mathrm{d}x \leq \int |\nabla\psi|^2 \, \mathrm{d}x \tag{1.17}$$

Proof. We will prove the result in one dimension. Let us write ψ as an integral function, namely

$$\psi(x) = \int_0^x f(t) \, \mathrm{d}t$$

Having written ψ in such a form, the inequality to prove is

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t) \, \mathrm{d}t\right)^2 \leq 4\int_0^\infty f(x)^2 \, \mathrm{d}x$$

By changing the variable $t \mapsto sx$ we get

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t) \, \mathrm{d}t\right)^2\right)^{\frac{1}{2}} = \left[\int_0^\infty \left(\int_0^1 f(sx) \, \mathrm{d}s\right)^2 \mathrm{d}x\right]^{\frac{1}{2}}$$

By Minkowski inequality

$$\left[\int_0^\infty \left(\int_0^1 f(sx) \, \mathrm{d}s\right)^2 \mathrm{d}x\right]^{\frac{1}{2}} \leq \int_0^1 \left(\int_0^\infty f(sx)^2 \, \mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}s$$

Then, changing one more time variable in the dx integral $x \mapsto \frac{1}{s}\xi$ we get

$$\int_0^1 \left(\int_0^\infty f(sx)^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \mathrm{d}s = \int_0^1 \left(\int_0^\infty f(\xi)^2 \,\mathrm{d}\xi\right)^{\frac{1}{2}} s^{-\frac{1}{2}} \,\mathrm{d}s = 2\left(\int_0^\infty f(\xi)^2 \,\mathrm{d}\xi\right)^{\frac{1}{2}}$$

we proved

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,\mathrm{d}t\right)^2\right)^{\frac{1}{2}} \leq 2\left(\int_0^\infty f(\xi)^2 \,\mathrm{d}\xi\right)^{\frac{1}{2}}$$

and squaring both members we recover the desired inequality. □

Remark. There exists a more general form of Hardy inequality in $L^p(\mathbb{R}^d)$ spaces, namely

$$\left\|\frac{f}{|x|}\right\|_{L^p(\mathbb{R}^n)} \leq \frac{p}{n-p}\,\|\nabla f\|_{L^p(\mathbb{R}^n)}$$

Remark. We furthermore remark that the Hardy inequality, equation (1.17), can be regarded as an uncertainty principle. Indeed, it states that a function cannot be concentrated around one point (the origin) unless its momentum is big, and vice-versa if its momentum is small then the function has to be spread in the space (see [16] for more details). Interestingly, the Hardy inequality was introduced in 1925 in the paper [29], two years before Heisenberg formulated the eponymous principle [30]: the history beyond the motivation and the background that led to Hardy to formulate the inequality is explained in the review paper [33].

Having introduced the Hardy class, let us consider some physically relevant aspects. In general, to being able to model the bosonic interactions we want our 2-body potential to be

- Repulsive on short distances;
- Attractive at some intermediate distances;
- Vanishing at infinite distances.

This translates into the necessity of having two constants r_0 and r_1 such that the potential V is positive[11] for $r < r_0$, negative for $r_0 < r < r_1$ and null for $r > r_1$:

$$\begin{cases} V(r) > 0 & \text{if } r < r_0 \\ V(r) < 0 & \text{if } r_0 < r < r_1 \\ V(r) = 0 & \text{if } r > r_1 \end{cases} \quad \text{where } r = |x|, x \in \mathbb{R}^d$$

Moreover, we introduce the constants a and b as

$$a = \sup_{0<r<r_0} V(r) \qquad b = -\inf_{r_0<r<r_1} V(r)$$

A graphical representation of the parameters r_0, r_1, a, b is showed in figure 1.2: we will refer to such parameters as *bounding parameters*.

If both a and b are finite, then the potential is limited[12]. Indeed, a vast class of Hardy potentials is given by the limited ones:

[11] We will commit a slight abuse of notation: thanks to spatial homogeneity, the potential is a function of just the modulus of its argument, $V(x) = V(|x|)$: sometimes we will put the modulus sign explicitly, other times we will just compute the potential in some $x \in \mathbb{R}^d$ omitting the $|\cdot|$.

[12] Recall that for $r > r_1$ no divergence can occur, since by definition $V(r) = 0$ for $r > r_1$.

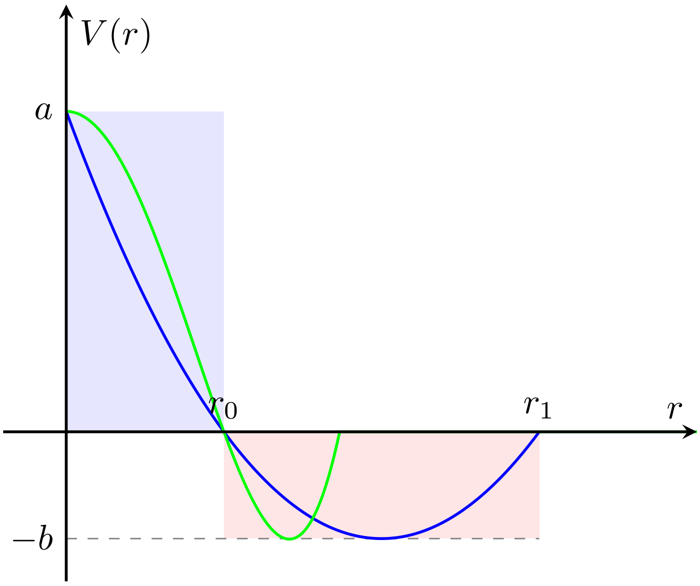


Figure 1.2: The parameters a, b, r_0 and r_1 determine the two rectangles that contain the potential. As an example, two potentials are sketched.

Proposition 1.2. Let V be a limited potential, $V \in L^\infty(\mathbb{R}^d)$. Then V is in Hardy class and the Hardy constant is given by

$$C_V = \|V\|_{L^\infty(\mathbb{R}^d)}$$

where the norm is given by

$$\|f\|_{L^\infty(\mathbb{R}^d)} = \inf\{C \geq 0 : |f(x)| \leq C \text{ almost everywhere }\}$$

Proof. We have

$$\begin{aligned}\|V\psi\|^2_{L^2(\mathbb{R}^d)} &= \int |V(x)\psi(x)|^2 \,\mathrm{d}x \leq C^2 \int |\psi(x)|^2 \,\mathrm{d}x \leq \\ &\leq C^2 \int |\psi(x)|^2 \,\mathrm{d}x + C^2 \int |\nabla\psi(x)|^2 \,\mathrm{d}x = C^2 \,\|\psi\|^2_{H^1(\mathbb{R}^d)}\end{aligned}$$

It is then easy to show that $C = \max\{a, b\}$. □

A first potential that we can consider in our physical model is the *step potential*, defined as

$$V(r) = \begin{cases} V_0 & \text{if } r < 0 \\ -\varepsilon & \text{if } r_0 < r < r_1 \\ 0 & \text{if } r > r_1 \end{cases} \quad \text{where } r = |x|, x \in \mathbb{R}^d$$

and represented in figure 1.3. Clearly, the bounding parameters are

$$r_0 \qquad r_1 \qquad V_0 = a \qquad \varepsilon = b$$

For such potential $\|V\|_{L^\infty(\mathbb{R}^d)} = \max\{V_0, \varepsilon\} = C_V$. This means

$$\|V\psi\|^2_{L^2(\mathbb{R}^d)} \leq C_V^2 \|\psi\|^2_{H^1(\mathbb{R}^d)} = \max\{V_0, \varepsilon\}^2 \|\psi\|^2_{H^1(\mathbb{R}^d)}$$

for all ψ.

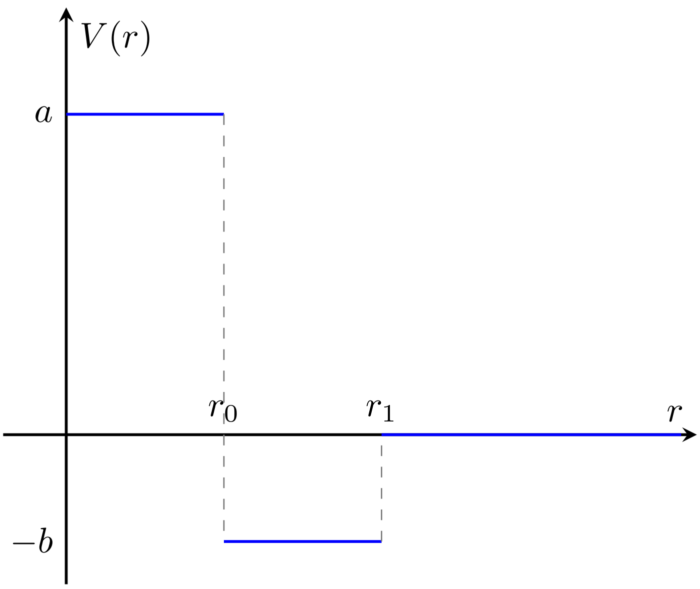


Figure 1.3: Step potential.

Lennard-Jones potential is an improvement compared to the step potential since it is smooth and is phenomenologically motivated, see [38] for instance. Analytic expression of such potential is

$$V_{\mathrm{LJ}}(r; \eta, \sigma) = 4\eta \left[\left(\frac{\sigma}{r}\right)^6 - \left(\frac{\sigma}{r}\right)^{12} \right]$$

where

1. parameter η is related to the depth of the potential well;
2. parameter σ is related to the radius of the particle in the hard sphere model.

Lennard-Jones potential is positive for $r < \sigma$: for small values of r it fastly diverges, $V \propto r^{-12}$. On the other hand, such potential presents a (negative) potential well for $r > \sigma$ and it is vanishing as $V \propto r^{-6}$ in the great distance ($r \gg 1$) limit, as can be seen in figure 1.4. In order to being able to include the Lennard-Jones potential in the (limited) Hardy class we mentioned above, we can consider a truncated Lennard-Jones potential instead of the actual one. Truncated version is given by

$$V(r) = \begin{cases} V_{\mathrm{LJ}}(r; \eta, \sigma) & \text{for } r \geq \lambda \\ V_{\mathrm{LJ}}(\lambda; \eta, \sigma) & \text{for } r < \lambda \end{cases}$$

and the plot of this function is shown in figure 1.4.

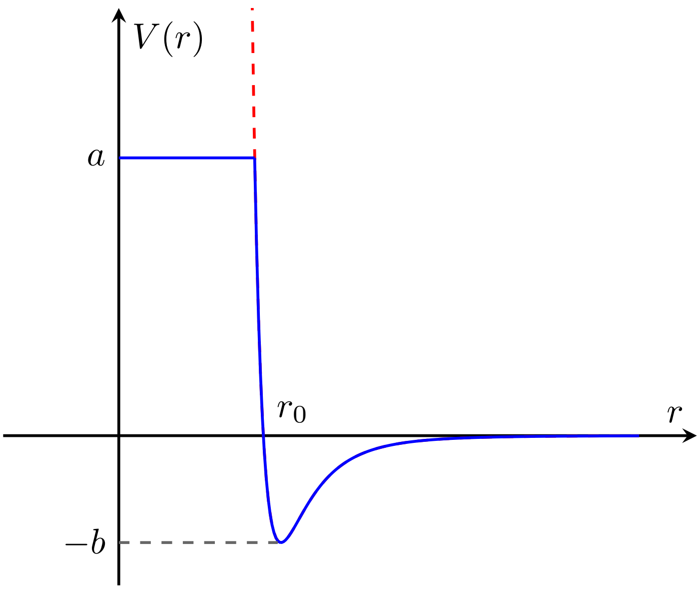


Figure 1.4: Truncated Lennard-Jones potential. Red dashed line represent the actual potential.

Hardy inequality for a given potential can be extended to field operators, as we show in the following proposition. We recall that, given two operators A, B acting on a Hilbert space $\mathcal{H}$, the notation

$$\mathsf{A} \leq \mathsf{B}$$

is used to indicate the following property:

$$\langle \varphi, \mathsf{A}\varphi \rangle \leq \langle \varphi, \mathsf{B}\varphi \rangle$$

for every $\varphi \in \mathrm{Dom}(\mathsf{A}) \cap \mathrm{Dom}(\mathsf{B})$ in the domains of the operators.

Proposition 1.3. Let V be a potential in the Hardy class, and let C_V be its Hardy constant. The field operator Ψ obeys the following inequality:

$$\int_{\mathbb{R}^d} V^2(x) \Psi^\dagger \Psi(x) \, \mathrm{d}x \leq C_V^2 \left(\int_{\mathbb{R}^d} \Psi^\dagger \Psi(x) \, \mathrm{d}x + \int_{\mathbb{R}^d} \nabla \Psi^\dagger \nabla \Psi(x) \, \mathrm{d}x \right)$$

Proof. Let us take an arbitrary $\varphi \in \mathcal{H}_N^+$, then

$$\psi(x) = \Psi(x)\varphi$$

Given the orthonormal basis $\{\varphi_j\}_j$, we call ψ_j the projection of ψ along the $j-th$ basis vector:

$$\psi_j(x) = \langle \varphi_j, \psi(x) \rangle_{\mathcal{H}_{N-1}^+}$$

Then we can write

$$\langle \varphi, \Psi^\dagger \Psi(x) \varphi \rangle_{\mathcal{H}_N^+} = \langle \psi(x), \psi(x) \rangle = \sum_{j=0}^{\infty} |\psi_j(x)|^2$$

$$\langle \varphi, \nabla\Psi^\dagger\nabla\Psi(x)\varphi\rangle_{\mathcal{H}_N^+} = \langle \nabla\psi(x), \nabla\psi(x)\rangle = \sum_{j=0}^{\infty} |\nabla\psi_j(x)|^2$$

For each j we can write the Hardy inequality: since we are dealing with the projections of the field operator computed against an arbitrary function φ we can use the traditional (i.e. functional) inequality, getting

$$\|V\psi_j\|_{L^2(\mathbb{R}^d)} \leq C_V \|\psi_j\|_{H^1(\mathbb{R}^d)}$$

that can be squared to get

$$\int V^2(x)\psi_j^2(x)\,\mathrm{d}x \leq C_V^2 \left(\int_{\mathbb{R}^d} \left(|\psi_j(x)|^2 + |\nabla\psi_j(x)|^2\right) \mathrm{d}x \right)$$

The above relation is valid for any j, therefore we can take the sum

$$\sum_{j=0}^{\infty} \int V^2(x)\psi_j^2(x)\,\mathrm{d}x \leq \sum_{j=0}^{\infty} C_V^2 \left(\int_{\mathbb{R}^d} \left(|\psi_j(x)|^2 + |\nabla\psi_j(x)|^2\right) \mathrm{d}x \right)$$

and therefore

$$\langle \varphi, \int_{\mathbb{R}^d} V^2(x)\Psi^\dagger\Psi(x)\,\mathrm{d}x\varphi\rangle_{\mathcal{H}_N^+} \leq C_V^2 \langle \varphi, \left(\int_{\mathbb{R}^d} \Psi^\dagger\Psi(x)\,\mathrm{d}x + \int_{\mathbb{R}^d} \nabla\Psi^\dagger\nabla\Psi(x)\,\mathrm{d}x \right) \varphi\rangle_{\mathcal{H}_N^+}$$

□

Given a potential V, we have already defined its interaction coefficients as the following quantity

$$V_{klmn} = \int_{\mathbb{R}^{2d}} \varphi_k^*(x)\varphi_l^*(y)V(x-y)\varphi_m(x)\varphi_n(y)\,\mathrm{d}x\,\mathrm{d}y$$

where φ_j are the single particle wavefunctions. Notice that this object contains information about both potentials, namely the confinement (one-body) and the interaction (two-body) one. The information of the former is given by the use of the single particles wavefunctions φ_j, while the latter appears explicitly into the expression.

It is easy to verify that the coefficients V_{klmn} obey the following symmetry

$$V_{klmn} = V_{mnkl}^*$$

Moreover, with our choice of $u(x)$, we have that the φ_j are real functions for all j values, therefore $V_{klmn} = V_{mnkl}$.

Remark. An explicit expression of V_{klmn} is cumbersome to compute: indeed, even for a simple step potential computations are long and there is little hope to find a closed and explicit formula for V_{klmn}. For instance, in dimension $d = 3$ each index is a 3-dimensional multiindex, and the interaction coefficient turns out to depend on $3 \times 4 = 12$ indices:

$$V_{klmn} = V_{\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}\begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}}$$

For potentials in Hardy class we have the following proposition, giving a bound for the value of the interaction coefficients (as long as the indices are bounded):

Proposition 1.4. Let $|k|, |l|, |m|, |n|$ be $< \Lambda$. The interaction coefficients V_{klmn} obey the following inequality

$$|V_{klmn}| \leq C_V (1 + d(2\Lambda + 1))^{\frac{1}{2}}$$

Proof. Starting from the definition of V_{klmn} we can write

$$\begin{aligned}|V_{klmn}| &\leq \int_{\mathbb{R}^{2d}} |\phi_k(y)||\phi_l(y)||\phi_m(x)||\phi_n(x)||V(x-y)|\,\mathrm{d}x\,\mathrm{d}y \leq \\ &\leq \int_{\mathbb{R}^d} |\phi_m(x)||\phi_n(x)| \left(\int_{\mathbb{R}^d} |\phi_k(y)||\phi_l(y)||V(x-y)|\,\mathrm{d}y\right)\mathrm{d}x\end{aligned}$$

Now, from Hölder inequality with $p = 2$ we have

$$\begin{aligned}\int_{\mathbb{R}^d} |V(x-y)\phi_k(y)||\phi_l(y)|\,\mathrm{d}y &\leq \sqrt{\int_{\mathbb{R}^d} |\phi_l(y)|^2\,\mathrm{d}y}\sqrt{\int_{\mathbb{R}^d} |V(x-y)|^2|\phi_k(y)|^2\,\mathrm{d}y} = \\ &= \|\phi_l\|_{L^2(\mathbb{R}^d)}\sqrt{\int_{\mathbb{R}^d} |V(x-y)|^2|\phi_k(y)|^2\,\mathrm{d}y}\end{aligned}$$

Recall that $\|\phi_l\|_{L^2(\mathbb{R}^d)} = 1$ therefore

$$|V_{klmn}| \leq \int_{\mathbb{R}^d} |\phi_m(x)||\phi_n(x)|\sqrt{\int_{\mathbb{R}^d} |V(x-y)|^2|\phi_k(y)|^2\,\mathrm{d}y}\,\mathrm{d}x$$

Then we can use the Hardy constant of the potential to estimate

$$\begin{aligned}\int_{\mathbb{R}^d} |\phi_m(x)||\phi_n(x)|\sqrt{\int_{\mathbb{R}^d} |V(x-y)|^2|\phi_k(y)|^2\,\mathrm{d}y}\,\mathrm{d}x &\leq \int_{\mathbb{R}^d} |\phi_m(x)||\phi_n(x)|C_V(1+\|\nabla\phi_k\|^2_{L^2(\mathbb{R}^d)})^{\frac{1}{2}}\,\mathrm{d}x \\ &\leq \|\phi_m\|_{L^2(\mathbb{R}^d)}\,\|\phi_n\|_{L^2(\mathbb{R}^d)}\,C_V(1+\|\nabla\phi_k\|^2_{L^2(\mathbb{R}^d)})^{\frac{1}{2}} \\ &= C_V(1+\|\nabla\phi_k\|^2_{L^2(\mathbb{R}^d)})^{\frac{1}{2}}\end{aligned}$$

We shall now estimate the gradient. The eigenvalue equation for the d-dimensional harmonic oscillator is

$$\left(-\frac{1}{2}\nabla^2 + \frac{1}{2}|x|^2\right)\phi_k = \varepsilon_k\phi_k \text{ where } \varepsilon_k = \sum_{i=1}^{d} k_i + \frac{d}{2}$$

We multiply both sides by ϕ_k, and then we take the integral obtaining

$$-\frac{1}{2}\int_{\mathbb{R}^d} (\nabla^2\phi_k)\phi_k\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d} |x|^2\phi_k^2\,\mathrm{d}x = \varepsilon_k\int_{\mathbb{R}^d}\phi_k^2\,\mathrm{d}x$$

Using the fact that $\int_{\mathbb{R}^d}\phi_k^2\,\mathrm{d}x = \|\phi_k\|_{L^2(\mathbb{R}^d)} = 1$ and integrating by parts the first term (boundary terms are vanishing) we get

$$\frac{1}{2}\int_{\mathbb{R}^d}(\nabla\phi_k)^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d}|x|^2\phi_k^2\,\mathrm{d}x = \varepsilon_k \Longrightarrow \frac{1}{2}\int_{\mathbb{R}^d}(\nabla\phi_k)^2\,\mathrm{d}x \leq \sum_{i=1}^{d} k_i + \frac{d}{2}$$

Since $|k| = \max_i k_i < \Lambda$ the estimate is

$$\|\nabla\phi_k\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}(\nabla\phi_k)^2\,\mathrm{d}x \leq 2d\Lambda + d$$

This yields to the result

$$|V_{klmn}| \leq C_V(1 + d(2\Lambda+1))^{\frac{1}{2}}$$

□

Notice that the bounding term is not depending on indexes, and therefore for the symmetrized interaction coefficients the same estimate holds:

$$|V_{(kl)mn}| = \frac{1}{2}|V_{klmn} + V_{lkmn}| \leq \frac{1}{2}\left(|V_{klmn}| + |V_{lkmn}|\right) \leq C_V \left(1 + d(2\Lambda + 1)\right)^{\frac{1}{2}}$$

For brevity sake we call the bounding term $\mathcal{B}_\Lambda$, and so

$$|V_{(kl)mn}| \leq \mathcal{B}_\Lambda$$

CHAPTER 2

Coherent states

Overview

Coherent states are usually introduced in two different ways:

- As eigenstates of the annihilation operator;
- As states that minimize the uncertainty in Heisenberg uncertainty principle.

The second property is the way in which coherent states were initially introduced by E. Schrödinger in 1926, i.e. as a "minimum uncertainty Gaussian wavepacket" [48], while a description in terms of the annihilation operator is more recent (1960) and due to R. Klauder [31]. Decades later, in 1963, R. J. Glauber in the attempt of explaining a problem related to the physics of interferometers[1] provided a fully quantum-mechanical description of the electromagnetic field in terms of coherence [26]. Two years later, F. T. Arecchi experimentally verified that a single-mode laser is in a coherent state with an unknown phase [6].

Despite being of great theoretical and mathematical interest, coherent states have a wide success also in applied fields. Indeed, the whole quantum optics and laser science field relies heavily on the concept of coherent state.

Our aim in this chapter is to define coherent states in a general fashion, following [17], by introducing a Lie group (the Weyl-Heisenberg group) and defining a translation operator acting on it. Then, a coherent state will be given by the application of such operator on any function in Schwartz space. By choosing as generating function the ground state of the harmonic oscillator we will get the so-called canonical coherent states: then, we will show that our definition fulfills the two properties requested above.

The expression of coherent states that we will use in the following chapters to compute coherent expectation values of operators will be of the form

$$\phi_\eta(z) = e^{z\cdot\eta - \frac{|\eta|^2}{2}}$$

We will provide an explanation on how it is possible to achieve the above shape starting from the canonical coherent states in the so-called Bargmann-Fock representation.

[1] More precisely the Hanbury Brown and Twiss (HBT) effect.

2.1 The Weyl-Heisenberg group

Historically, the definition of Weyl-Heisenberg group, which is motivated by Heisenberg commutation relations for position and momentum operators, goes back to H. Weyl's mathematical formulation of quantum kinematics in [54].

Let us consider the standard n-dimensional quantum mechanics operators in $L^2(\mathbb{R}^n)$

$$\mathsf{Q}_j = x_j \qquad \mathsf{P}_j = -i\hbar\frac{\partial}{\partial x_j}$$

where Q_j is the (j-th) position operator and P_j is the (j-th) momentum operator. Such operators are defined in the following domains

$$\operatorname{Dom}\mathsf{Q}_j = \left\{\psi \in L^2(\mathbb{R}^n) \text{ such that } x_j\psi(x) \in L^2(\mathbb{R}^n)\right\}$$

$$\operatorname{Dom}\mathsf{P}_j = \left\{\psi \in L^2(\mathbb{R}^n) \text{ such that } \frac{\partial\psi(x)}{\partial x_j} \in L^2(\mathbb{R}^n)\right\}$$

For brevity sake, we will denote as follows

$$\mathsf{Q} = (\mathsf{Q}_1, \dots, \mathsf{Q}_n) \qquad \mathsf{P} = (\mathsf{P}_1, \dots, \mathsf{P}_n)$$

The operators obey the *Heisenberg canonical commutation relation*

$$[\mathsf{Q}_i, \mathsf{P}_j] = -\delta_{ij} i\hbar$$

on the intersection of the domains, $\operatorname{Dom}\mathsf{Q} \cap \operatorname{Dom}\mathsf{P}$.

We can introduce the *Weyl-Heisenberg translation operator* as the following operator

$$\mathsf{T}(z) = \exp\left(\frac{i}{\hbar}(p \cdot \mathsf{Q} - q \cdot \mathsf{P})\right)$$

where $(q,p) \in \mathbb{R}^{2n}$ is a point on the phase space and the $\cdot$ denotes the linear combination $\sum_i p_i \mathsf{Q}_i$ (analogously for q and P). With some hypotesis on the Hilbert space[2] we have a product law for the above operators

$$\mathsf{T}(z_1)\mathsf{T}(z_2) = \exp\left(-\frac{i}{2\hbar}\sigma(z_1, z_2)\right)\mathsf{T}(z_1 + z_2) \tag{2.1}$$

where $z_1 = (q_1, p_1)$ and $z_2 = (q_2, p_2)$. Also, in the above formula $\sigma(\cdot,\cdot)$ denotes the *standard symplectic product*, defined as[3]

$$\sigma(\cdot,\cdot) : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R} \qquad \sigma(z_1, z_2) = z_1^{\mathsf{T}} \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix} z_2$$

Skew-symmetry of the symplectic product allows us to express Heisenberg commutation relation in

[2] In particular, Baker-Campbell-Haussdorff formula must hold, see [17] for further details.

[3] Even dimension is crucial to define the symplectic structure on a given vector space.

the integral form

$$\begin{aligned}
\mathsf{T}(z_1)\mathsf{T}(z_2) &= \exp\left(-\frac{i}{2\hbar}\sigma(z_1,z_2)\right)\mathsf{T}(z_1+z_2)\\
&= \exp\left(+\frac{i}{2\hbar}\sigma(z_2,z_1)\right)\mathsf{T}(z_2+z_1)\\
&= \exp\left(+\frac{i}{2\hbar}\sigma(z_2,z_1)+\frac{i}{\hbar}\sigma(z_2,z_1)-\frac{i}{\hbar}\sigma(z_2,z_1)\right)\mathsf{T}(z_2+z_1)\\
&= \exp\left(\frac{i}{\hbar}\sigma(z_2,z_1)\right)\mathsf{T}(z_2)\mathsf{T}(z_1)\\
&= \exp\left(-\frac{i}{\hbar}\sigma(z_1,z_2)\right)\mathsf{T}(z_2)\mathsf{T}(z_1)
\end{aligned}$$

We shall now focus on the reason why the operator is a translation one: indeed, the operator T satisfies

$$\mathsf{T}(z)\begin{pmatrix}\mathsf{Q}\\ \mathsf{P}\end{pmatrix}\mathsf{T}(z)^{-1} = \begin{pmatrix}\mathsf{Q}-q\\ \mathsf{P}-p\end{pmatrix} \tag{2.2}$$

To prove this relation, let us work component-wise: for the first row we have

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{it}{\hbar}(p\cdot\mathsf{Q}-q\cdot\mathsf{P})}\mathsf{Q}e^{-\frac{it}{\hbar}(p\cdot\mathsf{Q}-q\cdot\mathsf{P})}\right) = e^{\frac{it}{\hbar}(p\cdot\mathsf{Q}-q\cdot\mathsf{P})}[\mathsf{Q},p\cdot\mathsf{Q}-q\cdot\mathsf{P}]e^{-\frac{it}{\hbar}(p\cdot\mathsf{Q}-q\cdot\mathsf{P})} = -i\hbar q$$

and analogously for the second.

One-dimensional case

Let us put in dimension $d = 1$: let us define

$$\mathsf{e}_1 = -\frac{1}{2\hbar}\mathbf{1} \qquad \mathsf{e}_2 = -\frac{i}{\hbar}\mathsf{P} \qquad \mathsf{e}_3 = \frac{i}{\hbar}\mathsf{Q}$$

Easily one checks the following commutation relations:

$$[\mathsf{e}_2,\mathsf{e}_3] = 2\mathsf{e}_1 \qquad [\mathsf{e}_1,\mathsf{e}_2] = 0 \qquad [\mathsf{e}_1,\mathsf{e}_3] = 0$$

The operators e_j generate a 3-dimensional Lie algebra, the *Weyl-Heisenberg algebra*

$$\mathfrak{h}_1 = \mathrm{Span}\{\mathsf{e}_1,\mathsf{e}_2,\mathsf{e}_3\}$$

whose generic element is expressed using the coordinates (t,q,p)

$$\mathsf{W}_{(t,q,p)} = t\mathsf{e}_1 + q\mathsf{e}_2 + p\mathsf{e}_3$$

Lie bracket between two elements of the algebra is computed to be (using the shorthand $z = (q,p)$ and $z' = (q',p')$):

$$[\mathsf{W}_{(t,z)},\mathsf{W}_{(t',z')}] = 2\sigma(z,z')\mathsf{e}_1$$

Through the exponential mapping of the elements of $\mathfrak{h}_1$ we can generate the elements of the associated Lie group H(1): indeed, the exponential map $\exp : \mathfrak{h}_1 \to \mathrm{H}(1)$ maps the element of the algebra $\mathsf{W}_{(t,z)}$ into the the group element $G_{(t,z)}$

$$\exp : \mathfrak{h}_1 \to \mathrm{H}(1) \qquad \mathsf{W}_{(t,z)} \to \exp\left(\mathsf{W}_{(t,z)}\right) = G_{(t,z)}$$

Then, consider

$$\begin{aligned}\exp\left(\mathsf{W}_{(t,z)}\right)\exp\left(\mathsf{W}_{(t',z')}\right) &= \exp\left(\frac{1}{2}[\mathsf{W}_{(t,z)},\mathsf{W}_{(t',z')}] + \mathsf{W}_{(t,z)} + \mathsf{W}_{(t',z')}\right)\\ &= \exp\left(\mathsf{W}_{(t+t'+\sigma(z,z'),z+z')}\right)\end{aligned}$$

this expession allows us to deduce that the group H(1) is $\mathbb{R}^3$ endowed with the product

$$(t,z)\odot(t',z') = (t+t'+\sigma(z,z'), z+z')$$

Notice that:

- The identity is given by $(0,0,0)$, since

$$(0,0,0)\odot(t,q,p) = (t,q,p)\odot(0,0,0) = (t,q,p)$$

- Given any (t,q,p), the inverse element is $(-t,-q,-p)$

$$(t,q,p)\odot(-t,-q,-p) = \left(t-t-(q,p)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix}, q-q, p-p\right) = (0,0,0)$$

- The product $\odot$ is non-commutative.

In dimension n

We can easily generalize what seen in the $d=1$ case to a n dimensional case. We define the basis as

$$\mathsf{e}_1 = -\frac{1}{2\hbar}\mathbf{1} \qquad \mathsf{e}_j = -\frac{i}{\hbar}\mathsf{P}_j \text{ where } 1<j\leq n+1 \qquad \mathsf{e}_i = \frac{i}{\hbar}\mathsf{Q}_i \text{ where } n+1<i\leq 2n+1$$

The *n-th Weyl-Heisenberg Lie algebra* $\mathfrak{h}_n$ is a vector space of dimension $2n+1$ where each element can be written as

$$\mathsf{W}_{(t,q,p)} = t\mathsf{e}_1 + \sum_{j=2}^{n+1} q_j\mathsf{e}_j + \sum_{i=n+2}^{2n+1} p_i\mathsf{e}_i$$

Therefore, a coordinate system for the vector space is given by $(t,z)=(t,q,p)\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$. Lie bracket of two elements in $\mathfrak{h}_n$ is

$$[\mathsf{W}_{(t,q,p)},\mathsf{W'}_{(t,q,p)}] = 2\sigma(z,z')\mathsf{e}_1$$

By repeating the same steps done in the one-dimensional case, one finds that the *Weyl-Heisenberg group* H(n) is $\mathbb{R}^{2n+1}$ endowed with the following non-abelian product:

$$(t,z)\odot(t',z') = (t+t'+\sigma(z,z'), z+z')$$

We can define a unitary map $\rho:\mathrm{H}(n)\to L^2(\mathbb{R}^n)$ as

$$(t,z)\mapsto\rho(t,z) = e^{-\frac{it}{2\hbar}}\mathsf{T}(z)$$

This is the *Schrödinger representation*, which is a unitary representation[4] (i.e. a group homomorphism) of the Weyl-Heisenberg group over the (infinite dimensional) Hilbert space $L^2(\mathbb{R}^n)$.

[4] Let V be a vector space over the field K, and let $(G,\cdot)$. A *group representation* of G over the vector space V its the map

$$D: G\to \mathrm{GL}(V)$$

such that $D(g_1\cdot g_2) = D(g_1)D(g_2)$ for all $g_1,g_2\in G$, where $D(g)$ its an operator acting on the elements of V. The dimension of V, which can be also infinite, is called *dimension of the representation.*

If the representation map is injective, then the representation is said to be *faithful.*

A unitary representation is a group representation where the operators satisfy $D(g)^\dagger D(g) = D(g)D(g)^\dagger = \mathbf{1}$ for all $g\in G$.

Remark. Schrödinger representation is *irreducible*, meaning that it admits no invariant subspaces. The Stone-von Neumann theorem states that this is the unique irreducible representation of $\mathrm{H}(n)$ over a vector space, up to conjugation with an unitary operator (see for instance [25][28][45]).

Coherent states definition

We are almost ready to define what is a generic coherent state. Let $\psi \in L^2(\mathbb{R}^n)$: the Weil-Heisenberg translation operator acts on ψ as

$$(\mathsf{T}(z)\psi)(x) = e^{-\frac{i}{2\hbar}q\cdot p}e^{\frac{i}{\hbar}x\cdot p}\psi(x-q)$$

Similarly, let

$$\psi(k) = \frac{1}{(2\pi\hbar)^n}\int e^{-\frac{i}{\hbar}x\cdot k}\psi(x)\,\mathrm{d}x$$

be Fourier transform of $\psi(x)$: then

$$(\mathsf{T}(z)\psi)(k) = e^{\frac{i}{2\hbar}q\cdot p}e^{-\frac{i}{\hbar}q\cdot k}\psi(k-p)$$

Recall that the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^n)$ is given by

$$\mathcal{S}(\mathbb{R}^n) = \{f \in \mathcal{C}^\infty(\mathbb{R}^n) \mid \|f\|_{\alpha,\beta} < \infty \ \forall\ \alpha,\beta\}$$

where α,β are multiindices and

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n}|x^\alpha D^\beta[f(x)]| = \sup_{(x_1,\dots,x_n)\in\mathbb{R}^n}\left|x_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}\frac{\partial^{\beta_1+\dots+\beta_n}f(x)}{\partial x_1^{\beta_1}\dots x_n^{\beta_n}}\right|$$

is a semi-norm. Notice that $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$: moreover, since the $L^2(\mathbb{R}^n)$ basis $\{\varphi_j\}_j$ is entirely contained in $\mathcal{S}(\mathbb{R}^n)$, then $\mathcal{S}(\mathbb{R}^n)$ is densely contained in $L^2(\mathbb{R}^n)$.

Take any $\psi \in \mathcal{S}(\mathbb{R}^n)$: the *coherent state* associated to ψ and centered in the point $z=(q,p)\in\mathbb{R}^{2n}$ of the phase space is defined as

$$\psi_z(x) = (\mathsf{T}(z)\psi)(x)$$

The function ψ is a *generating function* of the coherent states. In the following we will specialize to the harmonic oscillator case and we will show that this definition of coherent state has all the desired features.

2.2 Canonical coherent states

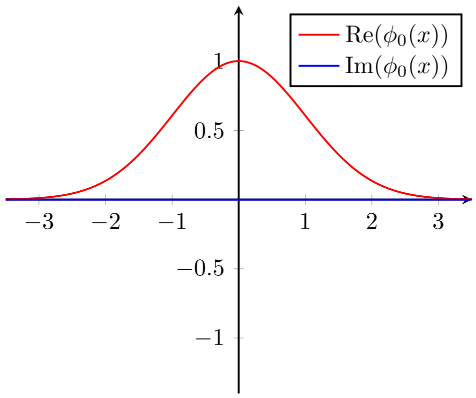
The case in which the generating function of the coherent states is the ground state of the harmonic oscillator, i.e. the standard Gaussian, is called the *canonical coherent states.* We take as reference state the n-dimensional Harmonic oscillator ground state

$$\phi_0(x) = (\pi\hbar)^{-\frac{n}{4}}\exp\left(-\frac{x^2}{2\hbar}\right)$$

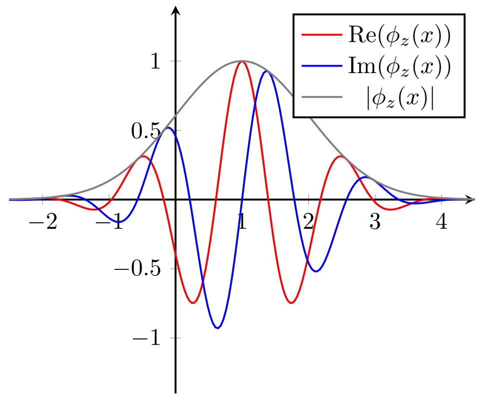
The coherent state associated to ϕ_0 (the canonical coherent state) has the form

$$\phi_z(x) = \left(\frac{1}{\pi\hbar}\right)^{-\frac{n}{4}}e^{-\frac{i}{2\hbar}q\cdot p}e^{\frac{i}{\hbar}x\cdot p}\exp\left(-\frac{(x-q)^2}{2\hbar}\right)$$

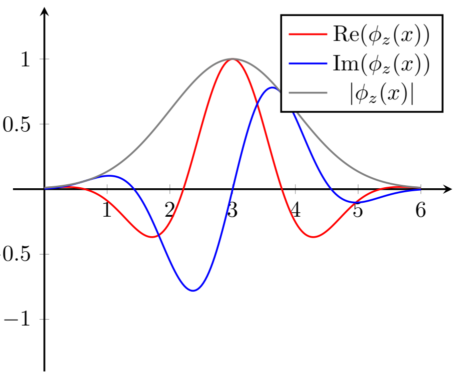
The coherent state ϕ_z is localized around the phase space point $z = (q, p)$: the localization has size $\sqrt{\hbar}$ in both the position and momentum directions[5]. The coherent state is in some sense the quantum analogue of the classical state of the system, represented by the point z in the phase space. In this regard, the action of the Weyl-Heisenberg group allows us to switch from the classical state to the quantum coherent state.



(a) $z = (0, 0)$



(b) $z = (1, 4)$



(c) $z = (3, 2)$

Figure 2.1: **(a).** Canonical coherent state centered in $z = (0, 0)$, corresponding to the ground state of the harmonic oscillator. **(b)** and **(c).** Canonical coherent states centered in two different points ($z = (1, 4)$ and $z = (3, 2)$ respectively). Notice how the modulus function is centered in the q value.

In the quantum harmonic oscillator is customary to define the *creation* and *destruction operators* as

$$\mathsf{a} = \frac{1}{\sqrt{2\hbar}}(\mathsf{Q} + i\mathsf{P}) \qquad \mathsf{a}^\dagger = \frac{1}{\sqrt{2\hbar}}(\mathsf{Q} - i\mathsf{P})$$

where both operators are defined on $\mathrm{Dom}(\mathsf{Q}) \cap \mathrm{Dom}(\mathsf{P})$. The operators obeys the following commu-

[5] In the sense that $\sqrt{\hbar}$ is the variance of the Gaussian, and so 99.7% of the coherent state is contained in the $x = q \pm 3\sqrt{\hbar}$ region.

tation relation, inherited by the non-commutativity of the position and momentum operators:

$$[\mathsf{a}_j, \mathsf{a}_k^\dagger] = \delta_{jk}$$

The Hamiltonian of the n dimensional harmonic oscillator can be written in terms of the ladder operators:

$$\mathsf{H} = \frac{1}{2}(\mathsf{P}^2 + \mathsf{Q}^2) = \hbar \sum_{j=1}^{n} \left(\mathsf{a}_j^\dagger \mathsf{a}_j + \frac{n}{2} \right)$$

It is easy to check that the ground state ϕ_0 is the eigenstate of the annihilation operator associated to the eigenvalue 0:

$$\mathsf{a}\phi_0 = \frac{(\pi\hbar)^{-\frac{n}{4}}}{\sqrt{2\hbar}} \left[x + \hbar \frac{\partial}{\partial x} \right] e^{-\frac{x^2}{2\hbar}} = \frac{(\pi\hbar)^{-\frac{n}{4}}}{\sqrt{2\hbar}} \left[x e^{-\frac{x^2}{2\hbar}} + \hbar e^{-\frac{x^2}{2\hbar}} \frac{x}{\hbar} \right] = 0$$

We can therefore ask ourselves if also the other coherent states we have defined are eigenstates of the annihilation operator: the answer is affirmative, as we shall see immediately.

Let us define the following complex number

$$\alpha_z = \frac{1}{\sqrt{2\hbar}}(q + ip) \tag{2.3}$$

then the following holds

$$\begin{aligned} \mathsf{T}(z)\mathsf{a}\mathsf{T}(z)^{-1} &= \frac{1}{\sqrt{2\hbar}} \mathsf{T}(z)(\mathsf{Q} + i\mathsf{P})\mathsf{T}(z)^{-1} \\ &= \frac{1}{\sqrt{2\hbar}}(\mathsf{Q} - q + i\mathsf{P} - ip) = \mathsf{a} - \alpha_z \end{aligned}$$

where equation (2.2) was used. Combining this equation together to $\phi_0 = \mathsf{T}^{-1}(z)\phi_z$ gives

$$(\mathsf{a} - \alpha_z)\phi_z = \mathsf{T}(z)\mathsf{a}\mathsf{T}(z)^{-1}\phi_z = \mathsf{T}(z)\mathsf{a}\phi_0 = 0$$

which gives us the desired result: every coherent state is eigenstate of the annihilation operator, and the corresponding eigenvalue is obtained mapping the point in which the coherent state is centered (q, p) into α_z using the map defined in equation (2.3):

$$\mathsf{a}\phi_z = \alpha_z \phi_z$$

We have, using BCH formula

$$\mathsf{T}(z) = \exp\left(\alpha \cdot \mathsf{a}^\dagger - \alpha^* \cdot \mathsf{a}\right) = \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(\alpha \cdot \mathsf{a}^\dagger) \exp\left(-\alpha^* \cdot \mathsf{a}\right) \tag{2.4}$$

therefore, we can use the above equation to write the z coherent state as[6]

$$\phi_z = \exp\left(-\frac{|\alpha|^2}{2} \right) \exp(\alpha \cdot \mathsf{a}^\dagger)\phi_0 \tag{2.5}$$

Coherent states are not orthogonal to each other: indeed, it is useful to compute the scalar product between any of them: using the $L^2(\mathbb{R}^n)$ product we have

$$\langle \phi_z, \phi_{z'} \rangle = \exp\left(i \frac{\sigma(z, z')}{2\hbar} \right) \exp\left(-\frac{|z - z'|^2}{4\hbar} \right) \tag{2.6}$$

[6] Recall that $\mathsf{a}\phi_0 = 0$ and so $\exp\left(-\alpha^* \cdot \mathsf{a}\right)\phi_0 = \exp\left(-\alpha^* 0\right)\phi_0 = \phi_0$.

We shall now proof equation (2.6). We have, using equation (2.4)

$$\langle \phi_0, \mathsf{T}(z)\phi_0 \rangle = e^{-\frac{|\alpha|^2}{2}} \langle \phi_0, e^{\alpha\cdot \mathsf{a}^\dagger} e^{-\alpha^*\cdot \mathsf{a}}\phi_0 \rangle = e^{-\frac{|z|^2}{4\hbar}} \left\| e^{-\alpha^*\cdot \mathsf{a}}\phi_0 \right\|^2_{L^2(\mathbb{R}^n)} \tag{2.7}$$

But as shown above $\mathsf{a}\phi_0 = 0$ and therefore we have $\left\| e^{-\alpha^*\cdot \mathsf{a}}\phi_0 \right\|^2_{L^2(\mathbb{R}^n)} = \|\phi_0\|^2_{L^2(\mathbb{R}^n)} = 1$

Moreover, using (2.1) we have

$$\mathsf{T}(z)\phi_{z'} = \mathsf{T}(z)\mathsf{T}(z')\phi_0 = \exp\left(-\frac{i}{2\hbar}\sigma(z,z')\right)\mathsf{T}(z+z')\phi_0 = \exp\left(-\frac{i}{2\hbar}\sigma(z,z')\right)\phi_{z+z'}$$

Using this results we can write the formula for the overlapping of two generic coherent states as

$$\langle \phi_z, \phi_{z'} \rangle = \langle \mathsf{T}(z)\phi_0, \mathsf{T}(z')\phi_0 \rangle = \exp\left(\frac{i}{2\hbar}\sigma(z,z')\right)\langle \phi_0, \mathsf{T}(z'-z)\phi_0 \rangle$$

and using (2.7) we can recover the product formula for two coherent states (equation (2.6)).

In dimension 1 the k-th eigenstate of the harmonic oscillator is obtained applying k times the creation operator to the ground state ϕ_0:

$$\varphi_k = \frac{1}{\sqrt{k!}}(\mathsf{a}^\dagger)^k\phi_0$$

The multiplying constant is added to have $\|\varphi_k\|_{L^2(\mathbb{R})} = 1$.

Then, expanding the exponential in the formula (2.5) we get the following well-known identity

$$\phi_z = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\cdot \mathsf{a}^\dagger}\phi_0 = e^{-\frac{|\alpha|^2}{2}}\left(1 + \alpha\cdot \mathsf{a}^\dagger + \frac{1}{2}(\alpha\cdot \mathsf{a}^\dagger)^2 + \dots\right)\phi_0 = e^{-\frac{|\alpha|^2}{2}}\sum_{k=0}^{\infty}\frac{\alpha^k}{\sqrt{k!}}\varphi_k \tag{2.8}$$

Equation (2.8) relates the coherent state centered in an arbitrary point to the eigenfunctions of the harmonic oscillator $\{\varphi_k\}_k$.

This relation can be generalized to arbitrary dimension: let $k = (k_1, \dots, k_n) \in \mathbb{N}^n$ be a multiindex, then the k-th eigenstate of the harmonic oscillator is

$$\varphi_k(x) = \varphi_{k_1}(x_1)\dots\varphi_{k_n}(x_n) = \prod_{j=1}^{n}\varphi_{k_j}(x_j)$$

Then, using the fact that each φ_{k_j} can be obtained by applying the creation operator to the ground state we get

$$\varphi_k = \prod_{j=1}^{n}\frac{(\mathsf{a}_j^\dagger)^{k_j}}{\sqrt{k_j!}}\phi_0$$

and this gives us

$$\phi_z = \exp\left(-\frac{|z|^2}{4\hbar}\right)\sum_{k_j=1}^{\infty}\frac{\alpha^k}{k!}\varphi_k$$

where for multiindices we have

$$\alpha^k = \alpha_1^{k_1}\dots\alpha_n^{k_n} \qquad k! = k_1!\dots k_n!$$

Coherent states dynamics

Remarkably, dynamics of the Harmonic oscillator preserves coherent states, namely the time evolution of a coherent state under the flow of the harmonic oscillator results in another coherent state with different phase. The ODE describing the classical harmonic oscillator $\ddot{q}+q=0$ is equivalent to the following first order system

$$\begin{cases} \dot{q}=p \\ \dot{p}=-q \end{cases} \tag{2.9}$$

Let $z_0=(q_0,p_0)$ be the initial datum in the phase space. Then, it is easy to show that the flow of equation (2.9) (i.e. the time evolution of the point under the dynamics of the system) is given by

$$z(t)=R_t z_0 \text{ where } R_t=\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

We can now switch to a quantum-mechanical treatment of the system: we will see how much the classical behaviour of the system will appear in the (quantum) coherent states dynamics.

Let us recall that in the Heisenberg picture time evolution of a generic operator X at time t is given by $\mathsf{X}(t)=\mathsf{U}(t)\mathsf{X}\mathsf{U}^\dagger(t)$ where $\mathsf{U}(t)=e^{-\frac{it}{\hbar}\mathsf{H}}$ is the unitary time evolution operator. Therefore, in quantum harmonic oscillator case time evolution of the operators is given by

$$\begin{pmatrix} \mathsf{Q}(t) \\ \mathsf{P}(t) \end{pmatrix}=e^{-\frac{it}{\hbar}\mathsf{H}}\begin{pmatrix} \mathsf{Q} \\ \mathsf{P} \end{pmatrix}e^{+\frac{it}{\hbar}\mathsf{H}}$$

It is easy to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \mathsf{Q}(t) \\ \mathsf{P}(t) \end{pmatrix}=\begin{pmatrix} -\mathsf{P}(t) \\ \mathsf{Q}(t) \end{pmatrix}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \mathsf{Q} \\ \mathsf{P} \end{pmatrix}$$

and therefore, the solution is given by

$$\begin{pmatrix} \mathsf{Q}(t) \\ \mathsf{P}(t) \end{pmatrix}=\exp\begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}\begin{pmatrix} \mathsf{Q} \\ \mathsf{P} \end{pmatrix}=R_{-t}\begin{pmatrix} \mathsf{Q} \\ \mathsf{P} \end{pmatrix}$$

Now, let us compute the time evolution of a coherent state centered (initially) in z_0:

$$\mathsf{U}(t)\phi_z=\mathsf{U}(t)\mathsf{T}(z_0)\mathsf{U}^\dagger(t)\mathsf{U}(t)\phi_0=\mathsf{T}(z_t)\mathsf{U}(t)\phi_0=\mathsf{T}(z_t)e^{\frac{-it}{\hbar}\mathsf{H}}\phi_0=e^{-it\frac{n}{2}}\phi_{z_t}$$

We found that a coherent state evolves into another coherent state, centered at a point which is the evolution of the original point under the classical flow of the system.

$$\phi_z(t)=e^{-it\frac{n}{2}}\phi_{z_t}$$

2.3 Bargmann-Fock representation

Bargmann-Fock representation is a possible representation of the coherent states which is well adaptable to the creation and annihilation operators of the harmonic oscillator. From now on we set $\hbar=1$. We begin by defining a new transform:

Let $u\in L^2(\mathbb{R}^n)$, $\|u\|_{L^2(\mathbb{R}^n)}=1$. Given a $\psi\in L^2(\mathbb{R}^n)$ we define its *Fourier-Bargmann transform* as

$$\mathcal{F}_u^{\mathcal{B}}\psi(z)=\langle u_z,\psi\rangle$$

If $u=\phi_0$, then $\mathcal{F}_u^{\mathcal{B}}(z)$ is called *standard Fourier-Bargmann transform* and it is denoted simply as $\mathcal{F}^{\mathcal{B}}\psi(z)\equiv\psi^\sharp(z)$.

We map a generic phase space point into the following complex variable:

$$(q, p) \mapsto \bar{z} = \frac{q - ip}{\sqrt{2}}$$

Before defining Bargmann-Fock space (following [25, 42]) we recall the notion of anti-analytic function: let us consider a function $f : \mathbb{C}^\ell \to \mathbb{C}$, given by $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$. We say that f is *anti-analytic* if $\frac{\partial f}{\partial z} = 0$. Since

$$\frac{\partial \bar{f}}{\partial \bar{z}} = \overline{\frac{\partial f}{\partial z}}$$

it follows that f is anti-analytic if and only if $\bar{f}$ is analytic. An anti-analytic function obeys the following modified Cauchy Riemann equations

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

Let $\mathcal{A}(\mathbb{C}^n)$ be the set of the anti-analytic functions $f : \mathbb{C}^n \to \mathbb{C}$. The *Bargmann-Fock space* is defined as

$$\mathcal{F}_B(\mathbb{C}^n) = \left\{ f \in \mathcal{A}(\mathbb{C}^n) \mid \int |f(\bar{z})|^2 e^{-|z|^2} \, \mathrm{d}z \wedge \mathrm{d}\bar{z} < +\infty \right\} \tag{2.10}$$

where the measure is

$$\mathrm{d}z \wedge \mathrm{d}\bar{z} = \pi^{-n} \, \mathrm{d}x \, \mathrm{d}y, \;\; z = x + iy$$

Such space is equipped with a scalar product that will be denoted as $\langle \cdot, \cdot \rangle$ and that is defined as

$$\langle f, g \rangle = \int f^*(\bar{z}) g(\bar{z}) e^{-|z|^2} \, \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

Remark. The use of antiholomorphic functions instead of holomorphic ones might seem strange: however, for historical reasons the creation operator is $\mathsf{a}^\dagger$ (one could equally develope a theory in which c is a creation operator and its adjoint $\mathsf{c}^\dagger$ is the annihilation operator). It is however possible to find equivalent definitions of Bargmann-Fock space that use holomorphic functions, for instance in [56].

Let us define the *Bargmann transform* as the following map from $L^2(\mathbb{R}^n)$ to Fock-Bargmann space $\mathcal{F}_B(\mathbb{C}^n)$

$$\psi \mapsto \mathcal{B}\psi = \psi^\sharp e^{\frac{p^2+q^2}{4}}$$

By redefining the Fourier-Bargmann transform[7], it is possible to get Bargmann transform an isometry from $L^2(\mathbb{R}^n)$ to Fock-Bargmann space $\mathcal{F}_B(\mathbb{C}^n)$.

It is not immediate to see that the Bargmann transform is anti-analytic (i.e. it carries no dependence on z), however we can get an explicit formula for the Bargmann transform: we have that

$$\langle \phi_z, \psi \rangle = e^{\frac{i}{2\hbar} p \cdot q} \int_{\mathbb{R}^n} \psi(x) (\phi_0(x - q))^* e^{-\frac{i}{\hbar} x \cdot p} \, \mathrm{d}x = e^{\frac{i}{2\hbar} p \cdot q} \int_{\mathbb{R}^n} \psi(x) (\phi_0(x - q))^* e^{-\frac{i}{\hbar} x \cdot p} \, \mathrm{d}x$$

and therefore

$$\mathcal{B}\psi(\bar{z}) = (\pi)^{-\frac{n}{4}} \int_{\mathbb{R}^n} \psi(x) \exp\left(-\left(\frac{x^2}{2} - \sqrt{2} x \cdot \bar{z} + \frac{\bar{z}^2}{2} \right) \right) \mathrm{d}x \tag{2.11}$$

[7] By choosing to put $(2\pi\hbar)^{\frac{n}{2}}$ instead of 1 as coefficient.

Using such representation is easy to see that $\mathcal{B}\psi(\bar{z})$ is an anti-holomorphic function since

$$\frac{\partial \mathcal{B}\psi}{\partial z} = 0$$

By introducing the *Bargmann kernel* as

$$\widehat{\mathcal{B}}(\bar{z}, x) = \left(\frac{1}{\pi}\right)^{\frac{n}{4}} \exp\left(-\left(\frac{x^2}{2} - \sqrt{2}x \cdot \bar{z} + \frac{\bar{z}^2}{2}\right)\right)$$

the Bargmann transform is simply given by the convolution of the kernel with the function:

$$\mathcal{B}[\psi](\bar{z}) = \int_{\mathbb{R}^n} \psi(x)\widehat{\mathcal{B}}(\bar{z}, x)\,\mathrm{d}x = \psi * \widehat{\mathcal{B}}(\bar{z}, \cdot)$$

Notice that Bargmann transform is linear thanks to the linearity of the integral, namely

$$\begin{aligned} \mathcal{B}[\lambda_1\psi_1 + \lambda_2\psi_2](\bar{z}) &= \int_{\mathbb{R}^n} (\lambda_1\psi_1(x) + \lambda_2\psi_2(x))\widehat{\mathcal{B}}(\bar{z}, x)\,\mathrm{d}x \\ &= \lambda_1 \int_{\mathbb{R}^n} \psi_1(x)\widehat{\mathcal{B}}(\bar{z}, x)\,\mathrm{d}x + \lambda_2 \int_{\mathbb{R}^n} \psi_2(x)\widehat{\mathcal{B}}(\bar{z}, x)\,\mathrm{d}x \\ &= \lambda_1\mathcal{B}[\psi_1](\bar{z}) + \lambda_2\mathcal{B}[\psi_2](\bar{z}) \end{aligned} \tag{2.12}$$

It is interesting to compute the Bargmann representation of the Harmonic oscillator. This representation will be simpler compared to the standard Schrödinger representation of the harmonic oscillator, however Stone-von Neumann theorem ensures that they are unitarily equivalent (see [56]).

We begin by stating the following identities (see [17])

$$\int_{\mathbb{R}^n} \frac{\partial\psi(x)}{\partial x}\widehat{\mathcal{B}}(\bar{z}, x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \psi(x)\left(\frac{x}{\hbar} - \frac{\sqrt{2}}{\hbar}\bar{z}\right)\widehat{\mathcal{B}}(\bar{z}, x)\,\mathrm{d}x$$

$$\partial_{\bar{z}} \int_{\mathbb{R}^n} \psi(x)\widehat{\mathcal{B}}(\bar{z}, x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \psi(x)\left(\frac{\sqrt{2}x}{\hbar} - \frac{\bar{z}}{\hbar}\right)\widehat{\mathcal{B}}(\bar{z}, x)\,\mathrm{d}x$$

which can be rewritten as

$$\mathcal{B}[\partial_x\psi(x)](\bar{z}) = \frac{1}{\hbar}\mathcal{B}[\psi(x)x](\bar{z}) - \frac{\sqrt{2}}{\hbar}\bar{z}\mathcal{B}[\psi(x)](\bar{z}) \tag{2.13}$$

$$\frac{\partial}{\partial\bar{z}}\mathcal{B}[\psi(x)](\bar{z}) = \frac{\sqrt{2}}{\hbar}\mathcal{B}[\psi(x)x](\bar{z}) - \frac{\bar{z}}{\hbar}\mathcal{B}[\psi(x)](\bar{z}) \tag{2.14}$$

By substituting equation (2.13) into equation (2.14) we get

$$\begin{aligned} \mathcal{B}[\partial_x\psi] &= -\frac{\sqrt{2}}{\hbar}\bar{z}\mathcal{B}[\psi] + \frac{1}{\sqrt{2}}\frac{\partial}{\partial\bar{z}}\mathcal{B}[\psi] + \frac{\sqrt{\bar{z}}}{\sqrt{2}\hbar}\mathcal{B}[\psi] \\ &= \frac{1}{\sqrt{2}}\left(\partial_{\bar{z}} - \frac{\bar{z}}{\hbar}\right)\mathcal{B}[\psi] \end{aligned} \tag{2.15}$$

and substituting equation (2.15) into equation (2.13) we get

$$\begin{aligned} \frac{1}{\hbar}\mathcal{B}[x\psi] &= \mathcal{B}[\partial_x\psi] + \frac{\sqrt{2}}{\hbar}\bar{z}\mathcal{B}[\psi] \\ &= \frac{1}{\sqrt{2}}\left(\partial_{\bar{z}} - \frac{\bar{z}}{\hbar}\right)\mathcal{B}[\psi] + \frac{\sqrt{2}}{\hbar}\bar{z}\mathcal{B}[\psi] \\ &= \frac{1}{\sqrt{2}}\left(\partial_{\bar{z}} + \frac{\bar{z}}{\hbar}\right)\mathcal{B}[\psi] \end{aligned}$$

Therefore, using the linearity property (2.12) we can get the Bargmann representation of position and momentum operators

$$\mathcal{B}(x\psi)(\bar{z}) = \frac{1}{\sqrt{2}}(\hbar\partial_{\bar{z}} + \bar{z})\mathcal{B}(\psi)(\bar{z})$$
$$\mathcal{B}(\hbar\partial_x\psi)(\bar{z}) = \frac{1}{\sqrt{2}}(\hbar\partial_{\bar{z}} - \bar{z})\mathcal{B}(\psi)(\bar{z})$$

which can be combined to get the Bargmann representation for the ladder operators

$$\mathcal{B}[\mathsf{a}^\dagger\psi](\bar{z}) = \bar{z}\mathcal{B}[\psi](\bar{z}) \qquad \mathcal{B}[\mathsf{a}\psi](\bar{z}) = \frac{\partial}{\partial\bar{z}}(\mathcal{B}[\psi](\bar{z}))$$

Interestingly, this is the way in which V. Fock himself introduced the creation and annihilation operators in [24]. CCR are preserved by such representation of the ladder operators:

$$[\mathsf{a}_k, \mathsf{a}^\dagger_{k'}]f(\bar{z}) = \mathsf{a}_k\mathsf{a}^\dagger_{k'}f(\bar{z}) - \mathsf{a}^\dagger_{k'}\mathsf{a}_k f(\bar{z}) = \frac{\partial}{\partial\bar{z}_k}\left(\bar{z}_{k'}f(\bar{z})\right) - \bar{z}_{k'}\frac{\partial}{\partial\bar{z}_k}f(\bar{z}) = \delta_{kk'}f(\bar{z})$$
$$[\mathsf{a}_k, \mathsf{a}_{k'}]f(\bar{z}) = \mathsf{a}_k\mathsf{a}_{k'}f(\bar{z}) - \mathsf{a}_{k'}\mathsf{a}_k f(\bar{z}) = \frac{\partial}{\partial\bar{z}_k}\frac{\partial}{\partial\bar{z}_{k'}}f(\bar{z}) - \frac{\partial}{\partial\bar{z}_{k'}}\frac{\partial}{\partial\bar{z}_k}f(\bar{z}) = 0$$
$$[\mathsf{a}^\dagger_k, \mathsf{a}^\dagger_{k'}]f(\bar{z}) = \mathsf{a}^\dagger_k\mathsf{a}^\dagger_{k'}f(\bar{z}) - \mathsf{a}^\dagger_{k'}\mathsf{a}^\dagger_k f(\bar{z}) = \bar{z}_k\bar{z}_{k'}f(\bar{z}) - \bar{z}_{k'}\bar{z}_k f(\bar{z}) = 0$$

Quantum harmonic oscillator has therefore the following Bargmann representation

$$\mathcal{B}[\mathsf{H}] = \hbar\bar{z}\cdot\frac{\partial}{\partial\bar{z}} + \frac{n\hbar}{2}$$

Remark. The basis functions $\{\varphi_j\}_j$ have a simple Bargmann representation

$$\mathcal{B}[\varphi_j](\bar{z}) = \left(\frac{1}{j!}\right)^{\frac{1}{2}}\bar{z}^j$$

where $j \in \mathbb{N}^n$ is a multiindex. Moreover $\{\mathcal{B}[\varphi_j](\bar{z})\}_j$ is an orthornormal basis of the Bargmann-Fock space $\mathcal{F}_B(\mathbb{C}^n)$ (see [17]).

Bargmann coherent states

Finally, we can proceed in computing the Bargmann representation of the canonical coherent states, which we will employ in the upcoming sections. We recall the general expression of the coherent state centered at the point $Z = (x, \xi)$ is

$$\phi_Z(y) = \left(\frac{1}{\pi}\right)^{\frac{n}{4}}\exp\left(-\frac{i}{2}x\cdot\xi\right)\exp\left(i\xi\cdot y\right)\exp\left(-\frac{(y-x)^2}{2}\right)$$

and therefore the Bargmann transform can be computed using equation (2.11), namely

$$\begin{aligned}
\mathcal{B}\phi_Z(\bar{z}) &= \left(\frac{1}{\pi}\right)^{\frac{n}{4}}\int_{\mathbb{R}^n}\phi_Z(y)\exp\left(-\frac{y^2}{2} + \sqrt{2}y\cdot\bar{z} - \frac{\bar{z}^2}{2}\right)\mathrm{d}y \\
&= \left(\frac{1}{\pi}\right)^{\frac{n}{2}}\int_{\mathbb{R}^n}\exp\left(-\frac{i}{2}x\cdot\xi\right)\exp\left(i\xi\cdot y\right)\exp\left(-\frac{(y-x)^2}{2}\right)\exp\left(-\frac{y^2}{2} + \sqrt{2}y\cdot\bar{z} - \frac{\bar{z}^2}{2}\right)\mathrm{d}y \\
&= \left(\frac{1}{\pi}\right)^{\frac{n}{2}}\exp\left(-\frac{\bar{z}^2}{2} - \frac{i}{2}x\cdot\xi - \frac{x^2}{2}\right)\int_{\mathbb{R}^n}\exp\left(-y^2 + y\cdot(\sqrt{2}\bar{z} + i\xi + x\right)\mathrm{d}y
\end{aligned}$$

Computing the integral one gets

$$\begin{aligned}\mathcal{B}\phi_Z(\bar{z}) &= \exp\left(\frac{1}{2}\left(\bar{z}^2 + \frac{x^2}{2} - \frac{\xi^2}{2} + i\xi\cdot x + \frac{2}{\sqrt{2}}x\cdot\bar{z} + \frac{2i}{\sqrt{2}}\xi\cdot\bar{z}\right)\right)\exp\left(-\frac{\bar{z}^2}{2} - \frac{i}{2}x\cdot\xi - \frac{x^2}{2}\right)\\ &= \exp\left(-\frac{x^2}{4} - \frac{\xi^2}{4} + \bar{z}\cdot\left(\frac{x+i\xi}{\sqrt{2}}\right)\right)\end{aligned}$$

By mapping the point Z to the complex number η as

$$Z \mapsto \bar{\eta} = \frac{x - i\xi}{\sqrt{2}}$$

we can finally write the Bargmann representation of the coherent state as

$$\mathcal{B}\phi_Z(\bar{z}) = e^{\bar{z}\cdot\bar{\eta} - \frac{|\eta|^2}{2}}$$

In a slight abuse of notation we will denote such coherent states simply as

$$\phi_\eta(\bar{z}) = e^{\bar{z}\cdot\eta - \frac{|\eta|^2}{2}}$$

and we will refer to them simply as *Bargmann coherent states*. Notice that this representation preserves all the important features that were present in the canonical coherent states, namely:

Eigenstate of annihilation operator. We showed before how on Fock-Bargmann space the Bargmann representation of the creation and annihilation operators is given by

$$\mathsf{a}_k(f) = \frac{\partial}{\partial\bar{z}_k}f(\bar{z}) \qquad \mathsf{a}_k^\dagger(f) = \bar{z}_k f(\bar{z})$$

It is easy to see that the Bargmann representation of coherent states is consistent with the fact that all the $\phi_\alpha(\bar{z})$ are the eigenstates of the annihilation operator

$$\mathsf{a}_k\phi_w(\bar{z}) = \frac{\partial}{\partial\bar{z}_k}\left(e^{w\cdot\bar{z} - \frac{1}{2}|w|^2}\right) = w_k\phi_w(\bar{z})$$

Product formula and normalization. Next, we can compute the product of two (possibly distinct) Bargmann coherent states

$$\begin{aligned}\langle\phi_\alpha,\phi_\beta\rangle &= \int \mathrm{d}z\wedge\mathrm{d}\bar{z}\ \overline{e^{\alpha\cdot\bar{z}}}e^{\beta\cdot\bar{z}}e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}e^{-|z|^2} =\\ &= \frac{1}{\pi^n}\int_{\mathbb{R}^{2n}}\mathrm{d}x\,\mathrm{d}y\ \overline{e^{(\alpha_x+i\alpha_y)\cdot(x-iy)}}e^{(\beta_x+i\beta_y)\cdot(x-iy)}e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}e^{-|x|^2-|y|^2} =\\ &= \frac{1}{\pi^n}\left(\int_{\mathbb{R}^n}\mathrm{d}x\ e^{(\alpha_x+\beta_x)x+i(\beta_y-\alpha_y)x}e^{-|x|^2}\right)\left(\int_{\mathbb{R}^n}\mathrm{d}y\ e^{(\alpha_y+\beta_y)y+i(\alpha_x-\beta_x)y}e^{-|y|^2}\right)e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)} =\\ &= \frac{1}{\pi^n}(\sqrt{\pi})^{2n}e^{\frac{1}{4}((\alpha_x+\beta_x)+i(\beta_y-\alpha_y))^2}e^{\frac{1}{4}((\alpha_y+\beta_y)+i(\alpha_x-\beta_x))^2}e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)} =\\ &= e^{\alpha_x\beta_x+\alpha_y\beta_y+i\alpha_x\beta_y-i\alpha_y\beta_x-\frac{1}{2}\alpha_x^2-\frac{1}{2}\alpha_y^2-\frac{1}{2}\beta_x^2-\frac{1}{2}\beta_y^2} =\\ &= e^{-\frac{1}{2}|\alpha-\beta|^2}e^{i\sigma(\alpha,\beta)}\end{aligned}$$

where

$$\sigma(\alpha,\beta) = \sigma((\alpha_x,\alpha_y),(\beta_x,\beta_y)) = \begin{pmatrix}\alpha_x\\ \alpha_y\end{pmatrix}\cdot\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\beta_x\\ \beta_y\end{pmatrix}$$

Notice that we recovered the same formula of the product of two canonical coherent states. In particular notice that the coherent states are normalized to 1 since

$$\langle\phi_\alpha,\phi_\alpha\rangle = e^{-\frac{1}{2}|\alpha-\alpha|^2}e^{i\sigma(\alpha,\alpha)} = 1$$

2.4 Minimal uncertainty

Another reason for which we can regard coherent states as "quantum states approximating classical behaviour" is that they have minimal uncertainty, namely satisfy Heisenberg uncertainty principle as an equality.

Heisenberg uncertainty principle[8] states that, given two self-adjoint operators A and B on an Hilbert space $\mathcal{H}$ the following inequality holds

$$\Delta_\psi \mathsf{A} \Delta_\psi \mathsf{B} \geq \frac{1}{2} \left| \langle [\mathsf{A},\mathsf{B}]\psi, \psi\rangle_{\mathcal{H}} \right|$$

for any $\psi \in \mathcal{H} \cap \mathrm{Dom}(\mathsf{A}) \cap \mathrm{Dom}(\mathsf{B})$ such that $\|\psi\|_{\mathcal{H}} = 1$, where

- $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is the scalar product of the Hilbert space;
- $\Delta_\psi \mathsf{A}$ is the *quantum variance*, defined as

  $$\Delta_\psi \mathsf{A} = \sqrt{\langle (\mathsf{A} - \langle \mathsf{A}\rangle_\psi \mathbf{1})^2 \rangle_\psi}$$

  where $\langle \mathsf{A}\rangle_\psi = \langle \psi, \mathsf{A}\psi\rangle_{\mathcal{H}}$ is the average;
- $[\mathsf{A},\mathsf{B}] = \mathsf{AB} - \mathsf{BA}$ is the operatorial commmutator;

The principle is general, in the sense that any couple of operators representing an observable can be put into the inequality. We can specialize to the case in which $\mathsf{A} = \mathsf{Q}$ and $\mathsf{B} = \mathsf{P}$: then, using the standard commutation relation we get

$$\Delta_\psi \mathsf{Q} \Delta_\psi \mathsf{P} \geq \frac{\hbar}{2}$$

where a normalized state $\|\psi\|_{\mathcal{H}} = 1$ was chosen.

Now we can see what happens if we choose ψ to be a canonical coherent state centered in a given point z: coherent mean values are computed as

$$\langle \phi_z, \mathsf{Q}\phi_z\rangle = \sqrt{\frac{\hbar}{2}} \langle \phi_z, (\mathsf{a} + \mathsf{a}^\dagger)\phi_z\rangle = \sqrt{\frac{\hbar}{2}} (\alpha_z + \alpha_z^*)$$

$$\langle \phi_z, \mathsf{P}\phi_z\rangle = \frac{1}{i}\sqrt{\frac{\hbar}{2}} \langle \phi_z, (\mathsf{a} - \mathsf{a}^\dagger)\phi_z\rangle = \sqrt{\frac{\hbar}{2}} \frac{\alpha_z - \alpha_z^*}{i}$$

Similarly, we can compute also the squared mean values:

$$\langle \phi_z, \mathsf{Q}^2\phi_z\rangle = \frac{\hbar}{2} \langle \phi_z, (\mathsf{a}^2 + (\mathsf{a}^\dagger)^2 + 2\mathsf{a}^\dagger\mathsf{a} + \mathbf{1})\phi_z\rangle = \frac{\hbar}{2}(\alpha_z^2 + (\alpha_z^*)^2 + 2|\alpha_z|^2 + 1)$$

$$\langle \phi_z, \mathsf{P}^2\phi_z\rangle = \frac{\hbar}{2} \langle \phi_z, \left(\frac{\mathsf{a} - \mathsf{a}^\dagger}{2}\right)^2 \phi_z\rangle = -\frac{\hbar}{2} \langle \phi_z, (\mathsf{a}^2 + (\mathsf{a}^\dagger)^2 - 2\mathsf{a}^\dagger\mathsf{a} - \mathbf{1}) = -\frac{\hbar}{2}(\alpha_z^2 + (\alpha_z^*)^2 - 2|\alpha_z|^2 - 1)$$

This allows us to compute the square of the quantum variances

$$(\Delta_{\phi_z}\mathsf{Q})^2 = \langle \mathsf{Q}^2\rangle - \langle \mathsf{Q}\rangle^2 = \frac{\hbar}{2}$$

$$(\Delta_{\phi_z}\mathsf{P})^2 = \langle \mathsf{P}^2\rangle - \langle \mathsf{P}\rangle^2 = \frac{\hbar}{2}$$

[8] The word "principle" survives for historical reasons, but this is actually a theorem about self-adjoint operators.

and therefore their product is

$$\Delta_{\phi_z}\mathsf{Q}\Delta_{\phi_z}\mathsf{P} = \frac{\hbar}{2}$$

As expected, Heisenberg uncertainty principle is satisfied as an equality by computing the averages and variances in canonical coherent states.

CHAPTER 3

Regularized theory

Overview

In this chapter the regularized theory is developed. To regularize the theory, a cutoff is chosen and just a finite number of terms in the evolution equations are considered. The regularized theory is studied in the Fock-Bargmann space on the coherent phase space $\mathbb{C}^\ell$, where ℓ is related to the value of the cutoff. The structure of this space is analyzed further in this chapter. Other tools are introduced, the most important being the Wick star product and the Wick symbol of an operator. In order to give a context for Wick star product, a review on general star products and their relation to quantization map choices is provided. Finally, the Gaussian thermal measure is introduced and the convergence in the $\beta \to +\infty$ limit is proven.

3.1 Canonical quantization and star products

In this section we will introduce the important concept of star product in the context of Wick quantization. In order to be able to understand the meaning of such operation we will briefly see which are the possibilities (at least, some among the many) to achieve the so-called canonical quantization of a system. We will follow mainly the review [14].

Canonical quantization and star products can be introduced in the framework of deformation quantization. In such a theory, non commutativity of operators in quantum mechanics is seen as a formal associative deformation of the pointwise product of the algebra of classical observables, which is given by $\mathcal{C}^\infty(M)$, the algebra of all complex-valued functions on a Poisson manifold M. The formal parameter of the deformation is an interpretation of Planck's constant $\hbar$: to give a better understanding in this section we will momentarily restore $\hbar$. Deformation quantization is a very universal method: indeed, the construction is possible for any Poisson manifold (see [32]).

Generally speaking, a quantization procedure consists in the choice of a map F that maps classical, commutative observables into quantum observables, represented by non-commutative operators on a Hilbert space[1]. Any quantization scheme should satisfy the following *classical limit condition*, meaning

[1] This is actually more complex than simply choosing a map. Indeed, one has to perform a pre-quantization procedure and then select a suitable subspace through the proper quantization map. Moreover, the procedure changes if it is performed on a vector space or on a general manifold: for a detailed description see chapters 22 and 23 of [28].

that for all classical observables f, g we have

$$\mathsf{F}(f)\mathsf{F}(g) = \mathsf{F}(fg) + o(\hbar) \qquad \mathsf{F}(f)\mathsf{F}(g) - \mathsf{F}(g)\mathsf{F}(f) = i\hbar\mathsf{F}(\{f,g\}) + o(\hbar^2)$$

where the symbol $\{\cdot,\cdot\}$ denotes the standard *Poisson bracket*, defined as the following bilinear map

$$\{\cdot,\cdot\} : \mathcal{C}^\infty \times \mathcal{C}^\infty \to \mathcal{C}^\infty \qquad f, g \mapsto \{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}$$

For our purposes it will be sufficient to choose f and g in the set of polynomials in given variables. We will denote the space of complex polynomials in the variables $x_1, \ldots, x_N$ as $\mathbb{C}[x_1, \ldots, x_N]$. Similarly, given N operators $\mathsf{A}_1, \ldots, \mathsf{A}_N$ we will denote the set of polynomial operators with complex coefficients as $\mathbb{C}[\mathsf{A}_1, \ldots, \mathsf{A}_N]$. Now we want to see which are the most common quantization schemes and what are their peculiarities.

Standard ordering

Consider the linear map $\rho_{\mathrm{s}} : \mathbb{C}[q,p] \to \mathbb{C}[\mathsf{Q},\mathsf{P}]$ acting as[2]

$$1 \mapsto \rho_{\mathrm{s}}(1) = \mathbf{1} \qquad q \mapsto \rho_{\mathrm{s}}(q) = \mathsf{Q} \qquad p \mapsto \rho_{\mathrm{s}}(p) = \mathsf{P} \qquad q^m p^n \mapsto \rho_{\mathrm{s}}(q^m p^n) = \mathsf{Q}^m\mathsf{P}^n$$

Now, let $f \in \mathbb{C}[q,p]$ and let ϕ be a smooth complex-valued function, $\phi \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C})$. Then it can be proved that the quantization of f assumes the following form

$$\rho_{\mathrm{s}}(f)\phi = \sum_{r=0}^{\infty} \frac{(-i\hbar)^r}{r!} \frac{\partial^r f}{\partial p^r}\bigg|_{p=0} \frac{\partial^r \phi}{\partial q^r}$$

Moreover, by defining the *standard star product* $\star_{\mathrm{s}}$ as

$$f \star_{\mathrm{s}} g = \rho_{\mathrm{s}}^{-1}(\rho_{\mathrm{s}}(f)\rho_{\mathrm{s}}(g)) = \sum_{r=0}^{\infty} \frac{(-i\hbar)^r}{r!} \frac{\partial^r f}{\partial p^r}\frac{\partial^r g}{\partial q^r}$$

it can be showed that this is a well-defined associative non-commutative product on the space $\mathbb{C}[q,p]$ obeying the classical limit, i.e.

$$f \star_{\mathrm{s}} g = fg - i\hbar\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} + o(\hbar^2)$$

Weyl ordering

From a physical point of view the standard ordering is not satisfactory: indeed, the monomial qp is mapped by ρ_{s} into the operator QP. However, such operator is not symmetric since

$$\langle \phi, \mathsf{QP}\psi\rangle = \langle (\mathsf{QP})^\dagger\phi, \psi\rangle = \langle \mathsf{P}^\dagger\mathsf{Q}^\dagger\phi, \psi\rangle = \langle \mathsf{PQ}\phi, \psi\rangle \neq \langle \mathsf{QP}\phi, \psi\rangle$$

To avoid this problem, the Weyl-Moyal quantization map is introduced: the action of this quantization map 'symmetrizes' the classical polynomial, therefore producing a symmetric operator. More precisely:

We define the *Weyl-Moyal quantization map* as the linear map $\rho_{\mathrm{wm}} : \mathbb{C}[q,p] \to \mathbb{C}[\mathsf{Q},\mathsf{P}])$ acting as

$$1 \mapsto \rho_{\mathrm{wm}}(1) = \mathbf{1} \qquad q \mapsto \rho_{\mathrm{wm}}(q) = \mathsf{Q} \qquad p \mapsto \rho_{\mathrm{wm}}(p) = \mathsf{P}$$

[2] It is sufficient to define the action of the quantization map on monomials and then use the linearity to extend it to polynomials.

$$q^m p^n \mapsto \rho_{\text{wm}}(q^m p^n) = \frac{1}{(m+n)!} \sum_{\sigma \in \text{Sym}_{n+m}} \mathsf{A}_{\sigma(1)} \dots \mathsf{A}_{\sigma(m+n)}$$

where each operator A_j is defined as

$$\mathsf{A}_j = \begin{cases} \mathsf{Q} & \text{if } 1 \leq j \leq m \\ \mathsf{P} & \text{if } m+1 \leq j \leq m+n \end{cases}$$

As stated above, the difference between standard and Weyl-Moyal quantization maps is that the latter provides a symmetrization of the operators: for instance, let us consider the monomial $q^2 p$. Standard quantization for such term would be

$$\rho_{\text{s}}(q^2 p) = \mathsf{Q}^2 \mathsf{P}$$

while applying the Weyl-Moyal map gives us the "symmetrized" version of the operator, namely

$$\rho_{\text{wm}}(q^2 p) = \frac{1}{3}(\mathsf{Q}^2\mathsf{P} + \mathsf{QPQ} + \mathsf{PQ}^2)$$

Notice that operators $\rho_{\text{wm}}(f)$ are symmetric, since

$$\rho_{\text{wm}}(f)^\dagger = \rho_{\text{wm}}(f^*)$$

and f is real, so $f = f^*$.

Weyl-Moyal and standard quantization maps are related to each other: it can be shown that for any $f \in \mathbb{C}[q,p]$ it holds

$$\rho_{\text{wm}}(f) = \rho_{\text{s}}\left(e^{\frac{\hbar}{2i}\frac{\partial^2}{\partial q \partial p}} f\right)$$

where $e^{\frac{\hbar}{2i}\frac{\partial^2}{\partial q \partial p}} : \mathbb{C}[q,p] \to \mathbb{C}[q,p]$ is an invertible map. Through the connection between ρ_{s} and ρ_{wm} written above, one can see that linearity and bijectivity of ρ_{s} is inherited by ρ_{wm}. Moreover, as in the previous case we can define a star product:

Let us define the *Weyl-Moyal star product* as

$$f \star_{\text{wm}} g = \rho_{\text{wm}}^{-1}(\rho_{\text{wm}}(f)\rho_{\text{wm}}(g)) = \sum_{r=0}^{\infty} \left(\frac{i\hbar}{2}\right)^r \frac{1}{r!} \sum_{s=0}^{r} \binom{r}{s} (-1)^{r-s} \frac{\partial^r f}{\partial q^s \partial p^{r-s}} \frac{\partial^r g}{\partial q^{r-s} \partial p^s}$$

Then, the following properties can be proved:

- $\star_{\text{wm}}$ is a well-defined non-commutative associative product on $\mathbb{C}[q,p]$, satisfying the classical limit
$$f \star_{\text{wm}} g = fg + \frac{i\hbar}{2}\{f,g\} + o(\hbar^2)$$
- $\star_s$ and $\star_{\text{wm}}$ are isomorphic in the following sense
$$\mathsf{D}(f \star_{\text{wm}} g) = (\mathsf{D}(f)) \star_{\text{s}} (\mathsf{D}(g))$$
- $\star_{\text{wm}}$ is hermitian with respect to pointwise complex conjugation:
$$(f \star_{\text{wm}} g)^* = g^* \star_{\text{wm}} f^*$$

Wick ordering

A third quantization map is given by the Wick ordering, and this is the scheme that we will actually use since it is related to the harmonic oscillator in the Bargmann representation.

We start by combining phase space variables in a very similar fashion to what we did in the Bargmann representation of coherent states, namely

$$z = \frac{q + ip}{\sqrt{2}}$$

Recall that in Fock-Bargmann space the annihilation operator is given by

$$(\mathsf{a}f)(\bar{z}) = \hbar \frac{\partial f}{\partial \bar{z}}$$

and its adjoint is

$$(\mathsf{a}^\dagger f)(\bar{z}) = \bar{z} f(\bar{z})$$

The space of polynomials in the $\bar{z}$ variable, $\mathbb{C}[\bar{z}]$, is a dense subspace of $\mathcal{F}_B(\mathbb{C}^\ell)$.

We define the *Wick quantization map* as the linear map $\rho_{\mathrm{w}} = \mathbb{C}[z, \bar{z}] \to \mathbb{C}[\mathsf{a}, \mathsf{a}^\dagger]$ acting as follows

$$1 \mapsto \rho_{\mathrm{w}}(1) = \mathbf{1} \qquad z \mapsto \rho_{\mathrm{w}}(z) = \mathsf{a} \qquad \bar{z} \mapsto \rho_{\mathrm{w}}(\bar{z}) = \mathsf{a}^\dagger \qquad \bar{z}^m z^n \mapsto \rho_{\mathrm{w}}(\bar{z}^m z^n) = (\mathsf{a}^\dagger)^m \mathsf{a}^n$$

For instance, the classical polynomial $z^2\bar{z} + \bar{z}^2 z \in \mathbb{C}[z, \bar{z}]$ is mapped through ρ_{w} in the following operator

$$\rho_{\mathrm{w}}(z^2\bar{z} + \bar{z}^2 z) = \mathsf{a}^\dagger \mathsf{a}^2 + (\mathsf{a}^\dagger)^2 \mathsf{a}$$

The action of ρ_{w} on a given polynomial can be summarized in two steps: first, quantizing the expression through the "substitutions"

$$\alpha \mapsto \mathsf{a} \qquad \bar{\alpha} \mapsto \mathsf{a}^\dagger$$

and second imposing the so-called *normal ordering*, in which all the $\mathsf{a}^\dagger$ operators appear on the left of the a. As for the standard quantization, it can be proved that the Wick quantization of f assumes the following form

$$\rho_{\mathrm{w}}(f) = \sum_{r=0}^{\infty} \frac{\hbar^r}{r!} \frac{\partial^r f}{\partial z^r}\bigg|_{z=0} \frac{\partial^r}{\partial \bar{z}^r}$$

for any $f \in \mathbb{C}[z, \bar{z}]$. Also in this case we can define a star product, the Wick star product

$$f \star_{\mathrm{w}} g = \rho_{\mathrm{w}}^{-1}(\rho_{\mathrm{w}}(f)\rho_{\mathrm{w}}(g)) = \sum_{r=0}^{\infty} \frac{\hbar^r}{r!} \frac{\partial^r f}{\partial z^r} \frac{\partial^r g}{\partial \bar{z}^r}$$

Wick star product obeys similar properties to the ones fulfilled by Weyl-Moyal product:

- $\star_{\mathrm{w}}$ is a well-defined non-commutative associative product on $\mathbb{C}[q, p]$, satisfying the classical limit

$$f \star_{\mathrm{w}} g = fg + \hbar \frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} + o(\hbar^2)$$

- $\star_{\mathrm{w}}$ and $\star_{\mathrm{wm}}$ are isomorphic in the following sense: there exists an operator

$$\mathsf{D}' = \exp\left(\frac{\hbar}{4}\left(\frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2}\right)\right)$$

  such that for all $f, g \in \mathbb{C}[q, p]$ it holds

$$\mathsf{D}'(f \star_{\mathrm{wm}} g) = (\mathsf{D}'(f)) \star_w (\mathsf{D}'(g))$$

- $\star_{\mathrm{w}}$ is hermitian with respect to pointwise complex conjugation:

$$(f \star_{\mathrm{w}} g)^* = g^* \star_{\mathrm{w}} f^*$$

Generalization to higher dimensions

All the above maps were introduced in the case of a single degree of freedom, i.e. quantization in $\mathbb{R}$, but of course they can all be generalized to higher dimensions. For instance, Wick star product in $\mathbb{R}^{2n}$ can be written as follows: by mapping the phase space coordinates (q_j, p_j) into n complex numbers as follows

$$z = \frac{q_j + ip_j}{\sqrt{2}}$$

one can write the formula for the product as

$$f \star_{\mathrm{w}} g = \sum_{r=0}^{\infty} \frac{\hbar^r}{r!} \sum_{k_1,\dots,k_r=1}^{n} \frac{\partial^r f}{\partial z_{k_1} \cdots \partial z_{k_r}} \frac{\partial^r g}{\partial \bar{z}_{k_1} \cdots \partial \bar{z}_{k_r}} \tag{3.1}$$

In the following we will use Wick star product only, therefore in order to have a lighter notation we will denote it simply by $\star$.

3.2 Wick bracket

Wick star product is a non-commutative, binary operation: we can combine two of them to obtain a skew-symmetric operator, the Wick bracket.

We define the *Wick bracket* of two functions $f, g \in \mathbb{C}[z, \bar{z}]$ as

$$\{\!\{f, g\}\!\} = f \star g - g \star f$$

This operator shares many properties of the standard Poisson bracket, such as

- Linearity: for all $a, b \in \mathbb{C}$ and for all $f, g, h \in \mathbb{C}[z, \bar{z}]$

$$\{\!\{af + bg, h\}\!\} = a\{\!\{f, h\}\!\} + b\{\!\{g, h\}\!\}$$

- Skew-symmetry: for all $f, g \in \mathbb{C}[z, \bar{z}]$ it holds $\{\!\{f, g\}\!\} = -\{\!\{g, f\}\!\}$
- Leibniz property (under $\star$): for all $f, g, h \in \mathbb{C}[z, \bar{z}]$

$$\{\!\{f, g \star h\}\!\} = \{\!\{f, g\}\!\} \star h + g \star \{\!\{f, h\}\!\}$$

- Jacobi identity: for all $f, g, h \in \mathbb{C}[z, \bar{z}]$

$$\{\!\{f, \{\!\{g, h\}\!\}\ \}\!\} + \{\!\{h, \{\!\{f, g\}\!\}\ \}\!\} + \{\!\{g, \{\!\{h, f\}\!\}\ \}\!\} = 0$$

It is not by chance that we can export the properties of Poisson brackets to the Wick bracket, since the latter can be seen as an algebraic deformation of the former as (see [8] for further details)

$$\{\!\{f, g\}\!\} = \{f, g\} + o(\partial^2)$$

The second order term can be explicitly written using equation (3.1), obtaining

$$\{\!\{f, g\}\!\} = \{f, g\} + \frac{1}{2} \sum_{kl} \left[\frac{\partial^2 f}{\partial z_k \partial z_l} \frac{\partial^2 g}{\partial \bar{z}_k \partial \bar{z}_l} - \frac{\partial^2 g}{\partial z_k \partial z_l} \frac{\partial^2 f}{\partial \bar{z}_k \partial \bar{z}_l} \right] + \dots \tag{3.2}$$

3.3 UV regularized theory

We have seen in the first chapter, and in particular in equation (1.16) that the Heisenberg equation for the time evolution of any annihilation operator a_k involves an infinite number of terms, resulting in a infinite system of ODEs. What we want to do now is to introduce a maximum value for the indices upon which the sum is performed: we call such value a cutoff and we will denote it with Λ. Then, choosing to sum the terms only if the norm (in a suitable sense that will be explained shortly) of the indices is smaller than Λ will gives us a regularized theory, namely a system in which the evolution of a given a_k is ruled by a finite number of equations. We begin by clarifying the concept of norm of a multi-index:

Let $k \in \mathbb{N}^d$ be a multi-index. We define the *multi-index norm* $|\cdot|$ on $\mathbb{N}^d$ as the following map

$$|\cdot| : \mathbb{N}^d \to \mathbb{N} \qquad k \mapsto |k| = \max_i \{k_i\}$$

In this norm the "sphere"[3] of radius Λ has a volume equal to

$$\sum_{k \in \mathbb{N}^d \;:\; |k| < \Lambda} 1 = \Lambda^d = \ell$$

Take for instance $d = 2, \Lambda = 3$, then the set $\{k \in \mathbb{N}^2 \;:\; |k| < 3\}$ has $9 = 3^2$ elements

$$(0,0) \quad (1,0) \quad (2,0) \quad (0,1) \quad (1,1) \quad (2,1) \quad (0,2) \quad (1,2) \quad (2,2)$$

Notice this is different from the usual norm on the multiindices space, namely $|\cdot|_{\text{m.i.}}$, which is defined as

$$|\cdot|_{\text{m.i.}} : \mathbb{N}^d \to \mathbb{N} \qquad k \mapsto |k|_{\text{m.i.}} = k_1 + \cdots + k_d$$

but the two norms are equivalent. Indeed,

$$|k|_{\text{m.i.}} \le |k| \le d|k|_{\text{m.i.}}$$

and

$$\frac{1}{d}|k| \le |k|_{\text{m.i.}} \le |k|$$

In the following we will adopt a simplified notation, putting $\sum_{k \in \mathbb{N}^d \;:\; |k| < \Lambda} \cdot = \sum_k^\Lambda \cdot$.

Having introduced the cutoff, we define the regularized number operator and Hamiltonian as

$$\mathsf{N}_\Lambda = \sum_k^\Lambda \mathsf{a}_k^\dagger \mathsf{a}_k \qquad \mathsf{H}_\Lambda = \sum_k^\Lambda \varepsilon_k \mathsf{a}_k^\dagger \mathsf{a}_k + \frac{1}{2}\sum_{klmn}^\Lambda V_{klmn} \mathsf{a}_k^\dagger \mathsf{a}_l^\dagger \mathsf{a}_m \mathsf{a}_n = \mathsf{H}_\Lambda^{(0)} + \mathsf{H}_\Lambda^{(\text{int})}$$

As claimed above, in the regularized theory we obtain a finite version of the evolution equation (1.16)

$$i\frac{\mathrm{d}\mathsf{a}_k^{(\Lambda)}(t)}{\mathrm{d}t} = [\mathsf{a}_k^{(\Lambda)}(t), \mathsf{H}] = \varepsilon_k \mathsf{a}_k^{(\Lambda)}(t) + \frac{1}{2}\sum_{lmn}^\Lambda v_{klmn} \mathsf{a}_l^{(\Lambda)\dagger}(t)\mathsf{a}_m^{(\Lambda)}(t)\mathsf{a}_n^{(\Lambda)}(t) \tag{3.3}$$

Notice that also in the regularized theory the number operator is conserved, namely we still have

$$[\mathsf{N}_\Lambda, \mathsf{H}_\Lambda] = 0$$

for every value $\Lambda > 1$.

[3] It is actually a d-dimensional hypercube.

Relation between Fock and Bargmann-Fock spaces

The introduction of the cutoff Λ allow us to discuss the relation between the physical bosonic Fock space of second quantization $\mathcal{F}_+$ and the Bargmann-Fock space $\mathcal{F}_B(\mathbb{C}^\ell)$ that was introduced in the first chapter (equation (2.10)).

We begin by building a finite dimensional linear subspace $\mathfrak{f}_\Lambda$ of the single particle Hilbert space $\mathcal{H}_1$: from the orthonormal basis $\{\varphi_j\}_j$ select all the Λ^d functions such that given φ_k it holds $|k| < \Lambda$. The selected functions will form the orthonormal basis for the subspace, that will be given by

$$\mathfrak{f}_\Lambda = \operatorname{span}\{\varphi_k \text{ such that } |k| < \Lambda\}$$

Notice that the dimension is finite, $\dim \mathfrak{f}_\Lambda = \Lambda^d$. Then, the symmetrized n-fold tensor product of $\mathfrak{f}_\Lambda$ is a subspace of $\mathcal{H}_+^N$ and the direct sum over all possible n values gives the desired subspace of the bosonic Fock space[4]

$$\bigoplus_{n=0}^{\infty} \Pi_+(\overbrace{\mathfrak{f}_\Lambda \otimes \cdots \otimes \mathfrak{f}_\Lambda}^{n \text{ times}}) \subset \mathcal{F}_+$$

Moreover, an isomorphism between $\mathcal{F}_B(\mathbb{C}^{\Lambda^d})$ and $\bigoplus_{n=0}^{\infty} \Pi_+(\overbrace{\mathfrak{f}_\Lambda \otimes \cdots \otimes \mathfrak{f}_\Lambda}^{n \text{ times}})$ can be built (see [25], section 1.6) and therefore

$$\mathcal{F}_B(\mathbb{C}^{\Lambda^d}) \simeq \bigoplus_{n=0}^{\infty} \Pi_+(\overbrace{\mathfrak{f}_\Lambda \otimes \cdots \otimes \mathfrak{f}_\Lambda}^{n \text{ times}}) \subset \mathcal{F}_+$$

Therefore, we can define an orthogonal projector $P_\Lambda : \mathcal{F}_+ \to \mathcal{F}_B(\mathbb{C}^n)$. Notice that we can obtain the regularized Hamiltonian as

$$\mathsf{H}_\Lambda = P_\Lambda \mathsf{H} P_\Lambda$$

We also have that

$$\mathsf{H}_\Lambda^{(0)} = P_\Lambda \mathsf{H}^{(0)} = \mathsf{H}^{(0)} P_\Lambda$$
$$\mathsf{N}_\Lambda = P_\Lambda \mathsf{N} = \mathsf{N} P_\Lambda$$

but in general interaction term does not preserve $\mathcal{F}_B(\mathbb{C}^n)$ and so

$$\mathsf{H}_\Lambda^{(\text{int})} \neq \mathsf{H}^{(\text{int})} P_\Lambda \qquad \text{and} \qquad [\mathsf{H}^{(\text{int})}, P_\Lambda] \neq 0$$

We define the *regularized field operator* as the truncated version of the full field operator (equation (1.8)), considering only Λ^d terms in the sum

$$\Psi_\Lambda(x,t) = \sum_k^\Lambda \mathsf{a}_k^{(\Lambda)}(t)\varphi_k(x)$$

To declutter the notation we will omit the Λ over the creation and destruction operators from now on. We will however continue to indicate it on all other quantities.

Now, let us take the coherent expectation value of equation (3.3), where ϕ_α are the Bargmann coherent states, obtaining

$$i\langle \phi_\alpha, \frac{\mathrm{d}\mathsf{a}_k(t)}{\mathrm{d}t}\phi_\alpha\rangle = \varepsilon_k \langle \phi_\alpha, \mathsf{a}_k(t)\phi_\alpha\rangle + \frac{1}{2}\sum_{lmn}^\Lambda V_{klmn}\langle \phi_\alpha, \mathsf{a}_l^\dagger(t)\mathsf{a}_m(t)\mathsf{a}_n(t)\phi_\alpha\rangle$$

[4] Recall that Π_+ is the bosonic projector, symmetrizing any N-particles state.

$$i\dot{a}_k(t,\alpha,\bar{\alpha}) = \varepsilon_k a_k(t,\alpha,\bar{\alpha}) + \frac{1}{2}\sum_{lmn}^{\Lambda} v_{klmn}\langle \phi_\alpha, \mathsf{a}_l^\dagger(t)\mathsf{a}_m(t)\mathsf{a}_n(t)\phi_\alpha\rangle$$

Here we are denoting the coherent expectation value[5] of destruction operators as

$$a_k(t,\alpha,\bar{\alpha}) = \langle \phi_\alpha, \mathsf{a}_k(t)\phi_\alpha\rangle$$

Remark. Unfortunately, for arbitrary times t we have that

$$\langle \phi_\alpha, \mathsf{a}_l^\dagger(t)\mathsf{a}_m(t)\mathsf{a}_n(t)\phi_\alpha\rangle \neq \langle \phi_\alpha, \mathsf{a}_l^\dagger(t)\phi_\alpha\rangle\langle \phi_\alpha, \mathsf{a}_m(t)\phi_\alpha\rangle\langle \phi_\alpha, \mathsf{a}_n(t)\phi_\alpha\rangle$$

Instead, the correct relation is

$$\langle \phi_\alpha, \mathsf{a}_l^\dagger(t)\mathsf{a}_m(t)\mathsf{a}_n(t)\phi_\alpha\rangle = \langle \phi_\alpha, \mathsf{a}_l^\dagger(t)\phi_\alpha\rangle \star \langle \phi_\alpha, \mathsf{a}_m(t)\phi_\alpha\rangle \star \langle \phi_\alpha, \mathsf{a}_n(t)\phi_\alpha\rangle$$

3.4 Coherent phase space

Let us recall that Λ is the cutoff introduced before, and put $\ell = \Lambda^d$.

The vector space $\mathbb{C}^\ell$ of (real) dimension $2\Lambda^d$ and equipped with linear coordinates $(\alpha_k)_{|k|<\Lambda}$ is called *coherent phase space*, since its points are the coherent states eigenvalues

$$\mathsf{a}_k\phi_\alpha = \alpha_k\phi_\alpha$$

For instance, for $\Lambda = 3$ and $d = 2$ we have that the coherent phase space is $\mathbb{C}^9$, and the corresponding 9 linear complex coordinates are given by

$$\alpha_{(0,0)} \quad \alpha_{(0,1)} \quad \alpha_{(1,0)} \quad \alpha_{(0,2)} \quad \alpha_{(2,0)} \quad \alpha_{(1,2)} \quad \alpha_{(2,1)} \quad \alpha_{(1,1)} \quad \alpha_{(2,2)}$$

It is possible to define a Poisson structure on the coherent phase space, where the Poisson tensor is given by

$$\Pi = -i\sum_k^\Lambda \frac{\partial}{\partial\alpha_k}\wedge\frac{\partial}{\partial\bar{\alpha}_k} = -\frac{i}{2}\sum_k^\Lambda\left(\frac{\partial}{\partial\alpha_k}\otimes\frac{\partial}{\partial\bar{\alpha}_k} - \frac{\partial}{\partial\bar{\alpha}_k}\otimes\frac{\partial}{\partial\alpha_k}\right)$$

It is a bit technical to show that Π is a Poisson tensor: to prove it we must introduce the Schouten-Nijenhuis bracket (we will follow [7]).

Let us consider a $(0,m)$ tensor A and a $(0,n)$ tensor B, whose expression in a local chart are

$$A(x) = \frac{1}{m!}A^{i_1\dots i_m}(x)\partial_{i_1}\wedge\cdots\wedge\partial_{i_m} \qquad B(x) = \frac{1}{n!}B^{j_1\dots j_n}(x)\partial_{j_1}\wedge\cdots\wedge\partial_{j_n}$$

The *Schouten-Nijenhuis bracket* between the two tensors is the map[6]

$$[\cdot,\cdot]_{SN} : \Gamma(\mathsf{T}M^{\wedge m})\otimes\Gamma(\mathsf{T}M^{\wedge n}) \to \Gamma(\mathsf{T}M^{\wedge m+n-1})$$

defined as

$$[A,B] = \frac{1}{(m+n-1)!}[A,B]^{k_1\dots k_{m+n-1}}\partial_{k_1}\wedge\cdots\wedge\partial_{k_{m+n-1}}$$

[5]This is actually the Wick symbol of the operator a_k. We will come back to the definition of Wick symbol in a further section.

[6]We denote the set of $(0,m)$ tensors as $\Gamma(\mathsf{T}M^{\wedge m})$.

Where the components $[A,B]^{k_1\dots k_{m+n-1}}$ of the tensor are given by

$$[A,B]^{k_1\dots k_{m+n-1}} = \frac{1}{(m-1)!n!}\varepsilon^{k_1\dots k_{m+n-1}}_{i_1\dots i_{m-1}j_1\dots j_n} A^{\nu i_1\dots i_{m-1}}\frac{\partial}{\partial x^\nu}B^{j_1\dots j_n} + \\ + \frac{(-1)^m}{m!(n-1)!}\varepsilon^{k_1\dots k_{m+n-1}}_{i_1\dots i_m j_1\dots j_{n-1}} B^{\nu j_1\dots j_{n-1}}\frac{\partial}{\partial x^\nu}A^{i_1\dots i_m}$$

and where ε represents the antisymmetric Kronecker symbol

$$\varepsilon^{i_1\dots i_p}_{j_1\dots j_p} = \det\begin{pmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_p} \\ \vdots & & \vdots \\ \delta^{i_p}_{j_1} & \cdots & \delta^{i_p}_{j_p} \end{pmatrix}$$

The Schouten-Nijenhuis bracket has the following properties:

- It extends Lie derivative, meaning that $[\cdot,\cdot]_{SN}$ with $m=n=1$ is the Lie bracket[7].
- It is *graded skew symmetric*,

$$[A,B] = -(-1)^{(|A|-1)(|B|-1)}[B,A]$$

- It is a *graded derivation* on $\Gamma(\mathsf{T}M^{\wedge\cdot})$:

$$[A, B\wedge C] = [A,B]\wedge C + (-1)^{|B|(|A|-1)}B\wedge[A,C]$$

- The Schouten–Nijenhuis bracket satisfies the following generalized Jacobi identity

$$(-1)^{(|A|-1)(|C|-1)}[A,[B,C]] + \text{ cyclic perm. of } A,B,C = 0$$

The reason why we have introduced this object is that provides a characterization of Poisson tensor: a tensor $\Pi\in\Gamma(\mathsf{T}M^{\wedge 2})$ is a Poisson tensor if and only if

$$[\Pi,\Pi]_{SN} = 0$$

[7]Given two vector fields, i.e. two $(0,1)$ tensors on a manifold M

$$X\in\Gamma(\mathsf{T}M) = \mathfrak{X}(M) \qquad Y\in\Gamma(\mathsf{T}M) = \mathfrak{X}(M)$$

their *Lie bracket* is the vector field defined as

$$[X,Y](f) = X(Y(f)) - Y(X(f))$$

We can obtain an expression in coordinates by choosing a local chart: suppose the vector fields can be written as

$$X = \sum_i X^i(x)\partial_i \qquad Y = \sum_i Y^j(x)\partial_j$$

then the Lie bracket in coordinates is

$$[X,Y] = \sum_{i=1}^n\sum_{j=1}^n \left(X^j(x)\partial_j Y^i(x) - Y^j(x)\partial_j X^i(x)\right)\partial_i$$

Lie bracket allows us to define *Lie derivative* of a vector field X along the flow of a vector field Y simply as

$$\mathcal{L}_Y X = [X,Y]$$

Notice that thanks to the graded skew-symmetricity the SN bracket when acting on $(0,2)$ tensors is symmetric.

At this point we just need to show that the Schouten–Nijenhuis bracket of the Π tensor with itself is null, but this can be easily seen using the definition of the bracket:

$$[\Pi,\Pi]^{k_1k_2k_3} = \frac{1}{2}\varepsilon^{k_1k_2k_3}_{i_1j_1j_2}\Pi^{\nu i_1}(x)\frac{\partial}{\partial x^\nu}\Pi^{j_1j_2}(x) + \frac{1}{2}\varepsilon^{k_1k_2k_3}_{i_1i_2j_1}\Pi^{\nu j_1}(x)\frac{\partial}{\partial x^\nu}\Pi^{i_1i_2}(x) = 0$$

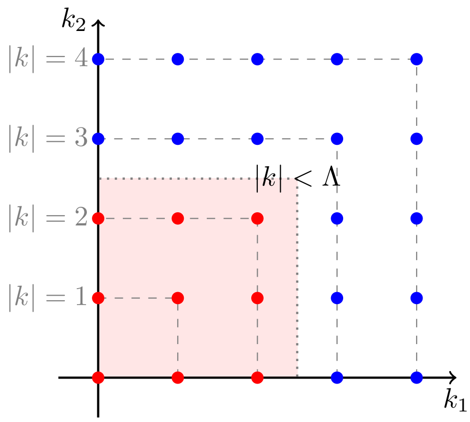


Figure 3.1: Choice of the cutoff Λ selects only some coordinates from the full lattice, determining the dimension on coherent phase space.

3.5 Wick symbol

From now on we will be working with regularized quantities. We will denote the space of truncated polynomials as

$$\mathbb{C}_\Lambda[A_1, A_2] = \left\{ \text{Polynomials of the form } \sum_{ij}^{\Lambda}\sum_{nm} c_{ij,nm}(A_1)_i^n(A_2)_j^m \right\}$$

where A_1 and A_2 are either complex variables or operators. In this setting the Wick quantization map can be restricted to a map $\rho_{\mathrm{w}} : \mathbb{C}_\Lambda[z,\bar{z}] \to \mathbb{C}_\Lambda[\mathsf{a},\mathsf{a}^\dagger]$.

Given an operator $\mathsf{F} \in \mathbb{C}_\Lambda[\mathsf{a},\mathsf{a}^\dagger]$, we define its *Wick symbol* as the expectation over coherent states

$$\sigma_W(\mathsf{F})(\alpha,\bar{\alpha}) = \langle \phi_\alpha, \mathsf{F}\phi_\alpha\rangle \tag{3.4}$$

where ϕ_α are the Bargmann coherent states on $\mathcal{F}_B(\mathbb{C}^\ell)$.

Notice that if F is an operator coming from the Wick quantization of a classical function f, then its Wick symbol is basically the inverse map, recovering the classical function starting from the quantum operator.

In particular, we have the following "basic" Wick symbols (see [42]):

$$\sigma_W(\mathsf{a}_k) = \langle \phi_\alpha, \mathsf{a}_k\phi_\alpha\rangle = \alpha_k$$

$$\sigma_W(\mathsf{a}_k^\dagger) = \langle \phi_\alpha, \mathsf{a}_k^\dagger \phi_\alpha \rangle = \bar{\alpha}_k$$
$$\sigma_W((\mathsf{a}_i^\dagger)^n \mathsf{a}_j^m) = \langle \phi_\alpha, (\mathsf{a}_i^\dagger)^n \mathsf{a}_j^m \phi_\alpha \rangle = \bar{\alpha}_i^n \alpha_j^m$$

We have the following inequalities for Wick symbols of products of creation and annihilation operators:

Proposition 3.1. The following inequalities hold:

$$\left[\sigma_W(\mathsf{a}_i \mathsf{a}_j \mathsf{a}_i^\dagger \mathsf{a}_j^\dagger)(\alpha, \bar{\alpha})\right]^{\frac{1}{2}} \leq 2 + 2|\alpha_i| + 2|\alpha_j| + |\alpha_i \alpha_j|$$

$$\left[\sigma_W(\mathsf{a}_i \mathsf{a}_i^\dagger)(\alpha, \bar{\alpha})\right]^{\frac{1}{2}} \leq 1 + |\alpha_i|$$

Proof. Both equations are proved by moving the creation operators to the left using CCR (equation (1.7)). For the first one we have

$$\begin{aligned}
\sigma_W(\mathsf{a}_i \mathsf{a}_j \mathsf{a}_i^\dagger \mathsf{a}_j^\dagger)(\alpha, \bar{\alpha}) &= \langle \phi_\alpha, \mathsf{a}_i \mathsf{a}_j \mathsf{a}_i^\dagger \mathsf{a}_j^\dagger \phi_\alpha \rangle \\
&= \langle \phi_\alpha, (1 + \delta_{ij} + \mathsf{a}_i^\dagger \mathsf{a}_i + \mathsf{a}_j^\dagger \mathsf{a}_j + \delta_{ij} \mathsf{a}_i^\dagger \mathsf{a}_j + \delta_{ij} \mathsf{a}_j^\dagger \mathsf{a}_i + \mathsf{a}_i^\dagger \mathsf{a}_j^\dagger \mathsf{a}_i \mathsf{a}_j) \phi_\alpha \rangle \\
&= 1 + \delta_{ij} + |\alpha_i|^2 + |\alpha_j|^2 + \delta_{ij} \alpha_i \bar{\alpha}_j + \delta_{ij} \alpha_j \bar{\alpha}_i + |\alpha_i|^2 |\alpha_j|^2 \\
&\leq 2 + |\alpha_i|^2 + |\alpha_j|^2 + \alpha_i \bar{\alpha}_j + \alpha_j \bar{\alpha}_i + |\alpha_i|^2 |\alpha_j|^2 \\
&= |\alpha_i + \alpha_j|^2 + |\alpha_i \alpha_j|^2 + 2
\end{aligned}$$

and using the sublinearity of the square root[8] we get

$$\begin{aligned}
\left[\sigma_W(\mathsf{a}_i \mathsf{a}_j \mathsf{a}_i^\dagger \mathsf{a}_j^\dagger)(\alpha, \bar{\alpha})\right]^{\frac{1}{2}} &\leq \sqrt{|\alpha_i + \alpha_j|^2 + |\alpha_i \alpha_j|^2 + 2} \\
&\leq |\alpha_i + \alpha_j| + |\alpha_i \alpha_j| + \sqrt{2} \\
&\leq |\alpha_i| + |\alpha_j| + |\alpha_i \alpha_j| + 2 \\
&\leq 2|\alpha_i| + 2|\alpha_j| + 2|\alpha_i \alpha_j| + 2
\end{aligned}$$

Similarly, for the second inequality we have

$$\sigma_W(\mathsf{a}_i \mathsf{a}_i^\dagger)(\alpha, \bar{\alpha}) = \langle \phi_\alpha, (\mathsf{a}_i^\dagger \mathsf{a}_i + 1) \phi_\alpha \rangle = |\alpha_i|^2 + 1$$

and therefore

$$\left[\sigma_W(\mathsf{a}_i \mathsf{a}_i^\dagger)(\alpha, \bar{\alpha})\right]^{\frac{1}{2}} \leq 1 + |\alpha_i|$$

□

Another Wick symbol that will turn out to be useful is the one for the number operator N, multiplied by a constant λ:

$$\begin{aligned}
\langle \phi_w, \lambda \mathsf{N} \phi_w \rangle &= \frac{\lambda}{\pi^n} \int e^{-|z|^2} \overline{\phi_w(\bar{z})} \sum_k \mathsf{a}_k^\dagger \mathsf{a}_k \phi_w(\bar{z}) \, \mathrm{d}z \wedge \mathrm{d}\bar{z} = \\
&= \frac{\lambda}{\pi^n} \int e^{-|z|^2} \overline{e^{w \cdot \bar{z} - \frac{1}{2}|w|^2}} \sum_k |w_k|^2 e^{w \cdot \bar{z} - \frac{1}{2}|w|^2} \, \mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&= \frac{\lambda |w|^2}{\pi^n} \int e^{-|z|^2} \overline{e^{w \cdot \bar{z} - \frac{1}{2}|w|^2}} e^{w \cdot \bar{z} - \frac{1}{2}|w|^2} \, \mathrm{d}z \wedge \mathrm{d}\bar{z} = \lambda |w|^2
\end{aligned}$$

[8] Meaning that for all $x, y \in \mathbb{R}^+$ we have

$$\sqrt{x} + \sqrt{y} \geq \sqrt{x + y}$$

When f is not a polynomial we have that we can define $\mathsf{F} = \mathsf{W}[f]$ as

$$\mathsf{F}(\psi)(\bar{z}) = \int f(\bar{z}, \alpha)\psi(\bar{\alpha})e^{-|\alpha|^2+\alpha\cdot\bar{z}}\,\mathrm{d}\alpha\wedge\mathrm{d}\bar{\alpha} \qquad f(\bar{z},\alpha) = \frac{\langle\phi_z, \mathsf{F}\phi_\alpha\rangle}{\langle\phi_z,\phi_\alpha\rangle}$$

In general the Wick symbol defined above is the diagonal Wick symbol (since the scalar product is taken between the same coherent states).

Remark. We can define also a more general symbol, the non diagonal one (following for instance [12]). The *non-diagonal Wick symbol* of the operator F is defined as

$$\sigma_W(\mathsf{F})(\bar{w}, z) = \frac{\langle\phi_w, \mathsf{F}\phi_z\rangle}{\langle\phi_w,\phi_z\rangle}$$

For $w = z$ this definition collapses to the one given in equation (3.4). Furthermore, since any entire function $K(\bar{w}, z)$ of the variables $\bar{w}$ and z is uniquely determined by its restriction to the diagonal $w = z$, then a Wick symbol is uniquely defined by its diagonal Wick symbol [9].

Given an operator F and its Wick symbol $\sigma_W(\mathsf{F})(w, \bar{w})$, we define the *anti-Wick symbol* of F (see [11]) as the function $\sigma_{AW}(\mathsf{F})(z, \bar{z})$ such that

$$\sigma_W(\mathsf{F})(\bar{w}, w) = \int_{\mathbb{C}^\ell} e^{-(z-w)(\bar{z}-\bar{w})}\sigma_{AW}(\mathsf{F})(\bar{z}, z)\,\mathrm{d}z\wedge\mathrm{d}\bar{z}$$

Notice that this expression is implicit, and for an arbitrary operator the existence of σ_{AW} is not guaranteed.

For instance, knowing that

$$\sigma_W(e^{-\lambda\mathsf{N}_\Lambda})(\bar{w}, w) = e^{-(1-e^{-\lambda})|w|^2}$$

we can check that

$$\sigma_{AW}(e^{-\lambda\mathsf{N}_\Lambda})(\bar{z}, z) = e^{\lambda\ell}e^{-(e^{\lambda}-1)|z|^2} \tag{3.5}$$

by making an explicit computation: let $z = x + iy$ and $w = a + ib$, where $x, y, a, b \in \mathbb{R}^\ell$, then[10]

$$\begin{aligned}\int_{\mathbb{C}^\ell} e^{-(z-w)(\bar{z}-\bar{w})}e^{\lambda\ell}e^{-(e^{-\lambda}-1)|z|^2}\,\mathrm{d}z\wedge\mathrm{d}\bar{z} &= \int_{\mathbb{R}^{2\ell}} \frac{\mathrm{d}x\,\mathrm{d}y}{\pi^\ell}e^{-x^2-y^2+(a-ib)(x+iy)+(a+ib)(x-iy)-a^2-b^2}\times \\ &\qquad\times e^{\lambda\ell}e^{-(e^{-\lambda}-1)(x^2+y^2)} \\ &= \frac{e^{-(a^2+b^2)}e^{\lambda\ell}}{\pi^\ell}\left(\int_{\mathbb{R}^\ell} e^{-e^{-\lambda}x^2+2ax}\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^\ell} e^{-e^{-\lambda}y^2+2by}\,\mathrm{d}y\right) \\ &= e^{\lambda\ell}e^{-|w|^2}e^{e^{\lambda}|w|^2} = e^{(e^{\lambda}-1)|w|^2}\end{aligned}$$

as expected.

The reason why we have introduced the anti-Wick symbol, and in particular the anti-Wick symbol of the cutoffed number operator is the following proposition, that will turn out to be useful to compute traces:

Proposition 3.2. For any $\lambda \in \mathbb{R}$ and for any Wick operator F the following trace formula holds

$$\frac{\mathrm{Tr}(\mathsf{F}e^{-\lambda\mathsf{N}_\Lambda})}{\mathrm{Tr}(e^{-\lambda\mathsf{N}_\Lambda})} = \int_{\mathbb{C}^\ell}\langle\phi_\alpha, \mathsf{F}\phi_\alpha\rangle\frac{\sigma_{AW}(e^{-\lambda\mathsf{N}_\Lambda})}{\mathrm{Tr}(e^{-\lambda\mathsf{N}_\Lambda})}\,\mathrm{d}\alpha\wedge\mathrm{d}\bar{\alpha}$$

[9] Also the proof that the function $(\bar{w}, z) \mapsto \sigma_W(\mathsf{F})(\bar{w}, z)$ is entire can be found in [25], section 2.7.

[10] See appendix for computation.

Proof. See [17]. □

We can use this proposition to compute the trace of the operator $e^{-\lambda \mathsf{N}_\Lambda}$: indeed, put $\mathsf{F} = \mathbf{1}$ and we get

$$1 = \int_{\mathbb{C}^\ell} \frac{\sigma_{AW}(e^{-\lambda \mathsf{N}_\Lambda})}{\mathrm{Tr}(e^{-\lambda \mathsf{N}_\Lambda})} \, \mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha}$$

from which we obtain, using equation (3.5)

$$\begin{aligned} \mathrm{Tr}(e^{-\lambda \mathsf{N}_\Lambda}) &= \int_{\mathbb{C}^\ell} \sigma_{AW}(e^{-\lambda \mathsf{N}_\Lambda}) \, \mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha} \\ &= e^{\lambda \ell} \int_{\mathbb{C}^\ell} e^{(e^\lambda - 1)|\alpha|^2} \, \mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha} \\ &= e^{\lambda \ell} \int_{\mathbb{C}^\ell} e^{(e^\lambda - 1)(x^2 + y^2)} \frac{1}{\pi^\ell} \, \mathrm{d}x \, \mathrm{d}y \\ &= e^{\lambda \ell} \left(\frac{\pi}{e^\lambda - 1} \right)^\ell \frac{1}{\pi^\ell} = \left(\frac{e^\lambda}{e^\lambda - 1} \right)^\ell \end{aligned} \tag{3.6}$$

We can combine the result of equation (3.5) and (3.6) to write

$$\frac{\mathrm{Tr}(\mathsf{F} e^{-\lambda \mathsf{N}_\Lambda})}{\mathrm{Tr}(e^{-\lambda \mathsf{N}_\Lambda})} = \int_{\mathbb{C}^\ell} \langle \phi_\alpha, \mathsf{F} \phi_\alpha \rangle (e^\lambda - 1)^\ell e^{-(e^\lambda - 1)|\alpha|^2} \, \mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha} \tag{3.7}$$

3.6 The effective field

Let us define the scalar Hamiltonian H_Λ as the coherent expectation value of the regularized Hamiltonian H_Λ

$$H_\Lambda(\alpha, \bar{\alpha}) = \langle \phi_\alpha, \mathsf{H}_\Lambda \phi_\alpha \rangle = \sum_k^\Lambda \varepsilon_k \bar{\alpha}_k \alpha_k + \frac{1}{2} \sum_{klmn}^\Lambda V_{klmn} \bar{\alpha}_k \bar{\alpha}_l \alpha_m \alpha_n$$

The Hamiltonian flow associated to H_Λ is given by the following finite system

$$i \dot{c}_k = \frac{\partial H_\Lambda}{\partial \bar{\alpha}_k}(c, \bar{c}) = \varepsilon_k c_k + \sum_{lmn}^\Lambda V_{klmn} \bar{c}_l c_m c_n$$

where the initial data is $c_k(0, \alpha, \bar{\alpha}) = \alpha_k$.

We define the *regularized effective field* as

$$\Psi_\Lambda^{(0)}(t, x) = \sum_k^\Lambda \mathsf{c}_k(t) \varphi_k(x) \tag{3.8}$$

where the operators c_k are such that $\sigma_W(\mathsf{c}_k(t))(\alpha, \bar{\alpha}) = c_k(t, \alpha, \bar{\alpha})$.

Now, substituting each operator c_k with its Wick symbol c_k in equation (3.8) gives us a scalar function

$$\psi_\Lambda(t, x) = \sum_k^\Lambda c_k(t, \alpha, \bar{\alpha}) \varphi_k(x)$$

which is the solution of the reduced scalar Hartree equation on $L^2(\mathbb{R}^d)$

$$i \frac{\mathrm{d}\psi_\Lambda}{\mathrm{d}t} = \frac{\partial \mathcal{E}}{\partial \bar{\psi}_\Lambda}(\psi_\Lambda, \bar{\psi}_\Lambda) \tag{3.9}$$

with $\mathcal{E}$ being the Hartree energy functional on $L^2(\mathbb{R}^d)$

$$\mathcal{E}(\psi, \bar{\psi}) = \langle \psi, \mathsf{h}\psi \rangle + \frac{1}{2}\langle \psi, V * |\psi|^2 \psi \rangle$$

Notice that the initial datum is given by

$$\psi_\Lambda(0, x) = \sum_k^\Lambda c_k(0, \alpha, \bar{\alpha})\varphi_k(x) = \sum_k^\Lambda \alpha_k \varphi_k(x)$$

The effective field that was just defined is the field operator whose Wick symbols obey the Hartree dynamics

$$\langle \phi_\alpha, \Psi_\Lambda^{(0)}(t, x)\phi_\alpha \rangle = \sum_k^\Lambda c_k(t, \alpha, \bar{\alpha})\varphi_k(x) = \psi_\Lambda(t, x)$$

In the next section we will introduce the Gaussian thermal measure, that will be used to distribute the initial data $\alpha \in \mathbb{C}^\ell$ and obtain an estimate for the deviation of the full quantum dynamics $\Psi(t)$ from the effective field $\Psi_\Lambda^{(0)}(t)$ (see proposition 4.8).

3.7 Gaussian measure convergence

Equation (3.7) motivates us to introduce a new measure, the thermal Gaussian measure. We begin by defining the following Gibbsian operator

$$\rho_\Lambda = \frac{e^{-\beta\omega \mathsf{N}_\Lambda}}{\mathrm{Tr}(e^{-\beta\omega \mathsf{N}_\Lambda})}$$

where[11] $\beta = \frac{1}{T}$, with the temperature T being our convergence parameter towards Hartree dynamics. Clearly, this operator has a unit trace

$$\mathrm{Tr}(\rho_\Lambda) = \mathrm{Tr}\left(\frac{e^{-\beta\omega \mathsf{N}_\Lambda}}{\mathrm{Tr}(e^{-\beta\omega \mathsf{N}_\Lambda})}\right) = \frac{\mathrm{Tr}(e^{-\beta\omega \mathsf{N}_\Lambda})}{\mathrm{Tr}(e^{-\beta\omega \mathsf{N}_\Lambda})} = 1$$

and equation (3.6) ensures that the trace is finite.

Given a generic operator F, using proposition 3.2 and equation (3.7) we can compute the following

$$\mathrm{Tr}(\mathsf{F}\rho_\Lambda) = \int_{\mathbb{C}^\ell} \sigma_W(\mathsf{F})(\alpha, \bar{\alpha})\, \mathrm{d}\mu(\alpha, \bar{\alpha})$$

where $\mathrm{d}\mu(\alpha, \bar{\alpha})$ is the *Gaussian thermal measure*, defined as

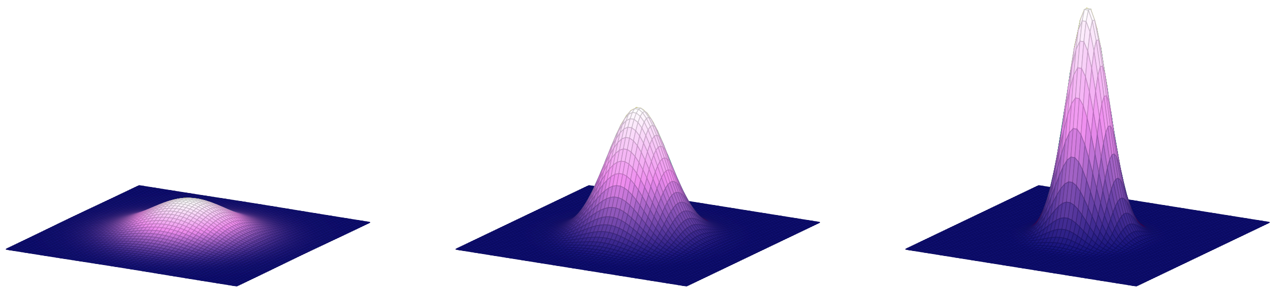
$$\mathrm{d}\mu(\alpha, \bar{\alpha}) = (e^{\beta\omega} - 1)^\ell e^{-(e^{\beta\omega}-1)|\alpha|^2}\, \mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha} = m_\beta(\alpha, \bar{\alpha})\, \mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha}$$

The Gaussian thermal measure concentrates around $\alpha = 0$ for temperature $T \to 0^+$: this can be seen clearly in the figure 3.2, where the function $m(\alpha, \bar{\alpha})$ is plotted for different values of β parameter.

More formally, consider the function

$$m_B(\alpha, \bar{\alpha}) = B^\ell e^{-B|\alpha|^2}$$

[11] Boltzmann constant k_B is put equal to one here and everywhere else.



(a) $\beta = 2$ (b) $\beta = 3$ (c) $\beta = 4.5$

Figure 3.2: Plot of function $m(\alpha, \bar{\alpha})$ for three increasing values of β.

where $B = e^{\beta\omega} - 1$. Notice that the integral $\int_{\mathbb{C}^\ell} \mathrm{d}\mu(\alpha, \bar{\alpha})$ is well-behaved in the limit, namely its value is not dependent from B:

$$\begin{aligned}\lim_{B\to+\infty} \int_{\mathbb{C}^\ell} \mathrm{d}\mu(\alpha, \bar{\alpha}) &= \lim_{B\to\infty} \int_{\mathbb{C}^\ell} m_B(\alpha, \bar{\alpha})\, \mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha} \\ &= \lim_{B\to\infty} \left(\frac{B}{\pi}\right)^\ell \left(\int_{\mathbb{R}^\ell} e^{-Bq^2}\, \mathrm{d}q\right) \left(\int_{\mathbb{R}^\ell} e^{-Bp^2}\, \mathrm{d}p\right) \\ &= \lim_{B\to\infty} \left(\frac{B}{\pi}\right)^\ell \left(\frac{\pi}{B}\right)^{\frac{\ell}{2}} \left(\frac{\pi}{B}\right)^{\frac{\ell}{2}} = 1\end{aligned}$$

It is easy to see that given an arbitrary function $g(\alpha, \bar{\alpha})$ we have

$$\begin{aligned}\lim_{B\to\infty} \int_{\mathbb{C}^\ell} g(\alpha, \bar{\alpha})\, \mathrm{d}\mu(\alpha, \bar{\alpha}) &= \lim_{B\to\infty} \int_{\mathbb{C}^\ell} m_B(\alpha, \bar{\alpha}) g(\alpha, \bar{\alpha})\, \mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha} \\ &= \lim_{B\to\infty} \int_{\mathbb{C}^\ell} \left(\frac{B}{\pi}\right)^\ell e^{-B(q^2+p^2)} g(q,p)\, \mathrm{d}p\, \mathrm{d}q = g(0,0)\end{aligned}$$

therefore

$$\lim_{B\to\infty} \left(\frac{B}{\pi}\right)^\ell e^{-B(q^2+p^2)} = \delta(q,p) = \delta(q)\delta(p) = \delta(q_1)\ldots\delta(q_\ell)\delta(p_1)\ldots\delta(p_\ell)$$

and in the $\beta \to +\infty$ limit the Gaussian thermal measure reproduces a Dirac delta centered in $\alpha = 0$, namely

$$\lim_{B\to\infty} m_B(\alpha, \bar{\alpha}) = \delta^{(\ell)}(\alpha)$$

3.8 Gaussian thermal norm

Let $\omega > 0$ be the lowest eigenvalue of h,

$$\omega = \inf\{\mathrm{Spec}(\mathsf{h})\}$$

In our case $\mathrm{Spec}(\mathsf{h}) = \{\frac{d}{2}, \frac{d}{2}+1, \frac{d}{2}+2, \ldots\}$ and so $\omega = \frac{d}{2}$, but for generality we will keep it unspecified.

We define the $\star$-*norm* of a field operator Φ as

$$\|\Phi\|_\star = \sqrt{\mathrm{Tr}\left(\rho_\Lambda \int_{\mathbb{R}^d} \Phi^\dagger(x)\Phi(x)\, \mathrm{d}^d x\right)}$$

Let us notice that for a generic operator F and its Wick symbols

$$\sigma_W[\mathsf{F}](\alpha,\bar{\alpha}) = f(\alpha,\bar{\alpha}) \qquad \sigma_W[\mathsf{F}^\dagger](\alpha,\bar{\alpha}) = \bar{f}(\alpha,\bar{\alpha})$$

the following inequality between Wick symbols holds:

$$\sigma_W[\mathsf{F}^\dagger\mathsf{F}](\alpha,\bar{\alpha}) = \bar{f}(\alpha,\bar{\alpha}) \star f(\alpha,\bar{\alpha}) = |f(\alpha,\bar{\alpha})|^2 + \sum_{n=1}^{\infty} \frac{\partial^n \bar{f}(\alpha,\bar{\alpha})}{\partial \alpha^n}\frac{\partial^n f(\alpha,\bar{\alpha})}{\partial \bar{\alpha}^n} \leq |f(\alpha,\bar{\alpha})|^2 \tag{3.10}$$

Now, decomposing the Φ operator as

$$\Phi(x) = \sum_k \varphi_k(x)\mathsf{d}_k \tag{3.11}$$

and considering the equation (3.10) we have the following bound, relating the $\star$-norm of the operator Φ with the sum of the norms of the Wick symbols of d_k:

$$\|\Phi\|_\star^2 \geq \sum_k^{\Lambda} \int_{\mathbb{C}^\ell} |\sigma_W(\mathsf{d}_k)|^2 \, \mathrm{d}\mu(\alpha,\bar{\alpha}) = \sum_k^{\Lambda} \|\sigma_W(\mathsf{d}_k)\|^2_{L^2(\mathrm{d}\mu)} \tag{3.12}$$

We can define another norm, sharper than $\|\cdot\|_\star$, motivated by the inequality (3.12):

Let Φ be as in equation (3.11). We define the *Gaussian thermal norm*, or simply *μ-norm*, as the following operator norm

$$\|\Phi\|_\mu^2 = \frac{1}{\Lambda^d}\sum_k^{\Lambda} \|\sigma_W(\mathsf{d}_k)\|^2_{L^2(\mathrm{d}\mu)}$$

From the above considerations, we have that the two norms introduced obey

$$\|\Phi\|_\mu^2 \leq \|\Phi\|_\star^2 \tag{3.13}$$

for every operator Φ.

Proposition 3.3. The following formula holds for all $\lambda \geq 0$

$$\int_{\mathbb{C}^\ell} |\alpha_k|^\lambda \, \mathrm{d}\mu(\alpha,\bar{\alpha}) = \frac{1}{B^{\frac{\lambda}{2}}}\Gamma\left(\frac{\lambda}{2}+1\right) \tag{3.14}$$

Proof. Let us prove this result: we can split the integral as

$$\int_{\mathbb{C}^\ell} |\alpha_k|^\lambda \, \mathrm{d}\mu(\alpha,\bar{\alpha}) = \int_{\mathbb{R}^{2\ell}} \frac{B^\ell}{\pi^\ell} e^{-B(\alpha_x^2+\alpha_y^2)}(\alpha_{kx}^2+\alpha_{ky}^2)^{\frac{\lambda}{2}} \, \mathrm{d}\alpha_x \, \mathrm{d}\alpha_y = \dots$$

where

$$\alpha_x = (\alpha_{x1},\dots,\alpha_{x\ell}) \in \mathbb{R}^\ell \qquad \alpha_y = (\alpha_{y1},\dots,\alpha_{y\ell}) \in \mathbb{R}^\ell$$

Now, for brevity put

$$I_j = \frac{B}{\pi}\int_{\mathbb{R}^2} e^{-B(\alpha_{xj}^2+\alpha_{yj}^2)} \, \mathrm{d}\alpha_{xj} \, \mathrm{d}\alpha_{yj}$$

then we have

$$\int_{\mathbb{C}^\ell} |\alpha_k|^\lambda \, \mathrm{d}\mu(\alpha,\bar{\alpha}) = I_1 I_2 \dots I_{k-1} I_{k+1} \dots I_\ell \left(\frac{B}{\pi}\int_{\mathbb{R}^2} e^{-B(\alpha_{xk}^2+\alpha_{yk}^2)} \left(\alpha_{xk}^2+\alpha_{yk}^2\right)^{\frac{\lambda}{2}} \, \mathrm{d}\alpha_{xk} \, \mathrm{d}\alpha_{yk}\right)$$

It is easy to show that for all $j \neq k$ it holds $I_j = 1$. Then, the integral inside round brackets can be evaluated as follows: switching to polar coordinates ($\xi = \sqrt{\alpha_{xk}^2 + \alpha_{yk}^2}$)

$$\frac{B}{\pi} \int_{\mathbb{R}^2} e^{-B(\alpha_{xk}^2 + \alpha_{yk}^2)} \left(\alpha_{xk}^2 + \alpha_{yk}^2\right)^{\frac{\lambda}{2}} \mathrm{d}\alpha_{xk} \, \mathrm{d}\alpha_{yk} = \frac{B}{\pi} 2\pi \int_0^\infty e^{-B\xi^2} \xi^{\lambda+1} \, \mathrm{d}\xi =$$

and then substituting $B\xi^2 = z$ we get the thesis[12]

$$= 2B \int_0^\infty \frac{1}{2\sqrt{B}} z^{-\frac{1}{2}} e^{-z} \left(\frac{z}{B}\right)^{\frac{\lambda+1}{2}} \mathrm{d}z = B^{-\frac{\lambda}{2}} \int_0^\infty e^{-z} z^{\frac{\lambda}{2}+1-1} \, \mathrm{d}z = B^{-\frac{\lambda}{2}} \Gamma\left(\frac{\lambda}{2} + 1\right)$$

□

Let us see two useful cases of the above general formula, equation (3.14):

- for $\lambda = 2$ we have
$$\int_{\mathbb{C}^\ell} |\alpha_k|^2 \, \mathrm{d}\mu(\alpha, \bar{\alpha}) = \frac{1}{B}$$
and by using proposition 3.2 we can also relate the result to the following trace:
$$\mathrm{Tr}(\mathsf{a}_k^\dagger \mathsf{a}_k \rho_\Lambda) = \frac{1}{B}$$
- for $\lambda = 4$ we have similarly
$$\int_{\mathbb{C}^\ell} |\alpha_k|^4 \, \mathrm{d}\mu(\alpha, \bar{\alpha}) = \frac{2}{B^2}$$
and so
$$\mathrm{Tr}((\mathsf{a}_k^\dagger)^2 (\mathsf{a}_k)^2 \rho_\Lambda) = \frac{2}{B^2}$$

Remark. The proof of proposition 3.3 can be easily adapted to prove the following result, since the result of the integral is completely independent from the indices of the α:

$$\int_{\mathbb{C}^\ell} |\overbrace{\alpha_k \alpha_l \dots \alpha_m}^{n \text{ terms}}|^\sigma \, \mathrm{d}\mu(\alpha, \bar{\alpha}) = \int_{\mathbb{C}^\ell} |\alpha_k|^{n\sigma} \, \mathrm{d}\mu(\alpha, \bar{\alpha}) \tag{3.15}$$

A fundamental invariance property of the thermal Gaussian measure is stated in the next proposition. Recall that, given an Hamiltonian vector field $X_H \in \mathfrak{X}(M)$ on a manifold M, we define its *flow* as the function solving the ODE related to the vector field, namely as the function

$$\Phi_{X_H}^t : \mathbb{R} \times M \to M \qquad (t, x) \mapsto \Phi_{X_H}^t(x)$$

such that

$$\frac{\mathrm{d}\Phi_{X_H}^t(x)}{\mathrm{d}t} = X(\Phi_{X_H}^t(x)) \qquad \Phi_{X_H}^0(x) = x_0$$

[12] Recall that the Euler's Γ function is defined as

$$\Gamma(n) = \int_0^\infty e^{-z} z^{n-1} \, \mathrm{d}z \text{ for all } n \in \mathbb{R}$$

The function is related to the factorial: when the argument is a natural number indeed one has

$$\Gamma(n+1) = n! \text{ for all } n \in \mathbb{N}$$

Proposition 3.4. Gaussian measure μ is invariant under the discrete Hartree flow,

$$\mathrm{d}\mu(\Phi^t_{H_\Lambda}(\alpha)) = \mathrm{d}\mu(\alpha) \text{ for all } t \geq 0$$

Proof. Measure can be written as a volume form on the coherent phase space:

$$\mathrm{d}\mu(\alpha) = \frac{1}{Z} e^{-B|\alpha|^2} \prod_k^\Lambda \mathrm{d}\alpha_k \wedge \mathrm{d}\bar{\alpha}_k$$

Since $\{N_\Lambda, H_\Lambda\} = 0$ we get

$$\mathrm{d}\mu(\Phi^t_{H_\Lambda}(\alpha)) = \frac{1}{Z} e^{-B|\alpha|^2} \prod_k^\Lambda \mathrm{d}(\Phi^t_{H_\Lambda}(\alpha))_k \wedge \mathrm{d}(\Phi^t_{H_\Lambda}(\bar{\alpha}))_k = \det(\mathrm{d}(\Phi^t_{H_\Lambda}(\alpha)))\, \mathrm{d}\mu(\alpha)$$

Recalling that $\Phi^t_{H_\Lambda}$ is a one-parameter group of symplectomorphisms we have $\det(\mathrm{d}(\Phi^t_{H_\Lambda}(\alpha))) = 1$ and we get the thesis. □

The above proposition has a number of consequences, for instance in the invariance of averages of Wick symbols, as shown in the following proposition:

Proposition 3.5. The following equation holds for all $\mathsf{F} \in \mathbb{C}_\Lambda[\mathsf{a}, \mathsf{a}^\dagger]$

$$\int_{\mathbb{C}^\ell} \sigma_W(\mathsf{F}(t))(\alpha, \bar{\alpha})\, \mathrm{d}\mu(\alpha) = \int_{\mathbb{C}^\ell} \sigma_W(\mathsf{F})(\alpha, \bar{\alpha})\, \mathrm{d}\mu(\alpha)$$

where $\mathsf{F}(t) \equiv \mathsf{F}(\mathsf{a}(t), \mathsf{a}^\dagger(t))$.

Proof. We have

$$\begin{aligned}
\int_{\mathbb{C}^\ell} \sigma_W(\mathsf{F}(t))(\alpha, \bar{\alpha})\, \mathrm{d}\mu(\alpha) &= \mathrm{Tr}((\mathsf{F}(t)\rho_\Lambda) \\
&= \mathrm{Tr}(e^{it\mathsf{H}_\Lambda} \mathsf{F} e^{-it\mathsf{H}_\Lambda} \rho_\Lambda) \\
&\overset{\diamondsuit}{=} \mathrm{Tr}(\mathsf{F} e^{-it\mathsf{H}_\Lambda} \rho_\Lambda e^{it\mathsf{H}_\Lambda}) \\
&\overset{\heartsuit}{=} \mathrm{Tr}(\mathsf{F} e^{-it\mathsf{H}_\Lambda} e^{it\mathsf{H}_\Lambda} \rho_\Lambda) \\
&= \mathrm{Tr}(\mathsf{F}\rho_\Lambda) \\
&= \int_{\mathbb{C}^\ell} \sigma_W(\mathsf{F})(\alpha, \bar{\alpha})\, \mathrm{d}\mu(\alpha)
\end{aligned}$$

where in $\diamondsuit$ we used cyclicity of the trace and in $\heartsuit$ we used $[\mathsf{H}_\Lambda, \rho_\Lambda] = 0$. □

3.9 Gaussian thermal measure and the Gibbs measure

In this short section we want to briefly discuss which is the relation between the Gaussian thermal measure μ that we have introduced above and the standard Gibbs measure g.

Recall that the Gibbs measure is defined as

$$\mathrm{d}g(\alpha, \bar{\alpha}) = \frac{e^{\lambda H(\alpha, \bar{\alpha})}\, \mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha}}{\int_{\mathbb{C}^\ell} e^{\lambda H(\alpha, \bar{\alpha})}\, \mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha}}$$

Let ω be as before, then ellipticity property for the Hamiltonian H implies $H(\alpha, \bar{\alpha}) \geq \omega|\alpha|^2$ and therefore

$$0 < e^{-\lambda H(\alpha, \bar{\alpha})} \leq e^{-\lambda \tau}$$

Now, let us put $e^{\lambda_0} = \lambda\omega + 1$: the anti-Wick symbol of $e^{-\lambda_0 \mathsf{N}_\Lambda}$ (equation (3.5)) then reads

$$\sigma_{AW}[e^{-\lambda_0 \mathsf{N}_\Lambda}](\alpha, \bar{\alpha}) = (\lambda\omega + 1)^\ell e^{-\lambda\omega|\alpha|^2}$$

We define another Gaussian measure, $\mathrm{d}m$, as

$$\mathrm{d}m(\alpha, \bar{\alpha}) = \frac{\sigma_{AW}(e^{-\lambda_0 \mathsf{N}})}{\mathrm{Tr}(e^{-\lambda_0 \mathsf{N}})} \,\mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha} = (\lambda\tau)^\ell e^{-\lambda\tau|\alpha|^2} \,\mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha}$$

It can be proved that there exists a constant C (see [44]), depending from ω, V and ℓ such that

$$\frac{(\lambda\omega)^{-\ell}}{\int_{\mathbb{C}^\ell} e^{\lambda H(\alpha,\bar{\alpha})} \,\mathrm{d}\alpha \wedge \mathrm{d}\bar{\alpha}} \leq C \int \mathrm{d}g(\alpha, \bar{\alpha}) = C$$

This implies the following inequality between norms

$$\|f\|_{L^2(\mathrm{d}g)} \leq \sqrt{C}\, \|f\|_{L^2(\mathrm{d}m)} = \sqrt{C}\, \|f\|_\mu$$

stating that the μ-norm and can be used to control the Gibbs norm.

In view of this inequality we are able to express the estimates about convergence towards Hartree dynamics, namely equations (4.5) and (4.6), using the Gibbs norm instead of the μ-norm.

3.10 A bound on the temperature

Let us consider the canonical Gibbs operator

$$\varrho_\Lambda = \frac{e^{-\beta \mathsf{H}_\Lambda}}{\mathrm{Tr}(e^{-\beta \mathsf{H}_\Lambda})}$$

where $\beta = \frac{1}{T}$. We want to see that imposing the following condition

$$\langle \mathsf{N}_\Lambda \rangle_{\varrho_\Lambda} = \mathrm{Tr}(\varrho_\Lambda \mathsf{N}_\Lambda) \leq N$$

namely that the average of the (regularized) number operator is not exceeding the number of particles N implies a bound on the temperature,

$$\langle \mathsf{N}_\Lambda \rangle_{\varrho_\Lambda} \leq N \iff T \in [0, T_C]$$

We aim to find such critical value T_C.

We begin by noticing that

$$\mathrm{Tr}(e^{-\beta\omega \mathsf{N}_\Lambda}) = \left(\frac{B+1}{B}\right)^\ell \leq e^{\frac{\ell}{B}}$$

and then

$$\mathrm{Tr}(\varrho_\Lambda \mathsf{N}_\Lambda) \leq \frac{\mathrm{Tr}(e^{-\beta\omega \mathsf{N}_\Lambda} \mathsf{N}_\Lambda)}{\mathrm{Tr}(e^{-\beta\chi \mathsf{N}_\Lambda^2})} = \frac{\frac{\ell}{B}}{\mathrm{Tr}(e^{-\beta\chi \mathsf{N}_\Lambda^2})}$$

where χ is a constant such that the inequality $\omega \mathsf{N}_\Lambda \leq \mathsf{H}_\Lambda \leq \chi \mathsf{N}_\Lambda^2$ is satisfied. The explicit expression for such quantity is $\chi = 27 + 2C_V$ where C_V is the Hardy constant of the potential (see [44] for a detailed proof). A lower bound for the denominator is given by

$$\mathrm{Tr}(e^{-\beta\chi \mathsf{N}_\Lambda^2}) \geq \ell \sum_{n=0}^{\infty} e^{-\beta\chi n^2} \geq \ell \int_0^\infty e^{-\beta\chi x^2} \,\mathrm{d}x = \frac{\ell}{2} \left(\frac{\pi}{\beta\chi}\right)^{\frac{1}{2}}$$

and therefore

$$\mathrm{Tr}(\varrho_\Lambda \mathsf{N}_\Lambda) \leq \frac{2}{B}\left(\frac{\beta\chi}{\pi}\right)^{\frac{1}{2}}$$

Notice how the bounding term is Λ independent, but rather it shows a dependence from T. We now impose

$$\frac{2}{B}\left(\frac{\beta\chi}{\pi}\right)^{\frac{1}{2}} \leq N$$

Now, both B and β are temperature-dependent objects, so we rewrite everything in terms of the former by using $\beta = \frac{1}{\omega}\log(B+1)$ and getting

$$\left(\frac{1}{NB}\right)^2 \leq \frac{1}{4}\frac{\omega\pi}{\chi\log(1+B)}$$

By rearranging the above equation we get the following inequality, which cannot however be analytically solved

$$\frac{\log(1+B)}{B^2} \leq \frac{1}{4}\frac{\omega\pi N^2}{\chi} \tag{3.16}$$

Even if we cannot write a solution explicitly, it is clear that the function $\frac{\log(1+B)}{B^2}$ is strictly decreasing in B, and therefore the inequality must be solved for $B \in [B_C, +\infty)$ for some B_C. Then, since also B itself is a decreasing function of T we get that $B \in [B_C, +\infty)$ implies $T \in [0, T_C]$, as stated in the beginning.

In order to write an explicit bound for T, we use the stronger condition

$$\frac{1}{\sqrt{B}} \leq \frac{1}{4}\frac{\omega\pi N^2}{\chi} \tag{3.17}$$

where

$$\frac{1}{\sqrt{B}} \geq \frac{\log(1+B)}{B^2} \text{ for all } B > 0$$

The resulting interval is

$$e^{\beta\omega} - 1 \geq \left(\frac{4\chi}{\omega\pi N^2}\right)^2$$

and therefore

$$0 \leq T \leq \frac{\omega}{\log\left(\left(\frac{4\chi}{\omega\pi N^2}\right)^2 + 1\right)} = T_C \tag{3.18}$$

We stress once more how the bound is independent from the cutoff Λ.

Remark. In [44] this computation is done for a more general confinement potential $u(x)$: indeed, the potential is left unspecified but the constraint

$$\text{There exists } c, q, \Omega > 0 \text{ and } p \in \mathbb{N} \text{ such that } c|x|^q \leq u(x) \leq \Omega|x|^p$$

is required. Notice that the harmonic potential that we chose satisfies such constraint for instance with $c = \frac{1}{2}$, $q = 2$, $\Omega = \frac{1}{2}$ and $p = 2$. Also with a generic p-potential the temperature is bounded, namely

$$0 \leq T \leq \frac{\omega}{\log\left(\left(\frac{p}{\Gamma(1/p)}\right)^2 \left(\frac{\chi_p}{\omega}\right)^{\frac{2}{p}} \frac{1}{N^2} + 1\right)} = T_C \tag{3.19}$$

where

$$\chi_p = 2(1+\Omega)3^p + 2C_V$$

The p-potential $u_p(x) = |x|^p$ in the limit $p \to +\infty$ reproduces the infinite well potential,

$$u_\infty(x) = \begin{cases} 0 \text{ if } |x| < 1 \\ +\infty \text{ elsewhere} \end{cases}$$

and also the critical temperature has a finite limit[13], $0 < \lim_{p\to+\infty} T_C < +\infty$.

Remark. The critical temperature appearing in equations (3.18) and (3.19) is smaller than the actual one, since in both cases they were derived using the inequality (3.17), which is stronger than the actual condition (3.16). Recall that we have used the stronger (and easier) inequality in order to get an analytic expression for T_C.

[13]Indeed, $\lim_{p\to+\infty} \frac{p}{\Gamma\left(\frac{1}{p}\right)} = 1$.

CHAPTER 4

Convergence to Hartree dynamics

Overview

In this last chapter the main result about the convergence towards Hartree dynamics is presented. The first part of the chapter is devoted to the discussion of bounds and preliminary results that will be used in the proof of the convergence theorem. In the second part the result is presented and proved: proof will follow the one presented in [44], using however some enhanced estimates. The proof is followed by a short comparison of the result with some of the existing literature, mainly [9, 21, 36]. Finally, a brief discussion about the application of the estimate to some physical potentials is carried out.

4.1 Preliminary results

We need to find the equations of motions of the Wick symbols of annihilation operators a. We have the following proposition:

Proposition 4.1. Let $\mathcal{L}_0$ be the Lie derivative along the flow of H_Λ

$$\mathcal{L}_0(\cdot) = \{\cdot, H_\Lambda\}$$

and let $\mathcal{L}_1$ be the following differential operator

$$\mathcal{L}_1 = \frac{1}{2}\sum_{ij}^{\Lambda}\left(\frac{\partial^2 H_\Lambda}{\partial\bar{\alpha}_i\partial\bar{\alpha}_j}\frac{\partial^2}{\partial\alpha_i\partial\alpha_j} - \frac{\partial^2 H_\Lambda}{\partial\alpha_i\partial\alpha_j}\frac{\partial^2}{\partial\bar{\alpha}_i\partial\bar{\alpha}_j}\right)$$

Then, a_k satisfies the following

$$\begin{cases} i\dot{a}_k = (\mathcal{L}_0 + \mathcal{L}_1)a_k \\ a_k(0) = \alpha_k \end{cases}$$

Proof. We begin with the operatorial Heisenberg equation for time evolution

$$i\frac{\mathrm{d}\mathsf{a}_k}{\mathrm{d}t} = [\mathsf{a}_k(t), \mathsf{H}_\Lambda(t)]$$

The coherent expectation value of such expression is given by

$$i\dot{a}_k(t) = \{\!\{a_k(t), H_\Lambda\}\!\}$$

Now, using equation (3.2) we can expand the Wick bracket as

$$\{\!\{a_k(t), H_\Lambda\}\!\} = \{a_k(t), H_\Lambda\} + \frac{1}{2}\sum_{ij}^{\Lambda}\left(\frac{\partial^2 H_\Lambda}{\partial\bar{\alpha}_i\partial\bar{\alpha}_j}\frac{\partial^2 a_k(t)}{\partial\alpha_i\partial\alpha_j} - \frac{\partial^2 H_\Lambda}{\partial\alpha_i\partial\alpha_j}\frac{\partial^2 a_k(t)}{\partial\bar{\alpha}_i\partial\bar{\alpha}_j}\right)$$

where all the terms containing derivatives of third order or higher are vanishing since H_Λ is a polynomial of degree 2 in the variables $\alpha, \bar{\alpha}$. □

We want to see how far is the evolution of the a_k from the effective field's c_k through the evaluation of the dynamics of the deviation term $\delta_k(t) = a_k(t) - c_k(t)$.

Proposition 4.2. Let $\delta_k(t) = a_k(t) - c_k(t)$. Then

$$\dot{\delta}_k(t) = \mathcal{L}_0\delta_k + \mathcal{L}_1 a_k$$

for $\delta_k(0) = 0$.

Proof. We have $i\dot{c}_k = \{c_k, H_\Lambda\} = \mathcal{L}_0 c_k$ and Lie derivative is linear, therefore

$$i\dot{\delta}_k = i(\dot{a}_k - \dot{c}_k) = \mathcal{L}_0(a_k - c_k) + \mathcal{L}_1 a_k = \mathcal{L}_0\delta_k + \mathcal{L}_1 a_k$$

□

Recall that the *pull-back* of a function f through the Hamiltonian flow $\Phi^t_{X_H}(x)$ is defined as

$$(\Phi^t_{X_H})^* f = f \circ \Phi^t_{X_H}$$

In the following we will be interested in the pullback through the flow of H_Λ, therefore we shall indicate $\Phi^t_{X_{H_\Lambda}}$ simply as Φ^t.

Proposition 4.3. The deviation term can be written as

$$\delta_k(t) = \int_0^t \left(\Phi^{t-s}\right)^* \mathcal{L}_1 a_k(s)\,\mathrm{d}s$$

Proof. We have

$$\delta_k(t) = \left(\Phi^t\right)^* \delta_k(0) + \int_0^t \left(\Phi^{t-s}\right)^* \mathcal{L}_1 a_k(s)\,\mathrm{d}s = \int_0^t \left(\Phi^{t-s}\right)^* \mathcal{L}_1 a_k(s)\,\mathrm{d}s$$

□

Proposition 4.4. The μ-norm of the deviation term is bounded by the term

$$\|\delta_k(t)\|_\mu^2 \leq \left(\int_0^t \|\mathcal{L}_1 a_k(s)\|_\mu\,\mathrm{d}s\right)^2$$

Proof. By definition of μ-norm we have

$$\|\delta_k(t)\|_\mu^2 = \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \int_{\mathbb{C}^\ell} \overline{\left(\Phi^{t-s}\right)^* \mathcal{L}_1 a_k(s)}\left(\Phi^{t-s'}\right)^* \mathcal{L}_1 a_k(s')\,\mathrm{d}\mu$$

We can apply Cauchy-Schwarz inequality to get the thesis

$$\begin{aligned}\|\delta_k(t)\|_\mu^2 &\leq \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \left\| \left(\Phi^{t-s}\right)^* \mathcal{L}_1 a_k(s) \right\|_\mu \left\| \left(\Phi^{t-s'}\right)^* \mathcal{L}_1 a_k(s') \right\|_\mu \\ &\overset{\diamondsuit}{\leq} \int_0^t \mathrm{d}s \int_0^t \mathrm{d}s' \, \|\mathcal{L}_1 a_k(s)\|_\mu \, \|\mathcal{L}_1 a_k(s')\|_\mu \\ &= \left(\int_0^t \|\mathcal{L}_1 a_k(s)\|_\mu \, \mathrm{d}s \right)^2\end{aligned}$$

where in ($\diamondsuit$) the invariance of the measure was used. □

Computation of the remainder

We want to estimate the remainder $\|\mathcal{L}_1 a_k\|_\mu$. Recall that we have

$$\mathcal{L}_1 a_q(s) = \frac{1}{2} \sum_{ij}^{\Lambda} \left(\frac{\partial^2 H_\Lambda}{\partial \bar{\alpha}_i \partial \bar{\alpha}_j} \frac{\partial^2 a_q(s)}{\partial \alpha_i \partial \alpha_j} - \frac{\partial^2 H_\Lambda}{\partial \alpha_i \partial \alpha_j} \frac{\partial^2 a_q(s)}{\partial \bar{\alpha}_i \partial \bar{\alpha}_j} \right)$$

Now, let $f(\alpha, \bar{\alpha}) = \sigma_W(\mathsf{F})(\alpha, \bar{\alpha})$, then the following identities that allow to express the derivative of any Wick symbol to the Wick symbol of a commutator can be proved

$$\frac{\partial f}{\partial \bar{\alpha}_k}(\alpha, \bar{\alpha}) = \langle \phi_\alpha, [\mathsf{a}_k, \mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger)] \phi_\alpha \rangle \tag{4.1}$$

$$\frac{\partial f}{\partial \alpha_k}(\alpha, \bar{\alpha}) = \langle \phi_\alpha, [\mathsf{F}(\mathsf{a}, \mathsf{a}^\dagger), \mathsf{a}_k^\dagger] \phi_\alpha \rangle \tag{4.2}$$

Second derivatives of H_Λ are

$$\frac{\partial^2 H_\Lambda}{\partial \bar{\alpha}_i \partial \bar{\alpha}_j} = \sum_{mn} V_{(ij)mn} \alpha_m \alpha_n$$

$$\frac{\partial^2 H_\Lambda}{\partial \alpha_i \partial \alpha_j} = \sum_{kl} V_{kl(ij)} \bar{\alpha}_k \bar{\alpha}_l$$

where

$$V_{(ij)mn} = \frac{V_{ijmn} + V_{jimn}}{2} \qquad V_{kl(ij)} = \frac{V_{kl(ij)} + V_{kl(ji)}}{2}$$

Using equations (4.1) and (4.2) we can compute

$$\begin{aligned}\frac{\partial^2 a_q(s)}{\partial \bar{\alpha}_i \partial \bar{\alpha}_j} &= \frac{\partial}{\partial \bar{\alpha}_i} \frac{\partial a_q(s)}{\partial \bar{\alpha}_j} = \frac{\partial}{\partial \bar{\alpha}_i} \langle \phi_\alpha, [\mathsf{a}_j, \mathsf{a}_q(s)] \phi_\alpha \rangle = \\ &= \langle \phi_\alpha, [\mathsf{a}_i, [\mathsf{a}_j, \mathsf{a}_q(s)]] \phi_\alpha \rangle = \langle \phi_\alpha, \Big(\mathsf{a}_i \mathsf{a}_j \mathsf{a}_q(s) - \mathsf{a}_j \mathsf{a}_q(s) \mathsf{a}_i - \mathsf{a}_i \mathsf{a}_q(s) \mathsf{a}_j + \mathsf{a}_q(s) \mathsf{a}_j \mathsf{a}_i \Big) \phi_\alpha \rangle \\ &= \langle \phi_\alpha, \mathsf{a}_i \mathsf{a}_j \mathsf{a}_q(s) \phi_\alpha \rangle - \alpha_i \langle \phi_\alpha, \mathsf{a}_j \mathsf{a}_q(s) \phi_\alpha \rangle - \alpha_j \langle \phi_\alpha, \mathsf{a}_i \mathsf{a}_q(s) \phi_\alpha \rangle + \alpha_i \alpha_j a_q(s)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 a_q(s)}{\partial \alpha_i \partial \alpha_j} &= \frac{\partial}{\partial \alpha_i} \frac{\partial a_q(s)}{\partial \alpha_j} = \frac{\partial}{\partial \alpha_i} \langle \phi_\alpha, [\mathsf{a}_q(s), \mathsf{a}_j^\dagger] \phi_\alpha \rangle \\ &= \langle \phi_\alpha, [[\mathsf{a}_q(s), \mathsf{a}_j^\dagger], \mathsf{a}_i^\dagger] \phi_\alpha \rangle = \langle \phi_\alpha, \Big(\mathsf{a}_q(s) \mathsf{a}_j^\dagger \mathsf{a}_i^\dagger - \mathsf{a}_i^\dagger \mathsf{a}_q(s) \mathsf{a}_j^\dagger - \mathsf{a}_j^\dagger \mathsf{a}_q(s) \mathsf{a}_i^\dagger + \mathsf{a}_i^\dagger \mathsf{a}_j^\dagger \mathsf{a}_q(s) \Big) \phi_\alpha \rangle \\ &= \langle \phi_\alpha \mathsf{a}_q(s) \mathsf{a}_j^\dagger \mathsf{a}_i^\dagger \phi_\alpha \rangle - \bar{\alpha}_i \langle \phi_\alpha \mathsf{a}_q(s) \mathsf{a}_j^\dagger \phi_\alpha \rangle - \bar{\alpha}_j \langle \phi_\alpha \mathsf{a}_q(s) \mathsf{a}_i^\dagger \phi_\alpha \rangle + \bar{\alpha}_i \bar{\alpha}_j a_q(s)\end{aligned}$$

Recall that

$$H_\Lambda(\alpha,\bar\alpha)=\sum_k^\Lambda \varepsilon_k\bar\alpha_k\alpha_k+\frac{1}{2}\sum_{klmn}^\Lambda V_{klmn}\bar\alpha_k\bar\alpha_l\alpha_m\alpha_n$$

and so by putting all together we get

$$\begin{aligned}\mathcal{L}_1 a_q(s)=&\frac{1}{2}\sum_{ijmn}V_{(ij)mn}\alpha_m\alpha_n\Big(\langle\phi_\alpha \mathsf{a}_q(s)\mathsf{a}_j^\dagger\mathsf{a}_i^\dagger\phi_\alpha\rangle-\bar\alpha_i\langle\phi_\alpha\mathsf{a}_q(s)\mathsf{a}_j^\dagger\phi_\alpha\rangle-\\&-\bar\alpha_j\langle\phi_\alpha\mathsf{a}_q(s)\mathsf{a}_i^\dagger\phi_\alpha\rangle+\bar\alpha_i\bar\alpha_j a_q(s)\Big)\\&-\frac{1}{2}\sum_{ijkl}V_{(ij)kl}\bar\alpha_k\bar\alpha_l\Big(\langle\phi_\alpha,\mathsf{a}_i\mathsf{a}_j\mathsf{a}_q(s)\phi_\alpha\rangle-\alpha_i\langle\phi_\alpha,\mathsf{a}_j\mathsf{a}_q(s)\phi_\alpha\rangle-\\&-\alpha_j\langle\phi_\alpha,\mathsf{a}_i\mathsf{a}_q(s)\phi_\alpha\rangle+\alpha_i\alpha_j a_q(s)\Big)\end{aligned}$$

By renaming the indices as $i,j\mapsto k,l$ in the first sum and $i,j\mapsto m,n$ in the second we can bring everything under the same sum

$$\begin{aligned}\mathcal{L}_1 a_q(s)=\frac{1}{2}\sum_{klmn}\Big[&V_{(kl)mn}\alpha_m\alpha_n\big(\langle\phi_\alpha,\mathsf{a}_q(s)\mathsf{a}_l^\dagger\mathsf{a}_k^\dagger\phi_\alpha\rangle-\bar\alpha_k\langle\phi_\alpha,\mathsf{a}_q(s)\mathsf{a}_l^\dagger\phi_\alpha\rangle-\bar\alpha_l\langle\phi_\alpha,\mathsf{a}_q(s)\mathsf{a}_k^\dagger\phi_\alpha\rangle+\bar\alpha_k\bar\alpha_l a_q(s)\big)\\&-V_{(mn)kl}\bar\alpha_k\bar\alpha_l\big(\langle\phi_\alpha,\mathsf{a}_m\mathsf{a}_n\mathsf{a}_q(s)\phi_\alpha\rangle-\alpha_m\langle\phi_\alpha,\mathsf{a}_n\mathsf{a}_q(s)\phi_\alpha\rangle-\\&-\alpha_m\langle\phi_\alpha,\mathsf{a}_n\mathsf{a}_q(s)\phi_\alpha\rangle+\alpha_m\alpha_n a_q(s)\big)\Big]\end{aligned}$$

Moreover, noticing that

$$V_{(kl)mn}=\frac{1}{2}\left(V_{klmn}+V_{lkmn}\right)=\frac{1}{2}\left(V_{klmn}+V_{klnm}\right)=V_{(mn)kl}$$

we can rewrite

$$\begin{aligned}\mathcal{L}_1 a_q(s)=\frac{1}{2}\sum_{klmn}V_{(kl)mn}\big(&\alpha_m\alpha_n\langle\phi_\alpha,\mathsf{a}_q(s)\mathsf{a}_l^\dagger\mathsf{a}_k^\dagger\phi_\alpha\rangle-\alpha_m\alpha_n\bar\alpha_k\langle\phi_\alpha,\mathsf{a}_q(s)\mathsf{a}_l^\dagger\phi_\alpha\rangle-\alpha_m\alpha_n\bar\alpha_l\langle\phi_\alpha,\mathsf{a}_q(s)\mathsf{a}_k^\dagger\phi_\alpha\rangle+\\&-\bar\alpha_k\bar\alpha_l\langle\phi_\alpha,\mathsf{a}_m\mathsf{a}_n\mathsf{a}_q(s)\phi_\alpha\rangle-\bar\alpha_k\bar\alpha_l\alpha_m\langle\phi_\alpha,\mathsf{a}_n\mathsf{a}_q(s)\phi_\alpha\rangle-\bar\alpha_k\bar\alpha_l\alpha_n\langle\phi_\alpha,\mathsf{a}_m\mathsf{a}_q(s)\phi_\alpha\rangle\big)\end{aligned}$$

Proposition 4.5. The following inequality holds

$$|\mathcal{L}_1 a_q(s)|^2\le\sigma_W(\mathsf{a}_q(s)\mathsf{a}_q^\dagger(s))(\alpha,\bar\alpha)(p(\alpha,\bar\alpha))^2$$

where p is the following polynomial

$$p(\alpha,\bar\alpha)=\sum_{klmn}|V_{(kl)mn}|\big(|\alpha_m\alpha_n|+|\alpha_k\alpha_m\alpha_n|+|\alpha_l\alpha_m\alpha_n|+|\alpha_m\alpha_k\alpha_n|+2|\alpha_k\alpha_l\alpha_m\alpha_n|\big)$$

Proof. Employing triangular inequality in the following form

$$|z_1+z_2+\cdots+z_n|\le|z_1|+|z_2|+\cdots+|z_n|\text{ for all }z_1,\dots,z_n\in\mathbb{C}$$

we write

$$\begin{aligned}|\mathcal{L}_1 a_q(s)|\le\frac{1}{2}\sum_{klmn}|V_{(kl)mn}|\big(&|\alpha_m\alpha_n|\left|\langle\phi_\alpha,\mathsf{a}_q(s)\mathsf{a}_l^\dagger\mathsf{a}_k^\dagger\phi_\alpha\rangle\right|+|\alpha_m\alpha_n|\left|\bar\alpha_k\langle\phi_\alpha,\mathsf{a}_q(s)\mathsf{a}_l^\dagger\phi_\alpha\rangle\right|+\\&+|\alpha_m\alpha_n\bar\alpha_l|\left|\langle\phi_\alpha,\mathsf{a}_q(s)\mathsf{a}_k^\dagger\phi_\alpha\rangle\right|+|\bar\alpha_k\bar\alpha_l|\,|\langle\phi_\alpha,\mathsf{a}_m\mathsf{a}_n\mathsf{a}_q(s)\phi_\alpha\rangle|+\\&+|\bar\alpha_k\bar\alpha_l\alpha_m|\,|\langle\phi_\alpha,\mathsf{a}_n\mathsf{a}_q(s)\phi_\alpha\rangle|+|\bar\alpha_k\bar\alpha_l\alpha_n|\,|\langle\phi_\alpha,\mathsf{a}_m\mathsf{a}_q(s)\phi_\alpha\rangle|\,\big)\end{aligned}$$

There are four type of coherent expectation values in this expression, and we shall estimate all of them: each of the following estimates will depend explicitly on the Wick symbol of $\mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)$, therefore for brevity we denote such quantity as $Q_\alpha \equiv \langle\phi_\alpha, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\phi_\alpha\rangle$. For the first one we can use proposition 3.1

$$\begin{aligned}\langle\phi_\alpha, \mathsf{a}_q(s)\mathsf{a}_k^\dagger\mathsf{a}_l^\dagger\phi_\alpha\rangle &= \langle\mathsf{a}_q^\dagger(s)\phi_\alpha, \mathsf{a}_k^\dagger\mathsf{a}_l^\dagger\phi_\alpha\rangle \\ &\leq \left\|\mathsf{a}_q^\dagger(s)\phi_\alpha\right\| \left\|\mathsf{a}_k^\dagger\mathsf{a}_l^\dagger\phi_\alpha\right\| \\ &\leq \sqrt{\langle\phi_\alpha, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\phi_\alpha\rangle}\sqrt{\langle\phi_\alpha, \mathsf{a}_k\mathsf{a}_l\mathsf{a}_k^\dagger\mathsf{a}_l^\dagger\phi_\alpha\rangle} \\ &\leq Q_\alpha^{\frac{1}{2}}(2+|\alpha_k|+|\alpha_l|+|\alpha_k\alpha_l|)\end{aligned}$$

and also for the second

$$\begin{aligned}\langle\phi_\alpha, \mathsf{a}_q(s)\mathsf{a}_l^\dagger\phi_\alpha\rangle &= \langle\mathsf{a}_q^\dagger(s)\phi_\alpha, \mathsf{a}_l^\dagger\phi_\alpha\rangle \\ &\leq \|\mathsf{a}_q(s)\phi_\alpha\| \left\|\mathsf{a}_l^\dagger\phi_\alpha\right\| \\ &\leq \sqrt{\langle\phi_\alpha, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\phi_\alpha\rangle}\sqrt{\langle\phi_\alpha, \mathsf{a}_l\mathsf{a}_l^\dagger\phi_\alpha\rangle} \\ &\leq Q_\alpha^{\frac{1}{2}}(1+|\alpha_l|)\end{aligned}$$

Instead, for the third and the fourth one we have simpler expressions:

$$\begin{aligned}|\langle\phi_\alpha, \mathsf{a}_m\mathsf{a}_n\mathsf{a}_q(s)\phi_\alpha\rangle| &= \left|\langle\mathsf{a}_m^\dagger\mathsf{a}_n^\dagger\phi_\alpha, \mathsf{a}_q(s)\phi_\alpha\rangle\right| \\ &= \left|\langle\mathsf{a}_q^\dagger(s)\phi_\alpha, \mathsf{a}_m\mathsf{a}_n\phi_\alpha\rangle\right| \\ &\leq \left\|\mathsf{a}_q^\dagger\right\| \|\mathsf{a}_m\mathsf{a}_n\| \\ &= Q_\alpha^{\frac{1}{2}}\sqrt{\langle\mathsf{a}_m\mathsf{a}_n\phi_\alpha, \mathsf{a}_m\mathsf{a}_n\phi_\alpha\rangle} \\ &= Q_\alpha^{\frac{1}{2}}|\alpha_m||\alpha_n|\end{aligned}$$

$$\begin{aligned}|\langle\phi_\alpha, \mathsf{a}_n\mathsf{a}_q(s)\phi_\alpha\rangle| &= \left|\langle\mathsf{a}_n^\dagger\phi_\alpha, \mathsf{a}_q(s)\phi_\alpha\rangle\right| \\ &= \left|\langle\mathsf{a}_q^\dagger(s)\phi_\alpha, \mathsf{a}_n\phi_\alpha\rangle\right| \\ &\leq \left\|\mathsf{a}_q^\dagger\right\| \|\mathsf{a}_n\| \\ &= Q_\alpha^{\frac{1}{2}}\sqrt{\langle\mathsf{a}_n\phi_\alpha, \mathsf{a}_n\phi_\alpha\rangle} \\ &= Q_\alpha^{\frac{1}{2}}|\alpha_n|\end{aligned}$$

We can put everything together and write

$$\begin{aligned}|\mathcal{L}_1 a_q(s)| \leq \frac{Q_\alpha^{\frac{1}{2}}}{2}\sum_{klmn}|V_{(kl)mn}|\big(&|\alpha_m\alpha_n|\,(2+|\alpha_k|+|\alpha_l|+|\alpha_k\alpha_l|)+|\alpha_m\alpha_n\bar{\alpha}_k|\,(1+|\alpha_l|)+ \\ &+|\alpha_m\alpha_n\bar{\alpha}_l|\,(1+|\alpha_k|)+|\bar{\alpha}_k\bar{\alpha}_l|\,|\alpha_m\alpha_n|+|\bar{\alpha}_k\bar{\alpha}_l\alpha_m|\,|\alpha_n|+|\bar{\alpha}_k\bar{\alpha}_l\alpha_n|\,|\alpha_m|\,\big) \\ \leq \frac{Q_\alpha^{\frac{1}{2}}}{2}\sum_{klmn}&|V_{(kl)mn}|\big(2|\alpha_m\alpha_n|+2|\alpha_k\alpha_m\alpha_n|+2|\alpha_l\alpha_m\alpha_n|+|\alpha_m\alpha_k\alpha_n|+4|\alpha_k\alpha_l\alpha_m\alpha_n|\big) \\ \leq Q_\alpha^{\frac{1}{2}}\sum_{klmn}&|V_{(kl)mn}|\big(|\alpha_m\alpha_n|+|\alpha_k\alpha_m\alpha_n|+|\alpha_l\alpha_m\alpha_n|+|\alpha_m\alpha_k\alpha_n|+2|\alpha_k\alpha_l\alpha_m\alpha_n|\big) \\ = Q_\alpha^{\frac{1}{2}}\;&p(\alpha)\end{aligned}$$

□

Proposition 4.6. Assuming $B \geq 1$ we have

$$\|\mathcal{L}_1 a_q(s)\|_\mu \leq 2^8 \sum_{klmn}^{\Lambda} |V_{klmn}|$$

Proof. From the previous proposition we have

$$\|\mathcal{L}_1 a_q(s)\|^2 \leq \int_{\mathbb{C}^\ell} \langle \phi_\alpha, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\phi_\alpha\rangle p^2(\alpha)\,\mathrm{d}\mu(\alpha,\bar{\alpha}) \leq \left\|p^2\right\|_\mu^2 \left[\int_{\mathbb{C}^\ell} \left|\langle \phi_\alpha, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\phi_\alpha\rangle\right|^2 \mathrm{d}\mu(\alpha,\bar{\alpha})\right]^{\frac{1}{2}}$$

We can compute the integral appearing on the right-hand side as follows: first, notice that from the invariance of the measure we can write

$$\begin{aligned}\int_{\mathbb{C}^\ell} \left|\langle \phi_\alpha, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\phi_\alpha\rangle\right|^2 \mathrm{d}\mu(\alpha,\bar{\alpha}) &= \int_{\mathbb{C}^\ell} \langle \phi_\alpha, \mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\mathsf{a}_q(s)\mathsf{a}_q^\dagger(s)\phi_\alpha\rangle\,\mathrm{d}\mu(\alpha,\bar{\alpha}) \\ &= \int_{\mathbb{C}^\ell} \langle \phi_\alpha, \mathsf{a}_q\mathsf{a}_q^\dagger\mathsf{a}_q\mathsf{a}_q^\dagger\phi_\alpha\rangle\,\mathrm{d}\mu(\alpha,\bar{\alpha})\end{aligned}$$

and then, using CCRs and proposition 3.3 we have

$$\begin{aligned}\int_{\mathbb{C}^\ell} \langle \phi_\alpha, \mathsf{a}_q\mathsf{a}_q^\dagger\mathsf{a}_q\mathsf{a}_q^\dagger\phi_\alpha\rangle\,\mathrm{d}\mu(\alpha,\bar{\alpha}) &= \int_{\mathbb{C}^\ell} \langle \phi_\alpha, (\mathsf{a}_q^\dagger\mathsf{a}_q^\dagger\mathsf{a}_q\mathsf{a}_q + 3\mathsf{a}_q^\dagger\mathsf{a}_q + 1)\phi_\alpha\rangle\,\mathrm{d}\mu(\alpha,\bar{\alpha}) \\ &= \int_{\mathbb{C}^\ell} (|\alpha_q|^4 + 3|\alpha_q|^2 + 1)\,\mathrm{d}\mu(\alpha,\bar{\alpha}) \\ &= 1 + \frac{3}{B} + \frac{2}{B^2}\end{aligned}$$

The norm of the polynomial p is instead harder to compute, and we will give an estimate for its upper bound: let

$$\left\|p^2\right\|_\mu^2 = \int_{\mathbb{C}^\ell} \left[\sum_{klmn} |V_{(kl)mn}|\big(|\alpha_m\alpha_n| + |\alpha_k\alpha_m\alpha_n| + |\alpha_l\alpha_m\alpha_n| + |\alpha_m\alpha_k\alpha_n| + 2|\alpha_k\alpha_l\alpha_m\alpha_n|\big)\right]^4 \mathrm{d}\mu(\alpha,\bar{\alpha})$$

and let us put $\mathcal{P}_{klmn} = |\alpha_m\alpha_n| + |\alpha_k\alpha_m\alpha_n| + |\alpha_l\alpha_m\alpha_n| + |\alpha_m\alpha_k\alpha_n| + 2|\alpha_k\alpha_l\alpha_m\alpha_n|$. Now, we know from proposition 1.4 that $|V_{klmn}| < \mathcal{B}_\Lambda$ and therefore

$$\begin{aligned}\left\|p^2\right\|_\mu^2 &= \int_{\mathbb{C}^\ell} \left(\sum_{klmn}^{\Lambda} |V_{klmn}|\mathcal{P}_{klmn}\right)^4 \mathrm{d}\mu(\alpha,\bar{\alpha}) \\ &\leq \int_{\mathbb{C}^\ell} \left(\sum_{klmn}^{\Lambda} \mathcal{B}\mathcal{P}_{klmn}\right)^4 \mathrm{d}\mu(\alpha,\bar{\alpha}) \\ &= \mathcal{B}^4 \int_{\mathbb{C}^\ell} \left(\sum_{klmn}^{\Lambda} \mathcal{P}_{klmn}\right)^4 \mathrm{d}\mu(\alpha,\bar{\alpha})\end{aligned}$$

Let us denote the quadruplets of multiindices as $I = klmn$, then opening the fourth power gives

$$\mathcal{B}^4 \int_{\mathbb{C}^\ell} \left(\sum_{klmn}^{\Lambda} \mathcal{P}_{klmn}\right)^4 \mathrm{d}\mu(\alpha,\bar{\alpha}) = \mathcal{B}^4 \int_{\mathbb{C}^\ell} \sum_{I}^{\Lambda}\sum_{I'}^{\Lambda}\sum_{I''}^{\Lambda}\sum_{I'''}^{\Lambda} \mathcal{P}_I\mathcal{P}_{I'}\mathcal{P}_{I''}\mathcal{P}_{I'''}\,\mathrm{d}\mu(\alpha,\bar{\alpha}) \tag{4.3}$$

Now, the product gives 70 terms but each of these terms has the same structure, namely it is the modulus of a product of a given number of α with different indices:

$$|\alpha_{k'}\alpha_{k''}\dots\alpha_{m'''}\alpha_{n'''}|$$

However, when integrating these terms over the phase space the indices are not relevant, and the integrals depends only on the number of α contained in each modulus. Therefore the whole equation (4.3) reduces to

$$\left\|p^2\right\|_\mu^2 = \mathcal{B}^4 \sum_I^\Lambda \sum_{I'}^\Lambda \sum_{I''}^\Lambda \sum_{I'''}^\Lambda \int (\mathcal{P}_I)^4 \, \mathrm{d}\mu(\alpha,\bar{\alpha}) = \mathcal{B}^4 \sum_I^\Lambda \sum_{I'}^\Lambda \sum_{I''}^\Lambda \sum_{I'''}^\Lambda J_{klmn}$$

where we have called J_{klmn} the following integral

$$\begin{aligned} J_{klmn} &= \int_{\mathbb{C}^\ell} (\mathcal{P}_{klmn})^4 \, \mathrm{d}\mu(\alpha,\bar{\alpha}) \\ &= \int_{\mathbb{C}^\ell} \left[|\alpha_m\alpha_n| + |\alpha_k\alpha_m\alpha_n| + |\alpha_l\alpha_m\alpha_n| + |\alpha_m\alpha_k\alpha_n| + 2|\alpha_k\alpha_l\alpha_m\alpha_n|\right]^4 \mathrm{d}\mu(\alpha,\bar{\alpha}) \end{aligned}$$

then it can be shown that for all $B \geq 1$

$$J_{klmn} \leq \frac{\mathcal{N}^4}{B^4} \tag{4.4}$$

where $\mathcal{N}$ is some integer constant, in our case $\mathcal{N} = 54$, see the remark at the end of the proof for an explanation. Then

$$\left\|p^2\right\|_\mu^2 \leq \mathcal{B}^4 \frac{\mathcal{N}^4}{B^4} \left(\sum_{klmn} 1\right)^4 = \left(\mathcal{B}_\Lambda \frac{\mathcal{N}}{B} \Lambda^{4d}\right)^4$$

The norm can be estimated as

$$\left\|\mathcal{L}_1 a_q(s)\right\|_\mu \leq \left\|p^2\right\|_\mu^{\frac{1}{2}} \left(1 + \frac{3}{B} + \frac{2}{B^2}\right)^{\frac{1}{2}} \leq \left(\frac{\mathcal{N}\mathcal{B}_\Lambda}{B}\Lambda^{4d}\right)\left(1 + \frac{3}{B} + \frac{2}{B^2}\right)^{\frac{1}{2}}$$

and it is simple to verify that for every $B \geq 1$ the inequality

$$\frac{1}{B}\left(1 + \frac{3}{B} + \frac{2}{B^2}\right)^{\frac{1}{2}} < \frac{\sqrt{6}}{B}$$

is satisfied. We conclude that

$$\left\|\mathcal{L}_1 a_q(s)\right\|_\mu \leq \mathcal{N}\mathcal{B}_\Lambda \Lambda^{4d} \frac{\sqrt{6}}{B}$$

□

Remark. We can motivate the bound of equation (4.4) as follows: as already said the phase space integral does not depend on the indices, but only on the number of α contained in each modulus. Therefore we can group the terms having the same number of α and we can moreover omit the indices: let us call A_j the generic term containing the product of j α factors, then

$$\begin{aligned} J_{klmn} &= \int_{\mathbb{C}^\ell} \left[|A_2| + 3|A_3| + 2|A_4|\right]^4 \mathrm{d}\mu(\alpha,\bar{\alpha}) \\ &= \int_{\mathbb{C}^\ell} \Big[|\alpha_k|^8 + 12|\alpha_k|^9 + 62|\alpha_k|^{10} + 180|\alpha_k|^{11} \\ &\qquad + 321|\alpha_k|^{12} + 360|\alpha_k|^{13} + 248|\alpha_k|^{14} + 96|\alpha_k|^{15} + 16|\alpha_k|^{16}\Big] \mathrm{d}\mu(\alpha,\bar{\alpha}) \\ &\propto \frac{1}{B^4} \\ &\leq \frac{7949784}{B^4} \end{aligned}$$

where[1] $7949784 = \Gamma(5) + \Gamma(6)(12+62) + \Gamma(7)(180+321) + \Gamma(8)(360+248) + \Gamma(9)(96+16)$. Since $\sqrt[4]{7949784} \simeq 53.09$ we put $\mathcal{N} = 54$.

4.2 Convergence to Hartree dynamics

We are ready to state and prove the main results, using the propositions developed up to now. Recall that $B = e^{\beta\omega} - 1$.

Proposition 4.7. The regularized effective field has a time-independent μ-norm, given by

$$\left\|\Psi_\Lambda^{(0)}(t)\right\|_\mu = \frac{1}{\sqrt{B}}$$

Proof. Measure μ is invariant under Hartree flow, thus

$$\left\|\Psi_\Lambda^{(0)}(t)\right\|_\mu = \left\|\Psi_\Lambda^{(0)}(0)\right\|_\mu = \left\|\sum_k^\Lambda \mathsf{c}_k(0)\varphi_k(x)\right\|_\mu$$

Then, using the μ-norm definition we have

$$\left\|\Psi_\Lambda^{(0)}(0)\right\|_\mu^2 = \frac{1}{\Lambda^d}\sum_k^\Lambda \|\alpha_k\|^2_{L^2(\mathrm{d}\mu)} = \frac{1}{B}$$

and we get the thesis. $\square$

Proposition 4.8 (Main result)**.** Let:

1. $\Psi(t)$ be the solution of the dynamics on Fock space, $i\dot{\Psi} = [\Psi, \mathsf{H}]$;
2. Ψ_Λ be the regularized field operator;
3. $\Psi_\Lambda^{(0)}$ be the effective field.

Then for $B > 1$, a cutoff $\Lambda \geq 1$ and positive time $t \geq 0$ we have the following estimate

$$\|\Psi(t) - \Psi_\Lambda(t)\|_\mu \leq \frac{4C_V\Lambda^{2d}}{B^{\frac{5}{2}}}t \tag{4.5}$$

where C_V is the Hardy constant of the interaction potential and d is the number of dimensions. The fluctuation around the effective field satisfies the following estimate

$$\left\|\Psi_\Lambda(t) - \Psi_\Lambda^{(0)}(t)\right\|_\mu \leq \frac{\mathcal{N}\sqrt{6}C_V(1 + d(2\Lambda+1)^{\frac{1}{2}})\Lambda^{4d}}{B}t \tag{4.6}$$

where $\mathcal{N}$ is an integer constant[2].

Proof. We begin by proving equation (4.5): let us define the operator Δ_Λ as the following difference

$$\Delta_\Lambda(t) = \Psi(t) - \Psi_\Lambda(t)$$

[1] To work with integers, here we are over estimating each $\Gamma\left(\frac{N}{2}\right)$ that appears with $\Gamma\left(\frac{N}{2}+\frac{1}{2}\right)$.
[2] The exact value will be determined in the proof.

Then, by definition of star norm

$$\|\mathsf{\Delta}_\Lambda(t)\|_\mu^2 \leq \frac{1}{\Lambda^d}\|\mathsf{\Delta}_\Lambda(t)\|_\star^2 = \frac{1}{\Lambda^d}\operatorname{Tr}\left(\frac{e^{-\beta\omega\mathsf{N}_\Lambda}}{\operatorname{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})}\int_{\mathbb{R}^d}\mathsf{\Delta}_\Lambda^\dagger(t,x)\mathsf{\Delta}_\Lambda(t,x)\right) \tag{4.7}$$

From a semigroup of operators argument we have (see for instance [20])

$$\mathsf{\Delta}_\Lambda(t) = e^{i\mathsf{H}_\Lambda t}\mathsf{\Delta}_\Lambda(0)e^{-i\mathsf{H}_\Lambda t} + \int_0^t e^{i\mathsf{H}_\Lambda(t-s)}[\mathsf{H}-\mathsf{H}_\Lambda,\mathsf{\Psi}(s)]e^{-i\mathsf{H}_\Lambda(t-s)}\,\mathrm{d}s$$

Using the sub-additivity of the norm[3] we can write the following inequality

$$\|\mathsf{B}_\Lambda(t)\|_\star \leq \left\|e^{i\mathsf{H}_\Lambda t}\mathsf{\Delta}_\Lambda(0)e^{-i\mathsf{H}_\Lambda t}\right\|_\star + \int_0^t\left\|e^{i\mathsf{H}_\Lambda(t-s)}[\mathsf{H}-\mathsf{H}_\Lambda,\mathsf{\Psi}(s)]e^{-i\mathsf{H}_\Lambda(t-s)}\right\|_\star\mathrm{d}s$$

Star norm is invariant under unitary conjugation of operators, indeed

$$\begin{aligned}\|\mathsf{B}_\Lambda(t)\|_\star &\leq \|\mathsf{\Delta}_\Lambda(0)\|_\star + \int_0^t\|[\mathsf{H}-\mathsf{H}_\Lambda,\mathsf{\Psi}(s)]\|_\star\,\mathrm{d}s = \\ &= \|\mathsf{\Delta}_\Lambda(0)\|_\star + \int_0^t\|(\mathsf{H}-\mathsf{H}_\Lambda)\mathsf{\Psi}(s)-\mathsf{\Psi}(s)(\mathsf{H}-\mathsf{H}_\Lambda)\|_\star\,\mathrm{d}s \\ &\leq \|\mathsf{\Delta}_\Lambda(0)\|_\star + \int_0^t\left(\|(\mathsf{H}-\mathsf{H}_\Lambda)\mathsf{\Psi}(s)\|_\star + \|\mathsf{\Psi}(s)(\mathsf{H}-\mathsf{H}_\Lambda)\|_\star\right)\mathrm{d}s\end{aligned} \tag{4.8}$$

The first term is vanishing: we have

$$\|\mathsf{\Delta}_\Lambda(0)\|_\star^2 = \operatorname{Tr}\left(\rho_\Lambda\int_{\mathbb{R}^d}\mathsf{\Delta}^\dagger(0,x)\mathsf{\Delta}(0,x)\,\mathrm{d}x\right)$$

and easily we have that[4]

$$\begin{aligned}\mathsf{\Delta}^\dagger(0,x)\mathsf{\Delta}(0,x) &= (\mathsf{\Psi}^\dagger(x)-\mathsf{\Psi}_\Lambda^\dagger(x))(\mathsf{\Psi}(x)-\mathsf{\Psi}_\Lambda(x)) \\ &= \mathsf{\Psi}^\dagger(x)\mathsf{\Psi}(x)-\mathsf{\Psi}^\dagger(x)P_\Lambda\mathsf{\Psi}(x)-\mathsf{\Psi}^\dagger(x)P_\Lambda\mathsf{\Psi}(x)+\mathsf{\Psi}^\dagger(x)P_\Lambda P_\Lambda\mathsf{\Psi}(x) \\ &= \mathsf{\Psi}^\dagger(x)\mathsf{\Psi}(x)-\mathsf{\Psi}^\dagger(x)P_\Lambda\mathsf{\Psi}(x) \\ &= P_\Lambda\mathsf{\Psi}^\dagger(x)(\mathbf{1}-P_\Lambda)\mathsf{\Psi}(x)P_\Lambda\end{aligned}$$

Therefore, the trace above reduces to

$$\begin{aligned}\|\mathsf{\Delta}_\Lambda(0)\|_\star^2 &= \operatorname{Tr}\left(\rho_\Lambda\int_{\mathbb{R}^d}P_\Lambda\mathsf{\Psi}^\dagger(x)(\mathbf{1}-P_\Lambda)\mathsf{\Psi}(x)P_\Lambda\,\mathrm{d}x\right) \\ &= \operatorname{Tr}\left(\rho_\Lambda P_\Lambda\int_{\mathbb{R}^d}\mathsf{\Psi}^\dagger(x)\mathsf{\Psi}(x)\,\mathrm{d}xP_\Lambda-\rho_\Lambda\int_{\mathbb{R}^d}P_\Lambda\mathsf{\Psi}^\dagger(x)P_\Lambda\mathsf{\Psi}(x)P_\Lambda\,\mathrm{d}x\right) \\ &= \operatorname{Tr}\left(\rho_\Lambda P_\Lambda\mathsf{N}P_\Lambda-\rho_\Lambda\mathsf{N}_\Lambda\right) = 0\end{aligned} \tag{4.9}$$

Let us now focus on the second term: using cyclicity of the trace (in the $\|\cdot\|_\star$ definition) we have that

$$\|(\mathsf{H}-\mathsf{H}_\Lambda)\mathsf{\Psi}(s)\|_\star = \|\mathsf{\Psi}(s)(\mathsf{H}-\mathsf{H}_\Lambda)\|_\star$$

[3] Meaning that $\|\mathsf{A}+\mathsf{B}\|_\star \leq \|\mathsf{A}\|_\star + \|\mathsf{B}\|_\star$.

[4] Recall that $(P_\Lambda)^2 = P_\Lambda$.

Therefore, let us compute the norm appearing on the right-hand side

$$\begin{aligned}\|\Psi(s)(\mathsf{H}-\mathsf{H}_\Lambda)\|_\star^2 &= \mathrm{Tr}\left(\frac{e^{-\beta\omega\mathsf{N}_\Lambda}}{\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})}(\mathsf{H}-\mathsf{H}_\Lambda)\int_{\mathbb{R}^d}\Psi(s,x)\Psi(s,x)\,\mathrm{d}x(\mathsf{H}-\mathsf{H}_\Lambda)\right)\\ &= \mathrm{Tr}\left(\frac{e^{-\beta\omega\mathsf{N}_\Lambda}}{\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})}(\mathsf{H}-\mathsf{H}_\Lambda)\mathsf{N}(\mathsf{H}-\mathsf{H}_\Lambda)\right)\\ &\overset{\diamondsuit}{=} \mathrm{Tr}\left(\frac{e^{-\beta\omega\mathsf{N}_\Lambda}}{\mathrm{Tr}(e^{-\beta\omega\mathsf{N}_\Lambda})}(\mathsf{H}-\mathsf{H}_\Lambda)^2\mathsf{N}_\Lambda\right)\end{aligned}$$

where in ($\diamondsuit$) we used the relation

$$(\mathsf{H}-\mathsf{H}_\Lambda)\mathsf{N}(\mathsf{H}-\mathsf{H}_\Lambda) = (\mathsf{HN}-P_\Lambda\mathsf{H}P_\Lambda\mathsf{N})(\mathsf{H}-\mathsf{H}_\Lambda) = (\mathsf{HN}-P_\Lambda\mathsf{H}P_\Lambda^2\mathsf{N})(\mathsf{H}-\mathsf{H}_\Lambda) = \mathsf{N}_\Lambda(\mathsf{H}-\mathsf{H}_\Lambda)^2$$

Now, since $\mathsf{H} = \mathsf{H}^{(0)} + \mathsf{H}^{(\mathrm{int})}$ we can write

$$\|\Psi(s)(\mathsf{H}-\mathsf{H}_\Lambda)\|_\star \le \left\|\Psi(s)(\mathsf{H}^{(0)}-\mathsf{H}_\Lambda^{(0)})\right\|_\star + \left\|\Psi(s)(\mathsf{H}^{(\mathrm{int})}-\mathsf{H}_\Lambda^{(\mathrm{int})})\right\|_\star$$

However, also here the first term on the right-hand side is vanishing:

$$\left\|\Psi(s)(\mathsf{H}^{(0)}-\mathsf{H}_\Lambda^{(0)})\right\|_\star = \mathrm{Tr}\left(\rho_\Lambda(\mathsf{H}^{(0)}-\mathsf{H}_\Lambda^{(0)})^2\mathsf{N}_\Lambda\right) = 0$$

since

$$P_\Lambda(\mathsf{H}^{(0)}-\mathsf{H}_\Lambda^{(0)})^2P_\Lambda = 0$$

and therefore

$$\|\Psi(s)(\mathsf{H}-\mathsf{H}_\Lambda)\|_\star \le \left\|\Psi(s)(\mathsf{H}^{(\mathrm{int})}-\mathsf{H}_\Lambda^{(\mathrm{int})})\right\|_\star$$

Then

$$\begin{aligned}\|\Psi(s)(\mathsf{H}-\mathsf{H}_\Lambda)\|_\star &\le \left[\mathrm{Tr}\left(\rho_\Lambda(\mathsf{H}^{(\mathrm{int})}-\mathsf{H}_\Lambda^{(\mathrm{int})})^2\mathsf{N}_\Lambda\right)\right]^{\frac{1}{2}}\\ &\le \left[\mathrm{Tr}\left(\rho_\Lambda\mathsf{N}_\Lambda^2\right)\right]^{\frac{1}{4}}\left[\mathrm{Tr}\left(\rho_\Lambda(\mathsf{H}^{(\mathrm{int})}-\mathsf{H}_\Lambda^{(\mathrm{int})})^4\right)\right]^{\frac{1}{4}}\\ &\le \left(\frac{\Lambda^d}{B}\right)^{\frac{1}{2}}\left[\mathrm{Tr}\left(\rho_\Lambda(\mathsf{H}^{(\mathrm{int})}-\mathsf{H}_\Lambda^{(\mathrm{int})})^4\right)\right]^{\frac{1}{4}}\end{aligned} \tag{4.10}$$

and notice that the bounding term is time independent. Therefore, combining equations (4.10), (4.8) and (4.9) we get

$$\|\Delta_\Lambda(t)\|_\star \le 2t\left(\frac{\Lambda^d}{B}\right)^{\frac{1}{2}}\left[\mathrm{Tr}\left(\rho_\Lambda(\mathsf{H}^{(\mathrm{int})}-\mathsf{H}_\Lambda^{(\mathrm{int})})^4\right)\right]^{\frac{1}{4}}$$

and using the inequality (4.7) we get

$$\|\Delta_\Lambda(t)\|_\mu \le 2t\left(\frac{1}{B}\right)^{\frac{1}{2}}\left[\mathrm{Tr}\left(\rho_\Lambda(\mathsf{H}^{(\mathrm{int})}-\mathsf{H}_\Lambda^{(\mathrm{int})})^4\right)\right]^{\frac{1}{4}} \tag{4.11}$$

We want to write an estimate for the term $\mathrm{Tr}\left(\rho_\Lambda(\mathsf{H}^{(\mathrm{int})}-\mathsf{H}_\Lambda^{(\mathrm{int})})^4\right)$: consider the following identities

$$\begin{aligned}P_\Lambda(\mathsf{H}^{(\mathrm{int})}-\mathsf{H}_\Lambda^{(\mathrm{int})})^2 = (P_\Lambda\mathsf{H}^{(\mathrm{int})}-P_\Lambda\mathsf{H}_\Lambda^{(\mathrm{int})})(\mathsf{H}^{(\mathrm{int})}-\mathsf{H}_\Lambda^{(\mathrm{int})}) &= P_\Lambda\left(\mathsf{H}^{(\mathrm{int})}\right)^2 - P_\Lambda\mathsf{H}^{(\mathrm{int})}P_\Lambda\mathsf{H}^{(\mathrm{int})}\\ &= P_\Lambda\mathsf{H}^{(\mathrm{int})}(\mathbf{1}-P_\Lambda)\mathsf{H}^{(\mathrm{int})}\end{aligned}$$

$$(\mathsf{H}^{(\mathrm{int})} - \mathsf{H}_\Lambda^{(\mathrm{int})})^2 P_\Lambda = (\mathsf{H}^{(\mathrm{int})} - \mathsf{H}_\Lambda^{(\mathrm{int})})(\mathsf{H}^{(\mathrm{int})} P_\Lambda - \mathsf{H}_\Lambda^{(\mathrm{int})} P_\Lambda) = \left(\mathsf{H}^{(\mathrm{int})}\right)^2 P_\Lambda - \mathsf{H}^{(\mathrm{int})} P_\Lambda \mathsf{H}^{(\mathrm{int})} P_\Lambda$$
$$= \mathsf{H}^{(\mathrm{int})}(\mathbf{1} - P_\Lambda)\mathsf{H}^{(\mathrm{int})} P_\Lambda$$

Combining the above equations we have

$$P_\Lambda(\mathsf{H}^{(\mathrm{int})} - \mathsf{H}_\Lambda^{(\mathrm{int})})^4 P_\Lambda = P_\Lambda \mathsf{H}^{(\mathrm{int})}(1 - P_\Lambda)\left(\mathsf{H}^{(\mathrm{int})}\right)^2 (1 - P_\Lambda)\mathsf{H}^{(\mathrm{int})} P_\Lambda$$

and then[5]

$$P_\Lambda(\mathsf{H}^{(\mathrm{int})} - \mathsf{H}_\Lambda^{(\mathrm{int})})^4 P_\Lambda = P_\Lambda \left|(\mathbf{1} - P_\Lambda)\mathsf{H}^{(\mathrm{int})}\right|^4 P_\Lambda$$

We are therefore able to write the following equality between traces

$$\operatorname{Tr}\left(\rho_\Lambda(\mathsf{H}^{(\mathrm{int})} - \mathsf{H}_\Lambda^{(\mathrm{int})})^4\right)^{\frac{1}{4}} = \operatorname{Tr}\left(\rho_\Lambda\left|(\mathbf{1} - P_\Lambda)\mathsf{H}^{(\mathrm{int})}\right|^4\right)^{\frac{1}{4}}$$

Now, since $0 \leq \left(\mathsf{H}^{(\mathrm{int})}\right)^2 \leq 4C_V^2 \mathsf{N}^4$ we have

$$\operatorname{Tr}\left(\rho_\Lambda\left|(\mathbf{1} - P_\Lambda)\mathsf{H}^{(\mathrm{int})}\right|^4\right) \leq 4C_V^2 \operatorname{Tr}\left(\rho_\Lambda \mathsf{H}^{(\mathrm{int})}(\mathbf{1} - P_\Lambda)\mathsf{N}^4(\mathbf{1} - P_\Lambda)\mathsf{H}^{(\mathrm{int})}\right)$$
$$\overset{\diamondsuit}{\leq} 4C_V^2 \operatorname{Tr}\left(\rho_\Lambda \mathsf{N}^4 \mathsf{H}^{(\mathrm{int})}(1 - P_\Lambda)\mathsf{H}^{(\mathrm{int})}\right)$$

where in ($\diamondsuit$) the commutation properties $[\mathsf{N}, P_\Lambda] = 0$ and $[\mathsf{H}^{(\mathrm{int})}, \mathsf{N}^4] = 0$ were used. Then

$$\operatorname{Tr}\left(\rho_\Lambda\left|(1 - P_\Lambda)\mathsf{H}^{(\mathrm{int})}\right|^4\right) \leq 2^4 C_V^4 \operatorname{Tr}\left(\rho_\Lambda \mathsf{N}^8\right) \leq 2^4 C_V^4 \operatorname{Tr}\left(\rho_\Lambda \mathsf{N}\right)^8 = 2^4 C_V^4 \left(\frac{\Lambda^d}{B}\right)^8$$

and we showed that

$$\operatorname{Tr}(\rho_\Lambda\left|(\mathbf{1} - P_\Lambda)\mathsf{H}^{(\mathrm{int})}\right|^4)^{\frac{1}{4}} \leq 2C_V \left(\frac{\Lambda^d}{B}\right)^2$$

Using this estimate in equation (4.11) gives

$$\left\|\Delta_\Lambda(t)\right\|_\mu \leq \frac{2t}{\sqrt{B}} 2C_V \left(\frac{\Lambda^d}{B}\right)^2$$

which is the result we wanted to prove, namely equation (4.5).

To prove equation (4.6), we have that deviation term fulfills

$$\left\|\delta_q(t)\right\|_{L^2(\mu)} \leq \int_0^t \left\|\mathcal{L}_1 a_q(s)\right\|_\mu \mathrm{d}s \leq \left\|\mathcal{L}_1 a_q\right\|_{L^2(\mu)} t$$

But then

$$\left\|\mathcal{L}_1 a_q\right\|_{L^2(\mu)} \leq 6\mathcal{B}_\Lambda \Lambda^{4d} \frac{\sqrt{6}}{B}$$

and therefore the μ-norm of the difference $\Psi_\Lambda(t) - \Psi_\Lambda^{(0)}(t)$ obeys

$$\left\|\Psi_\Lambda(t) - \Psi_\Lambda^{(0)}(t)\right\|_\mu = \left(\frac{1}{\Lambda^d}\sum_k^\Lambda \left\|\delta_k(t)\right\|^2_{L^2(\mathrm{d}\mu)}\right)^{\frac{1}{2}} \leq \left(\frac{1}{\Lambda^d}\sum_k^\Lambda \left\|\mathcal{L}_1 a_k\right\|^2_{L^2(\mathrm{d}\mu)} t\right)^{\frac{1}{2}} \leq \mathcal{N}\mathcal{B}_\Lambda \Lambda^{4d}\frac{\sqrt{6}}{B} t$$

[5] Recall that for an operator F we have $|\mathsf{F}|^2 = \mathsf{F}^\dagger \mathsf{F}$, and therefore $|\mathsf{F}|^4 = \mathsf{F}^\dagger \mathsf{F}\mathsf{F}^\dagger \mathsf{F}$.

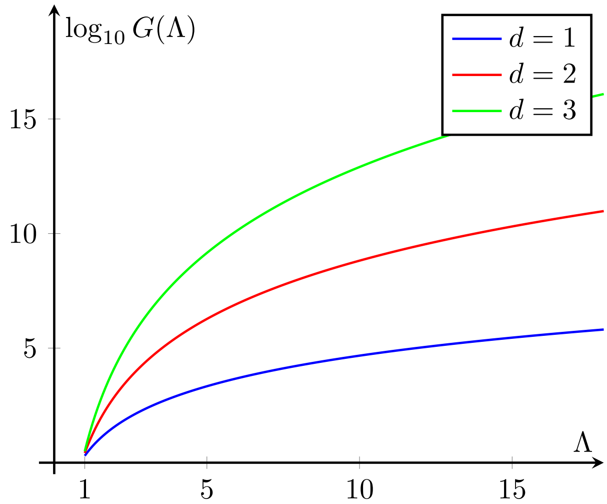


Figure 4.1: Behavior of $G(\Lambda)$ in different dimensions. Even if plotted as continuous lines, only the integer values of Λ are meaningful.

The bound can be written explicitly as

$$\begin{aligned}\left\|\Psi_\Lambda(t) - \Psi_\Lambda^{(0)}(t)\right\|_\mu &\leq \mathcal{N}\mathcal{B}_\Lambda \Lambda^{4d}\frac{\sqrt{6}}{B}t \\ &= \frac{\mathcal{N}\sqrt{6}C_V(1 + d(2\Lambda+1)^{\frac{1}{2}})\Lambda^{4d}}{B}t\end{aligned}$$

□

We can see the behavior of the bounding term with respect to the cutoff in figure 4.1, where the function

$$G(\Lambda) = (1 + d(2\Lambda+1)^{\frac{1}{2}})\Lambda^{4d}$$

is plotted for $d = 1, 2, 3$.

Notice moreover that both the estimates (4.5) and (4.6) are vanishing in the $T \to 0^+$ limit:

$$\|\Psi(t) - \Psi_\Lambda(t)\|_\mu \leq \frac{4C_V\Lambda^{2d}}{B^{\frac{5}{2}}}t \to 0 \text{ for } T \to 0^+$$

$$\left\|\Psi_\Lambda(t) - \Psi_\Lambda^{(0)}(t)\right\|_\mu \leq \frac{\mathcal{N}\sqrt{6}C_V(1 + d(2\Lambda+1)^{\frac{1}{2}})\Lambda^{4d}}{B}t \to 0 \text{ for } T \to 0^+$$

as well as $\left\|\Psi_\Lambda^{(0)}(t)\right\|_\mu$.

Comparison with existing literature

The approach presented here and also in [44] presents some novelties in the literature. The object studied in the standard literature, for instance [9, 36, 21], is the *one particle density operator* Γ :

$L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ associated to a Fock coherent state $\phi_{\sqrt{N}\theta} \in \mathcal{F}_+$ centered on a fixed $\theta \in L^2(\mathbb{R}^d)$, which is defined as

$$\Gamma(t,x,y) = \frac{\langle \phi_{\sqrt{N}\theta}, \Psi^\dagger(t,y)\Psi(t,x)\phi_{\sqrt{N}\theta}\rangle}{\langle \phi_{\sqrt{N}\theta}, \mathsf{N}\phi_{\sqrt{N}\theta}\rangle}$$

We can also rewrite such operator by introducing the projector P_θ onto the coherent state $\phi_{\sqrt{N}\theta}$ as

$$\Gamma(t,x,y) = \frac{1}{N}\operatorname{Tr}(P_\theta \Psi^\dagger(t,y)\Psi(t,x))$$

In N. N. Bogoliubov's original paper on superfluidity [13], the time dependent field operator $\Psi(t,x)$ is decomposed as the sum of two distinct terms, namely $\Psi^{(\mathrm{s})}(t,x) = \psi(t,x)\mathbf{1}$ where the scalar $\psi(t,x)$ is a solution of the Gross-Pitaevskii equation and the *normal fluid excitation* field $\Theta(t,x)$. Therefore

$$\Psi(t,x) = \Psi^{(\mathrm{s})}(t,x) + \Theta(t,x) = \psi(t,x)\mathbf{1} + \Theta(t,x)$$

The term $\Psi^{(\mathrm{s})}(t,x)$ has a one particle density operator given by

$$\Gamma_{\mathrm{s}}(t,x,y) = \frac{1}{N}\psi^\dagger(t,y)\psi(t,x)$$

Such Γ_{s} is the projector onto the single particle state $\psi(t)$ that solves the scalar Hartree equation $i\partial_t\psi(t) = (\mathsf{h} + V * |\psi|^2)\psi(t)$ with $\|\psi\|^2_{L^2} = N$.

The typical estimate that is obtained in the literature is on the growth of the difference between the one particle density operators Γ and Γ_{s} in a trace norm, and reads (see for instance [9])

$$\operatorname{Tr}|\Gamma(t) - \Gamma_{\mathrm{s}}(t)| \leq \frac{\exp c_1 \left(\exp\left(c_2 t\right)\right)}{\sqrt{N}}$$

for suitable constants $c_1, c_2 > 0$. We can make a short comparison between this result and the one presented in proposition 4.8.

Let us consider the deviation operator $\Delta^{(0)}_\Lambda = \Psi_\Lambda - \Psi^{(0)}_\Lambda$ and the operator

$$\delta\Gamma(t,x,y) = \operatorname{Tr}\left(\rho_\Lambda \left(\Delta^{(0)}_\Lambda\right)^\dagger \Delta^{(0)}_\Lambda\right)$$

It can be showed (see [44]) that the following inequality is fulfilled[6]

$$\operatorname{Tr}|\delta\Gamma| \geq \|\delta\Gamma\|_{\mathrm{HS}} \geq \frac{1}{\Lambda^d}\left\|\Delta^{(0)}_\Lambda\right\|_\star$$

and, thanks to equation (3.13) we get

$$\|\delta\Gamma\|_{\mathrm{HS}} \geq \frac{1}{\Lambda^d}\left\|\Delta^{(0)}_\Lambda\right\|_\star \geq \left\|\Delta^{(0)}_\Lambda\right\|^2_\mu$$

We therefore obtained a lower bound for $\|\delta\Gamma\|_{\mathrm{HS}}$, and the lower bound grows at most linearly in time as showed by proposition 4.8.

Moreover, let us introduce a cutoff in the single particle density operators as

$$\Gamma_\Lambda(t,x,y) = \operatorname{Tr}(\rho_\Lambda \Psi^\dagger_\Lambda(t,y)\Psi_\Lambda(t,x))$$

[6] Here $\|\cdot\|_{\mathrm{HS}}$ denotes the *Hilbert-Schmidt norm*, given by $\|\delta\Gamma(t,x,y)\|_{\mathrm{HS}} = \int_{\mathbb{R}^{2d}} \delta\Gamma(t,x,y)\,\mathrm{d}x\,\mathrm{d}y$.

$$\Gamma_\Lambda^{(0)}(t,x,y) = \mathrm{Tr}(\rho_\Lambda \Psi_\Lambda^{(0)\dagger}(t,y)\Psi_\Lambda^{(0)}(t,x))$$

with $\Gamma_\Lambda^{(0)}$ being the regularized effective one. Then it can be proved that (see [44] for details)

$$\left\|\Gamma_\Lambda - \Gamma_\Lambda^{(0)}\right\|_{\mathrm{HS}}^2 \le \left\|\Delta_\Lambda^{(0)}\right\|_\star^4 + 2\left\|\Delta_\Lambda^{(0)}\right\|_\star^3 \frac{\Lambda^d}{B} + 4\left\|\Delta_\Lambda^{(0)}\right\|_\star^{\frac{7}{2}} \left(\frac{\Lambda^d}{B}\right)^{\frac{1}{2}} + 2\left\|\Delta_\Lambda^{(0)}\right\|_\star \frac{\Lambda^d}{B}$$

Therefore, the quantity $\left\|\Gamma_\Lambda - \Gamma_\Lambda^{(0)}\right\|_{\mathrm{HS}}^2$ has an upper bound depending on the norm $\left\|\Psi_\Lambda - \Psi_\Lambda^{(0)}\right\|_\star$, which in turn is a lower bound for $\|\delta\Gamma\|_{\mathrm{HS}} = \mathrm{Tr}\left(\rho_\Lambda \left(\Psi_\Lambda - \Psi_\Lambda^{(0)}\right)^\dagger \left(\Psi_\Lambda - \Psi_\Lambda^{(0)}\right)\right)$

Remark. The field $\Psi^{(\mathrm{s})}(t,x)$ as defined in [13] can be decomposed as

$$\Psi^{(\mathrm{s})}(t,x) = \sum_k \left(\langle\varphi_k, \psi(t)\rangle_{L^2(\mathbb{R}^d)}\mathbf{1}\right)\varphi_k$$

By putting $\mathsf{f}_k = \langle\varphi_k, \psi(t)\rangle_{L^2(\mathbb{R}^d)}\mathbf{1}$, we have that the Wick symbol $\sigma_W(\mathsf{f}_k) = \langle\varphi_k, \psi(t)\rangle_{L^2(\mathbb{R}^d)}$, which is independent from α, is the projection along the basis $\{\varphi_k\}$ of the $\psi(t)$ solving the scalar Hartree equation (3.9) with $\Lambda = +\infty$. We can compare this to our definition of the regularized effective field: indeed, we have that the c_k, defined as the Wick symbols of the c_k appearing in the definition of $\Psi_\Lambda^{(0)}$, are the k-th components of the Fourier decomposition of the solution of the reduced scalar Hartree equation (3.9), with arbitrary initial data.

Application to potentials

To conclude, we can apply the estimate of equation (4.5) to the potentials mentioned in the first chapter. For this discussion we will limit ourselves to the $d = 3$ case.

Coulomb potential

For the three-dimensional Coulomb potential the optimal Hardy constant reads $C_V = 2$, therefore

$$\left\|\Psi_\Lambda(t) - \Psi_\Lambda^{(0)}(t)\right\|_\mu \le \mathcal{N}\mathcal{B}_\Lambda\Lambda^{4d}\frac{\sqrt{6}}{B}t = \frac{\mathcal{N}2\sqrt{6}(1+d(2\Lambda+1)^{\frac{1}{2}})\Lambda^{4d}}{B}t$$

Step potential and Lennard-Jones

Discussion for limited potentials is similar, so we group together the case of the step potential and the more accurate truncated Lennard-Jones potential. For a generic potential with bounding parameters a, b, r_0, r_1, where a, b, r_0 are finite and[7] $a > b$, the estimate of equation (4.5) reduces to

$$\left\|\Psi_\Lambda(t) - \Psi_\Lambda^{(0)}(t)\right\|_\mu \le \mathcal{N}\mathcal{B}_\Lambda\Lambda^{4d}\frac{\sqrt{6}}{B}t = \frac{\mathcal{N}a\sqrt{6}(1+d(2\Lambda+1)^{\frac{1}{2}})\Lambda^{4d}}{B}t$$

since in the described case the Hardy constant is precisely $C_V = a$.

[7]The condition $a > b$ implies that for both the step potential and the truncated Lennard-Jones we require that the repulsion is stronger than the short-range attractive force.

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