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Endpoint of Kerr-AdS superradiant instabilities and holography

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Abstract

Kerr-AdS black holes develop superradiant instabilities if their angular velocity $\Omega > 1$, as their Hawking quanta get exponentially amplified in a "Penrose-like" process. These black holes must then decay into some other stable solutions. In this work we review the mechanism behind the superradiant instabilities and discuss some proposals for their endpoint, focusing in particular on Revolving Black Holes and Grey Galaxies. These solutions are built by "storing" part of the energy and angular momentum outside of the black hole itself, respectively into its center of mass motion and into a cloud of far away, large angular momentum field excitations. They extend the gravitational phase space from the extremality bound for black holes down to the unitarity bound of the dual CFT. We will show how several holographic arguments provide useful information on such solutions and help us to construct them.

Contents

1 Introduction

It is a well known fact that, due to quantum effects, black holes are unstable. Hawking radiation is continuously emitted from the black hole horizon, taking away energy from the black hole; as the black hole shrinks, its temperature increases and the more and more power is emitted as Hawking radiation, leading eventually to the complete evaporation of the black hole. Black hole evaporation makes the study of black hole thermodynamics harder, since we have to deal with a thermodynamic system outside of thermodynamic equilibrium. The most obvious way in which we can solve this issue is by surrounding the black hole with mirrors, such that all the outgoing Hawking radiation will be reflected back and reabsorbed by the black hole. A new dynamical equilibrium is thus established between the black hole and a cloud of Hawking quanta that surrounds it: at any moment, as many Hawking quanta are emitted as those that fall again behind the horizon. Since there is no net flux of energy across the horizon, the black hole stays at a constant energy and thus at a constant size. Additionally, for large black holes the Hawking temperature is really small; therefore, the cloud of Hawking quanta surrounding the black hole is very "rarefied" and its effects can be safely neglected in most computations.

The natural way to introduce a mirror in the gravitational context is to put the black hole inside Anti-de Sitter (AdS) space. AdS space indeed acts as a confining box, where only particles with infinite energy can actually escape all the way to spatial infinity. Therefore, the above reasoning suggests that a black hole in AdS is a stable configuration, contrary to a black hole in flat space which evaporates. However, it was discovered 20 years ago that not all black holes in AdS are actually stable, due to a new kind of instabilities, the so-called "superradiant instabilities". In this work we will focus on superradiant instabilities of rotating Kerr-AdS black holes, since the behavior in the charged (and possibly rotating) case appear to be far more complex; additionally, we will stick for simplicity to AdS_4, though most of the considerations we will make should be easily generalizable in $D \geq 4$.

The main cause behind superradiant instabilities is a generalization of the Penrose process for waves. In the Penrose process, an observer can extract energy from the ergosphere of a rotating black hole without actually falling behind the horizon. It turns out that some of the Hawking quanta in the rotating cloud that surrounds the black hole in AdS can actually extract some energy from the black hole when they pass through the ergosphere; then, instead of escaping to infinity, they are again reflected back by the AdS confining potential towards the black hole and can extract some more energy, leading to an exponential growth of such modes. A heuristic argument explaining the mechanism with which waves can extract energy from the black hole was first proposed by Zel'dovich [1]. The idea is to modify the equations of motion of a field (in our case, a scalar field for simplicity) by taking into account the presence of a rotating black hole. Near the horizon, the black hole absorbs radiation, and this effect is captured by a simple dampening term added to the equations of motion. The rotation is then simply introduced by applying a Lorentz boost; while usually the equations of motion are Lorentz invariant and thus a boost would not modify them, we now have the dampening term, which transforms non-trivially. After the boost, the dampening gets an additional contribution proportional to the angular velocity, which can switch the sign of the dampening term, thus causing an exponential growth.

While the above heuristic method shows us that some modes will go superradiant, a more rigorous treatment of the problem is to explicitly compute the quasi-normal modes of fields in the Kerr-AdS background [2]. While the explicit calculation for a generic Kerr-AdS background is too complex, it becomes viable once we consider the case of a small, slowly rotating black hole. This way, we split the problem in two: the quasi-normal modes in the near-horizon region are the usual ones for a Kerr black hole; the modes far away from the horizon are almost the empty AdS ones. The two solutions must match in the intermediate region, not too close to the horizon and not too far away, allowing us to extract the quasi-normal modes. The result is that a Kerr-AdS black hole is unstable if its angular velocity satisfies $\Omega > 1$, i.e. if it rotates fast enough. Therefore, all the black holes between the onset of the instabilities at $\Omega = 1$ and the extremality bound for black holes are unstable. We will review the Zel'dovich heuristic argument and the quasi-normal modes computation in section 2.

After establishing that black holes with $\Omega > 1$ are unstable and thus decay, the question becomes to find the endpoints of such decays. The main tool that we will use in all our calculations is the AdS/CFT correspondence. The AdS/CFT correspondence is a much-studied and widely accepted conjectured duality between a D-dimensional theory of gravity with a negative cosmological constant — i.e. gravity in AdS_D — and a dual Conformal Field Theory (CFT) in $D-1$ dimensions living on the conformal boundary of the AdS_D (i.e. at spatial infinity). In particular, this is a weak-strong duality, meaning that as the gravity becomes weakly coupled the CFT becomes strongly coupled (and vice versa). The AdS/CFT was first proposed in the context of string theory, by considering stacks of D-branes in the decoupling limit [3]; however, it is generically expected that any theory of gravity on AdS_D admits a dual CFT on its $(D-1)$-dimensional boundary, however complex this CFT might be. Some evidence supporting this claim comes from comparing the symmetries of the two systems: the isometry group $SO(d,2)$ of AdS acts on the conformal boundary as conformal transformations, and indeed $SO(d,2)$ is also the conformal group in $D-1$ dimensions. Additionally, one can show that the 2-point functions of the CFT agree with the boundary-to-boundary 2-point functions of fields in the bulk, where the operators have the same spins and the conformal dimension in the CFT is related to the mass of the bulk field and the dimension of the spacetime. In particular, the 2-point functions in the bulk are constructed both using normalizable and non-normalizable modes (the latter blowing up at some point in the spacetime): normalizable modes are mapped to states in the CFT, while non-normalizable modes are mapped to the operators in the CFT, which can be thought as sources for the bulk fields [4]. Notice that the issue of superradiant instabilities and their interpretation in the dual CFT has raised many questions since the early days of holography, see e.g. [5].

Coming back to the endpoints, we need to find some new stable solutions that cover at least, in the $E-J$ phase diagram of Kerr-AdS, the whole region between the[1] $\Omega=1$ line down to the extremality line. However, the AdS/CFT correspondence suggests that these new solutions might actually cover a much bigger region. From the CFT point of view, there is no trace of the extremality bound; the only limit on states is instead the unitarity bound $E \geq J$, which is much weaker than extremality. In a generic CFT we should expect states down the unitarity bound; this seem in tension with the fact that there are no gravitational solutions (even unstable) below the extremality bound. We can therefore hope (and we will in fact show) that the new endpoints will actually solve this tension, providing gravitational states below the extremality bound and down to the unitarity bound.

As for actual endpoints, we will focus in section 3 on three candidates: black resonators, Revolving Black Holes (RBHs) and Grey Galaxies (GGs). The first proposed endpoints were the black resonators [6]. They are built by placing a black hole inside a geon or a boson star. Geons and boson stars are non-linear generalizations of normal modes for the gravitational field and a scalar field respectively; they are essentially solitons kept together by the confining effect of AdS. The idea then is that, as a mode goes superradiant and grows, one is left by a "combination" of such mode (which becomes solitonic as it grows) and the black hole; in particular, the resulting solution is left with the single helicoidal Killing vector $\partial_t + \Omega\partial_\varphi$. It turns out however that such solutions are still unstable [7], and thus they cannot be the actual endpoints of the instabilities. Nevertheless, they could still be intermediate, metastable steps in the decay.

RBHs and GGs have been instead proposed both in [8]. They share the same common idea: take away some of the energy and angular momentum from the black hole and "store it" into some other degrees of freedom outside the horizon, such that the angular velocity of the original black hole is brought down to $\Omega=1$. The difference between RBHs and GGs is then the choice of degrees of freedom in which to store the energy. RBHs are built by taking a black hole and setting its center of mass in motion, spinning around the center of AdS. In the semiclassical limit, a black hole can be seen as a spinning, heavy geodesic. One can then change a geodesic into another one by simply applying an isometry of the background; at the infinitesimal level, this means shifting the center of mass along the Killing vectors of AdS, i.e. exciting one of the black hole normal modes. The normal modes contribute to the one-loop determinant of the black hole; however, the key insight of RBHs is that one of the normal mode (with $\Delta E = \Delta J = 1$) gives a divergent contribution as $\Omega \to 1$. Therefore, this mode becomes populated at the macroscopic level, meaning that the black hole actually starts to rotate around the center of AdS.

GGs instead are built on the observation that the modes that become superradiant are actually really large angular momentum modes; due to the large angular momentum, they live

[1] Note that we always consider the microcanonical ensemble when constructing the $E-J$ phase diagram, that is $\Omega = \Omega(E,J)$.

really far away from the black hole itself and therefore their interaction with the black hole is essentially negligible. Therefore, GGs are built by considering a Kerr-AdS black hole and a cloud of large angular momentum mode in empty AdS in thermal equilibrium. It turns out that the cloud contribution to the partition function diverge as $\Omega \to 1$ and thus a macroscopic amount of energy and angular momentum — with once again $\Delta E = \Delta J$ — can be stored in the gas of modes. It turns out that GGs have a higher entropy than RBHs, and thus they dominate in the thermodynamic limit; additionally, since the black hole at their center has $\Omega = 1$, they are stable. Finally, both RBHs and GGs exist down to the unitarity bound, thus extending the gravitational $E - J$ phase diagram and solving the tension with the prediction from the CFT point of view.

Finally, notice that in the constructions of RBHs and GGs the crucial step is respectively to compute the one-loop determinant due to the normal modes and the partition function of the large angular momentum modes. In both cases, to simplify our calculations we employ the AdS/CFT correspondence, and perform the computations on the CFT side. In particular, the normal modes contribution to the partition function is obtained by starting with an ensemble of primary states with energy E and angular momentum J — i.e. the states corresponding to the Kerr-AdS black hole in the semiclassical limit — and acting with the conformal generators, which are simply the infinitesimal isometries of AdS. As for the partition function of the gas of large angular momentum modes, we instead use the fact that the normalizable modes in empty AdS are mapped to the states in the CFT, themselves mapped to operators in the CFT using the operator-state correspondence. In particular, the single-particle Hilbert space of a certain field is (essentially by definition) an irreducible representation of the isometry group of AdS; this group is simply the conformal group, whose irreducible representations are just the conformal families built by acting with derivatives on primaries. Therefore, computing the partition function of such modes is reduced to simply labeling all the descendants that can be built by starting from an operator in the CFT and by acting with derivatives. Holography and the AdS/CFT correspondence thus prove to be essential to simplify the construction of these new solutions. Although holography do not directly provide the explicit form of the solutions — which would require much more work — it is a valuable tool to obtain important physical information about the solutions in a relatively simple way, while also offering a radically new view on the gravitational phenomena under study.

2 Superradiant instabilities of Kerr-AdS

2.1 Black hole superradiance

The superradiant instabilities in Kerr-AdS come from the interaction between two well known physical phenomena related to rotating black holes and AdS space: the former is black hole superradiance and the latter is the fact that AdS space acts like a gravitational "box". Let us first focus on black hole superradiance. The idea behind black hole superradiance is essentially to generalize the Penrose process from particles to waves scattering with the black holes. Via the Penrose process, a body can enter into the ergosphere of a rotating black hole, eject some mass into the black hole and then come out with more energy than it started, effectively extracting energy and angular momentum from the rotating black hole. This is possible because the energy at spatial infinity is measured with respect to the timelike Killing vector ∂_t; this vector turns spacelike inside the ergosphere, making the energy of worldlines inside it negative. However, the actual black hole horizon is the surface at which the vector field $\partial_t + \Omega\partial_\varphi$ becomes null, and such surface lies inside the ergosphere. Thus, while in the ergosphere everything is forced to rotate in the same direction of the black hole, objects are still allowed to escape.

The way to generalize the Penrose process to waves has been first discussed by Zel'dovich in [1]; in this section we will review his heuristic argument, and integrate it with knowledge of the AdS normal modes in order to figure out which Kerr-AdS black holes are stable and which suffer from superradiant instabilities. Suppose for simplicity that we are studying the behavior of waves for a scalar field Φ subject to the usual Klein–Gordon wave equation:

$$\Box\Phi - m^2\Phi = 0\,. \tag{2.1}$$

We now wish to find a way to include the effect of the black hole on the scalar field without actually having to solve the above equation in the black hole background, find the modes of Φ, quantize them and pick the appropriate vacuum. Zel'dovich's idea is to approximate the black hole as a rotating body that absorbs radiation; we assume that the rotating body has radius $r_{\rm BH}$ and angular velocity Ω. Let us first turn off the angular velocity, so that we are just left with a static body that absorbs radiation. Near the surface of the body, we can effectively describe the absorption of waves falling behind the horizon by introducing a dampening coefficient $\alpha \geq 0$ and modifying the wave equation (2.1) as follows:

$$\Box\Phi + \alpha\frac{\partial\Phi}{\partial t} - m^2\Phi = 0\,. \tag{2.2}$$

Notice that this is exactly the same as adding a dampening to a harmonic oscillator. If we now switch back on the rotation, setting $\Omega \geq 0$, (2.2) will still be valid near the surface of the body, in a reference frame comoving with the surface itself. In order to go back to the static frame of an observer looking at the black hole from infinity, we just need to perform a Lorentz transformation. Let us assume without loss of generality that we are near a point of the horizon moving in the x direction at a speed $v = \Omega r_{\rm BH}$. Since the $\Box$ is already Lorentz invariant, we just need to Lorentz transform the dampening term, obtaining:

$$\Box'\Phi + \alpha\gamma\left(\partial'_t + \Omega r_{\rm BH}\partial'_x\right)\Phi - m^2\Phi = 0\,, \tag{2.3}$$

where the $'$ denote the static reference frame coordinates and $\gamma = 1/\sqrt{1-\Omega^2 r_{\rm BH}^2}$.

We now look for a cylindrical solution at infinity of the form

$$\Phi_{\omega,l_z} \sim e^{i(\omega t' - l_z \varphi)} = e^{i(\omega t' - l_z \arctan{(y'/x')})}\,, \tag{2.4}$$

with φ' the polar angle in the $x'y'$ plane, l_z the angular momentum of the mode and ω its energy. Plugging the above ansatz and zooming near the horizon (setting $x' \sim 0,\ y' \sim -r_{\rm BH}$, so that we are at the point moving in the x' direction) into (2.3), the dampening term becomes:

$$\alpha\gamma\left(\partial_t' + \Omega r_{\rm BH}\partial_x'\right)\Phi_{\omega,l_z} = \alpha\gamma i\left(\omega - \Omega l_z\right)\Phi_{\omega,l_z}\,. \tag{2.5}$$

Therefore, we see that if the condition

$$\Omega > \frac{\omega}{l_z} \tag{2.6}$$

is satisfied, the dampening term changes sign and becomes an enhancing term. This means that the mode scattering with the rotating black hole comes out with a bigger amplitude than it had coming in; this phenomenon is called superradiance.

The above discussion covers the first half of the physics of superradiant instabilities in Kerr-AdS, the phenomenon of black hole superradiance. The fact that we can throw a wave in a black hole and get a bigger wave back suggests a way to trigger an instability for the system [9]: if we can manage to reflect the wave once again back to the black hole with a “mirror”, the wave can become bigger and bigger in amplitude without stopping, rendering the whole system unstable. General Relativity gives us a really natural mirror to consider: Anti-de Sitter space (AdS). It is in fact well known that AdS space acts like a gravitational box, such that massive particles with finite energy cannot reach the conformal boundary at spatial infinity. We can see this by using the conservation of the Killing energy $E = p^0/z^2$ in Poincaré coordinates $ds^2 = (-dt^2 + dz^2 + \dots)/z^2$; for a finite initial energy, this implies $p^0 \sim z^2$ near the boundary. For a generic particle, we have

$$p_\mu p^\mu = p^0 p_0 + p^i p_i \geq g_{00}\left(p^0\right)^2 \sim z^2\,. \tag{2.7}$$

At the conformal boundary, $z = 0$ and hence $p_\mu p^\mu \geq 0$; therefore, no massive particle can reach the conformal boundary, and must then be reflected back to the center of AdS. In other words, the particle sees a “confining potential” that grows as it gets closer to the conformal boundary.

Let us now take a black hole and put it inside AdS, starting from a non-rotating Schwarzschild black hole. In flat space, due to Hawking radiation, we expect the Schwarzschild black hole to slowly radiate away, as more and more energy is carried away by the Hawking quanta. In AdS, however, the Hawking quanta are reflected back by the AdS wall, and thus fall back into the black hole; therefore there is a dynamic equilibrium of the Schwarzschild black hole with a cloud of Hawking quanta, contrary to what happens in flat space. Notice that this cloud, for non-superradiant black holes, carries negligible energy with respect to the black hole; hence we do not need to worry about backreaction and the Schwarzschild-AdS metric (obtained without accounting for this cloud) is still a good approximation of the real spacetime. If we now take a Kerr black hole and put it inside AdS, we should once again expect the Hawking quanta to form a cloud around the black hole; however, if some of them satisfy the superradiant condition (2.6), they will become more and more energetic, triggering a superradiant instability in the Kerr-AdS spacetime.

To find wether a black hole is stable or not, we need to find the ω and l_z of all of its modes and check whether there exist at least one mode satisfying (2.6). In particular, since we want to find the weakest condition possible on Ω, we should minimize the right hand side of (2.6). This suggest that we should focus on large l_z modes, for which the denominator on the right hand side is big. Modes with a large angular momentum mostly live really far away from the center of AdS, and are thus just weakly coupled to the black hole. They will therefore be essentially the same as the large l_z modes of empty AdS. Focusing for simplicity on the case of a massless scalar in empty AdS, we will show in section 2.2 that for such a field:

$$\omega_0 = 3 + 2n + l \,, \tag{2.8}$$

where $n \in \mathbb{N}^+$ is additional parameter labeling the modes, l is the total angular momentum of the mode and we added the subscript $_0$ to emphasize that it is the frequency in empty AdS. In particular, the black hole will begin to be unstable if any mode saturates the inequality (2.6):

$$\Omega > \frac{\omega_0}{l_z} = \frac{3 + 2n + l}{l_z} \,. \tag{2.9}$$

This first happens when Ω becomes bigger than the lowest possible value of the right hand side of the above equation. To achieve the lowest value, we can first off set $n = 0$. Then, since $|l_z| \leq l$, we set $l_z = l$; finally, we send $l \to +\infty$ to get rid of the 3. Therefore, a Kerr–AdS black hole will suffer from superradiant instabilities if:

$$\Omega > \lim_{l \to +\infty} \left[\frac{3 + 2n + l}{l_z} \right] \Bigg|_{n=0,\, l_z=l} = 1 \,. \tag{2.10}$$

2.2 Quasi-normal modes

While in section 2.1 we discussed a heuristic description of superradiant instabilities in Kerr-AdS, it would be nice if we could make our intuition more precise with some explicit computations. In this section, we will thus review the computation of quasi-normal modes in Kerr-AdS following [2]. To find the frequency ω of the quasi-normal modes we need to solve the wave equation for fields in the Kerr-AdS background. If we take the time dependence of the modes to be $\Phi \sim e^{-i\omega t}$, we see that if $\mathrm{Im}(\omega) < 0$ the mode is dampened in time, due to part of the wave falling into the black hole. If however we can find some modes for which $\mathrm{Im}(\omega) > 0$, these modes grow in time and thus signal the superradiant instability of the system. To make our life easier, we will focus in what follows on the quasi-normal modes of a scalar field Φ in the 4D background of Kerr-AdS_4. Additionally, we set the AdS length $\ell_{AdS} = 1$; it can later be reintroduced by dimensional analysis.

The Kerr-AdS spacetime is a vacuum solution of Einstein's equations with negative cosmological constant. The metric is as follows [10]:

$$ds^2 = -\frac{\Delta_r}{\rho^2}\left(dt - \frac{a}{\Sigma}\sin^2\theta d\varphi\right)^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta}{\rho^2}\sin^2\theta\left(a\,dt - \frac{r^2+a^2}{\Sigma}d\varphi\right)^2 , \quad (2.11)$$

with

$$\begin{aligned} \Delta_r &= (r^2+a^2)\left(1+r^2\right) - 2mr\,, & \Sigma &= 1-a^2\,, \\ \Delta_\theta &= 1 - a^2\cos^2\theta\,, & \rho^2 &= r^2 + a^2\cos^2\theta\,. \end{aligned} \quad (2.12)$$

The range of m and a, which parameterize all the possible solutions, are limited as follows. To avoid unphysical behaviors of the metric (such as the θ coordinate changing signature), we should impose $\Delta_\theta \geq 0$ and thus $0 \leq a \leq 1$. If we now look at the horizons of the solutions, the outer horizon is located at $r = r_+$, with r_+ denoting the largest root of the equation $\Delta_r(r_+) = 0$. The black hole reaches extremality when Δ_r has a double root at the outer horizon $r_+|_{\text{extr}}$, i.e. when $\partial_r\Delta_r(r_+|_{\text{extr}}) = 0$. If $r_+ < r_+|_{\text{extr}}$, naked singularities appear in the solutions; imposing $r_+ \geq r_+|_{\text{extr}}$ by solving the two equations explicitly yields:

$$r_+^2 \geq \frac{1}{6}\left(\sqrt{1+14a^2+a^4} - 1 - a^2\right) . \quad (2.13)$$

The extensive quantities characterizing the thermodynamics of the black hole, that is the energy E, the angular momentum J and the entropy S, are expressed in terms of the parameters a and m as follows [11]:

$$E = \frac{1}{G_N}\frac{m}{(1-a^2)^2}\,, \qquad J = \frac{1}{G_N}\frac{ma}{(1-a^2)^2}\,, \qquad S = \frac{\pi\left(r_+^2+a^2\right)}{G_N(1-a^2)}\,. \quad (2.14)$$

The intensive quantities corresponding to E and J, namely the (inverse) temperature β and the angular velocity Ω (with respect to a non-rotating frame at infinity[2]), are given by:

$$\beta = \frac{4\pi\left(r_+^2+a^2\right)}{r_+\left(1+a^2+3r_+^2-a^2/r_+^2\right)}\,, \qquad \Omega = \frac{a\left(1+r_+^2\right)}{r_+^2+a^2}\,. \quad (2.15)$$

Let us now attempt to solve for the quasi-normal modes of a scalar field Φ in the Kerr-AdS

[2] As noted in [11], in the literature other slightly different definitions of Ω have appeared; these measure the angular velocity with respect to rotating frames at infinity. However, while some choices (coupled with a different definition of E) can still satisfy the Quantum Statistical Relation [12], only the choices of (2.14) and (2.15) satisfy the first law of thermodynamics.

background (2.11); let us also assume that the scalar field is massless, so that the wave equation is simply the Klein–Gordon equation:

$$\Box\Phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) = 0\,. \tag{2.16}$$

Kerr-AdS is stationary and axisymmetric; this suggests that a good way to solve the above equation is to split:

$$\Phi\left(t,r,\theta,\varphi\right) = e^{-i\omega t + il_z\varphi}\widetilde{Y}_{ll_z}(\theta)R(r)\,. \tag{2.17}$$

Here ω is the frequency of the mode and l_z its angular momentum along the axis of rotation. We then introduce the function $\widetilde{Y}_{ll_z}(\theta)$ as generalized spherical harmonics in Kerr-AdS, with an additional parameter l in analogy to the total angular momentum of the usual spherical harmonics. Finally, the radial dependence of the wave function is captured by the function $R(r)$. If we now take the ansatz (2.17) and plug it into the Klein–Gordon equation, we can solve the equation by separating the variables, obtaining [2]:

$$\begin{aligned}\frac{\Delta_\theta}{\sin\theta}\left(\Delta_\theta\sin\theta\partial_\theta\widetilde{Y}_{ll_z}(\theta)\right) + \left(a^2\omega^2\cos^2\theta - \frac{l_z^2\Sigma^2}{\sin^2\theta} + A_{ll_z}\Delta_\theta\right)\widetilde{Y}_{ll_z}(\theta) = 0\,,\\ \Delta_r\partial_r\left(\Delta_r\partial_r R(r)\right) + \left[\omega^2(r^2+a^2)^2 - 2mal_z\omega r + a^2l_z^2 - \Delta_r\left(a^2\omega^2 + A_{ll_z}\right)\right]R(r) = 0\,,\end{aligned} \tag{2.18}$$

where A_{ll_z} is the separation constant, i.e. the eigenvalue of the angular part of the equation needed to separate the variables.

Additionally, we need to decide which boundary conditions to impose on the scalar field at infinity ($r \to +\infty$) and at the horizon ($r \to r_+$). Starting from the former, since AdS acts as reflecting box and we are not considering any sources at infinity, we should simply set:

$$\Phi \xrightarrow{r\to+\infty} 0\,. \tag{2.19}$$

As for $r \to r_+$, we generically find both ingoing and outgoing modes. However, since we are not interested in Hawking radiation and we are just looking for modes that are (partially) falling into the black hole, we will keep just the ingoing modes. More explicitly, we can switch to tortoise coordinates r_* — satisfying $dr_*/dr = (r^2+a^2)/\Delta_r$ — to "straighten" the light rays at the horizon and then impose that the scalar field behaves as follows [2]:

$$\Phi \xrightarrow{r_*\to-\infty} e^{-i\omega t - i(\omega - l_z\Omega)r_*}\,. \tag{2.20}$$

Trying to solve (2.18) analytically is practically an impossible task. In order to get some analytical results we will therefore need a few simplifying assumptions. We will assume the following:

- $m \ll 1/\omega$, that is the mode wavelength is much larger than the size of the black hole;

- $m \ll \ell_{AdS} = 1$, i.e. we have a small black hole with respect to the AdS radius;

these two approximations combined then give us a hierarchy of scales $m \ll 1 \ll 1/\omega$. Additionally, to simplify matter further, we also require:

- $a \ll m$ and $a \ll \ell_{AdS} = 1$, that is the black hole is slowly rotating.

The idea behind these approximations is to solve (2.18) in two different regimes: a near-horizon region such that $r - r_+ \ll 1/\omega$ and a far-away region with $r - r_+ \gg m$. In the near-horizon region, since the black hole is small, we are approximately in flat space and we can "forget" that we are in AdS; hence we are just left with the simpler task of finding the solution of the Klein–Gordon equation in the background of a Kerr black hole, rather than a Kerr-AdS one; the scalar modes in the Kerr background can for example be found in [13]. In the far-away region, instead, the effect of the black hole is negligible — since once again the black hole is small — and thus the problem simplify to finding the scalar modes in empty AdS, which are also well known [4]. The trick is then to consider the "intermediate" region where $m \ll r - r_+ \ll 1/\omega$: here both approximations are valid, and therefore we can match the two solutions. It is precisely this matching that allows us to extract the value of the frequencies ω. Finally, the slow rotation approximation allows us to simplify the angular equation in (2.18); in particular, it turns out that for $a \ll 1$ and $a\omega \ll 1$ the separation constant A_{ll_z} becomes the more "traditional":

$$A_{ll_z} = l(l+1) + \mathcal{O}(a^2, a^2\omega^2)\,. \tag{2.21}$$

Let us now solve (2.18) in the near-horizon region. First off, we can neglect the presence of the cosmological constant, and thus consider a Kerr background. Then, due to the slow rotation, $\omega a^2 \sim 0$, $a \ll m$ and $r \sim r_+ \sim m$. The radial part of (2.18) becomes[3]:

$$\Delta \partial_r \left(\Delta \partial_r R(r)\right) + r_+^4 (\omega - l_z \Omega)^2 R(r) - l(l+1)\Delta R(r) = 0\,, \tag{2.22}$$

where

$$\Delta = r^2 + a^2 - 2mr\,. \tag{2.23}$$

To solve the above equation, one needs to perform the change of variables

$$z = \frac{r - r_+}{r - r_-}\,, \qquad\qquad R(r) = z^{i\varpi}\,(1-z)^{l+1}\,F(z)\,, \tag{2.24}$$

[3] Notice that technically the Ω that appears in [2] is actually different from the Ω defined in (2.15) that we use in this work and that is used in [8]. This is exactly because, as explained in the footnote 2, one can measure Ω with respect to different rotating frames. In particular, [2] extract the angular velocity from the periodicity of the φ coordinate; in the coordinate system (2.11) this angular velocity is however measured with respect to a rotating frame at infinity [14]. In [8] the angular velocity is instead obtained by subtracting off the rotation at infinity, so that the frame at infinity is static. This might prompt some confusion on which Ω should appear in the $\Omega > 1$ condition obtained from the analysis of the quasi-normal modes. However, in the small, slowly rotating black hole limit the two definitions are equivalent; (2.22) thus holds true in this approximation with both definitions of Ω, and we will therefore stick with the definition used in [8] rather then the one used in [2].

where we introduce the superradiant factor

$$\varpi \equiv (\omega - l_z \Omega) \frac{r_+^2}{r_+ - r_-} \,, \tag{2.25}$$

where r_- is the inner horizon radius. This way (2.22) turns into the differential equation satisfied by the standard hypergeometric function ${}_2F_1$. After imposing the appropriate boundary conditions (no outgoing flux at the horizon), one can then expand for large radius (i.e. for r the "intermediate" region), obtaining the solution [2]:

$$R(r) \sim \Gamma(1 - 2i\varpi) \left[\frac{\Gamma(2l+1)}{\Gamma(l+1)\Gamma(l+1-2i\varpi)} \frac{r^l}{(r_+ - r_-)^l} + \frac{\Gamma(-2l-1)}{\Gamma(-l)\Gamma(-l-2i\varpi)} \frac{(r_+ - r_-)^{l+1}}{r^{l+1}} \right] . \tag{2.26}$$

Focusing now on the far-away region, we can simply forget about the presence of the small black hole — setting $a \sim m \sim 0$ — and solve (2.18) in the background of empty AdS. The radial equation this time becomes:

$$\left(r^2 + 1\right) \partial^2_{rr} R(r) + 2 \left(2r + \frac{1}{r}\right) \partial_r R(r) + \left(\frac{\omega^2}{1 + r^2} - \frac{l(l+1)}{r^2}\right) R(r) = 0 \,, \tag{2.27}$$

and it can be once again casted as a standard hypergeometric differential equation via the change of variables:

$$x = 1 + r^2 \,, \qquad R(r) = x^{\omega/2} \,(1 - x)^{1/2} \, F(x) \,. \tag{2.28}$$

It is well known that there are two types of solutions to the Klein–Gordon equation in AdS: normalizable and non-normalizable modes. In AdS/CFT, the former corresponds to states in the CFT, while the latter correspond to the insertion of operators in the CFT, which act as sources for the scalar field at the conformal boundary [4]. Since here we are interested only in solutions that go to zero at infinity (AdS acts as a box), we should just keep the normalizable modes[4]. This time we can explicitly write down $R(r)$ in the intermediate region by taking r to be small, obtaining:

$$R(r) \sim \left[\frac{(-1)^{l/2}\Gamma(-l - \frac{1}{2})}{\Gamma\left(\frac{2-l-\omega}{2}\right)\Gamma\left(\frac{2-l+\omega}{2}\right)} r^l + \frac{(-1)^{-3l/2}\Gamma(l + \frac{1}{2})}{\Gamma\left(\frac{3+l+\omega}{2}\right)\Gamma\left(\frac{3+l-\omega}{2}\right)} \frac{1}{r^{l+1}} \right] . \tag{2.29}$$

Now, if we were in empty AdS, the above expression would show us that, for a generic ω, there is a divergence of the mode as $r \to 0$. To get a physically sensible result with no divergence, we should therefore require one of the two Γ functions in the denominator to be infinite; this can

[4] As we reach the boundary $z \to 0$ of AdS_D, a scalar field behaves as $\Phi(z) \sim \Phi_{\text{norm}} z^{\Delta} + \Phi_{\text{non}} z^{D-1-\Delta}$, where Δ (a function of D and the mass) is the dimension of the corresponding CFT primary operator. If the field is massless, $\Delta = D - 1$. Hence the non-normalizable modes becomes constant on the boundary.

be achieved by setting[5]:

$$\omega = 3 + 2n + l \,. \tag{2.30}$$

In the AdS/CFT context, the above condition is simply telling us that the normalizable modes of the massless scalar are in one-to-one correspondence with the descendants of a scalar primary of scaling dimension $\Delta = 3$ [4]. In the case at hand, however, we are not in empty AdS, but there is a small black hole with radius $r = r_+ \ll 1$. Therefore, we can assume that the relation (2.30) gets slightly perturbed to:

$$\omega = \omega_0 + i\delta = 3 + 2n + l + i\delta \,, \tag{2.31}$$

where $\omega_0 = 3 + 2n + l$ is the empty AdS frequency and $\delta \ll 1$. Notice that we chose δ such that it contributes to the imaginary part of ω. This is because, due to the presence of the black hole, part of the wave will be absorbed (or amplified if there is superradiance), thus changing the amplitude of the wave (2.17). In particular, if $\delta < 0$ the amplitude will decrease exponentially in time (i.e. the wave falls into the black hole), while if $\delta > 0$ the amplitude grows exponentially in time, signaling the onset of superradiant instabilities.

To find δ, we can now proceed as follows. First, we Taylor expand (2.29) for $\delta \ll 1$, obtaining:

$$R(r) \sim \left[\frac{(-1)^{l/2}\Gamma(-l-\frac{1}{2})}{\Gamma\left(-\frac{1}{2}-l-n\right)\Gamma\left(\frac{5}{2}+n\right)} r^l + i\delta \frac{(-1)^{-3l/2+n+1} n!\,\Gamma(l+\frac{1}{2})}{2\,(2+l+n)!} \frac{1}{r^{l+1}} \right] . \tag{2.32}$$

If we now focus on the intermediate region where $m \ll r - r_+ \ll 1/\omega$, both the near-horizon solution (2.26) and the far-away solution above are valid at the same time. By direct comparison of the two expressions, we can finally obtain δ as a function of $\delta(l, n; r_+, r_-, a)$. The full analytic expression for δ is not particularly illuminating, so it will not be reported here (see [2] for more details). For our purposes, it is enough to know that

$$\delta \sim l_z \Omega - (3 + 2n + l) = l_z \Omega - \omega_0 \,, \tag{2.33}$$

up to a positive proportionality constant. Hence, the condition $\delta > 0$ for superradiant instabilities becomes

$$\Omega > \frac{\omega_0}{l_z} = \frac{3 + 2n + l}{l_z} \,, \tag{2.34}$$

which coincides with the heuristic estimates (2.9). Taking large angular momentum modes, we therefore recover the condition $\Omega > 1$ (see (2.10)) for the superradiant instabilities. Notice that while the above calculation focus on the quasi-normal modes of a scalar field, the same conclusion is expected to hold also for fields with spin [2]; since the Kerr-AdS solution appears within a theory with at least a graviton, all Kerr-AdS black holes with $\Omega > 1$ in any theory will be unstable.

[5] We could also set $\omega = -(3 + 2n + l)$, though here we focus only on the positive frequencies.

Additionally notice that the condition $\Omega = 1$ is saturated at $r_+^2 = a$; comparing with (2.13), we have $a \geq r_+^2|_{\text{extr}}$, and thus the $\Omega \geq 1$ bound is always saturated before the extremality bound. Therefore, in the energy-angular momentum $(E - J)$ phase diagram of Kerr-AdS, if we start from a black hole with $\Omega(E, J) = 1$, all the black holes with a lower mass E down to the extremality limit will be unstable. The $E - J$ phase diagram is reported below:

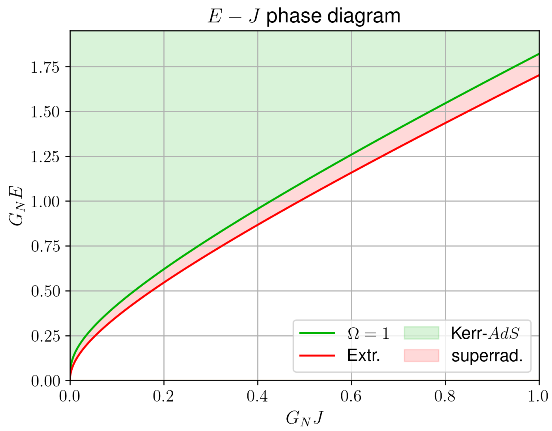

Figure 1: $E - J$ phase diagram of Kerr-AdS, in units of G_N.

While (2.10) does not tell us what happens to an unstable black hole — i.e. what is the endpoint of the instability — it suggests a way we could build such endpoints. In particular, since a black hole is unstable for $\Omega > 1$, we can imagine that the black hole will "shed" some of its energy and angular momentum until it reaches the value $\Omega = 1$. These additional energy and angular momentum will then have to be stored in some additional degrees of freedom outside the horizon. As we will discuss in section 3, this is indeed a way to construct some possible endpoints, with the extra energy and momentum stored either in the motion of the black hole center of mass (for the Revolving Black Boles of section 3.2) or in a thermal gas of particles surrounding the black hole (for the Grey Galaxies of section 3.3).

2.3 An AdS/CFT argument

In the two previous sections we have reviewed some aspects of superradiant instabilities in Kerr-AdS. While we have yet to discuss what exactly are the endpoints of these instabilities, we can expect them to be new solutions of the Einstein's equations in AdS. In particular these solutions must for sure replace the Kerr-AdS black holes in the $E-J$ phase diagram between the $\Omega = 1$ line (i.e. when instabilities appear) and the extremality bound of the black hole. However, nothing forbids these new solutions to enlarge the $E - J$ phase diagram beyond the extremality bound. From a holographic perspective the only bound for states in the $E - J$ phase diagram is the unitarity bound, while there is apparently no sign of the extremality bound. This suggests that the new endpoints of the superradiant instabilities might not only replace the Kerr-AdS

solution between the $\Omega = 1$ instability onset to the extremality bound, but additionally be valid solutions down to the unitarity bound $E = J$ [8]. We will now review how the unitarity bound is obtained for a CFT following [15].

The AdS/CFT correspondence states that a theory of quantum gravity on AdS_D is equivalent to a CFT$_{D-1}$ living on its conformal boundary; this is an example of strong/weak duality, since the gravitational theory is weakly coupled when the CFT is strongly coupled (and vice versa). In particular, there is a one-to-one mapping between the states of the gravitational theory and the states of the CFT, which themselves are in one-to-one correspondence with the local operators of the CFT itself. Let us now consider a Kerr-AdS black hole with mass E and angular momentum J. From the Bekenstein–Hawking area law, we know that the black hole carries an entropy and should therefore be thought as a huge ensemble of microstates. From the CFT point of view, the black hole should then correspond to an ensemble of states with roughly conformal dimension E and z component of the spin J. Note that here "roughly" means that the spacing between the conformal dimensions of the states has to be small enough so that in the gravitational theory at the semiclassical level they cannot be distinguished, thus reproducing the degeneracy of microstates captured by the area law. In other words, since quantum gravity effects kick in at energy $\sim M_{\mathrm{pl}}$, the spacing between the energy levels should be $\lesssim M_{\mathrm{pl}}$.

We can now ask ourselves for which values of E and J we expect states in the CFT to exist. In a generic CFT, the only requirement comes from unitarity. We will work in Euclidean signature, and we are thus free to pass to radial quantization. In radial quantization, the CFT lives on $\mathbb{R}^3$, but time evolves in an unusual manner: the CFT states live on spheres S^2, which time evolve outward from the center of $\mathbb{R}^3$ (corresponding to past infinity). The main advantage of radial quantization is the operator-state correspondence. Since any state can be brought to past infinity and thus live just at the origin of $\mathbb{R}^3$, any state can be obtained by acting on the CFT vacuum with a local operator at the origin. This establishes a correspondence between states and local operators, allowing us to work in either of the two perspectives depending on the situation.

Let us then consider an in-state $|E, J\rangle_{\{s\}} \equiv \mathcal{O}_{E,J_{\{s\}}}(0)\,|0\rangle$, where $\mathcal{O}_{E,J_{\{s\}}}(0)$ is a primary with conformal dimension E and spin $J > 0$. Here the subscript $_{\{s\}}$ denotes the explicit tensor indices of $\mathcal{O}_{E,J_{\{s\}}}$, which is an operator with spin; we will suppress these indices for clarity when possible. To define the norm of the state, we first need to define the out-state $\langle E, J| = \langle 0|\,[\mathcal{O}_{E,J}(0)]^\dagger$ living at infinity, or in other words the Hermitian conjugate, which is a non-trivial task in radial quantization. For a primary scalar field $\mathcal{O}$, the conjugate is defined as [15]

$$[\mathcal{O}_{E,J}(0)]^\dagger \equiv \lim_{|x|\to 0} |x|^{-2E} \mathcal{O}_{E,J}(\mathcal{R}(x))\,, \tag{2.35}$$

with $\mathcal{R}$ the spatial inversion operator sending the radius $|x| \to 1/|x|$ and the additional factor of $|x|^{-2E}$ needed to normalize $\langle E, J|E, J\rangle = 1$.

To find the unitarity bound, notice first that under the radial quantization inner product $P_\mu^\dagger = K_\mu$, where P_μ and K_μ are respectively the generators of translations and special conformal transformations. Additionally, we have that $[K_\mu, P_\nu] = 2i\,(\Delta\delta_{\mu\nu} - M_{\mu\nu})$, where Δ is the generator of scaling transformations and $M_{\mu\nu}$ are the generators of[6] $SO(D-1)$. Let us now consider the matrix of expectation values:

$$A_{\mu\nu} \equiv \frac{1}{2}\,{}_{\{s\}}\langle E, J|[P_\mu, K_\nu]\,|E, J\rangle_{\{s\}} \;, \tag{2.36}$$

where $\{s\}$ denotes explicitly the spin indices of the operator (that we previously suppressed for better readability). Due to $P_\mu^\dagger = K_\mu$, unitarity implies that $A = A^\dagger$, and hence that this matrix is positive definite $A \geq 0$, i.e. it has only eigenvalues $\lambda \geq 0$. We then rewrite

$$A_{\mu\nu} = i\,{}_{\{s\}}\langle E, J|\Delta\delta_{\mu\nu} - M_{\mu\nu}\,|E, J\rangle_{\{s\}} = E\delta_{\mu\nu} - \delta_{\{s\}\{t\}}\left(\Sigma^J_{\mu\nu}\right)_{\{s\}\{t\}} \;, \tag{2.37}$$

where $\Sigma^J_{\mu\nu}$ is the finite-dimensional spin J representation of the generator $M_{\mu\nu}$ of $SO(D-1)$. The unitarity condition then becomes

$$E \geq \delta_{\{s\}\{t\}}\left(\Sigma^J_{\mu\nu}\right)_{\{s\}\{t\}} \;. \tag{2.38}$$

For simplicity, let us focus on the case of a CFT$_3$, relevant for AdS_4. Since in $\left(\Sigma^J_{\mu\nu}\right)_{\{s\}\{s\}}$ both μ, ν are fixed it is convenient to rewrite it as[7]:

$$\begin{aligned}\left(\Sigma^J_{\mu\nu}\right)_{\{s\}\{t\}} &= \frac{1}{2}\left(\delta^\rho_\mu\delta^\sigma_\nu - \delta^\rho_\nu\delta^\sigma_\mu\right)\left(\Sigma^J_{\rho\sigma}\right)_{\{s\}\{t\}} = -\frac{1}{2}\left(\Sigma^1_{\mu\nu}\right)^{\rho\sigma}\left(\Sigma^J_{\rho\sigma}\right)_{\{s\}\{t\}} \\ &= -\frac{1}{2}\left(\Sigma^1_{\rho\sigma}\right)^{\mu\nu}\left(\Sigma^J_{\rho\sigma}\right)_{\{s\}\{t\}} \;,\end{aligned} \tag{2.39}$$

where we use the explicit form of the spin 1 generators. Using the fact that in $D-1=3$ we can dualize the antisymmetric rotation generators to the usual angular momentum vector — e.g. $M_{\mu\nu} = \epsilon_{\mu\nu\rho}M^\rho$ — we can further rewrite:

$$\left(\Sigma^J_{\mu\nu}\right)_{\{s\}\{t\}} = -\left(\Sigma^1_\alpha\right)^{\mu\nu}\left(\Sigma^J_\alpha\right)_{\{s\}\{t\}} \;. \tag{2.40}$$

Since Σ^1_α and Σ^J_α are just three dimensional angular momentum operators, we can simply use the well known quantum mechanical rules for adding angular momenta:

$$-\vec{L}\cdot\vec{S} = \frac{1}{2}\left(\vec{L}^2 + \vec{S}^2 - \left(\vec{L}+\vec{S}\right)^2\right) \;, \tag{2.41}$$

where we identify $\vec{L} \to \Sigma^1_\alpha$ and $\vec{S} \to \Sigma^J_\alpha$. Since $\vec{L}$ and $\vec{S}$ are respectively a spin 1 and spin J

[6] Since in this work we use D to denote the dimension of AdS_D, we will take the dimension of the CFT spacetime to be $D-1$.

[7] Since we are working in radial quantization and hence Euclidean signature, we are not being too careful with the position of the indices.

representation, we can decompose $\vec{L}+\vec{S}$ into a $1\otimes J=(J+1)\oplus J\oplus(J-1)$ spin representation. Hence, since our aim is to maximize $-\vec{L}\cdot\vec{S}$, we should just pick the spin $J-1$ for the $\vec{L}+\vec{S}$ operator, obtaining:

$$\left(\Sigma^J_{\mu\nu}\right)_{\{s\}\{t\}} \geq \frac{1}{2}\left(J(J+1)+2-(J-1)J\right)=J+1\,. \tag{2.42}$$

Therefore, unitarity of the boundary CFT_3 constrains the states of the gravitational theory on AdS to obey:

$$E\geq J+1\,. \tag{2.43}$$

Notice that, for other values of the dimension $D-1$ and different representations under $SO(D-1)$ (still with $J>0$), the above bound still applies up to terms of order $\mathcal{O}(D-1)$ (which are negligible for black hole states with macroscopic E and J). Finally, notice that in principle we could repeat the same reasoning by modifying (2.36) and adding more operators in the expectation value. However, it turns out that higher levels do not provide stronger bounds for $J\geq 1$, while in the spin $J=0,1/2$ case only the second level is needed [15].

Therefore, starting from a gravitational theory dual to a generic CFT we would expect that there would be states for all values of $E\geq J$, where we ignore from here on the order $\mathcal{O}(1)$ term. However, if we look at (2.14), we see that for a generic Kerr-AdS black hole:

$$E=\frac{1}{a}J\,. \tag{2.44}$$

For a fixed E and J, the parameter a is a function $a=a(E,J)$ in the range $0\leq a\leq 1$. The maximum $a_{\mathrm{extr}}=1$ of this function is achieved when the black hole is extremal. Additionally, $a_{\mathrm{extr}}=1$ only when $J=0$, i.e. the black hole is not rotating. Therefore, in the whole phase-space region

$$J\leq E\leq \frac{1}{a_{\mathrm{extr}}}J\,, \tag{2.45}$$

between extremality and the unitary bound, there are no black hole states, despite the fact that from the CFT point of view there is nothing preventing states in this region. In principle this fact could be interpreted as a special requirement that a boundary CFT must satisfy in order to be dual to a gravitational system. However, taking into account the superradiant instabilities of black holes with $\Omega>1$, the unitarity bound (2.43) hints at the presence of other gravitational states — different from Kerr-AdS — that would fill the $E-J$ phase diagram down to the unitarity bound $E=J$ and that could additionally be the endpoints of the superradiant instabilities (thus extending up to the $\Omega=1$ line).

Let us now recap what we learned from the previous sections by plotting once again the $E-J$ phase diagram of Kerr-AdS, this time taking into account the unitarity bound:

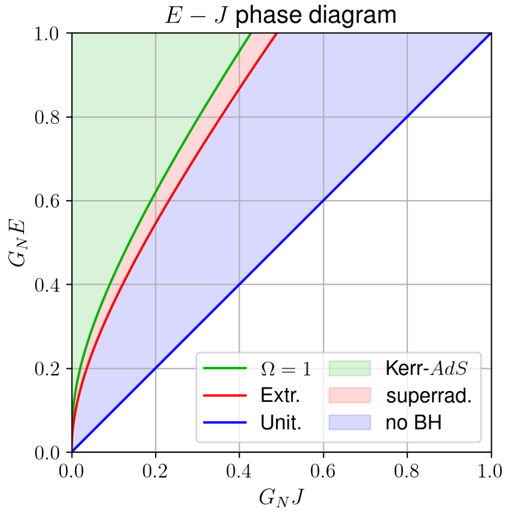

Figure 2: $E - J$ phase diagram of Kerr-AdS, in units of G_N, including the 45° line of the unitary bound.

From the plot we can distinguish three shaded regions. In the green shaded region, the angular velocity of the black hole is $\Omega < 1$, and hence we are in a phase dominated by stable Kerr-AdS black holes. If we lower the energy (at fixed J) and we cross the green line of $\Omega = 1$ (obtained via (2.15)), we reach the red shaded region with $\Omega > 1$; here Kerr-AdS solutions still exist but they suffer from superradiant instabilities. Therefore, some new solutions — the endpoints of such instabilities — should exist, at least up to extremality. The red line is instead extremality bound (2.13). Below the extremality bound, Kerr-AdS solutions develop naked singularities and are thus thought to be unphysical. However, from the point of view of the dual CFT, nothing forbids some new solutions to exist in this regime (the blue shaded region). The only hard limit below which we do not expect any solution is the unitary bound (the 45° blue line), below which the corresponding CFT states would not satisfy the unitarity bound (2.43), making the gravitational theory non-unitary. As explained in this section, the fact that no black hole solutions exist below extremality seems at odds with the fact that in a generic CFT we expect states down to the unitarity bound. In principle, this might be seen as a requirement on the CFT states for a generic CFT to be dual to a gravitational system. However, we know that new solutions must exist due to the superradiant instabilities, at least down to extremality. Therefore, it is not a stretch to conjecture that these gravitational solutions might extend below the extremality bound and down to the unitarity bound, thus reconciling the existence of states in a generic CFT below extremality with the gravitational description of the system.

3 Endpoints of the instability

3.1 Black resonators

After providing evidence for the superradiant instabilities of Kerr-AdS and hints of the possible solutions in section 2, it is now finally time to discuss the possible endpoints that could replace such instabilities. We will start now by briefly discussing black resonators following [6], the first of such proposals.

The idea behind black resonators is to create a "bridge" between the Kerr-AdS solution with other horizonless solutions of the Einstein equations in AdS, such as boson stars [16, 17] and geons [18, 19]. Boson stars and geons are essentially non-linear normal modes, i.e. solitonic solutions of the equations of motion with a certain energy ω and angular momentum l_z that are based once again on the fact that AdS acts as a box, reflecting outgoing waves back to the center. Therefore, at the non-linear level, it is possible that the fluctuations of any field will collapse together forming a bound state (possibly without a horizon), instead of escaping to infinity as usual. In particular, boson stars are built from fluctuations of a massless scalar field, while geons are solitons of the gravitational field itself. Notice that in both cases — due to the complexity of solving the equations of motion — the solutions have been constructed analytically only at the perturbative level, while there are only numerical solutions to the full non-linear problem. A key feature of both boson stars and geons is that these solutions are neither static nor stationary. They have only the single helicoidal Killing vector field $\partial_t + \Omega\partial_\varphi$ (while ∂_t and ∂_φ are not Killing vector separately), and therefore they are periodically rotating solutions that oscillate with a fixed frequency. Black resonators [6] are then constructed as solutions with a horizon and the single helicoidal vector field $\partial_t + \Omega\partial_\varphi$; in the zero horizon size limit and in the absence of matter, they reduce to geons. They can be intuitively thought as taking a black hole and putting it at the core of a geon, such that part of the energy and the angular momentum is stored in the soliton. Notice that, once again, these solutions have been constructed either perturbatively or numerically.

The way in which geons can provide an endpoint for the instability of the Kerr-AdS black holes goes as follows [20]. Let us consider a regular black hole on the onset of the superradiant instabilities, i.e. when $\Omega \gtrsim 1$. At that point, a large angular momentum mode with energy $\omega = \Omega l_z$ goes superradiant, due to (2.34). As the mode grows further, the black hole finds itself in a Bose condensate of the scalar mode

$$\Phi\left(t, r, \theta, \varphi\right) \sim e^{-i\omega t + i l_z \varphi} = e^{-i l_z (\Omega t - \varphi)} \, . \tag{3.1}$$

This mode thus breaks the symmetry of the Kerr-AdS solution under shifts of t and φ, while preserving the helicoidal Killing vector $\partial_t + \Omega\partial_\varphi$, the same Killing vector preserved by the geons. Therefore, it seems reasonable that the black hole with these growing "hair" simply decays to the corresponding black resonator.

Finally, we would like to study the thermodynamics of black resonators. While black resonators have been mostly constructed numerically and there is therefore no analytical formula for their entropy, a perturbative computation for low angular momentum $l = l_z$ scalar modes around a small E and J black hole has been performed in [6], yielding:

$$S_{\rm br} = 4\pi G_N E^2 \left[1 - \left(1 + \frac{1}{l}\right)\frac{J}{E}\right]^2 . \tag{3.2}$$

However, it turns out that black resonators are themselves unstable [21, 7] since they still have $\Omega > 1$, and thus they cannot be the true endpoint of the superradiant instabilities. They are regardless interesting solutions since, while unstable, they can possibly act as an intermediate step in the decay of the black holes: as the superradiant modes grow, the black hole first transitions into a state with only a helicoidal Killing vector (the black resonator), which later settles down into the final endpoint.

3.2 Revolving Black Holes

The two other possible endpoints that we will discuss are Revolving Black Holes (RBHs) and Grey Galaxies (GGs), which were both introduced in [8]. In both cases, the idea is to "store" part of the energy and angular momentum of the unstable $\Omega > 1$ black hole in some other degrees of freedom, in order to reduce the angular velocity of the black hole to $\Omega = 1$. The chosen degrees of freedom are however very different in the two cases; we will focus here on RBHs, developing an alternative approach to the one of [8], which follows a similar reasoning but is completely holographic. We will discuss Grey Galaxies in the next section 3.3.

The idea behind RBHs stems from noticing that Kerr-AdS black holes have a few normal modes, in addition to the quasi-normal modes associated to infalling waves. We can understand these modes by looking at the black hole from "far away", i.e. consider the black hole as a geodesic of a massive particle with spin sitting at the center of AdS. We can then "kick" the geodesic by applying AdS isometries, obtaining another black hole solution that is moving on a non-static geodesic. Since we just applied isometries, we have not excited any of the quasi-normal modes and there is no way for the new solution to dissipate the energy and angular momentum stored in its center of mass geodesic motion. Therefore, if we take a Kerr-AdS black hole and apply an infinitesimal isometry, we have constructed a normal mode of the Kerr-AdS solution. In order to see how these normal modes alter the partition function of the Kerr-AdS solution, we should therefore quantize the motion of a free particle with spin J and energy E in AdS_4.

As usual, the phase space of such a particle is simply given by the coset $SO(3, 2)/H$, where $SO(3, 2)$ is the group of isometries of AdS_4 and H is the subgroup of isometries leaving the particle trajectory and spin invariant. The quantization of this coset then produces the corresponding irreducible representation of the isometry group $SO(3, 2)$ describing the single-particle states. If the particle were spinless, then $H = SO(3) \times SO(2)$: the $SO(3)$ are rotations at fixed time around the center of the particle, while the $SO(2)$ are time translations[8]. If we now turn on spin, some of the $SO(3)$ rotations centered around the particle will rotate its spin and therefore they no longer leave the geodesic unchanged. However, the rotations in the Cartan $SO(2)$ of $SO(3)$ — i.e. the rotations around the axis of the particle spin — will leave both the trajectory and the particle spin untouched. Hence, the phase space of geodesics to be quantized is the group coset $SO(3,2)/\left(SO(2) \times SO(2)\right)$. The standard way to quantize such spaces is to consider coadjoint orbit quantization, i.e. constructing the coset by acting with $SO(3,2)$ on a chosen element of the coadjoint algebra of $\mathfrak{so}(3,2)$. With this construction we explicitly obtain a symplectic form on the coset, allowing us to construct the corresponding Hilbert space via e.g. geometric quantization. This method, often called the Kirillov's orbit method, gives a direct correspondence between quantization of coadjoint orbits and irreducible unitary representations of $SO(3,2)$, i.e. one-particle states [22].

Once we have the (quantized) normal modes of Kerr-AdS, we can compute the contributions of these normal modes to the black hole partition function using the Denef–Hartnoll–Sachdev (DHS) formula [23, 24]. This formula computes the one-loop determinant around an Euclidean saddle as an infinite product of factors, with each factor associated to a (quasi-)normal mode. In particular, the factor associated to a quasi-normal mode is just the partition function of a dampened quantum harmonic oscillators, while normal modes contribute with the partition function of a regular (non-dampened) harmonic oscillator. The main insight of [8] is that the contribution of one of the normal modes is divergent as $\Omega \to 1$. The one-loop correction due to this normal mode — which is usually subleading — thus becomes important, and an order $1/G_N$ of energy and angular momentum (i.e. comparable with the black hole E and J) can be stored in the normal mode, which becomes occupied macroscopically. From the geodesic description, this normal mode corresponds to making the black hole rotate infinitesimally around the center of AdS; since the mode gets occupied macroscopically, this indicates that the center of mass is actually rotating macroscopically around the center of AdS. The new solution obtained by taking the Kerr black hole and setting its center of mass in motion with a large amount of energy and angular momentum is thus called "Revolving Black Hole" (RBH). In particular, it can be shown by studying the wave equation of a massive spinning particle in AdS that the wavefunction of the center of mass is completely delocalized in φ, i.e. in the angular position of the black hole in the $\theta = \pi/2$ plane [8]; we are thus in the unusual situation of a macroscopic object (the black hole) whose wavefunction is inherently quantum in nature, and is not well approximated by a classical configuration.

The above derivation of RBHs appears in [8], where the AdS_4 geodesics and the black hole normal modes have been explicitly studied with the help of additional considerations coming from the easier AdS_3 case. In [8] it is also mentioned that one should be able to derive RBHs by

[8] This is because AdS_4 is a hyperboloid in $\mathbb{R}^{3,2}$, and the time translations come from the rotations in the two dimensions with negative signature.

using the AdS/CFT correspondence and working directly in the dual CFT. In the following we will flesh out this idea, finding the normal modes, rederiving the DHS formula for the one-loop determinant of the partition function and constructing the RBH thermodynamics by working directly in the CFT.

Generically we expect that a black hole on the gravity side is mapped to an ensemble of states in the CFT. The semiclassical approximation of the black hole partition function then corresponds to taking an ensemble of $\exp(S(E,J))$ highest spinning primary states $|E,J,J\rangle$ with energy E, total angular momentum J and angular momentum along the z axis $J_z = J$; loop corrections (which include the effects of the normal modes we are interested in) are then captured by the contributions of other primaries or of descendants. Approximating further the black hole as a massive, spinning geodesics is then equivalent to taking only one of these states $|E,J,J\rangle$. Kicking the geodesic by applying an AdS_4 isometry translates, in the CFT language, to applying a conformal transformation at the boundary; the normal modes obtained by applying infinitesimal isometries are then simply level 1 descendant states obtained by applying a single conformal generator to the "approximate" ensemble of just highest spin primaries $|E,J,J\rangle$.

In the Kerr-AdS_4 case, the relevant CFT_3 at the boundary is a CFT living on $S^2 \times \mathbb{R}$, where $\mathbb{R}$ is the temporal direction. After going to Euclidean time and applying a conformal transformation, the CFT_3 lives on $\mathbb{R}^3$ and thus we can apply radial quantization. We will approximate the Kerr-AdS black hole as an ensemble of highest spinning primaries $|E,J,J_z = J\rangle$ satisfying:

$$\begin{aligned} i\Delta\,|E,J,J\rangle &= E\,|E,J,J\rangle \ , \\ M^2\,|E,J,J\rangle &= J(J+1)\,|E,J,J\rangle \ , \\ M_z\,|E,J,J\rangle &= J\,|E,J,J\rangle \ . \end{aligned} \tag{3.3}$$

Here we keep explicit the fact that the states $|E,J,J\rangle$ have both $M^2 = J(J+1)$ and $M_z = J$, since they are highest spinning states. This is important since in black hole thermodynamics we care only about Δ and M_z, not M^2, since these are the extensive quantities kept fixed in the microcanonical ensemble. Alternatively, when computing the canonical partition function only Δ and M_z appear in the trace, not M^2.

We are now tasked with building level 1 descendants (i.e. the normal modes) of $|E,J,J\rangle$ by acting with the remaining conformal operator:

$$P_x,\, P_y,\, P_z,\, M_x,\, M_y\,; \tag{3.4}$$

notice that the K_μ annihilate the primaries, so they can be excluded from the analysis. Starting from the P_μ, since $[\Delta, P_\mu] = -iP_\mu$, we have:

$$i\Delta P_\mu\,|E,J,J\rangle = (E+1)P_\mu\,|E,J,J\rangle \ ; \tag{3.5}$$

the P_μ will therefore give rise to normal modes with energy $\Delta E = +1$. As for their behavior under M_z, using the well known fact that $[M_z, P_x \pm iP_y] = \pm (P_x \pm iP_y)$, we obtain:

$$\begin{aligned} M_z P_z |E, J, J\rangle &= J P_z |E, J, J\rangle \ , \\ M_z (P_x \pm iP_y) |E, J, J\rangle &= (J \pm 1)(P_x \pm iP_y) |E, J, J\rangle \ . \end{aligned} \tag{3.6}$$

Therefore the P_μ generate three normal modes, of energy $\Delta E = +1$ and angular momentum $\Delta J = 0, \pm 1$. These normal modes change the energy of the geodesics, thus moving it around in AdS_4.

As for M_x and M_y, we can similarly use the relations:

$$[\Delta, M_x \pm iM_y] = 0, \qquad\qquad [M_z, M_x \pm iM_y] = \pm (M_x \pm iM_y) \ ; \tag{3.7}$$

notice however that $(M_x + iM_y) |E, J, J\rangle = 0$, so we just need to consider $M_x - iM_y$. Therefore, we have:

$$\begin{aligned} i\Delta (M_x - iM_y) |E, J, J\rangle &= E (M_x - iM_y) |E, J, J\rangle \ , \\ M_z (M_x - iM_y) |E, J, J\rangle &= (J - 1)(M_x - iM_y) |E, J, J\rangle \ . \end{aligned} \tag{3.8}$$

Hence, we have found another normal mode with energy $\Delta E = 0$ and angular momentum $\Delta J_z = -1$, which modifies the spin of the black hole without changing its energy. This correspond to a rotation of the spacetime around the geodesics, leaving the trajectory of the massive particle untouched but modifying its spin.

To recap, we have three normal modes with energy $\Delta E = +1$ and angular momentum $\Delta J = 0, \pm 1$ and a normal mode with energy $\Delta E = 0$ and angular momentum $\Delta J = -1$, exactly the same as the ones obtained in [8] from the geodesics point of view. Notice that these modes are bosonic, and thus they can be excited multiple times; in particular, the modes with energy $\Delta E = 1$ can be excited (i.e. we can apply the corresponding operator again) infinitely many times, while the mode with $\Delta E = 0$ can only be excited $2J$ times, i.e. until the operator $(M_x - iM_y)$ hits the state $|E, J, -J\rangle$. Let us now compute the canonical partition function of the black hole. In the semiclassical approximation, i.e. considering only highest spinning primary operators, the partition function is:

$$\mathcal{Z} = \mathrm{Tr}\left[e^{-\beta H + \beta\Omega M_z}\right] \approx e^{S(E,J)} \langle E, J, J| e^{-\beta H + \beta\Omega M_z} |E, J, J\rangle = e^{S - \beta E + \beta\Omega J} \ , \tag{3.9}$$

where the entropy $S(E, J)$ counts the degeneracy of these spinning primary operators. Let us now refine the semiclassical approximation by taking into account the presence of the normal modes. If we focus for simplicity on a single (bosonic) normal mode of energy ΔE and angular momentum ΔJ generated by applying the operator $\mathcal{O}$ to $|E, J, J\rangle$, the canonical partition function becomes:

$$
\begin{aligned}
\mathcal{Z} = \mathrm{Tr}\left[e^{-\beta H+\beta\Omega M_z}\right] &\approx e^{S(E,J)} \sum_{n=0}^{+\infty} \langle E,J,J|\,(\mathcal{O}^n)^\dagger e^{-\beta H+\beta\Omega M_z} \mathcal{O}^n\,|E,J,J\rangle = \\
&= e^{S(E,J)-\beta E+\beta\Omega J}\left(\sum_{n=0}^{+\infty} e^{-\beta(\Delta E-\Omega\Delta J)n}\right) = \frac{1}{1-e^{-\beta(\Delta E-\Omega\Delta J)}} e^{S(E,J)-\beta E+\beta\Omega J}\,.
\end{aligned}
\tag{3.10}
$$

In other words, the effect of a single normal mode of energy ΔE and angular momentum ΔJ is to multiply the semiclassical black hole partition function by the partition function of a quantum harmonic oscillator of energy $\Delta E-\Omega\Delta J$; we have thus rederived the normal modes contribution to the DHS formula by using holographic considerations. If we have more than one mode, since all the modes are independent, we simply get the product of the various denominators. Notice that for the $\Delta E=0$ mode, we should just sum over $n\in[0,2J]$ rather than $n\in[0,+\infty]$; however, since the black hole has a large spin J, we can approximate the sum as a sum to $+\infty$ up to exponential accuracy in J [8]. Finally, in general normal and quasi-normal modes contribute to the partition function via the one-loop determinant computed around the saddle. Therefore, the above procedure is just an approximate, quicker way to obtain the one-loop contribution of the normal modes to the partition function, without having to actually expand the gravitational action around the saddle point and deal with e.g. gauge fixing and ghosts.

Let us now apply (3.10) to the Kerr-AdS case, where we have three normal modes with $\Delta E=1$, $\Delta J=0,\pm1$ and one with $\Delta E=0$, $\Delta J=-1$:

$$
\mathcal{Z} = \frac{1}{1-e^{-\beta(1-\Omega)}}\frac{1}{1-e^{-\beta(1+\Omega)}}\frac{1}{1-e^{-\beta}}\frac{1}{1-e^{\beta\Omega}} e^{S(E,J)-\beta E+\beta\Omega J}\,. \tag{3.11}
$$

As $\Omega\to1$, we see that the denominator associated with the $\Delta E=1$, $\Delta J=1$ mode goes to zero, making the partition function diverge. To better understand what is going on, we can compute the thermodynamic energy E_{norm}, angular momentum J_{norm} and entropy S_{norm} "stored" in the normal modes:

$$
\begin{aligned}
E_{\text{norm}} &= -\,\partial_\beta \log\mathcal{Z} + \frac{\Omega}{\beta}\partial_\Omega \log\mathcal{Z} - E\,,\\
J_{\text{norm}} &= \frac{1}{\beta}\partial_\Omega \log\mathcal{Z} - J\,,\\
S_{\text{norm}} &= \log\mathcal{Z} + \beta(E+E_{\text{norm}}) - \beta\Omega(J+J_{\text{norm}})\,.
\end{aligned}
\tag{3.12}
$$

Focusing on the near-superradiant regime $\Omega\sim1$, we have:

$$
\begin{aligned}
E_{\text{norm}} &= -\frac{1}{\beta(\Omega-1)} + \mathcal{O}\left((\Omega-1)^0\right)\,,\\
J_{\text{norm}} &= -\frac{1}{\beta(\Omega-1)} + \mathcal{O}\left((\Omega-1)^0\right)\,,\\
S_{\text{norm}} &= \mathcal{O}\left((\Omega-1)^0\right)\,.
\end{aligned}
\tag{3.13}
$$

The above equations are telling us that, as $\Omega \to 1$, macroscopic amounts of energy and angular momentum are being stored in the $\Delta E = 1$, $\Delta J = 1$ normal mode of the black hole (as shown by $E_{\text{norm}} = J_{\text{norm}} + \mathcal{O}\left((\Omega - 1)^0\right)$), i.e. the black hole center of mass starts moving around the center of AdS. More importantly, this means that we cannot keep increasing Ω to values $\Omega > 1$: if we continue to add energy and angular momentum to the system, they will simply increase the motion of the black hole center of mass, without actually being stored in the black hole itself. Additionally, the entropy associated to these mode is exactly zero, and not just approximately zero as it would seem from (3.13). This is because, given a certain E and J, there is just a single combination of conformal operators that can act on $|E, J, J\rangle$ to increase the energy of the system to $E + E_{\text{norm}}$ and the angular momentum of the system to $J + J_{\text{norm}}$. The non-zero entropy of (3.12) is then just an artifact of the thermodynamic approximation.

Therefore, we have inferred the existence of a new kind of solutions, obtained by taking a Kerr-AdS black hole and setting its center of mass in motion around the center of AdS, the Revolving Black Holes. In particular, since $\Omega \leq 1$, RBHs are stable, and thus they are valid candidates for the endpoint of the Kerr-AdS superradiant instabilities. Notice also that the black hole center of mass is not at a particular point in AdS; if we approximate it as a heavy particle, its wavefunction will be peaked around a particular radius r, but completely delocalized in φ [8]. This is an unusual situation, since we have massive classical object in a purely quantum superposition. In order to better understand how the $E-J$ phase space is modified by the RBH, let us consider a generic point (E, J) such that if we had a Kerr-AdS black hole it would have $\Omega > 1$. Let us now consider a RBH with a fraction $(1-x)$ of energy E and angular momentum J stored in the normal mode, and the remaining fraction x stored in the actual black hole itself. Using (2.14), this means that the actual black hole has the same a fixed while $m \to xm$. The total system has entropy:

$$S_{\text{tot}}(E, J; x) = S_{\text{Kerr-}AdS}\left(xE, xJ\right) , \tag{3.14}$$

since there is no entropy contribution coming from the normal mode. In principle the black hole could shed any fraction x of energy in the normal mode; the actual value of x is then the ones that is most thermodynamically stable, i.e. the value that maximizes $S_{\text{tot}}(E, J; x)$ while keeping $\Omega \leq 1$. To find x, we then simply take a derivative:

$$\partial_x S_{\text{tot}}(E, J; x) = 0 \qquad \Longrightarrow \qquad \Omega = 1\,. \tag{3.15}$$

This means that the most stable RBH solution is, as expected, the one obtained by storing just enough energy and angular momentum in the normal mode to avoid superradiance ($\Omega = 1$). In particular, since at $\Omega = 1$ we have $r_+^2 = a$, its entropy will be:

$$S_{\text{RBH}}(E, J) = S_{\text{Kerr-}AdS}\left(a, \sqrt{a}\right) = \frac{\pi a}{G_N(1-a)}\,. \tag{3.16}$$

Here a is the ratio

$$a = \frac{J_{\rm BH}}{E_{\rm BH}}\bigg|_{\Omega=1} \tag{3.17}$$

between the energy and angular momentum of the moving black hole. In particular, given a total energy E and J, in the RBH solution we have

$$E - J = (E_{\rm BH}|_{\Omega=1} + E_{\rm norm}) - (J_{\rm BH}|_{\Omega=1} + J_{\rm norm}) = (E_{\rm BH} - J_{\rm BH})|_{\Omega=1}\,, \tag{3.18}$$

since the normal mode carries $\Delta E = \Delta J$. Thus, using (2.14), we have:

$$(E_{\rm BH} - J_{\rm BH})|_{\Omega=1} = \frac{1}{G_N}\frac{\sqrt{a}}{2(1-a^2)^2}\,. \tag{3.19}$$

Let us also compare compare the RBH entropy $S_{\rm RBH}$ with the entropy $S_{\rm br}$ of a black resonator (3.2). Plugging (3.19) into (3.2), we have:

$$S_{\rm br} < 4\pi G_N\,(E-J)^2\,\frac{\pi}{G_N}\frac{a}{(1-a^2)^2} \approx \frac{\pi}{G_N}a\,, \tag{3.20}$$

where the inequality comes from neglecting the l dependent term in (3.2). Notice that we approximated $a \ll 1$, since (3.2) is valid for a small and slowly rotating black hole. Finally, comparing the above equation with the small a limit of (3.16), we see that:

$$S_{\rm br}(E,J) < S_{\rm RBH}(E,J)\,; \tag{3.21}$$

therefore, black resonators (which are already unstable) also have a lower entropy than RBHs. Notice that, while black resonators are unstable, they can still be intermediate metastable steps of the decay of superradiant black holes, arising as one of the superradiant mode grows. Additionally, there is nothing granting us that there are no other solutions, different from RBHs, which appear in the $\Omega > 1$ region and have a higher entropy than the RBHs. Indeed, in section 3.3 we will find such solutions, the so-called Grey Galaxies.

Finally, notice that the RBH construction is not just valid for values of E and J between the superradiant $\Omega > 1$ bound and extremality bound, but it can be extended to construct RBHs for any point with $E \geq J$. To do this, we simply start from a black hole with $\Omega = 1$ and start to store energy and angular momentum in the $\Delta E = 1, \Delta J = 1$ normal mode, creating a new family of RBHs that live on a 45° line in the $E - J$ phase diagram[9]. Since the $\Omega = 1$ curve exists for all values of $E \geq J$, with $E = J$ only at the origin $E = J = 0$, we can extend the RBH solutions to any point with $E \geq J$ phase diagram. In other words, the RBHs not only work as endpoints of the instabilities, but they fill the entire possible phase space up to the unitarity bound $E = J$ (which was previously empty). This realizes the heuristic prediction of

[9] The same exact construction holds for Grey Galaxies; see the discussion at the end of 3.3 and in particular figure 3.

section 2.3, and shows that the apparent absence of the extremal bound on the CFT side of the AdS/CFT duality is not a feature of the gravitational dual CFT, but rather it is simply due to the onset of new gravitational solutions below extremality which had not been previously taken into consideration.

3.3 Grey Galaxies

RBHs, however, are not the only possible endpoints for a Kerr-AdS black hole. As shown in [8], there exists another class of solutions that can replace superradiant Kerr-AdS black hole in the whole $E \geq J$ region: Grey Galaxies (GGs). We will now construct GGs following [8]. We however take a slightly different route in the computation of the partition function of the gas of large l modes (3.35): we will introduce plethystic exponentials to explicitly rewrite the multi-particle partition function in terms of the (easier) single-particle partition function and make a different approximation than [8] in order to reach (3.35).

These solutions share some similarities with both black resonators and RBHs. The idea is once again — similarly to RBHs — to store a fraction of the total energy and angular momentum outside of the black hole itself; the difference it that, instead of considering the motion of the black hole center of mass, the energy and angular momentum are stored in the cloud of Hawking radiation that naturally surrounds a black hole in AdS. Due to the confining potential seen by a particle in AdS, the Hawking quanta emitted by the horizon cannot escape to infinity, but they will bounce back and fall again behind the horizon. A dynamic equilibrium is established between the black hole itself and a cloud of Hawking radiation, with Hawking quanta being continuously emitted and absorbed by the black hole at an equal rate. If we ignore the backreaction of the cloud on the metric (which we know is fine for $\Omega < 1$, since nothing catastrophic happens), a Hawking quantum is then just a simple fluctuation of a field in the Kerr-AdS background; therefore the quanta in the cloud are just a collection of quasi-normal modes of the Kerr-AdS solution, and statistically will cover all the possible values of the energy ω and the angular momentum l of the quasi-normal modes.

As the black hole crosses the $\Omega = 1$ limit, (2.34) tells us that the first modes to go superradiant are the ones with angular momentum $l = l_z$ such that:

$$l \geq \frac{3}{\Omega - 1} \, . \tag{3.22}$$

Since $\Omega \approx 1$ at the onset of superradiant, the superradiant modes have a really high angular momentum $l \gg 1$. It can be shown that modes with $l = l_z \gg 1$ live really far away from the black hole, at a radius $r_l \sim \sqrt{l}$, and are sharply peaked at an angle $\theta = \pi/2$ [8]. Even without doing the actual math, we can understand these facts as follows. Modes with really high angular momentum will feel a really powerful centrifugal force; the larger the l, the further away the modes will orbit the black hole, and the squashing from the centrifugal force will focus most of the wave into the equatorial plane of rotation at $\theta = \pi/2$. The key idea behind GGs is to recognize that, since these modes live further away, they essentially live in empty AdS, interacting only very weakly with the black hole at its center. As these modes grow due to superradiance, they will form a gas of modes that carry energy and angular momentum comparable to the black hole itself, and thus they are no longer negligible when computing the thermodynamics of the whole system. Nevertheless, this gas of Hawking quanta essentially does not interact with the black hole since it is so far away. Therefore, we can consider the black hole and the gas of modes in empty AdS as independent system, compute their partition functions separately and then simply add their energies, entropies and angular momenta together. Since the gas of far-away modes is rotating in the equatorial plane $\theta = \pi/2$ and it is outside the horizon, it forms a "white" (i.e. visible) accretion disk around the black hole, hence the name "Grey Galaxies".

Finally, note that the approximation of the gas of modes as independent of the black hole is essentially what separates GGs from black resonators. In the black resonator case, one does indeed consider that a mode grows due to superradiance, while taking into account the full non-linear interaction of the mode with the black hole background; however, one picks just a single quasi-normal mode, rather than a gas of all the superradiant modes. One might worry that taking all the modes into account at the same time at the non-linear level would make the solutions too hard to study; the new insights of GGs is that, since these modes live far away from the black hole, the non-linearity can be simplified by taking the background to be empty AdS rather than Kerr-AdS.

Due to the very weak interaction between the cloud of superradiant modes and the black hole, to compute the thermodynamics of GGs we just need to take the result (2.14) for Kerr-AdS and add the contribution of a gas of large l modes in pure AdS. Once again, in order to compute the partition function of the modes in empty AdS, we can use the AdS/CFT correspondence and translate the problem in a CFT language. As explained previously, single-particle states in AdS form irreducible representations of the isometry group of AdS_4, which is the conformal group $SO(3,2)$ of the boundary. In the CFT, via the operator-state correspondence, each representation corresponds to a family of descendant operators obtained by acting with conformal generators on a primary operator $\mathcal{O}_{\Delta,s}$, where Δ is the conformal dimension of the operator (related to the mass of the AdS particle) and s is the spin of the operator (which coincides with the spin of the AdS particle). The partition function for the gas of modes can be decomposed as:

$$Z_{\text{gas}} = \text{Tr}\left(e^{-\beta H + \beta \Omega M_z}\right) = \sum_{\Delta,s} Z_{\Delta,s}\,, \qquad Z_{\Delta,s} = \text{Tr}_{\Delta,s}\left(e^{-\beta H + \beta \Omega M_z}\right)\,, \tag{3.23}$$

where we split the trace into the contribution of each conformal family with primary $\mathcal{O}_{\Delta,s}$. The trace $\text{Tr}_{\Delta,s}$ is a multi-particle trace; in the CFT language, this is a trace over the space of local operators obtained by taking an arbitrary number of tensor products of the descendants of the primary with dimension Δ and spin s. To simplify the calculations, it is convenient to rewrite the multi-particle trace in terms of a single-particle trace using the plethystic exponential:

$$Z_{\Delta,s} = \mathrm{PE}[Z_{\Delta,s}^{\mathrm{single}}](e^{\beta})\,, \qquad Z_{\Delta,s}^{\mathrm{single}} = \mathrm{Tr}_{\Delta,s}^{\mathrm{single}}\left(e^{-\beta H+\beta\Omega M_z}\right) \tag{3.24}$$

where the plethystic exponential is defined as follows

$$\mathrm{PE}[f](x) \equiv \exp\left[\sum_{k=1}^{+\infty}\frac{f(x^k)}{k}\right]\,. \tag{3.25}$$

The trace $\mathrm{Tr}_{\Delta,s}^{\mathrm{single}}$ is then a single-particle trace, meaning a trace over just the space of descendants of the conformal primary $\mathcal{O}_{\Delta,s}$.

All we have left to do is to find all the descendants of the primary $\mathcal{O}_{\Delta,s}$ with conformal dimension Δ and spin s. The descendants are built by acting with conformal generators on $\mathcal{O}_{\Delta,s}$; if we put $\mathcal{O}_{\Delta,s}$ at the origin, acting with conformal generators simply means acting with derivatives. In particular, it is convenient to separate the action of the derivatives into a trace and a traceless part as follows [8]:

$$\mathcal{O}_{\Delta,s}^{n,l} = (\partial_\nu\partial^\nu)^n\, C^{\mu_1\cdots\mu_l}\partial_{[\mu_1}\cdots\partial_{\mu_l]}\mathcal{O}_{\Delta,s}\,, \tag{3.26}$$

where $C^{\mu_1\cdots\mu_m}$ is a traceless antisymmetric tensor and $\mathcal{O}_{\Delta,s}^{n,l}$ denotes the descendant. Notice that in general it might happen that different combinations of derivatives give the same operator, or equivalently that there are some null states among the descendants. This is the case for short representations that satisfy the unitary bound $\Delta = 1+s$. Let us focus for now just on long representations with $\Delta > 1+s$, so that we do not have to worry about the null states. The operator $\mathcal{O}_{\Delta,s}^{n,l}$ has conformal dimension $\Delta_{n,l} = \Delta + 2n + l$, since each derivative operator adds $+1$ to the conformal dimension. As for the total angular momentum, the trace $\partial_\nu\partial^\nu$ is a scalar and thus does not contribute. As for the antisymmetric combination $\partial_{[\mu_1}\cdots\partial_{\mu_l]}$, it transforms in the spin l representation of $SO(3)$; thus we get a family of operators $\mathcal{O}_{\Delta,s}^{n,l}$ with total angular momentum $j \in |l-s|, \ldots, l+s-1, l+s$, and hence $j_z \in -j, \ldots, j-1, j$. The single-particle partition function thus reads:

$$Z_{\Delta,s}^{\mathrm{single}} = \sum_{n=0}^{+\infty}\sum_{l=l^*}^{+\infty}\sum_{j=|l-s|}^{l+s}\sum_{j_z=-j}^{j_z=j} e^{-\beta(\Delta+2n+l)+\beta\Omega j_z}\,, \tag{3.27}$$

where we sum only over modes with large angular momentum $l \gtrsim l^*$, where $l^* \gg 1$ is a fixed value. It is convenient to redefine $j = j' - l$, $j_z = j' - l - j'_z$, so that the partition function becomes (assuming $l^* > s$):

$$Z_{\Delta,s}^{\mathrm{single}} = \sum_{n=0}^{+\infty}\sum_{l=l^*}^{+\infty}\sum_{j'=-s}^{+s}\sum_{j'_z=0}^{2(l-j')} e^{-\beta(\Delta+2n)}e^{\beta l(\Omega-1)}e^{\beta\Omega j'}e^{-\beta\Omega j'_z}\,. \tag{3.28}$$

For large angular momenta, we can extend the extremum of the j_z' sum from $j_z' = 2(l-j')$ to $j_z' = +\infty$, since the exponential $e^{-\beta\Omega j_z'}$ is suppressed for $l \gg 1$. Next, notice that in the sum over l, due to the exponential $e^{\beta l(\Omega-1)}$, most of the contributions come from $l \sim 1/(\beta(1-\Omega)) \gg 1$; therefore, we can extend the sum from $l = l^*$ down to $l = 0$ without altering the result. We can then perform the sums over n, l and j_z', finding the single-particle partition function:

$$\begin{aligned} Z_{\Delta,s}^{\text{single}} &\approx \sum_{n=0}^{+\infty}\sum_{l=0}^{+\infty}\sum_{j'=-s}^{+s}\sum_{j_z'=0}^{+\infty} e^{-\beta(\Delta+2n)} e^{\beta l(\Omega-1)} e^{\beta\Omega j'} e^{-\beta\Omega j_z'} \\ &= \sum_{j'=-s}^{+s} \frac{1}{1-e^{-\beta(\Omega-1)}} \frac{e^{-\beta\Delta}}{1-e^{-2\beta}} \frac{1}{1-e^{-\beta\Omega}} e^{\beta\Omega j'} \,. \end{aligned} \tag{3.29}$$

Notice that we could also easily compute them sum over j', since

$$\sum_{j'=-s}^{+s} e^{\beta\Omega j'} = \frac{\sinh\left(\frac{\beta\Omega}{2}(2s+1)\right)}{\sinh\left(\frac{j\beta\Omega}{2}\right)} \,; \tag{3.30}$$

however, leaving the sum as it is shows us that the contribution of a single particle with spin s is the same as $2s+1$ spinless particle with conformal dimension $\Delta - \Omega s, \Delta - \Omega(s-1), \ldots, \Delta + \Omega s$.

To obtain the multi-particle trace (3.24) we simply use (3.25) as follows:

$$\log \mathcal{Z}_{\Delta,s} = \sum_{k=1}^{+\infty} \frac{1}{k} \frac{1}{1-e^{-k\beta(\Omega-1)}} \frac{1}{1-e^{-k\beta\Omega}} \sum_{j'=-s}^{+s} \frac{e^{-k\beta(\Delta-\Omega j')}}{1-e^{-2k\beta}} \tag{3.31}$$

The summand for large k goes as $e^{-k\beta(\Delta-\Omega j')}/k$, with $j' = -s, \ldots, s$; for a long representation with $\Delta > s+1$, the summand is thus highly suppressed. Therefore we can expand to first order in $\Omega \sim 1$ as follows:

$$\log \mathcal{Z}_{\Delta,s} \approx \sum_{k=1}^{+\infty} \frac{1}{k^2} \frac{1}{\beta(1-\Omega)} \frac{1}{1-e^{-k\beta}} \sum_{j'=-s}^{+s} \frac{e^{-k\beta(\Delta-j')}}{1-e^{-2k\beta}} \equiv -\frac{1}{\beta(1-\Omega)} \sum_{j'=-s}^{s} C_{\Delta-j'}(\beta) \,, \tag{3.32}$$

where we defined the function $C_\Delta(\beta)$ as the contribution of a single spinless particle of conformal dimension Δ. Notice also that short representations with $\Delta = s+1$ can be obtained by taking a long representation with $\Delta = s+1$ and subtracting the null states, which themselves form a long representation of conformal dimension $s+2$ and spin $s-1$ [8]. The single-particle trace of a short representation is thus simply the difference of the single-particle traces of a $(s+1, s)$ and a $(s+2, s-1)$ long representation. Using the following property of the plethystic exponential:

$$\text{PE}[f-g](x) = \text{PE}[f](x)\text{PE}[g]^{-1}(x) \,, \tag{3.33}$$

the multi-particle trace of a short representation is:

$$\log \mathcal{Z}^{\text{short}}_{s+1,s} = \log \mathcal{Z}_{s+1,s} - \log \mathcal{Z}_{s+2,s-1} = -\frac{1}{\beta(1-\Omega)}\left(C_1(\beta) + C_2(\beta)\right) . \tag{3.34}$$

To find the full partition function of the large l modes, we simply sum the contributions of all the particles in the bulk, obtaining:

$$\log \mathcal{Z}_{\text{gas}} = -\frac{1}{\beta(1-\Omega)} \sum_{\Delta,s} \sum_{j'=-s}^{s} C_{\Delta-j'}(\beta) \equiv -\frac{1}{\beta(1-\Omega)} C(\beta) , \tag{3.35}$$

where $C(\beta)$ is the summation of the Cs of all the fields in the bulk. Notice that $C(\beta) < 0$ by construction, and it is also possible to show that $C'(\beta) > 0$. From (3.35), we can finally extract the thermodynamic quantities of the (free) gas of large angular momentum modes:

$$\begin{aligned} E_{\text{gas}} &= -\frac{C(\beta)}{\beta^2(1-\Omega)^2} + \frac{C'(\beta)}{\beta(1-\Omega)} \approx -\frac{C(\beta)}{\beta^2(1-\Omega)^2} , \\ J_{\text{gas}} &= -\frac{C(\beta)}{\beta^2(1-\Omega)^2} , \\ S_{\text{gas}} &= \frac{\beta C'(\beta) - 2C(\beta)}{\beta(1-\Omega)} . \end{aligned} \tag{3.36}$$

At leading order, this gas of angular momentum modes carries an energy and angular momentum $E = J$, i.e. it behaves exactly as the normal mode with $\Delta E = \Delta J = 1$ in the RBH case (3.13). However, the gas carries additional entropy, which was not the case for the normal mode (which had null entropy). Grey galaxy solutions are then obtained by considering this gas of large l modes in thermodynamic equilibrium with the Kerr-AdS black hole. To find the equilibrium point, we assume that a fraction x of energy and angular momentum is stored into the black hole, and the remaining $(1-x)$ is stored in the gas; then we simply maximize the total entropy:

$$S_{\text{tot}}(E, J, x) = S_{\text{Kerr-}AdS}(xE, xJ) + S_{\text{gas}}((1-x)E, (1-x)J) . \tag{3.37}$$

However, while $S_{\text{Kerr-}AdS} \sim 1/G_N$, the gas entropy is subleading at $S_{\text{gas}} \sim 1/\sqrt{G_N}$ (when the energy of the gas is of order $E_{\text{gas}} \sim 1/G_N$); thus in the maximization we just need to maximize the Kerr-AdS entropy, neglecting the gas (i.e. the same as we did for the RBHs). Since the black hole entropy grows with the energy, we simply put as much energy as possible in the black hole while keeping it stable, i.e. we just set $\Omega = 1$ (once again, this is the same as for RBHs). A Grey Galaxy is thus given by a Kerr-AdS black hole at $\Omega = 1$ in equilibrium with a gas of modes with large angular momentum living really far away from the black hole; since for the gas $E_{\text{gas}} = J_{\text{gas}}$, we have families of GGs living on 45° lines in the $E - J$ phase diagram — starting from the $\Omega = 1$ line — for all the possible values of $E \geq J$ in the phase diagram. Since the entropy of the gas is greater than zero (though still subleading in $1/G_N$), the GGs dominate over the RBHs (and hence also over the black resonators, as explained in section 3.2) in the whole region between the $\Omega = 1$ line and the unitarity bound $E = J$.

Finally, let us plot again the $E - J$ phase diagram of the system (see figure 2), this time taking into accounts the existence of GGs:

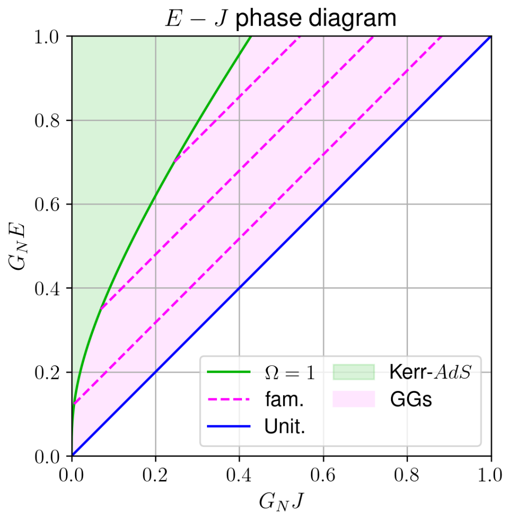

Figure 3: $E - J$ phase diagram of Kerr-AdS and Grey Galaxies, in units of G_N.

Comparing the above phase diagram with figure 2, we first off notice that there is no longer any mention of the extremality bound, and we are thus left with only two phases (i.e. two shaded regions). Above the $\Omega = 1$ (green) line, $\Omega < 1$ and thus the regular Kerr-AdS black holes are stable. Below the $\Omega > 1$ line, however, Kerr-AdS black holes develop superradiant instabilities and decay to GGs, which dominate over the whole pink shaded region between the $\Omega = 1$ line and the unitarity bound (blue line). We also highlighted a few particular families of GGs (the dashed pink lines). By "family" here we mean that we start from a Kerr-AdS black hole at $\Omega = 1$ and then we increase the total E and J of the system by storing the additional energy and angular momentum into a cloud of large angular momentum modes far away from the black hole. Since the cloud of modes carries $E = J$ (see (3.36)), a single family of GGs in the $E - J$ phase diagram is given by a 45° line starting from a particular black hole with $\Omega = 1$, parallel to the unitarity bound $E = J$. By attaching a 45° line to each point on the $\Omega = 1$ curve — i.e. by considering all the possible families of GGs — we can cover with the GGs phase the whole region between the onset of superradiant instabilities at $\Omega = 1$ and the CFT unitarity bound $E = J$. Finally, notice that since RBHs also have $E = J$ (from (3.16)) the same exact construction applies for RBHs. In particular, for every GG we have a RBH with the same energy and angular momentum, which is connected to the same $\Omega = 1$ Kerr-AdS black hole by the same 45° line; however, since the RBHs have a smaller entropy, there is no RBH-dominated phase and GGs are thermodynamically favored any time they exist.

At last, let us mention that in our analysis we simply assumed on physical grounds that the interaction between the black hole and the cloud of large l modes is negligible and hence that the two systems are independent. One might worry that as we put a large amount of energy and angular momentum $E_{\text{gas}} = J_{\text{gas}} \sim 1/G_N$, the back-reaction of the cloud on the spacetime might substantially modify the Kerr-AdS background and the interaction between the cloud and the black hole might no longer be negligible. However, it has been shown perturbatively in [8] that the back-reaction of the gas remains indeed small, and that the approximation of the gas living in empty AdS with no interaction with the Kerr-AdS black hole is still valid.

4 Conclusions

4.1 Summary of the work

In conclusion, we have reviewed the superradiant instabilities of Kerr black holes in an AdS background, and discussed the possible endpoints of such instabilities. We identified two new solutions proposed in [8] — Grey Galaxies and Revolving Black Holes — that are stable in the whole region between the onset of the superradiant instabilities down to the unitarity bound, with the former dominating at thermodynamic equilibrium. These solutions extend the gravitational phase space below the extremality bound for the black holes. Properties of these solutions have been investigated with the help of holographic arguments, which simplify considerably the calculations.

In section 2.1 and 2.2, we reviewed some evidence for the superradiant instabilities of Kerr-AdS black holes, starting with the heuristic argument proposed by Zel'dovich and then computing explicitly the quasi-normal modes in the Kerr-AdS background. These two approaches are complementary to each other. The study of quasi-normal modes of section 2.2 is more rigorous, but it requires the assumption of a small, slowly rotating black hole to make the computations manageable analytically. While one can use numerical methods to find the quasi-normal modes of bigger black holes, the heuristic argument of section 2.1 shows that the conclusions drawn from the quasi-normal modes analysis can be extended safely to black holes of all sizes, while additionally providing some more physical intuition of the process of superradiance, which is seen as arising from the combination of AdS acting as a reflecting box and the possibility of extracting energy from a rotating black hole via a generalized Penrose process.

Before discussing the possible endpoints of the black hole decays, we used the AdS/CFT correspondence in section 2.3 to establish which region in the $E - J$ phase space we expect to be dominated by the new solutions. We highlighted how there is some tension between the expectations derived from the gravitational and the CFT side of the duality: from the gravity point of view, the new solutions should extend from the onset of the instabilities down the extremality bound (2.13); from the CFT point of view, the solutions can in principle extend further, down to the much weaker unitarity bound (2.43). The explicit construction of GGs and RBHs — which indeed exist down to the unitarity bound — solve the tension between the gravitational theory and the CFT, providing the missing gravitational states in the gravitational phase diagram, which now covers the whole region $E \geq J$.

We then moved on to discuss in section 3 various attempts to construct the endpoints of the superradiant instabilities: black resonators, RBHs and GGs. Black resonators are built essentially by letting one of the superradiant modes grow, that is by combining geons and boson stars — generalized solitonic normal modes — with a black hole at their center. RBHs and GGs are instead built with the common idea of taking out some of the total E and J from the black hole itself and storing it into some other degrees of freedom in the system, respectively the motion of the black hole center of mass and the gas of large angular momentum Hawking quanta surrounding a black hole in AdS. While black resonators are themselves unstable due to still having a black hole with $\Omega > 1$ at their core, both RBHs and GGs manage to reduce the angular velocity of the black hole down to $\Omega = 1$. In both cases, an amount $\Delta E = \Delta J$ of energy and angular momentum is stored outside of the black hole horizon; in the $E - J$ phase diagram, we thus get families of solutions on 45° lines starting from $\Omega = 1$ Kerr-AdS black holes (see figure 3). In particular, RBHs in a single family carry zero additional entropy with respect to their common $\Omega = 1$ Kerr-AdS black hole entropy, since any point of the 45° line is reached by setting the center of mass in motion uniquely (i.e. applying only a unique AdS isometry). As for GGs, the $\Omega = 1$ Kerr-AdS black hole at their center is surrounded by a cloud of large l modes, which is by its nature a statistical system. Therefore the modes carry some additional (though subleading in G_N) entropy and hence GGs have a higher entropy than RBHs and dominate in the microcanonical ensemble.

For both RBHs and GGs, AdS/CFT considerations proved once again essential for simplifying the calculations. As for RBHs, the easiest way to compute the normal modes and their contribution to the one-loop determinant for the Kerr-AdS geometries was to study the descendants of the primaries associated to a Kerr-AdS black hole in the semiclassical approximation. As for GGs, the AdS/CFT correspondence was used to compute the partition function of the gas of modes, without actually having to solve for the wavefunctions of the fields and to compute their one-loop determinants. In principle, one might worry that the calculations on the CFT side are limited by the fact that the CFT is strongly coupled when the gravitational theory is weakly coupled. However, all our considerations were based purely on the conformal algebra itself, and thus work at any value of the CFT coupling. In particular, the unitary bound is obtained by evaluating commutators of the conformal algebra in generic states, and thus it holds for any CFT. As for RBHs, the normal modes are once again built by applying the conformal generators on an ensemble of primaries. In particular, we do not care about the specific details of this ensemble but just about the values of ΔE and ΔJ of the normal modes, which are fixed by conformal symmetry. Finally, as for GGs, the single-particle states in AdS simply form irreducible representations of the conformal group; therefore, even if we do not know the specific relation between the mass of a particle in AdS and the conformal dimension Δ of the corresponding primary (which will depend on the coupling), the thermodynamics of the gas of large l modes — given by (3.36) — stays the same.

4.2 Future directions

The introduction of GGs and RBHs lead to many possible generalizations and future research directions. The most important one is arguably to understand what happens when the black hole carries electromagnetic charge. It has been shown in [25] that charged black holes in AdS also suffer from instabilities, when the charge of the charged field is high enough. The first proposed endpoints in the case of charged, non-rotating black holes in AdS [26, 27] are hairy black holes whose hair are a Bose condensate of the charged field living in the vicinity of the black hole; as usual, these solutions have been built either perturbatively (for small black holes) or numerically. However, angular momentum and charge are not that different for black holes: it is indeed the case that some charged black holes can be uplifted to higher dimensional purely rotating solutions, where the "lower-dimensional" charge comes from the rotations of the higher-dimensional black hole in the compactified directions. While we did not derive GGs and RBHs in dimensions higher than 4 — i.e. with more than one angular momentum Cartans — their construction is general enough that we expect similar solutions to exist in higher dimensions. If we see charge as a "higher-dimensional" rotation, it is reasonable to expect that an analogue of GGs (and maybe even RBHs) should exists also for charged black holes. In particular, since the large l modes of the GGs live far away from the black hole and are weakly interacting with it, we might expect that these "electric GGs" consist of clouds of some different modes living yet again far away from the black hole at their center. These solutions will therefore be different from the hairy black holes [26, 27], whose charged condensates live close to the black hole.

Additionally, the analogous condition for the instabilities due to the charge is considerably less "clean" than the superradiant condition due to the angular momentum. In particular, a small charged black hole is unstable under a mode if $q\mu_q > \omega$ [8], where q is the charge of the mode, μ_q the electric potential of the black hole and ω the frequency of the mode; this expression however gets corrected as we take bigger and bigger black holes. This is in contrast with the superradiance condition $l_z\Omega > \omega$, which leads to $\Omega > 1$ for black hole of any size. By uplifting the solutions and considering the charge as angular momentum, one might then hope to find a better condition for the charge instability that works for black holes of any size.

Studying the behavior of unstable Kerr–Newman black holes in AdS is also of central importance for two (partially related) reasons: the possibility of finding new supersymmetric solutions and understanding the CFT dual of the endpoints. Let us discuss both of these reasons in one of the easiest setups, namely the duality between Type IIB supergravity on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ $SU(N)$ super-Yang–Mills theory on the 4D conformal boundary of AdS_5. In the dimensionally reduced gravitational theory on S^5, black holes can carry up to two angular momenta J_1, J_2 and three charges Q_1, Q_2, Q_3; the angular momenta come from the $SO(4)$ Cartans of the AdS_5

spatial rotations, while the charges come from the Cartans of the S^5 rotations. Restricting for simplicity to the subsector with $J_1 = J_2 \equiv J$ and $Q_1 = Q_2 = Q_3 \equiv Q$, black holes are labeled by the triplet (E, J, Q), with E the mass of the black holes. Supersymmetry and extremality actually impose two independent conditions on E, J and Q, and both must be imposed to obtain physical solutions (i.e extremal solutions with a real and not complex metric). Therefore, BPS black holes that are both supersymmetric and extremal do not exist for every value of J and Q, but only for a certain curve in the $J - Q$ plane; these black holes are called the Gutowski–Reall black holes [28]. Finding a generalization of GGs in this context therefore not only solves the issue of superradiant instabilities, since Gutowski–Reall black holes have $\Omega = 2$ [8]; additionally they might provide a new family of supersymmetric solutions in the whole $J - Q$ plane, similarly to how GGs provided new solutions in the whole $E - J$ phase diagram down to the extremality bound.

As for the CFT point of view, the advantage of the above setup is two-fold: first, the CFT dual ($SU(N)$ super-Yang–Mills) is known explicitly and well studied; second, supersymmetry allows for better control on both sides of the duality. In particular, regular black holes should be thought as a "quark-gluon plasma" on the CFT side [8]; the superradiant instabilities of the black holes then indicate that this plasma is not stable, and must decay into a new CFT phase. As for GGs, the gas of large l far away modes in GGs should correspond to a gas of independent, fast rotating "glue-balls" that are expelled by the quark-gluon plasma [8]. It would therefore be interesting to get a better understanding of what is actually happening on the CFT side, and to what phase of the CFT GGs correspond to. Additionally, while RBHs are not dominant thermodynamically, they exist as solutions and thus they should also correspond to some different phase on the CFT side. A possible approach to connect to the CFT side of the duality comes from index calculations and explicit microstate counting (see e.g. [29, 30, 31]); index computations can only "see" supersymmetric black holes, hence the importance of extending GGs (and possibly RBHs) to the supersymmetric case.

Note that there has been already some work in finding a generalization of GGs in the dimensional reduction of Type IIB supergravity on $AdS_5 \times S^5$ [32]. The advantage of this setup is that one can work on the uplifted solution — where charge is mapped to angular momentum — in order to leverage the construction of GGs (which deals with angular momentum only). In particular, the main proposal of [32] is that the hairy black holes that were speculated to be the endpoint of charged black holes are themselves unstable, and they decay into so-called "Dual Dressed Black Holes". These solutions are obtained by considering a black hole surrounded by one, two or three dual giant gravitons that live far away from the black hole and are thus weakly interacting with it. Dual giant graviton are probe D3 brane in the 10D theory that wrap an S^3 that moves inside the S^5 [33]; they are stabilized by their high angular momenta and thus live far away from the black hole. In the large N limit, they carry $E = Q_i$ (similarly to $E = J$ for large l modes of GGs), and thus we can store energy and charge in these dual giant gravitons, outside of the black holes, producing new stable endpoints [32].

Finally, besides an explicit construction of GGs and RBHs fully backreacted solutions, another interesting direction to explore is to get a better understanding of the decay of superradiant black holes into GGs. Numerical simulations do not give particularly conclusive results at the moment [8]. The issue is that, exactly because large l modes live far away and are thus interact weakly with the black hole, the decay of the black hole takes a really long time, making numerical computations challenging. It would therefore be interesting to have some better understanding of the decay, either by some more advanced numerical calculations or additional analytic insights. In particular, one might also explore the role of black resonators and RBHs in the decay. While the former are not stable and the latter have a lower entropy than GGs, they could still be intermediate step in the decay of superradiant black holes to GGs. For example, a superradiant black hole might first start to shed some of its energy and angular momentum into its center of mass motion rather than the large l modes, since they live far away, forming a RBH. This energy can then later be transferred to the large l modes, over a much longer period of time [8]. A further study of these fascinating phenomena will likely provide a deeper inside into the dynamics of black holes in AdS.

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