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Generalised Geometry, Type II Supergravities and Consistent Truncations

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Introduction

Since almost as early as its first appearance, it has been clear that string theory can be a suitable theory for the unification of general relativity with quantum mechanics. Indeed the quantum fluctuations of a closed fundamental string always contain a graviton in their spectrum. Apart from requiring gravity in its formulation, string theory has been proven to have several desirable properties. For example, the dimension of the string acts as a natural cut-off, making the theory free of ultraviolet divergences.
It is known that in order to include fermions in the theory, the presence of supersymmetry is required. This, together with the requirement of the absence of anomalies, allows one to fix the number of permitted space-time dimensions to $d = 10$. Whilst on the one hand this is somewhat appealing, it implies that a certain number of dimensions have to be 'compactified', with the size of the compactification manifold sufficiently small to make the theory compatible with the contemporary experimental evidences. Another important feature of (super-)string theory is that it has been shown that its low energy limit gives rise to supergravity - a theory with local supersymmetry. This theory, which is also a theory in its own right, can therefore be very useful both in understanding basic properties of string theory and in looking for its possible experimental evidences. It is then clear that 10-dimensional supergravity theories are particularly interesting in the context of string theory and string phenomenology. Despite all the good properties of these theories, both string theory and supergravity are not yet fully understood. For example, one of the problems is that it is not yet clear in what manner they can be compactified to successfully reproduce the effective theory in four dimensions represented by the Standard Model. This is just one example, but there are many more features which are still obscure. For this reason, string theory and supergravity are nowadays still intensely researched.
In recent years a process of geometrisation of string theory has been carried out. This process is performed by means of that part of geometry that falls under the name of 'generalised geometry'. Generalised geometry is essen-

tially the study of structures on a generalised tangent space $E \approx TM \oplus T^*M$ of a manifold M. It originated as an evolution of the main geometrical structures of mechanics in the work of Courant [Co1990] and was later further extended by the Hitchin's school (see e.g. [Hi2003, Gu2004, Hi2010]). At that time it was already known that supergravity could be reformulated with a larger symmetry group and with a structure reflecting the duality symmetries of string theory (see [CoStWa2011] and references therein), but generalised geometry gave a perfect framework upon which one could found these reformulations. For example, in the paper [CoStWa2011] it is shown how Type IIA and IIB supergravity theories, to leading order in the fermions, can be reformulated as generalised geometrical analogues of Einstein gravity. It turned out that many more features of supergravity theories can be described through its formalism. For example, (an extension of) generalised geometry can also be used to describe 11-dimensional supergravity compactified to a d-dimensional manifold ([CoStWa2013, CoStWa2013n2]). Furthermore, problems of compactification of string and supergravity theories, especially with fluxes, appear to have a clearer description and intelligibility in its language. More generally it is believed that a geometrical reformulation of string theory and supergravity will help us deal with their difficult aspects and extension, in the same manner that a geometric reformulation has helped us understand problems that range from the domain of mechanics to the ones of high energy theoretical physics.

In this thesis we will first describe some general features of generalised geometry, like the generalised tangent bundle and the natural geometric structures one can define on it, namely the natural metric, the Courant bracket, the Dorfman derivative and the generalised metric. Then we will introduce some of the aspects of Type IIA and IIB supergravity theories and show the relation that is present between the generalised geometrical structures and the field content and symmetries of the physical theories. Later we will describe, following closely the article by Waldram at al. [CoStWa2012], how the NSNS sector of Type IIA and Type IIB supergravity theories can be viewed as a gravitational theory for a generalised metric. This will lead us to the definition of a generalised Levi-Civita connection and to a brief discussion about generalised curvature operators.

After this we will consider the problem of consistent truncations ([DNP86]) and in particular we will focus on the generalised Scherk-Schwarz reduction (see [ScSc1979, LeStWa2014]). We will describe some general properties of this construction and derive some simple original results; we will then rederive with our formalism the known result on compactifications on compact Lie groups ([BaPoSa2015]). Then we will consider a known non-Lie grouplike solution and will show that it is actually obtained from a Lie group by

Inonu-Wigner contraction. Motivated by this analysis we will then try to understand the conditions under which similar contractions can be performed on other Lie groups in order to obtain non-Lie group homogeneous spaces. This will also contain original contributions.

Chapter 1

Generalised Geometry

In this chapter, we will introduce the generalised tangent bundle E and the main geometric structures that can be defined on it. After the introduction of the natural metric, we will then present the Courant bracket and the Dorfman derivative as suitable extensions to E of the Lie bracket and Lie derivative respectively: the first for integrability and the second for symmetry reasons. We will then introduce a 'generalised' metric and see how the resulting structure encodes the degrees of freedom of a metric and a two-form (or 'B-field'). In the end of the chapter, we will also present a generalisation of the construction of E used until that moment.
We hope that such an exposition will clarify the reason why the common geometric structures used in generalised geometry are legitimate. Moreover, by presenting in this chapter generalised geometry as a purely mathematical theory, i.e. that does not rely on any physical assumption, we would like to emphasise its striking feature of being able to perfectly describe the $NSNS$ sector of Type II supergravity theories; a feature that will be presented in chapter 2.
Finally, we would like to note that every time a new structure will be introduced, we will also consider its effect on the reduction of the frame bundle associated to E. This will turn out to be a very useful manner to keep track of the symmetries of the theory and to find connections with the related physical theories.

1.1 The natural structures of the generalised vector bundle

1.1.1 The natural metric

Let M be a differential manifold. Generalised geometry is the study of the geometrical structures that can be defined on the generalised tangent bundle $E \approx TM \oplus T^*M$, where TM and T^*M are the tangent and cotangent bundle associated to the manifold M respectively. We did not use the equality symbol because in general the generalised tangent bundle is defined as an extension of the tangent bundle via the cotangent, as we will see later in section 1.3.

For now let us consider the vector bundle $E = TM \oplus T^*M$ over M. In what follows we will denote sections of E with capital Latin letters ($X, Y, Z \in \Gamma(E)$), sections of TM with lower-case Latin letters ($v, w, z \in \Gamma(TM)$) and sections of T^*M with lower-case Greek letters ($\mu, \nu, \rho \in \Gamma(T^*M)$). We will call sections of E 'generalised vectors'. There is a natural metric defined on sections of E induced by the canonical pairing between vectors and covectors.

Definition 1.1 (Natural Metric)**.** If we denote by i_v the inner product on covariant tensor fields that associates each tensor T with the contraction of the vector v with the first component of T, we define the metric η as follows:

$$\eta(X,Y) := \frac{1}{2}(i_v\nu + i_w\mu) = \frac{1}{2}(\nu(v) + \mu(w))$$
$$\forall\ X = v + \mu, Y = w + \nu \in \Gamma(E)$$

If d is the dimension of M, then η has signature (d, d) and is therefore indefinite. One easy way to see it is as follows. If we write the generic generalised vector as a column vector where the first argument is the section of TM and the second the one of T^*M we can write:

$$\eta = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} \frac{1}{2}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix}$$

and clearly $\frac{1}{\sqrt{(2)}}\left(\begin{smallmatrix} \mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{smallmatrix}\right)$ is an orthogonal as well as symmetric matrix. This fact also implies that the group preserving this indefinite metric is $O(d, d)$. From this early fact we immediately see how this generalised tangent bundle is definitely not a generic vector bundle but instead is a very particular one. We now want to show in more details some of these particularities. To clarify our terminology we recall some definitions.

Definition 1.2 (3.9.1 [AbTo2011]). Let E, M, S three differential manifolds. A *fibre bundle* of typical fibre S is a differential surjective map $\pi : E \to M$ such that $\forall p \in M$ there exists a neighbourhood U of p and a diffeomorphism $\chi : \pi^{-1}(U) \to U \times S$ such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\chi"] \arrow[d, "\pi"'] & U \times S \arrow[dl, "\pi_1"] \\ U & \end{tikzcd}$$

where π_1 is the projection on the first coordinate.

Given a collection of pairs $\{(U_\alpha, \chi_\alpha)\}$ as above such that $\{U_\alpha\}$ forms an open covering of M, called *atlas*, we can find $\forall\ U_\alpha \cap U_\beta \neq \{\}$ maps $\psi_{\alpha\beta}$, called *transition functions*, that associate each $p \in U_\alpha \cap U_\beta$ with a diffeomorphism $\psi_{\alpha\beta}(p) : S \to S$ such that: $\chi_\alpha \circ \chi_\beta^{-1}(p, s) = (p, \psi_{\alpha\beta}(p)(s))\ \forall p \in U_\alpha \cap U_\beta$ and $s \in S$. The transition functions satisfy the *cocycle conditions*: $\psi_{\alpha\alpha}(p) = id_S$ and $\psi_{\alpha\beta}(p) \circ \psi_{\beta\gamma}(p) = \psi_{\alpha\gamma}(p)\ \forall\ p \in U_\alpha \cap U_\beta \cap U_\gamma$.

Definition 1.3 (3.9.6 and 3.9.9 [AbTo2011]). Given an action $\theta : G \times S \to S$ of a Lie group G on S, the fibre bundle $\pi : E \to M$ has a *G-structure* if there exists a family of differential maps $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ associated with the atlas $\{(U_\alpha, \chi_\alpha)\}$, called *transition functions* of the G-structure, such that the transition functions $\psi_{\alpha\beta}$ of the fibre bundle are given by the action of G: $\psi_{\alpha\beta}(p, s) = \theta(\phi_{\alpha\beta}(p), s) = \phi_{\alpha\beta}(p) \cdot s$.
A *principal fibre bundle* of structure group G is a fibre bundle with typical fibre G endowed with a G structure such that the action $\theta : G \times G \to G$ is given by the left translation.

For every vector bundle E (of rank n) there is an associated natural principal $GL(n)$ fibre bundle, called the *frame bundle*. It can be thought of as the bundle with typical fibre given by the ordered set of frames of the vector bundle. Specifically, chosen the canonical basis of $\mathbb{R}^n$ and fixed one point $p \in M$, there is one and only one frame of E_p, associated with each element of $GL(n)$, that corresponds to the image of the canonical basis of $\mathbb{R}^n$ via that element of $GL(n)$. A natural action of $GL(n)$ is then defined on the frames of the tangent bundle and it can be proved that this construction actually satisfies the assumptions of the definition of a principal fibre bundle (see e.g. [AbTo2011] Example 3.9.13).
In some situations we might also be interested in whether the structure group of the frame bundle can be consistently reduced to a subgroup of $GL(n)$, i.e. in whether we are able to find choices of local frames in the different patches that are connected with transition functions induced only by a subgroup

$G \subset GL(n)$. The possibility of the reduction of the structure group is clearly related to topological properties of the vector bundle. For example, if we are able to find a *global frame*, reducing therefore the structure group to the identity subgroup of $GL(n)$, the vector bundle is actually a trivial one, i.e. it is the direct product $M \times E_p$, for any $p \in M$ (see for example [Ko2011]). Sometimes the possibility of the reduction of the structure group can be conveniently encoded in the existence of some globally defined non-degenerate tensor fields, for then we can require to allow only transition functions that keep these tensor fields invariant.
Let us now consider again our generalised tangent bundle $E = TM \oplus T^*M$ and the natural metric η defined therein. We have already noted that the group preserving this indefinite metric is $O(d,d)$ - the structure group can therefore be reduced to $O(d,d)$. This reduction can be further extended as follows (see [Gu2004]). The highest exterior power can be decomposed as:

$$\bigwedge^{2d}(T_pM \oplus T_p^*M) = \bigwedge^d T_pM \otimes \bigwedge^d T_p^*M$$

where $\bigwedge^d(V)$ with $d \in \mathbb{N}$ indicates the space of the alternating d-contravariant tensors on the vector space V. We can therefore identify $\bigwedge^{2d}(T_pM \oplus T_p^*M)$ with $\mathbb{R}$ via the following natural pairing between $\bigwedge^d T_p^*M$ and $\bigwedge^d T_pM$: $(v^*, u) = det(v_i^*(u_j))$, with $v^* = v_1^* \wedge .. \wedge v_d^* \in \bigwedge^d T_p^*M$ and $u = u_1 \wedge ... \wedge u_d \in \bigwedge^d T_pM$. The number $1 \in \mathbb{R}$ then defines a canonical orientation on $T_pM \oplus T_p^*M$ and we can restrict the structure group to the subgroup of transition functions that preserve both η and the canonical orientation, i.e. construct an oriented atlas. Note that the orientability of the generalised tangent bundle depends only on the canonical pairing between vectors and 1-forms, in particular it does not imply the orientability of the underlying manifold.
We restricted in this very natural way the structure group of the generalised tangent bundle to $SO(d,d)$. Now consider its Lie algebra $\mathfrak{so}(d,d)$. It is clearly given by the anti-adjoint endomorphisms of the vector space $T_pM \oplus T_p^*M$, i.e. $\{O : T_pM \oplus T_p^*M \to T_pM \oplus T_p^*M \mid O^* = -O\}$. Using the splitting $T_pM \oplus T_p^*M$ we can write the matrix of O in block form as: $O = \left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right)$. Now, the adjoint is $O^* = \left(\begin{smallmatrix} P^* & R^* \\ Q^* & S^* \end{smallmatrix}\right)$, but we need to remember that the elements of the dual tangent space have now the first component belonging to T^*M and the second one to TM, because $(T_pM \oplus T_p^*M)^* \approx T_p^*M \oplus T_pM$. So, considering that O^* is a map $O^* : T_p^*M \oplus T_pM \to T_p^*M \oplus T_pM$, the condition $O^* = -O$ becomes: $S = -P^*$, $Q = -Q^*$ and $R = -R^*$. Recall that, from the definition of O, P is an endomorphism of T_pM, S of T_p^*M, while $Q : T_p^*M \to T_pM$ and $R : T_pM \to T_p^*M$. Calling them with more

standard names $P =: A$, $Q =: \beta$ and $R =: B$ we have that the most general element of $\mathfrak{so}(d,d)$ takes the form:

$$O = \begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix} \tag{1.1}$$

where $A \in End(T_pM)$, B is a two-form and β is a bi-vector. We have therefore recovered the fact that: $\mathfrak{so}(T_pM \oplus T_p^*M) = \bigwedge^2(T_pM \oplus T_p^*M) = End(T_pM) \oplus \bigwedge^2 T_p^*M \oplus \bigwedge^2 T_pM$, as long as the structure of vector space is concerned.

We can now find by exponentiation the most important subgroups of $SO(d,d)$, which together generate its connected component to the identity. Noting that $\left(\begin{smallmatrix} 0 & \beta \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ B & 0 \end{smallmatrix}\right)$ are nilpotent and that $\left(\begin{smallmatrix} A & 0 \\ 0 & -A^* \end{smallmatrix}\right)$ is diagonal, we immediately find, for $X = v + \mu \in T_pM \oplus T_p^*M$:

1. (B-transform): $exp(B) \cdot X = \left(\begin{smallmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} v \\ \mu \end{smallmatrix}\right) = v + (\mu - i_v B)$

2. (β-transform): $exp(\beta) \cdot X = \left(\begin{smallmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} v \\ \mu \end{smallmatrix}\right) = (v - i_\mu \beta) + \mu$

3. (GL(d)-action): $exp(A) \cdot X = \left(\begin{smallmatrix} exp(A) & 0 \\ 0 & exp(-A^*) \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} v \\ \mu \end{smallmatrix}\right) = exp(A)(v) + exp(-A^*)(\mu)$
   We can indeed extend this action to the full action of $GL(d)$, so that it, with both its two connected components, can be viewed as a subgroup of $SO(d,d)$.

What is shown here is quite remarkable: the natural structure group of the generalised tangent bundle can be thought of as an extension of the ordinary structure group of the usual tangent bundle, i.e. an extension of $GL(d)$, that also includes other 'symmetries' - β and B transforms.

1.1.2 Courant algebroid

Our process of 'natural reduction' of the structure group is however not done yet. In fact we know that on the tangent bundle there is a bilinear map that associates each pair of vector fields $v, w \in TM$ with another vector field $[v,w] \in TM$, namely the Lie bracket. This map is natural in the sense that it is invariant under the action of the diffeomorphisms (see [AMR1988]). Moreover it can be shown ([Gu2004] Prop 3.22) that these are also *all* the possible symmetries of the bracket. It is also well known that the Lie derivative on tensors, once restricted on vector fields, coincides with the Lie bracket. We would like to find something analogous to this construction also for the generalised vector bundle.

To guide ourselves in our quest for a generalised Lie bracket we will use another concept, which is also deeply related to the Lie bracket of vector fields: integrability. Recall that a k-dimensional ($k \leq d$) (smooth) *distribution* is a subbundle of the tangent bundle which is locally spanned by k-vector fields.

Definition 1.4 ([AbTo2011] 3.7.7)**.** An integral submanifold of a smooth k-dimensional distribution $\mathcal{D}$ is an immersed submanifold $S \hookrightarrow M$ such that $T_pS = \mathcal{D}_p \; \forall p \in S$. We will say that $\mathcal{D}$ is *integrable* if every $p \in M$ is contained in an integral submanifold of $\mathcal{D}$.
$\mathcal{D}$ is *completely integrable* if $\forall p \in M$ there exists a local chart (U, ϕ), $p \in U$, such that $\phi(U) = V^1 \times V^2 \subset \mathbb{R}^k \times \mathbb{R}^{d-k}$ and such that $\left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^k}\right)$ span $\mathcal{D}_p$ $\forall p \in U$.
A distribution $\mathcal{D}$ is *involutive* if for all pairs of local vector fields X, Y with value in $\mathcal{D}$ the vector field $[X, Y]$ is a local vector field with value in $\mathcal{D}$.

It is a classical result of differential geometry the fact that the following chain of implications holds:

$$\text{completely integrable} \Rightarrow \text{integrable} \Rightarrow \text{involutive} \overset{\text{Fr.}}{\Rightarrow} \text{completely integrable}$$

where the last implication takes the name of 'Frobenius theorem' (see e.g. [AbTo2011]). The point is that (complete) integrability is equivalent to closure under Lie bracket. Moreover a completely integrable distribution $\mathcal{D}$ defines a 'foliation' of M, composed of all the maximal integrable submanifolds of $\mathcal{D}$.
We want to develop a concept of integrability in the case of the generalised tangent bundle that is an extension of the one known for the tangent bundle. One of the simplest but non-trivial situations is the one given by the presence of an integrable distribution $\mathcal{D}$ on a smooth manifold M. Take the subbundle L of the tangent bundle generated by the integral distribution of a leaf Σ of the foliation induced by $\mathcal{D}$ in M, i.e. $L = T\Sigma$. Then the differential of the inclusion map $i : \Sigma \hookrightarrow M$ is a map $di : \Gamma(T\Sigma) \to \Gamma(TM)$ such that: $di([v, w]) = [di(v), di(w)] \; \forall \; v, w \in \Gamma(L)$. Moreover $[v, fw] = f[v, w] + di(v)(f)w \; \forall v, w \in \Gamma(L), f \in \mathcal{C}^\infty(M)$. These are actually the defining properties of what is called a *Lie algebroid*, i.e. a vector bundle L on M endowed with a Lie bracket and a bundle map $\tilde{a} : L \to TM$ that induces a map $a : \Gamma(L) \to \Gamma(TM)$, called *anchor*, that is a Lie algebra homomorphism and that also satisfies a Leibniz rule $[v, fw] = f[v, w] + a(v)(f)w \; \forall v, w \in \Gamma(L), f \in \mathcal{C}^\infty(M)$.
The reason why it is thought that Lie algebroids give the right extension of the construction is that they induce a 'generalised foliation' (in the sense of Sussmann [Su1973]) of the manifold M (see [Gu2004] or [Co1990]), i.e. the

distribution $a(L)$ still induces a foliation but now the dimension of the leafs can vary from point to point in the manifold (more specifically the dimension is a semi-continuous function on the manifold).
The problem for the generalised tangent bundle is that non-trivial brackets (that at least preserve some of the symmetries of the natural metric η) are plagued by the incapacity to satisfy the Jacobi identity.
Even if in general it is difficult to provide a significant bracket that gives a correspondence between closure and (generalised) integrability, there is a natural kind of subbundles of E for which the construction works.

These are the maximally isotropic distributions.

Definition 1.5. Let η be a bilinear form on the product $V \times V$ of a finite-dimensional vector space V over the field K. A subspace $W \subset V$ is called isotropic if $\eta|_{W\times W} = 0$. A subspace $P \subset V$ is called maximally isotropic if it is maximal with respect to inclusion, i.e. P is such that if it exists an isotropic vector space $W \subset V$ such that $P \subset W$, then $W = P$.

Remark 1. Note that for us K will always be $\mathbb{R}$, or exceptionally $\mathbb{C}$.

It is well known that all maximal isotropic subspaces of V have the same dimension (see Prop. 1.4.3 [Ch1996]). If m is the dimension of V, the common dimension r of all maximal isotropics is such that $r \leq [\frac{m}{2}]$ ($[\cdot]$ gives the closest integer from below)(from Prop. 1.3.2 [Ch1996]). Consider now our generalised tangent bundle. $\forall p \in M$ the tangent space T_pM has dimension d, whilst the vector space E_p has dimension $2d$. It is also clear that the tangent bundle is an isotropic subbundle of E (with respect to the natural metric η). Therefore, since $\mathrm{dim}TM = \frac{\mathrm{dim}E}{2}$, maximal isotropics of η have (the maximal) dimension d. Note that with this reasoning we also found that the simplest example of maximal isotropic of η in E is given by TM. It is worth noting that this is another feature of the particularity of the generalised tangent bundle: not only is there present a natural metric, but also this metric admits maximal isotropic subbundles of the maximal possible dimension; moreover the tangent bundle (as well as the cotangent one) is one of its maximal isotropics. This property of E is also strongly used in the discussion about spinors that appears in the appendix B.
Maximal isotropic subbundles of $E = TM \oplus T^*M$ were first studied by Courant in [Co1990], where they were given the name of almost-Dirac structures. In his paper of 1990 he also introduced the following bracket on sections of E:

Definition 1.6. The Courant bracket is a bilinear map $\Gamma(E)\times\Gamma(E) \to \Gamma(E)$

defined as follows:

$$\begin{aligned}&[\![X,Y]\!] = [v,w] + \mathcal{L}_v\nu - \mathcal{L}_w\mu - \frac{1}{2}\Big(d(i_v\nu - i_w\mu)\Big)\\&\forall X = v+\mu, Y = w+\nu \in \Gamma(E)\end{aligned}$$

This bracket is manifestly antisymmetric and, if we denote by π the natural projection $\pi : TM \oplus T^*M \to TM$, satisfies $\pi([\![X,Y]\!]) = [\![\pi(X),\pi(Y)]\!] = [v,w]$. As it was anticipated, the Courant bracket *does not* satisfy the Jacobi identity. We can encode the failure of the bracket to satisfy this identity in the following trilinear operator, called Jacobiator:

$$Jac(X,Y,Z) = [\![[\![X,Y]\!],Z]\!] + [\![[\![Y,Z]\!],X]\!] + [\![[\![Z,X]\!],Y]\!] \tag{1.2}$$

Gualtieri shows in his thesis [prop 3.27] that, for a maximal isotropic subbundle L of E, involutivity of the Courant bracket is equivalent to the condition $Jac|_L = 0$. In the case where the Courant bracket satisfies the Jacobi identity, it also becomes a Lie bracket and so it induces a Lie algebroid structure on the almost-Dirac structure L (and the related generalised foliation on M). An involutive almost-Dirac structure is called (integrable) Dirac structure.
Apart from being natural, maximal isotropic subbundles of E are relevant in some important situations. (Almost-)Dirac structures were first introduced to give a unifying description of symplectic and Poisson geometries. In this description pre-symplectic and 'almost-Poisson' structures were seen as maximally isotropic subbundles of E. The condition of integrability of these structures was then given in terms of the bracket in definition 1.6, i.e. the two-form defining the pre-symplectic structure is closed and the Poisson bracket satisfies the Jacobi identity if and only if the corresponding maximal isotopic subbundles are involutive with respect to the Courant bracket (see [Co1990] or [Gu2004]).
Later, in the work of Hitchin and Gualtieri ([Hi2003, Gu2004]), a complexified version of the generalised tangent bundle - $(TM \oplus T^*M) \otimes \mathbb{C}$ - was considered. Maximal isotropic subbundles of this vector bundle, i.e. complex (almost-)Dirac structures, were shown to unify, together with symplectic and Poisson geometries, also complex geometry. Even in that case the integrability condition was expressed in terms of closure under the Courant bracket.
The important point to understand for us is that the Courant bracket can be sensibly viewed as the extension of the Lie bracket to E. So, from now on, we will always think of the generalised tangent bundle as the triple $(E, \eta, [\![,]\!])$. This triple satisfies a number of properties that are the motivating examples for the following definition.

Definition 1.7 (Courant Algebroid)**.** (see [Gu2004] and reference therein) A *Courant algebroid* is a vector bundle E equipped with a non-degenerate symmetric bilinear form η, as well as a skew-symmetric bracket $[\![,]\!]$ on $\Gamma(E)$, and with a smooth bundle map $\pi : \Gamma(E) \to \Gamma(TM)$ called the *anchor*. This induces a natural differential operator $\mathbf{d} : \mathcal{C}^\infty(M) \to \Gamma(E)$ via the definition $\eta(\mathbf{d}f, X) = \frac{1}{2}\pi(X)f \ \forall \ f \in \mathcal{C}^\infty, X \in \Gamma(E)$. These structures must be compatible in the following sense:

1. $\pi([X,Y]) = [\pi(X), \pi(Y)] \ \ \forall X, Y \in \Gamma(E)$
2. $3Jac(X,Y,Z) = \mathbf{d}(\eta([\![X,Y]\!], Z) + \text{cyclic permutations}) \ \ \forall X, Y, Z \in \Gamma(E)$
3. $[\![X, fY]\!] = f[\![X,Y]\!] + \pi(X)(f)Y - \eta(X,Y)\mathbf{d}f \ \ \forall X, Y \in \Gamma(E), f \in \mathcal{C}^\infty$
4. $\pi \circ \mathbf{d} = 0, \ i.e. \ \eta(\mathbf{d}f, \mathbf{d}g) = 0 \ \ \forall f, g \in \mathcal{C}^\infty$
5. $\pi(X)(\eta(Y,Z)) = \eta([\![X,Y]\!] + \mathbf{d}\eta(X,Y), Z) + \eta(Y, [\![X,Z]\!] + \mathbf{d}\eta(X,Z))$
   $\forall X, Y, Z \in \Gamma(E)$

Remark 2. Note that in our case $\mathbf{d} = d$, i.e. it is the differential; in fact $\eta(df, X) = \frac{1}{2}\Big(df(\pi(X)) + 0\Big) = \frac{1}{2}\pi(X)(f)$. For a proof for the identities in 2, 3 and 5 see Prop.s 3.16, 3.17, 3.18 in [Gu2004]. So we can see that our triple $(E, \eta, [\![,]\!])$ is actually a Courant algebroid.

We saw that the symmetries (in the connected component to the identity) of η are generated by diffeomorphisms, B-transforms and β-transforms. Are these symmetries also symmetries of the whole Courant algebroid? I.e. are these symmetries also symmetries of the Courant bracket? We will give an answer to this question in two steps.

Proposition 1.1.1. *Let B be a two-form on M. The B-transform is a symmetry of the Courant bracket if and only if B is closed, i.e.*

$$[\![exp(-B)(X), exp(-B)(Y)]\!] = exp(-B)([\![X,Y]\!]) \ \ \forall X, Y \in \Gamma(E) \quad \Leftrightarrow \quad dB = 0 \tag{1.3}$$

Proof. Let us write $X, Y \in \Gamma(E)$ as $X = v + \mu$, $Y = w + \nu$. Recalling the definition of Courant bracket 1.6:

$$[\![X,Y]\!] = [v,w] + \mathcal{L}_v\nu - \mathcal{L}_w\mu - \frac{1}{2}\Big(d(i_v\nu - i_w\mu)\Big)$$

We have:

$$
\begin{aligned}
[\![exp(-B)(X), exp(-B)(Y)]\!] &= [\![X + i_v B, Y + i_w B]\!] \\
&= [\![X, Y]\!] + [\![i_v B, Y]\!] + [\![X, i_w B]\!] + \cancel{[\![i_v B, i_w B]\!]}_{=0} \\
&= [\![X, Y]\!] - \mathcal{L}_w(i_v B) - \frac{1}{2}d(-i_w i_v B) + \mathcal{L}_v(i_w B) - \frac{1}{2}d(i_v i_w B) \\
&= [\![X, Y]\!] + \Big(\mathcal{L}_v i_w - \mathcal{L}_w i_v\Big)B - d(i_v i_w B) \\
&= [\![X, Y]\!] + \Big(\cancel{\mathcal{L}_v i_w} - \mathcal{L}_w i_v\Big)B - \cancel{\mathcal{L}_v i_w B} + i_v d i_w B \\
&= [\![X, Y]\!] - \mathcal{L}_w i_v B + i_v \mathcal{L}_w B - i_v i_w (dB) \\
&= [\![X, Y]\!] + i_{[v,w]}B - i_v i_w (dB)
\end{aligned}
$$

Requiring therefore $[\![exp(-B)(X), exp(-B)(Y)]\!] = exp(-B)\Big([\![X, Y]\!]\Big)$ to hold $\forall X, Y \in \Gamma(E)$ is equivalent to require $i_v i_w (dB) = 0 \quad \forall v, w \in \Gamma(TM)$. The result then follows by the total antisymmetry of dB. $\square$

Remark 3. Note that we used implicitly some relations from ordinary differential geometry. These are the following (for a proof and discussion see e.g. [AMR1988] sec. 6.4)

$$\mathcal{L}_v \omega = (d i_v + i_v d)\omega \qquad \text{(Cartan's 'magic' formula)} \tag{1.4}$$

$$i_{[v,w]}\omega = (\mathcal{L}_v i_w - i_w \mathcal{L}_v)\omega \tag{1.5}$$

where ω is an arbitrary differential form on M and $v, w \in \Gamma(TM)$.

We saw that B-transforms with B closed and diffeomorphisms are common symmetries of η and $[\![,]\!]$. They are actually the *only* transformations having this property:

Theorem 1.1.2 ([Gu2004] 3.24)**.** *The group of orthogonal Courant automorphisms of E is the semidirect product of the group of diffeomorphisms of M, $Diff(M)$, with the abelian group under addition of closed two-forms, $\Omega^2_{closed}(M)$, $G = Diff(M) \ltimes \Omega^2_{closed}(M)$.*

Proof. Note that by definition the Courant bracket is invariant under the diffeomorphism group embedded in $SO(d,d)$. So if $f \in Diff(M)$, then $f_c := \begin{pmatrix} f_* & 0 \\ 0 & (f^*)^{-1} \end{pmatrix}$ is an orthogonal Courant automorphism. Suppose $F[\![X, Y]\!] = [\![F(X), F(Y)]\!]$ $\forall X, Y \in \Gamma(E)$. Then also $O := f_c^{-1} \circ F$ has this property. If $h \in \mathcal{C}^\infty(M)$ then from $O[\![hX, Y]\!] = [\![O(hX), O(Y)]\!]$ and from property 3 of the Courant algebroid we obtain the two members of the following equation respectively:

$$
\begin{aligned}
&hO([\![X, Y]\!]) - \pi(Y)(h)O(X) + \eta(X, Y)O(dh) = \\
&\qquad = hO([\![X, Y]\!]) - \pi(O(Y))(h)O(X) + \eta(O(X), O(Y))dh
\end{aligned}
$$

which by orthogonality is equivalent to

$$\pi(Y)(h)O(X) - \eta(X,Y)O(dh) = \pi(O(Y))(h)O(X) - \eta(X,Y)dh \qquad (1.6)$$

Restricted to $\Gamma(TM)$ (i.e. setting $X = v \in \Gamma(TM), Y = w \in \Gamma(TM)$) it becomes: $w(h)O(v) = \pi(O(w))(h)O(v)$ $\forall v, w \in TM$ and $\forall$ $h \in \mathcal{C}^\infty$. This implies $\pi(O(w)) = w$ $\forall w \in \Gamma(TM)$ and hence it allows us to write for the matrix form of O: $O = \left(\begin{smallmatrix} 1 & * \\ * & * \end{smallmatrix}\right)$. Similarly, if we restrict 1.6 to $\Gamma(T^*M)$ by setting $X = \lambda \in \Gamma(T^*M)$ and $Y = \mu \in \Gamma(T^*M)$ we find: $\pi(O(\mu)) = 0$ $\forall \mu \in \Gamma(T^*M)$ and the matrix form of O becomes: $O = \left(\begin{smallmatrix} 1 & 0 \\ * & * \end{smallmatrix}\right)$. With this result equation 1.6 becomes:

$$\eta(X,Y)O(dh) = \eta(X,Y)dh$$

which in turn implies $O = \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$, since h is arbitrary. But then by orthogonality O_{21} must be a two-form B. By proposition 1.1.1 for O to be a Courant automorphism B must in addition be closed. We have therefore proven that if F is an orthogonal Courant automorphism, then F takes the general form of $F = f_c \circ exp(B)$, with B closed. $\square$

In sight of theorem 1.1.2 we are now able to say that the requirement of a concept of involutivity in our formulation of the generalised tangent bundle has led us to further restrict its structure group from $SO(d,d)$ to $G = Diff(M) \ltimes \Omega^2_{closed}(M)$.
Let us go back for a while to the proposition 1.1.1. The calculation given in the proof showed that:

$$\begin{aligned} [\![exp(-B)(X), exp(-B)(Y)]\!] &= [\![X,Y]\!] + i_{[v,w]}B - i_v i_w(dB) \\ &= exp(-B)\Big([\![X,Y]\!] - i_v i_w(dB)\Big) \\ &= exp(-B)\Big([\![X,Y]\!]\Big) - i_v i_w(dB) \end{aligned}$$

So if we define a new bracket:

$$[\![X,Y]\!]_H := [\![X,Y]\!] - i_v i_w H \qquad \forall X = v+\mu, Y = y+\nu \in \Gamma(E) \quad (1.7)$$

with H a three-form on M, we see that

$$\begin{aligned} [\![exp(-B)(X), exp(-B)(Y)]\!]_H &= [\![exp(-B)(X), exp(-B)(Y)]\!] - i_v i_w H \\ &= exp(-B)\Big([\![X,Y]\!]_H\Big) - i_v i_w(dB) \end{aligned}$$

since $exp(-B)$ does not modify the vector component of a generalised vector. We conclude then that the 'twisted' bracket in 1.7 has the same symmetries

of the Courant bracket. One can also show that this new bracket, together with η, defines again a Courant algebroid for E if and only if $dH = 0$ (see [Gu2004] sec. 3.7 and references therein). Clearly involutivity for the untwisted Courant bracket does not imply involutivity for the twisted one. One easy way to see this is that the tangent bundle TM is Courant integrable if and only if $H = 0$.
What is relevant to us is that one can naturally introduce a closed three-form in the context of generalised geometry, which is also invariant under shifts by exact two-forms (because of the symmetry under B-transforms). This is a general feature of generalised geometry that will also appear later under different circumstances.

1.1.3 Generalised Lie Derivative

Since the Courant bracket does not in general satisfy the Leibniz rule, it can not be viewed as, or extended to, a derivation over the generalised tensor algebra. We recall that in the case of the standard tangent bundle we had the Lie bracket and the Lie derivative, which coincided on the space of the vector fields. Now, consider the group of diffeomorphisms of M: $Diff(M)$. As a group it is generated (modulo topological technicalities) by all its one-dimensional subgroups. It is well known that we can (from the $\exists!$ theorem of ordinary differential equations) think of each one-parameter subgroup of $Diff(M)$ as the flow of a vector field $v \in TM$, where, if $f_\lambda \in Diff(M)$, v is defined as $\frac{df_\lambda}{d\lambda} = v|_{f_\lambda}$. From this definition it is also clear that the Lie algebra of $Diff(M)$ is given by the Lie algebra of vector fields. The group of diffeomorphisms acts on the algebra of tensor fields via pull-back:

$$\begin{aligned} \theta : Diff(M) \times \mathcal{T}^r_s(M) &\longrightarrow \mathcal{T}^r_s(M) \\ (f_\lambda, t) &\longmapsto f^*_\lambda t \end{aligned}$$

Generators of this action are found making use of the following classical result, which we include here for completeness.

Theorem 1.1.3 (Lie Derivative Theorem)**.** *([AMR1988] 5.4.1)*
Consider a vector field $v \in \Gamma(TM)$ and a tensor field $t \in \mathcal{T}^r_s(M)$ both of class $\mathcal{C}^k$, $k \geq 1$, and let f_λ be the flow of v. Then, on the domain of the flow, we have:

$$\frac{d}{d\lambda} f^*_\lambda t = f^*_\lambda \mathcal{L}_v t \tag{1.8}$$

Evaluating equation 1.8 at $\lambda = 0$ we find:

$$\frac{d}{d\lambda} f^*_\lambda t\Big|_{\lambda=0} = \mathcal{L}_v t \tag{1.9}$$

which manifestly says that the generators for this action are given by the Lie derivatives of the corresponding tensors. We can therefore say that *the Lie derivatives generate the action of diffeomorphisms on M*. We have therefore seen that the Lie bracket induces the notion of integrability on M, whilst the Lie derivative generates the symmetries of the *Lie algebra* of vector fields on the manifold $(\Gamma(TM), [,])$.
What is the object that corresponds to the Lie derivative in our generalised tangent bundle? We want to find an object that generates the symmetries of the *Courant algebroid* $\Big(E, \eta, [\![,]\!]\Big)$. This in turn implies that this object must be a derivation, i.e. it must be bilinear and satisfy the Leibniz rule. We can then already see that it cannot be the Courant bracket, since it does not satisfy the Leibniz rule. As we already noticed that it is difficult to find a bracket that satisfies the Jacoby identity, it is likely that if such an object exists it will not be antisymmetric. Instead of giving a 'derivation' for such a generalised Lie derivative (we are not aware of the existence of such an approach), we will instead postulate a definition for this operator and then show that it satisfies the aforementioned properties. We will then try to give an argument that shows that this generalised Lie derivative is in a sense a 'natural' extension of the concept of Lie derivative to sections of the generalised tensor bundle of M.
The following bracket (Dorfman bracket) appears in the work of Dorfman [Do1987].

Definition 1.8 (Dorfman Derivative)**.**
The *generalised Lie derivative*, called also *Dorfman derivative*, is the following bilinear map on sections of the generalised tangent bundle E:

$$L: \qquad \Gamma(E) \times \Gamma(E) \longrightarrow \Gamma(E) \tag{1.10}$$

$$(X_{=v+\mu}, Y_{=w+\nu}) \longmapsto L_X Y := [v, w] + \mathcal{L}_v \nu - i_w(d\mu) \tag{1.11}$$

We will now describe the main properties of the Dorfman derivative in sight of the previous discussion.

Proposition 1.1.4.
The Dorfman derivative satisfies the following properties:

1. *(Symmetrization)* $L_X Y + L_Y X = 2d\Big(\eta(X, Y)\Big) \qquad \forall\ X, Y \in \Gamma(E)$
2. *(Leibniz rule)* $L_X\Big(L_Y Z\Big) = L_{(L_X Y)} Z + L_Y(L_X Z) \qquad \forall\ X, Y, Z \in \Gamma(E)$
3. *($B - transform$) The B-transform is a symmetry of the generalised Lie derivative if and only if the two-form B is closed.*

4. *(Prop. 3.18 [Gu2004]) Differentiation of the natural inner product can be expressed as follows:*

$$\pi(X)\eta(Y,Z) = \eta(L_XY,Z) + \eta(Y,L_XZ) \qquad \forall\, X,Y,Z \in \Gamma(E)$$

5. $L_X(fY) = f(L_XY) + \pi(X)(f)Y \qquad\qquad \forall\, X,Y \in \Gamma(E),\ \forall\, f \in \mathcal{C}^\infty$

Proof. 1. Let us write as usual $X = v+\mu, Y = w+\nu \in \Gamma(E)$. Then

$$\begin{aligned} L_XY + L_YX &= \cancel{([v,w]+[w,v])}|_{=0} + (\mathcal{L}_v\nu - i_w(d\mu) + \mathcal{L}_w\mu - i_v(d\nu)) \\ &\Big|\text{Cartan's formula} \\ &= \underline{\mathcal{L}_v\nu} - \cancel{\mathcal{L}_w\mu} + d(i_w\mu) + \cancel{\mathcal{L}_w\mu} - \underline{\mathcal{L}_v\nu} + d(i_v\nu) \\ &= d(i_w\mu + i_v\nu) = 2d\Big(\eta(X,Y)\Big) \end{aligned}$$

2. Defining X and Y as above and $Z = z+\lambda \in \Gamma(E)$ we can write for the different terms:

$$L_X(L_YZ) = [v,[w,z]] + (\mathcal{L}_v(\mathcal{L}_w\lambda - i_zd\nu) - i_{[w,z]}d\mu) \tag{1.12}$$

$$\begin{aligned} L_{L_XY}Z &= [[v,w],z] + \Big\{\mathcal{L}_{[v,w]}\lambda - i_z\Big(d(\mathcal{L}_v\nu - i_wd\mu)\Big)\Big\} \\ L_Y(L_XZ) &= [w,[v,z]] + \mathcal{L}_w\Big(\mathcal{L}_v\lambda - i_zd\mu\Big) - i_{[v,z]}d\nu \end{aligned}$$

Adding the last two equations together we get:

$$\begin{aligned} &L_{L_XY}Z + L_Y(L_XZ) = ([[v,w],z] + [w,[v,z]]) + \\ &\quad + ((\mathcal{L}_v\mathcal{L}_w - \cancel{\mathcal{L}_w\mathcal{L}_v})\lambda - i_z\{\mathcal{L}_vd\nu - d(i_wd\mu)\} + \cancel{\mathcal{L}_w\mathcal{L}_v\lambda} - \mathcal{L}_wi_zd\mu - i_{[v,z]}d\nu) \end{aligned}$$

where we used the fact that the Lie derivative is a homomorphism of Lie algebras and the fact that the differential commutes with the Lie derivative. Now we use the Jacobi identity for the Lie derivative in the vector term, the Cartan formula, the fact that $d^2 = 0$ and the relation $i_{[v,z]} = \mathcal{L}_vi_z - i_z\mathcal{L}_v$ (see remark 3), to write:

$$\begin{aligned} L_{L_XY}Z + L_Y(L_XZ) &= -[v,[z,w]] + \mathcal{L}_v\mathcal{L}_w\lambda - \cancel{i_z\mathcal{L}_vd\nu} + i_z\mathcal{L}_wd\mu + \\ &\quad - \mathcal{L}_wi_zd\mu - (\mathcal{L}_vi_z - \cancel{i_z\mathcal{L}_v})d\nu \\ &= [v,[w,z]] + \mathcal{L}_v(\mathcal{L}_w\lambda - i_zd\nu) - i_{[w,z]}d\mu \end{aligned}$$

which is exactly the same expression as the one in equation 1.12.

3. We have :

$$\begin{aligned} exp(-B)\cdot(L_XY) &= [v,w]+\mathcal{L}_v\nu - i_w(d\mu)+i_{[v,w]}B \\ &= [v,w]+\mathcal{L}_v\nu-\mathcal{L}_w\mu+d(i_w\nu)+\mathcal{L}_v(i_wB)-i_w(\mathcal{L}_vB) \end{aligned}$$

   Moreover we can also write:

$$\begin{aligned} L_{exp(-B)\cdot X}(exp(-B)\cdot Y) &= [v,w]+\mathcal{L}_v(\nu+i_wB)-i_wd(\mu+i_vB) \\ &= [v,w]+\mathcal{L}_v\nu+\mathcal{L}_v(i_wB)-i_w(d\mu)-i_w(d(i_vB)) \\ &= [v,w]+\mathcal{L}_v\nu+\mathcal{L}_v(i_wB)-\mathcal{L}_w\mu+d(i_w\mu)-i_w\mathcal{L}_vB+i_wi_v(dB) \end{aligned}$$

   We have therefore shown that, $\forall$ two form B, $exp(-B)\cdot(L_XY) = L_{exp(-B)\cdot X}(exp(-B)\cdot Y) - i_vi_w(dB)$ $\forall\, X,Y\in\Gamma(E)$. The result follows again by total antisymmetry of dB.

4. Writing $X=v+\mu, Y=w+\nu, Z=z+\lambda$ we can write:

$$\begin{aligned} &\eta(L_XY,Z)+\eta(Y,L_XZ) = \\ &\qquad = \frac{1}{2}(i_{[v,w]}\lambda + i_z(\mathcal{L}_v\nu - \cancel{i_w(d\mu)}) + i_{[v,z]}\nu + i_w(\mathcal{L}_v\lambda - \cancel{i_z(d\mu)})) \\ &\qquad = \frac{1}{2}([\mathcal{L}_v,i_w]\lambda + i_z(\mathcal{L}_v\nu)+[\mathcal{L}_v,i_z]\nu+i_w(\mathcal{L}_v\lambda)) \\ &\qquad = \frac{1}{2}(\mathcal{L}_v(i_w\lambda+i_z\nu)) \end{aligned}$$

5. It follows very easily by direct computation.

□

Proposition 1.1.4 confirms what we stated before: the Dorfman derivative satisfies the Leibniz property (property 2) but fails to be antisymmetric (property 1). Nevertheless we can see that this failure happens only up to an exact term; more specifically, the symmetrisation of the generalised Lie derivative equals the differential of the natural metric evaluated on the relevant generalised vector fields.
There is a strong relationship between the Courant bracket and the Dorfman derivative. We will collect its two most important features in the following:

Proposition 1.1.5.

1. *The antisymmetrisation of the Dorfman bracket is the Courant bracket, i.e.*

$$[\![X,Y]\!] = \frac{1}{2}\Big(L_XY - L_YX\Big)$$

2. *The Courant bracket and the Dorfman derivative satisfy the following identity:*

$$L_X [\![Y, Z]\!] = [\![L_X Y, Z]\!] + [\![Y, L_X Z]\!] \qquad \forall\, X, Y, Z \in \Gamma(E) \tag{1.13}$$

Proof. 1. The statement follows immediately from the definitions. Recall that for $X = v + \mu, Y = w + \nu \in \Gamma(E)$ we defined:

$$\begin{aligned} L_X Y &:= [v, w] + \mathcal{L}_v \nu - i_w d\mu = [v, w] + \mathcal{L}_v \nu - \mathcal{L}_w \mu + d(i_w \mu) \\ [\![X, Y]\!] &:= [v, w] + \mathcal{L}_v \nu - \mathcal{L}_w \mu - \frac{1}{2} d\big(i_v \nu - i_w \mu\big) \end{aligned}$$

2. Setting $X = v + \mu, Y = w + \nu, Z = z + \lambda \in \Gamma(E)$ we have:

$$L_X [\![Y, Z]\!] = [v, [w, z]] + \mathcal{L}_v (\mathcal{L}_w \lambda - \mathcal{L}_z \nu - \frac{1}{2} d(i_w \lambda - i_z \nu)) - i_{[w,z]}(d\mu)$$

Instead for the right hand side we have:

$$\begin{aligned} &[\![L_X Y, Z]\!] + [\![Y, L_X Z]\!] = \\ &\quad = [[v, w], z] + \Big(\underline{\mathcal{L}_{[v,w]}\lambda} - \underline{\mathcal{L}_z(\mathcal{L}_v \nu} - i_w d\mu)\Big) - \frac{1}{2} d\Big(i_{[v,w]}\lambda - i_z(\mathcal{L}_v \nu - i_w d\mu)\Big) + \\ &\qquad + [w, [v, z]] + \Big(\underline{\mathcal{L}_w(\mathcal{L}_v \lambda} - i_z d\mu) - \underline{\mathcal{L}_{[v,z]}\nu}\Big) - \frac{1}{2} d\Big(i_w(\mathcal{L}_v \lambda - i_z d\mu) - i_{[v,z]}\nu\Big) \\ &\quad = [v, [w, z]] + \Big(\underline{\mathcal{L}_v(\mathcal{L}_w \lambda - \mathcal{L}_z \nu)}\Big) + \mathcal{L}_z i_w d\mu - \mathcal{L}_w i_z d\mu + \\ &\qquad - \frac{1}{2} d\Big(\underbrace{i_{[v,w]}\lambda} - \overbrace{i_z \mathcal{L}_v \nu} + i_z i_w d\mu + \underbrace{i_w \mathcal{L}_v \lambda} - i_w i_z d\mu - \overbrace{i_{[v,z]}\nu}\Big) \\ &\quad = [v, [w, z]] + \Big(\mathcal{L}_v(\mathcal{L}_w \lambda - \mathcal{L}_z \nu)\Big) + \mathcal{L}_z i_w d\mu - \mathcal{L}_w i_z d\mu - \frac{1}{2} d\Big(\mathcal{L}_v(\underbrace{i_w \lambda} - \overbrace{i_z \nu}) + 2 i_z i_w d\mu\Big) \end{aligned}$$

The result then follows because $\mathcal{L}_z i_w d\mu - \mathcal{L}_w i_z d\mu - d\Big(i_z i_w d\mu\Big) = -i_{[w,z]} d\mu$; this can be easily shown using the Cartan formula.

□

Property 1 of proposition 1.1.4, together with property 1 of proposition 1.1.5, implies that the the Courant bracket and the Dorfman derivative differ from each other only up to an exact term, i.e.

$$[\![X, Y]\!] = L_X Y + d\eta(X, Y) \qquad \forall\, X, Y \in \Gamma(E) \tag{1.14}$$

Most importantly equation 1.13, together with property 4 of proposition 1.1.4, allows us to state that the generalised Lie derivative can be used to *generate the symmetries* of the Courant algebroid (inspired by [Ba2012]).[1]

[1] Recall that we stated before that the ordinary Lie derivative generates the symmetries of the *Lie algebra* $\big(TM, [\,]\big)$. Now we see that the generalised Lie derivative generates the symmetries of the *Courant algebroid* $\big(E, \eta, [\![\,]\!]\big)$.

This fact will be used later in the thesis, especially when will be dealing with supergravity. There is still one point that needs to be clarified. In fact the Dorfman derivative is a map $\Gamma(E) \times \Gamma(E) \to \Gamma(E)$, whilst the generators of the action of the group $G = Diff(M) \ltimes \Omega^2_{closed}(M)$ should correspond to a map: $\mathfrak{g} \times \Gamma(E) \to \Gamma(E)$, where $\mathfrak{g}$ is the Lie algebra of G. Here $\mathfrak{g} = \Gamma(TM) \oplus \Omega^2_{closed}(M)$, which appears to be different from $\Gamma(E)$. They can nevertheless be identified since a closed two-form is always locally equal to the differential of a one-form. In particular when we say that $X = v + \mu \in \Gamma(E)$ generates an element of $Diff(M) \ltimes \Omega^2_{closed}(M)$ we are really thinking of the element $\tilde{X} \in \mathfrak{g}$ associated with X through the identification $(v+\mu) \mapsto (v-d\mu)$ and at the Lie algebra action: $(v - d\mu) \cdot (w + \nu) = \mathcal{L}_v(w + \nu) - i_w d\mu$ [Hi2010], where the sum of the generators of a $GL(d)$ transformation and of a $(-B)$-transform are now evident.[2]
To complete the previous discussion we are only left with the question of the naturalness of the generalised Lie derivative. It is fair to make it clear at this point that we will only use the Dorfman derivative in the rest of the thesis. The reason relies on the fact that it will work as the generator of the symmetries of the NS-NS sector of the Type IIA and IIB supergravity theories. This is the physical naturalness. From a mathematical point of view the reason of the naturalness originates from the theory of the representation of Lie algebras. In order to explain this we first need to express the Dorfman derivative in an $O(d,d)$ covariant manner. First embed the action of the partial derivative operator into the generalised tangent bundle (see [CoStWa2011]) using the immersion $T^*M \to E$.[3] Recalling that $E^* \approx T^*M \times TM$ and that the partial derivative operator is a covariant tensor one can write:

$$\partial_M = \begin{cases} \partial_\mu & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d \end{cases}$$

Note that we will denote the generalised vector indices with capital Latin letters from the middle of the alphabet $(M, N, \ldots)$ and the usual (co)vector indices with lower-case Greek letters from the middle of the alphabet $(\mu, \nu, \rho, \ldots)$.

Proposition 1.1.6. *The generalised Lie derivative can be expressed in the following $O(d,d)$-covariant manner:*

$$\Big(L_X Y\Big)^M = X^N \partial_N Y^M + \Big(\partial^M X^N - \partial^N X^M\Big) Y_N \tag{1.15}$$

where the indices are risen and lowered with the natural inner product η.

[2] The Lie algebra bracket of $\mathfrak{g}$ is $[v - d\mu, w - d\nu]_{\mathfrak{g}} = [v, w]_{\Gamma(TM)} - \mathcal{L}_v d\nu + \mathcal{L}_w d\mu$

[3] Note that this map will still be present when we will consider the generalised tangent bundle as an extension of the tangent bundle by the cotangent one and it is in fact the pullback of the 'anchor map'.

Proof. The left hand side of equation 1.15 is already known to be, for $X = v + \mu, Y = w + \nu$:

$$\Big(L_X Y\Big)^M = \Big(\begin{smallmatrix} [v,w] \\ \mathcal{L}_v\nu - i_w(d\mu) \end{smallmatrix}\Big)^M$$

Recall that $\eta_{MN} = \frac{1}{2}\Big(\begin{smallmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\Big)$ so that $\eta^{MN} = 2\Big(\begin{smallmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\Big)$. This implies that if $\partial_N = \Big(\begin{smallmatrix} \partial_\mu \\ 0 \end{smallmatrix}\Big) \in \Gamma(E^*)$ then $\partial^R = \eta^{RN}\partial_N = 2\Big(\begin{smallmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\Big)\Big(\begin{smallmatrix} \partial_\mu \\ 0 \end{smallmatrix}\Big) = \Big(\begin{smallmatrix} 0 \\ 2\partial_\mu \end{smallmatrix}\Big) \in \Gamma(E)$. Then the first term of the right hand side of 1.15 reads:

$$X^N\partial_N Y^M = v^\mu\partial_\mu Y^M = \begin{pmatrix} v^\mu\partial_\mu w^\nu \\ v^\mu\partial_\mu\nu_\nu \end{pmatrix}$$

whilst the second part is:

$$\begin{aligned}
\Big(\partial^M X^N - \partial^N X^M\Big)Y_N &= \Big((\partial^M V^N)W_N - (\partial^N V^M)W_N\Big) \\
&= \begin{pmatrix} 0 - (\partial_N v^\mu)W^N \\ \Big(\begin{smallmatrix} 2\partial_\mu v^\nu & 2\partial_\mu\mu_\nu \end{smallmatrix}\Big)\frac{1}{2}\Big(\begin{smallmatrix} 0 & \delta^\rho_\nu \\ \delta^\nu_\rho & 0 \end{smallmatrix}\Big)\Big(\begin{smallmatrix} w^\rho \\ \nu_\rho \end{smallmatrix}\Big) - (\partial_N\mu_\mu)W^N \end{pmatrix} \\
&= \begin{pmatrix} -(\partial_\rho v^\mu)w^\rho \\ \Big(\begin{smallmatrix} 2\partial_\mu v^\rho & 2\partial_\mu\mu_\rho \end{smallmatrix}\Big)\frac{1}{2}\Big(\begin{smallmatrix} \nu_\rho \\ w^\rho \end{smallmatrix}\Big) - (\partial_\rho\mu_\mu w^\rho) \end{pmatrix} \\
&= \begin{pmatrix} -(\partial_\rho v^\mu)w^\rho \\ (\partial_\mu v^\rho)\nu_\rho + (\partial_\mu\mu_\rho)w^\rho - (\partial_\rho\mu_\mu)w^\rho \end{pmatrix} = \begin{pmatrix} -(\partial_\rho v^\mu)w^\rho \\ (\partial_\mu v^\rho)\nu_\rho - (i_w d\mu)_\mu \end{pmatrix}
\end{aligned}$$

Recalling the standard formula for the Lie derivative of a 1-form $(\mathcal{L}_v\mu)_\nu = v^\rho(\partial_\rho\mu_\nu) + (\partial_\nu v^\rho)\mu_\rho$, we can finally see that:

$$X^N\partial_N Y^M + \Big(\partial^M X^N - \partial^N X^M\Big)Y_N = \begin{pmatrix} [v,w]^\nu \\ (\mathcal{L}_v\nu)_\nu - (i_w d\mu)_\nu \end{pmatrix}$$

□

If we define the action of the Dorfman derivative on a function as the action of the Lie derivative of the vector component, i.e. $L_X f := \mathcal{L}_v f$, $\forall\, X = v + \mu \in \Gamma(E), f \in \mathcal{C}^\infty$, we can easily extend the action of the generalised Lie derivative to the whole tensor algebra of E (Willmore theorem). Let us now consider the expression of the generalised Lie derivative given in equation 1.15 and compare it with the standard Lie derivative. In the standard Lie derivative we have: $(\mathcal{L}_v w)^\mu = v^\rho\partial_\rho w^\mu - (\partial_\rho v^\mu)w^\rho$. In the second term of the right hand side of the former equation we can see the action of the $\mathfrak{gl}(d)$ matrix $a^\mu_\rho := -\partial_\rho v^\mu$. In the expression given in 1.15 we can instead see the action of $\mathfrak{so}(d,d)$; one way to see it is to notice the antisymmetric combination of indices that appears in the second part of the equation. A more direct approach is to look back at the proof of the proposition 1.1.6. In that proof we showed that the second part of the covariant expression of the Dorfman derivative can be expressed as:

$$\begin{aligned}\left(\partial^M X^N - \partial^N X^M\right) Y_N &= \begin{pmatrix} -(\partial_\rho v^\mu) w^\rho \\ (\partial_\mu v^\rho)\nu_\rho + (\partial_\mu \mu_\rho) w^\rho - (\partial_\rho \mu_\mu) w^\rho \end{pmatrix} \\ &= \begin{pmatrix} -(\partial_\rho v^\mu) & 0 \\ (\partial_\mu \mu_\rho - \partial_\rho \mu_\mu) & (\partial_\mu v^\rho) \end{pmatrix} \begin{pmatrix} w^\rho \\ \nu_\rho \end{pmatrix}\end{aligned}$$

Recalling the general form of an element of $SO(d,d)$ given in 1.1, we see that the previous equation yields the one in 1.1 if one sets: $A^\mu_\rho = -\partial_\rho v^\mu$, $B_{\mu\rho} = \partial_\mu \mu_\rho - \partial_\rho \mu_\mu$ and $\beta^{\mu\rho} = 0$. In particular, since $\beta = 0$ we see that the Lie algebra action connected with the generalised Lie derivative is actually the one of the symmetry group of the Courant algebroid, i.e. $G = Diff(M) \ltimes \Omega^2_{closed}(M)$.[4] To conclude this subsection we note that we were able to introduce two geometric objects acting on the generalised vector bundle. The first one is the Courant bracket, used in the literature to express integrability of distributions in E; we saw that this bracket corresponds to an extension of the Lie bracket to E. The second one is the Dorfman derivative, which is used to generate the symmetries of the Courant algebroid; this derivative can be viewed as an extension of the Lie derivative to sections of E (and relative tensors). While in the case of the standard tangent bundle the Lie derivative and the Lie bracket coincided when evaluated on vector fields, in the case of the generalised tangent bundle these objects are manifestly distinct.

1.2 Generalised Metric Structure

In the previous section we have only discussed structures that arise canonically from the definition of generalised vector bundle. We now want to make a step forward and add more structure in a manner that is akin to the introduction of a metric structure in an ordinary vector bundle. Let us introduce a generalised matrix $P^M_N \in \Gamma(E) \times \Gamma(E^*)$ such that:

$$P^M_N P^N_Q = \delta^M_Q \tag{1.16}$$

$$P^M_R \eta_{MN} P^N_S = \eta_{RS} \tag{1.17}$$

where clearly 1.17 implies that the matrix $P \in O(d,d)$. This new structure is called a 'generalised product structure' because it is the generalised geometric analogue of an ordinary product structure (see e.g.[Ko2011]). From the

[4] Note that this has to be the case, since we said that the generalised Lie derivatives generate the symmetries of the Courant algebroid.

defining properties 1.16 and 1.17 of P one can then construct the following projectors:

$$P^M_{\pm N} = \frac{1}{2}\left(\delta^M_N \pm P^M_N\right)$$

(It is clear that $P^2_\pm = P_\pm$ and that $P_+P_- = P_-P_+ = 0$). These projectors project E onto two subspaces: $C_\pm = \{P^M_{\pm N}V^N \quad \forall V \in \Gamma(E)\}$. We now want to show what the generic form for the P matrix is.

Proposition 1.2.1. *The generic[5] form for the P matrix defined by the conditions 1.16 and 1.17 is the following:*

$$P = \begin{pmatrix} g^{-1}B & g^{-1} \\ g - Bg^{-1}B & -Bg^{-1} \end{pmatrix} \tag{1.18}$$

where g is a non-degenerate symmetric covariant two-tensor and B is a two-form.

Proof. To prove the statement we will work in matrix notation. Let us make use of the splitting $E \approx TM \oplus T^*M$ writing P in block form: $P = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. Then the constraints 1.16 and 1.17 become respectively:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^2 + BC & AB + BD \\ CA + DC & CB + D^2 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

$$\begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^TC + C^TA & A^TD + C^TB \\ B^TC + D^TA & B^TD + D^TB \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

From these we can read off eight equations:

a) $A^2 + BC = \mathbb{1}$	$i)A^TC + C^TA = 0$
b) $AB + BD = 0$	ii) $B^TD + D^TB = 0$
c) $CA + DC = 0$	iii) $A^TD + C^TB = \mathbb{1}$
d) $CB + D^2 = \mathbb{1}$	iv) $B^TC + D^TA = \mathbb{1}$

It is clear that *iii*) and *iv*) are equivalent to each other. Now consider *iii*)

[5] We use the word ‘generic’ and not ‘general’ because of one subtlety that will appear in the proof.

and multiply it by D from the right:

$$\begin{aligned} A^T D^2 + C^T BD =_{\text{using } d)} A^T(\mathbb{1} - CB) + C^T BD &= D \\ &\big| \text{multiply by } C \\ &\Leftrightarrow A^T C - A^T CBC + C^T BDC = DC \\ &\big| \text{using } i) \\ &\Leftrightarrow C^T(-A + \underline{AB}C + \underline{BD}C) = DC \\ &\big| \text{using } b) \\ &\Leftrightarrow -C^T A = DC \\ &\big| \text{using } i) \\ &\Leftrightarrow A^T C = DC \end{aligned}$$

Now let us *assume* that C is invertible. Note that this is not the most general eventuality, but it is the generic case: a matrix is not invertible only if its determinant is equal to zero, i.e. only if its determinant takes value in a set of zero measure (one point in the entire real axis). With this assumption we are able to state that $A^T = D$. Making use of this fact and of relations $iii)$ and $d)$ we can also write:

$$D^2 + CB = A^T D + CB = \mathbb{1} = A^T D + C^T B$$

This implies that $CB = C^T B$. If we further assume that B is invertible we can also state $C^T = C$, i.e. C is symmetric. Taking the transpose of $d)$, using $A^T = D$, comparing it with $a)$, and recalling that C is symmetric and invertible, we also find $B^T = B$.

The three relations we have proven so far allow us to restrict to the relations $a)$-$c)$ only, because they cause $i)$ to be equivalent to $c)$, $ii)$ to $b)$, $iii)$ to $a)$ and $d)$ to $a)$. From $a)$ we see that A commutes with BC:

$$ABC = A(\mathbb{1} - A^2) = (\mathbb{1} - A^2)A = BCA$$

We can then prove that $b)$ is equivalent to $c)$. First write $AB + BD = AB + B^T A^T = 0$ and $CA + DC = CA + A^T C^T = 0$. Then $b) \Rightarrow c)$, because $BCA = ABC =_{b)} -B^T A^T C = -BA^T C^T$ and B is assumed to be invertible. Vice-versa $B^T(A^T C^T) =_{c)} -B(CA) = -ABC = -ABC^T$ and C is assumed to be invertible.

We have therefore been able to reduce the defining equations of P to the following constraints:

$$\begin{aligned} P &= \begin{pmatrix} A & B \\ C & A^T \end{pmatrix} \\ \text{with } C &= C^T \quad B = B^T \\ \text{and } A^2 + BC &= \mathbb{1} \quad AB = -(AB)^T \end{aligned}$$

Since B is invertible we can define the following matrix $\alpha := B^{-1}A$, i.e. $A = B\alpha$. In this way

$$B\alpha^T B = (B\alpha B)^T = (AB)^T = -AB = -B\alpha B$$

Since B is invertible this implies $\alpha^T = -\alpha$, i.e. α is antisymmetric. From $a)$ we get $B\alpha B\alpha + BC = \mathbb{1}$ and so

$$C = B^{-1} - \alpha B\alpha$$

and it is also clear that $A^T = (B\alpha)^T = \alpha^T B = -\alpha B$.
Calling the various objects with more standard names, i.e. $g^{-1} := B$ and B[6] $:= \alpha$ we see that we can generically write P as follows:

$$P = \begin{pmatrix} g^{-1}B & g^{-1} \\ g - Bg^{-1}B & -Bg^{-1} \end{pmatrix} \tag{1.19}$$

where B is an antisymmetric covariant tensor, i.e. a two-form, and g is a non degenerate symmetric covariant tensor, i.e. a metric. □

It is worthwhile at this point to note that, even though the symmetric covariant two-tensor g introduced by the generalised product structure P defines a metric on TM, there is no constraint that imposes a certain signature on it. We will then assume for now g to be Riemannian, but we will also keep the freedom of changing this signature to the one we prefer (usually Lorentzian) whenever it is needed.
Let us consider again the matrix P we have introduced so far. We can construct a covariant generalised two-tensor by simply lowering the upper index with the natural metric η. Let us define: $G_{MN} := \eta_{MR}P^R_N$. This matrix is symmetric, because, since $P^2 = \mathbb{1}$ and $P^T\eta P = \eta$, we can write:

$$G = \eta P = P^{-T}\eta = P^T\eta = P^T\eta^T = (\eta P)^T = G^T$$

Since both P and η are non-degenerate we can then view G as another metric on E, called the *generalised metric*. This metric has the following general form:

$$G = \eta P = \frac{1}{2}\begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} \tag{1.20}$$

and so the generalised metric is described by an ordinary metric together with a two-form. This implies that this metric describes $\frac{d(d+1)}{2} + \frac{d(d-1)}{2} = d^2$ degrees of freedom in contrast with the fact that an ordinary metric in $2d$ dimensions should have $\frac{2d(2d+1)}{2}$ free parameters. This fact is better explained by the following statement concerning the structure group induced by the generalised metric.

[6] Note that this B has nothing to do with the B used in the calculation for the proof.

Proposition 1.2.2. *The pair of metrics (η, G) is invariant under $O(d) \times O(d)$ transformations, i.e. the structure group of E induced by the couple of metrics (η, G) is $O(d) \times O(d) \subset O(d, d)$.*

Proof. We want to find the group of matrices whose elements satisfy the following relations:

$$A \in GL(2d) \mid \quad A^T \eta A = \eta \tag{1.21}$$
$$A^T G A = G \tag{1.22}$$

where $\eta = \frac{1}{2}\left(\begin{smallmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$ and G is as in 1.20. The expression for G is nevertheless quite cumbersome. The idea of the proof is then to try to find an easier system of equations to solve, instead of the one given by 1.21 and 1.22. Clearly 1.21 is the defining relation for an element of $O(d, d)$. If we were able to simplify the expression of G via an $O(d, d)$ transformation we would then be able to simplify the whole system as well. Noting the presence of a two-form in the definition of the generalised metric we can then try to make a B-transform. It turns out that a transform like this permits to considerably simplify the situation:

$$\frac{1}{2}\begin{pmatrix} \mathbb{1} & -B \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} =: G_0$$

We can adsorb this transformation as follows:

$$\begin{aligned} A^T G A &= A^T (exp(-B)^T G_0 exp(-B)) A \\ &= (exp(-B)A)^T G_0 (exp(-B)A) = (exp(-B)^T G_0 exp(-B)) \end{aligned}$$

This happens if and only if

$$\tilde{A}^T G_0 \tilde{A} := \Big(exp(-B)Aexp(B)\Big)^T G_0 \Big(exp(-B)Aexp(B)\Big) = G_0$$

Moreover, since both $exp(\pm B)$ belong to $O(d, d)$ we also have that $A^T \eta A = \eta$ if and only if $\tilde{A}^T \eta \tilde{A} = \eta$. Let us write $\tilde{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Because of the symmetry of the matrices η and G_0 from the system of equations:

$$\begin{cases} \tilde{A}^T \eta \tilde{A} = \eta \\ \tilde{A}^T G_0 \tilde{A} = G_0 \end{cases}$$

we only get six independent equations:

i) $a^T g a + c^T g^{-1} c = g$	α) $a^T c + c^T a = 0$
ii) $a^T g b + c^T g^{-1} d = 0$	β) $a^T d + c^T b = \mathbb{1}$
iii) $b^T g b + d^T g^{-1} d = g^{-1}$	γ) $b^T d + d^T b = 0$

We will from now on assume that a is invertible.[7] We can then proceed and express c^T in terms of a. Eq $\alpha)$ implies that $c^T = (-a^Tc)a^{-1}$. Then from $ii)$ we see that

$$a^T[gb - ca^{-1}g^{-1}d] = 0 \ \Leftrightarrow_{a\,invertible} gb = ca^{-1}g^{-1}d \ \Leftrightarrow \ b = g^{-1}ca^{-1}g^{-1}d$$

Now we can express d i terms of a:

$$c^Tb = (c^Tg^{-1}c)a^{-1}g^{-1}d =_{i)} (g - a^Tga)a^{-1}g^{-1}d = ga^{-1}g^{-1}d - a^Td$$

and so, using $\beta)$ we can say that $\mathbb{1} = a^Td + c^Tb = ga^{-1}g^{-1}d$, i.e. $\underline{d = gag^{-1}}$. Again via $ii)$, together with the new relation for d, we see that

$$\begin{aligned} 0 = a^Tgb + c^Tg^{-1}(gag^{-1}) &= a^Tgb + (c^Ta)g^{-1} \\ &=_{\alpha)} a^Tgb - a^Tcg^{-1} = a^T(gb - cg^{-1}) \end{aligned}$$

Since a is assumed to be invertible we then have $\underline{b = g^{-1}cg^{-1}}$.
We now show that the initial system of equations is equivalent to the following set of equations (if a is invertible):

$$\begin{cases} b = g^{-1}cg^{-1} \\ d = gag^{-1} \\ i)\, a^Tga + c^Tg^{-1}c = g \\ \alpha)\, a^Tc + c^Ta = 0 \end{cases}$$

We have already proven the first implication in the first part of the proof. For the second implication one can easily see that the new relations for b and d are sufficient to make $i)$ equivalent to $\beta)$ ($gb = cg^{-1}$ and $g^{-1}d = ag^{-1}$), $ii)$ to $\alpha)$ (in a similar manner), $iii)$ to the (transpose of) $\beta)$ and $\gamma)$ to $\alpha)$. To show the last equivalence (which is probably the less obvious one) let us express first a and c in terms of d and b: $a = g^{-1}dg$ and $c = gbg$. Then from $\alpha)$ we have: $0 = a^Tc + c^Ta = g(d^Tb + b^Td)g$.
The relations $b = g^{-1}cg^{-1}$ and $d = gag^{-1}$ say explicitly that the only two independent submatrices can be chosen to be a and c. We now want to show that the other two remaining equations ($i)$ and $\alpha)$) are equivalent to a system of equations for two independent matrices M and N that has the following form:

$$\begin{cases} M^TgM = g \\ N^TgN = g \end{cases}$$

[7]For a justification of this see the discussion in the proof of proposition 1.2.1

It is then clear that the group of matrices preserving both (η, G) is diffeomorphic to $O(d) \times O(d)$[8]

Consider

$$\begin{cases} a^T g a + c^T g^{-1} c = g \\ a^T (g g^{-1}) c + c^T (g^{-1} g) a = 0 \end{cases}$$

This is equivalent to:

$$\begin{aligned} & a^T g(a \pm g^{-1}c) + c^T g^{-1}(c \pm ga) = g \\ \Leftrightarrow \; & a^T g(a \pm g^{-1}c) + c^T(g^{-1}c \pm a) = g \\ \Leftrightarrow \; & a^T g(a \pm g^{-1}c) \pm c^T(a \pm g^{-1}c) = (a^T g \pm c^T)(a \pm g^{-1}c) \\ & = (a^T \pm c^T g^{-1}) g (a \pm g^{-1}c) \\ & = (a \pm g^{-1}c)^T g (a \pm g^{-1}c) = g \end{aligned}$$

The previous statement then follows defining $M := a + g^{-1}c$ and $N := a - g^{-1}c$.

We have proven that there is a diffeomorphism. We now want to show that the diffeomorphism induces also an isomorphism. The relation between the matrices A of the fundamental representation defined by 1.21 and 1.22 and the matrices $\tilde{A}$ is of the type: $A = U\tilde{A}U^{-1}$, with $U = exp(B)$, and so the two representations are isomorphic. We can then focus on $\tilde{A}$. We have shown that we can write it as: $\tilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & g^{-1}cg^{-1} \\ c & gag^{-1} \end{pmatrix}$. Now, the product between two matrices with this form is again a matrix with this form:

$$\begin{aligned} AA' &= \begin{pmatrix} a & g^{-1}cg^{-1} \\ c & gag^{-1} \end{pmatrix} \begin{pmatrix} a' & g^{-1}c'g^{-1} \\ c' & ga'g^{-1} \end{pmatrix} = \begin{pmatrix} aa' + g^{-1}cg^{-1}c' & ag^{-1}c'g^{-1} + g^{-1}ca'g^{-1} \\ ca' + gag^{-1}c' & cg^{-1}c'g^{-1} + gaa'g^{-1} \end{pmatrix} \\ &= \begin{pmatrix} a'' & g^{-1}c''g^{-1} \\ c'' & ga''g^{-1} \end{pmatrix} \end{aligned}$$

Finally, we know that $M := a + g^{-1}c, M' = a' + g^{-1}c', N = a - g^{-1}c$ and $N' = a' - g^{-1}c'$ are orthogonal. We now want to know whether even $M'' := a'' + g^{-1}c''$ and $N'' = a'' - g^{-1}c''$, with a'', c'' coming from the relation $AA' = A''$, are also orthogonal. We have indeed:

$$\begin{aligned} (a'' \pm g^{-1}c'') &= ((aa' + g^{-1}cg^{-1}c') \pm (g^{-1}ca' + ag^{-1}c')) \\ &= ((a \pm g^{-1}c)a' + (g^{-1}c \pm a)g^{-1}c') \\ &= (a \pm g^{-1}c)(a' \pm g^{-1}c') \end{aligned}$$

that directly shows that M'' and N'' are orthogonal. It actually shows much more: if $A'' = AA'$, then $N'' = NN'$ and $M'' = MM'$. We have therefore clearly proven the isomorphism. □

[8]Recall that we have chosen g to be Riemannian, but we could have made another choice of signature as well. In that case the group would have been $G = O(d-k, k) \times O(k, d-k)$ with $k \in \mathbb{Z}$, $0 \leq k \leq d$. The difference in signature between the first and the second orthogonal groups is such that $G \subset O(d, d)$.

Let us go back now to the discussion that appears before the proposition. Recall that in Riemannian geometry the structure group is reduced from $GL(d)$ to $O(d)$ by the presence of the (standard) metric. In generalised geometry, instead, we need to consider that our generalised tangent bundle is naturally equipped with the metric η. This metric reduces in a natural manner the structure group from $GL(2d)$ to $O(d,d)$. If, in addition to η, we introduce a generalised metric, the structure group is further reduced to $O(d) \times O(d)$. We have already seen previously in the chapter that we can view $O(d,d)$ as an extension of the group of diffeomorphisms; we are now induced to think of $O(d) \times O(d)$ as the generalised geometric analogous of $O(d)$. We can represent this chain of analogies in the following diagram:

$$\begin{array}{cccc} \textbf{GR} & O(d) & \subset & GL(d) \\ & \updownarrow & & \updownarrow \\ \textbf{GenG} & O(d) \times O(d) & \subset & O(d,d) \end{array}$$

As we will see in the next chapter, one of the important applications of generalised geometry to physics is the fact that it can describe the (bosonic) $NSNS$ sector of Type IIA and IIB supergravity theories as the generalised geometric analogue of general relativity, with the $O(d-1,1)$ structure replaced by an $O(d-1,1) \times O(1,d-1)$ structure.[9]

We would like to end this section with a remark concerning possible extensions of the theory that we have outlined until now. If one considers the expression for the generalised Lie derivative (but the same reasoning works equally well for the Courant bracket):

$$L_X Y = [v,w] + \mathcal{L}_v \nu - i_w(d\mu) \qquad X = v + \mu, Y = w + \nu \in \Gamma(E)$$

one can notice that the operators that appear in the one-form component of this formula are operators that are defined in total generality for the whole exterior algebra, i.e. for forms of any order. One can then try to extend the definition of the generalised vector bundle to the following vector bundle: $TM \oplus \bigwedge^p T^*M$, with p any non-negative integer. In this case we will have that the additional symmetries are given by the action of the abelian group under addition of closed $p+1$-forms: if $X = v + \alpha \in TM \oplus \bigwedge^p T^*M$ and if $A \in \bigwedge^{p+1} T^*M$ is closed, then the A-transform: $exp(A) \cdot X = v + \alpha + i_v A$ preserves the Courant bracket (see [Hi2003]). One can also consider more complicated generalised vector bundles in which the form part is composed of a direct sum of forms of different order, in which case one also needs to modify the definition of the generalised Lie derivative in order to conform it to the new symmetries of the new vector bundle. One can then

[9] See the previous footnote for an explanation of the signatures used here.

enter the domain of the so-called 'exceptional generalised geometry' in which the relevant structure can be related to the exceptional Lie groups (see [CoStWa2013, CoStWa2013n2]). What is important to us is that one can extend the construction we have made till now to include in the framework of generalised geometry also structure groups that are related to the exceptional Lie groups $E_{6(6)}, E_{7(7)}, E_{8(8)}$. What turns out is that also from them one can construct a generalised general relativity, that now is related to 11-dimensional supergravity.

1.3 Extension of E

We have seen that we can define a Courant algebroid on $TM \oplus T^*M$, namely that $TM \oplus T^*M$ can be endowed with the natural metric η and the Courant bracket $[\![,]\!]$ (in a compatible manner). Until now we have only worked with $E = TM \oplus T^*M$, but this description can be made slightly more flexible. We have indeed shown that the Courant algebroid structure is preserved by the group $Diff(M) \ltimes \Omega^2_{closed}(M)$. In particular, in addition to the symmetry induced by the invariance under change of coordinates, there is an additional symmetry (that does not commute with the diffeomorphisms) which is given by the (closed) B-transforms. We now want to redefine the structures that we have introduced until now in a local manner and then try to extend them via appropriate patching rules and see the consequences of this.
Let us introduce an open covering $\{U_{(i)}\}$ of our manifold M. We would like to be able to write the sections of E on each $U_{(i)}$ as the sum of a vector field with a one-form. For instance, on $U_{(i)}$ we could write a section $X \in \Gamma(E)$ as:

$$X(p) = v_{(i)}(p) + \mu_{(i)}(p) \qquad \forall p \in U_{(i)} \tag{1.23}$$

Since we are defining E starting from a local point of view we need to specify what happens in the case of a intersection of two elements of the open covering. We can surely set that for each $U_{(i)}, U_{(j)} \in \{U_{(i)}\}$ the following condition holds:

$$v_{(i)}(p) + \mu_{(i)}(p) = v_{(j)}(p) + \mu_{(j)}(p) \qquad \forall p \in U_{(i)} \cap U_{(j)}$$

In this case the definition of $X \in \Gamma(E)$ given by 1.23 holds globally and we can say that $E = TM \oplus T^*M$. On the other hand we could also make use of the additional symmetry that is intrinsic in the definition of the Courant algebroid and define E in such a way that in the intersection of two sets of the open covering we have:

$$v_{(i)}(p) + \mu_{(i)}(p) = v_{(j)}(p) + \mu_{(j)}(p) - i_{v_{(j)}(p)} d\Lambda_{(ij)}(p) \qquad \forall p \in U_{(i)} \cap U_{(j)}, \forall j \tag{1.24}$$

for an opportune set of patching one-forms $\{\Lambda_{ij}\}$ defined on the various non empty intersections $U_{(i)} \cap U_{(j)}$. Equation 1.24 says that the two expressions for the section of E have the same form, modulo a B-transform with $B = d\Lambda_{(ij)}$. Note that in the generic symmetry of the Courant algebroid the two-form must be closed, but since we are dealing with a local problem we can assume this to be exact (possibly even by choosing a suitable refinement of $\{U_{(i)}\}$). From equation 1.24 we can see that the vector part of the section of E is well defined in all patches $U_{(i)}$, whilst the form part does not define a proper section of T^*M, since it varies from patch to patch. It is also clear that the one-forms $\Lambda_{(ij)}$ have to satisfy: $\Lambda_{(ij)} = -\Lambda_{(ji)}\ \forall\, i, j$. In order for equation 1.24 to be consistent, $\{\Lambda_{(ij)}\}$ needs to satisfy another condition. Let us consider a triple intersection of patches $U_{(i)} \cap U_{(j)} \cap U_{(k)}$. Then, equation 1.24 implies:

$$\begin{aligned} v_{(i)} + \mu_{(i)} &= v_{(j)} + \mu_{(j)} - i_{v_{(j)}} d\Lambda_{(ij)} \\ &= v_{(k)} + \mu_{(k)} - i_{v_{(k)}} d\Lambda_{(ik)} = (v_{(j)} + \mu_{(j)} - i_{v_{(j)}} d\Lambda_{(kj)}) - i_{v_k} d\Lambda_{(ik)} \end{aligned}$$

This implies: $i_{v_{(i)}}(d\Lambda_{ij} + d\Lambda_{jk} + d\Lambda_{ki}) = 0$, which can be integrated to yield:

$$\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = d\Lambda_{(ijk)} \qquad \text{for a } \Lambda_{(ijk)} \in \mathcal{C}^\infty(M)$$

These conditions resemble the cocycle conditions for a $U(1)$-bundle and therefore one sometimes defines $exp(i\Lambda_{ijk}) =: g_{(ijk)} \in U(1)$ and writes:

$$\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = -i g^{-1}_{(ijk)} dg_{(ijk)} \qquad \text{in } U_{(i)} \cap U_{(j)} \cap U_{(k)} \tag{1.25}$$

It is clear that, even if one can locally (i.e. on each patch) write a section of our new generalised vector bundle E as the sum of a vector field and a one-form, this is not the general (global) form that a section of E will assume. We are still interested though in finding how to write a section of E as a section of $TM \oplus T^*M$, i.e. in finding an isomorphism between E and $TM \oplus T^*M$. In order to do that first recall that $\Lambda_{(ij)} = -\Lambda_{(ji)}$. We can therefore introduce a collection of two-forms $B = \{B_{(i)}\}$, where each $B_{(i)}$ is defined on $U_{(i)}$, such that:

$$d\Lambda_{(ij)} =: B_{(i)} - B_{(j)} \qquad \forall\, i, j \tag{1.26}$$

From this definition we see that, even if the different two-forms $B_{(i)}$ are only locally (i.e. on each patch) defined, the collection B defines a globally defined closed three-form H: $H|_{U_{(i)}} = dB_{(i)}\ \forall i$, a definition that is consistent thanks to 1.26. Now, let us consider the section $X \in \Gamma(E)$ that we defined in 1.23 on $U_{(i)}$. In the intersection $U_{(i)} \cap U_{(j)}$ with another patch it will take the form (recall that $v_{(i)} = v_{(j)}$):

$$v_{(i)} + \mu_{(i)} = v_{(j)} + \mu_{(j)} - i_{v_{(j)}} d\Lambda_{(ij)} = v_{(j)} + \mu_{(j)} + i_{v_{(j)}} B_{(j)} - i_{v_{(i)}} B_{(i)} \tag{1.27}$$

If we consider a new section $Y \in \Gamma(E)$ that on $U_{(i)}$ equals $v_{(i)} + \mu_{(i)} + i_{v_{(i)}}B_{(i)} \in \Gamma(TU_{(i)} \oplus T^*U_{(i)})$, then it will have the same form in all the patches thanks to equation 1.27. This section defines therefore a section of the whole $TM \oplus T^*M$. Moreover since for all patches $E|_{U_{(i)}} = TU_{(i)} \oplus T^*U_{(i)}$ and $\{v_{(i)} + \mu_{(i)} + i_{v_{(i)}}B_{(i)}\}$ generates $TU_{(i)} \oplus T^*U_{(i)}$ we have found and explicit isomorphism between E and $TM \oplus T^*M$. We would like to stress the fact that even thought there is an isomorphism between the two vector bundles, this isomorphism is *not* canonical, since it depends on the choice of the $B = \{B_{(i)}\}$.
Making use of all the symmetries of the Courant bracket in the way the patching is made, we ended up with a new definition of generalised vector bundle. The sections of this new bundle, after a choice of B, can be identified with sections of $TM \oplus T^*M$ of the form $X = v + \mu + i_vB = exp(-B)(X)$, where we used the definition of B-transform to write the last equality. Recalling the definition of the twisted Courant bracket 1.7, we can see that on the new generalised vector bundle there is a new Courant bracket given by the old Courant bracket $[\![,]\!]$ on $TM \oplus T^*M$ now twisted with a globally defined closed three-form $H = dB$.
Now consider again the generalised metric introduced in section 1.2. We know that it has the form:

$$G = \frac{1}{2}\begin{pmatrix} g - \tilde{B}g^{-1}\tilde{B} & \tilde{B}g^{-1} \\ -g^{-1}\tilde{B} & g^{-1} \end{pmatrix} \tag{1.28}$$

with g a metric and $\tilde{B}$ a two-form. The new patching of E now implies that $\tilde{B}$ satisfies the patching conditions 1.26 up to a sign. In particular, from proposition 2.3.1 it will be clear that the $\tilde{B}$ in 1.28 may be identified with *minus* the collection $B = \{B_{(i)}\}$ that defines the isomorphism between E and $TM \oplus T^*M$; thus a generalised metric defines a particular splitting of the generalised tangent bundle [GGP2010]. In particular a section X of E can now be represented as a section of $TM \oplus T^*M$ of the form: $X_{(i)} = v_{(i)} + \lambda_{(i)} - i_{v_{(i)}}\tilde{B}_{(i)}$; in this notation the patching 1.26 takes the form: $\Lambda_{(ij)} = B_{(i)} - B_{(j)} =: -\tilde{B}_{(i)} + \tilde{B}_{(j)}$.[10] To conclude this section we would like to note that the new structure we have just introduced is well known in the literature and can be explained in much more formal terms. The generalised tangent bundle can be defined as an extension of the tangent bundle via the cotangent bundle. In particular the following short exact sequence holds:

$$0 \longrightarrow T^*M \longrightarrow E \longrightarrow TM \longrightarrow 0$$

[10] In particular we could redefine $\tilde{B}$ in 1.28 to be minus itself, i.e. $\tilde{B} \mapsto -\tilde{B}$. In this way we will obtain the usual patching with sections of the form $X = v + \mu + i_v\tilde{B}$.

where the the first map is the natural inclusion and the second one is the natural projection and the extension depends on the patching one forms $\Lambda_{(ij)}$ as explained earlier. Then one defines a splitting, via the splitting lemma [MLa1975], making use of the homomorphism $exp(-B)|_{TM} : TM \to E$, after choosing local two-forms patched as in 1.26. Just for completeness we note that the relation 1.25 together with 1.26 makes B a 'connection structure on a gerbe'[Hi2001], the description of which is far beyond the scope of this thesis.

Chapter 2

Supergravity as Generalised Geometry

The analysis developed in the previous chapter was mainly mathematical in nature. In particular, we discussed the main features of the generalised vector bundle that will be relevant to us in the remaining of the thesis. Starting from this chapter, instead, we will try to explore some of the possible applications that this mathematical theory can have to physical theories. More specifically, in this chapter we will deal with Type IIA and Type IIB supergravity theories and show how the generalised geometry developed in the previous chapter can be used to 'geometrise' their $NSNS$ sector.
In the first part of the chapter, we will try to give a justification for the field content of the type II theories. The ideas and arguments we will use there are mainly taken from the first part of the course of String Theory of Professor Amihay Hanany that the author attended during the academic year 2015/2016 at Imperial College London. We will then focus on the bosonic degrees of freedom of the theories. We will give for them a 'pseudo-action' making use of the 'democratic formalism' of [BKORV01]. This formalism turns out to be well suited to the description of the RR-sectors of the type II supergravity theories in terms of generalised geometry.[1] Our use of it will nevertheless be just for a reason of completeness because we will then mainly focus on the common $NSNS$ sector. We will show that generalised geometry is able to naturally describe its field content, its symmetries and also its pseudo-action. The construction of the latter will be done by means of a 'generalised Levi-Civita' connection in a manner that is the closest analogue as possible with the construction used for the standard Einstein-Hilbert action

[1] More specifically the democratic formalism allows one to explicitely treat the RR-field strengths as $Spin(10, 10)$ spinors; see also the end of subsection 2.2.2

of ordinary geometry.

2.1 Field Content

2.1.1 Supersymmetry Multiplets

Let us start with supersymmetry. Supersymmetry is basically founded upon an extension of the usual theory of symmetries that allows one to include generators that are defined to satisfy, together with commutation relations, also anticommutation relations. In more formal terms this fact means that the ordinary Lie algebra of the symmetry generators becomes a 'graded' algebra. The graded algebra to which we are usually interested in, in supersymmetry (and also in supergravity), is the super-Poincaré algebra, i.e. a graded algebra extension of the Poincaré algebra.[2] Recall that the generators of the Poincaré group are: a vector P^μ and an antisymmetric tensor $M^{\mu\nu}$. These transform in tensorial representations of the Poincaré group. We know that there are also other possible representations: the spin representations. Extending to super-Poincaré allows one to include spinorial generators, i.e. generators, say $\{Q^I_\alpha\}$, that transform in the spin representation, where we could potentially have $I = 1, ..., N$ for a certain $N \in \mathbb{N}$ (extended supersymmetry). Since in supersymmetry we have (fermionic) spinorial generators, we can send, acting with an infinitesimal transformation, a boson to a fermion and vice-versa. This is a clear evidence that irreducible super-Poincaré representations[3] are not labelled by spin (or helicity, in the case of massless particles) any more. In particular, irreducible representations of the super-Poincaré algebra will be labelled by mass (like in the Poincaré case) and *super*spin. This allows having (super-)multiplets in the theory that contain states of different (ordinary) spin (or helicity).
We have already said that the $Q^I_\alpha, \forall\, I$, belong to a spin representation. For each space-time dimension d we can find the smallest spin representation that can be introduced in that dimension. If we call k_d the real dimension of this representation, k_d is equal to twice the (complex) dimension of a Dirac representation (that is equal to $2^{[\frac{d}{2}]}$), divided by a factor 2 if a Weyl condition can be applied, and divided by another factor 2 if an (independent) Majorana condition can be imposed (see [Po1998], Appendix B, for more

[2] There is also a theorem (Haag, Lopussaski and Sohnus) that states that the direct product of super-Poincaré and internal symmetries forms the most general group of symmetries for an S-matrix (at least in 4 dim)

[3] In four dimensions: we recall here that the notion of spin only makes sense in three or four dimensions where we can connect the (complexified) Lorentz algebra to the group SU(2).

details). For example in d=4, where the Weyl and Majorana conditions are known to be equivalent, and where the complex Dirac dimension is 4, we have $k_4 = 4$ real components. For simplicity, we will choose each of the N

$\mathbf{d}$	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{k_d}$	1	2	4	8	8	16	16	16	16	32	64

Table 2.1: Minimal real dimension k_d for a spin representation in d space-time dimensions. Table taken from [Po1998], Appendix B.

Q^I_α to transform in this smallest spin representation. We will consider in the following only massless multiplets because they describe the low energy limit degrees of freedom of the theory. Moreover we will call the number k_d the number of 'supercharges'. Now, in the supersymmetry algebra these spinorial generators satisfy some anticommutation relations. Chosen the momentum, for a massless multiplet these anticommutation relations allow one to show that half of the supersymmetry generators annihilate all the vectors in the representation, whilst the other half split in $\frac{k_d \times N}{4}$ creation and $\frac{k_d \times N}{4}$ annihilation operators, that act on the state of minimal helicity of the multiplet. Moreover in $d = 4$, where we can consider the Qs to be of spin $\frac{1}{2}$, each different creation operator increases the helicity by $\frac{1}{2}$. For example in $d = 4$, $N = 1$ we have 4 supercharges and one creator operator. Therefore, massless multiplets will be couples of massless particles with helicities $(\lambda, \lambda + \frac{1}{2})$. The crucial point is that there is a 'strong belief' that there are no interacting theories - with a *finite* number of fields[4] - for massless particles with helicity greater that 2.[5] [6] Consider a supersymmetric theory in $d = 4$. We have seen that we have in general $N \times 4$ supercharges; this implies the existence of $N (= \frac{k_4}{4} \times N)$ creation operators. If we do not want helicities λ with modulus greater than two, then we need to have: $-|\lambda_{max}| + \frac{N}{2} \leq |\lambda_{max}|$, i.e. $N \leq 2 \times 2|\lambda_{max}| = 8$. We can rephrase this result stating that we can not have more than $N \times k_4 = 32$ supercharges. If we now look back at table 2.1, we see that this also impose an upper dimension limit for the supersymmetry to be present in our theory. In particular, $d = 11$ appears to be the uppermost dimension that admits a supersymmetry multiplet. Let us consider explicitly the $N = 1$ supersymmetric theory in $d = 11$. This theory has

[4] There are, for example, non-local interacting theories in AdS that admit spins higher than 2, but they make use of an infinite number of fields; see e.g. [So2005]

[5] We are actually only able to construct renormalizable theories with massless particles that have maximum helicity equal to 1.

[6] There are several arguments for this statement. One of these looks at the 'soft-limit' of interacting particles, see Weinberg.

only one supermultiplet, that contains $2^{\frac{32}{4}} = 2^8 = 256$ polarisation modes. There is a very simple, but also very general, theorem in supersymmetry that states that the number of bosonic and fermionic degrees of freedom in a given multiplet are always equal to each other. This implies that in this multiplet there are 128 bosonic and 128 fermionic degrees of freedom.

2.1.2 Irreducible Representations

We are trying to find the field content of the supergravity theories. Until now we have only explored supersymmetric theories. Nevertheless, we can generically say that supergravity theories are supersymmetric theories that have, instead of global supersymmetry, a *local* supersymmetry. It can be shown that local supersymmetry implies general coordinate transformation invariance, and therefore it implies gravity; this is the reason of the name 'supergravity'. Nevertheless, since we are only considering questions about the field content of the theory, we can 'ignore' the difference between supersymmetry and supergravity and consider the supergravity multiplets as they were the same as the supersymmetry multiplets, i.e. irreducible representations of super-Poincaré. In fact, since we are very used to work with Poincaré representations, and not with the ones of super-Poincaré, we would also like to find the decompositions of the supermultiplets into irreducible representations of the Poincaré group. These are indeed the representations that tell us what kind of particles are present in the theory.
In order to decompose the supermultiplets into irreducible representations of the Poincaré group, we will make use of the little group theorem by Wigner. This theorem states that the irreducible representations of the Poincaré group $\mathcal{P}$, once the mass is fixed, are classified by the irreducible representations of the little group (the isotropy group) of the Lorentz subgroup $SO(d-1,1)$ of $\mathcal{P}$.[7] Now recall the Cartan classification of the simple Lie algebras. There are $A_n \approx \mathfrak{sl}(n{+}1,\mathbb{C})$, $B_n \approx \mathfrak{so}(2n{+}1,\mathbb{C})$, $C_n \approx \mathfrak{sp}(n,\mathbb{C})$ and $D_n \approx \mathfrak{so}(2n,\mathbb{C})$, and also the exceptional Lie algebras that we will not consider here. The index n in each series is the rank of the algebra, i.e. the dimension of the maximal Cartan subalgebra associated with that algebra (see e.g. [SaWe1986]). What is relevant to us here is that the irreducible representations for each of these simple algebras are classified by n positive *integers*, that represent what is called the 'highest weight' of the representation. We will therefore denote representations of $\mathfrak{so}(d-1,1,\mathbb{C}) \approx \mathfrak{so}(d,\mathbb{C})$ in the following manner:

[7] We recall that we are considering unitary representations of the Poincaré group $\mathcal{P}$. Since $\mathcal{P}$ is not compact its irreducible unitary representations are infinite-dimensional and therefore give rise to fields.

$[i_1, ..., i_n]$, where $i_j \in \mathbb{Z}_{\geq 0}$ and n is the rank of the the algebra, i.e. $n = \left[\frac{d}{2}\right]$.

	Representations of $\mathfrak{so}(k, \mathbb{C})$		
	Highest Weight	**Representation**	**Dim. of the Rep.**
$\mathbf{B_n}$	$[0, \ldots, 0]$	scalar	1
$\approx\mathfrak{so}(2n{+}1,\mathbb{C})$	$[1, 0, \ldots, 0]$	1-form (i.e. vector)	k
	$[0, 1, 0, \ldots, 0]$	2-form	$k(k-1)/2$
	$[0, ..., 1\vert_{jth}, 0, ..., 0]$	j-form	$\binom{k}{j}$
	$[0, \ldots, 0, 1]$	spinor	$2^{[\frac{k}{2}]}$
	$[2, 0, \ldots, 0]$	2-tensor symmetric traceless	$\frac{k(k+1)}{2} - 1$
	$[1, 0, \ldots, 0, 1]$	spinor-vector	$(k-1) \times 2^{[\frac{k}{2}]}$
$\mathbf{D_n}$	$[0, \ldots, 0]$	scalar	1
$\approx\mathfrak{so}(2n,\mathbb{C})$	$[1, 0, \ldots, 0]$	1-form (i.e. vector)	k
	$[0, ..., 1\vert_{jth}, 0, ..., 0]$	j-form	$\binom{k}{j}$
	$[0, \ldots, 0, 1, 1]$	(n−1)-form	$\binom{k}{n-1}$
	$[0, \ldots, 0, 1, 0]$	left-handed spinor	$2^{[\frac{k}{2}]-1}$
	$[0, \ldots, 0, 0, 1]$	right-handed spinor	$2^{[\frac{k}{2}]-1}$
	$[2, 0, \ldots, 0]$	2-tensor symmetric traceless	$\frac{k(k+1)}{2} - 1$
	$[1, 0, \ldots, 0, 1, 0]$	left-handed spinor-vector	$(k-1) \times 2^{[\frac{k}{2}]-1}$
	$[1, 0, \ldots, 0, 0, 1]$	right-handed spinor-vector	$(k-1) \times 2^{[\frac{k}{2}]-1}$

Table 2.2: Irreducible representations of $\mathfrak{so}(k, \mathbb{C})$ in the highest weight notation. Note that we found all the usual representations: vector, form, spinor, and also graviton (2^{nd} rank symmetric traceless) and gravitino (here indicated with 'spinor-vector').

In table 2.2 one can find the principal, lower dimensional, irreducible representations of $\mathfrak{so}(k, \mathbb{C})$ in the highest weight notation. Note that in even dimensions, where one can define a non-trivial equivalent of the 4-dimensional γ^5, one can find Weyl representations and therefore the irreducible spin representations split in even and odd. Let us go back to our problem: we need to find irreducible representations of the Poincaré group in d dimensions. We have already said that we are only left with the problem of finding the irreducible representations of the little group. In fact, since the little group for a massless particle is non-compact - and therefore does not admit any finite dimensional unitary representation - one usually considers only its maximally compact subgroup, which in our case is (a real form of) $\mathfrak{so}(d-2, \mathbb{C})$. Now consider our supermultiplet in $d = 11$. The (maximally compact subgroup of the) little group is generated by (a real form of) $\mathfrak{so}(9, \mathbb{C})$. We can then set $k = 9$ in the table 2.2 and find the various dimensions for the representations that are listed there. Since 9 is odd we are dealing with B_4 and we have:

1. $[1, 0, 0, 0]$: 9-dimensional vector representation
2. $[0, 1, 0, 0]$: 36-dimensional 2-form representation
3. $[0, 0, 1, 0]$: 84-dimensional 3-form representation
4. $[2, 0, 0, 0]$: 44-dimensional 2-tensor symmetric traceless representation
5. $[0, 0, 0, 1]$: 16-dimensional spinor representation
6. $[1, 0, 0, 1]$: 128-dimensional spinor-vector representation

where here 'dimension' means both the dimension of the irreducible representation of the little group and the on-shell degrees of freedom of the corresponding field. If we now recall that the $N = 1$, $d = 11$ supermultiplet contained 128 fermionic and 128 bosonic degrees of freedom, we see that there is one and only one possibility for the decomposition of the representations of super-Poincaré in irreducible representations of the subgroup $\mathcal{P}$:

$$N = 1, d = 11 \text{ Supermultiplet} \longmapsto ([2, 0, 0, 0], [0, 0, 0, 1], [1, 0, 0, 1])$$

Since this multiplet contains a graviton ($[2, 0, 0, 0]$), we will call this the 11-dimensional *supergravity multiplet*. We can only embed these on-shell degrees of freedom inside fields that include off-shell degrees of freedom. The field content of the $11d$ SUGRA multiplet will then be: $(g_{\mu\nu}, C_{\mu\nu\rho}, \psi^{\alpha}_{\mu})$, i.e. graviton, 3-form (bosons) and gravitino (fermion).
Till now we have only described the unique supermultiplet that is possible in $d = 11$. What we actually wanted at the beginning of the chapter was instead to study some supergravity theories in $d = 10$. Let us try to compactify one dimension to obtain a theory in $d = 10$ starting from the one in $d = 11$. More in detail we will use a toroidal compactification of one dimension to obtain a theory in $d = 10$ with a process known as 'dimensional reduction' (see also chapter 3). To clarify how it works let us start with a vector of $SO(9)$ A_μ. By dimensional reduction we will consider one of its component, say A_9, to be fixed[8] and the remaining 8 components, say A_i with $i = 1, \ldots, 8$, transforming in the $SO(8)$ vector representation. Now consider the graviton $g_{\mu\nu}$. Under dimensional reduction we will obtain g_{99}, g_{9i}, g_{ij}, i.e. a scalar, an $SO(8)$ vector and an $SO(8)$ graviton. For the three-form, instead, we will

[8]Actually transforming in the SO(1) representation, i.e. not transforming

only have an $SO(8)$ three-form and an $SO(8)$ two-form, since the index $\mu = 9$ can only appear once in the set of three indices, due to the total antisymmetry of $C^{(3)}$. Now consider the spinors. By dimensional counting we can see that the product of a spinor and a vector representation decomposes into the sum of a gravitino and a spinor: $[1,0,0,0] \otimes [0,0,0,1] = [1,0,0,1] + [0,0,0,1]$.[9] Again by dimensional counting and also by parity arguments we can also state that the dimensional reduction for the $SO(9)$ spinor works as follows: $[0,0,0,1]_9 \mapsto [0,0,1,0]_8 \oplus [0,0,0,1]_8$. We can then write:

$$\begin{aligned}
&[1,0,0,0]_9 \otimes \underline{[0,0,0,1]_9} \mapsto ([1,0,0,0]_8 \oplus [0,0,0,0]_8) \otimes \underline{([0,0,1,0]_8 \oplus [0,0,0,1]_8)} \\
&\quad = ([1,0,0,0]_8 \oplus [0,0,0,0]_8) \otimes [0,0,1,0]_8 \oplus ([1,0,0,0]_8 \oplus [0,0,0,0]_8) \otimes [0,0,0,1]_8 \\
&\quad = ([1,0,1,0]_8 \oplus [0,0,0,1]_8) \oplus [0,0,1,0]_8 \oplus ([1,0,0,1]_8 \oplus [0,0,1,0]) \oplus [0,0,0,1] \\
&[1,0,0,1]_9 \oplus [0,0,0,1]_9 \mapsto (?) \oplus ([0,0,1,0]_8 \oplus [0,0,0,1]_8)
\end{aligned}$$

We can clearly see that

$$[1,0,0,1]_9 \mapsto (?) = [1,0,0,1]_8 \oplus [1,0,1,0]_8 \oplus [0,0,1,0]_8 \oplus [0,0,0,1]_8$$

The following multiplet of $SO(8)$ representations is the result of the performed dimensional reduction:

$$[2000] \oplus [1000] \oplus [0000] \oplus [0011] \oplus [0100] \oplus [1010] \oplus [1001] \oplus [0010] \oplus [0001] \quad (2.1)$$

These representations are embedded (in the order above) in the following fields (as it could have been read directly from table 2.2): a graviton $g_{\mu\nu}$, a one-form $C^{(1)}_\mu$, a scalar ϕ, a three-form $C^{(3)}_{\mu\nu\rho}$, a two-form $B_{\mu\nu}$, a left-handed gravitino ψ^α_μ, a right-handed gravitino $\psi^{\alpha'}_\mu$, a left-handed fermion λ^α and a right-handed fermion $\lambda^{\alpha'}$. Since this multiplet contains a graviton we will call it: *(Type IIA) supergravity multiplet.*

Proposition 2.1.1. *The Type IIA supergravity multiplet can be factorised in the following manner:*[10]

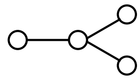
$$([1000] \oplus [0010])_L \otimes ([1000] \oplus [0001])_R \quad (2.2)$$

Proof. We have already seen what the mixed terms are. Consider now the square $V^2 := [1000] \otimes [1000]$. We have the tensor product of two vector

[9] This is a general feature that appears in every dimension. For example for $\mathfrak{so}(4,\mathbb{C}) \approx \mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C}) \approx \mathfrak{su}(2,\mathbb{C}) \oplus \mathfrak{su}(2,\mathbb{C})$, where we can use the standard spin notation (s_1, s_2) and the usual spin product rules, we have: $(\frac{1}{2},\frac{1}{2}) \otimes (\frac{1}{2},0) = (1,\frac{1}{2}) \oplus (0,\frac{1}{2})$. Note also that in the sum appears a spinor with a chiraity that is *opposite* to the one of the spinor in the product.

[10] Note that we will drop from now on all the commas in the highest weight notation.

representations. It is then well known that this decomposes in a direct sum of the antisymmetric part, the trace and the symmetric traceless part. We can then write: $V^2 = [2000] \oplus [0100] \oplus [0000]$.
Now consider the product $S^2 := [0010] \otimes [0001]$. We do not know how to calculate this expression yet. In order to calculate it we can use the decomposition of products of spin representations into sum of form representations as given in the Appendix B of [Po1998]. Otherwise we can make use of the following trick. The Dynkin diagram[11] for B_4 is the following:

D_4 

We can note that this diagram is symmetric under the permutations of the external nodes. Since each node in the diagram is associated with one of the entries in the highest weight notation, we can state that there is a symmetry under permutations of the first, third and fourth indices in the highest weight notation ('triality'). This means that $[0010] \otimes [0001] \mapsto [1000] \otimes [0010] = [1010] \oplus [0001]$. Permutating back the indices we obtain: $[0010] \otimes [0001] = [0011] \oplus [1000]$.[12] These ingredients allow us to write:

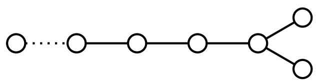
$$
\begin{aligned}
&([1000] \oplus [0010]) \otimes ([1000] + [0001]) = \\
&= ([1000] \otimes [1000])_{NSNS} \oplus ([1000] \otimes [0001] \oplus [0010] \otimes [1000])_{NSR-RNS} \oplus ([0010] \otimes [0001])_{RR}
\end{aligned}
$$

Where $NSNS$ stands for (Neveu-Schwarz)-(Neveu-Schwarz) sector, RR for Ramond-Ramond sector and $NSR - RNS$ are the mixed sectors. We can then write:

$$
\begin{aligned}
&([1000] \otimes [1000])_{NSNS} = [2000] + [0100] + [0000] \sim (g_{\mu\nu}, B_{\mu\nu}, \phi) \\
&([1000] \otimes [0001] \oplus [0010] \otimes [1000])_{NSR-RNS} = \\
&\qquad = [1001] \oplus [0010] \oplus [1010] \oplus [0001] \sim (\psi_\mu^{\alpha'}, \lambda^\alpha, \psi_\mu^\alpha, \lambda^\alpha) \\
&([0010] \otimes [0001])_{RR} = [0011] \oplus [1000] \sim (C^{(3)}_{\mu\nu\rho}, C^{(1)}_\mu)
\end{aligned}
$$

and this is the actual field content of the Type IIA supergravity multiplet. □

[11] The Dynkin diagrams are the diagrams used to classify the possible simple algebras; they can also be used to construct the Cartan matrix from which one can derive all the properties of the algebra representations, see [SaWe1986]. The Dynkin diagram for the D_n series is

D_n 

where there are as many nodes as the rank of the algebra.
[12] Which is actually the sum of a three form and a one form, as indicated in the appendix of [Po1998].

The factorisation given in proposition 2.1.1 indicates that the theory factorises in a left-handed component times a right handed component, yielding, therefore, a non-chiral theory. The factorisation in the chiral components is typical of any massless theory; recall for example the theory in $d = 4$ where the massless right and left-handed spinors are factorised. The multiplet appearing in each factor of the product in 2.2 is called 'vector multiplet'. We note that there is another possible independent product that can be performed with the vector multiplets we have found: the product of vector multiplets with the same chirality. Let us consider the field content of the theory that is obtained in this way:

$$([1000] \oplus [0001]) \otimes ([1000] \oplus [0001]) = \tag{2.3}$$

$$= (\underbrace{[2000]}_{\sim g_{\mu\nu}} \oplus \underbrace{[0100]}_{\sim B_{\mu\nu}} \oplus \underbrace{[0000]}_{\sim\phi})_{NSNS} \oplus 2(\underbrace{[1001]}_{\psi^{\alpha\prime}_{\mu}} \oplus \underbrace{[0010]}_{\lambda^{\alpha}})_{NSR-RNS} \oplus \tag{2.4}$$

$$\oplus\, (\underbrace{[0002]}_{\sim C^{\star(4)}_{\mu\nu\rho\sigma}} \oplus \underbrace{[0100]}_{\sim C^{(2)}_{\mu\nu}} \oplus \underbrace{[0000]}_{\sim C^{(0)}})_{RR} \tag{2.5}$$

where to calculate the Ramond-Ramond sector we used again the 'triality' propriety explained in the proof of proposition 2.1.1. The representation [0002] was not included in the table 2.2. It represents a self-dual four-form:[13] both its field strength and the Hodge dual of the latter are five-forms, since we are in $d = 10$, and they are set to be equal to each other. The self-duality condition reduces the propagating degrees of freedom of the representation by $\frac{1}{2}$.[14] The equation 2.3 defines another supergravity multiplet in $d = 10$, which is called *Type IIB supergravity multiplet*. Note that this multiplet is chiral by construction and it did not derive from a procedure of dimensional reduction. We have found in this way the field content of the two theories we were interested in.

To conclude this section we will make a few comments about these results. The first is that the $NSNS$-sectors of the TypeIIA and TypeIIB supergravity theories are exactly the same: they both derive by squaring a vector representation. The second fact is that, whilst in Type IIA we have spinors of both the chiralities, in Type IIB we have a 'doublet' of (gravi)fermions and a doublet of gravitinos, each doublet composed of spinors of the same chirality. Finally, both the Ramond-Ramond sectors are formed by the sum of form representations. While in Type IIA the forms are of even homogeneous degree, in Type IIB they are of odd homogeneous degree.

[13] Note that if [0002] is the self-dual then [0020] is the anti-self-dual, which clearly has the same number of degrees of freedom.

[14] We can also make a check of the degrees of freedom of this decomposition: [0001] has 2^{4-1} d.o.f., so it squares to 64; the sum of forms is instead: $\binom{8}{4}/2+28+1 = 35+28+1 = 64$.

2.2 Properties of the Theories

2.2.1 (Pseudo-)Action

Now that we know the field content of the theories, we would also like to write an action. In what follows, we will only focus on the bosonic part of the supermultiplet. There is a standard manner to write the kinetic term for the forms. Each form $C^{(i)}$, where i indicates the degree of the form, is associated with a field strength $F^{(i+1)} = dC^{(i)}$(+ an additional term in the case of RR forms); for example in electromagnetism we had the one-form potential A_μ, which was associated with the two-form $(dF)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The action is then written in the form: $\int F \wedge *F$, where the star indicates the Hodge dual. This kind of action can not be written, though, when the form is (anti-)self-dual, because in that case the action will be identically zero. In particular, because of the presence of the self-dual 5-form field strength, the construction of the action for type IIB theory is not straightforward and requires either to sacrifice manifest diffeomorphism invariance or to use auxiliary fields (see[PeSc1997, PSTDL]). For our purposes, however, we will use a different procedure that is somehow standard. We will write the action as if the five-form did not satisfy any self-duality relation and will then impose the self-duality constraint 'by hand' after the action has been varied. This procedure gives rise to what is called a 'pseudo-action', that reduces to be only a mnemonic object, used to remember the equations of motion.
Since for Type IIB we will use a pseudo-action, we will write a pseudo-action also for Type IIA. This approach allows for a unified treatment of the two theories that in addition has no Chern-Simons terms in the action - terms, topological in nature, that are instead present in the standard construction. With this procedure, these Chern-Simons terms will instead be hidden in the definition of the field strengths of the dual RR fields. The approach we are referring to is the one of the 'democratic formalism' (given in [BKORV01]) where forms of all the orders are treated 'democratically'. Democratically here means that forms of all the orders, odd for Type IIA and even for Type IIB, appear in the (extended) field content and that self-duality constraints, that allow keeping the degrees of freedom the same as in the original theory, are added again 'by hand' alongside a pseudo-action. More explicitely the bosonic content of the two theories is: $\left(g_{\mu\nu}, B_{\mu\nu}, \phi, \{A_{(n)}\}\right)$, where g is the metric, B is the $NSNS$ two-form, ϕ is the scalar ('dilaton') and $\{A_{(n)}\}$ is the democratic collection of forms, with $n \in \{0, 1, \ldots, 9\}$ and that is odd for TypeIIA and even for TypeIIB, in the so called 'A-basis'. This basis is just another manner of writing the RR degrees of freedom, which is related to the forms $\{C_{(n)}\}$ of the more standard 'C-basis' by $A_{(n-1)} := e^{-B} \wedge C_{(n-1)}$, $\forall\, n$.

The field strengths $\{F_{(n)}\}$ are related to the C-forms by $F_{(n)} = dC_{(n-1)} - dB \wedge C_{(n-1)}$, a relation that also explicitly includes the RR potential C. By using the 'A-basis' we can instead obtain a relation for F that depends on the potentials only through their derivatives: $F_{(n)} \equiv F^B_{(n)} = e^B \wedge dA_{(n-1)}$.
The bosonic pseudo-action can be written as follows (see [CoStWa2011] and [BKORV01]):

$$S_B = \frac{1}{2\kappa} \int \sqrt{-\mathrm{det} g}\Big[e^{-2\phi}\Big(\mathcal{R} + 4(\partial\phi)^2 - \frac{1}{12}H^2\Big) - \frac{1}{4}\sum_n \frac{1}{n!}(F^{(B)}_{(n)})^2\Big] \tag{2.6}$$

In equation 2.6 we can easily recognise the standard kinetic terms of the various fields, including the the Ricci scalar $\mathcal{R}$ for the metric g, the H^2, that stands for $H \wedge *H$ [15], for the $NSNS$ two-form B and analogous terms for the other forms. The additional constraints that need to be added to the pseudo-action 2.6 are:

$$F^{(B)}_{(n)} = (-1)^{[\frac{n}{2}]}(*F^{(B)})_{(10-n)} \qquad \forall\, n \tag{2.7}$$

The justification for the use of the A-basis of the democratic formalism to write the bosonic action will be given at the end of the following subsection.

2.2.2 Symmetries of the $NSNS$ sector and the RR sector

Apart from the pseudo-action we can also say something about the bosonic symmetries.[16] The $NSNS$ two-form B is a potential for the field strength $H = dB$. In general, H is only closed and so B is only locally defined. In particular shifts of the local Bs by exact two-forms give rise to the same field strength. This is very akin to what happens in electromagnetism: the four-potential A_μ is in general only locally defined and shifts by exact one-forms give rise to equivalent four-potentials (gauge transformations). This means that on different patches the A_μs can differ from each other, but in the intersections they are glued together by compatible 'transition functions' (whose differentials yield the patching). The cocycle (compatibility) conditions on the various transition functions then endow the manifold with the structure of a principal ($U(1)$) bundle, and A_μ defines a connection structure on this bundle. In the case of the $NSNS$ two-form B the construction is very similar, but now instead of 'transition functions' we have patching one-forms $\{\Lambda_{(ij)}\}$, defined for each intersection $U_{(i)} \cap U_{(j)}$ of two elements of

[15] Recall that to be integrated the integrand must be a volume form.
[16] (see [CoStWa2011], Section 2.2)

a given open covering $\{U_{(i)}\}$ of the manifold M. These patching one-forms are therefore such that $B_{(i)} - B_{(j)} = d\Lambda_{(ij)}$, $\forall\, i, j$. The cocycle conditions are now: $\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = d\Lambda_{(ijk)}$ on triple intersections of the form $U_{(i)} \cap U_{(j)} \cap U_{(k)}$. This is not a connection structure on a principal bundle any more, but instead a 'connection structure on a gerbe' [Hi2001]. The fields of the A-basis have a similar patching: $A^{(i)}_{(n)} = e^{-d\Lambda_{(ij)}} \wedge A^{(j)}_{(n)} + d\tilde{\Lambda}^{(ij)}_{(n-1)}$, where the $\{\tilde{\Lambda}^{(ij)}_{(n-1)}\}$ are other $(n-1)$-forms, clearly different from the one-forms $\Lambda_{(ij)}$. Finally, note that even if the gauge symmetry has been used for the patching, there is still a 'residual' gauge symmetry of the B-field, that consists of shifts by one forms whose differentials coincide in the two-fold intersections of the covering sets:

$$B'_{(i)} = B_{(i)} + d\lambda_{(i)}, \quad A'_{(i)} = e^{-d\lambda_{(i)}} A_{(i)} \quad \text{s.t. } d\lambda_{(i)} = d\lambda_{(j)} \text{ for } U_{(i)} \cap U_{(j)}$$

Local exact two-forms that coincide in the intersections of the patches define a global closed two form. Recalling that there is also a symmetry under diffeomorphisms and that this and the gauge symmetry of shifts by closed two forms do not commute with each other, one finds that the group of symmetries of the $NSNS$ bosonic sector is $G = Diff(M) \ltimes \Omega^2_{closed}(M)$. Moreover the symmetry generators have the form:

$$\delta_{v+\lambda} g = \mathcal{L}_v g, \quad \delta_{v+\lambda}\phi = \mathcal{L}_v \phi, \quad \delta_{v+\lambda} B_{(i)} = \mathcal{L}_v B_{(i)} + d\lambda_{(i)} \tag{2.8}$$

where we again recall that the Lie algebra $\mathfrak{g} = TM \oplus \Omega^2_{closed}(M)$ of G can be written as $TM \oplus T^*M$ by locally identifying closed two-forms with exact one-forms ($v + \lambda \mapsto v - d\lambda \in \mathfrak{g}$).

From what we have stated until now it should be already clear that there is a very close relationship between the $NSNS$ bosonic sector of the Type II supergravity theories and the generalised geometry we have presented in the former chapter. The patching that defines the field strength $H = dB$ in the Type II theories has exactly the same geometric structure as the patching of the generalised tangent bundle defined as an extension of the tangent bundle via the cotangent one introduced in section 1.3, where the collection $B = \{B_{(i)}\}$ defined a closed three form H on the entire generalised vector bundle. Moreover the symmetries of the Courant algebroid are exactly the same symmetries of the $NSNS$ sector: $Diff(M) \ltimes \Omega^2_{closed}(M)$. But there is much more. In section 1.2 we noted that the introduction of a generalised metric corresponds to the introduction of a pair of fields (g, B) that are a metric and a two-form and that the B defines an isomorphism between E and $TM \oplus T^*M$. This means that the B in the generalised metric can actually correspond to the B-field of Type II theories. The generalised metric seems to be able to unify two of the $NSNS$ fields in a single generalised geometric object G_{MN}. A strong argument in this direction is given by the fact that the generalised geometric formulation can also include the symmetries of these fields. We know that the symmetries of the Courant bracket are generated by the generalised Lie derivative. It turns out that the Dorfman derivative of the generalised metric encodes the action of the infinitesimal generators of $Diff(M) \ltimes \Omega^2_{closed}(M)$ on both g and B. We indeed know that we can write the generalised Lie derivative of the generalised metric as follows:

$$L_V G_{RS} = V^N \partial_N(G_{RS}) + (\partial_R V^N - \partial^N(\eta_{RQ}V^Q))G_{NS} + G_{RN}(\partial_S V^N - \partial^N(\eta_{SQ}V^Q))$$

If we then calculate, for example, this expression for the vector-vector component of G_{MN} (for which we are using the expression 1.20) we obtain (for $V = v + \lambda \in \Gamma(E) \approx \mathfrak{g}$):

$$\begin{aligned}
2L_V G_{\mu\nu} &= L_V(g - Bg^{-1}B)_{\mu\nu} \\
&= v^\rho \partial_\rho (g - Bg^{-1}B)_{\mu\nu} + \Big((\partial_\mu v^\rho, \partial_\mu \lambda_\rho) - (0, 2\partial_\rho(\frac{1}{2}\lambda_\mu))\Big) \cdot \begin{pmatrix} (g - Bg^{-1}B)_{\rho\nu} \\ (-g^{-1}B)^\rho_\nu \end{pmatrix} + \\
&\quad + \Big((g - Bg^{-1}B)_{\mu\rho}, (Bg^{-1})^\rho_\mu\Big) \cdot \left(\begin{pmatrix} \partial_\nu v^\rho \\ \partial_\nu \lambda_\rho \end{pmatrix} - \begin{pmatrix} 0 \\ \partial_\rho \lambda_\nu \end{pmatrix}\right) \\
&= v^\rho \partial_\rho (g - Bg^{-1}B)_{\mu\nu} + (\partial_\mu v^\rho)(g - Bg^{-1}B)_{\rho\nu} + (g - Bg^{-1}B)_{\mu\rho}(\partial_\nu v^\rho) + \\
&\quad - (\partial_\mu \lambda_\rho - \partial_\rho \lambda_\mu)(g^{-1}B)^\rho_\nu - (Bg^{-1})^\rho_\mu (\partial_\rho \lambda_\nu - \partial_\nu \lambda_\rho) \\
&= \big(\mathcal{L}_v(g - Bg^{-1}B)\big)_{\mu\nu} - (d\lambda)_{\mu\rho}(g^{-1}B)^\rho_\nu - (Bg^{-1})^\rho_\mu (d\lambda)_{\rho\nu}
\end{aligned}$$

One can also calculate:

$$\begin{aligned}
2L_V G_\mu^{\ \nu} &= L_V(Bg^{-1})_\mu^{\ \nu} \\
&= v^\rho \partial_\rho (Bg^{-1})_\mu^{\ \nu} + \left(\begin{pmatrix} \partial_\mu v^\rho \\ \partial_\mu \lambda_\rho \end{pmatrix} - \begin{pmatrix} 0 \\ \partial_\rho \lambda_\mu \end{pmatrix}\right)^T \begin{pmatrix} (Bg^{-1})_\rho^{\ \nu} \\ (g^{-1})^{\rho\nu} \end{pmatrix} + \begin{pmatrix} (g - Bg^{-1}B)_{\mu\rho} \\ (Bg^{-1})_\mu^{\ \rho} \end{pmatrix}^T \begin{pmatrix} 0 \\ -\partial_\rho v^\nu \end{pmatrix} \\
&= \mathcal{L}_v((Bg^{-1})_\mu^{\ \nu}) + (d\lambda)_{\mu\rho}(g^{-1})^{\rho\nu}
\end{aligned}$$

and similarly for the others. These are exactly the infinitesimal symmetry transformations one would have obtained by using the symmetry generators given in equation 2.8.

We can finally state that the generalised metric incorporates the degrees of freedom of the metric and B-field and that the symmetries of these two last objects are generated on the former by generalised Lie derivatives. At this point it is natural to ask oneself whether one can also build the (pseudo-) action making use of the generalised metric G_{MN}. Since, as we saw in chapter 1, the generalised metric induces an $O(d-1,1) \times O(1,d-1)$ structure on E, which is the generalised geometric analogue of the $O(d-1,1)$ structure used to construct general relativity, one can hope that he can also extend the procedure used to build the Einstein-Hilbert action to the case of the generalised metric. This is exactly what we are going to do in the remaining of this chapter.

To conclude this subsection let us consider the remaining bosonic fields. First, even though we said that generalised geometry seems to be able to unify two of the $NSNS$ fields in a single generalised geometric object G_{MN}, it can actually be easily extended (using a 'weighted' version of the generalised tangent bundle) to also include the dilaton and hence the whole field content of the $NSNS$ sector. We will deal with this extension in subsection 2.3.3.

Finally let us briefly deal with the RR sector. Note that we will follow here parts of [CoStWa2011]. One might wonder why we have made use of the democratic formalism when we wrote down the bosonic action in subsection 2.2.1. The reason is because, as it is known from studying the action of T-duality (see [BMZ1999, FOT2000, Ha2000, HuTo95]), the RR field strengths transform as $Spin(10,10)$ spinors and, as such, can be easily treated collectively and simmetrically in Type IIA and Type IIB with the language of generalised geometry. Before going forward, let us recall some notions about the $Spin(d,d)$ spinor representation. As it is explained in appendix B, the $O(d,d)$ Clifford algebra, i.e. $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$, can be realised on each coordinate patch $U_{(i)}$ of an open covering $\{U_{(i)}\}$ of M by indentifying spinors with weighted sums of forms $\psi_{(i)} \in \Gamma((\det T^*U_{(i)})^{-\frac{1}{2}} \otimes \bigwedge^\bullet T^*U_{(i)})$, with Clifford action:

$$X^A\Gamma_A\psi_{(i)} = i_{v_{(i)}}\psi_{(i)} + \lambda_{(i)} \wedge \psi_{(i)}$$

As it was explained in section 1.3, we defined the extesion of TM via T^*M by requiring that if $X \in \Gamma(E)$ that can be written in $E\Big|_{U_{(i)}}$ as $v_{(i)} + \lambda_{(i)}$ and in $E\Big|_{U_{(j)}}$ as $v_{(j)} + \lambda_{(j)}$, then in $E\Big|_{U_{(i)}\bigcap U_{(j)}}$ it holds:

$$v_{(i)} + \lambda_{(i)} = exp(d\Lambda_{(ij)})(v_{(j)} + \lambda_{(j)}) = v_{(j)} + \lambda_{(j)} - i_{v_{(j)}}d\Lambda_{(ij)}$$

where $exp(d\Lambda_{(ij)})$ indicates and action of a B-transform. This implied that we could introduce a collection of two-forms $B = \{B_{(i)}\}$, patched such that $B_{(i)} - B_{(j)} = d\Lambda_{(ij)}$, such that this B defined an isomorphism $\varphi_B : E \tilde{\to} TM \oplus T^*M$ as follows:

$$\varphi_B(X) = exp(-B_{(i)})(v_{(i)} + \lambda_{(i)}) = v_{(i)} + \lambda_{(i)} + i_{v_{(i)}}B_{(i)} \qquad \text{in } U_{(i)}\forall i$$

since in $U_{(i)}\bigcap U_{(j)}$ we have: $v_{(i)} + \lambda_{(i)} + i_{v_{(i)}}B_{(i)} = v_{(i)} + \lambda_{(j)} + i_{v_{(j)}}B_{(j)}$ $\forall i,j$. Now, as it is written in appendix B, the action of a B-transform (say B_{Tr}) on a spinor is such that: $B_{Tr} : (B_{(i)}, \psi_{(i)}) \mapsto exp(-B_{(i)})\psi_{(i)}$, where now $exp(B_{(i)})$

stands for:

$$exp(B_{(i)})\psi_{(i)} = (1 + B_{(i)} + \frac{1}{2}B_{(i)} \wedge B_{(i)} + \cdots) \wedge \psi_{(i)}$$

This means that, with our definition of the extended generalised tangent space E, a section of the spin bundle that on $U_{(i)}$ is $\psi_{(i)}$ and on $U_{(j)}$ is $\psi_{(j)}$ is such that, on $U_{(i)} \bigcup U_{(j)}$, it satisfies: $\psi_{(i)} = B_{Tr}(d\Lambda_{(ij)}, \psi_{(j)}) = exp(-d\Lambda_{(ij)})\psi_{(j)}$. Hence, given the collection $B = \{B_{(i)}\}$ as above, we have that

$$\psi^{(B)} := \tilde{\varphi}_B(\psi) = B_{Tr}(-B_{(i)}, \psi_{(i)}) = exp(B_{(i)})\psi_{(i)} \qquad \text{on } U_{(i)} \forall i$$

is a well defined spinor on the spin bundle on E (or, better, on the spin bundle isomorphic to the one on E via φ_B), since in $U_{(i)} \bigcap U_{(j)}$ we have: $exp(B_{(i)})\psi_{(i)} = exp(B_{(j)})\psi_{(j)}$ $\forall i, j$. Moreover, as we know from appendix B, the $Spin(d, d)$ spinor representation is not irreducible. We can project the spin representation into two irreducible spinor representations of the spin group: the ones generated by the chiral spinors. These representations are isomorphic (after we have made a choice of B) to weighted sums of even and odd forms respectively: $S^{\pm}_{\frac{1}{2}} \approx (\det T^*M)^{-\frac{1}{2}} \otimes \bigwedge^{even/odd} T^*M$.
Now let us go back to our RR fields. We can encode all the RR field strengths is a single polyform:

$$F^{(B)} := \sum_{(n=even/odd)} F_n^{(B)} = \sum_{(n=even/odd)} e^B \wedge dA_{(n-1)}$$

The patching of the $A_{(i)} = \sum_{m=odd/even} A^{(i)}_{(m)}$ on $U_{(i)} \bigcap U_{(j)}$, i.e. $A^{(i)}_{(n)} = e^{-d\Lambda_{(ij)}} \wedge A^{(j)}_{(n)} + d\tilde{\Lambda}^{(ij)}_{(n-1)}$, implies that the polyform $F_{(i)} = dA_{(i)}$ is patched as a spinor: $F_{(i)} = exp(-d\Lambda_{ij})F_{(j)}$. Hence, as generalised spinors:

$$F^{IIA/IIB} \in \Gamma(S^{\pm}_{\frac{1}{2}})$$

where the upper sign is for Type IIA and the lower for Type IIB. Furthermore the RR field strengths $F^B_{(n)}$ that appear in the action are symply F expressed in terms of the isomorphism $\tilde{\varphi}_B$: $F^{(B)} = exp(B_{(i)}) \wedge F_{(i)} = exp(B_{(i)}) \wedge \sum_{n=even/odd} dA^{(i)}_{(n-1)}$.
Unlike $F_{(i)}$, $A_{(i)}$ does not globally define a section of $S^{\pm}_{\frac{1}{2}}$ because of the additional gauge transformations $d\tilde{\Lambda}^{(ij)}_{(n-1)}$. In order to be able to 'geometrise' this additional gauge symmetry, one needs to make use of the $E_{d(d)}$ generalised geometry (see [CoStWa2013, CoStWa2013n2]).
Finally, as it is mentioned at the end of the appendix B, there is a natural $Spin(d, d)$-invariant bilinear on these spinor spaces, which is sometimes

called ‘Mukai pairing’. If ψ, ψ' are weighted poliforms with homogeneous n-form components: $\{\psi_{(n)}\}, \{\psi'_{(n)}\}$, the Mukai pairing is defined as follows:

$$\left\langle \psi, \psi' \right\rangle = \sum_{(n)} (-1)^{[(n+1)/2]} \psi_{(d-n)} \wedge \psi'_{(n)}$$

This will be used at the end of the chapter to rewrite the RR term of the action in a more generalised geometrical manner.

2.3 Geometrisation of the $NSNS$ Sector

Recall that the standard process used to describe general relativity is to find the Levi-Civita connection on TM and then to construct the Ricci tensor and scalar in order to build the action and equations of motion. In this section, we will try to replicate this process in the context of generalised geometry. We will therefore start with the notion of ‘generalised frames’ and then will try to define a ‘generalised Levi-Civita’ connection on E and to briefly discuss the possibility of constructing the generalised versions of the Riemann and Ricci tensors and of the Ricci scalar.

2.3.1 Generalised Frames

We first recall what a standard frame is. A frame is a (local) basis $\{\hat{e}_a\}$, $a = 1, \ldots, d$, of the tangent bundle TM. Clearly such an object exists in general only locally, because if there exists a frame of TM that is globally defined then the manifold is parallelisable and TM is a trivial vector bundle. A frame allows every section of TM to be written locally as: $V^\mu(p) = V^a(p)\, \hat{e}^\mu_a(p)$, $\forall\, p$ in a certain open subset $U \subset M$, where $\{V^a\}$ is a collection of scalar functions on U that represent $\forall\, p$ the coefficients of the vector V^μ with respect to the frame $\{\hat{e}_a\}$. As we mentioned at the beginning of chapter 1, the set of frames at each point in the manifold can be viewed as a $GL(d)$ principal bundle (the frame bundle). In particular any two frames $\{\hat{e}'_a\}$ and $\{\hat{e_a}\}$ are connected with a (local) $GL(d)$ transformation: $\hat{e}'_a = R_a{}^b\, \hat{e}_b$, i.e. $GL(d)$ is the structure group that relates the frames. If we introduce a Riemannian metric on M we can define the subbundle of the frame bundle defined by the frames that satisfy (orthonormal frames):

$$g_{\mu\nu}\, \hat{e}^\mu_a\, \hat{e}^\nu_b = \delta_{ab}$$

This reduces the structure group to $O(d)$, because any two orthonormal frames are connected with a (local) $O(d)$ transformation: $\hat{e}'_a = R_a{}^b\, \hat{e}_b$ such that $R^T R = \mathbb{1}$. Note that 'local transformation' here means that this $O(d)$ transformation can vary from point to point.[17]

Now consider generalised geometry. In this case we need to consider the generalised vector bundle $E \approx TM \oplus T^*M$. A generalised frame for E will be a (local) basis for it: $\{\hat{E}_A\}$, where we have $A = 1, \ldots, 2d$, since now the dimension of the vector bundle is twice the dimension of M. Recall that in generalised geometry we have a natural metric. The generalised geometric analogue for the standard $GL(d)$ bundle is therefore an $O(d,d)$ bundle composed of frames that satisfy: $\eta_{MN}\, \hat{E}^M_A \hat{E}^N_B = \eta_{AB}$, with frames connected with local $O(d,d)$ transformations. Finally the generalised geometric analogue of the standard $O(d)$ structure is the subbundle of the $O(d,d)$-frame bundle composed of 'doubly orthonormal' E-frames:

$$\begin{cases} \hat{E}^M_A \eta_{MN} \hat{E}^N_B = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \\ \hat{E}^M_A G_{MN} \hat{E}^N_B = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \end{cases}$$

where we also imposed to the frames to diagonalise both the metrics. It is clear that this further request does not modify the structure of the frame subbundle that will therefore have an $O(d) \times O(d)$ structure group (see Prop. 1.2.2).

Proposition 2.3.1. *A solution of the former system of equations is:*

$$\hat{E}^M_A = \begin{pmatrix} g^{-1}e_+ & -g^{-1}e_- \\ Bg^{-1}e_+ + e_+ & -Bg^{-1}e_- + e_- \end{pmatrix} = \begin{pmatrix} \hat{e}_+ & -\hat{e}_- \\ B\hat{e}_+ + e_+ & -B\hat{e}_- + e_- \end{pmatrix} \tag{2.9}$$

where $\{\hat{e}_{\pm\, a}\}$ are frames for the metric g and $\{e^a_\pm\}$ are their dual frames: $\hat{e}^\mu_{\pm\, a} e^b_{\pm\, \mu} = \delta^b_a$.[18] Moreover if the metric g is Riemannian the solution given in equation 2.9 is the only solution of the system.[19]

Proof. Instead of simply checking whether the solution given in 2.9 is a solution of the system we will give a constructive proof, that will also serve to

[17] This means that if our metric has Lorentzian signature the description of the tangent bundle via orthonormal frames explicitly realise the symmetry under local Lorentz transformations, typical of general relativity.

[18] Note that both e^a_μ and $e_{a\,\mu}$ are one-forms and that the Latin indices are risen and lowered with δ^{ab} and δ_{ab} in the Riemannian case and with the Minkowski metric in the Lorentzian case.

[19] See also section 3.2.1 for another proof of the Riemannian case.

show the second statement of the proposition and to clarify the sign of B in the isomorphism $E \approx TM \oplus T^*M$ induced by G. Recall that we can write:

$$(e^{-B})^T G_0 e^{-B} := \begin{pmatrix} \mathbb{1} & B \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ -B & \mathbb{1} \end{pmatrix} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix} = G$$

and that the $(-)B$-transform e^{-B} is an $O(d,d)$ transformation, i.e. it preserves η. Defining $\epsilon := e^{-B}\hat{E}$ we can rewrite the system of equations as:

$$\begin{cases} (\hat{E})^T (e^{-B})^T \eta e^{-B} (\hat{E}) = \epsilon^T \eta \epsilon = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \\ (\hat{E})^T (e^{-B})^T G_0 e^{-B} (\hat{E}) = \epsilon^T G_0 \epsilon = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \end{cases}$$

Writing $\epsilon = \sqrt{2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we get the following system of equations:

$$\begin{cases} i) \begin{pmatrix} a^T & c^T \\ b^T & d^T \end{pmatrix} \begin{pmatrix} c & d \\ a & b \end{pmatrix} = \begin{pmatrix} a^T c + c^T a & a^T d + c^T b \\ b^T c + d^T a & b^T d + d^T b \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \\ \\ ii) \begin{pmatrix} a^T & c^T \\ b^T & d^T \end{pmatrix} \begin{pmatrix} ga & gb \\ g^{-1}c & g^{-1}d \end{pmatrix} = \begin{pmatrix} a^T ga + c^T g^{-1}c & a^T gb + c^T g^{-1} d \\ b^T ga + d^T g^{-1} c & b^T gb + d^T g^{-1} d \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \end{cases}$$

Let us consider equations $i)_{(11)}$ and $ii)_{(11)}$. We can write them as:

$$\begin{aligned} &a^T g g^{-1} c + c^T g^{-1} g a = \mathbb{1} \\ &a^T g a + c^T g^{-1} c = \mathbb{1} \end{aligned}$$

Then subtracting the second from the first we get:

$$\begin{aligned} &a^T g(g^{-1}c - a) + c^T g^{-1}(ga - c) = 0 \\ &\Leftrightarrow a^T (c - ga) - c^T g^{-1}(c - ga) = 0 \\ &\Leftrightarrow (c^T g^{-1} - a^T)(c - ga) = 0 \\ &\Leftrightarrow (g^{-1}c - a)^T g (g^{-1}c - a) = 0 \end{aligned}$$

Clearly a solution of this equation is given by: $a = g^{-1}c$. Moreover we note that if g is Riemannian, i.e. positive definite, this has to be the solution. Similarly from equations $i)_{(22)}$ and $ii)_{(22)}$ we have

$$\begin{aligned} &b^T g g^{-1} d + d^T g^{-1} g b = -\mathbb{1} \quad \text{and} \quad b^T g b + d^T g^{-1} d = \mathbb{1} \\ &\Rightarrow b^T g(g^{-1}d + b) + d^T g^{-1}(gb + d) = 0 \\ &\Leftrightarrow b^T g(g^{-1}d + b) + d^T (b + g^{-1}d) = (b^T g + d^T)(b + g^{-1}d) = 0 \\ &\Leftrightarrow (b^T + d^T g^{-1}) g (b + g^{-1} d) = 0 \end{aligned}$$

and this has again $b = -g^{-1}d$ as a solution, which *has* to be the solution in the case g is positive definite. Note that the relations $a = g^{-1}c$ and

$b = -g^{-1}d$ make the systems $i)$ and $ii)$ equivalent. We can then focus only on $i)$. Summing and subtracting $i)_{(12)}$ with $i)_{(11)}$ we get (using $a = g^{-1}c$ and $b = -g^{-1}d$):

$$a^T(c \pm d) + c^T(a \pm b) = \mathbb{1} \;\Leftrightarrow\; a^T g(a \mp b) + c^T(a \pm b) = \mathbb{1}$$
$$\Leftrightarrow c^T(a \mp b + a \pm b) = \mathbb{1} \;\Leftrightarrow\; c^T a = \frac{\mathbb{1}}{2}$$

And in a similar way with $i)_{(22)}$ instead of $i)_{(11)}$:

$$(d^T \pm c^T)b + (b^T \pm a^T)d = -\mathbb{1} \;\Leftrightarrow\; (-b^T \pm a^T)gb + (b^T \pm a^T)d = -\mathbb{1}$$
$$\Leftrightarrow (b^T \mp a^T)d + (b^T \pm a^T)d = -\mathbb{1} \;\Leftrightarrow\; b^T d = \frac{-\mathbb{1}}{2}$$

We have therefore reduced the system of equations to

$$\begin{cases} c^T g^{-1} c = \frac{\mathbb{1}}{2} \\ d^T g^{-1} d = \frac{\mathbb{1}}{2} \end{cases}$$

together with the constraints $a = g^{-1}c$ and $b = -g^{-1}d$. The former system means that c and d are respectively proportional to the matrices of the dual of some orthonormal frames $\{\hat{e}_+\}$ and $\{\hat{e}_-\}$ for g, i.e. $(c)_{a\,\mu} = \frac{1}{\sqrt{2}} e_{+\,a\,\mu}$ and $(d)_{b\,\nu} = \frac{1}{\sqrt{2}} e_{-\,b\,\nu}$. This implies that we can write: $\epsilon = \left(\begin{smallmatrix} g^{-1}e_+ & -g^{-1}e_- \\ e_+ & e_- \end{smallmatrix}\right)$ and:

$$\hat{E} = e^B \epsilon = \left(\begin{smallmatrix} \mathbb{1} & 0 \\ B & \mathbb{1} \end{smallmatrix}\right)\left(\begin{smallmatrix} g^{-1}e_+ & -g^{-1}e_- \\ e_+ & e_- \end{smallmatrix}\right) = \left(\begin{smallmatrix} g^{-1}e_+ & -g^{-1}e_- \\ Bg^{-1}e_+ + e_+ & -Bg^{-1}e_- + e_- \end{smallmatrix}\right)$$

□

We have now found an explicit expression for a collection $\{\hat{E}_A\}$ of $2d$ generalised vectors that (locally) span the generalised tangent bundle and that are 'doubly orthonormal' in the sense expressed before. From the structure of the solution outlined in proposition 2.3.1 it is useful to split the capital index A in two sub-indices $a, \bar{a}$:

$$\begin{cases} A = a & \text{for } A = 1, \ldots, d \\ A = d + \bar{a} & \text{for } A = d+1, \ldots, 2d \end{cases}$$

We can then divide $\{\hat{E}_A\}$ in $\{\hat{E}^+_a\} \cup \{\hat{E}^-_{\bar{a}}\}$, where $\{\hat{E}^+_a\}$ are the orthonormal frames that correspond to the first 'block-column' of $\{\hat{E}_A\}$ in 2.9 and $\{\hat{E}^-_{\bar{a}}\}$ to the second. In other words can write (where we redefined $e_- \mapsto -e_-$ without losing any generality):

$$\hat{E}^+_a = \hat{e}^+_a + e^+_a - i_{\hat{e}^+_a} B \tag{2.10}$$
$$\hat{E}^-_{\bar{a}} = \hat{e}^-_{\bar{a}} - e^-_{\bar{a}} - i_{\hat{e}^-_{\bar{a}}} B \tag{2.11}$$

and from the proof of proposition 2.3.1 they satisfy:

$$\begin{array}{ll}\eta(\hat{E}^+_a, \hat{E}^-_{\bar{b}}) = 0 & \qquad G(\hat{E}^+_a, \hat{E}^-_{\bar{b}}) = 0 \\ \eta(\hat{E}^+_a, \hat{E}^+_b) = \eta_{ab} & \qquad G(\hat{E}^+_a, \hat{E}^+_b) = \eta_{ab} \\ \eta(\hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}}) = -\eta_{ab} & \qquad G(\hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}}) = \eta_{ab}\end{array}$$

where here we have chosen g to be of Lorentzian signature and where η_{ab} is the Minkowski metric. Moreover the previous system of equations is still satisfied if we apply separate *local* Lorentz transformations to the $\{\hat{E}^+_a\}$ and $\{\hat{E}^-_{\bar{a}}\}$, i.e. the set $\{\hat{E}'^+_a\} \cup \{\hat{E}'^-_{\bar{a}}\}$ given by:

$$\begin{cases} \hat{E}'^+_a = \Lambda^+_a{}^b \hat{E}^+_b & \text{s.t. } \Lambda^+ \in O(d-1,1) \\ \hat{E}'^-_{\bar{a}} = \Lambda^-_{\bar{a}}{}^{\bar{b}} \hat{E}^-_{\bar{b}} & \text{s.t. } \Lambda^- \in O(1,d-1) \end{cases}$$

is still a double orthonormal frame. This is another way to state that the structure group induced by the pair of metrics (η, G) in E is $O(d-1,1) \times O(1,d-1)$, as we already know from proposition 1.2.2.
The two collection of sections $\{\hat{E}^+_a\}$ and $\{\hat{E}^-_{\bar{a}}\}$ generate two subbundles C_+ and C_- of E; since $\eta(\hat{E}^+_a, \hat{E}^-_{\bar{b}}) = 0$ and $G(\hat{E}^+_a, \hat{E}^-_{\bar{b}}) = 0$ $\forall a, \bar{b} = 1, \ldots, d$, we have that $E = C_+ \oplus C_-$. We could also introduce a matrix $\mathbb{P}$ (like we have actually done in chapter 1) that satisfies $\mathbb{P}^2 = \mathbb{1}$, such that $C_\pm = \frac{1}{2}(\mathbb{1} \pm \mathbb{P})E =: \mathbb{P}_\pm E$ and then find the subgroup of $O(d,d)$ that separately preserves these two subbundles. We would again find:

$$O(p,q) \times O(q,p) \subset O(d,d) \qquad \text{with}\, p + q = d$$

The subbundles $C_\pm$, the projectors $\mathbb{P}_\pm$, the generalised metric G and the frames $\hat{E}^\pm_{a/\bar{a}}$ are all equivalent manners to define a metric g and a B field, i.e. to introduce in the geometric theory (part of) the field content of the $NSNS$ sector of the Type II theories.

Remark 4. We would like to stress that in the remaining of the chapter we will use a generalised metric that has the two form component equal to minus the one given in 1.20. In other words we will use the following generalised metric:

$$G = \frac{1}{2}\begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix} \tag{2.12}$$

that, together with η, admits the following generalised frames (according to proposition 2.3.1):

$$\hat{E}^+_a = \hat{e}^+_a + e^+_a + i_{\hat{e}^+_a} B \tag{2.13}$$

$$\hat{E}^-_{\bar{a}} = \hat{e}^-_{\bar{a}} - e^-_{\bar{a}} + i_{\hat{e}^-_{\bar{a}}} B \tag{2.14}$$

This means that a section X of E will be represented by the following section of $TM \oplus T^*M$: $X = v + \lambda + i_v B$.

2.3.2 Generalised Levi-Civita Connection

We recall that an affine connection on TM is an operator $\nabla : TM \times TM \to TM$ that is $\mathcal{C}^\infty(M)$-linear in the first component, $\mathbb{R}$-linear in the second component and that satisfies a Leibniz property: $\nabla(X, fY) \equiv \nabla_X(fY) = X(f)Y + f\nabla_X Y$.[20] Since ∇ is tensorial in the first component, we can equivalently define it in the following manner: $\nabla : TM \to T^*M \otimes TM \approx \Lambda^1(M) \otimes TM$ such that $\nabla(\alpha X + \beta Y) = \alpha\nabla X + \beta\nabla Y$ and $\nabla(fX) = f\nabla X + df \otimes X$, where $\alpha, \beta \in \mathbb{R}, f \in \mathcal{C}^\infty(M)$ and $X, Y \in \Gamma(TM)$. So it can be viewed as a map that associates a one-form with values in the tangent space with each section of TM. From its definition it can be shown that an affine connection is a local operator, i.e. its value in one point depends only on the values its arguments assume in a neighbourhood of that point[21]. We can therefore define it through its expressions in the coordinate bases of an atlas. In this way $\{\nabla_{\partial_\mu}(\partial_\nu) =: \Gamma^\rho_{\mu\nu}\partial_\rho\}$, with $\Gamma^\rho_{\mu\nu}$ called Christoffel symbols, completely determines the connection. We can also give a local expression for our second definition of connection. This is usually done by means of a non-coordinate basis (i.e. a frame) $\{\hat{e}_a\}$ of TM. It is then written: $\nabla\hat{e}_a = \omega^b{}_a\hat{e}_b$, where now the omegas are one-forms, called ‘connection one-forms’, and the connection expressed in this way is called, in the physics literature, ‘spin connection’. We can also easily find the relation between the Christoffel symbols and the connection one forms. We have:

$$\begin{aligned} i_{\partial_\mu}\nabla\hat{e}_a &= (i_{\partial_\mu}\omega^b{}_a)\hat{e}_b = \omega_\mu{}^b{}_a\hat{e}^\rho_b\partial_\rho \\ &= \nabla_{\partial\mu}(\hat{e}^\nu_a\partial_\nu) = \partial_\mu(\hat{e}^\rho_a)\partial_\rho + \hat{e}^\nu_a\Gamma^\rho_{\mu\nu}\partial_\rho \end{aligned}$$

and so we have found:

$$\omega_\mu{}^b{}_a\hat{e}^\rho_b = \partial_\mu(\hat{e}^\rho_a) + \hat{e}^\nu_a\Gamma^\rho_{\mu\nu} \tag{2.15}$$

which is sometimes referred to as ‘tetrad postulate’. Now let us go back to generalised geometry.

Definition 2.1. A generalised (affine) connection is a map

$$\begin{aligned} D:\ E &\longrightarrow E^* \otimes E \\ V^M &\longmapsto D_N V^M \end{aligned}$$

where we have expressed its image in coordinates, that satisfies the two following properties:

[20]This definition appeared for the first time in the paper [No1954].
[21]It is actually much more ‘locally defined’ than this, see [AbTo2011]

1. $D(\alpha X+\beta Y)=\alpha DX+\beta DY \;\; \forall\, \alpha,\beta\in\mathbb{R}, X,Y\in\Gamma(E);$ ($\mathbb{R}$-linearity)

2. $D(fX)=fDX+\begin{pmatrix} df\\ 0\end{pmatrix}\otimes X \;\; \forall\, f\in\mathcal{C}^\infty(M), X\in\Gamma(E).$ (Leibniz rule)

Note that in definition 2.1 we used the fact that since a section of E is of the form $X=\binom{v}{\lambda}$, with $v\in\Gamma(TM)$ and $\lambda\in\Gamma(T^*M)$, then an element of $\Gamma(E^*)$ is of the form $V^*=\binom{v^*}{\lambda^*}$, with $v^*\in\Gamma(T^*M)$ and $\lambda^*\in\Gamma(TM)$. So $\binom{df}{0}$ is consistently an element of E^* and in particular it is the image of $df\in\Gamma(T^*M)$ under the pull-back of the 'anchor map' onto the tangent space $\pi: E\to TM$, i.e. of $\pi^*: T^*M\to E^*$.
If we indicate with ξ_M a coordinate basis on E, i.e. (with $\mu,\tilde{\mu}=1,\dots,d$):

$$\begin{cases} \xi_M=\partial_\mu+i_{\partial_\mu}B & \text{if } M=\mu\\ \xi_M=dx^{\tilde{\mu}} & \text{if } M=d+\tilde{\mu}\end{cases}$$

then we define $D(\xi_M,\xi_N)\equiv D_{\xi_M}\xi_N\equiv D_M\xi_N =: \Gamma_M{}^R{}_N\xi_R$. Therefore, if $V\in\Gamma(E)$, $V=V^M\xi_M$, $V=v+\lambda$, we get

$$i_{\xi_M}D(V^N\xi_N)=i_{\xi_M}(V^N D\xi_N+\begin{pmatrix} dV^N\\ 0\end{pmatrix}\xi_N)=V^N\Gamma_M{}^R{}_N\xi_R+i_{\partial_\mu}(dV^R)\xi_R$$

This implies that: $D_MV^N=\partial_MV^N+\Gamma_M{}^N{}_PV^P$ (recall that $\partial_M=\binom{\partial_\mu}{0}$). We can also write this expression more explicitly:

$$\begin{cases} i)\ D_\mu V^N=\begin{pmatrix}\partial_\mu v^\nu+\Gamma_\mu{}^\nu{}_\rho v^\rho+\Gamma_\mu{}^{\nu\rho}\lambda_\rho\\ \partial_\mu\lambda_\nu+\Gamma_{\mu\nu\rho}v^\rho+\Gamma_{\mu\nu}{}^\rho\lambda_\rho\end{pmatrix}\\ ii)\ D^\mu V^N=\begin{pmatrix}0+\Gamma^{\mu\nu}{}_\rho v^\rho+\Gamma^{\mu\nu\rho}\lambda_\rho\\ 0+\Gamma^\mu{}_{\nu\rho}v^\rho+\Gamma^\mu{}_\nu{}^\rho\lambda_\rho\end{pmatrix}\end{cases}$$

Note that the gammas that appear in the relation $ii)$ are tensors (as opposed to the case of ordinary Christoffel symbols) because the expression is already tensorial since there are no partial derivatives.
Until now we have only talked about connections without mentioning any metric. We know that in the case of ordinary Riemannian geometry there is a unique connection that is compatible with the metric and torsion-free: the Levi-Civita connection. We would like to develop an analogous concept in the framework of generalised geometry. In ordinary geometry metric compatibility can be expressed as: $\nabla_\mu g=0$. This implies, for example, that orthogonal vectors at one point on M will remain orthogonal if parallel transported along a path in M (see e.g. [AbTo2011]), and, most importantly, the

following *differential condition*: $\partial_\mu[g_{\nu\rho}] := \nabla_\mu[g(\partial_\nu, \partial_\rho)] = \Gamma_{\mu\nu}{}^{\sigma} g_{\sigma\rho} + \Gamma_{\mu\rho}{}^{\sigma} g_{\nu\sigma}$. The ordinary torsion can be expressed in terms of the Chrstoffel symbols as: $T^\nu_{\mu\rho} = (\Gamma^\nu_{\mu\rho} - \Gamma^\nu_{\rho\mu})$, that is a tensorial object. Clearly imposing it to be zero corresponds to have the symmetry of the Christoffel symbols in their lower two indices (*algebraic condition*). We can also express the torsion with a more coordinate-invariant formula. Indeed:

$$\begin{aligned} v^\lambda \nabla_\lambda w^\mu - & w^\lambda \nabla_\lambda v^\mu - [v,w]^\mu = \\ &= v^\lambda(\partial_\lambda w^\mu + \Gamma^\mu_{\lambda\rho} w^\rho) - w^\lambda(\partial_\lambda v^\mu + \Gamma^\mu_{\lambda\rho} v^\rho) - (v^\lambda \partial_\lambda w^\mu - w^\lambda \partial_\lambda v^\mu) \\ &= \Gamma^\mu_{\lambda\rho} v^\lambda w^\rho - \Gamma^\mu_{\lambda\rho} w^\lambda v^\rho = T^\mu_{\lambda\rho} v^\lambda w^\rho \end{aligned}$$

Indicating with $[\,,\,]_\nabla$ the Lie bracket calculated using covariant derivatives (∇_μ) instead of partial derivatives (∂_μ), we can now write: $[v,w]^\mu_\nabla - [v,w]^\mu =: T^\mu_{\lambda\rho} v^\lambda w^\rho$. This definition extends also on general tensors:

$$(\mathcal{L}^\nabla_v - \mathcal{L}_v)\alpha^{\mu_1,\dots,\mu_p}_{\nu_1,\dots,\nu_q} = T(v)^{\mu_1}_{\rho}\,\alpha^{\rho,\dots,\mu_p}_{\nu_1,\dots,\nu_q} + \dots - T(v)^{\rho}_{\nu_1}\,\alpha^{\mu_1,\dots,\mu_p}_{\rho,\dots,\nu_p} - \dots - T(v)^{\rho}_{\nu_p}\,\alpha^{\mu_1,\dots,\mu_p}_{\nu_1,\dots,\rho}$$

where $T(v)^\mu_\nu := v^\lambda T^\mu_{\lambda\nu}$ and $\mathcal{L}^\nabla$ is the Lie derivative calculated using the covariant derivatives instead of partial derivatives. We will therefore formally write: $(\mathcal{L}^\nabla_v - \mathcal{L}_v) = T(v)$.
In generalised geometry the metric compatibility condition can be translated in:

$$D_M \eta = 0 \quad \text{and} \quad D_M G = 0 \tag{2.16}$$

In particular the first of these two equations implies that the generalised indices can be freely lowered and raised also when they are of tensors acted upon with generalised covariant derivatives. We can also define a generalised torsion tensor in a manner that is akin to the one we have just described in ordinary geometry. If L indicates the generalised Lie derivative,[22] we will then define the generalised torsion as:

$$T(V) := L^D_V - L_V \tag{2.17}$$

where L^D means that in the covariant expression of the generalised Lie derivative, given in 1.15, we are using the generalised covariant derivatives (D_M) instead of the partial derivatives (∂_M). We can also find an expression for the generalised torsion in terms of the generalised Christoffel symbols. We said that $T_{MN}{}^{P} V^M W^N := (L_V W)^P_D - (L_V W)^P$. Now:

$$\begin{aligned} (L_V W)^P_D &= V^N D_N W^P + (D^P V^N - D^N V^P) W_N \\ &= V^N(\partial_N W^P + \Gamma_N{}^P{}_R W^R) + (\partial^P V^N + \Gamma^{PN}{}_R V^R - \partial^N V^P - \Gamma^{NP}{}_R V^R) W_N \end{aligned}$$

[22] Recall that for $X = v + \lambda, Y = w + \mu \in \Gamma(E)$ we have $L_X Y = [v,w] + \mathcal{L}_v \mu - i_w(d\lambda)$ or, covariantly, $\left(L_X Y\right)^M = X^N \partial_N Y^M + \left(\partial^M X^N - \partial^N X^M\right) Y_N$

and since $(L_V W)^P = V^N \partial_N W^P + (\partial^P V^N - \partial^N V^P) W_N$ all the terms with partial derivatives cancel out and we have that:

$$T_{MN}^{\ \ \ P} V^M W^N = V^M W^N \Gamma_M{}^P{}_N + (\Gamma^{PM}{}_N - \Gamma^{MP}{}_N) V^N W_M \tag{2.18}$$

Let us consider one of the conditions of metric compatibility: $D_M \eta = 0$. This implies that:

$$\partial_M \eta^{NP} + \Gamma_M{}^N{}_Q \eta^{QP} + \Gamma_M{}^P{}_Q \eta^{NQ} = 0 \quad \Leftrightarrow \quad \Gamma_M{}^{NP} + \Gamma_M{}^{PN} = 0 \tag{2.19}$$

because $\partial_M \eta^{NP} = 0$, since the matrix of η is a constant. From equations 2.18 and 2.19 we can then write:

$$\begin{aligned} T^{MPN} &= \Gamma^{MPN} + \Gamma^{PNM} - \Gamma^{NPM} \\ &= \Gamma^{MPN} + \Gamma^{PNM} + \Gamma^{NMP} = 3\Gamma^{[MPN]} \end{aligned}$$

This argument shows that $T \in \Lambda^3 E \approx \Lambda^3(TM \oplus T^*M)$. This is somewhat different from the standard case where we have $T_{\text{std}} \in TM \otimes \Lambda^2(T^*M)$. In fact we have:

$$T^{MNP} = \{T^{\mu\nu\rho}, T^{\mu\nu}{}_\rho, T^\mu{}_{\nu\rho}, T_{\mu\nu\rho}\}$$

where $T^{\mu\nu\rho} \in \Lambda^3(TM), T^{\mu\nu}{}_\rho \in \Lambda^2(TM) \otimes T^*M, T^\mu{}_{\nu\rho} \in TM \otimes \Lambda^2(T^*M)$, which is the same tensor as the conventional torsion, and $T_{\mu\nu\rho} \in \Lambda^3(T^*M)$, which has the same tensorial properties of $H = dB$.

We have shown what two of the three required conditions on D, namely $D\eta = 0, T = 0$, imply in terms of generalised Christoffel symbols (the last one remaining is a differential condition that involves the generalised metric). Next, we would like to see explicitly whether the conditions for a generalised Levi-Civita connection determine it completely or not. In order to answer this question we will use the definition of a connection in terms of a spin connection. Consider for a moment the case of ordinary geometry. If $\{\hat{e}_a\}$ is a frame for g, then the metric compatibility condition implies that:

$$\begin{aligned} \nabla_\mu(g(\hat{e}_a, \hat{e}_b)) &= \nabla_\mu \eta_{ab} = 0 \\ &= (\nabla_\mu g)(\hat{e}_a, \hat{e}_b) + g(\omega_\mu{}^c{}_a \hat{e}_c, \hat{e}_b) + g(\hat{e}_a, \omega_\mu{}^c{}_b \hat{e}_c) = \omega_\mu{}^c{}_a \eta_{cb} + \omega_\mu{}^c{}_b \eta_{ac} \\ &= \omega_{\mu ba} + \omega_{\mu ab} \end{aligned}$$

Note that the metric compatibility implies an *algebraic* condition for the spin connection, while it implies a differential condition if one uses the Christoffel symbols. Moreover the fact that the matrices $(\omega_\mu)^b{}_a$, once the upper index is lowered by means of the Minkowski metric, are antisymmetric means that they are infinitesimal generators for the Lorentz group. The condition of

the connection to be torsionless implies instead that: $[\hat{e}_a, \hat{e}_b]^\nabla - [\hat{e}_a, \hat{e}_b] = 0$, which is a *differential* condition, whilst it was algebraic in the case of the Christoffel symbols.
In the case of generalised geometry we can consider the $E = C_+ \oplus C_-$ split frames, i.e. $\hat{E}_A = \{\hat{E}_a^+\} \cup \{\hat{E}_{\bar{a}}^-\}$. We will define a generalised spin-connection as follows: $D_M \hat{E}_A := \Omega_M{}^B{}_A \hat{E}_B$. With exactly the same procedure as before, we obtain that the generalised covariant derivative of a vector expressed in generalised frame indices can be written as:

$$D_M X^A = \partial_M X^A + \Omega_M{}^A{}_B X^B$$

We can write the former equation even more explicitly by decomposing the generalised vector under the splitting $E = C_+ \oplus C_-$, i.e. $X^A = \left(\begin{smallmatrix} X_+^a \\ X_-^{\bar{a}} \end{smallmatrix}\right)$, where $X_+^a \in C_+$ and $X_-^{\bar{a}} \in C_-$:[23]

$$\begin{cases} D_M X_+^a = \partial_M X_+^a + \Omega_M{}^a{}_b X_+^b + \Omega_M{}^a{}_{\bar{b}} X_-^{\bar{b}} \\ D_M X_-^{\bar{a}} = \partial_M X_-^{\bar{a}} + \Omega_M{}^{\bar{a}}{}_b X_+^b + \Omega_M{}^{\bar{a}}{}_{\bar{b}} X_-^{\bar{b}} \end{cases}$$

In the remaining of the section we would like to explicitly solve the constraints for a generalised Levi-Civita connection, expressed as a generalised spin-connection. We first deal with the metric compatibility. We know from subsection 2.3.1 that we can introduce $E = C_+ \oplus C_-$-split frames $\{\hat{E}_A\}$ such that $G(\hat{E}_A, \hat{E}_B) = \left(\begin{smallmatrix} \eta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{smallmatrix}\right) =: G_{AB}$ and $\eta(\hat{E}_A, \hat{E}_B) = \left(\begin{smallmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{smallmatrix}\right) =: \eta_{AB}$. Note that η_{AB} and G_{AB} are *constant* matrices. Metric compatibility, $D_M \eta = 0 = D_M G$, then implies:

$$\begin{aligned} D_M(G(\hat{E}_A, \hat{E}_B)) &= D_M G_{AB} = 0 \\ &= G(\Omega_M{}^C{}_A \hat{E}_C, \hat{E}_B) + G(\hat{E}_A, \Omega_M{}^C{}_B \hat{E}_C) = \Omega_M{}^C{}_A G_{CB} + G_{AC} \Omega_M{}^C{}_B \\ &= \begin{pmatrix} \Omega_M{}^c{}_a & \Omega_M{}^{\bar{c}}{}_a \\ \Omega_M{}^c{}_{\bar{a}} & \Omega_M{}^{\bar{c}}{}_{\bar{a}} \end{pmatrix} \begin{pmatrix} \eta_{cb} & 0 \\ 0 & \eta_{\bar{c}\bar{b}} \end{pmatrix} + \begin{pmatrix} \eta_{ac} & 0 \\ 0 & \eta_{\bar{a}\bar{c}} \end{pmatrix} \begin{pmatrix} \Omega_M{}^c{}_b & \Omega_M{}^c{}_{\bar{b}} \\ \Omega_M{}^{\bar{c}}{}_b & \Omega_M{}^{\bar{c}}{}_{\bar{b}} \end{pmatrix} \\ &= \begin{pmatrix} \Omega_{Mba} & \Omega_{M\bar{b}a} \\ \Omega_{Mb\bar{a}} & \Omega_{M\bar{b}\bar{a}} \end{pmatrix} + \begin{pmatrix} \Omega_{Mab} & \Omega_{Ma\bar{b}} \\ \Omega_{M\bar{a}b} & \Omega_{M\bar{a}\bar{b}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned} \tag{2.20}$$

and, considering that the case for η gives rise to the same calculation with

[23] $X = X^A \hat{E}_A = X_+^a \hat{E}_a^+ + X_-^{\bar{a}} \hat{E}_{\bar{a}}^-$.

$\eta_{\bar{a}\bar{b}} \mapsto -\eta_{\bar{a}\bar{b}}$, also:

$$\begin{aligned}
0 &= D_M(\eta(\hat{E}_A, \hat{E}_B)) = \\
&= \begin{pmatrix} \Omega_M{}^c{}_a & \Omega_M{}^{\bar{c}}{}_a \\ \Omega_M{}^c{}_{\bar{a}} & \Omega_M{}^{\bar{c}}{}_{\bar{a}} \end{pmatrix} \begin{pmatrix} \eta_{cb} & 0 \\ 0 & -\eta_{\bar{c}\bar{b}} \end{pmatrix} + \begin{pmatrix} \eta_{ac} & 0 \\ 0 & -\eta_{\bar{a}\bar{c}} \end{pmatrix} \begin{pmatrix} \Omega_M{}^c{}_b & \Omega_M{}^c{}_{\bar{b}} \\ \Omega_M{}^{\bar{c}}{}_b & \Omega_M{}^{\bar{c}}{}_{\bar{b}} \end{pmatrix} \\
&= \begin{pmatrix} \Omega_{Mba} & -\Omega_{M\bar{b}a} \\ \Omega_{Mb\bar{a}} & -\Omega_{M\bar{b}\bar{a}} \end{pmatrix} + \begin{pmatrix} \Omega_{Mab} & \Omega_{Ma\bar{b}} \\ -\Omega_{M\bar{a}b} & -\Omega_{M\bar{a}\bar{b}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned} \tag{2.21}$$

These relations show that the 'mixed components' of the spin connection vanish, since we have: $-\Omega_{M\bar{b}a} = \Omega_{Ma\bar{b}} = \Omega_{M\bar{b}a}$ and $-\Omega_{M\bar{a}b} = \Omega_{Mb\bar{a}} = \Omega_{M\bar{a}b}$. Moreover the diagonal submatrices of Ω_{MAB} are antisymmetric in the frame indices: $\Omega_{Mab} = -\Omega_{Mba}$ and $\Omega_{M\bar{a}\bar{b}} = -\Omega_{M\bar{b}\bar{a}}$; note that these are algebraic conditions and that they indicate that the Ω_M are generators of the Lorentz group.

Let us now consider the torsionless constraint on the generalise spin-connection. This constraint corresponds to the following statement:

$$T(\hat{E}_A, \hat{E}_B) \equiv L^D_{\hat{E}_A}\hat{E}_B - L_{\hat{E}_A}\hat{E}_B = 0 \qquad \forall A, B$$

In particular we want to work out the conditions on Ω_{MAB} that are imposed by the vanishing of the generalised torsion. Recalling the explicit covariant expression for the generalised Lie derivative, the first term reads off as (note that by definition $\hat{E}^N_A E_{B\,N} = \eta_{AB}$):

$$\begin{aligned}
L^D_{\hat{E}_A}\hat{E}_B &= \hat{E}^N_A D_N \hat{E}^M_B + (D^M \hat{E}^N_A - D^N \hat{E}^M_A) E_{B\,N} \\
&= \hat{E}^N_A \Omega_N{}^C{}_B \hat{E}^M_C + (\Omega^{MC}{}_A \hat{E}^N_C - \Omega^{NC}{}_A \hat{E}^M_C) E_{N\,B} \\
&= \Omega_A{}^C{}_B \hat{E}^M_C + \Omega^{MC}{}_A \eta_{CB} - \Omega_B{}^C{}_A \hat{E}^M_C
\end{aligned}$$

Since the natural metric η is non-degenerate we will use the fact that

$$T \equiv 0 \Leftrightarrow \eta(T_{AB}, \hat{E}_D) = 0 \qquad \forall\, \hat{E}_D, \forall\, A, B$$

Let us consider for now $\eta(L^D_{\hat{E}_A}\hat{E}_B, \hat{E}_D)$. We will consider the different possible split values ($\{A\} = \{a\} \cup \{\bar{a}\}$) of the generalised indices separately. Note that the off-diagonal submatrices of η_{AB} are zero.[24]

1. $\left(\scriptstyle A=a, B=b, D=d\right)$

$$\begin{aligned}
\eta(L^D_{\hat{E}_a}\hat{E}_b, \hat{E}_d) &= \Omega_a{}^C{}_b \hat{E}^M_C E_{M\,d} + \Omega^{MC}{}_a \eta_{Cb} E_{M\,d} - \Omega_b{}^C{}_a \hat{E}^M_C E_{M\,d} \\
&= \Omega_a{}^c{}_b \eta_{cd} + \Omega_d{}^c{}_a \eta_{cb} - \Omega_b{}^c{}_a \eta_{cd} \\
&= \Omega_{adb} + \Omega_{dba} - \Omega_{bda} = \Omega_{adb} + \Omega_{dba} + \Omega_{bad} = 3\Omega_{[adb]}
\end{aligned}$$

[24]Note that we will use an abuse of notation and indicate with $\hat{E}_a$ the C_+-split frames (instead of $\hat{E}^+_a$) and with $\hat{E}_{\bar{a}}$ the C_--split frames (instead of $\hat{E}^-_{\bar{a}}$)

2. $\left(_{A=\bar{a},B=\bar{b},D=\bar{d}}\right)$

$$\begin{aligned}\eta(L^D_{\hat{E}_{\bar{a}}}\hat{E}_{\bar{b}},\hat{E}_{\bar{d}}) &= \Omega_{\bar{a}}{}^{C}{}_{\bar{b}}\hat{E}^M_C E_{M\,\bar{d}} + \Omega^{MC}{}_{\bar{a}}\eta_{C\bar{b}}E_{M\,\bar{d}} - \Omega_{\bar{b}}{}^{C}{}_{\bar{a}}\hat{E}^M_C E_{M\,\bar{d}} \\ &= -\Omega_{\bar{a}}{}^{\bar{c}}{}_{\bar{b}}\eta_{\bar{c}\bar{d}} - \Omega_{\bar{d}}{}^{\bar{c}}{}_{\bar{a}}\eta_{\bar{c}\bar{b}} + \Omega_{\bar{b}}{}^{\bar{c}}{}_{\bar{a}}\eta_{\bar{c}\bar{d}} = -\Omega_{[\bar{a}\bar{d}\bar{b}]}\end{aligned}$$

3. $\left(_{A=\bar{a},B=b,D=d}\right)$

$$\begin{aligned}\eta(L^D_{\hat{E}_{\bar{a}}}\hat{E}_{b},\hat{E}_{d}) &= \Omega_{\bar{a}}{}^{C}{}_{b}\hat{E}^M_C E_{M\,d} + \Omega^{MC}{}_{\bar{a}}\eta_{Cb}E_{M\,d} - \Omega_{b}{}^{C}{}_{\bar{a}}\hat{E}^M_C E_{M\,d} \\ &= \Omega_{\bar{a}}{}^{c}{}_{b}\hat{E}^M_c E_{M\,d} + \Omega_{d}{}^{\bar{c}}{}_{\bar{a}}\eta_{\bar{c}b} - \Omega_{b}{}^{\bar{c}}{}_{\bar{a}}\hat{E}^M_{\bar{c}} E_{M\,d} = \Omega_{\bar{a}db}\end{aligned}$$

4. $\left(_{A=\bar{a},B=b,D=\bar{d}}\right)$

$$\eta(L^D_{\hat{E}_{\bar{a}}}\hat{E}_{b},\hat{E}_{\bar{d}}) = \Omega_{\bar{a}}{}^{\bar{c}}{}_{b}\hat{E}^M_c E_{M\,\bar{d}} + \Omega_{\bar{d}}{}^{\bar{c}}{}_{\bar{a}}\eta_{\bar{c}b} - \Omega_{b}{}^{\bar{c}}{}_{\bar{a}}\hat{E}^M_{\bar{c}} E_{M\,\bar{d}} = \Omega_{b\bar{d}\bar{a}}$$

5. $\left(_{A=a,B=\bar{b},D=d}\right)$

$$\eta(L^D_{\hat{E}_{a}}\hat{E}_{\bar{b}},\hat{E}_{d}) = \Omega_{a}{}^{\bar{c}}{}_{\bar{b}}\hat{E}^M_{\bar{c}} E_{M\,d} + \Omega_{d}{}^{c}{}_{a}\eta_{c\bar{b}} - \Omega_{\bar{b}}{}^{c}{}_{a}\hat{E}^M_{c} E_{M\,d} = -\Omega_{\bar{b}da}$$

6. $\left(_{A=a,B=\bar{b},D=\bar{d}}\right)$

$$\eta(L^D_{\hat{E}_{a}}\hat{E}_{\bar{b}},\hat{E}_{\bar{d}}) = \Omega_{a}{}^{\bar{c}}{}_{\bar{b}}\hat{E}^M_{\bar{c}} E_{M\,\bar{d}} + \Omega_{\bar{d}}{}^{c}{}_{a}\eta_{c\bar{b}} - \Omega_{\bar{b}}{}^{c}{}_{a}\hat{E}^M_{c} E_{M\,\bar{d}} = -\Omega_{a\bar{d}\bar{b}}$$

From the expressions we have found we can already say something very important. We note that, whilst every component of the generalised spin-connection with mixed (i.e. from both C_+ and C_-) frame indices appears in the above formulas, the components with indices that come from only one subbundle (i.e. only form C_+ or only from C_-) appear only in a totally antisymmetric combination. This means that, while the mixed components of the generalised spin-connection will be uniquely determined, the components with non-mixed indices will only have their totally antisymmetric combination fixed by the Levi-Civita conditions. Now, let us focus on the generalised Lie-derivative terms.

$$\begin{aligned}(L_{\hat{E}_A}\hat{E}_B)^M E_{M\,D} &= \hat{E}^N_A(\partial_N\hat{E}^M_B)E_{M\,D} + \hat{E}^M_D(\partial_M\hat{E}^N_A)E_{B\,N} - \hat{E}^N_B(\partial_N\hat{E}^M_A)E_{M\,D} \\ &= \hat{E}^N_A(\partial_N\hat{E}^M_B)E_{M\,D} + \hat{E}^M_D(\partial_M\hat{E}^N_A)E_{B\,N} + \hat{E}^N_B(\partial_N\hat{E}^M_D)E_{M\,A} \\ &= 3\hat{E}^N_{[A}(\partial_{|N|}\hat{E}^M_B)E_{D]\,M}\end{aligned}$$

where to go from the first to the second line we used the Leibniz rule and the fact that the matrix η_{AD} is constant. To work out this second constraint we will use the explicit expressions for the frames given in 2.13 and 2.14. For

example, we will have:
$\hat{E}_d^N = \left(\begin{smallmatrix} \hat{e}_d^{+\,\nu} \\ e^+_{d\,\nu}+\hat{e}_d^{+\,\rho}B_{\rho\nu}\end{smallmatrix}\right)$ and $E_a^M\partial_M\hat{E}_b^N = \left(\begin{smallmatrix} \hat{e}_a^{+\,\mu}(\partial_\mu\hat{e}_b^{+\,\nu}) \\ \hat{e}_a^{+\,\mu}\partial_\mu(e^+_{b\,\nu}+\hat{e}_b^{+\,\rho}B_{\rho\nu})\end{smallmatrix}\right)$. Then we have:

$$\begin{aligned}
&\eta(E_a^M\partial_M\hat{E}_b,\hat{E}_d) = \\
&\quad = \frac{1}{2}\Big(\hat{e}_a^{+\,\mu}(\partial_\mu\hat{e}_b^{+\,\nu})e^+_{d\,\nu} + \hat{e}_a^{+\,\mu}(\partial_\mu\hat{e}_b^{+\,\nu})\hat{e}_d^{+\,\rho}B_{\rho\nu} + \hat{e}_d^{+\,\nu}(\hat{e}_a^{+\,\mu}\partial_\mu e^+_{b\,\nu}) + \hat{e}_d^{+\,\nu}(\hat{e}_a^{+\,\mu}\partial_\mu\hat{e}_b^{+\,\rho})B_{\rho\nu} \\
&\qquad + \hat{e}_d^{+\,\nu}\hat{e}_b^{+\,\rho}(\hat{e}_a^{+\,\mu}\partial_\mu B_{\rho\nu})\Big) \\
&\quad |_{B-\text{antisymmetry}} \\
&\quad = \frac{1}{2}\Big(\hat{e}_a^{+\,\mu}(\partial_\mu\hat{e}_b^{+\,\nu})e^+_{d\,\nu} + (\hat{e}_a^{+\,\mu}\partial_\mu(e_b^{+\,\rho}g_{\rho\nu}))\hat{e}_d^{+\,\nu} + (\partial_\mu B_{\rho\nu})\,\hat{e}_a^{+\,\mu}\hat{e}_b^{+\,\rho}\hat{e}_d^{+\,\nu}\Big) \\
&\quad |_{\hat{e}_d^{+\,\nu}g_{\rho\nu}=e^+_{d\,\rho}} \\
&\quad = \frac{1}{2}\Big(2\hat{e}_a^{+\,\mu}(\partial_\mu\hat{e}_b^{+\,\nu})e^+_{d\,\nu} + \hat{e}_a^{+\,\mu}(\partial_\mu g_{\rho\nu})e_b^{+\,\rho}\hat{e}_d^{+\,\nu} + (\partial_\mu B_{\rho\nu})\,\hat{e}_a^{+\,\mu}\hat{e}_b^{+\,\rho}\hat{e}_d^{+\,\nu}\Big)
\end{aligned}$$

If we now take the totally antisymmetric part in $\{a,b,d\}$ of the previous relation the term involving the metric becomes zero. The term containing the B-field instead becomes: $\frac{1}{2}((\partial_\mu B_{\rho\nu})\,\hat{e}_{[a}^{+\,\mu}\hat{e}_b^{+\,\rho}\hat{e}_{d]}^{+\,\nu}) = \frac{1}{2}((\partial_{[\mu}B_{\rho\nu]})\,\hat{e}_{[a}^{+\,\mu}\hat{e}_b^{+\,\rho}\hat{e}_{d]}^{+\,\nu}) = \frac{1}{2}((\frac{1}{3}dB_{\mu\rho\nu})\,\hat{e}_{[a}^{+\,\mu}\hat{e}_b^{+\,\rho}\hat{e}_{d]}^{+\,\nu}) = \frac{1}{6}H_{abd}$.
The other term instead is:

$$\begin{aligned}
&\Big(\hat{e}_{[a}^{+\,\mu}(\partial_\mu\hat{e}_b^{+\,\nu})e^+_{d]\,\nu}\Big) = \\
&= \frac{1}{6}\Big((\hat{e}_a^{+\,\mu}\partial_\mu\hat{e}_b^{+\,\nu} - \hat{e}_b^{+\,\mu}\partial_\mu\hat{e}_a^{+\,\nu})e^+_{d\,\nu} + (\hat{e}_b^{+\,\mu}\partial_\mu\hat{e}_d^{+\,\nu} - \hat{e}_d^{+\,\mu}\partial_\mu\hat{e}_b^{+\,\nu})e^+_{a\,\nu} + (\hat{e}_d^{+\,\mu}\partial_\mu\hat{e}_a^{+\,\nu} - \hat{e}_a^{+\,\mu}\partial_\mu\hat{e}_d^{+\,\nu})\Big)e^+_{b\,\nu} \\
&\ |_{\text{Levi}-\text{Civita is torsion}-\text{free}} \\
&= \frac{1}{6}\Big((\hat{e}_a^{+\,\mu}\nabla_\mu\hat{e}_b^{+\,\nu} - \hat{e}_b^{+\,\mu}\nabla_\mu\hat{e}_a^{+\,\nu})e^+_{d\,\nu} + \text{cyclic permutations in a, b, d}\Big) \\
&= \frac{1}{6}\Big((\hat{e}_a^{+\,\mu}\omega_\mu{}^c{}_b\hat{e}_c^{+\,\nu} - \hat{e}_b^{+\,\mu}\omega_\mu{}^c{}_a\hat{e}_c^{+\,\nu})e^+_{d\,\nu} + \text{cyclic permutations}\Big) \\
&= \frac{1}{6}\Big((\omega_a{}^c{}_b\eta_{cd} - \omega_b{}^c{}_a\eta_{cd}) + (\omega_b{}^c{}_d\eta_{ca} - \omega_d{}^c{}_b\eta_{ca}) + (\omega_d{}^c{}_a\eta_{cb} - \omega_a{}^c{}_d\eta_{cb})\Big) \\
&= \frac{1}{3}\Big(\omega_{adb} + \omega_{bad} + \omega_{dba}\Big) = \omega^+_{[adb]}
\end{aligned}$$

where $\omega_\mu^{+a}{}_b$ is the ordinary Levi-Civita spin-connection for the metric g and the $+$ indicates that it is the one derived from the $\{\hat{e}_a^+\}$-frames. We have therefore that (after rearranging the indices):

$$\Omega_{[abd]} = \omega^+_{[abd]} - \frac{1}{6}H_{abd} \tag{2.22}$$

The case with all the barred indices is very similar. We have:

$$\hat{E}_{\bar{d}}^N = \begin{pmatrix} \hat{e}_{\bar{d}}^{-\,\nu} \\ -e^-_{\bar{d}\,\nu}+\hat{e}_{\bar{d}}^{-\,\rho}B_{\rho\nu}\end{pmatrix} \quad \text{and} \quad E_{\bar{a}}^M\partial_M\hat{E}_{\bar{b}}^N = \begin{pmatrix} \hat{e}_{\bar{a}}^{-\,\mu}(\partial_\mu\hat{e}_{\bar{b}}^{-\,\nu}) \\ \hat{e}_{\bar{a}}^{-\,\mu}\partial_\mu(-e^-_{\bar{b}\,\nu}+\hat{e}_{\bar{b}}^{-\,\rho}B_{\rho\nu})\end{pmatrix}$$

Similarly we get

$$\eta(\hat{E}_{\bar{a}}^N\partial_N\hat{E}_{\bar{b}},\hat{E}_{\bar{d}}) = \frac{1}{2}\Big(-2\hat{e}_{\bar{a}}^{-\,\mu}(\partial_\mu\hat{e}_{\bar{b}}^{-\,\nu})e^-_{\bar{d}\,\nu} - \hat{e}_{\bar{a}}^{-\,\mu}(\partial_\mu g_{\rho\nu})\hat{e}_{\bar{b}}^{-\,\rho}\hat{e}_{\bar{d}}^{-\,\nu} + (\partial_\mu B_{\rho\nu})\hat{e}_{\bar{a}}^{-\,\mu}\hat{e}_{\bar{b}}^{-\,\rho}\hat{e}_{\bar{d}}^{-\,\nu}\Big)$$

The antisymmetrisation works exactly as before, but now we have an extra minus sign in front of the first (and the second - which vanishes anyway) term. Recalling that there is an extra minus sign even in front of the Ω-term (but no extra sign in front of the H-term) we finally have:

$$\Omega_{[\bar{a}\bar{b}\bar{d}]} = \omega^{-}_{[\bar{a}\bar{b}\bar{d}]} + \frac{1}{6} H_{\bar{a}\bar{b}\bar{d}} \tag{2.23}$$

where now $\omega^{-\bar{a}}_{\mu\ \bar{b}}$ is the Levi-Civita spin-connection calculated with the $\{\hat{e}^{-}_{\bar{a}}\}$-frames.
It is clear that of the four remaining cases we are left to solve, only two are independent. This can be easily seen considering that the torsionless condition is the defining property of the mixed-indices generalised connection one-forms Ω. For example $-\Omega_{\bar{b}da} = (L^{D}_{\hat{E}_a}\hat{E}_{\bar{b}})^M \hat{E}_{d\,M} := 3\hat{E}^{N}_{[a}(\partial_{|N|}\hat{E}^{M}_{\bar{b}})\hat{E}_{d]\,M} = -3\hat{E}^{N}_{[\bar{b}}(\partial_{|N|}\hat{E}^{M}_{a})\hat{E}_{d]\,M} =: -(L^{D}_{\hat{E}_{\bar{b}}}\hat{E}_a)^M \hat{E}_{d\,M} = -\Omega_{\bar{b}da}$. So we are left with:

$$\begin{cases} i)\ \Omega_{\bar{a}db} = 3\hat{E}_{[\bar{a}}(\partial_{|N|}\hat{E}^{M}_{b})\hat{E}_{d]\,M} \\ ii)\ -\Omega_{ad\bar{b}} = 3\hat{E}_{[a}(\partial_{|N|}\hat{E}^{M}_{\bar{b}})\hat{E}_{\bar{d}]\,M} \end{cases}$$

Let us consider $ii)$.

$$\hat{E}^{N}_{a}\partial_N \hat{E}^{M}_{\bar{b}} = \begin{pmatrix} \hat{e}^{+\,\mu}_{a}(\partial_\mu \hat{e}^{-\,\nu}_{\bar{b}}) \\ \hat{e}^{+\,\mu}_{a}\partial_\mu(-e^{-}_{\bar{b}\,\nu} + \hat{e}^{-\,\rho}_{\bar{b}} B_{\rho\nu}) \end{pmatrix} \quad \text{and} \quad \hat{E}^{M}_{\bar{d}} = \begin{pmatrix} \hat{e}^{-\,\nu}_{\bar{d}} \\ -e^{-}_{\bar{d}\,\nu} + \hat{e}^{-\,\rho}_{\bar{d}} B_{\rho\nu} \end{pmatrix}$$

The particularity of the mixed-indices part is that the antisymmetrisation is not straightforward any more and we have to calculate explicitly all the contributing terms.

$$\begin{aligned} &\eta(\hat{E}^{N}_{a}\partial_N \hat{E}_{\bar{b}}, \hat{E}_{\bar{d}}) = \\ &\quad = \frac{1}{2}\big(-\hat{e}^{+\,\mu}_{a}(\partial_\mu \hat{e}^{-\,\nu}_{\bar{b}})e^{-}_{\bar{d}\,\nu} + \hat{e}^{+\,\mu}_{a}(\partial_\mu \hat{e}^{-\,\nu}_{\bar{b}})\hat{e}^{-\,\rho}_{\bar{d}} B_{\rho\nu} - \hat{e}^{-\,\nu}_{\bar{d}}(\hat{e}^{+\,\mu}_{a}\partial_\mu e^{-}_{\bar{b}\,\nu}) + \hat{e}^{-\,\nu}_{\bar{d}}\hat{e}^{+\,\mu}_{a}(\partial_\mu \hat{e}^{-\,\rho}_{\bar{b}})B_{\rho\nu} \\ &\qquad + \hat{e}^{-\,\nu}_{\bar{d}}\hat{e}^{-\,\rho}_{\bar{b}}\hat{e}^{+\,\mu}_{a}(\partial_\mu B_{\rho\nu})\big) \\ &\quad = \frac{1}{2}\big(-\hat{e}^{+\,\mu}_{a}(\partial_\mu \hat{e}^{-\,\nu}_{\bar{b}})e^{-}_{\bar{d}\,\nu} - \hat{e}^{+\,\mu}_{a}(\partial_\mu g_{\nu\rho})\hat{e}^{-\,\rho}_{\bar{b}}\hat{e}^{-\,\nu}_{\bar{d}} - \hat{e}^{+\,\mu}_{a}(\partial_\mu \hat{e}^{-\,\nu}_{\bar{b}})e^{-}_{\bar{d}\,\nu} + \hat{e}^{-\,\nu}_{\bar{d}}\hat{e}^{-\,\rho}_{\bar{b}}\hat{e}^{+\,\mu}_{a}(\partial_\mu B_{\rho\nu})\big) \\ &\quad = \frac{1}{2}\big(-2\hat{e}^{+\,\mu}_{a}(\partial_\mu \hat{e}^{-\,\nu}_{\bar{b}})e^{-}_{\bar{d}\,\nu} - \hat{e}^{+\,\mu}_{a}(\partial_\mu g_{\nu\rho})\hat{e}^{-\,\rho}_{\bar{b}}\hat{e}^{-\,\nu}_{\bar{d}} + \hat{e}^{-\,\nu}_{\bar{d}}\hat{e}^{-\,\rho}_{\bar{b}}\hat{e}^{+\,\mu}_{a}(\partial_\mu B_{\rho\nu})\big) \end{aligned}$$

And one can obtain in the same fashion (recall that there are always two terms that cancel out because of the antisymmetry of the B-field):

$$\begin{aligned} \eta(\hat{E}^{N}_{\bar{b}}\partial_N \hat{E}_{\bar{d}}, \hat{E}_a) &= \frac{1}{2}(\hat{e}^{-\,\mu}_{\bar{b}}(\partial_\mu \hat{e}^{-\,\nu}_{\bar{d}})e^{+}_{a\,\nu} - \hat{e}^{-\,\mu}_{\bar{b}}(\partial_\mu e^{-}_{\bar{d}\,\nu})\hat{e}^{+\,\nu}_{a} + \hat{e}^{+\,\nu}_{a}\hat{e}^{-\,\mu}_{\bar{b}}\hat{e}^{-\,\rho}_{\bar{d}}(\partial_\mu B_{\rho\nu})) \\ &= \frac{1}{2}(-\hat{e}^{-\,\mu}_{\bar{b}}\hat{e}^{-\,\nu}_{\bar{d}}(\partial_\mu e^{+}_{a\,\nu}) + \hat{e}^{-\,\mu}_{\bar{b}} e^{-}_{\bar{d}\,\nu}(\partial_\mu \hat{e}^{+\,\nu}_{a}) + \hat{e}^{+\,\nu}_{a}\hat{e}^{-\,\mu}_{\bar{b}}\hat{e}^{-\,\rho}_{\bar{d}}(\partial_\mu B_{\rho\nu})) \end{aligned}$$

$$\eta(\hat{E}^N_{\bar{d}}\partial_N\hat{E}_a, \hat{E}_{\bar{b}}) = \frac{1}{2}(-\hat{e}^{-}_{\bar{d}}{}^{\mu}(\partial_\mu \hat{e}^{+}_{a}{}^{\nu})e^{-}_{\bar{b}\nu} + \hat{e}^{-}_{\bar{d}}{}^{\mu}(\partial_\mu e^{+}_{a\nu})\hat{e}^{-}_{\bar{b}}{}^{\nu} + \hat{e}^{-}_{\bar{d}}{}^{\mu}\hat{e}^{+}_{a}{}^{\rho}\hat{e}^{-}_{\bar{b}}{}^{\nu}(\partial_\mu B_{\rho\nu}))$$
$$= \frac{1}{2}(\cancel{-\hat{e}^{-}_{\bar{d}}{}^{\mu}(\partial_\mu \hat{e}^{+}_{a}{}^{\nu})e^{-}_{\bar{b}\nu}} + \hat{e}^{-}_{\bar{d}}{}^{\mu}(\partial_\mu g_{\rho\nu})\hat{e}^{+}_{a}{}^{\rho}\hat{e}^{-}_{\bar{b}}{}^{\nu} + \cancel{\hat{e}^{-}_{\bar{d}}{}^{\mu}(\partial_\mu \hat{e}^{+}_{a}{}^{\nu})e^{-}_{\bar{b}\nu}} +$$
$$+ \hat{e}^{-}_{\bar{d}}{}^{\mu}\hat{e}^{+}_{a}{}^{\rho}\hat{e}^{-}_{\bar{b}}{}^{\nu}(\partial_\mu B_{\rho\nu}))$$

Adding these three terms together we get:

$$3\eta(\hat{E}^N_{[a}\partial_{|N|}\hat{E}_{\bar{b}}, \hat{E}_{\bar{d}}) =$$
$$= \tfrac{1}{2}(-2\hat{e}^{+}_{a}{}^{\mu}(\partial_\mu \hat{e}^{-}_{\bar{b}}{}^{\nu})e^{-}_{\bar{d}\nu} - \hat{e}^{+}_{a}{}^{\mu}(\partial_\mu g_{\nu\rho})\hat{e}^{-}_{\bar{b}}{}^{\rho}\hat{e}^{-}_{\bar{d}}{}^{\nu} - \hat{e}^{-}_{\bar{b}}{}^{\mu}\hat{e}^{-}_{\bar{d}}{}^{\nu}(\partial_\mu e^{+}_{a\nu}) + \hat{e}^{-}_{\bar{b}}{}^{\mu}(\partial_\mu \hat{e}^{+}_{a}{}^{\nu}))e^{-}_{\bar{d}\nu}$$
$$+ \hat{e}^{-}_{\bar{d}}{}^{\mu}(\partial_\mu g_{\rho\nu})\hat{e}^{+}_{a}{}^{\rho}\hat{e}^{-}_{\bar{b}}{}^{\nu} + (\partial_\mu B_{\rho\nu} + \partial_\rho B_{\nu\mu} + \partial_\nu B_{\mu\rho})\hat{e}^{+}_{a}{}^{\mu}\hat{e}^{-}_{\bar{b}}{}^{\rho}\hat{e}^{-}_{\bar{d}}{}^{\nu}) =$$
$$= \frac{1}{2}(-2\hat{e}^{+}_{a}{}^{\mu}(\partial_\mu \hat{e}^{-}_{\bar{b}}{}^{\nu})e^{-}_{\bar{d}\nu} - \hat{e}^{+}_{a}{}^{\mu}(\partial_\mu g_{\nu\rho})\hat{e}^{-}_{\bar{b}}{}^{\rho}\hat{e}^{-}_{\bar{d}}{}^{\nu} - \underline{\hat{e}^{-}_{\bar{b}}{}^{\mu}\hat{e}^{-}_{\bar{d}}{}^{\nu}(\partial_\mu g_{\nu\rho})\hat{e}^{+}_{a}{}^{\rho}}$$
$$- \underline{\cancel{\hat{e}^{-}_{\bar{b}}{}^{\mu}e^{-}_{\bar{d}\nu}(\partial_\mu \hat{e}^{+}_{a}{}^{\nu})}} + \cancel{\hat{e}^{-}_{\bar{b}}{}^{\mu}(\partial_\mu \hat{e}^{+}_{a}{}^{\nu})e^{-}_{\bar{d}\nu}} + \hat{e}^{-}_{\bar{d}}{}^{\mu}(\partial_\mu g_{\rho\nu})\hat{e}^{+}_{a}{}^{\rho}\hat{e}^{-}_{\bar{b}}{}^{\nu} + H_{a\bar{b}\bar{d}}) =$$
$$= \frac{1}{2}(-2\hat{e}^{+}_{a}{}^{\rho}(\partial_\rho \hat{e}^{-}_{\bar{b}}{}^{\nu})e^{-}_{\bar{d}\nu} - 2(\hat{e}^{+}_{a}{}^{\rho}\hat{e}^{-}_{\bar{b}}{}^{\mu}\hat{e}^{-}_{\bar{d}}{}^{\beta}g_{\beta\alpha})\frac{g^{\alpha\nu}}{2}(\partial_\mu g_{\rho\nu} + \partial_\rho g_{\nu\mu} - \partial_\nu g_{\mu\rho}) + H_{a\bar{b}\bar{d}}) =$$
$$= -\hat{e}^{+}_{a}{}^{\rho}((\partial_\rho \hat{e}^{-}_{\bar{b}}{}^{\nu}) + \hat{e}^{-}_{\bar{b}}{}^{\mu}\Gamma^{\nu}_{\mu\rho})e^{-}_{\bar{d}\nu} + \frac{H_{a\bar{b}\bar{d}}}{2}$$

but, recalling equation 2.15[25] we find:

$$-\Omega_{a\bar{d}\bar{b}} = 3\eta(\hat{E}^N_{[a}\partial_{|N|}\hat{E}_{\bar{b}}, \hat{E}_{\bar{d}]}) = -\hat{e}^{+}_{a}{}^{\rho}(\omega^{-}_{\rho}{}^{\bar{c}}{}_{\bar{b}}\,\hat{e}^{-}_{\bar{c}}{}^{\nu})e^{-}_{\bar{d}\nu} + \frac{1}{2}H_{a\bar{b}\bar{d}} = -\omega^{-}_{a\bar{d}\bar{b}} + \frac{1}{2}H_{a\bar{b}\bar{d}}$$

and therefore:

$$\Omega_{a\bar{d}\bar{b}} = \omega^{-}_{a\bar{d}\bar{b}} + \frac{1}{2}H_{a\bar{d}\bar{b}} \tag{2.24}$$

The other case is calculated in an analogous manner, but for matter of space we will omit the calculation and will only quote the result:

$$\Omega_{\bar{a}db} = \omega^{+}_{\bar{a}db} - \frac{1}{2}H_{\bar{a}db} \tag{2.25}$$

From this calculation we have found that the generalised Levi-Civita connection exists and that it depends on both the ordinary Levi-Civita connection (and therefore on the ordinary metric) and the three-form field strength $H = dB$. Even if this connection exists, it is *not* uniquely defined. In particular only the mixed index components of the generalised Levi-Civita spin-connection are uniquely defined, whilst the same-index components are only defined up to an arbitrary three-tensor, say $A^{\pm}$, i.e. one for each subbundle $C_{\pm}$, that satisfies $A^{+}_{[abd]} = 0 = A^{-}_{[\bar{a}\bar{b}\bar{d}]}$ and $A^{+}_{abd} = -A^{+}_{adb}$ and $A^{-}_{\bar{a}\bar{b}\bar{d}} = -A^{-}_{\bar{a}\bar{d}\bar{b}}$, where the first condition derives from the torsionless requirement and the second one from the metric-compatibility.

[25] $\omega_{\mu}{}^{b}{}_{a}\hat{e}^{\rho}_{b} = \partial_\mu(\hat{e}^{\rho}_{a}) + \hat{e}^{\nu}_{a}\Gamma^{\rho}_{\mu\nu}$

2.3.3 $NSNS$ Sector and Generalised Curvature

Until now, we have included in the generalised geometric description only the metric g and the B-field. Nevertheless, we had previously stated that the generalised geometric approach is able to describe the whole $NSNS$ bosonic sector of the Type II supergravity theories. We would then like to introduce the dilaton in our discussion and see how the results we have found so far are modified by its presence.
In order to add the dilaton we need to introduce an extra degree of freedom in the structure group.[26] This is achieved by considering a weighted version of the generalised vector bundle: $E \mapsto \tilde{E} := \det(T^*M) \otimes E$. Note that this extended generalised vector bundle has the same dimension as the previous one. The difference is that it has a natural principal bundle structure with fibre $O(d,d) \times \mathbb{R}^+$. One can indeed restrict its frame bundle to the bundle of conformal frames, i.e. $\{\hat{E}_A\}$ such that:

$$\eta(\hat{E}_A, \hat{E}_B) = \Phi^2 \eta_{AB}$$

for some frame-dependent conformal factor $\Phi \in \det(T^*M)$, so that changes of basis that preserve the natural metric up to an overall positive factor are now allowed. Tensors of $\tilde{E}$, that now are representations of $O(d,d) \times \mathbb{R}^+$, are tensors of E with definite weight (say p) under $\mathbb{R}^+$:

$$E^{\otimes n}_{(p)} = (\det(T^*M))^p \otimes E \ldots \otimes E$$

In this context it is clear that a choice of splitting for E, i.e. a choice of $B = \{B_{(i)}\}$ well patched, defines an isomorphism $\tilde{E} \approx \det(T^*M) \otimes (TM \oplus T^*M)$. Note that now generalised orthonormal frames are defined by requiring them to diagonalise both η and G up to an overall frame dependent conformal factor $\Phi \in \det(T^*M)$, i.e. $\hat{E}^M_A \eta_{MN} \hat{E}^N_B = \Phi^2 \, \eta_{AB}$ and $\hat{E}^M_A G_{MN} \hat{E}^N_B = \Phi^2 \, G_{AB}$. This clearly defines an $(O(p,q) \times O(q,p)) \times \mathbb{R}^+$ structure and is solved by simply rescaling by Φ the frames found in proposition 2.3.1. It is also clear how we can get an $O(p,q) \times O(q,p)$ structure from this: it is sufficient to impose to Φ to be a fixed density. We can then see that an $O(p,q) \times O(q,p)$-structure for $\tilde{E}$ defines not only a splitting of E, and so an isomorphism $\tilde{E} \approx \det(T^*M) \otimes (TM \oplus T^*M)$, but also a choice of Φ, and therefore an isomorphism $\tilde{E} \approx E \Rightarrow \tilde{E} \approx TM \oplus T^*M$. We can write the fixed conformal factor Φ as $\Phi =: e^{-2\phi}\sqrt{-\det g}$, because $\sqrt{-\det g}$ can be viewed as a global basis of $\det(T^*M)$ (since it transforms as a tensor density of weight $w = -1$) and $e^{-2\phi}$ as the coefficient of the section of $\det(T^*M)$ in this basis.
To summarise we have seen how an $O(d,p) \times O(p,d)$ structure determines an

[26] We will follow here quite closely the discussion in [CoStWa2011]

isomorphism between $\tilde{E}$ and $TM \oplus T^*M$, and therefore a choice of B and ϕ; we can represent this structure as the triple (η, G, Φ), with G the generalised metric and Φ the chosen $\det(T^*M)$-density.
Let us try to understand what the differences between $\tilde{E}$ and E in the construction of a generalised Levi-Civita connection are. We start again with a (conformally) orthonormal frame $\{\hat{E}_A\}$ for $\tilde{E}$ and define the generalised spin connection as follows: $D_M \hat{E}_A := \Omega_M{}^B{}_A \hat{E}_B$. For what concerns the metric compatibility, we need to consider that the $O(d,p) \times O(p,d)$-structure is specified by the triple (η, G, Φ). It is then natural to require

$$D\eta = DG = D\Phi = 0$$

If we consider again equations 2.20 and 2.21, we see that we cannot use any more the fact that $\eta(\hat{E}_A, \hat{E}_B)$ and $G(\hat{E}_A, \hat{E}_B)$ are constant to set their covariant derivative to zero, because now we have an extra Φ^2 factor: $\eta(\hat{E}_A, \hat{E}_B) = \Phi^2 \eta_{AB}$ and $G(\hat{E}_A, \hat{E}_B) = \Phi^2 G_{AB}$. Nevertheless, the additional condition $D\Phi = 0$ we have previously imposed keeps the value of their covariant derivative equal to zero, and thus the old results on $\Omega_M{}^B{}_A$ remain valid: $\Omega_M{}^b{}_{\bar{a}} = 0 = \Omega_M{}^{\bar{b}}{}_a$ and $\Omega_{Mab} = -\Omega_{Mba}$, $\Omega_{M\bar{a}\bar{b}} = -\Omega_{M\bar{b}\bar{a}}$.[27] Next, let us consider the torsionless condition. We first need to recall that tensors of $\tilde{E}$ are tensors of E with definite weight, say p, under $\det(T^*M)$. Recall that the action of the ordinary Lie derivative on weighted vector fields and one-forms is:

$$\begin{aligned} \mathcal{L}_v w^\mu &= v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu + p(\partial_\nu v^\nu) w^\mu \\ \mathcal{L}_v \lambda_\mu &= v^\nu \partial_\nu \lambda_\mu + (\partial_\mu v^\nu)\lambda_\nu + p(\partial_\nu v^\nu)\lambda_\mu \end{aligned}$$

where $v \in \Gamma(TM), w \in \Gamma\Big(\big(\det(T^*M)\big)^p \otimes TM\Big)$ and $\lambda \in \Gamma\Big(\big(\det(T^*M)\big)^p \otimes T^*M\Big)$. We can then see how the covariant expression of the Dorfman derivative is modified. From its definition: $L_X Y = \mathcal{L}_v w + \mathcal{L}_v \lambda - i_w(d\mu)$, if $X = v + \mu \in \Gamma(E)$ and $Y = w + \lambda \in \Gamma\Big(\big(\det(T^*M)\big)^p \otimes E\Big)$, we can see that the two new terms combine into:

$$p(\partial_\nu v^\nu) w^\mu + p(\partial_\nu v^\nu)\lambda_\mu = p(\partial_\nu v^\nu)(w^\mu + \lambda_\mu) = p(\partial_N X^N) Y^M$$

This implies that the general covariant form of the generalised Lie derivative on $\tilde{E}$, for generalised vectors of weight p, is:

$$(L_X Y)^M = X^N \partial_N Y^M + (\partial^M X^N - \partial^N X^M) Y_N + p(\partial_N X^N) Y^M \tag{2.26}$$

[27]This should not come as a surprise: imposing metric compatibility with an $O(p,q) \times O(q,p)$, $p+q = d$, structure implies that its connection one forms are infinitesimal generators of this group.

We are now ready to express the torsionless condition in the context of the extended generalised tangent bundle $\tilde{E}$. From equation 2.26 we can see that to determine completely the generalised Lie derivative it is sufficient to use generators for E in the first component and generators for $\tilde{E}$ in the second one. More concretely if $\{\hat{E}_A\}$ is now a conformally orthonormal frame for the $O(p,q)\times O(q,p)$-structure, then every $\hat{E}_A$ is a section of $\det(T^*M)\otimes E$, whilst $\{\Phi^{-1}\,\hat{E}_A\}$ are sections of E and are orthonormal frames that can be chosen to be the ones of proposition 2.3.1. Then, to completely determine the Dorfman derivative we can just calculate the expressions $L_{\Phi^{-1}\,\hat{E}_A}\hat{E}_B$, $\forall\, A,B$. Thus, the torsionless condition is:

$$
\begin{aligned}
&L^D_{\Phi^{-1}\,\hat{E}_A}\hat{E}_B = L_{\Phi^{-1}\,\hat{E}_A}\hat{E}_B \qquad \forall\, A,B\\
&\Leftrightarrow\ \eta(L^D_{\Phi^{-1}\,\hat{E}_A}\hat{E}_B,\hat{E}_C) = \eta(L_{\Phi^{-1}\,\hat{E}_A}\hat{E}_B,\hat{E}_C) \qquad \forall\, A,B,C\\
&\Leftrightarrow\ \eta(L^D_{\Phi^{-1}\,\hat{E}_A}\hat{E}_B,\Phi^{-1}\,\hat{E}_C) = \eta(L_{\Phi^{-1}\,\hat{E}_A}\hat{E}_B,\Phi^{-1}\,\hat{E}_C) \qquad \forall\, A,B,C
\end{aligned}
\tag{2.27}
$$

Let us explicitly calculate these expressions.

$$
\begin{aligned}
L_{\Phi^{-1}\,\hat{E}_A}\hat{E}_B &= (L_{\Phi^{-1}\,\hat{E}_A}\Phi)\Phi^{-1}\,\hat{E}_B + \Phi(L_{\Phi^{-1}\,\hat{E}_A}\Phi^{-1}\,\hat{E}_B)\\
&= ((\Phi^{-1}\,\hat{E}_A)^N\partial_N\Phi + (\partial_N(\Phi^{-1}\,\hat{E}_A)^N)\Phi)\Phi^{-1}\,\hat{E}_B + \Phi L_{\Phi^{-1}\,\hat{E}_A}\Phi^{-1}\,\hat{E}_B\\
&= (\partial_N\hat{E}^N_A)\Phi^{-1}\,\hat{E}_B + \Phi L_{\Phi^{-1}\,\hat{E}_A}\Phi^{-1}\,\hat{E}_B
\end{aligned}
$$

So $\eta(L_{\Phi^{-1}\,\hat{E}_A}\hat{E}_B,\Phi^{-1}\,\hat{E}_C) = (\partial_N\hat{E}^N_A)\eta_{BC} + \Phi\eta(L_{\Phi^{-1}\,\hat{E}_A}\Phi^{-1}\,\hat{E}_B,\Phi^{-1}\,\hat{E}_C)$. Note that the last term in this equation is exactly the same term we calculated in the previous subsection (times Φ) because the old frames of E take now the new formal expression: $\{\hat{E}_A\}\mapsto\{\Phi^{-1}\,\hat{E}_A\}$. For the other term, recalling that $D\Phi=0$ (and that $p=1$ for $\hat{E}_B$), we have instead:

$$
\begin{aligned}
&L^D_{\Phi^{-1}\,\hat{E}_A}\hat{E}^M_B =\\
&= (\Phi^{-1}\,\hat{E}_A)^N D_N\hat{E}^M_B + (D^M(\Phi^{-1}\,\hat{E}_A)^N - D^N(\Phi^{-1}\,\hat{E}_A)^M)E_{B\,N} + D_N(\Phi^{-1}\,\hat{E}^N_A)\hat{E}^M_B\\
&= \Phi^{-1}\,(\hat{E}^N_A\Omega_N{}^P{}_B\hat{E}^M_P + (\Omega^{MP}{}_A\hat{E}^N_P - \Omega^{NP}{}_A\hat{E}^M_P)E_{B\,N} + \Omega_N{}^P{}_A\hat{E}^N_P\hat{E}^M_B)
\end{aligned}
$$

So we get

$$
\begin{aligned}
\eta(L^D_{\Phi^{-1}\,\hat{E}_A}&\hat{E}_B,\Phi^{-1}\,\hat{E}_C) =\\
&= \hat{E}^N_A\Omega_N{}^P{}_B\eta_{PC} + \hat{E}^M_C\Omega_M{}^P{}_A\eta_{PB} - \hat{E}^N_B\Omega_N{}^P{}_A\eta_{PC} + \hat{E}^N_P\Omega_N{}^P{}_A\eta_{BC}\\
&= \Phi\,(3\Omega_{[ACB]} + \Omega_P{}^P{}_A\eta_{BC})
\end{aligned}
$$

where we used the fact that to pass from a frame index A to a coordinate index M we use the orthonormal frame of E $\{\Phi^{-1}\,\hat{E}_A\}$, and that $\Omega_{MAB} =$

$-\Omega_{MBA}$ because of the compatibility with η and Φ. We have eventually obtained that the torsionless condition is now equivalent to:

$$\begin{aligned}3\Omega_{[ACB]} + \Omega_{P}{}^{P}{}_{A}\eta_{BC} &= \eta(L_{\Phi^{-1}\hat{E}_A}\Phi^{-1}\hat{E}_B, \Phi^{-1}\hat{E}_C) + \Phi^{-1}(\partial_N \hat{E}^N_A)\eta_{BC} \\ &= 3(\Phi^{-1}\hat{E}^N_{[B})(\partial_{|N|}(\Phi^{-1}\hat{E}^M_C))(\Phi^{-1}E_{A]\,M}) + \Phi^{-1}(\partial_N \hat{E}^N_A)\eta_{BC}\end{aligned} \tag{2.28}$$

Since the two terms on each side of equation 2.28 have different properties of symmetry, we can split this in two separate equations:

$$3\Omega_{[ACB]} = 3(\Phi^{-1}\hat{E}^N_{[B})(\partial_{|N|}(\Phi^{-1}\hat{E}^M_C))(\Phi^{-1}E_{A]\,M}) \qquad \forall\, A, B, C \tag{2.29}$$

$$\Omega_{P}{}^{P}{}_{A}\eta_{BC} = \Phi^{-1}(\partial_N \hat{E}^N_A)\eta_{BC} \qquad \forall\, A, B, C \tag{2.30}$$

Equation 2.29 is the one we solved in the previous subsection. Equation 2.30 is instead a completely new condition. It implies that $\Omega_{D}{}^{D}{}_{A} = \Phi^{-1}\partial_N \hat{E}^N_A, \forall\, A$. Recalling the metric compatibility constraints, we have that it is equivalent to the two following equations: $\Omega_{d}{}^{d}{}_{a} = \Phi^{-1}\partial_N \hat{E}^{+\,N}_a, \forall\, a$ and $\Omega_{\bar{d}}{}^{\bar{d}}{}_{\bar{a}} = \Phi^{-1}\partial_N \hat{E}^{-\,N}_{\bar{a}}, \forall\, \bar{a}$. The vector component of $\hat{E}^{+\,N}_a$ is $e^{-2\phi}\sqrt{-\mathrm{det}g}\,\hat{e}^{+\,\mu}_a = \Phi\,\hat{e}^{+\,\mu}_a$ and similarly for $\hat{E}^{-\,N}_{\bar{a}}$. We have therefore:

$$\begin{aligned}\Omega_{d}{}^{d}{}_{a} = \Phi^{-1}\partial_N \hat{E}^{+\,N}_a = \Phi^{-1}\partial_\mu(e^{-2\phi}\sqrt{-\mathrm{det}g}\,\hat{e}^{+\,\mu}_a)& \\ &= -2\Phi^{-1}(\partial_\mu\phi)e^{-2\phi}\sqrt{-\mathrm{det}g}\,\hat{e}^{+\,\mu}_a + \Phi^{-1}e^{-2\phi}\partial_\mu(\sqrt{-\mathrm{det}g}\,\hat{e}^{+\,\mu}_a) \\ &= \Phi^{-1}\sqrt{-\mathrm{det}g}\,e^{-2\phi}(-2(\hat{e}^{+\,\mu}_a\partial_\mu\phi) + \nabla_\mu\hat{e}^{+\,\mu}_a) \\ &= -2\hat{e}^{+\,\mu}_a(\partial_\mu\phi) + \omega_{\mu}{}^{d}{}_{a}\hat{e}^{+\,\mu}_d \\ &= \omega_{d}{}^{d}{}_{a} - 2\partial_a\phi\end{aligned} \tag{2.31}$$

and similarly $\Omega_{\bar{d}}{}^{\bar{d}}{}_{\bar{b}} = \omega_{\bar{d}}{}^{\bar{d}}{}_{\bar{b}} - 2\partial_{\bar{b}}\phi$. We can now write the most general generalised Levi-Civita spin-connection on $\tilde{E}$. We note that these two last conditions are on the components with non-mixed indices of the Ωs, components that were left partially undetermined by the conditions found in subsection 2.3.2. These new conditions concern the contraction of the first two indices and are therefore completely independent of the ones we had previously found. To include them in the expression of the Ωs we need to remember that metric compatibility implies antisymmetry on the last two indices. We can then write: $\Omega_{abc} - \Omega_{[abc]} = \frac{-2}{9}(\eta_{ab}\partial_c\phi - \eta_{ac}\partial_b\phi)$, where the constant in front of the parentheses is such that the contraction with η^{ab} yields the result in 2.31. We can now summarise the results of this and the last subsection in the following:

Theorem 2.3.2. *Let $\tilde{E} = \det(T^*M) \otimes E$ be the weighted generalised tangent bundle and let (η, G, Φ) be an $O(p,q) \times O(q,p)$, $p+q=d$, structure on it. A generalised torsion free, metric compatible connection D on $\tilde{E}$ always exists, but it is not uniquely defined. Its expression in terms of a conformally-orthonormal frame $\{\hat{E}_A\} = \{\hat{E}^+_a\} \cup \{\hat{E}^-_{\bar{a}}\}$ can be written as:*

$$\begin{aligned}
D_a v_+^b &= \nabla_a v_+^b - \tfrac{1}{6} H_a{}^b{}_c v_+^c - \tfrac{2}{9}(\delta_a{}^b \partial_c \phi - \eta_{ac}\partial^b \phi) v_+^c + A^{+b}_{a\ \ c} v_+^c \\
D_{\bar{a}} v_+^b &= \nabla_{\bar{a}} v_+^b - \tfrac{1}{2} H_{\bar{a}}{}^b{}_c v_+^c \\
D_a v_-^{\bar{b}} &= \nabla_a v_-^{\bar{b}} + \tfrac{1}{2} H_a{}^{\bar{b}}{}_{\bar{c}} v_-^{\bar{c}} \\
D_{\bar{a}} v_-^{\bar{b}} &= \nabla_{\bar{a}} v_-^{\bar{b}} + \tfrac{1}{6} H_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}} v_-^{\bar{c}} - \tfrac{2}{9}(\delta_{\bar{a}}{}^{\bar{b}} \partial_{\bar{c}} \phi - \eta_{\bar{a}\bar{c}}\partial^{\bar{b}} \phi) v_-^{\bar{c}} + A^{-\bar{b}}_{\bar{a}\ \ \bar{c}} v_-^{\bar{c}}
\end{aligned}$$

where $A^\pm$ are arbitrary tensors that satisfy: $A^+_{[abc]} = 0 = A^-_{[\bar{a}\bar{b}\bar{c}]}$, $A^+_{abc} = -A^+_{acb}$, $A^-_{\bar{a}\bar{b}\bar{c}} = -A^-_{\bar{a}\bar{c}\bar{b}}$ and $A^{+a}_{a\ \ b} = 0 = A^{-\bar{a}}_{\bar{a}\ \ \bar{b}}$.

Remark 5. Note that the conditions on $A^\pm$ correspond to the constraints on the non-mixed indices components of $\Omega_A{}^B{}_C$.

Now that we have found the expression for the generalised Levi-Civita connection, it is natural to ask ourselves whether we can build a generalised analogous of the Riemann tensor and, more in general, of other curvature tensors from it. The analogous of the Riemann tensor would be (see [CoStWa2011]):

$$R(X,Y,Z) := [D_X, D_Y]Z - D_{[\![X,Y]\!]}Z$$

where $[\![\,,\,]\!]$ is the Courant bracket. One of the problems of this quantity is that it is *not* tensorial. Indeed:

$$R(fX, gY, hZ) := fgh([D_X, D_Y]Z - D_{[\![X,Y]\!]}Z) - \tfrac{1}{2} h\eta(X,Y) D_{(f\,dg - g\,df)} Z$$

and this shows that R is not tensorial unless its first two arguments are restricted to two η-orthogonal subspaces. For example $R \in ((C_\pm \otimes C_\mp) \otimes \mathfrak{o}(d,d))$ is a tensor. Nevertheless, according to theorem 2.3.2, the tensorial properties of R are not its only problem: since the generalised Levi-Civita connection is not unique, R is not unique either.[28] Since the uniqueness is essential for a physical theory to be developed, we will restrict ourselves to the use of only the unique operators that can be constructed from the generalised

[28]Note that this lack of uniqueness may also be related to the geometrical meaning of the generalised curvature: while the ordinary Riemann tensor encodes the failure of the parallel transport along an infinitesimal loop to be the identity map, it is not clear yet how the generalised Riemann operator should be interpreted.

Levi-Civita connection. The first set of such operators is composed of the ones involving mixed-indices and contractions of the non-mixed indices, i.e.:

$$
\begin{aligned}
D_{\bar a} v_+^b &= \nabla_{\bar a} v_+^b - \tfrac{1}{2} H_{\bar a}{}^b{}_c v_+^c \\
D_a v_-^{\bar b} &= \nabla_a v_-^{\bar b} + \tfrac{1}{2} H_a{}^{\bar b}{}_{\bar c} v_-^{\bar c} \\
D_a v_+^a &= \nabla_a v_+^a - 2(\partial_a \phi) v_+^a \\
D_{\bar a} v_-^{\bar a} &= \nabla_{\bar a} v_-^{\bar a} - 2(\partial_{\bar a} \phi) v_-^{\bar a}
\end{aligned}
$$

There are actually more operators than these. In appendix B it is explained how to introduce $Spin(d,d)$ spinor representations in E:[29] the Clifford algebra associated with η, i.e. $\{\Gamma_A, \Gamma_B\} = 2\eta_{AB}$, can be realised on each patch $U_{(i)}$ of the open covering $\{U_{(i)}\}$ of M by identifying spinors with weighted sums of forms $\Psi_{(i)} \in \Gamma((\det T^*U_{(i)})^{\frac{-1}{2}} \otimes \Lambda^{\cdot} T^*U_{(i)})$ with the Clifford action:

$$
X^A \Gamma_A \Psi_{(i)} = i_{v_{(i)}} \Psi_{(i)} + \lambda_{(i)} \wedge \Psi_{(i)} \qquad \text{for } X = v + \lambda \in \Gamma(E|_{U_{(i)}})
$$

where the patching works as follows: $\Psi_{(i)} = e^{d\Lambda_{ij}} \wedge \Psi_{(i)}$.[30] This means that $\Psi^{(B)} := e^{-B_{(i)}} \wedge \Psi_{(i)}$ is a well defined spinor of E. Note that, as explained in the appendix, the chiral spinors are associated with two $Spin(d,d)$ spinor bundles $S^{\pm}(E)$ that are isomorphic, when a splitting of E is chosen, to weighted sums of even or odd forms: $S^{\pm}(E) \approx (\det T^*M)^{\frac{-1}{2}} \otimes \Lambda^{\text{even/odd}} T^*M$. Finally we could also extend these spinors to spinors of $\tilde{E}$ by considering spinor representations of $Spin(d,d) \times \mathbb{R}^+$, that are weighted spinors of definite weight p.

Now, the $O(p,q) \times O(q,p)$-structure, i.e. the decomposition of $E = C_+ \oplus C_-$, allows for the introduction of $Spin(p,q)$ spinors. These are associated with the spinor bundles $S(C_\pm)$ associated with the $C_\pm$ subbundles. Let γ^a, $\gamma^{\bar a}$ the corresponding gamma matrices and $\epsilon^\pm \in \Gamma(S(C_\pm))$. We then have that, by definition, a generalised connection acts as:

$$
D_M \epsilon^+ = \partial_M \epsilon^+ + \tfrac{1}{4} \Omega_M{}^{ab} \gamma_{ab} \epsilon^+ \tag{2.32}
$$

$$
D_M \epsilon^- = \partial_M \epsilon^- + \tfrac{1}{4} \Omega_M{}^{\bar a \bar b} \gamma_{\bar a \bar b} \epsilon^- \tag{2.33}
$$

Then, according to what we stated before, there are four more uniquely determined operators:

$$
D_{\bar a} \epsilon^+ = \partial_{\bar a} \epsilon^+ + \tfrac{1}{4}(\omega_{\bar a bc} - \tfrac{1}{2} H_{\bar a bc}) \gamma^{bc} \epsilon^+ = (\nabla_{\bar a} - \tfrac{1}{8} H_{\bar a bc} \gamma^{bc}) \epsilon^+ \tag{2.34}
$$

$$
D_a \epsilon^- = \partial_a \epsilon^- + \tfrac{1}{4}(\omega_{a \bar b \bar c} + \tfrac{1}{2} H_{a \bar b \bar c}) \gamma^{\bar b \bar c} \epsilon^- = (\nabla_a + \tfrac{1}{8} H_{a \bar b \bar c} \gamma^{\bar b \bar c}) \epsilon^- \tag{2.35}
$$

[29] But the present argument is taken from [CoStWa2011]

[30] To understand why the patching works in this manner see, for instance, Example 2.10 in [Gu2004].

and

$$\begin{aligned}
\gamma^a D_a \epsilon^+ &= \gamma^a \Big(\partial_a + \tfrac{1}{4}\Omega_a{}^{bc}\gamma_{bc}\Big)\epsilon^+ \\
&= \gamma^a \Big(\partial_a + \tfrac{1}{4}(\omega_a{}^{bc} - \tfrac{1}{6}H_a{}^{bc} - \tfrac{2}{9}(\eta_{ab}\partial^c - \delta^c_a\partial^b)\phi + A^{+bc}_a)\gamma_{bc}\Big)\epsilon^+ \\
&= \Big(\gamma^a\nabla_a - \tfrac{1}{24}\gamma^{[a}\gamma^{bc]}H_{abc} - \tfrac{2}{36}(\gamma^b\partial^c - \gamma^c\partial^b)\phi\gamma_{bc} + \gamma^a\gamma^{bc}A^+_{abc}\Big)\epsilon^+ \\
&= \Big(\gamma^a\nabla_a - \tfrac{1}{24}\gamma^{abc}H_{abc} - \tfrac{1}{9}\gamma^b\partial^c\phi\tfrac{1}{2}(\gamma_b\gamma_c - \gamma_c\gamma_b) + \\
&\qquad + (\gamma^{abc} + \eta^{ab}\gamma^c - \eta^{ac}\gamma^b)A^+_{abc}\Big)\epsilon^+ \\
&\big|{\scriptstyle \gamma^b\frac{1}{2}(\gamma_b\gamma_c - \gamma_c\gamma_b) = \gamma^b\gamma_b\gamma_c - \gamma^b\eta_{bc} = 9\gamma_c} \\
&= \Big(\gamma^a\nabla_a - \tfrac{1}{24}\gamma^{abc}H_{abc} - \gamma^c\partial_c\phi + \gamma^{abc}A^+_{[abc]} + A^{+a}_{a\ \ c}\gamma^c + \eta^{ac}\gamma^b A^+_{acb}\Big)\epsilon^+ \\
&= \Big(\gamma^a\nabla_a - \tfrac{1}{24}\gamma^{abc}H_{abc} - \gamma^c\partial_c\phi\Big)\epsilon^+
\end{aligned} \tag{2.36}$$

and with the same calculation:

$$\gamma^{\bar{a}} D_{\bar{a}}\epsilon^- = \Big(\gamma^{\bar{a}}\nabla_{\bar{a}} + \tfrac{1}{24}\gamma^{\bar{a}\bar{b}\bar{c}}H_{\bar{a}\bar{b}\bar{c}} - \gamma^{\bar{c}}\partial_{\bar{c}}\phi\Big)\epsilon^- \tag{2.37}$$

These are all the possible uniquely defined operators. From them we can try to build four possible Ricci tensors:

$$\begin{aligned}
R_{a\bar{b}}v^a_+ &:= [D_a, D_{\bar{b}}]v^a_+ \\
R_{\bar{a}b}v^{\bar{a}}_- &:= [D_{\bar{a}}, D_b]v^{\bar{a}}_- \\
\tfrac{1}{2}R_{a\bar{b}}\gamma^a\epsilon^+ &:= [\gamma^a D_a, D_{\bar{b}}]\epsilon^+ \\
\tfrac{1}{2}R_{\bar{a}b}\gamma^{\bar{a}}\epsilon^- &:= [\gamma^{\bar{a}} D_{\bar{a}}, D_b]\epsilon^-
\end{aligned}$$

What turns out is that all these four equations actually define the same object. We cannot take any contraction of the generalised Ricci tensor. The Ricci scalar can nevertheless be defined in one of the following (equivalent) manners:

$$\begin{aligned}
-\tfrac{1}{4}S\epsilon^+ &:= (\gamma^a D_a\gamma^b D_b - D^{\bar{a}}D_{\bar{a}})\epsilon^+ \\
-\tfrac{1}{4}S\epsilon^- &:= (\gamma^{\bar{a}} D_{\bar{a}}\gamma^{\bar{b}} D_{\bar{b}} - D^a D_a)\epsilon^-
\end{aligned}$$

The fact that the generalised Ricci tensor and scalar are actually a tensor and a scalar can be seen by giving to them an explicit expression. In the gauge where $\hat{e}^+_a \equiv \hat{e}^-_{\bar{a}}$ these can be found to be (see [CoStWa2011]):

$$\begin{aligned}
R_{ab} &= \mathcal{R}_{ab} - \tfrac{1}{4}H_{acd}H_b{}^{cd} + 2\nabla_a\nabla_b\phi + \tfrac{1}{2}e^{2\phi}\nabla^c(e^{-2\phi}H_{cab}) \\
S &= \mathcal{R} + 4\nabla^2\phi - 4(\partial\phi)^2 - \tfrac{1}{12}H^2
\end{aligned}$$

From the second expression it is now clear that we can express the bosonic pseudo-action 2.6 as:

$$S_B = \frac{1}{2\kappa^2} \int \Big(\Phi S - \tfrac{\sqrt{-\det g}}{4} \sum_n \tfrac{1}{n!} (F_n^B)^2 \Big) \tag{2.38}$$

This finally shows what we promised at the beginning of the chapter. To conclude, we would like to make some final considerations. We see that without considering the extension $\tilde{E}$ of the generalised tangent bundle, and therefore including also the dilaton in our discussion, we could not have obtained all the uniquely defined operators needed to construct the Ricci tensor and scalar. This is another striking feature of the generalised geometry description: in order for the geometrisation of the $NSNS$ sector to be successful, we have to take into account the whole $NSNS$ field content inside the geometric construction.

In this chapter we described the geometrisation of the $NSNS$ sector. We introduced both a Ricci tensor and a Ricci scalar, whilst we have only used the second one. It can be shown that the Ricci tensor arises in the equations of motion obtained from the variation of the pseudo-action.

It is even possible a generalised covariant rewriting of the RR and fermionic sectors and a reformulation of all the equations of motions and supersymmetry variations in an explicit $O(9,1) \times O(1,9)$ covariant manner. These were not covered in the project and will take us too much space to be explained here. For a matter of completeness we will however disscuss very briefly how one can rewrite the RR term in the action via generalised geometry. Recall from the end of subsection 2.2.2 that the sum of the democratic RR field strengths can be treated collectively in the formalism of generalised geometry as a section of the chiral spin representation of $Spin(d,d)$, $F \in \Gamma(S^{\pm}_{\frac{1}{2}})$[31]. Now, an $O(p,q) \times O(q,p)$ structure (with $p+q=d$) provides two additional chirality operators (see [CoStWa2011] and references therein) $\Gamma^{(\pm)}$ on $Spin(d,d) \times \mathbb{R}^+$ spinors which one can define as:

$$\Gamma^{(+)} = \frac{1}{d!} \epsilon^{a_1 \cdots a_d} \Gamma_{a_1 \cdots a_d} \qquad \Gamma^{(-)} = \frac{1}{d!} \epsilon^{\bar{a}_1 \cdots \bar{a}_d} \Gamma_{\bar{a}_1 \cdots \bar{a}_d}$$

In the conformally orthonormal frame the Clifford action takes the form:

$$\Gamma_a \cdot \psi^{(B)} = i_{\hat{e}_a^+} \psi^{(B)} + e_a^+ \wedge \psi^{(B)} \qquad \Gamma_{\bar{a}} \cdot \psi^{(B)} = i_{\hat{e}_{\bar{a}}^+} \psi^{(B)} - e_{\bar{a}}^+ \wedge \psi^{(B)}$$

[31] One can actually straightforwardly take in consideration the weighted extension of E, $\tilde{E}$, and consider the $Spin(d,d) \times \mathbb{R}^+$ (chiral) spinors of weight p, i.e. sections of $S^{\pm}_{(p)} = (\det T^* M)^p \otimes S^{\pm}(E)$.

What is important for us is that, evaluated on the weighted n-form components of ψ, they become:

$$\Gamma^{(+)}\psi^{(B)}_{(n)} = (-1)^{[n/2]} * \psi^{(B)}_{(n)} \qquad \Gamma^{(-)}\psi^{(B)}_{(n)} = (-1)^{(d)}(-1)^{[(n+1)/2]} * \psi^{(B)}_{(n)}$$

Recalling the form of the Mukai pairing given at the end of section 2.2.2 we can finally state that the bosonic (pseudo-)action in the democratic formalism can be written in terms of generalised geometrical objects as follows:

$$S_B = \frac{1}{2\kappa^2} \int \Big(\Phi S + \tfrac{1}{4} \Big\langle F, \Gamma^{(-)} F \Big\rangle \Big) \tag{2.39}$$

Moreover, the self-duality conditions satisfied by the RR field strengths $F \in \Gamma(S^{\pm}_{\frac{1}{2}})$ become a chirality condition under the operator $\Gamma^{(-)}$: $\Gamma^{(-)}F = -F$. The interested reader can look up further details in the main references [CoStWa2011, CoStWa2012]. Finally, for a further reading about the generalised curvature operators one can for instance read [HoZw2012], where they are dealt in the very similar framework of 'double field theory'.

Chapter 3

Consistent Truncation

Like we mentioned in the introduction, it is known that consistency of the superstring theory on the worldsheet, and in particular the requirement of the absence of the conformal anomaly, requires having $d = 10$ space-time dimensions. There are only five string theories in 10-dimensions and all these theories have been proven to be related to an 11-dimensional theory, called 'M-theory', by 'dualities'. If we look at the supergravity theories as low-energy limits of string theories, their formulations in $d = 10$ and $d = 11$ acquire then a particular relevance. It is clear that, if these theories are really meant to describe our real world, it has to be possible to obtain from them a realistic effective theory in four dimensions. We need, therefore, to find a way to perform a sort of 'dimensional reduction'.

The idea that extra dimensions could be used to construct a unified theory of the fundamental interactions is not new and goes back to the times of Kaluza and Klein [Ka21Kl26]. In their works, they try to argue that electromagnetism and gravity together could be viewed as the effective four-dimensional theory of a five-dimensional pure gravitational theory. To obtain the effective theory in four dimensions they compactified the fifth dimension on a circle, and then sent its radius to zero.

Let us consider for a while this S^1-compactification. We can write the space-time coordinates as: $x^M = (x^\mu, x^4)$, with $\mu = 0, ..., 3$ and where x^4 is identified with $x^4 + 2\pi R_0$. Since x^4 is periodic, we can expand a generic scalar field $\Phi(x^M)$ in a Fourier series:

$$\Phi(x^M) = \sum_{n \in \mathbb{Z}} \phi_n(x^\mu) e^{in\frac{x^4}{R_0}}$$

Plugging this expression into the the Klein-Gordon equation for Φ, $\Box_5\Phi = M^2\Phi$, we obtain:

$$\Box_4\phi_n - \tfrac{n^2}{R_0^2}\phi_n = M^2\phi_n \quad \Leftrightarrow \quad \Box_4\phi_n = \underbrace{\left(M^2 + \tfrac{n^2}{R_0^2}\right)}_{m_n^2}\phi_n \tag{3.1}$$

This means that from a single scalar Φ in five dimensions we have obtained an infinite 'tower' of Kaluza-Klein (K-K) fields ϕ_n in four dimensions with masses m_n^2. In particular, from a massless scalar field in $d = 5$, i.e. with $M = 0$, we obtain another massless scalar ($m_0 = 0$) in four dimensions together with an infinite number of other fields, with masses $m_n = \frac{n}{R_0}$. If we now take the radius R_0 to be very small, we can neglect the massive modes and keep only the massless one. Note that the fact of retaining only the massless mode means that we are 'truncating' the field content of the full theory to a subset of it. Moreover, since from a massless scalar field $\Phi(x^M)$ in five dimensions we only get a massless scalar field in four dimensions $\phi_0(x^\mu)$, it is reasonable to state that the procedure of compactifying on a circle and then taking the small radius limit effectively corresponds to eliminating from the field(s) the dependence from the compactified coordinate. In sight of this, we can also see what happens to the metric. Consider a set of (dual) frames in five dimensions $\{e_N^A(x^M)\}$. Because of the local Lorentz invariance, it is possible to choose the following triangular parametrisation of the vielbeins [ScSc1979]

$$e_M^A(x^\nu) = \begin{pmatrix} e_\mu^a & A_\mu(x^\nu)\exp(\sigma(x^\nu)) \\ 0 & \exp(\sigma(x^\nu)) \end{pmatrix} \tag{3.2}$$

This implies that the five-dimensional metric can be written as:

$$\begin{aligned} ds^2 &= (g_{\mu\nu} + \exp(2\sigma)A_\mu A_\nu)dx^\mu dx^\nu + 2\exp(2\sigma)A_\mu dx^\mu dx^4 + \exp(2\sigma)dx^4 dx^4 \\ &= g_{\mu\nu}dx^\mu dx^\nu + \exp(2\sigma)(dx^4 + A_\mu dx^\mu)^2 \end{aligned}$$

The five-dimensional diffeomorphism invariance now reduces to four-dimensional diffeomorphism invariance plus the following gauge transformation:

$$\begin{cases} x^4 \mapsto x^4 - \lambda(x^\mu) \\ A_\mu \mapsto A_\mu + \partial_\mu\lambda \end{cases}$$

If one also calculates how the action is transformed assuming the independence of the fields from the compact dimension, one can easily find that there are no obstructions to this procedure and that the resulting four-dimensional theory contains the gravitational, gauge boson and scalar kinetic terms plus a

scalar dependent potential. Moreover, the resulting theory possesses a $U(1)$-gauge invariance. This process can easily be extended to the case in which k dimensions out of $d+k$ are compactified in a similar way. This means that the compactification now takes place on a k-dimensional torus $\mathbb{T}^k$. Looking at the complete metric G_{MN}, we see that now we get a d-dimensional metric $G_{\mu\nu}$, k vectors $G_{\mu i} = A^i_\mu$, $i = 1, ..., k$, and $\frac{k(k+1)}{2}$ scalars G_{ij}; the gauge group will now be the abelian $U(1)^k$.
From the toroidal dimensional reduction, we can understand some general phenomena typical of dimensional reductions. The first thing is that one usually *truncates* some degrees of freedom. For example, in the Kaluza-Klein theory the massive modes were eliminated taking the small radius limit. The second thing is that, in general, dimensional reductions give rise to gauge theories in the lower dimensions. The gauge group of the lower dimensional theory is related to the isometry group of the compactified manifold; for example in the toroidal case we had $\mathbb{T}^k \approx U(1)^k$. Finally, the process of a compactification followed by a truncation can usually be encoded in the restriction of the dependence of the fields on the compactified dimensions. For instance, in the K-K theory the dimensional reduction corresponded to eliminating this dependence from the fields.
We have said that a dimensional reduction is usually performed together with a truncation of the degrees of freedom of the theory. This means that the reduced theory is now different from the original one. We can then ask ourselves a natural question: when is a solution of the truncated system a solution of the full dimensional theory? If the solutions of the truncated system are also solutions of the complete theory we say that the truncation is *consistent*. To clarify the problem we can make an easy example. Let us consider a theory described by the following Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\lambda)^2 - \lambda\phi^2 \tag{3.3}$$

The Euler-Lagrange (E-L) equations of motion are easily found to be:

$$\begin{cases} \partial^2\phi = -2\lambda\phi \\ \partial^2\lambda = -\phi^2 \end{cases}$$

If we now set $\lambda \equiv 0$ the Lagrangian reduces to $\mathcal{L} = \frac{1}{2}(\partial\phi)^2$, with E-L equation $\partial^2\phi = 0$, whilst the former equations of motion become $\partial^2\phi = 0$ *and* $\phi^2 = 0$; the truncation is then inconsistent unless $\phi \equiv 0$ as well. On the other hand if we instead set $\phi \equiv 0$ we now get $\mathcal{L} = \frac{1}{2}(\partial\lambda)^2$. The equations of motion of the truncated Lagrangian are the same equations that one would have obtained if he had set $\phi \equiv 0$ in the non-truncated E-L equations, i.e. $\partial^2\lambda = 0$. Thus truncating ϕ is consistent, whilst truncating λ is inconsistent.

Now that we have understood the problem, let us go back to the question of when it is possible to have a consistent truncation. From the point of view of the restricted dependence of the various fields on the compactified coordinates $\{y^i\}$, we can see that this question can be translated to the following one: when can we factorise the $\{y^i\}$ dependence out from the action and transformation laws? There is no complete answer to this question yet, but there is a standard class of cases where this is always possible: (local) group manifolds. By local group manifolds we mean the quotients of the type: G/Γ, where G is a Lie group and Γ is a discrete, freely-acting subgroup of G [CoStWa2012]. Let us suppose that Γ acts on G from the left. The main reason why the truncation results to be consistent can be identified in the fact that such a (local) Lie group admits a *parallelisation* given by the generators of the right translation, i.e. given by the $\dim G$ left-invariant vector fields $\{\hat{e}_a(y)\}$. This is a very special kind of parallelisation because these frames satisfy:

$$[\hat{e}_a(y), \hat{e}_b(y)] = f_{ab}{}^c \hat{e}_c(y)$$

where $[\,,\,]$ is the Lie bracket and $f_{ab}{}^c$ are the structure *constants*. Their dual frames, given by the left-invariant one forms $\{e^a(y)\}$, satisfy the following Cartan structure equation: $de^a = -\frac{1}{2} f_{bc}{}^a (e^b \wedge e^c)$[1], where the $f_{bc}{}^a$ are still the structure constants associated with G. Let us consider again the generic triangular parametrisation of the vielbeins for the total metric G_{MN} of equation 3.2. In that case we assumed that all the fields were independent of y. The classical result of Scherk and Schwarz [ScSc1979], instead, says that it is possible to allow for a y-dependence, as long as this is encoded in the left-invariant vielbeins:

$$\begin{pmatrix} \omega_\mu^m(x) & A_\mu^\alpha(x)\Phi_\alpha^i(x) \\ 0 & \Phi_\alpha^i(x) \end{pmatrix} \mapsto$$
$$\mapsto \begin{pmatrix} \omega_\mu^m(x) & [A_\mu^a(x)\hat{e}_a^\alpha(y)][e_\alpha^b(y)\Phi_b^i(x)] \\ 0 & e_\alpha^a(y)\Phi_a^i(x) \end{pmatrix} =: \begin{pmatrix} \omega_\mu^m & (A')_\mu^\alpha(x,y)(\Phi')_\alpha^i(x,y) \\ 0 & (\Phi')_\alpha^i(x,y) \end{pmatrix}$$

where ω_μ^m are the vielbeins for the d-dimensional metric $g_{\mu\nu}$, and where $\mu = 1, ..., d$ and $\alpha, \beta = d+1, ..., d+k$ are spacetime indices, $m = 1, ..., d$ and $i, a, b = d+1, ..., d+k$ are tangent space indices.

In practice the Scherk and Schwarz reduction prescribes an ansatz for the allowed y-dependence. This ansatz requires the space-time indices related to

---

[1] Indeed we have:

$$de^a(\hat{e}_b, \hat{e}_c) = \hat{e}_b(e^a(\hat{e}_c)) - \hat{e}_c(e^a(\hat{e}_b)) - e^a([\hat{e}_b, \hat{e}_c]) = -f_{bc}{}^d \delta_d^a$$

the y-coordinates to be replaced by the corresponding tangent space indices and then to be contracted with the relative left-invariant vielbeins. Since we are imposing a restriction on the y-dependence, this also implies a restriction on the symmetry under general coordinate transformations. Using the prescription on the generators $\xi^M(x,y)$ of the $d+k$ diffeomorphisms themselves, we can now identify two different generators:

$$\begin{cases} \xi^\mu(x,y) \equiv \xi^\mu(x) \\ \tilde{\xi}^\alpha(x,y) = \hat{e}_a^\alpha(y)\tilde{\xi}^a(x) \end{cases}$$

If one now computes the resulting algebra he will find that $\{\xi^\mu(x)\}$ generates the d-dimensional general coordinates transformations, whilst for the other generators we have:

$$\begin{cases} [\tilde{\xi}_1(x,y), \xi_2(x)]^\alpha = \hat{e}_a^\alpha(y)(-\xi_2^\mu(x)\partial_\mu\tilde{\xi}_1^a(x)) := \hat{e}_a^\alpha(y)\xi_3^a(x) \\ [\tilde{\xi}_1(x,y), \tilde{\xi}_2(x,y)] = (\tilde{\xi}_1^a(x)\tilde{\xi}_2^b(x)\, f_{ab}{}^c)\hat{e}_a^\alpha(y) := \hat{e}_a^\alpha(y)\xi_3^a(x) \end{cases}$$

or, in other words, under d-dim diffeomorphisms $\tilde{\xi}_2^a \mapsto \tilde{\xi}_3^a(x) = -\xi^\mu(x)\partial_\mu\xi_2^a(x)$, i.e. they are *scalars*, and under the restricted transformations $\tilde{\xi}_2^a(x) \mapsto \tilde{\xi}_3^a(x) = f_{bc}{}^a\tilde{\xi}^b(x)\tilde{\xi}^c(x)$. This fact shows how important the fact that the $f_{ab}{}^c$ are *constants* is: if they were not constant we could not have factorised the y dependence out of these transformation laws. In particular, one could now also calculate the transformation laws of the x-dependent parts of the fields – e.g., if $(A')_\mu^\alpha(x,y) = \hat{e}_a^\alpha(y)A_\mu^a(x)$, the transformation of $A_\mu^a(x)$ – and see that the y dependence actually factors out of everywhere. Moreover $\delta A_\mu^a(x) \sim \partial_\mu\tilde{\xi}^a(x) + f^a_{\ bc}\tilde{\xi}^b(x)A_\mu^c(x)$, i.e. the $A_\mu^a(x)$ are now gauge potentials for the gauge group G. Finally the action (see equation (38) in [ScSc1979]), that results to be just the non abelian generalisation of the action of the traditional toroidal reduction plus a potential term, is, up to an factor, y-independent. This fact happens exactly because the $f_{ab}{}^c$ are constants. Requiring the invariance of the action under $\tilde{\xi}^\alpha(x,y)$ transformations can be shown to be equivalent to $\partial_\alpha[\hat{e}_a^\alpha \det(e)] = 0$. This, in turn, can be translated into the 'unimodularity' condition on the structure constants: $f_{ab}{}^b = 0$.[2]
We have given the idea of how, in the Scherk-Schwarz reduction, the y-dependence factors out of the action and transformation laws. This clearly shows that this truncation is consistent. Moreover, the ordinary Scherk-Schwarz reduction can be easily shown to be valid not only for pure gravity but also for gravity coupled to other fields [ScSc1979]. In particular, it can be used for supergravity theories. Finally, we note that we can also set $A_\mu^a \equiv 0$

[2]Note that this relation is often satisfied by Lie algebras; one remarkable example is given by semisimple Lie algebras.

in the Scherk-Schwarz ansatz for the full-dimensional vielbeins. In that case, we will obtain a split metric:[3]

$$
\begin{aligned}
ds^2_{d+k} = ds^2_d + ds^2_k &= g_{\mu\nu}dx^\mu dx^\nu + (\Phi^i_a(x)\delta_{ij}\Phi^j_b(x))e^a(y)e^b(y) \\
&= g_{\mu\nu}dx^\mu dx^\nu + h_{ab}(x)e^a(y)e^b(y)
\end{aligned}
$$

The matrix h_{ab} is the matrix that determines the scalar degrees of freedom in the truncated theory and is completely determined by the 'twist' matrix $\Phi^i_a(x)$.

Apart from the ordinary Scherk-Schwarz reduction, only a few other cases are known to admit a consistent truncation. For example, consistent reductions on S^7 and S^4 for 11-dimensional supergravity, S^5 for Type IIB and S^3 for the $NSNS$ sector of Type II supergravity, where the resulting gauge group G can be either $SU(2)$ (from the ordinary Scherk-Schwarz reduction) or $SU(2)\times SU(2)$. The idea of the work [LeStWa2014] is that all these cases can be described as the generalised geometrical analogue of the Scherk-Schwarz reduction. Indeed, in each of the previous cases the compactification manifold is equipped with a *global* doubly orthonormal generalised frame $\{\hat{E}_A\}$ of a generalised tangent bundle of the form $E \approx TM \oplus \Lambda^p(T^*M)$, with $p \in \mathbb{Z}$, that satisfies the following generalised geometric analogue of the Lie algebra relations:

$$
L_{\hat{E}_A}\hat{E}_B = X_{AB}{}^C\hat{E}_C \tag{3.4}
$$

with *constant* $X_{AB}{}^C$ and where L indicates the generalised Lie derivative. Since we know that the Dorfman derivative is generally not antisymmetric, equation 3.4 does not in general define a Lie algebra, but instead a Leibniz (or Loday) algebra. Such a parallelisation of E will be called from now on: '*generalised Leibniz parallelisation*'[4]. Since the $\{\hat{E}_A\}$ are globally defined, the generalised tangent bundle is trivial. It admits, therefore, a trivial spinor bundle (see appendix B) and also globally defined spinors. This suggests that these 'generalised Scherk-Schwarz' reductions preserve the number of supersymmetries. These ideas inspired the following conjecture [LeStWa2014]:

Conjecture 3.0.1. *Let be $\hat{E}_A$ be a global generalised frame for the generalised tangent bundle E of a manifold M that generates a Leibniz algebra accordingly to equation 3.4. Then, there is a consistent truncation on M preserving the same number of supersymmetries of the original theory. Moreover, the scalars of the theory are encoded by: $(\Phi')^I_M(x,y) = \Phi^I_B(x)E^B_M(y)$.*

[3]Note that these terms are also present in the general case. We set $A^a_\mu \equiv 0$ just for simplicity.

[4]See also definition 3.1.

In what follows we will only be concerned with $E \approx TM \oplus T^*M$. We will first show that the $SU(2) \times SU(2)$ gauging of an S^3 compactification comes from a generalised Scherk-Schwarz reduction. We will then discuss some general properties of the generalised Leibniz parallelisations and re-derive with our formalism the known result that on any (compact) Lie group G we can produce a $G \times G$ gauging. We will then focus on the generalised Leibniz parallelisation of $S^2 \times S^1$, found by De Felice [De2014], and show that this actually derives from a particular Inonu-Wigner contraction of the S^3 case. Finally, in the next chapter, we will try to explore the possibility of other 'particular' Inonu-Wigner contractions for general Lie groups, with the hope that this will eventually lead us to new examples of generalised Leibniz parallelisations.

3.1 Generalised Leibniz Parallelisation of S^3

In this section, we will present one particular case of the result in [LeStWa2014], which asserts that S^3 admits a generalised Leibniz parallelisation on $E \approx TM \oplus T^*M$. This parallelisation will actually turn out to generate a Lie algebra. Instead of the $\mathfrak{su}(2)$ that one would have obtained with the ordinary Scherk-Schwarz reduction, the resulting algebra will be $\mathfrak{su}(2) \times \mathfrak{su}(2)$. In this section, many general aspects of generalised Leibniz parallelisations on E will implicitly come out. In the following sections, we will then try to better understand how this example fits in with the bigger picture and how it can be possibly extended.

In order to find a generalised Leibniz parallelisation, we need to find a global generalised frame $\{\hat{E}_A\}$ for E that satisfies equation 3.4, i.e.: $L_{\hat{E}_A}\hat{E}_B = X_{AB}{}^C\hat{E}_C$, with constant $X_{AB}{}^C$. We recall from section 1.3 that the isomorphism between E and $TM \oplus T^*M$ depends on a choice of a collection of well patched two forms $B = \{B_{(i)}\}$, where each $B_{(i)}$ is defined on the patch $U_{(i)}$ of an open covering $\{U_{(i)}\}$ of M. Therefore, once the choice of B is made, a general section X of E is represented, through the isomorphism onto $TM \oplus T^*M$, by $X = v + \lambda + i_vB$, with $v \in \Gamma(TM)$, $\lambda \in \Gamma(T^*M)$. Now, let us again consider equation 3.4. Once the isomorphism between E and $TM \oplus T^*M$ is taken into account, the generalised Lie derivative can be expressed as: (from the proof of proposition 1.1.4):

$$\begin{aligned} &L_{\exp(-B)(\tilde{X})}\exp(-B)(\tilde{Y}) = \exp(-B)\Big([v,w] + \mathcal{L}_v\mu - i_w(d\lambda) - i_vi_wH\Big) \\ &X, Y \in \Gamma(E), \tilde{X} = v + \lambda, \tilde{Y} = w + \mu \in \Gamma(TM \oplus T^*M) \end{aligned} \tag{3.5}$$

It then appears that, if we want equation 3.5 to be satisfied, the choice of B cannot be arbitrary, but instead it has to be rigorously made in order to give rise to a well behaved three-form flux $H = dB$. Consider for a moment the vector part of equation 3.4. We need to find $2d = 2 \cdot 3 = 6$ vectors on S^3 that close into a Lie algebra. This is not a difficult quest: imagine to immerse S^3 in $\mathbb{R}^4$. Then the action of the rotation group in four-dimensions, i.e. $G = SO(4)$, spans the three-sphere. In other words S^3 is an orbit of the action of $SO(4)$ on $\mathbb{R}^4$. Since the dimension of G is $\frac{4\times 3}{2} = 6$, we can try to use the generators of this action on S^3 as the vector components of our generalised parallelisation.[5] Since we are looking at S^3 as the orbit of the action of $SO(4)$ on $\mathbb{R}^4$, it is then natural for us to use neither a parametric, nor a coordinate description of the manifold, but instead its definition through Cartesian equations. In particular we will use Cartesian coordinates $\{y\}$ on $\mathbb{R}^4$ subjected to the constraint: $\sum_{i=1}^4 y_i y_i = 1$. For the same reason we will induce on S^3 a metric from $\mathbb{R}^4$. We want to remark that the choice of the metric is very important in order to find a generalised Leibniz parallelisation. The right choice turns out to be the metric induced from the Euclidean one on $\mathbb{R}^4$, that is also equal to the standard round metric on the three-sphere:

$$\begin{cases} ds_3^2 = dy_1^2 + dy_2^2 + dy_3^2 + dy_4^2 \\ \sum_i y_i y_i = 1 \end{cases} \tag{3.6}$$

Let us find the generators of the $SO(4)$ action, which is actually given by the vector representation. Take the vector representation of the Lie algebra $\mathfrak{so}(4)$. This is composed of the matrices of components: $(v_{ij})_{ab} := (\delta_{ja}\delta_{ib} - \delta_{ia}\delta_{jb})$, i.e. of antisymmetric matrices. The components of the generators of the $SO(4)$ action on $\mathbb{R}^4$ are then given by:

$$[v_{ij}^{\#}(y)]_a = \frac{d}{dt}\Big[exp(tv_{ij})\Big]_{ab} y^b\Big|_{t=0} = [v_{ij}]_{ab} y^b = (\delta_{aj} y_i - \delta_{ai} y_j)$$

We can eventually say that the generators of this action are:

$$v_{ij}^{\#}(y) = (\delta_{aj} y_i - \delta_{ai} y_j)\partial_a = (y_i \partial_j - y_j \partial_i)$$

From the way we found them, viewing S^3 as an $SO(4)$ orbit, it is then clear that, even if they are expressed in coordinates of $\mathbb{R}^4$, these generators are

[5] Note that this gives an idea of the reason why to produce a generalised parallelisation of a d-sphere one makes use of a different generalised tangent bundle, namely $E' \approx TM \oplus \Lambda^{d-2}(T^*M)$, whose dimension is actually $d + \binom{d}{d-2} = d + \frac{d(d-1)}{2} = \frac{d(d+1)}{2}$. In that case we have that the d-sphere is related to the action of $SO(d+1)$ on $\mathbb{R}^{d+1}$, and $SO(d+1)$ is $\frac{d(d+1)}{2}$-dimensional.

actually vectors of the tangent bundle of S^3. To simplify the notation we will drop from now on the # superscript and will indicate them simply by v_{ij}. We can now show that the metric 3.6 has $SO(4)$ as a group of isometries. We will do this by showing that these generators are actually killing vectors for 3.6. Indeed we have: $\mathcal{L}_{v_{ij}}(g(\partial_k, \partial_l)) = \mathcal{L}_{v_{ij}}\delta_{kl} = 0$. Moreover, since $\mathcal{L}_{v_{ij}}\partial_k = \delta_{kj}\partial_i - \delta_{ki}\partial_j$, we have:

$$g(\delta_{kj}\partial_i - \delta_{ki}\partial_j, \partial_l) + g(\partial_k, \delta_{lj}\partial_i - \delta_{li}\partial_j) = \delta_{kj}\delta_{il} - \delta_{ki}\delta_{jl} + \delta_{ki}\delta_{lj} - \delta_{jk}\delta_{li} = 0$$

thus proving that $\mathcal{L}_{v_{ij}}g = 0$. The group $SO(4)$ is then the isometry group of S^3 and, since its action is clearly transitive, S^3 is a Riemannian homogeneous space (see e.g. [Pe2006]).
The only thing we are still left to specify is the three-form flux H. This will be chosen to be the following one:

$$H := \frac{1}{3}\,\epsilon_{i_1 i_2 i_3 i_4}\, y^{i_1}\, dy^{i_2} \wedge dy^{i_3} \wedge dy^{i_4}$$

where ϵ is the Levi-Civita tensor. We are now ready to prove the main statement of this section.

Proposition 3.1.1. *The following frame is a generalised Leibniz parallelisation for S^3:*

$$\hat{E}_{ij} := v_{ij} + \sigma_{ij} + i_{v_{ij}}B \tag{3.7}$$

with

$$v_{ij} = (y_i\partial_j - y_j\partial_i) \qquad \text{and} \qquad \sigma_{ij} = \star(dy_i \wedge dy_j) = \epsilon_{ijkl}\, y^k\, dy^l$$

where the star indicates the Hodge star operator and ϵ the Levi-Civita tensor.

Proof. Since the frames in 3.7 are labelled by two antisymmetric indices (ij) that range from 1 to 4, we see that they are 6 in total. Since they are linearly independent, they form a basis for E. Let us first verify that they are also globally defined. If $v_{ij}(y) = (y_i\partial_j - y_j\partial_i) = 0$ then $y_i = y_j = 0$. If also $\sigma_{ij} = \epsilon_{ijkl}\, y^k\, dy^l = \epsilon_{ij\hat{k}\hat{l}}(y^{\hat{k}}\, dy^{\hat{l}} - y^{\hat{k}}\, dy^{\hat{l}}) = 0$, where the hat indicates that the summation convention is not understood any more (recall that we only have four possible values for the indices), this in turn implies $y^k = y^l = 0$. Therefore $\hat{E}_{ij}(y) = 0$ if and only if $y^i = 0\,\forall\, i$. But this is impossible, since the constraint on the coordinates imposes: $\sum_i y_i y_i = 1$. So $\hat{E}_{ij}(y) \neq 0\,\forall\, y \in S^3$.
Let us now consider the Leibniz algebra relation. The vector part is straightforward and can be easily proven to give:

$$[v_{ij}, v_{kl}] = (\delta_{jk}\delta_i^m\delta_l^n - \delta_{jl}\delta_i^m\delta_k^n - \delta_{ik}\delta_j^m\delta_l^n + \delta_{il}\delta_j^m\delta_k^n)(y_m\partial_n - y_n\partial_m) =: C_{ij}{}^{mn}_{kl}v_{mn}$$

where $C_{ij\ kl}^{\ \ \ mn}$ are clearly the $\mathfrak{so}(4)$ structure constants. We are therefore left with:

$$L_{\hat{E}_{ij}} \hat{E}_{kl} = C_{ij\ kl}^{\ \ \ mn}(v_{mn} + i_{v_{mn}} B) + \Big[\underbrace{\Big(\mathcal{L}_{v_{ij}} \sigma_{kl}}_{J} + \underbrace{- i_{v_{kl}}(d\sigma_{ij})\Big) + i_{v_{kl}} i_{v_{ij}} H}_{I}\Big]$$

Now consider $I = i_{v_{kl}}(i_{v_{ij}} H - d\sigma_{ij})$. The first term in the parenthesis is:

$$\begin{aligned}
i_{v_{ij}}(\frac{1}{3}\,\epsilon_{i_1 i_2 i_3 i_4}\, y^{i_1}\, dy^{i_2} \wedge dy^{i_3} \wedge dy^{i_4}) &= \frac{1}{3}\,\epsilon_{i_1 i_2 i_3 i_4}\, y^{i_1} (dy^{i_2}(v_{ij}) \wedge dy^{i_3} \wedge dy^{i_4} + \\
&\quad - dy^{i_2} \wedge dy^{i_3}(v_{ij}) \wedge dy^{i_4} + dy^{i_2} \wedge dy^{i_3} \wedge dy^{i_4}(v_{ij})) \\
&= \frac{1}{3}\,\epsilon_{i_1 i_2 i_3 i_4}\, y^{i_1} \big((y_i \delta_j^{i_2} - y_j \delta_i^{i_2}) dy^{i_3} \wedge dy^{i_4} - (y_i \delta_j^{i_3} - y_j \delta_i^{i_3}) dy^{i_2} \wedge dy^{i_4} + \\
&\qquad + (y_i \delta_j^{i_4} - y_j \delta_i^{i_4}) dy^{i_2} \wedge dy^{i_3}\big) \\
&= \epsilon_{i_1 i_2 i_3 i_4}\, y^{i_1} \big((y_i \delta_j^{i_2} - y_j \delta_i^{i_2}) dy^{i_3} \wedge dy^{i_4}\big) \\
&= -\epsilon_{i_2 i_1 i_3 i_4}\, (y_i \delta_j^{i_2} - y_j \delta_i^{i_2})\, y^{i_1}\, dy^{i_3} \wedge dy^{i_4} \\
&= -\big(y_i \epsilon_{j k_1 k_2 k_3} - y_j \epsilon_{i k_1 k_2 k_3}\big)\, y^{k_1}\, dy^{k_2} \wedge dy^{k_3} \\
&= -\big(y_i \epsilon_{j k_1 k_2 k_3} - y_j \epsilon_{i k_1 k_2 k_3}\big)\, y^{[k_1}\, dy^{k_2} \wedge dy^{k_3]}
\end{aligned}$$

But the total antisymmetrisation of five indices out of four is zero, and so:

$$0 = 5\, y_{[i} \epsilon_{j k_1 k_2 k_3]} = (y_i \epsilon_{j k_1 k_2 k_3} - y_j \epsilon_{i k_1 k_2 k_3}) + \underbrace{y_{k_1} \epsilon_{i j k_2 k_3} - y_{k_2} \epsilon_{i j k_1 k_3} + y_{k_3} \epsilon_{i j k_1 k_2}}_{3 y_{[k_1} \epsilon_{|ij| k_2 k_3]}}$$

And hence we get:

$$\begin{aligned}
i_{v_{ij}} H &= 3 y_{k_1} \epsilon_{i j k_2 k_3}\, y^{[k_1}\, dy^{k_2} \wedge dy^{k_3]} \\
&= \cancel{3} y_{k_1} \epsilon_{i j k_2 k_3}\, (\frac{1}{\cancel{3}}[y^{k_1}\, dy^{k_2} \wedge dy^{k_3} + \underbrace{\cancel{y^{k_2}\, dy^{k_3} \wedge dy^{k_1}} + \cancel{y^{k_3}\, dy^{k_1} \wedge dy^{k_2}}}_{d \sum_i y_i y_i = 0\ \Rightarrow\ \sum_i y_i dy_i = 0}]) \\
&= \epsilon_{i j k_2 k_3} dy^{k_2} \wedge dy^{k_3}
\end{aligned}$$

The other term in I is therefore equal an opposite to this, since: $d\sigma_{ij} = d\Big(\star(dy_i \wedge dy_j)\Big) = d\Big(\epsilon_{i j k_2 k_3}\, y^{k_2}\, dy^{k_3}\Big) = \epsilon_{i j k_2 k_3}\, dy^{k_2} \wedge dy^{k_3}$. We have therefore proven that $I = 0$. We are now left with $J = \mathcal{L}_{v_{ij}} \sigma_{kl}$. But since the v_{ij} are killing vectors, the Lie derivative commutes with the Hodge star operator:

$$\begin{aligned}
\mathcal{L}_{v_{ij}} \star (dy^k \wedge dy^l) &= \star\big((\mathcal{L}_{v_{ij}} dy^k) \wedge dy^l + dy^k \wedge (\mathcal{L}_{v_{ij}} dy^l)\big) \\
&= \star\big((d\mathcal{L}_{v_{ij}} y^k) \wedge dy^l + dy^k \wedge (d\mathcal{L}_{v_{ij}} y^l)\big) \\
&= \star\big((dy_i \delta_j^k - dy_j \delta_i^k) \wedge dy^l + dy^k \wedge (dy_i \delta_j^l - dy_j \delta_i^l)\big) \\
&= (\delta_j^k \sigma_{il} - \delta_i^k \sigma_{jl} + \delta_j^l \sigma_{ki} - \delta_i^l \sigma_{kj}) = (\delta_j^k \sigma_{il} - \delta_i^k \sigma_{jl} - \delta_j^l \sigma_{ik} + \delta_i^l \sigma_{jk}) \\
&= (\delta_{jk} \delta_i^m \delta_l^n - \delta_{ik} \delta_j^m \delta_l^n - \delta_{jl} \delta_i^m \delta_k^n + \delta_{il} \delta_j^m \delta_k^n) \sigma_{mn} = C_{ij\ kl}^{\ \ \ mn} \sigma_{mn}
\end{aligned}$$

This proves that

$$L_{\hat{E}_{ij}}\hat{E}_{kl} = C_{ij}{}^{mn}_{kl}\Big(v_{mn} + \sigma_{mn} + i_{v_{mn}}B\Big)$$

which is what we wanted. □

We have found that these generalised frames generate an $\mathfrak{so}(4)$ Lie algebra. This is however not enough to assert that they form a Leibniz parallelisation: we also need the double orthonormality condition to be respected. One can easily check that these generalised vector fields are not doubly orthonormal. However if one considers, instead, their self-dual and antiself-dual combinations: $E^{\pm}_{ij} := E_{ij} + \frac{1}{2}\epsilon_{ijkl}E_{kl}$, he can find that only six of them are different from each other. Let us isolate these six generalised vectors:

$$\begin{aligned} E^{\pm}_1 &= E_{12} \pm E_{34}, \\ E^{\pm}_2 &= E_{14} \pm E_{23}, \\ E^{\pm}_3 &= E_{42} \pm E_{13} \end{aligned}$$

Then it is straightforward to show that they satisfy: $[E^{\pm}_a, E^{\pm}_b] = 2\epsilon_{abc}E^{\pm}_c$ and $[E^{\pm}_a, E^{\mp}_b] = 0$. Moreover $\eta(E^{\pm}_a, E^{\pm}_b) = \pm\delta_{ab}$, $G(E^{\pm}_a, E^{\pm}_b) = \delta_{ab}$ and $G(E^{\pm}_a, E^{\mp}_b) = \eta(E^{\pm}_a, E^{\mp}_b) = 0$. They are therefore the generalised parallelisation we were looking for.
This shows, as it is known, that $\mathfrak{so}(4) \approx \mathfrak{so}(3) \times \mathfrak{so}(3) = \mathfrak{su}(2) \times \mathfrak{su}(2)$. So we have eventually found the gauging we stated at the beginning of the section.

3.2 General properties of Leibniz parallelisations on E

Let us start with a clear definition of a Leibniz parallelisation on E.

Definition 3.1. Let be $E \approx TM \oplus T^*M$ the (standard) generalised vector bundle on the manifold M. A generalised Leibniz parallelisation on E is a global generalised frame $\{\hat{E}_A\}$ on E that is 'doubly orthonormal', in the sense that:

$$\eta(\hat{E}_A, \hat{E}_B) = \eta_{AB} \tag{3.8}$$

$$G(\hat{E}_A, \hat{E}_B) = G_{AB} \tag{3.9}$$

and that satisfies equation 3.4 with constant $X_{AB}{}^C$ for all A, B.

We will use here the following generalised metric:

$$G = \frac{1}{2}\begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix} \tag{3.10}$$

which differs from the one in equation 1.20 by a redefinition of: $B \mapsto -B$, in order to reproduce the isomorphism: $\phi : E \to TM \oplus T^*M$ such that $\phi(X) = v^X + \lambda^X + i_{v^X} B$.
The generalised Leibniz frame we gave for S^3 in section 3.1 generated actually a Lie algebra. Even if this in not always the case for general generalised tangent bundles (like the ones that appear in exceptional generalised geometry), for $E \approx TM \oplus T^*M$ this is *always* true. Indeed from point 1. of proposition 1.1.4 we have:

$$L_{\hat{E}_A}\hat{E}_B + L_{\hat{E}_B}\hat{E}_A = 2d\Big(\eta(\hat{E}_A, \hat{E}_B)\Big) = 0 \tag{3.11}$$

This means that $X_{AB}{}^D\hat{E}_D = L_{\hat{E}_A}\hat{E}_B = -L_{\hat{E}_B}\hat{E}_A = X_{BA}{}^D\hat{E}_D$. Thus contracting with $\eta(\hat{E}_C,\ )$ we find antisymmetry in the first two indices: $X_{ABC} = -X_{BAC}$. Recalling that the Dorfman derivative satisfies the Leibniz rule, we see that the Leibniz algebra relation reduces on (the standard) E to be a Lie algebra relation, i.e. the generalised structure constants $X_{AB}{}^C$ can be viewed as structure constants for a Lie algebra. Moreover, from equation 3.8, the point 4. of proposition 1.1.4 and equation 3.4, we can write

$$0 = L_{\hat{E}_A}\eta(\hat{E}_B, \hat{E}_C) = X_{AB}{}^D\eta_{DC} + \eta_{BD}X_{AC}{}^D = X_{ABC} + X_{ACB}$$

This, together with equation 3.11, implies that the X are totally antisymmetric: $X_{ABC} \equiv X_{[ABC]}$. Let us now consider again equation 3.4 once the isomorphism between E and $TM \oplus T^*M$ has been chosen. Then, from equation 3.5 we can write:[6]

$$\begin{aligned} L_{\exp(-B)(\tilde{E}_A)}\exp(-B)(\tilde{E}_B) &= \exp(-B)([v_A, v_B] + \mathcal{L}_{v_A}\lambda_B - i_{v_B}(d\lambda_A) - i_{v_A}i_{v_B}H) \\ &= X_{AB}{}^C(v_C + \lambda_C + i_{v_C}B) \end{aligned}$$

where we have written the frames $\{\hat{E}_A\}$ as $\{\hat{E}_A = \exp(-B)\tilde{E}_A = v_A + \lambda_A + i_{v_A}B\}$ in light of the isomorphism $E \approx TM \oplus T^*M$. If we now compare the vector components of the last equation we find $[v_A, v_B] = X_{AB}{}^C v_C$. This means that the *vector* components of the Leibniz parallelisation generate a Lie algebra with constant structure coefficients; they can therefore be viewed as generators of an action of a Lie group on M. Moreover, since $\{\hat{E}_A\}$ generates E, $\{v_A\}$ generates TM. This implies that, locally, the manifold is an orbit of

[6] Where the hat has been removed from the frames for notational convenience.

the Lie group action and, therefore, that M is locally a homogeneous space. It is then reasonable to focus our attention on homogeneous spaces. They can always be represented as coset spaces of the type: G/H, where both G, H are Lie groups and where H is a closed subgroup of G. In particular, G is a group that acts transitively on M and H is the 'isotropy subgroup' of the action of G on M. This means that, at each point in the manifold, there will be a subgroup of G isomorphic to H whose action is trivial on M. We called H 'the' isotropy subgroup because, if a manifold is connected, one can prove that if H is the isotropy group at $p \in M$ then it is also the isotropy group at every other point in M. We will from now on always assume that our manifold is connected (possibly by restricting ourselves to one of its connected components).

3.2.1 Double Orthonormality and Generalised Killing Frames

Double Orthonormality

We now want to solve the constraints given by the equations 3.8 and 3.9 in the case where the metric g on M is Riemannian. Note that this is the general case used in compactification problems. We have already found a solution of this problem in proposition 2.3.1. We will however solve this problem again, by means of a much easier proof, and will also explain some interesting consequences of the solution in the context of generalised Leibniz parallelisations.

Let $\{\hat{E}_A = v_A + \lambda_A + i_{v_A}B\}$[7] be a doubly orthonormal frame for E. We can then see that:

$$G(\hat{E}_A, \hat{E}_B) = (v_A^T, \lambda_A^T - (Bv_A)^T)\frac{1}{2}\begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1}\end{pmatrix}\begin{pmatrix} v_B \\ \lambda_B - Bv_B\end{pmatrix}$$
$$= (v_A^T, \lambda_A^T + v_A^T B)\frac{1}{2}\begin{pmatrix} gv_B - Bg^{-1}\lambda_B \\ g^{-1}\lambda_B\end{pmatrix} = \frac{1}{2}(v_A^T\, g\, v_B + \lambda_A^T\, g^{-1}\lambda_B)$$

and similarly:

$$\eta(\hat{E}_A, \hat{E}_B) = (v_A^T, \lambda_A^T - (Bv_A)^T)\frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0\end{pmatrix}\begin{pmatrix} v_B \\ \lambda_B - Bv_B\end{pmatrix}$$
$$= (v_A^T, \lambda_A^T + v_A^T B)\frac{1}{2}\begin{pmatrix} \lambda_B - Bv_B \\ v_B\end{pmatrix} = \frac{1}{2}(v_A^T\lambda_B + \lambda_A^T v_B)$$

[7] Recall that the inner product makes use of the contraction with the first covariant component of B, whilst the product of a matrix with a vector from the right makes use of the second component. Therefore, we have $i_v B \equiv -Bv$.

Let us now take into account the isomorphism between TM and T^*M induced by g. We can then write: $\lambda_A = i_{w_A} g \equiv g w_A$ for some $\{w_A\}$. This allows us to write:

$$G_{AB} = G(\hat{E}_A, \hat{E}_B) = \frac{1}{2}(v_A^T g v_B + w_A^T g w_B) = \frac{1}{2}(g(v_A, v_B) + g(w_A, w_B)) \quad (3.12)$$

$$\eta_{AB} = \eta(\hat{E}_A, \hat{E}_B) = \frac{1}{2}(v_A^T g w_B + w_A^T g v_A) = \frac{1}{2}(g(v_A, w_B) + g(w_A, v_B)) \quad (3.13)$$

Recall that, for the Riemannian case, we defined $G_{ab} = \eta_{ab} = \delta_{ab}$ and $G_{\bar{a}\bar{b}} = -\eta_{\bar{a}\bar{b}} = \delta_{\bar{a}\bar{b}}$. We then have:

$$\begin{aligned} 0 = G_{ab} - \eta_{ab} &= \frac{1}{2}(g(v_a, v_b) + g(w_a, w_b) - g(v_a, w_b) - g(w_a, v_b)) = \\ &= \frac{1}{2}(g(v_a - w_a, v_b - w_b)) \quad (3.14) \\ 0 = G_{\bar{a}\bar{b}} + \eta_{\bar{a}\bar{b}} &= \frac{1}{2}(g(v_{\bar{a}}, v_{\bar{b}}) + g(w_{\bar{a}}, w_{\bar{b}}) + g(v_{\bar{a}}, w_{\bar{b}}) + g(w_{\bar{a}}, v_{\bar{b}})) = \\ &= \frac{1}{2}(g(v_{\bar{a}} + w_{\bar{a}}, v_{\bar{b}} + w_{\bar{b}})) \quad (3.15) \end{aligned}$$

But since g is Riemannian, it is positive definite and thus equations 3.14 and 3.15, evaluated for $A = B$, imply: $v_a = w_a \,\forall\, a$ and $v_{\bar{a}} = -w_{\bar{a}} \,\forall \bar{a}$. Moreover, plugging this equation in the definitions 3.12 and 3.13 we get: $\delta_{ab} = g(v_a, v_b)$ and $\delta_{\bar{a}\bar{b}} = g(v_{\bar{a}}, v_{\bar{b}})$, thus implying that $\{v_a\}$ and $\{v_{\bar{a}}\}$ are both orthonormal frames for g. Finally these solutions automatically satisfy the mixed indices constraints because: $G_{a\bar{b}} = \frac{1}{2}(g(v_a, v_{\bar{b}}) + g(v_a, -v_{\bar{b}})) = 0$ and $\eta_{a\bar{b}} = \frac{1}{2}(g(v_a, -v_{\bar{b}}) + g(v_a, v_{\bar{b}})) = 0$.

Consequences We now want to point out an important consequence of this result. We know that a manifold is parellalisable if it admits a nowhere vanishing frame on it. In the generalised geometrical context we can introduce the notion of 'generalised parallelisability': a manifold will be considered generalised parallisable if it admits a nowhere vanishing generalised frame for its generalised tangent bundle. One can expect the generalised notion of parallelisability to be more general than the standard one, since each generalised vector is composed of twice the components of a standard vector field. This is actually the case in the context of exceptional generalised geometry, where it can be proved that *all* the spheres are generalised parallelisable (see [LeStWa2014]).[8] In the case of $E \approx TM \oplus T^*M$, instead, the requirement of double orthogonality requires the frames to be of the form: $\{\hat{E}_A\} = \{\hat{E}_a\} \cup \{\hat{E}_{\bar{a}}\}$, with $\hat{E}_a(x) = v_a(x) + i_{v_a(x)}(g + B)$ and

[8] As opposed to the case of ordinary geometry, where it is well known that the only spheres that are parallelisable are S^1, S^3 and S^7.

$\hat{E}_{\bar{a}} = v_{\bar{a}}(x) + i_{-v_{\bar{a}}(x)}(g - B)$. This implies that if $v_A(x) = 0$ then also $\hat{E}_A(x) = 0$. In particular, for E, (orthogonal) generalised parallelisability implies (orthogonal) ordinary parallelisability.
Even if generalised parallelisability implies ordinary parallelisability, we need to stress out that, in problems of consistent truncation, we look for generalised *Leibniz* parallelisations. In the ordinary case, the presence of a global frame $\{v_i\}$ for TM whose Lie brackets have constant structure coefficients, i.e. with $[v_i(x), v_j(x)] = f_{ij}{}^k v_k(x)$ with constant $f_{ij}{}^k$, implies that the manifold has a structure of (local) Lie group. To distinguish this parallelisation from the generalised one, we will call it '*Lie parallelisation*'. In the generalised case instead, a generalised Leibniz parallelisation still implies that the underlying manifold is parallelisable, but in general it will only be a (local) homogeneous space.

Example We now want to make a simple example of an application of the result we have just proven, that will exclude some (very) particular cases from the set of the possible generalised Leibniz parallelisations. Let M be a homogeneous space $M = G/H$ and let G be for simplicity a compact group. We can then endow it with a Riemannian bi-invariant metric and induce a metric on G/H in such a way that G acts by isometries on G/H, i.e. such that the generators of the action of G on G/H - induced by the left translation on G itself - are killing vectors (see e.g. [Pe2006]). We then have:

$$L_{\hat{E}_A}\hat{E}_B = [v_A, v_B] + \mathcal{L}_{v_A}\lambda_B - i_{v_B}(d\lambda_A - i_{v_A}H) + i_{[v_A,v_B]}B \tag{3.16}$$

Let us further suppose that, like in the case of S^3, $i_{v_A}H = d\lambda_A$.[9] Since the (bi-invariant) metric g on G induces an isomorphism between the tangent and the cotangent spaces, we can write $\{\lambda_A = i_{w_A}g\}$ for some vector fields $\{w_A\}$. Hence:

$$\begin{aligned}(\mathcal{L}_{v_A}\lambda_B)(v) &= \mathcal{L}_{v_A}(\lambda_B(v)) - \lambda_B(\mathcal{L}_{v_A}v)\\ &= \mathcal{L}_{v_A}(g(w_B, v)) - g(w_B, \mathcal{L}_{v_A}v)\\ &= \underbrace{\mathcal{L}_{v_A}(g)}_{=0}(w_A, v) + g(\mathcal{L}_{v_A}w_B, v) + \cancel{g(w_B, \mathcal{L}_{v_A}v)} - \cancel{g(w_B, \mathcal{L}_{v_A}v)}\\ &= (i_{(\mathcal{L}_{v_A}w_B)}g)(v)\end{aligned} \tag{3.17}$$

Since both $\{w_A\}$ and $\{v_A\}$ form a basis of TM we can always find a change of basis $R_A{}^C$ such that $w_A = R_A{}^C v_C$. In fact, from what we proved before,

[9] Note that, since H is closed, this implies that $\mathcal{L}_{v_A}H = 0$. Indeed: $0 = d(d\lambda_A - i_{v_A}H) = -di_{v_A}H = -\mathcal{L}_{v_A}H$.

this change of basis needs to have the form: $R_a{}^b = \delta_a{}^b$, $R_{\bar{a}}{}^{\bar{b}} = -\delta_{\bar{a}}{}^{\bar{b}}$, with the other components equal to zero. This, in particular, says that R is a constant matrix and therefore we can write:

$$\begin{aligned} X_{AB}{}^C(v_C + R_C{}^D i_{v_D} g + i_{v_C} B) = L_{\hat{E}_A}\hat{E}_B = [v_A, v_B] + & \underbrace{\mathcal{L}_{v_A}\lambda_B}_{=i_{[v_A, R_B{}^C v_C]}g} + i_{[v_A,v_B]}B \\ = X_{AB}{}^C(v_C + i_{v_C}B) + & X_{AC}{}^D R_B{}^C i_{v_D} g \end{aligned}$$

This can be true if and only if: $X_{AB}{}^C R_C{}^D - R_B{}^C X_{AC}{}^D = 0 \,\forall\, A, B, D$, or in other words, iff $[X_A, R] = 0$, where X_A is the A-th matrix of the adjoint representation. We note that if G were simple, then $[X_A, R] = 0, \forall\, A$ would require R, through the Schur's lemma, to be proportional to the identity. But this is clearly inconsistent with the form of R required by double orthogonality. Hence such a G cannot be simple. A similar fact will be also found under different circumstances, after the discussion about generalised killing vectors, which will be now addressed.

Generalised Killing Frames

Let us proceed with the next relevant property. Recall that an ordinary vector field $v(x)$ is killing if the Lie derivative of the metric g with respect to $v(x)$ is zero, i.e. if $\mathcal{L}_v g = 0$. Moreover, if a Lie parallelisation[10] is composed of killing vectors the (local) group acts upon itself by isometries. Consider now the generalised case. A section X of the generalised tangent bundle is called generalised killing if the generalised Lie derivative of the generalised metric G with respect to X is zero, i.e. if $L_X G = 0$. Let us now consider a generalised Leibniz frame $\{\hat{E}_A\}$ for E. We have already seen that $0 = L_{\hat{E}_A}\eta(\hat{E}_B, \hat{E}_C) = X_{AB}{}^D\eta_{DB} + \eta_{BD}X_{AC}{}^D$. Let us express this relation by explicitly splitting the indices in barred and unbarred:

$$\begin{cases} X_{Ab}{}^d\delta_{dc} + \delta_{bd}X_{Ac}{}^d = 0 & \text{for } B = b, C = c \\ -X_{A\bar{b}}{}^{\bar{d}}\delta_{\bar{d}\bar{c}} - \delta_{\bar{b}\bar{d}}X_{A\bar{c}}{}^{\bar{d}} = 0 & \text{for } B = \bar{b}, C = \bar{c} \\ -X_{Ab}{}^{\bar{d}}\delta_{\bar{d}\bar{c}} + \delta_{bd}X_{A\bar{c}}{}^d = 0 & \text{for } B = b, C = \bar{c} \end{cases}$$

So that $X_{Abc} = -X_{Acb}$, $X_{A\bar{b}\bar{c}} = -X_{A\bar{c}\bar{b}}$ and $X_{Ab\bar{c}} = X_{A\bar{c}b}$. Let us now consider the Dorfman derivative of the generalised metric. We have:

$$\begin{aligned} 0 = L_{\hat{E}_A}G(\hat{E}_B, \hat{E}_C) &= (L_{\hat{E}_A}G)(\hat{E}_B, \hat{E}_C) + G(X_{AB}{}^D\hat{E}_D, \hat{E}_C) + G(\hat{E}_B, X_{AC}{}^D\hat{E}_D) \\ &= (L_{\hat{E}_A}G)(\hat{E}_B, \hat{E}_C) + X_{AB}{}^D G_{DC} + G_{BD}X_{AC}{}^D \end{aligned}$$

[10]In the sense discussed before.

We then find:

$$0 = (L_{\hat{E}_A} G)(\hat{E}_b, \hat{E}_c) + \underbrace{X_{Ab}{}^{d}\delta_{dc} + \delta_{bd}X_{Ac}{}^{d}}_{=0} \;\Leftrightarrow\; (L_{\hat{E}_A} G)(\hat{E}_b, \hat{E}_c) = 0$$

$$0 = (L_{\hat{E}_A} G)(\hat{E}_{\bar{b}}, \hat{E}_{\bar{c}}) + \underbrace{X_{A\bar{b}}{}^{\bar{d}}\delta_{\bar{d}\bar{c}} + \delta_{\bar{b}\bar{d}}X_{A\bar{c}}{}^{\bar{d}}}_{=0} \;\Leftrightarrow\; (L_{\hat{E}_A} G)(\hat{E}_{\bar{b}}, \hat{E}_{\bar{c}}) = 0$$

$$0 = (L_{\hat{E}_A} G)(\hat{E}_b, \hat{E}_{\bar{c}}) + X_{Ab}{}^{\bar{d}}\delta_{\bar{d}\bar{c}} + \delta_{bd}X_{A\bar{c}}{}^{d} \;\Leftrightarrow\; (L_{\hat{E}_A} G)(\hat{E}_b, \hat{E}_{\bar{c}}) = -2X_{Ab\bar{c}}$$

These easy calculations show that the generalised Leibniz frames are generalised killing *if and only if* the mixed-indices components of the structure constants $X_{AB}{}^{C}$ vanish.
We can draw some useful conclusions from this fact. We already know that our generalised paralelisable manifolds is locally a homogeneous space. As such, it can always be written (locally) as G/H, where G is a group with structure constants given by $X_{AB}{}^{C}$ and where H is a closed subgroup of G. If the Leibniz parallelisation is also generalised killing, we can locally represent our manifold M as $M = G/H$ with G a *direct product* of Lie groups. Recall now that both $\{v_a\}$ and $\{v_{\bar{a}}\}$ are global frames for TM. The generalised killing relation tells us that each of these frames closes into a Lie algebra with constant structure coefficients, i.e. $[v_a, v_b] = X_{ab}{}^{c}v_c$ and $[v_{\bar{a}}, v_{\bar{b}}] = X_{\bar{a}\bar{b}}{}^{\bar{c}}v_{\bar{c}}$. Since each of these frames also generates TM, both of them are actually Lie parallelisations. This means that the manifold M can also be locally represented as a *Lie group*, say K. For consistency, the two Lie algebras have to be equal to each other. Since this implies that $G = K \times K$, and M can be expressed in both $M = G/H$ and $M = K$, we find that $M = K \times K/K$, and in particular that $H = K$.
Another consequence that we would like to show is that if a generalised Leibniz frame $\{\hat{E}_A\} = \{\hat{E}_a = v_a + i_{v_a}(g + b)\} \cup \{\hat{E}_{\bar{a}} = v_{\bar{a}} + i_{-v_{\bar{a}}}(g - B)\}$ is generalised killing, then both $\{v_a\}$ and $\{v_{\bar{a}}\}$ are composed of killing vectors. Indeed, since the mixed-indices components of $X_{AB}{}^{C}$ are zero, we have:

$$0 = \mathcal{L}_{v_a}(g(v_b, v_c)) = (\mathcal{L}_{v_a} g)(v_b, v_c) + \underbrace{X_{ab}{}^{d}\delta_{dc} + \delta_{bd}X_{ac}{}^{d}}_{=0} \quad \forall\, a, b, c \;\;\Rightarrow\;\; \mathcal{L}_{v_a} g = 0 \;\forall\, a$$

and similarly for the barred indices. This means that *such a* generalised Leibniz parallelisation on a (local) Lie group can only occur if the vector components of the generalised frames are killing vector fields.
Finally, we note that the relation $X_{ABC} = -X_{ACB}$ means that the Lie algebra of the group G in the coset representation of $M = G/H$ is a subalgebra of $O(d, d)$. The maximal compact subgroup of $O(d, d)$ is $O(d) \times O(d)$. We know that G is the direct product of a group times itself, i.e. $G = K \times K$. It is then clear that if K is compact and the dimension of K is grater then $\frac{\dim O(d)}{2}$, then each K needs to be a subgroup of $O(d)$, for maximality of $O(d) \times O(d)$ and the direct product structure of G.

3.2.2 Constructive Point of View

Now that we have proven some general features of the generalised Leibniz parallelisations, we would like to pause for a moment from the main argument and try to sketch an idea concerning a constructive manner of producing these generalised parallelisations. We note, however, that this 'constructive procedure' is far to be complete and therefore it only represents a first step in this direction.

The first phase of the construction is to find a parallelisable manifold that is (locally) a homogeneous space $M \approx G/H$; the group G needs to have twice the dimension of M. We note that the representation of M as a coset space is by no means unique; for instance, if K is another Lie group, we can also write $G = (G \times K)/(H \times K)$. By fixing the dimension of G, $\dim G = 2\dim M$, we also fix the dimension of H - in particular $\dim H = \dim M$ - together with their Lie algebras. These conditions are not enough, though, since we need the much stronger condition that requires the generators of the G-action on M to be nowhere vanishing and be separable in two sets of orthonormal frames for M. Recall that the space of vector fields on M is a Lie algebra for the Lie bracket. Then, these conditions can be easily seen to be equivalent to finding a set of global frames $\{v_a\}$ for TM with opportune anholonomy coefficients. More specifically we know that:

$$[v_a(x), v_b(x)] = C_{ab}{}^c(x) v_c$$

for some functions $C_{ab}{}^c(x)$, sometimes called 'anholonomy coefficients'. Then we can find a generalised Leibniz parallelisation if we can encode the x-dependence of $C_{ab}{}^c(x)$ in a suitable orthogonal, symmetric, point-dependent matrix $P(x)$ such that:

$$C_{ab}{}^c(x) = X_{ab}{}^c + X_{ab}{}^{\bar{c}} P_{\bar{c}}{}^c(x)$$

and

$$\begin{aligned}[P_{\bar{a}}{}^a v_{\bar{a}}, P_{\bar{b}}{}^b v_{\bar{b}}] &= ((v_{\bar{a}}(P_{\bar{b}}{}^c) P_c{}^{\bar{c}} - v_{\bar{b}}(P_{\bar{a}}{}^c)) P_c{}^{\bar{c}} + P_{\bar{a}}{}^a P_{\bar{b}}{}^b C_{ab}{}^c(x) P_c{}^{\bar{c}}) v_{\bar{c}} \stackrel{!}{=} \\ &\stackrel{!}{=} (X_{\bar{a}\bar{b}}{}^{\bar{c}} + X_{\bar{a}\bar{b}}{}^c P_c{}^{\bar{c}}(x)) v_{\bar{c}}\end{aligned}$$

with constant X coefficients. If these relations are satisfied we can define another set of global frames $\{v_{\bar{a}} := P_{\bar{a}}{}^a v_a\}$ and a metric g such that both $\{v_a\}$ and $\{v_{\bar{a}}\}$ are orthonormal frames for it. Indeed it is simply necessary (since these frames are globally defined) to define g with: $g(v_a, v_b) := \delta_{ab}$, because this will also imply $g(v_{\bar{a}}, v_{\bar{b}}) = \delta_{\bar{a}\bar{b}}$. Moreover the matrix P encodes the products of elements of the two frames: $g(v_{\bar{a}}, v_b) = P_{\bar{a}}{}^a \delta_{ab}$.

Note that until this point we have only dealt with the vector part of the generalised parallelisation. In fact, the one-form part comes *almost* for free:

one term is given by the dual frame, whilst the other is determined by the two form B. So, the only part we need to find is the two form B, or, better, the closed three form H, since B stands in general for a collection of two forms such that $dB = H$.
Suppose for a moment to have the desired couple of global orthonormal frames for M that close into a Lie algebra with constant coefficients. How can we find an H that permits to define a generalised Leibniz parallelisation? The answer is that the Leibniz relation $L_{\hat{E}_A}\hat{E}_B = X_{AB}{}^C\hat{E}_C$ can be viewed as a definition for such an H. More in the specific, since we assumed $[v_A, v_B] = X_{AB}{}^C v_C$, the vector part cancels from the Leibniz relation, that becomes:

$$\mathcal{L}_{v_A}\lambda_B - i_{v_B}(d\lambda_A - i_{v_A}H) = X_{AB}{}^D\lambda_D$$

Since $\{v_A\}$ are globally defined vector fields we can *define* H as follows:

$$i_{v_B}i_{v_A}H := X_{AB}{}^D\lambda_D + i_{v_B}d\lambda_A - \mathcal{L}_{v_A}\lambda_B \tag{3.18}$$

The problem of this definition is that it is redundant, since both $\{v_a\}$ and $\{v_{\bar{a}}\}$ are separately bases for TM. One can calculate that definition 3.18 leads to the following definitions:

$$H_{abc} = -X_{ab}{}^d\delta_{dc} - X_{ab}{}^{\bar{d}}g(v_{\bar{d}}, v_c) - X_{bc}{}^{\bar{d}}g(v_{\bar{d}}, v_a) - X_{ca}{}^{\bar{d}}g(v_{\bar{d}}, v_b) \tag{3.19}$$

$$H_{ab\bar{c}} = (\mathcal{L}_{v_b}g)(v_a, v_{\bar{c}}) - (\mathcal{L}_{v_a}g)(v_b, v_{\bar{c}}) - 2X_{ab}{}^{\bar{d}}\delta_{\bar{d}\bar{c}} - X_{ab}{}^D g(v_D, v_{\bar{c}}) \tag{3.20}$$

$$H_{a\bar{b}c} = (\mathcal{L}_{v_a}g)(v_c, v_{\bar{b}}) - (\mathcal{L}_{v_c}g)(v_a, v_{\bar{b}}) + 2X_{a\bar{b}}{}^d\delta_{dc} - X_{ca}{}^D g(v_D, v_{\bar{b}}) \tag{3.21}$$

$$H_{a\bar{b}\bar{c}} = (\mathcal{L}_{v_{\bar{b}}}g)(v_a, v_{\bar{c}}) - (\mathcal{L}_{v_{\bar{c}}}g)(v_a, v_{\bar{b}}) - 2X_{a\bar{b}}{}^{\bar{d}}\delta_{\bar{d}\bar{c}} + X_{\bar{b}\bar{c}}{}^D g(v_D, v_a) \tag{3.22}$$

$$H_{\bar{a}bc} = (\mathcal{L}_{v_c}g)(v_{\bar{a}}, v_b) - (\mathcal{L}_{v_b}g)(v_{\bar{a}}, v_c) + 2X_{\bar{a}b}{}^d\delta_{dc} + X_{bc}{}^D g(v_D, v_{\bar{a}}) \tag{3.23}$$

$$H_{\bar{a}b\bar{c}} = (\mathcal{L}_{v_{\bar{c}}}g)(v_b, v_{\bar{a}}) - (\mathcal{L}_{v_{\bar{a}}}g)(v_b, v_{\bar{c}}) - 2X_{\bar{a}b}{}^{\bar{d}}\delta_{\bar{d}\bar{c}} + X_{\bar{c}\bar{a}}{}^D g(v_D, v_b) \tag{3.24}$$

$$H_{\bar{a}\bar{b}c} = (\mathcal{L}_{v_{\bar{a}}}g)(v_c, v_{\bar{b}}) - (\mathcal{L}_{v_{\bar{b}}}g)(v_c, v_{\bar{a}}) + 2X_{\bar{a}\bar{b}}{}^d\delta_{dc} + X_{\bar{a}\bar{b}}{}^D g(v_D, v_c) \tag{3.25}$$

$$H_{\bar{a}\bar{b}\bar{c}} = X_{\bar{a}\bar{b}}{}^{\bar{d}}\delta_{\bar{d}\bar{c}} + X_{\bar{a}\bar{b}}{}^d g(v_d, v_{\bar{c}}) + X_{\bar{b}\bar{c}}{}^d g(v_d, v_{\bar{a}}) + X_{\bar{c}\bar{a}}{}^d g(v_d, v_{\bar{b}}) \tag{3.26}$$

For example:

$$\begin{aligned}
&i_{v_c}i_{v_b}i_{v_a}H = \\
&\quad = X_{ab}{}^d g(v_d, v_c) + X_{ab}{}^{\bar{d}}g(v_{\bar{d}}, v_c) + \underbrace{(d\lambda_a)(v_b, v_c)}_{= \underbrace{v_b(\lambda_a(v_c))}_{=0} - \underbrace{v_c(\lambda_a(v_b))}_{=0} - \lambda_a([v_b, v_c])} - \underbrace{\mathcal{L}_{v_a}(\lambda_b(v_c))}_{=0} + \lambda_b([v_a, v_c]) \\
&\quad = \cancel{X_{ab}{}^d\delta_{dc}} - X_{ab}{}^{\bar{d}}g(v_{\bar{d}}, v_c) - \cancel{X_{bc}{}^d\delta_{da}} - X_{bc}{}^{\bar{d}}g(v_{\bar{d}}, v_a) + X_{ac}{}^d\delta_{db} + X_{ac}{}^{\bar{d}}g(v_{\bar{d}}, v_b) \\
&\quad = -X_{ab}{}^d\delta_{dc} - X_{ab}{}^{\bar{d}}g(v_{\bar{d}}, v_c) - X_{bc}{}^{\bar{d}}g(v_{\bar{d}}, v_a) - X_{ca}{}^{\bar{d}}g(v_{\bar{d}}, v_b)
\end{aligned} \tag{3.27}$$

and

$$
\begin{aligned}
&i_{v_{\bar{c}}} i_{v_b} i_{v_a} H = X_{ab}{}^{d} g(v_d, v_{\bar{c}}) - X_{ab}{}^{\bar{d}} \delta_{\bar{d}\bar{c}} + (d\lambda_a)(v_b, v_{\bar{c}}) - \mathcal{L}_{v_a}(\lambda_b(v_{\bar{c}})) + \lambda_b([v_a, v_{\bar{c}}]) \\
&= X_{ab}{}^{d} g(v_d, v_{\bar{c}}) - X_{ab}{}^{\bar{d}} \delta_{\bar{d}\bar{c}} + \mathcal{L}_{v_b}(g(v_a, v_{\bar{c}})) - \cancel{v_{\bar{c}}(g(v_a, v_b))} - \lambda_a([v_b, v_{\bar{c}}]) - \mathcal{L}_{v_a}(g(v_b, v_{\bar{c}})) + X_{a\bar{c}}{}^{D} g(v_b, v_D) \\
&= [(\mathcal{L}_{v_b} g)(v_a, v_{\bar{c}}) - (\mathcal{L}_{v_a} g)(v_b, v_{\bar{c}})] + X_{ab}{}^{d} g(v_d, v_{\bar{c}}) - X_{ab}{}^{\bar{d}} \delta_{\bar{d}\bar{c}} + X_{ba}{}^{D} g(v_D, v_{\bar{c}}) + \cancel{X_{b\bar{c}}{}^{D} g(v_a, v_D)} + \\
&\quad - \cancel{X_{b\bar{c}}{}^{D} g(v_a, v_D)} - X_{ab}{}^{D} g(v_D, v_{\bar{c}}) - \underline{X_{a\bar{c}}{}^{D} g(v_b, v_D)} + \underline{X_{a\bar{c}}{}^{D} g(v_b, v_D)} \\
&= [\dots] + \cancel{X_{ab}{}^{d} g(v_d, v_{\bar{c}})} - X_{ab}{}^{\bar{d}} \delta_{\bar{d}\bar{c}} + \cancel{2} X_{ba}{}^{d} g(v_d, v_{\bar{c}}) + 2 X_{ba}{}^{\bar{d}} \delta_{\bar{d}\bar{c}} \\
&= [(\mathcal{L}_{v_b} g)(v_a, v_{\bar{c}}) - (\mathcal{L}_{v_a} g)(v_b, v_{\bar{c}})] - 2 X_{ab}{}^{\bar{d}} \delta_{\bar{d}\bar{c}} - X_{ab}{}^{D} g(v_D, v_{\bar{c}})
\end{aligned}
$$

The point is that all these definitions for H are consistent with one another. Indeed, recall that we defined $\{v_{\bar{a}} := P_{\bar{a}}{}^{a} v_a\}$, for a suitable orthogonal and symmetric tensor P. We then have:

$$
\begin{aligned}
P_c{}^{\bar{c}} H_{ab\bar{c}} &= \underbrace{(\mathcal{L}_{v_b} g)(v_a, v_c)}_{\cancel{\mathcal{L}_{v_b} g(v_a, v_c)} - X_{ba}{}^{D} g_{v_D, v_c} - X_{bc}{}^{D} g(v_a, v_D)} - (\mathcal{L}_{v_a} g)(v_b, v_c) - 2 X_{ab}{}^{\bar{d}} g(v_{\bar{d}}, v_c) - X_{ab}{}^{D} g(v_D, v_c) \\
&= -X_{ba}{}^{D} g_{v_D, v_c} - X_{bc}{}^{D} g(v_a, v_D) + \cancel{X_{ab}{}^{D} g(v_D, v_c)} + X_{ac}{}^{D} g(v_D, v_b) - 2 X_{ab}{}^{\bar{d}} g(v_{\bar{d}}, v_c) - \cancel{X_{ab}{}^{D} g(v_D, v_c)} \\
&= \underline{X_{ab}{}^{d} g(v_d, v_c)} + \cancel{X_{ab}{}^{\bar{d}} g(v_{\bar{d}}, v_c)} - \underline{X_{bc}{}^{d} \delta_{da}} - X_{bc}{}^{\bar{d}} g(v_{\bar{d}}, v_a) - X_{ca}{}^{d} \delta_{db} - X_{ca}{}^{\bar{d}} g(v_{\bar{d}}, v_c) - \cancel{2} X_{ab}{}^{\bar{d}} g(v_{\bar{d}}, v_c) \\
&= -X_{ab}{}^{d} \delta_{dc} - X_{ab}{}^{\bar{d}} g(v_{\bar{d}}, v_c) - X_{bc}{}^{\bar{d}} g(v_{\bar{d}}, v_a) - X_{ca}{}^{\bar{d}} g(v_{\bar{d}}, v_b) = H_{abc}
\end{aligned}
$$

so 3.19 is equivalent to 3.20. 3.20 is clearly equivalent to 3.21 noting that by total antisymmetry we have $H_{ab\bar{c}} = -H_{a\bar{c}b}$. Moreover 3.21 is equivalent to 3.22, since we have:

$$
\begin{aligned}
P_c{}^{\bar{c}} H_{a\bar{b}\bar{c}} &= (\mathcal{L}_{v_{\bar{b}}} g)(v_a, v_c) - P_c{}^{\bar{c}}(\mathcal{L}_{v_{\bar{c}}} g(v_{\bar{b}}, v_a)) - 2 X_{a\bar{b}}{}^{\bar{d}} g(v_{\bar{d}}, v_c) + P_c{}^{\bar{c}} X_{\bar{b}\bar{c}}{}^{D} g(v_D, v_a) \\
&= -X_{\bar{b}a}{}^{D} g(v_D, v_c) - X_{\bar{b}c}{}^{D} g(v_D, v_a) - 2 X_{a\bar{b}}{}^{\bar{d}} g(v_{\bar{d}}, v_c) + \\
&\qquad - P_c{}^{\bar{c}}[\underbrace{(\mathcal{L}_{v_{\bar{c}}} g)(v_{\bar{b}}, v_a)}_{\mathcal{L}_{v_{\bar{c}}}(g(v_{\bar{b}}, v_a)) - \cancel{X_{\bar{c}\bar{b}}{}^{D} g(v_D, v_a)} - X_{\bar{c}a}{}^{D} g(v_{\bar{b}}, v_D)} - \cancel{X_{\bar{b}\bar{c}}{}^{D} g(v_D, v_a)}] \\
&\Big| P_c{}^{\bar{c}} \mathcal{L}_{v_{\bar{c}}}(g(v_{\bar{b}}, v_a)) = \mathcal{L}_{v_c}(g(v_{\bar{b}}, v_a)) \\
&= -\underline{X_{\bar{b}a}{}^{D} g(v_D, v_c)} - \cancel{X_{\bar{b}c}{}^{D} g(v_D, v_a)} - \underline{2 X_{a\bar{b}}{}^{\bar{d}} g(v_{\bar{d}}, v_c)} - (\mathcal{L}_{v_c} g)(v_{\bar{b}}, v_a) - \cancel{X_{c\bar{b}}{}^{D} g(v_D, v_a)} + \\
&\qquad - X_{ca}{}^{D} g_{\bar{b}D} + X_{\bar{c}a}{}^{D} g_{D\bar{b}} P_c^{\bar{c}} \\
&= \underline{X_{a\bar{b}}{}^{d} \delta_{dc} - X_{a\bar{b}}{}^{\bar{d}} g(v_{\bar{d}}, v_c)} - (\mathcal{L}_{v_c} g)(v_{\bar{b}}, v_a) - X_{ca}{}^{D} g(v_{\bar{b}}, v_D) + \\
&\qquad + P_c{}^{\bar{c}} \underbrace{g([v_{\bar{c}}, v_a], v_{\bar{b}})}_{-\cancel{\mathcal{L}_{v_a}(g(v_{\bar{c}}, v_{\bar{b}}))} + (\mathcal{L}_{v_a} g)(v_{\bar{c}}, v_{\bar{b}}) + \underline{X_{a\bar{b}}{}^{D}(v_{\bar{c}}, v_D)}} \\
&= (\mathcal{L}_{v_a} g)(v_c, v_{\bar{b}}) - (\mathcal{L}_{v_c} g)(v_a, v_{\bar{b}}) + 2 X_{a\bar{b}}{}^{d} \delta_{dc} - X_{ca}{}^{D} g(v_D, v_{\bar{b}}) = H_{a\bar{b}c}
\end{aligned}
$$

and we can proceed like this till 3.26, thus proving that equation 3.18 is actually a consistent definition for a three-form $\tilde{H}$. Although this definition consistently defines a three-form, it is not clear whether it can be actually used to define the three-form flux H, since this has not only to be a three form, but has also to be closed. Note that we used the Leibniz relation to define H. This means that once we have found the two sets of global frames $\{v_a\} \cup \{v_{\bar{a}}\} =: \{v_A\}$ that satisfy $[v_A, v_B] = X_{AB}{}^{C} v_C$, we can build a generalised Leibniz parallelisation if and only if the three form defied in 3.18 is *closed*.

3.2.3 Lie Group Example

Let us now consider the simple case where our manifold M is a compact Lie group G. Note that this case is strongly related to the one in [BaPoSa2015]. In this case we have two natural actions of G onto itself: the right and left translations. Their generators, say $\{v_a\}$ and $\{v_{\bar{a}}\}$, being left and right invariant vector fields respectively, are globally defined. By definition, these vector fields satisfy:

$$
\begin{aligned}
[v_a, v_b] &= X_{ab}{}^{c} v_c \\
[v_{\bar{a}}, v_{\bar{b}}] &= X_{\bar{a}\bar{b}}{}^{\bar{c}} v_{\bar{c}}
\end{aligned}
$$

where both $X_{ab}{}^{c}$ and $X_{\bar{a}\bar{b}}{}^{\bar{c}}$ are the structure constants of the Lie group G. Since a left translation followed by a right translation gives the same result as the same right translation followed by the same left translation, in addition to $X_{ab}{}^{\bar{c}} = X_{\bar{a}\bar{b}}{}^{c} = 0$ we also have $X_{a\bar{b}}{}^{C} = 0$. This means that the structure constants[11] and the generators are well suited to being extended to a generalised Leibniz parallelisation. Since all the mixed-terms X vanish, we see that the future Leibniz parallelisation $\{\hat{E}_A\}$ is going to be composed of generalised killing vector fields, i.e. such that $L_{\hat{E}_A} G = 0, \forall\, \hat{E}_A$. This also implies that $\{v_A\}$ is composed of (ordinary) killing vectors for the metric g. Hence, this metric will have to be a bi-invariant (Riemannian) metric on G, whose existence is guaranteed by the fact that G is compact. Since all the requisites on the vector part of $\hat{E}_A$ are satisfied by $\{v_A\} := \{v_a\} \cup \{v_{\bar{a}}\}$, in order to prove that this construction admits a generalised Leibniz parallelisation we only need to prove that the three-form defined by 3.18 is closed. In the previous subsection we showed that all the particular definitions for H that range from 3.19 to 3.26 are equivalent. We can then choose the one we prefer the most. Let us take the one in 3.19. Since all the mixed-indices components of X are zero, we get: $H_{abc} := -X_{abc}$. Moreover from the calculation in 3.27 we can also see that: $H_{abc} = X_{ab}{}^{d}\delta_{dc} + d\lambda_a(v_b, v_c) + X_{ac}{}^{d}\delta_{db} = d\lambda_a(v_b, v_c)$, or, in other words, $i_{v_a} H = d\lambda_a$. Requiring $dH \equiv 0$ is equivalent to requiring $i_{v_d} dH = 0 \,\forall\, v_d$, and is therefore equivalent to:

$$
i_{v_d} dH = \mathcal{L}_{v_d} H - d(i_{v_d} H) = \mathcal{L}_{v_d} H - d(d\lambda_d) = \mathcal{L}_{v_d} H = 0 \qquad \forall\, v_d \in \{v_d\}
$$

[11] Recall that for a Leibniz parallelisation we need to have: $X_{a\bar{b}c} = X_{ac\bar{b}}$.

Let us evaluate this Lie derivative. Recalling that v_d is a killing vector and 3.17, we obtain:

$$\begin{aligned}
6\mathcal{L}_{v_d}H &= -X_{abc}\mathcal{L}_{v_d}(\lambda_a \wedge \lambda_b \wedge \lambda_c)\\
&= -X_{abc}[(X_{da}{}^{k})\lambda_k \wedge \lambda_b \wedge \lambda_c + (X_{db}{}^{k})\lambda_a \wedge \lambda_k \wedge \lambda_c + (X_{dc}{}^{k})\lambda_a \wedge \lambda_b \wedge \lambda_k]\\
&= X_{bc}{}^{a}X_{ka}{}^{d}\lambda_k \wedge \lambda_b \wedge \lambda_c + X_{abc}X_{db}{}^{k}\lambda_k \wedge \lambda_a \wedge \lambda_c - X_{abc}X_{dc}{}^{k}\lambda_k \wedge \lambda_a \wedge \lambda_b\\
&= X_{bc}{}^{l}X_{kl}{}^{d}\lambda_k \wedge \lambda_b \wedge \lambda_c + X_{ac}{}^{l}X_{kl}{}^{d}\lambda_k \wedge \lambda_a \wedge \lambda_c + X_{ab}{}^{l}X_{kl}{}^{d}\lambda_k \wedge \lambda_a \wedge \lambda_b\\
&= 3\underbrace{(X_{[bc}{}^{l}X_{k]l}{}^{d})}_{=0}\lambda_k \wedge \lambda_b \wedge \lambda_c
\end{aligned}$$

thanks to the Jacobi identity. Note that even if we are considering a Lie group manifold, according to the discussion is the last part of subsection 3.2.1, we are actually treating it as the homogeneous space $M = G_L \times G_R/G$.
To conclude this subsection we note that there are two more 'easy' cases where the H defined through 3.18 is closed. Let us suppose that only one of the two sets of global frames, say $\{v_a\}$, leads to generalised killing vectors, i.e. let us suppose that:

$$\begin{aligned}
(L_{\hat{E}_a}G)(\hat{E}_{\bar{b}}, \hat{E}_c) &= -2X_{a\bar{b}}{}^{d}\delta_{dc} = -2X_{ac}{}^{\bar{d}}\delta_{\bar{d}\bar{b}} = 0\\
(L_{\hat{E}_{\bar{a}}}G)(\hat{E}_{\bar{b}}, \hat{E}_c) &= -2X_{\bar{a}\bar{b}}{}^{d}\delta_{dc} = -2X_{\bar{a}c}{}^{\bar{d}}\delta_{\bar{d}\bar{b}} \neq 0
\end{aligned}$$

If this is possible then we can still define H such that:

$$\begin{aligned}
i_{v_a}H &= d\lambda_a\\
H_{abc} &= -X_{abc}
\end{aligned}$$

Since $X_{ac}{}^{\bar{d}}\delta_{\bar{d}\bar{b}} = 0$, this still implies $dH = 0$. Nevertheless, this is not something really new. Indeed, since the $\{v_a\}$ are global frames that close a Lie algebra with constant structure coefficients, they form a Lie parallelisation and, hence, our manifold M is still a (local) group manifold. The most straightforward example of this case is the one of a conventional flux compactification on a group manifold. Consider a bi-invariant (Riemannian) metric g on a (compact) group manifold G. We can then choose a frame for g composed of a set of generators of the left-translations $\{v_a\}$. This clearly provides a conventional (Lie) parallelisation for G composed of killing vectors. We can then construct a generalised Leibniz parallelisation as follows:

$$\hat{E}_A := \begin{cases} \hat{E}_a = v_a + \lambda_a + i_{v_a}B & a = 1, \dots, \mathrm{dim}G\\ \hat{E}_{\bar{a}} = v_{\bar{a}} - \lambda_{\bar{a}} + i_{v_{\bar{a}}}B & \bar{a} = \mathrm{dim}G + 1, \dots, 2\mathrm{dim}G \end{cases}$$

where we take $v_a \equiv v_{(a+\mathrm{dim}G)}$ and $i_{v_a}g = \lambda_a = \lambda_{(a+\mathrm{dim}G)}$ (both $\forall\, a$), i.e. we take the same vector components for the $\{\hat{E}_a\}$ and $\{\hat{E}_{\bar{a}}\}$. If we choose

$i_{v_a} H = d\lambda_a$, we can then easily calculate:

$$
\begin{aligned}
L_{\hat{E}_a}\hat{E}_b &= [v_a, v_b] + \mathcal{L}_{v_a}\lambda_b - \cancel{i_{v_b}(d\lambda_a)} - \cancel{i_{v_a} i_{v_b} H} + i_{[v_a,v_b]}B = X_{ab}{}^{c}(v_c + \lambda_c + i_c B) = X_{ab}{}^{c}\hat{E}_c \\
L_{\hat{E}_{\bar{a}}}\hat{E}_{\bar{b}} &= [v_{\bar{a}}, v_{\bar{b}}] - \mathcal{L}_{v_{\bar{a}}}\lambda_{\bar{b}} + i_{v_{\bar{b}}}(d\lambda_{\bar{a}}) - i_{v_{\bar{a}}} i_{v_{\bar{b}}} H + i_{[v_{\bar{a}},v_{\bar{b}}]}B = X_{\bar{a}\bar{b}}{}^{\bar{c}}(v_{\bar{c}} - 3\lambda_{\bar{c}} + i_{\bar{c}} B) = 2X_{\bar{a}\bar{b}}{}^{\bar{c}}\hat{E}_{\bar{c}} - X_{\bar{a}\bar{b}}{}^{c}\hat{E}_c \\
L_{\hat{E}_a}\hat{E}_{\bar{b}} &= [v_a, v_{\bar{b}}] - \mathcal{L}_{v_a}\lambda_{\bar{b}} - \cancel{i_{v_{\bar{b}}}(d\lambda_a)} - \cancel{i_{v_a} i_{v_{\bar{b}}} H} + i_{[v_a,v_{\bar{b}}]}B = X_{a\bar{b}}{}^{c}(v_c - \lambda_c + i_c B) = X_{a\bar{b}}{}^{\bar{c}}\hat{E}_{\bar{c}}
\end{aligned}
$$

If one further considers (similarly to what is done in [LeStWa2014]):

$$
\hat{E}_A := \begin{cases} \tilde{E}_a = \frac{1}{2}\left(\hat{E}_a + \hat{E}_{a+\mathrm{dim}G}\right) = v_a + i_{v_a}B \\ \tilde{E}^a = \frac{1}{2}\left(\hat{E}_a - \hat{E}_{a+\mathrm{dim}G}\right) = \lambda^a \qquad \text{for } a = 1, \ldots, \mathrm{dim}G \end{cases}
$$

he will find:

$$
\begin{aligned}
L_{\tilde{E}_a}\tilde{E}_b &= X_{ab}{}^{c}\tilde{E}_c - H_{abc}\tilde{E}^c \\
L_{\tilde{E}_a}\tilde{E}^b &= -X_{ac}{}^{b}\tilde{E}^c \\
L_{\tilde{E}^a}\tilde{E}^b &= 0
\end{aligned}
$$

This is the usual description of a conventional flux compactification and, hence, it defines a G gauging.
The other example is instead related to the dimension of the manifold M. Indeed in dimension $d = 3$ all the three-forms are closed. But there is also something more interesting going on: every (closed) orientable manifold of dimension $d = 3$ is parallelisable (see Stiefel's theorem, [ViFu2004], page 193). In sight of this, every homogeneous space of dimension three becomes relevant to the quest for generalised Leibniz parallelisations. Nevertheless, three-dimensional (Riemannian) homogeneous spaces have already been classifies (at least the simply connected ones) [Pa1996] and the only possible new example (related to $S^2 \times \mathbb{R}$) is already known and will be treated in the next section.

3.3 Another Example of Generalised Leibniz Parallelisability

We will now describe another example of generalised Leibniz parallelisability: the generalised parallelisability of $M = S^2 \times \mathbb{R}$, from which one can easily find the one of $S^2 \times S^1$. As opposed to the case of S^3, this manifold is only an homogeneous space and not a Lie group. As such, this represents a purely generalised geometric construction. We will present this example making explicit reference to the general features we have described in the previous sections, thus showing that it perfectly fits in that description of generalised

Leibniz parallelisability.
Let us start with the generalised parallelisation found by De Felice in his thesis [De2014]. He considered the following metric on $S^2 \times \mathbb{R}$:[12]

$$\begin{cases} g = dy_i dy_i + d\psi^2 \quad \text{with } i = 1,2,3 \\ y_i y_i = 1 \end{cases}$$

and the following three-flux[13]:

$$H := -\frac{1}{2}\epsilon_{ijk} y^i \, dy^j \wedge dy^k \wedge d\psi \tag{3.28}$$

where we used, again, constrained coordinates $\{y_i\}$ on the sphere S^2. Note that, apart from the constraint, this metric assumes the form of an Euclidean metric. De Felice noted that the following generalised vectors are globally defined:[14]

$$\begin{aligned} E_i &= v_i - i_{v_i} B + \lambda_i := \epsilon_{ijk} y_j \partial_k - i_{\epsilon_{ijk} y_j \partial_k} B + y_i d\psi \\ E_i' &= v_i' - i_{v_i'} B + \lambda_i' := y_i \partial_\psi - i_{y_i \partial_\psi} B + \epsilon_{ijk} y_j dy_k \end{aligned} \tag{3.29}$$

as it is evident from their explicit expressions:[15]

$$\begin{aligned} & v_1 = (\lambda_1')^\# = \begin{pmatrix} 0 \\ -y_3 \\ y_2 \\ 0 \end{pmatrix}, v_2 = (\lambda_2')^\# = \begin{pmatrix} y_3 \\ 0 \\ -y_1 \\ 0 \end{pmatrix}, v_3 = (\lambda_3')^\# = \begin{pmatrix} -y_2 \\ y_1 \\ 0 \\ 0 \end{pmatrix}, \\ & (\lambda_1)^\# = v_1' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ y_1 \end{pmatrix}, (\lambda_2)^\# = v_2' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ y_2 \end{pmatrix}, (\lambda_3)^\# = v_3' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ y_3 \end{pmatrix}, \end{aligned} \tag{3.30}$$

For example $E_1 = 0$ if and only if $v_1 = \lambda_1 = 0$, i.e. if and only if $y_j = 0$ for $j = 1,2,3$, which is impossible since $y_i y_i = 1$ by definition. De Felice [De2014] calculated that:

$$\begin{aligned} L_{E_i} E_j &= -\epsilon_{ijk} E_k \\ L_{E_i} E_j' &= -\epsilon_{ijk} E_k' \\ L_{E_i'} E_j' &= 0 \end{aligned}$$

[12] Note that everything we are going to say can actually be applied also to $S^2 \times S^1$. The only thing one needs to add is the compactification of the last coordinate, which is simply implemented by requiring: $\psi = \psi + 2\pi$.

[13] Note that it is sufficient to give the expression of H, since B is never explicitly used.

[14] Note that he used a B that is the opposite to the one used by us in subsection 3.2.1.

[15] Note that we are using the 'musical notation' indicating by $(\cdot)^\#$ the image of a one-form under the isomorphism between T^*M and TM given by the metric g, i.e. $\lambda \mapsto i_\lambda g^{-1} \in \Gamma(TM)$.

This shows that the Lie algebra of the group G in the coset representation of our homogeneous space $S^2 \times \mathbb{R} = G/H$ is $\mathfrak{g} = \mathfrak{iso}(3)$, i.e. the semidirect sum $\mathfrak{so}(3) \oplus_s \mathbb{R}^3$. As one can easily notice, the form of these generalised frames differs from the general form we presented in subsection 3.2.1. In fact these generalised vector fields satisfy:

$$\eta(E_i, E_j) = \eta(E'_i, E'_j) = 0 \quad \text{and} \quad \eta(E_i, E'_j) = \frac{1}{2}\delta_{ij}$$
$$G(E_i, E_j) = G(E'_i, E'_j) = \frac{1}{2}\delta_{ij} \quad \text{and} \quad G(E_i, E'_j) = 0$$

Indeed:

$$\eta(E_i, E_j) = \frac{1}{2}\Big(i_{y_j\partial_4}(\epsilon_{ikl}y_k dy_l) + i_{y_i\partial_4}(\epsilon_{jkl}y_k dy_l)\Big) = \frac{1}{2}\Big(\epsilon_{ikl}y_k y_j \cancel{dy_l(\partial_4)} + \epsilon_{jkl}y_k y_i \cancel{dy_l(\partial_4)}\Big) = 0$$
$$\eta(E'_i, E'_j) = \frac{1}{2}\Big(i_{\epsilon_{ikl}y_k\partial_l}(y_j dy_4) + i_{\epsilon_{jkl}y_k\partial_l}(y_i dy_4)\Big) = 0$$
$$\eta(E_i, E'_j) = \frac{1}{2}\Big(i_{\epsilon_{ikl}y_k\partial_l}(\epsilon_{jmn}y_m dy_n) + i_{y_j\partial_4}(y_i dy_4)\Big) = \frac{1}{2}\Big(\epsilon_{ikl}\, y_k\ \epsilon_{jml}\, y_m + y_j y_i\Big)$$
$$= \frac{1}{2}\Big((\delta_{ij}\delta_{km} - \cancel{\delta_{im}\delta_{kj}})y_k y_m + \cancel{y_j y_i}\Big) = \frac{1}{2}\delta_{ij}$$

and noting that $G(E_i, E_j) = g(v_i, v_j) + g^{-1}(\lambda_i, \lambda_j)$, and similarly for the others, we also have: $G(E_i, E_j) = G(E'_i, E'_j) = \eta(E_i, E'_j)$ and $G(E_i, E'_j) = \eta(E_i, E_j)$. We therefore see that the double orthonormality condition is not satisfied. We can nevertheless still try to use some linear combinations of E_i and E'_j in order to find doubly orthonormal frames. The right choice turns out to be:

$$E_i^{\pm} = E_i \pm E'_i$$

because

$$\eta(E_i^{\pm}, E_j^{\pm}) = \eta(E_i \pm E'_i, E_j \pm E'_j) = \eta(E_i, E_j) \pm \eta(E'_i, E_j) \pm \eta(E_i, E'_j) + \eta(E'_i, E'_j) = \pm\delta_{ij}$$
$$\eta(E_i^{\pm}, E_j^{\mp}) = \eta(E_i, E_j) \pm \eta(E'_i, E_j) \mp \eta(E_i, E'_j) - \eta(E'_i, E'_j) = \pm\delta_{ij} \mp \delta_{ij} = 0$$
$$G(E_i^{\pm}, E_j^{\pm}) = \delta_{ij}$$
$$G(E_i^{\pm}, E_j^{\mp}) = 0$$

Writing $E_i^{\pm} = v_1^{\pm} - i_{v_i^{\pm}}B + \lambda_i^{\pm}$ we can see that:

$$v_1^{\pm} = \begin{pmatrix} 0 \\ -y_3 \\ y_2 \\ \pm y_1 \end{pmatrix}, v_2^{\pm} = \begin{pmatrix} y_3 \\ 0 \\ -y_1 \\ \pm y_2 \end{pmatrix}, v_3^{\pm} = \begin{pmatrix} -y_2 \\ y_1 \\ 0 \\ \pm y_3 \end{pmatrix},\ (\lambda_1^{\pm})^{\#} = \begin{pmatrix} 0 \\ \mp y_3 \\ \pm y_2 \\ y_1 \end{pmatrix}, (\lambda_2^{\pm})^{\#} = \begin{pmatrix} \pm y_3 \\ 0 \\ \mp y_1 \\ y_2 \end{pmatrix}, (\lambda_3^{\pm})^{\#} = \begin{pmatrix} \mp y_2 \\ \pm y_1 \\ 0 \\ y_3 \end{pmatrix}$$

and therefore, as expected, $v_i^{\pm} = \pm(\lambda^{\pm})^{\#}$. Thus, using our former notation, we can write: $\{v_i^{+} \equiv v_a\}$ and $\{v_i^{-} \equiv v_{\bar{a}}\}$. Moreover the $\{v_i^{\pm}\}$ are actually frames, because one can easily see that: $g(v_i^\pm, v_j^\pm) = \delta_{ij}$. We can now try to calculate the matrix P of subsection 3.2.1. One finds:

$$
\begin{aligned}
g(v_1^+, v_1^-) &= y_3^2 + y_2^2 - y_1^2 = 1 - 2y_1^2 \\
g(v_1^+, v_2^-) &= -y_1 y_2 - y_1 y_2 = -2y_1 y_2 \\
g(v_1^+, v_3^-) &= -2y_1 y_3 \quad \ldots
\end{aligned}
$$

and more in general: $P_{ij} := g(v_i^+, v_j^-) = \delta_{ij} - 2y_i y_j$. This matrix is clearly symmetric and it also squares to the identity:

$$
(\delta_{ij} - 2y_i y_j)(\delta_{jk} - 2y_j y_k) = \delta_{ik} - 2y_i y_k - 2y_i y_k + 4y_i(y_j y_j) y_k = \delta_{ik}
$$

thus implying its orthogonality. One could calculate the Lie algebra generated by $\{v_i^\pm\}$ from scratch, but since part of the work has already be done by De Felice, we will make use of his results given in 3.29. We can quickly see that:

$$
\begin{aligned}
L_{E_i^\pm} E_j^\pm &= L_{E_i \pm E_i'}(E_j \pm E_j') = L_{E_i} E_j + \cancel{L_{E_i'} E_j'} \pm L_{E_i} E_j' \pm L_{E_i'} E_j = -\epsilon_{ijk}(E_k \pm 2E_k') \\
L_{E_i^\pm} E_j^\mp &= L_{E_i \pm E_i'}(E_j \mp E_j') = L_{E_i} E_j - \cancel{L_{E_i'} E_j'} \mp \underline{L_{E_i} E_j'} \pm \underline{L_{E_i'} E_j} = -\epsilon_{ijk}(E_k)
\end{aligned}
$$

and making use of the inverses of the definitions of $E_i^\pm$: $E_k = \frac{1}{2}\Big(E_k^+ + E_k^-\Big)$ and $E_k' = \frac{1}{2}\Big(E_k^+ - E_k^-\Big)$ we can write:

$$
\begin{aligned}
L_{E_i^\pm} E_j^\pm &= -\epsilon_{ijk}\Big(\frac{3}{2} E_k^\pm - \frac{1}{2} E_k^\mp\Big) \\
L_{E_i^+} E_j^- &= -\epsilon_{ijk}\Big(\frac{1}{2} E_k^+ + \frac{1}{2} E_k^-\Big)
\end{aligned}
$$

So, using the old notation: $\{\hat{E}_a \equiv E_i^+\}$ and $\{\hat{E}_{\bar{a}} \equiv E_i^-\}$, we can see that the structure constants have the following expressions:

$$
X_{abc} = -\frac{3}{2}\epsilon_{abc}; \quad X_{\bar{a}\bar{b}\bar{c}} = -\frac{3}{2}\epsilon_{\bar{a}\bar{b}\bar{c}}; \quad X_{a\bar{b}c} = -\frac{1}{2}\epsilon_{a\bar{b}c}; \quad X_{a\bar{b}\bar{c}} = -\frac{1}{2}\epsilon_{a\bar{b}\bar{c}}
$$

We can also see that the frames $\{v_i^\pm\}$ are not composed of killing vectors. Indeed:

$$
\begin{aligned}
X_{ab}{}^D g(v_D, v_c) + X_{ac}{}^D g(v_D, v_b) &= -\frac{3}{2}\cancel{\epsilon_{ab}{}^d \delta_{dc}} + \frac{1}{2}\epsilon_{ab}{}^{\bar{d}} P_{\bar{d}c} + \frac{1}{2}\epsilon_{ac}{}^{\bar{d}} P_{\bar{d}b} - \frac{3}{2}\cancel{\epsilon_{ac}{}^d \delta_{db}} \\
&= \frac{1}{2}\big(\epsilon_{ab\bar{d}}(\cancel{\delta_{\bar{d}c}} - 2y_{\bar{d}} y_c) + \epsilon_{ac\bar{d}}(\cancel{\delta_{\bar{d}b}} - 2y_{\bar{d}} y_b)\big) \\
&= -[\epsilon_{abc}(y_{\hat{c}}^2 - y_{\hat{b}}^2)] \neq 0 = \mathcal{L}_{v_a}(g(v_b, v_c))
\end{aligned}
$$

where the hat indicates that we are suppressing the summation convention and where to go from the second to the third line we used the fact that there are only three possible values for the indices. We now want to show that the three-flux H in 3.28 actually comes from definition 3.18 (as it is obvious, since 3.18 comes from the Leibniz relation). Indeed, using 3.19, we get:

$$\begin{aligned} H_{abc} &= -X_{bc}{}^{d}\delta_{da} - X_{ab}{}^{\bar{d}}g(v_{\bar{d}}, v_c) - X_{ca}{}^{\bar{d}}g(v_{\bar{d}}, v_b) - X_{bc}{}^{\bar{d}}g(v_{\bar{d}}, v_a) \\ &= \frac{3}{2}\underline{\epsilon_{abc}} - \frac{1}{2}\epsilon_{ab\bar{d}}(\underline{\delta_{\bar{d}c}} - 2y_{\bar{d}}y_c) - \frac{1}{2}\epsilon_{ca\bar{d}}(\underline{\delta_{\bar{d}b}} - 2y_{\bar{d}}y_b) - \frac{1}{2}\epsilon_{bc\bar{d}}(\underline{\delta_{\bar{d}a}} - 2y_{\bar{d}}y_a) \\ &= \epsilon_{ab\bar{d}}\, y_{\bar{d}}y_c + \epsilon_{ca\bar{d}}\, y_{\bar{d}}y_b + \epsilon_{bc\bar{d}}\, y_{\bar{d}}y_a \\ &= \epsilon_{abc}y_{\hat{c}}^2 + \epsilon_{cab}y_{\hat{b}}^2 + \epsilon_{bca}y_{\hat{a}}^2 = \epsilon_{abc}(y_{\hat{c}}^2 + y_{\hat{b}}^2 + y_{\hat{a}}^2) = \epsilon_{abc} \end{aligned}$$

Now, consider $H = \frac{1}{6}H_{abc}\lambda_a \wedge \lambda_b \wedge \lambda_c = \epsilon_{123}\lambda_1 \wedge \lambda_2 \wedge \lambda_3$. We then have:

$$\begin{aligned} H &= (-y_3dy_2 + y_2dy_3 + y_1dy_4) \wedge (y_3dy_1 - y_1dy_3 + y_2dy_4) \wedge (-y_2dy_1 + y_1dy_2 + y_3dy_4) \\ &= (-y_3^2dy_2dy_1 + y_1y_3dy_2dy_3 - y_2y_3dy_2dy_4 + y_2y_3dy_3dy_1 + y_2^2dy_3dy_4 + y_1y_3dy_4dy_1 - y_1^2dy_4dy_3)\wedge \\ &\qquad \wedge (-y_2dy_1 + y_1dy_2 + y_3dy_4) \\ &= -y_3^2dy_2dy_1dy_4 - \cancel{y_1y_2y_3dy_2dy_3dy_1} + y_1y_3^2dy_2dy_3dy_4 + y_2^2y_3dy_2dy_4dy_1 + \cancel{y_1y_2y_3dy_3dy_1dy_2} \\ &\quad + y_2y_3^2dy_3dy_1dy_4 - y_2^3dy_3dy_4dy_1 + y_1y_2^2dy_3dy_4dy_2 + y_1^2y_3dy_4dy_1dy_2 + y_1^2y_2dy_4dy_3dy_1 - y_1^3dy_4dy_3dy_2 \\ &= (y_3^3 + y_2^2y_3 + y_1^2y_3)dy_1dy_2dy_4 + (y_1y_3^2 + y_1y_2^2 + y_1^3)dy_2dy_3dy_4 - (y_2y_3^2 + y_2^3 + y_1^2y_2)dy_1dy_3dy_4 \\ &= (y_3dy_1dy_2 + y_1dy_2dy_3 - y_2dy_1dy_3)dy_4 \end{aligned}$$

which is minus the three-flux given in 3.28. This is because De Felice used minus our B to construct the isomorphism between E and $TM \oplus T^*M$ and $H = dB$.

3.4 First Considerations About Contractions

We have already shown that it is possible to have generalised Leibniz parallelisations on a (compact) Lie group manifold $M = G$. In that case, we represent our manifold as a homogeneous space of the type: $M = G \sim (G_L \times G_R)/G$, and use as vector components of the generalised frames the generators of the action on M induced by the left and right translations on G. The Leibniz relation then defines the three-flux as: $H = \frac{1}{3!}(-X_{abc})\lambda_a \wedge \lambda_b \wedge \lambda_c$, with X_{abc} the structure constants related to G. This construction is the one we built for S^3.

We also know that there are some other examples of generalised Leibniz parallelisations that are not built on group manifolds. As an example, we presented the Leibniz parallelisation of $S^2 \times \mathbb{R}$, which is strongly related to the one of $S^2 \times S^1$. In what follows we will show that this example is indeed related to the one on S^3 and that the relation is based on an Inonu-Wigner group contraction.

Let us start again with the three-sphere. Recall that, as a manifold, it is diffeomorphic to $SU(2)$, and hence we know that it is generalised Leibniz parallelisable in the standard manner: $S^3 \sim SU(2) \sim (SU(2)_L \times SU(2)_R)/SU(2)$. From an algebraic point of view we can also write: $SU(2) \sim (SU(2)_L \times SU(2)_R)/SU(2) := (SO(3) \times SO(3))/SO(3) \sim SO(4)/SO(3)$. The last coset representation is the one we actually used to construct our first generalised Leibniz parallelisation in section 3.1. Indeed, if we consider the action of $SO(4)$ on $\mathbb{R}^4$, given by the vector representation, the generators are $v_{ij} = y_i\partial_j - y_j\partial_i$, with $i, j = 1, ..., 4$. We know that these generators are also vector fields for S^3. Evaluating them at the north pole $\overrightarrow{y} = (0, 0, 0, 1)^T$ we find that, of the six vectors, only the three containing y_4 are not zero. The three vanishing vectors are v_{ab} with $a, b = 1, ..., 3$. If a generator vanishes at one point $p \in M$, it means that the action of the group elements generated by that vector field leaves that point invariant. Since $\{v_{ab}\}$ are actually generators of an $SO(3)$ subgroup of $SO(4)$, we can state that the isotropy subgroup at the north pole, and therefore also at every other point, is $SO(3)$. This confirms the diffeomorphism: $S^3 \approx SO(4)/SO(3)$. We can make a similar reasoning for $M = S^2 \times \mathbb{R}$. Indeed, we have shown that there is a transitive action of $G = SO(3) \ltimes \mathbb{R}^3$ on M, viewed as a manifold immersed in $\mathbb{R}^4$. Its generators (that are also generators of the action of G on $\mathbb{R}^4$) were: $v_a = \epsilon_{abc}y_b\partial_c$ and $v'_a = y_a\partial_4$ with $a, b, c = 1, ..., 3$. Considering the point $\overrightarrow{y} = (0, 0, 1, 0)$, i.e. at the north pole of S^2 and at the origin of $\mathbb{R}$, we find that $v_3 = v'_1 = v'_2 = 0$, and that these three together generate an $\mathfrak{iso}(2)$ Lie algebra. Therefore, we can view M as the coset space: $S^2 \times \mathbb{R} \approx (SO(3) \ltimes \mathbb{R}^3)/(SO(2) \ltimes \mathbb{R}^2)$. Since it is known that $S^d \approx SO(d + 1)/SO(d)$ we can see that: $(SO(3) \ltimes \mathbb{R}^3)/(SO(2) \ltimes \mathbb{R}^2) \approx (SO(3)/SO(2)) \times \mathbb{R}$.[16] Nevertheless, for dimensional reasons, we have to use the representation $S^2 \times \mathbb{R} \approx (SO(3) \ltimes \mathbb{R}^3)/(SO(2) \ltimes \mathbb{R}^2)$ in order to be able to construct a generalised Leibniz paralleisation.

Let us now briefly explain what a Inonu-Wigner contraction is (see [InWi1953] and also [Gi1974] for more details). The idea of contraction originated from the conviction that if a physical theory is a limiting case of another one, then also its group of symmetries must be a limiting case of the group of symmetries of the other. For example, if the Newtonian physics can be obtained from special relativity in the limit $c \to \infty$, then the Galilean group should be obtained in the same manner from the Poincaré group. An-

[16] This diffeomorphism is actually something non-trivial and is due to the fact that the normal $\mathbb{R}^n$ subgroups are abelian and to the fact that the subgroup of $SO(3)$ appearing in the quotient group is actually one dimensional (and therefore abelian).

other example is the relation between the de Sitter group and the Poincaré group. The idea of Inonu-Wigner was to consider a change of basis of the Lie algebra of the group that becomes singular under a certain limit, i.e. a certain $U(\epsilon)$ such that:

$$\begin{cases} \det\Big(U(\epsilon)\Big) \neq 0 & \text{for } \epsilon > 0 \\ \det\Big(U(\epsilon)\Big) = 0 & \text{for } \epsilon = 0 \end{cases}$$

The singularity of the transformation at $\epsilon = 0$ is essential if one wants to obtain a non isomorphic Lie group as a result of the $\lim_{\epsilon\to 0}$ operation. Consider the Lie algebra $\mathfrak{g}$ associated with G and let $d = \dim G$. Let also r be the rank of the singular transformation $U(0)$ resulting from the limiting procedure. We can choose a basis of $\mathfrak{g}$ such that $U(0)\mathfrak{g}$ is generated by the first r elements of the basis. More specifically, as *vector spaces*, we can split $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{h}$ such that $U(0)\mathfrak{r} = \mathfrak{r}$ and $U(0)\mathfrak{h} = 0$. We can then always represent $U(\epsilon)$, after a suitable (invertible) change of coordinates, in block form as: $U = \left(\begin{smallmatrix} \mathbb{1}+\epsilon V & 0 \\ 0 & \epsilon\mathbb{1} \end{smallmatrix}\right)$. Inonu and Wigner then found the conditions under which the limit under $\epsilon \to 0$ of the Lie algebra $\mathfrak{g}$ exists as follows. Divide the generators of $\mathfrak{g}$ under the splitting $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{h}$, i.e. set $\{I_\mu\} = \{I_\alpha\} \cup \{I_a\}$, with $\mu = 1, ..., d$, $\alpha = 1, ..., r$ and $a = r+1, ..., d$. Then $J_\mu := U(\epsilon)_\mu^\nu I_\nu$ is such that:

$$\begin{cases} J_\alpha = (\delta_\alpha^\beta + \epsilon V_\alpha^\beta) I_\beta \\ J_a = \epsilon I_a \end{cases}$$

We can then calculate:

$$\begin{aligned} [J_\alpha, J_\beta] &= [I_\alpha, I_\beta] + \epsilon([V_\alpha^\gamma I_\gamma, I_\beta] + [I_\alpha, V_\beta^\gamma I_\gamma]) + \epsilon^2 [V_\alpha^\gamma I_\gamma, V_\beta^\delta I_\delta] \\ &= X_{\alpha\beta}{}^\gamma I_\gamma + X_{\alpha\beta}{}^a I_a + \epsilon V_\alpha^\gamma (X_{\gamma\beta}{}^\delta I_\delta + X_{\gamma\beta}{}^a I_a) + \dots \\ &= X_{\alpha\beta}{}^\gamma J_\gamma + \frac{1}{\epsilon} X_{\alpha\beta}{}^a J_a + V_\alpha^\gamma X_{\gamma\beta}{}^a J_a + V_\beta^\gamma X_{\alpha\gamma}{}^a J_a + O(\epsilon) \end{aligned} \tag{3.31}$$

and

$$[J_\alpha, J_a] = [I_\alpha, \epsilon I_a] + \epsilon^2 [V_\alpha^\gamma I_\gamma, I_a] = X_{\alpha a}{}^b J_b + O(\epsilon) \tag{3.32}$$

$$[J_a, J_b] = \epsilon^2 [I_a, I_b] = O(\epsilon) \tag{3.33}$$

From equation 3.31 we can see that the limit exists if and only if $X_{\alpha\beta}{}^a = 0$, i.e. if and only if the subspace $\mathfrak{r}$ in the split of $\mathfrak{g}$ is a (closed) *subalgebra*. If this is the case, then the resulting algebra has $U(0)\mathfrak{h}$ as an invariant abelian subalgebra, since from equations 3.32 and 3.33 we have: $[U(0)\mathfrak{h}, U(0)\mathfrak{h}] \equiv 0$ and $[U(0)\mathfrak{g}, U(0)\mathfrak{h}] \subset U(0)\mathfrak{h}$. In other words, the resulting algebra is the semidirect sum: $U(0)\mathfrak{g} = \mathfrak{r} \oplus_s \mathbb{R}^{d-r}$. By exponentiating these new generators we see that, whilst the ones of $U(0)\mathfrak{r}$ remain unaltered and generate a subgroup $R \subset G$, the ones of $U(0)\mathfrak{h}$ for an infinite range of exponentiation parameter generate group elements 'infinitely' close to the identity. This is the origin of the name 'contraction': the resulting manifold consists in an infinitesimal neighbourhood of the subgroup R of G and hence the group G has been contracted. Going back to the limit $c \to \infty$ of special relativity, it can be shown that the Galilei group derives from a contraction of the Poicaré group with respect to its subgroup generated by spatial rotations and time translations.

Let us now proceed with our examples of generalised Leibniz parallelisability. The three-sphere was represented as $SO(4)/SO(3)$, whilst $M = S^2 \times \mathbb{R}$ as $(SO(3) \ltimes \mathbb{R}^2)/(SO(2) \ltimes \mathbb{R}^2) = ISO(3)/ISO(2)$. We also know that we can perform Inonu-Wigner contractions to obtain:

$$\begin{cases} SO(4) \longmapsto ISO(3) \\ SO(3) \longmapsto ISO(2) \end{cases}$$

At this point it is natural to suspect that M can be obtained by S^3 by Inonu-Wigner (I-W) contraction. Nevertheless we can notice a substantial difference between the procedure of an ordinary Inonu-Wigner contraction and the one we would need to obtain M from S^3. The former is indeed performed on *Lie groups*, while the second should be applied on a *homogeneous space*. This means that we should find a way to gain control on both the action group and its isotropy group. This is in general something complicated to do. Part of the difficulty is related to the problem of how group (and algebra) representations behave under I-W contractions. Indeed, it is clear that if we simply apply the singular transformation U on the representation generators $\tilde{I}_\mu$, i.e. $\tilde{J}_\alpha \mapsto \tilde{I}_\alpha$ and $\tilde{J}_a = \lim_{\epsilon \to 0} \epsilon \tilde{I}_a = 0$, we only get a representation of the subgroup R and not of the entire contracted group. Like I-W suggest, to 'save' a representation it is sometimes useful to perform some ϵ dependent transformations on it, before taking the limit. What we will do, instead, is trying to encode this operation in a limit procedure on the homogeneous space. To explain what this means we will use an example inspired by Example 1. in section 10.3 of [Gi1974]. Let us consider an n-sphere immersed in $\mathbb{R}^{n+1}$, i.e. described by the Cartesian equation: $y_i y_i = 1$. This manifold is acted upon by $SO(n+1)$ with generators: $v_{ij} = y_i \partial_j - y_j \partial_i$. We then take the 'north pole approximation', i.e. we take a limit where we constrain ourselves to an infinitesimal neighbourhood of the north pole : $\overrightarrow{y} = (0, \dots, 1)$. To perform this we rescale the coordinates $y_1, \dots, y_n$ by a factor of ϵ and then take the limit $\epsilon \to 0$. If we then plug this into the defining relation of the sphere, we find: $\lim_{\epsilon\to 0}(\sum_{i\neq n+1}\epsilon^2 y_i y_i) + y_{n+1}^2 = y_{n+1}^2 = 1$; this means that by taking this limit we imposed y_{n+1} to be equal to 1, whilst eliminating the constraint from the rest of the coordinates. Let us consider the group action generators:

$$\begin{cases} v_{ab} \mapsto \left(\epsilon y_a \frac{\partial}{\epsilon \partial^b} - \epsilon y_b \frac{\partial}{\epsilon \partial^a}\right) = v_{ab} \quad \text{for } a, b = 1, \ldots, n \\ v_{n+1\,a} \mapsto \frac{y_{n+1}}{\epsilon}\partial_a - \epsilon y_a \partial_{n+1} \end{cases}$$

we then see that to 'save' the construction we need to rescale $v_{n+1\,a} \mapsto \epsilon v_{n+1\,a}$, obtaining $v_{n+1\,a} = \partial_a$. In other words we need to perform a contraction of the action group $SO(n+1)$:

$$v_{ab} \mapsto v_{ab} \quad \text{and} \quad v_{n+1\,a} \mapsto \epsilon v_{n+1\,a}$$

Conversely we can also say that, after taking the contraction, the 'north pole' approximation *saves* the representation. The space we have obtained is $\mathbb{R}^n$, with action group $ISO(n)$. It is then clear that the isotropy group is $SO(n)$; a fact that is also easily verifiable evaluating the $ISO(n)$ generators at the origin of $\mathbb{R}^n$: $v_{ab}|_0 = 0$, $v_{n+1\,a}|_0 \neq 0$.
The merit of this example was that it was useful to show that it is indeed possible to make a contraction of a homogeneous space, through a contraction of its action group. The disappointing feature was, instead, that it only permitted to contract the action group but not the isotropy subgroup $SO(n)$. In our specific case instead, we would like to find a procedure that results in the effective contraction of both the action *and* the isotropy groups. It is worth noting that to have a contraction of the isotropy group we have to contract the action group as well, since, for a coset space of the type G/H to be a manifold, we need H to be a (closed) subgroup of G. Let us now go back to our S^3 contraction. We still want to take the contraction:

$$\begin{cases} v_{ab} \mapsto v_{ab} \\ v_{4\,a} \mapsto \epsilon v_{4\,a} \quad \text{with } a, b = 1, \ldots, 3 \end{cases}$$

but we need to save the representation in another manner. In this case, inspired by the work of Kim and Wigner [KiWi1987],[17] we will take the 'equatorial belt approximation'. This approximation is clearly the opposite limit to the one of the north pole approximation. To constrain ourselves on the equatorial belt we rescale the y_4-coordinate by ϵ. The defining relation of the sphere becomes then: $\lim_{\epsilon\to 0}(\sum_{i\neq 4} y_i y_i + \epsilon^2 y_4^2) = \sum_{i\neq 4} y_i y_i = 1$. This

[17] Note that in their work, even if they were dealing with little groups (i.e. with isotropy groups), they took usual I-W contractions.

implies that the resulting manifold is $S^2 \times \mathbb{R}$, as desired. This is because now y_4 is unconstrained, whilst the other variables still satisfy a sphere-like relation. Moreover, the representation is indeed saved, because:

$$v_{4\,a} \mapsto \lim_{\epsilon \to 0} \epsilon \Big(\epsilon y_4 \partial_a - \frac{y_a}{\epsilon} \partial_4 \Big) = y_a \partial_4$$

Note that these generators, together with v_{ab}, exactly coincide with the vector part of the generalised parallelisation of $S^2 \times \mathbb{R}$ of 3.30; thus, as before, evaluating these generators at the north pole of S^2 times the origin of $\mathbb{R}$, we see that the little group has been effectively contracted to $ISO(2)$.
From what we have said it is now clear that the $S^2 \times \mathbb{R}$ case can be obtained from the S^3 one by group contraction. We would like to remark that the manners in which a representation is saved are strongly representation-dependent. Therefore, from our point of view, they are less fundamental than the contraction itself. This is to mean that, even if we have obtained the two contractions by the same rescaling of the Lie algebra generators, but with different limits on the manifold itself, the difference between the two cases - north pole and equatorial belt approximations - has to be attributed to different contractions on the action group. This last fact will be properly dealt in the next chapter.

Chapter 4

Some particular Contractions of Homogeneous Manifolds

In the previous section, we provided a direct verification that the generalised Leibniz parallelisation of $S^2 \times \mathbb{R}$ comes from a contraction of the one of $S^3 \approx (SU(2)_L \times SU(2)_R)/SU(2)$. In that discussion, we were relying on a particular representation of the coset space and had, therefore, to save the representation through a limit procedure on the contracted manifold. Moreover, the resulting homogeneous space was strongly related to how we saved the representation. In particular, we obtained two different results for the north pole and the equatorial belt approximations respectively. In this section, we would like to start to investigate whether other contractions of homogeneous manifolds could lead to new examples of generalised Leibniz parallelisations. The homogeneous manifolds we are interested in are the ones isomorphic to a (compact) Lie group, i.e. of the type: $M = (G_L \times G_R)/G \approx G$, since we know that these are Leibniz parallelisable. There are probably some trivial contractions one can perform on them: contractions of the action group with respect to one of the direct product factors are likely to produce a manifold diffeomorphic to $\mathbb{R}^{\dim G}$. Here, instead, we would like to develop a formalism to understand when we can contract, together with the action group, also its isotropy group. It will turn out that these 'non-trivial' contractions are allowed if one considers as quotient group action and contraction subgroup some 'diagonal' subgroups of $(G_L \times G_R)$. Since we are considering a generic G we will not rely on any specific representation. We will find a criterion that allows us to determine whether such a 'mixed' contraction can provide a specific contraction of the isotropy group or not. Finally, we will study again the example of $G = SU(2)$ and, in addition to that, we will also argue that such a non-trivial contraction can also be performed on $SU(3)$, of which we will provide a particular example.

4.1 Quotient Manifold

Let G be a Lie group. Our starting point is the following space: $G_L \times G_R/G$, where we are taking the right coset by G of the group manifold $Q := G_L \times G_R$. This quotient can be divided in the following three cases:

1. Action on the right factor: $\phi_R : (g, (g_L, g_R)) \mapsto (g_L, g_R g^{-1})$
2. Action on the left factor: $\phi_L : (g, (g_L, g_R)) \mapsto (g_L g^{-1}, g_R)$
3. Mixed diagonal action: $\phi_{g_0} : (g, (g_L, g_R)) \mapsto (g_L g_0 g^{-1} g_0^{-1}, g_R g^{-1})$

Note that, in the case of the actions given by ϕ_R and ϕ_L, the subgroup G with respect to which we are taking the right quotient is a normal subgroup of the total group $G_L \times G_R$. Therefore, the quotient manifold in the first two cases inherits the structure of a group manifold from the total space. In particular the quotient manifold of case 1 is isomorphic to G_L and the one of case 2 is isomorphic to G_R.
In the last case, instead, we have an action by a subgroup which is not in general a normal subgroup and therefore, even if the resulting quotient manifold is diffeomorphic to a Lie group, the group product is not the one induced by the product on $G_L \times G_R$. Moreover it is clear that the third case also includes the case in which the image of the action is given by $[(g_L g_0 g^{-1} g_0^{-1}, g_R h_0 g^{-1} h_0^{-1})]$ after one has rescaled the group element $g \mapsto h_0 g h_0^{-1}$ and redefined $g_0 \mapsto g_0 h_0^{-1}$.
Even if the resulting manifold has not an evident group product, it has a left action of the group $G_L \times G_R$ given by left multiplication:

$$\Lambda_{LR} : ((g'_L, g'_R), [(g_L g_0 g^{-1} g_0^{-1}, g_R g^{-1})]) \mapsto [(g'_L g_L g_0 g^{-1} g_0^{-1}, g'_R g_R g^{-1})] \quad (4.1)$$

and similarly for the other two cases. In what follows it is sometimes useful to introduce the choice of gauge in which we choose to set the right factor equal to the group identity, making use of the quotient action. For example, for the generic group element of the quotient in 3 we will set $g = g_R$, obtaining: $[(g_L g_0 g^{-1} g_0^{-1}, g_R g^{-1})] \mapsto (g_L g_0 g_R^{-1} g_0^{-1}, e_R)$.
We can also translate the action in equation 4.1 in the language of this gauge:

$$\Lambda_{LR}^{gauge} : ((g'_L, g'_R), (g_L g_0 g_R^{-1} g_0^{-1}, e_R)) \mapsto (g'_L g_L g_0 (g'_R g_R)^{-1} g_0^{-1}, e_R) \quad (4.2)$$

Basically the left action of the left component becomes the left action on the manifold element, and the left action on the right component of Q becomes a sort of right action.
Similarly, in case 1 we can rewrite the left Q-action in this gauge: $(g'_L, g'_R) \cdot$

$(g_L, g_R g^{-1}) = (g'_L g_L, g'_R g_R g^{-1}) \underset{\text{gauge}}{\sim} (g'_L g_L, e_R)$, which is obviously the left multiplication in the left component G_L.
The origin of the homogeneous space is the projection of the identity of Q under the map $\pi : Q = G_L \times G_R \to G_L \times G_R / G = M$. This is given by:

$$\pi(e_L, e_R) = [(e_L g_0 g^{-1} g_0^{-1}, e_R g^{-1})] \underset{\text{gauge}}{\sim} (e_L g_0 e_R^{-1} g_0^{-1}, e_R) = (e_L, e_R) \quad (4.3)$$

4.2 Contractions for A=G

4.2.1 Contraction Submanifold

We now want to deal with contractions of the action group Q. In order to make a Inonu-Wigner contraction we need to select a subgroup $A \subset Q$ with respect to which we will perform it. In the case of $A = G$, as before, there are three different types of contractions we can perform:

1. $A = G_L \times \{e_R\}$
2. $A = \{e_L\} \times G_R$
3. $A = \{(a_0 a a_0^{-1}, a) | a \in G\}$, for a fixed $a_0 \in G$

It is clear that the subgroup A defines a submanifold in Q, which we will call *contraction submanifold* (CS). The left translation of Q induce a left translation in A as well, which is given by: $(Ad_{a_0}(a'), a') \cdot (Ad_{a_0}(a), a) = (Ad_{a_0}(a'a), a'a)$. This action is transitive on CS and induces vector fields that span its tangent bundle. These induced vector fields are as follows:

$$L_i^{\#}(Ad_{a_0}(a), a) = \frac{d}{dt}(a_0 exp(t\lambda_i) a a_0^{-1}, exp(t\lambda_i) a)|_{t=0} =: (Ad_{a_0}(\tilde{\lambda}_i(a)), \tilde{\lambda}_i(a)) \quad (4.4)$$

where $\{\lambda_i\}_{1 \leq i \leq \dim G}$ form a basis for the Lie algebra $\mathfrak{g}$ of G. In particular, for a linear group, these become: $(a_0 \lambda_i a a_0^{-1}, \lambda_i a)$.
Next we want to see what happens to the CS after we take the quotient. Clearly:

$$\pi((Ad_{a_0}(a), a)) = [(Ad_{a_0}(a) Ad_{g_0}(g^{-1}), a g^{-1})] \underset{\text{gauge}}{\sim} (Ad_{a_0}(a) Ad_{g_0}(a^{-1}), e_R)$$

The left action becomes:

$$\begin{aligned}\Lambda_{LR}|_A :&(Ad_{a_0}(a'), a') \cdot [(Ad_{a_0}(a) Ad_{g_0}(g^{-1}), a g^{-1})] = \\ &= [(Ad_{a_0}(a'a) Ad_{g_0}(g^{-1}), a' a g^{-1})] \underset{\text{gauge}}{\sim} (Ad_{a_0}(a'a) Ad_{g_0}((a'a)^{-1}), e_R)\end{aligned}$$

The generators of this action become, in this gauge, the following vector fields:

$$d\pi(L_i^{\#})([(Ad_{a_0}(a)Ad_{g_0}(g^{-1}), ag^{-1})]) \underset{\text{gauge}}{\sim}$$
$$\underset{\text{gauge}}{\sim} \frac{d}{dt}(a_0 exp(t\lambda_i)aa_0^{-1}g_0a^{-1}exp(-t\lambda_i)g_0^{-1}, e_R)|_{t=0} \quad (4.5)$$

which in the case of a linear group become

$$(a_0\lambda_i aa_0^{-1}g_0a^{-1}g_0^{-1} - a_0aa_0^{-1}g_0a^{-1}\lambda_i g_0^{-1}, 0)$$

Now assume that we are dealing with connected groups. Then, in order to study the kind of homogeneous space Q'/G' we get after the contraction of Q with respect to A, we only need to know what the action group and the isotropy group at one single point in the manifold are. Since there is a preferred point in a homogeneous space, i.e. the origin, we will investigate these properties at this point (recall that $\pi(e_L, e_R) \underset{\text{gauge}}{\sim} (e_L, e_R)$). For simplicity, assume from now on that we are dealing with a linear group. At the origin, the generators of the left action in CS become:

$$d\pi(L_i^{\#})(e_L, e_R) \underset{\text{gauge}}{=} (a_0\lambda_i a_0^{-1} - g_0\lambda_i g_0^{-1}, 0) \quad (4.6)$$

Remark 6. We want to remark that all the matrices of the adjoint representation $Ad_{a_0} : \mathfrak{g} \to \mathfrak{g}$ (for $a_0 \in G$) are always automorphisms of the Lie algebra $\mathfrak{g}$ of G. In particular $Ad_{a_0}(\mathfrak{g}) = \mathfrak{g}\ \forall a_0 \in G$ and Ad_{a_0} sends bases into bases.

4.2.2 Total Tangent Space and its Contractions

The remark allows us to choose as a basis for the tangent space a completion of the set of the vectors in 4.4. More specifically, the following vectors span the tangent space TQ of the Q-manifold restricted to CS:

$$TQ\Big|_{CS} = \{(Ad_{a_0}(\tilde{\lambda}_i(a)), \tilde{\lambda}_i(a)), (Ad_{a_0}(\tilde{\lambda}_i(a)), -\tilde{\lambda}_i(a))\} = V_{CS} \oplus \overline{V}_{CS} \quad (4.7)$$

and we want to underline the fact that the first dim$G$ vectors span the integral distribution of CS. Note that, at the identity, these vectors become:

$$T_{(e_L,e_R)}Q = \{(Ad_{a_0}(\lambda_i), \lambda_i), (Ad_{a_0}(\lambda_i), -\lambda_i)\} \quad (4.8)$$

We further note that the action by left translation of Q on itself is transitive and that also its generators form a basis for its tangent bundle TQ. The action can be divided in two commuting actions:

$$\Lambda_{LR}((g'_L, e_R), (g_L, g_R)) = (g'_L g_L, g_R)$$
$$\Lambda_{LR}((e_L, g'_R), (g_L, g_R)) = (g_L, g'_R g_R)$$

So the generators of this action are:

$$l_i(g_L, g_R) = \frac{d}{dt}(exp(t\lambda_i)g_L, g_R)|_{t=0} = (\tilde{\lambda}_i(g_L), 0)$$
$$r_i(g_L, g_R) = \frac{d}{dt}(g_L, exp(t\lambda_i)g_R)|_{t=0} = (0, \tilde{\lambda}_i(g_R))$$

If the basis $\{\lambda_i\}$ satisfies the commutation relations: $[\lambda_i, \lambda_j] = f_{ij}{}^k \lambda_k$, then even these vector fields satisfy the same commutation relations, with the same structure *constants* $f_{ij}{}^k$:

$$[l_i(q), l_j(q)] = f_{ij}{}^k l_k(q)$$
$$[r_i(q), r_j(q)] = f_{ij}{}^k r_k(q) \qquad q \in Q$$

We can so deduce that even the vectors $\tilde{\lambda}_i(g)$ satisfy the same relations:

$$[\tilde{\lambda}_i(g), \tilde{\lambda}_j(g)] = f_{ij}{}^k \tilde{\lambda}_k(g) \tag{4.9}$$

Let us now study the commutation relations of the basis in 4.7. We have:

$$[(Ad_{a_0}(\tilde{\lambda}_i(a)), \tilde{\lambda}_i(a)), (Ad_{a_0}(\tilde{\lambda}_j(a)), \tilde{\lambda}_j(a))] = f_{ij}{}^k (Ad_{a_0}(\tilde{\lambda}_k(a)), \tilde{\lambda}_k(a))$$
$$[(Ad_{a_0}(\tilde{\lambda}_i(a)), -\tilde{\lambda}_i(a)), (Ad_{a_0}(\tilde{\lambda}_j(a)), -\tilde{\lambda}_j(a))] = f_{ij}{}^k (Ad_{a_0}(\tilde{\lambda}_k(a)), \tilde{\lambda}_k(a))$$
$$[(Ad_{a_0}(\tilde{\lambda}_i(a)), \tilde{\lambda}_i(a)), (Ad_{a_0}(\tilde{\lambda}_j(a)), -\tilde{\lambda}_j(a))] = f_{ij}{}^k (Ad_{a_0}(\tilde{\lambda}_k(a)), -\tilde{\lambda}_k(a))$$

This is because of equation 4.9 together with the fact that $Ad_{a_0} : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of the Lie algebra $\mathfrak{g}$. So, with respect to the splitting of equation 4.7, we have that:

$$[V_{CS}, V_{CS}] \subset V_{CS}$$
$$[\overline{V}_{CS}, \overline{V}_{CS}] \subset V_{CS}$$
$$[V_{CS}, \overline{V}_{CS}] \subset \overline{V}_{CS}$$

From these equations we can see that V_{CS} is a subalgebra of TQ, whilst $\overline{V}_{CS}$ is only a subspace of TQ. Recall that we are going to take the contraction with respect to the group that is generated by the vectors in V_{CS}.
After the contraction we have the following algebra:

$$[V_{CS}, V_{CS}] \subset V_{CS} \tag{4.10}$$
$$[\overline{V'}_{CS}, \overline{V'}_{CS}] = 0 \tag{4.11}$$
$$[V_{CS}, \overline{V'}_{CS}] \subset \overline{V'}_{CS} \tag{4.12}$$

where $\overline{V'}_{CS}$ is the subspace that contains the contracted vectors. Note that now it actually forms an *invariant abelian subalgebra* of the algebra of the vector fields over the contracted Q (say Q_c).

4.2.3 Results of the Contraction

Now that the contraction is performed on the total manifold Q, we want to see its effects on the quotient manifold M. We know that the left translation on A induces an action on the quotient manifold and therefore generators of the former action induce generators of the induced action (by projection), see equation 4.5. These vector fields evaluated at the origin for a matrix Lie group become the vectors in equation 4.6.

In what follows we will only consider the cases 3 at the beginning of section 4.1 and 3 at the beginning of subsection 4.2.1. This is because they are the most interesting cases; for a discussion of the remaining cases, see appendix A. Depending on the choices of the contraction and quotient groups (i.e. choices of a_0 and g_0) we can find the following three cases:

1. $Ad_{a_0}(\lambda_i) - Ad_{g_0}(\lambda_i)$ never equal to zero
2. $Ad_{a_0}(\lambda_i) - Ad_{g_0}(\lambda_i)$ always equal to zero
3. $Ad_{a_0}(\lambda_i) - Ad_{g_0}(\lambda_i) = 0 \quad \text{for } 1 \leq i \leq k < \mathrm{d} = dimG$
   $Ad_{a_0}(\lambda_i) - Ad_{g_0}(\lambda_i) \neq 0 \quad \text{for } k+1 \leq i \leq d$

Remark 7. Note that once the map $Ad_{a_0} - Ad_{g_0} : \mathfrak{g} \to \mathfrak{g}$ is chosen, its kernel is also chosen. Now, since till now the basis $\{\lambda_i\}$ has been completely arbitrary, we can choose without loosing generality the first k vectors to span its kernel and the remaining $d-k$ to be in the complement of the kernel. That is assumed to be done in case 3.

1. When $Ad_{a_0} - Ad_{g_0}$ has trivial kernel the tangent space of the CS (i.e. V_{CS}) span the horizontal subspace (at the identity). By dimensional arguments (recall that we chose the contraction group to be equal to G) its complement (i.e. $\overline{V'}_{CS}$) span the vertical subspace. Therefore, the isotropy group after the contraction is isomorphic to $\mathbb{R}^d$, and so the quotient manifold is isomorphic to $A = G$.

2. When $Ad_{a_0} - Ad_{g_0}$ is always zero it means that we are contracting the manifold Q by the same group with respect to which we are quotienting it out. Therefore the left translation of A onto itself is projected to the trivial action in the quotient manifold. The tangent space to CS at the

origin span the vertical subspace and so the isotropy group is A, and the quotient manifold is diffeomorphic to $\mathbb{R}^d$.

3. It is easy to see that the vector fields in $\overline{V'}_{CS}$ can be projected in a similar way as the ones in V_{CS} yielding (at the origin): $(Ad_{a_0}(\lambda_i) + Ad_{g_0}(\lambda_i), 0)$, where one has to notice the difference in the relative sign when compared to vectors in equation 4.6. Since, by dimensional counting, the vertical space needs to have other $d-k$ basis vectors, in the same spirit of remark 7, we can choose the basis $\{\lambda_i\}$ in such a way that the first k vectors annihilate $Ad_{a_0} - Ad_{g_0}$ and the last $d-k$ annihilate $Ad_{a_0} + Ad_{g_0}$. (In this way the vectors $(Ad_{a_0}(\lambda_i), \lambda_i)$ for $1 \leq i \leq k <$ d and $(Ad_{a_0}(\lambda_i), -\lambda_i)$ for $k+1 \leq i \leq d$ (which are contracted) span the vertical subspace at the origin).
Call $\{\tau_a\}|_{1\leq a\leq k}$ the generators of the kernel of $Ad_{a_0} - Ad_{g_0}$. In order for the contracted quotient to be a manifold we need the isotropy group to be a closed subgroup of the action group. Note that, after the contraction, this is satisfied iff the generators of the kernel of $Ad_{a_0} - Ad_{g_0}$ close a subalgebra of the algebra $\mathfrak{a} = \mathfrak{g}$.
We are now only left with the existence question, i.e. given a subalgebra $\mathfrak{p} \in \mathfrak{a}$ when can we find a $(a_0, g_0) \in G \times G$ such that the kernel of $Ad_{a_0} - Ad_{g_0}$ coincides with the subalgebra $\mathfrak{p}$ with respect to which we want to contract $\mathfrak{g}$.
We can easily find some necessary conditions, which are useful to restrict our field of research. Looking at the equation $Ad_{a_0}(\tau_a) = Ad_{g_0}(\tau_a)$ $\forall a$ we see that the only important parameter is $b_0 := g_0^{-1}a_0$ because the equation is equivalent to $Ad_{b_0}(\tau_a) = \tau_a$ $\forall a$. We also note that, upon exponentiation, this is also equivalent to:
$$exp(t\tau_a)b_0exp(-t\tau_a) = b_0$$
which implies:
$$[\tau_a, b_0] = 0 \quad \forall a \tag{4.13}$$
(Note that in a similar way we can also find: $\{\lambda_i, b_0\} = 0$ $\forall k+1 \leq i \leq d$).

4.3 Examples

We will now use this formalism with the only example discussed so far: $G = SU(2)$. Let us use as generators the antihermitian $\lambda_i = i\sigma_i$, where σ_i are the Pauli matrices. We want to describe the three examples we know, i.e. S^3, $\mathbb{R}^3$ and $S^2 \times \mathbb{R}$, in a unified manner. Consider the total manifold $Q =$

$SU(2)\times SU(2)$ and let us take the quotient of it with respect to the diagonal subgroup $G = \{(g,g)\in Q | g\in G\}$ (this means that we are setting $g_0 = e_G$).

1. The case $M = SU(2)\sim S^3$ is clearly obtained without taking any contraction. It has already been described in the past and we are not going into details here. We just note that the left action of Q in our gauge, since $g_0 = e_G$, becomes $(g'_L, g'_R)\cdot(g_L g_R^{-1}, e_R)\mapsto(g'_L(g_L g_R^{-1})(g'_R)^{-1}, e_R)$, i.e. the (direct) product of left and right translations.

2. The case $M\sim\mathbb{R}^3$ is obtained by choosing as the contraction group the same group as the diagonal subgroup we are taking the quotient with. We simply set $a_0 = g_0$. This case gives a result which is the same result as the north pole approximation.

3. We want to obtain an isotropy group isomorphic to $ISO(2)$ by contraction. We note that the vector fields: $(Ad_{a_0}(\lambda_1),\lambda_1)$, $(Ad_{a_0}(\lambda_2),-\lambda_2)_{contr.}$ and $(Ad_{a_0}(\lambda_3),-\lambda_3)_{contr.}$ form a subalgebra of the total algebra. We now only need to find an $a_0\in SU(2)$ such that:

$$a_0\lambda_1 a_0^{-1} = \lambda_1 \tag{4.14}$$
$$a_0\lambda_2 a_0^{-1} = -\lambda_2 \tag{4.15}$$
$$a_0\lambda_3 a_0^{-1} = -\lambda_3 \tag{4.16}$$

This system has a unique solution given by $a_0 = i\sigma_1$ and our basis for the vertical subspace at the origin takes the form: $\{(\lambda_1,\lambda_1)$, $(-\lambda_2,\lambda_2)_{contracted}$, $(-\lambda_3,\lambda_3)_{contracted}\}$.

Remark 8. The last example pointed out one fact that is general: except in the case where G is abelian, there always exists at least one non-trivial subgroup of H with respect to which we can contract H. Indeed, if we take $g_0^{-1}a_0 = b_0 = exp(\lambda_j)$ for some fixed j, then it will certainly commute with λ_j, thus implying $\dim\Big(ker(Ad_{b_0} - Ad_{e_G})\Big)\geq 1$.

4. If what we said till now makes any sense, then we can try to make new predictions. Let us try to work with $G = SU(3)$. Consider the Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}, \lambda_2 = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}, \lambda_3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}, \lambda_4 = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},$$
$$\lambda_5 = \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}, \lambda_6 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}, \lambda_7 = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}, \lambda_8 = \tfrac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}.$$

Consider now the following $SU(3)$ matrix: $b_0 = \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. It is clear that $b_0^{-1} = b_0 = b_0^\dagger$. It is also clear that $b_0 \lambda_a b_0^{-1} = \lambda_a$ $\forall a = 1, 2, 3, 8$. Moreover we have $b_0 \lambda_k b_0^{-1} = -\lambda_k$ $\forall k = 4, 5, 6, 7$. So we should be able to contract the isotropy group $SU(3)$ with respect to the subgroup generated by exponentiation of: $\{\lambda_1, \lambda_2, \lambda_3, \lambda_8\}$. The first three clearly span an $SU(2)$ subgroup, while the last one commutes with all the other three (and obviously with itself). Since all the matrices are traceless the subgroup should be: $S(U(2) \times U(1))$. This means that the resulting manifold will be: $(SU(3) \ltimes \mathbb{R}^8)/(S(U(2) \times U(1)) \ltimes \mathbb{R}^4)$.[1]

[1]Note that one could naively think this is diffeomorphic to $SU(3)/S(U(2) \times U(1)) \times \mathbb{R}^4 \approx G_1(\mathbb{C}^3) \times \mathbb{C}^2$, where $G_1(\mathbb{C}^3)$ is the complex Grassmannian of the complex lines (or planes) in $\mathbb{C}^3$ (see e.g. [Pe2006]). However we think this is not the case, since $S(U(2) \times U(1))$ is not abelian.

Conclusion

In this thesis, we have seen some of the roles of generalised geometry in supergravity theories. In particular, we have seen how generalised geometry is well suited to geometrically describing the large group of symmetries of Type II supergravities and how it can be used to view these theories as a generalised geometrical analogue of Einstein's gravity ([CoStWa2011, CoStWa2012]). All this gave evidence that the generalised tangent bundle E is a good bundle to work with while dealing with supergravity theories. Using a better fitted geometrical formalism not only permits one to rewrite the theories in a more compact manner but can also be useful in controlling its difficult features and unclear aspects. An example of this is the problem of when a truncation is consistent. Generalised geometry has been proven to be a useful tool in understanding why some 'mysterious' consistent reductions were possible. Its generalised Scherk-Schwarz reduction ([LeStWa2014]) can be used to perform all the known consistent truncations and possibly many more. In this work, we analysed in some detail what general features a generalised Leibniz parallelisation should have if $E \approx TM \oplus T^*M$ and studied two explicit examples: a Lie group and a homogeneous space. Then, having showed that these are connected by a non-standard Inonu-Wigner contraction ([InWi1953]) of a Lie group viewed as a homogeneous space, we tried to understand how such non-standard contractions can be performed and provided a new example of that. We have not proven that this new example leads to a generalised Leibniz parallelisation. We believe nevertheless that, since a contraction is essentially a limit process on a Lie group, such a process is likely to provide the desired result, i.e. a Leibniz parallelisation.

In this thesis, we have only covered the basic material on the subject of generalised geometry. Apart from a little discussion in the appendix B, we have not covered spinors (see e.g. [Ch1996, Gu2004, Ko2011]). They are clearly essential to describing a supergravity theory. Once they are taken into account one can describe the entire Type II theories in a $O(d-1,1)\times O(1,d-1)$ covariant manner ([CoStWa2011]). Other possible extensions of the formalism we presented here are given by the so-called exceptional generalised geometries (see

[CoStWa2013, CoStWa2013n2]). These theories have been proven to be well suited to describing compactifications of the 11-dimensional supergravity in a manner that is similar to the one conventional generalised geometry uses to describe the Type II supergravities. There are also problems that are still open. To give a few examples we can mention the geometrical interpretation of the generalised Riemann tensor, the question of generalised holonomy and clearly the proof of the conjecture in 3.0.1 (of [LeStWa2014]), not to mention the discovery of a necessary condition for a manifold to admit consistent truncation.
What is most striking for us is nevertheless how a purely mathematical theory, developed to understand the geometric structures of mechanics, turned out to perfectly describe supergravity theories. The purely mathematical development of chapter 1 was indeed meant to emphasise this fact: whilst in the construction of generalised geometry we only gave some consideration to the generalised vector bundle − in particular we never mentioned supersymmetry − the resulting theory turned out to automatically describe a *supersymetric* theory.

Appendices

Appendix A

A.1 The Other Cases

In this appendix we will briefly deal with the cases that are not dealt in chapter 4; the result of this discussion will simply be that no interesting feature arises.
Consider the following quotient action:

$$\phi_R : (g, (g_L, g_R)) \mapsto (g_L, g_R g^{-1}) \underset{\text{gauge}}{\sim} (g_L, e_R)$$

The left translation on Q translates then in the quotient manifold as follows:

$$\Lambda_{LR} : ((g'_L, g'_R), [(g_L, g_R g^{-1})]) \mapsto [g'_L g_L, g'_R g_R g^{-1}] \underset{\text{gauge}}{\sim} (g'_L g_L, e_R)$$

So, basically, the vertical space is generated by the right part of the left translation, while the horizontal space is generated by the left one. Consider now the subgroup $A \in Q$, $A = \{(a_0 a a_0^{-1}, a) | a \in G\}$. This is projected on the quotient manifold M as follows:

$$\pi(a_0 a a_0^{-1}, a) = [(a_0 a a_0^{-1}, a g^{-1})] \underset{\text{gauge}}{\sim} (a_0 a a_0^{-1}, e_R)$$

From this we can easily see that, upon projection, the manifold Q and its submanifold A coincide (recall that for a_0 fixed, the conjugation is an inner automorphism of the Lie group). The action on M induced by the restriction of the left translation onto A is:

$$\begin{aligned}(Ad_{a_0}(a'), a') \cdot \pi(Ad_{a_0}(a), a) &:= \pi((Ad_{a_0}(a'), a') \cdot (Ad_{a_0}(a), a)) \\ &= [(Ad_{a_0}(a'a), a'ag^{-1})] \underset{\text{gauge}}{\sim} (Ad_{a_0}(a'a), e_R)\end{aligned}$$

which is clearly isomorphic to the left translation on the left component. This implies that its generators span the tangent space of the quotient (group) manifold M and are never equal to zero. Since their complement $(Ad_{a_0}(\tilde{\lambda}_i(q)), -\tilde{\lambda}_i(q))$ are projected to the same vector fields, it is more convenient to choose as a basis for TQ the following one:

$$\{(Ad_{a_0}(\tilde{\lambda}_i(q)), \tilde{\lambda}_i(q)), r_i(q) := (0, \tilde{\lambda}_i(q))\}$$

which clearly maintains good commutation relations, i.e. with the same structure constants. Moreover $r_i(q)$ form a basis for the vertical space. The isotropy group is therefore isomorphic, after the contraction, to $\mathbb{R}^d$. We found that contraction with respect to a diagonal G-subgroup gives rise to a quotient manifold M that is isomorphic to G.
Similarly a contraction with respect to $A = G_L \times \{e_R\}$ gives rise to the same result, while the one with respect to $A = \{e_L\} \times G_R$ gives an isotropy group isomorphic to G and an M diffeomorphic to $\mathbb{R}^d$.
We also note that, since the construction is symmetric in the exchange of the left with the right factor, we have analogous results if we use the quotient action given by:

$$\phi_L : (g, (g_L, g_R)) \mapsto (g_L g^{-1}, g_R) \underset{\text{gauge}}{\sim} (e_L, g_R)$$

Finally we would like to deal with the diagonal quotient and left/right contraction. We already know that the generic element in M is:

$$[(g_L g_0 g^{-1} g_0^{-1}, g_R g^{-1})] \underset{\text{gauge}}{\sim} (g_L g_0 g_R^{-1} g_0^{-1}, e_R)$$

Consider the contraction submanifold given by:

$$A = \{e_L\} \times G_R = \{(e_L, g_R) | g_R \in G\}$$

The projection takes the form:

$$\pi(e_L, g_R) = [(e_L g_0 g^{-1} g_0^{-1}, g_R g^{-1})] \underset{\text{gauge}}{\sim} (g_0 g_R^{-1} g_0^{-1}, e_R)$$

In particular, the integral distribution of CS is never projected to a zero vector (recall that the induced left translation on A corresponds to a translation of the right factor and therefore sends $(g'_R, g_R) \mapsto (g'_R g_R)$. This means that $TCS = \{(0, \tilde{\lambda}_i(q))\}$ coincides with the horizontal distribution. We can choose as generators of the vertical distribution the vector fields: $(Ad_{g_0}(\tilde{\lambda}_i(q)), \tilde{\lambda}_i(q))$, which are then contracted. Therefore the isotropy group becomes isomorphic to $\mathbb{R}^d$ and M to G.
For $A = G_L \times \{e_R\}$, $\pi|_A \underset{\text{gauge}}{=} id_A$. Therefore $TCS = \{(\tilde{\lambda}_i(q), 0)\}$ forms a basis for the horizontal distribution and we can use as a basis for the vertical distribution: $\{(\tilde{\lambda}_i(q), Ad_{g_0^{-1}}(\tilde{\lambda}_i(q)))\}$, and the result is the same as before.

Appendix B

B.1 Spinors

In the development of chapter 1, we outlined some general properties of some geometrical objects that can be defined on the generalised tangent bundle E. There is still at least one important object that can be introduced: a spin structure. This argument was not dealt in the project on which this thesis is based. Nevertheless, for a reason of completeness, we think it is worth at least mentioning the fact that spinors can naturally be included in the contest of generalised geometry. This appendix is therefore meant to give the basic notions about the Clifford algebra and spin representations that are related to our discussion. It will essentially be a summary of the principal results that can be found in the main reference about the argument [Ch1996], and thought to be an easier reference for a reader that comes from a physics background. For this reason many results are only quoted, in order to simplify the discussion and to make it easier to follow the construction of the general theory. Moreover, we will also use some results from [Gu2004].

In general, in order to be able to introduce spinors on a manifold we first need to identify a special orthogonal group. This implies that we need to have a metric, which enables one to introduce an orthogonal group, and an orientation, which in turn enables one to restrict this structure group to the special orthogonal group. In the specific case of the generalised vector bundle, we have seen that we can introduce these two structures 'for free': the canonical pairing between vectors and covectors allows one to define the natural metric η and the orientation without any further assumption. This implies that an $SO(d,d)$ comes naturally together with the generalised vector bundle. Secondly, for a manifold to admit a spin bundle it is necessary that some topological conditions are satisfied. In the special case of E, however, where the relevant metric η has signature (d, d), it can be shown that a $Spin(d, d)$ structure always exists (see Props 2.27 and 2.28 in [Gu2004] and reference therein). In the rest of the subsection, we would like to introduce the main ideas regarding the Clifford algebra and its relevant representations that are useful to deal with spinors in supergravity.

To define a Clifford algebra one usually makes use of quadratic forms. These forms are closely related to bilinear forms.

Definition B.1 (Section 1.2 [Ch1996])**.** Let V be a vector space over a field K.[1] A quadratic form on V is a map $Q : V \to K$ with the following properties:

1. $Q(ax) = a^2 Q(x)\ \forall\, a \in K, x \in V$
2. The mapping $(x, y) \mapsto Q(x+y) - Q(x) - Q(y)$ is a bilinear form B on $V \times V$, called the bilinear form associated to Q (which is symmetric by definition).

Proposition B.1.1 (I.2.2 [Ch1996])**.** *Let B_0 be any bilinear form on $V \times V$. Then $x \mapsto B_0(x, x)$ is a quadratic form on V and any quadratic form may be represented in this manner.*

Proof. The first statement is easily proven using the definitions. Let $\{x_1, ..., x_n\}$ be a basis of V. For the second part one can show (for example by induction) that

$$Q(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i^2 Q(x_i) + \sum_{i<j} a_i a_j B(x_i, x_j)$$

and we may define a bilinear form B_0 on $V \times V$ as follows:

$$B_0(\sum_{i=1}^{n} a_i x_i, \sum_{j=1}^{n} a'_j x_j) = \sum_{i=1}^{n} a_i a'_i Q(x_i) + \sum_{i<j} a_i a'_j B(x_i, x_j)$$

and clearly $Q(x) = B_0(x, x)\ \forall\, x \in V$.[2] □

Proposition B.1.2 (I.3.2 [Ch1996])**.** *Assume that the bilinear form B is non-degenerate. Let N be an isotropic subspace of V of dimension r. Then there exists an isotropic subspace P of dimension r such that $N \cap P = \{0\}$ and $N + P$ is not isotropic. If $\{x_1, ..., x_r\}$ is a basis of N and P has the properties stated above, there is a basis $\{y_1, ..., y_r\}$ of P such that $B(x_i, y_j) = \delta_{ij}$, for $1 \leq i, j \leq r$.*

[1] We, as opposed to the discussion given in [Ch1996], will only consider fields of characteristic 0, usually $\mathbb{R}$

[2] Note that this B_0 such that $B_0(x, x) = Q(x)\ \forall\, x \in V$, as opposed to B, does not have to be symmetric.

It is clear that in this notation the orthogonal group is the set of linear mappings $s : V \to V$ such that $Q(s \cdot x) = Q(x)$, i.e. $B(s \cdot x, s \cdot y) = B(x, y)$. We now want to introduce the Clifford algebra. Recall that if A is an algebra, the ideal generated in A by a subset $X \subset A$ is the set of all elements of A which are sums of products of the form $a \cdot x \cdot a'$, with $a, a' \in A, x \in X$.

Definition B.2 (Section 2.1 [Ch1996])**.** Let Q be a quadratic form as above. Let T be the tensor algebra of the vector space V and I the ideal generated in T by the elements $x \otimes x - Q(x) \cdot 1 \ \forall x \in V$. Then the factor algebra $C = T/I$ is called the Clifford algebra of the quadratic form Q.

The proposition we will quote in the following will allow us to choose a convenient representation of the Clifford algebra.

Proposition B.1.3 (II.1.1 and II.1.2 [Ch1996])**.** *Let ϕ be a linear mapping $\phi : V \to C'$, where C' is an algebra over K. Assume that $(\phi(x))^2 = Q(x) \cdot 1$ for $x \in V$. Then ϕ may be extended to a homomorphism ψ of C into C'. If $\phi(V)$ generates C', then $\psi(C) = C'$. If $\{x_1, ..., x_n\}$ is a basis of V, the set*

$$\mathcal{B} = \{x_{i_1}...x_{i_h} \quad \text{for } 1 \leq i_1 < i_2 < ... < i_h \leq n, i_j \in \mathbb{N} \quad \forall j\}$$

is a basis of C, which is therefore of dimension 2^n.

Relation between the Clifford and the exterior algebras. Consider the exterior algebra $\bigwedge$ on V. Recall that it admits a basis of decomposable tensors, which are homogeneous. Since the ideal I with respect to which we take the quotient of the tensor algebra T to create the Clifford algebra C is generated by a sum of elements of even degree (i.e. 0 and 2), every element of I may be written as a sum of elements of $I_+ := I \cap T_{\text{even}}$ and $I_- := I \cap T_{\text{odd}}$. Then C is the direct sum of $C_+ := T/I_+$ and $C_- := T/I_-$. If λ is a linear function on V, there exists an antiderivation δ of $\bigwedge$ such that $\delta x = \lambda(x) \cdot 1$ for $x \in V$; δ is homogeneous of degree -1 and $\delta^2 = 0$. If B_0 is a bilinear form on $V \times V$ such that $B_0(x, x) = Q(x) \ \forall x \in V$, let us denote by δ_x the antiderivation of $\bigwedge$ such that $\delta_x \cdot y = B_0(x, y) \cdot 1 \ \forall y \in V$ and let us denote also by L_x the operator of left (exterior) multiplication by x in $\bigwedge$. We can define a linear map ϕ as follows:

$$\begin{aligned} \phi : & V \longrightarrow \mathfrak{C}_{\bigwedge} (= C') \\ & x \longmapsto L'_x = L_x + \delta_x \end{aligned}$$

where $\mathfrak{C}_{\bigwedge}$ is the algebra of the endomorphisms of the vector space $\bigwedge$. This map satisfies the following property: $(\phi(x))^2 = L_x\delta_x + \delta_x L_x = Q(x) \cdot I$, where I is the identity map and $x \in V$. For, $L_x^2 = \delta_x^2 = 0$ and, if $u \in \bigwedge$, we have

$$\delta_x L_x \cdot u = \delta_x(x \wedge u) = (\delta_x x) \wedge u - x \wedge (\delta_x u) = Q(x)u - L_x \delta_x u$$

From proposition B.1.3 we can then state that there is an homomorphism $\psi : C \to \mathfrak{C}_{\bigwedge}$. Moreover if $x \in V$, since $\delta_x \cdot 1 = 0$, we have $\phi(x) \cdot 1 = L'_x \cdot 1 = x$ and so $x \mapsto L'_x$ is an isomorphism of V. We can then think of V as a subspace of C and write $x^2 = Q(x) \cdot 1$ if $x \in V$ or, similarly:

$$xy + yx = \{x, y\} = B(x, y) \cdot 1 \quad \text{for } x, y \in V$$

We can then define a linear map $\theta : C \to \bigwedge$ such that $\theta(u) = \psi(u) \cdot 1 \; \forall\, u \in \bigwedge$. Since, from proposition B.1.3, $\dim C = 2^n = \dim \bigwedge$, θ is a *linear isomorphism* which coincides with the identity on $K \cdot 1$ and on V. Consider the image of a product in C of h elements of V under the homomorphism ψ:

$$\begin{aligned}\psi(x_1 ... x_h) &= (L_{x_1} + \delta_{x_1}) ... (L_{xh} + \delta_{x_h}) \\ &= L_{x_1} ... L_{x_h} + \sum_{k=-h}^{h-1} \Phi_k\end{aligned}$$

where Φ_k is of degree k [3]. It follows that $\theta(x_1 ... x_h) = x_1 \wedge ... \wedge x_h + \sum_{k=0}^{h-1} \xi_k$, where ξ_k is of degree k. In particular, since L_x is homogeneous of degree 1 and δ_x of degree -1, Φ_k will always be of degree $(h-r)+(-r) = h-2r$. This means that, for h even (odd), only terms of even (odd) degree k will appear in the sum. The next proposition follows from what we have said until now.

Proposition B.1.4 (II.1.6 [Ch1996])**.** *Let there be given a bilinear form B_0 on $V \times V$ such that $Q(x) = B_0(x, x) \; \forall\, x \in V$. We can then identify the underlying vector space of the Clifford algebra C with that of the exterior algebra $\bigwedge$ of V in such a way that, $\forall\, x \in V$, the operator of left multiplication by x in C is $L_x + \delta_x$ defined as above. Moreover we have that $C_+ = \sum_{k \,\text{even}} \bigwedge_k$ and $C_- = \sum_{k \,\text{odd}} \bigwedge_k$.*

We now want to show that, in the specific case that is of concern to us, we can define a faithful representation of the Clifford algebra into a particular exterior algebra (cfr. with the proof of Prop II.2.1 in [Ch1996]). Suppose that V is of *even* dimension $2d$, that the bilinear form B associated with Q is non-degenerate and that the dimension of its maximal isotropic subspaces is d. Note that with these assumptions the center of the Clifford algebra coincides with the field K [Prop II.2.1][4]. Let N and P be two maximal isotropic subspaces of V which are supplementary to each other. Let $\{x_1, ... x_d\}$ and $\{y_1, ... y_d\}$ be bases of N and P respectively such that $B(x_i, y_j) = \delta_{ij}$ for

[3] Which means that it sends an homogeneous element of degree l into one of degree $l + k$, $l \in \mathbb{Z}_{\geq 0}$

[4] I.e. in this case the Clifford algebra is a 'central simple' algebra.

$1 \leq i, j \leq r$ (recall proposition B.1.2). Let B_0 the bilinear form on $V \times V$ defined by the conditions:

$$B_0(x_i, x_j) = B_0(y_i, y_j) = B_0(x_i, y_j) = 0 \text{ and } B_0(y_i, x_j) = \delta_{ij} \quad \forall 1 \leq i, j \leq d$$

This bilinear form is such that $B_0(x, x) = Q(x)\ \forall x \in V$. Therefore, it can be used to identify the space C to the underlying vector space of the exterior algebra $\bigwedge$ on V (proposition B.1.4). Let us call $\bigwedge^N$ and $\bigwedge^P$ the exterior subalgebras of $\bigwedge$ generated by N and P respectively. Since, by construction, if $x \in N$ $\delta_x(\bigwedge^N) = \{0\}$ (B_0 vanishes on $N \times N$), it follows that $\forall\, u \in \bigwedge^N$ $xu = x \wedge u$ and the algebra $\bigwedge^N$ is identical to the subalgebra of C generated by N. We can see in a similar way that $\bigwedge^P$ is a subalgebra of C. If we define $f := y_1...y_d = y_1 \wedge ... \wedge y_d$, we have $y_i f = 0\ \forall\, i$ (f corresponds to a top form on P) and so the set of elements of the form $x_{i_1}...x_{i_k} f$ form a basis of the left ideal Cf, i.e. for $u \in \bigwedge^N$ the map $u \mapsto uf$ is a linear isomorphism of $\bigwedge^N$ with Cf, which is therefore of dimension 2^d. Let us define the following (linear) representation of C into the space $\mathfrak{E}_{\bigwedge}$ of the endomorphisms of $\bigwedge$:

$$\begin{aligned} \rho : C &\longrightarrow \mathfrak{E}_{\bigwedge} \\ w &\longmapsto \rho(w) \quad \text{such that} \quad wuf = (\rho(w) \cdot u) f \quad \forall\, u \in \bigwedge^N \end{aligned}$$

Let us describe this representation. Since $\bigwedge^N$ can be viewed as a subalgebra of C and since B_0 vanishes on both $N \times N$ and $N \times P$, it is clear that if $w \in \bigwedge^N$ then $\rho(w)$ is the operator of left multiplication by w in $\bigwedge^N$. Similarly we can deduce that if $x \in N$ and $v \in \bigwedge^P$ then $xv = x \wedge v$. This in turn implies that $uf = u \wedge f\ \forall\, u \in \bigwedge^N$. Now consider the case where $y \in P$. Since B_0 vanishes also on $P \times P$ we can write: $(\rho(y)u)f = yuf = (\delta_y + L_y)u \wedge f = \delta_y(u \wedge f) = (\delta_y u)f$, since f is a top form in $\bigwedge^P$ and δ_y is an antiderivation.

Proposition B.1.5 (In the proof of Prop. II.2.1 [Ch1996])**.**
The representation ρ is a faithful representation. In particular if σ and σ_1 are elements of the basis $\mathcal{B} = \{x_{i_1}...x_{i_h} \text{ for } 1 \leq i_1 < i_2 < ... < i_h \leq n, i_j \in \mathbb{N}\ \ \forall j\}$ of C^N there is one $w \in C$ such that $\rho(w)\sigma = \sigma_1$ and $\rho(w)\sigma' = 0$ if $\sigma' \in \mathcal{B}$ and $\sigma' \neq \sigma$; moreover $\rho(C)$ has dimension 2^{2d}.

Proof. We know that $\rho(f)$ is homogeneous of degree $-d$ and that, defined $e := x_d...x_1$, we have $\rho(f)e = 1$. If σ is a product of h elements, let τ be the product of the elements of the basis of N not appearing in σ. Then:

$$\begin{cases} \rho(\tau)\sigma' = \pm e & \text{if } \sigma' = \sigma \\ \rho(\tau)\sigma' = 0 & \text{if length of } \sigma' \geq \text{length of } \sigma \text{ and they are} \\ & \text{products of 2 different sets of elements of } N \\ \rho(\tau)\sigma' = \text{homog. of degree} < d & \text{if length of } \sigma' < \text{length of } \sigma \end{cases}$$

Then $w = \pm\sigma_1 f\tau$ has the desired properties. From what we said it follows also that $\rho(C)$ sends the basis of $\bigwedge^N$ to itself, and so it is equal to the algebra of all vector-space endomorphisms of $\bigwedge^N$ and is therefore of dimension 2^{2d}. $\square$

In the case of the generalised vector bundle we have the decomposition $E \approx TM \oplus T^*M$, where we saw that TM and T^*M are maximally isotropic subbundles of E with respect to η. We can then identify N with T^*M and P with TM.[5] The space of representation of ρ is therefore the space of differential forms on M. Let us choose a local chart on M. We can then choose as a basis of TM $\{\partial_1, ..., \partial_d\}$ and as a basis of T^*M $\{dx_1, ..., dx_d\}$. According to the definition of the bilinear form B_0 we see that $\delta_{dx_i}\partial_j = 0$ and $\delta_{\partial_i} dx_j = \delta_{ij}$. We can see that in our case the antiderivation δ coincides with the inner product on differential forms. In conclusion we have the following representation of the Clifford algebra: if $X = v + \mu \in \Gamma(E)$ and $\omega \in \bigwedge^\bullet(M)$ we have

$$\Gamma_X \cdot \omega := \rho(X) \cdot \omega = i_v\omega + \mu \wedge \omega$$

and this defines a faithful representation. It is now also clear why we have just used the gamma to label this representation: if $\{X_1, \ldots, X_n\}$ is a basis for E, $\{\Gamma_{X_1}, \ldots, \Gamma_{X_n}\}$ generates the (representation of the) Clifford algebra and we have: $\{\Gamma_{X_i}, \Gamma_{X_j}\} \cdot \omega = \eta(X_i, X_j)\omega$.

B.1.1 The Clifford Group

Definition B.3. Let the bilinear form B associated with Q be non-degenerate and let G be the orthogonal group of Q. The Clifford group of G, denoted by Γ, is the group of *invertible* elements $s \in C$ such that $sxs^{-1} \in V \ \forall x \in V$. The linear automorphism $\chi(s) : x \mapsto sxs^{-1}$ is a linear representation of Γ, called the vector representation.

Prop II.3.1 of [Ch1996] then states that, for dim$V$ even, $\chi(\Gamma) = G$. We shall denote by Γ^+ the group $\Gamma \cap C_+$, and set $G^+ = \chi(\Gamma^+)$. Proposition II.3.3 of [Ch1996] states that G^+ is the group of operations of determinant 1 in G, i.e. it is the special orthogonal group.
Let us define α^T as the linear map of the tensor algebra T of V into itself that reverts the order of the products in the decomposable tensors (e.g. $\alpha^T(t_1 \otimes t_2) = t_2 \otimes t_1$) extended to the whole tensor algebra by linearity.

[5] All these identifications are to be understood as taken pointwise on M, i.e. each T_pM is identified with P, etc.

Since the generators of the ideal used to define the Clifford algebra are left invariant by the action of α^T (they are homogeneous of even degree), α^T naturally induces an automorphism of C, that is called *main automorphism* of C. Since α^T preserves the homogeneous degree of the decomposable tensors, we also have that $\alpha(C_\pm) = C_\pm$.

Proposition B.1.6 (II.3.5 [Ch1996])**.** *If $s \in \Gamma$, then $\alpha(s) \in \Gamma$ and $\alpha(s)s$ is an element of the center of C*

Proof. Let $x \in V$. We can clearly write: $sx = (sxs^{-1})s = (\chi(s) \cdot x)s$. Applying α to both sides of the equation we find:

$$x\alpha(s) = \alpha(x)\alpha(s) =_{eq} \alpha(s)\alpha(\underbrace{\chi(s) \cdot x}_{\in V}) = \alpha(s)\chi(s) \cdot x = \alpha(s)sxs^{-1}$$

i.e. $x\alpha(s)s = \alpha(s)sx$, which is exactly what we wanted. □

Since in our case (even dimension of V) the center of C is $K \cdot 1$, this proposition shows that $\alpha(s)s \in K \cdot 1$ whenever $s \in \Gamma$. We can therefore define the following *norm homomorphism*[6]: $\alpha(s)s =: \lambda(s) \cdot 1 \ \forall s \in \Gamma$, where clearly $\lambda(s) \in K$ and $\lambda(c \cdot 1) = c^2$ if $c \in K^*$[6] and $\lambda(x) = Q(x)$ if $x \in V$, $Q(x) \neq 0$.

Definition B.4. We will denote by Γ_0 the group of elements $s \in \Gamma$ such that $\lambda(s) = 1$ and call $\Gamma_0^+ = \Gamma_0 \cap \Gamma^+$ the reduced Clifford group of Γ.

B.1.2 Spinors

[Section 2.4] It is known that all irreducible representations of the simple algebra C are equivalent. Let us select one of them, say ρ, and call its space of representation S the space of spinors of Q. The representation ρ is called spin representation of C. This representation induces several representations of different spaces, namely ρ_+ of C_+, ρ_Γ of Γ, ρ_{Γ^+} of Γ^+ and $\rho_{\Gamma_0^+}$ of Γ_0^+. All these representations are also called *spin representations*. Nevertheless, we are particularly interested in the spin representations of the special orthogonal group and so will focus on ρ_{Γ^+} and $\rho_{\Gamma_0^+}$.
[7]'Consider now the representation ρ^+ of C_+. This representation is either simple or the sum of two simple representations' (Prop. II.2.3 [Ch1996]). 'If

[6]The homomorphism is between Γ and the multiplicative group of invertible elements of the center (K) of C, say K^*.

[7]Quoted from the discussion before proposition II.4.2 in [Ch1996]. Note that here the term 'simple' means irreducible.

C_+ is not simple, then C_+ has two inequivalent simple representations, and both must occus in ρ^+, since ρ^+ is faithful. In that case, ρ^+ is the sum of two inequivalent simple representations. It follows that S may be represented in one and only one way as the sum of two subspaces each of which yields a simple representation of C_+. These two spaces are then called the spaces of *half-spinors*, and the corresponding representations of C_+ the half-spin representations. The representations of Γ^+ and Γ_0^+ induced by the half-spin representations of C_+ are called the half-spin representations of these groups'.

Proposition B.1.7 (II.4.2 and II.4.3)**.** *The spin representation $\rho_{\Gamma_0^+}$ (or ρ_{Γ^+}) of Γ_0^+ (Γ^+) is either simple or the sum of two simple representations. If C_+ is not a simple algebra, then the half-spin representations of Γ_0^+ (Γ^+) are simple and they are inequivalent to each other.*

Consider again our specific case in which $E \approx TM \oplus T^*M$. This is a direct sum of two maximally isotropic subspaces with respect to the metric η. Let us define at each point $p \in M$ in the manifold f_p as the product in C on the elements of some basis of T_pM, then Cf_p is a minimal left ideal of C, and we have $Cf_p = C^{T^*_pM}$, where $C^{T^*_pM}$ is the subalgebra of C generated by T^*_pM. There is a representation ρ of C on C^{T^*M} such that $vuf = (\rho(v) \cdot u)f$ $\forall v \in C, u \in C^{T^*M}$. We will choose this representation to be the one described previously in the appendix. Since Cf_p $\forall p \in M$ is a minimal left ideal, ρ is irreducible. We may therefore take the space of spinors S to be C^{T^*M}, which may be identified with the exterior algebra of T^*M; moreover $C^{T^*M}_\pm = C^{T^*M} \cap C_\pm$ may be identified with the even (+) and odd (-) elements of C^{T^*M}. Clearly for $x \in V$ we have: $\rho(x) : C^{T_pM}_\pm \to C^{T_pM}_\mp$; this implies that, for $u \in C_+$, $\rho(u)$ maps $C^{T_pM}_+$ and $C^{T_pM}_-$ into themselves. One can then deduce that the spin representation ρ_+ of C_+ is not irreducible. In this case, from prop B.1.7 we can deduce that the half-spin representations of Γ^+ and Γ_0^+ on $C^{T^*M}_+$ and $C^{T^*M}_-$ are inequivalent irreducible representations.

Let us now summarise the findings of this section using a terminology that is more understandable to physicists. $SO(d,d)$ spinors can be represented by elements of the exterior algebra of T^*M. The Clifford algebra (i.e. the 'gamma matrices' in the physics terminology) is represented by the endomorphisms $\rho(x) \equiv \Gamma_x$ acting on $\bigwedge^\bullet T^*M$, with $x \in E$, and where the bilinear form is obviously η. The identity component of $Spin(d,d)$ is Γ_0^+. The vector representation $\chi : s \mapsto sxs^{-1}$, $s \in Spin(d,d)$, $x \in E$ is the homomorphism that defines the double cover of $SO(d,d)$ by $Spin(d,d)$. Since $\mathfrak{so}(d,d) = \bigwedge^2(TM \oplus T^*M)$ sits naturally inside the Clifford algebra we can see its action on the spin representation [Gu2004]. It turns out that the action of a B, β and $GL(d)$-transforms are as follows (see [Gu2004] examples 2.10-2.12):

1. If $B = \frac{1}{2}B_{ij}e^i \wedge e^j$: $B \cdot \phi = (-B \wedge \phi)$
   This exponentiates to: $e^{-B}\phi = (1 - B + \frac{1}{2}B \wedge B + ...)\phi$

2. If $\beta = \frac{1}{2}\beta^{ij}e_i \wedge e_j$: $\beta \cdot \phi = \frac{1}{2}\beta^{ij}i_{e_j}(i_{e_i}\phi) = i_\beta\phi$.
   This exponentiates to $e^\beta\phi = (1 + i_\beta + \frac{1}{2}i_\beta^2 + ...)\phi$.

3. For $GL(d)$ the problem is that the diagonal $GL(d)$ subgroup of $SO(d,d)$ has two fibres $GL_1(d)$ and $GL_2(d)$ in $Spin(d,d)$ under the homomorphism (covering map) χ. If we consider their common connected intersection $GL^+(d)$ we have, for $A = A_i^j e^i \otimes e_j$ an endomorphism of TM: $A \cdot \phi = \frac{1}{2}A_i^j(i_{e_j}(e^i \wedge \phi) - e^i \wedge i_{e_j}\phi) = \frac{1}{2}A_i^j\delta_j^i\phi - A_i^j e^i \wedge i_{e_j}\phi = -A^*\phi + \frac{1}{2}(TrA)\phi$.
   By exponentiation the action of $GL^+(d)$ on the exterior algebra of T^*M is by: $g \cdot \phi = \sqrt{\det g}(g^*)^{-1}\phi$. This means that as a $GL^+(d)$ representation the spinor representation decomposes as:

$$S = \bigwedge^{\bullet}(T^*M) \otimes (\det TM)^{1/2}$$

This last fact shows that the spin representation is actually *almost* equal to the exterior algebra - i.e. modulo a little technicality.

B.2 Bilinear Pairing on Spinors

Let α be the main automorphism of C. C^{T^*M} is isomorphic to the exterior algebra of T^*M. Consider f as above and $u, v \in \bigwedge(T^*M)$. Then we have:

$$\alpha(uf)vf = \alpha(f)\alpha(u)vf = (-1)^{\frac{d(d-1)}{2}}f\alpha(u)vf$$

since $f = \partial_1...\partial_d$ is the product in C of d elements of a (local) basis of TM. We have $\alpha(u)v \in \bigwedge^\bullet(T^*M) \approx C^{T^*M}$ and $f\alpha(u)vf = (\rho(f) \cdot \alpha(u)v)f$, where ρ is the usual representation. Let e be the product of the elements of a (local) basis $\{dx_1, ..., dx_d\}$ of T^*M. $\rho(f)$ is of degree $-d$ and so it maps upon 0 every homogeneous element of degree $< d$ of C^{T^*M}. So, if pe is the homogeneous component of degree d of $\alpha(u)v$, then $\rho(f) \cdot \alpha(u)v = (-1)^{\frac{d(d-1)}{2}}p \cdot 1$. Let us denote by $\beta(u,v)e$ the homogeneous component of degree d of $\alpha(u)v$, whence: $\alpha(uf)vf = \beta(u,v)f$, where β is a bilinear form on $C^{T^*M} \times C^{T^*M}$.

Proposition B.2.1 (III.2.1 and following discussion in [Ch1996]). *The bilinear form $\beta : C^{T^*M} \times C^{T^*M} \to K$ such that $\alpha(uf)vf = \beta(u,v)f$ satisfies the following properties:*

1. *β is a bilinear invariant of the spin representation of Γ_0^+;*
2. *β is non-degenerate;*
3. *β has the following antisymmetry property: $\beta(v,u) = (-)^{\frac{d(d-1)}{2}}\beta(u,v)$.*

Proof.

1. If $s \in \Gamma$ we have $\beta(\rho(s)\cdot u, \rho(s)\cdot v) = \alpha(suf)svf = \alpha(uf)\alpha(s)svf = \lambda(s)\alpha(uf)vf = \lambda(s)\beta(u,v)$, by the definition of the norm homomorphism λ.
2. If $0 \neq u \in \bigwedge(T^*M)$, then $\alpha(u) \neq 0$ and there exists an $v \in \bigwedge(T^*M)$ such that $\alpha(u)v = e$, i.e. such that $\beta(u,v) = 1$.
3. Since α^2 is the identity we have that $\alpha(v)u = \alpha(\alpha(u)v) = \alpha(e)p = (-1)^{\frac{d(d-1)}{2}}pe$, where pe is the homogeneous component of degree d of $\alpha(u)v$.

$\square$

Note that point 1 of proposition B.2.1 implies that the B-transform is a symmetry of the bilinear pairing because $B \in \mathfrak{so}(d,d)$.
To conclude consider the generalised vector bundle as an extension of the tangent bundle via the cotangent one as in section 1.3. Let $\{\Lambda_{(ij)}\}$ be the patching one-forms and define $B = \{B_{(i)}\}$ the patching two-forms that satisfy $B_{(i)} - B_{(j)} = d\Lambda_{(ij)}\ \forall\, i,j$. This B defines an isomorphism between spinors of E and spinors of $(\det T^*M)^{-1/2} \otimes \bigwedge^\bullet(T^*M)$. Let $\{\Psi_{(i)}\}$ be $TM \oplus T^*M$ spinors defined on each patch of $\{U_{(i)}\}$. The patching rules introduced in section 1.3 then imply that $\Psi_{(i)} = e^{-d\Lambda_{(ij)}}\Psi_{(j)}$. Then $e^{+B_{(i)}} \wedge \Psi_{(i)}$ are well defined spinors of E (or, better, they are the image under the isomorphism between E and $TM \oplus T^*M$ defined by B). This is clear from the action of B-transform on spinors written in the previous subsection.

Bibliography

[AbTo2011] M. Abate, F. Tovena, *Geometria Differenziale.* Unitext, Springer-Verlag Italia, 2011

[AMR1988] R. Abraham, J.E. Marsden, T.S. Ratiu *Manifolds, Tensor Analysis and Applications.* Applied Mathematical Sciences, volume 75. Springer-Verlag, 1988

[BaPoSa2015] A. Baguet, C.N. Pope and H. Samtleben *Consistent Pauli reduction on group manifolds* arXiv:1510.08926v1 [hep-th]

[Ba2012] D. Baraglia *Leibniz algebroids, twisting and exceptional generalised geometry* Journal of Geometry and Physics, Volume 62, Issue 5, May 2012, 903–934 [arXiv:math.DG/1101.0856v2]

[BKORV01] E. Bergshoeff, R. Kallosh, T. Ortin, D. Roest, A. Van Proeyen *New formulations of D=10 supersymmetry and D8-O8 domain walls.* Class. Quant. Grav. , 3359-3382 (2001) [hep-th/0103233]

[BMZ1999] D. Brace, B. Morariu, B. Zumino *T duality and Ramond-Ramond backgrounds in the matrix model* Nucl. Phys. **B549**, 181-193 (1999) [hep-th/9811213]

[Ch1996] C. Chevalley, *The Algebraic Theory of Spinors and Clifford Algebras. Collected Works.,* volume 2. Springer Verlag, 1996

[CoStWa2011] A. Coimbra, C. Strickland-Constable and D. Waldram, *Supergravity as Generalised Geometry I: Type II Theories.* JHEP **1111** (2011), 091 [arXiv:1107.1733 [hep-th]]

[CoStWa2012] A. Coimbra, C. Strickland-Constable and D. Waldram, *Generalised Geometry and type II Supergravity.* Fortschritte der Physik, Vol. 60, (2012) 982-986, doi:10.1002/prop.201100096

[CoStWa2013] A. Coimbra, C. Strickland-Constable and D. Waldram *Supergravity as Generalised Geometry II: $E_{d(d)} \times \mathbb{R}^+$ and M theory* arXiv:1212.1586 [hep-th]

[CoStWa2013n2] A. Coimbra, C. Strickland-Constable and D. Waldram *$E_{d(d)} \times \mathbb{R}^+$ Generalised Geometry, Connections and M theory.* arXiv:1112.3989 [hep-th]

[CoWe1988] T. Courant, A. Weinstein, *Beyond Poisson structures.* Actions hamiltoniens de groupes. Troisième théorème de Lie (Lyone, 1986), Travaux en Cours, Vol. 27, Hermann, Paris, 1988, 39-49

[Co1990] T.J. Courant, *Dirac manifolds.* Trans. Amer. Math. Soc. **319** (1990), 631-661

[De2014] O De Felice, *Generalised Geometry, Parallelisations and Consistent Truncations.* `http://www.imperial.ac.uk/media/imperial-college/research-centres-and-groups/theoretical-physics/msc/dissertations/2014/Oscar-de-Felice-Dissertation.pdf`, (2014)

[Do1987] I. Ya. Dorfman, *Dirac structures of integrable evolution equations* Physics Lett. **A 125**, 240 (1987)

[DNP86] M.J. Duff, B.E.W. Nilsson, C.N. Pope *Kaluza-Klein Supergravity* Phys. Rept. **130** 1 (1986)

[FOT2000] M. Fukuma, T. Oota, H. Tanaka *Comments on T dualities of Ramond-Ramond potentials on tori* Prog. Theor. Phys. **103** (2000) 425-446 [hep-th/9907132]

[Gi1974] R. Gilmore *Lie Groups, Lie Algebras, and Some of Their Applications* John Wiley & Sons (1974)

[Ha2000] S.F. Hassan *SO(d, d) transformations of Ramond-Ramond fields and space-time spinors* Nucl. Phys. **B583** (2000) 431-453 [hep-th/9912236] S.F. Hassan *T duality, space-time spinors and RR fields in curved backgrounds* Nucl. Phys.**B568** (2000) 145-161 [hep-th/9907152]

[Hi2001] N.J. Hitchin, *Lectures on special Lagrangian submanifolds.* arXiv:math/9907034

[Hi2003] N. Hitchin, *Generalised Calabi-Yau manifolds.* Quart. J. Math. Oxford Ser. **54**, 281 (2003) [arXiv:math.dg/0209099]

[Hi2010] N. Hitchin, *Lectures on generalized geometry* arXiv:math.DG/1008.0973v1

[HoZw2012] O. Hohm and B. Zwiebach, *On the Riemann tensor in double field theory.* J. High Energ. Phys. (2012) 2012: 126. doi:10.1007/JHEP05(2012)126 [arXiv:1112.5296v3 [hep-th]]

[HuTo95] C.M. Hull and P.K. Townsend *Unity of superstring dualities* Nucl. Phys. **B438** (1995) 109 [hep-th/9410167]

[InWi1953] E. Inonu and E.P. Wigner *On the Contraction of Groups and their Representations* Proc. Natl. Acad. Sci. USA. 1953 Jun; 39(6): 510–524.

[Ka21Kl26] T. Kaluza *On the Problem of Unity in Physics* Sitzungsber.Preuss.Akad.Wiss.Berlin (Math.Phys.) 1921 (1921) 966-972 O. Klein *Quantum Theory and Five-Dimensional Theory of Relativity.* (In German) Z.Phys. 37 (1926) 895-906

[KiWi1987] Y. S. Kim and E. P. Wigner *Cylindrical group and massless particles* Journal of Mathematical Physics 28, 1175 (1987); doi: 10.1063/1.527824

[Ko2011] P. Koerber, *Lectures on Generalized Complex Geometry for Physicists.* Fortsch.Phys. **59** (2011), 169-242 [arXiv:1006.1536v2 [hep-th]]

[GGP2010] M. Gabella, J.P. Gauntlett, E. Palti et al. *AdS5 Solutions of Type IIB Supergravity and Generalized Complex Geometry* Commun. Math. Phys. (2010) 299: 365. doi:10.1007/s00220-010-1083-y

[Gu2004] M. Gualtieri, *Generalised Complex Geometry.* Oxford University DPhil thesis (2004) [arXiv:math.DG/0401221] and [arXiv:math.DG/0703298]

[LeStWa2014] K. Lee, C. Strickland-Constable and D. Waldram, *Spheres, generalised parallelisability and consistent truncations.*

[MLa1975] S. Mac Lane *Homology.* Reprint of the 1975 edition, Springer Classics in Mathematics, ISBN 3-540-58662-8, p.16

[No1954] K. Nomizu *Invariant Affine Connections on Homogeneous Spaces.* American Journal of Mathematics, Vol. 76, No. 1 (Jan., 1954), pp. 33-65

[PSTDL] P. Pasti, D. Sorokin and M. Tonin *Duality symmetric actions with manifest space-time symmetries* Phys. Rev **D52** (1995) 4277-4281, hep-th/9506109 P. Pasti, D. Sorokin and M. Tonin *On Lorentz invariant actions for chiral p-forms.* Phys. Rev. **D55** (1997) 6292-6298, hep-th/9611100 P. Pasti, D. Sorokin and M. Tonin *Covariant action dor a D=11 five-brane with the chiral fields.* Phys. Lett. **B398** (1997) 41, hep-th/9701037 G. Dall'Agata, K. Lechner and M. Tonin *D=10, N=IIB supergravity: Lorentz-invariant actions and duality.* JHEP **9807** (1998) 017 hep-th/9806140

[Pa1996] V. Patrangenaru *Classifying 3- and 4-dimensional homogeneous Riemannian manifolds by Cartan triples* Pacific Journal of Mathematics, Vol. 173 (1996), No. 2, 511–532, DOI: 10.2140/pjm.1996.173.511

[PeSc1997] M. Perry and J.H. Schwarz *Interacting chiral gauge fields in six dimensions and Born-Infeld theory.* Nucl. Phys. **B489** (1997) 47-64, hep-th/9611065

[Pe2006] P. Petersen *Riemannian Geometry* Springer Science +Business Media, LLC (2006)

[Po1998] J. Polchinski *String Theory: Volume 2, Superstring Theory and Beyond.* Cambridge Monographs on Mathematical Physics, Cambridge University Press, 1998

[SaWe1986] D.H. Sattinger, O.L. Weaver *Lie groups and algebras with applications to physics, geometry and mechanics.* Pp 215. DM 98. 1986. ISBN 3-540-96240-9 (Springer)

[ScSc1979] J. Scherk and J.H. Schwarz *How to Get Masses from Extra Dimensions* Nucl. Phys. B **153**, 61 (1979)

[So2005] D. Sorokin *Introduction to the Classical Theory of Higher Spins* arXiv:hep-th/0405069v3

[Su1973] H. J. Sussmann, *Orbits of Families of Vector Fields and Integrability of Distributions.* Trans. Amer. Math. Soc. **180** (1973): 171-88

[ViFu2004] O. Ya. Viro, D. B. Fuchs *Topology II. Homotopy and Homology. Classical Manifolds* V.A. Rokhlin, S.P. Novikov (Eds.), Springer-Verlag (2004)