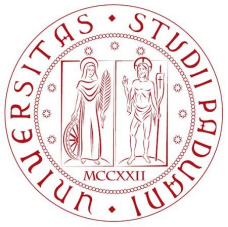

**Università degli Studi di Padova**

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# Out-of-equilibrium quasi-stationary states in ultracold atomic gases

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CONTENTS

# INTRODUCTION

In this thesis, we want to study one of the several questions left open when we approach to a non-equilibrium dynamical theory. More precisely, we are going to focus on ultracold atomic gases, which are an extraordinary tool in order to enlighten one of the *dark side* of this topic. Their importance, not only in condensed matter physics, is greatly remarked in literature [14, 33, 45].

They can help us in facing the following problem: if we consider the equilibrium statistical mechanics scenario, we always refer to the coupling of our system with a large reservoir [25]. Its role basically consists in the energy, or particles, exchange: thanks to it, our system can reach a thermal equilibrium. As noted in [33], the reservoir itself has to be intended in thermal equilibrium; a rigorous development of this topic has to specify how the reservoir could have reached its equilibrium state. A superficial answer can involve another, bigger resevoir, but this is a *regressus ad infinitum* scheme.

Then, the main question is if a lonely, isolated and many-body system can evolve from an out-of-equilibrium state to a thermal equilibrium one.

In the classical world, this relaxation has been deeply explored, with really burdensome computation and effective theory; for transport phenomena in neutral fluids, we mention the Chapman-Enskog [10] series expansion of collisional integral. For classical plasmas, the situation is substantially more complicated, because of the long-range nature of interactions between particles; physicists are not always able to fully derive all the dynamical properties. More precisely, there is not a whole theory capable to embrace all the phenomena developing in plasma [12]. However, despite some really awkward points, the classical procedure of equilibrium relaxation is heading towards an entire comprehension.

On the other side, we have the quantum world: here, the relaxation towards a certain thermal equilibrium state is still an open question [45]. This lack of understanding usually reveals itself in a great variety of techniques, whose connection point is not always clear [34]. Hence, the building-up of a quantum non-equilibrium theory seems to have an internal lack of universality. This point can't be underestimate: it is a relevant topic in many physical fields, as again underlined by Langen [33]. For example, it seems to be a key-point in cosmology and inflation studies; we mention the [44] where, quoting the abstract, *We finally argue that many*

*questions of the thermal history of the universe should be addressed in terms of prethermalization.*

The problem of relaxation towards a thermal equilibrium appears also in high-energy, nuclear and condensed matter physics. Moreover, the time evolution of an out-of-equilibrium quantum system and its interaction with an external ambient is an interesting point when we talk about quantum information and classical limits of quantum mechanics [57].

The usefulness of ultracold atomic gases must be searched from an experimental point of view. We easily realize that the experimental feasibility of an isolated quantum system which is, at the same time, accessible to the experiment is a really demanding challenge. Ultracold atomic gases have revealed themselves as a really powerful tool, which combines the possibility of a great isolation from the external enviroment and the simultaneous accessibility for measure devices.

From 1995, when the Bose-Einstein condensation was observed in atomic vapours of $^{87}Rb$ [2], a plenty of experimental techniques was developed in order handle atomic gases.

Nowadays, by means of particular configuration of magneto-optical traps, we also manage to study really exotic configurations; for example, in [50] we have chains of cesium atoms that reproduce a 1D Ising model. The tunability of all parameters allows us to realize a great variety of physical situation.

In this thesis, we're going to analyze the out-of-equilibrium behaviour of ultracold atomic gases by means of the Boltzmann-Vlasov equation. This a well-known equation derived by a truncation of a BBGKY hierarchy; it describes the phase-space dynamics of the distribution function for a collisionless system. Then, in the first chapter, we are going to study how this equation can be obtained, both for Bose gases and Fermi ones. We will present the derivation through two different approaches: for Bose gases we set up the derivation on [56]. We start from the microscopic dynamics equation, i.e the Schroedinger one, and, under certain assumptions, we derive a kinetic equation.

For Fermi gas, on the other side, we give a heuristic derivation based on the quasi-particle concept which can allow us to better focus on the physical concepts and validity conditions of our approximation.

The second chapter consists in an original derivation of the dimensional reduction method for Boltzmann-Vlasov equation. Basically, the core idea consists in developing a method through which one could derive a 1D Boltzmann-Vlasov equation moving from the original 3D one; in reduced equation, the only memory

of the original problem consists in a proper scaling of certain parameter. We will show it in detail.

The physical reason behind this approach lies in the will of taking advantage from the lighter computational and analytical workload of a 1D system, without losing the proximity to a real 3D physical system. This idea is widely used in condensed matter problems: setting a proper confinement potential only on a transverse plane, we reasonably can suppose that all the relevant dynamical features of the condensate will evolve on the longitudinal axis. We will formulate a variational approach for Boltzmann-Vlasov equation by means of a formal analogy with quantum mechanics, especially with the Hartree-Fock method.

The third chapter contains the dynamical analysis of the 1D Boltzmann-Vlasov equation; actually we will face the *linearized* Vlasov equation. Our analysis makes use of ideas and methods from classical plasma theory, due to the fact that the Boltzmann-Vlasov equation for ultracold atomic gases and the Vlasov-Poisson one are similar.

An important achievement is the demonstration of Landau damping occurring in ultracold atomic gases; it is the classical collisionless phenomenon and one of the clearest example of the particle-wave interaction. A self-consistent derivation of it is given. Moreover, we will be able to derive an analytical form for the dispersion relation in the case of weakly damped waves perturbing our medium.

We think that one of the interesting point of this thesis is how, moving from physical systems (plasmas and ultracold atomic gases), that seem to be really different from each other, we manage to describe them using almost the same formalism and theoretical tools.

We also give some analytical results for more complex situation as, for example, two-peaked distribution functions; we will focus on an analysis concerning the linear instability.

The last chapter is devoted to the presentation of a numerical approach towards the Boltzmann-Vlasov equation. We will present the semilagrangian method that we have exploited in order to derive a numerical confirmation of Landau damping.

# 1

## KINETIC EQUATION FOR ULTRACOLD ATOMIC GASES

In this opening chapter, the main topic we are going to develop and deepen concerns the dynamics of ultracold and dilute atomic gases. A complete treatment of a dynamical theory of these systems go obviously beyond the scope of this chapter and, moreover, of this thesis. On the other hand, we must have understood what are the concepts and the theoretical tools we will handle in the following.

Fixing our mind on this *epistemological* issue, we are able to understand what are the mathematical tools we have to build in order to give a faithful account of physical phenomena. From a very general point of view, a *dynamical theory* is a physical theory by which we can study the time evolution of a certain system. The physics offers a great variety of dynamical theories, as classical electrodynamics or quantum mechanics but, as underlined in [12], they all have some precise features in common.

A well-established theory has to describe the state of our system at any time instant. In other words, we have to specify all the generalized position and momentum coordinates; if we consider the elementary quantum mechanics, the state is given by a vector $|\Psi\rangle$ belonging to a proper Hilbert space. Its representative is a square-integrable function $\Psi(\mathbf{r})$, which we refer to as *wave function*.

So, we can say that the state of a physical system is defined by giving the numerical values of a certain set of variables. Then, we need a certain set of equations which specifies the evolution in time of these variables. For a quantum system, we have to consider the Schroedinger equation.

Following the path developed in [12], we have to choose the level through which we want to enhance the theory. We can study a quantum system of $N$ identical particles: it is described by means of a $N$-particle wave function, whose time evolution is given by solving the many-body Schroedinger's equation.

On the contrary, according to a classical picture, if $N$ is large, it's worthwhile to reconsider the problem from a statistical point of view. This can be done by means of the distribution function[25]

$f(\mathbf{r}, \mathbf{p}, t)$; it gives the particle density in a certain point of a six-dimensional $(\mathbf{r}, \mathbf{p})$-space at time $t$. The dynamics of $f(\mathbf{r}, \mathbf{p}, t)$ is given by the well-known Boltzmann equation.

In this chapter, we want to study the possibility of connecting these two levels; in other words, we will analyze a sort of statistical limit of the $N$-body Schroedinger's equation and we will be able to get a quantum analog of the classical Boltzmann equation. The chance of building up this connection has attracted a lot of attention over the years, by the reason of an apparent paradox.

It lies on the properties of the motion equation under time reversal: the many-body quantum dynamics is ruled by the Schroedinger's equation, which is a time-reversible hyperbolic one. On the other side, if the number of particles is large enough, the dynamical features are often well depicted by the Boltzmann equation, which is a time-irreversible one. Balescu [7] offers a great review about this topic, also involving the attempts to approach the problem by means of a BBGKY quantum hierachy and its truncation. We will also face this approach in the following.

We divide the treatise in two main section: we will show two different approach, the first one for ultracold Bose gases, the other one for Fermi gases.

## 1.1 AN APPROACH FOR BOSE GASES

The path we will follow is the one presented in the breakthrough article by Zaremba, Nikuni and Griffin [56], subsequently recovered and slightly reformulated in the same authors' book [55], where they make use of the Green's functions formalism. Another landmark is the first of three papers by Kirkpatrick and Dorfman [29] concerning the transport phenomena in Bose-condensed gases at finite temperature.

So, let us begin by writing the time evolution of a generic field operator through the Heisenberg equation with a $N$-body Hamiltonian [14, 15]:

$$
\begin{aligned}
& i\hbar \frac{\partial \hat{\Psi}}{\partial t}(\mathbf{r}, t) = \left[\hat{\Psi}, \hat{H}\right] \\
& = \left[ -\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}) + \frac{1}{2}\int d\mathbf{r}'\, \hat{\Psi}^\dagger(\mathbf{r}', t) V(\mathbf{r} - \mathbf{r}') \hat{\Psi}(\mathbf{r}', t) \right] \hat{\Psi}(\mathbf{r}, t) \, .
\end{aligned}
$$

Due to the fact we are dealing with Bose gases, we consider the canonical commutation rules given by

$$
\begin{cases}
\left[\hat{\Psi}(\mathbf{r}), \hat{\Psi}^\dagger(\mathbf{r}')\right] = \delta(\mathbf{r} - \mathbf{r}') \\
\left[\hat{\Psi}(\mathbf{r}), \hat{\Psi}(\mathbf{r}')\right] = 0 \, .
\end{cases}
\tag{1.1}
$$

Considering the equation (1.1), we have to specify the meaning of the integral which describes the atom-atom interaction. Our initial aim consists in writing a generalized Gross-Pitaevskii (GP) equation for the condensate; in order to do this, we have to replace the field operator $\hat{\Psi}(\mathbf{r},t)$ with a sort of *condensate wavefunction*, in a way we are going to specify [29]. However, this replacement leads to a poor approximation when short interatomic distances are involved.

The assumption of a dilute and ultracold gas is thus crucial [14]: in this case we know that only the binary collisions at low energies play a significant dynamical role, while configurations with three or more particles simultaneously interacting can be neglected. In other words, diluteness implies that the range of interatomic forces $x_0$ is smaller than the average distance between particles, a parameter fixed by the gas density, i.e $d = \frac{1}{n^3}$ where $n = \frac{N}{V}$. We can handle both $n$ and $d$ as finite quantities, as we know it is possible to carry the system to the thermodynamical limit. In formula, we get

$$\boxed{n|a|^3 \ll 1 \, .} \tag{1.2}$$

The diluteness assumption helps us to simplify the interatomic integral: thanks to it, we can consider interatomic spacing large enough to let us writing the physical quantities in an asymptotic form. We are now able to express the interatomic potential in a form that is not strictly related with the precise structure of microscopic interaction. In this situation, the collisions are characterized by a single parameter: using the elementary theory of quantum scattering, we find that it is the $s$-wave scattering length $a_{\mathrm{s}}$. Moreover, the exact microscopic interaction can be replaced by an effective one:

$$V(\mathbf{r}-\mathbf{r}') = g\delta(\mathbf{r}-\mathbf{r}') \, , \tag{1.3}$$

where the effective coupling constant can be defined as

$$g = \frac{4\pi\hbar a_s^2}{m} \, . \tag{1.4}$$

The (1.1) can be rewritten as the following one:

$$i\hbar\frac{\partial\hat{\Psi}}{\partial t}(\mathbf{r},t) = \left[\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r})\right]\hat{\Psi}(\mathbf{r},t) + g\hat{\Psi}^\dagger(\mathbf{r},t)\hat{\Psi}(\mathbf{r},t)\hat{\Psi}(\mathbf{r},t) \, . \tag{1.5}$$

We consider a thermal average on a non-equilibrium statistical ensemble with a broken (gauge) simmetry, which leads us to a non-zero mean value of the field operator. We represented the complex function $\Phi(\mathbf{r},t)$ by means of the condensate density $n_0(\mathbf{r},t)$ and a phase $\theta(\mathbf{r},t)$ corresponding to the assumption of broken simmetry occurring. Therefore, the *condensate wavefunction* is defined as

$$\left\langle\hat{\Psi}(\mathbf{r},t)\right\rangle_{\rho_t} = \Phi(\mathbf{r},t) = \sqrt{n_0(\mathbf{r},t)}\, e^{i\theta(\mathbf{r},t)} \, , \tag{1.6}$$

where $\rho_t$ is the proper non-equilibrium density matrix. A generalization to the non-uniform and time-depedent case of the Bogoliubov's prescription leads us to the following splitting [14, 42]:

$$\hat{\Psi}(\mathbf{r},t) = \Phi(\mathbf{r},t) + \tilde{\Psi}(\mathbf{r},t) \;, \tag{1.7}$$

with the constraint $\left\langle \tilde{\Psi}(\mathbf{r},t) \right\rangle_{\rho_t} = 0$. In order to get the first important equation, we have to make the thermal average on both sides of (1.1) together with splitting (1.7). The three-field correlation $\left\langle \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi} \right\rangle$ gives us back

$$\begin{aligned} \left\langle \hat{\Psi}^\dagger \hat{\Psi} \hat{\Psi} \right\rangle_{\rho_t} &= \left\langle (\Phi_0^* + \tilde{\Psi}^\dagger)(\Phi_0 + \tilde{\Psi})(\Phi_0 + \tilde{\Psi}) \right\rangle_{\rho_t} \\ &= n_0\Phi_0 + 2\tilde{n}\Phi_0 + g\left[ \tilde{m}\Phi_0^* + \left\langle \tilde{\Psi}^\dagger \tilde{\Psi} \tilde{\Psi} \right\rangle \right] , \end{aligned} \tag{1.8}$$

where $\tilde{n} = \left\langle \tilde{\Psi}^\dagger(\mathbf{r},t)\tilde{\Psi}(\mathbf{r},t) \right\rangle$ is the non-equilibrium non-condensate density, while $\tilde{m} = \left\langle \tilde{\Psi}(\mathbf{r},t)\tilde{\Psi}(\mathbf{r},t) \right\rangle$ is the off-diagonal one. Therefore, the condensate dynamics is given by the Zaremba-Nikuni-Griffin (ZNG) equation:

$$\boxed{\begin{aligned} i\hbar\frac{\partial \Phi_0}{\partial t} = \left[ -\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + gn_0 + 2g\tilde{n} \right]\Phi_0 + \\ + g\left[ \tilde{m}\Phi_0^* + \left\langle \tilde{\Psi}^\dagger \tilde{\Psi} \tilde{\Psi} \right\rangle \right] , \end{aligned}} \tag{1.9}$$

where we the argument of function are position coordinate and time.

We want to understand the dynamics of the non-condensate part. The analogous of ZNG equation for the non-condensate particles can be obtained from (1.1) simply by a substitution of the field operator $\hat{\Psi}$ with its corresponding splitted form defined in (1.7).

We find

$$\begin{aligned} i\hbar\frac{\partial(\Phi_0 + \tilde{\Psi})}{\partial t} = \left[ \frac{\hbar^2}{2m}\nabla^2 + V_{ext} \right](\Phi_0 + \tilde{\Psi}) + g\Big[ n_0\Phi_0 + 2n_0\tilde{\Psi} + \\ + \Phi_0^*\tilde{\Psi}\tilde{\Psi} + \tilde{\Psi}^\dagger\tilde{\Psi}\tilde{\Psi} + n_0\tilde{\Psi}^\dagger + 2\Phi\tilde{\Psi}^\dagger\tilde{\Psi}_0 \Big] , \end{aligned}$$

and, consequently,

$$\begin{aligned} i\hbar\frac{\partial \tilde{\Psi}}{\partial t} = -i\hbar\frac{\partial \Phi_0}{\partial t} + \left[ -\frac{\hbar^2}{2m}\nabla^2 + V_{ext} \right]\left( \Phi_0 + \tilde{\Psi} \right) + g\Big[ n_0\Phi_0 + \\ + 2n_0\tilde{\Psi} + \Phi_0^*\tilde{\Psi}\tilde{\Psi} + \phi_0^2\tilde{\Psi}^\dagger + 2\Phi_0\tilde{\Psi}^\dagger\tilde{\Psi} + \tilde{\Psi}^\dagger\tilde{\Psi}\tilde{\Psi} \Big] . \end{aligned}$$

By inserting the (1.9) in the last equation, we get the Heisenberg equation of motion for the non-condensate field operator:

$$i\hbar\frac{\partial\tilde{\Psi}}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V_{ext} + 2gn\right]\tilde{\Psi} - 2g\tilde{n}\tilde{\Psi} + g\Phi_0^2\tilde{\Psi}^\dagger + g\Phi_0^*\Big(\tilde{\Psi}\tilde{\Psi} - \tilde{m}\Big) + \\ + 2g\Phi_0\Big(\tilde{\Psi}^\dagger\tilde{\Psi} - \tilde{n}\Big) + g\left[\tilde{\Psi}^\dagger\tilde{\Psi}\tilde{\Psi} - \Big\langle\tilde{\Psi}^\dagger\tilde{\Psi}\tilde{\Psi}\Big\rangle\right], \tag{1.10}$$

where we have defined $n = n_0 + \tilde{n}$ as the total density.

This is the exact equation of motion for the field operator $\tilde{\Psi}(\mathbf{r}, t)$. In order to get a simpler equation, we consider the theoretical scheme developed in [26, 29]. The key-point consists in the use of the so-called quantum mechanical distribution function (QMDF), whose the best known example is the Wigner's one [35, 54]:

$$\hat{f}(\mathbf{r}, \mathbf{p}, t) = \int d^3\mathbf{r}' e^{i\frac{\mathbf{p}\cdot\mathbf{r}'}{\hbar}}\tilde{\Psi}^\dagger\left(\mathbf{r} + \frac{\mathbf{r}'}{2}, t\right)\tilde{\Psi}\left(\mathbf{r} - \frac{\mathbf{r}'}{2}, t\right). \tag{1.11}$$

The expectation value of a generic operator $\hat{A}(t)$ can be written in terms of the density matrix trace:

$$\begin{aligned}\Big\langle\hat{A}(t)\Big\rangle_{\rho_t} &= Tr\Big[\rho(t_0), \hat{A}(t)\Big] \\ &= Tr\Big[\rho(t_0), \hat{U}^\dagger(t, t_0)\hat{A}(t_0)\hat{U}(t, t_0)\Big] \\ &= Tr\Big[\tilde{\rho}(t, t_0)\hat{A}(t_0)\Big],\end{aligned} \tag{1.12}$$

where $\hat{U}(t, t_0)$ is the unitary time-evolution operator, which satisfies the well-known equation $i\frac{d\hat{U}}{dt} = \hat{H}_{\text{tot}}\hat{U}$ and $\tilde{\rho}(t, t_0) = \hat{U}(t, t_0)\rho(t_0)\hat{U}^\dagger(t, t_0)$. In the last line of (1.12) we used the cyclic property of trace. So, the expectation value of the Wigner operator is given by

$$f(\mathbf{r}, \mathbf{p}, t) = Tr\Big[\tilde{\rho}(t, t_0)\hat{f}(\mathbf{r}, \mathbf{p}, t_0)\Big] \tag{1.13}$$

and must be interpreted as *the distribution function for out-of-condensate atoms.*

We can get the equation of motion for the distribution function by a time derivative:

$$\begin{aligned}\frac{\partial f}{\partial t}(\mathbf{r}, \mathbf{p}, t) &= \frac{\partial}{\partial t}Tr\Big[\tilde{\rho}(t, t_0)\hat{f}(\mathbf{r}, \mathbf{p}, t_0)\Big] \\ &= Tr\left[\frac{\partial\tilde{\rho}}{\partial t}(t, t_0)\hat{f}(\mathbf{r}, \mathbf{p}, t_0)\right] \\ &= \frac{1}{i\hbar}Tr\Big[\hat{H}_{\text{tot}}, \tilde{\rho}(t, t_0)\Big]\hat{f}(\mathbf{r}, \mathbf{p}, t_0) \\ &= \frac{1}{i\hbar}Tr\tilde{\rho}(t, t_0)\Big[\hat{f}(\mathbf{r}, \mathbf{p}, t_0), \hat{H}_{\text{tot}}\Big].\end{aligned}$$

The Hamiltonian $\hat{H}_{\rm tot}$ can be split up into different terms[15, 29, 56] which reproduce the dynamics of (1.10):

$$\begin{aligned}
\hat{H}_{tot}(t) &= \hat{H}_{\rm mf}(t) + \hat{H}_I(t) \\
\hat{H}_I(t) &= \hat{H}_I^{(1)} + \hat{H}_I^{(2)} + \hat{H}_I^{(3)} + \hat{H}_I^{(4)} \\
\hat{H}_{\rm mf} &= \int d^3\mathbf{r}\, \tilde{\Psi}^\dagger \left[ -\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\mathbf{r}) + 2gn(\mathbf{r}t) \right] \tilde{\Psi} \\
\hat{H}_I^{(1)} &= \int d^3\mathbf{r} \left[ \hat{C}_1(\mathbf{r},t)\tilde{\Psi}^\dagger + \hat{C}_1^*(\mathbf{r},t)\tilde{\Psi} \right] \\
\hat{H}_I^{(2)} &= \frac{g}{2}\int d^3\mathbf{r} \left[ \Phi_0^2(\mathbf{r},t)\tilde{\Psi}^\dagger\tilde{\Psi}^\dagger + \Phi_0^{*2}(\mathbf{r},t)\tilde{\Psi}\tilde{\Psi} \right] \\
\hat{H}_I^{(3)} &= g\int d^3\mathbf{r} \left[ \Phi_0^*(\mathbf{r},t)\tilde{\Psi}^\dagger\tilde{\Psi}\tilde{\Psi} + \Phi_0(\mathbf{r},t)\tilde{\Psi}^\dagger\tilde{\Psi}^\dagger\tilde{\Psi} \right] \\
\hat{H}_I^{(4)} &= \frac{g}{2}\int d^3\mathbf{r}\, \tilde{\Psi}^\dagger\tilde{\Psi}^\dagger\tilde{\Psi}\tilde{\Psi} - 2g\int d^3\mathbf{r}\, \tilde{n}(\mathbf{r},t)\tilde{\Psi}^\dagger\tilde{\Psi}\ ,
\end{aligned} \tag{1.14}$$

where $\hat{C}_1 = -g\Big[2\tilde{n}\Phi_0 + \tilde{m}\Phi_0^* + \Big\langle \tilde{\Psi}^\dagger\tilde{\Psi}\tilde{\Psi} \Big\rangle \Big]$. Among all these terms, in the absence of condensate, the interesting ones are $\hat{H}_{\rm mf}$ (the Hartree-Fock term) and $\hat{H}_I^{(4)}$. The equation of motion for $f(\mathbf{r},\mathbf{p},t)$ becomes

$$\boxed{\frac{\partial f}{\partial t} = \frac{1}{i\hbar} Tr\tilde{\rho}(t,t_0)\Big[\hat{f}(t_0), \hat{H}_{\rm mf}\Big] + \frac{1}{i\hbar} Tr\tilde{\rho}(t,t_0)\Big[\hat{f}(t_0), \hat{H}_I\Big]} \tag{1.15}$$

with $\hat{f}(t_0) = \hat{f}(\mathbf{r},\mathbf{p},t_0)$. The first term of the R.H.S gives us back the so-called free streaming operator, while the second one describes the collisional dynamics of the system. By means of a plane-wave expansion of the field operator, we get

$$\begin{cases} \tilde{\Psi}(\mathbf{r},t_0) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{r}}\hat{a}_{\mathbf{p}} \\ \\ \tilde{\Psi}^\dagger(\mathbf{r},t_0) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{r}}\hat{a}_{\mathbf{p}}^\dagger \end{cases}$$

where we consider the convention $\hbar = 1$. The Fourier transform of the Wigner's operator $\hat{f}(\mathbf{r},\mathbf{p},t_0)$ is given by

$$\begin{aligned}
\hat{f}(\mathbf{r},\mathbf{p},t_0) &= \int d^3\mathbf{r}'\, e^{i\mathbf{p}\cdot\mathbf{r}}\tilde{\Psi}^\dagger\left(\mathbf{r}+\frac{\mathbf{r}'}{2}\right)\tilde{\Psi}\left(\mathbf{r}-\frac{\mathbf{r}'}{2}\right) \\
&= \frac{1}{V}\sum_{\mathbf{p}_1,\mathbf{p}_2} e^{i(\mathbf{p}_2-\mathbf{p}_1)\cdot\mathbf{r}}\hat{a}_{\mathbf{p}_1}^\dagger\hat{a}_{\mathbf{p}_2}\int d^3\mathbf{r}'\, e^{i\mathbf{p}\cdot\mathbf{r}'}e^{-i\mathbf{p}_1\cdot\frac{\mathbf{r}'}{2}}e^{i\mathbf{p}_1\cdot\frac{\mathbf{r}'}{2}} \\
&= \sum_{\mathbf{p}_1,\mathbf{p}_2} e^{i(\mathbf{p}_2-\mathbf{p}_1)\cdot\mathbf{r}'}\hat{a}_{\mathbf{p}_1}^\dagger\hat{a}_{\mathbf{p}_2}\delta_{\mathbf{p}-\frac{1}{2}\mathbf{p}_1-\frac{1}{2}\mathbf{p}_2} \\
&= \sum_q e^{i\mathbf{q}\cdot\mathbf{r}}\hat{a}^\dagger_{\mathbf{p}-\frac{1}{2}\mathbf{q}}\hat{a}_{\mathbf{p}+\frac{1}{2}\mathbf{q}}\ ,
\end{aligned}$$

where $\mathbf{q} = \mathbf{p}_2 - \mathbf{p}_1$. So, from (1.15) we have

$$\boxed{\frac{\partial f}{\partial t} = \frac{1}{i\hbar}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}\, Tr\,\tilde{\rho}(t,t_0)\left[\hat{a}^\dagger_{\mathbf{p}+\frac{1}{2}\mathbf{q}}\hat{a}_{\mathbf{p}-\frac{1}{2}\mathbf{q}},\ \hat{H}_{\rm tot}\right].} \tag{1.16}$$

This enlightens a hierarchical structure for the dynamics of the quantum distribution function, as found in [29]. However, there is a non-trivial difference concerning the theoretical framework: their equations were placed in a local rest frame for the fluid, while we work in the laboratory frame. This choice implies the fact that the excitation energies are defined in relation with condensate atoms one, using the Bogoliubov's quasiparticle formalism [15].

We have to specify the ordering scheme through which we can highlight the most relevant terms of $\hat{H}_{\rm tot}(t)$. The simplest way is to consider the coupling constant $g$ (and therefore the s-wave scattering length) as a *smallness parameter*, i.e $g \sim \epsilon$. So, a term is small if it is $\sim \sigma(\epsilon^{\alpha})$ with $\alpha > 0$.

Hence, we limit ourselves to the case of $T > T_{\rm BEC}$, so the only terms we retain are $\hat{H}_{\rm mf}(t)$ and $\hat{H}_I^{(4)}$. By recalling [55, 56], we can show that the second term is proportional to $g^2$, so we can consider it small. Therefore, the most relevant term for the dynamics is the mean-field one.

Indeed, we have

$$C_{22}[f] \equiv -\frac{i}{\hbar} Tr\, \tilde{\rho}(t,t_0)\Big[\hat{f}(\mathbf{r},\mathbf{p},t_0), \hat{H}^{(4)}(t)\Big]\,, \tag{1.17}$$

where two assumptions are now crucial:

- We suppose that $\hat{H^{(4)}}_I$ describes essentially a collisional process, then its time scale is much smaller than other time scales in the system. All significant dynamical effects reveal themselves at times $t'$ very close to $t$: the system has no memory of its initial conditions.

- The hydrodynamical variables vary slowly in time and space [12, 25]. For example, with $t' \neq t$ and $\mathbf{r}' \neq \mathbf{r}$, we consider $\tilde{n}(\mathbf{r}',t) \simeq \tilde{n}(\mathbf{r},t)$.

Thanks to plane-wave expansion, the Hamiltonian can be written as follows

$$\hat{H}_I^{(4)}(t) \simeq \frac{g}{2V} \sum_{\substack{\mathbf{p}_1,\mathbf{p}_2 \\ \mathbf{p}_3,\mathbf{p}_4}} \delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4} \hat{a}^{\dagger}_{\mathbf{p}_1}\hat{a}^{\dagger}_{\mathbf{p}_2}\hat{a}_{\mathbf{p}_3}\hat{a}_{\mathbf{p}_4} - 2g\hat{n}\sum_{\mathbf{p}} \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\mathbf{p}}\,.$$

The second term preserves the particles number, so it commutes with every operator which keeps it unchanged; due to this fact, we consider only the first term of $\hat{H}_I^{(4)}(t)$:

$$C_{22}[f] = -i\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}\, Tr\; \tilde{\rho}(t,t_0)\Bigg[\hat{a}^{\dagger}_{\mathbf{p}+\frac{1}{2}\mathbf{q}}\hat{a}_{\mathbf{p}-\frac{1}{2}\mathbf{q}}, \frac{g}{2V}\times$$
$$\times \sum_{\substack{\mathbf{p}_1,\mathbf{p}_2 \\ \mathbf{p}_3,\mathbf{p}_4}} \delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4} \hat{a}^{\dagger}_{\mathbf{p}_1}\hat{a}^{\dagger}_{\mathbf{p}_2}\hat{a}_{\mathbf{p}_3}\hat{a}_{\mathbf{p}_4}\Bigg]\,.$$

The non-zero contribution from the commutator can be enumerated thanks to a generalization of Wick's theorem for statistical physics [36]. Moreover,

$$\begin{aligned}\left\langle \hat{a}^\dagger_{\mathbf{p}_1}\hat{a}_{\mathbf{p}_2}\right\rangle_{\tilde{\rho}_t} &= Tr\,\tilde{\rho}(t_0)\hat{U}^\dagger(t,t_0)\hat{a}^\dagger_{\mathbf{p}_1}\hat{a}_{\mathbf{p}_2}\hat{U}(t,t_0)\\ &\simeq e^{i(E_{\mathbf{p}_1}-E_{\mathbf{p}_2})(t-t_0)}\left\langle \hat{a}^\dagger_{\mathbf{p}_1}\hat{a}_{\mathbf{p}_2}\right\rangle_{\tilde{\rho}_{t_0}}\\ &\simeq e^{i(E_{\mathbf{p}_1}-E_{\mathbf{p}_2})(t-t_0)}\delta_{\mathbf{p}_1,\mathbf{p}_2}f(\mathbf{r},\mathbf{p},t)\ .\end{aligned} \tag{1.18}$$

Finally we get

$$\begin{aligned}C_{22}[f] =&\frac{\pi g^2}{V^2}\sum_{\substack{\mathbf{p}_1,\mathbf{p}_2\\ \mathbf{p}_3,\mathbf{p}_4}}\delta(E_{\mathbf{p}_1}+E_{\mathbf{p}_2}-E_{\mathbf{p}_3}-E_{\mathbf{p}_4})\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}_4}\times\\ &\times\Big[\delta_{\mathbf{p},\mathbf{p}_1}+\delta_{\mathbf{p},\mathbf{p}_2}-\delta_{\mathbf{p},\mathbf{p}_3}-\delta_{\mathbf{p},\mathbf{p}_4}\Big]\times\\ &\times\Big[f_1f_2(1+f_3)(1+f_4)+(1+f_1)(1+f_2)f_3f_4\Big]\end{aligned} \tag{1.19}$$

and taking the continuum limit, i.e $\frac{1}{V}\sum_{\mathbf{p}} \longrightarrow \int \frac{d^3\mathbf{p}}{(2\pi)^3}$ the collisional integral $C_{22}[f]$ takes an appearance that enlighten its precise physical meaning:

$$\begin{aligned}C_{22}[f] = \frac{2g^2}{(2\pi)^5\hbar^7}\int d^3\mathbf{p}_2d^3\mathbf{p}_3d^3\mathbf{p}_4\delta(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)\delta(E_{\mathbf{p}}+\mathbf{E}_{\mathbf{p}_2}-\mathbf{E}_{\mathbf{p}_3}-\mathbf{E}_{\mathbf{p}_4})\times\\ \times\Big[f_1f_2(1+f_3)(1+f_4)+(1+f_1)(1+f_2)f_3f_4\Big]\ .\end{aligned}$$

The collisional integral is proportional to $g^2$, so it is a small quantity compared to the Hartree-Fock term $\hat{H}_{\mathrm{mf}}$. From (1.16), we have

$$\boxed{\frac{\partial f}{\partial t}+\frac{i}{\hbar}\sum_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{r}}Tr\,\tilde{\rho}(t,t_0)\left[\hat{a}^\dagger_{\mathbf{p}+\frac{1}{2}\mathbf{q}}\hat{a}_{\mathbf{p}-\frac{1}{2}\mathbf{q}},\hat{H}_{\mathrm{mf}}\right]=\sigma(\epsilon)} \tag{1.20}$$

where $\hat{H}_{\mathrm{mf}}=\int d^3\mathbf{r}\tilde{\mathbf{\Psi}}^\dagger[T+U_{\mathrm{mf}}]\tilde{\mathbf{\Psi}}$. Let us now consider only the kinetic term $T=\frac{\hbar^2}{2m}\nabla^2$, whose contribution can be computed more easily than mean-field potential one. The commutator $[\hat{a}^\dagger_{\mathbf{p}_1}\hat{a}_{\mathbf{p}_2},\hat{T}]$ leads us to

$$\begin{aligned}\frac{\partial f}{\partial t}+\frac{i}{\hbar}\sum_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{r}}Tr\,\tilde{\rho}(t,t_0)\Big(E_{\mathbf{p}-\frac{1}{2}\mathbf{q}}-E_{\mathbf{p}+\frac{1}{2}\mathbf{q}}\Big)\times\\ \times\left\langle\hat{a}^\dagger_{\mathbf{p}-\frac{1}{2}\mathbf{q}}\hat{a}_{\mathbf{p}+\frac{1}{2}\mathbf{q}}\right\rangle_{\tilde{\rho}_t}+\ldots=\sigma(\epsilon)\ .\end{aligned} \tag{1.21}$$

We can rewrite the energy difference by means of

$$E_{\mathbf{p}-\frac{1}{2}\mathbf{q}}-E_{\mathbf{p}+\frac{1}{2}\mathbf{q}}\simeq\frac{\partial E(\mathbf{p})}{\partial\mathbf{p}}\cdot\frac{\mathbf{q}}{m}+\sigma(|\mathbf{q}|^3) \tag{1.22}$$

and, from (1.20), we get the

$$\frac{\partial f}{\partial t} + \frac{i}{\hbar}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}\left[\frac{\partial E}{\partial \mathbf{p}}\cdot\mathbf{q} + \sigma(|\mathbf{q}|^3)\right]\left\langle \hat{a}^\dagger_{\mathbf{p}-\frac{1}{2}\mathbf{q}}\hat{a}_{\mathbf{p}+\frac{1}{2}\mathbf{q}}\right\rangle_{\tilde{\rho}_t} + \ldots = \sigma(\epsilon)$$

$$\frac{\partial f}{\partial t} + \frac{i}{\hbar}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}\frac{\partial E_{\mathbf{p}}}{\partial \mathbf{p}}\cdot\mathbf{q}\, e^{i\Delta E_{\mathbf{p}}(t-t_0)}\left\langle \hat{a}^\dagger_{\mathbf{p}-\frac{1}{2}\mathbf{q}}\hat{a}_{\mathbf{p}+\frac{1}{2}\mathbf{q}}\right\rangle_{\tilde{\rho}_0} + \ldots = \sigma(\epsilon)$$

$$\frac{\partial f}{\partial t} + \frac{i}{\hbar}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}\,\frac{\partial E_{\mathbf{p}}}{\partial \mathbf{p}}\cdot\mathbf{q}\, f\left(\mathbf{p}+\frac{1}{2}\mathbf{q},\mathbf{r},t\right) + \ldots = \sigma(\epsilon)\ .$$

As in [29], the last line can be rewritten as

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}} f(\mathbf{r},\mathbf{p},t) + \ldots = 0 \tag{1.23}$$

if we are working at this first order on $g$ (or, equivalently, in the s-wave scattering length).

For the analytical details about the term involving $[\hat{a}^\dagger_{\mathbf{p}_1}\hat{a}_{\mathbf{p}_2}, U_{\mathrm{mf}}]$, we report only the result, that can be obtained by a similar reasoning:

$$\boxed{\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}} f(\mathbf{r},\mathbf{p},t) - \nabla_{\mathbf{r}} U_{\mathrm{mf}}\cdot\nabla_{\mathbf{p}} f = 0\ .} \tag{1.24}$$

The mean-field potential is defined as

$$U_{\mathrm{mf}}(\mathbf{x},t) = V_{\mathrm{ext}}(\mathbf{r}) + 2gn(\mathbf{r},t) \tag{1.25}$$

where we have removed the *tilde* above $n(\mathbf{r},t)$, because we are dealing with an ultracold gas, but above $T_{\mathrm{BEC}}$. From elementary notions of the classical kinetic theory[25], the particle density can be defined as

$$n(\mathbf{r},t) = \int \frac{d\mathbf{p}}{(2\pi\hbar)^3}\ f(\mathbf{r},\mathbf{p},t)\ . \tag{1.26}$$

We return on this definition extensively in the next chapter.

## 1.2 AN APPROACH FOR FERMI GAS

In this section we want to derive a kinetic equation for an ultracold Fermi gas. This equation is similar to the Bose one, except for a numeric factor in the mean field potential. The approach we are going to propose is a phenomenological one, based on Landau's two-fluids theory [31, 38].

We will focus on physical assumption, instead of a rigorous analytical development; because of this choice, we will have to consider some *ad-hoc* assumptions in the following; a detailed treatment of this topic can be found in [15]

This section is divided in two steps: initially we specify the concept of quasiparticles[1] for a Fermi fluid system; then, thanks to it, we write a kinetic equation for a *quasiparticle gas*.

### 1.2.1 *The concept of quasi-particle*

Let us begin by considering an interacting Fermi gas without specifying anything else about it; our approach is based on the so-called *adiabatic switching-on* of the interaction [15]. This approximation is founded on a connection between the states of the physical interacting Fermi system and the ones of a non-interacting fermionic gas.

Since we assume to know everything about the non-interacting system, the interaction between particles can be thought as switched on infinitely slowly. Due to a series of *adiabatic* transformations, an initial non-interacting state, described by a ditribution function $f^{(0)}(\mathbf{p})$ transforms into an interacting system state. For example, we can consider a time-dependent Hamiltonian[15] given by

$$\hat{H} = \hat{H}_0 + e^{-\epsilon|t|}\hat{H}_1 \tag{1.27}$$

where $\hat{H}_0$ is the non-interacting Hamiltonian, while $\hat{H}_1$ is the interacting one. However, there is no physical reason assuring that all interacting states (or better, eigenstates) can be produced in this way; an example of this impasse can be found in the superconductors features [36].

This is the main *ad-hoc* assumption: we impose that all the interacting states can be realized by a series of adiabatic transformations of a non-interacting state, labelled by its distribution function $f^{(0)}(\mathbf{p})$. We consider this statement as the definition of a *normal Fermi fluid* [38].

1 As we briefly outlined in the previous section, Kirkpatrick and Dorfman [29] inferred the kinetic equations for a Bose gas exploiting precisely the (Bogoliubov's) quasiparticle formalism.

Afterwards, an excited state of the real system can be realized in the following way; we start by considering a particle added to the ideal distribution function $f^{(0)}(\mathbf{p})$, whose state is given by the ket $|\mathbf{p}\rangle$. Through this picture, an excited state of the interacting Fermi system is created; its momentum is $\mathbf{p}$, because of the $\mathbf{p}$-conservation in the collisions; as the interaction increases, the particle slowly perturbes the surrounding fluid particles. When the interaction is totally turned on, a particle moves together with the distortion it created: in the field theory formalism[15], one cay refer to this particle as a *dressed* one: it carries a sort of self-energy cloud. If we consider it as an independent physical object, we will call it a *quasiparticle*. So, the above excited state corresponds to the real ground state plus a quasiparticle of momentum $\mathbf{p}$.

The concept of *quasiparticle* is well-defined only in the proximity of the Fermi surface. Indeed, near to this area, the quasiparticles lifetime becomes enough long to allow an adiabatic expansion; the switching-on time must not be too fast, in order to keep the transformations adiabatic. It can't be neither too slow: in this case we will have an irreversible transformation. Indeed, if the switchting-on time is longer than quasiparticle lifetime, when we reach the physical value of the interaction, the state will be already decayed.

We now define the system *excitation*: at the end of the transformation, we get a state of the interacting system whose distribution function is $f(\mathbf{r}, \mathbf{p})$. So, the departure from the ideal situation is

$$\delta f(\mathbf{r}, \mathbf{p}) = f(\mathbf{r}, \mathbf{p}) - f^{(0)}(\mathbf{p}) \ . \tag{1.28}$$

We are dealing with ultracold atomic gases, then only the near-Fermi-surface modes can be excited. We have to underline [38] that the real physical quantity is not $f(\mathbf{r}, \mathbf{p})$, but $\delta f(\mathbf{r}, \mathbf{p})$: a quasiparticle is an unstable physical entity, i.e. it necessarily undergoes to damping collisional processes. Therefore, it does not make much sense to talk of equilibrium distribution function in a range where quasiparticles are unstable.

According to Landau [31], we introduce the relation between the quasiparticle distribution function $f(\mathbf{r}, \mathbf{p})$ and the energy of the system. By considering a small departure $\delta f$ from the non-interacting state, the energy functional $E[f]$ is defined as follows

$$E[f] = E_0 + \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} \delta f(\mathbf{r}, \mathbf{p}) + \sigma(\delta f^2) \tag{1.29}$$

where $\epsilon_{\mathbf{p}}$ is the first functional derivative of $E$. Then, if we are dealing with a one extra quasiparticle state, its energy is $E_0 + \epsilon_{\mathbf{p}}$, where $\epsilon_{\mathbf{p}}$ is the quasiparticle energy; hence, the quasiparticle energy is an additive quantity, up to second order on $\delta f$.

We refer to $\epsilon_{\mathbf{p}}$ as a first functional derivative: we have to require its continuity through the Fermi surface. We assume this property as a part of *normal Fermi system* definition.

In order to give a statistical description of our system, the grancanonical ensemble is a natural choice: the chemical potential $\mu$ is fixed, rather than energy. Thus, the proper thermodynamical potential is the Gibbs one (or the Gibbs free energy) [25] and the departure from the ground state one is given by

$$G - G_0 = E - E_0 - \mu(N - N_0)$$

where $N_0$ is the number of particle in the ground state (or in a certain proper non-interacting excited state). In order to get an equation similar to (1.29), we write

$$N - N_0 = \sum_{\mathbf{p}} \delta f(\mathbf{r}, \mathbf{p}) \ . \tag{1.30}$$

By recalling the (1.29), we have

$$G - G_0 = \sum_{\mathbf{p}} (\epsilon_{\mathbf{p}} - \mu)\delta f(\mathbf{r}, \mathbf{p}) + \sigma(\delta f^2) \ .$$

This is a series expansion of the Gibbs potential functional, in power of $\delta f$; we know that $\delta f(\mathbf{r}, \mathbf{p}) = \pm 1$ in a $\delta$-thick shell[2]. On the other side, also $(\epsilon_{\mathbf{p}} - \mu)$ is a $\delta$-order quantity, so we have to retain the $\delta f^2$ term:

$$G - G_0 = \sum_{\mathbf{p}} (\epsilon_{\mathbf{p}} - \mu)\delta f(\mathbf{r}, \mathbf{p}) + \frac{1}{2} \sum_{\mathbf{p},\mathbf{p}'} g_{\mathbf{p}\mathbf{p}'} \delta f(\mathbf{p}) \delta f(\mathbf{p}') + \sigma(\delta f^3) \ , \tag{1.31}$$

where $g_{\mathbf{p},\mathbf{p}'}$ is the second derivative of $G[f]$; we assume the continuity of $g_{\mathbf{p},\mathbf{p}'}$ as a part of *normal Fermi fluid*. The physical meaning of $g_{\mathbf{p},\mathbf{p}'}$ is understood by interpreting it as the interaction energy between a $|\mathbf{p}\rangle$-quasiparticle and a $|\mathbf{p}'\rangle$-one.

We assumed a homogeneous medium, in order to simplify the notation. In the next section we will consider an inhomogeneous one in order to get a heuristic derivation of Boltzmann-Vlasov equation for an ultracold Fermi atomic gas.

### 1.2.2 *Kinetic equations for Fermi gas: a heuristic path*

If we consider a distribution function $f(\mathbf{r}, \mathbf{p})$, we need to specify the concept of local energy for a quasi particle. By recalling the (1.31), we have

$$\tilde{\epsilon}(\mathbf{p}) - \mu = [\epsilon(\mathbf{p}) - \mu] + \sum_{\mathbf{p}'} g_{\mathbf{p},\mathbf{p}'} \delta f(\mathbf{p}) \ . \tag{1.32}$$

2 The switching-on of the interaction implies a certain displacement of the Fermi surface, namely $\delta$.

Because of the medium distortion, we can think of $\tilde{\epsilon}(\mathbf{r},\mathbf{p})$ as *local energy.*

The gradient

$$\nabla_{\mathbf{r}}\tilde{\epsilon}(\mathbf{r}) = \nabla_{\mathbf{r}}\left[\sum_{\mathbf{p},\mathbf{p}'} g_{\mathbf{p},\mathbf{p}'}\delta f(\mathbf{r},\mathbf{p})\right]$$

has the physical meaning of the force felt by the $|\mathbf{p}\rangle$-quasiparticle, due to medium distortion. The *local-equilibrium* distribution function is defined by

$$\tilde{f}^{(0)}(\mathbf{p}) = f^{(0)}(\tilde{\epsilon}(\mathbf{p}-\mu) \tag{1.33}$$

where the $f^{(0)}$ is the usual Fermi-Dirac distribution function. From a quantum point of view, we have to consider the Heisenberg uncertainty principle. If we can't know, at the same time, both position and momentum, then a physical entity as the ditribution function $f(\mathbf{r},\mathbf{p},t)$ has no physical sense. It can take a precise meaning if we take into account only *macroscopic* perturbations: this implies that wavevectors and frequencies have to remain much smaller than the atomic ones.

Let us consider now the Fourier transform of the distribution function $f(\mathbf{r},\mathbf{p},t)$. The perturbation is such that the only relevant response is the linear one so, due to linearity, we can deal with one single mode:

$$f(\mathbf{r},\mathbf{p},t) = f^{(0)}(\mathbf{p}) + \delta f(\mathbf{p})\, e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}\ .$$

The uncertainty on momentum is $\hbar\mathbf{q}$ while on the energy is $\hbar\omega$; at finite temperature the Fermi surface has a $k_BT$-thick on energy space and a $\frac{k_B}{v_F}$ on momentum one. So, we can neglect the fluctuations given by Heisenberg principle if

$$\begin{cases} \hbar|\mathbf{q}|v_F \ll k_BT \\ \hbar\omega \ll k_BT \end{cases} \tag{1.34}$$

where $v_F = p_F\frac{d^2\epsilon}{dp^2}$.

As remarked in [38], the classical regime depicted by (1.34) is too restrictive. By a re-interpretation of distribution function [31], one can give the less restrictive conditions:

$$\boxed{\begin{cases} \hbar|\mathbf{q}|v_F \ll \mu \\ \hbar\omega \ll \mu \end{cases}} \tag{1.35}$$

Under these set of conditions, we have to intend the distribution function as the probability of finding a couple, in contrast with the usual meaning of probability of finding a particle with a certain coordinate on the phase space. More precisely, we can recall

what we said concerning a Bose gas, when we have introduced the quantum Wigner's distribution function. The Fourier transform of Wigner distirbution will involve the couple $\hat{c}^{\dagger}_{\mathbf{p}+\hbar\frac{\mathbf{q}}{2}}\hat{c}_{\mathbf{p}-\hbar\frac{\mathbf{q}}{2}}$. Under the (1.35), we have to place the distribution function in the Wigner semiclassical picture of statistical mechanics [35, 54].

The local energy of the system can be defined by means of a series expansion [31, 38]:

$$\begin{aligned} E &= E_0 + \int d\mathbf{r}\delta E(\mathbf{r}) \\ \delta E(\mathbf{r}) &= \sum_{\mathbf{p}} \epsilon_{\mathbf{p}}\delta f(\mathbf{r},\mathbf{p}) + \frac{1}{2}\sum_{\mathbf{p},\mathbf{p}'} g_{\mathbf{p}\mathbf{p}'}\delta f(\mathbf{r},\mathbf{p})\delta f(\mathbf{r},\mathbf{p}) \ . \end{aligned} \tag{1.36}$$

The interaction energy is defined by means of the 2-particle correlation function $g_{\mathbf{p},\mathbf{p}'} = \int d^3\mathbf{r}' f(\mathbf{X},\mathbf{X}')$, where $\mathbf{X}$-coordinate points out a coordinate in the phase space, i.e $\mathbf{X} = (\mathbf{r},\mathbf{p})$. We now assume that gas is dilute enough that the only relevant kind of collisions are the binary ones. This means we are only interested in collisional phenomena given by the short-range interatomic forces; then, we replace $\delta f(\mathbf{r}',\mathbf{p}')$ with $\delta f(\mathbf{r},\mathbf{p}')$.

From (1.36), the local energy for a quasiparticle is given by

$$\tilde{\epsilon}(\mathbf{p}) = \epsilon(\mathbf{p}) + \sum_{\mathbf{p}'} g_{\mathbf{p},\mathbf{p}'}\delta f(\mathbf{r},\mathbf{p}') \ . \tag{1.37}$$

By considering the quasiparticles as independent, the dynamics of every single particle is given by a mean-field hamiltonian similar to the one in (1.14). Then, we apply the standard method [12, 25] based on the following of a small phase-space element flow, under the dynamics given by its hamiltonian; this will lead us to a Boltzmann equation for the distribution function $f(\mathbf{r},\mathbf{p})$:

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{p}}\tilde{\epsilon}(\mathbf{r},\mathbf{p})\cdot\nabla_{\mathbf{r}} f - \nabla_{\mathbf{r}}\tilde{\epsilon}(\mathbf{r},\mathbf{p})\cdot\nabla_{\mathbf{p}} f = 0 \tag{1.38}$$

One has to go in the proximity of Fermi surface, so as to consider only excited quasiparticles. The kinetic equation for excited quasiparticles can be obtained by considering

$$f(\mathbf{r},\mathbf{p},t) = f^{(0)}(\tilde{\epsilon}) + \delta f(\mathbf{r},\mathbf{p},t) \tag{1.39}$$

where $\tilde{\epsilon} = \tilde{\epsilon}(\mathbf{r},\mathbf{p})$. By replacing $f(\mathbf{r},\mathbf{p},t)$ in (1.38) with (1.39), we finally get

$$\boxed{\frac{\partial \delta f}{\partial t} + \nabla_{\mathbf{p}}\tilde{\epsilon}(\mathbf{r},\mathbf{p})\cdot\nabla_{\mathbf{r}}\delta f - \nabla_{\mathbf{r}}\tilde{\epsilon}(\mathbf{r},\mathbf{p},t)\cdot\nabla_{\mathbf{p}}\delta f = 0 \ .} \tag{1.40}$$

We strongly remark that we have neglected the binary collisions, too. Indeed, we have already pointed out that we put our system in a non-collisional and non-dissipative framework when we decided to consider the mean-field Hamiltonian:

$$H(\mathbf{r},\mathbf{p}) = \frac{|\mathbf{p}|^2}{2m} + \left[V_{\text{ext}}(\mathbf{r}) + g\int d^3\mathbf{p}\ \delta f(\mathbf{r},\mathbf{p},t)\right] \tag{1.41}$$

The main and more difficult point consists in the choice and method through which we want to describe the force felt by a quasiparticle. A detailed account of this topic, as usual, can be found in the classical books on quantum many-body systems [15, 36].

# 2

## DIMENSIONAL REDUCTION OF BOLTZMANN-VLASOV EQUATION

In the previous chapter we have shown how, starting from the Schroedinger's equation, we manage to get a quantum analog of classical collisionless Boltzmann-Vlasov equation for an ultracold and dilute Bose gas:

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} f - \nabla_{\mathbf{r}} U_{\mathrm{mf}}(\mathbf{r}, t) \cdot \nabla_{\mathbf{p}} f = 0 \tag{2.1}$$

where $U_{\mathrm{mf}} = V_{\mathrm{ext}}(\mathbf{r}) + 2gn(\mathbf{r}, t)$. All the interesting quantities are defined by means of the distribution function $f(\mathbf{r}, \mathbf{p}, t)$ (1.26).

We only considered terms at the first order on coupling parameter[1] $g$: this choice *identifies* the collisionless regime, because the collisional integral (1.19) is quadratic in $g$, while the free-streaming term, described by the self-consistent Hartree-Fock Hamiltonian $\hat{H}_{\mathrm{mf}}$ of (1.14) is linear and non-dissipative.

The (2.1) is an equation which lives in a 6-dimensional phase-space, then its integration is a demanding challenge, both from analytical and computational point of view. Many analytical techniques have been explored in order to give a detailed depiction of gas dynamical behaviour. In [27, 39], a particular scaling ansatz has been used to include in the dynamical evolution the dissipative effects made up by collisions, while [28] analyse the collisional damping of a Bose gas by means of a variational method. One can try to numerically integrate the (2.1), but this choice implies a great computational effort. The other choice consists in solving a much simpler problem, the two-dimensional phase-space analog of (2.1): the working load is lighter, but a pure one-dimensional problem in configuration space is not completely a real physical situation.

In this chapter we explore the possibility of a third path, which manages to connect the closeness to physical reality and a lighter computational effort.

1 We recall that the coupling parameter is defined as $g = \frac{4\pi a \hbar^2}{m}$ where $a$ is the s-wave scattering length. We chose $g$ (or equivalently $a$) as the *smallness parameter* of the problem.

This can be achieved by means of the so-called *dimensional reduction* scheme: under few assumptions, a 6-dimensional phase-space equation can be simplified to a 2-dimensional one, where all the memories of the original problem are dumped in a redefinition of certain proper parameter.

This idea has been enforced in condensed-matter topics; an enlightening example can be found in [49].

We initially describe the idea of the dimensional reduction of Boltzmann-Vlasov equation from a heuristic point, presenting the founding physical idea and a simple example.

## 2.1 THE CORE IDEA, AND A SIMPLE EXAMPLE

If we start by considering our system made of an ultracold and diluted atomic gas in a usual 3D configuration space, the main assumption consists in switching on a strong confining external potential $V_{\text{ext}}$ along two directions of the configuration space. For example, we can have

$$V_{ext}(\mathbf{r}) = \frac{1}{2} m\omega_{\perp}^2 \left(x^2 + y^2\right) . \tag{2.2}$$

This is not a *Gedankenexperiment*: several developments and improvements occured in the last twenty years, concerning ultracold atomic gas experimental studies. Hence, atomic gases can be handled by means of well-known and widely used experimental techniques and the tunability of many parameters allows us to realize a multitude of different physical situation, among which the one we have explained above.

Among all possible choices of external confining potential, a very simple one is the following

$$V_{\text{ext}}(\mathbf{r}) = \begin{cases} +\infty & \text{if } x \in [-a; +a] \wedge y \in [-b; +b] \\ 0 & \text{elsewhere} \end{cases} \tag{2.3}$$

which correspond to the physical situation of a gas held in a closed vessel.

Due to this strong confinement on the transverse plane, we expect that interesting features of the system dynamics will be found only along the z-axis. By following this idea, distribution function $f(\mathbf{r}, \mathbf{p}, t)$ can be factorised in the following way

$$\boxed{f(\mathbf{r}, \mathbf{p}, t) = \varphi(z, p_z, t)\chi(x, y, p_x, p_y)} \tag{2.4}$$

where the time dependence is imposed only on the z-factor.

This splitting of the distribution function will be considered again in the next section, where we will propose a more rigorous treatment of the dimensional reduction method. The time-dependence of the distribution function is given only by $\varphi(z, p_z, t)$;

this a strong assumption and a future development of this topic will have to involve the relaxation of this constraint.

The key point consists in an appropriate choice for $f(\mathbf{r}, \mathbf{p}, t)$ in order to give a good account of the transverse plane dynamics.

If the potential is like (2.3), a $f(\mathbf{r}, \mathbf{p}, t)$ can be choosen as follows:

$$f(\mathbf{r}, \mathbf{p}, t) = \beta\delta(p_x)\delta(p_y)\varphi(z, P_z, t) \tag{2.5}$$

where $\beta$ is a constant. The two Dirac $\delta$-functions fix the problem of energy conservation, while it is reasonable to assume a uniform probability distribution function on the transverse plane.

By means of a substitution and a straightforward integration, the Boltzmann-Vlasov equation can be written as

$$\frac{\partial f}{\partial t} + \left[\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}} - \nabla_{\mathbf{r}}\left(V_{\text{ext}}(\mathbf{r}) + 2g\int d^3\tilde{\mathbf{p}}\; f(\mathbf{r}, \tilde{\mathbf{p}}, t)\right)\cdot\nabla_{\mathbf{p}}\right] f = 0\ ,$$

with the normalization constraint

$$\int_\Omega d^3\mathbf{r}d^3\mathbf{p} f(\mathbf{r}, \mathbf{p}, t) = N \tag{2.6}$$

where $N$ is the total number of particle, while $\mathbf{\Omega}$ is the phase space. Since $N$ is a dimensionless quantity, $[f] = [L]^{-3}[P]^{-3}$. By inserting (2.5) in the normalization integral, we get $[\varphi] = [L]^{-1}[P]^{-1}$ and consequently, $[\beta] = [L]^{-2}$. In order to write a 2-dimensional Boltzmann-Vlasov equation, we replace the 6-dimensional phase space distribution $f(\mathbf{r}, \mathbf{p}, t)$ with its factorization (2.5):

$$\frac{\partial}{\partial t}(\varphi\chi) + \left[\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}} - \nabla_{\mathbf{r}}\left(V_{\text{ext}}(\mathbf{r}) + 2g\int d^3\tilde{\mathbf{p}}\; \chi\varphi(z, \tilde{p_z}, t)\right)\cdot\nabla_{\mathbf{p}}\right](\varphi\chi) = 0\ .$$

Since $\chi = \beta\delta(p_x)\delta(p_y)$, the terms can be arranged as follows:

$$\begin{aligned}
&\frac{\partial}{\partial t}(\varphi\chi) = \chi\frac{\partial\varphi}{\partial t}\ ,\\
&\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}}(\varphi\chi) = \chi\frac{p_z}{m}\frac{\partial\varphi}{\partial z}\ ,\\
&\nabla_{\mathbf{p}}(\varphi\chi) = \sum_{i=1}^{3}\frac{\partial}{\partial p_i}\big[\varphi(z, p_z, t)\chi]\hat{\mathbf{e}}_i = \chi\frac{\partial\varphi}{\partial p_z}\hat{\mathbf{e}}_z\ ,
\end{aligned}$$

where $\hat{\mathbf{e}}_i$ is the i-axis unit vector. The mean-field term is slightly more complicated; it gives us a precise physical constraint about the effectiveness of the factorization. We have

$$\begin{aligned}
&\nabla_{\mathbf{r}}\left[V_{\text{ext}} + 2g\beta\int d^3\tilde{\mathbf{p}}\varphi(z, \tilde{p}_z, t)\delta(\tilde{p}_x)\delta(\tilde{p}_y)\right]\cdot\chi\frac{\partial\varphi}{\partial p_z}\hat{\mathbf{e}}_z =\\
&= 2g\beta^2\delta(p_x)\delta(p_y)\frac{\partial}{\partial z}\left[\int d\tilde{p}_x\delta(\tilde{p}_x)\int d\tilde{p}_y\delta(\tilde{p}_y)\int d\tilde{p}_z\varphi(z, \tilde{p}_z, t)\right]\frac{\partial\varphi}{\partial p_z}\\
&= 2g\beta^2\delta(p_x)\delta(p_y)\int d\tilde{p}_z\ \frac{\partial\varphi}{\partial z}(z, \tilde{p}_z, t)\frac{\partial\varphi}{\partial p_z}\ .
\end{aligned}$$

By an integration over $dp_x$ and $dp_y$, we get

$$\beta\delta(p_x)\delta(p_y)\left[\frac{\partial\varphi}{\partial t}+\frac{p_z}{m}\frac{\partial\varphi}{\partial z}-2g\beta\int d\tilde{p}_z\,\frac{\partial\varphi}{\partial z}(z,\tilde{p}_z,t)\frac{\partial\varphi}{\partial p_z}\right]=0$$

$$\int dp_x dp_y\delta(p_x)\delta(p_y)\left[\frac{\partial\varphi}{\partial t}+\frac{p_z}{m}\frac{\partial\varphi}{\partial z}-2g\beta\int d\tilde{p}_z\,\frac{\partial\varphi}{\partial z}(z,\tilde{p}_z,t)\frac{\partial\varphi}{\partial p_z}\right]=0\,.$$

By recalling that $\delta$ is defined in such a way that $\int_{\mathbb{R}} dx\ \delta(x)=1$, we finally get

$$\boxed{\frac{\partial\varphi}{\partial t}+\frac{p_z}{m}\frac{\partial\varphi}{\partial z}-2g\beta\int d\,\tilde{p}_z\frac{\partial\varphi}{\partial z}(z,\tilde{p}_z,t)\frac{\partial\varphi}{\partial p_z}=0} \tag{2.7}$$

where $[\beta]=\frac{1}{[L]^2}$. So, one can define $g_{1\mathrm{d}}=g\beta$. This is the scaling of the coupling parameter which conceals all the memories of the original problem (the one given by (2.1).
We can give an order of magnitude for the *dimensional scaling parameter*; indeed, due to the normalization constraint, we have

$$n(\mathbf{r},t)=\int d^3\mathbf{p}\,f(\mathbf{r},\mathbf{p},t)\quad\Longrightarrow\quad\int_\Omega d^3\mathbf{r}d^3\mathbf{p}\ f(\mathbf{r},\mathbf{p},t)=(\text{const.})\,.$$

Then, we write

$$\begin{aligned}(\text{const.})&=\beta\int_{\text{box}}dxdy\underbrace{\int dp_xdp_y\ \delta(p_x)\delta(p_x)\delta(p_y)}_{=1}\int dzdp_z\ f(z,p_z,t)\\&\sim\beta L^2_{\text{box}}\int dzdp_z\ f(z,p_z,t)\,,\end{aligned}$$

suggesting that the $\beta$ parameter has to be proportional to the reciprocal square of a certain characteristic length of the problem; in this case, it is the confining box length. We are going to see that this is a typical situation.

## 2.2 A VARIATIONAL APPROACH

We now consider the problem of dimensional reduction from a more general point of view, approaching it in a self consistent way.

In order to do this, we have to go beyond the simple *box potential* defined in (2.3), giving an account of a more realistic situation. The experimental techniques developed in order to handle ultracold atomic gases make a wide use of magneto-optical traps [9, 43]. This large set of experimental methods opens the door to the possibility of isolating and manipulating this kind of system. There is a plenty of feasible confining configurations, i.e. many choices for $V_{\text{ext}}(\mathbf{r})$. For example, in [49] it was considered the problem of studying the dynamics of a cigar-shaped condensate through the dimensional reduction method.

We start by considering the Boltzmann-Vlasov equation for a Bose gas as in (2.1), and we define the differential operator $\hat{L}$ as

$$\hat{L} = -\frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{r}} + \nabla_{\mathbf{r}} \left[ V_{\text{ext}}(\mathbf{r}) + 2g \int d^3\tilde{\mathbf{p}}\, F(\mathbf{r}, \tilde{\mathbf{p}}, t) \right] \cdot \nabla_{\mathbf{p}} \,, \quad (2.8)$$

where we use the capital letters for the distribution function $F(\mathbf{r}, \mathbf{p}, t)$ defined in 6-dimensional phase space.

A collisionless dynamics can be depicted through Hamiltonian formalism, thanks to the non-dissipative behaviour.

A classical mean-field Hamiltonian can be write as follows

$$H = \frac{|\mathbf{p}|^2}{2m} + V_{ext}(\mathbf{r}) + 2g \int d^3\tilde{\mathbf{p}} f(\mathbf{r}, \tilde{\mathbf{p}}, t) \quad (2.9)$$

and consequently

$$\begin{aligned} \hat{L} &= -\nabla_{\mathbf{p}} H \cdot \nabla_{\mathbf{r}} + \nabla_{\mathbf{r}} H \cdot \nabla_{\mathbf{p}} \\ &= \sum_{i=1}^{3} \left[ -\frac{\partial H}{\partial p_i} \frac{\partial}{\partial x_i} + \frac{\partial H}{\partial x_i} \frac{\partial}{\partial p_i} \right] \end{aligned} \quad (2.10)$$

Hence, the (2.1) can be expressed in a compact form:

$$\frac{\partial F}{\partial t}(\mathbf{r}, \mathbf{p}, t) = \hat{L} F(\mathbf{r}, \mathbf{p}, t) \quad (2.11)$$

We are going to obtain a variational principle thanks to interesting formal analogies between the Liouville's equation for a classical distribution function [7, 12, 25] and the Schroedinger's one. Although we will get formulae which resemble the well-known quantum mechanics ones, we have to keep in mind that this is only a formal analogy, in the sense that it won't imply any of typical quantum phenomena.

For example, suppose we were able to solve the (2.11). Then the mean value of any dynamical variable $A(\mathbf{r}, \mathbf{p}, t)$ can be computed as

$$\langle A(t) \rangle_{F(t)} = \int_{\Omega} d^3\mathbf{r} d^3\mathbf{p}\, A(\mathbf{r}, \mathbf{p}, t) F(t) \,, \quad (2.12)$$

where $F(t) = F[\mathbf{r}(t), \mathbf{p}(t), t]$. Therefore, in order to describe system dynamics, one has to solve the (2.11). This is a Liouville equation for the $F(t)$: it is homogeneous, linear, and involving only first-order partial derivatives; these features imply the important property that every function of solutions is itself a solution. This fact is linked to the possibility of giving a Hamiltonian depiction of motion, and consequently to our dealing with a collisionless problem.

Because of this property, $F(t)$ can be defined as follows:

$$\boxed{F(t) = \Big| u(t) \Big|^2 \,,} \quad (2.13)$$

where $u(t)$ is complex function defined up to a phase $\theta(t)$, i.e. $u(t) = |u|e^{i\theta}$. We don't have to worry about this arbitrary phase, since it won't affect the physics of the system because of (2.12) and because every dynamics variable is a function only of position and momentum, and not of their derivatives. Therefore, due to this arbitrary choice of the phase, we can assume that $\theta$ satisfies the (2.11), such that $u(t)$ must obey to the following equation:

$$\frac{\partial u}{\partial t}(\mathbf{r}, \mathbf{p}, t) = \hat{L}u(\mathbf{r}, \mathbf{p}, t) \ . \tag{2.14}$$

We can rewrite the (2.12) as

$$\begin{aligned} \langle A(t) \rangle_{F(t)} &= \int_\Omega d^3\mathbf{r} d^3\mathbf{p} A \Big| u(t) \Big|^2 \\ &= \Big\langle u(t), Au(t) \Big\rangle_{\mathbb{L}_2(\Omega)} \ , \end{aligned} \tag{2.15}$$

where

$$\Big\langle f, g \Big\rangle_{\mathbb{L}_2(\Omega)} = \int_\Omega d^3\mathbf{r} d^3\mathbf{p} \ f^* g \tag{2.16}$$

is the inner product in $\mathbb{L}_2(\Omega)$, i.e the space of square-modulus integrable functions.

The pair (2.14) and (2.15) establishes a formal analogy with quantum mechanics. By multiplying (2.14) by the imaginary unit, we will find the Schroedinger's equation; on the other hand, we have to keep in mind that this analogy is only a formal one: we can consider the (2.15) in quantum formalism, but the formula $\langle u, Au \rangle$ gives us a transition probability (in other words, it's a matrix element of a quantum operator $A$), and we must take the square modulus in order to get the mean value.

Despite this conceptual difference, a formulation of time-dependent Hartree-Fock-like method can be stated in the following way.

Let us consider a proper functional of $u(t)$, defined as

$$\boxed{\mathcal{F}[u] = \left\langle u, \left[ \hat{L} - \frac{\partial}{\partial t} \right] u \right\rangle_{\mathbb{L}_2(\Omega)} \ ,} \tag{2.17}$$

with the usual constraint $\langle u, u \rangle = 1$. In terms of $F(t)$, we find

$$\begin{aligned} \mathcal{F}[u] &= \left\langle u, \left[ \hat{L} - \frac{\partial}{\partial t} \right] u \right\rangle \\ &= \left\langle |u|e^{i\theta}, \left[ \hat{L} - \frac{\partial}{\partial t} \right] |u|e^{i\theta} \right\rangle \\ &= \left\langle |u|^2 e^{i\theta}, \left[ \hat{L} - \frac{\partial}{\partial t} \right] e^{i\theta} \right\rangle \\ &= i \int d\Omega f(t) \left[ \hat{L} - \frac{\partial}{\partial t} \right] \theta(t) \ , \end{aligned} \tag{2.18}$$

where $d\Omega = d^3\mathbf{r}d^3\mathbf{p} = \prod_{i=1}^{3} dx_i\, dp_i$. The (2.18) underlines an important warning concerning the $u(t)$-phase $\theta(t)$: we must not set it to zero *a priori*, or $\mathcal{F}$ becomes an identically null functional, with an obvious non-physical meaning.

We require that the solutions of $\partial_t u = \hat{L}u$ are the stationary points (i.e curves) of the functional $\mathcal{F}[u]$ for independent variations of $u^*$ and $u$. The proof is not difficult: we can consider the first variation on $u^*$; because of the normalization constraint, we also include a proper Lagrange multiplier $\lambda$. Therefore, we define

$$\tilde{\mathcal{F}}[u,\lambda] = \mathcal{F}[u] + \lambda\left(\langle u,u\rangle_{\mathbb{L}_2(\Omega)} - 1\right). \tag{2.19}$$

We fulfill a variation on (2.19) one term by one. So, by considering the first term, i.e. $\mathcal{F}[u]$, we have

$$\begin{aligned}
\delta\mathcal{F}[u,h] &= \frac{d}{d\epsilon}\mathcal{F}[(u+\epsilon h)^*, u]\bigg|_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\left\langle u+\epsilon h,\ \left[\hat{L} - \frac{\partial}{\partial t}\right]u\right\rangle_{\mathbb{L}_2(\Omega)}\bigg|_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\left[\mathcal{F}[u] + \epsilon\int_\Omega d^3\mathbf{r}d^3\mathbf{p}\ h^*\left(\hat{L} - \frac{\partial}{\partial t}\right)u\right]_{\epsilon=0} \\
&= \int d\Omega\ h^*\left[\hat{L} - \frac{\partial}{\partial t}\right]u\,.
\end{aligned}$$

For the term proportional to Lagrange multiplier, we find that

$$\begin{aligned}
&(\lambda + d\lambda)\left[\int d\Omega\ (u+\epsilon h)^*\, u - 1\right] = \\
&= \lambda\,\langle u,u\rangle + \lambda\epsilon\int d\Omega h^* u - \lambda + d\lambda\left[\int d\Omega (u+\epsilon h)^* u - 1\right] \\
&= \lambda\epsilon\int d\Omega h^* u + d\lambda\left[\int d\Omega u^* u - 1\right],
\end{aligned}$$

where we make use of the normalization constrain $\langle u,u\rangle$ in going from second line to the last line. At the end, we have

$$\begin{aligned}
\delta\mathcal{F}[u,h,\lambda,d\lambda] = \int d\Omega h^*\left[\hat{H} - \frac{\partial}{\partial t} + \lambda\right]u + \\
+\ d\lambda\left[\int d\Omega u^* u - 1\right].
\end{aligned} \tag{2.20}$$

This variation turns out to be zero for any direction $(h, d\lambda)$ if

$$\begin{cases}
\int_\Omega d^3\mathbf{r}d^3\mathbf{p}\ u^* u = 1 \\
\left[\hat{L} - \dfrac{\partial}{\partial t} + \lambda\right]u = 0\,.
\end{cases} \tag{2.21}$$

The second equation implies that the dependence from $\lambda$ emerges only through a factor $e^{\lambda t}$. The next step consists in showing that $\lambda$ does not affect the dynamics, so it can be removed by considering $\lambda = 0$. Indeed, by an integration over the whole phase space, we have

$$\begin{aligned} \int d\Omega\, u^* \left[\hat{L} - \frac{\partial}{\partial t} + \lambda\right] u = 0 \\ \int d\Omega\, u^* \left[\hat{L} - \frac{\partial}{\partial t}\right] u + \lambda = 0\ . \end{aligned} \tag{2.22}$$

So, $\lambda = i\mathbb{R}$. The $\lambda$ contribution is only made of a phase factor $e^{\lambda t}$ which can be dropped. Hence (2.14) naturally arises from the variational method presented here.

The practical usefulness of a variational principle does not consist in the derivation of (2.1), but rather on the searching for its approximate solutions under peculiar physical situation.

In the following section we are going to explore a physical situation which can occur in experiments that handle ultracold atomic gas (see for example [14, 49]).

## 2.3 THE HARMONIC CONFINEMENT

By means of laser cooling techniques, the gas particles can be trapped in a harmonic potential, whose analytical form is given by

$$V_{\text{ext}}(\mathbf{r}) = \frac{1}{2} m\omega^2 (x^2 + y^2) \tag{2.23}$$

and we take the (2.4) as trial function. Moreover, we assume the following form for the transverse factor of $u(t)$:

$$u(\mathbf{r}, \mathbf{p}, t) = \varphi(z, p_z, t)\chi(x, y, p_x, p_y)$$

and

$$\chi(x, y, p_x, p_y) = \frac{1}{\pi\sigma\gamma} e^{-\frac{x^2+y^2}{2\sigma^2}} e^{-\frac{p_x^2+p_y^2}{2\gamma^2}} \tag{2.24}$$

where $\sigma^2 = \frac{\hbar}{m\omega}$. We interpret $\sigma$ as the characteristic length of the external trapping potential; this is a tunable parameter in the experiments.

If the confinement is strong enough to assume that, on the X-Y plane, the particles fill up only the ground state, the corresponding $|x\rangle$-representation eigenfunction is the $0^{\text{th}}$-order Hermite polynomial, i.e. a gaussian function. We can move from $|x\rangle$ to $|p\rangle$ by means of a Fourier transform, but it is a well-known fact that the

Fourier transform of a gaussian function is once again a gaussian one. Precisely, the Fourier transform of $e^{-d^2x^2}$ is given by

$$\begin{aligned}\mathfrak{F}\Big[e^{-d^2x^2}\Big] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dx\; e^{-d^2x^2}e^{ikx} \\ &= \frac{1}{d\sqrt{2}}\exp\left(-\frac{\omega^2}{4d^2}\right).\end{aligned}$$

Then, the bigger is $d$, i.e the narrower is the original Gaussian, the wider is its Fourier transform. The parameters $\sigma$ and $\gamma$ specify the width of the corresponding gaussian function, so they define the position (and momentum) dispersion. Because of the Fourier transform property, they are not independent each other but the smaller is the dispersion on position, the bigger is the momentum one.

We replace $u(t)$ in (2.17) with (2.24):

$$\begin{aligned}\mathcal{F}[u] &= \left\langle u, \left[\hat{L} - \frac{\partial}{\partial t}\right]u\right\rangle \\ &= \int_\Omega d^3\mathbf{r}d^3\mathbf{p}(\varphi\chi)^*\Bigg[-\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}} + \nabla_{\mathbf{r}}\Bigg(V_{\text{ext}} + \\ &\qquad + 2g\int d^3\tilde{\mathbf{p}}|\varphi\chi|^2\Bigg)\cdot\nabla_{\mathbf{p}} - \frac{\partial}{\partial t}\Bigg]\varphi\chi\,.\end{aligned}$$

In order to evaluate the functional (2.17) with the trial function given by (2.4) and (2.24), we are going to consider the one term by one.

1. **Temporal derivative term**
   Here, and in the following, the integral $\int d^3\mathbf{r}\,d^3\mathbf{p}\,(\bullet)$ is intended to be performed over the whole phase space.

$$\begin{aligned}&\int_\Omega d^3\mathbf{r}d^3\mathbf{p}\,u^*(\mathbf{r},\mathbf{p},t)\frac{\partial u}{\partial t}\;(\mathbf{r},\mathbf{p},t) \\ &= \frac{1}{\pi^2\sigma^2\gamma^2}\int dxdydp_xdp_y\; e^{-\frac{x^2+y^2}{\sigma^2}}e^{-\frac{p_x^2+p_y^2}{\gamma^2}}\times \\ &\qquad\qquad \times\int dzdp_z\varphi(z,p_z,t)\frac{\partial\varphi}{\partial t}(z,p_z,t) \\ &= \int dzdp_z\varphi(z,p_z,t)\frac{\partial\varphi}{\partial t}(z,p_z,t)\,,\end{aligned}$$

   where we make use of the well-known result

$$\int_0^{+\infty} dx\,xe^{-\alpha x^2} = \frac{1}{2\alpha}\,.$$

2. **Kinetic term**

$$\begin{aligned}&\int d^3\mathbf{r}d^3\mathbf{p}\,u^*(\mathbf{r},\mathbf{p},t)\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}}u(\mathbf{r},\mathbf{p},t) = \\ &= \int_\Omega d^3\mathbf{r}d^3\mathbf{p}\,u^*(\mathbf{r},\mathbf{p},t)\left[\varphi\frac{p_x}{m}\frac{\partial\chi}{\partial x} + \varphi\frac{p_y}{m}\frac{\partial\chi}{\partial y} + \chi\frac{p_z}{m}\frac{\partial\varphi}{\partial z}\right].\end{aligned}$$

Now, for the $z$-component, we get

$$\begin{aligned}
&\int_\Omega d^3\mathbf{r}d^3\mathbf{p}(\chi\varphi)^*\frac{p_z}{m}\chi\frac{\partial\varphi}{\partial z}\\
&=\int dzdp_z\,\varphi^*\frac{p_z}{m}\frac{\partial\varphi}{\partial z}\int dxdydp_xdp_y\chi^*\chi\\
&=\int dzdp_z\varphi^*(z,p_z,t)\frac{p_z}{m}\frac{\partial\varphi}{\partial z}(z,p_z,t)\;,
\end{aligned}$$

while for one of the transverse component

$$\begin{aligned}
&\int d^3\mathbf{r}d^3\mathbf{p}\,\varphi^*\varphi\chi^*\frac{p_x}{m}\frac{\partial\chi}{\partial x}=\\
&=\int dzdp_z\,\varphi^*\varphi\int dxdydp_xdp_y\,\chi^*\frac{p_x}{m}\frac{\partial\chi}{\partial x}\\
&=\int dzdp_z\,\varphi^*\varphi\int dxdydp_xdp_z\,\frac{xp_x}{m\sigma_\perp^2}\chi^*\chi\\
&=0\;.
\end{aligned}$$

3. **External trapping term**

$$\begin{aligned}
&\int_\Omega d^3\mathbf{r}d^3\mathbf{p}\,u^*(\mathbf{r},\mathbf{p},t)\nabla_{\mathbf{r}}V_{\rm ext}(\mathbf{r})\cdot\nabla_{\mathbf{p}}u(\mathbf{r},\mathbf{p},t)=\\
&=\int_\Omega d^3\mathbf{r}d^3\mathbf{p}\,u^*(\mathbf{r},\mathbf{p},t)\nabla_{\mathbf{r}}\left[\frac{m\omega^2}{2}(x^2+y^2)\right]\cdot\nabla_{\mathbf{p}}u(\mathbf{r},\mathbf{p},t)\\
&=\int_\Omega d^3\mathbf{r}d^3\mathbf{p}\,u^*(\mathbf{r},\mathbf{p},t)\left[m\omega^2x\frac{\partial u}{\partial p_x}+m\omega^2y\frac{\partial u}{\partial p_y}\right]\\
&=-\int_\Omega d^3\mathbf{r}d^3\mathbf{p}\,u^*(\mathbf{r},\mathbf{p},t)\left[m\omega^2\frac{xp_x}{\gamma^2}+m\omega^2\frac{yp_y}{\gamma^2}\right]u(\mathbf{r},\mathbf{p},t)\\
&=0\;,
\end{aligned}$$

where in the last we consider again the parity of gaussian function.

4. **Mean field term**

This the most awkward term, but we approach it by means of the same ploy exploited above.

$$\begin{aligned}
&\int_\Omega d^3\mathbf{r}d^3\mathbf{p}\,u^*(\mathbf{r},\mathbf{p},t)\;\nabla_{\mathbf{r}}V_{\rm mf}(\mathbf{r})\cdot\nabla_{\mathbf{p}}u(\mathbf{r},\mathbf{p},t)=\\
&=\int_\Omega d^3\mathbf{r}d^3\mathbf{p}\,u^*(\mathbf{r},\mathbf{p},t)\;\nabla_{\mathbf{r}}\left[2g\int d\tilde{p}|u(\mathbf{r},\tilde{\mathbf{p}},t)|^2\right]\cdot\nabla_{\mathbf{p}}u(\mathbf{r},\mathbf{p},t)\;,
\end{aligned}$$

where the spatial gradient of the mean-field potential is given by

$$\begin{aligned}
&\nabla_{\mathbf{r}}\left[\int d\tilde{p}_xd\tilde{p}_ye^{-\frac{1}{\gamma^2}(\tilde{p}_x^2+\tilde{p}_y^2)}\int d\tilde{p}_z|\varphi(z,\tilde{p}_z,t)|^2\frac{1}{\pi^2\sigma^2\gamma^2}e^{-\frac{1}{\sigma^2}(x^2+y^2)}\right]=\\
&=-\frac{2}{\pi\sigma^4}e^{-\frac{1}{\sigma^2}(x^2+y^2)}\int d\tilde{p}_z|\varphi(z,\tilde{p}_z,t)|^2\Big(x\hat{\mathbf{e}}_x+y\hat{\mathbf{e}}_y\Big)+\\
&\qquad+\left[\frac{e^{-\frac{1}{\sigma^2}(x^2+y^2)}}{\pi\sigma^2}\int d\tilde{p}_z\frac{\varphi^*\varphi}{\partial z}\right]\hat{\mathbf{e}}_z\;.
\end{aligned}$$

The inner product of this vector field and the momentum gradient of $u(\mathbf{r},\mathbf{p},t)$ gives us back three terms. We consider the $z$-axis one, which leads us to the dimensional scaling of the coupling parameter:

$$\frac{2g}{\pi\sigma^2}\int_\Omega d^3\mathbf{r}d^3\mathbf{p}u^*(\mathbf{r},\mathbf{p},t)e^{-\frac{1}{\sigma^2}(x^2+y^2)}\int d\tilde{p}_z\,\frac{\partial|\varphi|^2}{\partial z}\frac{\partial u}{\partial p_z}u(\mathbf{r},\mathbf{p},t) =$$
$$= \frac{2g}{\pi^3\sigma_\perp^4\gamma_\perp^2}\int dxdydp_xdp_y\,e^{-\frac{2}{\sigma^2}(x^2+y^2)}e^{-\frac{1}{\gamma^2}(p_x^2+p_y^2)}\times$$
$$\times\int dzdp_z\varphi^*\int d\tilde{p}_z\frac{\partial|\varphi|^2}{\partial z}(z,\tilde{p},t)\frac{\partial\varphi}{\partial p_z}$$
$$= \frac{g}{\pi\sigma_\perp^2}\int dzdp_z\,\varphi^*(z,p_z,t)\int d\tilde{p}_z\frac{\partial}{\partial z}|\varphi(z,\tilde{p}_z,t)|^2\frac{\partial\varphi}{\partial p_z}(z,p_z,t)\,.$$

In order to compute the contribution of the transverse component, we again take advantage of the parity property of our trial function:

$$-\frac{2g}{\pi\sigma_\perp^4}\int_\Omega d^3\mathbf{r}d^3\mathbf{p}\,u^*(\mathbf{r},\mathbf{p},t)xe^{-\frac{1}{\sigma^2}(x^2+y^2)}\times$$
$$\times\int d\tilde{p}_z\,|\varphi(z,\tilde{p}_z,t)|^2\frac{\partial u}{\partial p_x}(\mathbf{r},\mathbf{p},t)\propto$$
$$\propto\int_\Omega d\mathbf{x}d\mathbf{p}\,xp_xe^{-\frac{x^2+y^2}{\sigma^2}}e^{-\frac{p_x^2+p_y^2}{\gamma^2}}\varphi^*(z,p_z,t)\int dzdp_z\varphi^*\times$$
$$\times\int d\tilde{p}_z\,|\varphi(z,\tilde{p}_z,t)|^2\varphi$$
$$= 0\,.$$

Finally, the functional $\mathcal{F}[u]=\mathcal{F}[\varphi]$ can be written as follows:

$$\mathcal{F}[\varphi]=\int_\Omega d^3\mathbf{r}d^3\mathbf{p}\varphi^*\left[-\frac{p_z}{m}\frac{\partial}{\partial z}+\frac{g}{\pi\sigma^2}\frac{\partial}{\partial z}\int d\tilde{p}_z|\varphi(z,\tilde{p}_z,t)|^2\frac{\partial}{\partial p_z}-\frac{\partial}{\partial t}\right]\varphi\,.$$

Subsequently, by means of a variation on $\varphi^*$, we obtain the

$$\frac{\partial\varphi}{\partial t}=\left[-\frac{p}{m}\frac{\partial}{\partial z}+\frac{g}{\pi\sigma^2}\frac{\partial}{\partial z}\int d\tilde{p}_z|\varphi(z,\tilde{p}_z,t)|^2\frac{\partial}{\partial p_z}\right]\varphi\,. \qquad (2.25)$$

However, this is not the 1D Boltzmann-Vlasov equation we are looking forIn order to get a reduced equation for the distribution function, we have to remember that $F(\mathbf{r},\mathbf{p},t)$ is defined as the square modulus of the $u(\mathbf{x},\mathbf{p},t)$. Now, we can move from the (2.25) to a 1-dimensional equation for $f(z,p_z,t)$ simply by multipying the (2.25) by $\varphi^*$ and adding it to its complex conjugate.

Finally, we get an equation for $f=f(z,p_z,t)$ where all the memories of the original system are hidden in a proper parameter scaling:

$$\boxed{\frac{\partial f}{\partial t}+\frac{p}{m}\frac{\partial f}{\partial z}-\frac{g}{\pi\sigma_\perp^2}\frac{\partial}{\partial z}\int d\tilde{p}_zf(z,\tilde{p}_z,t)\frac{\partial f}{\partial p_z}=0\,.} \qquad (2.26)$$

and we can define $g_{1D} = \frac{g}{\pi\sigma_\perp^2}$ which is dimensionally consistent with what we have found in the first section.

As we said at the beginning of this chapter, we get an a quasi-2D[2] Boltzmann-Vlasov equation. We only use a Hartree-Fock like variational principle and a simple assumption on the confining potential; moreover, this potential can be handled with precision and its characteristic parameters are tunable and deeply involved in the reducted dynamics. In the calculation above, we write the (2.26) for a dilute Bose gas under a harmonic confining potential, mostly thanks to parity property of gaussian functions in (2.24).

We highlight the redundance of our choice of considering both a gaussian function on position and momentum. Indeed, we can equally obtain the (2.26) by assuming a gaussian behaviour only for the position on the transverse plane, and nothing else.

2 we are referring to the phase-space dimension, not to configuration space one.

# 3

# DYNAMICAL ANALYSIS OF BOLTZMANN-VLASOV EQUATION

## 3.1 LINEARIZATION AND LANDAU CONTOUR

In this chapter, we want to study the linear dynamical stability of the solutions of collisionless Boltzmann-Vlasov equation. We will highlight how the Landau-Vlasov plasma theory can be exploited in order to understand some features of the dynamical behaviour of ultracold atomic gases.

We have already explored the dimensional reduction approximation method, whose really relevant result consists in leading us to the 1D Boltzmann-Vlasov equation.

Due to the non-linearity of (2.26), we study the dynamical behaviour of small pertubations acting on a stationary distribution; this distribution is described by a proper distribution function $f_0(p)$; every spatially uniform and stationary distribution function is a solution of (2.26). Hence, we consider

$$f(x,p,t) = f_0(p) + \delta f(x,p,t) \ . \tag{3.1}$$

The smallness of $\delta f$ is specified by the following well-known relation[12, 25]:

$$\frac{\delta f}{f^{(0)}} \approx -\frac{\lambda}{L} \ , \tag{3.2}$$

where $L$ is the characteristic length of the system, while $\lambda = \bar{v}\tau$ is quantity related to the mean-free path of a gas particle. If one wants to see in the experiments out-of-equilibrium quasi-stationary states, the key quantity is the collision time $\tau$. In order to work in the collisionless regime, we have to assume that the characteristic time $t_p$ of the considered phenomenon is much smaller than $\tau$, i.e. $t_p \ll \tau$.

We get the linearized equation simply by replacing $f(x,p,t)$ with (3.1) in (2.26):

$$\boxed{\frac{\partial \delta f}{\partial t} + \frac{p}{m}\frac{\partial \delta f}{\partial x} - g\int d\tilde{p}\frac{\partial \delta f}{\partial x}(x,\tilde{p},t)\frac{\partial f_0}{\partial p}(p) = 0 \ .} \tag{3.3}$$

Let us assume that $\delta f$ can be expressed in terms of its Fourier transform:

$$\begin{aligned} \delta f(x,p,t) &= \int dk d\omega \; \delta\tilde{f}(k,p,\omega) e^{i(kx-\omega t)} \\ \delta\tilde{f}(k,p,\omega) &= \frac{1}{(2\pi)^2} \int dx dp \; \delta f(x,p,t) e^{-i(kx-\omega t)} \; . \end{aligned} \tag{3.4}$$

At this point, the pattern we want to follow is the one developed by Landau in his work about the linear response of a plasma. Therefore, we consider the ultracold gas behaviour as a Cauchy initial value problem: this approach will naturally lead us to complex frequency values and to the rediscovery of a *Landau damping* for this system, which seems so far from plasma and long range interaction system.

In order to put an initial value problem in the right term, we have to specify not only the diffential equation, whose solution will give us the dynamics of the pertubation, but also an initial condition. So, for the (3.3), we can consider the initial condition given by

$$\delta f(x,p,t=0) = \delta f_0(x,p) \; . \tag{3.5}$$

This problem could be solved by means of the Laplace-transform method [30, 37], but we think it could be more useful to set it in a way which allows us to take advantage from the Fourier-transform method [53].

The first problem to face concerns the temporal domain over which the Fourier transform and the Laplace one can be defined. In order to exploit the first one, we have to enlarge the range to $-\infty < t < +\infty$, while the problem we fixed in (3.3) and (3.5) covers the $0 < t < +\infty$. This can be done considering the following equation:

$$\boxed{\frac{\partial \delta f}{\partial t} + \frac{p}{m}\frac{\partial \delta f}{\partial x} - g \int d\tilde{p} \frac{\partial \delta f}{\partial x}(x,\tilde{p},t) \frac{\partial f_0}{\partial p}(p) = \delta f_0 \delta(t) \; ,} \tag{3.6}$$

with the prescription $\delta f(x,p,t) = 0$ for $t < 0$. The $\delta$-function plays the role of an impulsive force acting only at $t = 0$, which brings the system to the initial state defined by (3.5).

By means of (3.4) we get the following equation in the Fourier space:

$$\left( -\omega + \frac{p}{m}k \right) \delta\tilde{f} - gk \int d\tilde{p} \; \delta\tilde{f}(k,\tilde{p},\omega) \frac{\partial f_0}{\partial p}(p) = \frac{\delta\tilde{f}_0}{2\pi i} \tag{3.7}$$

and, consequently,

$$\begin{aligned} \delta\tilde{f}(k,p,\omega) = \frac{1}{2\pi i k} \frac{\delta\tilde{f}_0}{\left( \frac{p}{m} - \frac{\omega}{k} \right)} + \frac{g \frac{\partial f_0}{\partial p}(p)}{\left( \frac{p}{m} - \frac{\omega}{k} \right)} \times \\ \times \int d\tilde{p} \; \delta\tilde{f}(k,\tilde{p},\omega) \; . \end{aligned} \tag{3.8}$$

On the other hand, one can also try to compute an analytical expression for the integral involving the perturbation $\delta\tilde{f}(k,p,\omega)$; we get

$$\int dp\delta\tilde{f}(k,p,\omega) = -\frac{i}{2\pi k\epsilon(k,\omega)}\int dp\ \frac{\delta\tilde{f}_0}{\frac{p}{m}-\frac{\omega}{k}}\ . \tag{3.9}$$

In the equation above, we have defined

$$\boxed{\epsilon(k,\omega) = 1 - g\int dp\ \frac{\frac{\partial f_0}{\partial p}(p)}{\left(\frac{p}{m}-\frac{\omega}{k}\right)}\ .} \tag{3.10}$$

In the following, this function will be called *dielectric function*, in analogy with plasma physics.

Now, we easily realize that, if we manage to compute the integral in (3.9), we can also determine the $\delta\tilde{f}(k,p,\omega)$ by means of the (3.8). In order to reconstruct the evolution in time of perturbation[37], we only need to compute the integral:

$$\delta f(k,p,t) = \int_{\Gamma_\omega} d\omega\ e^{-i\omega t}\delta\tilde{f}(k,p,\omega)\ , \tag{3.11}$$

where $\Gamma_\omega$ is a proper integration contour in the complex frequencies plane, yet to be determined. The key-point concerns the proper choice of the integration contour: we must handle this issue very carefully.

We start by observing that if the contour $\Gamma_\omega$ lies above all the possible singularities, we can shift it towards $\Im\omega \longrightarrow +\infty$ so the (3.11) gives $\delta f(k,p,t) = 0$ and satisfies the causality requirement given by $\delta f(x,p,t) = 0$ for $t < 0$, as shown in 3.1.

At this point, we can modify the integration contour [53] into the one shown in 3.2 and, consequently, we can write the (3.11) as

$$\begin{aligned}\delta f(k,p,t) &= \int_{\Gamma_\omega} d\omega\ e^{-i\omega t}\delta\tilde{f}(k,p,\omega) \\ &= \lim\left[\int_A^B + \int_B^C + \int_C^D\right] d\omega\, e^{-i\omega t}\delta\tilde{f}(k,p,\omega)\ .\end{aligned}$$

The contributions from the vertical segment vanish, because the integrand oscillates with an arbitrarily high frequency. On the other hand, also the contribution given by $BC$ is null: we are considering a function $e^{\Im\omega t}$, where $\Im\omega > 0$ and $t < 0$.

Considering $t > 0$, without changing the integral, we can modify the integration path as $\Gamma'_\omega$ or $\Gamma''_\omega$. In the latter case, it is evident that the only contributions to $\delta f(k,p,t)$ come from the poles of $\delta\tilde{f}(k,p,\omega)$.

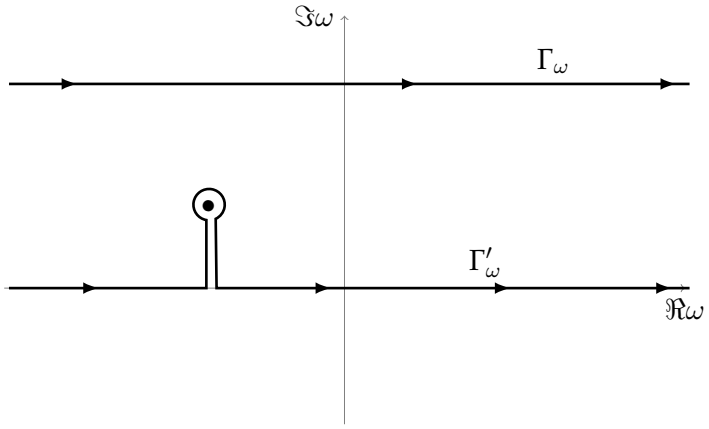


Figure 3.1: Acceptable integration contours in the plane of complex frequency $\omega$ which satisfy the causality requirement. If we shift $\Gamma_\omega$ downward, we must be careful about the prescription of not crossing the singularities. In that case, we are going to consider a contour similar to $\Gamma'_\omega$.

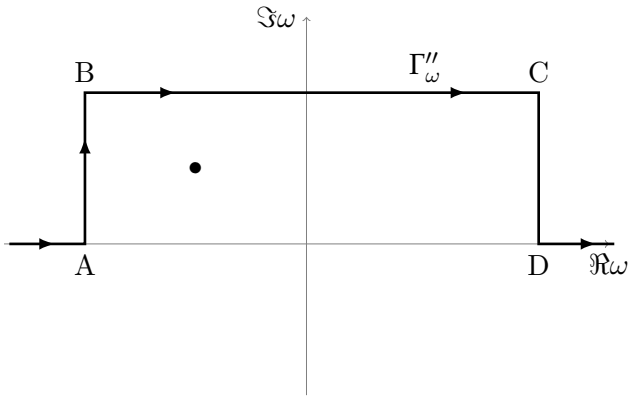


Figure 3.2: $\Gamma''_\omega$ is the contour in which we have deformed $\Gamma_\omega$. We suppose an infinite distance between $AB$ and $BC$.

In order to modify the integration contour from $\Gamma_\omega$ or $\Gamma'_\omega$ to $\Gamma'''_\omega$ in 3.3, we have to analytically continue the function $\epsilon(k,\omega)$ which appears in (3.9) from upper half plane of complex frequency to the lower one . We suppose that $k > 0$ and that $f_0(p)$ is a well-behaved function, so its first derivate won't give us issues about poles or other singularities. The poles od $\delta f(k,p,\omega)$ are thus identified by the dispersion relation $\epsilon(k,\omega) = 0$.

In order to analytically continue $\epsilon(k,\omega)$, we have to specify the integration path. First, we observe that the integral is naturally defined on the real-$p$ axis; for $\Im\omega > 0$ all poles thus lies above the real-$p$ axis. On the other side, we need to extend the domain of $\epsilon(k,\omega)$ towards real values and negative imaginary part ones, by means of analytical continuation. In these case we have to modify

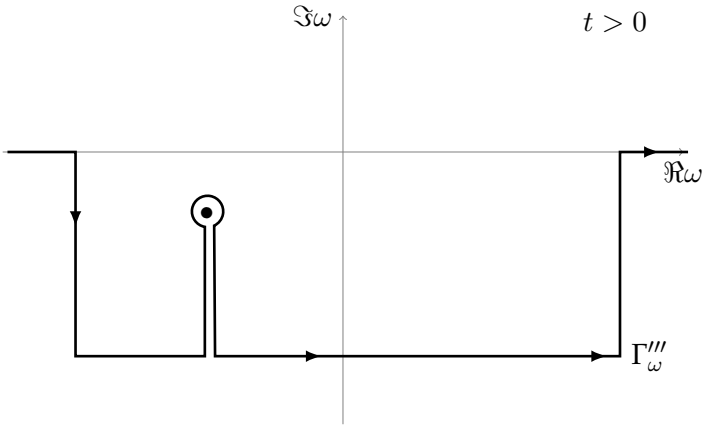


Figure 3.3: An integration contour for $t > 0$.

the path so that it does not cross any singularity. We enumerate the three different possibilities given by the sign of $\Im\omega$.

1. **Positive imaginary part**

   If we have $\Im\omega > 0$, all the singularities of $\epsilon(k, \omega)$ obviously lie above the real axis, which can be taken as integration path. The situation is depicted in 3.4

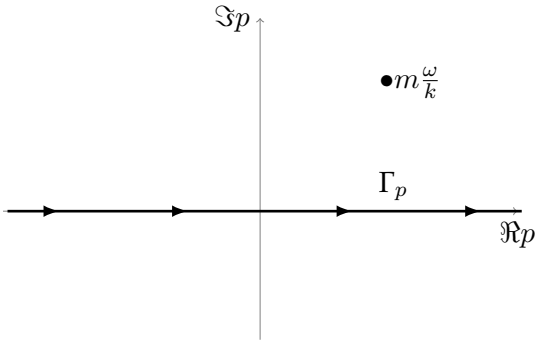


Figure 3.4: Integration path in the case $\Im\omega > 0$.

We find that

$$\boxed{\epsilon(k,\omega) = 1 - g\int_{-\infty}^{+\infty} dp\, \frac{\partial f_0}{\partial p}(p)\left(\frac{p}{m} - \frac{\omega}{k}\right)^{-1}.} \qquad (3.12)$$

2. **Null imaginary part**

   Now, we have a simple pole on the real axis. Hence, the path must be deformed so that it passes below the singularity, as in 3.5.

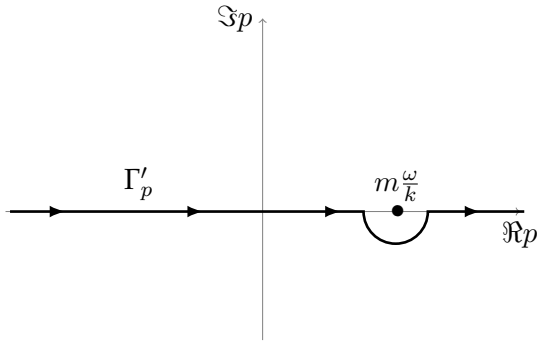


Figure 3.5: Integration path in complex-$p$ plane, for the case $\Im\omega = 0$.

We get

$$\boxed{\epsilon(k,\omega) = 1 - g\mathcal{P}\int dp\,\frac{\partial f_0}{\partial p}(p)\left(\frac{p}{m} - \frac{\omega}{k}\right)^{-1} - i\pi g\frac{\partial f_0}{\partial p}\bigg|_{p=\frac{m\omega}{k}}},$$

(3.13)

where the $\mathcal{P}$ denotes the Cauchy principal value.

3. **Negative imaginary part**

   If we have understood the pattern we are following, it's easy to convince ourselves both concerning the form of the integration path 3.6 and the form of the dielectric function.

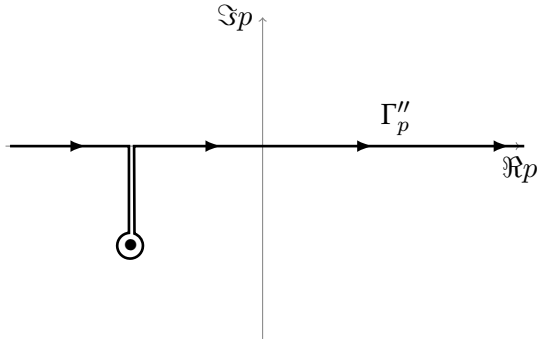


Figure 3.6: Here we consider the integration path for the case $\Im\omega < 0$.

We have then:

$$\boxed{\epsilon(k,\omega) = 1 - g\int dp\,\frac{\partial f_0}{\partial p}(p)\left(\frac{p}{m} - \frac{\omega}{k}\right)^{-1} - 2i\pi g\frac{\partial f_0}{\partial p}\bigg|_{p=\frac{m\omega}{k}}},$$

(3.14)

We underline the fact that in (3.14) the contribution from the residue computed in the pole $\omega = \frac{m\omega}{k}$ is double compared to the one in (3.13): the reason lies in the different paths we have considered in these two situation. In the figure 3.5 we have only a semi-circle indenation, while in 3.6 we have a complete tour around the pole.

So, in order to conclude this topic, we remark the Landau prescription on dielectric function definition: the integration contour has to avoid the pole in $\frac{\omega}{k}$ passing below it, and this implies the three possible situation we have just enumerated above. We can label the Landaur contour with $\mathcal{L}$ and , and one can tautly sum up that three cases in the formula

$$1 - g \int_{\mathcal{L}} dp \, \frac{\partial f_0}{\partial p} \left( \frac{p}{m} - \frac{\omega}{k} \right)^{-1} = 0 \, . \tag{3.15}$$

From this equation we can extract the wave-mode frequencies, but its usefulness does not stop here, as we are going to see.

## 3.2 LINEAR STABILITY ANALYSIS OF BOLTZMANN-VLASOV EQUATION

### 3.2.1 *Stability of a single-peaked distribution*

This short section is devoted to the proof of a really simple theorem about the stability of single-peaked distribution functions.

As previously said, a spatially uniform and stationary distribution $f_0(p)$ solves the (2.26); obviously, the existence of a stationary state does not imply, in any way, its stability. We also remark that a stationary state of the 1D Boltzmann-Vlasov equation (2.26) is a quasi-stationary one for the original 3D equation.

This theorem states that *if the spatially uniform and time-independent distribution function $f_0(p)$ has only one maximum and $g > 0$, then the system can't sustain unstable wave modes.*

The proof is standard [30, 37, 53], and it proceeds by contradiction; hence, we assume that a single-peaked distribution can display unstable wave modes.

By recalling the formal dispersion relation

$$1 - g \int dp \, \frac{\partial f_0}{\partial p}(p) \left( \frac{p}{m} - \frac{\omega}{k} \right)^{-1} = 0 \, ,$$

we write the following equation

$$1 - g \int dp \, \frac{\partial f_0}{\partial p}(p) \frac{\left( \frac{p}{m} - \frac{\omega_R}{k} + i\frac{\omega_I}{k} \right)}{\left( \frac{p}{m} - \frac{\omega_R}{k} \right) + \left( \frac{\omega_I}{k} \right)^2} = 0 \, . \tag{3.16}$$

This complex equation can be splitted as follows

$$\begin{cases} \Re\epsilon(k,\omega) \equiv 1 - g \int dp \, \dfrac{\partial f_0}{\partial p}(p) \dfrac{\frac{p}{m} - \frac{\omega_R}{k}}{\left(\frac{p}{m} - \frac{\omega_R}{k}\right) + \left(\frac{\omega_I}{k}\right)^2} = 0 \\ \Im\epsilon(k,\omega) \equiv -g \int dp \, \dfrac{\partial f_0}{\partial p}(p) \dfrac{\frac{\omega_I}{k}}{\left(\frac{p}{m} - \frac{\omega_R}{k}\right) + \left(\frac{\omega_I}{k}\right)^2} = 0 \, . \end{cases} \tag{3.17}$$

At this point we make of use of our hypothesis on $f_0(p)$. If we have supposed there is a maximum in $p_0$, then

$$\left(\frac{p_0}{m} - \frac{p}{m}\right) \frac{\partial f_0}{\partial p}(p) \geq 0 \, .$$

Moreover, since $\Re\epsilon(k,\omega) = 0$ and $\Im\epsilon(k\omega) = 0$, we can write

$$\Re\epsilon(k,\omega) - \frac{1}{\omega_I}\left(k\frac{p_0}{m} - \omega_R\right)\Im\epsilon(k,\omega) = 0 \, .$$

By replacing $\Re\epsilon$ and $\Im\epsilon$ with the definitions in (3.17), we finally reach the

$$1 + g \int dp \, \frac{\partial f_0}{\partial p}(p) \frac{\left(\frac{p_0}{m} - \frac{p}{m}\right)}{\left(\frac{p}{m} - \frac{\omega_R}{k}\right)^2 + \left(\frac{\omega_I}{k}\right)^2} = 0 \, . \tag{3.18}$$

This is absurd, indeed the integrand function in (3.18) is everywhere positive, so the equation can't be satisfied and the initial assumption is fake.

It's interesting to underline that this stability theorem is really similar to the Gardner one in plasma physics [19]: despite a different dispersion relation, the proof proceeds in the same way and we find another remarkable connection between the plasma dynamics and the ultracold atomic gases one.

### 3.2.2 *Weakly damped waves: the Landau damping in atomic gases*

Another typical situation in plasma physics which can be generalized to ultracold atomic gases is that of weakly damped waves (or, eventually, weakly unstable modes), i.e situation where $|\Im\omega| \ll \Re\omega$. From a strictly physical point of view, this means that the waves amplitude varies very little in a time period. Here, we are going to consider weakly damped waves in an ultracold atomic gas.

We have to keep in mind the warning given by the Landau prescription: $\omega$ is a complex variable, then, in order to lighten the notation, in the following we will write $\Re\omega = \omega_R$ and $\Im\omega = \omega_I$.

Because of the assumption $|\omega_I| \ll |\omega_R|$, a Taylor expansion of the dielectric function around the real value $\omega_R$ leads to

$$\begin{aligned} \epsilon(k,\omega) &= \epsilon(k,\omega_R + i\omega_I) \\ &\simeq \epsilon(k,\omega_R) + i\omega_I \frac{\partial\epsilon}{\partial\omega_R}(k,\omega_R) \ . \end{aligned} \tag{3.19}$$

Since we are considering a small imaginary part frequency, we have to make use of the dispersion relation in the form (3.13), namely

$$\epsilon(k,\omega) = \underbrace{1 - g\mathcal{P}\int dp\ \frac{\partial f_0}{\partial p}(p)\left(\frac{p}{m} - \frac{\omega}{k}\right)^{-1}}_{=\epsilon_R(k,\omega)} \underbrace{- i\pi g \frac{\partial f_0}{\partial p}\bigg|_{p\frac{m\omega}{k}}}_{=\epsilon_I(k,\omega)} \ .$$

Placing it in (3.19), we obtain

$$\begin{aligned} \epsilon(k,\omega) &\simeq \epsilon_R(k,\omega_R) + i\epsilon_I(k,\omega_R) + \omega_I\left[i\frac{\partial\epsilon_R}{\partial\omega_R}(k,\omega_R) - \frac{\partial\epsilon_I}{\partial\omega_R}(k,\omega_R)\right] \\ &= 1 - g\mathcal{P}\int dp\ \frac{\frac{\partial f_0}{\partial p}(p)}{\left(\frac{p}{m} - \frac{\omega_R}{k}\right)} - i\pi g\frac{\partial f_0}{\partial p}(p)\bigg|_{p=\frac{m\omega_R}{k}} + \\ &\quad + \ i\omega_I\frac{\partial}{\partial\omega_R}\left[-g\mathcal{P}\int dp\ \frac{\frac{\partial f_0}{\partial p}(p)}{\left(\frac{p}{m} - \frac{\omega_R}{k}\right)}\right] + \omega_I\frac{\partial}{\partial\omega_R}\left[\pi g\frac{\partial f_0}{\partial p}\bigg|_{p=\frac{m\omega_R}{k}}\right] . \end{aligned} \tag{3.20}$$

where, in the first line, we make use of the Cauchy-Riemann conditions on $\epsilon(k,\omega)$. By a comparison of the real part of L.H.S and R.H.S of (3.20), we have

$$1 - g\mathcal{P}\int dp\ \frac{\frac{\partial f_0}{\partial p}(p)}{\left(\frac{p}{m} - \frac{\omega_R + i\omega_I}{k}\right)} = 1 - g\mathcal{P}\int dp\ \frac{\frac{\partial f_0}{\partial p}(p)}{\left(\frac{p}{m} - \frac{\omega_R}{k}\right)} + \\ +\ \omega_I \frac{\partial}{\partial \omega_R}\left[\pi g \frac{\partial f_0}{\partial p}\bigg|_{p=\frac{m\omega_R}{k}}\right]. \tag{3.21}$$

If $\frac{\omega_I}{\omega_R} \ll 1$, then the last term of(3.21) vanishes.

We can equate the real and imaginary part to zero. Focusing on the real part, we obtain

$$\boxed{1 - g\mathcal{P}\int dp\ \frac{\partial f_0}{\partial p}(p)\left(\frac{p}{m} - \frac{\omega_R}{k}\right)^{-1} = 0\ .} \tag{3.22}$$

Then, coming back to (3.20), we finally reach the following equation for the imaginary part of $\omega$:

$$\boxed{\omega_I = -\frac{\pi \frac{\partial f_0}{\partial p}\left(\frac{m\omega_R}{k}\right)}{\frac{\partial}{\partial \omega_R}\left[\mathcal{P}\int dp\ \frac{\partial f_0}{\partial p}(p)\left(\frac{p}{m} - \frac{\omega}{k}\right)^{-1}\right]\Big|_{\omega_I=0}}\ .} \tag{3.23}$$

At this point, we consider the Maxwell-Boltzmann distribution function

$$f_0(p) = \frac{A}{\sqrt{\pi}p_T} e^{-\frac{p^2}{p_T^2}} \tag{3.24}$$

where $A = \frac{2\pi\hbar N}{L}$ is the normalization constant, $L$ is the length of axial domain and $p_T$ the momentum dispersion. In order to compute $\omega_R$ we use the (3.22), with an extra assumption: we assume that the phase speed $\frac{\omega_R}{k}$ is much larger than the typical thermal one, i.e $\frac{p}{m} \ll \frac{\omega_R}{k}$. Hence, by means of a binomial expansion, we write:

$$-\frac{1}{\frac{p}{m} - \frac{\omega_R}{k}} = \frac{1}{\frac{\omega_R}{k}} + \frac{1}{\left(\frac{\omega_R}{k}\right)^2}\frac{p}{m} + \frac{1}{\left(\frac{\omega_R}{k}\right)^3}\left(\frac{p}{m}\right)^2 + \cdots\ . \tag{3.25}$$

It can be noticed that (3.24) is an even function, so its derivative is an odd function. Hence, we understand that only the odd powers of $\frac{p}{m}$ will contribue to this computation; namely, after the setting $\Re\epsilon(k, \omega_R + i\omega_I) = 0$, we have

$$1 = g\int dp \left(\frac{2A}{\sqrt{\pi}p_T^3} p e^{-\frac{p^2}{p_T^2}}\right)\left[\left(\frac{k}{\omega_R}\right)^2\frac{p}{m} + \left(\frac{k}{\omega_R}\right)\left(\frac{p}{m}\right) + \cdots\right]. \tag{3.26}$$

An easy calculation of these gaussian integrals leads us to

$$\begin{aligned} 1 &= \frac{gA}{m}\left(\frac{k}{\omega}\right)^2\left[1+\frac{3p_T^2}{2m^2}\left(\frac{k}{\omega}\right)^2\right]+\cdots \\ \omega_R^2 &= \frac{gA}{m}\left[1+\frac{3p_T^2}{2m^2}\left(\frac{k}{\omega}\right)^2\right]k^2+\cdots \\ &\simeq \frac{gA}{m}\left[1+\frac{3p_T^2}{2m^2}\frac{m}{gA}\right]k^2 \ . \end{aligned} \tag{3.27}$$

In going to the last line we have supposed $\mathbf{p_T} \ll \sqrt{\mathbf{mgA}}$. We finally find:

$$\omega_R \simeq \sqrt{\frac{gA}{m}\left(1+\frac{3p_T^2}{mgA}\right)}k \ . \tag{3.28}$$

In order to compute also $\omega_I$, we simply need to use (3.23), an expression in which we know everything. So, we arrive at the following:

$$\omega_I = -\sqrt{\pi}\frac{mg^2A^2}{p_T^3}e^{-3/2}e^{-\frac{mgA}{p_T^2}} \ . \tag{3.29}$$

We immediately observe that the negative $\omega_I$ corresponds to perturbation damping: we rediscover the Landau damping in the ultracold atomic gases. This phenomenon occuring in collisionless systems is a well-known and understood fact in plasma physics [32]; it is considered the most striking example of particle-wave interaction phenomena. Indeed, damping in a collisionless system can appear quite misleading: in this context, damping implies that the wave (in this case a *matter wave*) loses a part of its energy. In order to understand the mechanism [12] of this energy loss, we consider a single-peaked distribution $f_0(p_x)$, where $p_x$ equates the phase velocity of matter wave, i.e. $\frac{\omega}{k}$. Moreover, let us suppose that the first derivative of $f_0(p_x)$ is negative; then, there are more particles which move slightly slower than the wave compared to the ones moving slightly faster.

By interpreting the matter wave as a potential propagating with a $\frac{\omega}{k}$ velocity, its trough manages to capture particles moving with a slightly lower velocity. Hence these particles continue to move with the wave. At this point, we understand that particles with a speed slightly lower than the matter wave one are accelerated, while particles slightly faster than the wave are slowed down. If the first derivative of $f_0(p_x)$ is negative, we have more accelerated particles than slowed down ones; this means that the matter wave has transferred a net amount of its energy to the atomic gas.

### 3.2.3 *Stability of a double-peaked distribution*

By means of the Gardner-like theorem about single-peaked distribution, we are not able to say anything about more complex si-

tuation; for example, we can consider the problem of two-stream instability, a common topic in plasma physics. So, we basically want to study the linear stability of a double-peaked distribution, which is the minimum request for displaying this kind of instability. We will answer to this problem by means of the Nyquist theorem [30, 37, 53]: it will make us reach a criterion of stability similar to Penrose's one [41].

In the following, we are going to assume a uniform initial distribution, i.e $f_0 = f_0(p)$; hence, in this section, we study the (linear) stability of a uniform equilibrium. Non-spatially uniform distributions need more subtle techniques.

We can understand that the key-point lies in the properties of the dielectric function $\epsilon(k, \omega)$ (3.10). More precisely, if we want to show the existence of unstable modes, we have to search for the zeros of dielectric function in upper half-plane of complex frequencies.

As underlined by [37], the number of $\epsilon(k, \omega)$ zeros can be computed by the following integral

$$N = \frac{1}{2\pi i} \int_{\Gamma_\omega} d\omega \, \frac{1}{\epsilon(k,\omega)} \frac{\partial \epsilon}{\partial \omega}(k, \omega) \, , \tag{3.30}$$

where the integration contour $\Gamma_\omega$ must be intended as counterclockwise and we assume that $\frac{\partial \epsilon}{\partial \omega}$ has no poles inside the enclosed area of the complex frequency plane. This is a possible statement of the Nyquist theorem. In proximity of a simple zero the dielectric function can be expanded in a Taylor series [37], i.e

$$\epsilon(k, \omega) = 0 + \left. \frac{\partial \epsilon}{\partial \omega} \right|_{\omega_0} (\omega - \omega_0) + \cdots \tag{3.31}$$

but, at the same time, it holds

$$\frac{\partial \epsilon}{\partial \omega}(k, \omega) = \left. \frac{\partial \epsilon}{\partial \omega} \right|_{\omega_0} + \left. \frac{\partial^2 \epsilon}{\partial \omega^2} \right|_{\omega_0} (\omega - \omega_0) + \cdots \tag{3.32}$$

Therefore, at the first order, one can write

$$\frac{1}{\epsilon(k,\omega)} \, \frac{\partial \epsilon}{\partial \omega} \simeq \frac{1}{\omega - \omega_0} \, . \tag{3.33}$$

By means of the residues theorem, we see that the integration of (3.33) leads us to

$$\int_{\Gamma_\omega} d\omega \, \frac{1}{\omega - \omega_0} = 2\pi i \sum_{\omega_n} \mathrm{Res}[f(\omega_n)] = 2\pi i \, .$$

Obviously, this is not a rigorous proof of (3.30), but only a heuristic way to show its reason. If the function displays a finite number of poles within the contour $\Gamma_\omega$, then [30] we have to exploit a more

general equation than the (3.30). To be torough, we report it as it appears in [30]:

$$\int_{\Gamma_\omega} d\omega \, \frac{1}{\epsilon(k,\omega)} \, \frac{\partial \epsilon}{\partial \omega}(k,\omega) = 2\pi i (N_0 - N_P) \, , \tag{3.34}$$

where $N_0$ is the number of zeros and $N_P$ simple poles one.

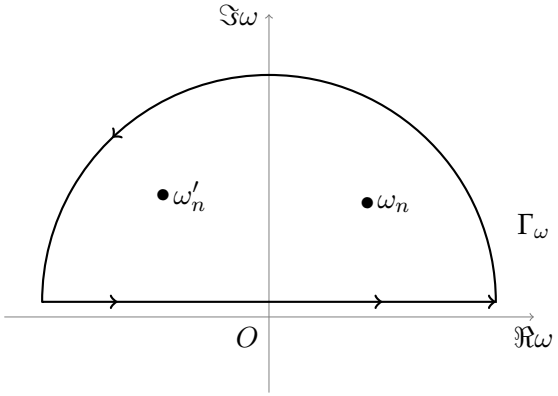


Figure 3.7: This is the proper integration contour which enlightens the presence of unstable mode $\omega_n$ and $\omega_n'$.

As noted by [37], if we close the contour on frequency plane, we can map it into a contour on the complex plane for the dielectric function. Indeed, we see that

$$N = \frac{1}{2\pi i} \int_{\Gamma_\omega} d\omega \, \frac{1}{\epsilon} \frac{\partial \epsilon}{\partial \omega}(k,\omega) = \frac{1}{2\pi i} \int_{\Gamma_\epsilon} d\epsilon \, \frac{1}{\epsilon(k,\omega)} \, , \tag{3.35}$$

where $\Gamma_\epsilon$ is the contour we get by evaluating $\epsilon(k,\omega)$ at every point of $\Gamma_\omega$ on the relative plane. We understand that, if we move from the contour in 3.7 to $\epsilon$-plane one, we will find unstable modes only if $\Gamma_\epsilon$ encloses the origin. A possible scenario is sketched in the figure 3.8.

We can specify some features of $\Gamma_\epsilon$: we split it into two parts, one given by the $\infty$-radius semicircle in the upper half-plane, the other one along the real axis. For the first part we have to make use of the dispersion relation in the form (3.12), while for second one of (3.13). If we suppose a well-behaved initial distribution function, from the (3.12) simply vanishes, we get the

$$\lim_{|\omega| \to +\infty} \epsilon(k,\omega) = 1 \, . \tag{3.36}$$

For the real axis, we consider again the (3.13), but the property of regularity of $f_0(p)$ implies that $\frac{\partial f_0}{\partial p}(p)$ as $p \to \infty$; then, we come back to (3.36), i.e $\epsilon(k,\infty) = 1$. We can say something more: if we keep considering a well-behaved distribution function, we see that for $\omega \to +\infty$ we have $\Im > 0$, so we approach $\epsilon = 1$ from the

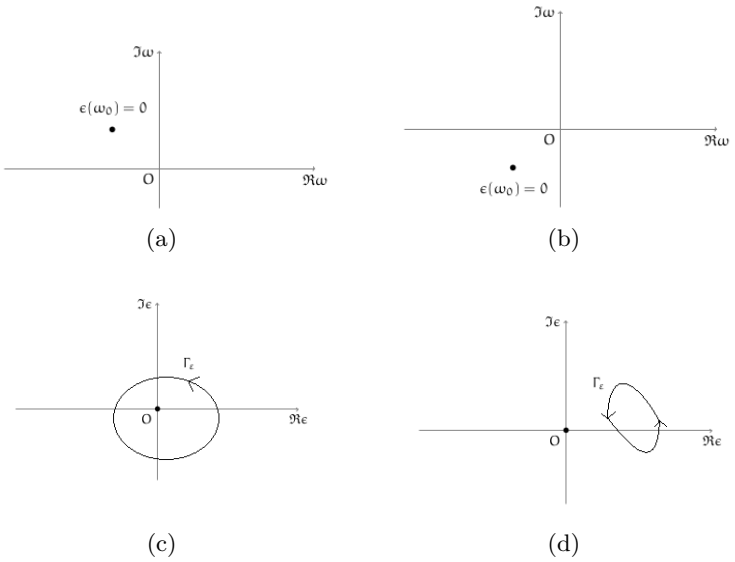


Figure 3.8: In this figure, we give two examples of mapping the contour from frequency to epsilon complex plane. In the first case we have one zero of dielectric function on upper half-plane, so $\Gamma_\epsilon$ embraces the origin once. On the right, we have a zero with negative imaginary part, so the contour on $\epsilon$-plane won't embrace the origini. If we have more than one zero we will obviously have more complicated configuration.

upper half-plane, while or $\omega \to -\infty$ we have the contrary. The situation is depicted in the 3.9.

Now, we realise what is the simplest contour that implies instability. Indeed, we can close $\Gamma_\epsilon$ in the way showed in 3.10. This choice immediately exhibits a problem: due to the sense of $\Gamma_\epsilon$, we find

$$N = \frac{1}{2\pi i} \int_{\Gamma_\epsilon} d\epsilon \, \frac{1}{\epsilon(k,\omega)} = -1 \, ,$$

which is a clearly nonsense result, so the system which leads us to that contour has to be stable.

We see that the system stability is connected to the times the contour $\Gamma_\epsilon$ crosses the real axis of complex $\epsilon$-plane; as we have seen above, one crossing implies stability, therefore we have to require at least three crossings. In order to find how many real axis crossings occur in our system, we simple have to solve the equation $\Im\epsilon = 0$.

It's a remarkable fact that this simple reasoning has led us to an alternative proof the Gardner's-like stability theorem. Indeed, when we consider the (3.13), a zero of the dielectric function imaginary part corresponds to a stationary point of the initial distribution $f_0(p)$. One real axis crossing corresponds to a single-peak

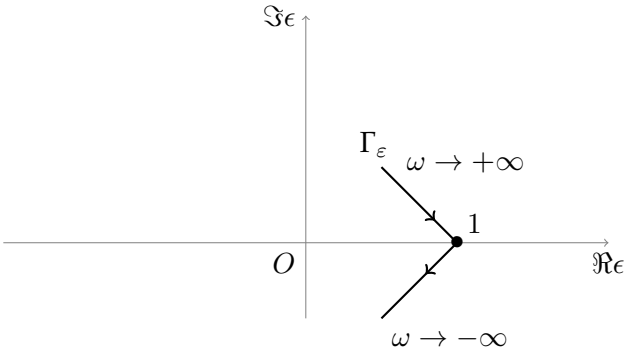


Figure 3.9: The integration contour $\Gamma_\epsilon$ when we approach $\epsilon(k, \infty)$.

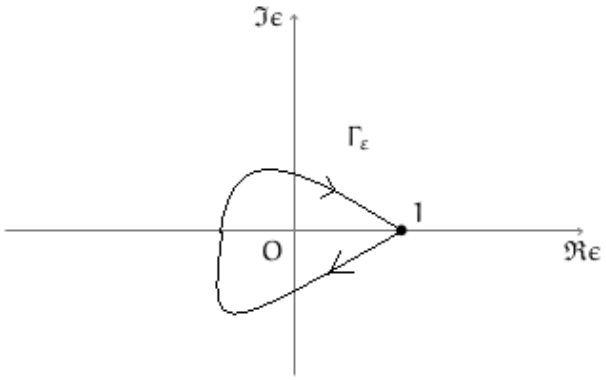


Figure 3.10: This is the simplest integration contour we can draw according to the prescription in 3.9. Unfortunately, it leads to a non-sense result.

distribution function, then, as we already know, the system is stable: there can't be unstable modes in this situation.

Nevertheless, in order to show if the system can display unstable modes, we have to go beyond a simple reasoning concerning how many times the integration contour crosses the real axis. In other words it is a necessary, but not sufficient, condition. We must not forget that the integration contour must embrace the origin of the $\epsilon$ plane. So, if we consider a double-peaked initial distribution function, with a minimum in $p_0$. The key-point consists in com-

puting the sign of $\Re\epsilon(k, p_0)$, where $p_0 = \frac{m\omega_0}{k}$. So, from the (3.13), we simply get the

$$\begin{aligned}\Re\epsilon\left(k, \omega = \frac{kp_0}{m}\right) &= 1 - g\mathcal{P}\int_{-\infty}^{+\infty} dp\ \frac{\partial f_0}{\partial p}(p)\left(\frac{p}{m} - \frac{p_0}{m}\right)^{-1} \\ &= 1 - gm\mathcal{P}\int_{-\infty}^{+\infty} dp\ \frac{1}{p - p_0}\left[\frac{\partial f_0}{\partial p}(p) - \frac{\partial f_0}{\partial p}(p_0)\right] \\ &= 1 - gm\mathcal{P}\int_{-\infty}^{+\infty} dp\ \frac{f_0(p) - f_0(p_0)}{(p - p_0)^2}\ ,\end{aligned} \tag{3.37}$$

where in going from the first line to the second one, we made use f $\frac{\partial f_0}{\partial p}(p_0) = 0$. From the second to the last one we integrated by parts.

The system is unstable only if the contour embraces the origin, so we have to require that $\Re\epsilon(k, p_0) < 0$, i.e. we get

$$\boxed{\mathcal{P}\int_{-\infty}^{+\infty} dp \frac{f_0(p) - f_0(p_0)}{(p - p_0)^2} > \frac{1}{gm}\ .} \tag{3.38}$$

Finally, it's surely interesting a remark about this *Penrose criterion* [37, 41, 53]: the (3.38) is a necessary and sufficient condition for ultracold atomic gases linear instability.

We think that another interesting is about its *experimental value*. In classical plasmatheory , the appearance of this criterion is slightly different: the R.H.S of (3.38) is zero. This implies that, if the initial distribution function has a hole, i.e an area where's there no particle[1], then (3.38) assures the existence of unstable modes. In the same way[53], the criterion we have found can relate the deepness of the minimum (compared to the maximum position) and the occurring of the two stream instability.

We want to conclude this chapter giving a simple example where we are able to analytically specify specify the range of instability wave modes. The situation we are going to analyze is given by a double Dirac $\delta$-function distribution, i.e:

$$f_0(p) = \frac{A}{2}\Big[\delta(p - p_0) + \delta(p + p_0)\Big] \tag{3.39}$$

where $A$ can be intended as a normalization factor.

Now, exploiting the same ideas we've followed in order to get the (3.38), we start by searching the zeros of dielectric function with $\Im\omega > 0$. The first step consists in solving the

$$1 - g\mathcal{P}\int_{-\infty}^{+\infty} dp\ \frac{\partial f_0}{\partial p}(p)\left(\frac{p}{m} - \frac{\omega}{k}\right) = 0\ . \tag{3.40}$$

1 We have to remember that we are talking of homogeneous distribution, so we are talking of a no-particle area in the momentum space.

Recalling the derivation properties of the $\delta$-function, i.e.

$$\int_{\mathbb{R}} dx\delta'(x)\Phi(x) = -\int_{\mathbb{R}} dx\delta(x)\Phi'(x)$$

and renaming $c = \frac{\omega}{k}$, we get the equation

$$c^4 - \left(2\frac{p_0^2}{m^2} + \frac{gA}{m}\right)c^2 + \left(\frac{p_0}{m}\right)^4 - \frac{gA}{m}\left(\frac{p_0}{m}\right) = 0\,. \tag{3.41}$$

By solving it, we get

$$c_\pm^2 = \frac{1}{m^2}\left[p_0^2 + \frac{1}{2}mgA \pm \frac{1}{2}\sqrt{mgAp_0^2 + m^2g^2A^2}\right]\,. \tag{3.42}$$

For $c_+^2$ the R.H.S is obviously always positive, then, in this case, we have stability.

There's something different when we consider $c_-^2$. The dispersion relations are given by $\omega = \pm\sqrt{c_-^2}k$, but we immediately realize that $c_-^2$ can be a negative value. In this case, we are certain there a zero of $\epsilon(k,\omega)$ with $\Im\omega > 0$ (moreover, the frequency, in this case, is purely imaginary).

Therefore, the stability criterion for the double Dirac $\delta$ distribution is $c_-^2 < 0$, then

$$p_0^2 + \frac{1}{2}mgA - \frac{1}{2}\sqrt{8mgAp_0^2 + m^2g^2A^2} < 0 \tag{3.43}$$

which leads us to a precise range of unstable wave modes:

$$\boxed{-\sqrt{mgA} < p_0 < \sqrt{mgA}\,.} \tag{3.44}$$

This is an ideal situation, but also a simple way to enlighten our approach to linear stability analysis.

# 4

# A NUMERICAL APPROACH TOWARDS THE BOLTZMANN-VLASOV EQUATION

## 4.1 THE SCHEME: SEMI-LAGRANGIAN APPROACH

In the previous chapter we have described an analysis concerning the 1D collisionless Boltzmann-Vlasov equation applied to ultracold and dilute atomic gases. We have shown that, moving from the classical kinetic theory framework, one can apply to this kind of system the formalism of Landau-Vlasov plasma theory; it has been a fruitful approach, since we have pulled out a set of analytical results with a solid theoretical background.

For example, from a Landau-like analysis of the linearized Boltzmann -Vlasov equation, we have derived a theorem about the stability of single-humped distribution functions and the occurring of a typical phenomenon in collisionless system, which displays a particle-wave interaction: the Landau damping, with a precise analytical result, the equations (3.29) and (3.28).

We recall that, especially for (3.28), it must hold a precise condition of effectiveness: this involves the momentum dispersion of Maxwell-Boltzmann initial distribution

$$\boxed{p_T \ll \sqrt{mgA}\,.} \tag{4.1}$$

In this last chapter, we aim to search for a numerical confirmation of all the results we have achieved till now.

The equation we want to solve solve numerically is

$$\boxed{\frac{\partial f}{\partial t} + \frac{p}{m}\frac{\partial f}{\partial x} + G(x,t)\frac{\partial f}{\partial p} = 0} \tag{4.2}$$

where, in order to lighten up the notation, we have defined

$$G(x,t) = -g \int dp\, \frac{\partial f}{\partial x}(x,p,t)\,.$$

In order to implement a proper code, we borrow the core idea from the plasma physics methods, as in the previous chapter. We want to study the phase-space dynamics of the distribution function; therefore we begin by considering that numerical methods which imply the building up of a phase-space grid.

Another key-point consists in the so-called *operator splitting procedure* [47, 51]. It consists in moving from the (4.2) and splitting the differential operator

$$\frac{p}{m}\frac{\partial}{\partial x} + G(x,t)\frac{\partial}{\partial p}$$

into two different parts: one concerning the motion on position space, the other on the momentum one. Hence, from a single equation (4.2), we obtain a system composed by two partial differential equations, i.e.

$$\begin{cases} \dfrac{\partial f}{\partial t} + \dfrac{p}{m}\dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial t} + G(x,t)\dfrac{\partial f}{\partial p} = 0 \end{cases} \tag{4.3}$$

This method requires to solve not the original (4.2) for a whole time step $\Delta t$, but to solve separately the equations (4.3), each one for a whole time step, or for another different increment we will specify in the following.

This is the heart of the so called *semi-lagrangian method* for solving the Vlasov equation. Its first and effective implementation was made by Cheng and Knorr [11] on the 1D Vlasov-Poisson system, which is really similar to our (4.2).

If it's true that, by means of operator splitting method, we now have to handle a PDE-system, it's equally true that the splitted equations are much simpler than the starting one. We give a brief comment about the splitting error, following [51].

Essentially, we assume that our equation has the form

$$\frac{\partial f}{\partial t} = (A+B)f$$

where $A$ and $B$ are two differential operator. After a time step, we will have the formal solution given by

$$f(t+\Delta t) = e^{(A+B)\Delta t} f(t) \ .$$

If we write down each formal solution of splitted equation, taken separetely, we have $f(t+\Delta t) = e^{A\Delta t} f(t)$ and $f(t+\Delta t) = e^{B\Delta t} f(t)$.

The standard splitting method operator [51] consists in solving on one time step first $\partial_t f = Af$, then $\partial_t f = Bf$. On the whole time step, one gets the

$$\tilde{f}(t+\Delta t) = e^{B\Delta t} e^{A\Delta t} f(t) \ . \tag{4.4}$$

If the operators commute, the splitting method is exact. If they don't commute (this is our case), we can reduce the splitting error by three different steps: first, an integration on space (or $A$) of a half-time step, then a whole time step integration on momentum

(or $B$) and finally we repeat the first point. This is the so-called *Strang splitting method* and corresponds to the formal solution:

$$\tilde{f}(t+\Delta t) = e^{\frac{A}{2}\Delta t} e^{B\Delta t} e^{\frac{A}{2}\Delta t} f(t) \; . \tag{4.5}$$

The advantage of the *Strang splitting method* lies in a proposition whose proof can be found in [51]. It demonstrates that the standard splitting method is of the first order in time, while the Strang splitting method is of order two in time.

Another crucial point concerns the solution of both equations in (4.3). One of the strong point of the splitting operator method is that we have to handle two flux-conservative equations; this feature really simplifies the algorithm development.

Indeed, by following [52], we consider a more general problem given by the equation

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{X}} \cdot \left[ U(\mathbf{X}, t) \right] = 0 \; . \tag{4.6}$$

We introduce the characteristic curves of the equation above, which are the solutions of the following dynamical system

$$\frac{d\mathbf{X}}{dt} = U(\mathbf{X}, t) \tag{4.7}$$

. We denote the characteristic curve by $\mathbf{X} = \mathbf{X}(t, \mathbf{x}, s)$: it must be intended as the solution at time $t$, whose value is $x$ at time $s$. It's easy to realize that $f$ in (4.6) is conserved along these curves. Indeed, we have

$$\begin{aligned} \frac{d}{dt}\left[ f(\mathbf{X}(t), t \right] &= \frac{\partial f}{\partial t} + \frac{d\mathbf{X}}{dt} \cdot \nabla_{\mathbf{X}} f \\ &= \frac{\partial f}{\partial t} + U(\mathbf{x}(t), t) \cdot \nabla_{\mathbf{X}} f \\ &= 0 \; . \end{aligned} \tag{4.8}$$

Hence, as made, for example, in [52], we can move along this curve, namely

$$\boxed{f(\mathbf{X}(t, \mathbf{x}\; s), t) = f(\mathbf{X}(s, \mathbf{x}\; s), t) = f(\mathbf{x}, t) \; .} \tag{4.9}$$

This is the crucial property to be exploited in the semi-lagrangian algorithm; indeed, we can now define a set of grid points $\{\mathbf{x}_m\}_{m=1,\ldots,N}$ and, given the value of the function at any grid point, the evolution in time can be computed by means of

$$f(\mathbf{x}_m, t_n + \Delta t) = f(\mathbf{X}(t_n - \Delta t; \mathbf{x}_m, t_n + \Delta t), t_n - \Delta t) \; . \tag{4.10}$$

For every mesh point $\mathbf{x}_m$, the semilagrangian method consists in two steps:

- Compute the initial point of the curve ending on $\mathbf{x}_m$, i.e, following the notation in [52], the

$$\mathbf{X}(t_m - \Delta t; \mathbf{x}_m, t_n + \Delta t) \ .$$

- Then, we compute the evolved value of function $f$

$$f(\mathbf{X}(t_n - \Delta t; \mathbf{x}_m, t_n + \Delta t), t_n - \Delta t)$$

by means of a certain interpolation method. Here, we make use of the cubic spline functions[11].

We have then split the integration of (4.2) in two different problems: the first one about the solution of (4.7), the second one, on the other side, consists in an interpolation of the new values of the distribution function.

The (4.2) can be numerically integrated by considering a phase space grid, namely $f_{ij} = f(x_j, p_i)$ and then by proceeding as follows:

1. **Half time step evolution on position space**

   We solve the first one of (4.3):

$$\begin{aligned} f^*_{ij} = f^*(x_j, p_i) &= f^n\left(x_j - p_i \frac{\Delta t}{2}, p_i\right) \\ &= f^n(x_j - \delta_i \Delta x, v_i) \ , \end{aligned} \tag{4.11}$$

   where $\delta_i = p_i \frac{\Delta t}{2\Delta x}$

2. **A whole time step evolution on momentum space**

   The initial condition of this evolution is given by $f^*$ we have computed in 1:

$$\begin{aligned} f^{**}_{ij} &= f^{**}(x_j, p_i) = f^*\left(x_j, p_i - G^*_j \Delta t\right) \\ &= f^*(x_j, p_i - \delta_j \Delta p) \ , \end{aligned} \tag{4.12}$$

   where $\delta_j = G^*_j \frac{\Delta t}{\delta p}$ and $G^*_j = -g \frac{\partial}{\partial x} \int d\tilde{p} f^* \Big|_{x=x_j}$.

3. **Another half time step evolution on position axis**

   We simply have to repeat the first point, taking as initial condition $f^{**}$.

In this way, we have reduced the integration of (4.2) to an interpolation problem.

Following the original work of Cheng and Knorr [11], we make use of the cubic spline interpolation method, with periodic boundary condition. For a detailed description of this method, which can be generalized for more difficult situation [18, 52], we refer to

[24, 47]. Here, we sketch the simplest version of this method.

Basically, we have to face the following problem: given the set of $f(x_j, v_i)$ on the grid points a certain instant, we want to compute the value

$$\hat{f}_i \equiv f(x_i + \delta\Delta x) \tag{4.13}$$

where, for clarity, we write down only one variable. Another remark is about the value of $\delta$, whose definition was given above; we require its value to be in the range $0 < \delta < 1$. This gives two conditions on the increments:

$$\begin{cases} |p_{\max}|\Delta t < 2\Delta x \\ \\ |G_{\max}|\Delta t < \Delta p \end{cases} \tag{4.14}$$

These conditions can be considered as a sort of *Courant stability condition* for our algorithm[47].

Now, if we suppose a constant spacing on the grid, we can write $\hat{f}_i$ by means of its original value and its first derivative, labelled by $s_i$:

$$\boxed{\begin{aligned} \hat{f}_i = \Big[s_i\delta(1-\delta)^2 - s_{i+1}(1-\delta)\delta^2\Big]\Delta x + f_i(1-\delta)^2(1+2\delta) + \\ + f_{i+1}\delta^2\Big(3-2\delta\Big)\,. \end{aligned}} \tag{4.15}$$

Since the constant spacing must be intended separately on position coordinate and on momentum one, $\Delta x$ and $\Delta p$ can be different.

The first derivative is computed by solving the following linear system[24]:

$$s_{i+1} + 4s_i + s_{i-1} = \frac{3}{\Delta x}\Big(f_{i+1} - f_{i-1}\Big)\,.$$

The last point concerns the grid boundary condition. We choose periodic boundary conditions on position, i.e

$$\boxed{\begin{aligned} f_N &= f_0, & s_N &= s_0 \\ f_{-1} &= f_{N-1}, & s_{-1} &= s_{N-1} \end{aligned}} \tag{4.16}$$

Concerning the momentum shifting, we force to zero the distribution function at the extreme value of $p$, as made by [11]:

$$\boxed{\begin{aligned} f_N = f_{-1} = 0 \\ s_N = s_{-1} = 0 \end{aligned}} \tag{4.17}$$

All the subscript index in (4.17) and (4.16) must be intended as a C-style array notation.

We now present the results about the numerical confirmation of what found in the previous chapter.

As initial condition we consider a Maxwell-Boltzmann distribution function:

$$f_0(p) = \frac{A}{\sqrt{\pi} p_T} \exp\left( - \frac{p^2}{p_T^2} \right), \tag{4.18}$$

where, as said in the previous chapter, $A$ is a normalization factor, while $p_T$ is the momentum dispersion. Indeed, if we take the limit $p_T \to 0$, we get a Dirac $\delta$-function[48], an extremely localized distribution function.

We impose the following normalization constraint

$$\int_\Omega dx dp \; f(x,p,t) = 1 \; , \tag{4.19}$$

where the integral subscript $\Omega$ intends an integration carried on the whole phase-space. In the case of $f_0(p)$, we simply find $1 = A \cdot \frac{1}{\text{Vol}}$ with [Vol] being the volume enclosing the gas; then, for a 1D system it is the length of enclosing box, labelled by $L$. For problems with higher dimensionality, we will find $[A] \simeq [L]^{-d}$. In order to better focus on the physical meaning of the results we present here, we don't plot the distribution function $f(x,p,t)$, but the particle density $n(x,t)$ (or $\rho(x,t)$):

$$n(x,t) = \int_{-p_{\max}}^{+p_{\max}} dp \; f(x,p,t) \; . \tag{4.20}$$

It is a well-known fact that there's no interesting dynamics given by the Maxwell-Boltzmann distribution, so the numerical integration of (4.2) with $f(x,p,t) = f_0(p)$ has to be intended as a sort of first and simple validation of the code.

We expect that there's no time evolution if we start from a Maxwell-Boltzmann distribution function. This is exactly what happens: the Maxwell-Boltzmann distribution is a stationary state of our system. In 4.1 we see the probability conservation during the dynamics.

We have to test our code on something more interesting, which can allows us to see the occurring of the Landau damping. Hence, we consider the perturbed initial distribution

$$f(x,p,t) = f_0(p) + \delta f(x,p,t) \; ,$$

where $f_0(p)$ is given by (4.18). If we perturb it with a sinusoidal wave, we will have the following initial condition

$$f(x,p,t=0) = f_0(p)\Big[1 + \alpha \cos(\bar{k} x)\Big] \; . \tag{4.21}$$

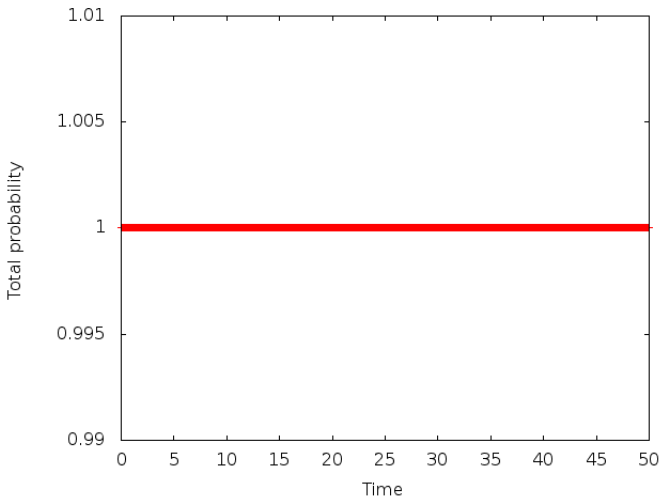


Figure 4.1: Here we see how the total probability (4.19) is preserved during the dynamics.

This choice is a classical one, followed by many authors working with the Vlasov-Poisson equations [11, 16, 17].

This initial distribution(4.21) is depicted in 4.2.

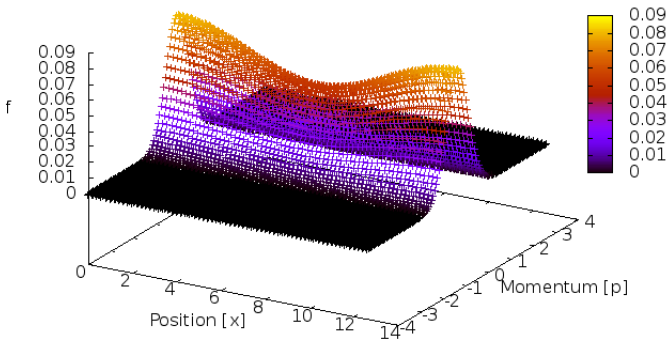


Figure 4.2: Graphic representation of (4.21). We take $\alpha = 0.2$, $\bar{k} = 0.5$, $g = 0.1$, $p_T = \sqrt{0.5}$

The integration of the 1D collisionless Boltzmann-Vlasov equation with this initial data, shows a clear damping of the initial perturbed distribution (see 4.3) . Hence, ultracold atomic gases in collisionless regime display the Landau damping, that can be faithfully described by our code.

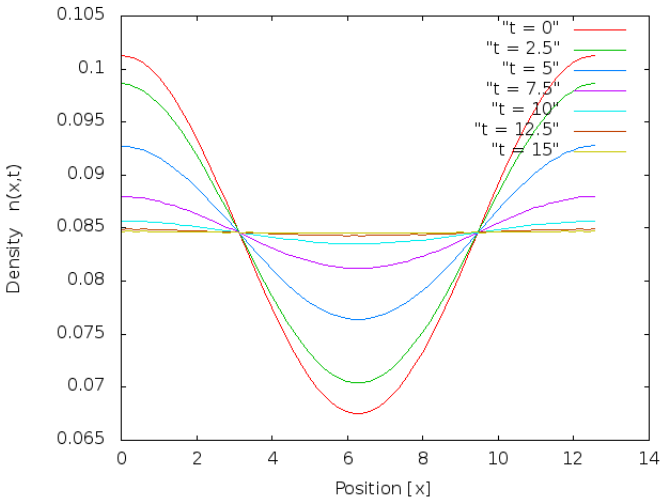


Figure 4.3: Strong evidence of Landau damping, with initial condition given by (4.21) in 4.2. We underline that the relaxation process leads the system exactly at the value given by the corresponding unperturbed Maxwellian distribution.

The next step we want to check concerns the effectiveness of the relation found in the previous chapter

$$\begin{cases} \omega_R = \sqrt{\dfrac{gA}{m}\left(1 + \dfrac{3p_T^2}{2mgA}\right)} \\ \\ \omega_I = -\sqrt{\pi} e^{-\frac{3}{2}} \dfrac{mg^2A^2}{p_T^3} \exp\left(-\dfrac{mgA}{p_T^2}\right) \end{cases}$$

with the constraint that $p_T < \sqrt{mgA}$.

These relations can be used in order to express an analytical form for the distribution function, i.e.

$$f(x,p,t) = f_0(p)\left[1 + 2\alpha e^{kc_I t}\cos(\bar{k}c_R t)\cos(\bar{k}x)\right]. \qquad (4.22)$$

If we plot the evolution of perturbation amplitude given by the numerical integration and the one given by (4.22) we observe a discrepancy 4.4.

We are confident that the problem lies in the computational method we have chosen. Indeed the semi-lagrangian algorithm, as underlined in [11, 16, 18], lacks of conservativity. The plot 4.5 clearly displays this problem: the total probability has to be a prime integer, but the dynamics does not conserve it. The non-conservation of total probability implies that the normalization coefficient varies with the time. By recalling the (4.22) and the form of $\omega_I$, we see that $A$ is the argument of an exponential function which, in turn, has to be the argument of another exponential.

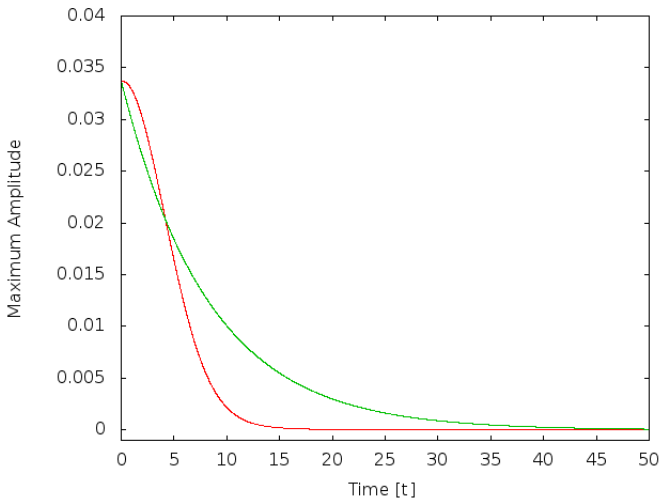


Figure 4.4: The red line is the numerical output, while the green one to the analytical result. The value of the parameter are the same of 4.2.

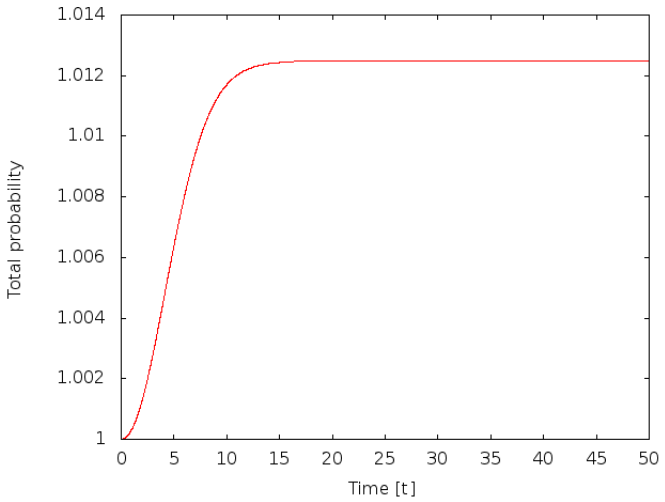


Figure 4.5: We clearly see a lack of conservation concerning the total probability. This is a known problem of the semi-lagrangian algorithm.

Hence, also a small error can propagate itself and causes a considerable final discrepancy.

# CONCLUSIONS

At the end of this thesis work, it is worthwhile to outline the achieved results and the limits we have to face and overtake in the future developments of this theoretical (and numerical) work on ultracold atomic gases.

The first point concerns the dimensional reduction of Boltzmann-Vlasov equation: it is a procedure which greatly simplify the study of Boltzmann-Vlasov equation. Thanks to this method, we have been able to give a good account of the dynamical properties by means of a simpler equation. Moreover, the main assumption is strictly related to a common physical situation: the switching on of a strong confining potential on a transverse plane. This situation is not only a theoretical abstraction; on the contrary, the experimental work on Bose gases during the last years [33] shows us that this situation is experimentally feasible.

Our approach can thus be validated by means of experimental techniques already used for condensed Bose gases. We remark the possibility of testing the effectiveness of the dimensional reduction method in configurations different from the one that we considered: for example, one could want to study a 2D problem by setting a confinement only along a single axis.

Another strong point of this thesis concerns the dynamical analysis of (linearized) Boltzmann-Vlasov equation. From a methodological point of view, we underlined how this analysis has been carried on by means of plasma theory techniques. More precisely, after a standard linearization of the Boltzmann-Vlasov equation, we had to face a series of subtleties that are the same ones of Landau-Vlasov plasma theory; a striking example is the connection between the pathologies of the dielectric function and the dispersion relation.

We think it's interesting that, by the same formalism, we can describe the (linear) dynamics of two systems, to all appearances really different; indeed, ultracold atomic gases can display certain stability properties which are well-known phenomena for plasmas. We have shown how single-peaked distribution functions can't sustain growing wave modes, so they are stable: basically, this is the analog of the Gardner's theorem for classical plasmas. Moreover the real interesting point is that we also proved the occurring of the classical wave-particle interaction phenomenon for collisionless system, the Landau damping.

Our hope surely consists in an experimental confirmation of this predicted dynamical behaviour: the great tunability and isolation from the external environment can ease this kind of experimental study. The techniques of magneto-optical trapping, widely used in handling different configurations of condensed Bose gases, can be used for an experimental realization of a two-peaked distribution function; then, by varying the relevant parameters, as the distribution depth between the peaks, we can explore the range of unstable wave modes. From this point of view, we surely have to consider the Penrose-like criterion as a good tool for addressing the laboratory work to the proper values of involved parameters.

Finally, we briefly discuss the impasse encountered at the end of our work. In order to numerically integrate the Boltzmann-Vlasov equation, we have implemented a semilagrangian algorithm, based on the work of Cheng and Knorr. In the field of Vlasov-Poisson system, it gives back good results, while here we observed against a great discrepancy between analytical and numerical results. We have not been able to numerically confirm the dispersion relations (3.28) and (3.29). This is certainly the weak point of this thesis but, on the other side, we are confident that the underlying problem is a numerical one.

We have underestimated the importance of the non-conservation of total probability: this is a well-known problem of semilagrangian algorithm, whose solution can be found in the implementation of a conservative algorithm [16, 17] or in an improvement of the semilagrangian one [13]. Concerning the conservative algorithm, we have to remark that, probably, one has to go beyond the original proposal by Fijalkow: its linear approximation of distribution function on the phase space cells seems to be too rough for our purpose. Following the conservative path, the right way could be the one proposed in [18].

We surely have to keep in mind this numerical issue, but we must not forget that, on a qualitative level, the numerical integration displays a behaviour that was theoretically predicted. Then, this enlightens the effectiveness of our approach and the possibility of more in-depth theoretical analysis, for example about a possible non-linear stage of landau damping[30, 37].

Another key point concerns the numerical study about linear instability of the situations where we lose the possibility of achieving analytical results. So, once the algorithm problems will be fixed, we could proceed to explore the two-stream instability and more complex situation, with higher dimensionality and different confining potentials.

By means of our original approach, we are confident that this kind of system will make us achieve good results and a solid understanding of some features of out-of-equilibrium dynamics.

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