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Geometric and dynamic phase-space structure of a class of nonholonomic integrable systems with symmetries

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Introduction

This Thesis concerns some mechanical systems subject to nonholonomic constraints, in particular we deal with a typical example of nonholonomic constraint: the pure rolling motion of a sphere on a surface.

Nonholonomic mechanics is an active and interesting field that has applications and links with robotics and control theory. From a geometrical point of view, nonholonomic systems are defined by non-integrable distributions.

This is still a very open research field: the nonholonomic world is deeply different from the langrangian and variational worlds hence, for example, Noether Theorem does not hold and there is not a strict connection between symmetries and conservation laws.

Furthermore, integrability of nonholonomic systems is still largely not understood. In this regard, some help can arrive from symmetries: if the system can be reduced by symmetry, under the action of a compact group, and the reduced system exhibits periodic dynamics, then the reconstructed dynamics is quasi-periodic.

In this situation, the research in nonholonomic mechanics focus also on the study of classes of systems. This Thesis regards the geometric and dynamic aspects of a particular class of nonholonomic systems: we study the rolling motion of a homogeneous sphere inside a convex surface of revolution that rotates around its vertical axis of symmetry.

The sphere rolling on a convex surface of revolution is a classical nonholonomic system, whose study began already with Routh [36]. In particular, the case in which the surface is at rest has been deeply explored, see for instance [27, 40, 18]. In that case, after a $SO(3) \times S^1$ -reduction, we are left with a 4-dimensional reduced system and it is proved that the motions of this reduced system are periodic and the reconstructed dynamics is quasi-periodic on tori up to dimension 3.

The proof makes use of 3 first integrals of the reduced system: the energy and 2 other quantities that are built with the use of techniques that go back to Routh.

While this problem is well understood, the study of the case in which the surface uniformly rotates has only recently begun.

Let us stress the fact that the introduction of the rotation in the nonholonomic systems can have important consequences.

First of all, the rotation of the surface can produce a totally different dynamics, as we will illustrate by considering 2 classical and elementary examples: the sphere on the rotating horizontal plane and the sphere in the rotating vertical cylinder.

Moreover, when the surface rotates, the nonholonomic constraint is not linear anymore, but affine (linear nonhomogeneous) in the velocities and this has the consequence that the energy of the system is not conserved. This is a general problem for nonholonomic systems with affine constraints and has only recently been solved with the introduction of the so called *moving energy* (see [18]) that is a new type of first integral that replaces the energy.

Therefore to consider rotating surfaces raises new questions about the integrability and the

dynamics of the system.

For what concerns the sphere rolling inside a rotating convex surface of revolution, the system, as in the case at rest, can be $SO(3) \times S^1$ -reduced to a 4-dimensional system. This reduced system has 3 first integrals: in [18], it is proved the existence of a conserved moving energy, while reference [9] proves the existence of other 2 first integrals that generalize those of the case at rest.

In [18], these 3 first integrals are used to prove the periodicity of the dynamics in the 4-dimensional reduced system (and hence the quasi-periodicity, on tori up to dimension 3, of the reconstructed dynamics) for small values of the angular velocity of the rotating surface. The proof makes use of continuation techniques from the case at rest.

Hence this Thesis aims to realize a more detailed study of the rotating case, in order to obtain stronger and more general results, without the restriction to small values of the angular velocity of the surface.

Following the approach of the previous works, we make use of the 3 first integrals to study the reduced dynamics. We prove that

Proposition *The 3 first integrals of the reduced system give a submersion from the reduced phase space to \mathbb{R}^3 , except at the equilibria of the reduced system.*

Then we consider the following case: the surface profile is described by a function $\Psi(\frac{r^2}{2})$ where r is the distance from the vertical axis of the surface. We say that the profile is *superquadratic* for x big enough if $\lim_{x \rightarrow \infty} \frac{x}{\Psi(x)} = 0$. We prove that

Proposition *The level sets of the moving energy are compact if the profile is superquadratic for x big enough.*

Hence, since we have a submersion given by 3 first integrals and one of them has compact level sets, we can prove that

Theorem *If the profile is superquadratic for x big enough, the $SO(3) \times S^1$ -reduced dynamics is periodic except possibly on the critical fibers of the 3 first integrals fibration (which correspond to the equilibria of the reduced system and their level sets). Correspondingly, the unreduced dynamics is quasi-periodic.*

In order to have a better understanding of the dynamics we study in detail the case in which the surface is a paraboloid of revolution. First, we realize an analytical work to explicitly determine the 3 first integrals and then we use them in the framework of a numerical analysis of the reduced system.

Let us call E the moving energy, \mathcal{Y}_1 and \mathcal{Y}_2 the other 2 first integrals. We restrict the study to the common level sets of \mathcal{Y}_1 and \mathcal{Y}_2 , obtaining a family of systems with 1 degree of freedom, that parametrically depend on the values of \mathcal{Y}_1 and \mathcal{Y}_2 .

Let us call $\Sigma_{\bar{y}_1, \bar{y}_2}$ the common level sets of \mathcal{Y}_1 and \mathcal{Y}_2 . We prove that

Proposition *The restriction of the $SO(3) \times S^1$ -reduced system to each $\Sigma_{\bar{y}_1, \bar{y}_2}$ is a lagrangian system, with Lagrangian of the type $L = T - \mathcal{V}$: a kinetic term T quadratic in the velocities and an effective positional potential \mathcal{V} .*

Hence we have a family of 2-parameters lagrangian systems with 1 degree of freedom.

We study the dynamics of these systems, starting from their equilibria which are given by the critical points of the effective potential.

By means of a numerical analysis, we show that, for certain values of \mathcal{Y}_1 and \mathcal{Y}_2 , the effective potential \mathcal{V} can exhibit up to 3 critical points, 2 minima and 1 maximum, that correspond to 2 stable equilibria and 1 unstable equilibrium, with its stable and unstable manifolds, of the reduced system.

For the $SO(3)$ -reduced system (namely, the motion of the center of the sphere in the paraboloid), these equilibria are horizontal circular orbits at constant height on the paraboloid. Hence we discover the existence of a family (parametrized by the values of \mathcal{Y}_1 and \mathcal{Y}_2) of unstable horizontal circular orbits on the paraboloid. This is a completely unexpected behavior, totally different from the situation with the surface at rest.

This Thesis has the following structure:

In Chapter 1 we recall some facts about nonholonomic mechanics: we consider in particular the case in which the nonholonomic constraint is affine in the velocities. In particular, we show how such those systems are defined and how to write their equations of motion. Eventually we illustrate 2 elementary examples: the sphere on the rotating horizontal plane and the sphere in the rotating vertical cylinder.

In Chapter 2, given the key role of the first integrals in the study of nonholonomic systems, we illustrate some known facts about the conservation mechanisms in nonholonomic mechanics: in particular we talk about energy, momenta and moving energy. In order to apply these notions, we use the 2 examples of Chapter 1 and compute the first integrals of those 2 systems.

In Chapter 3 we study the system composed of a sphere which rolls without sliding inside a rotating convex surface of revolution, obtaining the results we have revealed above. The last part of this Chapter, namely Section 3.3, is devoted to the study of the case in which the surface is a paraboloid: in particular we illustrate the results of the numerical analysis concerning the critical points of \mathcal{V} .

In Chapter 4 there are some short conclusions with future developments.

In Chapter 5 there are the Appendices with some technical facts.

Chapter 1

Nonholonomic systems

1.1 Introduction

In this chapter, we will recall some elementary notions about the nonholonomically constrained mechanical systems. There exists an extensive literature on this subject: see for example [35, 31, 34, 11].

In particular, during our presentation of nonholonomic dynamics, we will study the nonholonomic constraints which are linear non-homogeneous, i.e. affine, in the velocities.

An important example of linear homogeneous nonholonomic constraint appears when studying systems formed by a rigid body that rolls without slinding on a surface at rest; on the other hand, when a rigid body rolls on a moving surface, the nonholonomic constraint is affine.

1.2 Nonholonomic systems with affine constraints

Let us start from a *holonomic ideal mechanical system*, defined on a n -dimensional smooth manifold \hat{Q} ,¹ which is the configuration manifold, endowed with local coordinates $q_i \in Q \subseteq \mathbb{R}^n$, $i = 1 \dots n$, and consider its tangent bundle $T\hat{Q}$, which is the phase space, with local coordinates $(q, \dot{q}) \in Q \times \mathbb{R}^n$. Let us assume that the Lagrangian $\hat{L} : T\hat{Q} \rightarrow \mathbb{R}$ of the system has the form

$$\hat{L} = \hat{T} + \hat{b} - \hat{V} \circ \pi \quad (1.1)$$

where \hat{T} is a positive definite quadratic form on $T\hat{Q}$, \hat{b} is a 1-form on \hat{Q} , \hat{V} is a function on \hat{Q} and π is the tangent bundle projection $\pi : T\hat{Q} \rightarrow \hat{Q}$. We interpret \hat{T} as the kinetic energy, \hat{V} as the potential energy of the positional forces that act on the system, and \hat{b} as the generalized potential of the gyrostatic forces that act on the system. If it is not differently specified, we will always consider this type of Lagrangians.

The equations of motion of the system are given by *Lagrange's Equations* which have the well known expression (in coordinates):

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad i = 1 \dots n$$

For further reading on Lagrangian mechanics, see [4, 31, 3, 35, 33].

Now we consider the presence of some *nonholonomic constraints* acting on the system, starting from the case of constraints which are linear and homogeneous in the velocities.

¹Here and in the rest of the Thesis, we will denote global objects with "hat symbols" and their local representatives with the same letters without the hat.

Definition A *linear homogeneous nonholonomic constraint* of rank r , $2 \leq r < n$ is the restriction of the velocities to a non-integrable smooth distribution \mathcal{D} on Q of constant rank r .

Let us recall some notions from differential geometry that will be useful to understand the objects we are dealing with; for further reading, see [1, 30, 37].

Definition A smooth *distribution* of rank r on a smooth manifold \hat{Q} is a subset $\hat{\mathcal{D}} \subset T\hat{Q}$ such that for all $\hat{q} \in \hat{Q}$

1. $\hat{\mathcal{D}}_{\hat{q}} \equiv \hat{\mathcal{D}} \cap T_{\hat{q}}\hat{Q}$ is a r -dimensional subspace of $T_{\hat{q}}\hat{Q}$.
2. For every $\hat{Y} \in \hat{\mathcal{D}}_{\hat{q}}$ there exists a vector field \hat{X} on \hat{Q} such that $\hat{X}_{\hat{q}} = \hat{Y}$.

Definition Let $\hat{\mathcal{D}}$ be a distribution on \hat{Q} . An *integral manifold* of $\hat{\mathcal{D}}$ is an immersed submanifold $\hat{S} \hookrightarrow \hat{Q}$ such that $T_{\hat{q}}\hat{S} = \hat{\mathcal{D}}_{\hat{q}}$ for all $\hat{q} \in \hat{Q}$. A distribution is *integrable* if there exists an integral manifold passing through each $\hat{q} \in \hat{Q}$.

Locally, a linear homogeneous nonholonomic constraint is given by a system of $k = n - r$ linearly independent smooth 1-forms on Q . This system defines by annihilation the distribution \mathcal{D} , which is called *constraint distribution*. Hence, in coordinates, the fibers \mathcal{D}_q can be described as the kernel of a $k \times n$ matrix $S(q)$, which depends smoothly on q and has everywhere rank k

$$\mathcal{D}_q = \ker S(q) = \{\dot{q} \in T_q Q : S(q)\dot{q} = 0\} \quad (1.2)$$

The constraint distribution $\hat{\mathcal{D}}$ can also be regarded as a submanifold, a subbundle, $\hat{D} \subset T\hat{Q}$ of dimension $2n - k = n + r$, which is called the *constraint manifold*. In coordinates

$$D = \{(q, \dot{q}) \in TQ : S(q)\dot{q} = 0\}$$

Let us now consider the *affine*, namely linear non-homogeneous, nonholonomic constraints: at each configuration $\hat{q} \in \hat{Q}$, the velocities of the system are restricted to be in an affine subspace $\hat{\mathcal{M}}_{\hat{q}} \subset T_{\hat{q}}\hat{Q}$. Specifically, we deal with a nonintegrable distribution $\hat{\mathcal{D}}$ on \hat{Q} of constant rank r , just like in the linear homogeneous case, and a vector field \hat{Z} on \hat{Q} , such that, at each $\hat{q} \in \hat{Q}$,

$$\hat{q} \in \hat{\mathcal{M}}_{\hat{q}} = \hat{\mathcal{D}}_{\hat{q}} + \hat{Z}(\hat{q}) \quad (1.3)$$

The affine distribution $\hat{\mathcal{M}}$ can also be regarded as a submanifold, which is called the *constraint manifold*, $\hat{M} \subset T\hat{Q}$ of dimension $n + r$, which is an affine subbundle of the tangent bundle $T\hat{Q}$.

The case of linear homogeneous constraints is recovered when \hat{Z} is a section of $\hat{\mathcal{D}}$, i.e. $\hat{M} = \hat{\mathcal{D}}$. It is also clear that \hat{Z} is defined up to a section of $\hat{\mathcal{D}}$.

1.3 The d'Alembert Principle and the reaction force

We follow the generalment agreement of literature and work with "ideal" constraints, i.e. those constraints that satisfy the *d'Alembert Principle* (for references see [4, 34, 35, 32, 5, 2]). D'Alembert Principle states that, given the system in a certain configuration, *the reaction forces that the constraint can exert are those that annihilate any possible virtual displacement*

of the system from that configuration compatible with the constraint.

Let us apply this Principle to the mechanical systems subject to affine nonholonomic constraints (the homogeneous case can be studied as a particular case).

For such these systems, it is assumed that a virtual displacement from a configuration $\hat{q} \in \hat{Q}$ compatible with the constraint is a vector \hat{q} tangent to the configuration manifold in \hat{q} and in $\hat{D}_{\hat{q}}$, even when the constraint is affine (see [34, 32] for explanations).

Therefore d'Alembert Principle states that if (q, \dot{q}) is in D then the reaction force $R(q, \dot{q})$ satisfies

$$R(q, \dot{q}) \cdot \dot{q} = 0 \quad \forall \dot{q} \in D_q \quad \forall q \in Q \quad (1.4)$$

Under this hypothesis, it can be proved the following proposition (see [13] for a more complete statement)

Proposition *Under the ideality hypothesis, there is a unique function*

$$\hat{R}_{\hat{L}\hat{M}} : \hat{M} \rightarrow \hat{D}^\circ$$

that associates an ideal reaction force $\hat{R}_{\hat{L}\hat{M}}(\hat{q}, \hat{q})$ to each constrained kinematic state $(\hat{q}, \hat{q}) \in \hat{M}$, with the property that the restriction to \hat{M} of the Lagrange equations with the reaction force

$$\left(\frac{d}{dt} \frac{\partial \hat{L}}{\partial \hat{q}} - \frac{\partial \hat{L}}{\partial \hat{q}} \right) \Big|_{\hat{M}} = \hat{R}_{\hat{L}, \hat{M}} \quad (1.5)$$

defines a dynamical system on \hat{M} .

Partial proof The complete proof of this fact can be found in [13] and follows the classical approach of [2]; here, let us just give a sketch of the computation of the reaction force.

Working in coordinates $(q, \dot{q}) \in Q \times \mathbb{R}^n$, we consider the local representative of the Lagrangian, namely

$$L(q, \dot{q}) = \frac{1}{2} \dot{q} \cdot A(q) \dot{q} + b(q) \cdot \dot{q} - V(q) \quad (1.6)$$

where $A(q)$ is the kinetic matrix, a $n \times n$ symmetric nonsingular matrix, and $b(q) \in \mathbb{R}^n$.

For what concerns \mathcal{D} , the local representative of the constraint distribution \hat{D} , we can denote its fibers as the kernel of a $k \times n$ matrix $S(q)$ with constant rank $k = n - r$. So we have

$$\mathcal{D}_q = \{ \dot{q} \in T_q Q = \mathbb{R}^n : S(q) \dot{q} = 0 \}$$

If \mathcal{M} is the local representative of \hat{M} , M that of \hat{M} and $Z : Q \rightarrow \mathbb{R}^n$ that of \hat{Z} , we have

$$\dot{q} \in \mathcal{M}_q \Leftrightarrow \exists u \in \text{Ker } S(q) : \dot{q} = u + Z(q)$$

that is to say

$$\dot{q} \in \mathcal{M}_q \Leftrightarrow S(q) [\dot{q} - Z(q)] = 0$$

Thus

$$M = \{ (q, \dot{q}) \in Q \times \mathbb{R}^n : S(q) \dot{q} + s(q) = 0 \}$$

where

$$s(q) = -S(q)Z(q) \in \mathbb{R}^k$$

Given the local representative $t \mapsto (q_t, \dot{q}_t, R(q_t, \dot{q}_t))$ of a curve in $M \times \mathcal{D}^\circ$ that satisfies

$$\left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) (q_t, \dot{q}_t) = R(q_t, \dot{q}_t) \quad (1.7)$$

then, for all t ,

$$S(q_t)\dot{q}_t + s(q_t) = 0 \quad (1.8)$$

and, in view of (1.4), $R(q_t, \dot{q}_t) \in \text{range } S(q_t)^T$. Thus there exists a curve $t \mapsto \lambda_t \in \mathbb{R}^k$ such that

$$R(q_t, \dot{q}_t) = S(q_t)^T \lambda_t$$

So it is clear that we have to determine the coefficients λ_t in order to compute R_t . Given that the Lagrangian has the form (1.6), we can write

$$\left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) = A(q)\ddot{q} + l(q, \dot{q})$$

where $l \in \mathbb{R}^n$ is given by ²

$$l_i = \frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \dot{q}_j - \frac{\partial L}{\partial q_i} \quad i = 1, \dots, n$$

Then eq. (1.7) can be rewritten as

$$A(q_t)\ddot{q}_t + l(q_t, \dot{q}_t) = S(q_t)^T \lambda_t$$

which gives

$$\ddot{q}_t = A(q_t)^{-1} [S(q_t)^T \lambda_t - l(q_t, \dot{q}_t)] \quad (1.9)$$

Now, let us derive eq. (1.8) with respect to t . We obtain

$$S(q_t)\ddot{q}_t + \sigma(q_t, \dot{q}_t) = 0$$

where

$$\sigma_a = \frac{\partial S_{ai}}{\partial q_j} \dot{q}_i \dot{q}_j + \frac{\partial s_a}{\partial q_j} \dot{q}_j \quad a = 1, \dots, r \quad i = 1, \dots, n$$

Finally, in view of (1.9), we have

$$S(q_t)A(q_t)^{-1} [S(q_t)^T \lambda_t - l(q_t, \dot{q}_t)] + \sigma(q_t, \dot{q}_t) = 0$$

that can be solved because $SA^{-1}S^T$ is invertible by the hypothesis made on S and A :

$$\lambda_t = [S(q_t)A(q_t)^{-1}S(q_t)^T]^{-1} [S(q_t)A(q_t)^{-1}l(q_t, \dot{q}_t) - \sigma(q_t, \dot{q}_t)] \quad (1.10)$$

So, locally, the restriction to M of the reaction force $R_t = S(q_t)^T \lambda_t$, where λ_t is given by (1.10), corresponds to

$$R_{L,M} = S^T(SA^{-1}A^T)^{-1}(SA^{-1}l - \sigma)|_M \quad (1.11)$$

□

In the following we will need also to use the unrestricted version of $R_{L,M}$: let us simply call it R .

It has to be said that, when computing $R_{L,M}$, usually we do not use equation (1.11), namely we do not compute each single term (S, A, l, σ) , but we obtain it performing the computation that leads to that equation.

²Here and in the rest of the Thesis we make use of the convention of summation over repeated indexes.

Some remarks on the reaction force

It is important to notice that, while the constraints have kinematical nature, equation (1.11) shows that the reaction force depends on some dynamical features of the system, such as its mass distribution (contained in the kinetic matrix) and the potential of the active forces, the latter appearing in the computation of l . In other words, different systems, with different Lagrangians, can be constrained in the same way, but there will be different reaction forces. Let us see this aspect in a basic example: the "nonholonomic particle" (see [35, 13]). Such a system has configuration manifold $Q = \mathbb{R}^3$ with coordinates $q = (x, y, z) \in \mathbb{R}^3$ and Lagrangian

$$L(q, \dot{q}) = \frac{1}{2} \|\dot{q}\|^2 - V(q)$$

with an unspecified potential energy V . Let us consider the nonholonomic homogeneous constraint

$$\dot{z} + x\dot{y} - y\dot{x} = 0$$

In this example, $n = 3$ and $k = 1$, the distribution \mathcal{D} has constant rank 2 and is the kernel of $S(q) = (-y, x, 1)$, i.e it is spanned by the vector fields $\partial_x + y\partial_y$ and $\partial_y - x\partial_x$. The constraint manifold D is diffeomorphic to $Q \times \mathbb{R}^2$ with global coordinates $(x, y, z, \dot{x}, \dot{y})$. The determination of the reaction force, following equation (1.11), is rather simple since we have $A = \mathbb{1}$ and $l = V'$. Then, a straightforward computation, [13], shows that

$$R_{L,D} = S^T (SS^T)^{-1} V' \Big|_D = \frac{1}{1+x^2+y^2} \begin{pmatrix} y^2 & -xy & -y \\ -xy & x^2 & x \\ -y & x & 1 \end{pmatrix} V' \Big|_D$$

So it is clear that the form of the potential energy modifies the reaction force, while the constraint manifold D is not system-dependent.

This kind of considerations will be very relevant for conservation laws and first integrals, as it will be clear in Chapter 2.

1.4 The equations of motion

In this Section we will see how to write the equations of motion for a mechanical system with nonholonomic constraints.

Clearly, one could use Lagrange's equations (1.5), but that could be somehow complicated. There exists another way: the use of *Hamel's equations* which are useful because they let us study the system in terms of angular velocity, as we will see. These equations and, in particular, the techniques involved have a long history and are still a subject of study: see [26, 34, 6, 5] for further reading.

Then, since in this Thesis we will specifically study the pure rolling motion of a sphere, we will take into account the *cardinal equations* of motion for a rigid body. For further reading on the rigid body dynamics and on the Cardinal equations, see [4, 22, 31, 3, 28, 20].

In this section, we will work in coordinates in order to have a clearer notation.

1.4.1 Hamel's equations: velocities and quasivelocities

Hamel's equations emerge when we try to write the equations (1.5) not using ordinary velocities, namely tangent lifted coordinates \dot{q} , but a linear pointwise combination of them, called

quasivelocities

$$(q, \dot{q}) \mapsto (q, w(q, \dot{q})) = (q, \Omega(q)\dot{q})$$

where $\Omega(q) \in GL(n, \mathbb{R})$. We will denote the inverse transformation by $(q, w) \mapsto (q, \Sigma(q)w)$ with $\Sigma(q)^{-1} = \Omega(q)$.

The use of quasivelocities is aimed to simplify the description of the dynamics of the system: sometimes using ordinary velocities and Lagrange's equations can be not effective. For example, let us think about the rigid body and Euler equations that describe the evolution of the angular momentum in terms of the angular velocity; it is clear that this description is more "user-friendly" than, for example, writing Lagrange's equations using the time derivatives of Euler's angles.

The use of quasivelocities has also some inconveniences: since we are not using tangent lifted coordinates, we can not expect to preserve the Lagrangian form of the equations of motion, but we meet something more complex and cumbersome.

In literature, Hamel's equations are used as a typical approach to write down the equations of motions of the nonholonomic systems.

Let us see how Hamel's equations are defined in presence of a nonholonomic constraint which is affine in the velocities³ and defines a constraint manifold M , with the notation of Sections 1.2 and 1.3.

Proposition *In terms of (q, w) , equations (1.5) become the so called **Hamel's equations** which are the restriction to the constraint manifold $\tilde{M} = \tilde{S}w + s$, where $\tilde{S} = S\Sigma$, of the following system*

$$\begin{cases} \dot{q} = \Sigma(q)w \\ \dot{w} = \tilde{A}^{-1}(\tilde{R} - \tilde{l}) \end{cases} \quad (1.12)$$

where $\tilde{A} = \Sigma^T A \Sigma$ and A is the kinetic matrix of the Lagrangian L , $\tilde{R} = \Sigma^T R$, $\tilde{l}_k = \frac{\partial^2 \tilde{L}}{\partial q_h \partial w_k} \Sigma_{hl} w_l + \frac{\partial \tilde{L}}{\partial w_j} \gamma_{hi}^j \Sigma_{ik} \Sigma_{hl} w_l - \frac{\partial \tilde{L}}{\partial q_i} \Sigma_{ik}$ and $\gamma_{hi}^j = \left[\frac{\partial \Omega_{ji}}{\partial q_h} - \frac{\partial \Omega_{jh}}{\partial q_i} \right]$ with $k, h, l, j, i = 1, \dots, n$.

Proof Let us define

$$\tilde{L}(q, w(q, \dot{q})) := L(q, \dot{q})$$

and compute

$$\begin{aligned} \frac{\partial L}{\partial q_i} &= \frac{\partial \tilde{L}}{\partial q_i} + \frac{\partial \tilde{L}}{\partial w_j} \frac{\partial w_j}{\partial q_i} = \frac{\partial \tilde{L}}{\partial q_i} + \frac{\partial \tilde{L}}{\partial w_j} \frac{\partial \Omega_{jh}^j}{\partial q_i} \dot{q}_h \\ \frac{\partial L}{\partial \dot{q}_i} &= \frac{\partial \tilde{L}}{\partial w_j} \frac{\partial w_j}{\partial \dot{q}_i} = \frac{\partial \tilde{L}}{\partial w_j} \Omega_{ji} \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} &= \frac{d}{dt} \left[\frac{\partial \tilde{L}}{\partial w_j} \right] \Omega_{ji} + \frac{\partial \tilde{L}}{\partial w_j} \frac{d \Omega_{ji}}{dt} = \\ &= \frac{\partial^2 \tilde{L}}{\partial w_h \partial w_j} \frac{\partial w_h}{\partial q_k} \dot{q}_k \Omega_{ji} + \frac{\partial^2 \tilde{L}}{\partial w_h \partial w_j} \frac{\partial w_h}{\partial \dot{q}_k} \ddot{q}_k \Omega_{ji} + \frac{\partial^2 \tilde{L}}{\partial q_h \partial w_j} \dot{q}_h \Omega_{ji} + \frac{\partial \tilde{L}}{\partial w_j} \frac{\partial \Omega_{ji}}{\partial q_h} \dot{q}_h \end{aligned}$$

Using $w_j = \Omega_{jh} \dot{q}_h$, $\dot{q}_h = \Sigma_{hj} w_j$, $\frac{\partial w_h}{\partial q_k} \dot{q}_k + \frac{\partial w_h}{\partial \dot{q}_k} \ddot{q}_k = \dot{w}_h$ we obtain

$$\frac{\partial L}{\partial q_i} = \frac{\partial \tilde{L}}{\partial q_i} + \frac{\partial \tilde{L}}{\partial w_j} \frac{\partial \Omega_{jh}}{\partial q_i} \Sigma_{hl} w_l$$

³The use of Hamel's equations is more general than this one.

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial^2 \tilde{L}}{\partial w_h \partial w_j} \Omega_{ji} \dot{w}_h + \frac{\partial^2 \tilde{L}}{\partial q_h \partial w_j} \Omega_{ji} \Sigma_{hl} w_l + \frac{\partial \tilde{L}}{\partial w_j} \frac{\partial \Omega_{ji}}{\partial q_h} \Sigma_{hl} w_l$$

Hence we have

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \frac{\partial^2 \tilde{L}}{\partial w_h \partial w_j} \Omega_{ji} \dot{w}_h + \frac{\partial^2 \tilde{L}}{\partial q_h \partial w_j} \Omega_{ji} \Sigma_{hl} w_l + \frac{\partial \tilde{L}}{\partial w_j} \left[\frac{\partial \Omega_{ji}}{\partial q_h} - \frac{\partial \Omega_{jh}}{\partial q_i} \right] \Sigma_{hl} w_l - \frac{\partial \tilde{L}}{\partial q_i}$$

Now we define

$$\gamma_{hi}^j := \left[\frac{\partial \Omega_{ji}}{\partial q_h} - \frac{\partial \Omega_{jh}}{\partial q_i} \right]$$

Let us notice that the coefficients γ_{hi}^j are anti-symmetric in the (i, h) indices. The unrestricted equations (1.5) become

$$\frac{\partial^2 \tilde{L}}{\partial w_h \partial w_j} \Omega_{ji} \dot{w}_h + \frac{\partial^2 \tilde{L}}{\partial q_h \partial w_j} \Omega_{ji} \Sigma_{hl} w_l + \frac{\partial \tilde{L}}{\partial w_j} \gamma_{hi}^j \Sigma_{hl} w_l - \frac{\partial \tilde{L}}{\partial q_i} = R_i$$

Let us multiply both sides by Σ

$$\frac{\partial^2 \tilde{L}}{\partial w_k \partial w_h} \dot{w}_h + \frac{\partial^2 \tilde{L}}{\partial q_h \partial w_k} \Sigma_{hl} w_l + \frac{\partial \tilde{L}}{\partial w_j} \gamma_{hi}^j \Sigma_{ik} \Sigma_{hl} w_l - \frac{\partial \tilde{L}}{\partial q_i} \Sigma_{ik} = R_i \Sigma_{ik}$$

and define

$$\tilde{l}_k := \frac{\partial^2 \tilde{L}}{\partial q_h \partial w_k} \Sigma_{hl} w_l + \frac{\partial \tilde{L}}{\partial w_j} \gamma_{hi}^j \Sigma_{ik} \Sigma_{hl} w_l - \frac{\partial \tilde{L}}{\partial q_i} \Sigma_{ik}$$

$$\tilde{R} := \Sigma^T R$$

$$\tilde{A}_{kh} := (\Sigma^T A \Sigma)_{kh} = \frac{\partial^2 \tilde{L}}{\partial w_k \partial w_h}$$

where A is the kinetic matrix of L . Hence we have

$$\tilde{A}_{kh} \dot{w}_h + \tilde{l}_k = \tilde{R}_k$$

Since both A and Σ are invertible matrices, so it is \tilde{A} and then we can solve the previous equation for \dot{w}

$$\dot{w} = \tilde{A}^{-1}(\tilde{R} - \tilde{l})$$

Hence the dynamical system, in terms of the coordinates q and the quasivelocities w , is given by the system

$$\begin{cases} \dot{q} = \Sigma(q)w \\ \dot{w} = \tilde{A}^{-1}(\tilde{R} - \tilde{l}) \end{cases}$$

which has to be restricted to the constraint manifold \tilde{M} .

□

1.4.2 The cardinal equations

This Thesis concerns a specific class of nonholonomic systems that involve a homogeneous sphere rolling without sliding on a surface. Therefore, since we are going to deal with the rigid body dynamics, we can derive the equations of motion for such systems, namely equations (1.5) by means of the *cardinal equations*.

Proposition *Consider a rigid body with centre of mass C , subject to some external active forces and some reaction forces due to constraints, and consider an arbitrary point P ; the equations of motion of the rigid body are given by the **cardinal equations** which are*

$$mA_C = F \quad (1.13)$$

$$\frac{d\mathcal{M}_P}{dt} = N_P + mV_C \times V_P \quad (1.14)$$

where A_C is the acceleration of the centre of mass, F is the resultant of the active and reactive forces, \mathcal{M}_P is the angular momentum of the body with respect to P , N_P is the resultant torque of all the active and reactive forces with respect to P , V_C and V_P are respectively the velocity of C and the velocity of P .

Note that, when applying the cardinal equations to the study of a mechanical system subject to a nonholonomic constraint which is affine in the velocities, we have to restrict the equations (1.13) and (1.14) to the constraint manifold.

1.5 Examples

We now explore 2 classical examples of nonholonomic systems: the sphere rolling on a rotating horizontal plane and the sphere rolling inside a rotating vertical cylinder. These examples are interesting because they represent borderline cases of the main subject of this Thesis, namely the sphere rolling inside a convex surface of revolution.

Besides these 2 systems have been deeply studied and understood, see for instance [36, 34, 35, 9, 7] and references therein.

They show how much particular and unpredictable the nonholonomic dynamics can be.

In these examples we want to obtain the equations of motion using Hamel's equations methods.

1.5.1 The sphere on the rotating horizontal plane

Let us consider the following system: a homogeneous sphere of mass m and radius $a > 0$, rolling without sliding on a plane that rotates with angular velocity \vec{k} with respect an axis orthogonal to the plane itself.

Let us consider a fixed reference frame $\{O, e_x, e_y, e_z\}$, with the z -axis coincident with the rotation axis, namely $\vec{k} = ke_z$, $k \in \mathbb{R}$. The sphere is subject to the holonomic constraint of moving in the plane $z = 0$, namely its centre of mass C moves in the plane $z = 0$. So the configuration manifold is the 5 dimensional manifold $\hat{Q} = \mathbb{R}^2 \times SO(3)$ where, in particular, we have $(x, y) \in \mathbb{R}^2$ which are the coordinates of the centre of mass, while we indicate with \mathcal{R} the generic element of $SO(3)$. Let us call $(x, y, \mathcal{R}) = q \in Q$. Then, when doing computations, we need to consider the Euler's angles (φ, θ, ψ) as a possible parametrization of $SO(3)$.

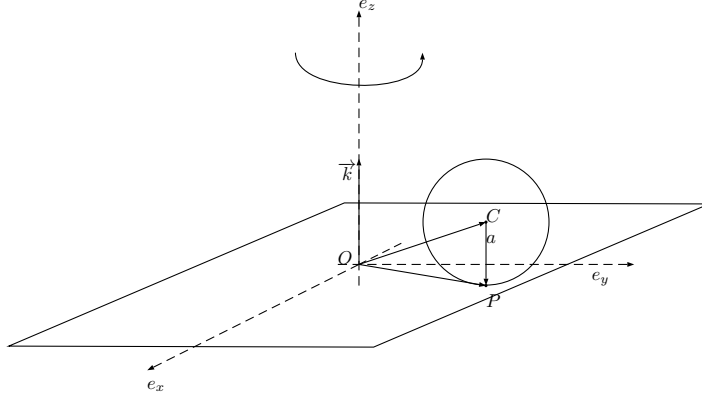


Figure 1.1: The sphere on the plane

Let us write the Lagrangian of the system, in terms of the spatial angular velocity of the sphere ⁴

$$L : Q \times \mathbb{R}^2 \times \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$L(q, \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I}{2}(\omega_x^2 + \omega_y^2 + \omega_z^2)$$

where I is the moment of inertia relative to its centre of mass, i.e. $I = \frac{2}{5}ma^2$.

Now we impose the nonholonomic constraint of rolling without sliding on the rotating plane. The pure rolling constraint means that the instantaneous point of contact P moves with the plane

$$V_C + \omega \times CP = ke_z \times OP$$

We have

$$OC = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}, \quad V_C = \begin{pmatrix} \dot{x} \\ \dot{y} \\ 0 \end{pmatrix}$$

$$CP = \begin{pmatrix} 0 \\ 0 \\ -a \end{pmatrix}, \quad OP = \begin{pmatrix} x \\ y \\ -a \end{pmatrix}$$

and the constraint means that

$$\begin{cases} \dot{x} - a\omega_y + ky = 0 \\ \dot{y} + a\omega_x - kx = 0 \end{cases} \quad (1.15)$$

The equations (1.15) are exactly of the type $\tilde{S}(q)w + s(q) = 0$ and define the constraint manifold \tilde{M} , with

$$w = (\dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z) = \Omega(q)\dot{q}, \quad \Omega(q) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin\theta\sin\varphi & \cos\varphi \\ 0 & 0 & 0 & -\cos\varphi\sin\theta & \sin\varphi \\ 0 & 0 & 1 & \cos\theta & 0 \end{pmatrix}$$

⁴This Lagrangian is written in terms of quasivelocities, namely, using the notation of the Proposition of Hamel's equations, it corresponds to \tilde{L} .

$$\tilde{S}(q) = \begin{pmatrix} 1 & 0 & 0 & -a & 0 \\ 0 & 1 & a & 0 & 0 \end{pmatrix}, \quad s(q) = \begin{pmatrix} ky \\ -kx \end{pmatrix}$$

So now, following the procedure we have seen before, we are ready to compute \tilde{l} and \tilde{R} and to write Hamel's equations (1.12).

Let us better define the constraint manifold \tilde{M} . It can be parametrized in terms of $(q, \omega_x, \omega_y, \omega_z)$ using the system (1.15) which gives $\dot{x} = a\omega_y - ky$ and $\dot{y} = -a\omega_x + kx$ ⁵. Therefore the constraint manifold is 8-dimensional

$$\begin{aligned} \tilde{M} &= \{(q, \dot{x}, \dot{y}, \omega) \in Q \times \mathbb{R}^2 \times \mathbb{R}^3 : \dot{x} = a\omega_y - ky, \quad \dot{y} = -a\omega_x + kx\} \\ \tilde{M} &\simeq Q \times \mathbb{R}^3 \end{aligned}$$

Hamel's equations are

$$\begin{aligned} \dot{\mathcal{R}} &= \tilde{\omega}\mathcal{R} \\ \dot{x} &= a\omega_y - ky \\ \dot{y} &= -a\omega_x + kx \\ \dot{\omega}_x &= \frac{akm}{I+ma^2}(a\omega_y - ky) \\ \dot{\omega}_y &= \frac{akm}{I+ma^2}(-a\omega_x + kx) \\ \dot{\omega}_z &= 0 \end{aligned} \tag{1.16}$$

The first 3 equations are those that lets us pass from the angular velocity to the rotation matrix in $SO(3)$ (parametrized by Euler's angles), with

$$\omega = (\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3, \quad \tilde{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

We can notice that the last 5 equations do not have $\mathcal{R} \in SO(3)$ and decouple from the first 3 equations. Hence we can study the $SO(3)$ -reduced system defined on $\mathbb{R}^2 \times \mathbb{R}^3$ whose equations of motion are exactly the last 5 equations of the system (1.16), while $\dot{\mathcal{R}} = \tilde{\omega}\mathcal{R}$ are the reconstruction equations.

We can perform this reduction because the Lagrangian and the constraint are invariant under the tangent lift of the action of $SO(3)$: that is because the system is invariant under the rotation of the axis of the reference frame which is solidal with the sphere. This simmetry will characterise the next example and the main subject of the Thesis, i.e. the sphere rolling on the convex surface of revolution (see Chapter ??), so we will use this reduction procedure again.

Therefore we work in the 5-dimensional manifold $\tilde{M}_5 = \tilde{M}/SO(3)$.

It can be noticed that ω_z is a first integral for the system and could be used to obtain a further restriction, by arbitrarily fixing its value, but at the moment this fact is not very relevant. Anyway, the value of ω_z does not influence the evolution of $(x, y, \omega_x, \omega_y)$.

Let us integrate the last 5 equations of (1.16) in order to study the dynamics of the system, both in the rotating and in the fixed case, the latter being obtained simply putting $k = 0$

⁵We can make other choices, for example we can pick $\omega_x = (kx - \dot{y})/a$ and $\omega_y = (\dot{x} + ky)/a$ and work with the remaining coordinates $(q, \dot{x}, \dot{y}, \omega_z)$.

everywhere.

- $k = 0$

When the horizontal plane is at rest, the angular velocity ω is conserved, i.e. $\dot{\omega} = 0$, while the centre of mass of the sphere moves according to

$$x(t) = a\omega_y t + x_0$$

$$y(t) = -a\omega_x t + y_0$$

where $x_0 = x(0)$ and $y_0 = y(0)$, while ω_x and ω_y are the constant values of the x and y components of the angular velocity. Clearly this is the parametrization of a straight line.

- $k \neq 0$

When the plane rotates, the motion of the centre of mass is given by

$$x(t) = \frac{-5kx_0 + 7a\omega_{x_0} + 7(kx_0 - a\omega_{x_0})\cos(\frac{2}{7}kt) + 7(a\omega_{y_0} - ky_0)\sin(\frac{2}{7}kt)}{2k}$$

$$y(t) = \frac{-5ky_0 + 7a\omega_{y_0} + 7(ky_0 - a\omega_{y_0})\cos(\frac{2}{7}kt) + 7(kx_0 - a\omega_{x_0})\sin(\frac{2}{7}kt)}{2k}$$

where $x_0 = x(0)$, $y_0 = y(0)$, $\omega_{x_0} = \omega_x(0)$, $\omega_{y_0} = \omega_y(0)$. The trajectory is a circle with centre (x_c, y_c) in

$$x_c = \frac{-5kx_0 + 7a\omega_{x_0}}{2k}$$

$$y_c = \frac{-5ky_0 + 7a\omega_{y_0}}{2k}$$

and radius R with

$$R^2 = \frac{49}{4k^2} [k^2(x_0^2 + y_0^2) - 2ak(x_0\omega_{x_0} + y_0\omega_{y_0}) + a^2(\omega_{x_0}^2 + \omega_{y_0}^2)]$$

Hence the 2 cases are very different from each other: when the plane is fixed, the sphere rolls along a *straight line*, according to the initial conditions, while, when the plane rotates, the sphere rolls along a *circular trajectory*.

1.5.2 The sphere in the rotating cylinder

Our second example concerns a heavy homogeneous sphere rolling without slidding on the internal surface of a cylinder of radius R which rotates uniformly around its axis, which stays vertical.

As done before, let us call m the mass of the sphere, a its radius and \vec{k} the angular velocity of the cylinder with respect a fixed reference frame $\{O, e_x, e_y, e_z\}$ with the origin in the cylinder axis and such that $\vec{k} = ke_z$, $k \in \mathbb{R}$. The system is subject to the holonomic constraint that the center of the sphere lies on a cylinder with radius $R - a$.

The configuration manifold is the 5 dimensional manifold $\hat{Q} = S^1 \times \mathbb{R} \times SO(3)$ and we call $q = (\alpha, z, \mathcal{R}) \in Q$.

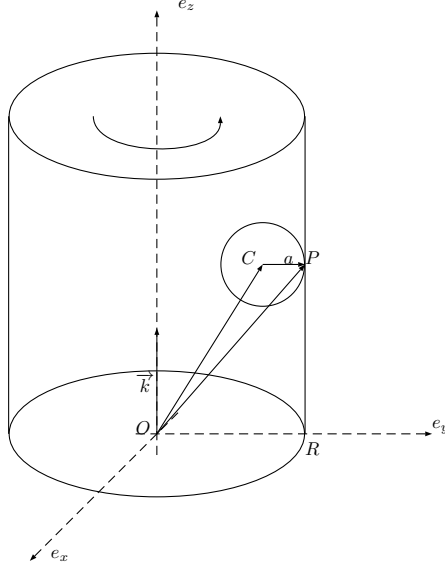


Figure 1.2: The sphere in the cylinder

The geometry of the problem suggests the use of cylindrical coordinates for the position of C , i.e.

$$OC = \begin{pmatrix} (R-a)\cos\alpha \\ (R-a)\sin\alpha \\ z \end{pmatrix} \quad (\alpha, z) \in S^1 \times \mathbb{R}$$

The Lagrangian is

$$L : Q \times \mathbb{R}^2 \times \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$L(q, \dot{\alpha}, \dot{z}, \omega_x, \omega_y, \omega_z) = \frac{m}{2} [(R-a)^2 \dot{\alpha}^2 + \dot{z}^2] + \frac{I}{2} (\omega_x^2 + \omega_y^2 + \omega_z^2) - mgz$$

with $I = \frac{2}{5}ma^2$ and g is the gravitational acceleration.

Now we impose the nonholonomic constraint that the sphere rolls without sliding in the rotating cylinder

$$V_C + \omega \times CP = ke_z \times OP$$

where

$$V_C = \begin{pmatrix} -\dot{\alpha}(R-a)\sin\alpha \\ \dot{\alpha}(R-a)\cos\alpha \\ \dot{z} \end{pmatrix}, \quad CP = \begin{pmatrix} a\cos\alpha \\ a\sin\alpha \\ 0 \end{pmatrix}, \quad OP = \begin{pmatrix} R\cos\alpha \\ R\sin\alpha \\ z \end{pmatrix}$$

which gives

$$\begin{cases} \dot{\alpha}(R-a) - kR + a\omega_z = 0 \\ \dot{z} + a\omega_x\sin\alpha - a\omega_y\cos\alpha = 0 \end{cases} \quad (1.17)$$

Once again, equations (1.17) have the form $\tilde{S}(q)w + s(q) = 0$ and define the constraint manifold \tilde{M} , with

$$w = (\dot{\alpha}, \dot{z}, \omega_x, \omega_y, \omega_z) = \Omega(q)\dot{q}, \quad \Omega(q) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin\theta\sin\varphi & \cos\varphi \\ 0 & 0 & 0 & -\cos\varphi\sin\theta & \sin\varphi \\ 0 & 0 & 1 & \cos\theta & 0 \end{pmatrix}$$

$$\tilde{S}(q) = \begin{pmatrix} R-a & 0 & 0 & 0 & a \\ 0 & 1 & a\sin\alpha & -a\cos\alpha & 0 \end{pmatrix}, \quad s(q) = \begin{pmatrix} -kR \\ 0 \end{pmatrix}$$

As in the previous example, the system (1.17) gives us a possible parametrization of \tilde{M} with $(q, \omega_x, \omega_y, \omega_z)$, given that $\dot{\alpha} = (kR - a\omega_z)/(R - a)$ and $\dot{z} = a(\omega_y\cos\alpha - \omega_x\sin\alpha)$. Therefore the constraint manifold is 8-dimensional

$$\tilde{M} = \{(q, \dot{\alpha}, \dot{z}, \omega) \in Q \times \mathbb{R}^2 \times \mathbb{R}^3 : \dot{\alpha} = (kR - a\omega_z)/(R - a), \dot{z} = a(\omega_y\cos\alpha - \omega_x\sin\alpha)\}$$

$$\tilde{M} \simeq Q \times \mathbb{R}^3$$

Next we compute \tilde{l} and \tilde{R} and write Hamel's equations (1.12).

Once again, just like the previous example, the equations for $\dot{\mathcal{R}}$ decouple from the others: we can make use of the $SO(3)$ invariance and pass to a 5-dimensional phase space $\tilde{M}_5 = \tilde{M}/SO(3)$ which is diffeomorphic to $S^1 \times \mathbb{R} \times \mathbb{R}^3$. Let us work in \tilde{M}_5 .

The equations of motion defined on \tilde{M}_5 are

$$\begin{aligned} \dot{\alpha} &= \frac{kR - a\omega_z}{R - a} \\ \dot{z} &= a(\omega_y\cos\alpha - \omega_x\sin\alpha) \\ \dot{\omega}_x &= \left(\frac{mIa}{I + ma^2}\right)\sin\alpha \left[g - a\left(\frac{kR - a\omega_z}{R - a}\right)(\omega_x\cos\alpha + \omega_y\sin\alpha) \right] \\ \dot{\omega}_y &= \left(\frac{mIa}{I + ma^2}\right)\cos\alpha \left[-g + a\left(\frac{kR - a\omega_z}{R - a}\right)(\omega_x\cos\alpha + \omega_y\sin\alpha) \right] \\ \dot{\omega}_z &= 0 \end{aligned} \tag{1.18}$$

From the analysis of the system (1.18) it can be noticed that there exists a conjugation between the dynamics in the cylinder at rest, which can be obtained by putting $k = 0$ in all of (1.18), and that one in the rotating cylinder.

Let us call $(\hat{L}, \hat{Q}, \hat{M}_5)|_{k=0}$ the dynamical system corresponding to the cylinder at rest and $(\hat{L}, \hat{Q}, \hat{M}_5)|_{k \neq 0}$ the dynamical system corresponding to the rotating cylinder.

Proposition *The dynamics of $(\hat{L}, \hat{Q}, \hat{M}_5)|_{k \neq 0}$ is conjugate to the dynamics of $(\hat{L}, \hat{Q}, \hat{M}_5)|_{k=0}$ by the following diffeomorphism*

$$\mathcal{C} : \tilde{M}_5|_{k \neq 0} \rightarrow \tilde{M}_5|_{k=0}$$

$$\mathcal{C} : (\alpha, z, \omega_x, \omega_y, \omega_z) \mapsto \left(\alpha, z, \omega_x, \omega_y, \omega_z - \frac{kR}{a} \right)$$

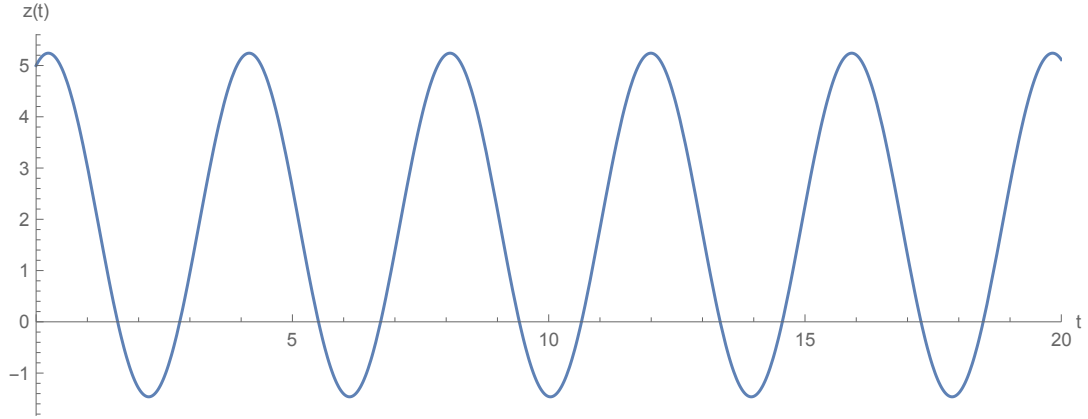
Proof A straightforward computation shows that the push-forward under \mathcal{C} of the vector field (1.18) is equal to the substitution $k = 0$ in all of the system (1.18).

Therefore, from a dynamical point of view, the system with the cylinder at rest and that one with the rotating cylinder are the same: contrary to the previous example of Section 1.5.1, we do not expect substantial differences when the surface rotates.

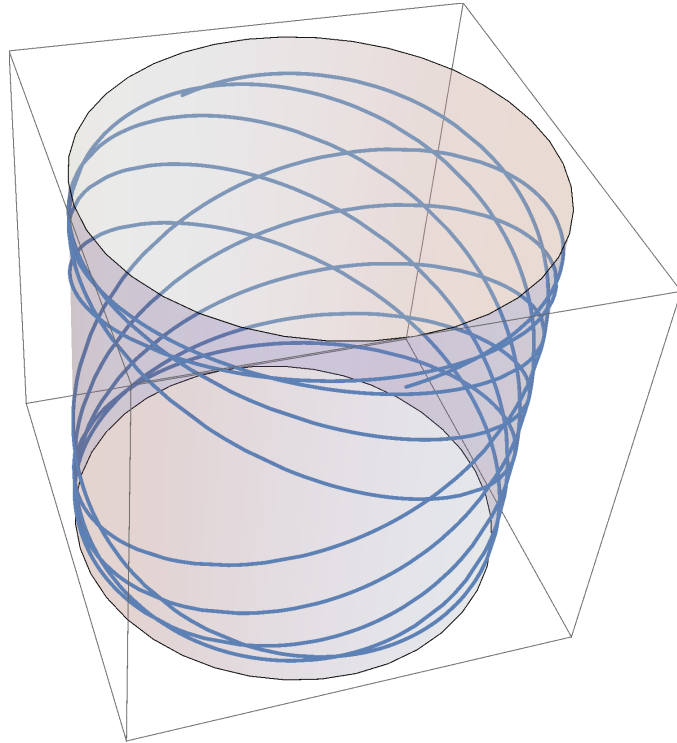
The integration of (1.18) gives us an interesting look at the dynamics of the system. Let us consider the initial data for $\omega_z(t)$, i.e. $\omega_{z_0} = \omega_z(0)$. If we choose $\omega_{z_0} - \frac{kR}{a} = 0$, the sphere falls under the action of gravity: when the cylinder is at rest we just have to take $\omega_{z_0} = 0$. If we choose $\omega_{z_0} - \frac{kR}{a} \neq 0$, both in the case $k \neq 0$ and $k = 0$, the sphere does not fall but oscillates

between a highest and a lowest altitude. Sometimes (see e.g. [9] and [23]) this behavior is linked to a phenomenon that can be observed while playing basketball or golf: it may happen that the ball almost enters into the basket or into the hole, but then jumps out of it, going upwards.

The following Figures illustrate an example of the motion of a sphere in a cylinder with $\omega_{z_0} - \frac{kR}{a} \neq 0$: Figure 1.3(a) shows the oscillations of z between 2 altitudes, while Figure 1.3(b) shows the corresponding motion of the centre of mass of the sphere in the cylinder.



(a) Oscillations of $z(t)$



(b) The corresponding motion in the cylinder

Figure 1.3: Example of the dynamics in the cylinder with $\omega_{z_0} - \frac{kR}{a} \neq 0$

Chapter 2

Conservation of energy and momenta

2.1 Introduction

First integrals are very important for the study of a dynamical system. Speaking of holonomic Lagrangian Mechanics, there exists a powerful mechanism to produce first integrals: the *Noether Theorem* associates conserved quantity to the symmetries of the system (see [4]). Briefly, the Noether Theorem guarantees that if the Lagrangian $\hat{L} : T\hat{Q} \rightarrow \mathbb{R}$ of a dynamical system is invariant under the tangent-lift of a free action of a k -dimensional Lie Group on \hat{Q} , then there exist k independent first integrals which are linear in the conjugate momenta.

When dealing with nonholonomic systems, the relation between conserved quantities and symmetries of the system is more complicated. In fact, not all the symmetries of a system give conserved momenta, as it is in the holonomic case. In particular, even if the system is time-independent, there can be energy dissipation: this means that nonholonomic systems are out of the lagrangian and variational worlds.

Nowadays, it can be said that the mechanisms (if there exist some *mechanisms*) which link symmetries of the nonholonomic systems to conserved quantities are still not completely understood.

In the next Chapter we will study a certain class of nonholonomic systems subject to affine constraints and that analysis will rest on the use of first integrals. In particular we will make use of a *moving energy*: this is a new type of first integral that has only recently been introduced (see [18]) to replace the energy for such those systems.

So in this Chapter we will illustrate some results concerning the first integrals of systems subject to affine nonholonomic constraints, following [13, 14, 15, 16, 17].

In the following we will use $(\hat{L}, \hat{Q}, \hat{M})$ to refer to a mechanical system with Lagrangian $\hat{L} : T\hat{Q} \rightarrow \mathbb{R}$ of the form (1.1), subject to nonholonomic constraints which are affine in the velocities, defined on an affine subbundle $\hat{M} = \hat{D} + \hat{Z}$ that corresponds to a submanifold $\hat{M} \subset T\hat{Q}$, according to the notation of Section 1.2. The corresponding holonomic system is (\hat{L}, \hat{Q}) .

2.2 The reaction-annihilator distribution

First of all, let us introduce the *reaction-annihilator distribution* \hat{R}° (see [14]). The need for this object is due to the following argument: in the first Chapter we stated that the ideality condition means that the constraint can exert all forces that lie in \hat{D}° , the annihilator of \hat{D} , but the computation we made for $\hat{R}_{L,M}$, namely Eq. (1.11), shows that it may happen that only a subset of all the possible reaction forces is actually exerted. In coordinates, that is to

say that the map

$$S^T(SA^{-1}A^T)^{-1}(SA^{-1}l - \sigma)|_{\mathcal{M}_q} : \mathcal{M}_q \rightarrow \mathcal{D}_q^\circ$$

is not necessarily surjective. So when the system is in a certain configuration $\hat{q} \in \hat{Q}$, with any possible velocity $\hat{q} \in \hat{\mathcal{M}}_{\hat{q}}$, the reaction forces are in the set

$$\hat{\mathbf{R}}_{\hat{q}} = \bigcup_{\hat{q} \in \hat{\mathcal{M}}_{\hat{q}}} \hat{R}_{\hat{L}, \hat{M}}(\hat{q}, \hat{q})$$

with

$$\hat{\mathbf{R}}_{\hat{q}} \subseteq \hat{\mathcal{D}}_{\hat{q}}^\circ$$

Definition [13] *The reaction-annihilator distribution $\hat{\mathbf{R}}^\circ$ of a nonholonomic system with affine constraints $(\hat{L}, \hat{Q}, \hat{M})$ is the distribution on \hat{Q} whose fiber $\hat{\mathbf{R}}_{\hat{q}}^\circ$ at $\hat{q} \in \hat{Q}$ is the annihilator of $\hat{\mathbf{R}}_{\hat{q}}$.*

In other words, given a vector field \hat{X} on \hat{Q}

$$\hat{X}_{\hat{q}} \in \hat{\mathbf{R}}_{\hat{q}}^\circ \Leftrightarrow \langle \hat{R}_{\hat{L}, \hat{M}}(\hat{q}, \hat{q}), \hat{X}_{\hat{q}} \rangle = 0 \quad \forall \hat{q} \in \hat{Q}, \quad \hat{q} \in \hat{\mathcal{M}}_{\hat{q}}$$

where $\langle \cdot, \cdot \rangle$ is the cotangent-tangent pairing. Since the reaction force depends on the characteristics of the system, the same is true for $\hat{\mathbf{R}}^\circ$, while this is not true for $\hat{\mathcal{D}}$.

Being a section of $\hat{\mathbf{R}}^\circ$ is a weaker condition than being a section of $\hat{\mathcal{D}}$:

$$\hat{\mathcal{D}}_{\hat{q}} \subseteq \hat{\mathbf{R}}_{\hat{q}}^\circ \quad \forall \hat{q} \in \hat{Q}$$

The importance of \mathbf{R}° becomes clear when considering the conservation of energy and momenta of the system.

2.3 Conservation of energy

Let us start from the *energy*, sometimes called *Jacobi integral*. Given a holonomic system (\hat{L}, \hat{Q}) , the energy is the function

$$\hat{E}_{\hat{L}}(\hat{q}, \hat{q}) := \langle \hat{p}_{\hat{L}}, \hat{q} \rangle - \hat{L}(\hat{q}, \hat{q}) \quad (2.1)$$

where $\hat{p}_{\hat{L}}$ is the momentum 1-form relative to \hat{L} , namely in coordinates $p_L = \frac{\partial L}{\partial \dot{q}}$. For a holonomic system, the conservation of the energy is granted if and only if the Lagrangian is time independent, namely if the system is invariant under time translations. This is not necessarily true when the constraint is nonholonomic.

The energy $\hat{E}_{\hat{L}, \hat{M}}$ of a nonholonomic system $(\hat{L}, \hat{Q}, \hat{M})$ is defined as the restriction $\hat{E}_{\hat{L}}|_{\hat{M}}$ to the constraint manifold \hat{M} of the $\hat{E}_{\hat{L}}$ of (\hat{L}, \hat{Q})

$$\hat{E}_{\hat{L}, \hat{M}} := \hat{E}_{\hat{L}}|_{\hat{M}}$$

Proposition [13] *For a nonholonomic mechanical system with affine constraints $(\hat{L}, \hat{Q}, \hat{M})$, the energy $\hat{E}_{\hat{L}, \hat{M}}$ is a first integral if and only if \hat{Z} is a section of $\hat{\mathbf{R}}^\circ$.*

Proof We work in coordinates. If $t \mapsto (q_t, \dot{q}_t)$ is a motion of the system, then

$$\begin{aligned} \frac{d}{dt} E_{L,M}(q_t, \dot{q}_t) &= \ddot{q}_t \left(\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial \dot{q}} \right) (q_t, \dot{q}_t) + \dot{q}_t \cdot \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \right) (q_t, \dot{q}_t) = \\ &= \dot{q}_t \cdot R_{L,M}(q_t, \dot{q}_t) \end{aligned}$$

Now $\dot{q} \in \mathcal{M}_q$ means that $\dot{q} = u + Z(q)$ with $u \in \mathcal{D}_q$, then

$$\dot{q}_t \cdot R_{L,M}(q_t, \dot{q}_t) = Z(q_t) \cdot R_{L,M}(q_t, \dot{q}_t)$$

given that $R_{L,M}$ is ideal and hence annihilates \mathcal{D} . Therefore $E_{L,M}$ is a first integral if and only if, for all $q \in Q, \dot{q} \in \mathcal{M}_q, Z(q) \cdot R_{L,M}(q, \dot{q}) = 0$, that is $Z(q) \in \mathbb{R}_q^\circ$ for all $q \in Q$. □

Let us notice that if the constraint is homogeneous, namely if \hat{Z} is a section of $\hat{\mathcal{D}}$, the energy is always conserved, given that $\hat{\mathcal{D}}_{\hat{q}} \subseteq \hat{\mathbb{R}}_{\hat{q}}^\circ$ for all $\hat{q} \in \hat{Q}$: this is a well known fact, see [34]. On the contrary, the conservation of energy is not always granted for systems with nonholonomic affine constraints.

2.4 Conservation of momenta

Let us now take into account the symmetries of the system and see if they produce first integrals.

We need to recall some notions about Lie Groups and their action (for further reading, see [1, 30]).

Let us consider a Lie Group G and a smooth action $\hat{\Psi}$ of G on the configuration manifold \hat{Q}

$$\hat{\Psi} : G \times \hat{Q} \rightarrow \hat{Q} \quad (g, \hat{q}) \mapsto \hat{\Psi}(g, \hat{q}) = \hat{\Psi}_g(\hat{q})$$

The tangent lift $\hat{\Psi}^{T\hat{Q}}$ of the action $\hat{\Psi}$ is the action of G on $T\hat{Q}$ given by

$$\hat{\Psi}^{T\hat{Q}} : G \times T\hat{Q} \rightarrow T\hat{Q} \quad (g, \hat{q}, \hat{q}) \mapsto \hat{\Psi}^{T\hat{Q}}(g, \hat{q}, \hat{q}) = (\hat{\Psi}_g(\hat{q}), T_{\hat{q}}\hat{\Psi}_g \cdot \hat{q})$$

where $T_{\hat{q}}\hat{\Psi}_g$ is the tangent lift of $\hat{\Psi}_g$, in coordinates $T_q\Psi_g \cdot \dot{q} = \Psi'_g(q) \cdot \dot{q}$ with $\Psi'_g(q) = \frac{\partial \Psi_g}{\partial q}$.

Let us denote $\hat{\xi}_\eta = \frac{d}{dt} \hat{\Psi}_{\exp(t\eta)}|_{t=0}$ the *infinitesimal generator* of the action associated to an element $\eta \in \mathfrak{g}$, the Lie Algebra of G . The associated η *component of the momentum map* of $\hat{\Psi}^{T\hat{Q}}$ is, by definition, the function

$$\hat{J}_{\hat{\xi}_\eta} : T\hat{Q} \rightarrow \mathbb{R} \quad \hat{J}_{\hat{\xi}_\eta} := \langle \hat{p}_{\hat{L}}, \hat{\xi}_\eta \rangle \quad (2.2)$$

In coordinates, $J_{\xi_\eta} = \frac{\partial L}{\partial \dot{q}} \cdot \xi_\eta$.

The infinitesimal generator of the tangent lift of the action, $\hat{\Psi}^{T\hat{Q}}$, relative to $\eta \in \mathfrak{g}$, is defined by $\hat{\xi}_\eta^{T\hat{Q}} = \frac{d}{dt} \hat{\Psi}_{\exp(t\eta)}^{T\hat{Q}}|_{t=0}$ and is equal to the tangent lift of $\hat{\xi}_\eta$. In coordinates, $\xi_\eta^{TQ} = \xi_i^\eta \partial_{q_i} + \dot{q}_j \frac{\partial \xi_i^\eta}{\partial \dot{q}_j} \partial_{\dot{q}_i}$.

Proposition [13] *Given a nonholonomic system $(\hat{L}, \hat{Q}, \hat{M})$, if \hat{L} is invariant under the tangent lift $\hat{\Psi}^{T\hat{Q}}$ of an action $\hat{\Psi}$ of a Lie Group G on \hat{Q} , namely $\hat{L} \circ \hat{\Psi}_g^{T\hat{Q}}|_{\hat{M}} = \hat{L}|_{\hat{M}} \quad \forall g \in G$,*

then for any $\eta \in \mathfrak{g}$, the momentum $\hat{J}_{\xi_\eta}|_{\hat{M}}$ is a first integral of the system if and only if its generator is a section of $\hat{\mathbf{R}}^\circ$.

Proof We work in coordinates. For any motion $t \mapsto (q_t, \dot{q}_t)$

$$\begin{aligned} \frac{d}{dt} J_{\xi_\eta}|_M(q_t, \dot{q}_t) &= \left(\frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}_i} \right] \xi_i^\eta + \dot{q}_j \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \xi_i^\eta}{\partial q_j} \right)(q_t, \dot{q}_t) = \\ &= \left(\frac{\partial L}{\partial q_i} \xi_i^\eta + \dot{q}_j \frac{\partial L}{\partial \dot{q}_i} \frac{\partial \xi_i^\eta}{\partial q_j} + R_i \xi_i^\eta \right)(q_t, \dot{q}_t) = \xi_\eta^{TQ}(L)|_M + R_{L,M} \cdot \xi_\eta \end{aligned}$$

The invariance of L gives $\xi_\eta^{TQ}(L)|_M = 0$, hence $J_{\xi_\eta}|_M$ is a first integral of the system if and only if, at each $q \in Q$, ξ_η annihilates all reaction forces $R_{L,M}(q, \dot{q})$ with $\dot{q} \in \mathcal{M}_q$, i.e. $\xi_\eta(q) \in \mathbf{R}_q^\circ$.

□

More generally, one can look for the conditions for the conservation of the momentum $\hat{J}_{\hat{X}} := \langle \hat{p}_{\hat{L}}, \hat{X} \rangle$ associated to a generic vector field \hat{X} on \hat{Q} .

Proposition [17] *Given a nonholonomic system $(\hat{L}, \hat{Q}, \hat{M})$ and a vector field \hat{X} on \hat{Q} . Then any two of the following three conditions imply the third:*

- i) \hat{X} is a section of $\hat{\mathbf{R}}^\circ$.
- ii) $\hat{X}^{T\hat{Q}}(\hat{L})|_{\hat{M}} = 0$.
- iii) $\hat{J}_{\hat{X}}|_{\hat{M}}$ is a first integral of $(\hat{L}, \hat{Q}, \hat{M})$.

The proof goes as that one of the previous proposition and can be found in [17].

So in general one need not to start from the infinitesimal generators of a symmetry group action to try to build conserved momenta, but it is clear that if the system we are studying exhibits some symmetries, those generators are the natural candidates since they automatically satisfy the second condition of the previous proposition, namely the invariance of the Lagrangian.

2.5 The moving energy

It can happen that neither the energy nor the momentum associated to a symmetry group of the system are first integrals. In that case, one can look for a first integral by combining the two: the so called *moving energy*.

In this section we follow [17]. Let us start with a general definition that does not depend on the presence of symmetries of the system.

Definition [17] *Given a nonholonomic system $(\hat{L}, \hat{Q}, \hat{M})$, the **moving energy** generated by a vector \hat{X} on \hat{Q} is the restriction to \hat{M} of the function*

$$\hat{E}_{\hat{L}, \hat{X}} := \hat{E}_{\hat{L}} - \langle \hat{p}_{\hat{L}}, \hat{X} \rangle = \hat{E}_{\hat{L}} - \hat{J}_{\hat{X}} \quad (2.3)$$

where $\hat{E}_{\hat{L}}$ is the energy of (\hat{L}, \hat{Q}) , see (2.1), $\hat{p}_{\hat{L}}$ is the momentum associated to \hat{L} and $\langle \cdot, \cdot \rangle$ is the cotangent-tangent pairing.

Since $\hat{\mathcal{M}} = \hat{\mathcal{D}} + \hat{\mathcal{Z}}$, the conditions for the conservation of a moving energy are expressed by the following statement.

Proposition [17] *Any of the following three conditions imply the third:*

- i) $\hat{X} - \hat{Z}$ is a section of \hat{R}° .
- ii) $\hat{X}^{T\hat{Q}}(\hat{L})|_{\hat{M}} = 0$.
- iii) $\hat{E}_{\hat{L}, \hat{X}}|_{\hat{M}}$ is a first integral of $(\hat{L}, \hat{Q}, \hat{M})$.

Proof We work in coordinates. We make use of the results of the previous propositions. For any motion $t \mapsto (q_t, \dot{q}_t)$ of the system

$$\begin{aligned} \frac{d}{dt} E_{L,M}(q_t, \dot{q}_t) &= Z(q_t) \cdot R_{L,M}(q_t, \dot{q}_t) \\ \frac{d}{dt} (p_L \cdot X)(q_t, \dot{q}_t) &= \left(X^{TQ}(L) + R_{L,M} \cdot X \right)(q_t, \dot{q}_t) \end{aligned}$$

where X^{TQ} is the tangent lift of X . Therefore

$$\frac{d}{dt} E_{L,X}|_M = R_{L,M} \cdot (Z - X) + X^{TQ}(L)|_M$$

Thus, at each point $q \in Q$, the vanishing in all of \mathcal{M}_q of any two among $\frac{d}{dt} E_{L,X}$, $R \cdot (Z - X)$ and $X^{TQ}(L)$ implies the vanishing of the third in all of \mathcal{M}_q . In particular, $R_{L,M} \cdot (Z - X) = 0$ in all of \mathcal{M}_q means that $Z - X$ belongs to the fiber at q of \mathbb{R}° .

□

As in the case of momenta, one usually takes into account the infinitesimal generators of the symmetry group action, if they are present, also when it comes to the construction of moving energies. Therefore the previous proposition leads to the following statement.

Proposition [17] *Given a nonholonomic system $(\hat{L}, \hat{Q}, \hat{M})$, if \hat{L} is invariant under the tangent lift $\hat{\Psi}^{T\hat{Q}}$ of an action $\hat{\Psi}$ of a Lie Group G on \hat{Q} then for any $\eta \in \mathfrak{g}$, $\hat{E}_{\hat{L}, \hat{\xi}_\eta}|_{\hat{M}}$ is a first integral of $(\hat{L}, \hat{Q}, \hat{M})$ if and only if $\hat{\xi}_\eta - \hat{Z}$ is a section of \hat{R}° .*

The proof goes exactly as that one of the previous proposition.

2.6 Some remarks

We have seen that nonholonomic systems with affine constraints can present a variable behavior when it comes to the conservation of energy and momenta, in particular in presence of symmetries.

The *reaction-annihilator distribution*, which has a strongly system-dependent nature, plays a central role in the conservation mechanisms of these systems. This means, for example, that different time independent systems (with different geometry, active forces, mass distribution), subject to the same nonholonomic affine constraint and with the same symmetries, can have or not have the energy (2.1) as first integral. Besides, we find the same variety of situations when it comes to the conservation of the momentum maps (2.2) associated to the symmetries of the systems.

The recent introduction of the moving energies seems to partially solve this kind of problems and suggests that, perhaps, *energy* is not a suitable concept for nonholonomic systems.

On the other hand, given that the distribution $\hat{\mathcal{D}}$ depends only on the constraint and not on the system (i.e. the Lagrangian), the so called *horizontal generators*, namely those vector fields \hat{X} that are sections of $\hat{\mathcal{D}}$ (recall that $\hat{\mathcal{D}} \subseteq \hat{\mathbb{R}}^\circ$), are in this respect special because, according to the conditions of the propositions we have seen, they generate conserved momenta for all the nonholonomic systems $(\hat{L}, \hat{M}, \hat{Q})$ with invariant Lagrangian $\hat{X}(\hat{L})|_{\hat{M}} = 0$. An analogous argument holds for the moving energies: those generators \hat{X} such that $\hat{X} - \hat{Z}$ is a section of $\hat{\mathcal{D}}$, generate a conserved moving energy for all nonholonomic systems $(\hat{L}, \hat{M}, \hat{Q})$ with invariant Lagrangian $\hat{X}(\hat{L})|_{\hat{M}} = 0$.

2.7 Examples

Let us reconsider the examples of Sections 1.5.1 and 1.5.2 in order to illustrate the concepts we have just exposed: in particular, these examples clearly show that the conservation mechanisms are strongly system-dependent because of the role of $\hat{\mathbb{R}}^\circ$.

2.7.1 The sphere on the rotating horizontal plane

Let us consider the sphere rolling without sliding on the rotating plane of the example in Section 1.5.1, defined on $\tilde{M} \simeq Q \times \mathbb{R}^3$. We focus on the conservation of energy, momenta and moving energy.

Given the absence of the potential energy in the Lagrangian L , the energy is equal to the restriction to the manifold \tilde{M} of

$$E_{L, \tilde{M}} = \frac{m}{2} [(a\omega_y - ky)^2 + (-a\omega_x + kx)^2] + \frac{I}{2} (\omega_x^2 + \omega_y^2 + \omega_z^2)$$

Now if we derive $E_{L, \tilde{M}}$ along the flow of the vector field (1.16) we obtain

$$\frac{d}{dt} E_{L, \tilde{M}} = k^2 \frac{mIa(x\omega_y - y\omega_x)}{I + ma^2}$$

Hence the energy is conserved only in the non rotating case, i.e. when $k = 0$.

Now let us consider the S^1 -symmetry of the system, i.e. its symmetry under rotation around the e_z axis of the plane and the sphere. The infinitesimal generator of these rotations that corresponds to the Lie algebra element $\eta \in \mathbb{R}$, is

$$\xi_\eta = \begin{pmatrix} -\eta y \\ \eta x \\ 0 \\ 0 \\ \eta \end{pmatrix}, \quad \eta \in \mathbb{R}$$

Its associated momentum is

$$J_{\xi_\eta}|_{\tilde{M}} = m\eta(kx^2 - ax\omega_x + ky^2 - ay\omega_y + I\omega_z)$$

and it is not conserved

$$\frac{d}{dt}J_{\xi_\eta}|_{\tilde{M}} = \xi_\eta \cdot \tilde{R}|_{\tilde{M}} = mIak\eta \frac{x\omega_y - y\omega_x}{I + ma^2}$$

Therefore let us consider a moving energy: we need to find a generator ξ_η such that $\xi_\eta - Z$ is a section of \mathbb{R}° . That generator is ξ_k , as it is easy to check.

Hence the moving energy $E_{L,\xi_k}|_{\tilde{M}} = E_{L,\tilde{M}} - J_{\xi_k}|_{\tilde{M}}$

$$E_{L,\xi_k}|_{\tilde{M}} = -\frac{mk^2}{2}(x^2 + y^2) + \frac{I + ma^2}{2}(\omega_x^2 + \omega_y^2) + \frac{I\omega_z}{2}(\omega_z - k)$$

is a first integral of the system.

We can notice that the moving energy that we have found is $SO(3)$ -invariant and so can be projected to \tilde{M}_5 . $E_{L,\xi_k}|_{\tilde{M}_5}$ has exactly the same expression of $E_{L,\xi_k}|_{\tilde{M}}$ because of the absence of $\mathcal{R} \in SO(3)$ in the latter.

2.7.2 The sphere in the rotating cylinder

Let us consider the sphere rolling without sliding on the rotating cylinder of the Section 1.5.2, defined on $\tilde{M} \simeq Q \times \mathbb{R}^3$.

The expression for the energy $E_{L,\tilde{M}}$ is quite complex, but what really matters is that it is conserved. In fact, it is easy to check that the affine part of the constraint, i.e. the vector field Z , is a section of \mathbb{R}° .

Let us now consider the S^1 -symmetry of the system, namely its invariance under rotation along the vertical axis of the cylinder, that is e_z . The infinitesimal generator of these rotations, associated to the Lie algebra element $\eta \in \mathbb{R}$, is

$$\xi_\eta = \begin{pmatrix} \eta \\ 0 \\ 0 \\ 0 \\ \eta \end{pmatrix}, \quad \eta \in \mathbb{R}$$

and it is easy to check that is a section of \mathbb{R}° for all $\eta \in \mathbb{R}$, also if the cylinder is at rest, namely $k = 0$.

Therefore the associated momentum

$$J_{\xi_\eta}|_{\tilde{M}} = mkR\eta(R - a) + \eta\omega_z(I - ma(R - a))$$

is conserved.

Hence we need not to build a moving energy.

As in the previous example, the 2 first integrals, the energy and the S^1 -momentum, are $SO(3)$ -invariant and can be projected to \tilde{M}_5 : their expressions do not change in \tilde{M}_5 since they do not contain $\mathcal{R} \in SO(3)$.

As stated in Section 1.5.2, there exists a conjugation between the system at rest and the rotating one. The fact that the energy and the S^1 -momentum are first integrals of both systems can be explained as due to the existence of that conjugation.

Chapter 3

The sphere in the rotating cup

3.1 Introduction

In this Chapter we will study the system composed of a homogeneous sphere which rolls without sliding inside a convex surface of revolution which rotates around its axis, which is vertical, under the action of the weight force.

The study of the case in which the surface is at rest has been performed in [36, 34, 40, 27, 18]. In particular, in [27], the system, which is defined on a 8-dimensional constraint manifold, is reduced by its $SO(3) \times S^1$ -symmetry to a 4-dimensional system and this reduced system has 3 independent first integrals: the energy and other 2 quantities. It is proved that the common level sets of these 3 first integrals, in the reduced space, are compact and connected, and hence are closed curves, so that the reduced dynamics is periodic ([27] uses a different argument to prove periodicity, the compactness of the level sets is proved in [18]). Next, thanks to a reconstruction result due to Field and Krupa (see [21, 29, 27, 18, 11]), it is proved that the unreduced dynamics is quasi-periodic on tori up to dimension 3.

The study of the case in which the surface rotates has only recently begun.

Just like in the case with the surface at rest, the rotating system has a $SO(3) \times S^1$ -symmetry and can be reduced to a 4-dimensional manifold. The existence of 3 first integrals is known also in this case.

Specifically, the energy is not conserved, since the nonholonomic constraint is linear non-homogeneous in the velocities, but there is a conserved moving energy whose existence is proved in [19] (an expression for a moving energy for a homogeneous sphere which rolls on an arbitrary rotating surface appeared later in [8]). The existence of 2 first integrals, that generalize the first integrals of the case at rest, has been observed in [9].

These 3 first integrals can be used to prove the periodicity of the reduced dynamics, if their level sets are regular and compact. Reference [19], with a qualitative argument, using continuation techniques, proves the periodicity of the reduced dynamics, at least for small values of the angular velocity of the surface.

The aim of this Thesis is to investigate this problem in a more general approach, without the restriction to small values of the angular velocity, and to study in detail the dynamics of the reduced system.

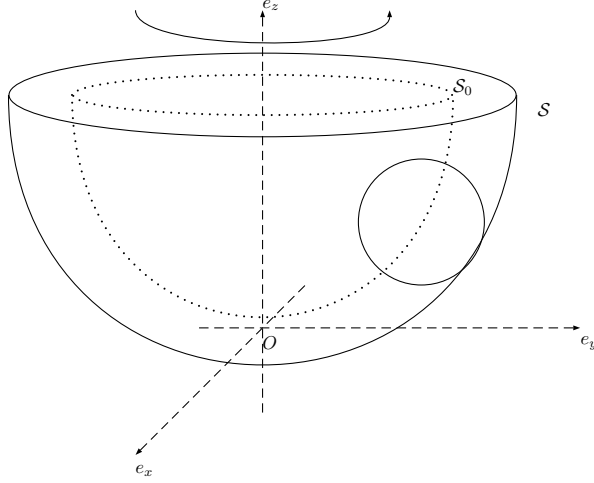


Figure 3.1: The sphere in the rotating cup

3.2 General profile

3.2.1 The system

Let us consider a fixed reference frame $\Sigma = \{O, e_x, e_y, e_z\}$ and a homogeneous sphere, with mass $m > 0$ and radius $a > 0$, whose centre C is subject to the holonomic constraint of moving in a convex surface of revolution \mathcal{S}_0 with vertical axis coincident with e_z .

More precisely, we assume that the surface is obtained by rotating around the z -axis the graph of an even, convex and smooth function $\phi : \mathbb{R} \rightarrow \mathbb{R}$

$$z = \phi(\sqrt{x^2 + y^2})$$

Next, we introduce the nonholonomic affine constraint: the sphere rolls without sliding on a surface \mathcal{S} , whose points have normale distance a from those of \mathcal{S}_0 , which rotates around its vertical axis with constant angular velocity $\vec{k} = ke_z$.

Using polar coordinates, namely $(r, \beta) \in \mathbb{R}_{>0} \times S^1$, the surface is described by $z = \phi(r)$ and the centre of mass of the sphere is

$$OC = \begin{pmatrix} r \cos \beta \\ r \sin \beta \\ \phi(r) \end{pmatrix}$$

The configuration manifold is the 5-dimensional manifold $\hat{Q} = \mathbb{R}^2 \times SO(3)$ and, as done in the other examples, let us call $(r, \beta, \mathcal{R}) = q \in Q$, where $\mathcal{R} \in SO(3)$.

When writing the Lagrangian of the system, we can get rid of the mass m of the sphere; hence we have

$$L : Q \times \mathbb{R}^2 \times \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$L = \frac{1}{2} \dot{r}^2 [1 + (\phi'(r))^2] + \frac{1}{2} r^2 \dot{\beta}^2 + \frac{1}{2} \|\omega\|^2 - g\phi(r)$$

where $I = \frac{2}{5}a^2$ is the moment of inertia of the sphere.

Now let us call P the point of the sphere in contact with the surface and impose the nonholonomic constraint of rolling without sliding

$$V_C + \omega \times CP = ke_z \times OP$$

with $V_C = \frac{d}{dt}OC$, $CP = an$ where n is the exterior normal to the surface,

$$CP = \frac{a}{\sqrt{1 + (\phi'(r))^2}} \begin{pmatrix} \phi'(r)\cos\beta \\ \phi'(r)\sin\beta \\ -1 \end{pmatrix}$$

and $OP = OC + CP$.

The constraint gives 2 independent equations, namely

$$\begin{cases} a\omega_x - a\cos\beta(k - \omega_z)\phi'(r) - [-\dot{r}\sin\beta + r\cos\beta(k - \dot{\beta})]\sqrt{1 + (\phi'(r))^2} = 0 \\ a\omega_y - a\sin\beta(k - \omega_z)\phi'(r) - [\dot{r}\cos\beta + r\sin\beta(k - \dot{\beta})]\sqrt{1 + (\phi'(r))^2} = 0 \end{cases} \quad (3.1)$$

that can be solved to obtain a parametrization of the constraint manifold.

We have started with $Q \times \mathbb{R}^2 \times \mathbb{R}^3$, then we apply the nonholonomic constraint and we can use the system (3.1) to parametrize the constraint manifold \tilde{M} : we choose $(q, \dot{r}, \dot{\beta}, \omega_z)$ with

$$\begin{cases} \omega_x = \cos\beta(k - \omega_z)\phi'(r) + \frac{[-\dot{r}\sin\beta + r\cos\beta(k - \dot{\beta})]}{a}\sqrt{1 + (\phi'(r))^2} \\ \omega_y = \sin\beta(k - \omega_z)\phi'(r) + \frac{[\dot{r}\cos\beta + r\sin\beta(k - \dot{\beta})]}{a}\sqrt{1 + (\phi'(r))^2} \end{cases}$$

Therefore the constraint manifold $\tilde{M} \subset Q \times \mathbb{R}^2 \times \mathbb{R}^3$ is 8-dimensional

$$\tilde{M} \simeq Q \times \mathbb{R}^2 \times \mathbb{R} \ni (q, \dot{r}, \dot{\beta}, \omega_z)$$

We obtain the equations of motion of the system as Hamel's equations; they are quite cumbersome and we do not present them here (see Appendix A). What really matters is that the system is invariant under the $SO(3) \times S^1$ -action. The $SO(3)$ and S^1 actions commute and so can be reduced in stages.

The $SO(3)$ -symmetry is exactly the same one that we met in Examples 1.5.1 and 1.5.2 and the reduction can be easily realized by excluding the equations $\dot{\mathcal{R}} = \tilde{\omega}\mathcal{R}$ and by performing the quotient $Q/SO(3) = \mathbb{R}^2$: hence we work in the 5-dimensional space $\tilde{M}_5 \simeq \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}$ with coordinates $(r, \beta, \dot{r}, \dot{\beta}, \omega_z)$.

The S^1 -symmetry is due to the fact that the Lagrangian and the constraint are invariant under the rotation of the surface and the sphere around the vertical axis e_z . The lifted S^1 -action is free at all points of \tilde{M}_5 except in the origin of $\mathbb{R}^2 \times \mathbb{R}^2$ (the sphere is standing on the bottom of the surface spinning around its vertical). Hence we exclude that *singular stratum*: we consider $\mathbb{R}^2 \setminus \{0, 0\} \times \mathbb{R}^2 \setminus \{0, 0\} \times \mathbb{R}$ and perform the S^1 quotient which gives us a 4-dimensional manifold.

We could use $(r, \dot{r}, \dot{\beta}, \omega_z)$ as coordinates in the reduced 4-dimensional space, but, following [27], we prefer using different coordinates: we embed the $SO(3) \times S^1$ -reduced space in \mathbb{R}^5 as manifold

$$\tilde{M}_4 = \{p \in \mathbb{R}^5 : p_0 \geq 0, p_1 \geq 0, 4p_0p_1 = p_2^2 + p_3^2, p_0^2 + p_1^2 > 0\}$$

with

$$p_0 = \frac{\dot{r}^2 + r^2\dot{\beta}^2}{2} \quad p_1 = \frac{r^2}{2} \quad p_2 = r\dot{r} \quad p_3 = r^2\dot{\beta} \quad p_4 = \omega \cdot n \quad (3.2)$$

where n is the exterior normal to the surface.

$(p_0, p_1, p_2, p_3, p_4)$ are the generators of the invariant polynomials of the S^1 -action: the invariant polynomials are usually used to perform the singular reduction (for further reading on singular reduction, see [10, 11]). They would let us to study the singular stratum too, where the lifted

S^1 -action is not free, but here we use them to study the *regular stratum* \tilde{M}_4 .

The use of the polynomials (3.2) gives us a fundamental advantage because the equations of motion of the reduced system, written in terms of those polynomials, let us build 2 first integrals, as we will show in the next Section.

It is convenient (see [19]) to describe the surface profile with $\Psi : \mathbb{R} \rightarrow \mathbb{R}$

$$\Psi\left(\frac{r^2}{2}\right) = \phi(r)$$

Hence, in terms of the polynomials (3.2), the surface is described by $\Psi(p_1)$.

Let us now show the equations of motion of the $SO(3) \times S^1$ -reduced system, defined on \tilde{M}_4 .

Proposition *The equations of motion of the reduced system defined on \tilde{M}_4 are the following*

$$\begin{aligned}\dot{p}_0 &= p_2 G_0(p_0, p_1, p_2, p_3, p_4) \\ \dot{p}_1 &= p_2 \\ \dot{p}_2 &= G_2(p_0, p_1, p_2, p_3, p_4) \\ \dot{p}_3 &= p_2 [G_3(p_1) p_4 + k g_3(p_1)] \\ \dot{p}_4 &= p_2 [G_4(p_1) p_3 + k g_4(p_1)]\end{aligned}\tag{3.3}$$

with

$$G_0(p_0, p_1, p_2, p_3, p_4) = \frac{a I p_3 \Psi''(p_1) [k + p_4 \mathcal{F}(p_1)] + \Psi'(p_1) \mathcal{F}(p_1) [\Psi'(p_1) (-2A p_0 + I k p_3) - a^2 g - A p_2^2 \Psi''(p_1)]}{A \mathcal{F}(p_1)^3}$$

$$G_2(p_0, p_1, p_2, p_3, p_4) = \frac{-a I k p_3 \Psi'(p_1) + \mathcal{F}(p_1) [2A p_0 - I p_3 (k + a p_4 \Psi'(p_1)) - 2p_1 \Psi'(p_1) (a^2 g + A p_2^2 \Psi''(p_1))]}{A \mathcal{F}(p_1)^3}$$

$$G_3(p_1) = \frac{a I [\Psi'(p_1) + 2p_1 \Psi''(p_1)]}{A \mathcal{F}(p_1)^2} \quad g_3(p_1) = \frac{I [\mathcal{F}(p_1)^3 + a \Psi'(p_1) + 2a p_1 \Psi''(p_1)]}{A \mathcal{F}(p_1)^3}$$

$$G_4(p_1) = \frac{\Psi'(p_1)^3 - \Psi''(p_1)}{a \mathcal{F}(p_1)^2} \quad g_4(p_1) = \frac{[\mathcal{F}(p_1) + a \Psi'(p_1)] [\Psi'(p_1) + 2p_1 \Psi''(p_1)]}{a \mathcal{F}(p_1)^3}$$

where

$$A = a^2 + I \quad \mathcal{F}(p_1) = \sqrt{1 + 2p_1 \Psi'(p_1)^2}$$

Proof See Appendix A.

If we put $k = 0$, we restore the equations found in [27].

The equilibria of system (3.3) are the points $p \in \tilde{M}_4$ such that

$$p_2 = 0 \quad -a I k p_3 \Psi'(p_1) + \mathcal{F}(p_1) [2A p_0 - I p_3 (k + a p_4 \Psi'(p_1)) - 2a^2 g p_1 \Psi'(p_1)] = 0 \tag{3.4}$$

In the following we will refer with "reduced system" to the $SO(3) \times S^1$ -reduced system.

3.2.2 The first integrals of the reduced system

We aim to study the dynamics of the reduced system by means of its first integrals, following the treatment of the case at rest.

Since the unreduced system is subject to an affine nonholonomic constraint, the energy is not necessarily conserved. It can be seen that Z , the affine part of the constraint, is not a section of \mathbb{R}° ; hence the energy of the system is not conserved.

We compute the moving energy $E_{L,\xi_k} = E_{L,\tilde{M}} - J_{\xi_k}|_{\tilde{M}}$, where $E_{L,\tilde{M}}$ is the energy of the unreduced system, $J_{\xi_k}|_{\tilde{M}}$ is the restriction to \tilde{M} of the conjugate momentum associated to the infinitesimal generator ξ_k of the S^1 -symmetry: this moment is not conserved because ξ_η is not a section of \mathbb{R}° for any $\eta \in \mathbb{R}$.

This moving energy is conserved, since $Z - \xi_k$ is a section of \mathbb{R}° .

The computations performed to check if Z , ξ_η and $Z - \xi_k$ are sections of \mathbb{R}° are performed with *Mathematica* because they are not straightforward.

Since $E_{L,\xi_k}|_{\tilde{M}}$ is $SO(3) \times S^1$ -invariant, it is a first integral for the reduced system as well. Let us call E the expression of the moving energy in \tilde{M}_5 .

Proposition *The moving energy E*

$$E = g\Psi(p_1) + \frac{Ap_0}{a^2} + \frac{Ik^2 p_1 \left(1 + \frac{2a\Psi'(p_1)}{\mathcal{F}(p_1)}\right)}{a^2} - \frac{Ik^2}{2\mathcal{F}(p_1)^2} + \frac{I(k^2 + p_4^2)}{2} + \frac{Ap_2^2 \Psi'(p_1)^2}{2a^2} - \frac{Ik p_3 \left(1 + \frac{a\Psi'(p_1)}{\mathcal{F}(p_1)}\right)}{a^2} \quad (3.5)$$

is a first integral for the reduced system (3.3).

Proof See Appendix B.

If we put $k = 0$ we restore the energy integral of the non rotating system, see [27].

Next, following [27] we can build 2 more first integrals. The procedure follows a classical approach that dates back to Routh, [36] (see also [34, 40, 9]).

Proposition *The reduced system defined on \tilde{M}_4 has 2 smooth first integrals which are functionally independent in all of \tilde{M}_4 and are linear in the angular velocity k of the rotating convex surface.*

Proof We start from the equations of motion for (p_1, p_3, p_4) of system (3.3)

$$\begin{cases} \dot{p}_1 = p_2 \\ \dot{p}_3 = p_2[G_3(p_1)p_4 + kg_3(p_1)] \\ \dot{p}_4 = p_2[G_4(p_1)p_3 + kg_4(p_1)] \end{cases} \quad (3.6)$$

Using the fact that $\dot{p}_1 = p_2$, we can pass from system (3.6) to

$$\begin{cases} \frac{dp_3}{dp_1} = G_3(p_1)p_4 + kg_3(p_1) \\ \frac{dp_4}{dp_1} = G_4(p_1)p_3 + kg_4(p_1) \end{cases} \quad (3.7)$$

The system (3.7) is a non-autonomous linear non-homogeneous differential system for $(p_3(p_1), p_4(p_1))$.

The functions $G_3(p_1)$, $g_3(p_1)$, $G_4(p_1)$ and $g_4(p_1)$ are defined and smooth in an interval $\mathcal{I} = (-l, +\infty)$ where $l > 0$ is the first zero of $\mathcal{F}(p_1)$ to the left of the origin.

The general integral of system (3.7) is

$$p_3 \quad : \quad p_1 \mapsto K_1 \Pi_3(p_1) + K_2 \Sigma_3(p_1) + \tilde{\varphi}_3(p_1)$$

$$p_4 \quad : \quad p_1 \mapsto K_1 \Pi_4(p_1) + K_2 \Sigma_4(p_1) + \tilde{\varphi}_4(p_1)$$

where K_1, K_2 are real constants, $(\Pi_3(p_1), \Pi_4(p_1))$ and $(\Sigma_3(p_1), \Sigma_4(p_1))$ are 2 independent solutions of the homogeneous system associated to (3.7) (it is given by putting $k = 0$), $(\tilde{\varphi}_3(p_1), \tilde{\varphi}_4(p_1))$ is a particular solution of (3.7). These solutions are defined and smooth in the whole interval \mathcal{I} .

The particular solution is linear in the angular velocity k : let us show this property.

Call

$$\Phi(p_1) := \begin{pmatrix} \Pi_3(p_1) & \Sigma_3(p_1) \\ \Pi_4(p_1) & \Sigma_4(p_1) \end{pmatrix}$$

and

$$kv(p_1) := k \begin{pmatrix} g_3(p_1) \\ g_4(p_1) \end{pmatrix}$$

The matrix $\Phi(p_1)$ non singular because the solutions of the homogeneous system associated to (3.7) are independent in the interval of existence \mathcal{I} . That is to say that

$$D(p_1) := \det \left[\begin{pmatrix} \Pi_3(p_1) & \Sigma_3(p_1) \\ \Pi_4(p_1) & \Sigma_4(p_1) \end{pmatrix} \right] \neq 0 \quad \forall p_1 \in \mathcal{I}$$

Furthermore $D(p_1)$ is a constant:

$$\frac{d}{dp_1} D(p_1) = \Pi_3'(p_1) \Sigma_4(p_1) + \Pi_3 \Sigma_4'(p_1) - \Pi_4'(p_1) \Sigma_3(p_1) - \Pi_4(p_1) \Sigma_3'(p_1) =$$

$$= \Pi_4(p_1) G_3(p_1) \Sigma_4(p_1) + \Pi_3(p_1) G_4(p_1) \Sigma_3(p_1) - G_4(p_1) \Pi_3(p_1) \Sigma_3(p_1) - \Pi_4(p_1) G_3(p_1) \Sigma_4(p_1) = 0$$

where we have used the equations (3.7).

Let us call D the constant determinant.

Using the variation of constants method (see [12]), a particular solution $\tilde{\varphi}(p_1) = (\tilde{\varphi}_3(p_1), \tilde{\varphi}_4(p_1))$ is given by

$$\tilde{\varphi}(p_1) = \Phi(p_1) \int_{\mathcal{I}} (\Phi^{-1}(\bar{p}_1) kv(\bar{p}_1)) d\bar{p}_1 = k [\Phi(p_1) \int_{\mathcal{I}} (\Phi^{-1}(\bar{p}_1) v(\bar{p}_1)) d\bar{p}_1]$$

and therefore is linear in k . Let us highlight this fact, writing for the particular solution

$$k \begin{pmatrix} \Gamma_3(p_1) \\ \Gamma_4(p_1) \end{pmatrix} := \begin{pmatrix} \tilde{\varphi}_3(p_1) \\ \tilde{\varphi}_4(p_1) \end{pmatrix}$$

The general integral is then

$$\begin{pmatrix} p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} \Pi_3(p_1) & \Sigma_3(p_1) \\ \Pi_4(p_1) & \Sigma_4(p_1) \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} + k \begin{pmatrix} \Gamma_3(p_1) \\ \Gamma_4(p_1) \end{pmatrix}$$

Therefore let us compute $K_1 = \mathcal{Y}_1(p_1, p_3, p_4)$ and $K_2 = \mathcal{Y}_2(p_1, p_3, p_4)$

$$\mathcal{Y}_1(p_1, p_3, p_4) := \frac{1}{D} [p_3 \Sigma_4(p_1) - p_4 \Sigma_3(p_1)] + \frac{k}{D} [\Sigma_3(p_1) \Gamma_4(p_1) - \Sigma_4(p_1) \Gamma_3(p_1)] \quad (3.8)$$

$$\mathcal{Y}_2(p_1, p_3, p_4) := \frac{1}{D} [p_4 \Pi_3(p_1) - p_3 \Pi_4(p_1)] + \frac{k}{D} [\Pi_4(p_1) \Gamma_3(p_1) - \Pi_3(p_1) \Gamma_4(p_1)]$$

The functions $\mathcal{Y}_1(p_1, p_3, p_4)$ and $\mathcal{Y}_2(p_1, p_3, p_4)$ are first integrals of the reduced system (3.3).

In fact, reconsidering the time dependence of the polynomials,

$$\begin{aligned} \frac{d}{dt} \mathcal{Y}_1(p_1, p_3, p_4) &= \frac{1}{D} [\dot{p}_3 \Sigma_4(p_1) + p_3 \Sigma_4'(p_1) \dot{p}_1 - \dot{p}_4 \Sigma_3(p_1) - p_4 \Sigma_3'(p_1) \dot{p}_1] + \\ &\quad + \frac{k}{D} [\Sigma_3'(p_1) \dot{p}_1 \Gamma_4(p_1) + \Sigma_3(p_1) \Gamma_4'(p_1) \dot{p}_1 - \Sigma_4'(p_1) \dot{p}_1 \Gamma_3(p_1) - \Sigma_4(p_1) \Gamma_3'(p_1) \dot{p}_1] \end{aligned}$$

and substituting

$$\begin{aligned}
\dot{p}_1 &= p_2 \\
\dot{p}_3 &= p_2[G_3(p_1)p_4 + kg_3(p_1)] \\
\dot{p}_4 &= p_2[G_4(p_1)p_3 + kg_4(p_1)] \\
\Pi'_3(p_1) &= G_3(p_1)\Pi_4(p_1) \\
\Pi'_4(p_1) &= G_4(p_1)\Pi_3(p_1) \\
\Sigma'_3(p_1) &= G_3(p_1)\Sigma_4(p_1) \\
\Sigma'_4(p_1) &= G_4(p_1)\Sigma_3(p_1) \\
\Gamma'_3(p_1) &= G_3(p_1)\Gamma_4(p_1) + g_3(p_1) \\
\Gamma'_4(p_1) &= G_4(p_1)\Gamma_3(p_1) + g_4(p_1)
\end{aligned}$$

we obtain $\frac{d}{dt}\mathcal{Y}_1(p_1, p_3, p_4) = 0$. Analogously $\frac{d}{dt}\mathcal{Y}_2(p_1, p_3, p_4) = 0$.

Finally, \mathcal{Y}_1 and \mathcal{Y}_2 are functionally independent in all of \tilde{M}_4 , as it is easy to check since $\Pi_3(p_1)\Sigma_4(p_1) - \Sigma_3(p_1)\Pi_4(p_1) = D \neq 0$.

□

Let us rewrite the 2 first integrals

$$\mathcal{Y}_1(p_1, p_3, p_4) = \mathcal{J}_1(p_1, p_3, p_4) + ky_1(p_1)$$

$$\mathcal{Y}_2(p_1, p_3, p_4) = \mathcal{J}_2(p_1, p_3, p_4) + ky_2(p_1)$$

where

$$\mathcal{J}_1(p_1, p_3, p_4) = \frac{1}{D}[p_3\Sigma_4(p_1) - p_4\Sigma_3(p_1)], \quad \mathcal{J}_2(p_1, p_3, p_4) = \frac{1}{D}[p_4\Pi_3(p_1) - p_3\Pi_4(p_1)]$$

and

$$ky_1(p_1) = \frac{k}{D}[\Sigma_3(p_1)\Gamma_4(p_1) - \Sigma_4(p_1)\Gamma_3(p_1)], \quad ky_2(p_1) = \frac{k}{D}[\Pi_4(p_1)\Gamma_3(p_1) - \Pi_3(p_1)\Gamma_4(p_1)]$$

Hence the first integrals \mathcal{Y}_1 and \mathcal{Y}_2 are composed of \mathcal{J}_1 and \mathcal{J}_2 , that are the first integrals of the non rotating case, found in [27]¹, and 2 terms ky_1 , ky_2 that are linear in the angular velocity k and depend on p_1 .

Now we demonstrate that the moving energy E , \mathcal{Y}_1 and \mathcal{Y}_2 are functionally independent in all of \tilde{M}_4 but the equilibria of the system (3.3) which are given by (3.4).

Proposition *The map $(E, \mathcal{Y}_1, \mathcal{Y}_2) : \tilde{M}_4 \rightarrow \mathbb{R}^3$ is a submersion in all of \tilde{M}_4 but at the equilibria of (3.3).*

Proof We follow exactly the same proof for the non rotating case of the work [19]. In \tilde{M}_4 we have

$$\mathcal{F}(p_0, p_1, p_2, p_3) = 0, \quad p_0 \geq 0, \quad p_1 \geq 0, \quad p_0^2 + p_1^2 > 0$$

for $\mathcal{F}(p_0, p_1, p_2, p_3) = \frac{p_3^2 + p_2^2}{2} - 2p_0p_1$. We introduce Lagrange multipliers $\mu, \lambda, \lambda_1, \lambda_2$ and show that, at each point of S_4 , the equation

$$\mu d\mathcal{F} + \lambda dE + \lambda_1 d\mathcal{Y}_1 + \lambda_2 d\mathcal{Y}_2 = 0$$

¹In [27], the first integrals are $D\mathcal{J}_1$ and $-D\mathcal{J}_2$, but it is essentially the same result.

has only the trivial solution $\mu = \lambda = \lambda_1 = \lambda_2 = 0$. Let us consider the function $\mathcal{G} = \mu\mathcal{F} + \lambda E + \lambda_1\mathcal{Y}_1 + \lambda_2\mathcal{Y}_2$ where the Lagrange multipliers have to be thought as parameters. Now let us start from

$$\frac{\partial \mathcal{G}}{\partial p_0} = \frac{A}{a^2}\lambda - 2\mu p_1 \quad \frac{\partial \mathcal{G}}{\partial p_2} = \frac{A\Psi'(p_1)^2}{a^2}p_2\lambda + p_2\mu$$

and they vanish simultaneously if and only if either $\mu = \lambda = 0$ or $p_2 = 0$ and $\lambda = 2\frac{a^2}{A}p_1\mu$. The first case leads to $\lambda_1 d\mathcal{Y}_1 + \lambda_2 d\mathcal{Y}_2 = 0$ that has the only solution $\lambda_1 = \lambda_2 = 0$ since \mathcal{Y}_1 and \mathcal{Y}_2 are functionally independent in M_4 . So we consider $p_2 = 0$ and $\lambda = 2\frac{a^2}{A}p_1\mu$ with nonzero λ and μ . We may assume $\mu = 1$. Now let us consider $\frac{\partial \mathcal{G}}{\partial p_3}$ and $\frac{\partial \mathcal{G}}{\partial p_4}$, evaluated at $p_2 = 0$ and $\lambda = 2\frac{a^2}{A}p_1$

$$\begin{aligned} \frac{\partial \mathcal{G}}{\partial p_3} \Big|_{p_2=0, \lambda=2\frac{a^2}{A}p_1} &= -2kp_1 + p_3 + \frac{\lambda_1 \Sigma_4(p_1) - \lambda_2 \Pi_4(p_1)}{D} \\ \frac{\partial \mathcal{G}}{\partial p_4} \Big|_{p_2=0, \lambda=2\frac{a^2}{A}p_1} &= \frac{2a^2 I p_1}{A} \left(p_4 + \frac{k}{\mathcal{F}(p_1)} \right) + \frac{\lambda_2 \Pi_3(p_1) - \lambda_1 \Sigma_3(p_1)}{D} \end{aligned}$$

We put $\frac{\partial \mathcal{G}}{\partial p_3} \Big|_{p_2=0, \lambda=2\frac{a^2}{A}p_1} = 0$ and $\frac{\partial \mathcal{G}}{\partial p_4} \Big|_{p_2=0, \lambda=2\frac{a^2}{A}p_1} = 0$ and obtain the following linear system

$$\frac{1}{D} \begin{pmatrix} \Sigma_4(p_1) & -\Pi_4(p_1) \\ -\Sigma_3(p_1) & \Pi_3(p_1) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 2kp_1 - p_3 \\ -\frac{2a^2 I p_1}{A} \left(p_4 + \frac{k}{\mathcal{F}(p_1)} \right) \end{pmatrix}$$

that we solve for λ_1, λ_2 . Given that the matrix is non singular, this system has a unique solution: let us call it $(\bar{\lambda}_1, \bar{\lambda}_2)$. So we have the following relations

$$\begin{cases} \frac{\bar{\lambda}_1 \Sigma_4(p_1) - \bar{\lambda}_2 \Pi_4(p_1)}{D} = 2kp_1 - p_3 \\ \frac{\bar{\lambda}_1 \Sigma_3(p_1) - \bar{\lambda}_2 \Pi_3(p_1)}{D} = \frac{2a^2 I p_1}{A} \left(p_4 + \frac{k}{\mathcal{F}(p_1)} \right) \end{cases} \quad (3.9)$$

Now we compute $\frac{\partial \mathcal{G}}{\partial p_1}$ evaluated at $p_2 = 0, \lambda = 2\frac{a^2}{A}p_1, \lambda_1 = \bar{\lambda}_1, \lambda_2 = \bar{\lambda}_2$. Let us start from $\bar{\lambda}_1 \frac{\partial \mathcal{Y}_1}{\partial p_1} + \bar{\lambda}_2 \frac{\partial \mathcal{Y}_2}{\partial p_1}$, recalling the system (3.9). In order to obtain a clearer result, we omit the dependence on p_1 .

$$\begin{aligned} \frac{\partial \mathcal{Y}_1}{\partial p_1} &= \frac{1}{D} [p_3 G_4 \Sigma_3 - p_4 G_3 \Sigma_4] + \frac{k}{D} [G_3 \Sigma_4 \Gamma_4 + \Sigma_3 (G_4 \Gamma_3 + g_4) - G_4 \Sigma_3 \Gamma_3 - \Sigma_4 (G_3 \Gamma_4 + g_3)] = \\ &= \frac{1}{D} [p_3 G_4 \Sigma_3 - p_4 G_3 \Sigma_4 + k(\Sigma_3 g_4 - \Sigma_4 g_3)] \end{aligned}$$

Analogously for $\frac{\partial \mathcal{Y}_2}{\partial p_1}$

$$\frac{\partial \mathcal{Y}_2}{\partial p_1} = \frac{1}{D} [p_4 G_3 \Pi_4 - p_3 G_4 \Pi_3 + k(\Pi_4 g_3 - \Pi_3 g_4)]$$

Therefore, a straightforward computation shows that

$$\bar{\lambda}_1 \frac{\partial \mathcal{Y}_1}{\partial p_1} + \bar{\lambda}_2 \frac{\partial \mathcal{Y}_2}{\partial p_1} = \frac{\bar{\lambda}_1 \Sigma_3 - \bar{\lambda}_2 \Pi_3}{D} (p_3 G_4 + k g_4) - \frac{\bar{\lambda}_1 \Sigma_4 - \bar{\lambda}_2 \Pi_4}{D} (p_4 G_3 + k g_3)$$

that, thanks to (3.9), gives

$$\bar{\lambda}_1 \frac{\partial \mathcal{Y}_1}{\partial p_1} + \bar{\lambda}_2 \frac{\partial \mathcal{Y}_2}{\partial p_1} = \frac{2a^2 I p_1}{A} \left(p_4 + \frac{k}{\mathcal{F}(p_1)} \right) (p_3 G_4 + k g_4) - (2kp_1 - p_3) (p_4 G_3 + k g_3)$$

This result does not depend on the specific form of the solutions of the system (3.7). Finally we compute $\frac{\partial \mathcal{G}}{\partial p_1}$ at $p_2 = 0$ and $\lambda = 2\frac{a^2}{A}p_1$

$$\begin{aligned}\frac{\partial \mathcal{G}}{\partial p_1} \Big|_{p_2=0, \lambda=2\frac{a^2}{A}p_1} &= \frac{\partial \mathcal{F}}{\partial p_1} + \left(2\frac{a^2}{A}p_1\right) \frac{\partial E}{\partial p_1} \Big|_{p_2=0} + \bar{\lambda}_1 \frac{\partial \mathcal{Y}_1}{\partial p_1} + \bar{\lambda}_2 \frac{\partial \mathcal{Y}_2}{\partial p_1} = \\ &= \frac{aIk p_3 \Psi'(p_1)}{A\mathcal{F}(p_1)} - 2p_0 + \frac{Ip_3}{A} (k + ap_4 \Psi'(p_1)) + 2\frac{a^2}{A} gp_1 \Psi'(p_1)\end{aligned}$$

Hence, for $p_2 = 0$, the condition $\frac{\partial \mathcal{G}}{\partial p_1} = 0$ is exactly the equilibrium condition given by (3.4). Therefore, the restriction to S_4 of the functions $E, \mathcal{Y}_1, \mathcal{Y}_2$ are functionally independent at all points but the equilibria of the system (3.3). □

We stress that the results of this Section apply to all profiles, in fact our computations are independent on the specific expression of the solutions of (3.7), in other words we do not need to specify the profile of the surface.

It is also clear that system (3.7) leads to the construction of 2 first integrals both for the fixed and the rotating case, the former being obtained by putting $k = 0$.

Nevertheless, it has to be said that, with certain profiles, system (3.7) can not be solved and hence \mathcal{Y}_1 and \mathcal{Y}_2 can not be explicitly computed: a case in which this is possible is with a paraboloid of revolution, as we will see in the following.

3.2.3 Boundedness of the level sets of the moving energy

In Section 3.2.2 we have shown that $(E, \mathcal{Y}_1, \mathcal{Y}_2) : \tilde{M}_4 \rightarrow \mathbb{R}^3$ is a submersion in all of \tilde{M}_4 except on the equilibria of the reduced system. Therefore if we can say something about the topology of the intersections of the level sets of these 3 first integrals, we can obtain informations about the orbits of the reduced system.

Let us consider the reduced moving energy (3.5).

The expression (3.5) is given by a sum of terms, but some of them can have positive or negative sign and therefore it is not clear if its level sets are bounded. Let us examine the moving energy term by term

- $\frac{Ik^2}{2\mathcal{F}(p_1)^2}$ is positive and bounded because $0 < \frac{1}{\mathcal{F}(p_1)^2} \leq 1$
- $g\Psi(p_1) + \frac{Ap_0}{a^2} + \frac{Ik^2 p_1}{a^2} + \frac{Ip_4^2}{2} + \frac{Ap_2^2 \Psi'(p_1)^2}{2a^2}$ is a sum of positive terms
- $-\frac{Akp_3}{a^2}$ has undefined sign
- $\frac{Ip_4}{\mathcal{F}(p_1)}$ has undefined sign

Hence the problematic term is $-\frac{Akp_3}{a^2} + \frac{Ip_4}{\mathcal{F}(p_1)}$ because if we fix a finite value for the moving energy, this term can diverge to $-\infty$ while the positive sum $g\Psi(p_1) + \frac{Ap_0}{a^2} + \frac{Ik^2 p_1}{a^2} + \frac{Ip_4^2}{2} + \frac{Ap_2^2 \Psi'(p_1)^2}{2a^2}$ can diverge to $+\infty$.

We can show that if the profile satisfies a certain condition, the moving energy has bounded level sets.

Definition The profile $\Psi(p_1)$ is said to be *superquadratic*² for p_1 big enough if $p_1 = o(\Psi(p_1))$ for $p_1 \rightarrow \infty$, namely

$$\lim_{p_1 \rightarrow \infty} \frac{p_1}{\Psi(p_1)} = 0$$

Proposition Given a profile superquadratic for p_1 big enough, the moving energy (3.5) has bounded level sets in S_4 .

Proof Let us consider the moving energy given by (3.5). Since $\frac{Ik^2}{2}(-1 + \frac{1}{\mathcal{F}(p_1)^2})$ is positive and bounded and $\frac{Ap_2^2\Psi'(p_1)^2}{2a^2} \geq 0$, we have

$$E \geq g\Psi(p_1) - \frac{Akp_3}{a^2} + \frac{Ikp_4}{\mathcal{F}(p_1)} + \frac{Ap_0}{a^2} + \frac{Ik^2p_1}{a^2} + \frac{Ip_4^2}{2}$$

Since in \tilde{M}_4 and hence $4p_1p_0 = p_2^2 + p_3^2$ gives $p_3^2 \leq 4p_0p_1$,

$$E \geq g\Psi(p_1) - \frac{Ak\sqrt{4p_0p_1}}{a^2} + \frac{Ikp_4}{\mathcal{F}(p_1)} + \frac{Ap_0}{a^2} + \frac{Ik^2p_1}{a^2} + \frac{Ip_4^2}{2}$$

Next $I = A - a^2$ and so

$$\begin{aligned} E &\geq \left[g\Psi(p_1) - k^2p_1 \right] + \left[\frac{Ikp_4}{\mathcal{F}(p_1)} + \frac{Ip_4^2}{2} \right] + \frac{A}{a^2} \left[p_0 + k^2p_1 - 2\sqrt{p_0k^2p_1} \right] = \\ &= \left[g\Psi(p_1) - k^2p_1 \right] + \left[\frac{Ikp_4}{\mathcal{F}(p_1)} + \frac{Ip_4^2}{2} \right] + \frac{A}{a^2} \left[\sqrt{p_0} - \sqrt{k^2p_1} \right]^2 = \\ &= f(p_1) + g(p_4) + [h(p_0, p_1)]^2 \end{aligned}$$

with

$$f(p_1) = g\Psi(p_1) - k^2p_1 \quad g(p_4) = \frac{Ikp_4}{\mathcal{F}(p_1)} + \frac{Ip_4^2}{2} \quad h(p_0, p_1) = \sqrt{p_0} - \sqrt{k^2p_1}$$

We conclude that

$$f(p_1) + g(p_4) \leq E - [h(p_0, p_1)]^2 \leq E$$

Now if $\Psi(p_1)$ is superquadratic for p_1 big enough, then

$$f(p_1) > 0 \quad \text{for } p_1 \gg 1 \quad \text{and} \quad \lim_{p_1 \rightarrow +\infty} f(p_1) = +\infty$$

Moreover

$$g(p_4) > 0 \quad \text{for } p_4 \gg 1 \quad \text{and} \quad \lim_{p_4 \rightarrow +\infty} g(p_4) = +\infty$$

So that $f(p_1) + g(p_4) \leq E$ implies the boundedness of both p_1 and p_4 : \exists constants $l_E > 0$ and $m_E > 0$ such that

$$p_1 \leq l_E \quad |p_4| \leq m_E$$

Furthermore, for $1 \ll p_1 \leq l_E$ and $1 \ll |p_4| \leq m_E$, the inequality

$$[h(p_0, p_1)]^2 \leq E - f(p_1) - g(p_4)$$

²The term "superquadratic" is due to the fact that $p_1 = r^2/2$.

gives $[h(p_0, p_1)]^2 \leq E$, namely

$$\frac{\sqrt{A}}{a} \left[\sqrt{p_0} - \sqrt{k^2 p_1} \right] \leq \sqrt{E}$$

Hence

$$\sqrt{p_0} \leq \frac{a}{\sqrt{A}} \sqrt{E} + \sqrt{k^2 p_1} \leq \frac{a}{\sqrt{A}} \sqrt{E} + k \sqrt{l_E} := \sqrt{n_E}$$

and so

$$p_0 \leq n_E$$

Finally, the boundedness of p_1 and p_0 implies that of p_3 and p_2 , given that $4p_0 p_1 = p_2^2 + p_3^2$. □

Hence, since the reduced moving energy, (3.5), $E : \tilde{M}_4 \rightarrow \mathbb{R}$ is a continuous function, its level sets are closed. Therefore, with a profile which is superquadratic for p_1 big enough, those level sets are *compact*.

In the next Section we will use this fact to show the periodicity of the dynamics of the reduced system.

3.2.4 Periodic dynamics of the reduced system

Let us prove the following statement.

Theorem *Given a profile superquadratic for p_1 big enough, the orbits of the $SO(3) \times S^1$ -reduced system are periodic except possibly on the critical fibers of the $(E, \mathcal{Y}_1, \mathcal{Y}_2)$ fibration, which correspond to the equilibria of the reduced system and their level sets.*

Proof Let us consider a profile superquadratic for p_1 big enough. In Section 3.2.3 we have seen that the moving energy (3.5) has compact level sets if the profile is superquadratic for p_1 big enough.

Besides \mathcal{Y}_1 and \mathcal{Y}_2 are smooth functions and their level sets in \tilde{M}_4 are hence closed.

Let us consider \tilde{M}_4 without the equilibria (3.4) of the reduced system and the equilibria level sets. Let us call these critical fibers \bar{e} . If we exclude \bar{e} from \tilde{M}_4 , we are left with the other closed level sets of the 3 first integrals.

Hence the intersections of the level sets of the 3 first integrals $E, \mathcal{Y}_1, \mathcal{Y}_2$ are compact in $\tilde{M}_4 \setminus \bar{e}$, because the intersections of the compact level sets of E with the closed level sets of \mathcal{Y}_1 and \mathcal{Y}_2 are compact.

So we have 3 functionally independent first integrals with compact common fibers in the 4-dimensional space $\tilde{M}_4 \setminus \bar{e}$. Therefore the connected component of these common fibers are closed curves and the reduced dynamics is periodic on them. □

For what concerns the unreduced dynamics, we can state the following fact.

Corollary *Given a profile superquadratic for p_1 big enough, the dynamics of the unreduced system is generically quasi-periodic on tori up to dimension 3.*

Proof This fact is proved by means of the argument of Field and Krupa (see [21, 29, 27, 18, 11]).

Briefly: the group $SO(3) \times S^1$ acts freely on the constraint manifold \tilde{M} , after the exclusion of the singular stratum where the action of S^1 is not free; the reduced dynamics is generically periodic and hence its reconstructed dynamics is generically quasi-periodic on tori up to dimension 3, that is the rank of $SO(3) \times S^1$. □

In the next Section, in order to better understand the characteristics of the reduced dynamics, we will study the case in which the profile is a paraboloid of revolution. For that specific profile we will be able to explicitly compute the first integrals. In so doing, we will also give us information about the dynamics on a profile which is not superquadratic for p_1 big enough.

3.3 The rotating paraboloid

3.3.1 The system

Let us consider the case of a paraboloid of revolution, namely the surface is described by

$$z = \frac{r^2}{l}$$

where l is a parameter that determines the width of the profile. We assume $l > 2a$ (a is the radius of the sphere), so as to exclude that the sphere gets stuck in the bottom of the paraboloid.

We aim to study the dynamics of the $SO(3) \times S^1$ -reduced system, by means of the 3 first integrals we have seen in Section 3.2.2.

All we have stated in the previous Sections remain true, except for the boundedness of the moving energy.

Since we need to reduce the number of parameters of the system, we perform a scaling of the variables: we consider r/a and $t\sqrt{\frac{g}{l}}$.

Hence the Lagrangian of the system is

$$L = \frac{1}{2} [\dot{r}^2 (1 + 4\chi^2 r^2) + r^2 \dot{\beta}^2] + \frac{||\omega||^2}{5} - r^2$$

where $\chi = a/l$ is the only parameter, namely the ratio between the radius of the sphere and the width parameter of the paraboloid, with $0 < \chi < 1/2$.

As done in the general case, we reduce the system under $SO(3) \times S^1$ and we obtain the 4-dimensional manifold \tilde{M}_4 immersed in \mathbb{R}^5 , defined by means of the polynomials (3.2).

Proposition *The equation of motion of the reduced system defined on \tilde{M}_4 are the following*

$$\begin{aligned} \dot{p}_0 &= -\frac{2p_2[5+4\chi^2(7p_0-kp_3)]}{7(1+8\chi^2p_1)} \\ \dot{p}_1 &= p_2 \\ \dot{p}_2 &= -\frac{2[2k\chi p_3 + \sqrt{1+8\chi^2p_1}(-7p_0+10p_1+kp_3+2\chi p_3 p_4)]}{7(1+8\chi^2p_1)^{3/2}} \\ \dot{p}_3 &= p_2 \left[\frac{4\chi p_4}{7(1+8\chi^2p_1)} + k \frac{4(1+8\chi^2p_1)^{3/2} + 4\chi}{7(1+8\chi^2p_1)^{3/2}} \right] \\ \dot{p}_4 &= p_2 \left[\frac{8\chi^3 p_3}{1+8\chi^2 p_1} + k \frac{4\chi^2 + 2\chi \sqrt{1+8\chi^2 p_1}}{(1+8\chi^2 p_1)^{3/2}} \right] \end{aligned} \tag{3.10}$$

Proof To prove this statement it is sufficient to rewrite the equations of system (3.3), with the following substitutions

$$\begin{aligned}
p_0 &= a^2 p_0 \\
p_1 &= a^2 p_1 \\
p_2 &= a^2 p_2 \\
p_3 &= a^2 p_3 \\
\Psi(p_1) &= 2a^2 p_1 / l \\
\Psi'(p_1) &= 2/l \\
\Psi''(p_1) &= 0 \\
g &= l
\end{aligned}$$

□

The equilibria of system (3.10) are the points $p \in \tilde{M}_4$ such that

$$p_2 = 0 \quad 2k\chi p_3 + \sqrt{1 + 8\chi^2 p_1}(-7p_0 + 10p_1 + kp_3 + 2\chi p_3 p_4) = 0$$

In the next Section we will study the first integrals of the $SO(3) \times S^1$ -reduced system (we will refer to it by "reduced system"); in particular we will explicitly compute the 2 first integrals $\mathcal{Y}_1(p_1, p_3, p_4)$ and $\mathcal{Y}_2(p_1, p_3, p_4)$ given by (3.8).

3.3.2 The first integrals of the reduced system

Let us start from the moving energy E of the reduced system.

Proposition *The moving energy E*

$$E = \frac{7}{5}p_0 + \left(2 + \frac{2k^2}{5}\right)p_1 + \frac{k^2}{5(1 + 8\chi^2 p_1)} + \frac{14\chi^2}{5}p_2^2 - \frac{7k}{5}p_3 + \frac{2k}{5\sqrt{1 + 8\chi^2 p_1}}p_4 + \frac{1}{5}p_4^2 \quad (3.11)$$

is a first integral of the reduced system (3.10).

Proof To prove this statement it is sufficient to repeat the computations that lead to the expression (3.5), namely to compute the energy of the system and the S^1 -momentum.

□

Next we follow the procedure we have seen in Section 3.2.2 to compute 2 other first integrals.

Proposition *The reduced system has 2 first integrals of the form (3.8) with*

$$\begin{aligned}
\Pi_3(p_1) &= \cosh\left[\frac{\ln(1+8\chi^2 p_1)}{\sqrt{14}}\right] \\
\Pi_4(p_1) &= \sqrt{14}\chi \sinh\left[\frac{\ln(1+8\chi^2 p_1)}{\sqrt{14}}\right] \\
\Sigma_3(p_1) &= \sinh\left[\frac{\ln(1+8\chi^2 p_1)}{\sqrt{14}}\right] \\
\Sigma_4(p_1) &= \sqrt{14}\chi \cosh\left[\frac{\ln(1+8\chi^2 p_1)}{\sqrt{14}}\right] \\
\Gamma_3(p_1) &= \frac{4p_1}{13} - \frac{11}{52\chi^2} \\
\Gamma_4(p_1) &= \frac{1}{26\chi} + \frac{4\chi p_1}{13} - \frac{1}{\sqrt{1+8\chi^2 p_1}}
\end{aligned} \tag{3.12}$$

Proof Starting from system (3.10) and considering the equations for p_1, p_3, p_4 , we obtain system (3.7) with

$$\begin{aligned}
G_3(p_1) &= \frac{4\chi}{7(1+8\chi^2 p_1)} \\
G_4(p_1) &= \frac{8\chi^2}{1+8\chi^2 p_1} \\
\Gamma_3(p_1) &= \frac{4(1+8\chi^2 p_1)^{3/2} + 4\chi}{7(1+8\chi^2 p_1)^{3/2}} \\
\Gamma_4(p_1) &= \frac{4\chi^2 + 2\chi\sqrt{1+8\chi^2 p_1}}{(1+8\chi^2 p_1)^{3/2}}
\end{aligned}$$

If we repeat the computations of Section 3.2.2, we can prove that \mathcal{Y}_1 and \mathcal{Y}_2 have the expressions given by (3.8) with the substitutions (3.12). □

As above these 2 first integrals and the moving energy (3.11) form a submersion from \tilde{M}_4 to \mathbb{R}^3 , except on the equilibria (3.10) of the reduced system.

However we are not able to state the boundedness of the level sets of the moving energy, since the profile is not superquadratic for p_1 big enough.

3.3.3 Dynamics in the paraboloid

In this Section we want to obtain some information about the reduced dynamics in the rotating paraboloid by means of the 3 first integral of the system.

We want to study the restriction of the dynamics to the 2-dimensional common level sets of \mathcal{Y}_1 and \mathcal{Y}_2 ; these level sets are characterised by the values \bar{y}_1, \bar{y}_2 of the 2 first integrals, hence let us call them $\Sigma_{\bar{y}_1, \bar{y}_2}$.

We will show that the restricted dynamics is lagrangian with a Lagrangian of mechanical form $L = T - \mathcal{V}$, where T is the kinetic energy and \mathcal{V} is an effective positional potential. Furthermore we will use standard techniques, like the analysis of the phase portrait, to study the restricted dynamics.

Excluding the bottom of the paraboloid

First of all let us describe the level sets $\Sigma_{\bar{y}_1, \bar{y}_2}$ in the reduced manifold. Let us use the following coordinates to parametrize the reduced system

$$(r, \dot{r}, \dot{\beta}, \omega_z) \in \mathbb{R}_{>0} \times \mathbb{R}^3$$

These coordinates exclude $r = 0$ which means the bottom of the paraboloid and all the motions that pass through that point. Let us call $\bar{M}_4 := \mathbb{R}_{>0} \times \mathbb{R}^3$ (notice that $\bar{M}_4 \simeq \tilde{M}_4|_{p_1 > 0}$).

Proposition In \bar{M}_4

- i) Each $\Sigma_{\bar{y}_1, \bar{y}_2}$ is a 2-dimensional submanifold that can be parametrized with (r, \dot{r}) .
- ii) The restriction of the reduced system to each $\Sigma_{\bar{y}_1, \bar{y}_2}$ is a lagrangian system, with 1 degree of freedom, with Lagrangian of the mechanical form $L = T - \mathcal{V}$, where we can interpret T as a kinetic energy and \mathcal{V} as an effective positional potential.

Proof

- i) Let us rewrite the first integrals \mathcal{Y}_1 and \mathcal{Y}_2 in terms of $(r, \dot{r}, \dot{\beta}, \omega_z)$, using the relations (3.2), and fix 2 values (\bar{y}_1, \bar{y}_2) for the 2 integrals. Next we obtain expressions for $\dot{\beta}$ and ω_z as functions of r with \bar{y}_1 and \bar{y}_2 as parameters.

$$\begin{aligned} \dot{\beta} &= \frac{2k}{13} + \frac{1}{r^2} \left[-\frac{11k}{52\chi^2} + \bar{y}_1 \cosh \left[\frac{\log(1+4\chi^2 r^2)}{\sqrt{14}} \right] + \bar{y}_2 \sinh \left[\frac{\log(1+4\chi^2 r^2)}{\sqrt{14}} \right] \right] \\ \omega_z &= \frac{5k+20k\chi^2 r^2 - 13\chi^2(2\bar{y}_1 + \sqrt{14}\bar{y}_2) \cosh \left[\frac{\log(1+4\chi^2 r^2)}{\sqrt{14}} \right] + 13k\sqrt{1+4\chi^2 r^2} - 13\chi^2(\sqrt{14}\bar{y}_1 + 2\bar{y}_2) \sinh \left[\frac{\log(1+4\chi^2 r^2)}{\sqrt{14}} \right]}{13\chi\sqrt{1+4\chi^2 r^2}} \end{aligned} \quad (3.13)$$

System 3.13 is the graph expression of $\Sigma_{\bar{y}_1, \bar{y}_2}$.

Let us rename the expressions in (3.13)

$$\begin{aligned} \dot{\beta} &= f_{\dot{\beta}}(r; \bar{y}_1, \bar{y}_2) \\ \omega_z &= f_{\omega_z}(r; \bar{y}_1, \bar{y}_2) \end{aligned}$$

Hence $\Sigma_{\bar{y}_1, \bar{y}_2}$ can be parametrized in \bar{M}_4 by $(r, \dot{r}) \in \mathbb{R}_{>0} \times \mathbb{R}$

$$(r, \dot{r}) \mapsto \left(\begin{array}{c} r \\ \dot{r} \\ \dot{\beta} = f_{\dot{\beta}}(r; \bar{y}_1, \bar{y}_2) \\ \omega_z = f_{\omega_z}(r; \bar{y}_1, \bar{y}_2) \end{array} \right) \in \Sigma_{\bar{y}_1, \bar{y}_2} \subset \bar{M}_4 \quad (3.14)$$

- ii) Let us write the moving energy E (3.11) in \bar{M}_4 using the relations (3.2) and restrict it to $\Sigma_{\bar{y}_1, \bar{y}_2}$ with the parametrization given by (3.14): the restriction has the following form

$$E|_{\Sigma_{\bar{y}_1, \bar{y}_2}} = T(r, \dot{r}) + \mathcal{V}(r; \bar{y}_1, \bar{y}_2)$$

where $T(r, \dot{r}) = \frac{7}{10} (1 + 4\chi^2 r^2) \dot{r}^2$ is a kinetic term and $\mathcal{V}(r; \bar{y}_1, \bar{y}_2)$ is an effective positional potential that depends on the parameters (\bar{y}_1, \bar{y}_2) , whose expression is given in

Appendix C.

This is the energy of the following Lagrangian

$$L : \mathbb{R}_{>0} \times \mathbb{R} \rightarrow \mathbb{R}$$

$$L(r, \dot{r}; \bar{y}_1, \bar{y}_2) = T(r, \dot{r}) - \mathcal{V}(r; \bar{y}_1, \bar{y}_2) \quad (3.15)$$

Moreover if we compute the Lagrange equation for \ddot{r} we obtain the equation for \dot{v}_r of system (5.2) restricted to $\Sigma_{\bar{y}_1, \bar{y}_2}$ with the parametrization given by (3.14) (see Appendix D for more details).

□

Hence, the restriction to the submanifolds $\Sigma_{\bar{y}_1, \bar{y}_2}$ produces a family of lagrangian systems that depend on the values of the first integrals $\mathcal{Y}_1, \mathcal{Y}_2$.

These systems can also be regarded as hamiltonian systems and the Hamiltonian is given by $H = T + \mathcal{V}(\bar{y}_1, \bar{y}_2)$. This is a remarkable fact, given that nonholonomic systems are far from being lagrangian (and hamiltonian), as we have already noticed. There is an active study field in this regard aimed to the research of hamiltonian structures in nonholonomic systems: see for instance [19, 24].

The critical points of the effective potential

Let us study the dynamics of the lagrangian systems defined on the level sets $\Sigma_{\bar{y}_1, \bar{y}_2}$. Let us notice that, since the lagrangian systems have 1 degree of freedom, we can show and analyse their phase portraits in $\mathbb{R}_{>0} \times \mathbb{R} \ni (r, \dot{r})$.

This study can also give us information about the case at rest, whose dynamics has never been studied in this detail.

We start studying the equilibria of these lagrangian systems: they are given by the critical points of $\mathcal{V}(r)$.

Since the expression of the effective potential is quite cumbersome, the study of its critical points is performed numerically with the use of *Mathematica*. This numerical analysis shows that there is a great difference between the case at rest and the rotating case. In fact, as we will show, at least for our choice of the values \bar{y}_1, \bar{y}_2 , we see that, when $k = 0$, there is always 1 minimum point for $\mathcal{V}(r)$, while when $k \neq 0$ the effective potential can present from 0 up to 3 critical points. In fact, in the rotating case, $\mathcal{V}(r)$ has generically 1 or 3 critical points: in this last case, there are 2 minima and 1 maximum that give 2 stable equilibria and 1 unstable equilibrium that we will call r_{s_1}, r_{s_2} and r_u . Sometimes, instead, $\mathcal{V}(r)$ has 0 or 2 critical points: it occurs when the lower stable equilibrium r_{s_1} reaches the bottom of the paraboloid and disappears; as we will show this fact is related to the presence of a cusp at $r = 0$ in the diagram of the critical point of $\mathcal{V}(r)$.

All these equilibria correspond, in the $SO(3)$ -reduced system (namely, the motion of the centre of mass of the sphere), to horizontal circular orbits with constant r , i.e. constant height on the paraboloid.

It is particularly interesting the presence of a family of unstable equilibria r_u : they are hyperbolic equilibria with their stable and unstable manifolds which give, in the $SO(3)$ -reduced system, horizontal circular orbits and motions spiraling back and forth to them.

As we are going to show, the dynamics scenario is very variegated with the appearance of

critical points with a bifurcation mechanism, as we change the parameters values we are dealing with.

In this Paragraph we show the position of the critical points of $\mathcal{V}(r)$, for different cases. We proceed by fixing different values \bar{y}_1 and \bar{y}_2 , while we vary the angular velocity k . We keep $\chi = 0.2$ for all this analysis.

In this study we take into account the fact that the dynamics with angular velocity k and $-k$ are conjugated (obviously).

Proposition *If X_k is the $SO(3) \times S^1$ -reduced vector field given by (3.3), then for all $k \in \mathbb{R}$*

$$C_*X_k = X_{-k}$$

where the conjugation \mathcal{C} is given by

$$\mathcal{C} : \tilde{M}_4 \rightarrow \tilde{M}_4$$

$$\mathcal{C} : (p_0, p_1, p_2, p_3, p_4) \mapsto (p_0, p_1, p_2, -p_3, -p_4)$$

This conjugation acts on the first integrals $\mathcal{Y}_{1,2}$ (3.8) in this way

$$C_*\mathcal{Y}_{1,2} = -\mathcal{Y}_{1,2}|_{-k}$$

Proof Straightforward computation acting on (3.3) and (3.8).

□

Therefore we will study the system for 25 couples of values (\bar{y}_1, \bar{y}_2) , showed in Figure 3.2. We call each couple of values $\bar{Y}^{(i)} := (\bar{y}_1^{(i)}, \bar{y}_2^{(i)})$, $i = 1, \dots, 25$. For each $\bar{Y}^{(i)}$ we vary k , but it is sufficient to consider $k \geq 0$ given the conjugation exposed above.

The results are showed in the following pages, in Figures from 3.3 to 3.15(f).

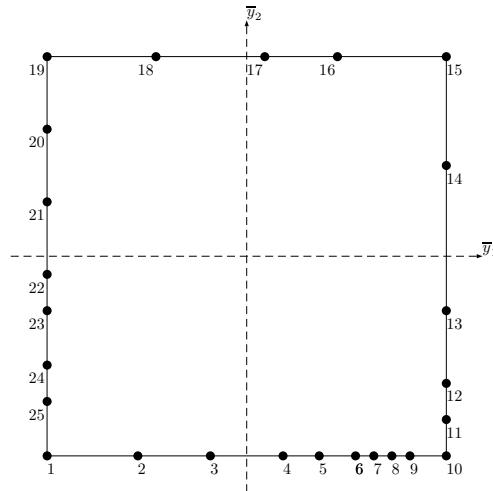


Figure 3.2: Different values of $(\bar{y}_1, \bar{y}_2) = \bar{Y}^{(i)}$. The range is $\bar{y}_1 \in [-100, 100]$, $\bar{y}_2 \in [-150, 150]$

Let us start with a first example, relative to the couple $\bar{Y}^{(1)}$, and let us analyse it in detail in order to better understand the other cases too.

Figure 3.3 shows a rich situation (notice that the r -axis has logarithmic scale in order to have a clearer image). We see the presence of line of minima r_{s_1} , but we also notice an "island" of critical points of $\mathcal{V}(r)$ which is bounded by a higher line of minima r_{s_2} and a lower line of maxima r_u .

The appearance of these critical points is characterized by a bifurcation mechanism due to the fact that r_u and r_{s_2} join to make a single minimum, passing through a horizontal flex point. This bifurcation diagram, which is sometimes called *imperfect pitchfork*, is interesting because it is not general; for further reading on bifurcations, see [25, 39].

Figures 3.4(a), 3.4(b), 3.4(c) illustrate the profile of the effective potential for different values of k , showing the presence of either 1 or 3 critical points, while Figures 3.5(a), 3.5(b), 3.5(c) show the corresponding phase portraits.

Figures from 3.6(a) to 3.6(h) show how the position of the critical points changes as we move from $\bar{Y}^{(2)}$ to $\bar{Y}^{(10)}$: we vary \bar{y}_1 and maintain \bar{y}_2 constant.

We can notice that the tip of the island becomes lower as we change \bar{y}_1 . Furthermore it appears a cusp at $r = 0$ in the line of the minima r_{s_1} : for a certain value of k , r_{s_1} approaches the bottom of the paraboloid, i.e. $r = 0$, and then moves away from it. Since each equilibrium corresponds to a horizontal circular orbit at constant height on the paraboloid, what happens is that the stable horizontal circular orbit corresponding to r_{s_1} "falls" towards the bottom for a certain value of k , becomes an equilibrium when it reaches $r = 0$ and then reappears and rises up, as we change k . We can not analyse in detail what happens in the cusp, since $V(r)$ (see Appendix C for its expression) is divergent for $r = 0$.

Figures from 3.6(d), 3.6(e), 3.6(f) show what happens when we pass from $\bar{Y}^{(6)}$ to $\bar{Y}^{(8)}$: the line of minima r_{s_1} approaches the line of maxima r_u ; these 2 lines connect in $\bar{Y}^{(7)}$ (when r_{s_1} coincides with r_u , $\mathcal{V}(r)$ has a flex point); then the line of r_u breaks into 2 distinct parts.

Figures 3.7 and 3.10 show in detail the situation in $\bar{Y}^{(7)}$ and $\bar{Y}^{(10)}$.

Figures from 3.13(a) to 3.13(e) show that the cusp does not move when \bar{y}_2 is constant and we pass from $\bar{Y}^{(11)}$ to $\bar{Y}^{(15)}$. Next, Figures from 3.14(a) to 3.14(d) show that the cusp moves when we vary \bar{y}_2 passing from $\bar{Y}^{(16)}$ to $\bar{Y}^{(19)}$. In all of these cases, there is only the line of minima r_{s_1} , namely $\mathcal{V}(r)$ has only a critical point (a minimum), except in the cases $\bar{Y}^{(11)}$ and $\bar{Y}^{(12)}$, Figures 3.13(a) and 3.13(b), that are characterised by the presence of a small bifurcation region near $k = 20$ with the presence of 3 critical points: this region disappears passing from $\bar{Y}^{(12)}$ to $\bar{Y}^{(13)}$.

Figures from 3.15(a) to 3.15(f) show the appearance of the island of r_u and r_{s_2} , as we pass from $\bar{Y}^{(20)}$ to $\bar{Y}^{(25)}$, namely by changing \bar{y}_2 and keeping \bar{y}_1 constant.

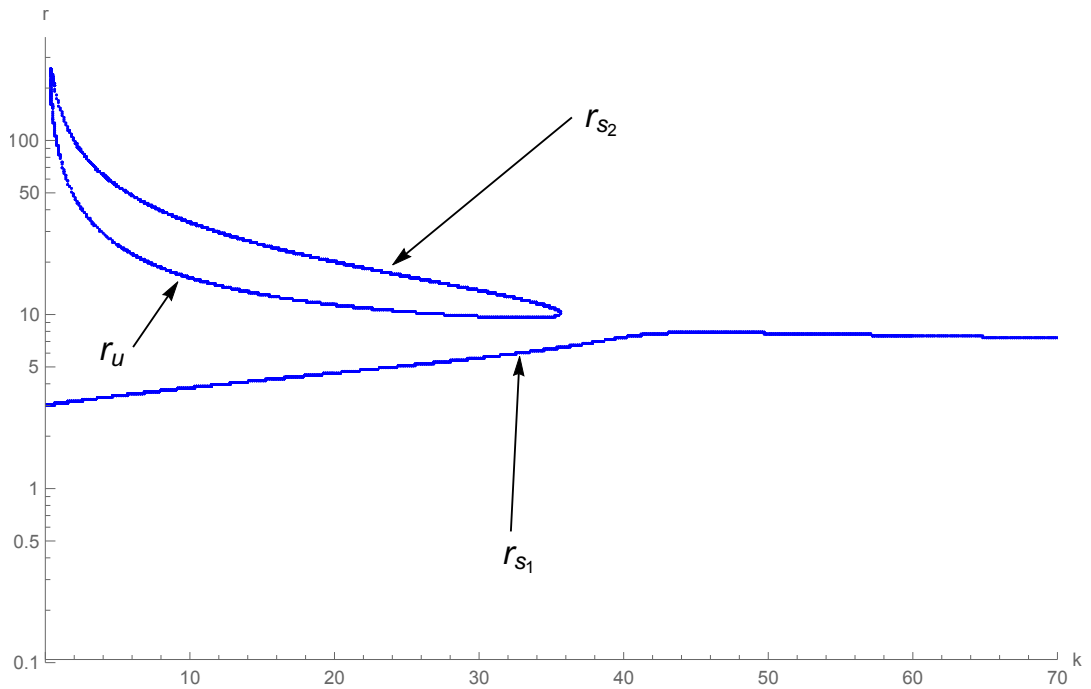
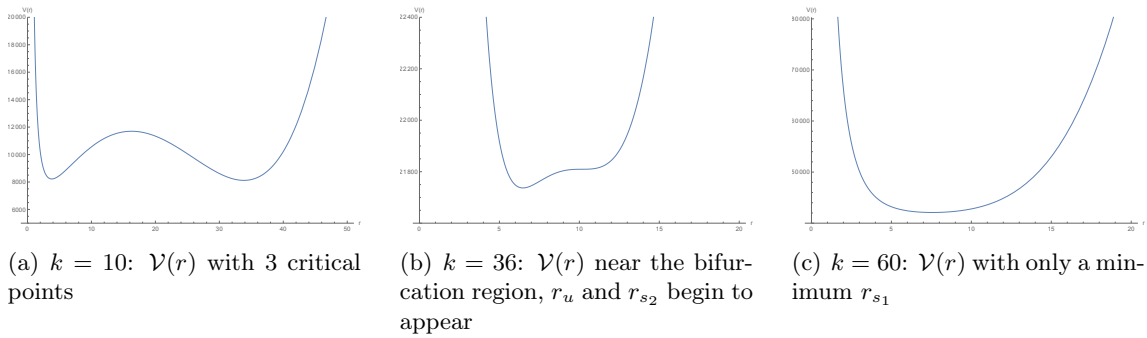


Figure 3.3: Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(1)}$

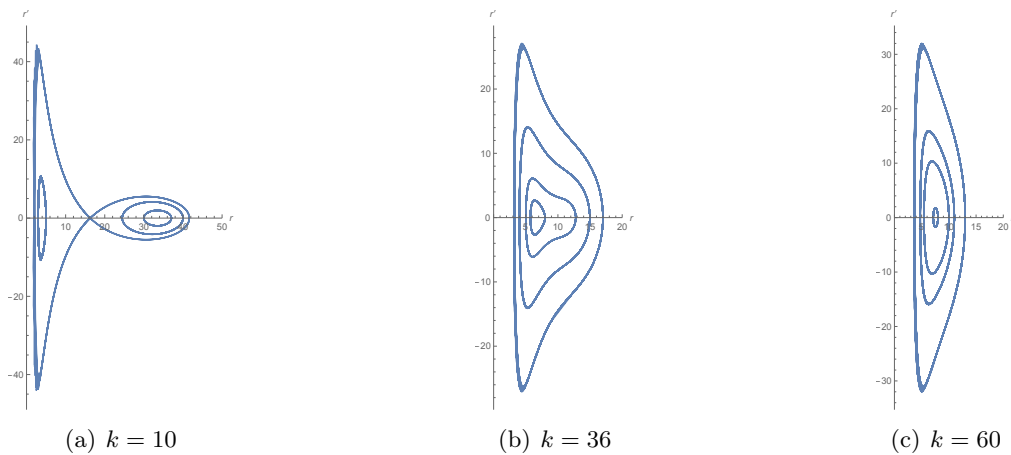


(a) $k = 10$: $\mathcal{V}(r)$ with 3 critical points

(b) $k = 36$: $\mathcal{V}(r)$ near the bifurcation region, r_u and r_{s_2} begin to appear

(c) $k = 60$: $\mathcal{V}(r)$ with only a minimum r_{s_1}

Figure 3.4: The effective potential $\mathcal{V}(r)$

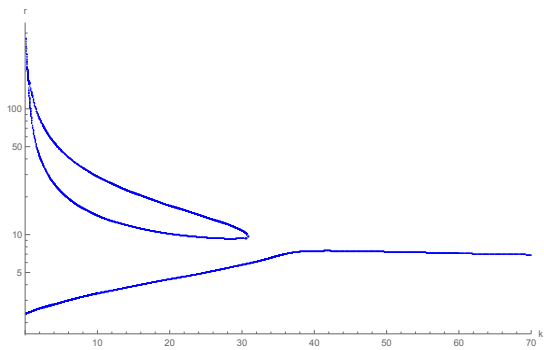


(a) $k = 10$

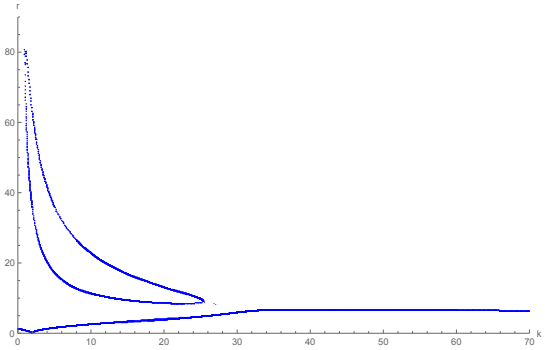
(b) $k = 36$

(c) $k = 60$

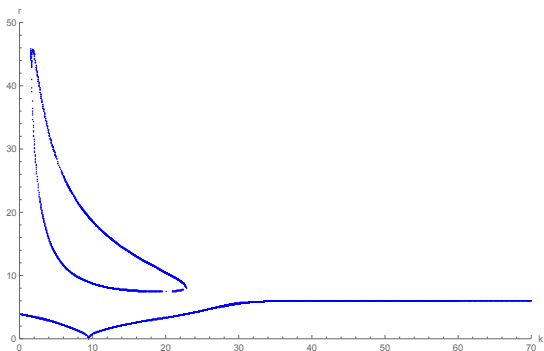
Figure 3.5: The phase portrait



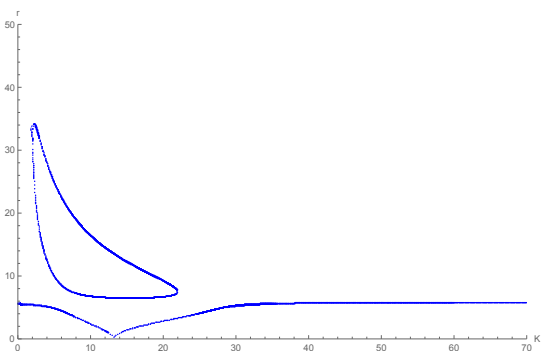
(a) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(2)}$



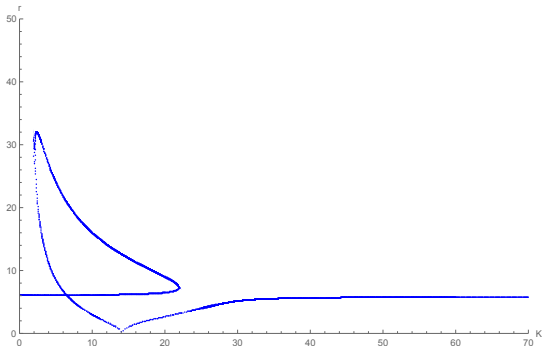
(b) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(3)}$



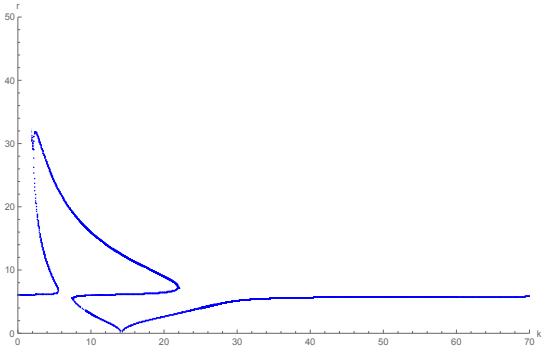
(c) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(4)}$



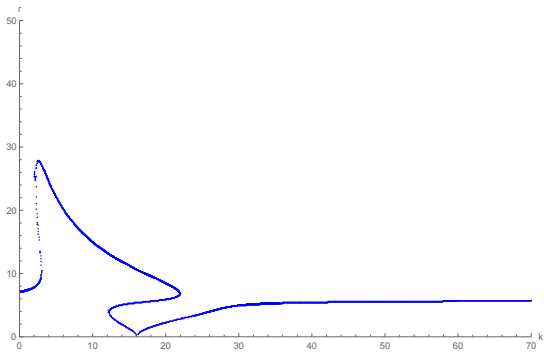
(d) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(6)}$



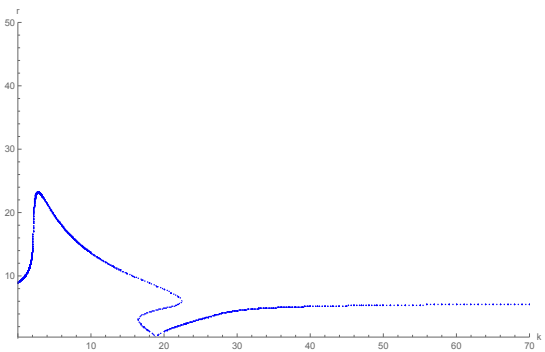
(e) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(7)}$



(f) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(8)}$



(g) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(9)}$



(h) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(10)}$

Figure 3.6: Critical points of $\mathcal{V}(r)$ from $\bar{Y}^{(2)}$ to $\bar{Y}^{(10)}$

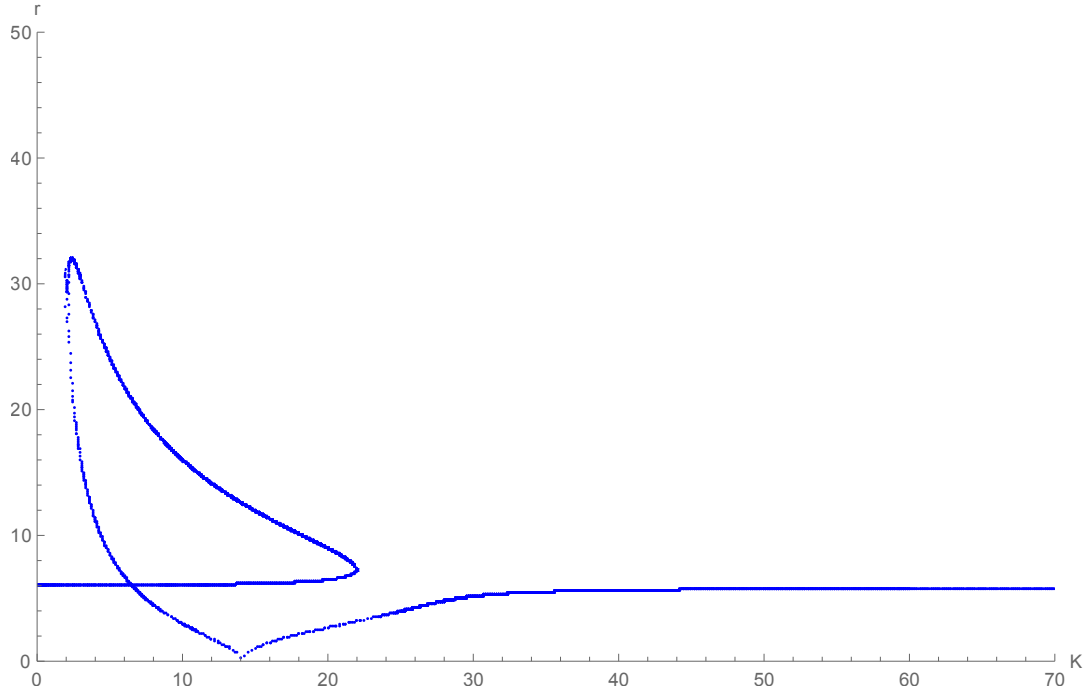
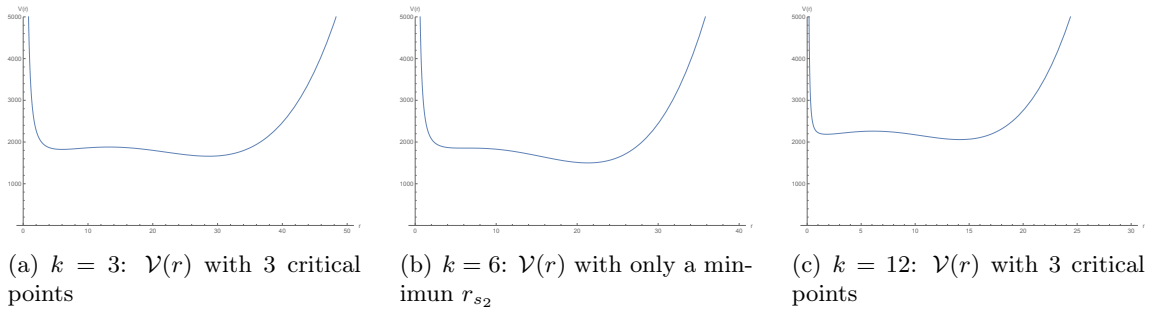


Figure 3.7: Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(7)}$

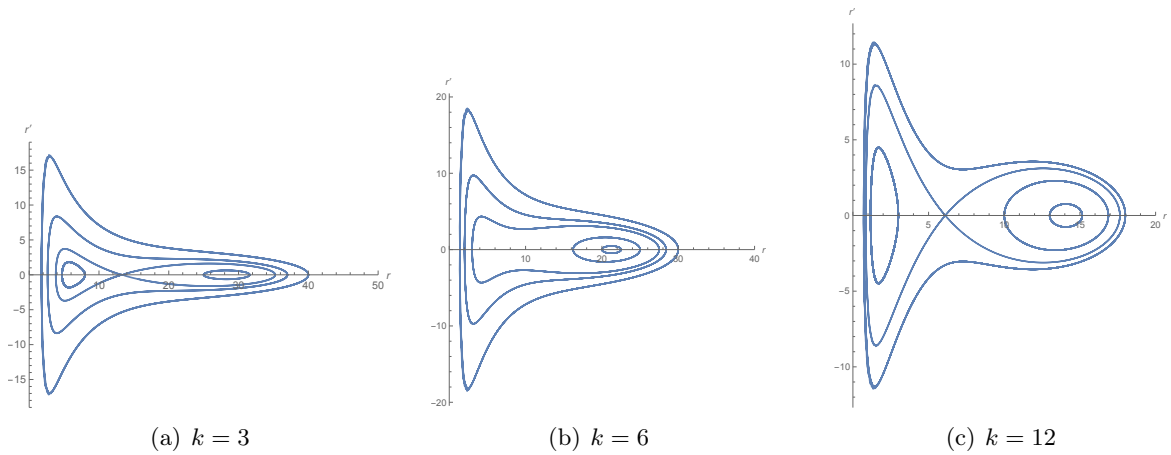


(a) $k = 3$: $\mathcal{V}(r)$ with 3 critical points

(b) $k = 6$: $\mathcal{V}(r)$ with only a minimum r_{s_2}

(c) $k = 12$: $\mathcal{V}(r)$ with 3 critical points

Figure 3.8: The effective potential $\mathcal{V}(r)$



(a) $k = 3$

(b) $k = 6$

(c) $k = 12$

Figure 3.9: The phase portrait

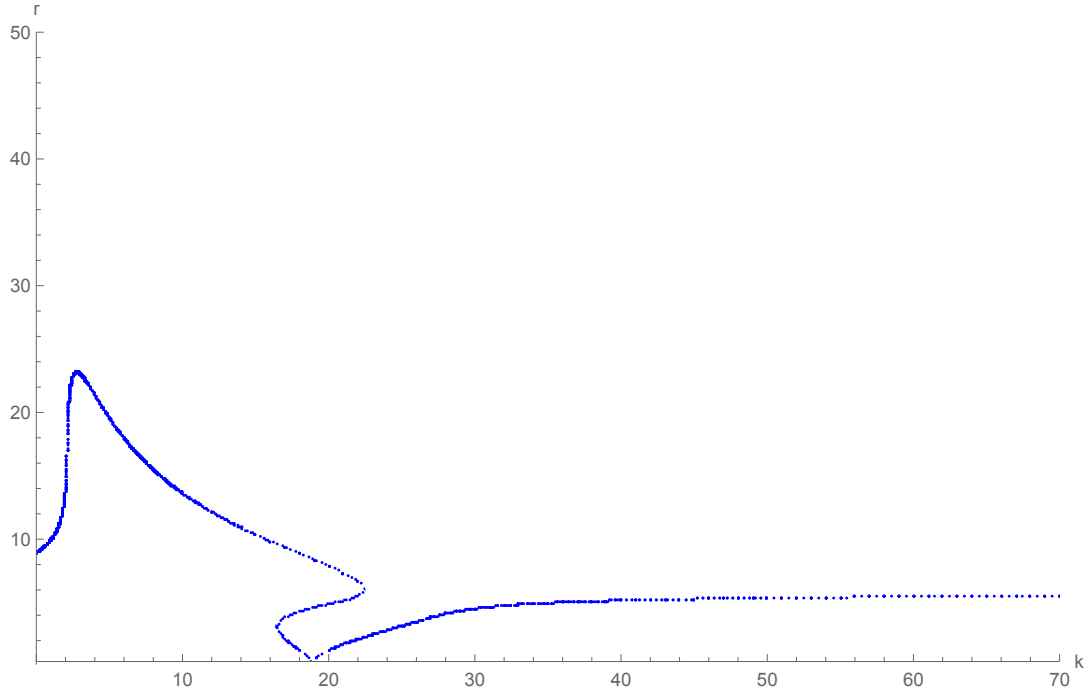
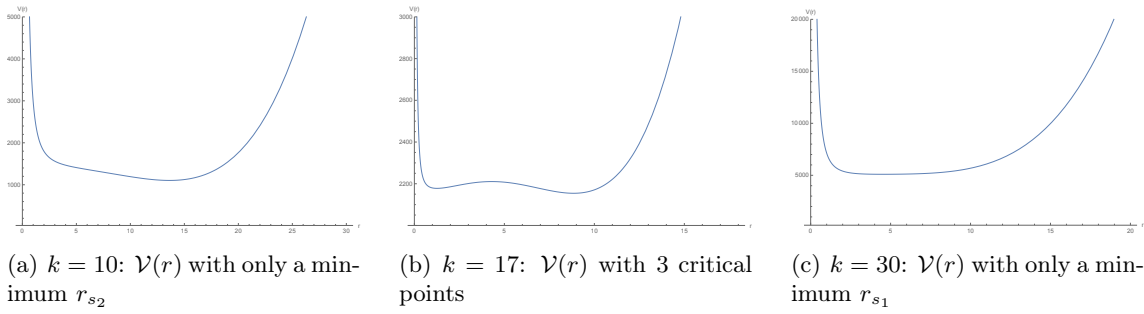


Figure 3.10: Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(10)}$

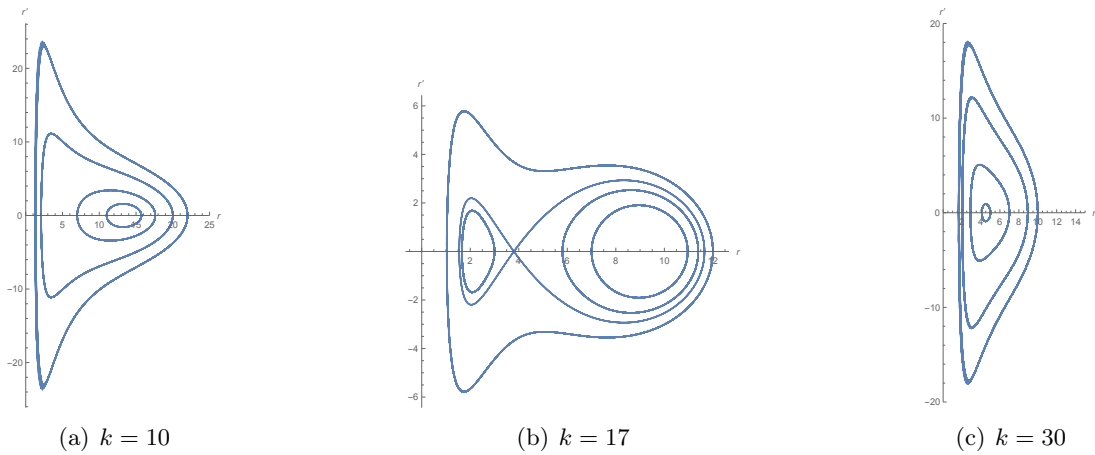


(a) $k = 10$: $\mathcal{V}(r)$ with only a minimum r_{s2}

(b) $k = 17$: $\mathcal{V}(r)$ with 3 critical points

(c) $k = 30$: $\mathcal{V}(r)$ with only a minimum r_{s1}

Figure 3.11: The effective potential $\mathcal{V}(r)$

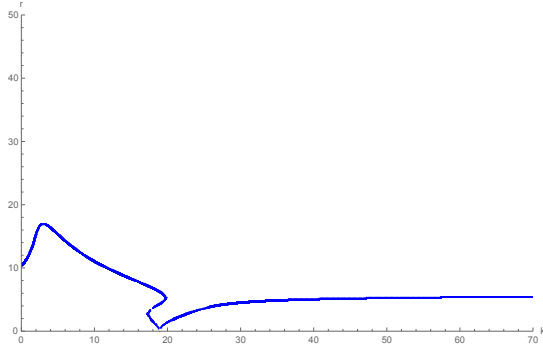


(a) $k = 10$

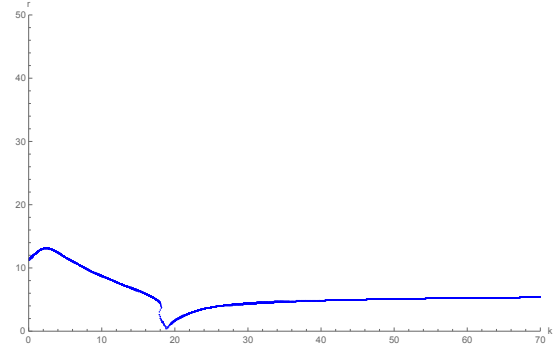
(b) $k = 17$

(c) $k = 30$

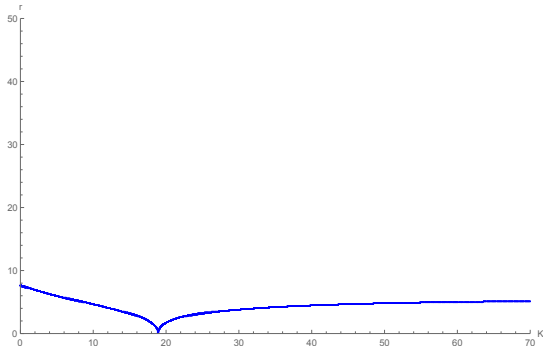
Figure 3.12: The phase portrait



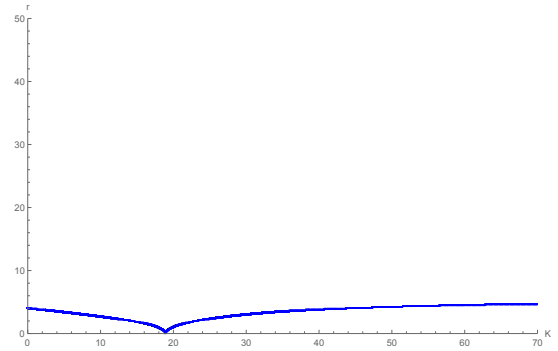
(a) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(11)}$



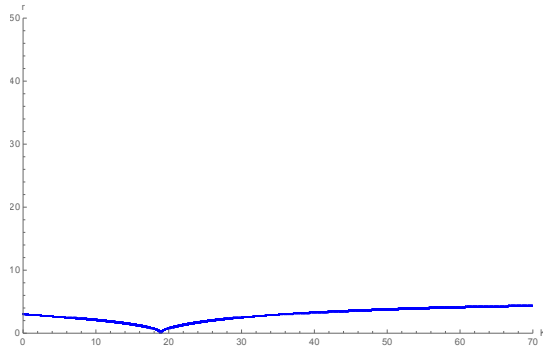
(b) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(12)}$



(c) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(13)}$

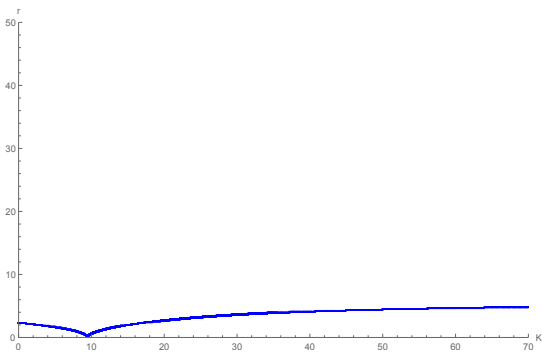


(d) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(14)}$

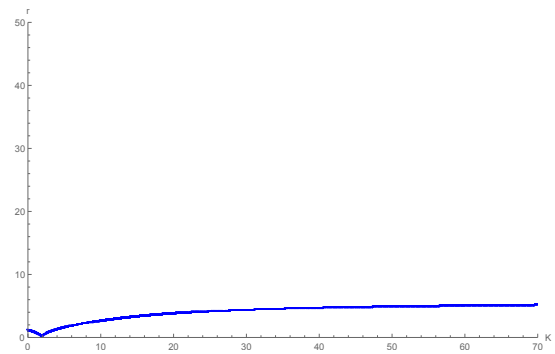


(e) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(15)}$

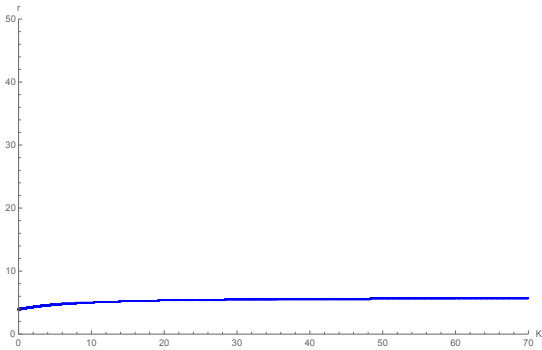
Figure 3.13: Critical points of $\mathcal{V}(r)$ from $\bar{Y}^{(11)}$ to $\bar{Y}^{(15)}$



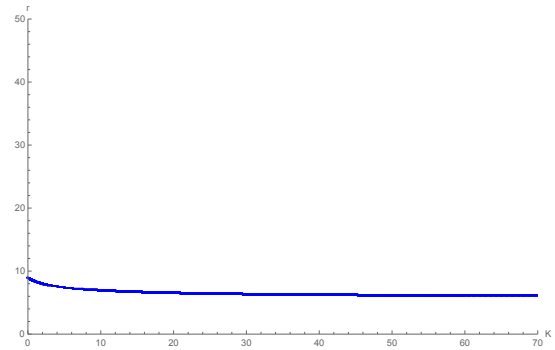
(a) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(16)}$



(b) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(17)}$

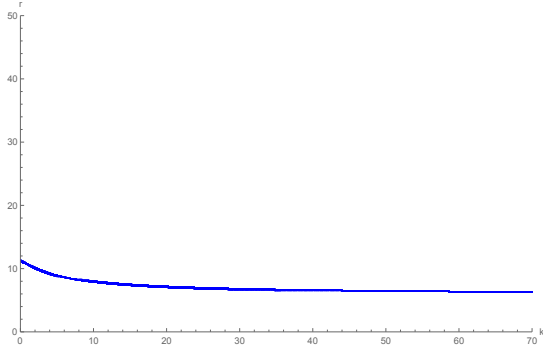


(c) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(18)}$

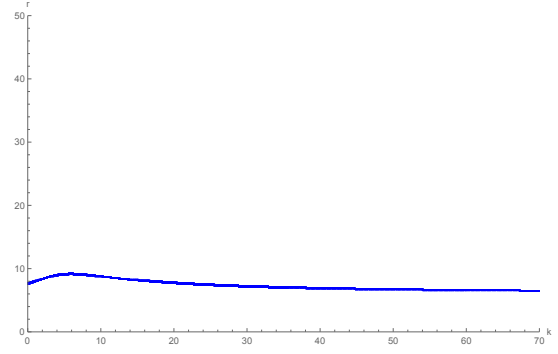


(d) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(19)}$

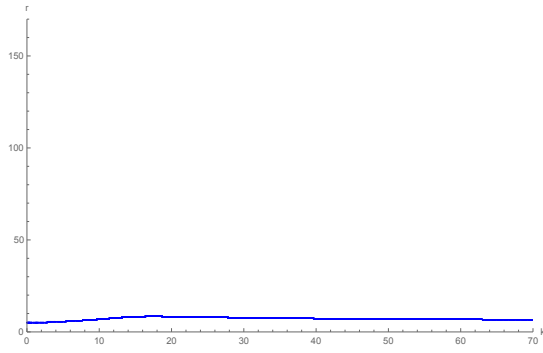
Figure 3.14: Critical points of $\mathcal{V}(r)$ from $\bar{Y}^{(16)}$ to $\bar{Y}^{(19)}$



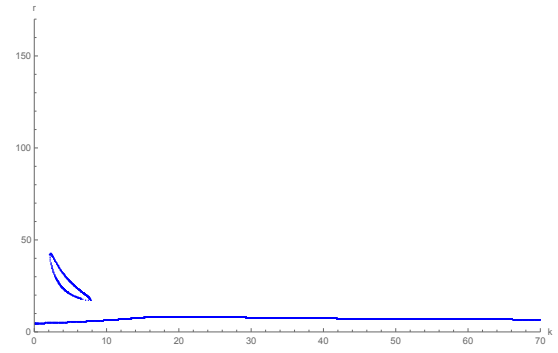
(a) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(20)}$



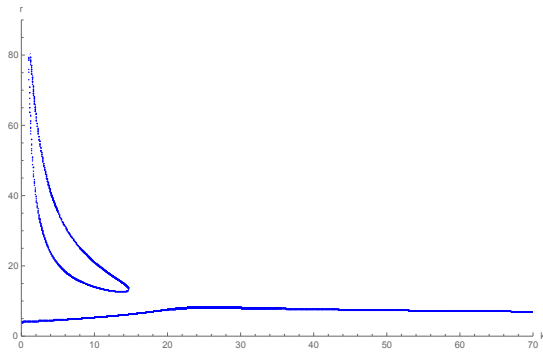
(b) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(21)}$



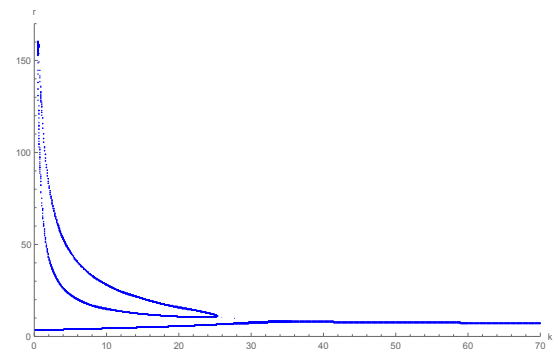
(c) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(22)}$



(d) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(23)}$



(e) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(24)}$



(f) Critical points of $\mathcal{V}(r)$ in $\bar{Y}^{(25)}$

Figure 3.15: Critical points of $\mathcal{V}(r)$ from $\bar{Y}^{(20)}$ to $\bar{Y}^{(25)}$

Chapter 4

Conclusions

This Thesis shows that it is possible to perform a detailed study of the dynamics for the system composed of a homogeneous sphere which rolls without sliding on a convex rotating surface. This study rests on the use of the symmetries and of the first integrals of the system, expanding the techniques used by previous works in the case with the surface at rest.

In particular, we show the fundamental role of the moving energy which substitutes the energy as first integral in the rotating case, namely when the nonholonomic constraint is linear non-homogeneous. The moving energy is used to prove that the dynamics is generically quasi-periodic, for a certain type of profile of the surface.

In the case in which the profile is a paraboloid of revolution, the study of the reduced dynamics, restricted to the level sets $\Sigma_{\bar{y}_1, \bar{y}_2}$ of the first integrals $\mathcal{Y}_1, \mathcal{Y}_2$, shows some unexpected results. We discover that the systems defined on $\Sigma_{\bar{y}_1, \bar{y}_2}$ are lagrangian systems. Moreover we show the appearance of a family of unstable horizontal circular orbits for the motion of the centre of the sphere in the paraboloid: this is a completely different situation from the case at rest. This study is preliminar and is the starting point for a deeper and more completed analysis. In particular, a future work should investigate the quasi-periodicity of the dynamics for all types of profile and also the nature of the critical fibers of the $(E, \mathcal{Y}_1, \mathcal{Y}_2)$ submersion.

Besides it should analyse the nature of \mathcal{Y}_1 and \mathcal{Y}_2 . For $k = 0$, these 2 first integrals are proved to be gauge momenta (see [16] [38]); if they were gauge momenta also in the rotating case, it could be explained the Lagrangian nature of the systems defined on $\Sigma_{\bar{y}_1, \bar{y}_2}$ (see [24]).

For what concerns the analysis performed with the paraboloid, our analysis strongly suggests the periodicity of the reduced dynamics (and hence the quasi-periodicity of the unreduced dynamics) for a wide range of values of the first integrals \mathcal{Y}_1 and \bar{Y}_2 . Therefore a future work should investigate the presence of something different, by considering more values of \bar{y}_1, \bar{y}_2 and also different widths χ of the surface (we study only the case $\chi = 0.2$). Furthermore it should take into account the bottom of the paraboloid and all the motions through that point.

Chapter 5

Appendices

5.1 Appendix A

We prove that the equations of motion for the $SO(3) \times S^1$ -reduced system are given by (3.3). Let us obtain Hamel's equations of the unreduced 8-dimensional system, defined on $\tilde{M} \simeq Q \times \mathbb{R}^2 \times \mathbb{R}$ which is parametrized with $(q, \dot{r}, \dot{\beta}, \omega_z)$.

The affine nonholonomic constraint is defined by $\tilde{S}(q)w + s(q) = 0$ with

$$w = (\dot{r}, \dot{\beta}, \omega_x, \omega_y, \omega_z)$$

$$\tilde{S}(q) = \begin{pmatrix} \sqrt{1 + (\phi'(r))^2} & 0 & a \sin \beta & -a \cos \beta & 0 \\ 0 & r \sqrt{1 + (\phi'(r))^2} & a \cos \beta & a \sin \beta & a \phi'(r) \end{pmatrix}$$

$$s(q) = \begin{pmatrix} 0 \\ -k(a\phi'(r) + r\sqrt{1 + (\phi'(r))^2}) \end{pmatrix}$$

Hamel's equations are

$$\dot{\mathcal{R}} = \check{\omega} \mathcal{R}$$

$$\dot{r} = v_r$$

$$\dot{\beta} = v_\beta$$

$$\dot{v}_r = \frac{1}{A\check{\mathcal{F}}(r)^2} [I r (\phi'(r))^2 v_\beta (v_\beta - k) + r v_\beta (v_\beta A - I k) - \phi'(r) (a^2 g + a I (k - \omega_z) v_\beta \check{\mathcal{F}}(r) + A v_r^2 \phi''(r))]$$

$$\dot{v}_\beta = \frac{v_r}{A r \check{\mathcal{F}}(r)} [-2 A v_\beta + (\phi'(r))^2 (I k - 2 A v_\beta) + I r \phi'(r) (k - v_\beta) \phi''(r) + I k + I a (k - \omega_z) \check{\mathcal{F}}(r) \phi''(r)]$$

$$\dot{\omega}_z = \frac{a \phi'(r) v_r}{A (\check{\mathcal{F}}(r))^3} [k + k \phi'(r)^2 + a (k - \omega_z) \check{\mathcal{F}}(r) \phi''(r) + r \phi'(r) (k - v_\beta \phi''(r))]$$

(5.1)

with

$$A = I + a^2, \quad \check{\mathcal{F}}(r) = \sqrt{1 + (\phi'(r))^2}$$

We perform the usual $SO(3)$ -reduction by cutting off the first 3 equations, namely $\dot{\mathcal{R}} = \check{\omega} \mathcal{R}$, and passing to the 5-dimensional space $\tilde{M}_5 \simeq \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \ni (r, \beta, \dot{\beta}, \omega_z)$.

Next we perform the S^1 -reduction by means of the invariant polynomials (3.2).

Let us call the map $\pi : \tilde{M}_5 \rightarrow \tilde{M}_4$, $\pi : (r, \beta, \dot{r}, \dot{\beta}, \omega_z) \mapsto (p_0, p_1, p_2, p_3, p_4)$. Let us notice that, given our parametrization of \tilde{M} , and consequently of \tilde{M}_5 , p_4 has the following expression

$$p_4 = \frac{-\omega_z + (k - \omega_z)(\phi'(r))^2}{\check{\mathcal{F}}(r)} + \frac{r\phi'(r)(k - \dot{\beta})}{a}$$

So if we perform the push-forward of the last 5 equations of system (5.1) under the map π and we substitute the profile function $\phi(\sqrt{2p_1})$ with $\Psi(p_1)$, we obtain the equations (3.3). This last computation is not straightforward and has been performed with *Mathematica*.

5.2 Appendix B

Let us obtain the expression (3.5) of the moving energy restricted to \tilde{M}_4 .

We start from the unreduced system, defined on the 8-dimensional constraint manifold \tilde{M} .

We compute the energy $E_{L, \tilde{M}}$

$$E_{L, \tilde{M}} = g\phi(r) + \frac{\mathbb{I}[\omega_z^2 + (\phi'(r))^2(k - \omega_z)^2]}{2} + \frac{A\check{\mathcal{F}}(r)^2\dot{r}^2}{2a^2} + \frac{\mathbb{I}r(k - \omega_z)\phi'(r)\check{\mathcal{F}}(r)(k - \dot{\beta})}{a} + \frac{r^2[\mathbb{I}k^2 + \mathbb{I}(\phi'(r))^2(k - \dot{\beta})^2 - 2\mathbb{I}k\dot{\beta} + A\dot{\beta}^2]}{2a^2}$$

where

$$A = \mathbb{I} + a^2, \quad \check{\mathcal{F}}(r) = \sqrt{1 + (\phi'(r))^2}$$

We compute the S^1 -momentum $J_{\xi_k}|_{\tilde{M}}$ relative to the infinitesimal generator $\xi_k = (0, k, 0, 0, k)$, where k is the angular velocity of the convex surface,

$$J_{\xi_k}|_{\tilde{M}} = k(\mathbb{I}\omega_z + r^2\dot{\beta})$$

The unreduced moving energy is

$$\begin{aligned} E_{L, \xi_k}|_{\tilde{M}} &= E_{L, \tilde{M}} - J_{\xi_k}|_{\tilde{M}} = \\ &= g\phi(r) + \frac{\mathbb{I}[\omega_z^2 - 2k\omega_z + (\phi'(r))^2(k - \omega_z)^2]}{2} + \frac{A\check{\mathcal{F}}(r)^2\dot{r}^2}{2a^2} + \frac{\mathbb{I}r(k - \omega_z)\phi'(r)\check{\mathcal{F}}(r)(k - \dot{\beta})}{a} + \frac{r^2[\mathbb{I}k^2 + \mathbb{I}(\phi'(r))^2(k - \dot{\beta})^2 + A\dot{\beta}(\dot{\beta} - 2k)]}{2a^2} \end{aligned}$$

This is a first integral for the unreduced system (5.1), as it can be checked by means of *Mathematica*, since it is not a simple computation.

Since the moving energy $E_{L, \xi_k}|_{\tilde{M}}$ is $SO(3) \times S^1$ -invariant, it can be projected to \tilde{M}_4 and is a first integral for the reduced system as well. The $SO(3)$ -reduction is immediate, since $E_{L, \xi_k}|_{\tilde{M}}$ does not contain $\mathcal{R} \in SO(3)$. The S^1 -reduction is performed by rewriting the moving energy in terms of the polynomials (3.2).

For completeness, let us rewrite the energy (we make use of the function Ψ for the profile)

$$E_{L, \tilde{M}_4} = g\Psi(p_1) + \frac{Ap_0}{a^2} + \frac{\mathbb{I}k^2p_1(1 + \frac{2a\Psi'(p_1)}{\mathcal{F}(p_1)})}{a^2} - \frac{\mathbb{I}k^2}{2\mathcal{F}(p_1)^2} + \frac{\mathbb{I}(k^2 + p_4^2)}{2} + \frac{Ap_2^2\Psi'(p_1)^2}{2a^2} - \frac{\mathbb{I}kp_3(1 + \frac{a\Psi'(p_1)}{\mathcal{F}(p_1)})}{a^2}$$

and the momentum

$$J_{\xi_k}|_{\tilde{M}_4} = \frac{k}{a\mathcal{F}(p_1)^2} [a\mathcal{F}(p_1)^2(\mathbb{I}k + p_3) - \mathbb{I}ak - \mathbb{I}\mathcal{F}(p_1)(ap_4 - 2kp_1\Psi'(p_1) + p_3\Psi'(p_1))]$$

Their difference gives for the reduced moving energy $E := E_{L, \xi_k}|_{\tilde{M}_4}$ the expression given by (3.5).

Again, by means of *Mathematica*, it can be checked that E is a first integral for the reduced system (3.3).

5.3 Appendix C

The effective potential $\mathcal{V}(r; \bar{y}_1, \bar{y}_2)$ is

$$\begin{aligned} \mathcal{V}(r; \bar{y}_1, \bar{y}_2) = & \frac{1}{27040r^2} \left[9646ds + \frac{847k^2}{\chi^4} + 4732d^2 f(r)^{1-\sqrt{\frac{2}{7}}} + 4732s^2 f(r)^{1+\sqrt{\frac{2}{7}}} + \right. \\ & + \frac{52skf(r)^{1+\sqrt{\frac{1}{14}}(-77+4\sqrt{14}r^2\chi^2)} - 52dkf(r)^{1-\sqrt{\frac{1}{14}}(77+4\sqrt{14}r^2\chi^2)}}{\chi^2} + \\ & \left. + \frac{32r^2 \left[169\chi^2(5r^2-7ds\chi^2) + k^2(212+3\chi^2r^2-169\chi^2+4r^4\chi^4) \right]}{\chi^2} \right] \end{aligned}$$

where

$$d = \bar{y}_1 - \bar{y}_2, \quad s = \bar{y}_1 + \bar{y}_2, \quad f(r) = 1 + 4\chi^2r^2$$

5.4 Appendix D

In the constraint manifold $\tilde{M} \simeq SO(3) \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ the equations of motion are the following

$$\dot{\mathcal{R}} = \check{\omega}\mathcal{R}$$

$$\dot{r} = v_r$$

$$\dot{\beta} = v_\beta$$

$$\dot{v}_r = \frac{r}{7(1+4\chi^2r^2)} \left[-10 - 28\chi^2v_r^2 - 2kv_\beta + 7v_\beta^2 + 4\chi v_\beta \sqrt{1+4\chi^2r^2}(\omega_z - k) + 8\chi^2v_\beta r^2(v_\beta - k) \right]$$

$$\dot{v}_\beta = \frac{2v_r}{7r(1+4\chi^2r^2)} \left[k + 2\chi \sqrt{1+4\chi^2r^2}(k - \omega_z) + 8\chi^2r^2(k - 4v_\beta) - 7v_\beta \right]$$

$$\dot{\omega}_z = \frac{10\chi r v_r}{7(1+4\chi^2r^2)^{3/2}} \left[k + 2\chi \sqrt{1+4\chi^2r^2}(k - \omega_z) + 4\chi^2r^2(2k - v_\beta) \right]$$

(5.2)

The equations of (5.2) are Hamel's equations of the system and are obtained in the same way we have seen in Appendix A, in the case of a general profile.

After the $SO(3) \times S^1$ -reduction, in $\bar{M}_4 \ni (r, \dot{r}, \dot{\beta}, \omega_z)$ the equations of motion are those for $(r, \dot{r}, \dot{\beta}, \omega_z)$.

In $\Sigma_{\bar{y}_1, \bar{y}_2}$, parametrized with (r, \dot{r}) , the equations of motion are the third and the fourth equation of system (5.2) with the substitution given by (3.14): the latter coincides with the Lagrange equation for \ddot{r} of the Lagrangian (3.15).

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