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Composite Higgs and Axions: Towards a Natural Theory

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To *all* my family and, especially, to my aunt Paola

Abstract

We review some attempts to solve the Naturalness problem - namely the the lightness of the Higgs mass in relation to the scale of new physics - and the strong CP problem, that is the measured CP-invariance in the strong sector despite the presence of the so-called θ term. In particular, in the first part we study the so-called composite Higgs models as a solution to the Naturalness problem, focussing our attention on a particular model, the *minimal linear sigma model*. In the second part we review one of the most convincing explanation to the strong CP problem, the so-called axions. There are essentially three ways to introduce axions in a beyond Standard Model Lagrangian; after a short summary, we focus on the so-called KSVZ model and verify why it is safe from possible fine-tuning problems. Finally, we investigate the possibility to solve both the problems in an unique framework, the so-called *axion minimal linear sigma model*. These models, unlike the KSVZ model, suffers fine-tuning problems that suggest to associate the arising Nambu-Goldstone boson to a more massive axion-like particle, rather than an axion.

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Introduction

The Higgs boson was the last unverified part of the Standard Model (SM) of particle physics and its discovery in 2012 has completed the particle content predicted by the SM decades ago. In particular, it has confirmed the simplest possible picture of electroweak symmetry breaking (EWSB). Shortly, the so-called *Higgs mechanism* splits an elementary doublet of scalar particles into an unphysical sector, providing the longitudinal polarization to the vector bosons W and Z, and a single physical spin zero particle, the Higgs particle. Moreover, thanks to this mechanism, it has also been possible to understand how the SM fermions get a mass without violating the SM gauge symmetry, providing in this way a unified picture of mass generation.

However, there are several hints that seem to tell us that this is not the end of the story. To start with, the SM does not contain (a complete description of) gravity. Putting aside gravity, we can glimpse signals of physics beyond Standard Model (BSM) in the SM itself. Ironically, the main reason that has driven physicists in searching BSM physics at TeV scale has been precisely the particle that has ultimately confirmed the SM picture, the Higgs boson, and in particular the lightness of the Higgs mass. Its value has been questioned in relation to the so-called *Naturalness problem*, namely the reason of the large separation of scales between the electroweak scale and the scale of new physics. The interest towards the Naturalness problem starts in 1976 thanks to two works of Gildener [1] and Weinberg [2]. They revealed a conceptual difficulty in the context of grand unified theories (GUT): one-loop quantum corrections were found to give contributions to the Higgs mass proportional to the mass of the superheavy states, of the order of $M_{\mathrm{GUT}} = 10^{14}\,\mathrm{GeV}$. They noticed that keeping a large separation of scales between the electroweak scale and M_{GUT} required fine-tuning the parameters of the theory of more than 10^{-24}. This is nothing but a one specific realization of the Naturalness problem. However, the question did not get relevance until the 1980, when Veltman published an influential paper [3] emphasizing the problem. In 1981 Witten pointed out how *supersymmetry*, formerly introduced but not to achieve this goal, could solve the Naturalness problem. Since then, this has become one of the most studied puzzles in particle physics and one of the driving motivations to explore physics BSM.

Related to the problem of lightness of the Higgs mass, there is another fact which could signal the presence of physics BSM: it is the so-called *Yukawa hierarchy*. We have said that the Higgs mechanism has provided an explanation of the generation of SM fermion masses in a unique language but this is not exactly fair. Looking at the Yukawa couplings more carefully, we can notice that there is a (very) large hierarchy between the Yukawa couplings: precisely, there are more than 9 order of magnitude between the masses of the lightest fermions (the neutrinos) and the heaviest one(the top quark). In principle this is not a problem, as the Yukawa couplings are free parameters in the SM. However, such a hierarchy would seem to indicate a not so common origin for the fermion masses. These considerations have led many physicists to believe that the Higgs scheme is nothing but "a convenient parametrization of our ignorance concerning the dynamics of spontaneous symmetry breaking" [4]. This belief is supported by the fact that Higgs would represent the unique example of elementary spin zero particle in nature, while in other known phenomena of spontaneous symmetry breaking (SSB) its role is played by composite excitations[1]. Supersymmetry would justify elementary scalars but no direct or indirect hints of them have been found so far. This suggested decades ago [5], [6] a dynamical nature for the Higgs particle as a pseudo-Nambu-Goldstone boson (pNGB) which would explain in this way the lightness of the Higgs mass. This class of models is known as *composite Higgs models* (CHMs), because the Higgs arises as a bound state of a new strong force. The important point is that, despite its composite nature, the Higgs can behave as an elementary particle, matching in this way what we experimentally see, through the mechanism of *vacuum misalignment*, firstly introduced by Georgi [5]. The idea underlying these models is that the complex Higgs doublet arises as a pNGB of a spontaneous symmetry breaking (SSB) of a group G to a group H, where G and H are specified by the model. H is supposed to contain the full EW symmetry or, more precisely, the SM custodial symmetry, namely the $SO(4)$ symmetry of SM scalar sector. The reason is that the custodial symmetry is able to protect the SM relation $\rho = 1$ at tree-level. Experimentally, $\rho = 1$ is valid at percent level and the deviations are well described by SM loop effects. Furthermore, the coset G/H has to be as large as to contain the four degrees of freedom of the Higgs multiplet. These models offer also a natural explanation for the Yukawa hierarchy: the SM fermion masses are generated at tree-level through the so-called *partial fermion compositeness* hypothesis, introduced by Kaplan [6].

The fine-tuning problems in the SM do not finish here. The strong sector, described by the quantum chromo-dynamics (QCD), seems to be experimentally invariant under the CP symmetry. However, as pointed out by t'Hooft in the 80's, the QCD Lagrangian, because of its non-abelian nature, contains a CP-violating term, the so-called θ term. The trouble is that this

[1]It is the case of pions in QCD.

term produces a neutron magnetic dipole moment and, as we do not see it experimentally, we are forced to set $\theta \sim 10^{-9}$ without any apparent reason and this, again, seems to be too artificial. The problem has been known as *strong CP problem.* There were made several proposes to solve the strong CP problem. Kaplan [7] understood that a massless up quark could allow to rotate away the θ term. However, this possibility was soon excluded by experimental data. Forty years ago, inspired by the massless up quark solution, Helen Quinn and Roberto Peccei suggested a dynamical solution to the strong CP problem [8], [9]. They postulated that the full Lagrangian of the SM was invariant under an additional global chiral $U(1)$ symmetry that became known as $U(1)_{\text{PQ}}$ symmetry. They demonstrated that the SSB of this symmetry drives dynamically the θ parameter to zero. Because of the SSB, there is an associated pseudo-NGB (pNGB) in the theory, the so-called *axion* [10], [11]. However, the PQWW model proposed by Peccei, Quinn, Weinberg and Wilczek was soon ruled out by experimental data. The main trouble was that the PQ axion arises as a linear combination of neutral phases of the two Higgses present in the model and this forces the SSB scale f_a to be equal to the electroweak scale v but there were found no axions at that scale. However, the underlying idea of the PQ model, that is the dynamical adjustment of the θ angle, works for any scale. This suggested to consider models in which the PQ breaking scale was independent of the EW scale v. The first to introduce these models were Kim, Shifman, Vainshtein and Zakharov (KSVZ model) [12], [13]. They enlarged the SM particle spectrum with an extra heavy fermion sector, needed to have a further $U(1)_{\text{PQ}}$ symmetry in the Lagrangian, and an extra complex singlet (under the whole SM group) σ in which phase is contained the axion. Another attempt to introduce axions in a BSM Lagrangian was done by Dine, Fischler, Srednicki and Zhitnitsky (DFSZ model) [14], [15]. This model is a mixture between the PQWW model and the KSVZ model. In fact, here the axion arises as a linear combination of the phases of two Higgses and one (singlet) complex scalar field, that is essential to push the axion scale f_a over v. The main difference between these two models is that, because of the two Higgses, the DFSZ axion possesses tree-level couplings with the ordinary quarks and leptons while the KSVZ model does not. In both these models $f_a \gg v$ and the resulting axions are very light ($m_a \sim 1/f_a$), very weakly coupled (coupling $\sim 1/f_a$) and very long lived ($\tau(a \to 2\gamma) \sim f_a^5$), and for this they are called *invisible axions.*

Putting aside the Naturalness problems - to which the cosmological constant problem should be added - other troubles related to incontrovertible experimental facts such as dark matter, dark energy, baryon asymmetry, neutrino masses and neutrino oscillations hard testing the SM. However, none of these, strong CP problem included, prevents the new physics scale to be at very high energies. The only argument able to do this is the hierarchy problem, as it predicts the scale of new physics to be at TeV scale.

The aim of this thesis is to explain in more detail the Naturalness and the strong CP problem and to analyse composite Higgs models and axions as respective ways out to them. We will start (chapter 1) showing why the SM cannot be considered the ultimate theory of Nature, focusing on the Naturalness problem and one of its possible solutions, the CHMs. Then, in the chapter 2, we will analyse the effective field theories (EFTs) method, which is the educated language for the BSM physics. The chapter 3 will be devoted to the study of the composite Higgs scenario and its main features; in the chapter 4 we will study in more detail a particular CHM, the minimal linear sigma model with a $SO(5)/SO(4)$ SSB pattern [16]. We will show the predictions given by the pNGB nature of the Higgs, and one of the possible realizations of the partial fermion compositeness hypothesis. The second part of the thesis will be devoted to the study of the strong CP problem; in particular we will explain why this is considered a problem (chapter 5) and we will study the axions as its possible solution (chapter 6), focusing our attention on the main axion models: the (ruled-out) PQWW model, the KSVZ model and the DFSZ model. Finally, in the chapter 7 we will investigate the possibility to solve both the Naturalness and the strong CP problems, studying a possible extension of the minimal linear sigma model [17], adding a new complex scalar field in order to mimic the KSVZ solution to the strong CP problem. However, we will see that, if we want to keep in the low TeV range the $SO(5)/SO(4)$ breaking scale in order to not invalidate the solution of the Naturalness problem, the identification of the scalar particle as a KSVZ axion creates naturalness problems: these models are highly fine-tuned. This leads to consider the possibility to associate the (angular part of the) scalar field to a more massive axion-like particle (ALP) such as a 1 GeV axion with an associated scale of ~ 200 TeV, that may show up in collider searches. However, the ALP hypothesis could not help in addressing the strong CP problem. Instead, the ALPs appear in many models of BSM physics, such as string theory, as pNGBs associated to the breaking of $U(1)$ symmetries.

Chapter 1

The SM is *Not* the End of the Story

A complete description of gravity is missing in the SM and this requires the latter to be extended. Actually, the SM contains a partial description of gravity but this description, based on the perturbation theory, is valid up to around the Planck mass, because the effective gravity coupling grows like $g_G \simeq E/M_P$. Therefore some new physics must emerge much below M_P to stop the growth of the coupling strength [18]. This means that the SM, given that it has a finite cut-off, i.e. its validity is limited up to a finite energy scale, is for sure an effective field theory. Since the SM is an effective field theory, there is no reason to require the renormalizability of the Lagrangian. The most general Lagrangian will be then a sum of infinite local operators O_i invariant under the gauged $SU(3)_c \times SU(2)_L \times U(1)_Y$ (and, obviously, the Lorentz) symmetry:

$$\mathcal{L}_{\text{eff}} = \sum_i c_i O_i. \tag{1.1}$$

Since the Lagrangian has mass dimension 4, if an operator O_i has dimension d_i the respective coefficient c_i must have dimension $4 - d_i$. We can rewrite the SM Lagrangian at the Λ scale as

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{SM}} + \sum_i \frac{1}{\Lambda^{d_i-4}} \hat{c}_i O_i \tag{1.2}$$

where $\mathcal{L}_{\text{SM}}$ is the usual SM $d \leq 4$ Lagrangian, we have made the coefficients dimensionless and $d_i > 4$. This way of proceeding is peculiar of effective field theories; we will return on this topic in the next chapter. However, we can already understand at this level why the Nature is approximately well described by a renormalizable theory, without renormalizability being a principle: all the operators with a dimension larger than 4 have a suppression factor that *hides* their phenomenological impact, at least up to the Λ scale.

Just to make an example, the *unique* $d = 5$ term that can be constructed by imposing the SM gauge symmetry is the Weinberg operator [19]

$$\frac{c}{\Lambda}\left(\bar{l}_L H^c\right)\left(l_L^c H^c\right) \tag{1.3}$$

where l_L denotes the lepton doublet, l_L^c its charge conjugate, H is the Higgs doublet and $H^c = i\sigma^2 H^*$. The important point is that this term violates the lepton number, an accidental symmetry of the SM, and, for $\Lambda \simeq 10^{14}$ GeV and $c = O(1)$, it generates neutrino masses of correct magnitude. This simple argument could explain why, for instance, the lepton number-violating processes are suppressed with respect to the lepton number-conserving, without having an underlying (accidental) symmetry. However, and this is the point, following this line of reasoning, we expect the Higgs mass term, the only SM operator with a dimension less than 4, to be of order

$$c\Lambda^2 H^\dagger H \equiv \frac{1}{2} m_h^2 H^\dagger H. \tag{1.4}$$

But if we take $\Lambda \simeq 10^{14}$, we are forced to set $c \sim 10^{-28}$ in order to reproduce the correct value of the Higgs mass ($m_h = 125$ GeV). The question is: why is this number so small? Or, in other words, what is the reason of the enormous hierarchy between the electroweak scale and the scale of new physics? This is the essence of the Naturalness problem, that we are going to develop in more detail in the next section. However, we can already appreciate the contradiction: the problem is based on the same logic by which its phenomenological virtues, that is the suppression of $d > 4$ operators, were established. In order to keep c of order one, we are forced to conclude that Λ is low, in the TeV range, such that a light enough Higgs is obtained "Naturally". This implies that the new physics at the cut-off must be now non-generic, since we cannot rely on a large Λ.

1.1 The Naturalness Problem

Since the problem we are going to describe is based on what we mean with "Natural", it is important to define the meaning of Naturalness of a theory. The definition of Naturalness is due to 't Hooft [20]:

Let us consider a theory valid up to a maximum energy Λ and make all its parameters dimensionless by measuring them in units of Λ. The theory is said to be Natural if all its parameters are of order one. A parameter is allowed to be much smaller than one only if setting it to zero increases the symmetry of the theory.

In the second statement is contained the point of the argument. A small parameter is not necessarily problematic because, thanks to a theorem of

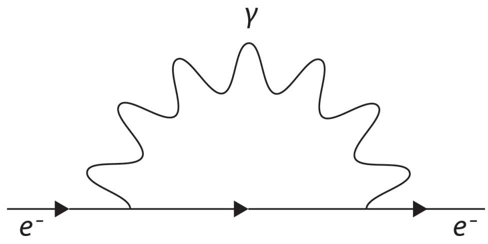

Figure 1.1: The figure shows the one-loop contribution to the electron mass. In principle the amplitude is quadratically divergent but, because of the underlying axial symmetry, the divergence that shows up is logarithmic.

QFT, the symmetries of the classical action must be symmetries of the full quantum action, as well[1]. This means that if a parameter of a theory is protected by a symmetry, in the sense that by setting it to zero the symmetry increase, it will not receive large quantum corrections from the quantum theory because the symmetry has to be restored in the limit when the parameter goes to 0. It is instructive to give an example of the idea underlying the principle. The QED Lagrangian with one fermion is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}\left(i\not{D} - m\right)\psi. \tag{1.5}$$

In the limit $m \to 0$ the symmetry group is enlarged by a further $U(1)_A$ symmetry, where A stays for axial, that acts on left and right spinors as

$$\psi_L \to e^{i\alpha}\psi_L \qquad \psi_R \to e^{-i\alpha}\psi_R. \tag{1.6}$$

Because of this symmetry, all the contributions to the fermion mass arising at loop level, just like the one depicted in figure 1.1, must be necessarily proportional to the fermion mass so that in the limit $m \to 0$ they can disappear, preserving the chiral symmetry. A priori we could have two contributions to the fermion mass: $m \sim a_1 m \log\Lambda + a_2\Lambda$ but, as we know that in the limit $m \to 0$ they have to disappear, a_2 must be necessarily zero and, in fact, performing the calculation we find $m \sim m\log\Lambda$. Depending on the logarithm of the upper cut-off scale, the fermion mass is therefore *protected* by large quantum corrections.

This is precisely what does not happen in the SM with the Higgs. Since the Higgs mass term is not protected by any symmetries, it receives large quantum corrections that push m_H close to Λ. The absence of a symmetry is linked to the spin-zero nature of the Higgs boson. In fact, all the other particles with spin 1/2 or higher have a symmetry able to protect their masses (the chiral symmetry for spin 1/2, the gauge symmetry for spin 1 gauge

[1] Actually, there is a exception due to the so-called anomalies but this is not the case of interest to us.

bosons). This makes particularly sense if we notice that the symmetries arising in a massless theory allow us to eliminate degrees of freedom from the classical theory and any quantum correction to these masses necessarily would restore them, leading to an inconsistency (the degrees of freedom would change in the transition from the classical to the quantum theory). Instead, a spin 0 particle, massive or massless, possesses the same degrees of freedom, so it is not needed to protect them by quantum corrections [21].

Returning to the Higgs mass, the idea is that some physics should appear at the TeV scale in order to stop the growth of the Higgs mass due to (1.4). We do not know if the Nature respects the Naturalness criterion, but there are several examples in physics where the Nature has chosen to be "Natural". For instance, the electromagnetic contribution to the difference between the charged and the neutral pion mass is

$$M^2_{\pi^+} - M^2_{\pi^0} = \frac{3\alpha}{4\pi}\Lambda^2 \tag{1.7}$$

where Λ here is the ultraviolet cut-off scale of the effective theory of pions. The request that the difference calculated with (1.7) does not exceed the experimental observed value of $(M^2_{\pi^+} - M^2_{\pi^0})_{\rm exp} \sim 1225\,\text{MeV}$ implies that Λ must be smaller than 850 MeV. In fact, before that scale, there exists a particle, the ρ meson, so the effective description of the pions has to be changed in order to reproduce the correct value of $M^2_{\pi^+} - M^2_{\pi^0}$.

Another example historically relevant it is the discovery of the charm quark by Glashow, Iliopoulos and Maiani [22]. At the energies of the kaon mass we can calculate, with the effective theory valid at this scale, the mass difference between the K^0_L and K^0_S states[2]. The result is:

$$\frac{M_{K^0_L} - M_{K^0_S}}{M_{K^0_L}} = \frac{G_F^2 f_K^2}{6\pi^2}\sin^2\theta_c\Lambda^2 \tag{1.8}$$

where $f_K = 114\,\text{MeV}$ is the decay constant and $\sin\theta_c = 0.22$ is the Cabibbo angle. If we require that the theoretical prevision does not exceed the experimental value of $(M_{K^0_L} - M_{K^0_S})/M_{K^0_L} \sim 7\times 10^{-15}$, we find $\Lambda < 2$ GeV. In fact, before reaching this energy, a new particle - the charm quark - modifies the high energy behaviour of the theory, through the so-called GIM mechanism.

The idea with the Higgs mass is exactly the same: in order to keep the theoretical prediction of the Higgs mass lower than its experimental value without violate the Naturalness principle, we find that the scale at which should appear the new physics, as anticipated above, is around the TeV. We can reformulate the problem in the same fashion as the $K - \bar{K}$ system by estimating the leading order contributions to the Higgs mass due to the one-loop diagrams shown in Fig. 1.2, containing the top quark, the W, Z

[2] The subscripts stay for long and short and they refers to the the lifetime of the two states.

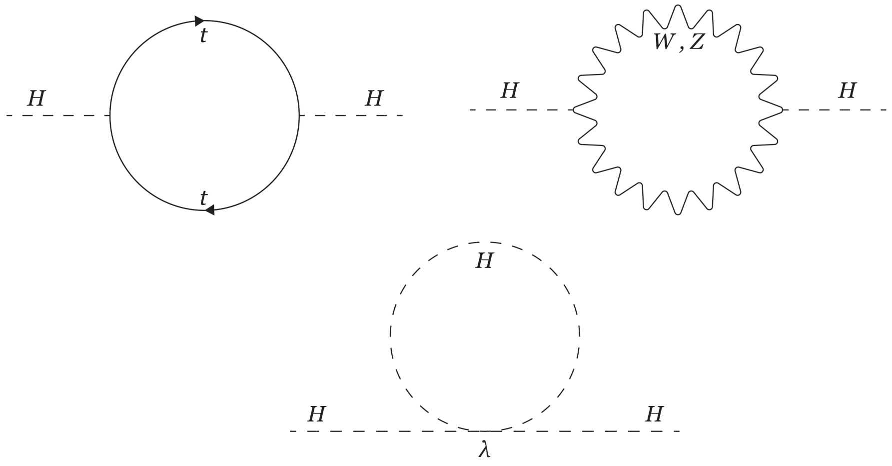

Figure 1.2: The figure shows the highest contributions to the Higgs mass coming from top (upper left), gauge bosons (upper right) and Higgs boson (bottom) loop diagrams.

bosons and the Higgs boson. The calculation of the four diagrams depicted in Fig. 1.2 reads

$$\delta m_H^2 \sim \frac{3G_F}{4\sqrt{2}\pi^2}\left(4m_t^2 - 2m_W^2 - m_Z^2 - m_H^2\right)\Lambda^2. \tag{1.9}$$

We find, comparing the prediction given by (1.9) with the experimental value, $\Lambda \sim$ TeV. Therefore, the expected scale of new physics, provided that the Naturalness principle is really a principle, is around the TeV.

1.2 A Possible Solution to the Naturalness Problem

The Naturalness problem was not fully considered until 1980, when Witten pointed out how supersymmetry could solve the problem. However, no direct or indirect hint of supersymmetry have been found so far. This led physicists to consider other kind of solutions. In this work we will explore one of the most cogent solution to it, the so-called composite Higgs models. The Higgs boson is viewed in this picture as a bound state of a new strong force. Since the basic idea underlying the model is to mimic what does happen in QCD with the pions, we briefly recall how, starting by the QCD Lagrangian at high energies, the effective theory of pions shows up. The QCD can be viewed at high energies as a free theory near its conformal fixed point. This means that the QCD at high energies does not have dimensional couplings. To be more precise, there are dimensional couplings, which are the quark masses, but they are protected by chiral symmetry, as we have seen before, so their dependence from the ultraviolet (UV) scale is the same as if they were dimensionless. Let $\Lambda_{\rm UV}$ be the scale where the QCD can be viewed as a free theory (we can take, for instance, $\Lambda_{\rm UV} = m_Z$ where m_Z is the mass of the Z boson). Under the renormalization group, the coupling constants run and, thanks to the fact that there are no dimensional couplings, the theory develops an exponentially suppressed (with respect to $\Lambda_{\rm UV}$) scale $\Lambda_{\rm QCD}$ which is not present in the UV theory[3]. The appearance of a scale without any dimensional couplings in the theory (and, therefore, without any scales) is called *dimensional transmutation*. It turns out that this scale is *insensitive* to the $\Lambda_{\rm UV}$ scale. Now, at $\Lambda_{\rm QCD}$ a large number of bound states appears, the hadrons. Looking at the hadrons' spectrum we notice that the pion masses are lower than those of other hadrons: the reason is that, in the limit $m_u = m_d \to 0$, the pions are the NGBs of the spontaneous symmetry breaking $(SU(2)_L \times SU(2)_R)/SU(2)_V$. However, the $SU(2)_L \times SU(2)_R$ symmetry is softly broken by the gauging of the $U(1)_{\rm em}$ generator. This soft breaking is responsible of a (small) mass for the pions that become in this way pNGBs.

Returning to the Higgs, the idea of the composite Higgs models is precisely this: in this picture the Higgs is a pNGB of a spontaneously broken symmetry (with a soft breaking term given by the gauging of the EW group), in such a way that its mass is naturally lower than other resonances' masses. However, as pointed out by [18], the Goldstone symmetry is in itself of no help in addressing the Naturalness problem. What is important to solve the Naturalness problem is the strongly coupled nature of the underlying UV theory by which the Higgs mass is stabilized through dimensional transmutation.

[3]It is often referred to it as the *confinement* scale

Chapter 2

Effective Field Theories

In composite Higgs models there are no attempts to formulate an underlying UV theory of the EW mechanism. Instead, through an effective field theory (EFT) approach, they try to make predictions about the low energy effects of some UV completion of the theory, without worrying about what this theory is. In this sentence there is the essence of the EFT approach and the reason why it is so powerful: it means that we can forget about details of the UV theory because the quantum fluctuations associated with the high energy physics will affect the low energy theory only through the values of a few of its parameters. The explanation of this unusual and counterintuitive simplification is due to Kennet Wilson [23]. He has given in particular a new point of view about the meaning of renormalization. The construction is based on the functional integral approach to the field theory. Suppose we have a quantum field theory of a scalar field ϕ valid up to a scale Λ and we are interested in the physics at some lower scale $E \ll \Lambda$. The generating functional can be formally written as

$$Z[J] = \int \mathcal{D}\phi \, e^{i\int[\mathcal{L}+J\phi]} = \int \prod_k d\phi(k) \, e^{i\int[\mathcal{L}+J\phi]} \tag{2.1}$$

where in the second equality we have written the field ϕ in terms of its Fourier components. To impose a sharp ultraviolet cut-off Λ we restrict the number of the integration variables displayed in (2.1). That is, we integrate only over $\phi(k)$ with $|k| < \Lambda$ and set $\phi(k) = 0$ for $k > \Lambda$. We now divide the remaining fields $\phi(k)$ into two groups by choosing a cut-off $b\Lambda$ with $b < 1$ [24], [25]

$$\phi = \phi_H + \phi_L \tag{2.2}$$

with

$$\phi_H = \begin{cases} \phi(k) & \text{for} \quad b\Lambda \le |k| < \Lambda \\ 0 & |k| < b\Lambda \end{cases} \tag{2.3}$$

$$\phi_L = \begin{cases} 0 & \text{for} \quad b\Lambda \le |k| < \Lambda \\ \phi(k) & |k| < b\Lambda \end{cases} . \tag{2.4}$$

The functional integral now reads

$$\int \mathcal{D}\phi_L \mathcal{D}\phi_H \, e^{i\int[\mathcal{L}(\phi_L,\phi_H)+J(\phi_L+\phi_H)]} \equiv \int \mathcal{D}\phi_L \, e^{i\int[\mathcal{L}_{\text{eff}}+J\phi_L]} \tag{2.5}$$

where we have defined the *effective action*

$$e^{i\int \mathcal{L}_{\text{eff}}} \equiv \int \mathcal{D}\phi_H \, e^{i\int[\mathcal{L}(\phi_L,\phi_H)+J\phi_H]}. \tag{2.6}$$

In this way the heavy degrees of freedom disappear and we remain with the only low fields. This process is called *integrating out* a field. Taking the parameter b close to 1 and iterating this procedure one can describe the result of integrating over the high momentum degrees of freedom of a field theory as a trajectory or a flow in the space of all possible Lagrangians. For historical reasons, these continuously generated transformations of Lagrangians are referred to as the *renormalization group.* Actually, they do form a semi-group, rather than a group, because the operation of integrating out degrees of freedom is not invertible. If we perform the integral (in most cases this can be done only perturbatively) in (2.6) the resulting Lagrangian is an infinite sum of local interactions described by operators $\mathcal{O}_i$ that depend only on the fields ϕ_L and that respect the symmetries of the full theory:

$$\mathcal{L}_{b\Lambda}\left[\phi_L\right] = \sum_i g_i \mathcal{O}_i \left[\phi_L\right] \tag{2.7}$$

where d_i is the mass dimension of the operator $\mathcal{O}$ and the couplings g_i as we have anticipated depend only the couplings of the full theory and on the scale Λ. We can extract the scale dependence of the couplings g_i by defining the dimensionless couplings

$$\lambda_i = \Lambda^{d_i-4} g_i. \tag{2.8}$$

Since $b\Lambda$ is the only characteristic scale of the process we expect the g_i's to be of order 1. Now, for a process at the scale $b\Lambda$ we can estimate[1] the modulus of the operator $\mathcal{O}$ as:

$$\int d^4x \, \mathcal{O} = (b\Lambda)^{d_i-4} \tag{2.9}$$

so that the i'th term is of order

$$\left(\frac{b\Lambda}{\Lambda}\right)^{d_i-4} = b^{d_i-4}. \tag{2.10}$$

Now comes the point. In the low energy regime, that is when $b \to 0$, the importance of the terms depend on the dimension d_i of the operator. In particular:

[1] This estimate of the integral is valid as long as the coupling is sufficiently weak so that the energy of the operator can be estimated starting by its kinetic term and it is $b\Lambda$.

- if $d_i > 4$ the respective term flows to zero and for this it is termed irrelevant;
- if $d_i < 4$ the respective term flows to infinity and it is called relevant;
- if $d_i = 4$ the operator is equally important at all scales and it is termed marginal; for the marginal operators it is needed a more detailed analysis in order to establish their flow.

In the language of the renormalization group we are following the evolution of coefficients near the free-field fixed point[2]. We now understand why the only operators needed to describe the low energy physics are the relevant ones, for which $d_i < 4$ (and some $d_i = 4$ operator) that are nothing but the renormalizable terms in the language of the old-fashioned perturbation theory. This simple argument explains why a renormalizable theory, such as the QED, is a good (almost perfect) approximation of the Nature: whatever the Lagrangian of QED was at its fundamental scale, as long as its couplings are sufficiently weak, it must be described at the energies of our experiments by a *renormalizable* effective Lagrangian. To conclude, in this new picture the situation is reversed with respect to the old-fashioned renormalizability: we always expect the non-renormalizable terms to appear at some level so they are not a problem anymore. But now there is a new type of problem: the superrenormalizable terms. For instance, a mass term of dimension 2 is expected to appear with a coefficient of order $\lambda\Lambda^2$. Without a fine-tuning or some symmetry that would forbid this term, the field would have a mass of order Λ. But this is a contradiction: the field, that is by definition a low energy degree of freedom, should disappear since its mass is near to the cut-off [25]. This is precisely the Naturalness problem in the language of the renormalization group.

2.1 The Sigma Model

The sigma model provides a simple introduction to effective Lagrangians because all the relevant manipulations can be explicitly demonstrated. In fact here the Goldstone boson fields, the pions, which are the only relevant fields in the low energy limit, are present at all stages of the calculation. The sigma model was originally introduced by Gell-Mann and Levy (1960) to derive in a convincing manner the formula of Goldberger and Treiman. However, it is useful to regard it as a "toy model" in order to show the main features of the effective field theories. Shortly, it is based on a Lagrangian invariant under $SO(4)$ spontaneously broken to the $SO(3)$ group, leading

[2]The Lagrangian with all the couplings put to zero is a fixed-point of the renormalization group, in the sense that it remains unchanged under the renormalization group's flow.

to 3 Goldstone bosons in the coset $SO(4)/SO(3)$. By integrating out the remaining heavy field,[3] we obtain an effective field theory describing the Goldstone's dynamics. The resulting Lagrangian is non-linear (because the kinetic term is not linear in the fields, as we will see) and it is made of infinite terms. In fact, the theory is not renormalizable, but this is not problem since we know that it is an effective one, the sigma model (that is, with the heavy field reintroduced) being the UV completion.

The sigma model Lagrangian is [26]

$$\mathcal{L} = \bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}\partial_\mu\pi\partial^\mu\pi + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - g\bar{\psi}(\sigma + i\tau\cdot\pi\gamma_5)\psi + \frac{\mu^2}{2}(\sigma^2+\pi^2) - \frac{\lambda}{4}(\sigma^2+\pi^2)^2 \tag{2.11}$$

where $\pi = (\pi_1, \pi_2, \pi_3)$ is the pion triplet, the τ's are the Pauli matrices and ψ is a fermion field. For $\mu^2 > 0$, the $SO(4)$ symmetry is spontaneously broken to $SO(3)$, leading to three Goldstone bosons in the coset $SO(4)/SO(3)$ (the three pions); this leads σ and ψ to acquire a mass. In order to show the salient features of the effective Lagrangians, let us rewrite the sigma model Lagrangian in a more convenient form defining

$$\Sigma = \sigma\mathbb{1} + i\tau\cdot\pi. \tag{2.12}$$

The Lagrangian (2.11) is rewritten in terms of the Σ fields, as

$$\mathcal{L} = \frac{1}{4}\,\mathrm{Tr}\left(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right) + \frac{\mu^2}{4}\,\mathrm{Tr}\left(\Sigma^\dagger\Sigma\right) - \frac{\lambda}{16}\left[\mathrm{Tr}\left(\Sigma^\dagger\Sigma\right)\right]^2 + \\ + i\bar{\psi}\partial\!\!\!/\psi - g\left(\bar{\psi}_L\Sigma\psi_R + \bar{\psi}_R\Sigma^\dagger\psi_L\right) \tag{2.13}$$

The model is invariant under the $SU(2)_L \times SU(2)_R$ transformations[4]:

$$\psi_L \to L\psi_L, \qquad \psi_R \to R\psi_R, \qquad \Sigma \to L\Sigma R^\dagger \tag{2.14}$$

for $L, R \in SU(2)$. This is the so called linear representation because the kinetic term $\frac{1}{4}\,\mathrm{Tr}\left(\partial_\mu\Sigma\partial^\mu\Sigma^\dagger\right)$ is linear in the fields. From now, we will forget the fermionic terms as they do not really add anything to our discussion on the effective Lagrangians. However, there are other ways to display the content of the sigma model besides the above linear representation. For instance, the so-called exponential representation is defined via [5]

$$\Sigma = \sigma + i\tau\cdot\pi = \rho U \equiv \rho e^{i\tau\cdot\frac{\pi'}{v}}. \tag{2.15}$$

[3] Originally, it corresponded to a spinless meson called σ, from which the name of the model.

[4] $SU(2)_L \times SU(2)_R$ and $SO(4)$ are isomorphic at the level of their algebra, therefore the symmetry remains the same of (2.11), as one expects.

[5] This ansatz is suggested by the CCWZ (Coleman, Callan, Wess, Zumino) formalism, as we will see later.

Using this form, the Lagrangian is rewritten as

$$\mathcal{L} = \frac{1}{2}\partial_\mu \rho \partial^\mu \rho + \frac{\rho^2}{4}\,\mathrm{Tr}\left[\partial_\mu U \partial^\mu U^\dagger\right] + \frac{\mu^2}{2}\rho^2 - \frac{\lambda}{4}\rho^4 \tag{2.16}$$

where we have used the fact that the Maurer-Cartan form $U^\dagger \partial^\mu U$ (and clearly $U\partial^\mu U^\dagger$ too) belongs to $su(2)$, so it is traceless. We can read from (2.15) the transformation properties of U under $SU(2)_L \times SU(2)_R$:

$$U \to LUR^\dagger. \tag{2.17}$$

We notice that in this form - and this is a general feature of the NGBs - the relevant degrees of freedom (the π''s) transform in a non-linear way under $SU(2)_L \times SU(2)_R$. Furthermore we can easily read that the radial field ρ picks out a VEV, while the angular fields π' have no-potential: they are the NGBs of the theory.

Before entering in more general questions about the effective Lagrangians, we just stop one moment on the redefinition (2.15). There is nothing, a priori, which says us that such a redefinition will not change the theory (amplitudes, etc.): in fact, it can be easily checked that the Feynman diagrams of the same process are different for each representation. However, the amplitudes remain the same: this is essentially the content of a powerful field theoretic theorem, proved first by Haag [27] and then reviewed by Callan, Coleman, Wess, Zumino [28], [29]. It states that if two fields are related non-linearly, for example $\phi = \chi F(\chi)$ with $F(0) = 1$, then the same experimental observables result if one calculates with the field ϕ using $\mathcal{L}(\phi)$ or instead with χ using $\mathcal{L}(\chi F(\chi))$. Since we do not need to worry about this redefinition, we are ready to show what the low energy limit of the sigma model is, by integrating out the heavy field ρ, which is the topic of the next section.

2.2 Integrating Out Heavy Fields

In order to show the main consequences of integrating out a field and to read the low energy regime of the sigma model, we need to integrate out the heavy field ρ, in the sense explained at the beginning of the chapter. This time we do not integrate the Fourier components of the field but the entire field with a mass M larger than the scale E of the process. As we have already outlined, the crucial point is that all the effects of heavy particles can be incorporated into a few constants that depends on the mass of the heavy particle. This is essentially the content of the so-called *decoupling theorem*, firstly proved by Appelquist and Carazzone [30]. The theorem states that

all effects of the heavy particles appear either as a renormalization of the coupling constants in the theory or else are suppressed by powers of the heavy particle mass, provided that the remaining low energy theory is renormalizable.

The most famous example is the Fermi interaction $\mathcal{L} \sim G_F/\sqrt{2}(\bar{\psi}\psi)(\bar{\psi}\psi)$ postulated by Enrico Fermi in order to explain the β decay. We now know that it is a low energy description of electroweak interactions and indeed it derives from the electroweak Lagrangian by integrating out the heavy bosons $W^{\pm}$, Z. By comparing the Fermi Lagrangian with the Glashow-Weinberg-Salam model we find $G_F/\sqrt{2} = g^2/8m_W^2$; as we can see the constant is renormalized by an inverse power of the W boson mass.

We now apply these ideas to the sigma model. Firstly, in order to explicit the SSB, we rewrite the Lagrangian (2.16) as

$$\mathcal{L} = \frac{1}{2}\partial_\mu \rho \partial^\mu \rho + \frac{\rho^2}{4} \operatorname{Tr}\left[\partial_\mu U \partial^\mu U^\dagger\right] - \frac{\lambda}{4}\left(\rho^2 - v^2\right)^2 \tag{2.18}$$

where $v^2 = \mu^2/\lambda$. There are two ways to perform the integration of the field ρ. The first method consists in integrating out the field by following the original definition, that is performing the integration of the heavy field in the path integral formalism as explained in the introduction of the chapter. The second method consists in solving the equation of motion for ρ order by order in the coupling λ and then put the resulting field in the Lagrangian[6]. We will perform the calculation with the second method, as it is more immediate than the path integral one and it will be useful in most of the calculations of this thesis. The equation of motion for ρ reads

$$\Box\rho + \lambda\rho\left(\rho^2 - v^2\right) - \frac{\rho}{2}t = 0 \tag{2.19}$$

where we have defined $t \equiv \operatorname{Tr}\left[\partial_\mu U \partial^\mu U^\dagger\right]$. Now we expand ρ in powers of $1/\lambda^n$

$$\rho = \rho_0 + \frac{\rho_1}{\lambda} + \cdots \tag{2.20}$$

and we solve the equation order by order in λ putting this expansion in the equation of motion for ρ:

$$\begin{aligned} \lambda^1 : &\qquad \left(\rho_0^3 - v^2\rho_0\right) = 0 \implies \rho_0^2 = v^2, \\ \lambda^0 : &\qquad \Box\rho_0 + 2\rho_0^2\rho_1 + \rho_1\left(\rho_0^2 - v^2\right) - \rho_0\frac{t}{2} = 0 \implies \rho_1 = \frac{t}{4v} \end{aligned} \tag{2.21}$$

and so on. In conclusion the ρ field, at the order $1/\lambda$, is

$$\rho = v + \frac{1}{\lambda}\frac{t}{4v} + o\left(\frac{1}{\lambda}\right). \tag{2.22}$$

The effective Lagrangian at the order $1/\lambda$ finally reads

$$\begin{aligned} \mathcal{L} &= \frac{v^2}{4}\operatorname{Tr}\left[\partial_\mu U \partial^\mu U^\dagger\right] + \frac{1}{16\lambda}\left[\operatorname{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)\right]^2 + o\left(\frac{1}{\lambda}\right) = \\ &= \frac{v^2}{4}\operatorname{Tr}\left[\partial_\mu U \partial^\mu U^\dagger\right] + \frac{v^2}{8m_\rho^2}\left[\operatorname{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)\right]^2 + o\left(\frac{1}{\lambda}\right) \end{aligned} \tag{2.23}$$

[6] However, one should be careful about this second method, because it is not always applicable.

where we have introduced the mass of the ρ particle $m_\rho^2 = 2\lambda v^2$. It is now clear that the net result of integrating out the field ρ is a low energy effective Lagrangian written as expansion in powers of $\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)$. We notice that the effective Lagrangian preserves the original $SU(2)_L \times SU(2)_R$: this is true at all orders. However, this is not an interesting case since we know what the UV completion of the theory is, i.e. the sigma model. The effective Lagrangian technique is instead extremely useful when we do not know the full theory but we know that the theory is invariant under some symmetry group G spontaneously broken to H. Suppose, for example, we are dealing with an unknown theory with the same $SU(2)_L \times SU(2)_R$ symmetry spontaneously broken to $SU(2)_V$. What we have to do is to write the most general effective Lagrangian in the relevant degrees of freedom at low energy, i.e. the NGBs, consistent with this symmetry, in terms of unknown coefficients that have to be determined phenomenologically. In our case we have

$$\begin{aligned}\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \cdots =& \frac{F^2}{4}\,\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \alpha_1 \left[\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)\right]^2 + \\ &+ \alpha_2\,\mathrm{Tr}\left(\partial_\mu U \partial_\nu U^\dagger\right)\mathrm{Tr}\left(\partial^\mu U \partial^\nu U^\dagger\right) + \cdots\end{aligned} \tag{2.24}$$

where F is the typical energy scale of the theory[7]. We remark that we just needed to specify the symmetry group in order to write the most general effective Lagrangian describing the low energy physics. This is essentially the content of the CCWZ formalism that we are going to show in the next section, that generalizes this method to a generic group G spontaneously broken to H. As explained in the introduction, the important point is that, since the coefficient of an operator of dimension d (which carries d derivatives, since U has no dimension) is $1/\Lambda^{(d-4)}$ where Λ is an upper scale that depends on the theory, at sufficiently low energies the matrix elements of most of these terms are very small because proportional to q^d/M^d, so we can neglect them leaving us with a very few terms. This last way of proceeding, that is to parametrize how the new physics can shows up by writing the most general Lagrangian consistent with the symmetries of the underlying UV theory, is the so-called *bottom-up* approach, in contrast with the previous one, where, starting by a well known UV complete theory (the linear σ model) we calculate its low energy regime by integrating out the heavy field, which is called *top-down* approach.

2.3 CCWZ Formalism

At the end of the previous section we have shown the effective Lagrangian of a theory with a SSB $SO(4)/SO(3)$. However, as we have outlined, the whole

[7] For instance, in the case of pions it is the pion decay constant.

discussion is general and it does not depend on the symmetry group of the Lagrangian. In fact, in 1960 Coleman, Callan, Wess, Zumino [28], [29] developed a formalism that allows to write, in a completely model-independent fashion, general low energy effective Lagrangians for strongly - or weakly coupled - theories characterised by a generic G/H symmetry breaking pattern, describing the Goldstone bosons associated with the symmetry breaking and the heavy resonances. We briefly recall, following [18], [28], [29], the content and the main ideas of the so-called CCWZ formalism.

Let G be the global symmetry group of the theory. We assume that the vacuum state is only invariant under a subgroup $H \in G$ leading to a $G \to H$ spontaneous symmetry breaking and thus to the appearance of massless NGB's in the coset G/H. We will refer to T^A, with $A = (1, \cdots \dim[G])$, as the generators of the Lie algebra of G. Once we choose a reference vacuum F, we can split the generators (with respect to F) into unbroken ($a = 1, \cdots, \dim[H]$) and broken ($\hat{a} = \hat{1}, \cdots, \dim[G/H]$):

$$T^A = \{T^a, \hat{T}^{\hat{a}}\}. \tag{2.25}$$

By definition, F is invariant only under H:

$$(T^a)^{ij} F^j = 0 \qquad (\hat{T}^{\hat{a}})^{ij} F^j \neq 0 \tag{2.26}$$

where the i, j indices run over the representation space of the fields of the theory. The starting point is to identify the correct degrees of freedom that describe the massless NGBs. Since the NGBs have no mass and no potential, this suggests that they are related to the representative vacuum F^i by a local G transformation. Following this idea, it makes particularly sense to perform the ansatz

$$\Phi^i = \left[e^{i\theta_A(x)T^A}\right]^{ij} F^j \tag{2.27}$$

where we have denoted with Φ^i the fields of the theory living in a multiplet of G. The $\theta_A(x)$ are suitable candidates to be the NGBs; in fact, for θ_A constant, $e^{i\theta_A T^A}$ belongs to G, and, as G is the symmetry group of the Lagrangian, these degrees disappear from the potential part of the Lagrangian, so they have no potential and consequently zero mass. Their appearance in the Lagrangian is completely due to the kinetic term. However, because of (2.26), not all these fields are physical. In order to explain why, we remind that a generic $g \in G$ element can be decomposed uniquely into a product of the form:

$$g = e^{i\xi^{\hat{a}}\hat{T}^{\hat{a}}} e^{iu^a T^a} \tag{2.28}$$

with $e^{iu^a T^a} \in H$. Applying this decomposition to (2.27) we find

$$\Phi^i = \left[e^{i\theta_A(x)T^A}\right]^{ij} F^j = \left[e^{i\xi^{\hat{a}}(x)\hat{T}^{\hat{a}}}\right]^{ik} \left[e^{iu^a(x)T^a}\right]^{kj} F^j = \left[e^{i\xi^{\hat{a}}(x)\hat{T}^{\hat{a}}}\right]^{ij} F^j \tag{2.29}$$

where in the last equality we have used (2.26). This is consistent with what we already know: the NGBs are as many as the broken generators of the group G and indeed $\hat{a} = 1, \cdots, \dim[G/H]$. In order to get canonically normalized (and dimensionals) NGB fields it is useful to make the definition

$$\xi^{\hat{a}}(x) \equiv \frac{\sqrt{2}}{f}\pi^{\hat{a}}(x) \tag{2.30}$$

where f is the scale of the spontaneous symmetry breaking and the π's are the canonically normalized NGB fields. The so-called Goldstone matrix $e^{i\xi^{\hat{a}}(x)\hat{T}^{\hat{a}}}$ is the basic object on which the CCWZ construction is based:

$$U[\pi] \equiv \exp\left[i\frac{\sqrt{2}}{f}\pi^{\hat{a}}(x)\hat{T}^{\hat{a}}\right]. \tag{2.31}$$

Let us now see how the group G acts on the Goldstone bosons. Following the original paper of Coleman, Wess and Zumino, we notice that, for any element $g_0 \in G$, we have

$$g_0 e^{i\xi^{\hat{a}}(x)\hat{T}^{\hat{a}}} = e^{i\xi'^{\hat{a}}(x)\hat{T}^{\hat{a}}} e^{iu'^{a}(x)T^{a}} \tag{2.32}$$

where

$$\xi' = \xi'(\xi; g_0), \qquad u' = u'(\xi; g_0). \tag{2.33}$$

Because of (2.32), the action of an element $g \in G$ on the Goldstone matrix $U[\pi]$ reads:

$$g \cdot U[\pi] = U[\pi^{(g)}] \cdot h[\pi; g] \tag{2.34}$$

with $U[\pi^{(g)}]$ and $h[\pi; g]$ specified by the structure of the group. The transformation of the Goldstone bosons π is then defined implicitly by

$$U[\pi^{(g)}] = g \cdot U[\pi] \cdot h^{-1}[\pi; g]. \tag{2.35}$$

We notice that this is consistent with the transformation law of Φ:

$$\Phi' = U[\pi^{(g)}]^{ij} F^j = \left(g \cdot U[\pi] \cdot h^{-1}[\pi; g]\right)^{ij} F^j = \left(g \cdot U[\pi]\right)^{ij} F^j = g^{ij}\Phi^j \tag{2.36}$$

where in the second equality we have used the fact that, by definition, $h^{ij}F^j = F^i$. Looking at (2.35), it is important to notice that the transformation acting on the fields π is local, since $h^{-1}[\pi; g]$ depends on fields π. Furthermore the $\pi^{\hat{a}}$'s belong to a non-linear representation of the group G, in the sense that the transformation cannot be written as:

$$\pi'^{\hat{a}} = A^{\hat{a}\hat{b}}\pi^{\hat{b}}$$

for some linear matrix A independent from π. However, (2.35) provides a representation of G, and in fact it respects the group multiplication rule. In order to verify it, we just need to observe that

$$\begin{aligned} U\left[\left(\pi^{(g_2)}\right)^{(g_1)}\right] &= g_1 \cdot U[\pi^{(g_2)}] \cdot h^{-1}[\pi^{(g_2)}; g_1] = \\ &= g_1 \cdot g_2 \cdot U[\pi] \cdot h^{-1}[\pi; g_2] \cdot h^{-1}[\pi^{(g_2)}; g_1] = \\ &= g_1 \cdot g_2 \cdot U[\pi] \cdot h^{-1}[\pi; g_1 \cdot g_2] = U\left[\pi^{(g_1 g_2)}\right] \end{aligned} \tag{2.37}$$

where in the third equality we have used the fact that h is itself a group representation, so $h^{-1}(g_1 g_2) = h^{-1}(g_2) \cdot h^{-1}(g_1)$.

However, there is a case in which the NGBs transform linearly, that is when $g \in H$. To see this, we notice that the G algebra decomposes as

$$\begin{aligned} [T^a, T^b] &= i f^{ab}_c T^c + i f^{ab}_{\hat{c}} \hat{T}^{\hat{c}} = i f^{ab}_c T^c \equiv (t^a_{\mathrm{Ad}})^b_c T^c \\ [T^a, \hat{T}^{\hat{b}}] &= i f^{a\hat{b}}_c T^c + i f^{a\hat{b}}_{\hat{c}} \hat{T}^{\hat{c}} = i f^{a\hat{b}}_{\hat{c}} \hat{T}^{\hat{c}} \equiv (t^a_\pi)^{\hat{b}}_{\hat{c}} \hat{T}^{\hat{c}} \\ [\hat{T}^{\hat{a}}, \hat{T}^{\hat{b}}] &= i f^{\hat{a}\hat{b}}_c T^c + i f^{\hat{a}\hat{b}}_{\hat{c}} \hat{T}^{\hat{c}} \end{aligned} \tag{2.38}$$

where in the first line we have used the fact that the T^a's are the generators of the subgroup H, so their commutator cannot contain generators that do not belong to H, so $f^{ab}_{\hat{c}} = 0$. From this and from the antisymmetric properties of the structure constants it follows that $0 = f^{ab}_{\hat{c}} = -f^{a\hat{b}}_c$, thanks to which the right side of the second line can be written as depending only on the broken generators. In the second line we have defined t^a_π because, as we now show, it is the representation in which the NGBs $\pi^{\hat{a}}$ transform under H. Firstly, it can be easily seen that t^a_π satisfies the commutation relations of the H algebra so it is at least a representation of H. Now, in order to show that the NGBs transform under H with (the exponential of) t^a_π, we notice that for $g = e^{i\alpha^a T^a} \in H$ we have

$$\begin{aligned} g \cdot U[\pi] = g \cdot U[\pi] \cdot g^{-1} \cdot g &= \exp\left[i\frac{\sqrt{2}}{f}\pi^{\hat{a}}(x) g \hat{T}^{\hat{a}} g^{-1}\right] \cdot g = \\ = \exp\left[i\frac{\sqrt{2}}{f}\pi^{\hat{a}}(x)\left(e^{i\alpha^a t^a_\pi}\right)^{\hat{a}\hat{b}} \hat{T}^{\hat{b}}\right] \cdot g &= U\left[\left(e^{i\alpha^a t^a_\pi}\right)^{\hat{b}\hat{a}} \pi^{\hat{b}}\right] \cdot g \end{aligned} \tag{2.39}$$

where we have used the exponentiated version of (2.38). Comparing (2.39) with (2.35), as $g \in H$, we find

$$\pi^{(g)}_{\hat{a}} = (e^{i\alpha^a t^a_\pi})_{\hat{b}\hat{a}} \pi_{\hat{b}}. \tag{2.40}$$

On the other hand, the transformations along the broken generators cannot be written explicitly on the $\pi^{\hat{a}}$'s. Their action is relatively simple only on the Goldstone matrix, as we can see from (2.35). In fact, what Coleman-Callan-Wess-Zumino did was to find objects that could transform in a more controlled way than $U[\pi]$. The fundamental objects employed in the CCWZ construction are the $p_\mu^{\hat{a}}$, v_μ^a symbols. They are defined by decomposing on the $\mathcal{G}$ algebra the Maurer-Cartan form:

$$iU[\pi]^{-1} \cdot \partial_\mu U[\pi] = p_\mu^{\hat{a}} \hat{T}^{\hat{a}} + v_\mu^a T^a \equiv p_\mu + v_\mu. \tag{2.41}$$

Now and in what follows, we will forget about the π dependence of h but we remember that g is a global transformation, conversely h depends on x because of (2.34). Under G, the Maurer-Cartan form transforms as

$$\begin{aligned} iU[\pi^{(g)}]^{-1} \cdot \partial_\mu U[\pi^{(g)}] &= ih \cdot U[\pi]^{-1} \cdot g^{-1} \left[g \cdot \partial_\mu U[\pi] \cdot h^{-1} + g \cdot U[\pi] \cdot \partial_\mu h^{-1} \right] \\ &= ih \cdot U[\pi]^{-1} \cdot \partial_\mu U[\pi] \cdot h^{-1} + ih \cdot \partial_\mu h^{-1} = \\ &= h \cdot p_\mu \cdot h^{-1} + h \cdot v_\mu \cdot h^{-1} + h \cdot \partial_\mu h^{-1}. \end{aligned} \tag{2.42}$$

Since $ih \cdot \partial_\mu h^{-1}$ is itself a Maurer-Cartan form associated to the subgroup H, it contains only the T^a generators. It follows that p_μ, v_μ transform as

$$\begin{aligned} p_\mu &\to h \cdot p_\mu \cdot h^{-1} \\ v_\mu &\to h \cdot v_\mu \cdot h^{-1} + ih \cdot \partial_\mu h^{-1}. \end{aligned} \tag{2.43}$$

We see from (2.43) that $p_\mu^{\hat{a}}$ transforms exactly like π:

$$p_\mu^{\hat{a}} \to p_\mu^{(g),\hat{a}} = (e^{i\alpha^a t_\pi^a})^{\hat{b}\hat{a}} p_\mu^{\hat{b}}. \tag{2.44}$$

However, and this is the point, p_μ, differently from π, transforms with t_π^a under the *whole* G, not just under the subgroup H. For what concerns the v_μ^a symbols, they transform as if they were gauge fields associated with a local H invariance. So they can be employed to construct covariant derivatives and field strength but they cannot be inserted directly in the operators.

The CCWZ prescription is to construct G-invariant operators by combining p_μ and v_μ symbols and their derivatives. The remarkable fact is that all the operators can be constructed in this way [8]. Since, as we see from (2.43), p_μ and v_μ transform under G with *local* H transformations, we need to worry about building H invariants with the standard group theory tools and the full G invariance will follow automatically. So, for example the two derivatives operators can be obtained by contracting p symbols with H invariant tensors. The most general combination of such operators defines the so called 2-derivative non-linear σ-model Lagrangian. In general there can be more H invariant tensors. However, for H compact one always exists (the $\delta^{\hat{a}\hat{b}}$) and the corresponding term is given by

$$\mathcal{L}^{(2)} = \frac{f^2}{4} p_\mu^{\hat{a}} \delta^{\hat{a}\hat{b}} p^{\mu,\hat{b}}. \tag{2.45}$$

[8] With the exception of the Wess-Zumino-Witten term which signals the presence of an anomaly in G.

It provides the Goldstone bosons kinetic terms ($f^2/4$ has been added in order to have the correct normalization) plus an infinite set of two derivative interactions, which are controlled by the unique parameter f. If $\delta^{\hat{a}\hat{b}}$ is the only invariant, and this is the case for the $SO(N)/SO(N-1)$ symmetry breaking pattern, then all the 2-derivative interactions can be predicted in terms of the unique constant f. This means that, whatever the UV theory is, provided it is based on the symmetry breaking pattern $SO(N)/SO(N-1)$, it leads, at the 2-derivative level, to the same Lagrangian and physical predictions. So the results obtained in the linear sigma model example are completely general, in spite of the fact that the linear sigma model is just one possible UV realization of the low energy theory.

Now, before continuing with the CCWZ construction, we just stop one moment on the calculation of the $p_\mu^{\hat{a}}$. In general, they depend hardly on the specific group G and H but sometimes (actually very often in physics) we can simplify the kinetic term. It is the case when G/H is a so-called symmetric coset, i.e. the G algebra has an automorphism (an example is the chiral group $SU(N)_L \times SU(N)_R$)

$$\begin{aligned} T^a &\to T^a, \\ \hat{T}^{\hat{a}} &\to -\hat{T}^{\hat{a}}. \end{aligned} \tag{2.46}$$

Firstly, we notice that the automorphism (2.46) implies, from (2.38), that $f^{\hat{a}\hat{b}}_{\hat{c}} = 0$. Furthermore, under the automorphism, $U \to U^\dagger$ and viceversa. This implies that

$$U\partial_\mu U^\dagger = -p_\mu^{\hat{a}}\hat{T}^{\hat{a}} + v_\mu^a T^a \equiv -p_\mu + v_\mu \tag{2.47}$$

where the dependence on π is made implicit. Taking the difference between (2.41) and (2.47) we find

$$p_\mu = \frac{1}{2}\left(U^\dagger\partial_\mu U - U\partial_\mu U^\dagger\right). \tag{2.48}$$

The kinetic term is therefore

$$\mathrm{Tr}[p_\mu p^{\mu\dagger}] = 2\,\mathrm{Tr}\left[\partial_\mu U\partial_\mu U^\dagger\right] - 2\,\mathrm{Tr}\left[U^\dagger U^\dagger\partial_\mu U\partial_\mu U\right] = 2\,\mathrm{Tr}\left[\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right] \tag{2.49}$$

where we have defined

$$\Sigma = U\tilde{U}^\dagger = U^2 \tag{2.50}$$

where $\tilde{U}$ is the image of U under the automorphism and, as we have said, $\tilde{U} = U^\dagger$. From the U transformation we can deduce the Σ transformation:

$$\Sigma = U\tilde{U}^\dagger \to gUh^{-1}h\tilde{U}^\dagger\tilde{g}^\dagger = g\Sigma\tilde{g}^\dagger. \tag{2.51}$$

Thus, for symmetric spaces we can construct a Goldstone matrix U transforming linearly under G (in general, because of the π dependence in h, only

p_μ enters in the kinetic term). This is precisely what we made explicitly with the sigma model in the previous section. In fact, in the case of the chiral group broken to his vectorial part: $(SU(N)_L \times SU(N)_R)/SU(N)_V$, the broken generators are the axial one $A^{\hat{a}} \equiv L^{\hat{a}} - R^{\hat{a}}$ so the automorphism sends

$$A = L - R \to -A = R - L \tag{2.52}$$

i.e., the automorphism interchanges R with L. We can read from (2.51) the Σ transformation ($g = g_L g_R = g_R g_L$):

$$\Sigma \to g_L g_R \Sigma g_L^\dagger g_R^\dagger \tag{2.53}$$

that is equivalent to say that Σ transform as[9]

$$\Sigma \to g_L \Sigma g_R^\dagger, \tag{2.54}$$

which is the U transformation law in the sigma model.

Until now we have been considering a *global* G group. However, the discussion can be easily generalized to a *local* G group and this is the case we are interested in, as the SM symmetry group is local. Denoting with $A_{\mu A}$ with $A = 1, \cdots, \dim[G]$ the gauge fields (there is one for each generator), then the basic object to construct invariant operators is the generalized Maurer-Cartan form

$$U[\pi]^{-1}\left(A_\mu + i\partial_\mu\right) U[\pi] \equiv p_\mu\left[\pi, A\right] + v_\mu\left[\pi, A\right] \tag{2.55}$$

where, as usual, $A_\mu = A_\mu^A T^A$, $p_\mu = p_\mu^{\hat{a}} T^{\hat{a}}$, $v_\mu = v_\mu^a T^a$. We can make this decomposition because, as A_μ belongs to the G algebra, the action of $U[\pi]^{-1}$ and $U[\pi]$ on A_μ will always produce terms belonging to the G algebra. It can be easily demonstrated that it all works as in the global case with the substitution of the standard derivative with the covariant one. For instance, the generalized p and v symbols transform as

$$\begin{aligned} p_\mu\left[\pi, A\right] &\to h \cdot p_\mu\left[\pi, A\right] \cdot h^{-1} \\ v_\mu\left[\pi, A\right] &\to h \cdot v_\mu\left[\pi, A\right] \cdot h^{-1} + ih \cdot \partial_\mu h^{-1} \end{aligned} \tag{2.56}$$

and the two-derivative term in the case of a symmetric coset reads

$$\mathcal{L}^{(2)} = \frac{f^2}{4}\,\mathrm{Tr}\left[D_\mu \Sigma D_\mu \Sigma^\dagger\right]. \tag{2.57}$$

It is also possible - and it will be the case of interest to us - to gauge a *subgroup* of the whole G. Formally this is achieved by gauging all the fields A_μ^A and decoupling the unwanted ones sending the coupling strength to zero.

[9] In the sense that objects invariant under (2.53) are also invariant under (2.54) and vice versa.

Chapter 3

The Composite Higgs Scenario

Until now we have been describing the motivations that led to the develop of the composite Higgs idea. Instead, this chapter will be devoted to the study of the main features of the composite Higgs scenario, shared by almost all composite Higgs models.

In analogy with QCD, we have a global group G spontaneously broken to a subgroup H at the confinement scale f (the analogue of the Λ_{QCD} scale), leading to NGBs in the coset G/H; the Higgs is one of these NGBs. The subgroup H is assumed to contain the EW group and G is assumed to be large enough to contain at least one Higgs doublet. However, if the symmetry was exact the Higgs would be exactly massless, it would have no potential and its VEV would be unobservable. In fact, it is essential that G be *explicitly* broken. In composite Higgs models this is achieved by relying on two different sources of (small) G breaking:

- a G breaking comes from the gauging of G_{EW} contained in the unbroken H group, just like the gauging of the electromagnetic group $U(1)_{\text{em}}$ provides a small mass to pions;
- another source of small G breaking is induced by the coupling with the fermions of the theory that are the SM fermions plus an extra sector of heavy fermions. In fact, in general the SM fermions live in multiplets only of the EW group, so they cannot be coupled to the Higgs without breaking the G symmetry. These Yukawa-like couplings are indispensable also to give a mass to SM fermions.

The Higgs mass is therefore controlled by the explicit breaking of the symmetry. However, as we have previously outlined, the explicit breaking has to be small, in order to keep the Higgs lighter than other resonances and to not destabilize the hierarchy among f and the Λ_{UV} scale. The situation is pictorially represented in Fig. 3.1. In the case of an exact G symmetry the VEV can be rotated on the reference vacuum $\vec{F}$, with $|F| = f$, by a G transformation, therefore its projection (that corresponds to the NGBs'

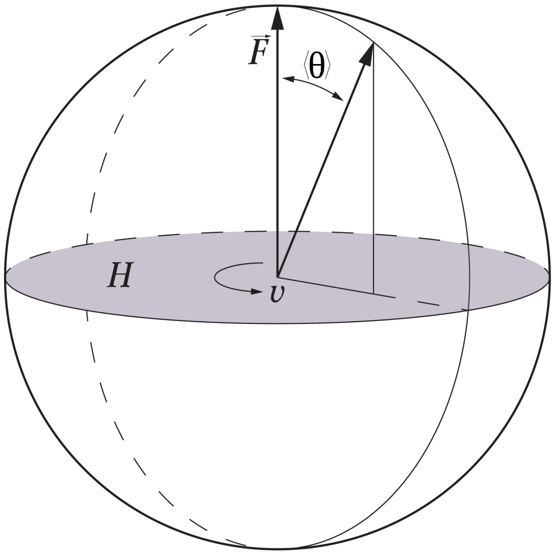

Figure 3.1: A geometrical illustration of the EWSB through vacuum misalignment in the case $G/H = SO(3)/SO(2)$ [18].

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VEV) is zero. On the other hand, taking the G breaking into account means that the Goldstone fields now develop a potential and their VEV $\langle\theta\rangle$ is not arbitrary anymore. This causes the breaking of G_{EW} contained in the unbroken H giving rise to the EWSB [18]. Summarizing, in the theory there are two scales:

- the scale f of SSB of G;
- the scale v of the EWSB that breaks G_{EW} to $U(1)_{\text{em}}$.

Now comes the point. If the so called misalignment angle $\langle\theta\rangle$ is small, then the composite Higgs boson can behave as an elementary one. This condition is usually expressed in the study of composite Higgs models by defining the parameter ξ as

$$\xi \equiv \frac{v^2}{f^2} = \sin^2\langle\theta\rangle \ll 1. \tag{3.1}$$

The limit $\xi \to 0$, at fixed v, corresponds to send the scale f to infinity, i.e. to decouple the composite sector from the low energy physics. The parameter ξ regulates the deviations of the CHM predictions from the SM ones. In fact, as we will see, the theory systematically reduces to the SM for $\xi \to 0$ and the composite Higgs behaves effectively as an elementary particle. This mechanism depicted is called *vacuum misalignment* and it has been described for the first time by Kaplan and Georgi [5], [31].

3.1 Partial Fermion Compositeness

We have already outlined in the previous chapters that, beyond the reasons related to the Naturalness and triviality problems of the Higgs, another trouble of the SM that has driven physicists to find an extension of the latter is the Yukawa hierarchy: the Weinberg-Salam model does not explain the patterns seen in masses and mixing of fermions. We remind that there is nothing that forbids us to adjust ad hoc the Yukawa couplings in order to obtain the measured fermion masses, as they are free parameters in the SM. However, in the same spirit as the Higgs mass, one would like to obtain an understanding from some underlying physics. There have been many attempt to address this problem during the last decades. In composite Higgs models the Yukawa hierarchy is explained by the *partial fermion compositeness* hypothesis, firstly introduced by Kaplan [6]. Let us see how the mechanism works.

At the scale f, where f in the notation above is the confinement scale of the new strong force, the strong sector confine, develops the Higgs boson and, on top of it, generates a set of resonances with typical mass f[1] [18]. These (heavy) fermionic resonances mix with SM fermions, through bilinear operators, generating a seesaw-like mechanism. This is why the mechanism is called "partial fermion compositeness": the SM fermions are linear superimpositions of elementary (the light fermions) and composite (the heavy resonances) degrees of freedom. However, it is important to remind that in all the CHMs there are no attempts to construct a microscopical description of the EW mechanism. These models try to understand which are the low energy implications of some UV theory, regardless of what the theory is. Therefore the bilinear couplings should be regarded as effective couplings emerging at the infrared (IR) scale f after their evolution under the renormalization group down to the IR region. Moreover, as the bilinear operators must be invariant under $SU(3)_c \times SU(2)_L \times U(1)_Y$, the heavy fermions must carry the same $SU(2)_L \times U(1)_Y$ quantum numbers of the SM fermions and transform in the fundamental of $SU(3)_c$. It is useful to give an example, in order to see how the mechanism works. Denoting with Q and T the heavy partners of the quark (here the top), doublet with hypercharge 1/6 and singlet with hypercharge 2/3 respectively, the mass term of the SM quark-heavy quark system reads

$$\mathcal{L}_{\text{mass}} = -f\bar{Q}^u Q^u - f\bar{Q}^d Q^d - f\bar{T}T - \Lambda_Q\,(\bar{q}_L Q + h.c.) - \Lambda_T\,(\bar{t}_R T + h.c.) \quad (3.2)$$

where the subscripts u and d stand for up and down components of the heavy-quark doublet. We notice that the masses of the Q and T heavy fermions are the same: this is because they are part of a larger multiplet of a larger group which contains the unbroken SM group. In order to read the

[1]This is exactly what happens in QCD with baryons.

fermion masses we write $\mathcal{L}_{\text{mass}}$ in matrix notation (we consider only the top sector, the bottom being the same)

$$\mathcal{L}_{\text{mass,t}} = \begin{pmatrix} \bar{t}_L & \bar{Q}^u_L & \bar{T}_L \end{pmatrix} \begin{pmatrix} 0 & \Lambda_Q & 0 \\ 0 & f & 0 \\ \Lambda_T & 0 & f \end{pmatrix} \begin{pmatrix} t_R \\ Q^u_R \\ T_R \end{pmatrix} + h.c. \tag{3.3}$$

The eigenvalues are $\{f, f, 0\}$: from this mechanism arise two heavy fermions and one massless fermion that can be identified with the top quark. However, the above discussion needs to be extended. The limitations are essentially two: firstly, the SM fermions arise exactly massless, and we know that they have a small, but non zero mass. This can be easily solved by changing the bilinear couplings, as we will see in more detail in the composite Higgs model discussed in the next chapter. Secondly, in most the group used in CHMs there is no space to contain the SM quark representations $\mathbf{2}_{1/6}$, $\mathbf{1}_{2/3}$, $\mathbf{1}_{-1/3}$, so we cannot couple all the heavy fermions with the SM fermions. In particular, the problem arises because of the hypercharge. We show how the problem is solved performing the calculation for a specific SSB $SO(5)/SO(4)$ because it is the case we are interested in. In the $SO(5)/SO(4)$ model it is possible to identify the hypercharge with the diagonal generator of $SU(2)_R$ embedded in $SO(4) \sim SU(2)_L \times SU(2)_R$ but it turns out that with this choice not all the SM quark representation are contained in $SO(5)$. An extension of the global symmetry group of the composite sector is required in order to implement the partial compositeness. The simplest possibility is to add a new unbroken $U(1)_x$, extending the breaking pattern to

$$SO(5) \times U(1)_x \to SO(4) \times U(1)_x. \tag{3.4}$$

In this way we can identify the hypercharge as

$$Y = \Sigma^3_R + x. \tag{3.5}$$

This is enough to have correct hypercharge assignments. In fact, since the heavy fermion $SO(5)$ fiveplet $\psi^{(2/3)}$, which contains the two $SU(2)_L$ doublets Q and X and the singlet T, decomposes under $SO(4)$ representations as

$$\psi^{(2/3)} = (X, Q, T) \tag{3.6}$$

where X, Q, T possess $\Sigma^3_R = \{1/2, -1/2, 0\}$ charge respectively and the superscript 2/3 indicates the x charge, then, under $G_{\text{EW}} \equiv SU(2)_L \times U(1)_Y$, the $SO(5)$ fiveplet decomposes as

$$\mathbf{5}_{\frac{2}{3}} \to \mathbf{4}_{\frac{2}{3}} \oplus \mathbf{1}_{\frac{2}{3}} \to \mathbf{2}_{\frac{7}{6}} \oplus \mathbf{2}_{\frac{1}{6}} \oplus \mathbf{1}_{\frac{2}{3}}. \tag{3.7}$$

Here the subscript stays for the hypercharge, the first arrow indicates the decomposition of the fundamental of $SO(5)$ in $SO(4)$ representations, the second indicates the decomposition of the fundamental of $SO(4)$ in $SU(2)_L \times U(1)_Y$ representations. We immediately recognize that the two last terms have the correct quantum numbers in order to be coupled to q_L and t_R respectively. In the next chapter we will see a specific realization of the partial compositeness hypothesis in a more realistic CHM.

3.2 The Abelian Composite Higgs Model

In order to see how the composite Higgs model works, we will develop the ideas of the previous sections in a simple $SO(3)/SO(2)$ sigma model that provides an abelian composite Higgs [18].

Let ϕ be a triplet of real scalar fields with the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^T \partial^\mu \phi - \lambda \left(\phi^T \phi - f^2\right)^2 . \tag{3.8}$$

The Lagrangian is invariant under the $SO(3)$ transformation:

$$\phi \to O\phi, \qquad O = e^{i\alpha^A T^A} \in SO(3) \tag{3.9}$$

where the $T^A = (T, T^{\hat{a}})$ generators are taken to be

$$T = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad T^{\hat{a}} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \right\} \tag{3.10}$$

where, in the notation of the previous chapter, the "hat" index runs over the broken generators. In fact, the potential is minimized by $\phi^T \phi = f^2$, so the field ϕ acquires a VEV that in our convention is taken to be

$$F = \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}. \tag{3.11}$$

As explained in the CCWZ construction, the NGBs can be associated to the fluctuations along the broken generators, so we can redefine the field ϕ, by performing the ansatz

$$\phi = e^{i\frac{\sqrt{2}}{f}\pi^{\hat{a}} T^{\hat{a}}} \begin{pmatrix} 0 \\ 0 \\ f + \sigma(x) \end{pmatrix} \equiv U[\pi] \begin{pmatrix} 0 \\ 0 \\ f + \sigma(x) \end{pmatrix} \tag{3.12}$$

where $U[\pi]$ is the Goldstone matrix. In this simple case we can calculate explicitly it. The result reads

$$U[\pi] = \begin{bmatrix} \delta^{\hat{a}\hat{b}} - \left(1 - \cos\frac{\pi}{f}\right)\frac{\pi^{\hat{a}}\pi^{\hat{b}}}{\Pi^2} & \sin\frac{\pi}{f}\frac{\pi^{\hat{a}}}{\pi} \\ -\sin\frac{\pi}{f}\frac{\pi^{\hat{a}}}{\pi} & \cos\frac{\pi}{f} \end{bmatrix} \tag{3.13}$$

where $\pi = \sqrt{\pi_1^2 + \pi_2^2}$.

Actually, it turns out that the expression (3.13) holds for any $SO(N) \to SO(N-1)$ breaking, provided that we choose the broken generators have non vanishing entries in the last line and column [18]. Definitely, with this redefinition, ϕ reads

$$\phi = (v + \sigma(x)) \begin{pmatrix} \sin \frac{\pi}{f} \frac{\pi^1}{\pi} \\ \sin \frac{\pi}{f} \frac{\pi^2}{\pi} \\ \cos \frac{\pi}{f} \end{pmatrix}. \tag{3.14}$$

Putting (3.2) into the Lagrangian we obtain

$$\begin{aligned}\mathcal{L} = & \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma - \lambda \left(\sigma^2 + 2f\sigma\right)^2 + \\ & + \frac{1}{2}\left(1 + \frac{\sigma}{f}\right)^2 \left[\frac{f^2}{\pi^2} \sin^2 \frac{\pi}{f} \partial_\mu \pi^{\hat{a}} \partial^\mu \pi^{\hat{a}} + \frac{f^2}{4\pi^4}\left(\frac{\pi^2}{f^2} - \sin^2 \frac{\pi}{f}\right) \partial_\mu \pi^2 \partial^\mu \pi^2\right] = \\ = & \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma - 4\lambda f^2 \sigma^2 - 4\lambda f^2 \sigma^3 - \lambda \sigma^4 + \\ & + \frac{1}{2}\left(1 + \frac{\sigma}{f}\right)^2 \left[\frac{f^2}{\pi^2} \sin^2 \frac{\pi}{f} \partial_\mu \pi^{\hat{a}} \partial^\mu \pi^{\hat{a}} + \frac{f^2}{4\pi^4}\left(\frac{\pi^2}{f^2} - \sin^2 \frac{\pi}{f}\right) \partial_\mu \pi^2 \partial^\mu \pi^2\right].\end{aligned} \tag{3.15}$$

A few comments are in order:

- as we see, since $\phi^i \phi^i = (\sigma + f)^2$, the Goldstone bosons have no potential, all their dependence in the Lagrangian comes from the kinetic term, which is non linear, as we mentioned before;
- the σ field acquires, as expected, a mass: $m^2 = 8\lambda f^2$;
- since the symmetry group of the Lagrangian has three generators, three symmetries are expected. However, after the redefinition of the field ϕ by exploiting its radial and angular parts, and after that the σ field takes the VEV f, the symmetries associated with the broken generators act in a non-linear way on the NGBs, as we saw in the CCWZ construction; in general they can not be explicitly written in a finite form. Conversely, the symmetry associated with the unbroken generator remains linear when acting on the $\pi^{\hat{a}}$:

$$\pi^{\hat{a}} = \left[e^{i\alpha\sigma_2}\right]^{\hat{a}\hat{b}} \pi^{\hat{b}} \Longleftrightarrow \Phi^i = \left[e^{i\sqrt{2}\alpha T}\right]^{ij} \Phi^j \tag{3.16}$$

  where σ_2 is the generator of rotations in two dimensions.

In order to make our toy model more similar to the Higgs model, we define the complex field $H = (\pi^1 + i\pi^2)/\sqrt{2}$. Integrating out the heavy field, the Lagrangian at the leading order reads

$$
\begin{aligned}
\mathcal{L}_{\rm H} =& \frac{1}{2}\left[\frac{f^2}{|H|^2}\sin^2\frac{\sqrt{2}|H|}{f}\partial_\mu H^\dagger \partial^\mu H + \right. \\
&+ \left. \frac{f^2}{4|H|^4}\left(\frac{2|H|^2}{f^2} - \sin^2\frac{\sqrt{2}|H|}{f}\right)\partial_\mu |H|^2 \partial^\mu |H|^2\right].
\end{aligned} \tag{3.17}
$$

The last step to make the model more similar to the physical Higgs is to gauge the unbroken $U(1)$ symmetry, in order to implement the usual Higgs mechanism. The gauging of the unbroken $U(1)$ is made by replacing the usual derivative with the covariant one:

$$
\partial_\mu \phi \to D_\mu \phi = \left(\partial_\mu - \sqrt{2} ie A_\mu T\right)\phi. \tag{3.18}
$$

The gauging, since it selects one generator among three, breaks $SO(3)$ explicitly to $SO(2)$. Using (3.16), the covariant derivative acting on the NGBs reads

$$
D_\mu \pi^{\hat a} = \left(\partial_\mu - \sqrt{2} ie A_\mu \sigma_2\right)^{\hat a \hat b}\pi^{\hat b}. \tag{3.19}
$$

If we switch to the complex notation by introducing the field $H = (\pi^1 - i\pi^2)/\sqrt{2}$, it is easy to show that the covariant derivative reads

$$
D_\mu H = \partial_\mu H - ie A_\mu H. \tag{3.20}
$$

By substituting ∂_μ with D_μ in the Lagrangian (3.17) we obtain

$$
\begin{aligned}
\mathcal{L}_{\rm H,\ gauged} =& \frac{1}{2}\left[\frac{f^2}{|H|^2}\sin^2\frac{\sqrt{2}|H|}{f} D_\mu H^\dagger D^\mu H + \right. \\
&+ \left. \frac{f^2}{4|H|^4}\left(\frac{2|H|^2}{f^2} - \sin^2\frac{\sqrt{2}|H|}{f}\right)\partial_\mu |H|^2 \partial^\mu |H|^2\right]
\end{aligned} \tag{3.21}
$$

where we have used the fact that

$$
\begin{aligned}
D_\mu |H|^2 = D_\mu\left(H^\dagger H\right) &= \left(\partial_\mu H^\dagger + ie A_\mu H^\dagger\right) H + H^\dagger\left(\partial_\mu H - ie A_\mu H\right) = \\
&= \left(\partial_\mu H^\dagger\right) H + H^\dagger\left(\partial_\mu H\right) = \partial_\mu |H|^2.
\end{aligned} \tag{3.22}
$$

Breaking explicitly $SO(3)$ to $SO(2)$ implies that the symmetries associated with the broken generators are not symmetries anymore. The main consequence is that terms that in principle were not permitted as they violated the symmetries, now can appear. In our case, it can be shown that the symmetry associated with the broken generators is, at infinitesimal level

$$
\pi^{\hat a} \to \pi^{\hat a} + \pi \cot\frac{\pi}{f}\alpha^{\hat a} + \left(\frac{f}{\pi} - \cot\frac{\pi}{f}\right)\alpha^{\hat b}\pi^{\hat b}\frac{\pi^{\hat a}}{\pi}. \tag{3.23}
$$

By expanding around 0 $\cot\frac{\pi}{f} \sim \frac{f}{\pi}$ we see that the vacuum configuration 0 is transformed into a constant field:

$$\pi^{\hat{a}} \to \pi^{\hat{a}} + f\alpha^{\hat{a}} + \left(\frac{f}{\pi} - \frac{f}{\pi}\right)\alpha^{\hat{b}}\pi^{\hat{b}}\frac{\pi^{\hat{a}}}{\pi} = \pi^{\hat{a}} + f\alpha^{\hat{a}} \quad \Rightarrow \quad 0^{\hat{a}} \to 0^{\hat{a}} + f\alpha^{\hat{a}}. \tag{3.24}$$

This is the famous shift symmetry that characterizes the Goldstone bosons. As a consequence, a potential term is forbidden - as it would not allow a shift symmetry. But if the symmetry is explicitly broken the quantum Lagrangian is no longer forced to preserve the symmetry - as it is no longer a symmetry - and a potential can be generated. It turns out that in our case at one-loop level a potential is generated, and this implies that the Higgs picks a VEV:

$$|H| = \frac{V}{\sqrt{2}}. \tag{3.25}$$

After picking out the VEV, the Higgs mechanism can start its work: one component becomes massive, the other is the would-be Goldstone boson eaten by the A_μ field. We can read from the Lagrangian (3.17) the term that gives mass to the A_μ field

$$\mathcal{L}_{\text{mass,A}} = \frac{1}{2}\frac{e^2 f^2}{|H|^2}\sin^2\frac{\sqrt{2}|H|}{f}A_\mu A^\mu |H|^2; \tag{3.26}$$

evaluating H on its VEV, we obtain a mass term for A_μ:

$$m_A = ef\sin\frac{V}{f} \equiv ev \tag{3.27}$$

where we have defined the scale v of the $U(1)$ symmetry breaking. As we see, in this case the scale v is not "elementary": it derives from an another scale, and this is expressed by the formula

$$v = f\sin\frac{V}{f} \tag{3.28}$$

from which we find

$$\xi \equiv \frac{v^2}{f^2} = \sin^2\frac{V}{f}. \tag{3.29}$$

Chapter 4

The Minimal Linear σ Model for the Goldstone Higgs

In the last chapter we showed the consequences of the composite Higgs scenario in a model that did not contain the EW gauge group. Now we want to extend the procedure to a more realistic model, where the unbroken subgroup H contains the full electroweak group G_{EW}. The *minimal* possibility is to consider a sigma model with a spontaneous symmetry breaking $SO(5)/SO(4)$. It is the minimal because the broken generators are 4, and the emerging NGBs can be associated to the four real components of the usual Higgs doublet. To be more precise, this is the minimal model that respects the so-called *custodial symmetry*, namely the $SO(4)$ symmetry of the scalar sector, broken only by the gauging of the G_{EW} group. The importance of the custodial symmetry relies on the fact that it ensures the $\rho = 1$ relation at tree-level. In fact, an even more minimal model would be the $SU(3) \to SU(2) \times U(1)$ breaking, but, as it does not respect the custodial symmetry, the $\rho = 1$ relation is not protected. The mechanism that generates the light Higgs works essentially as in the $SO(3)/SO(2)$ model.

In this chapter we will study a *linear* sigma model, following the paper [16]. The idea underlying the model is to construct a renormalizable model which in its scalar part is a linear sigma model in which it is included a new scalar particle σ. The main difference with traditional composite Higgs models is that, as the theory is renormalizable, this model keeps open two possibilities: it can be considered either as an ultimate model made out of elementary fields, or as a renormalizable version of a deeper dynamics, much as the linear σ model is to QCD. The model contains also heavy fermions coupled to SM fermions, in order to give a natural light mass to the latter. As we said, this is provided by the partial compositeness hypothesis. The heavy fermions will be coupled to the fiveplet ϕ, through $SO(5)$ preserving interactions, and to SM fermions, through $SO(5)$ breaking terms.

The Lagrangian can be written as the sum of three terms describing respectively the pure gauge, scalar and fermionic sectors

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_s + \mathcal{L}_f. \tag{4.1}$$

$\mathcal{L}_g$ does not need any discussion, as it contains the kinetic terms of the SM gauge bosons and it has the same form as the SM gauge boson Lagrangian. The aim of the remaining sections is to study the fermionic and scalar sectors.

4.1 The Scalar Sector

The analysis will be performed following the paper [16]. In this section we will concentrate on the scalar sector. The scalar Lagrangian describing the scalar-gauge and the scalar-scalar interaction is

$$\mathcal{L}_s = \frac{1}{2}(D_\mu\phi)^T(D_\mu\phi) - V(\phi) \tag{4.2}$$

where the covariant derivative acting on ϕ reads

$$D_\mu\phi = \left(\partial_\mu + ig\Sigma_L^i W_\mu^i + ig'\Sigma_R^3 W_\mu^3\right) \tag{4.3}$$

and Σ_L^i, Σ_R^3 are the generators of $SU(2)_L$ and $SU(2)_R$ subgroups of $SO(4)$ group embedded in $SO(5)$ [18]. We already know that three among five components of ϕ will be associated to the longitudinal components of the SM gauge bosons - denoted below with $\pi^i, i = 1,2,3$ - while the other two will correspond to the Higgs h, the other NGB, and to the heavy resonance σ. For simplicity, we will present the results in the unitary gauge, in which $\pi^i = 0$:

$$\phi = (\pi^1, \pi^2, \pi^3, h, \sigma) \to (0, 0, 0, h, \sigma). \tag{4.4}$$

The potential $V(\phi)$ is made of two parts: a part of spontaneous symmetry breaking $SO(5)/SO(4)$ and a part of explicit breaking of $SO(5)$ but $SO(4)$ preserving. In the unitary gauge[1] the potential reads

$$\begin{aligned}
&V(h,\sigma) = V_{SSB} + V_{EB},\\
&V_{SSB} = \lambda(\sigma^2 + h^2 - f^2)^2,\\
&V_{EB} = a_1 f^2\sigma^2 + a_2 f\sigma h^2 - \beta f^2 h^2 - \alpha f^3\sigma + a_3\sigma^2 h^2 + a_4 f\sigma^3 + a_5 h^4 + a_6\sigma^4.
\end{aligned} \tag{4.5}$$

The first part depends, as usual, from the two parameters controlling the SSB. The second part contains a priori eight terms so, at a first sight, the total number of parameters needed to describe the theory seems to be ten.

[1] To go in a general gauge it suffices to replace h^2 with the $SO(4)$ invariant combination $h^2 + \pi^i\pi^i$.

Actually, two of them can be reabsorbed via a redefinition of the remaining eight[2]. It can be easily shown that, performing the substitutions

$$\lambda \to \lambda + \lambda' \qquad f^2 \to f^2 + f'^2 \tag{4.6}$$

with

$$\lambda' = -a_6 \qquad f'^2 = \frac{a_1 + 2a_6}{2(\lambda - a_6)} f^2, \tag{4.7}$$

the σ^2, σ^4 terms go away from the Lagrangian. For completeness we show below how the other parameters change under these transformations

$$\begin{aligned}
&\alpha \to \alpha \left(\frac{a_1 + 2a_6}{2(\lambda - a_6)}\right)^{\frac{3}{2}} \equiv \alpha', \\
&\beta \to \beta + \beta \frac{a_1 + 2a_6}{2(\lambda - a_6)} + 2\lambda \frac{a_1 + 2a_6}{2(\lambda - a_6)} - 2a_6 - 2a_6 \frac{a_1 + 2a_6}{2(\lambda - a_6)} \equiv \beta', \\
&a_2 \to a_2 \sqrt{1 + \frac{a_1 + 2a_6}{2(\lambda - a_6)}} \equiv a_2', \\
&a_3 \to a_3 - 2a_6 \equiv a_3', \\
&a_4 \to a_4 \sqrt{1 + \frac{a_1 + 2a_6}{2(\lambda - a_6)}} \equiv a_2', \\
&a_5 \to a_5 - a_6 \equiv a_5'.
\end{aligned} \tag{4.8}$$

Under the redefinitions (4.6), (4.7) the potential changes as

$$V(h, \sigma) = \lambda(\sigma^2 + h^2 - f^2)^2 - \alpha' f^3 \sigma - \beta' f^2 h^2 + a_2' f \sigma h^2 + a_3' \sigma^2 h^2 + a_4' f \sigma^3 + a_5' h^4. \tag{4.9}$$

From now on, following a procedure already adopted in [32], we will continue the analysis keeping only the α' and β' terms, as these are the only strictly necessary soft breaking terms in order to absorb the divergences generated by one-loop Coleman-Weinberg potential, as we will see later. Renaming α' and β' with α and β respectively, the potential finally reads

$$V(h, \sigma) = \lambda(\sigma^2 + h^2 - f^2)^2 - \alpha f^3 \sigma - \beta f^2 h^2. \tag{4.10}$$

For an appropriate region of parameters both the h and σ fields acquire a non-vanishing VEV:

$$\begin{cases} \dfrac{\partial V}{\partial h} = 4\lambda(v_\sigma^2 + v^2 - f^2)v - 2\beta f^2 v = 0 \\ \dfrac{\partial V}{\partial \sigma} = 4\lambda(v_\sigma^2 + v^2 - f^2)v_\sigma - \alpha f^3 = 0 \end{cases} \implies \begin{cases} v_\sigma = f \dfrac{\alpha}{2\beta} \\ v^2 = f^2 \left(1 - \dfrac{\alpha^2}{4\beta^2} + \dfrac{\beta}{2\lambda}\right). \end{cases} \tag{4.11}$$

[2]The minus sign in $-\beta f^2 h^2$ is to have $\beta > 0$, inequality that will be clear later, whereas the minus sign in front of $\alpha f^3 \sigma$ serves to have a positive v_σ.

The $SO(5)$ VEV is therefore corrected by the β term in the potential:

$$v^2 + v_\sigma^2 = f^2\left(1 + \frac{\beta}{2\lambda}\right). \tag{4.12}$$

From (4.11), (4.12) it follows that both $f^2 > 0$ and $f^2 < 0$ are in principle allowed, but if we want to interpret the Higgs h as a NGB in the limit $\alpha, \beta \to 0$ it is needed that $f^2 > 0$. According to (4.11), the positivity of v^2 and the $|v| < |v_\sigma|$ constraint lead respectively to

$$\begin{aligned} \alpha^2 &< 4\beta^2\left(1 + \frac{\beta}{2\lambda}\right), \\ \alpha^2 &> 2\beta^2\left(1 + \frac{\beta}{2\lambda}\right). \end{aligned} \tag{4.13}$$

We notice that, for $\beta \ll \lambda$, in order to get $v^2 \ll f^2$, (4.11) requires a fine-tuning such that $\alpha/2\beta$ is very close to unity. Now we want to expand the σ and h fields around their minima in order to find mass eigenstates and eigenvalues. Performing the substitutions $h \equiv \hat{h} + v$, $\sigma \equiv \hat{\sigma} + v_\sigma$ the scalar potential reads

$$V = V_{mass} + V_{int} \tag{4.14}$$

with

$$V_{int} = \lambda\hat{h}^4 + \lambda\hat{\sigma}^4 + 4v\lambda\hat{h}^3 + 4\lambda v_\sigma\hat{\sigma}^3 + +4v_\sigma\lambda\hat{\sigma}\hat{h}^2 + 2\lambda\hat{\sigma}^2h^2 + 4v\lambda\hat{h}\hat{\sigma}^2 \tag{4.15}$$

and

$$\begin{aligned} V_{mass} &= 4\lambda v^2\hat{h}^2 + (4\lambda v_\sigma^2 + \beta f^2)\hat{\sigma}^2 + 8\lambda v v_\sigma \hat{h}\hat{\sigma} = \\ &= \begin{pmatrix}\hat{h} & \hat{\sigma}\end{pmatrix} \begin{pmatrix} 4\lambda v^2 & 4\lambda v v_\sigma \\ 4\lambda v v_\sigma & 4\lambda v_\sigma^2 + \beta f^2 \end{pmatrix} \begin{pmatrix}\hat{h} \\ \hat{\sigma}\end{pmatrix}. \end{aligned} \tag{4.16}$$

In order to diagonalize the mass matrix we recall that, given a symmetric matrix

$$S = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{4.17}$$

the angle of the corresponding rotation matrix R that diagonalizes S

$$R = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \tag{4.18}$$

is such that

$$\tan 2\gamma = \frac{2b}{c-a}. \tag{4.19}$$

In our case we obtain, using (4.12),

$$\tan 2\gamma = \frac{4vv_\sigma}{3v_\sigma^2 - v^2 - f^2}. \tag{4.20}$$

The (physical) mass eigenstates are therefore

$$\begin{aligned} h_{\rm phys} &= \hat{h}\cos\gamma - \hat{\sigma}\sin\gamma \\ \sigma_{\rm phys} &= \hat{\sigma}\cos\gamma + \hat{h}\sin\gamma. \end{aligned} \tag{4.21}$$

From now on, for simplicity, the notation $h_{\rm phys}$, $\sigma_{\rm phys}$ will be traded by h and σ, respectively, unless otherwise specified. The mass eigenvalues are given by

$$m^2_{\text{heavy, light}} = 4\lambda f^2\left[\left(1+\frac{3}{4}\frac{\beta}{\lambda}\right) \pm \left(1+\frac{\beta}{2\lambda}\left(1+\frac{\alpha^2}{2\beta^2}+\frac{\beta}{8\lambda}\right)\right)^{\frac{1}{2}}\right]. \tag{4.22}$$

The positivity of mass square eigenvalues implies some constraints on the parameters. For $f^2 > 0$, $m^2_{\rm heavy}$ is positive if

$$1+\frac{3}{4}\frac{\beta}{\lambda} > 0 \implies 3\beta + 4\lambda > 0; \tag{4.23}$$

$m^2_{\rm light}$ is positive if

$$\begin{aligned} &1+\frac{3}{4}\frac{\beta}{\lambda} > \sqrt{1+\frac{\beta}{2\lambda}\left(1+\frac{\alpha^2}{2\beta^2}+\frac{\beta}{8\lambda}\right)} \implies 2\beta^2 + 4\beta\lambda - \lambda\frac{\alpha^2}{\beta} > 0 \\ &\implies \frac{1}{4\beta\lambda}\left(4\beta^2 + 2\frac{\beta^3}{\lambda} - \alpha^2\right) > 0 \implies \beta > 0 \end{aligned} \tag{4.24}$$

where in the last implication we have used (4.13). We can now understand the overall minus sign of the term $-\beta f^2\sigma^2$ in (4.10): if it was overall positive (as for instance for $f^2 < 0$ and $\beta > 0$) we would have had $v = 0$, i.e. no spontaneous symmetry breaking. It is instructive to get the masses of the heavy and light eigenstates in the limit when the $SO(5)$ symmetry breaking is small, i.e. $\beta/4\lambda \ll 1$. Defining $x \equiv \beta/4\lambda$ and expanding for $x \ll 1$:

$$\begin{aligned} m^2_{\text{heavy, light}} =& 4\lambda f^2\left[1+3x \pm \left(1+2x\left(1+\frac{\alpha^2}{2\beta^2}+\frac{x}{2}\right)\right)^{\frac{1}{2}}\right] \simeq \\ &\simeq 4\lambda f^2\left[1+3x \pm \left(1+x\left(1+\frac{\alpha^2}{2\beta^2}\right)\right)\right] \simeq \\ &\simeq 4\lambda f^2\left[1+3x \pm \left(1+x\left(3-2\frac{v^2}{f^2}\right)\right)\right] \end{aligned} \tag{4.25}$$

where we have used

$$\frac{\alpha^2}{4\beta^2} = 1 - \frac{v^2}{f^2} + 2x \simeq 1 - \frac{v^2}{f^2}. \tag{4.26}$$

Therefore in this limit the mass eigenstates are

$$\begin{cases} m^2_{\rm heavy} = 8\lambda f^2 + 2\beta(3f^2 - v^2) + O\left(\dfrac{\beta}{4\lambda}\right) \\ m^2_{\rm light} = 2\beta v^2 + O\left(\dfrac{\beta}{4\lambda}\right). \end{cases} \tag{4.27}$$

4.2 Scalar-Gauge Bosons Couplings and Tree-Level Decays

In the unitary gauge, the kinetic scalar Lagrangian written in terms of the unrotated fields $\hat{h}, \hat{\sigma}$ reads

$$\mathcal{L}_{s,kin} = \frac{1}{2}\partial_\mu\hat{\sigma}\partial^\mu\hat{\sigma} + \frac{1}{2}\partial_\mu\hat{h}\partial^\mu\hat{h} + \frac{g^2}{4}(\hat{h}+v)^2 W^+_\mu W^{\mu,-} + \frac{g^2+g'^2}{8}(\hat{h}+v)^2 Z_\mu Z^\mu \tag{4.28}$$

that fixes the VEV v to be the electroweak scale defined from the W mass:

$$v = 246\,\text{GeV}. \tag{4.29}$$

The kinetic term (4.28) and the scalar potential (4.14) provide the scalar Lagrangian

$$\begin{aligned}\mathcal{L}_s = &\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}m_\sigma^2\sigma^2 - \frac{1}{2}m_h^2h^2 - \\ &- \lambda(h^2+\sigma^2)(h^2+\sigma^2+4(vh+v_\sigma\sigma)\cos\gamma + 4(v\sigma - v_\sigma h)\sin\gamma) + \\ &+ \frac{g^2}{4}(h\cos\gamma + \sigma\sin\gamma + v)^2 W^+_\mu W^{\mu-} + \\ &+ \frac{g^2+g'^2}{8}(h\cos\gamma + \sigma\sin\gamma + v)^2 Z_\mu Z^\mu\end{aligned} \tag{4.30}$$

where h and σ are written in the mass basis. Rearranging the terms in parenthesis and introducing the gauge boson masses $m_W^2 = g^2v^2/4$, $m_Z^2 = (g^2+g'^2)v^2/4$ the Lagrangian reads

$$\begin{aligned}\mathcal{L}_s = &\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu h\partial^\mu h - \frac{1}{2}m_\sigma^2\sigma^2 - \frac{1}{2}m_h^2h^2 - \lambda(h^2+\sigma^2)^4 \\ &- 4\lambda(v\cos\gamma - v_\sigma\sin\gamma)(h^3+h\sigma^2) - 4\lambda(v\sin\gamma + v_\sigma\cos\gamma)(\sigma^3+h^2\sigma) + \\ &+ \left(1+\frac{h}{v}\cos\gamma + \frac{\sigma}{v}\sin\gamma\right)^2\left(m_W^2 W^+_\mu W^{\mu-} + \frac{1}{2}m_Z^2 Z_\mu Z^\mu\right).\end{aligned} \tag{4.31}$$

Now, in order to calculate the scalar tree-level decays, it is useful rewrite the Lagrangian in terms of observables quantities. The Lagrangian possesses four independent parameters that can be expressed in terms of four observables which we choose to be

$$G_F \equiv \frac{1}{\sqrt{2}v^2}, \qquad m_h, \qquad m_\sigma, \qquad \sin\gamma. \tag{4.32}$$

The system of equations can be solved analytically; firstly, v and v_σ in terms of the four physical parameters (4.32) read

$$\begin{aligned}v^2 &= \frac{1}{\sqrt{2}G_F}, \\ v_\sigma &= \frac{v\sin(2\gamma)(m_\sigma^2 - m_h^2)}{2(m_\sigma^2\sin^2\gamma + m_h^2\cos^2\gamma)}\end{aligned} \tag{4.33}$$

and the four parameters of the scalar potential are found to be

$$\begin{aligned}
\lambda &= \frac{\sin^2\gamma m_\sigma^2}{8v^2}\left(1+\frac{m_h^2}{m_\sigma^2}\cot^2\gamma\right),\\
\frac{\beta}{4\lambda} &= \frac{m_h^2 m_\sigma^2}{m_\sigma^4\sin^2\gamma + m_h^4\cos^2\gamma - 2m_h^2m_\sigma^2},\\
\frac{\alpha^2}{4\beta^2} &= \frac{(m_\sigma^2-m_h^2)^2\sin^2(2\gamma)}{4\left(m_\sigma^4\sin^2\gamma + m_h^4\cos^2\gamma - 2m_h^2m_\sigma^2\right)},\\
f^2 &= \frac{v^2(m_\sigma^4\sin^2\gamma + m_h^4\cos^2\gamma - 2m_h^2m_\sigma^2)}{(m_\sigma^2\sin^2\gamma + m_h^2\cos^2\gamma)^2}.
\end{aligned} \tag{4.34}$$

From the equation for f we can extract the parameter ξ that expresses the vacuum misalignment

$$\xi \equiv \frac{v^2}{f^2} = \frac{(m_\sigma^2\sin^2\gamma + m_h^2\cos^2\gamma)^2}{(m_\sigma^4\sin^2\gamma + m_h^4\cos^2\gamma - 2m_h^2m_\sigma^2)}. \tag{4.35}$$

In the limit when $m_h^2 \ll m_\sigma^2$ (which derives from $\beta/4\lambda \ll 1$, $v^2/f^2 \ll 1$) it reads

$$\begin{aligned}
\xi &= \frac{(m_\sigma^2\sin^2\gamma + m_h^2\cos^2\gamma)^2}{(m_\sigma^4\sin^2\gamma + m_h^4\cos^2\gamma - 2m_h^2m_\sigma^2)}\\
&\simeq \left(\sin^2\gamma + 2\cos^2\gamma\frac{m_h^2}{m_\sigma^2}\right)\left(1+2\frac{m_h^2}{m_\sigma^2\sin^2\gamma}\right)\\
&\simeq \left[\sin^2\gamma + 2\frac{m_h^2}{m_\sigma^2}(2-\sin^2\gamma)\right]
\end{aligned} \tag{4.36}$$

that inverted gives

$$\sin^2\gamma \simeq \frac{v^2}{f^2} - 4\frac{m_h^2}{m_\sigma^2}. \tag{4.37}$$

We see that in the limit $m_h/m_\sigma \ll 1$ the mixing angle γ coincides and the parameter ξ

$$\sin\gamma = \frac{v}{f} \equiv \sqrt{\xi}. \tag{4.38}$$

We now turn to the tree-level decay rate of h and σ into SM bosons. The relevant couplings are

$$\begin{aligned}
g_{h\to W^+W^-} &= \left(\frac{2\cos\gamma}{v}\right)m_W^2,\\
g_{h\to ZZ} &= \left(\frac{\cos\gamma}{v}\right)m_Z^2,\\
g_{\sigma\to W^+W^-} &= \left(\frac{2\sin\gamma}{v}\right)m_W^2,\\
g_{\sigma\to ZZ} &= \left(\frac{\sin\gamma}{v}\right)m_Z^2,\\
g_{\sigma\to hh} &= 4\lambda\left(v\sin\gamma + v_\sigma\cos\gamma\right).
\end{aligned} \tag{4.39}$$

We see that the couplings of h to gauge bosons differ from the SM by $\cos\gamma$, so the decay rate, that is proportional to the square of the couplings, is the SM decay rate times a $\cos^2\gamma$:

$$\begin{aligned}\Gamma(h \to W^+W^-) &= \Gamma_{SM}(h \to W^+W^-)\cos^2\gamma,\\ \Gamma(h \to ZZ) &= \Gamma_{SM}(h \to ZZ)\cos^2\gamma\end{aligned} \tag{4.40}$$

where

$$\begin{aligned}\Gamma_{SM}(h \to W^+W^-) &= \frac{1}{4\pi}\frac{m_W^4}{m_h v^2}\sqrt{1-4\frac{m_W^2}{m_h^2}}\left(3+\frac{1}{4}\frac{m_h^4}{m_W^4}-\frac{m_h^2}{m_W^2}\right),\\ \Gamma_{SM}(h \to ZZ) &= \frac{1}{8\pi}\frac{m_Z^4}{m_h v^2}\sqrt{1-4\frac{m_Z^2}{m_h^2}}\left(3+\frac{1}{4}\frac{m_h^4}{m_Z^4}-\frac{m_h^2}{m_Z^2}\right).\end{aligned} \tag{4.41}$$

The σ decay rate into SM bosons is the same as (4.40) with the replacement $m_h \leftrightarrow m_\sigma$ because h and σ are both scalars and have the same couplings so phase space and amplitudes are the same:

$$\begin{aligned}\Gamma(\sigma \to W^+W^-) &= \frac{1}{4\pi}\frac{m_W^4}{m_\sigma v^2}\cos^2\gamma\sqrt{1-4\frac{m_W^2}{m_\sigma^2}}\left(3+\frac{1}{4}\frac{m_\sigma^4}{m_W^4}-\frac{m_\sigma^2}{m_W^2}\right),\\ \Gamma(\sigma \to ZZ) &= \frac{1}{8\pi}\frac{m_Z^4}{m_\sigma v^2}\cos^2\gamma\sqrt{1-4\frac{m_Z^2}{m_\sigma^2}}\left(3+\frac{1}{4}\frac{m_\sigma^4}{m_Z^4}-\frac{m_\sigma^2}{m_Z^2}\right).\end{aligned} \tag{4.42}$$

In the limit $\frac{m_{W,Z}^2}{m_\sigma^2} \ll 1$ the mass dependent part reads

$$\begin{aligned}&\frac{m_{W,Z}^4}{m_\sigma}\sqrt{1-4\frac{m_{W,Z}^2}{m_\sigma^2}}\left(3+\frac{1}{4}\frac{m_\sigma^4}{m_{W,Z}^4}-\frac{m_\sigma^2}{m_{W,Z}^2}\right)=\\ &=\frac{m_{W,Z}^4}{m_\sigma}\left(1-2\frac{m_{W,Z}^2}{m_\sigma^2}\right)\left(3+\frac{1}{4}\frac{m_\sigma^4}{m_{W,Z}^4}-\frac{m_\sigma^2}{m_{W,Z}^2}\right)\simeq\frac{1}{4}m_\sigma^3+O\left(\frac{m_{W,Z}^2}{m_\sigma^2}\right)\end{aligned} \tag{4.43}$$

therefore

$$\begin{aligned}\Gamma(\sigma \to W^+W^-) &= \frac{\sqrt{2}G_F}{16\pi}m_\sigma^3\sin^2\gamma\left[1+O\left(\frac{m_W^2}{m_\sigma^2}\right)\right],\\ \Gamma(\sigma \to ZZ) &= \frac{\sqrt{2}G_F}{32\pi}m_\sigma^3\sin^2\gamma\left[1+O\left(\frac{m_Z^2}{m_\sigma^2}\right)\right]\end{aligned} \tag{4.44}$$

where we have substituted $1/v^2 = \sqrt{2}G_F$.

For the decay $\sigma \to hh$ the calculation is similar. The phase space is the same as the decay into gauge bosons, the only difference is the amplitude

$$\mathcal{M} = 4\lambda(v\sin\gamma + v_\sigma\cos\gamma). \tag{4.45}$$

In the end, $\Gamma_{\sigma\to hh}$ reads

$$\Gamma = \frac{|\mathcal{M}|^2}{8\pi}\frac{|\vec{p}|}{m_\sigma^2} = \frac{2\lambda^2 m_\sigma}{\pi}\sqrt{1-4\frac{m_h^2}{m_\sigma^2}}\left(\frac{v}{m_\sigma}\sin\gamma + \frac{v_\sigma}{m_\sigma}\cos\gamma\right)^2. \tag{4.46}$$

Since, when $m_h^2/m_\sigma^2 \ll 1$

$$\begin{aligned}
\lambda &= \frac{\sin^2\gamma m_\sigma^2}{8v^2}\left[1+O\left(\frac{m_h^2}{m_\sigma^2}\right)\right],\\
v_\sigma &= v\cot\gamma\left[1+O\left(\frac{m_h^2}{m_\sigma^2}\right)\right],\\
(v\sin\gamma + v_\sigma\cos\gamma)^2 &= \frac{v^2}{\sin^2\gamma}\left[1+O\left(\frac{m_h^2}{m_\sigma^2}\right)\right],\\
\sqrt{1-4\frac{m_h^2}{m_\sigma^2}} &= 1+O\left(\frac{m_h^2}{m_\sigma^2}\right),
\end{aligned} \tag{4.47}$$

$\Gamma_{\sigma\to hh}$, in this limit, reads

$$\Gamma_{\sigma\to hh} = \frac{\sqrt{2}G_F}{32\pi}m_\sigma^3\sin^2\gamma\left[1+O\left(\frac{m_h^2}{m_\sigma^2}\right)\right]. \tag{4.48}$$

The experimental values of the couplings between the physical Higgs and the SM gauge bosons, which are modified by a factor of $\cos^2\gamma$ with respect to the SM ones, restrict the parameter space of the theory. In particular, they translate on a bound on γ:

$$\begin{aligned}
\sin^2\gamma &< 0.18 \qquad \text{for} \qquad m_h < m_\sigma,\\
\sin^2\gamma &> 0.82 \qquad \text{for} \qquad m_h > m_\sigma.
\end{aligned} \tag{4.49}$$

Moreover, the interpretation of the Higgs as pNGB also narrows the allowed region in the parameter space. The Fig. 4.1 shows the region of allowed values in the $(m_\sigma, \sin^2\gamma)$ plane. In fact, once we fix m_h and v, the pair $m_\sigma, \sin^2\gamma$ define completely the scalar sector so each point of the graphic corresponds to a distinct physical situation. In particular:

- the red region corresponds to $f^2 < 0$, i.e. no $SO(5)$ breaking;
- the orange region corresponds to $|v| > |v_\sigma|$, that is when the roles of h and σ have switched and σ becomes the pNGB;
- the white regions are excluded by Higgs data on the couplings with SM gauge bosons;
- the physical regions, that is when the Higgs arises as a pNGB, are depicted in light blue (where $m_\sigma > m_h$) and blue (where $m_\sigma < m_h$). Nevertheless, we will restrict the discussion below only on the light blue region because in the blue region the interpretation of the Higgs as a pNGB requires a fine-tuning of the parameters.

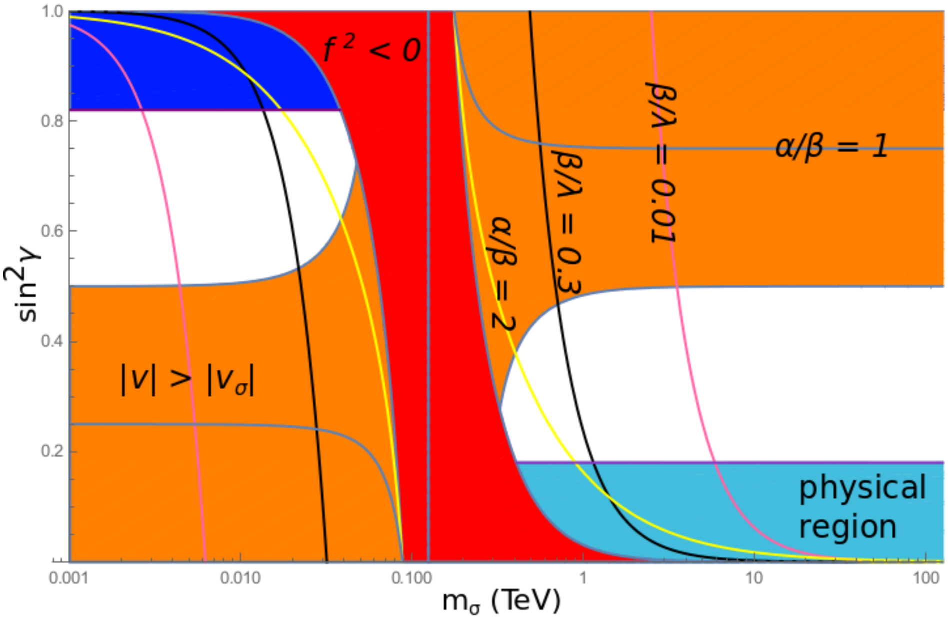

Figure 4.1: The figure shows the parameter space as a function of $(m_\sigma, \sin^2\gamma)$. These are the only parameters needed to completely specify the scalar potential, as we have fixed v and m_h to their observed values. The red region corresponds to $f^2 < 0$, the orange region corresponds to when σ becomes the pNGB. The white regions are excluded by Higgs data on the Higgs decay into gauge bosons (that allow the region below the horizontal line at $\sin^2\gamma = 0.18$ and $\sin^2\gamma = 0.82$). Thus there remain available the light blue region and the blue region. Nevertheless, this last region, in which $m_\sigma < m_h \sim 0.125\,\mathrm{TeV}$, requires a fine-tuning of the parameters in order to allow an interpretation of the Higgs as pNGB, so we will focus our discussion on the light blue region. In the figure are also shown some curves for fixed values of β/λ and α/β [16]

.

4.3 The Fermionic Sector

The fermionic sector, as we already pointed out, is important for two different reasons:

- it transmits, together with the gauge bosons, the explicit breaking of $SO(5)$ to the composite sector, inducing a potential at one-loop level for the h field with a non-trivial minimum, breaking the SM electroweak symmetry at a scale $v \neq f$ and providing a mass for h.
- it provides a mass for the SM quarks, through the mechanism of the partial compositeness.

However, the fermionic Lagrangian is strictly model-dependent, as there are several ways to introduce exotic fermions. The set-up considered here follows the paper [16]. Summarising, the particle content of the model is the following:

- four different types of heavy (exotic) fermions[3] either in the fundamental representation of $SO(5)$, which we denote as ψ, or singlets denoted by χ; there are two for each one, corresponding to the two $U(1)_x$ charges needed to have correct hypercharge assignments;
- SM fermions; for simplicity we will restrict our discussion to the third generation of quarks, however the generalization to the real case of three families is trivial;
- the scalar field ϕ, which contains the Higgs field h, the three longitudinal components of SM gauge bosons and the heavy field σ. By construction ϕ couples only to heavy exotic fermions in an $SO(5)$ invariant way. The sources of $SO(5)$ breaking lie instead in the electroweak gauge interactions and in the mixing terms between heavy exotic fermions and SM fermions.

The first step to construct the fermionic Lagrangian is to enlarge the $SO(5)$ symmetry group with a further (exact) $U(1)_x$ symmetry. In fact, as we outlined in the previous chapter, this is essential if want to rely on the partial fermion compositeness to give a mass to SM fermions. The SSB pattern therefore reduces to

$$SO(5) \times U(1)_x \to SO(4) \times U(1)_x \sim SU(2)_L \times SU(2)_R \times U(1)_x. \quad (4.50)$$

The hypercharge is then defined via

$$Y = \Sigma^3_R + x. \quad (4.51)$$

[3]These fermions are the heavy resonances in the case of strongly coupled regime.

Two different $U(1)_x$ charges are compatible with SM hypercharge assignments: 2/3 and −1/3. For this we will consider two different copies of heavy fermions for each representation, differentiated by $U(1)_x$, as they are necessary to induce mass terms for both the top and bottom quarks. Summarising, the four heavy fermions considered are

$$\begin{aligned} &\psi^{(2/3)} \sim (X, Q, T^{(5)}), \qquad \psi^{(-1/3)} \sim (Q', X', B^{(5)}), \\ &\chi^{(2/3)} \sim T^{(1)}, \qquad \chi^{(-1/3)} \sim B^{(1)} \end{aligned} \tag{4.52}$$

where the superscripts (2/3), (−1/3) indicate the $U(1)_x$ charges and we have decomposed the $SO(5)$ representations in terms of $SU(2)$ representations. In particular, $X^{(')}, Q^{(')}$ denote the two different $SU(2)_L$ doublets contained in the fundamental of $SO(5)$, $T^{(5)}$, $B^{(5)}$ are $SU(2)$ singlets while $T^{(1)}$, $B^{(1)}$ are $SO(5)$ singlets (and, consequently, $SU(2)$ singlets). We remind that, in our decomposition of $SO(5)$ in $SU(2)$ representations, the first doublet has $\Sigma^3_R = 1/2$ while the second one has $\Sigma^3_R = -1/2$. In this way the heavy fermions have the right quantum numbers to be coupled to the SM fermions[4]. For example X, Q are $SU(2)_L$ doublets and possess, respectively

$$\begin{aligned} &Y(X) = \Sigma^3_R + x = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}, \\ &q_{em}(X_u) = \Sigma^3_L + Y(X) = \Sigma^3_L + \Sigma^3_R + x = \frac{1}{2} + \frac{1}{2} + \frac{2}{3} = \frac{5}{3}, \\ &q_{em}(X_d) = \Sigma^3_L + Y(X) = \Sigma^3_L + +\Sigma^3_R + x = -\frac{1}{2} + \frac{1}{2} + \frac{2}{3} = \frac{2}{3}, \end{aligned} \tag{4.53}$$

$$\begin{aligned} &Y(Q) = \Sigma^3_R + x = -\frac{1}{2} + \frac{2}{3} = \frac{1}{6}, \\ &q_{em}(Q_u) = \Sigma^3_L + Y(Q) = \Sigma^3_L + \Sigma^3_R + x = \frac{1}{2} - \frac{1}{2} + \frac{2}{3} = \frac{2}{3}, \\ &q_{em}(Q_d) = \Sigma^3_L + Y(Q) = \Sigma^3_L + +\Sigma^3_R + x = -\frac{1}{2} - \frac{1}{2} + \frac{2}{3} = -\frac{1}{3}. \end{aligned} \tag{4.54}$$

$T_{(1,5)}$ are $SO(4)$ (therefore $SU(2)_L$ too) singlets so:

$$\begin{aligned} &Y(T_{(1,5)}) = \Sigma^3_R + x = \frac{2}{3}, \\ &q_{em}(T_{(1,5)}) = \Sigma^3_L + Y(T_{(1,5)}) = \frac{2}{3}. \end{aligned} \tag{4.55}$$

For $\psi^{(-1/3)}$ and $\chi^{(-1/3)}$ the calculations are the same except for the x charge: $x(\psi^{(-1/3)}) = x(\chi^{(-1/3)}) = -1/3$. The tables 4.3 show the quantum numbers of the particles in the model.

[4]The SM fermions are taken to be $U(1)_x$ singlets so that some of them have the same quantum numbers than the SM ones.

Charge/Field	X	Q	$T_{1,5}$
Σ^3_R	1/2	-1/2	0
$SU(2)_L \times U(1)_Y$	(2, 7/6)	(2, 1/6)	(1, 2/3)
x	2/3	2/3	2/3
q_{EM}	$X^u = 5/3$ $X^d = 2/3$	$Q^u = 2/3$ $Q^d = -1/3$	2/3

Charge/Field	Q'	X'	$B_{1,5}$
Σ^3_R	1/2	- 1/2	0
$SU(2)_L \times U(1)_Y$	(2, 1/6)	(2, -5/6)	(1, -1/3)
x	-1/3	-1/3	-1/3
q_{EM}	$Q'^u = 2/3$ $Q'^d = -1/3$	$X'^u = -1/3$ $X'^d = -4/3$	-1/3

Charge/Field	H	$\tilde{H}$
Σ^3_R	1/2	-1/2
$SU(2)_L \times U(1)_Y$	(2, 1/2)	(2, -1/2)
x	0	0
q_{EM}	$H^u = 1$ $H^d = 0$	$\tilde{H}^u = 0$ $\tilde{H}^d = 1$

Charge/Field	q_L	t_R	b_R
Σ^3_R	1/6	2/3	-1/3
$SU(2)_L \times U(1)_Y$	(2, 1/6)	(1, 2/3)	(1, -1/3)
x	0	0	0
q_{EM}	$u_L = 2/3$ $d_L = -1/3$	2/3	$-1/3$

Table 4.1: The tables show the quantum numbers of the particles in the model; q_L is the usual SM quark doublel and t_R, b_R are the singlets of SM top and bottom quarks, respectively. We notice that $Q^{(')}$,$B^{(1,5)}$,$T^{(1,5)}$ have the correct hypercharge and electric charge assignments in order to be coupled to the SM fermions.

With this choice of charges, the most general fermionic Lagrangian invariant under $SU(2)_L \times U(1)_Y$ is given by

$$
\begin{aligned}
\mathcal{L}_F = & \, \bar{q}_L i\not{D} q_L + \bar{t}_R i\not{D} t_R + \bar{b}_R i\not{D} b_R \\
& + \bar{\psi}^{(2/3)} \left(i\not{D} - M_5 \right) \psi^{(2/3)} + \bar{\psi}^{(-1/3)} \left(i\not{D} - M_5' \right) \psi^{(-1/3)} \\
& + \bar{\chi}^{(2/3)} \left(i\not{D} - M_1 \right) \chi^{(2/3)} + \bar{\chi}^{(-1/3)} \left(i\not{D} - M_1' \right) \chi^{(-1/3)} \\
& - \Big[y_1 \bar{\psi}_L^{(2/3)} \phi \chi_R^{(2/3)} + y_2 \bar{\psi}_R^{(2/3)} \phi \chi_L^{(2/3)} + \\
& + y_1' \bar{\psi}_L^{(-1/3)} \phi \chi_R^{(-1/3)} + y_2' \bar{\psi}_R^{(-1/3)} \phi \chi_L^{(-1/3)} + \\
& + \Lambda_1 \left(\bar{q}_L \Delta_{2\times 5}^{(2/3)} \right) \psi_R^{(2/3)} + \Lambda_2 \bar{\psi}_L^{2/3} \left(\Delta_{5\times 1}^{(2/3)} t_R \right) + \Lambda_3 \bar{\chi}_L^{(2/3)} t_R + \\
& + \Lambda_1' \left(\bar{q}_L \Delta_{2\times 5}^{(-1/3)} \right) \psi_R^{(-1/3)} + \Lambda_2' \bar{\psi}_L^{(-1/3)} \left(\Delta_{5\times 1}^{(-1/3)} b_R \right) + \\
& + \Lambda_3' \bar{\chi}_L^{(-1/3)} b_R + \text{h.c.} \Big] \, .
\end{aligned}
\tag{4.56}
$$

The couplings $\Lambda_{(1,2)}^{(\prime)}$ can be regarded, in the case of strongly interacting regime, as effective couplings arising at the confinement scale after their evolution under the renormalization group. On the other hand, the Δ's denote matrices connecting $SO(5)$ and $SU(2)_L \times U(1)_Y$ representations; for instance, the Λ_1 coupling explicitly reads

$$
\begin{pmatrix} \bar{t}_L & \bar{b}_L \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X_R^u \\ X_R^d \\ Q_R^u \\ Q_R^d \\ T_R^5 \end{pmatrix} = \bar{q}_L Q_R . \tag{4.57}
$$

It is useful rewrite the Lagrangian (4.56) in $SU(2)_L$ components. To this purpose we remind that the complex Higgs doublet is related to the real fourplet representation by [18]

$$
\phi = \begin{pmatrix} H \\ \tilde{H} \\ \sigma \end{pmatrix} . \tag{4.58}
$$

Then, the Lagrangian reads

$$
\begin{aligned}
\mathcal{L}_F =& \bar{q}_L i\not{D} q_L + \bar{t}_R i\not{D} t_R + \bar{b}_R i\not{D} b_R + \bar{Q}\left(i\not{D} - M_5\right) Q + \bar{X}\left(i\not{D} - M_5\right) X + \\
&+ \bar{T}^{(5)}\left(i\not{D} - M_5\right) T^{(5)} + \bar{T}^{(1)}\left(i\not{D} - M_1\right) T^{(1)} + \bar{Q}'\left(i\not{D} - M_5'\right) Q' + \\
&+ \bar{X}'\left(i\not{D} - M_5'\right) X' + \bar{B}^{(5)}\left(i\not{D} - M_5'\right) B^{(5)} + \bar{B}^{(1)}\left(i\not{D} - M_1'\right) B^{(1)} - \\
&- \Big[y_1 \left(\bar{X}_L H T_R^{(1)} + \bar{Q}_L H^c T_R^{(1)} + \bar{T}_L^{(5)} \sigma T_R^{(1)}\right) + \\
&+ y_2 \left(\bar{T}_L^{(1)} H^\dagger X_R + \bar{T}_L^{(1)} H^{c,\dagger} Q_R + \bar{T}_L^{(1)} \sigma T_R^{(5)}\right) \\
&+ y_1' \left(\bar{X}_L' H^c B_R^{(1)} + \bar{Q}_L' H B_R^{(1)} + \bar{B}_L^{(5)} \sigma B_R^{(1)}\right) + \\
&+ y_2' \left(\bar{B}_L^{(1)} H^{c,\dagger} X_R' + \bar{B}_L^{(1)} H^\dagger Q_R' + \bar{B}_L^{(1)} \sigma B_R^{(5)}\right) \\
&+ \Lambda_1 \bar{q}_L Q_R + \Lambda_2 \bar{T}_L^{(5)} t_R + \Lambda_3 \bar{T}_L^{(1)} t_R + \\
&+ \Lambda_1' \bar{q}_L Q_R' + \Lambda_2' \bar{B}_L^{(5)} b_R + \Lambda_3' \bar{B}_L^{(1)} b_R + \text{h.c.} \Big].
\end{aligned}
\tag{4.59}
$$

We can also rewrite the mass part of (4.59) in a more compact form defining a fermionic vector whose components are ordered by their electrical charge:

$$
\Psi = (X^u, \mathcal{T}, \mathcal{B}, X^{d'}) \tag{4.60}
$$

where $\mathcal{T}$ and $\mathcal{B}$ are the six components, respectively top-like and bottom-like, fermions:

$$
\begin{aligned}
\mathcal{T} &= \left(t, Q^u, X^d, T^{(5)}, T^{(1)}, Q^{u'}\right) \\
\mathcal{B} &= \left(b, Q^{d'}, X^{u'}, B^{(5)}, B^{(1)}, Q^d\right).
\end{aligned}
\tag{4.61}
$$

The fermionic mass terms can then be written as

$$
\mathcal{L}_{mass} = -\bar{\Psi}_L \mathcal{M}(h,\sigma) \Psi_R \tag{4.62}
$$

where $\mathcal{M}(h,\sigma)$ is the block diagonal 14×14 matrix

$$
\mathcal{M}(h,\sigma) = \text{diag}\left(M_5, \mathcal{M}^{\mathcal{T}}(h,\sigma), \mathcal{M}^{\mathcal{B}}(h,\sigma)\, M_5'\right), \tag{4.63}
$$

$$
\mathcal{M}^{\mathcal{T}} = \begin{pmatrix}
0 & \Lambda_1 & 0 & 0 & 0 & \Lambda_1' \\
0 & M_5 & 0 & 0 & y_1 \frac{h}{\sqrt{2}} & 0 \\
0 & 0 & M_5 & 0 & y_1 \frac{h}{\sqrt{2}} & 0 \\
\Lambda_2 & 0 & 0 & M_5 & y_1 \sigma & 0 \\
\Lambda_3 & y_2 \frac{h}{\sqrt{2}} & y_2 \frac{h}{\sqrt{2}} & y_2 \sigma & M_1 & 0 \\
0 & 0 & 0 & 0 & 0 & M_5'
\end{pmatrix} \tag{4.64}
$$

and $\mathcal{M}^{\mathcal{B}}(h,\sigma) = \mathcal{M}^{\mathcal{T}}(h,\sigma)$ with the substitutions $(y_i\Lambda_i, M_i) \leftrightarrow (y_i^{'}\Lambda_i^{'}, M_i^{'})$.

We notice that the mass matrix (4.63) is block diagonal. This is consistent with the fact that the Lagrangian (4.62) must preserve, by construction, the $U(1)_{\rm em}$ symmetry. The mixing of the fermions with the same charge does not violate charge conservation so it can occur and indeed it is what happens because of (4.64). Also, we notice that, even in the limit of vanishing Yukawa couplings, the exotic fermions get mixed via the $SO(5)$ breaking couplings.

The matrices can be diagonalized analytically in some interesting limit, in general they have to be diagonalized numerically. However, we can estimate the top and bottom masses by integrating out the heavy fields and retaining only the highest order contribution. In the calculation we will neglect all the covariant derivatives as they are responsible of next-to-leading-order corrections. Since the matrix is a block diagonal matrix, the equations of motion will be decoupled for each block. We will perform the calculation for the top mass, the calculation of the bottom mass being the same. As we have to integrate all the heavy fields, it is convenient define an "heavy vector" T

$$T \equiv \begin{pmatrix} Q^u \\ X^d \\ T^{(5)} \\ T^{(1)} \\ Q'^u \end{pmatrix} \tag{4.65}$$

and write the equation of motion for T. The Lagrangian for t and T is

$$\mathcal{L}_{\rm mass,tT} = -\begin{pmatrix} \bar{t}_L & \bar{T}_L \end{pmatrix} \begin{pmatrix} 0 & \alpha^T \\ \beta & \gamma(h,\sigma) \end{pmatrix} \begin{pmatrix} t_R \\ T_R \end{pmatrix} + h.c. \tag{4.66}$$

with

$$\alpha = \begin{pmatrix} \Lambda_1 \\ 0 \\ 0 \\ 0 \\ \Lambda_1' \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 \\ 0 \\ \Lambda_2 \\ \Lambda_3 \\ 0 \end{pmatrix}, \qquad \gamma = \begin{pmatrix} M_5 & 0 & 0 & y_1\frac{h}{\sqrt{2}} & 0 \\ 0 & M_5 & 0 & y_1\frac{h}{\sqrt{2}} & 0 \\ 0 & 0 & M_5 & y_1\sigma & 0 \\ y_2\frac{h}{\sqrt{2}} & y_2\frac{h}{\sqrt{2}} & y_2\sigma & M_1 & 0 \\ 0 & 0 & 0 & 0 & M_5' \end{pmatrix}. \tag{4.67}$$

The equations of motion for T are

$$\begin{cases} \gamma T_R + \beta t_R = 0 \\ \gamma^\dagger T_L + \alpha^* t_L = 0 \\ \bar{T}_R \gamma^\dagger + \bar{t}_R \beta^\dagger = 0 \\ \bar{T}_L \gamma + \bar{t}_L \alpha^T = 0. \end{cases} \tag{4.68}$$

Substituting (4.68) in (4.66) we find

$$
\begin{aligned}
-\mathcal{L}_{\text{mass,tT}} &= \bar{t}_L \alpha^T T_R + \bar{T}_L \beta t_R + \bar{T}_L \gamma T_R + h.c. = \\
&= -\bar{t}_L \alpha^T \gamma^{-1} \beta t_R + \bar{t}_L \alpha^T \gamma^{-1} \beta t_R - \bar{t}_L \alpha^T \gamma^{-1} \beta t_R + h.c. = \\
&= \bar{t_L} \left(-\alpha^T \gamma^{-1} \beta \right) t_R + h.c.
\end{aligned}
\tag{4.69}
$$

from which we can read the leading order (LO) contribution to the top mass

$$
m_t = \alpha^T \gamma^{-1} \beta = \frac{y_1 \Lambda_1 \Lambda_3 - y_1 y_2 \Lambda_1 \Lambda_2 \sigma / M_5}{M_1 M_5 - y_1 y_2 (h^2 + \sigma^2)} \frac{h}{\sqrt{2}}. \tag{4.70}
$$

As expected, the bottom mass can be inferred from (4.70) by performing the substitutions

$$
\{y_1, y_2, \Lambda_1, \Lambda_2, \Lambda_3, M_1, M_5\} \to \{y_1', y_2', \Lambda_1', \Lambda_2', \Lambda_3', M_1', M_5'\}. \tag{4.71}
$$

4.4 The Coleman-Weinberg One-Loop Potential

As we have introduced an explicit breaking of the $SO(5)$ symmetry by gauging the EW group and by coupling the $SO(5)$ scalar fiveplet with the heavy fermions, the $SO(5)$ breaking terms are not forbidden and this provides an Higgs mass term at one-loop level. In order to show it, we recall that the divergent part of the Coleman-Weinberg (CW) one-loop potential is given by [33]

$$
V^{\text{CW}} = -\frac{1}{64\pi^2} \left(\text{Tr} \left[\mathcal{M} \mathcal{M}^\dagger \right] \Lambda^2 - \text{Tr} \left[\left(\mathcal{M} \mathcal{M}^\dagger \right)^2 \right] \log \frac{\Lambda^2}{\mu^2} \right) \tag{4.72}
$$

where Λ is the ultraviolet cut-off, μ a generic renormalization scale and $\mathcal{M}$ is the mass matrix of the theory. The CW potential exhibits a quadratic and a logarithmic divergence. This means that the divergent terms arising from the fermionic sector and the gauge sector have to be renormalized by introducing the respective counterterms in the tree-level Lagrangian. We start from the contribution due to fermions. By using (4.63), (4.64) we find

$$
\begin{aligned}
\text{Tr} \left[\mathcal{M} \mathcal{M}^\dagger \right] &= M_5^2 + \text{Tr} \left[\mathcal{M}^{\mathcal{T}} \mathcal{M}^{\mathcal{T}\dagger} \right] + \text{Tr} \left[\mathcal{M}^{\mathcal{B}} \mathcal{M}^{\mathcal{B}\dagger} \right] + M_5'^2, \\
\text{Tr} \left[\left(\mathcal{M} \mathcal{M}^\dagger \right)^2 \right] &= M_5^4 + \text{Tr} \left[\left(\mathcal{M}^{\mathcal{T}} \mathcal{M}^{\mathcal{T}\dagger} \right)^2 \right] + \text{Tr} \left[\left(\mathcal{M}^{\mathcal{B}} \mathcal{M}^{\mathcal{B}\dagger} \right)^2 \right] + M_5'^4
\end{aligned}
\tag{4.73}
$$

The contributions due to top and bottom quarks do not mix since the mass matrix is block diagonal because of the underlying electric charge symmetry. Moreover, since the mass matrix $\mathcal{M}^{\mathcal{T}}$ differs from $\mathcal{M}^{\mathcal{B}}$ by the substitution of the parameters with the respective primed, we can perform the calculation only for the top sector, and the bottom sector will come automatically. We start from the quadratically divergent contribution that reads

$$\begin{aligned}&\mathrm{Tr}\left[\mathcal{M}^{\mathcal{T}}\mathcal{M}^{\mathcal{T}\dagger}\right]=\\&=M_1^2+3M_5^2+M_5'^2+\Lambda_1^2+\Lambda_1'^2+\Lambda_2^2+\Lambda_3^2+(y_1^2+y_2^2)(h^2+\sigma^2)=\\&=M_1^2+3M_5^2+M_5'^2+\Lambda_1^2+\Lambda_1'^2+\Lambda_2^2+\Lambda_3^2+(y_1^2+y_2^2)\phi^T\phi\end{aligned}\tag{4.74}$$

where in the last equality we have reintroduced the fields π^i. Consequently

$$\mathrm{Tr}\left[\mathcal{M}\mathcal{M}^\dagger\right]=M_1^2+5M_5^2+2\Lambda_1^2+\Lambda_2^2+\Lambda_3^2+(y_1^2+y_2^2)\phi^T\phi+\{\}\leftrightarrow\{\}'\tag{4.75}$$

where with $\{\}\leftrightarrow\{\}'$ we mean that the same contributions arise from the bottom sector substituting the unprimed couplings with the primed ones. We see that the quadratically divergent term is $SO(5)$ invariant and we do not need to add counterterms to the tree level potential.

On the other hand, the logarithmic contribution arising from the top sector is

$$\mathrm{Tr}\left[\left(\mathcal{M}^{\mathcal{T}}\mathcal{M}^{\mathcal{T}\dagger}\right)^2\right]=d_1+d_2\sigma+d_3h^2+d_4\left(\phi^T\phi\right)+d_5\left(\phi^T\phi\right)^2\tag{4.76}$$

where[5]

$$\begin{aligned}d_2&=4\left(y_1M_1+y_2M_5\right)\Lambda_2\Lambda_3,\\d_3&=+y_2^2\Lambda_1^2-2y_1^2\Lambda_2^2,\\d_4&=4y_1y_2M_1M_5+2\left(y_1^2\Lambda_2^2+y_2^2\Lambda_3^2\right)+2\left(y_1^2+y_2^2\right)\left(M_1^2+M_5^2\right),\\d_5&=y_1^4+y_2^4\end{aligned}\tag{4.77}$$

where we have used the fact a σ^2 divergence can be transformed in an h^2 divergence. In fact, calling d_3' the coefficient of the σ^2 term we have

$$d_3'\sigma^2+d_3h^2+d_4\left(\phi^T\phi\right)=(d_3-d_3')h^2+(d_3'+d_4)\left(\phi^T\phi\right).\tag{4.78}$$

A similar analysis should be done for the gauge bosons loops. However, following the paper [32], we have neglected the $SO(5)$ breaking terms arising from the gauge bosons loops because we are interested to modifications of the potential due to the fermionic sector because the potential arising from the gauge sector is negligible with respect to it. In the end, since the divergences arising from the bottom sector are the same, the appropriate terms needed to absorb the divergences arising at one-loop level are the α and β terms in (4.10).

[5]We do not show d_1 as it is a constant term.

The last remark is about the choice to introduce counterterms related only to the one-loop potential. In fact, as the full renormalizability of the theory requires the presence of all gauge invariant operators of dimension equal to or smaller than four, there is nothing that forbids the appearance of other $SO(5)$ breaking terms at two or more loops, the renormalization procedure may thus require to include further symmetry breaking counterterms beyond those considered; however, as they arise at more than one-loop, their finite contributions are weighted by comparatively negligible coefficients and therefore they can be omitted in this analysis [16].

4.5 Higgs and σ Coupling to Gluons

Other two observables that we can calculate related to the strong sector are the scalar decays into gauge bosons $h \to gg$, $\sigma \to gg$. We start recalling what the SM predicts. In the SM the Higgs decay into two gluons arises at one-loop level, through fermions loops. The Feynman diagram of the process is depicted in Fig. 4.2 and the corresponding amplitude is given by

$$\mathcal{A}_h = -i\frac{\alpha_s}{\pi} g_h^{\rm SM} \epsilon_\mu^a \epsilon_\nu^b (p \cdot k\, g^{\mu\nu} - p^\mu k^\nu)\delta^{ab} \tag{4.79}$$

where $g_h^{\rm SM}$ is a scale dependent function that characterize the amplitude strength. We notice that the shape of the amplitude is consistent with the fact that all the effective interactions arising from a gauge-invariant theory have to be gauge-invariant. In fact, it allows to reconstruct the effective interaction $hG^a_{\mu\nu}G^{\mu\nu a}$, which is the only gauge invariant way to couple the Higgs with two gluons. A straightforward calculation shows that

$$g_h^{SM} = \sum_i \frac{y_i}{\sqrt{2}} \frac{1}{m_i} I\left(\frac{m_h^2}{m_i^2}\right) \tag{4.80}$$

where m_i is the quark mass, $y_i \equiv \sqrt{2}m_i/v$ is the corresponding Yukawa coupling and

$$I\left(\frac{q^2}{m^2}\right) = \int_0^1 dx \int_0^{1-x} dz \frac{1-4xz}{1-xz\frac{q^2}{m^2}} \sim \begin{Bmatrix} \frac{1}{3} & \text{for} & m^2 \gg q^2 \\ 0 & \text{for} & m^2 \ll q^2 \end{Bmatrix}. \tag{4.81}$$

Because of (4.81) the amplitude is dominated by the top quark; for instance, the most relevant contribution after the top quark comes from the bottom quark, whose integral is $I\left(q^2/m^2\right) \sim 10^{-2}$, so it can be neglected. In conclusion, in the SM we have

$$g_h^{\rm SM} = \frac{y_t}{\sqrt{2}} \frac{1}{m_t} I\left(\frac{m_h^2}{m_t^2}\right) \sim \frac{1}{3v}. \tag{4.82}$$

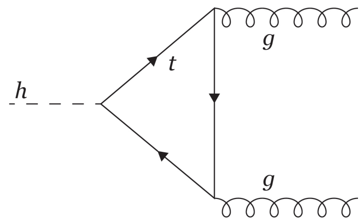

Figure 4.2: The Higgs decay into two gluons, mediated by the top quark. There are two topological independent diagrams: one is depicted in the figure, the other one is the same with the fermionic arrows inverted.

Now we turn to the MLσM. Here the $h \to gg$ coupling arises via fermions loops as well with the difference that now the fermions running in the loop are the top quark plus all the heavy fermions[6] because for them $I\left(m_h^2/M^2\right) \sim 1/3$. In order to make explicit the Yukawa-like coupling it is useful rewrite the Lagrangian (4.62) as

$$-\mathcal{L}_{\text{mass}} = \bar{\Psi}_L \bar{\mathcal{M}} \Psi_R + \hat{h} \bar{\Psi}_L \frac{\partial \bar{\mathcal{M}}}{\partial h} \Psi_R + \hat{\sigma} \bar{\Psi}_L \frac{\partial \bar{\mathcal{M}}}{\partial \sigma} \Psi_R + \text{h.c.} \tag{4.83}$$

where we have defined

$$\bar{\mathcal{M}} \equiv \mathcal{M}(v, v_\sigma), \qquad \frac{\partial \bar{\mathcal{M}}}{\partial h} \equiv \left.\frac{\partial \mathcal{M}(h,\sigma)}{\partial h}\right|_{\substack{h=v \\ \sigma=v_\sigma}}, \qquad \frac{\partial \bar{\mathcal{M}}}{\partial \sigma} \equiv \left.\frac{\partial \mathcal{M}(h,\sigma)}{\partial h}\right|_{\substack{h=v \\ \sigma=v_\sigma}} \tag{4.84}$$

and we have used the fact that the mass matrix $\mathcal{M}(h,\sigma)$ is linear in $\hat{h}$ and $\hat{\sigma}$[7]. In the form (4.83) we immediately recognize that the matrices $\partial\mathcal{M}/\partial h$ and $\partial\mathcal{M}/\partial\sigma$ are the Yukawa-like couplings. However, neither fermions nor scalars are in their mass eigenstates, so we have to perform both the rotations. Starting by the fermions, the rotation leads to

$$-\mathcal{L}_{\text{mass}} = M^i \bar{\Psi}_L^i \Psi_R^i + \hat{h} \bar{\Psi}_L^i (Y_h)^{ij} \Psi_R^j + \hat{\sigma} \bar{\Psi}_L^i (Y_\sigma)^{ij} \Psi_R^j + \text{h.c.} \tag{4.85}$$

where $Y_{h,\sigma}$ are the Yukawa couplings arising after the rotation in the fermion mass basis. On the other hand, the rotation of the scalar fields leads to

$$\mathcal{L}_{\text{Yuk}} = h \bar{\Psi}_L \left[Y_h \cos\gamma - Y_\sigma \sin\gamma\right] \Psi_R + \sigma \bar{\Psi}_L \left[Y_h \sin\gamma + Y_\sigma \cos\gamma\right] \Psi_R + h.c. \tag{4.86}$$

Therefore in this model the effective couplings $g_{h,\sigma}^{\text{CH}}$ (in the scalar mass basis) are given by

$$\begin{aligned} g_h^{\text{CH}}(m_h^2) &\equiv g_{\hat{h}}(m_h^2) \cos\gamma - g_{\hat{\sigma}}(m_h^2) \sin\gamma = \\ &= \sum_i \frac{1}{m_i} I\left(\frac{m_h^2}{m_i^2}\right) \left[(Y_h)_{ii} \cos\gamma - (Y_\sigma)_{ii} \sin\gamma\right], \end{aligned} \tag{4.87}$$

[6]They carry color charge and live in the fundamental of $SU(3)_c$, so they are coupled to gluons as the SM fermions.

[7]We recall that $\hat{h}$ and $\hat{\sigma}$ are the (unphysical) unrotated scalar fields.

$$g_{\sigma}^{\rm CH}(m_\sigma^2) \equiv g_{\hat{h}}(m_\sigma^2)\sin\gamma + g_{\hat{\sigma}}(m_\sigma^2)\cos\gamma = \\ = \sum_i \frac{1}{m_i} I\left(\frac{m_\sigma^2}{m_i^2}\right)\left[(Y_h)_{ii}\sin\gamma + (Y_\sigma)_{ii}\cos\gamma\right] \tag{4.88}$$

where the i index, as we have said, runs over all the fermions for which the integral $I\left(q^2/m_i^2\right)$ is considerably different from zero and m_i's are their masses. In order to perform the calculation we notice that

$$\frac{\partial}{\partial h}\log\left(\det\mathcal{M}\right)\Big|_{\substack{h=v\\ \sigma=v_\sigma}} = \frac{\partial}{\partial h}\log\left(\prod_i m_i(h,\sigma)\right)\Big|_{\substack{h=v\\ \sigma=v_\sigma}} = \\ = \sum_i \frac{\partial}{\partial h}\log\left(m_i(h,\sigma)\right)\Big|_{\substack{h=v\\ \sigma=v_\sigma}} = \sum_i \frac{(Y_h)_{ii}}{m_i} \tag{4.89}$$

and the same for Y_σ. In the first equivalence we have used the fact that the determinant is invariant under a change of basis. This is useful because in this way we do not need to go in the fermion mass basis. Now, $g_{\hat{h}}^{\rm CH}(m_h^2)$ is given by

$$g_{\hat{h}}^{\rm CH}(m_h^2) = \sum_i (Y_h)_{ii}\frac{1}{m_i} I\left(\frac{m_h^2}{m_i^2}\right) \sim \frac{1}{3}\frac{\partial}{\partial h}\left[\log\left(\det\mathcal{M}\right) - \log m_b\right]\Big|_{\substack{h=v\\ \sigma=v_\sigma}} = \\ = \frac{1}{6}\frac{\partial}{\partial h}\left[\log\left(\det\mathcal{M}\mathcal{M}^\dagger\right) - \log m_b m_b^*\right]\Big|_{\substack{h=v\\ \sigma=v_\sigma}} \tag{4.90}$$

where we have used (4.81) and subtracted the contribution arising from the bottom quark, as for it the integral (4.81) is almost zero. On the other hand $g_{\hat{\sigma}}^{\rm CH}(m_h^2)$ is given by[8]

$$g_{\hat{\sigma}}^{\rm CH}(m_h^2) = \sum_i (Y_\sigma)_{ii}\frac{1}{m_i} I\left(\frac{m_\sigma^2}{m_i^2}\right) \sim \frac{1}{3}\frac{\partial}{\partial \sigma}\left[\log\left(\det\mathcal{M}\right) - \log m_b\right]\Big|_{\substack{h=v\\ \sigma=v_\sigma}} = \\ = \frac{1}{6}\frac{\partial}{\partial \sigma}\left[\log\left(\det\mathcal{M}\mathcal{M}^\dagger\right) - \log m_b m_b^*\right]\Big|_{\substack{h=v\\ \sigma=v_\sigma}} . \tag{4.91}$$

Now, as

$$\frac{\partial}{\partial h}\log\left(\det\mathcal{M}\mathcal{M}^\dagger\right) = \frac{4}{v}, \tag{4.92}$$

$$\frac{\partial}{\partial \sigma}\log\left(\det\mathcal{M}\mathcal{M}^\dagger\right) = \frac{2y_2\Lambda_2}{y_2\Lambda_2 v_\sigma - M_5\Lambda_3} + \frac{2y_2'\Lambda_2'}{y_2'\Lambda_2' v_\sigma - M_5'\Lambda_3'}, \tag{4.93}$$

[8] We notice that we are calculating $g_{\hat{\sigma}}$ valued at m_h^2.

$$\frac{\partial}{\partial h}\left(\log m_b m_b^*\right)\Big|_{\substack{h=v\\ \sigma=v_\sigma}} = \frac{2}{v} - \frac{4vy_1'y_2'}{y_1'y_2'(v^2+v_\sigma^2)-M_1'M_5'}, \tag{4.94}$$

$$\frac{\partial}{\partial \sigma}\left(\log m_b m_b^*\right)\Big|_{\substack{h=v\\ \sigma=v_\sigma}} = \frac{2y_2'\Lambda_2'}{y_2'\Lambda_2'v_\sigma - M_5'\Lambda_3'} - \frac{4v_\sigma y_1'y_2'}{y_1'y_2'(v^2+v_\sigma^2)-M_1'M_5'}, \tag{4.95}$$

$g_{\hat{h}}^{\rm CH}(m_h^2)$ and $g_{\hat{\sigma}}^{\rm CH}(m_h^2)$ read

$$g_{\hat{h}}^{\rm CH}(m_h^2) = \frac{1}{3v} + \frac{2vy_1'y_2'}{3y_1'y_2'(v^2+v_\sigma^2)-3M_1'M_5'} = \frac{1}{3v} - \frac{2vy_1'y_2'}{3M_1'M_5'} + O\left(\frac{1}{(M_1'M_5')^2}\right), \tag{4.96}$$

$$\begin{aligned} g_{\hat{\sigma}}^{\rm CH}(m_h^2) &= \frac{y_2\Lambda_2}{3y_2\Lambda_2 v_\sigma - 3M_5\Lambda_3} + \frac{2v_\sigma y_1'y_2'}{3y_1'y_2'(v^2+v_\sigma^2)-3M_1'M_5'} = \\ &= -\frac{1}{3}\frac{y_2\Lambda_2}{M_5\Lambda_3} + O\left(\frac{v_\sigma}{M_1'M_5'}, \frac{v_\sigma}{M_5^2}\right). \end{aligned} \tag{4.97}$$

In the end, the effective coupling in the scalar mass basis is given by

$$g_h = g_{\hat{h}}(m_h^2)\cos\gamma - g_{\hat{\sigma}}(m_h^2)\sin\gamma \sim \frac{1}{3v}\cos\gamma - \frac{1}{3}\frac{y_2\Lambda_2}{M_5\Lambda_3}\sin\gamma. \tag{4.98}$$

The important point is that in the limit $m_t \gg m_h$ the hgg effective coupling is exactly as in the SM. The contribution arising from the heavy fermions tends to cancel out for masses larger than v.

With analogous procedure, we can obtain the σgg amplitude. The difference with the previous case is that now the top quark is lighter or comparable in mass to σ and this means that that the integral (4.81) cannot be approximated to 1/3. What we have to do is to subtract the "false" top quark contribution from the total contribution and then add it without approximating the integral:

$$\begin{aligned} g_{\hat{h}}^{\rm CH}(m_\sigma^2) &= \frac{1}{6}\frac{\partial}{\partial h}\left[\log\left(\det \mathcal{M}\mathcal{M}^\dagger\right) - \log m_t m_t^* - \log m_b m_b^*\right]\Big|_{\substack{h=v\\ \sigma=v_\sigma}} + \\ &+ \frac{1}{2}I\left(\frac{m_\sigma^2}{m_t^2}\right)\frac{\partial}{\partial h}\left[\log m_t m_t^*\right]\Big|_{\substack{h=v\\ \sigma=v_\sigma}} = \\ &= -\frac{2}{3}v\left(\frac{y_1y_2}{M_1M_5} + \frac{y_1'y_2'}{M_1'M_5'}\right) + \frac{1}{v}I\left(\frac{m_\sigma^2}{m_t^2}\right) + O\left(\frac{vv_\sigma^2}{M_1^2M_5^2}, \frac{vv_\sigma^2}{M_1'^2M_5'^2}\right) \end{aligned} \tag{4.99}$$

where the derivative of the top mass logarithm can be read as usual from (4.94) by substituting the unprimed couplings with the primed ones. Analogously $g_{\hat{\sigma}}^{\rm CH}(m_\sigma^2)$ reads

$$\begin{aligned} g_{\hat{\sigma}}^{\rm CH}(m_\sigma^2) = & \frac{1}{6}\frac{\partial}{\partial\sigma}\left[\log\left(\det\mathcal{M}\mathcal{M}^\dagger\right) - \log m_t m_t^*) - \log m_b m_b^*\right]\Big|_{\substack{h=v\\ \sigma=v_\sigma}} + \\ & + \frac{1}{2} I\left(\frac{m_\sigma^2}{m_t^2}\right)\frac{\partial}{\partial\sigma}\left[\log m_t m_t^*\right]\Big|_{\substack{h=v\\ \sigma=v_\sigma}} = \\ & = -\frac{2}{3}v_\sigma\left(\frac{y_1 y_2}{M_1 M_5} + \frac{y_1' y_2'}{M_1' M_5'}\right) - \frac{y_2\Lambda_2}{M_5\Lambda_3} I\left(\frac{m_\sigma^2}{m_t^2}\right) + \\ & + O\left(\frac{v v_\sigma^2}{M_1^2 M_5^2}, \frac{v v_\sigma^2}{M_1'^2 M_5'^2}\right) \end{aligned} \tag{4.100}$$

Finally the σgg effective coupling is given by

$$\begin{aligned} g_\sigma^{\rm CH}(m_\sigma^2) \equiv & \, g_{\hat{h}}(m_\sigma^2)\sin\gamma + g_{\hat{\sigma}}(m_\sigma^2)\cos\gamma \sim \\ & \sim \left[\frac{1}{v} I\left(\frac{m_\sigma^2}{m_t^2}\right) - \frac{2}{3}v\left(\frac{y_1 y_2}{M_1 M_5} + \frac{y_1' y_2'}{M_1' M_5'}\right)\right]\sin\gamma + \\ & + \left[-\frac{y_2\Lambda_2}{M_5\Lambda_3} I\left(\frac{m_\sigma^2}{m_t^2}\right) - \frac{2}{3}v_\sigma\left(\frac{y_1 y_2}{M_1 M_5} + \frac{y_1' y_2'}{M_1' M_5'}\right)\right]\cos\gamma. \end{aligned} \tag{4.101}$$

Summarising, the hgg decay is dominated by the top quark for very heavy fermions, while the heavy sector has a more significant impact on σgg transitions.

Chapter 5

The Strong CP Problem

In the 1970's the strong interactions had a puzzling problem, which became particularly clear with the development of the QCD. The QCD Lagrangian is:

$$\mathcal{L}_{\text{QCD}} = \sum_{i=1}^{6} \bar{\psi}_i \left(i\not{D} - m_i \right) \psi_i - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} \tag{5.1}$$

with $D_\mu = \partial_\mu - ig_s G^a_\mu T^a$ where the T^a's are the $SU(3)$ generators, the sum over the color indices is understood and the i index runs over the six quarks. It is immediate to see that in the limit when $m_i \to 0$ the Lagrangian (5.1) is invariant under $U(6)_L \times U(6)_R$ where the subscripts stay for left and right. Actually, since the only up and down quark masses are smaller than the typical scale of the formation of hadrons, that is $m_u, m_d \ll \Lambda_{\text{QCD}}$, only a chiral $U(2)_L \times U(2)_R$ is a very good approximate global symmetry of strong interactions. This symmetry, however, is not manifest in the hadrons' spectrum. In order to explain why, we need to decompose the symmetry group in his vectorial and axial parts:

$$U(2)_L \times U(2)_R = SU(2)_V \times SU(2)_A \times U(1)_V \times U(1)_A \tag{5.2}$$

where the V and A are the vector and axial respectively. The $U(1)_V$ subgroup corresponds to the conservation of baryon number and it is an exact symmetry of QCD, as also the mass term is invariant under $U(1)_V$. On the other hand, one would expect the remaining $SU(2)_V \times SU(2)_A \times U(1)_A$ to be a good approximate symmetry of QCD, because of the smallness m_u, m_d compared to the Λ_{QCD} scale. However, this is true only with the vectorial part of symmetry, while the axial part $SU(2)_A \times U(1)_A$ is not seen in the spectrum. Indeed, there are several reasons (both theoretical and experimental) that lead us to believe that the QCD global symmetry group $U(2)_L \times U(2)_R$ is spontaneously broken by the vacuum condensate $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \neq 0$. Most of all is the fact that it is present, in the hadron spectrum, an approximate multiplet of light particles - the pions - whose mass vanishes in the limit $m_u, m_d \to 0$ so they are suitable candidates to be the NGBs of the broken

$SU(2)_A$ symmetry. Remarkably, however, there is no corresponding pseudoscalar state with vanishing mass, in the same limit, associated with the broken $U(1)_A$ symmetry. Weinberg [34] first understood that the problem is related to the *anomalous nature* of the $U(1)_A$ symmetry. Shortly, the reason why there are no approximate NGBs associated with this abelian chiral symmetry is that, as a consequence of the anomaly, this is really not a symmetry of the full quantum theory. The important point is that, because of the anomaly, at one-loop level the QCD Lagrangian has a further term:

$$\mathcal{L}_{\text{eff}} = \theta \frac{\alpha_s}{8\pi} G_{a\mu\nu} \tilde{G}^{a\mu\nu}. \tag{5.3}$$

However, although this feature of chiral symmetry was pointed out before the advent of QCD as a theory of strong interactions, the resolution of $U(1)_A$ problem required a better understanding of QCD vacuum [35], [36]. In fact,the right-hand side of the equation is a total divergence

$$\begin{aligned} G^{\mu\nu a}\tilde{G}^a_{\mu\nu} &= \partial_\mu K^\mu, \\ K^\mu &= \epsilon^{\mu\nu\rho\sigma} A^a_\nu \left[G^a_{\rho\sigma} - \frac{2}{3} g_s f^{abc} A^b_\rho A^c_\sigma \right] \end{aligned} \tag{5.4}$$

so it would seem that, since a total divergence does not affect the equations of motion, this could give no help in addressing the $U(1)_A$ problem. But, as pointed out by 't Hooft, the QCD vacuum is richer than expected. In fact, as the QCD is a *non abelian* gauge theory, the different vacuum configurations of the gauge field A^a_μ cannot be mapped into the trivial vacuum $A^a_\mu = 0$, i.e. they belong to different topological classes[1]. Taking into account (as we should) all the topological different QCD vacua[2], the total divergence is not irrelevant anymore. The point is that this term - also called the θ term - leads to an *enormous* neutron electric dipole moment ($d_n = e\theta m_q/m_N^2$), unless we set the parameter θ to be less than 10^{-9} [38].

The problem is actually made worse because it is fomented by the EW sector. In principle the Yukawa matrix is neither hermitian nor diagonal and, in order to go in the physical mass basis, we need to perform a rotation of quarks that leads in general to a complex diagonal matrix:

$$\mathcal{L}_{\text{mass}} = -m_i e^{-i\beta_i} \bar{q}^i_L q^i_R + h.c. \tag{5.5}$$

In order to make the masses real we need to absorb the phases by redefining

[1] Shortly, in non abelian groups we can always extract an $SU(2)$ subgroup and the gauge field restricted to the $SU(2)$ subgroup $A^i_\mu(x)$ can be viewed as a map from the contour of the spacetime to the SU(2) group; this map is nothing but the third homotopy group of the three sphere and its different topological classes are labelled by an integer number: $\pi_3(S^3) = Z$ [37].

[2] It is essential in order to have a gauge invariant vacuum.

the quark fields with a chiral rotation

$$\begin{aligned} q_L^i &\to e^{-i\frac{\beta_i}{2}} q_L^i \\ q_R^i &\to e^{+i\frac{\beta_i}{2}} q_R^i. \end{aligned} \tag{5.6}$$

In the next section we will show that this redefinition shifts the vacuum angle θ by

$$\theta \to \bar{\theta} = \theta + \sum_i \beta_i = \theta + \operatorname{Arg}\det Y \tag{5.7}$$

where Y is the Yukawa matrix still not diagonalized and the second equality comes from the fact that trace and determinant are invariant under a change of basis. The strong CP problem is then:

why the combination of QCD and EW parameters which make up $\bar{\theta}$ is so small?

We notice that, putting the QCD θ parameter to zero does not provide CP conservation since it is already broken in the EW sector, so the symmetry does not increase and, following the definition of Naturalness, the theory is not natural. We stress again that, as $\bar{\theta}$ is a free parameter of the theory, in principle any value of $\bar{\theta}$ is equally likely. Anyway, in the same spirit as the Higgs mass, one would like to obtain an understanding from some underlying physics of why this number is so small.

5.1 Chiral Transformations and $U(1)_A$ Anomaly

We have said that the resolution of the $U(1)_A$ problem starts recognizing that the $U(1)_A$ symmetry is anomalous. In order to explain why, we briefly review the main features of the classical axial symmetry. Firstly it is useful to derive the Noether's theorem in a more convenient way. Let us consider a Lagrangian $\mathcal{L}[\phi]$ depending from a set of field ϕ and a transformation of fields $\phi \to \phi' = \phi + \epsilon(x) f[\phi]$. Now, expanding $\mathcal{L}[\phi']$ to the first order in ϵ we find

$$\mathcal{L}' \equiv \mathcal{L}[\phi', \partial\phi'] = \mathcal{L}[\phi, \partial\phi] + \frac{\partial \mathcal{L}'}{\partial \epsilon}\epsilon + \frac{\partial \mathcal{L}'}{\partial(\partial_\mu \epsilon)}\partial_\mu \epsilon. \tag{5.8}$$

If $\phi \to \phi + \epsilon f[\phi]$ with $\epsilon = \text{const}$ is a symmetry of the Lagrangian then $\frac{\partial \mathcal{L}}{\partial \epsilon(x)} = 0$, and we are left with

$$\mathcal{L}[\phi', \partial\phi'] = \mathcal{L}[\phi, \partial\phi] + \frac{\partial \mathcal{L}'}{\partial(\partial_\mu \epsilon)}\partial_\mu \epsilon. \tag{5.9}$$

As

$$J^\mu \equiv \frac{\partial \mathcal{L}'}{\partial(\partial_\mu \epsilon)} = \frac{\partial \mathcal{L}'}{\partial(\partial_\mu \phi)}\phi f[\phi], \tag{5.10}$$

we immediately recognize from the second equality that J^μ is exactly the Noether's current of the symmetry transformation. Now, the QCD Lagrangian for one quark is

$$\mathcal{L} = -\frac{1}{4} G^{\mu\nu a} G^a_{\mu\nu} + \bar{\psi} \left(i \not{D} - m \right) \psi. \tag{5.11}$$

Performing a chiral rotation of the fermion field

$$\psi \to e^{-i\alpha(x)\gamma_5} \psi \qquad \text{or} \qquad \begin{cases} \psi_L \to e^{+i\alpha(x)} \psi_L, \\ \psi_R \to e^{-i\alpha(x)} \psi_R \end{cases} \tag{5.12}$$

the Lagrangian transform consequently as

$$\mathcal{L}' = \mathcal{L} + 2im\alpha \, \bar{\psi}\gamma_5\psi + \left(\bar{\psi}\gamma^\mu\gamma^5\psi \right) \partial_\mu \alpha(x). \tag{5.13}$$

We see that the Lagrangian is invariant if $m = 0$, and, in this case, the associated (conserved) Noether current is the axial current:

$$J^\mu_5 = \frac{\partial \mathcal{L}'}{\partial(\partial_\mu \alpha)} = \bar{\psi}\gamma^\mu\gamma^5\psi. \tag{5.14}$$

This is true at a classical level. Anyway, in the first chapter we have said that, thanks to a theorem of QFT, the symmetries of the classical action must be also symmetries of the full quantum action, but this is not completely exact. In fact, the demonstration of this theorem relies on the fact that the functional measure of the path integral is invariant under the transformation, but this does not always happen. As pointed out by Fujikawa [39], under a chiral transformation the functional measure changes, leading to the non-conservation of the quantum chiral current. It turns out that, because of this non-invariance, the divergence of the axial current reads

$$\partial_\mu J^\mu_5 = 2im \, \bar{\psi}\gamma_5\psi - \frac{g^2}{16\pi^2} G^{\mu\nu a} \tilde{G}^a_{\mu\nu}. \tag{5.15}$$

This is the anomaly associated with the transformation (5.12). It is also useful to derive the anomaly associated with the more general chiral transformation

$$\begin{aligned} \psi_L &\to e^{-i\alpha n_L} \psi_L, \\ \psi_R &\to e^{-i\alpha n_R} \psi_R. \end{aligned} \tag{5.16}$$

For this purpose, we notice that the left-handed and right-handed currents are related to the chiral current (5.14) by

$$\begin{aligned} \frac{1}{n_L} \partial_\mu J^\mu_L &= \partial_\mu \left(\bar{\psi}_L \gamma^\mu \psi_L \right) = \frac{1}{2} \partial_\mu \left(\bar{\psi}\gamma^\mu\psi \right) - \frac{1}{2} \partial_\mu \left(\bar{\psi}\gamma^\mu\gamma^5\psi \right) = -\frac{1}{2} \partial_\mu J^\mu_5, \\ \frac{1}{n_R} \partial_\mu J^\mu_R &= \partial_\mu \left(\bar{\psi}_R \gamma^\mu \psi_R \right) = \frac{1}{2} \partial_\mu \left(\bar{\psi}\gamma^\mu\psi \right) + \frac{1}{2} \partial_\mu \left(\bar{\psi}\gamma^\mu\gamma^5\psi \right) = +\frac{1}{2} \partial_\mu J^\mu_5 \end{aligned} \tag{5.17}$$

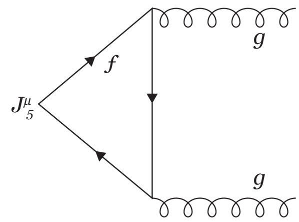

Figure 5.1: The triangle diagram responsible of the anomaly. Here f is the fermion coupled to the gluons g. It can be shown that *all* the contribution to the anomaly is given by this diagram, so the anomaly is *one-loop exact* [40].

where in the last equality we have used the fact that the vector current $\bar{\psi}\gamma^\mu\psi$ is conserved. Since the anomaly associated with the chiral current (5.14) is (5.15), the divergence of the chiral current associated to (5.16), using (5.17), is

$$\partial_\mu J_5^\mu = 2im\,\bar{\psi}\gamma_5\psi - \frac{g^2}{32\pi^2}\left(-n_L + n_R\right) G^{\mu\nu a}\tilde{G}^a_{\mu\nu}. \tag{5.18}$$

Setting $n_L = -n_R = -1$, we recover (5.15).

Now, because of the anomaly, the Lagrangian (5.11) (taking $m = 0$ for simplicity) at one-loop level contains extra anomalous term

$$\mathcal{L} = -\frac{1}{4}G^{\mu\nu a}\tilde{G}^a_{\mu\nu} + \bar{\psi}i\not{D}\psi - \frac{g^2}{32\pi^2}\left(-n_L + n_R\right) G^{\mu\nu a}\tilde{G}^a_{\mu\nu} \tag{5.19}$$

provided that the fields transform as in (5.16). From this last equation we can understand why (5.6), needed to ensure real quark masses, leads to a shift in the θ parameter by $\sum_i \beta_i$. In fact, taking $n^i_L = -n^i_R = \beta_i/2$ the term in parenthesis is $\sum_i \beta_i = \operatorname{Arg}\det Y$.

It is important to remark for the sequent discussion that the same result can be derived with the perturbative approach. In fact, it can be shown that the anomaly arises because of the so-called triangle diagram, depicted in Fig. 5.1.

The last part of this section is devoted to particularize the previous discussion to SM gauge bosons. The SM gauge group is $SU(3)_c \times SU(2)_L \times U(1)_Y$ so we have to consider all the couplings of the chiral current with all the gauge fields. The SM quarks transform under different representations of the SM group:

$$\begin{aligned} q_L &= (u_L, d_L) \in (\mathbf{3}, \mathbf{2})_{\mathbf{1/6}}, \\ u_R &\in (\mathbf{3}, \mathbf{1})_{\mathbf{2/3}}, \\ d_R &\in (\mathbf{3}, \mathbf{1})_{\mathbf{-1/3}} \end{aligned} \tag{5.20}$$

where we have considered for simplicity only one family. A chiral transformation of the quarks

$$\begin{aligned} q'_L &= e^{-i\alpha n_q} q_L, \\ u'_R &= e^{-i\alpha n_u} u_R, \\ d'_R &= e^{-i\alpha n_d} d_R \end{aligned} \tag{5.21}$$

leads, because of (5.18), to an anomaly associated with the gluon field $G^a_{\mu\nu}$

$$\mathcal{L}_{\text{anomaly},G} = -\frac{g^2}{32\pi^2} \left(-2n_q + n_u + n_d\right) G^{\mu\nu a} \tilde{G}^a_{\mu\nu} \tag{5.22}$$

where the 2 is due to the fact that $SU(3)_c$ "sees" q_L as made by two fields (u_L and d_L). In the same way, the anomaly associated to $W^{\mu i}$ reads

$$\mathcal{L}_{\text{anomaly},W} = -\frac{g^2}{32\pi^2} \left(-3n_q\right) W^{\mu\nu i} \tilde{W}^i_{\mu\nu} \tag{5.23}$$

where the 3 is due to the fact that there are three copies of q_L (it belongs to the 3 of $SU(3)_c$) and u_R and d_R are not present here because they are singlets under $SU(2)_L$ so the chiral transformation does not produce anomalous couplings with W^i_μ. Finally, the anomaly associated to the $U(1)_Y$ field B_μ reads

$$\mathcal{L}_{\text{anomaly},B} = -\frac{g^2}{16\pi^2} \left(-6n_q Y_q^2 + 3n_u Y_u^2 + 3n_d Y_d^2\right) \tilde{B}^{\mu\nu} B_{\mu\nu} \tag{5.24}$$

where 6 and 3 are the degeneracies of q_L and u_R, d_R fields respectively, that is $U(1)_Y$ sees q_L as 6 copies of fermions with the same charge. The 1/16 differs from 1/32 because in the chiral current (5.18) we have implicitly assumed that the generators of $SU(3)_c$ are normalized as

$$\text{Tr}[G^{\mu\nu} G_{\mu\nu}] = \frac{1}{2} G^{\mu\nu a} G_{\mu\nu a} \tag{5.25}$$

and the same for $W^{\mu\nu}$. On the other hand $\text{Tr}[B^{\mu\nu} B_{\mu\nu}] = B^{\mu\nu} B_{\mu\nu}$ since the group is abelian, so the $\frac{1}{2}$ factor is not present.

We conclude this section with a remark: the gauge group of the SM is chiral, therefore there could be anomalies in the associated conserved chiral currents. This would be a disaster, since it can be shown that if the gauge group of the SM were broken at a quantum level this would bring to the non-unitarity of the theory. However, it turns out that *all* the anomalies cancel: the SM is anomaly-free. With the previous discussion we can easily show how this happens for the anomaly associated to the interaction between three $U(1)_Y$ chiral currents, that is through the triangle diagram in Fig. 5.1 with three $U(1)_Y$ currents on the three vertices. Introducing the leptons $l_L = (\nu_L, e_L)$ and e_R and transforming the fields in (5.21) and the leptons with $n_j \equiv Y_j$, where Y_j is the hypercharge and j runs over all the fields of the one family-SM, the parenthesis in (5.24) is zero:

$$-6Y_q^3 + 3Y_u^3 + 3Y_d^3 - 2Y_l^3 + Y_e^3 = -\frac{1}{36} + \frac{8}{9} - \frac{1}{9} + \frac{1}{4} - 1 = 0. \qquad (5.26)$$

The discussion can be easily generalized to the real case of three families of quarks and leptons.

Chapter 6

A Chiral Solution to the Strong CP Problem: Axions

There have been made several attempts to solve the strong CP problem. Before going in the details of the axion solution, perhaps one of the most cogent solutions to it, it is instructive to analyse the different solutions proposed during the second half of 900'. We start from the simplest possibility: spontaneously broken CP. If CP is a symmetry of nature which is spontaneously broken, then one can set $\theta = 0$ at the Lagrangian level [41]. However, if CP is spontaneously broken θ gets induced back at loop level, so one needs, in general, to ensure that θ vanishes also at one-loop level. Although models exist where this is accomplished, experimental data are in excellent agreement with the CKM model - a model where CP is explicitly, not spontaneously broken.

The second possibility, explored by 't Hooft, is a massless up quark. We can easily read from the discussion in section 5.1 how a massless fermion can allow to rotate the θ-vacua away. In fact, transforming the massless up quark as

$$u \to u' = e^{-i\frac{\theta}{2}\gamma_5} u \tag{6.1}$$

the QCD Lagrangian is invariant but, because of the chiral nature of the rotation, an anomaly shows up:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} G_{\mu\nu a} G^{\mu\nu a} + \bar{u} \not{D} u + \theta \frac{g^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{\mu\nu a} \to \\ \mathcal{L} &= -\frac{1}{4} G_{\mu\nu a} G^{\mu\nu a} + \bar{u} \not{D} u + \frac{g^2}{32\pi^2} (\theta - \theta) G^a_{\mu\nu} \tilde{G}^{\mu\nu a} \end{aligned} \tag{6.2}$$

and the θ term goes away. For some time the massless up quark possibility was taken seriously. The reason is that, even if the Lagrangian mass for the up quark is zero, the 't Hooft determinental interaction may generate a useful up quark mass for chiral perturbation [42]. Now it is clear that this possibility is ruled out.

The most attractive solution has been proposed by Peccei and Quinn [8], [9]. Inspired by the massless up quark case, they extended the scalar sector of the SM by adding an other Higgs doublet. As we will see later, the potential for two Higgs doublets can be chosen in such a way that the symmetry group of the Lagrangian is enlarged by an additional global chiral $U(1)$ symmetry with respect to the SM symmetry group, the so-called $U(1)_{PQ}$ symmetry. Obviously, the existence of an exact $U(1)_{PQ}$ symmetry would solve the strong CP problem because it all would work as in the massless up quark case. However, this symmetry cannot be exact, since we do not see it in the particle spectrum. What Peccei and Quinn showed is that, even if $U(1)_{PQ}$ is spontaneously broken (with a soft breaking term given by the anomaly), the parameter θ is dynamically driven to zero. However, there is a price to pay: a spontaneously broken $U(1)_{PQ}$ symmetry leads to the appearance, in the particle spectrum, of a pNGB - pseudo because the anomaly softly breaks $U(1)_{PQ}$ the symmetry. The axion[1] - this is the name that has been given to the pNGB - provides, with its VEV, to clean up the θ term.

In the original paper, Peccei and Quinn start showing how the mechanism works in a toy model and then they generalize the discussion to a more realistic two-Higgs doublet model (2HDM), formerly introduced by Weinberg [43]. The idea is that we can rotate away the θ term even if the quark masses are included in the Lagrangian, provided that at least one fermion gets its entire mass from a Yukawa coupling to a scalar field, so that the full Lagrangian can possess a single chiral $U(1)$ invariance. The Lagrangian of the toy model contains one fermion field ψ belonging to some non trivial representation of the color group and one complex scalar field ϕ:

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4}G^a_{\mu\nu}G^{\mu\nu a} + i\bar{\psi}\gamma_\mu D^\mu\psi + g\bar{\psi}\left[\phi\left(\frac{1+\gamma_5}{2}\right) + \phi^*\left(\frac{1-\gamma_5}{2}\right)\right]\psi \\ & + \theta\frac{g^2}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{\mu\nu a} + \partial_\mu\phi^*\partial^\mu\phi - \mu^2(\phi^*\phi) - h(\phi^*\phi)^2\end{aligned} \tag{6.3}$$

with $\mu^2 < 0$ and $h > 0$. The Lagrangian is invariant under the chiral symmetry $U(1)_{PQ}$:

$$\begin{cases}\psi \to e^{-i\gamma_5\sigma}\psi, \\ \phi \to e^{+2i\sigma}\phi.\end{cases} \tag{6.4}$$

Under the transformations (6.4) the classical Lagrangian remains unchanged but, as we have seen, the quantum Lagrangian contains an anomalous term:

$$\mathcal{L}_{\text{eff}} = -2\sigma\frac{g^2}{32\pi^2}G^a_{\mu\nu}\tilde{G}^{\mu\nu a}. \tag{6.5}$$

[1]The name derives from a detergent brand, because it "cleans" the θ term.

Therefore the net result of the chiral rotation is a shift of the θ parameter by

$$\theta \to \theta - 2\sigma. \tag{6.6}$$

However, the transformation (6.4) leads, in general, to a complex fermion mass. In order to understand the solution proposed by Peccei and Quinn we need to write the scalar field in radial-angular notation:

$$\phi = \frac{\rho}{\sqrt{2}} e^{i\frac{a}{f_a}} \tag{6.7}$$

where a is the NGB associated to the broken $U(1)_{\mathrm{PQ}}$ generator and f_a is the scale of the spontaneous symmetry breaking. Now, the Haag theorem allows us to rotate away the axion field a from the Yukawa coupling by performing the transformations

$$\begin{aligned} \psi_R &\to e^{-i\frac{a}{2f_a}} \psi_R, \\ \psi_L &\to e^{+i\frac{a}{2f_a}} \psi_L. \end{aligned} \tag{6.8}$$

With these transformations the Lagrangian (6.3) becomes

$$\begin{aligned} \mathcal{L} = &-\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} + i\bar{\psi}\gamma_\mu D^\mu \psi + \frac{\partial_\mu a}{2f_a} \left[\bar{\psi}_R \gamma^\mu \psi_R - \bar{\psi}_L \gamma^\mu \psi_L\right] + \\ &+ \frac{g}{\sqrt{2}} \rho \bar{\psi}\psi + \left(\theta - \frac{a}{f_a}\right) \frac{g^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{\mu\nu a} + \frac{1}{2} \partial_\mu \rho \partial^\mu \rho - \frac{\mu^2}{2} \rho^2 - \frac{h}{4} \rho^4. \end{aligned} \tag{6.9}$$

If there were not an anomaly in the chiral current, the ρ field would acquire a VEV (providing a mass for ψ) while the axion would be exactly massless and his vev would be unobservable. However, the anomaly breaks explicitly the $U(1)_{\mathrm{PQ}}$ symmetry. As a consequence, the shift symmetry is not a symmetry anymore and it cannot protect the axion from taking a VEV and a mass. What Peccei and Quinn showed is that the periodicity of the pseudoscalar density expectation value $\langle G\tilde{G} \rangle$ in the relevant theta parameter $(\theta - a/f_a)$ forces the axion to pick out the VEV

$$\langle a \rangle = \theta f_a. \tag{6.10}$$

This is enough to solve the strong CP problem since, when expressing the Lagrangian in terms of the physical field $a_{\mathrm{phys}} = a - \langle a \rangle$, the θ parameter is dynamically driven to zero. Furthermore, the softly breaking term, that is the only term providing an effective potential for the axion, gives it a mass as well:

$$m_a^2 \equiv \langle \frac{\partial^2 V_{\mathrm{eff}}}{\partial a^2} \rangle = \frac{1}{f_a} \frac{g^2}{32\pi^2} \frac{\partial}{\partial a} \langle G^a_{\mu\nu} \tilde{G}^{\mu\nu a} \rangle_{|\langle a \rangle = f_a \theta}. \tag{6.11}$$

The calculation of the axion mass was first done explicitly by current algebra techniques by Bardeen and Tye [44]. This formula states that the axion mass arises because of the soft breaking due to the anomaly: it comes from instantonic contributions. The above discussion can be generalized to the case in which the transformation of the fermions is more general than (6.8). In this case the axion coupling with gauge bosons can be written generically as

$$-\zeta \frac{a}{f_a} \frac{g^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{\mu\nu a} \tag{6.12}$$

where ζ is a model-dependent parameter. It all works as in the previous case: the axion now picks the VEV

$$\langle a \rangle = \frac{f_a}{\zeta} \theta \tag{6.13}$$

and the θ term goes away from the Lagrangian when we write the Lagrangian in terms of the physical axion $a_{\text{phys}} = a - \langle a \rangle$.

Now that the mechanism is clear we can start introducing axions in more realistic models. Basically, there have been studied three types of axion models, depending on the way they enter into the Lagrangian:

- the Peccei-Quinn-Weinberg-Wilczek (PQWW) model, where the axion is introduced as a linear combination of the phases in a 2HDB [8], [9];
- the Kim-Shifman-Vainshtein-Zakharov (KSVZ) model [12], [13], where the axion is the phase of a complex scalar field, just like the toy model described above;
- the Dine-Fishler-Srednicki-Zhithnitsky (DFSZ) model [14], [15], where the axion is introduced as a linear combination of the phase of the SM Higgs and a complex scalar field.

We will describe briefly all these models in the next sections.

6.1 PQWW Model

As we have anticipated, the first model proposed by Peccei and Quinn relies on a 2HDB first introduced by Weinberg [43]. The most general renormalizable potential for two Higgs doublets with the reflection symmetry $\Phi_i \to -\Phi_i$ (and, obviously, the SM custodial $SO(4)$ symmetry) is

$$\begin{aligned} V(\Phi_1, \Phi_2) = &- \sum_i \mu_i^2 \Phi_i^\dagger \Phi_i + \sum_{i,j} a_{ij} (\Phi_i^\dagger \Phi_i)(\Phi_j^\dagger \Phi_j) + \\ &+ \sum_{i,j} b_{ij} (\Phi_i^\dagger \tau^a \Phi_i)(\Phi_j^\dagger \tau^a \Phi_j) + \sum_{i,j} c_{ij} (\Phi_i^\dagger \Phi_j)(\Phi_i^\dagger \Phi_j) \end{aligned} \tag{6.14}$$

where $i = 1, 2$. Hermiticity requires that μ_i^2 be real, a_{ij} and b_{ij} be real and symmetric, while c_{ij} needs only be hermitian. Actually this potential, because of the c_{ij} term, does allow just a $U(1)$ symmetry and we are forced to associate it to the SM $U(1)_Y$. But if we set $c_{ij} = 0$ - and this is what Peccei and Quinn did - we can easily see that the symmetry group enlarges by an extra $U(1)$ symmetry that we can associate to the $U(1)_{\rm PQ}$. We can write the two independent $U(1)$ transformations as

$$U(1)_Y: \qquad \Phi_1 \to e^{\frac{i\alpha}{2}}\Phi_1, \qquad \Phi_2 \to e^{-\frac{i\alpha}{2}}\Phi_2 \tag{6.15}$$

$$U(1)_{PQ}: \qquad \Phi_1 \to e^{i\beta\Gamma_1}\Phi_1, \qquad \Phi_2 \to e^{i\beta\Gamma_2}\Phi_2 \tag{6.16}$$

where Γ_1 and Γ_2 are the PQ charges of Φ_1 and Φ_2. We will see in a moment that their ratio is fixed by the model, if we require the latter to be consistent with the SM. The last ingredient is the choice of the Yukawa couplings, needed to give a mass to fermions. The Yukawa couplings are needed also to "transform" the $U(1)_{\rm PQ}$ symmetry in a chiral symmetry, just like the toy model in the previous section. Therefore they have to be written so that the global $U(1)_{PQ}$ symmetry and, obviously, the custodial $SU(2)_L \times U(1)_Y$ symmetry are preserved. This is achieved by coupling Φ_1 to d_R and e_R and Φ_2 to u_R:

$$\mathcal{L}_{\rm Yukawa} = \Gamma^u \bar{Q}_L \Phi_1 d_R + \Gamma^d \bar{Q}_L \Phi_2 u_R + \Gamma^l \bar{l}_L \Phi_1 e_R + \text{h.c.} \tag{6.17}$$

that is invariant under the $U(1)_{PQ}$ symmetry if the fermions transform as

$$Q_L \to Q_L, \qquad u_R \to e^{-i\beta\Gamma_2}u_R, \qquad d_R \to e^{-i\beta\Gamma_1}d_R. \tag{6.18}$$

The next step is to identify the axion component in the phases of Φ_1 and Φ_2. For this purpose we remind that the axion is the NGB of the spontaneously broken $U(1)_{PQ}$, hence it must shift under $U(1)_{PQ}$:

$$a \to a + \beta\lambda. \tag{6.19}$$

where λ is a generic constant. The neutral part of the Higgs doublets can be parametrized as

$$\Phi_1^0 = \frac{1}{\sqrt{2}}(v_1 + \rho_1)e^{i\frac{P_1}{v_1}}, \qquad \Phi_2^0 = \frac{1}{\sqrt{2}}(v_2 + \rho_2)e^{i\frac{P_2}{v_2}}. \tag{6.20}$$

One linear combination of P_1 and P_2 is absorbed by the Z boson and becomes the longitudinal component of it, the other one is the axion. To find this linear combination let us apply a rotation in the (P_1, P_2) space:

$$\begin{cases} z = -P_1 \sin\theta + P_2 \cos\theta \\ a = P_1 \cos\theta + P_2 \sin\theta \end{cases}, \tag{6.21}$$

$$\begin{cases} P_1 = -z \sin\theta + a \cos\theta \\ P_2 = z \cos\theta + a \sin\theta \end{cases}. \tag{6.22}$$

Substituting (6.22) in (6.20) and imposing (6.16) we find

$$\begin{aligned} e^{i\frac{a\cos\theta}{v_1}} &\to e^{i(\beta\Gamma_1+\frac{a\cos\theta}{v_1})}, \\ e^{i\frac{a\sin\theta}{v_2}} &\to e^{i(\beta\Gamma_2+\frac{a\sin\theta}{v_2})} \end{aligned} \tag{6.23}$$

where we have written only the part containing the axion field a. Comparing (6.23) with (6.19)

$$\begin{aligned} \frac{v_1}{\cos\theta}\Gamma_1 &= \lambda, \\ \frac{v_2}{\sin\theta}\Gamma_2 &= \lambda \end{aligned} \tag{6.24}$$

or, equivalently

$$\begin{aligned} \cos\theta &= \frac{v_1\Gamma_1}{\sqrt{v_1^2\Gamma_1^2+v_2^2\Gamma_2^2}}, \\ \sin\theta &= \frac{v_2\Gamma_2}{\sqrt{v_1^2\Gamma_1^2+v_2^2\Gamma_2^2}}. \end{aligned} \tag{6.25}$$

The condition that the NGB z is absorbed into the Z boson determines Γ_1 and Γ_2. Since the Z boson charge Q_Z can be taken as $(I_3 - Y)$ we have $Q_Z(\Phi_1^0) = -1/2 - 1/2 = -1$, $Q_Z(\Phi_2^0) = 1/2 + 1/2 = 1$. Under the $U(1)_Z$ transformation $z \to z + \gamma\lambda'$. Repeating the argument given above for the axion and substituting the $U(1)_{PQ}$ with the $U(1)_Z$ transformation

$$\begin{aligned} e^{-i\frac{z\sin\theta}{v_1}} &\to e^{i(\gamma Q_Z(\Phi_1^0)-\frac{z\sin\theta}{v_1})}, \\ e^{i\frac{z\cos\theta}{v_2}} &\to e^{i(\gamma Q_Z(\Phi_2^0)+\frac{z\cos\theta}{v_2})} \end{aligned} \tag{6.26}$$

we find

$$\begin{aligned} \cos\theta &= \frac{v_2 Q_Z(\Phi_2^0)}{\sqrt{v_1^2[Q_Z(\Phi_1^0)]^2+v_2^2[Q_Z(\Phi_2^0)]^2}} = \frac{v_2}{\sqrt{v_1^2+v_2^2}}, \\ \sin\theta &= -\frac{v_1 Q_Z(\Phi_1^0)}{\sqrt{v_1^2[Q_Z(\Phi_1^0)]^2+v_2^2[Q_Z(\Phi_2^0)]^2}} = \frac{v_1}{\sqrt{v_1^2+v_2^2}} \end{aligned} \tag{6.27}$$

that implies

$$\tan\theta = \frac{v_1}{v_2}. \tag{6.28}$$

Finally, comparing (6.28) with (6.25),

$$\frac{\Gamma_1}{\Gamma_2} = \frac{v_2^2}{v_1^2}. \tag{6.29}$$

As we have anticipated, we do not have the freedom of choosing both Γ_1 and Γ_2. Defining $x = \frac{v_2}{v_1}$ and choosing $\Gamma_1 = x$, $\Gamma_2 = \frac{1}{x}$, we can finally read the axion content of the two Higgs doublets:

$$\Phi_1 = \frac{v_1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} e^{ix\frac{a}{v}}, \qquad \Phi_2 = \frac{v_2}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix} e^{i\frac{a}{xv}} \tag{6.30}$$

where $v = \sqrt{v_1^2 + v_2^2}$. We can rewrite the $U(1)_{PQ}$ transformations (6.18) (6.19) as

$$\begin{aligned} &a \to a + \beta v, \\ &u_R \to e^{-i\frac{\beta}{x}} u_R, \\ &d_R \to e^{-i\beta x} d_R. \end{aligned} \tag{6.31}$$

The Noether current associated to this symmetry reads

$$J^\mu_{PQ} = f_a \partial^\mu a + \frac{1}{x}\bar{u}_R \gamma^\mu u_R + x\bar{d}_{R_i}\gamma^\mu d_R + +x\bar{l}_R \gamma^\mu l_R \tag{6.32}$$

from which we can read the ζ parameter

$$\zeta = \left(x + \frac{1}{x}\right). \tag{6.33}$$

However, as we outlined, the PQWW axion is ruled out. The main phenomenological trouble is that its physics is tied with the electroweak symmetry breaking scale, and the experimental data do not agree with the prediction of the model. Just to make an example, Bardeen [45] estimated the branching ratio

$$\text{BR}\left(K^+ \to \pi^+ + a\right) \simeq 3 \times 10^{-5} \left(x + \frac{1}{x}\right)^2 \tag{6.34}$$

which is well above the bound

$$\text{BR}\left(K^+ \to \pi^+ + \text{nothing}\right) < 3.8 \times 10^{-8} \tag{6.35}$$

found by [46].

6.2 Invisible Axion Models

Despite it is a sensible assumption to suppose that the $U(1)_{PQ}$ breaking scale f_a is the same as the electroweak scale $v = 246$ GeV, this is not necessary. The dynamical adjustment of the strong CP angle, as previously seen, is independent from the scale f_a. The difference is that, if $f_a \gg v$, the axion becomes very light, very weakly coupled and very long lived. Thus, these axions are, apparently, *invisible*. Because $f_a \gg v$ in the invisible axion models, the $U(1)_{PQ}$ symmetry must be broken by an $SU(2) \times U(1)_Y$ singlet complex scalar field σ. The axion is then, essentially, the phase of σ. That is, the scalar field can be written as:

$$\sigma = \frac{1}{\sqrt{2}}(f + \rho)e^{i\frac{a}{f_a}g(v^i, f_a)} \tag{6.36}$$

where $g(v^i, f_a)$ is a function that describes how much axion content is present into the phase of the σ field and it depends on the scales present in the model. For instance, in the toy model described in the previous section $g(v^i) = 1$, in the PQWW model we have $g(v^i, f_a) = v_2/(vv_1)$. The important point is that in these models, because of $f_a \gg v$, $|g(v^i, f_a)| \simeq 1$, that is the axion is primarily composed of the σ field. The invisible axions are classified into two types depending on whether or not they have a tree-level coupling to ordinary SM fermions. The KSVZ axions are hadronic axions with only induced coupling to SM fermions. The DFSZ axions arise in model where axions naturally couples to SM fermions already at tree level. Now we are going to see, in more detail, these models.

6.2.1 KSVZ Model

In the KSVZ model the ordinary quarks and leptons are PQ singlets. The $SU(2) \times U(1)_Y$ singlet field σ, however, interacts with new heavy quarks X^i (which carry $U(1)_{\text{PQ}}$ charge) via Yukawa interaction. The minimal model is achieved by adding, in the SM particle spectrum, one heavy fermion X and one complex scalar field σ that do not couple with SM fermions, with only a coupling between the Higgs and the σ. The Yukawa coupling for the minimal KSVZ model then reads

$$\mathcal{L}_{KSVZ} = -y_\sigma \sigma \bar{X}_L X_R - y_\sigma^* \sigma^\dagger \bar{X}_R X_L. \tag{6.37}$$

The most general scalar potential containing the SM custodial $SO(4)$ as well as the $U(1)_{\text{PQ}}$ symmetry is

$$\mathcal{V}(\phi, \sigma) = \lambda_\phi \left[\phi^\dagger \phi - v_\phi^2\right]^2 + \lambda_\sigma \left[\sigma^\dagger \sigma - v_\sigma^2\right]^2 + \lambda_{\phi\sigma}(\phi^\dagger \phi)(\sigma^\dagger \sigma). \tag{6.38}$$

The Yukawa coupling (6.37) allows for the presence of two independent $U(1)$ symmetries that we can choose to be:

$$\begin{aligned} U(1)_{\text{PQ}} : \quad & X \to e^{i\gamma_5 \alpha} X, \qquad \sigma \to e^{-2i\alpha} \sigma, \\ U(1)_X : \quad & X \to e^{i\beta} X \end{aligned} \tag{6.39}$$

The $U(1)_{\text{PQ}}$ is the chiral symmetry needed to rotate away the θ term, provided of course that Q belongs to a non trivial representation of $SU(3)_c$, while $U(1)_X$ gives the X-type baryon-number conservation. We notice that these two symmetries are respected by the scalar potential (6.38) and, obviously, by the SM Lagrangian. Also, for a finite range of parameters we have $\langle\phi\rangle \neq 0$, $\langle\sigma\rangle \neq 0$, so that the spontaneous symmetry breaking can take place and give mass to heavy fermions - through the σ particle - and to SM fermions, through the usual Higgs mechanism; furthermore the existence of the spontaneously broken $U(1)_{\text{PQ}}$ implies the appearance of the axion a. The interactions of KSVZ axion with the gauge bosons arises, as we have

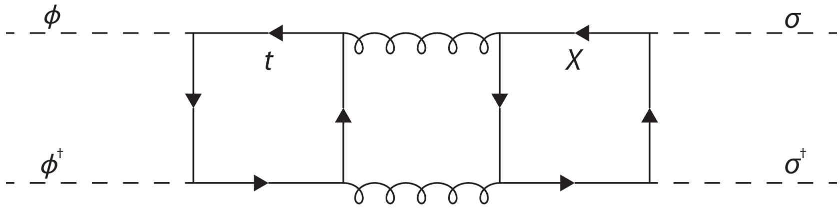

Figure 6.1: The first non-zero diagram contributing to the coupling $(\phi^\dagger\phi)(\sigma^\dagger\sigma)$ is the three-loop diagram depicted in figure.

anticipated, at a loop-level as a result of the chiral anomaly. The coupling is induced, as usual, by transferring the axion dependence from the σ field to the X field via

$$X \to e^{-i\frac{a}{2f_a}\gamma_5} X \tag{6.40}$$

and the anomaly then reads

$$\mathcal{L}_{\text{eff}} = -\frac{a}{f}\left[\frac{\alpha_s^2}{8\pi} G_{\mu\nu a}\tilde{G}^{\mu\nu a} + 3\frac{\alpha_{\text{em}}^2}{4\pi} q_X^2 F_{\mu\nu}\tilde{F}^{\mu\nu}\right]. \tag{6.41}$$

Through the anomaly the axions couple with the quarks and leptons, as well.

We now open a parenthesis on the parameter $\lambda_{\phi\sigma}$. In the original paper [12], [13], the authors decide to set $\lambda_{\phi\sigma} = 0$. This choice from the point of view of the axion physics, as pointed out also by the authors, does not change anything because the coupling $\lambda_{\phi\sigma}$ involves the radial parts of the scalar fields. However, as this term has dimension 4, there is nothing that forbids his appearance at loop level, and, in fact, it is easy to show that it shows up. The point is that this term, as we will see in the next chapter, pushes the lower scale of the theory (the electroweak VEV v), near the upper scale (the scale of the axion f_a), unless we fine-tune the parameter $\lambda_{\phi\sigma}$. However, it turns out that, at least in this case, $\lambda_{\phi\sigma} = 0$ is in a certain sense protected by large quantum corrections because of the fact that the heavy and the SM fermionic sectors are decoupled[2]. In fact, the first non-zero diagram contributing to the $\lambda_{\phi\sigma} = 0$ coupling is the three-loop diagram depicted in Fig. 6.1.

We can estimate the contribution given by the heavy fermions by looking at the right part of the diagram, depicted in Fig. 6.2. Taking for simplicity on shell gluons, y_σ real and $M \gg p^i$ where p^i are the external momenta, the amplitude of the diagram 6.2 at the first order in the external momenta reads

$$\mathcal{M} = \frac{5}{48\pi^2}\frac{g_s^2 y_\sigma^2}{M^2}\left[(p_3\cdot p_4)\eta^{\mu\nu} - p_3^\nu p_4^\nu\right]\delta^{ab}\epsilon_\mu^a(p_3)\epsilon_\nu^b(p_4) \tag{6.42}$$

[2]Independently from this discussion we remark that, as the $\lambda_{\phi\sigma}$ term is renormalizable, a full renormalizable Lagrangian should contain it.

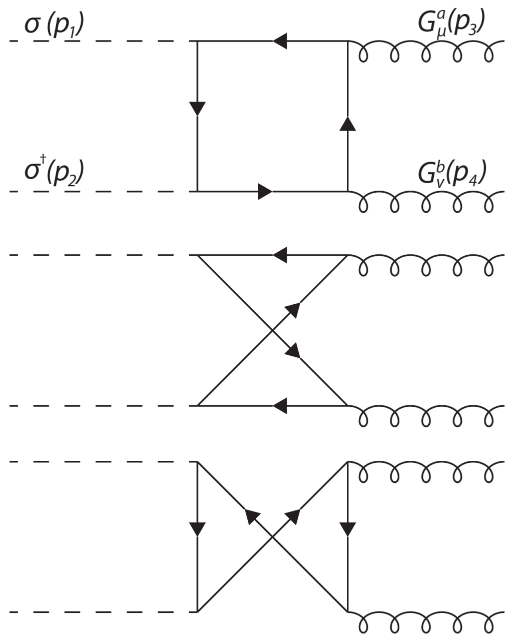

Figure 6.2: Three of the six topological independent diagrams that contribute to $\lambda_{\phi\sigma}$. The other three graphs are the same with the fermion lines inverted.

where M is the heavy fermion mass arising after the SSB of the σ particle and we have set for simplicity $p_2 = 0$. We see that, for large M, the contribution to $\lambda_{\phi\sigma}$ arising at quantum level is negligible. The Lorentz and group structure of the amplitude (and therefore also the $1/M^2$ coefficient for dimensional analysis) could be easily predicted by noticing that the only dimension-6 gauge invariant operator containing the scalar field σ and the gluons is

$$\mathcal{L}_{\text{eff}} = c_1 \sigma^\dagger \sigma G^a_{\mu\nu} G^{\mu\nu a} \tag{6.43}$$

that is nothing but the configuration-space version of (6.42).

6.2.2 DFSZ model

In this class of models, the interactions of the axions with quarks and leptons arises already at tree-level. To achieve this, they must carry $U(1)_{PQ}$ charge, so one again needs two Higgs fields Φ_1 and Φ_2 that are coupled, via Yukawa interaction, to down and up quarks, respectively. The model was originally proposed by Dine, Fishler, Srednicky, Zhithnitsky [14], [15]. They added a complex field σ, coupled only to Φ_1 and Φ_2, which carries the most part of the axion field, in the sense explained at the beginning of the section. For this, even if leptons and quarks are coupled via Yukawa interaction to the axion, they feel the effects of the axions mainly through the interaction that the σ field has with Φ_1 and Φ_2. The Yukawa coupling is the same as the PQWW model

$$\mathcal{L}_{\text{Yukawa}} = \Gamma^u \bar{Q}_L \Phi_1 d_R + \Gamma^d \bar{Q}_L \Phi_2 u_R + \Gamma^l \bar{l}_L \Phi_1 e_R + \text{h.c.} \tag{6.44}$$

The Lagrangian possesses four independent $U(1)$ symmetries that can be associated to the SM $U(1)_Y$, the $U(1)_{\text{PQ}}$ chiral symmetry, the baryon number $U(1)_B$ and the lepton number $U(1)_L$. The scalar potential has to be chosen so that it respects these symmetries, i.e. it has to be chosen in order to have the $U(1)_{\text{PQ}}$ and, obviously, the custodial $SO(4)$ symmetry. With these requests, the most general scalar potential containing Φ_1, Φ_2, σ is

$$\begin{aligned}\mathcal{V}(\Phi_1, \Phi_2, \sigma) &= \lambda_1(|\Phi_1|^2 - v_1^2)^2 + \lambda_2(|\Phi_2|^2 - v_2^2)^2 + \lambda(|\sigma|^2 - f_a^2)^2 \\ &+ (a|\Phi_1|^2 + b|\Phi_2|^2)|\sigma|^2 + c(\Phi_1^T \tau_2 \Phi_2 \sigma^2 + h.c.) + d|\Phi_1^T \tau_2 \Phi_2|^2 + e|\Phi_1^\dagger \Phi_2|^2\end{aligned} \tag{6.45}$$

where the $SU(2)_L$ indices are understood. The only term not obviously invariant under $SU(2)_L$ is $\Phi_1^T \tau_2 \Phi_2$. However,

$$\Phi_1^T \tau_2 \Phi_2 \to \Phi_1^T U^T \tau_2 U \Phi_2 = \Phi_1^T \tau_2 \tau_2 U^T \tau_2 U \Phi_2 = \Phi_1^T \tau_2 U^\dagger U \Phi_2 = \Phi_1^T \tau_2 \Phi_2 \tag{6.46}$$

where we have used the fact that $\tau_2 U \tau_2 = U^*$. We notice that, setting $c = 0$, we would have three independent $U(1)$: the c-term serves to reduce the number from three to two. The $U(1)_{\text{PQ}}$ on the scalar fields reads

$$\Phi_1 \to e^{i\alpha X_1}\Phi_1, \qquad \Phi_2 \to e^{i\alpha X_2}\Phi_2, \qquad \sigma \to e^{i\alpha X_\sigma}\sigma. \tag{6.47}$$

The interaction term $\Phi_1^T \tau_2 \Phi_2 \sigma^2$ gives a condition on the $U(1)_{\text{PQ}}$ charges:

$$X_1 + X_2 + 2X_\sigma = 0. \tag{6.48}$$

Now we want to find the axion content in the σ field. We will proceed as in the PQWW model. Firstly, we need to explicit the neutral angular parts of the Higgses and the angular part of the complex field σ:

$$\begin{aligned}\Phi_1^0 &= \frac{1}{\sqrt{2}}(v_1 + \rho_1) e^{i\frac{P_1}{v_1}}, \\ \Phi_2^0 &= \frac{1}{\sqrt{2}}(v_2 + \rho_2) e^{i\frac{P_2}{v_2}}, \\ \sigma &= \frac{1}{\sqrt{2}}(f + \rho) e^{i\frac{P_\sigma}{f}}.\end{aligned} \tag{6.49}$$

Now, what we have to do is to apply a rotation in the field space

$$\begin{pmatrix} P_1 \\ P_2 \\ P_\sigma \end{pmatrix} = \begin{bmatrix} \cos\theta\cos\gamma & -\sin\theta & -\sin\gamma\cos\theta \\ \sin\theta\cos\gamma & \cos\theta & -\sin\theta\sin\gamma \\ \sin\gamma & 0 & \cos\gamma \end{bmatrix} \begin{pmatrix} a \\ z \\ z' \end{pmatrix} \tag{6.50}$$

where, in the same notation used for the PQWW model, z is the NGB eaten by the Z boson, a is the axion and z' is the remaining field that will not partecipate in the subsequent discussion. The matrix in (6.50) is a composition of two rotations: one in the (P_1, P_2) plane, the other in the (P_1, P_σ) plane. The reason is that the only fields coupled to the Z boson are Φ_1 and Φ_2, so that a linear combination of them is eaten by the Z. On the other hand, the axion will arise as a linear combination of the PQWW axion and the phase of the σ field. Indeed θ is the same as the PQWW model so z takes the same form too:

$$z = -\frac{v_1 P_1 + v_2 P_2}{v}. \tag{6.51}$$

This fixes[3] the charges in (6.47)

$$\begin{aligned} X_1 &= \frac{v_2^2}{v^2} \\ X_2 &= \frac{v_1^2}{v^2} \\ X_\sigma &= -\frac{1}{2} \end{aligned} \tag{6.52}$$

where $v = \sqrt{v_1^2 + v_2^2}$. In order to determine the γ angle we need to impose that a shifts under $U(1)_{\text{PQ}}$, that is a is the NGB of the spontaneously broken $U(1)_{\text{PQ}}$: $a \to a + \beta\lambda$. We find

$$\begin{aligned} \sin\gamma &= -\frac{v f_a}{\sqrt{v^2 f_a^2 + 4 v_1^2 v_2^2}} \\ \cos\gamma &= \frac{2 v_1 v_2}{\sqrt{v^2 f_a^2 + 4 v_1^2 v_2^2}}. \end{aligned} \tag{6.53}$$

The axion field is then

$$a = P_1 \cos\theta \cos\gamma + P_2 \sin\theta \sin\gamma + P_\sigma \sin\gamma = \frac{2 v_1 v_2 (v_1 P_2 + v_2 P_1) - v^2 f_a P_\sigma}{v\sqrt{v^2 f_a^2 + 4 v_1^2 v_2^2}}, \tag{6.54}$$

so in this case the g-factor reads

$$g(v_1, v_2, f_a) = -\frac{v}{\sqrt{v^2 + 4\frac{v_1^2 v_2^2}{f_a^2}}}. \tag{6.55}$$

In the limit $f \gg v_1, v_2$

$$a = -P_\sigma + \frac{2 v_1 v_2}{f_a v^2}(v_1 P_2 + v_2 P_1) \tag{6.56}$$

and $|g(v^i, f_a)| = 1$ i.e. the axion is primarily composed of the σ field, as it is appropriate for invisible axion models.

[3] Actually, as in the PQWW axion, the model fixes only the ratio, and we are free to make a choice: here we make the same choice as the PQWW model, that is $X_1 + X_2 = 1$.

Chapter 7

The Minimal Axion Minimal Linear σ Model

The MLσM studied in the fourth chapter can be considered an optimal framework where to look for a solution to the strong CP problem. The important point is that we can rely on the heavy fermion sector, introduced in the context of composite Higgs models in order to explain the Yukawa hierarchy, to rotate away the QCD θ term. Indeed, extending the scalar spectrum with an additional complex scalar field, singlet under the global $SO(5)$, the symmetry content of the model is supplemented by an extra $U(1)_{\rm PQ}$ symmetry, that is, in a certain region of the parameter space, spontaneously broken, leading to a further Nambu-Goldstone boson. Depending on the scale of the spontaneous symmetry breaking, the NGB can be associated to a KSVZ axion, or a more massive ALP. The particle content of the model is therefore the same as the MLσM plus a scalar complex field s that we will write in the usual exponential form:

$$s \equiv \frac{r}{\sqrt{2}} e^{ia/f_a} \tag{7.1}$$

where f_a is the scale of the SSB of the axion. We will follow the paper [17] for the analysis of scalar and fermionic sector. The complete renormalizable Lagrangian for the axion minimal linear σ model (AMLσM) can be written as the sum of three terms describing respectively the pure gauge, fermionic and scalar sector:

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_f + \mathcal{L}_s. \tag{7.2}$$

The gauge Lagrangian is the same as the SM:

$$\mathcal{L}_g = -\frac{1}{4} G^{\mu\nu a} G^a_{\mu\nu} - \frac{1}{4} W^{\mu\nu i} W^i_{\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} + \frac{\alpha_s}{8\pi} \theta G^{\mu\nu a} \tilde{G}^a_{\mu\nu} \tag{7.3}$$

with the sum over $SU(3)_c$ and $SU(2)_L$ indices is understood. We postpone the discussion on the fermionic and scalar Lagrangian in the next sections.

7.1 The Fermionic Lagrangian

Assuming no couplings of s with the SM fermions, the most general fermion Lagrangian coupled to the scalar particle s is

$$\begin{aligned}\mathcal{L}_F =& \bar{q}_L i\not{D} q_L + \bar{t}_R i\not{D} t_R + \bar{b}_R i\not{D} b_R \\ &+ \bar{\psi}^{(2/3)}\left(i\not{D} - M_5\right)\psi^{(2/3)} + \bar{\psi}^{(-1/3)}\left(i\not{D} - M_5^{'}\right)\psi^{(-1/3)} \\ &+ \bar{\chi}^{(2/3)}\left(i\not{D} - M_1\right)\chi^{(2/3)} + \bar{\chi}^{(-1/3)}\left(i\not{D} - M_1^{'}\right)\chi^{(-1/3)} \\ &- \Big[y_1\bar{\psi}_L^{(2/3)}\phi\chi_R^{(2/3)} + y_2\bar{\psi}_R^{(2/3)}\phi\chi_L^{(2/3)} + \\ &+ y_1^{'}\bar{\psi}_L^{(-1/3)}\phi\chi_R^{(-1/3)} + y_2^{'}\bar{\psi}_R^{(-1/3)}\phi\chi_L^{(-1/3)} \\ &+ \Lambda_1\left(\bar{q}_L\Delta_{2\times 5}^{(2/3)}\right)\psi_R^{(2/3)} + \Lambda_2\bar{\psi}_L^{2/3}\left(\Delta_{5\times 1}^{(2/3)}t_R\right) + \Lambda_3\bar{\chi}_L^{(2/3)}t_R \\ &+ \Lambda_1^{'}\left(\bar{q}_L\Delta_{2\times 5}^{(-1/3)}\right)\psi_R^{(-1/3)} + \Lambda_2^{'}\bar{\psi}_L^{(-1/3)}\left(\Delta_{5\times 1}^{(-1/3)}b_R\right) + \\ &+ \Lambda_3^{'}\bar{\chi}_L^{(-1/3)}b_R + h.c.\Big] + \mathcal{L}_{Fs}\end{aligned} \tag{7.4}$$

with

$$\begin{aligned}\mathcal{L}_{Fs} =& -\Big[z_1\bar{\chi}_R^{(2/3)}\chi_L^{(2/3)}s + \tilde{z}_1\bar{\chi}_R^{(2/3)}\chi_L^{(2/3)}s^* + z_5\bar{\psi}_R^{(2/3)}\psi_L^{(2/3)}s + \\ &+ \tilde{z}_5\bar{\psi}_R^{(2/3)}\psi_L^{(2/3)}s^* + z_1^{\prime}\bar{\chi}_R^{(-1/3)}\chi_L^{(-1/3)}s + \tilde{z}_1^{\prime}\bar{\chi}_R^{(-1/3)}\chi_L^{(-1/3)}s^* + \\ &+ z_5^{\prime}\bar{\psi}_R^{(-1/3)}\psi_L^{(-1/3)}s + \tilde{z}_5^{\prime}\bar{\psi}_R^{(-1/3)}\psi_L^{(-1/3)}s^*\Big].\end{aligned} \tag{7.5}$$

As we have outlined, this is the most general Lagrangian invariant under $SO(5)\times U(1)_x$[1]. However, as it will be clear after the discussion on the Coleman-Weinberg potential, the Lagrangian (7.4) does not allow a further $U(1)_{PQ}$ symmetry, so, if we want to insist in the interpretation of (the angular part of) s as an axion we have to restore the $U(1)_{PQ}$ symmetry by setting the $U(1)_{PQ}$ breaking terms to zero, after making a choice of PQ charges.

In complete analogy with the MLσM, we can rewrite the fermionic mass term as

$$\mathcal{L}_{mass} = -\bar{\Psi}_L \mathcal{M}(h,\sigma,s)\Psi_R \tag{7.6}$$

where Ψ is the fermionic multiplet defined in the MLσM and $\mathcal{M}(h,\sigma,s)$ is the block diagonal 14×14 matrix:

$$\begin{aligned}&\mathcal{M}(h,\sigma,s) = \mathrm{diag}\left(M_5(s), \mathcal{M}^{\mathcal{T}}(h,\sigma,s), \mathcal{M}^{\mathcal{B}}(h,\sigma,s)\, M_5^{\prime}(s)\right), \\ &\mathcal{M}^{\mathcal{B}}(h,\sigma,s) = \mathcal{M}^{\mathcal{T}}(h,\sigma,s) \quad \text{with} \quad (y_i,\Lambda_i,M_i)\leftrightarrow(y_i^{\prime},\Lambda_i^{\prime},M_i^{\prime}),\end{aligned} \tag{7.7}$$

[1] As the scalar field s is an $SO(5)\times U(1)_x$ singlet, we can define the $U(1)_x$ charges in the same way as the MLσM

$$\mathcal{M}^{\mathcal{T}} = \begin{pmatrix} 0 & \Lambda_1 & 0 & 0 & 0 & \Lambda'_1 \\ 0 & M_5(s) & 0 & 0 & y_1\frac{h}{\sqrt{2}} & 0 \\ 0 & 0 & M_5(s) & 0 & y_1\frac{h}{\sqrt{2}} & 0 \\ \Lambda_2 & 0 & 0 & M_5(s) & y_1\sigma & 0 \\ \Lambda_3 & y_2\frac{h}{\sqrt{2}} & y_2\frac{h}{\sqrt{2}} & y_2\sigma & M_1(s) & 0 \\ 0 & 0 & 0 & 0 & 0 & M'_5(s) \end{pmatrix} \tag{7.8}$$

with

$$M_1(s) = M_1 + z_1 s + \tilde{z}_1 s^*, \qquad M_5(s) = M_5 + z_5 s + \tilde{z}_5 s^* \tag{7.9}$$

We notice that the mass matrix (7.8) is in form identical to the mass matrix (4.64), with the difference that now the mass term of the heavy fermions receives a contribution from the coupling with the complex field s. Exactly as in the MLσM, the mixing between the various species will provide a natural explanation for the lightness of the SM quark masses.

7.2 The Scalar Lagrangian and the CW Potential

The scalar part of the Lagrangian contained in (7.2) describing scalar-gauge and scalar-scalar interactions reads

$$\mathcal{L}_s = (D_\mu\phi)^T (D^\mu\phi) + \partial_\mu s^* \partial^\mu s - V(\phi, s) \tag{7.10}$$

with

$$V(\phi, s) = V^{\text{SSB}}(\phi, s) + V^{\text{CW}}(\phi, s) \tag{7.11}$$

where V^{CW} is the Coleman-Weinberg potential, that we are going to calculate in a moment, and V^{SSB} is the most general potential with the symmetry breaking pattern $(SO(5) \times U(1)_{\text{PQ}})/SO(4)$:

$$V^{\text{SSB}} = \lambda\left(\phi^T\phi - f^2\right)^2 + \lambda_s\left(2s^*s - f_s^2\right)^2 - 2\lambda_{s\phi}\left(s^*s\right)\left(\phi^T\phi\right). \tag{7.12}$$

In the unitary gauge, with the same notation as the MLσM, the Lagrangian reads

$$\begin{aligned}\mathcal{L}_s = &\frac{1}{2}\partial_\mu h\partial^\mu h + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{h^2}{4}\left[g^2 W_\mu^+ W^{\mu,-} + \frac{g^2+g'^2}{2}Z_\mu Z^\mu\right] + \frac{1}{2}\partial_\mu r\partial^\mu r \\ &+ \frac{r^2}{2f_a^2}\partial_\mu a\partial^\mu a - \lambda\left(h^2+\sigma^2-f^2\right)^2 - \lambda_s\left(r^2-f_s^2\right)^2 + \lambda_{s\phi}r^2\left(h^2+\sigma^2\right).\end{aligned} \tag{7.13}$$

The Coleman-Weinberg potential has the same form as the MLσM, the only difference being the fermion masses. We insist on the fact that, as the full Lagrangian (7.4) does not allow an $U(1)_{\text{PQ}}$ symmetry, we expect that the CW potential breaks it explicitly - since it is not a symmetry. However, the idea is to keep all the terms in order to make the discussion as much general as possible and only in the end to set some of them to zero in order to restore the $U(1)_{\rm PQ}$ symmetry. We will only show the results, as the calculation is the same as (4.75) with the difference that now $M_{1,5}$ are complex and depend on the field s. The quadratic divergence reads

$$\begin{aligned}\mathrm{Tr}\left[\mathcal{M}\mathcal{M}^\dagger\right] =&M_1(s)M_1^*(s)+5M_5(s)M_5^*(s)+2\Lambda_1^2+\Lambda_2^2+\Lambda_3^2+\\&+\left(y_1^2+y_2^2\right)\phi^T\phi+\{\}\leftrightarrow\{\}'\end{aligned} \tag{7.14}$$

where $\{\}\leftrightarrow\{\}'$ means that the same contributions arise from the bottom sector and they are obtained as usual by substituting the unprimed coupling with the corresponding primed ones. Now, in general $M_i(s)M_i^*(s)$ produces $U(1)_{\rm PQ}$ breaking terms but, as we have said, once we choose specific $U(1)_{\rm PQ}$ charges, they disappear. This is true for all the $U(1)_{\rm PQ}$ breaking terms that arise at a quantum level so we will forget about them, as we already know that they will cancel out. On the other hand, the logarithmic divergence reads

$$\begin{aligned}\mathrm{Tr}\left[\left(\mathcal{M}\mathcal{M}^\dagger\right)^2\right] =&\, d_0+d_1(s^*s)+d_2\left(\phi^T\phi\right)+d_3(s^*s)^2+d_4\left(\phi^T\phi\right)^2+\\&+d_5\left(\phi^T\phi\right)(s^*s)+\tilde{d}_1\sigma+\tilde{d}_2h^2.\end{aligned} \tag{7.15}$$

This calculation will serve to understand the counterterms needed to absorb the divergences so we show below the only divergent terms:

$$\begin{aligned}\tilde{d}_1&=4\left(y_1M_1+y_5M_5\right)+\{\}\leftrightarrow\{\}'\\ \tilde{d}_2&=y_2^2\Lambda_1^2-2y_1^2\Lambda_2^2+\{\}\leftrightarrow\{\}'.\end{aligned} \tag{7.16}$$

We can now understand the reason why we have taken all the terms: for a certain choice of parameters[2] the sigma tadpole cancels out and we remain with a minimal (in terms of numbers of parameters) model.

Now we turn to the CW potential generated by gauge bosons loops. In fact in this theory, differently from the MLσM, we cannot neglect the soft breaking terms arising from them. The reason is that, in order to give a consistent mass to the Higgs, we need at least two soft breaking terms and the fermionic sector provides only one term. The gauge boson mass term, in matrix notation, reads

$$\begin{aligned}&\begin{pmatrix}W_\mu^+ & W_\mu^- & Z_\mu\end{pmatrix}\mathcal{M}_g^2\begin{pmatrix}W^{\mu+}\\W^{\mu-}\\Z^\mu\end{pmatrix}\equiv\\ &\equiv\begin{pmatrix}W_\mu^+ & W_\mu^- & Z_\mu\end{pmatrix}\begin{pmatrix}0 & \frac{h^2g^2}{8} & 0\\ \frac{h^2g^2}{8} & 0 & 0\\ 0 & 0 & \frac{h^2(g^2+g'^2)}{8}\end{pmatrix}\begin{pmatrix}W^{\mu+}\\W^{\mu-}\\Z^\mu\end{pmatrix},\end{aligned} \tag{7.17}$$

[2]For instance taking $M_1=M_5=M_1'=M_5'=0$.

therefore

$$\begin{aligned}\mathrm{Tr}[\mathcal{M}_g^2] &= \frac{(g^2+g'^2)}{8}h^2 \\ \mathrm{Tr}[(\mathcal{M}_g^2)^2] &= \frac{1}{64}\left[2g^4+(g^2+g'^2)^2\right]h^4\end{aligned} \tag{7.18}$$

and both break explicitly the $SO(5)$ symmetry.

Summarising, once a specific PQ charge assignment is assumed, provided of course that it respects the $U(1)_{\mathrm{PQ}}$ symmetry, all the Coleman-Weinberg $U(1)_{\mathrm{PQ}}$ breaking coefficients cancel and the only terms that we have to introduce in order to ensure the renormalizability of the model are σ, h^2, h^4. In particular we can choose the charges in such a way that $\tilde{d}_1$ disappear and only h^2, h^4 have to be added.

7.3 The $U(1)_{\mathrm{PQ}}$ Invariant Lagrangian and the Minimal Model

The Lagrangian (7.4) describes the most general couplings between heavy fermions and the complex scalar field s. However, as we have said, such a Lagrangian does not allow a $U(1)_{\mathrm{PQ}}$ symmetry. For instance, the terms[3]

$$M_5\bar{\psi}_R\psi_L^{(2/3)}, \qquad z_5\bar{\psi}_R^{(2/3)}\psi_L^{(2/3)}s, \qquad \tilde{z}_5\bar{\psi}_R^{(2/3)}\psi_L^{(2/3)}s^* \tag{7.19}$$

are not simultaneously invariant under $U(1)_{\mathrm{PQ}}$. In fact, if the fields transform under $U(1)_{\mathrm{PQ}}$ as

$$\psi_R^{(2/3)'} = e^{i\alpha n_R}\psi_R^{(2/3)}, \qquad \psi_L^{(2/3)'} = e^{i\alpha n_L}\psi_L^{(2/3)}, \qquad s' = e^{i\alpha n_s}s, \tag{7.20}$$

then, in order (7.19) to be invariant under (7.20), the following conditions should be satisfied

$$\begin{aligned}&-n_{\psi_R}+n_{\psi_L}=0,\\ &-n_{\psi_R}+n_{\psi_L}+n_s=0,\\ &-n_{\psi_R}+n_{\psi_L}-n_s=0.\end{aligned} \tag{7.21}$$

Assuming a non vanishing charge n_s, that is essential in order to solve the strong CP problem[4], these conditions cannot be simultaneously satisfied, in particular we cannot have a bare mass term for all the heavy fermions. In general there are a lot of charge assignments that allow a $U(1)_{\mathrm{PQ}}$ symmetry. However, these different possible choices can be restricted by relying on

[3] We will forget the superscripts 2/3 or -1/3 unless otherwise indicated. Moreover, we will show the calculation for $\psi^{(2/3)}$ but the above discussion clearly holds for all the species $\psi^{(-1/3)}$ $\chi^{(2/3)}$, $\chi^{(-1/3)}$

[4] n_s is nothing but the difference between the left and right charges of the heavy fermions; if $n_s = 0$, left and right fermions would transform with the same charge and there would not be any anomalies.

strongly physical conditions. Firstly, a $U(1)_{\rm PQ}$ symmetry requires that the following conditions must be simultaneously satisfied[5]:

$$
\begin{aligned}
-n_{\psi_L} + n_{\chi_R} &= 0 \quad \text{or} \quad y_1 = 0, \\
-n_{\psi_R} + n_{\chi_L} &= 0 \quad \text{or} \quad y_2 = 0, \\
-n_{\chi_R} + n_{\chi_L} &= 0 \quad \text{or} \quad M_1 = 0, \\
-n_{\chi_R} + n_{\chi_L} + n_s &= 0 \quad \text{or} \quad z_1 = 0, \\
-n_{\chi_R} + n_{\chi_L} - n_s &= 0 \quad \text{or} \quad \tilde{z}_1 = 0, \\
-n_{\psi_R} + n_{\psi_L} &= 0 \quad \text{or} \quad M_5 = 0, \\
-n_{\psi_R} + n_{\psi_L} + n_s &= 0 \quad \text{or} \quad z_5 = 0, \\
-n_{\psi_R} + n_{\psi_L} - n_s &= 0 \quad \text{or} \quad \tilde{z}_5 = 0, \\
n_{\psi_R} &= 0 \quad \text{or} \quad \Lambda_1 = 0, \\
n_{\psi_L} &= 0 \quad \text{or} \quad \Lambda_2 = 0, \\
n_{\chi_L} &= 0 \quad \text{or} \quad \Lambda_3 = 0
\end{aligned}
\tag{7.22}
$$

where we have chosen the charge of the SM fermions (the reason is explained below) and the ϕ field containing the Higgs field to be zero and the "or" means that either the interaction conserve the $U(1)_{\rm PQ}$ symmetry or the corresponding parameter must be put to zero. We can rely on the following physical conditions to restrict the possible choices of charges:

- the partial fermion compositeness states that the mass terms for the SM quarks have to originate at tree level. Since, generalizing to this model the result obtained in the MLσM, the top mass reads

$$m_t = y_1 \Lambda_1 \frac{\Lambda_3 - y_2 \Lambda_2 \sigma / M_5(s)}{M_1(s) M_5(s) - y_1 y_2 (h^2 + \sigma^2)} \frac{h}{\sqrt{2}}, \tag{7.23}$$

  this condition implies that we must have $y_1 \neq 0$, $\Lambda_1 \neq 0$ and either $\Lambda_3 \neq 0$ or $y_2 \neq 0 \wedge \Lambda_2 \neq 0$;

- the axion has to be coupled to at least one of the heavy fermions in order to solve the strong CP problem, and this translates in the fact that at least one among M_1 and M_5 must be zero;

- if we want to forbid the axion coupling to the SM fermions then they have to be necessarily PQ singlets.

By fixing[6] $n_s = 1$, the minimal (in terms of number of parameters) scenario with the requirements above is with the choice of charges shown in the table 7.3. The table shows two equivalent choices of $U(1)_{\rm PQ}$ charges that deliver the same physics. Definitely in this specific set up the terms

[5] We show the calculations only for the top sector, being the calculation for the bottom the same.

[6] the model leaves the freedom to fix one the the charges.

n_{q_L}	n_{t_R}	n_{ψ_L}	n_{ψ_R}	n_{χ_L}	n_{χ_R}
0	0	+1	0	0	+1
0	0	−1	0	0	−1

Table 7.1: $U(1)_{\rm PQ}$ charge assignment used in the model that minimizes the number of parameters. The assignment can be trivially extended to the bottom sector.

not allowed by symmetries (and therefore required to be set to zero) are Λ_2, M_1, M_5 and, depending on which choice between $n_{\psi_L} = n_{\chi_R} = \pm 1$, either $\tilde{z}_1, z_5$ or $z_1, \tilde{z}_5$ are not allowed, respectively. As expected, all the divergent terms that provide an explicit breaking of the $U(1)_{\rm PQ}$ symmetry are now zero. Moreover, with this specific choice we have

$$\begin{aligned} \tilde{d}_1 &= 0, \\ \tilde{d}_2 &= y_2^2 \Lambda_1^2. \end{aligned} \tag{7.24}$$

We see that no sigma tadpole contribution is generated and the only divergent terms are h^2 and h^4: these are all the only terms that we need to add in the tree-level Lagrangian in order to absorb the divergences generated by the one-loop CW potential. We conclude this section with two comments:

- in calculating the charge of heavy fermions we assigned charge $n_s = 1$ to the complex scalar field. However, from the point of view of the scalar and fermionic sector, it does not matter what the value of n_s is. Instead n_s becomes relevant in the coupling of the axion with the heavy fermions. The explicit calculation is shown in the next section.

- as we have said, this is the charge assignment that minimizes the number of parameters. In fact, if we take for instance the SM fermions charged under $U(1)_{\rm PQ}$ then it is possible to redefine the charges so that M_1 (or M_5) and Λ_2 are now allowed. This would bring to the appearance of the $\tilde{d}_1$ in the one-loop CW potential, so it would be needed a further σ tadpole term in the tree-level Lagrangian.

7.4 Axion Coupling to Gauge Bosons and Fermions

The aim of this section is to calculate the axion coupling to gauge bosons, through the chiral anomaly, and to heavy fermions, through the chiral current. We will use the results of the section 5.1. As already done for the PQWW model, we can make a redefinition of the fields by inserting the axion in the phase of the PQ charged fields:

$$
\begin{aligned}
s &\to e^{i n_s a/f_a} s, \\
\psi_L^{(2/3)} &\to e^{i n_{\psi_L} a/f_a} \psi_L^{(2/3)}, \\
\chi_R^{(2/3)} &\to e^{i n_{\chi_R} a/f_a} \chi_R^{(2/3)}, \\
\psi_L^{(-1/3)} &\to e^{i n_{\psi'_L} a/f_a} \psi_L^{(-1/3)}, \\
\chi_R^{(-1/3)} &\to e^{i n_{\chi'_R} a/f_a} \chi_R^{(-1/3)}.
\end{aligned}
\tag{7.25}
$$

In order to calculate the couplings, we have to remember that the fields live in multiplets of $SO(5) \times SU(3)_c$. So, for instance in the case of the coupling of the fermions to $G^a_{\mu\nu}$ we have to take into account that there are 5 copies of ψ_L, ψ'_L, 1 copy of χ_R, χ'_R, etc. Generalizing the results of (5.19), (5.22), (5.23), (5.24) the coupling of the axion to gauge bosons reads

$$
\mathcal{L}_{a,gb} = -\frac{\alpha_s}{8\pi}\frac{a}{f_a} c_{agg} G^a_{\mu\nu}\tilde{G}^{\mu\nu a} - \frac{\alpha_2}{8\pi}\frac{a}{f_a} c_{aW^iW^i} W^i_{\mu\nu}\tilde{W}^{\mu\nu i} - \frac{\alpha_1}{8\pi}\frac{a}{f_a} c_{aBB} B_{\mu\nu}\tilde{B}^{\mu\nu}
\tag{7.26}
$$

where

$$
\begin{aligned}
&c_{agg} = 5\left(n_{\psi_L} + n_{\psi'_L}\right) - \left(n_{\chi_R} + n_{\chi'_R}\right), \\
&c_{aW^iW^i} = 6\left(n_{\psi_L} + n_{\psi'_L}\right), \\
&c_{aBB} = 2\Big[3n_{\psi_L}\left(2Y_X^2 + 2Y_Q^2 + Y_{T_5}^2\right) + 3n_{\psi'_L}\left(2Y_{X'}^2 + 2Y_{Q'}^2 + Y_{B_5}^2\right) + \\
&-3n_{\chi_R} Y_{T_1}^2 - 3n_{\chi'_R} Y_{B_1}^2\Big].
\end{aligned}
\tag{7.27}
$$

The 5 factor in c_{agg} arises because, as we have said, there are 5 copies of ψ_L, ψ'_L; the 6 factor in $c_{aW^iW^i}$ arises because there are 3 copies ($SU(3)_c$ triplet) times 2 copies (ψ_L contains two different $SU(2)_L$ doublets); the 3 factors in c_{aBB} are due to the fact that we have to consider the 3 copies of the $SU(3)_c$ triplet ψ_L, and (Y, X, T_5) is the usual decomposition of $SO(5)$ fiveplet under $SU(2)_L \times U(1)_Y$, so there are 2 copies of Y_X, two of Y_Q, etc. In order to read the axion coupling to the physical gauge field bosons we have to perform the Weinberg rotation

$$
\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & \sin\theta_w \\ -\sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}.
\tag{7.28}
$$

c_{agg}	$c_{a\gamma\gamma}$	c_{aZZ}	c_{aWW}	$c_{a\gamma Z}$
8	112/3	49.3	108.1	17.8

Table 7.2: Numerical coefficients c_{aii}.

Now, for every a_1, a_2 we have[7]

$$\begin{aligned}
&a_1 g^2 W^3_{\mu\nu}\tilde{W}^{\mu\nu 3} + a_2 g'^2 B_{\mu\nu}\tilde{B}^{\mu\nu} = \\
&= a_1 g^2 \left(\cos^2\theta_w Z_{\mu\nu}\tilde{Z}^{\mu\nu} + \sin^2\theta_w F_{\mu\nu}\tilde{F}^{\mu\nu} + 2\cos\theta_w \sin\theta_w A_{\mu\nu}\tilde{Z}^{\mu\nu}\right) + \\
&+ a_2 g'^2 \left(\cos^2\theta_w F_{\mu\nu}\tilde{F}^{\mu\nu} + \sin^2\theta_w Z_{\mu\nu}\tilde{Z}^{\mu\nu} - 2\cos\theta_w \sin\theta_w A_{\mu\nu}\tilde{Z}^{\mu\nu}\right).
\end{aligned} \tag{7.29}$$

By using the relation $e = g\sin\theta_w = g'\cos\theta_w$, (7.29) reads

$$\begin{aligned}
&a_1 g^2 W^3_{\mu\nu}\tilde{W}^{\mu\nu 3} + a_2 g'^2 B_{\mu\nu}\tilde{B}^{\mu\nu} = \\
&= e^2\left(\frac{a_1}{\tan^2\theta_w} + a_2\tan^2\theta_w\right) Z_{\mu\nu}\tilde{Z}^{\mu\nu} + e^2\left(a_1 + a_2\right) F_{\mu\nu}\tilde{F}^{\mu\nu} + \\
&+ 2e^2\left(\frac{a_1}{\tan\theta_w} - a_2\tan\theta_w\right) A_{\mu\nu}\tilde{Z}^{\mu\nu}.
\end{aligned} \tag{7.30}$$

Using this result, the axion couplings to gauge bosons are given by

$$\begin{aligned}
\mathcal{L}_{a,gb} = &-\frac{\alpha_s}{8\pi}\frac{a}{f_a} c_{agg} G^a_{\mu\nu}\tilde{G}^{\mu\nu a} - \frac{\alpha_{\text{em}}}{8\pi}\frac{a}{f_a} c_{aWW} W^+_{\mu\nu}\tilde{W}^{\mu\nu -} - \\
&-\frac{\alpha_{\text{em}}}{8\pi}\frac{a}{f_a} c_{a\gamma\gamma} F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{\alpha_{\text{em}}}{8\pi}\frac{a}{f_a} c_{aZZ} Z_{\mu\nu}\tilde{Z}^{\mu\nu} - \frac{\alpha_{\text{em}}}{8\pi}\frac{a}{f_a} c_{a\gamma Z} F_{\mu\nu}\tilde{Z}^{\mu\nu}
\end{aligned} \tag{7.31}$$

where

$$\begin{aligned}
c_{aW^+W^-} &= 2 c_{aW^iW^i}, \\
c_{a\gamma\gamma} &= c_{aW^3W^3} + c_{aBB}, \\
c_{aZZ} &= \frac{1}{\tan^2\theta_w} c_{aW^3W^3} + \tan^2\theta_w\, c_{aBB}, \\
c_{a\gamma Z} &= 2\frac{1}{\tan\theta_w} c_{aW^3W^3} - \tan\theta_w\, c_{aBB}.
\end{aligned} \tag{7.32}$$

The table 7.2 shows the numerical results of the couplings c_{aii} considering the charge assignments of the previous section.

[7] we are considering here only the pure kinetic part of $W^3_{\mu\nu}W^{\mu\nu 3}$ because we are interested to the coupling of the axion with two gauge bosons and the self-interaction between gauge bosons would give rise to a coupling of the axion with more than two gauge bosons

We now turn to the coupling with the heavy fermions. We show the calculation for one fermion species, as the generalization to the others is trivial[8]:

$$\mathcal{L}_{\psi_L} = +\bar{\psi}\left(i\not{D} - M_5\right)\psi - y_1\bar{\psi}_L\phi\chi_R - z_5\bar{\psi}_R\psi_L s - \tilde{z}_5\bar{\psi}_R\psi_L s^* + \text{h.c.} \tag{7.33}$$

Under (7.25) the Lagrangian gets modified as follows

$$\begin{aligned}\mathcal{L}_{\psi_L} \to &\mathcal{L}_{\psi_L} + n_{\psi_L}\frac{\partial_\mu a}{2f_a}\bar{\psi}\gamma^\mu\gamma^5\psi + in_{\psi_L}\frac{a}{f_a}M_5\bar{\psi}_L\psi_R + \\ &+ iy_1\left(n_{\psi_L} - n_{\chi_R}\right)\frac{a}{f_a}\bar{\psi}_L\phi\chi_R - iz_5\left(n_{\psi_L} + n_s\right)\frac{a}{f_a}\bar{\psi}_R\psi_L s \\ &- i\tilde{z}_5\left(n_{\psi_L} - n_s\right)\frac{a}{f_a}\bar{\psi}_R\psi_L s^* + h.c.\end{aligned} \tag{7.34}$$

where we have used the fact that

$$\begin{aligned}&-n_{\psi_L}\frac{\partial_\mu a}{f_a}\bar{\psi}_L\gamma^\mu\psi_L = n_{\psi_L}\frac{a}{f_a}\partial_\mu\left(\bar{\psi}_L\gamma^\mu\psi_L\right) = \\ &= -n_{\psi_L}\frac{a}{2f_a}\partial_\mu\left(\bar{\psi}\gamma^\mu\gamma^5\psi\right) = n_{\psi_L}\frac{\partial_\mu a}{2f_a}\partial_\mu\left(\bar{\psi}\gamma^\mu\gamma^5\psi\right).\end{aligned} \tag{7.35}$$

7.5 The Scalar Potential

The full scalar potential, as stated in the previous chapter, is given by

$$V(h,\sigma,r) = \lambda\left(h^2+\sigma^2-f^2\right)^2 + \lambda_s\left(r^2-f_s^2\right)^2 - \lambda_{s\phi}r^2\left(h^2+\sigma^2\right) - \beta f^2h^2 + \gamma h^4 \tag{7.36}$$

where $-\beta f^2h^2$ $+\gamma h^4$ are introduced to absorb the divergences arising from the one loop CW potential. The VEVs are found by solving the equations

$$\begin{cases}\dfrac{\partial V}{\partial h} = 4\lambda h\left(h^2+\sigma^2-f^2\right) - 2\beta f^2h + 4\gamma h^3 - 2\lambda_{s\phi}r^2h = 0 \\ \dfrac{\partial V}{\partial\sigma} = 4\lambda\sigma\left(h^2+\sigma^2-f^2\right) - 2\lambda_{s\phi}r^2\sigma = 0 \\ \dfrac{\partial V}{\partial r} = 4\lambda_s r\left(r^2-f_s^2\right) - 2\lambda_{s\phi}r\left(h^2+\sigma^2\right) = 0.\end{cases} \tag{7.37}$$

The result reads

$$\begin{cases}v_h^2 = \dfrac{\beta}{2\gamma}f^2 \\ v_\sigma^2 = \left(1-\dfrac{\lambda_{s\phi}^2}{4\lambda\lambda_s}\right)^{-1}\left[f^2\left(1-\dfrac{\beta}{2\gamma}+\dfrac{\beta}{2\gamma}\dfrac{\lambda_{s\phi}^2}{4\lambda\lambda_s}\right)+\dfrac{f_s^2}{2}\dfrac{\lambda_{s\phi}}{\lambda}\right] \\ v_r^2 = \left(1-\dfrac{\lambda_{s\phi}^2}{4\lambda\lambda_s}\right)^{-1}\left[f_s^2+\dfrac{f^2}{2}\dfrac{\lambda_{s\phi}}{\lambda_s}\right] \equiv f_a^2\end{cases} \tag{7.38}$$

[8] We will omit the superscript (2/3).

where with the last equation we mean that the scale f_a defined in (7.1) is exactly the VEV v_r. In this way the kinetic term arising after the SSB from the coupling

$$\frac{r^2}{2f_a^2}\partial_\mu a \partial^\mu a \tag{7.39}$$

is canonically normalized. Assuming all parameters non vanishing we need to take into account some physical constraints on the parameters. In particular:

- it is needed that $\lambda > 0$ and $\lambda_s > 0$, as it is appropriate for all the spontaneously symmetry breaking potentials, in order to have a potential bounded from below;
- the positivity of v_h^2, v_σ^2, v_r^2 implies that β and γ have to have the same sign and

$$\lambda_{s\phi}^2 < 4\lambda\lambda_s; \tag{7.40}$$

- by fixing the Higgs VEV $v_h = 246\,\text{GeV}$, we can extract, from the experimental bound on the ξ parameter $\xi = v^2/f^2 < 0.18$, a bound on the ratio β/γ:

$$\frac{\beta}{\gamma} = 2\xi < 0.36. \tag{7.41}$$

We can now write the Lagrangian in terms of the physical fields

$$\begin{cases} h = \hat{h} + v_h \\ \sigma = \hat{\sigma} + v_\sigma \\ r = \hat{r} + v_r \end{cases} \tag{7.42}$$

in order to read the mass eigenvalues and eigenstates. The general solution can be studied only numerically; on the other hand simple analytic expressions can be obtained in two specific frameworks:

- integrating out the heaviest scalar degree of freedom, that is the radial scalar field $\hat{r}$, and studying the LO terms of the Lagrangian;
- assuming $f_s \sim f$, expanding perturbatively in the small β, $\lambda_{s\phi}$ parameters.

We start from the integration of the heavy field r. For this purpose we notice that the mass of the heaviest scalar receives a leading order (LO) contribution proportional to:

$$m_3 \propto \sqrt{8\lambda_s} f_s. \tag{7.43}$$

We see that there are two ways to integrate out the $\hat{r}$ field that represent two physically different scenarios:

- we can integrate out by expanding $\hat{r}$ in powers of $\frac{1}{\lambda_s}$ assuming $\lambda_s >> 1$, that corresponds to assume an UV strong interacting regime. Moreover in this regime, as the Yukawa couplings with the exotic fermions do not depend on λ_s, the decoupling of the field $\hat{r}$ does not have any impact on the spectrum of the exotic fermions, that are still present in the low energy spectrum;
- we can also integrate out the field $\hat{r}$ by assuming $f_s \gg f$. In this second scenario, as the mass of the heavy fermions depends on f_s, the exotic fermion sector decouples at the same time as the heaviest scalar field.

Before analysing the two cases separately, it is useful to look at what happens in general when we integrate out the heavy field $\hat{r}$, and then to detail the discussion to the two specific cases. Integrating out the heavy field $\hat{r}$ leads, in the scalar sector, to an effective potential that, at the leading order in the appropriate expansion parameter, can be written as

$$V_R^{\rm LO} = \lambda_R \left(h^2 + \sigma^2 - f_R^2\right)^2 - \beta_R f_R^2 h^2 + \gamma h^4 \tag{7.44}$$

where

$$\lambda_R = k_\lambda \lambda, \qquad \beta_R = \frac{k_\lambda}{k_f}\beta, \qquad f_R^2 = \frac{k_f}{k_\lambda} f^2. \tag{7.45}$$

We notice that $\beta_R f_R^2 = \beta f^2$, that is the h^2 (and h^4 as well) term is not renormalized by the integration, as one expects since they do not depend on the field $\hat{r}$. The VEVs read

$$v_h^2 = \frac{\beta_R}{2\gamma} f_R^2, \qquad v_\sigma^2 = f_R^2 \left(1 - \frac{\beta_R}{2\gamma}\right). \tag{7.46}$$

Defining the physical fields

$$h = \hat{h} + v_h, \qquad \sigma = \hat{\sigma} + v_\sigma \tag{7.47}$$

we can extract from (7.44) the mass matrix

$$\begin{aligned} V_{\rm mass} &= \left(4\lambda_R v_h^2 + 4\gamma v_h^2\right) \hat{h}^2 + \left(4\lambda_R v_\sigma^2\right) \hat{\sigma}^2 + \left(8\lambda_R v_h v_\sigma\right) \hat{h}\hat{\sigma} = \\ &= \frac{1}{2} \begin{pmatrix} \hat{h} & \hat{\sigma} \end{pmatrix} \left[8\lambda_R \begin{pmatrix} (1+\gamma/\lambda_R)\, v_h^2 & v_h v_\sigma \\ v_h v_\sigma & v_\sigma^2 \end{pmatrix}\right] \begin{pmatrix} \hat{h} \\ \hat{\sigma} \end{pmatrix} \end{aligned} \tag{7.48}$$

where we have used $\beta_R f_R^2 = 2\gamma v_h^2$. The mass eigenstates are therefore

$$m_{1,2}^2 = 4\lambda_R \left[\left(1 + \frac{\gamma}{\lambda_R}\right) v_h^2 + v_\sigma^2 \pm \sqrt{\left(1 + \frac{\gamma}{\lambda_R}\right)^2 v_h^4 + 2\left(1 - \frac{\gamma}{\lambda_R}\right) v_h^2 v_\sigma^2 + v_\sigma^4}\right] \tag{7.49}$$

while the mixing angle is given by

$$\tan 2\theta = \frac{2 v_h v_\sigma}{v_\sigma^2 - (1+\gamma/\lambda_R)\, v_h^2}. \tag{7.50}$$

The positivity of the mass square eigenvalues is guaranteed by imposing that trace and determinant of the mass matrix are both positive. This, together with the fact that we must have $v_h^2 > 0$, leads to the conditions:

$$\lambda_R > 0 \qquad \gamma > 0 \qquad \beta_R > 0. \tag{7.51}$$

We can now explore in detail the two limits $\lambda_s \gg 1$ and $f_s \gg f$ separately. Starting by $\lambda_s \gg 1$, the appropriate expansion of the r field in terms of λ_s reads

$$r = r_0 + \frac{1}{\lambda_s} r_1 + o\left(\frac{1}{\lambda_s}\right). \tag{7.52}$$

where r_0 and r_1 are found by solving the equation of motion for $\hat{r}$ order by order in λ_s. The equation of motion is

$$\Box r + \frac{r}{f_s^2}\partial_\mu a \partial^\mu a - 4\lambda_s r\left(r^2 - f_s^2\right) + 2\lambda_{s\phi} r\left(h^2+\sigma^2\right) = 0. \tag{7.53}$$

Putting (7.52) in (7.53) we find

$$\begin{aligned}
\lambda_s^1: \qquad & r_0^2 = f_s^2, \\
\lambda_s^0: \qquad & \frac{1}{f_s}\partial_\mu a \partial^\mu a - 8 f_s^2 r_1 + 2\lambda_{s\phi} f_s\left(h^2+\sigma^2\right) = 0 \implies \\
& r_1 = \frac{1}{8 f_s^3}\partial_\mu a \partial^\mu a + \frac{\lambda_{s\phi}}{4 f_s}\left(h^2+\sigma^2\right).
\end{aligned} \tag{7.54}$$

Using (7.52) and (7.54), we can see that the LO potential is of the form (7.44) with

$$k_\lambda = 1 \qquad k_f = \left(1 + \frac{1}{2}\frac{\lambda_{s\phi}}{\lambda}\frac{f_s^2}{f^2}\right). \tag{7.55}$$

On the other hand, the next to leading order (NLO) Lagrangian reads

$$\delta\mathcal{L}_s = \frac{1}{\lambda_s}\left[\frac{1}{f_s} r_1 \partial_\mu a \partial^\mu a - 4 f_s^2 r_1^2 + 2\lambda_{s\phi} f_s r_1\left(h^2+\sigma^2\right)\right] = \frac{1}{\lambda_s} 4 f_s^2 r_1^2. \tag{7.56}$$

The positivity of f_R^2 translates into a constraint on the coupling $\lambda_{s\phi}$:

$$\lambda_{s\phi} > -2\lambda\frac{f^2}{f_s^2}. \tag{7.57}$$

In the limit $\lambda_s \gg 1$ the masses the mixing angle are

$$
\begin{aligned}
m_{1,2}^2 &= 4\lambda f^2 \left[k_f + \frac{\beta}{2\lambda} \pm \sqrt{k_f^2 - \frac{\beta}{\lambda} k_f + \frac{\beta^2}{\gamma\lambda} + \frac{\beta^2}{4\lambda^2}} \right], \\
\tan 2\theta &= \frac{\sqrt{2\beta\left(k_f - \frac{\beta}{2\gamma}\right)}}{\sqrt{\gamma\left(k_f - \frac{\beta}{\gamma} - \frac{\beta}{2\lambda}\right)}}
\end{aligned}
\tag{7.58}
$$

with k_f given by (7.55); in the limit for small β, these formulas reduce to

$$
\begin{aligned}
m_1^2 &= 4\beta f^2 \left(1 - \frac{\beta}{2\gamma}\right), \\
m_2^2 &= 8\lambda f^2 \left(1 + \frac{\beta^2}{4\gamma\lambda}\right) + 4\lambda_{s\phi} f_s^2, \\
\tan 2\theta &= \sqrt{\frac{2\beta}{\gamma}\left(1 - \frac{\beta}{2\gamma}\right)}.
\end{aligned}
\tag{7.59}
$$

We now turn to the other relevant limit, $f_s \gg f$. In order to perform the calculation, it is useful define the dimensionless variable $\tilde{r}$

$$
r = f_s \tilde{r}. \tag{7.60}
$$

In terms of $\tilde{r}$ the scalar Lagrangian reads

$$
\mathcal{L} \sim \frac{f_s^2}{2} \partial_\mu \tilde{r} \partial^\mu \tilde{r} + \frac{\tilde{r}^2}{2} \partial_\mu a \partial^\mu a - V(h, \sigma, r) \tag{7.61}
$$

and

$$
V(h, \sigma, r) \sim \lambda \left(h^2 + \sigma^2 - f^2\right)^2 + \lambda_s f_s^4 \left(\tilde{r}^2 - 1\right)^2 - \lambda_{s\phi} f_s^2 \tilde{r}^2 \left(h^2 + \sigma^2\right) \tag{7.62}
$$

where the $\sim$ symbols mean that we have neglected the kinetic terms of h and σ, the h^2 and h^4 terms because they will not participate in the subsequent discussion. The EOM (7.53) then becomes

$$
f_s^2 \Box \tilde{r} + \tilde{r} \partial_\mu a \partial^\mu a - 4\lambda_s f_s^4 \tilde{r} \left(\tilde{r}^2 - 1\right) + 2\lambda_{s\phi} f_s^2 \tilde{r} \left(h^2 + \sigma^2\right) = 0. \tag{7.63}
$$

The appropriate expansion parameter is f/f_s therefore the $\tilde{r}$ expansion in this case reads

$$
\tilde{r} = \tilde{r}_0 + \frac{f}{f_s} \tilde{r}_1 + \frac{f^2}{f_s^2} \tilde{r}_2 + \cdots \tag{7.64}
$$

Putting (7.64) in (7.63) we find:

$$
\begin{aligned}
\tilde{r}_0 &= 1, \\
\tilde{r}_1 &= 0, \\
\tilde{r}_2 &= \frac{\lambda_{s\phi}}{4\lambda f^2} (h^2 + \sigma^2).
\end{aligned}
\tag{7.65}
$$

Now we can return to the dimensionless r:

$$r \equiv f_s \tilde{r} = f_s \tilde{r}_0 + \frac{f^2}{f_s^2}\tilde{r}_2 + \cdots \equiv r_0 + \frac{f}{f_s} r_1 + \cdots \tag{7.66}$$

where

$$\begin{aligned} r_0 &= f_s, \\ r_1 &= \frac{\lambda_{s\phi}}{4\lambda_s f}(h^2+\sigma^2), \end{aligned} \tag{7.67}$$

Therefore the first order contributions to k_λ and k_f in the limit $f_s \gg f$ are

$$\begin{aligned} k_\lambda &= \left(1 - \frac{1}{4}\frac{\lambda_{s\phi}^2}{\lambda\lambda_s}\right) \\ k_f &= \left(1 + \frac{1}{2}\frac{\lambda_{s\phi}}{\lambda}\frac{f_s^2}{f^2}\right) \end{aligned} \tag{7.68}$$

while the NLO Lagrangian reads

$$\begin{aligned} \delta\mathcal{L}_s^{\text{NLO}} =& \frac{\lambda_{s\phi}}{4\lambda_s}\frac{h^2+\sigma^2}{f_s^2}\left[(\partial_\mu a)(\partial^\mu a) + \frac{\lambda_{s\phi}}{4\lambda_s}(h^2+\sigma^2)^2\right] + \\ &+ \frac{\lambda_{s\phi}^2}{32\lambda_s^2 f_s^2}\partial_\mu(h^2+\sigma^2)\partial^\mu(h^2+\sigma^2). \end{aligned} \tag{7.69}$$

In the limit $f_s \gg f$ (that is, $k_f \gg 1$) the expressions for the two lightest mass eigenvalues and for their mixing read

$$m_1^2 = 4\beta f^2\left(1 - \frac{\beta}{\gamma}\frac{\lambda}{\lambda_{s\phi}}\frac{f^2}{f_s^2}\right), \tag{7.70}$$

$$m_2^2 = 4\lambda_{s\phi} f_s^2\left(1 + 2\frac{\lambda}{\lambda_{s\phi}}\frac{f^2}{f_s^2}\right), \tag{7.71}$$

$$\tan 2\theta = 2\sqrt{\frac{\beta}{\gamma}\frac{\lambda}{\lambda_{s\phi}}}\frac{f}{f_s} \tag{7.72}$$

where we have assumed for simplicity $\lambda_{s\phi} \ll 4\lambda\lambda_s$.

It is also possible to study analytically the particle spectrum without integrating the field r, by assuming $f_s \sim f$ and $\beta, \lambda_{s\phi} \ll 1$. A straightforward calculation shows that in this limit the mass matrix reads

$$2f^2\begin{pmatrix} 4(\gamma+\lambda)\frac{\beta}{2\gamma} & 4\lambda\sqrt{\frac{\beta}{2\gamma}-\frac{\beta^2}{4\gamma^2}} & 0 \\ 4\lambda\sqrt{\frac{\beta}{2\gamma}-\frac{\beta^2}{4\gamma^2}} & 4\lambda(1-\frac{\beta}{2\gamma}) + 2\lambda_{s\phi}\frac{f_s^2}{f^2} & -2\lambda_{s\phi}\frac{f_s}{f} \\ 0 & -2\lambda_{s\phi}\frac{f_s}{f} & 4\lambda_s\frac{f_s^2}{f^2} + 2\lambda_{s\phi} \end{pmatrix} + O(\beta\lambda_{s\phi}, \lambda_{s\phi}^2). \tag{7.73}$$

Now, the eigenvalues equation for a symmetric matrix

$$S = \begin{pmatrix} a & d & 0 \\ d & b & e \\ 0 & e & c \end{pmatrix} \tag{7.74}$$

is

$$x^3 - \mathrm{Tr}[S]x^2 + (ab + bc + ac - d^2 - e^2)x + cd^2 + ae^2 - abc = 0. \tag{7.75}$$

In our case $e^2 \propto \lambda_{s\phi}^2$ so we can neglect it and the eigenvalues are

$$\begin{aligned} m_1^2 &= \frac{1}{2}\left(a + b - \sqrt{(a-b)^2 + 4d^2}\right) = 4\beta f^2 \left(1 - \frac{\beta}{2\gamma}\right) + \mathcal{O}\left(\beta^3, \beta^2\lambda_{s\phi}\right) \\ m_2^2 &= \frac{1}{2}\left(a + b + \sqrt{(a-b)^2 + 4d^2}\right) = 8\lambda f^2 \left(1 + \frac{1}{2}\frac{\lambda_{s\phi}}{\lambda}\frac{f_s^2}{f^2}\right) + \mathcal{O}\left(\beta^2, \beta\lambda_{s\phi}, \lambda_{s\phi}^2\right) \\ m_3^2 &= c = 8\lambda_s f_s^2 \left(1 + \frac{1}{2}\frac{\lambda_{s\phi}}{\lambda}\frac{f^2}{f_s^2}\right) + \mathcal{O}\left(\beta\lambda_{s\phi}, \lambda_{s\phi}^2\right) \end{aligned} \tag{7.76}$$

while the mixing angles are given by

$$\begin{aligned} \tan 2\theta_{12} &\sim \frac{2d}{b-a} \sim \sqrt{\frac{2\beta}{\gamma}}\left(1 + \mathcal{O}\left(\beta, \lambda_{s\phi}\right)\right), \\ \tan 2\theta_{23} &\sim \frac{2e}{c-b} = \lambda_{s\phi}\frac{f f_s}{\lambda f^2 - \lambda_s f_s^2}\left(1 + \mathcal{O}\left(\beta\lambda_{s\phi}, \lambda_{s\phi}^2\right)\right). \end{aligned} \tag{7.77}$$

The analytic results of these two limits will be compared with the numerical results in the next section.

7.6 Numerical Analysis

The previous analysis has been done by integrating out the field r. If we want to analyse the complete scalar particle spectrum, a numerical analysis is needed. In general, the scalar mass matrix $\mathcal{M}_s^2$ is real and it has to be diagonalised by an orthogonal transformation

$$\mathrm{diag}(m_1^2, m_2^2, m_3^2) = U(\theta_{12}, \theta_{13}, \theta_{23})^T \mathcal{M}_s^2 \, U(\theta_{12}, \theta_{13}, \theta_{23}) \tag{7.78}$$

where θ_{ij} is the rotation angle in the (i, j) sector. The corresponding scalar mass eigenstates $\varphi_1, \varphi_2, \varphi_3$ are defined by

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = U(\theta_{12}, \theta_{13}, \theta_{23})^T \begin{pmatrix} \hat{h} \\ \hat{\sigma} \\ \hat{r} \end{pmatrix}. \tag{7.79}$$

We can identify the eigenvector with the lowest mass, φ_1, with the Higgs particle; just like the MLσM, the coefficients of the mixing provide a modification of the SM Higgs couplings. In particular, the φ_1 couplings to SM gauge bosons can be deduced from the couplings of $\hat{h}$, as $\hat{\sigma}$ and $\hat{r}$ are singlets under the SM gauge group. The composition of $\hat{h}$ in terms of φ_i is given by:

$$\hat{h} = c_{12}c_{13}\varphi_1 + c_{13}s_{12}\varphi_2 + s_{13}\varphi_3 \equiv C_1\varphi_1 + C_2\varphi_2 + C_3\varphi_3 \tag{7.80}$$

where $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$ and we have defined the coefficients C_i for shortness. In this way, the couplings with the SM gauge bosons in terms of the physical fields read

$$\frac{g^2}{4}v_h^2\left(\frac{\hat{h}}{v_h}+1\right)^2 W_\mu^+W^{\mu-} = m_W^2\left(C_1\frac{\varphi_1}{v_h}+C_2\frac{\varphi_2}{v_h}+C_3\frac{\varphi_3}{v_h}+1\right)^2 W_\mu^+W^{\mu-}, \tag{7.81}$$

$$\frac{g^2+g'^2}{8}v_h^2\left(\frac{\hat{h}}{v_h}+1\right)^2 Z_\mu Z^\mu = \frac{m_Z^2}{2}\left(C_1\frac{\varphi_1}{v_h}+C_2\frac{\varphi_2}{v_h}+C_3\frac{\varphi_3}{v_h}+1\right)^2 Z_\mu Z^\mu \tag{7.82}$$

therefore the couplings of the Higgs particle are modified by a factor of C_1. Clearly, when we integrate out the heaviest field r, we expect $C_1 \sim 1+\mathcal{O}(\epsilon)$, thanks to the mechanism of the vacuum misalignment.

We can now turn to analyse the scalar potential parameter space. The scalar potential possesses 7 independent parameters: 5 dimensionless coefficient $\lambda, \lambda_s, \beta, \gamma, \lambda_{s\phi}$ and two scales f, f_s. Actually, by fixing the Higgs VEV and the Higgs mass to their experimental values, we can express the β and γ parameters as functions of the remaining 5 parameters $\lambda, \lambda_s, \lambda_{s\phi}, f, f_s$. In this numerical analysis we have taken $f = 2\,\text{TeV}$ that corresponds to $\xi \sim 0.015$, well inside the experimental bound on $\xi \equiv v^2/f^2 < 0.18$. By varying the values of the remaining 4 parameters we have studied the behaviour of the masses m_2, m_3 and of the coefficients C_1, C_2, C_3. In Fig. 7.1 the masses $m_{2,3}$ are shown as a function of $\lambda_{s\phi}$ (upper left), $\lambda = \lambda_s$ (upper right) or λ (lower). In all the plots we have considered three different values of f_s, $f_s = 1\,\text{TeV}$, $f_s = 10^3\,\text{TeV}$, $f_s = 10^6\,\text{TeV}$. In the upper left plot we have taken $\lambda = \lambda_s = 10$, and the numerical results follow the analytic expressions found in the previous section: in particular m_3, represented by the red dashed line, is independent from $\lambda_{s\phi}$ as expected from (7.43); m_2, represented by the blue continue line, shows an increasing behaviour with a constant slope, according to (7.59). We notice that, for $f_s = 1\,\text{TeV}$, the roles of the eigenstates are exchanged: m_2 becomes the highest eigenvalue. In the upper right plot we have chosen $\lambda_{s\phi} = 0.1$ and $\lambda = \lambda_s$. The brown area is excluded from the constraint in (7.40). In the proximity of it, the factor $[1-\lambda_{s\phi}/(4\lambda\lambda_s)]^{-1}$ becomes wider and the analytic prediction given by (7.59) deviates from the numerical result. Away from it, the red dashed line well represents the linear behaviour given by (7.43), the blue continue one is constant, as expected from (7.59). Also here, for $f_s = 1\,\text{TeV}$, m_2 becomes the highest eigenvalue. Finally, in the lower plot we have taken $\lambda_{s\phi} = 0.1$ and $\lambda_s = 10$. We can see that, according to (7.43), m_3 is independent from λ; on the other hand m_2, according to (7.59), is almost independent from λ for large f_s and has a linear dependence for $f_s \sim f$.

The behaviour of the mixing coefficients C_1^2, C_2^2, C_3^2 is shown in Fig. 7.2 where two different plots are shown corresponding to the values of $f_s = 1\,\text{TeV}$, $f_s = 3\,\text{TeV}$. The green-dot-dashed line describes C_1^2, the blue-continue line C_2^2 and the red-dashed line C_3^2. As expected, in both the plots $C_1 \sim 1$, i.e. the largest contribution to $\hat{h}$ is given by φ_1. The contributions to $\hat{h}$ given by φ_2, φ_3 are much smaller, at the level of 1% at most. This features are shared throughout the parameter space. The only difference is that, for $fs > f$, the largest contamination is given by ϕ_2, while for $fs < f$ is given by φ_3, as it is confirmed by (7.77).

7.7 QCD Axion or ALP? A Fine-Tuning Problem

In this section we will compare the prediction given by the MAMLσM with the present axion data in order to investigate the possibility to associate the angular part of s to a QCD axion or a more massive ALP. ALPs appear in many models of BSM physics, such as string theory, as pNGBs associated to the breaking of $U(1)$ symmetries. The properties of these particles are similar to that of axions, but their mass and coupling to photons are not related, making the corresponding parameter space larger than the axion case. On the other hand, QCD axions must satisfy more stringent conditions. In particular, in the case of QCD axion, the axion mass m_a and the SSB scale f_a are related by

$$m_a f_a \approx m_\pi f_\pi \tag{7.83}$$

where $m_\pi \sim 135\,\text{MeV}$ is the pion mass and $f_\pi \sim 94\,\text{MeV}$ is the pion decay constant. The axion coupling to photons is bounded from both astrophysical and terrestrial data and they depend on the axion mass. This upper bounds can be summarized as

$$\begin{aligned}
|g_{a\gamma\gamma}| &\lesssim 7 \cdot 10^{-11}\,\text{GeV}^{-1} && \text{for} \quad m_a \lesssim 10\,\text{meV}, \\
|g_{a\gamma\gamma}| &\lesssim \cdot 10^{-10}\,\text{GeV}^{-1} && \text{for} \quad 10\,\text{meV} \lesssim m_a \lesssim 10\,\text{eV}, \\
|g_{a\gamma\gamma}| &\ll \cdot 10^{-12}\,\text{GeV}^{-1} && \text{for} \quad 10\,\text{eV} \lesssim m_a \lesssim 0.1\,\text{GeV}, \\
|g_{a\gamma\gamma}| &\lesssim \cdot 10^{-3}\,\text{GeV}^{-1} && \text{for} \quad 0.1\,\text{GeV} \lesssim m_a \lesssim 1\,\text{TeV}.
\end{aligned} \tag{7.84}$$

In the MAMLσM model[9]

$$g_{a\gamma\gamma} = \frac{\alpha_{em}}{2\pi} \frac{c_{a\gamma\gamma}}{f_a} \tag{7.85}$$

[9] We have set the VEV f_s

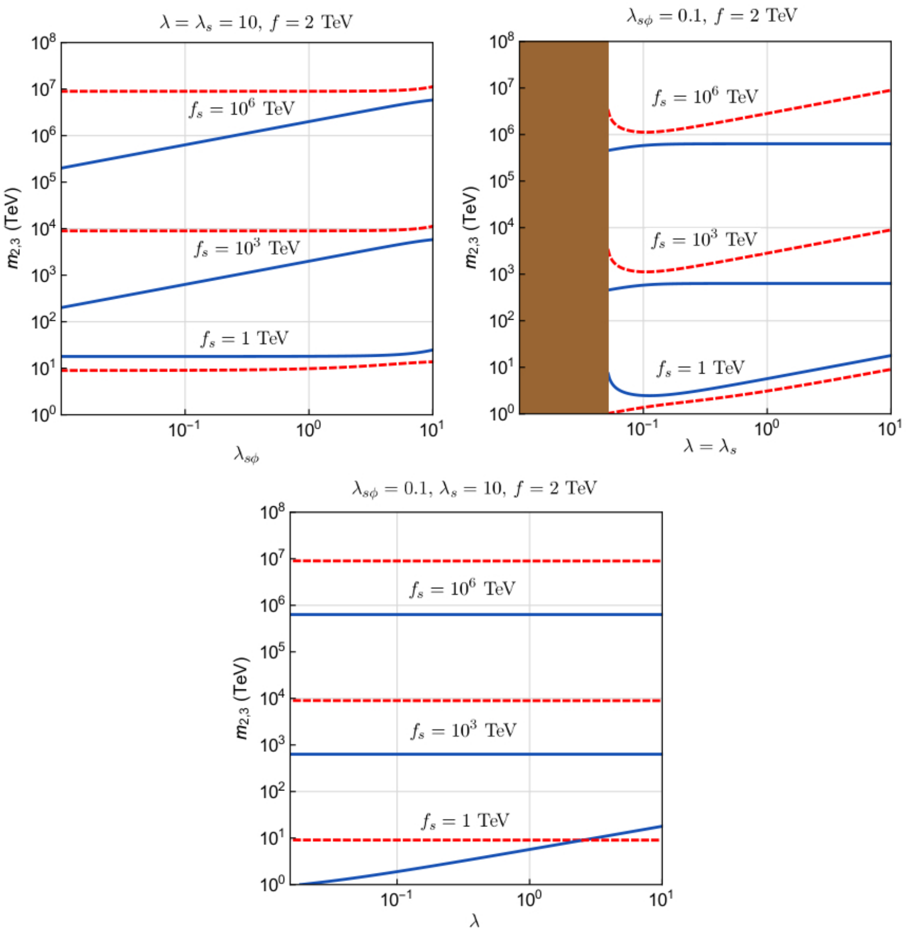

Figure 7.1: In the graphics we have shown the profiles of the scalar masses $m_{2,3}$ as a function of $\lambda_{s\phi}$ (upper left) $\lambda = \lambda_s$ (upper right) and λ (lower). The red-dashed line represents the heaviest scalar m_3, while the blue-continue line the next-to-heaviest with mass m_2. Three different values of f_s are shown: $f_s = 1\,\text{TeV},\ f_s = 10^3\,\text{TeV},\ f_s = 10^6\,\text{TeV}$ [17].

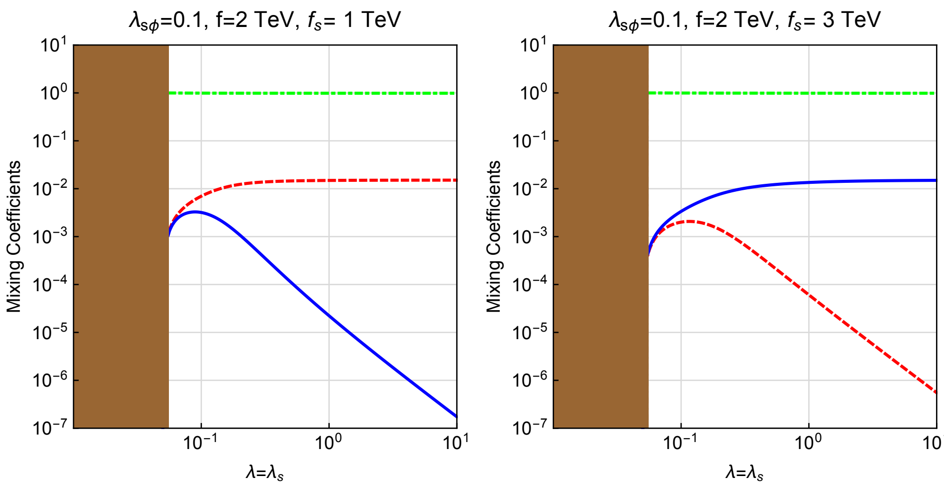

Figure 7.2: The profiles of the coefficients squared C_1^2, C_2^2, C_3^2 as a function of $\lambda = \lambda_s$. The other parameters are chosen at fixed values: $f = 2\,\text{TeV}$, $\lambda_{s\phi} = 0.1$, $f_s = 1\,\text{TeV}$ on the left and $f = 2\,\text{TeV}$, $\lambda_{s\phi} = 0.1$, $f_s = 3\,\text{TeV}$ on the right. The green-dot-dashed line describes C_1^2 the blue-continue line C_2^2 and the red-dashed line C_3^2. The brown area is excluded from the constraint in (7.40) [17].

where $c_{a\gamma\gamma} = 112/3$, therefore the bounds in (7.84) are translated in bounds on the scale f_a:

$$\begin{aligned} f_a &\gtrsim 2 \cdot 10^7\,\text{GeV} \quad \text{for} \quad m_a \lesssim 10\,\text{meV}, \\ f_a &\gtrsim 10^7\,\text{GeV} \quad \text{for} \quad 10\,\text{meV} \lesssim m_a \lesssim 10\,\text{eV}, \\ f_a &\gg 10^9\,\text{GeV} \quad \text{for} \quad 10\,\text{eV} \lesssim m_a \lesssim 0.1\,\text{GeV}, \\ f_a &\gtrsim 1\,\text{GeV} \quad \text{for} \quad 0.1\,\text{GeV} \lesssim m_a \lesssim 1\,\text{TeV}. \end{aligned} \tag{7.86}$$

From astrophysical data we know that the axion mass has to belong to the meV − eV window. It follows that, in this model, a consistent QCD axion can only be generated if the scale f_s[10] associated to the SSB of the $U(1)_{\text{PQ}}$ symmetry is

$$f_s > 3.7 \times 10^8\,\text{GeV}. \tag{7.87}$$

The resulting axion is therefore an invisible axion, as such a large f_s scale strongly suppresses all the couplings with SM fermions and gauge bosons, preventing any possible detection at colliders or at low-energy (flavour) experiments. The difference between the traditional axion models resides partially in the couplings $c_{a\gamma\gamma}$ and c_{agg} for which the MAMLσM predicts a very

[10]We are confusing the axion scale f_a with the VEV of the radial component r, as they are related by (7.38), and are similar in first approximation.

sharp value for the ratio:

$$\frac{c_{a\gamma\gamma}}{c_{agg}} = \frac{14}{3}. \tag{7.88}$$

For much lighter values of the f_s scale, instead, the astrophysical bounds on $g_{a\gamma\gamma}$ can be satisfied only assuming that the axion mass and its characteristic scale f_s are not correlated. This corresponds to the ALP scenario: differently from the QCD axion, an ALP has a mass that is independent from its characteristic scale f_s, because of the additional sources of soft symmetry breaking. Therefore in this scenario, as the PQ symmetry is not exact, the strong CP problem could not be solved. In this case, consistent values for the ALP mass and the f_s scale that pass the astrophysical bounds are $m_a \sim 1\,\text{GeV}$ and $f_s \sim 200\,\text{TeV}$. Also, with these values the ALP can evade detection at colliders, because of its decay into gauge bosons inside the detector [17].

However, as we now show, we do not have the freedom to fix both the scales f and f_s, unless fine-tuning the parameter $\lambda_{s\phi}$. We have seen, eq. (7.38), that the scale f is renormalized by a factor $\sqrt{\lambda_{s\phi}} f_s$. This means that, unless of adjusting $\lambda_{s\phi}$ ad hoc, (and this seems to be too artificial) f is pushed near the upper scale f_s. Therefore if we want to keep f in the TeV range in order to not invalidate our solution to the Naturalness problem, we are forced to set f_s at the same scale. But then, as this value does not satisfies (7.87), the strong CP problem cannot be solved. In particular, the σ particle has to be regarded as an ALP, rather than an axion so that it has not to satisfy the bound (7.87). We notice that the problem cannot be avoided: also taking $\lambda_{s\phi} = 0$ at a Lagrangian level, the contributions arising at *one-loop level* would give it back, with a value considerably different from zero, differently from the analysis performed in the KSVZ model. This is due to the fact that the couplings responsible of the mixing between heavy fermions and SM fermions (the Λ's) act as a portal between the s and ϕ sectors. On the other hand, in the original KSVZ model the problem does not arise because heavy and SM fermions are decoupled (at tree-level) and the only way that have to interact is through gluons, as we have seen previously.

The previsions given by the model for the NGB nature of the Higgs translate, just like in the MLσM, in a modification of the Higgs couplings to gauge bosons by a factor of C_1, the analogue of $\cos\gamma$ in the MLσM. Moreover, as the mass of the exotic fermions, arising after the SSB of the s particle, is controlled by f_s, a large f_s allows to rely on the partial fermion compositeness in order to explain the Yukawa hierarchy.

Conclusions: Coming Soon!

In this thesis we have analysed some attempts to extend the SM Lagrangian in order to solve the Naturalness and the strong CP problem. In particular, in the first part we have studied a MLσM [16] in which the Higgs arises a pNGB of a SSB $SO(5)/SO(4)$. This is enough to explain the lightness of the Higgs mass, providing a solution to the hierarchy problem. The model mimics a class of models, known as composite Higgs models, in which the Higgs arises as a bound state of a new strong force. Nevertheless, unlike the CHMs, the theory is renormalizable and UV complete. This keeps open two possibilities: either the theory is weakly coupled or it is strongly coupled. In the latter case, with the σ particle integrated out, the model falls in the category of the non-linear realizations and the Higgs arises as a bound state. On the other hand, even though the theory is UV complete, the weakly coupled regime would necessitate further explanations, as the Naturalness problem for the Higgs mass has replaced with that of the σ particle.

The scalar sector of the theory contains the fiveplet ϕ, whose components can be associated to the three longitudinal polarization of the gauge bosons, the Higgs particle and the σ particle. In particular, its potential depends on four independent parameters, that can be parametrized by the Higgs mass m_h, the EWSB scale v, the σ particle mass m_σ and the mixing angle $\sin\gamma$. Fixing m_h and v to their experimental values the parameter space is completely defined by the pair $(m_\sigma, \sin\gamma)$. Following [16] we have identified the area in the $(m_\sigma, \sin\gamma)$ plane in which the Higgs can be interpreted as a pNGB, taking into account also the experimental bounds on $\sin\gamma$. In fact, we have seen how the pNGB nature of the Higgs translates in a modification of the Higgs couplings to gauge bosons with respect to the SM predicted values. In particular, the Higgs couplings to W and Z are weighted down by $\cos\gamma$. Moreover, just like the CHMs, the model provides an explanation of the Yukawa hierarchy, relying on the *partial fermion compositeness* mechanism. This has been achieved by adding in the particle spectrum a set of new vector-like heavy fermions that couple to the scalar fiveplet ϕ, through $SO(5)$ invariant interactions, and to (the third generation of) SM fermions, through $SO(5)$ breaking terms. Moreover, the soft breaking of the $SO(5)$ symmetry in the fermionic sector, through the α and β parameters, makes the Higgs a pNGB, rather than a NGB, giving it a mass. In particular, we have chosen the parameters in order to absorb the divergences generated from the one-loop Coleman-Weinberg potential. Finally, we have calculated the h and σ decays into two gluons. The former is dominated by the top quark for very heavy fermions, matching in this way the prediction given by the SM. On the other hand, the heavy sector has a more significant impact on σgg transitions.

In the second part we have dealt with some aspects of the strong CP problem and discussed one of the most convincing solution to it, the *axions*. In fact, after the SSB of the $U(1)_{\rm PQ}$ symmetry, the axion VEV provides a dynamical adjustment of the θ parameter. After a short review on the different ways to introduce axions in a BSM Lagrangian, we have focussed on the so-called KSVZ model. In particular, following the paper [17], we have analysed the possibility to extend the MLσM for the Goldstone Higgs by adding a scalar field s, singlet under the whole $SO(5)$, and extending the SSB pattern of the MLσM with a further $U(1)_{\rm PQ}$ symmetry in order to solve the strong CP problem. The possibility is suggested from the fact that the heavy fermionic sector can allow to rotate away the QCD θ term, just like in the KSVZ model. In this model the content of the fermionic sector is the same as the MLσM, with the difference that now the heavy fermions couple to s, as well. These interactions provide also a mass for the heavy fermions, which arises after the SSB of the s particle. On the other hand, the presence of the scalar particle s enlarges the scalar particle spectrum and the potential now depends on 7 parameters: the two SSB scales f and f_s, the couplings $\lambda, \lambda_s, \lambda_{s\phi}$ and the two parameters β and γ arising from the Coleman Weinberg mechanism through the fermionic sector. Moreover, introducing few general requirements, we have seen that this is the potential with the minimal number of parameters needed to absorb the divergences generated by the CW potential. Following [17], we have obtained analytic expressions for the scalar masses by integrating out the highest degree of freedom in two limits: $f_s \gg f$ and $\lambda_s \gg 1$. The latter limit corresponds to the strongly interacting regime. Furthermore, we have also calculated the masses in the regime $f_s \sim f$ in the limit of small parameters $\beta, \lambda_{s\phi} \ll 1$. These limits have served to compare them with the numerical analysis of the parameter space. This latter analysis has been done by identifying 2 parameters with the physical Higgs mass and the Higgs VEV, just like in the MLσM, and varying the remaining 5 [17].

The analytical and numerical analysis of the parameter space show that for $f, f_s > 1\,\text{TeV}$ the heaviest scalar degrees of freedom are unlikely to give signals at the present and future LHC run. On the other hand, the non-linearity of the EWSB mechanism, as in the MLσM, leads to deviations from the SM predictions in Higgs and gauge boson sectors that are more experimentally viable in the future.

Finally, we have investigated the possibility to associate the (angular part of the) scalar particle s either to an axion or to a more massive ALP. In the first case the axion mass is expected in the [meV, keV] range. This, together with the strong bounds present on the axion coupling to two photons, prevent the axion scale $f_a \sim f_s$ to be less than 10^8 GeV. The trouble is that, in this case, unless fine-tuning $\lambda_{s\phi}$, the scales f and f_s, because of this parameter, are unavoidably related and the lower scale f is pushed near the upper scale f_s. In particular, this is due to the tree-level couplings between SM fermions and heavy fermions, which are not present in the original KSVZ model. In the latter we saw how this prevents $\lambda_{s\phi}$ from taking large quantum corrections. This means that, since the solution to the Naturalness problem is valid only if f is in the low TeV range, the price to retain valid the MAMLσM as possible explanation to it is to lower our claim on the solution of the strong CP problem: we cannot rely on a large f_s anymore and, as the present axion data prevent f_s from being less than $\sim 10^8$ TeV, if we want to keep f_s lower than this value we are forced to regard (the angular part of) s as a more massive ALP, rather than an axion. In fact, the ALPs are free from cosmological bounds and a more massive 1 GeV ALP with an associate scale of $\sim$ 200 TeV range is possible. However, as we have outlined, the ALP hypothesis could be of no help in addressing the strong CP problem.

The Naturalness problem and the strong CP problem are without doubt two of the most studied puzzles in particle physics and two of the most strong motivations in searching physics BSM. These questions share the same underlying guiding idea: nothing is by chance. However, we are not at all sure if these problems are really problems. After all there could be anthropic reasons explaining why these numbers are so small. Actually, the anthropic hypothesis seems to be available for the Naturalness problem but not for the strong CP problem. As discussed by Peccei and Wilczek, in fact, a Universe where CP is violated strongly seems as viable as a Universe where it is not.

The history has often taught that behind some philosophical questions there are physical answers. Beyond the personal opinions, one thing is certain: at least for the Naturalness problem, time has come for the question to be settled by experimental data.

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