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BACHELOR DEGREE IN MATHEMATICS

Bachelor Thesis

Stability vs. complexity of ecosystems: a random matrix approach.

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Contents

Introduction

In this thesis we face from a mathematical point of view the ecology issues of *stability* and *complexity* of ecosystems and underline the relevance of discussing their relation when studying living networks, where interactions between species are of fundamental importance in the evolution of the system.

Chapter 1: in the first chapter we will hark back to Robert May's work, which was an analytical investigation of complex systems using the mathematical tool of random matrices in the limit when the number of species n is large, and we will see that his studies led to a well defined result: any large complex ecosystem, as the number of species gets large or for increasing values of connectance, tends to be unstable.

The question is that this result is in stark contrast with the empirical evidence because, in reality, many biodiversity-rich ecosystems exist. Hence we will find that we cannot model natural networks assuming random variables but we have to use stochastic methods or add some extra hypothesis over the model.

Chapter 2: This last solution will be our approach in the second chapter of this thesis, where, as shown in [1], we will focus on *large sparse* ecosystems, so networks with a large number of species where the interactions between them are few, and we will model this ecology problem mathematically using the random matrix theory: we will describe the system using a Lotka-Volterra system of coupled differential equations and face the question of *feasibility*, i.e. the problem of finding a solution with no vanishing species.

We will see that, working under the extra block-structure assumption over the matrix of interactions under study, there exists an explicit threshold, depending on the considered parameters and reflecting the strength of the interactions, which guarantees the existence of a positive equilibrium as the number of species gets large.

Chapter 3: Finally in the third chapter we will focus on the issue of stability: we will investigate the conditions that guarantee the global stability of the equilibrium solutions and see that feasibility and global stability occur simultaneously.

Chapter 1

Stability vs. complexity of ecosystems

In this chapter we introduce the concept of *complex systems*, that are systems composed of many elements that interact with each other: first we show that if we want to model them we have to analyze the interactions between species, and this idea leads us to consider a Lotka-Volterra system of coupled differential equations; then we introduce Robert May's work, that is regarded as one of the most relevant in the development of theoretical ecology since it was one of the first attempts to investigate the stability of large ecosystems.

1.1 Complex ecosystems

The relation between complexity and stability of ecosystems is an issue of scientific and mathematical interest since long time and it has often been questioned in the investigation of the behavior of living systems.
Theoretical models evolved in order to account the interactions occurring within individuals of the same species or of different ones: early studies argued that increased complexity enhanced ecosystems stability, but these results were then judged incomplete and heterogeneous by later studies, which came to opposite conclusions, suggesting that simple ecosystems were more stable than complex ones, as highlighted in [2].
Nowadays there are still many open questions because, in the attempt to model living systems, we go towards various sets of problems, starting from the possible different meanings of *complexity* and *stability*.
These two concepts can be analyzed from many points of view: when we will consider a system, we will call it *stable* if all variables return to the initial equilibrium after they've been perturbed from it; considering the *complexity* of an ecosystem we could evaluate the species richness (the number of species), the connectance (which can be defined as the ratio of the number

of actual interactions and the number of all possible interactions), the interaction strength (the mean magnitude of the interaction: the size of the effect of the density of one species on the growth rate of another one) or evenness (a measure of biodiversity which quantifies how equal the community is numerically).
General living systems are affected by different degrees of complexity because they're made up of many components interacting: the approach, when we try to shed light on this type of problems, is not to follow the behavior of each small constituent of the system, but to give a description of the global phenomenon. This because a huge number of degrees of freedom would be required to describe all the single individuals and because to understand the system in its wholeness it is not enough to frame it as a sum of elements.
The central idea of this thesis is that to analyze dynamics of a *complex system*, which is a system whose components are correlated to other ones through networks, we have to investigate interactions, that are strongly present and crucially shape the system's evolution.

In living systems many species coexist together so interactions between different ones must be considered if we want to model them: this idea naturally leads to the formulation of coupled and non linear dynamical systems and this is why we will examine a Lotka-Volterra system of coupled differential equations.
The Lotka-Volterra equations, also known as the *predator–prey equations*, are a pair of first-order non linear differential equations, frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey. The populations change through time according to the pair of equations:

$$\begin{cases} \dot{x} = (a - by)x \\ \dot{y} = (cx - d)y \end{cases}$$

with $a, b, c, d \geq 0$.
The idea is that the two populations have growth rates that depend linearly on the other one: when predators are too abundant preys will have a negative growth because of the intense predation, when preys are few, predators will decay in number.
The Lotka-Volterra system was the archetypal model which pushed the idea of studying ecosystems by the investigation of the network dynamics and it was used by Robert May, who played a key role in the development of theoretical ecology through the 1970s and 1980s, in the attempt to understand what makes an ecosystem stable.
In the following section we will hark back to these studies and will try to investigate May's paradox.

1.2 May's work and paradox

The first attempt to investigate the stability of large ecosystems came from Robert May, whose work is considered one of the most influential in theoretical ecology.
As we can see in [3], he analyzed the relation between stability and complexity studying dynamics in the neighborhood of the equilibrium point and, by Taylor expansion, he obtained the equation:

$$\dot{x} = Ax \tag{1.1}$$

where x is the $n \times 1$ column vector of the disturbed populations x_j and $A = (a_{jk})$ is the $n \times n$ interaction matrix where the element a_{jk} describes the effect of species k on species j near equilibrium.
In this model he made some assumptions:

- he took the diagonal elements of A as $a_{ii} = -1$ for all $i \in [n] = \{1, \ldots, n\}$, meaning that, if disturbed from the equilibrium, each species would return at it with a characteristic damping time equal to -1: so when the species are isolated from the others the system would not diverge;
- he assumed that A, instead of being computed from concrete data, is a random matrix with connectance C, which is defined as the probability that any pair of species would interact, probability which is measured as the percentage of non zero elements in the matrix. So he postulated that the non-diagonal elements of A are zero with probability $1 - C$, while with probability C they are drawn from a random number distribution $\mathcal{P}$, which is chosen to be of mean value 0 and mean square value α, where α expresses the average interactions "strength".

Therefore to study the stability of System (1.1), May searched for the eigenvalues of A with negative real part and found that the system is almost certain stable (with $\mathcal{P}(n, \alpha, C) \to 1$) if

$$\alpha < (nC)^{-1/2}$$

and almost certain unstable (with $\mathcal{P}(n, \alpha, C) \to 0$) if

$$\alpha > (nC)^{-1/2}.$$

This result can be interpreted in two ways:

- for every fixed level of connectance C, if n is sufficiently large, we have instability, and the transition form stability to instability is very sharp for $n \gg 1$;

- for a fixed number of species n, the system is stable up to a certain critical level of connectance C and then, as this increases, suddenly becomes unstable.

This is what generates May's paradox: he proved that any large complex ecosystem, no matter the form of the random variables used, tends to be unstable as the number of species gets large or for increasing values of connectance, but this result disagrees with the empirical evidence because it is unquestionable the existence of many biodiversity-rich ecosystems.
The critical point of his work is that natural ecosystems are not exactly randomly selected ones. For simple systems, up to a certain degree, we can see that stability and complexity are almost directly proportional: for example, in the prey/predator model described by the Lotka-Volterra equations, there are few species interacting with each other and in general this kind of system tends to be unstable.
However, in reality, population dynamics are much more complex: they can show fluctuations due to immigration and emigration of the population as well as external forcing factors that can change from year to year including the weather, the abundance of competitors and predators or the amount of food available. Furthermore all natural ecosystems passed through millions of years of natural selection, so they are characterized by the self-emergence on very large scale of non trivial spatial structures, known as *spatial patterns*.
Therefore, since living systems are too large and complex to be approached in a deterministic way, to give a more realistic description we should use stochastic models and regard species interactions as the result of a long optimization process. Hence, to model this problem, we will focus on the spatial pattern of *sparsity*, which recent studies has highlighted to be widespread in living systems, and we will see that for System (1.1), under the assumptions of a sparse matrix A and with the extra hypothesis of being a block matrix, properties of stability will be present only above a certain critical level of connectance.

Chapter 2

Large sparse ecosystems

2.1 Preliminary notions

In this section we are going to introduce some definitions and to state some propositions from matrix theory that will be crucial in the development of our analysis.
We first recall two different operations between matrices: we will use them in the following chapter to define the matrix that describes the interactions between species.

Definition 2.1. (Hadamard product). *For two matrices A and B of the same dimension $m \times n$, the Hadamard product $A \circ B$ is a matrix of the same dimension as the operands, with elements given by: $(A \circ B)_{ij} = (A)_{ij}(B)_{ij}$.*

For example the Hadamard product for a 3×3 matrix A with a 3×3 matrix B is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}\, b_{11} & a_{12}\, b_{12} & a_{13}\, b_{13} \\ a_{21}\, b_{21} & a_{22}\, b_{22} & a_{23}\, b_{23} \\ a_{31}\, b_{31} & a_{32}\, b_{32} & a_{33}\, b_{33} \end{bmatrix}$$

Definition 2.2. (Kronecker product). *If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is the $pm \times qn$ block matrix:*

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

In the following chapter we are going to analyze a Lotka-Volterra system assuming some extra hypothesis over the matrix under study, so to explain the characterization of the model we give two definitions and recall some results from matrix theory that we will need.

Definition 2.3. (Adjacency matrix). *An adjacency matrix is a square matrix used to represent a finite graph. The elements of the matrix indicate whether pairs of vertices are adjacent or not in the graph.*

Definition 2.4. (Hermitization matrix). *The Hermitization matrix associated to a $n \times n$ matrix A is* $\mathcal{H}(A) = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$.

Observation 2.1. *$\mathcal{H}(A)$ has a symmetric spectrum; the singular values of A, with associated left and right singular vectors u and v, are the non-negative eigenvalues of $\mathcal{H}(A)$, with associated eigenvector* $w = \begin{pmatrix} u \\ v \end{pmatrix}$.
Moreover, since $\|A\|$ is the largest singular value of A, it is the largest eigenvalue of $\mathcal{H}(A)$.

Observation 2.2. *If v is a vector, then $\|v\|$ is its Euclidean norm; if A is a matrix then $\|A\|$ stands for its spectral norm and $\|A\|_F = \sqrt{\sum_{ij} |A_{ij}|^2}$ is its Frobenius norm; if φ is a function from some space Σ to $\mathbb{R}$, then* $\|\varphi\| = \sup_{x\in\Sigma} |\varphi(x)|$.

We state a proposition that we are going to use in the development of our analysis: we give an estimate of the spectral norm of $\frac{\Delta \circ A}{\sqrt{d}}$.

Proposition 2.1. *Assume that A is a $n \times n$ matrix with i.i.d. $\mathcal{N}(0,1)$ entries, that Δ is a $n \times n$ adjacency matrix of a d-regular graph, that $d \geq \log(n)$. Then there exists a constant $\kappa > 0$ independent from n (one can take for instance $\kappa = 22$) such that*

$$\mathbb{P}\left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \geq \kappa \right) \xrightarrow[n\to\infty]{} 0.$$

In particular, let $\delta \in (0,1)$ be fixed and $\alpha = \alpha(n) \xrightarrow[n\to\infty]{} \infty$. Then

$$\mathbb{P}\left(\left\| \frac{\Delta \circ A}{\alpha\sqrt{d}} \right\| \leq 1 - \delta \right) \xrightarrow[n\to\infty]{} 1.$$

This result, which proof can be found in [4] and is crucially based on the fact that $d_n \geq \log(n)$ and that the entries of A are $\mathcal{N}(0,1)$, will be useful in the following analysis because, under the hypothesis of the main theorem of this thesis, this estimate will hold.

2.2 Feasibility of large sparse ecosystems

In this section we will first present the question of feasibility of the foodweb, then we will state the main result of this thesis and we will give its proof.

We consider a large ecosystem (foodweb) with n species, where the abundances follow a Lotka-Volterra system of coupled differential equations and

we examine the question of *feasibility* of the foodweb, that is the existence of an equilibrium solution of the system where no species disappears.
We assume that each species interacts with $d = d_n$ other species and that their interaction coefficients are independent random variables.
We establish that for a given range of d there exists an explicit threshold, depending on n and d and reflecting the strength of the interactions, which guarantees the existence of a positive equilibrium as the number of species n gets large and we use a Lotka-Volterra (LV) system to model a given foodweb with n species.
Let $x_n = (x_{k(t)})_{k\in[n]}$ be the vector of the abundances of the various species at time $t \geq 0$ and suppose that its components are connected by the coupled equations:

$$\frac{dx_k(t)}{dt} = x_k(t)\left(r_k - x_k(t) + \sum_{l=1}^{n} M_{kl}x_l(t)\right) \quad \text{for} \quad k \in [n],$$

where r_k is the intrinsic growth of species k and $M = (M_{kl})$ is a large sparse (mostly composed of zeros) random matrix, accounting for the interactions between species.

At the equilibrium we have $\frac{dx_n}{dt} = 0$, so x_n is solution of the system

$$x_k(t)\left(r_k - x_k(t) + \sum_{l=1}^{n} M_{kl}x_l(t)\right) = 0 \quad \text{for} \quad x_k \geq 0 \quad \text{and} \quad k \in [n]. \tag{2.1}$$

Our aim is to determine the existence of a *feasible* solution x_n where all the components of the vector of abundances are $x_k > 0$ for all $k \in [n]$ (that is a scenario with no vanishing species). In this latter case System (2.1) becomes:

$$x_k = r_k + \sum_{l=1}^{n} M_{kl}x_l \quad \text{for} \quad x_k > 0 \quad \text{and} \quad k \in [n].$$

Harking back to May's work we consider intrinsic growths $r_k = 1$ for all $k \in [n]$ so that the system under study will be:

$$\frac{dx_k(t)}{dt} = x_k(t)\left(1 - x_k(t) + \sum_{l=1}^{n} M_{kl}x_l(t)\right) \quad \text{for} \quad k \in [n] \tag{2.2}$$

and since we look for the equilibrium solution, by imposing $\frac{dx_k(t)}{dt} = 0$ and considering the positive solutions, we have that the system becomes:

$$x_n = \mathbf{1}_n + M_n x_n,$$

where $\mathbf{1}_n$ is the $n \times 1$ vector with components 1.

As said before, we consider the parameter of *connectance* as the percentage of non-zero entries in the interaction matrix M_n and, supposing the nature of interactions as random, we will see that, as claimed by May's complexity/stability theory, sparse ecosystems lead to stable equilibrium: as highlighted by recent studies [5] foodwebs can be very sparse.
Living systems are composed of interacting entities, such as genes, individuals and species with the ability to rearrange and tune their own interactions in order to achieve a desired output: several studies indicate that interaction networks in living systems possess a non-random architecture characterised by the emergence of recurrent patterns and regularities and, from the analysis of different biological networks, emerged that a widespread pattern is sparsity, i.e. the percentage of the active interactions (connectivity) scales inversely proportional to the system size.

Motivated by these studies we are going to focus on sparse ecosystems and to encode this sparsity we consider a d_n-regular graph with n vertices and its associated $n \times n$ adjacency matrix $\Delta_n = (\Delta_{ij})$:

$$\Delta_{ij} := \begin{cases} 1 & \text{if there is an edge pointing from i to j,} \\ 0 & \text{otherwise.} \end{cases}$$

In this graph each vertex has d_n edges pointing from a vertex $k \in [n]$ to i, and has d_n other edges pointing from i to a vertex $l \in [n]$ therefore Δ_n has d_n non-zero entries per row and per column and $n \times d_n$ non-zero entries overall.
We assume that the interaction matrix M_n is in the form:

$$M_n = \frac{\Delta_n \circ A_n}{\alpha_n \sqrt{d_n}} \tag{2.3}$$

where A_n is a $n \times n$ matrix with independent Gaussian $\mathcal{N}(0,1)$ entries, $\Delta \circ A_n = (\Delta_{ij} A_{ij})$ is the Hadamard Product between Δ_n and A_n, $(\alpha_n)_{n\geq 1}$ is a positive sequence.

Block permutation matrix model (BPMM)
To develop our analysis we want to give an extra structure to the system under investigation: we assume that $d \geq \log(n)$ (that is a necessary condition if we want to work under the hypothesis of Proposition 2.1) and we suppose that the matrix which describes the interactions between species is given by (2.3) with the extra hypothesis that the matrix Δ_n is a *block-permutation* adjacency matrix defined as:

$$\Delta_n = P_\sigma \otimes J_d = (P_{ij} J_d)_{i,j\in[m]} \tag{2.4}$$

where $\otimes$ is the Kronecker matrix product, $J_d = \mathbf{1}_d \mathbf{1}_d^{\mathrm{T}}$ is the $d \times d$ matrix of ones and $P_\sigma = (P_{ij})_{i,j\in[m]}$ is the permutation matrix associated to a permutation $\sigma \in S_m$:

$$P_{i,j} = \begin{cases} 1 & \text{if } j = \sigma(i), \\ 0 & \text{else.} \end{cases}$$

Under these assumptions we are ready to state the main result of this thesis:

Theorem 2.1. *Let A_n be a $n \times n$ matrix with i.i.d. $\mathcal{N}(0,1)$ entries and Δ_n given by BPMM model; assume that $\alpha_n \xrightarrow[n\to\infty]{} \infty$ and denote by*

$$\alpha_n^* = \sqrt{2 \log n}.$$

Let $x_n = (x_k)_{k\in[n]}$ be the solution of

$$x_n = \mathbf{1}_n + \frac{1}{\alpha_n \sqrt{d_n}} (\Delta_n \circ A_n) x_n. \tag{2.5}$$

Then

1. *If there exists $\varepsilon > 0$ such that eventually $\alpha_n \leq (1-\varepsilon)\alpha_n^*$, then*

$$\mathbb{P}\left\{ \min_{k\in[n]} x_k > 0 \right\} \xrightarrow[n\to\infty]{} 0,$$

2. *If there exists $\varepsilon > 0$ such that eventually $\alpha_n \geq (1+\varepsilon)\alpha_n^*$, then*

$$\mathbb{P}\left\{ \min_{k\in[n]} x_k > 0 \right\} \xrightarrow[n\to\infty]{} 1.$$

Observation 2.3. *We will assume $\alpha_n \to \infty$ since, as highlighted in* [6], *a feasible solution is unlikely to exist if $\alpha_n \equiv \alpha$ is a constant.*

Observation 2.4. *We are requiring $d_n \geq \log(n)$ and, since A's entries are $\mathcal{N}(0,1)$, we are under the hypothesis of Proposition (2.1) which provides an estimate of $\left\|\frac{\Delta\circ A}{\sqrt{d}}\right\|$: it holds that for $\alpha = \alpha_n \xrightarrow[n\to\infty]{} \infty$ and for fixed $\delta \in (0,1)$*

$$\mathbb{P}\left(\left\| \frac{\Delta \circ A}{\alpha\sqrt{d}} \right\| \leq 1 - \delta \right) \xrightarrow[n\to\infty]{} 1,$$

so the matrix $I_n - \frac{\Delta_n \circ A_n}{\alpha_n\sqrt{d_n}}$ is invertible.

2.3 Proof of Theorem 2.1

In this section our aim is to prove Theorem 2.1.
The Equation under study is (2.5) and it can be rewritten it as

$$\left(I - \frac{\Delta \circ A}{\alpha\sqrt{d}}\right)x = \mathbf{1}.$$

In the previous observation we showed that $I - \frac{\Delta \circ A}{\alpha\sqrt{d}}$ is invertible so, calling $Q = \left(I - \frac{\Delta \circ A}{\alpha\sqrt{d}}\right)^{-1}$, the equation becomes $x = Q\mathbf{1}$.

We now recall the following property of the Neumann series: if T is a bounded linear operator and the Neumann series converges in the operator norm, then $I - T$ is invertible and its inverse is $(I-T)^{-1} = \sum_{l=0}^{\infty} T^l$.

We know that, thanks to the normalization term $\frac{1}{\sqrt{d}}$, the term $\frac{\Delta \circ A}{\alpha\sqrt{d}}$ has a bounded norm, so the property of the Neumann series does hold, hence:

$$\sum_{l=0}^{\infty}\left(\frac{\Delta \circ A}{\alpha\sqrt{d}}\right)^l = \left(I - \frac{\Delta \circ A}{\alpha\sqrt{d}}\right)^{-1} = Q,$$

so Equation (2.5) becomes

$$x_k = \mathbf{e}_k^{\mathrm{T}} Q\mathbf{1} = \sum_{l=0}^{\infty} \mathbf{e}_k^{\mathrm{T}} \left(\frac{\Delta \circ A}{\alpha\sqrt{d}}\right)^l \mathbf{1}.$$

.
We define

$$Z_k = \mathbf{e}_k^{\mathrm{T}}\left(\frac{\Delta \circ A}{\sqrt{d}}\right)\mathbf{1} \quad \text{and} \quad R_k = \mathbf{e}_k^{\mathrm{T}} \sum_{l=2}^{\infty} \frac{1}{\alpha^{l-2}}\left(\frac{\Delta \circ A}{\sqrt{d}}\right)^l \mathbf{1},$$

.
so we have

$$x_k = 1 + \frac{Z_k}{\alpha} + \frac{R_k}{\alpha^2}. \tag{2.6}$$

We observe that Z_k are i.i.d. $\mathcal{N}(0,1)$ and define $\hat{M} = \min_{k\in[n]} Z_k$; so from Equation (2.6) we get

$$1 + \frac{1}{\alpha}\hat{M} + \frac{1}{\alpha^2}\min_{k\in[n]} R_k \leq \min_{k\in[n]} x_k \leq 1 + \frac{1}{\alpha}\hat{M} + \frac{1}{\alpha^2}\max_{k\in[n]} R_k. \tag{2.7}$$

Now recall that $\alpha^* = \sqrt{2\log(n)}$ and denote by $\beta_n^* = \alpha_n^* - \frac{1}{2\alpha_n^*}\log(4\pi\log(n))$ and $G(x) = e^{-e^{-x}}$ a Gumble distributed random variable.
Then, as shown in [6], it holds that

$$\mathbb{P}(\alpha_n^*(\hat{M}_n + \beta_n^*) \geq x) \xrightarrow[n\to\infty]{} G(x)$$

and so $\frac{\hat{M}+\beta^*}{\alpha^*} = o_P(1)$, namely $\mathbb{P}\left(\frac{\hat{M}+\beta^*}{\alpha^*} \ll 1\right) = 1$.

We observe that

$$\frac{\beta_n^*}{\alpha_n^*} = 1 + \frac{1}{2\alpha_n^{*2}} \log(4\pi \log(n)) = 1 + \frac{1}{4\log(n)} \log(4\pi \log(n)) \xrightarrow[n\to\infty]{} 1$$

and if we focus on the first inequality of System (2.7), we have that it holds if and only if

$$1 + \frac{\alpha_n^*}{\alpha_n}\left(\frac{\hat{M}}{\alpha_n^*} + \frac{1}{\alpha_n\alpha_n^*} \min_{k\in[n]} R_k\right) = 1 + \frac{\alpha_n^*}{\alpha_n}\left(\frac{\hat{M}+\beta_n^*}{\alpha_n^*} - \frac{\beta_n^*}{\alpha_n^*} + \frac{1}{\alpha_n\alpha_n^*} \min_{k\in[n]} R_k\right)$$

and the same relation holds for the second inequality.
Thus in the limit when $n \to \infty$, System (2.7) is

$$\begin{aligned} 1 + \frac{\alpha_n^*}{\alpha_n}\left(o_P(1) - 1 + \frac{1}{\alpha_n\alpha_n^*} \min_{k\in[n]} R_k\right) &\le \min_{k\in[n]} x_k \\ &\le 1 + \frac{\alpha_n^*}{\alpha_n}\left(o_P(1) - 1 + \frac{1}{\alpha_n\alpha_n^*} \max_{k\in[n]} R_k\right) \end{aligned} \tag{2.8}$$

We now state the following:

Lemma 2.1. *Under the assumptions of Theorem 2.1, the following convergence holds:*

$$\frac{\max_{k\in[n]} R_k}{\alpha_n\sqrt{2\log(n)}} \xrightarrow[n\to\infty]{\mathcal{P}} 0 \quad \text{and} \quad \frac{\min_{k\in[n]} R_k}{\alpha_n\sqrt{2\log(n)}} \xrightarrow[n\to\infty]{\mathcal{P}} 0.$$

Then, using this last Lemma and that $o_P(1) \to 0$ for $n \to \infty$, from System (2.8) we get $\min\limits_{k\in[n]} x_k = 1 - \frac{\alpha_n^*}{\alpha_n}$.
Thus, searching for a feasible solution, we have that $\min\limits_{k\in[n]} x_k > 0$ if and only if $\alpha_n > \alpha_n^*$, that is equivalent to $\alpha_n \ge (1+\varepsilon)\alpha_n^*$ for $\varepsilon > 0$, which is the thesis of the theorem.

Our aim then will be proving Lemma 2.1. To do that we proceed by steps:

1. we prove the validity of replacing R_k with a truncated version $\tilde{R}_k$: here we will use the property of Sub-Gaussianity which follows from Lipschitzianity and, since $R_k(A)$ fails to be Lipschitz (it has quadratic higher order terms), we provide a truncated version;

2. we prove that $A \mapsto \tilde{R}_k(A)$ is Lipschitz, so that there exists a real constant K such that $\left|\tilde{R}_k(A) - \tilde{R}_k(B)\right| \le K\|A-B\|_F$;

3. we give a uniform estimate for $\mathbb{E}\tilde{R}_k(A)$, so we find a constant $C > 0$ such that $\sup_{k\in[n]} \left|\mathbb{E}\tilde{R}_k(A)\right| \leq C$ for all $n \geq n_1$;

and then we will have the tools to conclude the proof of Lemma 2.1.

1) Truncation

Consider $R_k = \mathbf{e}_k^{\mathrm{T}} \sum_{l=2}^{\infty} \frac{1}{\alpha^{l-2}} \left(\frac{\Delta\circ A}{\sqrt{d}}\right)^l \mathbf{1}$ and take $\eta \in (0,1)$ and κ as in Proposition 2.1; we define the smooth function $\varphi : \mathbb{R}^+ \longrightarrow [0,1]$ such that

$$\varphi(x) := \begin{cases} 1 & \text{if } x \in [0, \kappa + 1 - \eta], \\ 0 & \text{if } x \geq \kappa + 1. \end{cases}$$

Denote by

$$\varphi_d(A) := \varphi\left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|\right)$$

then, according to Proposition 2.1, it holds that

$$\mathbb{P}\Big(\varphi_d(A) = 1\Big) = \mathbb{P}\left(\left\|\frac{\Delta\circ A}{\sqrt{d}}\right\| \leq \kappa\right) = 1 - \mathbb{P}\left(\left\|\frac{\Delta\circ A}{\sqrt{d}}\right\| > \kappa\right) \xrightarrow[n\to\infty]{} 1.$$

So if we introduce the truncated value

$$\tilde{R}_k(A) = \varphi_d(A) R_k(A)$$

then

$$\begin{aligned} \mathbb{P}\Big(\max_k R_k(A) \neq \max_k \tilde{R}_k(A)\Big) &\leq \mathbb{P}\Big(\exists k \in [n], R_k(A) \neq \tilde{R}_k(A)\Big) \\ &\leq \mathbb{P}(\varphi_d(A) < 1) \\ &\leq \mathbb{P}\left(\left\|\frac{\Delta\circ A}{\sqrt{d}}\right\| \geq \kappa\right) \xrightarrow[n\to\infty]{} 0, \end{aligned}$$

so we deduce that

$$\frac{\max_{k\in[n]} R_k(A) - \max_{k\in[n]} \tilde{R}_k(A)}{\alpha_n \alpha_n^*} \xrightarrow[n\to\infty]{\mathcal{P}} 0,$$

consequently to prove Lemma 2.1 it will be enough to prove that

$$\frac{\max_{k\in[n]} \tilde{R}_k(A)}{\alpha_n \alpha_n^*} \xrightarrow[n\to\infty]{\mathcal{P}} 0. \tag{2.9}$$

Analogous for the minimum.

2) Lipschitz property
Denote by $R_k(A) = \sum_{l=2}^{\infty} \rho_{k,l}(A)$ and $\tilde{R}_k(A) = \sum_{l=2}^{\infty} \tilde{\rho}_{k,l}(A)$, where

$$\rho_{k,l}(A) = \mathbf{e}_k^{\mathrm{T}} \frac{1}{\alpha^{l-2}} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^l \mathbf{1} \quad \text{and} \quad \tilde{\rho}_{k,l}(A) = \varphi_d(A) \rho_{k,l}(A). \tag{2.10}$$

We now state and give the proof of the following:

Lemma 2.2. *Let $\kappa > 0$ as in Proposition 2.1, $\delta \in (0,1)$ and n_0 such that $\forall n \geq n_0$,*

$$\frac{\kappa + 1}{\alpha_n} \leq 1 - \delta.$$

For $l \geq 2$ and $n \geq n_0$, the function $\tilde{\rho}_{k,l} : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$ is K_l-Lipschitz, i.e.

$$\left| \tilde{\rho}_{k,l}(A) - \tilde{\rho}_{k,l}(B) \right| \leq K_l \|A - B\|_F,$$

where $K_l = K_l(\kappa, n_0, \delta) > 0$ is a constant independent from k, d and $n \geq n_0$. Moreover, $K := \sum_{l \geq 2} K_l < \infty$. In particular, the function $\tilde{R}_k$ is K-Lipschitz:

$$\left| \tilde{R}_k(A) - \tilde{R}_k(B) \right| \leq K \|A - B\|_F.$$

Proof. The proof proceeds in three steps:

Step 1
First we consider the hermitization matrix of $\Delta \circ A$ in the case when $\mathcal{H}(\Delta \circ A)$ has a simple spectrum, i.e. each eigenvalue appears with multiplicity 1, so for Observation 2.1 we know that $\|\Delta \circ A\|$ is the largest eigenvalue of $\mathcal{H}(\Delta \circ A)$.

We compute :

$$\left\| \nabla \tilde{\rho}_{k,l}(A) \right\| = \sqrt{\sum_{i,j=1}^{n} \left| \partial_{ij} \tilde{\rho}_{k,l}(A) \right|^2}$$

and

$$\partial_{ij} \tilde{\rho}_{k,l}(A) = \partial_{ij} \big(\varphi_d(A) \rho_{k,l}(A) \big) = \big(\partial_{ij} \varphi_d(A) \big) \rho_{k,l}(A) + \varphi_d(A) \partial_{ij} \rho_{k,l}(A).$$

Therefore, defining $S_{1,ij} = \big(\partial_{ij} \varphi_d(A) \big) \rho_{k,l}(A)$ and $S_{2,ij} = \varphi_d(A) \partial_{ij} \rho_{k,l}(A)$, we have

$$\sum_{i,j=1}^{n} \left| \partial_{ij} \tilde{\rho}_{k,l}(A) \right|^2 \leq 2 \sum_{i,j=1}^{n} \left| S_{1,ij} \right|^2 + 2 \sum_{i,j=1}^{n} \left| S_{2,ij} \right|^2.$$

We first focus on the term $S_{1,ij}$.

We compute

$$\partial_{ij}\varphi_d(A) = \frac{1}{\sqrt{d}}\varphi'\Big(\Big\|\frac{\Delta\circ A}{\sqrt{d}}\Big\|\Big)\partial_{ij}\|\Delta\circ A\|.$$

Then if u and v are respectively the left and right normalized singular vectors associated to the largest singular value $\|\Delta\circ A\|$ of $\Delta\circ A$, we know that $\mathcal{H}(\Delta\circ A)w = \|\Delta\circ A\|w$, with $w = \begin{pmatrix} u \\ v \end{pmatrix}$ and $\|w\| = 2$.

Now we consider the following theorem, which proof can be found in [7]:

Theorem 2.2. *Let $B \in M_k$ and $E = (e_{ij}) \in M_k$ and suppose that λ is a simple eigenvalue of B. Let x and y be, respectively, right and left eigenvectors of B corresponding to λ. Then $\lambda(t)$ is differentiable at $t = 0$ and*

$$\frac{d\lambda(t)}{dt}\Big|_{t=0} = \frac{y^T E x}{y^T x}.$$

From this Theorem, denoting by $\partial_{ij} = \frac{\partial}{\partial A_{ij}}$, we deduce that $\|\Delta\circ A\|$ is differentiable and

$$\partial_{ij}\|\Delta\circ A\| = \begin{cases} \frac{1}{\|w\|}(u^{\mathrm{T}}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}v + v^{\mathrm{T}}\mathbf{e}_j\mathbf{e}_i^{\mathrm{T}}u) = u^{\mathrm{T}}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}v & \text{if } \Delta_{ij}\neq 0, \\ 0 & \text{else.} \end{cases}$$

Therefore

$$S_{1,ij} = \big(\partial_{ij}\varphi_d(A)\big)\rho_{k,l}(A) = \begin{cases} u^{\mathrm{T}}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}v\;\varphi'\Big(\frac{\|\Delta\circ A\|}{\sqrt{d}}\Big)\frac{\mathbf{1}}{\sqrt{d}}\rho_{k,l}(A) & \text{if } \Delta_{ij}\neq 0, \\ 0 & \text{else.} \end{cases}$$

Then, for $i\in[n]$, we denote by

$$\mathcal{I}_i = \{j\in[n],\ \Delta_{ij} = 1\}.$$

Note that $\mathrm{card}(\mathcal{I}) = d$ (in our hypothesis each species interacts exactly with d other species) and, remembering that u,v are unit vectors and the definition of $\rho_{k,l}(A)$, we have

$$\begin{aligned}
\sum_{i,j\in[n]}|\,S_{1,ij}\,|^2 &= \sum_{i\in[n]}\sum_{j\in\mathcal{I}_i}\Big|u^{\mathrm{T}}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}v\;\varphi'\Big(\Big\|\frac{\Delta\circ A}{\sqrt{d}}\Big\|\Big)\frac{\mathbf{1}}{\sqrt{d}}\rho_{k,l}(A)\Big|^2 \\
&\leq \Big|\varphi'\Big(\Big\|\frac{\Delta\circ A}{\sqrt{d}}\Big\|\Big)\frac{\mathbf{1}}{\sqrt{d}}\rho_{k,l}(A)\Big|^2\sum_{i\in[n]}|u^{\mathrm{T}}\mathbf{e}_i|\sum_{j\in[n]}|\mathbf{e}_j^{\mathrm{T}}v| \\
&= \Big|\varphi'\Big(\Big\|\frac{\Delta\circ A}{\sqrt{d}}\Big\|\Big)\frac{\mathbf{1}}{\sqrt{d}}\rho_{k,l}(A)\Big|^2 \\
&= \Big|\varphi'\Big(\Big\|\frac{\Delta\circ A}{\sqrt{d}}\Big\|\Big)\mathbf{e}_k^{\mathrm{T}}\frac{\mathbf{1}}{\alpha^{l-2}}\Big(\frac{\Delta\circ A}{\sqrt{d}}\Big)^l\frac{\mathbf{1}}{\sqrt{d}}\Big|^2. \qquad (2.11)
\end{aligned}$$

We observe that, since we are modeling the problem with $\Delta \circ A$ block matrix with only d non zero entries per row, then $(\Delta \circ A)^l$ remains a block matrix with only d non zero entries per row. We denote by

$$\mathcal{J}_{k,l} = \left\{ p \in [n], \left[(\Delta \circ A)^l \right]_{kp} \neq 0 \right\}$$

and by $\mathbf{1}^{\mathcal{J}_{k,l}}$ the $n \times 1$ vector with all zero entries except the ones corresponding to $\mathcal{J}_{k,l}$ that are set to 1. It holds then that $\|\mathcal{J}_{k,l}\| = \sqrt{d}$ and that $\mathbf{e}_k^{\mathrm{T}}(\Delta \circ A)^l \mathbf{1} = \mathbf{e}_k^{\mathrm{T}}(\Delta \circ A)^l \mathbf{1}^{\mathcal{J}_{k,l}}$.
Then we observe that $\varphi'\big(\| \frac{\Delta \circ A}{\sqrt{d}} \|\big) = 0$ if $\| \frac{\Delta \circ A}{\sqrt{d}} \| \geq \kappa + 1$, so

$$\left| \varphi'\left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \right) \right|^2 \leq \|\varphi'\|_\infty^2.$$

.

Proceeding with (2.11), we have

$$\begin{aligned}
&\left| \varphi'\left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \right) \mathbf{e}_k^{\mathrm{T}} \left(\frac{\Delta \circ A}{\alpha\sqrt{d}} \right)^{l-2} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^2 \frac{\mathbf{1}^{\mathcal{J}_{k,l}}}{\sqrt{d}} \right|^2 \\
&\qquad \leq \left| \varphi'\left(\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \right) \right|^2 \| \mathbf{e}_k^{\mathrm{T}} \|^2 \left\| \left(\frac{\Delta \circ A}{\alpha\sqrt{d}} \right)^{l-2} \right\|^2 \left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|^4 \left\| \frac{\mathbf{1}^{\mathcal{J}_{k,l}}}{\sqrt{d}} \right\|^2 \\
&\qquad \leq \|\varphi'\|_\infty^2 \left\| \frac{\Delta \circ A}{\alpha\sqrt{d}} \right\|^{2(l-2)} \left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|^4 \\
&\qquad \leq \|\varphi'\|_\infty^2 (1-\delta)^{2(l-2)} (1+\kappa)^4, \qquad (2.12)
\end{aligned}$$

where in the final inequality we used that κ is taken as in Proposition 2.1 so, with probability tending to 1, it holds that

$$\left\| \frac{\Delta \circ A}{\sqrt{d}} \right\| \leq \kappa \quad \text{and} \quad \left\| \frac{\Delta \circ A}{\alpha\sqrt{d}} \right\| \leq 1 - \delta.$$

Finally we obtained

$$\sum_{i,j=1}^{n} \left| S_{1,ij} \right|^2 \leq \|\varphi'\|_\infty^2 (1-\delta)^{2(l-2)} (1+\kappa)^4. \qquad (2.13)$$

Now we focus on the term $S_{2,ij}$.
It holds that if $j \notin \mathcal{I}_i$ then $\partial_{ij}\rho_{k,l}(A) = 0$, while if $j \in \mathcal{I}_i$:

$$\partial_{ij}\rho_{k,l}(A) = \frac{1}{\alpha^{l-2}(\sqrt{d})^l} \sum_{p=0}^{l-1} \mathbf{e}_k^{\mathrm{T}} (\Delta \circ A)^p \mathbf{e}_i \mathbf{e}_j^{\mathrm{T}} (\Delta \circ A)^{l-1-p} \mathbf{1}$$

so we compute

$$\begin{aligned}
&\sum_{i\in[n]}\sum_{j\in\mathcal{I}_i}\left|\,\partial_{ij}\rho_{k,l}(A)\,\right|^2\\
&\qquad\leq\frac{l}{\alpha^{2(l-2)}d^l}\Bigg(\sum_{i\in[n]}\sum_{j\in\mathcal{I}_i}\left|\,\mathbf{e}_k^{\mathrm{T}}(\Delta\circ A)^{l-1}\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}\mathbf{1}\,\right|^2\\
&\qquad\quad+\sum_{p=0}^{l-2}\sum_{i\in[n]}\sum_{j\in\mathcal{I}_i}|\mathbf{e}_k^{\mathrm{T}}(\Delta\circ A)^p\mathbf{e}_i\mathbf{e}_j^{\mathrm{T}}(\Delta\circ A)^{l-1-p}\mathbf{1}|^2\Bigg)\\
&\qquad=\frac{l}{\alpha^{2(l-2)}d^l}\Bigg(d\sum_{i\in[n]}\left|\,\big[(\Delta\circ A)^{l-1}\big]_{k,i}\,\right|^2\\
&\qquad\quad+d\sum_{p=0}^{l-2}\sum_{i\in[n]}\sum_{j\in\mathcal{I}_i}\left|\,\big[(\Delta\circ A)^p\big]_{k,i}\mathbf{e}_j^{\mathrm{T}}(\Delta\circ A)^{l-1-p}\frac{\mathbf{1}}{\sqrt{d}}\,\right|^2\Bigg)\\
&\qquad\leq\frac{l}{\alpha^{2(l-2)}d^{l-1}}\Bigg(\Big[(\Delta\circ A)^{l-1}\big((\Delta\circ A)^{l-1}\big)^{\mathrm{T}}\Big]_{k,k}\\
&\qquad\quad+\sum_{p=0}^{l-2}\sum_{i\in[n]}\left|\,\big[(\Delta\circ A)^p\big]_{k,i}\,\right|^2\sum_{j\in\mathcal{I}_i}\left|\,\mathbf{e}_j^{\mathrm{T}}(\Delta\circ A)^{l-1-p}\frac{\mathbf{1}}{\sqrt{d}}\,\right|^2\Bigg). \qquad (2.14)
\end{aligned}$$

We define $T := \sum_{j\in\mathcal{I}_i}\left|\mathbf{e}_j^{\mathrm{T}}(\Delta\circ A)^{l-1-p}\frac{\mathbf{1}}{\sqrt{d}}\right|^2$ and show that

$$T \leq \left\|\,\Delta\circ A\,\right\|^{2(l-1-p)}. \qquad (2.15)$$

Let $\mathbf{1}^{\mathcal{I}_i}$ be the $n\times 1$ vector with 0 entries everywhere except the ones belonging to $\mathcal{I}_i$ that are set to 1, and define $I_{\mathcal{I}_i} = diag\big(\mathbf{1}^{\mathcal{I}_i}(k), k\in[n]\big)$ so that

$$T = \frac{\mathbf{1}^{\mathrm{T}}}{\sqrt{d}}\big[(\Delta\circ A)^{l-1-p}\big]^{\mathrm{T}} I_{\mathcal{I}_i}(\Delta\circ A)^{l-1-p}\frac{\mathbf{1}}{\sqrt{d}}.$$

Notice that, since we are modeling with block matrices, $(\Delta\circ A)^{l-1-p} = (P_\tau\otimes\mathbf{1}_d\mathbf{1}_d^{\mathrm{T}})\circ B$ for some $\tau\in S_m$ and some $n\times n$ matrix B and there exists a $d\times d$ block B_i of $(\Delta\circ A)^{l-1-p}$ such that $\big[(\Delta\circ A)^{l-1-p}\big]^{\mathrm{T}} I_{\mathcal{I}_i}(\Delta\circ A)^{l-1-p}$ is a matrix with zero everywhere except a $d\times d$ block $B_i^{\mathrm{T}}B_i$ on the diagonal; so we have

$$T = \frac{\mathbf{1}_d^{\mathrm{T}}}{\sqrt{d}}B_i^{\mathrm{T}}B_i\frac{\mathbf{1}_d}{\sqrt{d}} \leq \|B_i^{\mathrm{T}}B_i\| \leq \|B_i\|^2 \leq \|(\Delta\circ A)^{l-1-p}\|^2 \leq \|(\Delta\circ A)\|^{2(l-p-1)}.$$

Therefore Equation (2.15) is proved and we can go back to (2.14) and have

$$\begin{aligned}
&\sum_{i\in[n]}\sum_{j\in\mathcal{I}_i}\left|\,\partial_{ij}\rho_{k,l}(A)\,\right|^2 &(2.16)\\
&\qquad\le\frac{l}{\alpha^{2(l-2)}d^{l-1}}\Bigg(\left\|\,(\Delta\circ A)^{l-1}\,\right\|^2\\
&\qquad\qquad+\sum_{p=0}^{l-2}\left[\left((\Delta\circ A)^p\right)^{\mathrm{T}}(\Delta\circ A)^p\right]_{kk}\|\Delta\circ A\|^{2(l-p-1)}\Bigg)\\
&\qquad\le\frac{l}{\alpha^{2(l-2)}d^{l-1}}\left(\|\Delta\circ A\|^{2(l-1)}+\sum_{p=0}^{l-2}\|\Delta\circ A\|^{2p}\|\Delta\circ A\|^{2(l-p-1)}\right)\\
&\qquad=\frac{l^2}{\alpha^{2(l-2)}d^{l-1}}\|\Delta\circ A\|^{2(l-1)}\\
&\qquad=l^2\left\|\frac{\Delta\circ A}{\alpha\sqrt{d}}\right\|^{2(l-2)}\left\|\frac{\Delta\circ A}{\sqrt{d}}\right\|^2. &(2.17)
\end{aligned}$$

Hence we get

$$\begin{aligned}
\sum_{i,j=1}^n|S_{2,ij}|^2&=\sum_{i\in[n]}\sum_{j\in\mathcal{I}_i}\left|\,\varphi_d(A)\partial_{ij}\rho_{k,l}(A)\,\right|^2\\
&\le l^2|\varphi_d(A)|^2\left\|\frac{\Delta\circ A}{\alpha\sqrt{d}}\right\|^{2(l-2)}\left\|\frac{\Delta\circ A}{\sqrt{d}}\right\|^2\\
&\le l^2(1-\delta)^{2(l-2)}(1+\kappa)^2. &(2.18)
\end{aligned}$$

Therefore combining (2.13) and (2.18) we have:

$$\begin{aligned}
\sum_{i,j=1}^n\left|\,\nabla\tilde{\rho}_{k,l}(A)\,\right|&\le\sqrt{2\sum_{i,j=1}^n\left|\,S_{1,ij}\,\right|^2+2\sum_{i,j=1}^n\left|\,S_{2,ij}\,\right|^2}\\
&\le\sqrt{2\|\varphi'\|_\infty^2(1-\delta)^{2(l-2)}(1+\kappa)^4+2l^2(1-\delta)^{2(l-2)}(1+\kappa)^2}\\
&\le 2(1-\delta)^{l-2}(1+\kappa)^2\left(\|\varphi'\|_\infty^2+l\right)=:K_l.
\end{aligned}$$

So we obtained the local estimation $|\nabla\tilde{\rho}_{k,l}(A)\,|\le K_l$, where K_l is independent from k, d, n, that is the Lipschitz property for the single matrix in the case when $\mathcal{H}(\Delta\circ A)$ has a simple spectrum.

Step 2

Next step we want to prove is is that the Lipschitz property holds along the segment $[A,B]$, with A and B matrices such that $\mathcal{H}(\Delta\circ A)$ and $\mathcal{H}(\Delta\circ B)$ have a simple spectrum.

We consider the interpolation matrix for $t\in[0,1]$

$$A_t=(1-t)A+tB.$$

Since the eigenvalues are continuous, then there exists $\varepsilon > 0$ such that $\mathcal{H}(\Delta \circ A_t)$ has a simple spectrum for $t \in [0, \varepsilon) \cup (1-\varepsilon, 1]$ and the number of eigenvalues of $\mathcal{H}(\Delta \circ A)$ remains constant for $t \in [0,1]$ except for a finite number of points $\{t_l : 1 \leq l \leq L\}$.
Now if we consider the interval (t_{l-1}, t_l) we have:

$$\begin{aligned}
\left| \tilde{\rho}_{k,l}(A_{t_l}) - \tilde{\rho}_{k,l}(A_{t_{l-1}}) \right| &= \left| \lim_{\tau \nearrow t_l} \int_{t_{l-1}}^{\tau} \frac{d}{dt} \tilde{\rho}_{k,l}(A_t)\, dt \right| \\
&= \left| \lim_{\tau \nearrow t_l} \int_{t_{l-1}}^{\tau} \nabla \tilde{\rho}_{k,l}(A_t) \circ \frac{d}{dt}(A_t)\, dt \right| \\
&\leq \lim_{\tau \nearrow t_l} \int_{t_{l-1}}^{\tau} \|\nabla \tilde{\rho}_{k,l}(A_t)\| \times \|B - A\|_F\, dt \\
&\leq K_l (t_l - t_{l-1)} \|B - A\|_F
\end{aligned}$$

and iterating this process for all the intervals we get

$$\begin{aligned}
\left| \tilde{\rho}_{k,l}(B) - \tilde{\rho}_{k,l}(A) \right| &\leq \sum_{l=1}^{L+1} \left| \tilde{\rho}_{k,l}(A_{t_l}) - \tilde{\rho}_{k,l}(A_{t_{l-1}}) \right| \\
&\leq \sum_{l=1}^{L+1} K_l (t_l - t_{l-1}) \|B - A\|_F \\
&= K_l \|B - A\|_F.
\end{aligned}$$

Thus we proved the Lipschitz property of the segment $[A, B]$.

Step 3
Finally we want to prove that, under the hypothesis over the model BPMM, the set of matrices $(\Delta \circ A)$ such that $\mathcal{H}(\Delta \circ A)$ has a simple spectrum is dense in the set of matrices $(\Delta \circ A, A \in \mathbb{R}^{n \times n})$.
We define

$$\Pi = P_\sigma \otimes I_d$$

where I_d is the $d \times d$ identity matrix, P_σ is the permutation matrix used to define the model BPMM and $\otimes$ is the Kronecker product, so Π is a $n \times n$ permutation matrix and it holds that $\Pi\Pi^{\mathrm{T}} = \Pi^{\mathrm{T}}\Pi = I_n$.
We also define

$$D_A = (\Delta \circ A)\Pi^{\mathrm{T}}$$

that is a block diagonal matrix with m $d \times d$ blocks $\left(A_{(\mu)}\right)_{\mu \in [m]}$ on the diagonal. We observe that $D_A \Pi = \Delta \circ A$ and that, since

$$D_A D_A^{\mathrm{T}} = (\Delta \circ A)\Pi^{\mathrm{T}}\Pi(\Delta \circ A)^{\mathrm{T}} = (\Delta \circ A)(\Delta \circ A)^{\mathrm{T}},$$

then D_A and $\Delta \circ A$ have the same singular values; therefore $\mathcal{H}(\Delta \circ A)$ and D_A have the same eigenvalues and, if they have simple spectrum, they have

it simultaneously simple.
Now let

$$A_{(\mu)} = U_{(\mu)}\Lambda_{(\mu)}V_{(\mu)}$$

be the singular value decomposition (SVD) of the blocks of matrix D_A, with $U_{(\mu)}$ and $V_{(\mu)}$ unitary matrix and $\Lambda_{(\mu)}$ diagonal matrix.
We consider a small ε-perturbation of $\Lambda_{(\mu)}$ into $\Lambda^{\varepsilon}_{(\mu)}$ such that all the $\Lambda^{\varepsilon}_{(\mu)}$'s have distinct diagonal elements, ε-close to the $\Lambda^{\varepsilon}_{(\mu)}$'s.
Denote by

$$A^{\varepsilon}_{(\mu)} = U_{(\mu)}\Lambda^{\varepsilon}_{(\mu)}V_{(\mu)}$$

and let D^{ε}_A be the block diagonal matrix with blocks $\left(A^{\varepsilon}_{(\mu)}\right)_{\mu\in[m]}$: then $\mathcal{H}(D^{\varepsilon}_A)$ is arbitrarily close to $\mathcal{H}(D_A)$ and has a simple spectrum.
We have that $D^{\varepsilon}_A\Pi$ is ε-close to $\Delta \circ A$ and it holds that if $\Delta_{ij} = 0$ then $(D^{\varepsilon}_A\Pi)_{ij} = 0$. If we define the matrix A^{ε} such that

$$[A^{\varepsilon}]_{ij} = \begin{cases} [D^{\varepsilon}_A\Pi]_{ij} & \text{if } \Delta_{ij} = 1, \\ A_{ij} & \text{else,} \end{cases}$$

then we have

$$[A^{\varepsilon} - A]_{ij} = \begin{cases} [D^{\varepsilon}_A\Pi]_{ij} - A_{ij} & \text{if } \Delta_{ij} = 1, \\ 0 & \text{else,} \end{cases}$$

so that

$$\|\Delta \circ A^{\varepsilon} - \Delta \circ A\|_F = \|A^{\varepsilon} - A\|_F \xrightarrow[\varepsilon\to 0]{} 0.$$

That is the proof of the density of the set of matrices $(\Delta \circ A)$ such that $\mathcal{H}(\Delta \circ A)$ has a simple spectrum in the set of matrices $(\Delta \circ A, A \in \mathbb{R}^{n\times n})$.

Finally we can conclude: if we consider the two matrices $\Delta \circ A$ and $\Delta \circ B$ given by our model BPMM and $D^{\varepsilon}_A = \Delta \circ A^{\varepsilon}$ and $D^{\varepsilon}_B = \Delta \circ B^{\varepsilon}$ then, for the continuity of $C \mapsto \rho_{\tilde{k,l}}(C)$, we have:

$$|\tilde{\rho}(B^{\varepsilon}) - \tilde{\rho}(B)| \le \tilde{\rho}(\|B^{\varepsilon} - B\|) \xrightarrow[\varepsilon\to 0]{} 0$$

$$|\tilde{\rho}(A^{\varepsilon}) - \tilde{\rho}(A)| \le \tilde{\rho}(\|A^{\varepsilon} - A\|) \xrightarrow[\varepsilon\to 0]{} 0$$

thus

$$\begin{aligned} |\tilde{\rho}_{k,l}(B) - \tilde{\rho}_{k,l}(A)| &\le |\tilde{\rho}_{k,l}(B^{\varepsilon}) - \tilde{\rho}_{k,l}(B)| + |\tilde{\rho}_{k,l}(A^{\varepsilon}) - \tilde{\rho}_{k,l}(A)| + |\tilde{\rho}_{k,l}(B^{\varepsilon}) - \tilde{\rho}_{k,l}(A^{\varepsilon})| \\ &\le K_l\|B^{\varepsilon} - A^{\varepsilon}\|_F \xrightarrow[\varepsilon\to 0]{} 0. \end{aligned}$$

This concludes the proof of the Lipschitz property of $A \mapsto \tilde{R}_k(A)$. $\square$

3) Uniform estimate for $\mathbb{E}\tilde{R}_k(A)$
We now state and prove two propositions that we will need to find a uniform estimate for $\mathbb{E}\tilde{R}_k(A)$.

Proposition 2.2. *Under the assumptions of Lemma 2.2, the following estimate holds true:*

$$\mathbb{E}\max_{k\in[n]}\big(\tilde{R}_k - \mathbb{E}\tilde{R}_k\big) \le K\sqrt{2\log(n)}.$$

Proof. By applying Tsirelson-Ibragimov-Sudakov inequality to $\tilde{R}_k(A)$, as we can see in [8], we obtain the following exponential estimate:

$$\mathbb{E}e^{\lambda(\tilde{R}_k(A)-\mathbb{E}\tilde{R}_k(A))} \le e^{\frac{\lambda^2K^2}{2}} \quad \forall\lambda\in\mathbb{R}.$$

Now using this estimate and the Jensen inequality we obtain:

$$\begin{aligned}
\exp\left(\lambda\,\mathbb{E}\max_{k\in[n]}\big(\tilde{R}_k - \mathbb{E}\tilde{R}_k\big)\right) &\le \mathbb{E}\exp\left(\lambda\max_{k\in[n]}\big(\tilde{R}_k - \mathbb{E}\tilde{R}_k\big)\right)\\
&\le \sum_{k=1}^{n}\mathbb{E}e^{\lambda(\tilde{R}_k(A)-\mathbb{E}\tilde{R}_k(A))}\\
&\le \sum_{k=1}^{n} e^{\frac{\lambda^2K^2}{2}}\\
&= n\,e^{\frac{\lambda^2K^2}{2}}. \qquad (2.19)
\end{aligned}$$

Hence for $\lambda > 0$,

$$\mathbb{E}\max_{k\in[n]}\big(\tilde{R}_k - \mathbb{E}\tilde{R}_k\big) \le \frac{1}{\lambda}\log\big(n\,e^{\frac{\lambda^2K^2}{2}}\big) = \frac{1}{\lambda}\left[\log n + \frac{\lambda^2K^2}{2}\right] =: \Phi(\lambda).$$

Now we optimize in λ: we compute

$$\Phi'(\lambda) = \frac{-\log n}{\lambda^2} + \frac{K^2}{2} = 0 \iff \lambda^* = \frac{\sqrt{2\log n}}{K},$$

so we get $\Phi(\lambda^*) = K\sqrt{2\log n}$, that is the desired estimate. □

Proposition 2.3. *Under the assumptions of Theorem 2.1, there exists $n_1 \in \mathbb{N}$ and a constant $C > 0$ such that for all $n > n_1$*

$$\sup_{k\in[n]}\big|\,\mathbb{E}\tilde{R}_k(A)\,\big| \le C.$$

Proof. We recall that $\Delta \circ A$ is a $n \times n$ block permutation matrix with m blocks $(A^{(\mu)})_{\mu \in [m]}$ of size $d \times d$.
For a given block $A^{(\mu)}$ we denote by $\mu_1, \ldots, \mu_d$ the d indices corresponding to the rows of the block $A^{(\mu)}$ in $\Delta \circ A$. We denote by $\mathbf{1}^{(\mu)}$ the $n \times 1$ vector with ones for the indices $(\mu_i)_{i \in [d]}$ and zeros elsewhere and by exchangeability we have

$$\mathbb{E}\tilde{R}_{\mu_k}(A) = \mathbb{E}\tilde{R}_{\mu_1}(A)$$

for all $k \in [d]$.

Therefore we have

$$\begin{aligned}
\left| \mathbb{E}\tilde{R}_{\mu_k}(A) \right| &= \left| \frac{1}{d} \sum_{i=1}^{d} \mathbb{E}\tilde{R}_{\mu_i}(A) \right| \\
&= \left| \frac{1}{d} \sum_{i=1}^{d} \mathbb{E}\left(\varphi_d(A) \mathbf{e}_{\mu_i}^{\mathrm{T}} \sum_{l=2}^{\infty} \frac{1}{\alpha^{l-2}} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^l \mathbf{1} \right) \right| \\
&= \left| \frac{1}{d} \mathbb{E}\left(\varphi_d(A) \mathbf{1}^{(\mu)\mathrm{T}} \sum_{l=2}^{\infty} \frac{1}{\alpha^{l-2}} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^l \mathbf{1} \right) \right| \\
&\leq \mathbb{E} \left| \varphi_d(A) \sum_{l=2}^{\infty} \frac{\mathbf{1}^{(\mu)\mathrm{T}}}{\alpha^{l-2}\sqrt{d}} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^l \frac{\mathbf{1}}{\sqrt{d}} \right|. \qquad (2.20)
\end{aligned}$$

We observe that $(\Delta \circ A)^l$ is a block matrix composed of m blocks $d \times d$ and among the d row

$$\left(\left[(\Delta \circ A)^l \right] \right)_{i \in \{\mu_1, \ldots, \mu_d\}, j \in [n]}$$

there exist $\nu_1, \ldots \nu_d$ (consecutive) indices such that the only non-zero entries are

$$\left(\left[(\Delta \circ A)^l \right] \right)_{i \in \{\mu_1, \ldots, \mu_d\}, j \in \{\nu_1, \ldots, \nu_d\}}$$

thus, denoting by $\mathbf{1}^{(\nu)}$ the $n \times 1$ vector of ones for the indices $(\nu_i)_{i \in [d]}$ and zero elsewhere, we have

$$\mathbf{1}^{(\mu)\mathrm{T}} (\Delta \circ A)^l \mathbf{1} = \mathbf{1}^{(\mu)\mathrm{T}} (\Delta \circ A)^l \mathbf{1}^{(\nu)}.$$

So we have that

$$\begin{aligned}
\frac{1}{\alpha^{l-2}} \left| \frac{\mathbf{1}^{(\mu)\mathrm{T}}}{\sqrt{d}} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^l \frac{\mathbf{1}}{\sqrt{d}} \right| &= \frac{1}{\alpha^{l-2}} \left| \frac{\mathbf{1}^{(\mu)\mathrm{T}}}{\sqrt{d}} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^l \frac{\mathbf{1}^{(\nu)}}{\sqrt{d}} \right| \\
&\leq \frac{1}{\alpha^{l-2}} \left\| \frac{\mathbf{1}^{(\mu)\mathrm{T}}}{\sqrt{d}} \right\| \left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|^l \left\| \frac{\mathbf{1}^{(\nu)}}{\sqrt{d}} \right\| \\
&\leq \left\| \frac{\Delta \circ A}{\alpha\sqrt{d}} \right\|^{l-2} \left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|^2.
\end{aligned}$$

Then, taking $\kappa > 0$ as in Proposition 2.1, $\delta \in (0,1)$, $n_0 \in \mathbb{N}$ as in Lemma 2.2, it holds that

$$\begin{aligned}
\sum_{l=2}^{\infty} \frac{1}{\alpha^{l-2}} & \left| \varphi_d(A) \frac{\mathbf{1}^{(\mu)\mathrm{T}}}{\sqrt{d}} \left(\frac{\Delta \circ A}{\sqrt{d}} \right)^l \frac{\mathbf{1}}{\sqrt{d}} \right| \\
& \leq \varphi_d(A) \sum_{l=2}^{\infty} \left\| \frac{\Delta \circ A}{\alpha\sqrt{d}} \right\|^{l-2} \left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|^2 \\
& = \varphi_d(A) \left\| \frac{\Delta \circ A}{\sqrt{d}} \right\|^2 \sum_{l=0}^{\infty} \left\| \frac{\Delta \circ A}{\alpha\sqrt{d}} \right\|^l \\
& \leq (1+\kappa)^2 \sum_{l=0}^{\infty} (1-\delta)^l \\
& = \frac{(1+\kappa)^2}{\delta}.
\end{aligned}$$

Now we plug this estimate into (2.20) and get

$$\left| \mathbb{E}\tilde{R}_{\mu_k}(A) \right| \leq \frac{(1+\kappa)^2}{\delta}.$$

Since this estimate is uniform over $\mu_1, \ldots, \mu_d$ and over all the blocks $\big(A^{(\mu)}\big)$, then the proposition is proved with $C = \frac{(1+\kappa)^2}{\delta}$.

$\square$

Finally we are ready to prove Lemma 2.1 and, as shown previously, it is sufficient to prove the convergence of the term in (2.9):

Proof. Since $\tilde{R}_i(A)$'s are exchangeable, $\mathbb{E}\tilde{R}_k(A) = \mathbb{E}\tilde{R}_1(A)$. Notice that $\max\limits_{k\in[n]} \tilde{R}_k(A) - \tilde{R}_1(A) \geq 0$, hence using the Markov inequality we have

$$\begin{aligned}
\mathbb{P}\bigg\{ & \frac{\max_{k\in[n]} \tilde{R}_k(A) - \tilde{R}_1(A)}{\alpha\sqrt{2\log n}} \geq \varepsilon \bigg\} \\
& \leq \frac{\mathbb{E}\big(\max_{k\in[n]} \tilde{R}_k(A) - \tilde{R}_1(A)\big)}{\varepsilon\alpha\sqrt{2\log n}} \\
& = \frac{\mathbb{E}\big(\max_{k\in[n]} (\tilde{R}_k(A) - \mathbb{E}\tilde{R}_k(A) + \mathbb{E}\tilde{R}_k(A)) - \tilde{R}_1(A)\big)}{\varepsilon\alpha\sqrt{2\log n}} \\
& = \frac{\mathbb{E}\big(\max_{k\in[n]} (\tilde{R}_k(A) - \mathbb{E}\tilde{R}_k(A))\big)}{\varepsilon\alpha\sqrt{2\log n}} \\
& \leq \frac{K\sqrt{2\log n}}{\varepsilon\alpha\sqrt{2\log n}} \\
& = \frac{K}{\varepsilon\alpha}
\end{aligned}$$

by Proposition 2.2. Therefore we have that

$$\frac{\max_{k\in[n]} \tilde{R}_k(A) - \tilde{R}_1(A)}{\alpha\sqrt{2\log n}} \xrightarrow[n\to\infty]{\mathcal{P}} 0. \tag{2.21}$$

Now we show that

$$\frac{\tilde{R}_1(A)}{\alpha\sqrt{2\log n}} \xrightarrow[n\to\infty]{\mathcal{P}} 0. \tag{2.22}$$

By Proposition 2.3 we have the uniform estimate $\sup_{k\in[n]} \left|\mathbb{E}\tilde{R}_k(A)\right| \leq C$, so we have $\mathbb{E}\tilde{R}_1(A) = \mathcal{O}(1)$ and then

$$\frac{\mathbb{E}\tilde{R}_1(A)}{\alpha\sqrt{2\log n}} \xrightarrow[n\to\infty]{\mathcal{P}} 0.$$

Applying Poincare's inequality in its extension to Lipschitz functionals as shown in [9] to the Lipschitz functional $A \mapsto \tilde{R}_1(A)$, that is Lipshitz by the previous proof, we can bound $\tilde{R}_1(A)$'s variance by K^2 and using the Chebyschev inequality, we obtain

$$\begin{aligned}
\mathbb{P}\left(\left|\frac{\tilde{R}_1(A) - \mathbb{E}\tilde{R}_1(A)}{\alpha\sqrt{2\log n}}\right| > \delta\right) &= \mathbb{P}\left(\left|\tilde{R}_1(A) - \mathbb{E}\tilde{R}_1(A)\right| > \delta\alpha\sqrt{2\log n}\right) \\
&\leq \frac{Var\left(\tilde{R}_1(A)\right)}{2\delta^2\alpha^2\log n} \\
&\leq \frac{K^2}{2\delta^2\alpha^2\log n} \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

Hence by this convergence and since $\mathbb{E}\tilde{R}_1(A) = \mathcal{O}(1)$, Equation (2.22) holds and combining it with (2.21) finally yields:

$$\frac{\max_{k\in[n]} \tilde{R}_k(A)}{\alpha\sqrt{2\log n}} \xrightarrow[n\to\infty]{\mathcal{P}} 0.$$

Therefore, since we proved that the convergence of (2.9) holds, we have established the first part of Lemma 2.1 and one can prove the second assertion similarly. □

Since we know that Theorem 2.1 follows from this Lemma, then the proof is concluded.

Chapter 3

Stability of equilibria for ecosystems

Aside from the question of feasibility, arises the question of *stability* for a large complex system describing the time evolution of the abundances of the various species of a foodweb: how likely a perturbation of the solution will return to the equilibrium? In this chapter we investigate this problem and obtain two main results: first we find the conditions for the existence of a unique, globally stable and non-negative solution for the system under study; then, restricting to the case of feasibility, we find that if a feasible solution exists, then it is globally stable.

3.1 Preliminary notions

In this section we highlight some notions that will be necessary in the following analysis of stability. We first distinguish the concepts of stability, asymptotic stability, global stability and Volterra-Lyapunov stability.
Given $f : \Lambda \subseteq \mathbb{R}^n \to \mathbb{R}$, we consider the general ordinary differential equation:

$$\dot{x} = f(x)$$

and suppose that x^* is an equilibrium, i.e. the point satisfies $f(x^*) = 0$. Now we take an initial point $x(0)$ in the neighborhood of x^* and define:

Definition 3.1. (Stability). *An equilibrium x^* is said to be stable if for any neighborhood U of x^*, there exists a neighborhood W of x^* such that any orbit initiating in W at time $t = 0$ remains in U for all $t \geq 0$ (i.e., $x(0) \in W$ implies $x(t) \in U$ for all $t \geq 0$). It is said to be* asymptotically stable *if it is stable and the orbit converges to x^* (i.e., $x(t) \to x^*$ for all $x(0) \in W$ as $t \to \infty$).*

Observation 3.1. *If x^* is not stable, it is said to be unstable. Note that x^* is not stable if $x(t)$ does not remain in U, even if $x(t) \to x^*$ as $t \to \infty$.*

We define the basin of attraction *of x^* as the set of points $x(0)$ satisfying $x(t) \to x^*$ as $t \to \infty$.*

Definition 3.2. (Global stability). *When the basin of attraction of x^* is the whole state space and x^* is stable, x^* is said to be globally stable.*

In the following section we will focus on the analysis of the stability of Equation (2.2) and we will search for an equilibrium solution x_n living in the state place $\left(\mathbb{R}^+ \setminus \{0\}\right)^n$, so we will say that x_n is a globally stable equilibrium if it is asymptotically stable and the neighborhood W can be taken as the whole state place $\left(\mathbb{R}^+ \setminus \{0\}\right)^n$.

Now we introduce a new concept of stability for matrices and state Takeuchi and Adachi's Theorem, that we will need in the proof of a theorem in the next section:

Definition 3.3. (Volterra-Lyapunov stability). *A $n \times n$ real matrix B is Volterra-Lyapunov stable if there exists a $n \times n$ matrix D such that $DB + B^{\mathrm{T}}D$ is negative definite.*

Theorem 3.1. (Takeuchi and Adachi). *Let $A = (a_{ij})$ be a $n \times n$ real matrix. If A is Volterra-Lyapunov stable, the LV system*

$$\dot{x_i} = x_i\left(b_i + \sum_{j=1}^{n} a_{ij}x_j\right), \quad i = 1, \dots, n$$

has a non-negative and globally stable equilibrium point x^* for each $b \in \mathbb{R}^n$.

The proof of this Theorem can be found in [11].

Now we state two theorems that we will need in the next section: first we recall the Spectral method for Lyapunov stability, that we are going to use as a condition to study the stability of our system by investigating the eigenvalues of its Jacobian matrix; then we claim Bauer and Fike's theorem, that we will use in the next section to compare the spectra of the matrices $\mathcal{J}(x_n) = \mathrm{diag}(x_n)$ and of the Jacobian matrix of the system evaluated in x_n.

Theorem 3.2. (Spectral method for Lyapunov stability). *Consider the non linear vector field $X \in C^2(\mathbb{R}^m, \mathbb{R}^m)$ and the differential equation $\dot{x} = X(x)$ such that $x = 0$ is an equilibrium, then:*

1. *if $\mathrm{Re}(\mathrm{Spect}(X'(x))) < 0$, then x^* is an asymptotically stable equilibrium for $\dot{x} = X(x)$;*
2. *if there exists j' such that $\mathrm{Re}(j') > 0$, then x^* is unstable;*

3. *if* $\mathrm{Re}(\mathrm{Spect}(X'(x))) \leq 0$ *and there exists* j' *such that* $\mathrm{Re}(j') = 0$ *we can't deduce if the equilibrium is stable or not.*

Theorem 3.3. (Bauer and Fike). *Let* $A \in M_n$ *be diagonalizable, and suppose that* $A = S\Lambda S^{-1}$*,in which* S *is non singular and* Λ *is diagonal. Let* $E \in M_n$ *and let* $\|\cdot\|$ *be a matrix norm on* M_n *that is induced by an absolute norm on* $\mathbb{C}^n$*. If* λ *is an eigenvalue of* $A+E$*, there is an eigenvalue* μ *of* A *such that*

$$|\lambda - \mu| \leq \|S\|\|S^{-1}\|\|E\| = \kappa(S)\|E\|$$

in which $\kappa(\cdot)$ *is the condition number with respect to the matrix norm* $\|\cdot\|$*.*

3.2 Stability results

We complement the result of Theorem 2.1 by addressing the question of stability in the context of a Lotka-Volterra system and prove that under the domain of positivity of x_n, so under the condition $\alpha_n \geq (1+\varepsilon)\alpha_n^*$, feasibility and global stability occur simultaneously.
We recall that for the Spectral method of Lyapunov stability the equilibrium solution x_n of the LV System (2.2) is stable if the Jacobian matrix of the system evaluated in x_n, that is

$$\mathcal{J}(x_n) = \mathrm{diag}(x_n)\big(-I_n + M_n\big) \tag{3.1}$$

has all its eigenvalues with negative real part.

Now we state and give the proof of a theorem which gives the conditions for the existence of a unique, globally stable and non-negative solution for System (2.1):

Theorem 3.4. *Let* $d_n \geq \log(n)$*,* $\alpha_n \xrightarrow[n\to\infty]{} \infty$*, and* Δ_n *the adjacency matrix of a* d_n*-regular graph.*
Then, with probability going to one as $n \to \infty$*, Equation* (2.1) *admits a unique non-negative solution* x_n*.*
Moreover, this solution is a globally stable equilibrium.

Proof. We first prove that the matrix $M - I$ is Volterra-Lyapunov stable, so that there exists a positive definite diagonal matrix D such that $DB + B^{\mathrm{T}}D$ is negative definite: we take $D = I$ and then $I(M-I) + (M-I)^{\mathrm{T}}I = M + M^{\mathrm{T}} - 2I$. Since $M + M^{\mathrm{T}}$ is hermitian, the condition that the matrix $M + M^{\mathrm{T}} - 2I$ has all negative eigenvalues is satisfied if the spectral radius $\rho(M + M^{\mathrm{T}}) < \rho(2I) = 2 \iff \rho(M) < 1$.
Since for every matrix norm $\|\cdot\|$ and for every matrix A it holds that $\rho(A) \leq \|A\|$, then

$$\mathbb{P}\big(\rho(M) < 1\big) \geq \mathbb{P}\big(\|M\| < 1\big) \xrightarrow[n\to\infty]{} 1$$

for Proposition 2.1.
Therefore with probability tending to one matrix $M + M^{\mathrm{T}} - 2I$ is negative definite, so $M - I$ is Lyapunov stable.
Hence, according to Takeuchi and Adachi's theorem, with probability tending to one as $n \to \infty$, this LV system has a unique non-negative and globally stable equilibrium. □

This result holds under hypothesis that are less restrictive than the ones of Theorem 2.1 and gives the conditions for the existence of a unique, globally stable and non-negative solution: therefore the solution we are considering may have zero components (corresponding to vanishing species).
If now we restrict our hypothesis assuming the BPMM and work under the domain of positivity of the solutions, we have the following:

Proposition 3.1. *Let $d_n \geq \log(n)$, $\alpha_n \xrightarrow[n\to\infty]{} \infty$, and assume that Δ_n is given by the model BPMM. Denote by Σ_n the spectrum of the Jacobian matrix $\mathcal{J}(x_n)$ given by (3.1).*
Assume that there exists $\varepsilon > 0$ such that eventually $\alpha_n \geq (1+\varepsilon)\alpha_n^*$. Then:

1. *The probability that the equilibrium x_n is feasible and globally stable converges to one,*

2. *The spectrum Σ_n asymptotically coincides with $-diag(x_n)$ in the sense that:*
$$\max_{\lambda\in\Sigma_n} \min_{k\in[n]} |\lambda + x_k| \xrightarrow[n\to\infty]{\mathcal{P}} 0,$$

3. *Moreover,*
$$\max_{\lambda\in\Sigma_n} \mathrm{Re}(\lambda) \leq -(1-l^+) + o_P(1) \quad \textit{where} \quad l^+ := \limsup_{n\to\infty} \frac{\alpha_n^*}{\alpha_n} < 1. \tag{3.2}$$

Proof.

1. Under these hypothesis, Theorem 3.4 holds, so there exists a unique, globally stable and non-negative equilibrium solution to Equation (2.2). Moreover, since there exists $\varepsilon > 0$ such that $\alpha_n > (1+\varepsilon)\alpha_n^*$, then, by Theorem 2.1, x_n is a feasible solution with probability tending to one for $n \to \infty$.

2. We first establish the following estimates
$$\begin{cases} \min\limits_{k\in[n]} x_k \geq 1 - l^+ - o_P(1) \\ \max\limits_{k\in[n]} x_k \leq 1 + l^+ + o_P(1). \end{cases} \tag{3.3}$$

The first inequality comes from Proposition 2.1, where we found that

$$\min_{k\in[n]} x_k \geq 1 + \frac{\alpha_n^*}{\alpha_n}\bigg(-1 + o_P(1) + \frac{1}{\alpha_n\alpha_n^*}\min_{k\in[n]} R_k\bigg)$$

and so, according to Lemma 2.1, we obtain the result.
In the same way from the decomposition of x_n as

$$x_k = 1 + \frac{Z_k}{\alpha} + \frac{R_k}{\alpha^2}$$

we have

$$\begin{aligned} 1 + \frac{\alpha_n^*}{\alpha_n}\bigg(o_P(1) - 1 + \frac{1}{\alpha_n\alpha_n^*}\min_{k\in[n]} R_k\bigg) &\leq \max_{k\in[n]} x_k \\ &\leq 1 + \frac{\alpha_n^*}{\alpha_n}\bigg(o_P(1) - 1 + \frac{1}{\alpha_n\alpha_n^*}\max_{k\in[n]} R_k\bigg). \end{aligned}$$

Therefore, since for the previous proofs it holds that $\frac{M_n-\beta_n^*}{\alpha_n^*} \xrightarrow[n\to\infty]{\mathcal{P}} 0$ and $\frac{\max_{k\in[n]}R_k}{\alpha_n^*\alpha_n} \xrightarrow[n\to\infty]{\mathcal{P}} 0$ and we have that $\frac{\beta_n^*}{\alpha_n^*} = o_P(1)$ (where we recall that in general $\alpha = o_P(1)$ if and only if $\mathbb{P}(\alpha \ll 1) = 1$), then we get

$$\begin{aligned} \max_{k\in[n]} x_k &\leq 1 + \frac{M_n}{\alpha_n} + \frac{\max_{k\in[n]}R_k}{\alpha_n^2} \\ &= 1 + \frac{\alpha_n^*}{\alpha_n}\bigg(\frac{M_n - \beta_n^*}{\alpha_n^*} + \frac{\beta_n^*}{\alpha_n^*} + \frac{\max_{k\in[n]}R_k}{\alpha_n^*\alpha_n}\bigg) \\ &= 1 + l^+ + o_P(1), \end{aligned} \tag{3.4}$$

so the second inequality is also established.
Now we recall the definition given in (3.1) of $J(x_n)$ and we use the Bauer and Fike's Theorem to compare the spectra of the matrices $\mathcal{D}(x_n) = \text{diag}(x_n)$ and $\mathcal{J}(x_n) = \mathcal{D}(x_n)+\mathcal{E}(x_n)$ with $\mathcal{E}(x_n) = \text{diag}(x_n)M_n$. Since $\text{diag}(x_n)$ is a diagonal matrix and the elements on the diagonal are the components of vector x_n, then the eigenvalues of $\mathcal{D}(x_n)$ are $-x_k$ for $k \in [n]$. So if λ is an eigenvalue of $\mathcal{J}(x_n) = \mathcal{D}(x_n) + \mathcal{E}(x_n)$, then for Bauer and Fike's Theorem there exists an eigenvalue $-x_k$ of

$\mathcal{D}(x_n)$ such that

$$\begin{aligned} |\lambda + x_k| &\leq ||\mathcal{E}(x_n)|| \\ &= \left\| \mathrm{diag}(x_n) \frac{\Delta_n \circ A_n}{\alpha_n \sqrt{d}} \right\| \\ &\leq \frac{1}{\alpha_n} ||\mathrm{diag}(x_n)|| \left\| \frac{\Delta_n \circ A_n}{\sqrt{d}} \right\| \\ &\leq \frac{1}{\alpha_n} (1 + l + o_P(1))(2 + o_p(1)) \\ &= o_P(1). \end{aligned} \tag{3.5}$$

3. we have $\mathrm{Re}(\lambda) + x_k \leq |\lambda + x_k| = o_P(1)$ so using the first estimate of (3.3), we have $\mathrm{Re}(\lambda) \leq -\min_{k \in [n]} x_k + o_P(1) \leq -(1 - l^+) + o_P(1)$.

$\square$

As a consequence of (3.2), since $l^+ < 1$, we have that $\max_{\lambda \in \Sigma_n} \mathrm{Re}(\lambda) < 0$, so, for the Spectral method for Lyapunov stability, x_n is an asymptotically stable equilibrium and for any $x_n(0) \in (\mathbb{R}^{+*})^n$, the orbit $x_n(t)$ converges to the equilibrium x_n at an exponential convergence rate.

Conclusions

The complexity vs. stability issue is a topic of major importance within the mathematical modeling of large ecosystems. Sometimes referred to as the complexity/stability paradox, this issue has been central in ecology at least since the seminal work of Robert May. May's approach starts from the following consideration: the species interaction networks - which may be very complex - is the key feature shaping the dynamics of the system, however, due to the complexity of the network and the huge number of degrees of freedom involved, the interaction strengths and the network itself cannot be determined or measured. To overcome these difficulties, May proposed a stochastic approach, and modeled the interaction network as random matrix A whose entries are i.i.d. variables set as $\mathcal{N}(0,1)$. The matrix A is meant to describe the effect of each species on the others. Within this framework, May studied the stability of the linearized system $\dot{x} = Ax$, and proved that it tends to be unstable as the number of species gets large or for increasing connectivity. Thus, according to May's model, large/complex ecosystems should be unstable. Such a conclusion collides with nature where complexity and stability are proportionally linked. The contrast between (a simple and with reasonable assumptions) model and empirical observations is known as the complexity/stability paradox or May's paradox. What generates May's paradox is that his analysis neglects the structure of the network, which cannot be regarded as random but rather should be considered as the result of a long optimization process due to evolutive pressure. Therefore when we try to describe living systems we have to include some structure of the interaction network, which is believed to be the feature driving the dynamics of ecosystems and that ultimately brings to the emergence of time or spatial patterns in ecosystems. Thus in the second chapter of the thesis we focused on one of these, sparsity, and studied large sparse ecosystems. First of all we presented some definitions and recollected some results from matrix theory; then we analyzed the question of feasibility of a Lotka-Volterra ecosystem: the equation under investigation is

$$x_n = \mathbf{1}_n + M_n x_n$$

where x_n is the vector of the abundances of the species and M_n is a sparse random matrix defined as

$$M_n = \frac{\Delta_n \circ A_n}{\alpha_n \sqrt{d_n}},$$

where the sparsity is encoded by the block permutation matrix Δ_n based on an underlying d_n-regular graph, and the randomness by i.i.d. random variables for the non-null entries of the matrix of interactions A_n, α_n is a positive sequence and $1/\sqrt{d_n}$ is a standard normalization term.
Our main conclusion is that a sharp phase transition occurs at $\alpha_n^* = \sqrt{2\log(n)}$: above this threshold a feasible solution exists while below it it does not. Our approach of proof crucially relies on the technical assumption over the matrix Δ_n given by BPMM, which somehow concentrates the non-null entries of the sparse interaction matrix into localized blocks.
Finally in the third chapter we concentrated on the question of stability, first recalling some notions useful in the subsequent analysis and then showing that under the hypothesis of Theorem 2.1 feasibility and global stability occur simultaneously.

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